



On the (strong) persistence property, (nearly) normally torsion-freeness, and normality of monomial ideals with a view toward Combinatorics

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Présentée par Mehrdad NASERNEJAD

Intitulé

Sur la (forte) propriété de persistance, (presque) normalement sans torsion, et normalité des idéaux monomiaux en vue de la combinatoire

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On the (strong) persistence property, (nearly) normally torsion-freeness,
and normality of monomial ideals with a view toward Combinatorics

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November 2023

Abstract

The main aim of this work is to study several topics in monomial ideals. Generally, monomial ideals play a fundamental role in investigating the connection between commutative algebra and combinatorics. The relation between these two disciplines allows us to utilize techniques and methods in commutative algebra to solve combinatorial problems, and vice versa. So commutative algebraists have started exploring the properties of finite simple (hyper)graphs through monomial ideals. In particular, the study of square-free monomial ideals from both the algebraic and combinatorial points of view has become a very energetic and lively field of research in commutative algebra. In this direction, we focus in this text on some titles of combinatorial commutative algebra, which are related to the persistence property, strong persistence property, symbolic strong persistence property, normally torsion-freeness, nearly normally torsion-freeness, normality, and ideals arising from graphs and hypergraphs.

Key Words: Monomial ideals, Associated primes, Persistence property, Normally torsion-freeness, Normality, Graphs, Hypergraphs.

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Introduction

Let R be a commutative Noetherian ring and I be an ideal of R . A well-known result of Brodmann [21] showed that the sequence $\{\text{Ass}_R(R/I^k)\}_{k \geq 1}$ of associated prime ideals is stationary for large k . That is, there exists a positive integer k_0 such that $\text{Ass}_R(R/I^k) = \text{Ass}_R(R/I^{k_0})$ for all $k \geq k_0$. The minimal such k_0 is called the *index of stability* of I and $\text{Ass}_R(R/I^{k_0})$ is called the *stable set* of associated prime ideals of I , which is denoted by $\text{Ass}^\infty(I)$. A natural question arises in the context of Brodmann's Theorem is as follows:

(*) Is it true that $\text{Ass}_R(R/I) \subseteq \text{Ass}_R(R/I^2) \subseteq \cdots \subseteq \text{Ass}_R(R/I^k) \subseteq \cdots$?

McAdam [86] presented an example which says, in general, the above question has negative answer. We say that an ideal I of R satisfies the *persistence property* if it holds true in (*), i.e., $\text{Ass}_R(R/I^k) \subseteq \text{Ass}_R(R/I^{k+1})$ for all $k \geq 1$. Along this argument, an ideal I of R has the *strong persistence property* if $(I^{k+1} :_R I) = I^k$ for all $k \geq 1$. Ratliff showed in [113] that $(I^{k+1} :_R I) = I^k$ for all large k . Note that the strong persistence property implies the persistence property, however, the converse is not true.

Suppose now that I is a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K and $x_1, \dots, x_n$ are indeterminates. Question (*) does not have an affirmative answer for all monomial ideals, see [61] and [95] for counterexamples. Moreover, Kaiser, Stehlík, and Škrekovski [76] showed that not all square-free monomial ideals have the persistence property. However, recently, by applying combinatorial methods, several papers have been published for finding the classes of monomial ideals which satisfy the persistence property, refer to [50], [45], and [66] for more information. So, we have the following subquestion:

(†) Which classes of monomial ideals have the persistence property?

Question (†) has affirmative answers in the following cases:

- (1) I is the edge ideal of a simple graph (see [88, Theorem 2.15]).
- (2) I is the cover ideal of a perfect graph (see [45, Corollary 5.11]).
- (3) I is a polymatroidal ideal (see [66, Proposition 3.3]).

(4) I is the edge ideal of a finite graph with loops (see [118, Corollary 2]).

Furthermore, it is known by [88] that all edge ideals of finite simple graphs satisfy the strong persistence property. It has already been shown in [65, Proposition 2.4] that every polymatroidal ideal has the strong persistence property. In addition, according to [4], [3], and [104], the cover ideals of the following imperfect graphs are normal and satisfy the strong persistence property:

- (1) Cycle graphs of odd orders.
- (2) Wheel graphs of even orders.
- (3) Helm graphs of odd orders with greater than or equal to 5.

Along this argument, the concept of symbolic strong persistence property was presented in [118]. An ideal I in a commutative Noetherian ring R has the *symbolic strong persistence property* if $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k \geq 1$, where $I^{(k)} = \bigcap_{\mathfrak{p} \in \text{Min}(I)} (I^k R_{\mathfrak{p}} \cap R)$ denotes the k -th symbolic power of I . It has been shown in [118] that the strong persistence property implies the symbolic strong persistence property, however, little is known for the other classes of monomial ideals which satisfy the symbolic strong persistence property.

The *edge ideal* corresponding to a graph G has been introduced by Villarreal [132] and is generated by the monomials $x_i x_j$, where $\{i, j\}$ is an edge of G , and the Alexander dual of it, denoted by $I(G)^\vee$, is called the *cover ideal* corresponding to G . Next, path ideals of graphs were first introduced by Conca and De Negri [30] in the context of monomial ideals of linear type. An ideal I in a commutative Noetherian ring R is called *normally torsion-free* if $\text{Ass}(R/I^k) \subseteq \text{Ass}(R/I)$ for all $k \geq 1$. A few examples of normally torsion-free monomial ideals appear from graph theory. Already, in [126], it has been proved that a finite simple graph G is bipartite if and only if its edge ideal is normally torsion-free. Moreover, by [48], it is well-known that the cover ideals of bipartite graphs are normally torsion-free. In addition, in [66], it has been verified that every transversal polymatroidal ideal is normally torsion-free. However, normally torsion-free square-free monomial ideals have been studied in [50, 126, 128], but little is known for the normally torsion-free monomial ideals which are not square-free, for example see [110].

This manuscript has 11 chapters. Chapter 1 is devoted to introducing the required definitions and notations of commutative algebra and (hyper)graph theory used throughout the text to keep this work as self-contained as possible.

In Chapter 2, we argue on the persistence property of monomial ideals. In fact, several results related to stable sets and index of stability of monomial and square-free monomial ideals are given. In particular, we investigate the relation between persistence property and the Alexander dual of square-free monomial ideals. We

also discuss the properties of two classes of monomial ideals, that is, unisplit and separable monomial ideals, such that we show that every monomial ideal in these two classes (i) has the (strong) persistence property, (ii) has a strongly superficial element, and (iii) is Ratliff-Rush closed. We close this chapter with a discussion about persistence property for associated primes of a family of ideals.

In Chapter 3, we are interested in studying the strong persistence property of monomial ideals. We start with looking at the associated primes of powers of cover ideals of three imperfect graphs, i.e., odd cycle graphs, wheel graphs of even orders, and helm graphs of odd orders with greater than or equal to 5, such that we show their cover ideals have the strong persistence property. Next, we introduce the notion of superficial ideals. We continue with investigating the relation between superficiality and normality. We terminate this chapter with giving a class of primary monomial ideals do not satisfy the strong persistence property.

The purpose of Chapter 4 is to probe the concept of symbolic strong persistence property. We first state several results on the relation between symbolic strong persistence property and monomial operations such as weighting, contraction, and monomial localization. After that, we investigate some consequences related to the relation between the strong persistence property and symbolic strong persistence property. In particular, we prove that every square-free monomial ideal satisfies the symbolic strong persistence property.

The focus of Chapter 5 is to identify some classes of normally torsion-free monomial ideals. Indeed, we first study the normally torsion-freeness of the path ideals associated to directed trees. After that, we consider the relation between the normally torsion-freeness and the Alexander dual of square-free monomial ideals. Next, we examine the normally torsion-freeness under some monomial operations such as deletion, contraction, expansion, monomial localization, and weighting. We finally finish this chapter with a discussion related to normally torsion-freeness of ideals arising from neighborhood hypergraphs and their transversal hypergraphs. In particular, we inspect the normally torsion-freeness of dominating ideals of cycles.

The aim of Chapter 6 is to settle the other classes of nearly normally torsion-free monomial ideals. To do this, we first explore the graphs whose cover ideals are nearly normally torsion-free monomial ideals. We next turn our attention to studying the t -spread principal Borel ideals, and present several classes of t -spread principal Borel ideals which are nearly normally torsion-free. We close this chapter with an argument on the nearly normally torsion-freeness of closed neighborhood ideals and dominating ideals of some classes of trees and cycles.

In Chapter 7, we concentrate on the normality of monomial ideals. In the first step, we show that the cover ideals of three classes of imperfect graphs are normal, that is, odd cycle graphs, wheel graphs of even orders, and helm graphs of odd orders

with greater than or equal to 5. We next consider the normality of monomial ideals under some monomial operations like deletion, contraction, monomial localization, expansion, and weighting. We also present an argument on the normality of linear combinations of normal monomial ideals. After that, we verify the cover ideals of Jahangir's graphs are normal. We next show that the cover ideals of theta graphs are normal. We enrich this chapter with giving some results regarding to the normality of closed neighborhood ideals and dominating ideals of complete bipartite graphs. This chapter will be terminated by a discussion about the normality of dominating ideals of h -wheel graphs.

Chapter 8 deals with the combinatorial aspects. In fact, we are going to answer two questions. Suppose that $V = \{1, \dots, n\}$ is a non-empty set of n elements, $\mathcal{A} = \{A_1, \dots, A_m\}$ a non-empty family of m non-empty subsets of V , and $2 \leq d \leq n$ a positive integer. In the first question, we investigate the question arises whether there exists a d -uniform hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = V$ where $\mathcal{A}$ is the set whose elements are the minimal vertex cover sets of $\mathcal{H}$. Now, assume that $\mathcal{V} = \{1, \dots, n\}$ is a non-empty set of n elements, $\mathcal{S} = \{S_1, \dots, S_m\}$ a non-empty set of m non-empty subsets of $\mathcal{V}$. In the second question, we inspect the question arises whether there exists an undirected finite simple graph G with $V(G) = \mathcal{V}$ where $\mathcal{S}$ is the set whose elements are the minimal dominating sets of G .

Chapter 9 is concerned with max-path ideals and matroidal path ideals. In the first part, we start introducing the definitions of maximal paths and path cover sets. Also, we state several properties of these notions. Next, we present the definition of max-path ideals, and argue on the graphs which their max-path ideals have certain heights. In the second part, we consider a class of polymatroidal ideals, that is, matroidal path ideals of graphs. In particular, we explore the Cohen-Macaulay property and limit depth of matroidal path ideals.

In Chapter 10, we first provide a necessary and sufficient condition whether the graded maximal ideal appears in the set of associated prime ideals of a monomial ideal such that we only employ the unique minimal set of monomial generators. Next, we turn our attention to studying the nature of corner-elements of monomial ideals and discussing the supports of corner-elements related to monomial ideals. As application, we verify that under certain conditions, every unmixed König ideal is normally torsion-free. Furthermore, we show that every square-free transversal polymatroidal ideal is normally torsion-free.

Chapter 11 offers a lot of open questions in this field. In fact, our aim is to provide a perspective for this scope for future research.

Throughout this text, K is a field and $R = K[x_1, \dots, x_n]$ is a polynomial ring in the variables $x_1, \dots, x_n$. For a monomial ideal I in R , we denote the unique minimal set of monomial generators of I by $\mathcal{G}(I)$. The symbol $\mathbb{N}$ (respectively, $\mathbb{N}_0$, $\mathbb{Q}$, $\mathbb{Q}_{\geq 0}$)

will always denote the set of positive integers (respectively, nonnegative integers, rational numbers, nonnegative rational numbers).

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Chapter 1

Preliminaries

In this chapter, we collect all basic and crucial materials for this manuscript coming from commutative algebra and combinatorics. We summarize them in the following sections.

1.1 Commutative Algebra

In this section, we recall some definitions and notions from commutative algebra, which can be found in [60], [133], and [125].

Let R be a commutative Noetherian ring and I be an ideal of R . A prime ideal $\mathfrak{p} \subset R$ is an *associated prime* of I if there exists an element v in R such that $\mathfrak{p} = (I :_R v)$, where $(I :_R v) = \{r \in R \mid rv \in I\}$. The *set of associated primes* of I , denoted by $\text{Ass}_R(R/I)$, is the set of all prime ideals associated to I . When there is no confusion about the underlying ring, we will denote the set of associated primes of I simply by $\text{Ass}(R/I)$ or $\text{Ass}(I)$, see [125, Definition 9.32 and Remarks 9.33].

A presentation of an ideal I as intersection $I = \bigcap_{i=1}^k \mathfrak{q}_i$ with each $\mathfrak{q}_i$ is a primary ideal is called a *primary decomposition* of I . Let $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i$ for each $i = 1, \dots, k$. The primary decomposition is called a *minimal primary decomposition* if none of the $\mathfrak{q}_i$ can be omitted in this intersection, and also $\mathfrak{p}_i \neq \mathfrak{p}_j$ for all $1 \leq i \neq j \leq k$. If $I = \bigcap_{i=1}^k \mathfrak{q}_i$ is a minimal primary decomposition of I , then we have $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$. The minimal members of $\text{Ass}_R(R/I)$ are called the *minimal primes* of I , and $\text{Min}(I)$ denotes the set of minimal prime ideals of I . Moreover, the associated primes of I which are not minimal are called the *embedded primes* of I , see [125, Definition 4.15] for more information.

In particular, if I is a square-free monomial ideal, then $\text{Ass}_R(R/I) = \text{Min}(I)$, see [60, Corollary 1.3.6].

Let I be an ideal of R and $\text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Given an integer $n \geq 1$, the

n -th *symbolic power* of I is defined to be the ideal $I^{(n)} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, where $\mathfrak{q}_i$ is the primary component of I^n corresponding to $\mathfrak{p}_i$, see [133, Definition 4.3.22].

A well-known result of Brodmann [21] showed that the sequence $\{\text{Ass}_R(R/I^k)\}_{k \geq 1}$ of associated prime ideals is stationary for large k . That is, there exists a positive integer k_0 such that $\text{Ass}_R(R/I^k) = \text{Ass}_R(R/I^{k_0})$ for all $k \geq k_0$. The minimal such k_0 is called the *index of stability* of I , is denoted by $\text{astab}(I)$, and $\text{Ass}_R(R/I^{k_0})$ is called the *stable set* of associated prime ideals of I , which is denoted by $\text{Ass}^\infty(I)$, refer to [69] in order to see some examples.

An ideal $I \subset R$ satisfies the *persistence property* if $\text{Ass}_R(R/I^k) \subseteq \text{Ass}_R(R/I^{k+1})$ for all positive integers k .

An ideal $I \subset R$ has the *strong persistence property* if $(I^{k+1} :_R I) = I^k$ for all positive integers k .

If I and J are two ideals in R , then we say that J is a *superficial ideal* for I if the following conditions are satisfied:

- (i) $\mathcal{G}(J) \subseteq \mathcal{G}(I)$, where $\mathcal{G}(L)$ denotes the unique minimal set of monomial generators of a monomial ideal L ;
- (ii) $(I^{k+1} :_R J) = I^k$ for all positive integers k .

It should be noted that an ideal has a superficial ideal if and only if it has the strong persistence property, refer to [112] for more information.

An ideal $I \subset R$ satisfies the *symbolic strong persistence property* if $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all positive integers k , where $I^{(k)}$ denotes the k -th symbolic power of I , see [118].

An ideal $I \subset R$ is called *normally torsion-free* if $\text{Ass}_R(R/I^k) \subseteq \text{Ass}_R(R/I)$ for all $k \geq 1$, see [60, Definition 1.4.5].

In particular, if I is a square-free monomial ideal, then I is normally torsion-free if and only if $I^k = I^{(k)}$ for all $k \geq 1$ (cf. [60, Theorem 1.4.6]).

A monomial ideal I in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K is called *nearly normally torsion-free* if there exist a positive integer k and a monomial prime ideal $\mathfrak{p}$ such that $\text{Ass}_R(R/I^m) = \text{Min}(I)$ for all $1 \leq m \leq k$, and $\text{Ass}_R(R/I^m) \subseteq \text{Min}(I) \cup \{\mathfrak{p}\}$ for all $m \geq k+1$, see [9, Definition 2.1].

Let R be a unitary commutative ring and I an ideal in R . An element $f \in R$ is *integral* over I , if there exists an equation

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0 \quad \text{with } c_i \in I^i.$$

The set of elements $\bar{I}$ in R which are integral over I is the *integral closure* of I . The ideal I is *integrally closed*, if $I = \bar{I}$, and I is *normal* if all powers of I are integrally closed, refer to [60, Definition 1.4.1].

The notion of integrality for a monomial ideal I can be described in a simpler way as following: a monomial $u \in R = K[x_1, \dots, x_n]$ is integral over $I \subset R$ if and only if there exists an integer k such that $u^k \in I^k$, see [60, Theorem 1.4.2].

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K . An ideal $I \subset R$ is called a *monomial ideal* if it admits a generating set consisting of monomials. Furthermore, I is said to be a *square-free monomial ideal* if it is generated by square-free monomials.

Let $\mathcal{G}(I)$ be the unique minimal generating set of a monomial ideal I of a polynomial ring R . The *support* of a monomial $u \in R$, denoted by $\text{supp}(u)$, is the set of variables that divide u (cf. [133, Definition 6.1.5]). Moreover, for a monomial ideal I , we set $\text{supp}(I) = \bigcup_{u \in \mathcal{G}(I)} \text{supp}(u)$. Given a square-free monomial ideal $I \subset R$, the *Alexander dual* of I , denoted by $I^\vee$, is given by

$$I^\vee = \bigcap_{u \in \mathcal{G}(I)} (x_i : x_i \in \text{supp}(u)),$$

refer to [133, Definition 6.3.38 and Theorem 6.3.39].

1.2 Graph theory

In this section, we provide some necessary definitions related to graph theory, which can be found in [18] and [137].

Let $G = (V(G), E(G))$ be a finite simple undirected graph with the vertex set $V(G)$ and the edge set $E(G)$. The *degree* of a vertex $v \in V(G)$ is denoted by $\deg(v)$ and it represents the number of vertices adjacent to v . A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H of G is called an *induced subgraph* if it contains all the edges of G that have both vertices in H .

A set $T \subseteq V(G)$ is called a *vertex cover* of G if it intersects every edge of G non-trivially. A vertex cover is called *minimal* if it does not properly contain any other vertex cover of G . For each vertex $v \in V(G)$, the *closed neighborhood* of v in G is defined as follows:

$$N_G[v] = \{u \in V(G) : \{u, v\} \in E(G)\} \cup \{v\}.$$

When there is no confusion about the underlying graph, we will denote $N_G[v]$ simply by $N[v]$. A subset $S \subseteq V(G)$ is called a *dominating set* of G if $S \cap N[v] \neq \emptyset$ for all $v \in V(G)$. A dominating set is called *minimal* if it does not properly contain any other dominating set of G . A minimum dominating set of G is a minimal dominating set with the smallest size. The *dominating number* of G , denoted by $\gamma(G)$, is the size of its minimum dominating set, that is,

$$\gamma(G) = \min\{|S| : S \text{ is a minimal dominating set of } G\}.$$

The dominating sets and domination numbers of graphs are well-studied topics in graph theory. Further information on dominations in graphs can be found in [53].

In general, square-free monomial ideals can be associated with graphs in many different ways. We recall some commonly known definitions in this context. Let $V(G) = \{x_1, x_2, \dots, x_n\}$ and $R = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K . The *edge ideal* of G , denoted by $I(G)$, is

$$I(G) = (x_i x_j : \{x_i, x_j\} \in E(G)).$$

The *cover ideal* of G , denoted by $J(G)$, is

$$J(G) = (\prod_{x_i \in T} x_i : T \text{ is a minimal vertex cover of } G).$$

In [122], the *closed neighborhood ideal* of G has been introduced as

$$NI(G) = (\prod_{x_j \in N[x_i]} x_j : x_i \in V(G)).$$

Moreover, in [122], the *dominating ideal* of G is defined as

$$DI(G) = (\prod_{x_i \in S} x_i : S \text{ is a minimal dominating set of } G).$$

It is a well-recognized fact that $J(G)$ is the Alexander dual of $I(G)$ (cf. [60, Lemma 9.1.4]). It is shown in [122, Lemma 2.2] that $DI(G)$ is the Alexander dual of $NI(G)$.

A *path* in $G = (V(G), E(G))$ is a sequence of distinct vertices $x_{i_1}, \dots, x_{i_t}$ such that $\{x_{i_j}, x_{i_{j+1}}\} \in E(G)$ for $j = 1, \dots, t-1$. The *length* of a path is the number of edges in it. If a graph G itself is a path on n vertices, then it is denoted by P_n . A *cycle*, denoted by C_n , is a graph with $V(C_n) = \{x_1, \dots, x_n\}$ and

$$E(C_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\} \cup \{\{x_1, x_n\}\}.$$

A graph G is said to be *chordal* if it has no induced cycles of length greater than three. Equivalently, a graph G is chordal if and only if it admits a perfect elimination ordering of its vertices. A *perfect elimination ordering* of G is an ordering $v_1, \dots, v_n$ of $V(G)$ with the property that for each i, j , and l , if $i < j$, $i < l$ with $v_l, v_j \in N_G[v_i]$, then $v_l \in N_G[v_j]$. Note that chordal graphs are sometimes referred to as *triangulated graphs*.

Let $G = (V(G), E(G))$ be a finite simple graph. The graph G is called *perfect* if for any induced subgraph G_S , with $S \subseteq V(G)$, we have $\chi(G_S) = \omega(G_S)$, where $\omega(H)$ denotes the size of the largest clique of a graph H , see [45, Definition 5.1]. An *imperfect graph* is a graph that is not perfect.

In fact, according to [18, Theorem 14.18], a graph is perfect if and only if it contains no odd cycle of length at least five, or its complement, as an induced subgraph. Families of imperfect graphs include:

- (1) Cycle graphs C_{2n+1} ;
- (2) Fullerenes (which by definition contain an odd 5-cycle);
- (3) King graphs $K_{m,n}$ with $\min\{m, n\} \geq 4$;
- (4) Helm graphs H_n for odd $n \geq 5$;
- (5) Wheel graphs W_{2n} .

1.3 Hypergraph theory

In this section, we recall some notions and concepts from hypergraph theory, which can be found in [15] and [133]. A finite *hypergraph* $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ on a vertex set $V(\mathcal{H}) = \{x_1, x_2, \dots, x_n\}$ is a collection of edges $E(\mathcal{H}) = \{E_1, \dots, E_m\}$ with $E_i \subseteq V(\mathcal{H})$ for all $i = 1, \dots, m$. The *incidence matrix* of $\mathcal{H}$ is a $n \times m$ matrix $A = (a_{ij})$ such that $a_{ij} = 1$ if $x_i \in E_j$, and $a_{ij} = 0$ otherwise. A hypergraph $\mathcal{H}$ is called *simple*, if $E_i \subseteq E_j$ implies $i = j$. Simple hypergraphs are also known as *clutters*. Moreover, if $|E_i| = d$ for all $i = 1, \dots, m$, then $\mathcal{H}$ is called a *d-uniform* hypergraph. A 2-uniform hypergraph $\mathcal{H}$ is just a finite simple graph.

Let $V(\mathcal{H}) = \{x_1, x_2, \dots, x_n\}$ and $R = K[x_1, \dots, x_n]$ be a polynomial ring in n variables over a field K . The *edge ideal* of $\mathcal{H}$ is given by

$$I(\mathcal{H}) = \left(\prod_{x_j \in E_i} x_j : E_i \in E(\mathcal{H}) \right).$$

A subset $W \subseteq V(\mathcal{H})$ is a *vertex cover* or *transversal* of $\mathcal{H}$ if $W \cap E_i \neq \emptyset$ for all $i = 1, \dots, m$. A transversal W is *minimal* if no proper subset of W is a transversal of $\mathcal{H}$. The family of minimal transversals of $\mathcal{H}$ constitutes a simple hypergraph on $V(\mathcal{H})$ called the *transversal hypergraph* of $\mathcal{H}$, and denoted by $T_r(\mathcal{H})$. The cover ideal of the hypergraph $\mathcal{H}$, denoted by $J(\mathcal{H})$, is given by

$$J(\mathcal{H}) = \left(\prod_{x_i \in W} x_i : W \text{ is a minimal transversal of } \mathcal{H} \right).$$

In combinatorial commutative algebra, the transversals of a hypergraph are referred to as covers, and that is why the ideal $J(\mathcal{H})$ is usually called the cover ideal of $\mathcal{H}$. Moreover, similar to the case of edge ideal of graphs, the cover ideal $J(\mathcal{H})$ is the Alexander dual of $I(\mathcal{H})$, that is, $J(\mathcal{H}) = I(\mathcal{H})^\vee$, consult [133, Theorem 6.3.39].

Let G be a graph with $V(G) = \{x_1, \dots, x_n\}$. The *neighborhood hypergraph* of G , denoted by $\mathcal{N}(G)$, is a hypergraph on the vertex set $V(G)$ with edges $N[x_i]$ for all $i = 1, \dots, n$. Note that the edge ideal of $\mathcal{N}(G)$ is the closed neighborhood ideal $NI(G)$. A reader may find a confusing that we use the word “closed” with ideal but

not with hypergraphs. It is because we want to keep the standard notations given in [15] and the naming of ideal given in [122].

The *t-path hypergraph* of G , denoted by $\mathcal{H}_t(G)$, is a hypergraph on the vertex set $V(G)$ whose edges are all distinct t -paths in G . The edge ideal of $\mathcal{H}_t(G)$ is called the *t-path ideal* of G and is denoted by $I_t(G)$. When $t = 2$, then $I_2(G)$ is simply the *edge ideal* $I(G)$ of G . If there is no t -path in G , then we set $I_t(G) = 0$.

A *simplicial complex* Δ on a set of vertices $V(\Delta) = \{x_1, \dots, x_n\}$ is a collection of subsets of $V(\Delta)$, with the property that if $F \in \Delta$, then all subsets of F are also in Δ , including the empty set. An element of Δ is called a *face* of Δ , and the maximal faces of Δ with respect to inclusion are called *facets* of Δ , see [60, Page 15]. The *facet ideal* of Δ is given by

$$I(\Delta) = \left(\prod_{x_i \in F} x_i : F \text{ is a facet of } \Delta \right).$$

The set of all facets of Δ can be viewed as a simple hypergraph. In this case, the definition of the edge ideal of a hypergraph consisting of all the facets of Δ is the same as the facet ideal of Δ .

Chapter 2

On the persistence property of monomial ideals

2.1 Stable set and index of stability of a monomial ideal

Let R be a commutative Noetherian ring and I be an ideal of R . A well-known result of Brodmann [21] showed that the sequence $\{\text{Ass}_R(R/I^k)\}_{k \geq 1}$ of associated prime ideals is stationary for large k . That is, there exists a positive integer k_0 such that $\text{Ass}_R(R/I^k) = \text{Ass}_R(R/I^{k_0})$ for all $k \geq k_0$. The minimal such k_0 is called the *index of stability* of I and $\text{Ass}_R(R/I^{k_0})$ is called the *stable set of associated prime ideals* of I , which is denoted by $\text{Ass}^\infty(I)$. In recent researches, by using combinatorial tools, many papers have been published in order to describe the stable set of associated prime ideals of a monomial ideal and a square-free monomial ideal, see [14], [27], [50], and [66] for more information. In this section, we study these notions. It should be noted that the results of this section can be found in [81] and [82].

2.1.1 On the stable set of monomial ideals

The following result is a key lemma for us to show our main results. Notice that throughout this section, $R = K[x_1, \dots, x_n]$ stands for a polynomial ring over a field K .

Lemma 2.1.1. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be nonzero monomial prime ideals of R . Then there exists a monomial ideal I of R such that $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.*

Proof. Without loss of generality, we may assume that $|\mathcal{G}(\mathfrak{p}_i)| \leq |\mathcal{G}(\mathfrak{p}_j)|$ for all

$1 \leq i < j \leq m$. Fix $d \in \mathbb{N}$ and put $I := \mathfrak{p}_1^d \cap \mathfrak{p}_2^{2d} \cap \mathfrak{p}_3^{4d} \cap \cdots \cap \mathfrak{p}_m^{2^{m-1}d}$. We will prove that $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. In order to achieve this claim, firstly, we show that

$\bigcap_{j=1, j \neq i}^m \mathfrak{p}_j^{2^{j-1}d} \not\subseteq \mathfrak{p}_i^{2^{i-1}d}$ for all $i = 1, \dots, m$. Suppose to the contrary that there exists

$i \in \mathbb{N}$ with $1 \leq i \leq m$ such that $\bigcap_{j=1, j \neq i}^m \mathfrak{p}_j^{2^{j-1}d} \subseteq \mathfrak{p}_i^{2^{i-1}d}$. Hence, we consider the following three cases:

Case 1. Let $\mathfrak{p}_2^{2d} \cap \mathfrak{p}_3^{4d} \cap \cdots \cap \mathfrak{p}_m^{2^{m-1}d} \subseteq \mathfrak{p}_1^d$. Then $\mathfrak{p}_2 \cap \mathfrak{p}_3 \cap \cdots \cap \mathfrak{p}_m \subseteq \mathfrak{p}_1$, and thus there exists $i \in \mathbb{N}$ with $2 \leq i \leq m$ such that $\mathfrak{p}_i \subseteq \mathfrak{p}_1$, and so $\mathcal{G}(\mathfrak{p}_i) \subseteq \mathcal{G}(\mathfrak{p}_1)$. Also, we have $|\mathcal{G}(\mathfrak{p}_1)| \leq |\mathcal{G}(\mathfrak{p}_i)|$. Hence, $\mathcal{G}(\mathfrak{p}_1) = \mathcal{G}(\mathfrak{p}_i)$, and therefore $\mathfrak{p}_1 = \mathfrak{p}_i$, which is a contradiction.

Case 2. Let $\bigcap_{j=1, j \neq i}^m \mathfrak{p}_j^{2^{j-1}d} \subseteq \mathfrak{p}_i^{2^{i-1}d}$ with $2 \leq i \leq m-1$. For $j = i+1, \dots, m$, since $\mathfrak{p}_i \neq \mathfrak{p}_j$ and $|\mathcal{G}(\mathfrak{p}_i)| \leq |\mathcal{G}(\mathfrak{p}_j)|$, there exists $x_{\ell_j} \in \mathcal{G}(\mathfrak{p}_j) \setminus \mathcal{G}(\mathfrak{p}_i)$. Also, for $j = 1, \dots, i-1$, we choose an arbitrary element x_{ℓ_j} in $\mathcal{G}(\mathfrak{p}_j)$. Now, set $v := \prod_{j=1, j \neq i}^m x_{\ell_j}^{2^{j-1}d}$.

Then $v \in \bigcap_{j=1, j \neq i}^m \mathfrak{p}_j^{2^{j-1}d}$, and so $v \in \mathfrak{p}_i^{2^{i-1}d}$. Therefore, there exists $u \in \mathcal{G}(\mathfrak{p}_i^{2^{i-1}d})$ which

divides v . Let $u = \prod_{t=1}^{2^{i-1}d} x_{k_t}$ with $x_{k_t} \in \mathcal{G}(\mathfrak{p}_i)$ for all $t = 1, \dots, 2^{i-1}$. Since $x_{\ell_j} \notin \mathcal{G}(\mathfrak{p}_i)$

for all $j = i+1, \dots, m$ and u divides v , we have $\prod_{t=1}^{2^{i-1}d} x_{k_t}$ divides $x_{\ell_1}^d x_{\ell_2}^{2d} \cdots x_{\ell_{i-1}}^{2^{i-2}d}$.

Then $\deg(\prod_{t=1}^{2^{i-1}d} x_{k_t}) \leq \deg(x_{\ell_1}^d x_{\ell_2}^{2d} \cdots x_{\ell_{i-1}}^{2^{i-2}d})$, and so $2^{i-1}d \leq d + 2d + \cdots + 2^{i-2}d$.

Hence, $2^{i-1} \leq 1 + 2 + \cdots + 2^{i-2}$, which is a contradiction.

Case 3. Let $\bigcap_{j=1}^{m-1} \mathfrak{p}_j^{2^{j-1}d} \subseteq \mathfrak{p}_m^{2^{m-1}d}$ with $m \geq 2$. If $1 \leq j \leq m-1$, then since $\mathfrak{p}_j \neq \mathfrak{p}_m$

and $|\mathcal{G}(\mathfrak{p}_j)| \leq |\mathcal{G}(\mathfrak{p}_m)|$, there exists $x_{\ell_j} \in \mathcal{G}(\mathfrak{p}_m) \setminus \mathcal{G}(\mathfrak{p}_j)$. Now, put $v := \prod_{j=1}^{m-1} x_{\ell_j}^{2^{j-1}d}$.

Then $v \in \bigcap_{j=1}^{m-1} \mathfrak{p}_j^{2^{j-1}d}$, and so $v \in \mathfrak{p}_m^{2^{m-1}d}$. Therefore, there exists $u \in \mathcal{G}(\mathfrak{p}_m^{2^{m-1}d})$ such

that u divides v . Then $\deg(u) \leq \deg(v)$, and so $2^{m-1}d \leq d + 2d + \cdots + 2^{m-2}d$. Hence, $2^{m-1} \leq 1 + 2 + \cdots + 2^{m-2}$, which is a contradiction.

Therefore, $\bigcap_{j=1, j \neq i}^m \mathfrak{p}_j^{2^{j-1}d} \not\subseteq \mathfrak{p}_i^{2^{i-1}d}$ for all $i = 1, \dots, m$, and hence

$$\mathfrak{p}_1^d \cap \mathfrak{p}_2^{2d} \cap \mathfrak{p}_3^{4d} \cap \cdots \cap \mathfrak{p}_m^{2^{m-1}d},$$

is an irredundant presentation of I . Also, in view of [133, Corollary 6.1.8], every $\mathfrak{p}_i^{id}$ is a $\mathfrak{p}_i$ -primary ideal for $i = 1, \dots, m$. This implies that $\mathfrak{p}_1^d \cap \mathfrak{p}_2^{2d} \cap \mathfrak{p}_3^{4d} \cap \cdots \cap \mathfrak{p}_m^{2^{m-1}d}$

is a minimal primary decomposition of I , and hence

$$\text{Ass}_R(R/I) = \text{Ass}_R(R/\mathfrak{p}_1^d \cap \mathfrak{p}_2^{2d} \cap \mathfrak{p}_3^{4d} \cap \cdots \cap \mathfrak{p}_m^{2^{m-1}d}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\},$$

for all $d \in \mathbb{N}$, as claimed. $\square$

Remark 2.1.2. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be nonzero monomial prime ideals of R such that $|\mathcal{G}(\mathfrak{p}_i)| \leq |\mathcal{G}(\mathfrak{p}_j)|$ for all $1 \leq i < j \leq m$. It follows from the proof of Lemma 2.1.1 that, for all $d \in \mathbb{N}$,

$$\text{Ass}_R(R/\mathfrak{p}_1^d \cap \mathfrak{p}_2^{2d} \cap \mathfrak{p}_3^{4d} \cap \cdots \cap \mathfrak{p}_m^{2^{m-1}d}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}.$$

This means there exist infinite monomial ideals with associated prime $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.

Proposition 2.1.3. Let $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and $\mathfrak{B} = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$ be two sets of monomial prime ideals of R such that $\mathfrak{B} \subseteq \mathfrak{A}$. Then there exist monomial ideals I and J of R with the following properties:

(i) $\text{Ass}_R(R/I) = \mathfrak{A}$, $\text{Ass}_R(R/J) = \mathfrak{B}$; and,

(ii) $I \subseteq J$, $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$.

Proof. Set $I := \mathfrak{p}_1 \cap \mathfrak{p}_2^2 \cap \mathfrak{p}_3^4 \cap \cdots \cap \mathfrak{p}_m^{2^{m-1}}$. By virtue of Remark 2.1.2, we obtain $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\} = \mathfrak{A}$. Now, let $\sum$ be the set of all R -submodules R/I of the form K/I , where $\text{Ass}_R(K/I) \subseteq \mathfrak{A} \setminus \mathfrak{B}$ and $K \supseteq I$ is a monomial ideal of R . Clearly, $\sum \neq \emptyset$, and hence $\sum$ has a maximal element. Suppose that J is a monomial ideal of R such that J/I is a maximal element in $\sum$.

We first show that $\text{Ass}_R(R/J) \subseteq \mathfrak{B}$. To do this, suppose that $\mathfrak{p}$ is an arbitrary element of $\text{Ass}_R(R/J)$. Then there exists a monomial v of R such that $\mathfrak{p} = (J :_R v)$. Thus, $R/\mathfrak{p} \cong (Rv + J)/J$. Set $F := Rv + J$. It is clear that F is a monomial ideal and that $\text{Ass}_R(F/J) = \{\mathfrak{p}\}$. On the other hand, the following exact sequence

$$0 \rightarrow J/I \rightarrow F/I \rightarrow F/J \rightarrow 0,$$

induces the following inclusions

$$\text{Ass}_R(F/I) \subseteq \text{Ass}_R(J/I) \cup \text{Ass}_R(F/J) \subseteq (\mathfrak{A} \setminus \mathfrak{B}) \cup \{\mathfrak{p}\}.$$

Suppose, on the contrary, that $\mathfrak{p} \in \mathfrak{A} \setminus \mathfrak{B}$. Then $\text{Ass}_R(F/I) \subseteq \mathfrak{A} \setminus \mathfrak{B}$. Hence, $F/I \in \sum$. By maximality of J/I , we have $F/I = J/I$. Thus, $v \in J$, which is a contradiction. Therefore, $\mathfrak{p} \notin \mathfrak{A} \setminus \mathfrak{B}$, and so $\mathfrak{p} \in \text{Ass}_R(F/J)$. Thus, $\mathfrak{p} \in \mathfrak{B}$. Since $\mathfrak{p} \notin \mathfrak{A} \setminus \mathfrak{B}$, we get $\mathfrak{p} \in \mathfrak{B}$. Therefore, $\text{Ass}_R(R/J) \subseteq \mathfrak{B}$.

Now, we show that $\mathfrak{A} \setminus \mathfrak{B} \subseteq \text{Ass}_R(J/I)$. Assume that $\mathfrak{q}$ is an arbitrary element in $\mathfrak{A} \setminus \mathfrak{B}$. Then $\mathfrak{q} \notin \mathfrak{B}$, and hence $\mathfrak{q} \notin \text{Ass}_R(R/J)$. On the other hand, the exact sequence

$$0 \rightarrow J/I \rightarrow R/I \rightarrow R/J \rightarrow 0,$$

implies the inclusion $\text{Ass}_R(R/I) \subseteq \text{Ass}_R(J/I) \cup \text{Ass}_R(R/J)$. Thus, $\mathfrak{q} \in \text{Ass}_R(J/I)$. Therefore, $\mathfrak{A} \setminus \mathfrak{B} \subseteq \text{Ass}_R(J/I)$.

Finally, we show that $\mathfrak{B} \subseteq \text{Ass}_R(R/J)$. Suppose that $\mathfrak{b}$ is an arbitrary element in $\mathfrak{B}$. Since $\mathfrak{B} \subseteq \mathfrak{A}$, we deduce that $\mathfrak{b} \notin \mathfrak{A} \setminus \mathfrak{B}$, and thus $\mathfrak{b} \notin \text{Ass}_R(J/I)$. On the other hand, we have $\text{Ass}_R(R/I) \subseteq \text{Ass}_R(J/I) \cup \text{Ass}_R(R/J)$. Thus, $\mathfrak{b} \in \text{Ass}_R(R/J)$. We therefore obtain $\mathfrak{B} \subseteq \text{Ass}_R(R/J)$, and the proof is complete. $\square$

Theorem 2.1.4. *Let $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and $\mathfrak{B} = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$ be two arbitrary sets of monomial prime ideals of R . Then there exist monomial ideals I and J of R with the following properties:*

- (i) $\text{Ass}_R(R/I) = \mathfrak{A} \cup \mathfrak{B}$, $\text{Ass}_R(R/J) = \mathfrak{B}$; and,
- (ii) $I \subseteq J$, $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$.

Proof. Set $\mathfrak{D} := \mathfrak{A} \cup \mathfrak{B}$. Then, by Proposition 2.1.3, there exist monomial ideals I and J of R such that $I \subseteq J$, $\text{Ass}_R(R/I) = \mathfrak{D} = \mathfrak{A} \cup \mathfrak{B}$, $\text{Ass}_R(R/J) = \mathfrak{B}$, and $\text{Ass}_R(J/I) = \mathfrak{D} \setminus \mathfrak{B}$. Since $\mathfrak{D} \setminus \mathfrak{B} = \mathfrak{A} \setminus \mathfrak{B}$, we obtain $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$. $\square$

The following theorem has been proved also in [14, Theorem 1.2].

Theorem 2.1.5. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be nonzero monomial prime ideals of R . Then there exists a monomial ideal I of R such that $\text{Ass}^\infty(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.*

Proof. By virtue of [62, Corollary 2.2], there exists some $d \in \mathbb{N}$ such that for $I := \mathfrak{p}_1^d \cap \mathfrak{p}_2^{2d} \cap \mathfrak{p}_3^{4d} \cap \dots \cap \mathfrak{p}_m^{2^{m-1}d}$, we have

$$I^k = \mathfrak{p}_1^{kd} \cap \mathfrak{p}_2^{2kd} \cap \mathfrak{p}_3^{4kd} \cap \dots \cap \mathfrak{p}_m^{2^{m-1}kd},$$

for all $k \in \mathbb{N}$. Now, by Remark 2.1.2, we obtain

$$\text{Ass}_R(R/I^k) = \text{Ass}_R(R/\mathfrak{p}_1^{kd} \cap \mathfrak{p}_2^{2kd} \cap \mathfrak{p}_3^{4kd} \cap \dots \cap \mathfrak{p}_m^{2^{m-1}kd}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\},$$

for all $k \in \mathbb{N}$. In fact, the index of stability of I is equal to 1. Therefore, $\text{Ass}^\infty(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, as desired. $\square$

Proposition 2.1.6. *Let $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and $\mathfrak{B} = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$ be two sets of monomial prime ideals of R such that $\mathfrak{B} \subseteq \mathfrak{A}$. Then there exist monomial ideals I and J of R with the following properties:*

- (i) $\text{Ass}^\infty(I) = \mathfrak{A}$, $\text{Ass}_R(R/J) = \mathfrak{B}$; and,
- (ii) $I \subseteq J$, $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$.

Proof. By Theorem 2.1.5, there exists an ideal I such that $\text{Ass}^\infty(I) = \text{Ass}_R(R/I) = \mathfrak{A}$. Now, by using a method similar to the one has been used in the proof of Proposition 2.1.3, one can complete the proof. $\square$

Corollary 2.1.7. *Let $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and $\mathfrak{B} = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$ be two arbitrary sets of monomial prime ideals of R . Then there exist monomial ideals I and J of R such that*

$$(i) \text{ Ass}^\infty(I) = \mathfrak{A} \cup \mathfrak{B}, \text{ Ass}_R(R/J) = \mathfrak{B}; \text{ and,}$$

$$(ii) \text{ } I \subseteq J, \text{ Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}.$$

Proof. Set $\mathfrak{D} := \mathfrak{A} \cup \mathfrak{B}$. By Theorem 2.1.5, there exists an ideal I such that $\text{Ass}^\infty(I) = \text{Ass}_R(R/I) = \mathfrak{D}$. Now, by using a method similar to the one has been used in the proof of Theorem 2.1.4, one can complete the proof. $\square$

2.1.2 On the stable set of square-free monomial ideals

In this subsection, we investigate the stable set of a square-free monomial ideal.

Proposition 2.1.8. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be nonzero monomial prime ideals of R such that they are generated by disjoint non-empty subsets of $\{x_1, \dots, x_n\}$. Then there exists a square-free monomial ideal I such that $\text{Ass}_R(R/I^k) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ for all $k \in \mathbb{N}$. Moreover, $\text{Ass}^\infty(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.*

Proof. Clearly, in the polynomial ring $R = K[x_1, \dots, x_n]$, the sequence $x_1, \dots, x_n$ is a permutable regular sequence. Hence, by [113, Remark 2.8], for all positive integers $k_1, \dots, k_m$, we have the following equalities

$$\text{Ass}_R(R/\mathfrak{p}_1^{k_1} \dots \mathfrak{p}_m^{k_m}) = \bigcup_{i=1}^m \text{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}.$$

Also, by virtue of [113, Theorem 2.7], we have $\mathfrak{p}_1^{k_1} \dots \mathfrak{p}_m^{k_m} = \mathfrak{p}_1^{k_1} \cap \dots \cap \mathfrak{p}_m^{k_m}$ for all positive integers $k_1, \dots, k_m$. Set $I := \mathfrak{p}_1 \dots \mathfrak{p}_m = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_m$. Then I is a square-free monomial ideal and $\text{Ass}_R(R/I^k) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ for all $k \in \mathbb{N}$. Moreover, we have $\text{Ass}^\infty(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. $\square$

Remark 2.1.9. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be prime ideals as in Proposition 2.1.8. Then $I := \mathfrak{p}_1 \dots \mathfrak{p}_m$ is a square-free monomial ideal such that $\text{Ass}_R(R/I^k) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ for all $k \in \mathbb{N}$ and $\text{Ass}^\infty(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$.*

Example 2.1.10. *Assume that $\mathfrak{p}_1 = (x_1, x_3)$, $\mathfrak{p}_2 = (x_2, x_4, x_5)$, and $\mathfrak{p}_3 = (x_6)$ are monomial prime ideals of $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$. Put*

$$I := \mathfrak{p}_1 \mathfrak{p}_2 \mathfrak{p}_3 = (x_1 x_2 x_6, x_1 x_4 x_6, x_1 x_5 x_6, x_2 x_3 x_6, x_3 x_4 x_6, x_3 x_5 x_6).$$

Then, by Proposition 2.1.8, we deduce that $\text{Ass}_R(R/I^k) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$ for all $k \in \mathbb{N}$. Also, one has $\text{Ass}^\infty(I) = \{\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3\}$.

Theorem 2.1.11. *Let $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ be a set of nonzero monomial prime ideals of R such that they are generated by disjoint non-empty subsets of $\{x_1, \dots, x_n\}$. Also, suppose that $\{\mathfrak{A}_1, \dots, \mathfrak{A}_r\}$ is a partition of $\mathfrak{A}$. Then there exist square-free monomial ideals $I_1, \dots, I_r$ such that, for all positive integers $k_1, \dots, k_r, d$,*

$$(i) \text{ Ass}_R(R/I_i^{k_i}) = \mathfrak{A}_i;$$

$$(ii) \text{ Ass}_R(R/I_1^{k_1 d} \cdots I_r^{k_r d}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}; \text{ and,}$$

$$(iii) \text{ Ass}^\infty(I_1^{k_1} \cdots I_r^{k_r}) = \text{Ass}^\infty(I_1^{k_1}) \cup \cdots \cup \text{Ass}^\infty(I_r^{k_r}).$$

Proof. For all $i = 1, \dots, r$, suppose that $\mathfrak{A}_i = \{\mathfrak{p}_{i,1}, \dots, \mathfrak{p}_{i,t_i}\}$. Hence, we get $\bigcup_{i=1}^r \mathfrak{A}_i = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. Set $I_i := \mathfrak{p}_{i,1} \cdots \mathfrak{p}_{i,t_i}$ for all $i = 1, \dots, r$. By Proposition 2.1.8, we have $\text{Ass}_R(R/I_i^{k_i}) = \{\mathfrak{p}_{i,1}, \dots, \mathfrak{p}_{i,t_i}\}$ for all $k_i \in \mathbb{N}$ and $i = 1, \dots, r$. Also, we derive that $I_1^{k_1} \cdots I_r^{k_r} = \prod_{i=1}^r \prod_{j=1}^{t_i} \mathfrak{p}_{i,j}^{k_i}$. Thus, by [113, Remark 2.8], we obtain

$$\begin{aligned} \text{Ass}_R(R/(I_1^{k_1} \cdots I_r^{k_r})^d) &= \text{Ass}_R(R/\prod_{i=1}^r \prod_{j=1}^{t_i} \mathfrak{p}_{i,j}^{k_i d}) \\ &= \bigcup_{i=1}^r \bigcup_{j=1}^{t_i} \text{Ass}_R(R/\mathfrak{p}_{i,j}^{k_i d}) \\ &= \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}, \end{aligned}$$

for all positive integers $k_1, \dots, k_r, d$. It follows now from Proposition 2.1.8 that $\text{Ass}^\infty(I_i^{k_i}) = \{\mathfrak{p}_{i,1}, \dots, \mathfrak{p}_{i,t_i}\}$ for all $i = 1, \dots, r$. So, the other assertion is obvious and the proof is complete. $\square$

2.1.3 Associated primes of square-free monomial ideals

We begin with the following lemma, which will be used in the proof of Theorem 2.1.15

Lemma 2.1.12. *Let S be a commutative ring and let $a_1, \dots, a_m$ be elements constituting a permutable S -sequence. Let J be an ideal generated by monomials in $a_{t+1}, \dots, a_m$ for some $t \in \mathbb{N}$ with $1 \leq t \leq m-1$. Then $(J :_S a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}) = J$ for all $n_1, n_2, \dots, n_t \in \mathbb{N}$.*

Proof. We proceed by induction on t . In the case in which $t = 1$ follows from [77, Lemma 3]. Now, suppose, inductively, that $t > 1$ and that the result has been proved for t . Then, by using Lemma 3 in [77] in conjunction with the inductive hypothesis, we obtain the following equalities

$$(J :_S a_1^{n_1} a_2^{n_2} \cdots a_{t+1}^{n_{t+1}}) = ((J :_S a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}) :_S a_{t+1}^{n_{t+1}}) = (J :_S a_{t+1}^{n_{t+1}}) = J.$$

This completes the inductive step and the proof. $\square$

Corollary 2.1.13. *Let I be a monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ and J be an arbitrary ideal of R . Also, suppose that $x_{\ell_j} \notin \bigcup_{i=1}^m \text{supp}(u_i)$ for all $j = 1, \dots, t$ with $1 \leq t \leq n - 1$. If $v := x_{\ell_1}^{n_1} \cdots x_{\ell_t}^{n_t}$ with $n_1, n_2, \dots, n_t \in \mathbb{N}$, then $(vJ + I :_R v) = J + I$.*

Proof. By virtue of Lemma 2.1.12, one has $(I :_R v) = I$. Hence, we deduce that

$$(vJ + I :_R v) = (vJ :_R v) + (I :_R v) = J + I,$$

and the proof is over. $\square$

Proposition 2.1.14. *Let I and J be (not necessary monomial) ideals of a commutative Noetherian ring S . Also, suppose that I is a radical ideal of S . Then $(I :_S J)$ is a radical ideal and $(I :_S J) = (I :_S \sqrt{J})$.*

Proof. It is clear $(I :_S J) \subseteq \sqrt{(I :_S J)}$. For the reverse inclusion, suppose that $h \in \sqrt{(I :_S J)}$. Then $h^\ell J \subseteq I$ for some $\ell \in \mathbb{N}$. Assume that $J = (y_1, \dots, y_k)$. Then $h^\ell y_i \in I$ for all $i = 1, \dots, k$. Hence, $(hy_i)^\ell \in I$ for all $i = 1, \dots, k$. Since I is a radical ideal, it follows that $hJ \subseteq I$, and so $h \in (I :_S J)$. Therefore, $(I :_S J) = \sqrt{(I :_S J)}$.

Now, we prove that $(I :_S J) = (I :_S \sqrt{J})$. Clearly, $(I :_S \sqrt{J}) \subseteq (I :_S J)$. Suppose that h is an arbitrary element in $(I :_S J)$. Then $hJ \subseteq I$. Assume that $\sqrt{J} = (z_1, \dots, z_r)$. Then there exist $\beta_1, \dots, \beta_r \in \mathbb{N}$ such that $z_i^{\beta_i} \in J$ for all $i = 1, \dots, r$. Hence, $hz_i^{\beta_i} \in I$ for all $i = 1, \dots, r$. Therefore, $(hz_i)^{\beta_i} \in I$ for all $i = 1, \dots, r$. Since I is a radical ideal, we have $\sqrt{I} = I$. Hence, $h\sqrt{J} \subseteq I$, and so $h \in (I :_S \sqrt{J})$. Consequently, $(I :_S J) = (I :_S \sqrt{J})$. $\square$

In the following theorem we study the associated prime ideals of a square-free monomial ideal of $R = K[x_1, \dots, x_n]$.

Theorem 2.1.15. *Let I be a square-free monomial ideal of R , $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, and $\mathfrak{p} \in \text{Ass}_R(R/I)$. Then there exists a square-free monomial v of R with $\text{supp}(v) \subseteq \bigcup_{i=1}^m \text{supp}(u_i)$ such that $\mathfrak{p} = (I :_R v)$.*

Proof. On account of [60, Corollary 1.3.10], there exists a monomial w of R such that $\mathfrak{p} = (I :_R w)$. In view of Lemma 2.1.12, we may assume that $w = x_{\ell_1}^{\alpha_1} \cdots x_{\ell_t}^{\alpha_t}$, where the x_{ℓ_i} 's are distinct, $\alpha_1, \dots, \alpha_t$ are positive integers, and $\text{supp}(w) \subseteq \bigcup_{i=1}^m \text{supp}(u_i)$. By setting $v := x_{\ell_1} \cdots x_{\ell_t}$, we show that $\mathfrak{p} = (I :_R v)$. Clearly, $\text{supp}(w) = \text{supp}(v)$. Since I is a square-free monomial ideal, by Proposition 2.1.14, we have the following equalities

$$\mathfrak{p} = (I :_R w) = (I :_R \sqrt{w}) = (I :_R v),$$

as claimed. $\square$

The next example shows that Theorem 2.1.15 can fail for monomial ideals in general.

Example 2.1.16. Consider the ideal $I = (x_1^2x_2^3, x_1x_3^2, x_2^3x_3)$ in the polynomial ring $R = K[x_1, x_2, x_3]$, where K is a field and x_1, x_2, x_3 are indeterminates. Then, by computation in Macaulay2, we obtain

$$\text{Ass}_R(R/I) = \{(x_1, x_2), (x_1, x_3), (x_2, x_3)\}.$$

Assume in contrary that there exists a square-free monomial v with $\text{supp}(v) \subseteq \{x_1, x_2, x_3\}$ such that $\mathfrak{p} = (I :_R v)$. Then we have the following cases:

Case 1. $v = x_1$. Then $(I :_R v) = (x_1x_2^3, x_2^3x_3, x_3^2)$.

Case 2. $v = x_2$. Then $(I :_R v) = (x_1^2x_2^2, x_1x_3^2, x_2^2x_3)$.

Case 3. $v = x_3$. Then $(I :_R v) = (x_2^3, x_1x_3)$.

Case 4. $v = x_1x_2$. Then $(I :_R v) = (x_1x_2^2, x_2^2x_3, x_3^2)$.

Case 5. $v = x_1x_3$. Then $(I :_R v) = (x_2^3, x_3)$.

Case 6. $v = x_2x_3$. Then $(I :_R v) = (x_1x_3, x_2^2)$.

Case 7. $v = x_1x_2x_3$. Then $(I :_R v) = (x_2^2, x_3)$.

Since $(I :_R v)$ must be a prime monomial ideal, we observe all of them are impossible.

Proposition 2.1.17. Let I and J be two square-free monomial ideals of R such that $\text{Ass}_R(R/I) = \text{Ass}_R(R/J)$. Then $I = J$.

Proof. Since I and J are square-free monomial ideals, by [60, Corollary 1.3.6], we have $\text{Min}(I) = \text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\} = \text{Ass}_R(R/J) = \text{Min}(J)$. We therefore obtain $I = \bigcap_{i=1}^t \mathfrak{p}_i = J$. $\square$

Note that Proposition 2.1.17 can fail for monomial ideals in general.

Example 2.1.18. Consider the ideals $I = (x_1^2, x_2^3, x_3^4)$ and $J = (x_1^4, x_2^3, x_3^2)$ in the polynomial ring $R = K[x_1, x_2, x_3]$, where K is a field and x_1, x_2, x_3 are indeterminates. It follows from [60, Proposition 1.3.7] that $\text{Ass}_R(R/I) = \text{Ass}_R(R/J) = \{(x_1, x_2, x_3)\}$, while $I \neq J$.

Suppose that $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and $\mathfrak{B} = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$ are two sets of monomial prime ideals of R such that the minimal elements of these sets with respect to inclusion are the same. Now, one can ask the question whether there exist square-free monomial ideals I and J of R such that $\text{Ass}_R(R/I) = \mathfrak{A}$ and $\text{Ass}_R(R/J) = \mathfrak{B}$?

In the following, we provide a partial answer to the above question.

Proposition 2.1.19. *Let $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and $\mathfrak{B} = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$ be two sets of monomial prime ideals of R such that the elements in each of these sets are minimal with respect to inclusion and that $\mathfrak{B} \subseteq \mathfrak{A}$. Then there exist unique square-free monomial ideals I and J of R with the following properties:*

- (i) $\text{Ass}_R(R/I) = \mathfrak{A}$, $\text{Ass}_R(R/J) = \mathfrak{B}$; and,
- (ii) $I \subseteq J$, $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$.

Proof. Set $I := \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_m$. Since $I = \sqrt{I}$, by [60, Corollary 1.2.5], we obtain I is a square-free monomial ideal and also

$$\text{Min}(I) = \text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\} = \mathfrak{A}.$$

Now, assume that $\sum$ is the set of all R -submodules R/I of the form K/I , where $\text{Ass}_R(K/I) \subseteq \mathfrak{A} \setminus \mathfrak{B}$ and $K \supseteq I$ is a square-free monomial ideal of R . Clearly, $\sum \neq \emptyset$, and hence $\sum$ has a maximal element. Suppose that J is a square-free monomial ideal of R such that J/I is a maximal element in $\sum$. We first show that $\text{Ass}_R(R/J) \subseteq \mathfrak{B}$. To do this, suppose that $\mathfrak{p}$ is an arbitrary element of $\text{Ass}_R(R/J)$. Then, by Theorem 2.1.15, there exists a square-free monomial v of R such that $\mathfrak{p} = (J :_R v)$. Thus, $R/\mathfrak{p} \cong (Rv + J)/J$. Set $F := Rv + J$. It is clear that F is a square-free monomial ideal and that $\text{Ass}_R(F/J) = \{\mathfrak{p}\}$. Now, in view of Proposition 2.1.3, we have $\text{Ass}_R(R/J) = \mathfrak{B}$ and $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$.

Set $J_1 := \mathfrak{p}'_1 \cap \dots \cap \mathfrak{p}'_t$. Since $J_1 = \sqrt{J_1}$, again by [60, Corollary 1.2.5], we obtain J_1 is a square-free monomial ideal such that $\text{Min}(J_1) = \text{Ass}_R(R/J_1) = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$. On account of $\text{Ass}_R(R/J) = \text{Ass}_R(R/J_1) = \mathfrak{B}$, Proposition 2.1.17 implies that $J = \mathfrak{p}'_1 \cap \dots \cap \mathfrak{p}'_t$, and the proof is done. $\square$

Example 2.1.20. *Let $\mathfrak{B} = \{(x_2), (x_4)\}$ and $\mathfrak{A} = \{(x_2), (x_4), (x_1, x_3)\}$ be two sets of monomial prime ideals of the polynomial ring $R = K[x_1, x_2, x_3, x_4]$, where K is a field and x_1, x_2, x_3, x_4 are indeterminates. Now, by setting*

$$I := (x_2) \cap (x_4) \cap (x_1, x_3) = (x_1x_2x_4, x_2x_3x_4),$$

and

$$J := (x_2) \cap (x_4) = (x_2x_4),$$

we obtain $\text{Ass}_R(R/I) = \{(x_2), (x_4), (x_1, x_3)\}$, $\text{Ass}_R(R/J) = \{(x_2), (x_4)\}$, and $\text{Ass}_R(J/I) = \{(x_1, x_3)\}$.

Now, we state the main result of this subsection in the next theorem.

Theorem 2.1.21. *Let $\mathfrak{A} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and $\mathfrak{B} = \{\mathfrak{p}'_1, \dots, \mathfrak{p}'_t\}$ be two arbitrary sets of monomial prime ideals of R such that the elements of $\mathfrak{A} \cup \mathfrak{B}$ are minimal with respect to inclusion. Then there exist unique square-free monomial ideals I and J of R with the following properties:*

(i) $\text{Ass}_R(R/I) = \mathfrak{A} \cup \mathfrak{B}$, $\text{Ass}_R(R/J) = \mathfrak{B}$; and,

(ii) $I \subseteq J$, $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$.

Proof. Set $\mathfrak{D} := \mathfrak{A} \cup \mathfrak{B}$. Then, by Proposition 2.1.19, there exist unique square-free monomial ideals $I := \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m \cap \mathfrak{p}'_1 \cap \cdots \cap \mathfrak{p}'_t$ and $J := \mathfrak{p}'_1 \cap \cdots \cap \mathfrak{p}'_t$ of R such that $I \subseteq J$, $\text{Ass}_R(R/I) = \mathfrak{D} = \mathfrak{A} \cup \mathfrak{B}$, $\text{Ass}_R(R/J) = \mathfrak{B}$, and $\text{Ass}_R(J/I) = \mathfrak{D} \setminus \mathfrak{B}$. Since $\mathfrak{D} \setminus \mathfrak{B} = \mathfrak{A} \setminus \mathfrak{B}$, we deduce that $\text{Ass}_R(J/I) = \mathfrak{A} \setminus \mathfrak{B}$. This finishes the proof. $\square$

Example 2.1.22. Let $\mathfrak{B} = \{(x_4), (x_1, x_3), (x_2, x_3)\}$ and $\mathfrak{A} = \{(x_4), (x_5), (x_1, x_3)\}$ be two sets of monomial prime ideals in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$, where K is a field and x_1, x_2, x_3, x_4, x_5 are indeterminates. Now, set

$$J := (x_4) \cap (x_1, x_3) \cap (x_2, x_3) = (x_3x_4, x_1x_2x_4),$$

and

$$I := (x_4) \cap (x_5) \cap (x_1, x_3) \cap (x_2, x_3) = (x_1x_2x_4x_5, x_3x_4x_5).$$

As the elements of $\mathfrak{A} \cup \mathfrak{B} = \{(x_4), (x_5), (x_1, x_3), (x_2, x_3)\}$ are minimal with respect to inclusion, by Theorem 2.1.21, we get $\text{Ass}_R(R/J) = \{(x_4), (x_1, x_3), (x_2, x_3)\}$, $\text{Ass}_R(R/I) = \{(x_4), (x_5), (x_1, x_3), (x_2, x_3)\}$, and $\text{Ass}_R(J/I) = \{(x_5)\}$.

2.1.4 Associated prime ideals under squeezing

In this subsection, we investigate the relation among associated prime ideals of three nested ideals. Suppose that $J_1 \subseteq I \subseteq J_2$ are three ideals of R such that $\text{Ass}_R(R/J_1) = \text{Ass}_R(R/J_2)$. Then can we conclude that $\text{Ass}_R(R/J_1) = \text{Ass}_R(R/I) = \text{Ass}_R(R/J_2)$? As we shall show in Example 2.1.25, there is no general result in this context even for monomial ideals. Although, there exists a similar result for minimal prime ideals which we state in Theorem 2.1.23. We also show that if J_1 and J_2 are two square-free monomial ideals of R and $J_1 \subseteq I \subseteq J_2$, then the answer of the above question is affirmative in this situation which we give in Proposition 2.1.24.

Theorem 2.1.23. (Squeeze theorem for minimal prime ideals) *Let I , J_1 , and J_2 be ideals of a commutative Noetherian ring S such that $J_1 \subseteq I \subseteq J_2$ and that $\text{Min}(J_1) = \text{Min}(J_2) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Then $\text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$.*

Proof. Let $I = Q_1 \cap \cdots \cap Q_m$ be a minimal primary decomposition of I such that $\sqrt{Q_i} = \mathfrak{q}_i$ for all $i = 1, \dots, m$, and also let $\text{Min}(I) = \{\mathfrak{q}_{i_1}, \dots, \mathfrak{q}_{i_s}\}$. Suppose to the contrary that $s \neq t$. Then $t < s$ or $s < t$. First assume that $t < s$. Since $\sqrt{I} = \bigcap_{j=1}^s \mathfrak{q}_{i_j}$ and $\sqrt{J_1} = \sqrt{J_2} = \bigcap_{i=1}^t \mathfrak{p}_i$, we have $\bigcap_{j=1}^s \mathfrak{q}_{i_j} = \bigcap_{i=1}^t \mathfrak{p}_i$. Also, since $\bigcap_{j=1}^s \mathfrak{q}_{i_j} \subseteq \mathfrak{p}_1$, there exists $j \in \mathbb{N}$ with $1 \leq j \leq s$ such that $\mathfrak{q}_{i_j} \subseteq \mathfrak{p}_1$. It follows from the minimality of $\mathfrak{p}_1$ that $\mathfrak{p}_1 = \mathfrak{q}_{i_j}$. Without loss of generality, we may assume that

$\mathfrak{p}_1 = \mathfrak{q}_{i_1}$. Also, $\bigcap_{j=1}^s \mathfrak{q}_{i_j} \subseteq \mathfrak{p}_2$. Hence, there exists $1 \leq j \leq s$ such that $\mathfrak{q}_{i_j} \subseteq \mathfrak{p}_2$. Suppose to the contrary that $\mathfrak{q}_{i_j} = \mathfrak{q}_{i_1}$. Then $\mathfrak{p}_1 \subseteq \mathfrak{p}_2$, and, by the minimality of $\mathfrak{p}_2$, we obtain $\mathfrak{p}_1 = \mathfrak{p}_2$, which is a contradiction. Hence, $\mathfrak{q}_{i_j} \neq \mathfrak{q}_{i_1}$. Without loss of generality, we may assume that $\mathfrak{q}_{i_j} = \mathfrak{q}_{i_2}$. Then $\mathfrak{p}_2 = \mathfrak{q}_{i_2}$. Continuing this procedure yields $\mathfrak{p}_j = \mathfrak{q}_{i_j}$, where $j = 1, \dots, t$. Because $\bigcap_{j=1}^s \mathfrak{q}_{i_j} = \bigcap_{i=1}^t \mathfrak{p}_i$, this implies that $\mathfrak{q}_{i_1} \cap \dots \cap \mathfrak{q}_{i_t} \subseteq \mathfrak{q}_{i_{t+1}} \cap \dots \cap \mathfrak{q}_{i_s}$. Hence, there exists $1 \leq k \leq t$ such that $\mathfrak{q}_{i_k} \subseteq \mathfrak{q}_{i_s}$, and, by the minimality of $\mathfrak{q}_{i_s}$, we deduce that $\mathfrak{q}_{i_k} = \mathfrak{q}_{i_s}$, which is a contradiction. In a similar way, we conclude that the hypothesis $s < t$ leads to a contradiction. We therefore get $s = t$, and so $\text{Min}(I) = \{\mathfrak{q}_{i_1}, \dots, \mathfrak{q}_{i_s}\} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. $\square$

Proposition 2.1.24. *Let J_1 and J_2 be square-free monomial ideals of R and I be an ideal of R such that $J_1 \subseteq I \subseteq J_2$ and that*

$$\text{Ass}_R(R/J_1) = \text{Ass}_R(R/J_2) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}.$$

Then

$$\text{Ass}_R(R/I) = \text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}.$$

Proof. According to Proposition 2.1.17, it follows that $J_1 = J_2$. This yields that $I = J_1 = J_2$, and so $\text{Ass}_R(R/I) = \text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. $\square$

Note that Proposition 2.1.24 can fail for monomial ideals in general.

Example 2.1.25. *Consider the ideals $J_1 = (x_1, x_2, x_4)^3$, $J_2 = (x_1, x_2, x_4)$, and*

$$I = (x_1, x_2, x_4)^3 + (x_1, x_2, x_3, x_4)(x_1, x_2, x_4)^2,$$

in the polynomial ring $R = K[x_1, x_2, x_3, x_4]$, where K is a field and x_1, x_2, x_3, x_4 are indeterminates. Then $J_1 \subseteq I \subseteq J_2$ and, by computation with Macaulay2, we have

$$\text{Ass}_R(R/J_1) = \text{Ass}_R(R/J_2) = \{(x_1, x_2, x_4)\},$$

while $\text{Ass}_R(R/I) = \{(x_1, x_2, x_4), (x_1, x_2, x_3, x_4)\}$.

2.1.5 Some recursive formulas for associated prime ideals

We begin with the following lemma, which will be used in the proof of Theorem 2.1.27.

Lemma 2.1.26. *Let R be a polynomial ring with coefficients in a field K and indeterminates $x_1, \dots, x_n$. Also, let $Q = (x_{i_1}^{a_1}, \dots, x_{i_r}^{a_r})$ be a monomial ideal of R and $h = x_{j_1}^{b_1} \dots x_{j_s}^{b_s}$ be a monomial of R such that $x_{i_1}, \dots, x_{i_r}, x_{j_1}, \dots, x_{j_s} \in \{x_1, \dots, x_n\}$ and $a_1, \dots, a_r, b_1, \dots, b_s \in \mathbb{N}$. Then*

$$\text{Ass}_R(R/hQ) = \{(x_{i_1}, \dots, x_{i_r}), (x_{j_1}), \dots, (x_{j_s})\}.$$

Proof. Since $\text{Ass}_R(R/Q) = \{(x_{i_1}, \dots, x_{i_r})\}$, we have $(x_{i_1}, \dots, x_{i_r}) = (Q :_R v)$ for some monomial v in R . Since $(hQ :_R hv) = (Q :_R v)$ and $(hQ :_R hv) \in \text{Ass}_R(R/hQ)$, we deduce that $(x_{i_1}, \dots, x_{i_r}) \in \text{Ass}_R(R/hQ)$. In addition, since $(x_{j_k}) \in \text{Min}(hQ)$ for all $k = 1, \dots, s$, we obtain $(x_{j_1}), \dots, (x_{j_s}) \in \text{Ass}_R(R/hQ)$. Therefore, one has

$$\{(x_{i_1}, \dots, x_{i_r}), (x_{j_1}), \dots, (x_{j_s})\} \subseteq \text{Ass}_R(R/hQ).$$

For completing the proof, we show the reverse inclusion. To do this, set $L := \text{supp}(h) \cap \{x_{i_1}, \dots, x_{i_r}\}$. Suppose that $c_k := a_k + b_k$ when $x_{i_k} \in L$, and also $c_k := a_k$ when $x_{i_k} \notin L$, for all $k = 1, \dots, r$. Now, put $Q' := (x_{i_1}^{c_1}, x_{i_2}^{c_2}, \dots, x_{i_r}^{c_r})$. We verify that $hQ = hR \cap Q'$. It is easy to see that the unique minimal set of monomial generators of hQ is $\mathcal{G}(hQ) = \{hx_{i_1}^{a_1}, \dots, hx_{i_r}^{a_r}\}$. Also, the unique minimal set of monomial generators of $hR \cap Q'$ is

$$\mathcal{G}(hR \cap Q') = \{\text{lcm}(h, x_{i_1}^{c_1}), \text{lcm}(h, x_{i_2}^{c_2}), \dots, \text{lcm}(h, x_{i_r}^{c_r})\}.$$

If $x_{i_k} \notin L$, for some $1 \leq k \leq r$, then $c_k = a_k$, and so $\text{lcm}(h, x_{i_k}^{c_k}) = hx_{i_k}^{a_k}$. Also, if $x_{i_k} \in L$, for some $1 \leq k \leq r$, then $c_k = a_k + b_k$, and so $\text{lcm}(h, x_{i_k}^{c_k}) = hx_{i_k}^{a_k}$. Thus, $\mathcal{G}(hR \cap Q') = \mathcal{G}(hQ)$, and therefore $hQ = hR \cap Q'$. Since the following R -homomorphism

$$\theta : R/hR \cap Q' \rightarrow R/hR \oplus R/Q',$$

given by $\theta(r + hR \cap Q') = (r + hR, r + Q')$ for $r \in R$, is monomorphism, we deduce that

$$\text{Ass}_R(R/hR \cap Q') \subseteq \text{Ass}_R(R/hR) \cup \text{Ass}_R(R/Q'),$$

and also

$$\text{Ass}_R(R/hQ) \subseteq \{(x_{i_1}, \dots, x_{i_r}), (x_{j_1}), \dots, (x_{j_s})\}.$$

This finishes the proof. $\square$

Theorem 2.1.27. *Let I be a monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Also, assume that there exists a monomial $h = x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ such that $h|u_i$ for all $i = 1, \dots, m$. By setting $J := (u_1/h, \dots, u_m/h)$, we have*

$$\text{Ass}_R(R/I) = \text{Ass}_R(R/J) \cup \{(x_{j_1}), \dots, (x_{j_s})\}.$$

Proof. Since $h|u_i$ for all $i = 1, \dots, m$, we have $I = hJ$. Assume that $J = Q_1 \cap \cdots \cap Q_k$ is a minimal primary decomposition of J such that $\sqrt{Q_i} = \mathfrak{p}_i$ for all $i = 1, \dots, k$. Then we get $I = h(\bigcap_{i=1}^k Q_i) = \bigcap_{i=1}^k (hQ_i)$, and so the following R -homomorphism

$$\phi : R/I \rightarrow R/hQ_1 \oplus \cdots \oplus R/hQ_k,$$

given by $\phi(r + I) = (r + hQ_1, \dots, r + hQ_k)$ for $r \in R$, is monomorphism. Therefore,

$$\text{Ass}_R(R/I) \subseteq \text{Ass}_R(R/hQ_1) \cup \cdots \cup \text{Ass}_R(R/hQ_k).$$

Now, Lemma 2.1.26 implies that $\text{Ass}_R(R/hQ_i) = \{\mathfrak{p}_i, (x_{j_1}), \dots, (x_{j_s})\}$ for all $i = 1, \dots, k$. Thus, we obtain

$$\text{Ass}_R(R/I) \subseteq \{\mathfrak{p}_1, \dots, \mathfrak{p}_k, (x_{j_1}), \dots, (x_{j_s})\}.$$

For completing the proof, we need to show that

$$\{\mathfrak{p}_1, \dots, \mathfrak{p}_k, (x_{j_1}), \dots, (x_{j_s})\} \subseteq \text{Ass}_R(R/I).$$

To do this, choose $\mathfrak{p} \in \text{Ass}_R(R/J)$. Then $\mathfrak{p} = (J :_R v)$ for some monomial v in R . Since $(hJ :_R hv) = (J :_R v)$ and also $(hJ :_R hv) \in \text{Ass}_R(R/I)$, it is easy to see that $\{\mathfrak{p}_1, \dots, \mathfrak{p}_k\} \subseteq \text{Ass}_R(R/I)$.

On the other hand, since for all $\ell \in \{1, \dots, s\}$, $(x_{j_\ell}) \in \text{Min}(I)$, one can conclude that $\{(x_{j_1}), \dots, (x_{j_s})\} \subseteq \text{Ass}_R(R/I)$, and the proof is done. $\square$

Corollary 2.1.28. *Let I be a monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Also, assume that there exists a monomial $h = x_{j_1}^{\alpha_1} \cdots x_{j_t}^{\alpha_t}$ in R such that $h|u_i$ for all $i = 1, \dots, m$. By setting $J := (u_1/h, \dots, u_m/h)$, we have*

$$\text{Min}(I) = \{(x_{j_1}), (x_{j_2}), \dots, (x_{j_t})\} \cup \{\mathfrak{p} \in \text{Min}(J) \mid \text{supp}(\mathfrak{p}) \cap \{x_{j_1}, \dots, x_{j_t}\} = \emptyset\}.$$

Remark 2.1.29. ([133, Exercise 6.1.26]) *Let $R' = K[x_2, \dots, x_n]$ and $R = R'[x_1]$ be polynomial rings over a field K and $x_1, \dots, x_n$ are indeterminates. If I' is an ideal of R' and $\mathfrak{p} \in \text{Ass}_R(R/(I', x_1))$, then*

- (a) $\mathfrak{p} = x_1 R + \mathfrak{p}' R$, where $\mathfrak{p}'$ is a prime ideal of R' ; and,
- (b) $\mathfrak{p}'$ is an associated prime of R'/I' .

In the following theorem we provide a generalization of Remark 2.1.29.

Theorem 2.1.30. *Suppose that I with $\mathcal{G}(I) = \{u_1, \dots, u_{m-1}, x_{j_1}^{\alpha_1} \cdots x_{j_r}^{\alpha_r}\}$ is a monomial ideal of R such that $x_{j_t} \notin \text{supp}(u_i)$ for all $i = 1, \dots, m-1$ and $t = 1, \dots, r$. By setting $J := (u_1, \dots, u_{m-1})$, we have*

$$\text{Ass}_R(R/I) = \{\mathfrak{p} \mid \mathfrak{p} = \mathfrak{q} + x_{j_t} R \text{ for all } t = 1, \dots, r \text{ and } \mathfrak{q} \in \text{Ass}_R(R/J)\}.$$

Proof. Let $J = Q_1 \cap \cdots \cap Q_s$ be a minimal primary decomposition of J such that $\sqrt{Q_i} = \mathfrak{p}_i$ for all $i = 1, \dots, s$. For simplicity, we put $h := x_{j_1}^{\alpha_1} \cdots x_{j_r}^{\alpha_r}$. Then we have

$$I = J + hR = hR + \bigcap_{i=1}^s Q_i = \bigcap_{i=1}^s (hR + Q_i).$$

Also, there exists an R -monomorphism

$$\theta : R/I \rightarrow \bigoplus_{i=1}^s R/hR + Q_i,$$

given by $\theta(r + I) = (r + hR + Q_1, \dots, r + hR + Q_s)$ for $r \in R$. Hence,

$$\text{Ass}_R(R/I) \subseteq \bigcup_{i=1}^s \text{Ass}_R(R/hR + Q_i).$$

Next, fix $1 \leq i \leq s$, and let $Q_i = (x_{i_1}^{c_1}, \dots, x_{i_k}^{c_k})$ with $c_1, \dots, c_k \in \mathbb{N}$. Since

$$Q_i + hR = Q_i + \bigcap_{t=1}^r (x_{j_t}^{\alpha_t}) = \bigcap_{t=1}^r (Q_i + (x_{j_t}^{\alpha_t})),$$

there exists an R -monomorphism

$$\phi : R/Q_i + hR \rightarrow \bigoplus_{t=1}^r R/Q_i + (x_{j_t}^{\alpha_t}),$$

given by $\phi(r + Q_i + hR) = (r + Q_i + (x_{j_1}^{\alpha_1}), \dots, r + Q_i + (x_{j_r}^{\alpha_r}))$ for $r \in R$, and so

$$\text{Ass}_R(R/Q_i + hR) \subseteq \bigcup_{t=1}^r \text{Ass}_R(R/Q_i + (x_{j_t}^{\alpha_t})).$$

Since $Q_i + (x_{j_t}^{\alpha_t}) = (x_{i_1}^{c_1}, \dots, x_{i_k}^{c_k}, x_{j_t}^{\alpha_t})$ is a $(x_{i_1}, \dots, x_{i_k}, x_{j_t})$ -primary ideal, for all $t = 1, \dots, r$, it is easy to see that

$$\text{Ass}_R(R/Q_i + (x_{j_t}^{\alpha_t})) = (x_{i_1}, \dots, x_{i_k}, x_{j_t}) = \mathfrak{p}_i + (x_{j_t}).$$

Therefore, we get

$$\text{Ass}_R(R/I) \subseteq \{\mathfrak{p} \mid \mathfrak{p} = \mathfrak{q} + x_{j_t}R \text{ for all } t = 1, \dots, r \text{ and } \mathfrak{q} \in \text{Ass}_R(R/J)\}.$$

For the reverse inclusion, suppose that $\mathfrak{q} \in \text{Ass}_R(R/J)$. Then there exists monomial v in R such that $\mathfrak{q} = (J :_R v)$. On account of Lemma 2.1.12, we may assume that $\text{supp}(v) \subseteq \bigcup_{i=1}^{m-1} \text{supp}(u_i)$. Fix $1 \leq t \leq r$ and put $h' := x_{j_1}^{\alpha_1} \cdots x_{j_{t-1}}^{\alpha_{t-1}} x_{j_t}^{\alpha_t-1} x_{j_{t+1}}^{\alpha_{t+1}} \cdots x_{j_r}^{\alpha_r}$. Since $x_{j_s} \notin \bigcup_{i=1}^{m-1} \text{supp}(u_i)$, where $1 \leq s \leq r$, by Lemma 2.1.12, it is easy to see that $(J :_R v) = (J :_R v h')$ and $(x_{j_t}) = (x_{j_t} :_R v)$. We therefore get the following equalities

$$\begin{aligned} \mathfrak{q} + (x_{j_t}) &= (J :_R v) + (x_{j_t} :_R v) \\ &= (J :_R v) + (x_{j_t} h' :_R v h') \\ &= (J :_R v) + (h :_R v h') \\ &= (J :_R v h') + (h :_R v h') \\ &= (J + hR :_R v h') \\ &= (I :_R v h'). \end{aligned}$$

Also, since $(I :_R v h') \in \text{Ass}_R(R/I)$, we derive that $\mathfrak{q} + (x_{j_t}) \in \text{Ass}_R(R/I)$. Thus, $\{\mathfrak{p} \mid \mathfrak{p} = \mathfrak{q} + x_{j_t}R \text{ for all } t = 1, \dots, r \text{ and } \mathfrak{q} \in \text{Ass}_R(R/J)\} \subseteq \text{Ass}_R(R/I)$. $\square$

Corollary 2.1.31. *Suppose that I with $\mathcal{G}(I) = \{u_1, \dots, u_{m-1}, w_1, \dots, w_r\}$ is a monomial ideal of R such that $\text{supp}(w_i) \cap \text{supp}(w_j) = \emptyset$ for all $1 \leq i \neq j \leq r$ and $\text{supp}(w_i) \cap \text{supp}(u_j) = \emptyset$ for all $1 \leq i \leq r$ and $1 \leq j \leq m-1$. Then the set of associated prime ideals of I is equal to the following set*

$$\{\mathfrak{q} + \sum_{i=1}^r z_i R \mid \text{for all } \mathfrak{q} \in \text{Ass}_R(R/J) \text{ and } z_i \in \text{supp}(w_i) \text{ with } i = 1, \dots, r\},$$

where $J := (u_1, \dots, u_{m-1})$.

Proof. We proceed by induction on r . In the case in which $r = 1$, the claim can be deduced from Theorem 2.1.30. Now, suppose, inductively, that $r > 1$ and that result has been proved for r . Set $J_1 := (u_1, \dots, u_{m-1}, w_1, \dots, w_{r-1})$. Theorem 2.1.30 implies that

$$\text{Ass}_R(R/I) = \{\mathfrak{q}' + z_r R \mid \text{for all } \mathfrak{q}' \in \text{Ass}_R(R/J_1) \text{ and } z_r \in \text{supp}(w_r)\}.$$

Moreover, our inductive hypothesis yields that the set $\text{Ass}_R(R/J_1)$ is equal to

$$\{\mathfrak{q} + \sum_{i=1}^{r-1} z_i R \mid \text{for all } \mathfrak{q} \in \text{Ass}_R(R/J) \text{ and } z_i \in \text{supp}(w_i) \text{ with } i = 1, \dots, r-1\}.$$

This completes the inductive step and the proof. $\square$

Example 2.1.32. *Consider the monomial ideal*

$$I = (x_1^3 x_2^2 x_3, x_1^2 x_2^2 x_3 x_4, x_1^2 x_2^4 x_3^2 x_5, x_1 x_2^2 x_6^3 x_7, x_1 x_2^2 x_8 x_9^2),$$

in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$, where K is a field and $x_1, \dots, x_9$ are indeterminates. Now, put $h := x_1 x_2^2$ and

$$J := (x_1^2 x_3, x_1 x_3 x_4, x_1 x_2^2 x_3^2 x_5, x_6^3 x_7, x_8 x_9^2).$$

By Theorem 2.1.27, we can conclude that

$$\text{Ass}_R(R/I) = \text{Ass}_R(R/J) \cup \{(x_1), (x_2)\}.$$

Next, set $J_1 := (x_1^2 x_3, x_1 x_3 x_4, x_1 x_2^2 x_3^2 x_5)$, $w_1 := x_6^3 x_7$, and $w_2 := x_8 x_9^2$. Corollary 2.1.31 shows that $\text{Ass}_R(R/J)$ is given by

$$\{(\mathfrak{q}, x_6, x_8), (\mathfrak{q}, x_6, x_9), (\mathfrak{q}, x_7, x_8), (\mathfrak{q}, x_7, x_9) \mid \text{for all } \mathfrak{q} \in \text{Ass}_R(R/J_1)\}.$$

Also, put $h := x_1 x_3$ and $J_2 := (x_1, x_4, x_2^2 x_3 x_5)$. By Theorem 2.1.27, one derives that

$$\text{Ass}_R(R/J_1) = \text{Ass}_R(R/J_2) \cup \{(x_1), (x_3)\}.$$

Again, set $J_3 := (x_1, x_4)$. By Corollary 2.1.31, we obtain

$$\text{Ass}_R(R/J_2) = \{(\mathbf{q}', x_2), (\mathbf{q}', x_3), (\mathbf{q}', x_5) \mid \text{for all } \mathbf{q}' \in \text{Ass}_R(R/J_3)\}.$$

Finally, $\text{Ass}_R(R/J_3) = \{(x_1, x_4)\}$, and so the procedure has been terminated. Then we have

$$\text{Ass}_R(R/J_2) = \{(x_1, x_4, x_2), (x_1, x_4, x_3), (x_1, x_4, x_5)\},$$

and

$$\text{Ass}_R(R/J_1) = \{(x_1, x_4, x_2), (x_1, x_4, x_3), (x_1, x_4, x_5), (x_1), (x_3)\}.$$

Therefore, $\text{Ass}_R(R/J)$ is given by

$$\begin{aligned} \text{Ass}_R(R/J) = \{ & (x_1, x_4, x_2, x_6, x_8), (x_1, x_4, x_3, x_6, x_8), (x_1, x_4, x_5, x_6, x_8), (x_1, x_6, x_8), \\ & (x_3, x_7, x_8), (x_3, x_6, x_8), (x_1, x_4, x_2, x_6, x_9), (x_1, x_4, x_3, x_6, x_9), \\ & (x_1, x_4, x_5, x_6, x_9), (x_1, x_6, x_9), (x_3, x_6, x_9), (x_1, x_4, x_2, x_7, x_8), \\ & (x_1, x_4, x_3, x_7, x_8), (x_1, x_4, x_5, x_7, x_8), (x_1, x_7, x_8), (x_1, x_4, x_5, x_7, x_9), \\ & (x_1, x_7, x_9), (x_3, x_7, x_9), (x_1, x_4, x_2, x_7, x_9), (x_1, x_4, x_3, x_7, x_9)\}. \end{aligned}$$

Ultimately, we conclude that $\text{Ass}_R(R/I)$ is given by

$$\begin{aligned} \text{Ass}_R(R/I) = \{ & (x_1, x_4, x_2, x_6, x_8), (x_1, x_4, x_3, x_6, x_8), (x_1, x_4, x_5, x_6, x_8), (x_1), \\ & (x_2), (x_1, x_6, x_8), (x_3, x_6, x_8), (x_1, x_4, x_2, x_6, x_9), (x_1, x_4, x_3, x_6, x_9), \\ & (x_1, x_4, x_5, x_6, x_9), (x_1, x_6, x_9), (x_3, x_6, x_9), (x_1, x_4, x_2, x_7, x_8), \\ & (x_1, x_4, x_3, x_7, x_8), (x_1, x_4, x_5, x_7, x_8), (x_1, x_7, x_8), (x_1, x_4, x_5, x_7, x_9), \\ & (x_1, x_7, x_9), (x_3, x_7, x_9), (x_1, x_4, x_2, x_7, x_9), (x_1, x_4, x_3, x_7, x_9) \\ & (x_3, x_7, x_8)\}. \end{aligned}$$

Moreover, direct computation with Macaulay2 shows that our result is true.

2.2 Persistence property and the Alexander dual

We first recall some definitions from [54] and [137] which will be needed in the following. It should be noted that the results of this section can be found in [102].

Definition 2.2.1. A directed edge of a graph is an assignment of a direction to an edge of a graph. If $\{w, u\}$ is an edge, we write (w, u) to denote the directed edge, where the direction is from w to u . A graph is a directed graph if each edge has been

assigned a direction. A path of length t in a directed graph is a sequence of vertices $i_1, \dots, i_{t+1}$ such that $e_j = (i_j, i_{j+1})$ is a directed edge for $j = 1, \dots, t$.

Fix a positive integer t and a directed graph G . The path ideal of G of length t is the following monomial ideal

$$I_t(G) := (x_{i_1} \cdots x_{i_{t+1}} : i_1, \dots, i_{t+1} \text{ is a path of } G \text{ of length } t),$$

and also the Alexander dual of $I_t(G)$ is defined as follows:

$$I_t(G)^\vee := \bigcap_{i_1, \dots, i_{t+1} \text{ is a path of } G \text{ of length } t} (x_{i_1}, \dots, x_{i_{t+1}}).$$

A tree T can be considered as a directed graph by choosing any vertex of T to be the root of the tree, and assigning to each edge the direction “away” from the root. Since T is a tree, the assignment of a direction will always be possible. A rooted tree T is a tree with one vertex chosen as root. If tree T has no such root, then we say that T is unrooted.

As an example, consider $T_1 = (V(T_1), E(T_1))$ and $T_2 = (V(T_2), E(T_2))$, where $V(T_1) = V(T_2) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(T_1) = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_2, v_5)\}$, and $E(T_2) = \{(v_1, v_2), (v_1, v_3), (v_2, v_4), (v_2, v_5)\}$ in the following graphs. Then the tree T_1 , the left graph in Figure 1, is an example of a tree which is not rooted, while the tree T_2 , the right graph in Figure 1, is rooted at the vertex v_1 .

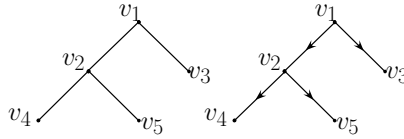


Figure 1

It has already been proved in [16, Theorem 5.8] and [17, Theorem 3.1] (when $t = 2$) the following theorem.

Theorem 2.2.2. *Let $K_{1,n}$ be a star graph with the vertex set $\{z, x_1, \dots, x_n\}$ and centre z . Let J_2 be the corresponding Alexander dual of $I_2(K_{1,n})$. Then*

- (1) $(z, x_1, \dots, x_n) \in \text{Ass}(R/J_2^s)$ for $s \geq n - 1$; and,
- (2) $(z, x_1, \dots, x_n) \notin \text{Ass}(R/J_2^s)$ for $s < n - 1$,

where $(z, x_1, \dots, x_n) \subset K[z, x_1, \dots, x_n]$ is the maximal ideal.

Next, in [16, Conjecture 6.5], made up the following conjecture.

Conjecture 2.2.3. *Let G be an unrooted tree on the vertex set $V_G = \{z, x_1, \dots, x_n\}$ with the edge set E_G which is*

$$\{zx_1, zx_2, \dots, zx_k, x_1x_{k+1}, x_2x_{k+2}, \dots, x_kx_{2k}, x_{k+1}x_{2k+1}, x_{k+2}x_{2k+2}, \dots, x_{2k}x_{3k}, \dots\}.$$

Then

- (1) $(z, x_1, \dots, x_n) \in \text{Ass}(R/J_{2\ell}^s)$ for $s \geq k - 1$; and,
- (2) $(z, x_1, \dots, x_n) \notin \text{Ass}(R/J_{2\ell}^s)$ for $s < k - 1$,

where 2ℓ is the length of the longest even path of G .

In this section, we provide an affirmative answer to this conjecture. To do this, we begin with the following definition.

Definition 2.2.4. *A tree is said to be starlike if exactly one of its vertices has degree greater than two. This vertex is called the center of the starlike.*

We also say that T is a symmetric starlike tree if T is a starlike tree and $d(u, v) = d(u', v')$ for all $u, u', v, v' \in L_T$ with $u \neq v$ and $u' \neq v'$.

For instance, consider the following trees. Then the left graph in Figure 2 is an example of a starlike tree with center z , while the right graph in Figure 2 is an example of a symmetric starlike tree with center z .

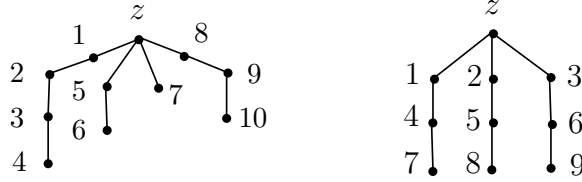


Figure 2

In the following, we probe the persistence property of the Alexander dual of path ideals. For this purpose, we need the following lemma.

Lemma 2.2.5. *Suppose that I is a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$, $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, and $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$. Suppose also that there exists a variable $x_t \in \{x_1, \dots, x_n\}$ such that $x_t \in \mathfrak{p}_i$ for all $i = 1, \dots, s$. Then there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$ such that $u_i = x_t$.*

Proof. Since I is a square-free monomial ideal, by [60, Corollary 1.3.6], it follows that $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_s$, and hence $x_t \in I$. On account of [60, Proposition 1.1.5], there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$ such that $u_i \in \mathcal{G}(I)$ and u_i divides x_t . Because u_i is a monomial in R , this implies that $u_i = x_t$, as claimed. $\square$

Theorem 2.2.6. *Let T be an unrooted starlike tree on the vertex set $\{z, 1, \dots, n\}$ with center z . Also, let I be the monomial ideal corresponding to T which is generated by the paths of maximal lengths, and corresponding Alexander dual J . Then the ideal J has the persistence property.*

Proof. Suppose that $|L_T| = k$. It is routine to check that the path $i_1, \dots, i_t$ has maximal length in T if and only if $i_1, i_t \in L_T$. This implies that

$$I = (x_{i_1} \cdots x_{i_j} x_z x_{i_{j+1}} \cdots x_{i_t} : i_1, i_t \in L_T).$$

We note that I has exactly $\binom{k}{2}$ minimal generators. Thus, one can easily show that

$$J = \bigcap_{i_1, i_t \in L_T} (x_{i_1}, \dots, x_{i_j}, x_z, x_{i_{j+1}}, \dots, x_{i_t}),$$

which is a minimal primary decomposition of J in $R' = K[x_z, x_1, \dots, x_n]$ over a field K . In view of Lemma 2.2.5, we deduce that $x_z \in \mathcal{G}(J)$. Assume that $\mathcal{G}(J) = \{x_z, u_1, \dots, u_s\}$. Since $\mathcal{G}(J)$ is a minimal generating set for J , we obtain $\gcd(x_z, u_j) = 1$ for all $j = 1, \dots, s$. Now, let $t \in \mathbb{N}$ be an arbitrary element and pick $\mathfrak{p} \in \text{Ass}_{R'}(R'/J^t)$. Hence, there exists a monomial v in R' such that $\mathfrak{p} = (J^t :_{R'} v)$. Our aim is now to establish $\mathfrak{p} = (J^{t+1} :_{R'} x_z v)$. In the light of $J = x_z R' + \sum_{j=1}^s u_j R'$, it follows that $J^{t+1} = J^t x_z R' + J^t \sum_{j=1}^s u_j R'$, and so

$$(J^{t+1} :_{R'} x_z v) = (J^t x_z :_{R'} x_z v) + \left(\sum_{j=1}^s J^t u_j :_{R'} x_z v \right).$$

Since $(J^t x_z :_{R'} x_z v) = (J^t :_{R'} v)$ and $(\sum_{j=1}^s J^t u_j :_{R'} x_z v) = \sum_{j=1}^s (J^t u_j :_{R'} x_z v)$, we can derive the following equality

$$(J^{t+1} :_{R'} x_z v) = (J^t :_{R'} v) + \sum_{j=1}^s (J^t u_j :_{R'} x_z v).$$

For completing the proof, it is enough to prove that $(J^t u_j :_{R'} x_z v) \subseteq \mathfrak{p}$ for any $j = 1, \dots, s$. To do this, fix j with $1 \leq j \leq s$ and consider the following two cases:

Case 1. $u_1^{\alpha_1} \cdots u_s^{\alpha_s} \in \mathcal{G}(J^t)$ with $\alpha_1 + \cdots + \alpha_s = t$ and $\alpha_1, \dots, \alpha_s \in \mathbb{N}_0$. Due to $(u_j, x_z) = 1$ for all $j = 1, \dots, s$, one has $(u_1^{\alpha_1} \cdots u_s^{\alpha_s} u_j :_{R'} x_z v) = (u_1^{\alpha_1} \cdots u_s^{\alpha_s} u_j :_{R'} v)$. Thus, $(u_1^{\alpha_1} \cdots u_s^{\alpha_s} u_j :_{R'} x_z v) \subseteq \mathfrak{p}$.

Case 2. $u_1^{\alpha_1} \cdots u_s^{\alpha_s} x_z^\beta \in \mathcal{G}(J^t)$ with $\beta \in \mathbb{N}$, $\alpha_1 + \cdots + \alpha_s + \beta = t$, and $\alpha_1, \dots, \alpha_s \in \mathbb{N}_0$. It follows that $(u_1^{\alpha_1} \cdots u_s^{\alpha_s} x_z^\beta u_j :_{R'} x_z v) = (u_1^{\alpha_1} \cdots u_s^{\alpha_s} x_z^{\beta-1} u_j :_{R'} v)$. Hence, $(u_1^{\alpha_1} \cdots u_s^{\alpha_s} x_z^\beta u_j :_{R'} x_z v) \subseteq \mathfrak{p}$.

Thus, $\mathfrak{p} \in \text{Ass}_{R'}(R'/J^{t+1})$, and so the ideal J has the persistence property, as desired. $\square$

In the next example we describe the major goal of Theorem 2.2.6.

Example 2.2.7. Suppose that T is the unrooted starlike tree with center z , as the left graph in Figure 2. By using the notations similar that we used in the proof of Theorem 2.2.6, we obtain

$$I = (x_4x_3x_2x_1x_zx_8x_9x_{10}, x_4x_3x_2x_1x_zx_7, x_5x_6x_zx_7, x_4x_3x_2x_1x_zx_5x_6, \\ x_5x_6x_zx_8x_9x_{10}, x_7x_zx_8x_9x_{10}),$$

and so

$$J = (x_4, x_3, x_2, x_1, x_z, x_8, x_9, x_{10}) \cap (x_4, x_3, x_2, x_1, x_z, x_7) \cap (x_5, x_6, x_z, x_7) \\ \cap (x_4, x_3, x_2, x_1, x_z, x_5, x_6) \cap (x_5, x_6, x_z, x_8, x_9, x_{10}) \cap (x_7, x_z, x_8, x_9, x_{10}).$$

According to Theorem 2.2.6, the ideal J has the persistence property.

Corollary 2.2.8. Suppose that T is an unrooted symmetric starlike tree on the vertex set $V(T) = \{z, 1, \dots, n\}$ with center z and the following edge set

$$E(T) = \{\{z, i\}, \{kj + i, kj + k + i\} \mid i = 1, \dots, k, j = 0, \dots, m - 1\},$$

such that $n = k(m + 1)$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Suppose also that $I_{2m+2}(T)$ is the path ideal corresponding to T of length $2m + 2$ and corresponding Alexander dual J_{2m+2} . Then the ideal J_{2m+2} has the persistence property.

Let us illustrate Corollary 2.2.8 with an example. Suppose that T is the unrooted symmetric starlike tree with center z , as shown in Figure 2, on the vertex set $V(T) = \{z, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then we have

$$I_6(T) = (x_7x_4x_1x_zx_2x_5x_8, x_7x_4x_1x_zx_3x_6x_9, x_8x_5x_2x_zx_3x_6x_9).$$

It follows that the Alexander dual of $I_6(T)$, denoted by J_6 , is given by

$$J_6 = (x_7, x_4, x_1, x_z, x_2, x_5, x_8) \cap (x_7, x_4, x_1, x_z, x_3, x_6, x_9) \cap (x_8, x_5, x_2, x_z, x_3, x_6, x_9) \\ = (x_z, x_8x_9, x_7x_9, x_5x_9, x_4x_9, x_2x_9, x_1x_9, x_7x_8, x_6x_8, x_4x_8, x_3x_8, x_1x_8, x_6x_7, \\ x_5x_7, x_3x_7, x_2x_7, x_5x_6, x_4x_6, x_2x_6, x_1x_6, x_4x_5, x_3x_5, x_1x_5, x_3x_4, x_2x_4, x_2x_3, \\ x_1x_3, x_1x_2).$$

Now, Corollary 2.2.8 implies that the ideal J_6 has the persistence property. On the other hand, one can see this result in another way. In fact, the ideal

$$L = (x_8x_9, x_7x_9, x_5x_9, x_4x_9, x_2x_9, x_1x_9, x_7x_8, x_6x_8, x_4x_8, x_3x_8, x_1x_8, \\ x_6x_7, x_5x_7, x_3x_7, x_2x_7, x_5x_6, x_4x_6, x_2x_6, x_1x_6, x_4x_5, x_3x_5, x_1x_5, \\ x_3x_4, x_2x_4, x_2x_3, x_1x_3, x_1x_2),$$

is generated by quadratic square-free monomials, and so it is the edge ideal of another graph. By [88, Theorem 2.15], one can conclude that L has the persistence property, and [50, Lemma 3.4] yields that J_6 has the persistence property.

We now explore when the unique homogenous maximal ideal appears. To achieve this, we first recall the definition of the expansion operator on monomial ideals which has been stated in [13], and then apply it as a criterion for the persistence property of monomial ideals.

Definition 2.2.9. *Let K be a field and R be the polynomial ring over a field K in the variables $x_1, \dots, x_n$. Fix an ordered n -tuple $(i_1, \dots, i_n)$ of positive integers, and consider the polynomial ring $R^{(i_1, \dots, i_n)}$ over K in the variables*

$$x_{11}, \dots, x_{1i_1}, x_{21}, \dots, x_{2i_2}, \dots, x_{n1}, \dots, x_{ni_n}.$$

Let $\mathfrak{p}_j$ be the monomial prime ideal $(x_{j1}, x_{j2}, \dots, x_{ji_j}) \subseteq R^{(i_1, \dots, i_n)}$ for all $j = 1, \dots, n$. Attached to each monomial ideal $I \subset R$ with a set of monomial generators $\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_m}\}$, where $\mathbf{x}^{\mathbf{a}_i} = x_1^{a_i(1)} \dots x_n^{a_i(n)}$ and $a_i(j)$ denotes the j -th component of the vector $\mathbf{a}_i = (a_i(1), \dots, a_i(n))$ for all $i = 1, \dots, m$. We define the expansion of I with respect to the n -tuple $(i_1, \dots, i_n)$, denoted by $I^{(i_1, \dots, i_n)}$, to be the monomial ideal

$$I^{(i_1, \dots, i_n)} = \sum_{i=1}^m \mathfrak{p}_1^{a_i(1)} \dots \mathfrak{p}_n^{a_i(n)} \subseteq R^{(i_1, \dots, i_n)}.$$

We simply write R^* and I^* , respectively, rather than $R^{(i_1, \dots, i_n)}$ and $I^{(i_1, \dots, i_n)}$.

For example, consider $R = K[x_1, x_2, x_3]$ and the ordered 3-tuple $(1, 3, 2)$. Then we have $\mathfrak{p}_1 = (x_{11})$, $\mathfrak{p}_2 = (x_{21}, x_{22}, x_{23})$, and $\mathfrak{p}_3 = (x_{31}, x_{32})$. So, for the monomial ideal $I = (x_1 x_2, x_3^2)$, the ideal $I^* \subseteq K[x_{11}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}]$ is $\mathfrak{p}_1 \mathfrak{p}_2 + \mathfrak{p}_3^2$, namely

$$I^* = (x_{11}x_{21}, x_{11}x_{22}, x_{11}x_{23}, x_{31}^2, x_{31}x_{32}, x_{32}^2).$$

To prove Lemma 2.2.12, one needs frequently use the following proposition and lemma. We state them here for ease of reference.

Proposition 2.2.10. ([13, Proposition 1.2]) *Let I be a monomial ideal, and consider an (irredundant) primary decomposition $I = Q_1 \cap \dots \cap Q_m$ of I . Then $I^* = Q_1^* \cap \dots \cap Q_m^*$ is an (irredundant) primary decomposition of I^* . In particular,*

$$\text{Ass}(S^*/I^*) = \{\mathfrak{p}^* : \mathfrak{p} \in \text{Ass}(S/I)\}.$$

Lemma 2.2.11. ([13, Lemma 1.1]) *Let I and J be monomial ideals in a polynomial ring S . Then*

- (i) $f \in I^*$ if and only $\pi(f) \in I$, for all $f \in S^*$;

$$(ii) \ (I + J)^* = I^* + J^*;$$

$$(iii) \ (IJ)^* = I^* J^*;$$

$$(iv) \ (I \cap J)^* = I^* \cap J^*;$$

$$(v) \ (I : J)^* = (I^* : J^*);$$

$$(vi) \ \sqrt{I^*} = \sqrt{I}^*;$$

(vii) If the monomial ideal Q is $\mathfrak{p}$ -primary, then Q^* is $\mathfrak{p}^*$ -primary.

To prove Theorem 2.2.13 we require the following auxiliary lemma.

Lemma 2.2.12. *Let I be a monomial ideal of R . Then I has the persistence property if and only if I^* has.*

Proof. For necessity, consider $\mathfrak{p} \in \text{Ass}_R(R/I^n)$ for an arbitrary $n \in \mathbb{N}$. According to Proposition 2.2.10, one has $\text{Ass}_{R^*}(R^*/(I^n)^*) = \{\mathfrak{p}^* : \mathfrak{p} \in \text{Ass}_R(R/I^n)\}$. It follows that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/(I^n)^*)$. Due to Lemma 2.2.11(iii), we have $(I^n)^* = (I^*)^n$. Since I^* has the persistence property, one can conclude that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/(I^*)^{n+1})$. Again, by Lemma 2.2.11(iii), we deduce that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/(I^{n+1})^*)$. By virtue of Proposition 2.2.10, we get $\mathfrak{p} \in \text{Ass}_R(R/I^{n+1})$. This implies that I has the persistence property. Sufficiency follows from in a similar way, and the proof is complete. $\square$

We are now ready to establish Theorem 2.2.13.

Theorem 2.2.13. *Let T be an unrooted starlike tree on the vertex set $\{z, 1, \dots, n\}$ with center z . Also, let I be the monomial ideal corresponding to T which is generated by the paths of maximal lengths, and corresponding Alexander dual J . If $\deg_T z = k$, then*

$$(1) \ (x_z, x_1, \dots, x_n) \in \text{Ass}_{R'}(R'/J^s) \text{ for } s \geq k - 1; \text{ and,}$$

$$(2) \ (x_z, x_1, \dots, x_n) \notin \text{Ass}_{R'}(R'/J^s) \text{ for } s < k - 1,$$

where $(x_z, x_1, \dots, x_n)$ is the unique homogeneous maximal ideal in the polynomial ring $R' = K[x_z, x_1, \dots, x_n]$.

Proof. Since $\deg_T z = k$, the graph $T \setminus \{z\}$ has exactly k connected components, say $L_1, \dots, L_k$, where each component is a path graph with $|V(L_i)| = h_i$ for each $i = 1, \dots, k$. Put $h_0 := 0$ and for all $i = 1, \dots, k$, let

$$V(L_i) := \{h_1 + \dots + h_{i-1} + j : j = 1, \dots, h_i\}.$$

This implies that $E(T)$ is given by

$$\left\{ \left\{ z, \sum_{t=1}^{i-1} h_t + 1 \right\}, \left\{ \sum_{t=1}^{i-1} h_t + j, \sum_{t=1}^{i-1} h_t + j + 1 \right\} : i = 1, \dots, k, j = 1, \dots, h_i - 1 \right\}.$$

Now, set $\mathfrak{p}_i := (x_{h_1+\dots+h_{i-1}+j}, x_{h_1+\dots+h_{i-1}+j+1} : j = 1, \dots, h_i - 1)$ for all $i = 1, \dots, k$. Then one can conclude that

$$J = \bigcap_{i,j \in \{1, \dots, k\}, i \neq j} (\mathfrak{p}_i + x_z R' + \mathfrak{p}_j),$$

and so $J = x_z R' + \bigcap_{i,j \in \{1, \dots, k\}, i \neq j} (\mathfrak{p}_i + \mathfrak{p}_j)$. It is routine to check that

$$\bigcap_{i,j \in \{1, \dots, k\}, i \neq j} (\mathfrak{p}_i + \mathfrak{p}_j) = \sum_{j=1}^k \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_{j-1} \cap \widehat{\mathfrak{p}}_j \cap \mathfrak{p}_{j+1} \cap \dots \cap \mathfrak{p}_k,$$

where $\widehat{\mathfrak{p}}_j$ means that this term is omitted. As $\mathfrak{p}_i$ and $\mathfrak{p}_j$, for every $i, j \in \{1, \dots, k\}$ with $i \neq j$, are generated by disjoint sets, it follows that $\mathfrak{p}_i \cap \mathfrak{p}_j = \mathfrak{p}_i \mathfrak{p}_j$, and so

$$J = x_z R' + \sum_{j=1}^k \mathfrak{p}_1 \cdots \mathfrak{p}_{j-1} \widehat{\mathfrak{p}}_j \mathfrak{p}_{j+1} \cdots \mathfrak{p}_k.$$

Now, consider the following monomial ideal

$$\mathfrak{a} := x_z R' + \sum_{j=1}^k (x_1 \cdots x_{j-1} \widehat{x}_j x_{j+1} \cdots x_k) R',$$

where $\widehat{x}_j$ means that this term is omitted. In order to use Lemma 2.2.12, set $\mathfrak{p}_{k+1} := x_z R'$. This implies that J is the expansion of $\mathfrak{a}$ with respect to the $(k+1)$ -tuple $(h_1, h_2, h_3, \dots, h_k, 1)$. According to [17, Lemma 2.5], when $t = 2$, we obtain $\mathfrak{a} = J_2(G)$, where $G = K_{1,n}$ is the star graph on the vertex set $\{z, 1, \dots, n\}$ with center z and corresponding Alexander dual $J_2(G)$. Therefore, the result follows immediately from Lemma 2.2.12, [13, Proposition 1.2], and Theorem 2.2.2. $\square$

We can employ Theorem 2.2.13 to give an affirmative answer to Conjecture 2.2.3 in the following corollary.

Corollary 2.2.14. *Suppose that T is an unrooted symmetric starlike tree on the vertex set $V(T) = \{z, 1, \dots, n\}$ with center z and the following edge set*

$$E(T) = \{\{z, i\}, \{kj + i, kj + k + i\} \mid i = 1, \dots, k \text{ and } j = 0, \dots, m - 1\},$$

such that $n = k(m + 1)$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Suppose also that $I_{2m+2}(T)$ is the path ideal corresponding to T of length $2m + 2$ and corresponding Alexander dual J_{2m+2} . Then

- (1) $(x_z, x_1, \dots, x_n) \in \text{Ass}_{R'}(R'/J_{2m+2}^s)$ for $s \geq k - 1$; and,
- (2) $(x_z, x_1, \dots, x_n) \notin \text{Ass}_{R'}(R'/J_{2m+2}^s)$ for $s < k - 1$,

where $(x_z, x_1, \dots, x_n)$ is the unique homogeneous maximal ideal in the polynomial ring $R' = K[x_z, x_1, \dots, x_n]$.

The following example illuminates what happens in Corollary 2.2.14.

Example 2.2.15. Suppose that T is the unrooted symmetric starlike tree, as shown in Figure 3, on the vertex set $V(T) = \{z, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$.

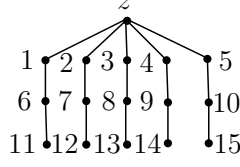


Figure 3

Hence, we obtain the path ideal $I_6(T)$ is given by

$$\begin{aligned} & (x_{11}x_6x_1x_zx_2x_7x_{12}, x_{11}x_6x_1x_zx_3x_8x_{13}, x_{11}x_6x_1x_zx_4x_9x_{14}, x_{11}x_6x_1x_zx_5x_{10}x_{15}, \\ & x_{12}x_7x_2x_zx_3x_8x_{13}, x_{12}x_7x_2x_zx_4x_9x_{14}, x_{12}x_7x_2x_zx_5x_{10}x_{15}, x_{13}x_8x_3x_zx_4x_9x_{14}, \\ & x_{13}x_8x_3x_zx_5x_{10}x_{15}, x_{14}x_9x_4x_zx_5x_{10}x_{15}). \end{aligned}$$

It follows that the Alexander dual of $I_6(T)$ is given by

$$\begin{aligned} J_6 &= (x_{11}, x_6, x_1, x_z, x_2, x_7, x_{12}) \cap (x_{11}, x_6, x_1, x_z, x_3, x_8, x_{13}) \\ &\cap (x_{11}, x_6, x_1, x_z, x_4, x_9, x_{14}) \cap (x_{11}, x_6, x_1, x_z, x_5, x_{10}, x_{15}) \\ &\cap (x_{12}, x_7, x_2, x_z, x_3, x_8, x_{13}) \cap (x_{12}, x_7, x_2, x_z, x_4, x_9, x_{14}) \\ &\cap (x_{12}, x_7, x_2, x_z, x_5, x_{10}, x_{15}) \cap (x_{13}, x_8, x_3, x_z, x_4, x_9, x_{14}) \\ &\cap (x_{13}, x_8, x_3, x_z, x_5, x_{10}, x_{15}) \cap (x_{14}, x_9, x_4, x_z, x_5, x_{10}, x_{15}). \end{aligned}$$

Due to Theorem 2.2.13, one can conclude that

- (1) $(x_z, x_1, \dots, x_{15}) \in \text{Ass}_{R'}(R'/J_6^s)$ for $s \geq 4$; and,
- (2) $(x_z, x_1, \dots, x_{15}) \notin \text{Ass}_{R'}(R'/J_6^s)$ for $s < 4$,

where $(x_z, x_1, \dots, x_{15})$ is the unique homogeneous maximal ideal in the polynomial ring $R' = K[x_z, x_1, \dots, x_{15}]$.

We conclude this argument with giving another class of graphs which the path ideals generated by path of length two have the persistence property. To do this, we first recall the definition of strongly superficial elements and then introduce a class of graphs which are called centipede graphs. Moreover, it should be noted that the following results can be found in [97].

Definition 2.2.16. Let I be an ideal in a commutative ring S , and let $k \in \mathbb{N}$. An element x in S is called a *superficial element of degree k for I* if $x \in I^k$ and there exists $c \in \mathbb{N}$ such that $(I^{n+k} :_S x) \cap I^c = I^n$ for all $n \geq c$. If $(I^{n+k} :_S x) = I^n$ for all $n \in \mathbb{N}$, we say that x is a *strongly superficial element of degree k for I* (see [90, 4.1.2]).

The following proposition is essential for us in order to complete the proof of Theorem 2.2.19.

Proposition 2.2.17. Suppose that I is an arbitrary ideal in a commutative Noetherian ring S such that has a strongly superficial element of degree one. Then I has the persistence property.

Proof. Let u be a strongly superficial element of degree one for I . Then $(I^{n+1} :_S u) = I^n$ for all $n \in \mathbb{N}$. Since $(I^{n+1} :_S I) \subseteq (I^{n+1} :_S u)$, it follows that $(I^{n+1} :_S I) = I^n$ for all $n \in \mathbb{N}$. For completing the proof, assume that $I = (v_1, \dots, v_t)$. Choose an arbitrary $m \in \mathbb{N}$ and consider $\mathfrak{p} \in \text{Ass}_S(S/I^m)$. Then there exists an element c in S such that $\mathfrak{p} = (I^m :_S c)$. This implies that

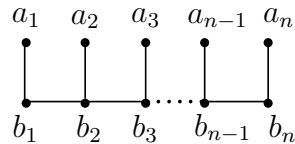
$$\begin{aligned} \mathfrak{p} &= ((I^{m+1} :_S I) :_S c) \\ &= (I^{m+1} :_S Ic) \\ &= \bigcap_{i=1}^t (I^{m+1} :_S v_i c). \end{aligned}$$

One can conclude that there exists a positive integer $1 \leq j \leq t$ such that $\mathfrak{p} = (I^{m+1} :_S v_j c)$, and so $\mathfrak{p} \in \text{Ass}_S(S/I^{m+1})$. Therefore, I has the persistence property, as claimed. $\square$

In the next definition, we present a class of graphs which we will use in Theorem 2.2.19.

Definition 2.2.18. The *centipede graph* W_n with $n \in \mathbb{N}$, as shown in the figure below, is the graph on the vertex set $\{a_1, \dots, a_n\} \cup \{b_1, \dots, b_n\}$. The set of edges of the centipede graph is given by

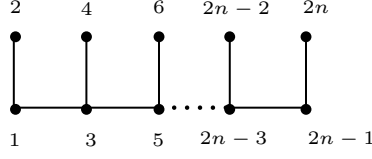
$$E(W_n) = \{\{a_i, b_i\} : 1 \leq i \leq n\} \cup \{\{b_j, b_{j+1}\} : 1 \leq j \leq n-1\}.$$



We now state another main result of this section in the following theorem.

Theorem 2.2.19. Let W_n , for some $n \in \mathbb{N}$ with $n \geq 2$, be a centipede graph with corresponding path ideal $I_2(W_n)$. Then $I_2(W_n)$ has the persistence property.

Proof. To simplify the notation, we consider the following centipede graph on the vertex set $V(W_n) = \{1, 2, 3, \dots, 2n-2, 2n-1, 2n\}$ and also set $J := I_2(W_n)$.



It is routine to verify that

$$\begin{aligned} \mathcal{G}(J) = & \{x_{2k-1}x_{2k}x_{2k+1}, x_{2k-1}x_{2k+1}x_{2k+2} : k = 1, \dots, n-1\} \\ & \cup \{x_{2r-1}x_{2r+1}x_{2r+3} : r = 1, \dots, n-2\}. \end{aligned}$$

Clearly, the ideal J has exactly $3n-4$ distinct minimal generators. Assume that $\mathcal{G}(J) = \{u_1, \dots, u_k\}$ with $k := 3n-4$, such that $u_1 := x_1x_2x_3$, $u_2 := x_1x_3x_4$, $u_3 := x_1x_3x_5$, $u_4 := x_3x_5x_7$, $u_5 := x_3x_5x_6$, and $u_6 := x_3x_4x_5$. Here and in the following, our aim is to show that $(J^{m+1} :_R u_1) = J^m$ for all $m \in \mathbb{N}$. To do this, choose an arbitrary $m \in \mathbb{N}$. Then we have the following equalities

$$\begin{aligned} (J^{m+1} :_R u_1) &= (J^m J :_R u_1) \\ &= \sum_{i=1}^k (J^m u_i :_R u_1) \\ &= J^m + \sum_{i=2}^k (J^m u_i :_R u_1). \end{aligned}$$

For completing the proof, we show that $(J^m u_i :_R u_1) \subseteq J^m$ for all $i = 2, \dots, k$. To achieve this, choose an arbitrary element $v \in (J^m u_i :_R u_1)$ for some $2 \leq i \leq k$. Hence, $vu_1 \in J^m u_i$, and so there exists a generator of J^m such as $u_1^{a_1} \cdots u_k^{a_k}$ with $a_1 + \cdots + a_k = m$, such that $u_i u_1^{a_1} \cdots u_k^{a_k}$ divides vu_1 . So, there is a monomial w in R such that $vu_1 = w u_i u_1^{a_1} \cdots u_k^{a_k}$. If $a_1 \geq 1$, then $v = w u_i u_1^{a_1-1} \cdots u_k^{a_k}$. By $u_i u_1^{a_1-1} \cdots u_k^{a_k} \in J^m$, one can conclude that $v \in J^m$. Thus, we assume that $a_1 = 0$, and so $a_2 + \cdots + a_k = m$. Since $x_2 \notin u_j$ for all $j = 2, \dots, k$, it follows that $vx_1x_3 = w' u_i u_2^{a_2} \cdots u_k^{a_k}$ such that $w = x_2 w'$ for some monomial w' in R . Hence, one may consider the following cases:

Case 1. $u_i = u_2$ or $u_i = u_3$. Accordingly, $v = w' x_4 u_2^{a_2} \cdots u_k^{a_k}$ or $v = w' x_5 u_2^{a_2} \cdots u_k^{a_k}$, and so $v \in J^m$.

Case 2. $u_i = u_4$. Then $vx_1 = w' x_5 x_7 u_2^{a_2} \cdots u_k^{a_k}$. Hence, we have to consider the following subcases:

Subcase 2.1. $a_2 \geq 1$ or $a_3 \geq 1$. Therefore, $v = w' x_4 u_4 u_2^{a_2-1} u_3^{a_3} \cdots u_k^{a_k}$ or $v = w' x_5 u_4 u_2^{a_2} u_3^{a_3-1} \cdots u_k^{a_k}$. This implies that $v \in J^m$.

Subcase 2.2. $a_2 = 0$ and $a_3 = 0$. Then $vx_1 = w' x_5 x_7 u_4^{a_4} \cdots u_k^{a_k}$. Due to $x_1 \notin u_j$ for all $j = 4, \dots, k$, it follows that $v = w'' x_5 x_7 u_4^{a_4} \cdots u_k^{a_k}$ such that $w' = x_1 w''$ for some monomial w'' in R . One can conclude that $v \in J^m$.

Case 3. $u_i = u_r$ with $r \geq 5$. Then $vx_1x_3 = w'u_ru_2^{a_2} \cdots u_k^{a_k}$. We now consider the following subcases:

Subcase 3.1. $a_2 \geq 1$ or $a_3 \geq 1$. The proof is similar to Subcase 2.1.

Subcase 3.2. $a_2 = 0$ and $a_3 = 0$. This implies that $vx_1x_3 = w'u_ru_4^{a_4} \cdots u_k^{a_k}$. On account of $x_1 \notin u_j$ for all $j = 4, \dots, k$, it follows that $vx_3 = w''u_ru_4^{a_4} \cdots u_k^{a_k}$ such that $w' = x_1w''$ for some monomial w'' in R . If $a_4 \geq 1$ or $a_5 \geq 1$ or $a_6 \geq 1$, then $v = w''x_5x_7u_ru_4^{a_4-1} \cdots u_k^{a_k}$ or $v = w''x_5x_6u_ru_4^{a_4}u_5^{a_5-1} \cdots u_k^{a_k}$ or $v = w''x_4x_5u_ru_4^{a_4}u_5^{a_5}u_6^{a_6-1} \cdots u_k^{a_k}$, and so $v \in J^m$. If $a_i = 0$ for all $i = 4, 5, 6$, according to $x_3 \notin u_j$ for all $j = 7, \dots, k$, one can conclude that $v = w_1u_7^{a_7} \cdots u_k^{a_k}$ such that $w''u_r = x_3w_1$ for some monomial w_1 in R . Thus, $v \in J^m$.

Consequently, we obtain $(J^{m+1} :_R u_1) = J^m$ for all $m \in \mathbb{N}$, and so u_1 is a strongly superficial element of degree one for the ideal J . Due to Proposition 2.2.17, one can derive that J has the persistence property, as desired. $\square$

2.3 Condition $(\sharp)$, and unisplit and separable monomial ideals

Our main aim in this section is to introduce the other classes of monomial ideals with the persistence property which call them unisplit and separable monomial ideals. In fact, we show that every ideal I in this class has the following properties:

- (i) I has the (strong) persistence property.
- (ii) I has a strongly superficial element.
- (iii) I is Ratliff-Rush closed.

To do this, we state condition $(\sharp)$ on monomial ideals and show that a monomial ideal I is unisplit (or separable) if and only if it satisfies condition $(\sharp)$. Moreover, we prove that monomial ideals with condition $(\sharp)$ have the properties (i)-(iii). In addition, note that the results of this section can be found in [99].

Throughout this section, $R = K[x_1, \dots, x_n]$ stands for a polynomial ring over a field K . We first begin with the following notions. Let $G = (V(G), E(G))$ be a finite simple graph on the vertex set $[n] := \{1, \dots, n\}$. Recall that the *edge ideal* associated to G is the monomial ideal

$$I(G) = (x_i x_j : \{i, j\} \in E(G)) \subset R = K[x_1, \dots, x_n],$$

and the *cover ideal* associated to G is the monomial ideal

$$J(G) = \bigcap_{\{i, j\} \in E(G)} (x_i, x_j) \subset R = K[x_1, \dots, x_n].$$

Let I be a monomial ideal of R generated in one degree. We say that I is *polymatroidal* if the following “exchange condition” is satisfied: For monomials $u = x_1^{a_1} \cdots x_n^{a_n}$ and $v = x_1^{b_1} \cdots x_n^{b_n}$ belonging to $\mathcal{G}(I)$ and, for each i with $a_i > b_i$, one has j with $a_j < b_j$ such that $x_j u / x_i \in \mathcal{G}(I)$ (see [60, Definition 12.6.1]).

In what follows, we present two classes of monomial ideals such that are none of them, but we will show that satisfying the persistence property.

We define unisplit monomial ideals in the following definition.

Definition 2.3.1. *Let I be a monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Then we say that I is a unisplit monomial ideal if there exists $i \in \mathbb{N}$ with $1 \leq i \leq m$ such that each monomial u_j has no common factor with u_i for all $j \in \mathbb{N}$ with $1 \leq j \leq m$ and $j \neq i$. We call u_i as split generator.*

Example 2.3.2. *Consider the ideal $I = (x_3^2 x_5 x_6^3, x_1^3 x_2^2 x_4^4, x_1^6 x_2^3 x_7^4, x_2^2 x_7^4 x_4^5)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$. It is easy to see that I is a unisplit monomial ideal of R with $x_3^2 x_5 x_6^3$ is the split generator.*

We next give the definition of separable monomial ideals.

Definition 2.3.3. *Let I be a monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Then we say that I is a separable monomial ideal if there exist $i \in \mathbb{N}$ with $1 \leq i \leq m$ and monomials g and w in R such that $w \neq 1$, $u_i = wg$, $\gcd(w, g) = 1$, and for all $j \in \mathbb{N}$ with $1 \leq j \neq i \leq m$, $\gcd(u_j, u_i) = w$.*

Example 2.3.4. *Consider the following monomial ideal*

$$I = (x_1 x_2 x_3^3 x_4^5, x_1^2 x_2^3 x_3^4 x_5^3 x_6^5, x_1^2 x_2 x_3^5 x_5^2 x_6, x_1^3 x_2^2 x_3^6 x_4^4),$$

in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$. Then, by setting

$$u_1 := x_1 x_2 x_3^3 x_4^5, u_2 := x_1^2 x_2^3 x_3^4 x_5^3 x_6^5, u_3 := x_1^2 x_2 x_3^5 x_5^2 x_6, u_4 := x_1^3 x_2^2 x_3^6 x_4^4,$$

$i := 1$ and $w := x_1 x_2 x_3^3$, one can easily check that I is a separable monomial ideal of R .

It is necessary to note that the set of unisplit monomial ideals and the set of separable monomial ideals are disjoint.

Here, we study some properties of monomial ideals which we need in the following. We begin with the remark below.

Remark 2.3.5. *Let $I_1, \dots, I_n$ be monomial ideals of R . Then it is easy to see that $(I_1 + \cdots + I_n :_R u) = (I_1 :_R u) + \cdots + (I_n :_R u)$ for all monomials u in R .*

Proposition 2.3.6. *Let I and J be two ideals of a commutative Noetherian ring S and u be an arbitrary element in S . Then $(uJ + I :_S u) = J + (I :_S u)$. Moreover, if u is a non-zero-divisor in S , then $(uJ :_S u) = J$.*

Proof. Suppose that h is an arbitrary element in $(uJ + I :_S u)$. Then $hu \in uJ + I$. Hence, there exist $\alpha \in J$ and $\beta \in I$ such that $hu = \alpha u + \beta$. Thus, $h - \alpha \in (I :_S u)$, and so $h \in J + (I :_S u)$. Therefore, $(uJ + I :_S u) \subseteq J + (I :_S u)$. For the reverse inclusion, let h be an arbitrary element in $J + (I :_S u)$. Then $h = \alpha + \beta$ for some $\alpha \in J$ and $\beta \in (I :_S u)$. Since $hu = \alpha u + \beta u$, it follows that $hu \in uJ + I$. Hence, $h \in (uJ + I :_S u)$. Therefore, $J + (I :_S u) \subseteq (uJ + I :_S u)$, and so the proof is complete.

Now, suppose that $I = 0$ and u is a non-zerodivisor in S . Then $(I :_S u) = 0$, and so $(uJ :_S u) = J$, as required. $\square$

In the next definition, we introduce condition $(\sharp)$ on monomial ideals and we show that any monomial ideal that satisfying condition $(\sharp)$ has the persistence property and is Ratliff-Rush closed. Also, we prove that it has a strongly superficial element.

Definition 2.3.7. Suppose that I is a monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. We say that I satisfies condition $(\sharp)$ if there exists a positive integer i with $1 \leq i \leq m$ such that

$$(u_1^{\alpha_1} \cdots u_{i-1}^{\alpha_{i-1}} \widehat{u_i^{\alpha_i}} u_{i+1}^{\alpha_{i+1}} \cdots u_m^{\alpha_m} u_j :_R u_i) = u_1^{\alpha_1} \cdots u_{i-1}^{\alpha_{i-1}} \widehat{u_i^{\alpha_i}} u_{i+1}^{\alpha_{i+1}} \cdots u_m^{\alpha_m} (u_j :_R u_i),$$

for all $j = 1, \dots, m$ with $j \neq i$ and $\alpha_1, \dots, \alpha_m \in \mathbb{N}_0$, where $\widehat{u_i^{\alpha_i}}$ means that this term is omitted and $\mathbb{N}_0$ is the set of nonnegative integers.

Now, we state the first main result of this section. To accomplish this, one has to frequently apply the following proposition.

Proposition 2.3.8. ([60, Proposition 1.2.2]) Let I and J be monomial ideals. Then $(I : J)$ is a monomial ideal, and

$$(I : J) = \bigcap_{v \in \mathcal{G}(J)} (I : v).$$

Moreover, $\{u/\gcd(u, v) : u \in \mathcal{G}(I)\}$ is a set of generators of $(I : v)$.

Theorem 2.3.9. Let I be a monomial ideal satisfies condition $(\sharp)$. Then I is either a unisplit monomial ideal or a separable monomial ideal.

Proof. Without loss of generality, we may assume that $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ is the unique minimal set of monomial generators of I such that

$$(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j :_R u_1) = u_2^{\alpha_2} \cdots u_m^{\alpha_m} (u_j :_R u_1),$$

for all $j = 2, \dots, m$ and $\alpha_2, \dots, \alpha_m \in \mathbb{N}_0$. Consider $k, j \in \{2, \dots, m\}$. By [60, Proposition 1.2.2], the generator of $(u_k u_j :_R u_1)$ has the form $\frac{u_k u_j}{\gcd(u_k u_j, u_1)}$. On the other hand, we have $(u_k u_j :_R u_1) = u_k (u_j :_R u_1) = u_j (u_k :_R u_1)$. Due to [60,

Proposition 1.2.2], the generator of $(u_j :_R u_1)$ (respectively, $(u_k :_R u_1)$) has the form $\frac{u_j}{\gcd(u_j, u_1)}$ (respectively, $\frac{u_k}{\gcd(u_k, u_1)}$). This implies that

$$\frac{u_k u_j}{\gcd(u_k u_j, u_1)} = u_k \frac{u_j}{\gcd(u_j, u_1)} = u_j \frac{u_k}{\gcd(u_k, u_1)}.$$

Accordingly, for each $k, j \in \{2, \dots, m\}$, it follows that

$$\gcd(u_k u_j, u_1) = \gcd(u_j, u_1) = \gcd(u_k, u_1).$$

Thus, there exists a fixed monomial such as w in R such that $\gcd(u_j, u_1) = w$ for all $j = 2, \dots, m$. So, we have the following two cases:

Case 1. $w = 1$. Then based on Definition 2.3.1, one can deduce that I is a unisplit monomial ideal.

Case 2. $w > 1$. Assume that $u_1 = wg$ for some monomial g in R . By Definition 2.3.3, in order to show that I is a separable monomial ideal, it is sufficient to prove that $\gcd(w, g) = 1$. To do this, suppose that $\gcd(w, g) > 1$, and seek a contradiction. Then there is a monomial $x_i^{\alpha_i}$ in R such that $x_i \in \{x_1, \dots, x_n\}$, $\alpha_i \in \mathbb{N}$, $g = x_i^{\alpha_i} v_1$ and $w = x_i^{\alpha_i} v_2$ for some monomials v_1 and v_2 in R . Choose two arbitrary generators such as u_i and u_j in $\mathcal{G}(I)$. Then $u_i = wh_1$ and $u_j = wh_2$ for some monomials h_1 and h_2 in R . This gives that $u_i u_j = w^2 h_1 h_2$, and so $x_i^{\alpha_i} w$ divides $\gcd(u_i u_j, u_1)$, this is the required contradiction. $\square$

In the next theorems, we prove that any unisplit (or separable) monomial ideal satisfies condition $(\sharp)$.

Theorem 2.3.10. *Every unisplit monomial ideal of R satisfies condition $(\sharp)$.*

Proof. Let I be a monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Since I is a unisplit monomial ideal, without loss of generality, we may assume that each of monomials $u_2, \dots, u_m$ has no common factor with u_1 . By [60, Proposition 1.2.2], the generator of $(u_i :_R u_1)$ is the monomial of the form $\frac{u_i}{\gcd(u_i, u_1)} = u_i$ for all $i = 2, \dots, m$. Again, by [60, Proposition 1.2.2], the generator of $(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i :_R u_1)$, for all $i = 2, \dots, m$, has the following form

$$\frac{u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i}{\gcd(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i, u_1)}.$$

By virtue of $\gcd(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i, u_1) = 1$, for all $i = 2, \dots, m$, it follows that

$$\frac{u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i}{\gcd(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i, u_1)} = u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i.$$

This implies that, for all $i = 2, \dots, m$, we have

$$(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i :_R u_1) = u_2^{\alpha_2} \cdots u_m^{\alpha_m} (u_i :_R u_1).$$

This yields that I satisfies condition $(\sharp)$, as desired. $\square$

Theorem 2.3.11. *Every separable monomial ideal of R satisfies condition $(\sharp)$.*

Proof. Let I be a separable monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ such that $u_j := x_1^{c_{j,1}} \cdots x_n^{c_{j,n}}$ for all $j = 1, \dots, m$ and $c_{j,1}, \dots, c_{j,n} \in \mathbb{N}_0$. Now, without loss of generality, we may assume that there exist monomials g and $w := x_1^{\beta_1} \cdots x_n^{\beta_n}$ with $\beta_1, \dots, \beta_n \in \mathbb{N}_0$, in R such that $u_1 = wg$, $\gcd(w, g) = 1$, and $\gcd(u_j, u_1) = w$ for all $j = 2, \dots, m$. Due to [60, Proposition 1.2.2], the generator of $(u_j :_R u_1)$ has the form $\frac{u_j}{\gcd(u_j, u_1)}$ for all $j = 2, \dots, m$. Furthermore, $\frac{u_j}{\gcd(u_j, u_1)} = \frac{u_j}{w}$ for all $j = 2, \dots, m$. Again, according to [60, Proposition 1.2.2], the generator of $(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j :_R u_1)$, for all $j = 2, \dots, m$ and $\alpha_2, \dots, \alpha_m \in \mathbb{N}_0$, is the monomial of the following form

$$\frac{u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j}{\gcd(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j, u_1)}.$$

For completing the proof, it is enough to verify that $\gcd(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j, u_1) = w$ for all $j = 2, \dots, m$. In fact, for all $j = 2, \dots, m$, one has

$$u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j = \prod_{t=1}^n x_t^{c_{2,t}\alpha_2 + \cdots + c_{j-1,t}\alpha_{j-1} + c_{j,t}(\alpha_j + 1) + c_{j+1,t}\alpha_{j+1} + \cdots + c_{m,t}\alpha_m}.$$

Since $\gcd(w, g) = 1$ and $\gcd(u_j, u_1) = w$ for all $j = 2, \dots, m$, one can conclude that

$$c_{1,t} \leq c_{2,t}\alpha_2 + \cdots + c_{j-1,t}\alpha_{j-1} + c_{j,t}(\alpha_j + 1) + c_{j+1,t}\alpha_{j+1} + \cdots + c_{m,t}\alpha_m,$$

for all $t = 1, \dots, n$ with $\beta_t \neq 0$. This implies that $\gcd(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j, u_1) = w$ for all $j = 2, \dots, m$. Therefore, we get

$$(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j :_R u_1) = u_2^{\alpha_2} \cdots u_m^{\alpha_m} (u_j :_R u_1),$$

for all $j = 2, \dots, m$, and so I satisfies condition $(\sharp)$, as claimed. $\square$

Theorems 2.3.9, 2.3.10, and 2.3.11 characterize any monomial ideal that satisfies condition $(\sharp)$.

In the following, we give some combinatorial aspects. In fact, we study at the intersection of graph theory and commutative algebra by relating powers of square-free monomial ideals in polynomial rings to paths of finite simple graphs. We then show that certain classes of these path ideals satisfy condition $(\sharp)$.

Let G be a finite simple graph on the vertex set $[n] := \{1, \dots, n\}$. Then the sequence of distinct vertices $i_1, \dots, i_r$ is called a *maximal path* when $i_1, \dots, i_r$ is a path and furthermore, for all $j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_r\}$, none of $j, i_1, \dots, i_r$ and $i_1, \dots, i_r, j$ is a path.

Definition 2.3.12. ([123, Definition 3.1]) *Let G be a graph on the vertex set $[n] := \{1, \dots, n\}$. Then the ideal*

$$PI(G) := (x_{i_1} \cdots x_{i_t} : i_1, \dots, i_t \text{ is a maximal path in } G),$$

in $R = K[x_1, \dots, x_n]$ is called a max-path ideal of G .

Theorem 2.3.13. *Let T be a starlike tree on the vertex set $\{z\} \cup [n]$ with the center z . Then $PI(T)^\vee$ has the persistence property, where $I^\vee$ means the Alexander dual ideal of I .*

Proof. Let L_T be the set of vertices of degree one in T . It is routine to check that the path $i_1, \dots, i_t$ is a maximal path in T if and only if $i_1, i_t \in L_T$. This implies that

$$PI(T) = (x_{i_1} \cdots x_{i_j} x_z x_{i_{j+1}} \cdots x_{i_t} : i_1, i_t \in L_T).$$

Thus, one can conclude that

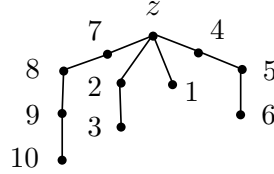
$$PI(T)^\vee = \bigcap_{i_1, i_t \in L_T} (x_{i_1}, \dots, x_{i_j}, x_z, x_{i_{j+1}}, \dots, x_{i_t}).$$

One can easily observe that this is a minimal primary decomposition of $PI(T)^\vee$ in the polynomial ring $R' = K[x_z, x_1, \dots, x_n]$ over the field K . By Lemma 2.2.5, it follows that $x_z \in \mathcal{G}(PI(T)^\vee)$. Assume that $\mathcal{G}(PI(T)^\vee) = \{x_z, u_1, \dots, u_s\}$. Since $\mathcal{G}(PI(T)^\vee)$ is a minimal set, this implies that $\gcd(x_z, u_j) = 1$ for all $j = 1, \dots, s$. Due to Definition 2.3.1, $PI(T)^\vee$ is a unisplit monomial ideal, and so has the persistence property, as desired. $\square$

In the next example we illustrate the major goal of Theorem 2.3.13.

Example 2.3.14. *Let T be a tree on the vertex set $V(T) := \{z\} \cup [10]$ and the edge set $E(T) := \{\{1, z\}, \{2, z\}, \{2, 3\}, \{4, z\}, \{4, 5\}, \{5, 6\}, \{7, z\}, \{7, 8\}, \{8, 9\}, \{9, 10\}\}$.*

By figure below, it is easy to verify that T is starlike with the center z .



This implies that

$$PI(T) = (x_4 x_5 x_6 x_z x_7 x_8 x_9 x_{10}, x_1 x_z x_4 x_5 x_6, x_1 x_z x_7 x_8 x_9 x_{10}, x_2 x_3 x_z x_4 x_5 x_6, x_1 x_z x_2 x_3),$$

and so

$$PI(T)^\vee = (x_4, x_5, x_6, x_z, x_7, x_8, x_9, x_{10}) \cap (x_1, x_z, x_4, x_5, x_6) \cap (x_1, x_z, x_7, x_8, x_9, x_{10}) \\ \cap (x_2, x_3, x_z, x_7, x_8, x_9, x_{10}) \cap (x_2, x_3, x_z, x_4, x_5, x_6) \cap (x_1, x_z, x_2, x_3).$$

According to Theorem 2.3.13, it follows that $PI(T)^\vee$ has the persistence property.

The following theorem is concerned with the separable monomial ideals.

Theorem 2.3.15. *Suppose that T is a starlike tree on the vertex set $\{z\} \cup [n]$ with the center z and $L_T = \{j_1, \dots, j_r\}$ is the set of vertices of degree one in T . Suppose also that $z, i_{1,k}, i_{2,k}, \dots, i_{t,k}, j_k$ is the path between z and j_k for all $k = 1, \dots, r$. Then, for all $s = 1, \dots, r$, the ideal*

$$J_s := (x_{j_s} x_{i_{1,s}} x_{i_{2,s}} \cdots x_{i_{t,s}} x_z x_{i_{1,k}} x_{i_{2,k}} \cdots x_{i_{t,k}} x_{j_k} : k \in \{1, \dots, r\} \setminus \{s\}),$$

has the persistence property.

Proof. Without loss of generality, we need only prove the assertion for the case $s = 1$. To do this, set $w := x_{j_1} x_{i_{1,1}} x_{i_{2,1}} \cdots x_{i_{t,1}} x_z$ and $u_k := x_{i_{1,k}} x_{i_{2,k}} \cdots x_{i_{t,k}} x_{j_k}$ for all $k = 2, \dots, r$. This implies that $\mathcal{G}(J_1) = \{wu_2, wu_3, \dots, wu_r\}$. It is routine to verify that $\gcd(w, u_2) = 1$ and $\gcd(wu_2, wu_j) = w$ for all $j = 3, \dots, r$. Then J_1 is a separable monomial ideal, and so has the persistence property. $\square$

We conclude our argument with the following example which clarifies the main aim of Theorem 2.3.15.

Example 2.3.16. *Let T be the starlike tree which has stated in Example 2.3.14. By putting $j_1 := 10$, we deduce that the following ideal has the persistence property*

$$J_1 := (x_{10}x_9x_8x_7x_zx_1, x_{10}x_9x_8x_7x_zx_2x_3, x_{10}x_9x_8x_7x_zx_4x_5x_6).$$

2.3.1 Strongly superficial elements and Ratliff-Rush closed ideals

Our main goal in this subsection is to show that any monomial ideal that satisfying condition $(\sharp)$ has the persistence property. To do this, we begin with the following theorem. Recall that an ideal I in a commutative Noetherian ring S has the strong persistence property if $(I^{k+1} :_S I) = I^k$ for all $k \in \mathbb{N}$. It is important to note that if I has the strong persistence property, then I has the persistence property. On the other hand, there exist some monomial ideals which have the persistence property, but do not necessary have the strong persistence property. As an example, consider the ideal I generated by monomials $x_1x_2x_3$, $x_1x_2x_4$, $x_1x_3x_5$, $x_1x_4x_6$, $x_1x_5x_6$, $x_2x_3x_6$, $x_2x_4x_5$, $x_2x_5x_6$, $x_3x_4x_5$, and $x_3x_4x_6$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$. Using [88, Example 2.18] shows that I has the persistence property. On the other hand, we obtain $(I^3 :_R I) \neq I^2$, and so I has no the strong persistence property.

Theorem 2.3.17. *Let J be a monomial ideal satisfies condition $(\sharp)$. We then have $(J^{k+1} :_R J) = J^k$ for all $k \in \mathbb{N}$, i.e., J has the strong persistence property.*

Proof. Without loss of generality, suppose that $\mathcal{G}(J) = \{u_1, \dots, u_m\}$ such that

$$(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j :_R u_1) = u_2^{\alpha_2} \cdots u_m^{\alpha_m} (u_j :_R u_1),$$

for all $j = 2, \dots, m$ and $\alpha_2, \dots, \alpha_m \in \mathbb{N}_0$. Since $J^k \subseteq (J^{k+1} :_R J)$ for all $k \in \mathbb{N}$, we need only show that $(J^{k+1} :_R J) \subseteq J^k$ for all $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$. As $J = \sum_{j=1}^m u_j R$, this implies that

$$(J^{k+1} :_R J) = \bigcap_{j=1}^m (J^{k+1} :_R u_j).$$

Note also that $J^{k+1} = J^k \left(\sum_{i=1}^m u_i R \right) = \sum_{i=1}^m J^k u_i$. It follows from Remark 2.3.5 that

$$(J^{k+1} :_R J) = \bigcap_{j=1}^m (J^{k+1} :_R u_j) = \bigcap_{j=1}^m \sum_{i=1}^m (J^k u_i :_R u_j).$$

On the other hand, according to Proposition 2.3.6, one can conclude that

$$\sum_{i=1}^m (J^k u_i :_R u_1) = (J^k u_1 :_R u_1) + \sum_{i=2}^m (J^k u_i :_R u_1) = J^k + \sum_{i=2}^m (J^k u_i :_R u_1).$$

However, by Remark 2.3.5, for $i \in \mathbb{N}$ with $2 \leq i \leq m$, we have the following equalities

$$\begin{aligned} (J^k u_i :_R u_1) &= \left(\sum_{\alpha_1 + \dots + \alpha_m = k} u_1^{\alpha_1} \cdots u_m^{\alpha_m} u_i :_R u_1 \right) \\ &= \sum_{\alpha_1 + \dots + \alpha_m = k} (u_1^{\alpha_1} \cdots u_m^{\alpha_m} u_i :_R u_1) \\ &= \sum_{\alpha_1 + \dots + \alpha_m = k, \alpha_1 > 0} (u_1^{\alpha_1} u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i :_R u_1) \\ &\quad + \sum_{\alpha_2 + \dots + \alpha_m = k} (u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i :_R u_1). \end{aligned}$$

Due to Proposition 2.3.6, for $i \in \mathbb{N}$ with $2 \leq i \leq m$, one can deduce that

$$\begin{aligned} (J^k u_i :_R u_1) &= \sum_{\alpha_1 + \dots + \alpha_m = k, \alpha_1 > 0} u_1^{\alpha_1 - 1} u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_i R \\ &\quad + \sum_{\alpha_2 + \dots + \alpha_m = k} u_2^{\alpha_2} \cdots u_m^{\alpha_m} (u_i :_R u_1). \end{aligned}$$

It is routine to check that $(J^k u_i :_R u_1) \subseteq J^k$ for $i \in \mathbb{N}$ with $2 \leq i \leq m$, and hence $\sum_{i=2}^m (J^k u_i :_R u_1) \subseteq J^k$. This implies that $\sum_{i=1}^m (J^k u_i :_R u_1) = J^k$. Since

$\bigcap_{j=1}^m \sum_{i=1}^m (J^k u_i :_R u_j) \subseteq \sum_{i=1}^m (J^k u_i :_R u_1)$, it follows that $(J^{k+1} :_R J) \subseteq J^k$. Therefore, $(J^{k+1} :_R J) = J^k$, as required. $\square$

Remarks 2.3.18. (1) Note that Theorem 2.3.17 can fail for a monomial ideal that is not a unisplit monomial ideal. As an example, consider the following monomial ideal

$$I = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_1x_4x_5^2, x_1^2x_2x_5),$$

in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. Then I is not a unisplit monomial ideal. On the other hand, one can conclude that

$$(I^2 :_R I) = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_1x_4x_5^2, x_1^2x_2x_5, x_1x_2x_3x_4x_5).$$

Therefore, we get $(I^2 :_R I) \neq I$.

(2) Observe that Theorem 2.3.17 can fail for a monomial ideal that is not a separable monomial ideal. As an example, consider the ideal I generated by monomials $x_1x_2x_3$, $x_1x_2x_4$, $x_1x_3x_5$, $x_1x_4x_6$, $x_1x_5x_6$, $x_2x_3x_6$, $x_2x_4x_5$, $x_2x_5x_6$, $x_3x_4x_5$ and $x_3x_4x_6$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$. Then I is not a separable monomial ideal. On the other hand, we obtain $(I^3 :_R I) \neq I^2$.

Lemma 2.3.20 is essential for us in order to prove Theorem 2.3.22. To verify Lemma 2.3.20, one requires the subsequent lemma.

Lemma 2.3.19. ([90, Lemma 2.2]) Let I^* denote the eventual stable value of the ascending chain $(I^2 : I) \subseteq (I^3 : I^2) \subseteq \dots$.

- (1) If J is an ideal containing I , then $J \subseteq I^*$ if and only if there is an $n \geq 1$ with $J^n = I^n$.
- (2) If J is an ideal with $I \subseteq J \subseteq I^*$, then $A^*(J) = A^*(I)$.
- (3) If S is a multiplicatively closed subset of R disjoint from I , then $(I_S)^* = (I^*)_S$.
- (4) $I \subseteq I^* \subseteq (I)_a$.
- (5) For all large n , $I^{n*} = I^n$.
- (6) For $n \geq 1$, $(I^*)^n \subseteq I^{n*}$.
- (7) For $n \geq 1$, I^{n*} is the eventual stable value of the increasing chain

$$(I^{n+1} : I) \subseteq (I^{n+2} : I^2) \subseteq (I^{n+3} : I^3) \subseteq \dots$$

- (8) If $\mathfrak{p} \in \text{Ass}(R/I^*)$, then $\mathfrak{p} \in \text{Ass}(R/(I^*)^n)$ for all $n \geq 1$. Also, $\text{Ass}(R/I^*) \subseteq A^*(I)$.

Lemma 2.3.20. Let J be a monomial ideal satisfies condition $(\sharp)$ and let J^* denote the eventual stable value of the ascending chain

$$(J^2 :_R J) \subseteq (J^3 :_R J^2) \subseteq \dots \subseteq (J^{i+1} :_R J^i) \subseteq \dots$$

Then $J^{k*} = J^k$ for all $k \in \mathbb{N}$. Moreover, $J^* = J$.

Proof. Fix $k \in \mathbb{N}$. Due to Lemma 2.3.19, J^{k*} is the eventual stable value of the following increasing chain

$$(J^{k+1} :_R J) \subseteq (J^{k+2} :_R J^2) \subseteq \cdots \subseteq (J^{k+i} :_R J^i) \subseteq \cdots.$$

Thus, it is enough to show that $(J^{k+i} :_R J^i) = J^k$ for all $i \in \mathbb{N}$. To achieve this, we proceed by induction on i . By Theorem 2.3.17, we have $(J^{k+1} :_R J) = J^k$. Now, suppose inductively that $i > 1$ and that the assertion is proved for i . Then, according to Theorem 2.3.17 and the inductive hypothesis, we have the following equalities

$$(J^{k+i+1} :_R J^{i+1}) = ((J^{k+i+1} :_R J) :_R J^i) = (J^{k+i} :_R J^i) = J^k.$$

This completes the inductive step and the proof. $\square$

To show Theorem 2.3.22, one has to repeatedly apply the next proposition. Moreover, recall that an ideal is called a *regular ideal* if it contains a non-zero-divisor element.

Proposition 2.3.21. ([90, Proposition 4.15]) *Suppose that $0 \leq m < n$. Then*

- (1) $(I^{n*} : I^{m*}) = (I^{n*} : I^m) = I^{n-m*}.$
- (2) *For $n \geq 1$, let J be an ideal with $J \subseteq I^{n*}$. If $\mathfrak{p} \in \text{Ass}(R/J)$, then $\mathfrak{p} \in \text{Ass}(I^{n-1*}/J).$*
- (3) $\text{Ass}(R/I^*) \subseteq \text{Ass}(R/I^{2*}) \subseteq \text{Ass}(R/I^{3*}) \subseteq \cdots.$
- (4) *For $n \geq 1$, $S_n(I) \subseteq \text{Ass}(I^{n*}/I^n).$*

Theorem 2.3.22. *Let J be a monomial ideal satisfies condition $(\sharp)$. We then have the following statements.*

- (i) *For positive integers r and s with $r < s$, $(J^s :_R J^r) = J^{s-r}.$*
- (ii) *Let J' be an ideal of R such that $J' \subseteq J^n$ for some $n \in \mathbb{N}$. Then*

$$\text{Ass}_R(R/J') \subseteq \text{Ass}_R(J^{n-1}/J').$$

- (iii) $\text{Ass}_R(R/J) \subseteq \text{Ass}_R(R/J^2) \subseteq \cdots \subseteq \text{Ass}_R(R/J^i) \subseteq \cdots.$
- (iv) $\text{Ass}_R(R/J) \subseteq \text{Ass}^\infty(J).$

Proof. As every nonzero ideal of R is regular, the statements (i), (ii), and (iii) follow from Lemma 2.3.20 and Proposition 2.3.21.

(iv) It follows immediately from Proposition 2.3.21(4) that $\text{Ass}_R(R/J^n) \setminus \text{Ass}^\infty(J)$ is a subset of $\text{Ass}_R(J^{n*}/J^n)$ for all $n \in \mathbb{N}$, we can derive from Lemma 2.3.20 that $\text{Ass}_R(R/J^n) \subseteq \text{Ass}^\infty(J)$, and so $\text{Ass}_R(R/J) \subseteq \text{Ass}^\infty(J).$ $\square$

Corollary 2.3.23. *Every ideal satisfies condition $(\sharp)$ has the persistence property.*

Remark 2.3.24. *It is necessary to note that the inclusion $\text{Ass}_R(R/I) \subseteq \text{Ass}^\infty(I)$ in part (iv) of Theorem 2.3.22, can be strict. As an example, let I be the following square-free monomial ideal*

$$I = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2),$$

in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. Then, by [50, Example 3.14], one can deduce that

$$\text{Ass}_R(R/I) = \{(x_1, x_3), (x_2, x_5), (x_3, x_5), (x_2, x_4), (x_1, x_4)\},$$

while

$$\text{Ass}^\infty(I) = \{(x_1, x_3), (x_2, x_5), (x_3, x_5), (x_2, x_4), (x_1, x_4), (x_1, x_2, x_3, x_4, x_5)\}.$$

This implies that $\text{Ass}_R(R/I) \subsetneq \text{Ass}^\infty(I)$.

According to Corollary 2.3.23, Theorem 2.3.9, Example 2.3.2, and Example 2.3.4, we presented two classes of monomial ideals such that are none of edge ideals, cover ideals, and polymatroidal ideals, but satisfy the persistence property.

Here, we explore the height of any monomial ideal that satisfying condition $(\sharp)$ in Remarks 2.3.25 and 2.3.26.

Remark 2.3.25. *Let I be a separable monomial ideal of R . Then $\text{ht}I = 1$.*

Proof. First proof. In the first proof we describe the set of associated prime ideals of I . To achieve this, without loss of generality, one may assume that $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ such that $u_1 = wg$ for some monomials $w, g \in R$, $\gcd(w, g) = 1$ and $\gcd(u_1, u_j) = w$ for all $j = 2, \dots, m$. Put $J := (u_1/w, \dots, u_m/w)$ and suppose that $w = x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ with $b_1, \dots, b_s \in \mathbb{N}$. By virtue of Theorem 2.1.27, it follows that

$$\text{Ass}_R(R/I) = \text{Ass}_R(R/J) \cup \{(x_{j_1}), \dots, (x_{j_s})\}.$$

The claim is therefore an immediate consequence of definition of height of an ideal.

Second proof. With the same assumptions and notations in the first proof, one can conclude that $I = wJ$. Since $w \neq 1$, the assertion follows from the Krull's Principal Ideal Theorem. $\square$

Remark 2.3.26. *Let I with $\mathcal{G}(I) = \{u_1, \dots, u_{m-1}, x_{j_1}^{\alpha_1} \cdots x_{j_r}^{\alpha_r}\}$ be a unisplit monomial ideal of R such that $x_{j_t} \notin \text{supp}(u_i)$ for all $i = 1, \dots, m-1$ and $t = 1, \dots, r$. By setting $J := (u_1, \dots, u_{m-1})$, we then have $\text{ht}I = \text{ht}J + 1$.*

Proof. According to Theorem 2.1.30, one can conclude that

$$\text{Ass}_R(R/I) = \{\mathfrak{p} \mid \mathfrak{p} = \mathfrak{q} + x_{j_t}R \text{ for all } t = 1, \dots, r \text{ and } \mathfrak{q} \in \text{Ass}_R(R/J)\}.$$

The claim follows from this fact that each $\mathfrak{q} \in \text{Ass}_R(R/J)$ is a monomial prime ideal. $\square$

The concept of superficial element was first introduced by P. Samuel in [119] for the study of multiplicities of primary ideals in local rings. In fact, it is well-known that superficial elements are not only well-behaved with respect to multiplicities but also with respect to Hilbert polynomials. Strongly superficial elements have important role in commutative algebra (see [127]), refer to Definition 2.2.16.

Theorem 2.3.27. *Suppose that I satisfies condition $(\sharp)$. Then I has a strongly superficial element of degree 1.*

Proof. Without loss of generality, we may assume that $\mathcal{G}(I)$ is the unique minimal set of monomial generators of I such that

$$(u_2^{\alpha_2} \cdots u_m^{\alpha_m} u_j :_R u_1) = u_2^{\alpha_2} \cdots u_m^{\alpha_m} (u_j :_R u_1),$$

for all $j = 2, \dots, m$ and $\alpha_2, \dots, \alpha_m \in \mathbb{N}_0$. As we proved in Theorem 2.3.17, it follows that $(I^{k+1} :_R u_1) = I^k$ for all $k \in \mathbb{N}$. This implies that u_1 is a strongly superficial element of degree 1 for I , as claimed. $\square$

We now introduce a class of Ratliff-Rush closed ideals. Let I be an arbitrary ideal in a commutative Noetherian ring S . From the maximal condition on ideals of S , it follows that there exist ideals I^* in S which are maximal with respect to the condition

$$I^{*n} = I^n \text{ for all large } n.$$

Ratliff and Rush, in [114, Theorem 2.1], proved that if I is a regular ideal, then there exists a unique such I^* , which can be presented in terms of I as follows:

$$I^* := \bigcup_{n \in \mathbb{N}} (I^{n+1} :_S I^n).$$

In fact, the eventual stable value of the ascending chain

$$(I^2 :_S I) \subseteq (I^3 :_S I^2) \subseteq \cdots \subseteq (I^{i+1} :_S I^i) \subseteq \cdots$$

is I^* .

Definition 2.3.28. *The ideal I^* is called the Ratliff-Rush ideal associated with I or the Ratliff-Rush closure of I . A regular ideal I for which $I^* = I$ is called Ratliff-Rush closed.*

Corollary 2.3.29. *Suppose that I satisfies condition $(\sharp)$. Then I is a Ratliff-Rush closed ideal.*

Proof. According to Theorem 2.3.22(i), it follows that $I^* = \bigcup_{k \in \mathbb{N}} (I^{k+1} :_R I^k) = I$. This implies that I is a Ratliff-Rush closed ideal, as claimed. $\square$

2.3.2 Towards constructing the other classes

In this subsection, our goal is to introduce a method for constructing new monomial ideals which have the persistence property based on the monomial ideals which we know have the persistence property. For this purpose, we provide a criterion for the persistence property. To do this, we refer to the notion of the expansion operator on monomial ideals which has been stated in Definition 2.2.9, and employ it as a criterion for the persistence property of monomial ideals.

Lemma 2.3.30. *Let I be a monomial ideal of R . Then I^* has the persistence property if and only if I has.*

Proof. For necessity, consider $\mathfrak{p} \in \text{Ass}_R(R/I^n)$ for an arbitrary $n \in \mathbb{N}$. According to Proposition 2.2.10, it follows that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/I^{n*})$. Due to Lemma 2.2.11(iii), we have $I^{n*} = (I^*)^n$. Since I^* has the persistence property, one can conclude that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/(I^*)^{n+1})$. Again, by Lemma 2.2.11(iii), we deduce that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/I^{n+1*})$. By virtue of Proposition 2.2.10, one has $\mathfrak{p} \in \text{Ass}_R(R/I^{n+1})$. This implies that I has the persistence property. Sufficiency follows from in a similar way, and the proof is done. $\square$

Next proposition is necessary for us in order to clarify Example 2.3.32.

Proposition 2.3.31. *Let $I \subset R = K[x_1, \dots, x_n]$ be a monomial ideal which has the persistence property, and let $u = x_{i_1}^{a_1} \cdots x_{i_r}^{a_r}$ be a monomial in R with $a_1, \dots, a_r \in \mathbb{N}$. Then uI has the persistence property.*

Proof. Let n be an arbitrary positive integer. Due to Theorem 2.1.27, one has

$$\text{Ass}_R(R/(uI)^n) = \text{Ass}_R(R/I^n) \cup \{(x_{i_1}), \dots, (x_{i_r})\}.$$

By our hypothesis, it follows that $\text{Ass}_R(R/I^n) \subseteq \text{Ass}_R(R/I^{n+1})$. Again, by Theorem 2.1.27, we obtain

$$\text{Ass}_R(R/(uI)^{n+1}) = \text{Ass}_R(R/I^{n+1}) \cup \{(x_{i_1}), \dots, (x_{i_r})\}.$$

Hence, $\text{Ass}_R(R/(uI)^n) \subseteq \text{Ass}_R(R/(uI)^{n+1})$, and so uI has the persistence property, as desired. $\square$

The following example illuminates our method for constructing new monomial ideals which have the persistence property based on the monomial ideals which we know have the persistence property.

Example 2.3.32. *Let J be the following monomial ideal*

$$\begin{aligned} J := (& x_{11}x_{21}^2x_{31}^2x_{41}x_{51}, x_{11}x_{21}^2x_{31}^2x_{41}x_{52}, x_{11}x_{21}^2x_{31}^2x_{42}x_{51}, x_{11}x_{21}^2x_{31}^2x_{42}x_{52}, \\ & x_{12}x_{21}^2x_{31}^2x_{41}x_{51}, x_{12}x_{21}^2x_{31}^2x_{41}x_{52}, x_{12}x_{21}^2x_{31}^2x_{42}x_{51}, x_{12}x_{21}^2x_{31}^2x_{42}x_{52}, \\ & x_{21}^2x_{31}^3x_{51}, x_{21}^2x_{31}^3x_{52}, x_{11}x_{21}x_{31}^3x_{41}x_{51}, x_{11}x_{21}x_{31}^3x_{41}x_{52}, x_{11}x_{21}x_{31}^3x_{42}x_{51}, \\ & x_{11}x_{21}x_{31}^3x_{42}x_{52}, x_{12}x_{21}x_{31}^3x_{41}x_{51}, x_{12}x_{21}x_{31}^3x_{41}x_{52}, x_{12}x_{21}x_{31}^3x_{42}x_{51}, \\ & x_{12}x_{21}x_{31}^3x_{42}x_{52}, x_{21}x_{31}^2x_{51}^2x_{61}x_{71}, x_{21}x_{31}^2x_{51}^2x_{62}x_{71}, x_{21}x_{31}^2x_{51}^2x_{63}x_{71}, \\ & x_{21}x_{31}^2x_{51}x_{52}x_{61}x_{71}, x_{21}x_{31}^2x_{51}x_{52}x_{62}x_{71}, x_{21}x_{31}^2x_{51}x_{52}x_{63}x_{71}, \\ & x_{21}x_{31}^2x_{52}^2x_{61}x_{71}, x_{21}x_{31}^2x_{52}^2x_{62}x_{71}, x_{21}x_{31}^2x_{52}^2x_{63}x_{71}), \end{aligned}$$

in the polynomial ring $R = K[x_{11}, x_{12}, x_{21}, x_{31}, x_{41}, x_{42}, x_{51}, x_{52}, x_{61}, x_{62}, x_{63}, x_{71}]$. We claim that J has the persistence property. To do this, put $\mathfrak{p}_1 := (x_{11}, x_{12})$, $\mathfrak{p}_2 := (x_{21})$, $\mathfrak{p}_3 := (x_{31})$, $\mathfrak{p}_4 := (x_{41}, x_{42})$, $\mathfrak{p}_5 := (x_{51}, x_{52})$, $\mathfrak{p}_6 := (x_{61}, x_{62}, x_{63})$, and $\mathfrak{p}_7 := (x_{71})$. It is routine to check that

$$J = \mathfrak{p}_1\mathfrak{p}_2^2\mathfrak{p}_3^2\mathfrak{p}_4\mathfrak{p}_5 + \mathfrak{p}_2^2\mathfrak{p}_3^3\mathfrak{p}_5 + \mathfrak{p}_1\mathfrak{p}_2\mathfrak{p}_3^3\mathfrak{p}_4\mathfrak{p}_5 + \mathfrak{p}_2\mathfrak{p}_3^2\mathfrak{p}_5^2\mathfrak{p}_6\mathfrak{p}_7.$$

Now, consider the following monomial ideal

$$I := (x_1x_2^2x_3^2x_4x_5, x_2^2x_3^3x_5, x_1x_2x_3^3, x_4x_5, x_2x_3^2x_5^2x_6x_7),$$

in the polynomial ring $R_1 = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7]$. It is easy to see that J is the expansion of I with respect to the 7-tuple $(2, 1, 1, 2, 2, 3, 1)$. In order to prove our claim, set $I_1 := (x_1x_2x_4, x_2x_3, x_1x_3x_4, x_5x_6x_7)$ and $u := x_2x_3^2x_5$. Since I_1 is a unisplit monomial ideal, it follows that I_1 has the persistence property and, by Proposition 2.3.31, one can conclude that $I = uI_1$ has the persistence property. Now, Lemma 2.3.30 implies that J has the persistence property, as required.

2.4 Persistence property for associated primes of a family of ideals

Throughout this section, R is a commutative Noetherian ring. Additionally, the results of this section can be found in [98]. We start with the following definition.

Definition 2.4.1. Let Φ be a family of ideals of R . Then we say that Φ has the persistence property if there exists a relation $\leq$ on Φ such that $(\Phi, \leq)$ is a partially ordered set with the following properties:

- (i) For all $\mathfrak{b} \in \Phi$, the set $(\bigcup_{\mathfrak{a} \in \Phi} \text{Ass}_R(R/\mathfrak{a})) \cap V(\mathfrak{b})$ is finite, where for an ideal $\mathfrak{c}$ of R , $V(\mathfrak{c})$ is the set of prime ideals contain $\mathfrak{c}$.
- (ii) For all ideals $\mathfrak{a}, \mathfrak{b} \in \Phi$ with $\mathfrak{a} \leq \mathfrak{b}$, we have $\text{Ass}_R(R/\mathfrak{a}) \subseteq \text{Ass}_R(R/\mathfrak{b})$.

For an ideal $\mathfrak{c}$ of R , consider the family $\Phi_{\mathfrak{c}} = \{\mathfrak{c}^n\}_{n \geq 1}$ of ideals of R . Clearly, Brodmann's Theorem shows that, for all $\mathfrak{c}^m \in \Phi_{\mathfrak{c}}$, the set $(\bigcup_{\mathfrak{a} \in \Phi_{\mathfrak{c}}} \text{Ass}_R(R/\mathfrak{a})) \cap V(\mathfrak{c}^m) = \bigcup_{n \geq 1} \text{Ass}_R(R/\mathfrak{c}^n)$ is finite. It is necessary to note that in the above definition the partially order $\leq$ is not reverse inclusion in general. In this section, we provide two families of ideals (with partially order relations) which have the persistence property. We now state the first example constructed on R -regular sequences.

Theorem 2.4.2. Let n be a positive integer with $n \leq \text{depth}(R)$. Put

$$\Phi = \{(x_1, \dots, x_n) : x_1, \dots, x_n \text{ is an } R\text{-regular sequence}\}.$$

Then $(\Phi, \supseteq)$ satisfies the persistence property.

Proof. To prove the first part of Definition 2.4.1, suppose that $\mathfrak{p}$ is an arbitrary element in $(\bigcup_{\mathfrak{a} \in \Phi} \text{Ass}_R(R/\mathfrak{a})) \cap V(\mathfrak{b})$, for some $\mathfrak{b} \in \Phi$. Then there exists $\mathfrak{a} \in \Phi$ such that $\mathfrak{p} \in \text{Ass}_R(R/\mathfrak{a}) \cap V(\mathfrak{b})$. Let $z_1, \dots, z_n$ be an R -sequence generating the ideal $\mathfrak{b}$ and $y_1, \dots, y_n$ be an R -sequence generating the ideal $\mathfrak{a}$. In the local ring $R_{\mathfrak{p}}$, we have $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{a}R_{\mathfrak{p}})$, and so $y_1/1, \dots, y_n/1$ is a maximal $R_{\mathfrak{p}}$ -sequence contained in $\mathfrak{p}R_{\mathfrak{p}}$. Thus, $\text{depth}R_{\mathfrak{p}} = n$. Now, $z_1/1, \dots, z_n/1$ is an $R_{\mathfrak{p}}$ -sequence contained in $\mathfrak{p}R_{\mathfrak{p}}$, and so it must be a maximal such sequence, and we must have $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{b}R_{\mathfrak{p}})$. Therefore, $\mathfrak{p} \in \text{Ass}_R(R/\mathfrak{b})$, which is a finite set.

Now, consider the relation $\leq$ on Φ such that, for all $\mathfrak{a}, \mathfrak{b} \in \Phi$, $\mathfrak{a} \leq \mathfrak{b}$ if and only if $\mathfrak{a} \supseteq \mathfrak{b}$. Then $(\Phi, \leq)$ is a partially ordered set.

Assume that $\mathfrak{a}$ and $\mathfrak{b}$ are two ideals in Φ such that $\mathfrak{a} \leq \mathfrak{b}$. Then there exist R -regular sequences $x_1, \dots, x_n$ and $y_1, \dots, y_n$ such that

$$\mathfrak{a} := (x_1, \dots, x_n) \supseteq \mathfrak{b} := (y_1, \dots, y_n).$$

Hence, there exists an $n \times n$ matrix $A = (a_{ij})$ such that $y_j = \sum_{i=1}^n a_{ij}x_i$ for all j with $1 \leq j \leq n$. Let α be the determinant of A . Then, by [47, Theorem 3.4], the determinantal map $f_{\mathfrak{b}}^{\mathfrak{a}} : R/\mathfrak{a} \rightarrow R/\mathfrak{b}$ given by multiplication in α is an R -monomorphism. Hence, $\text{Ass}_R(R/\mathfrak{a}) \subseteq \text{Ass}_R(R/\mathfrak{b})$. This means that Φ satisfies the second part of Definition 2.4.1. $\square$

Here, we present the second example constructed on d -sequences. The theory of d -sequences was introduced by Huneke in [72].

Definition 2.4.3. ([72, Definition 1.1]) A sequence $x_1, \dots, x_n$ of elements of R is called a d -sequence on R if

(i) $x_i \notin Rx_1 + \dots + Rx_{i-1} + Rx_{i+1} + \dots + Rx_n$ for all $i = 1, \dots, n$; and,

(ii) for all $k \geq i + 1$ and all $i \in \mathbb{N}_0$ which $x_0 = 0$,

$$((x_0, \dots, x_i) :_R x_{i+1}x_k) = ((x_0, \dots, x_i) :_R x_k).$$

It is an unconditioned strong d -sequence (u.s.d.-sequence) on R if $x_1^{a_1}, \dots, x_n^{a_n}$ is a d -sequence in any order for all positive integers $a_1, \dots, a_n$.

Now, let $x = x_1, \dots, x_n$ be a sequence of elements of R and consider the family

$$\Phi_x := \{(x_1^{a_1}, \dots, x_n^{a_n}) : a_1, \dots, a_n \in \mathbb{N} \text{ with } a_1, \dots, a_n \geq 2\},$$

of ideals of R . In the following, for two arbitrary elements $\mathbf{a} = (x_1^{a_1}, \dots, x_n^{a_n})$ and $\mathbf{b} = (x_1^{b_1}, \dots, x_n^{b_n})$ of Φ_x , we say that $\mathbf{a} \leq \mathbf{b}$ if and only if $b_i \leq a_i$ for all $i = 1, \dots, n$. Clearly, $(\Phi_x, \leq)$ is a partially ordered set. Note that if $\mathbf{a} \leq \mathbf{b}$, then $\mathbf{a} \supseteq \mathbf{b}$, but the converse is not true. So, the relation $\leq$ in $(\Phi_x, \leq)$ is not the reverse inclusion.

Theorem 2.4.4. Let n be a positive integer and $x = x_1, \dots, x_n$ be a u.s.d.-sequence on R . Then, by using the above notation, $(\Phi_x, \leq)$ has the persistence property.

Proof. Suppose that $\mathbf{a} = (x_1^{a_1}, \dots, x_n^{a_n})$ is an arbitrary element of Φ_x . For $i \in \mathbb{N}$ with $1 \leq i \leq n$, consider the following homomorphism

$$f : \frac{x_i^{a_i-1}R + \sum_{j=1}^n x_j^{a_j}R}{\sum_{j=1}^n x_j^{a_j}R} \longrightarrow \frac{R}{\sum_{j=1, j \neq i}^n x_j^{a_j}R + x_i^{a_i+1}R},$$

given by $f(\alpha + \sum_{j=1}^n x_j^{a_j}R) := \alpha x_i + \sum_{j=1, j \neq i}^n x_j^{a_j}R + x_i^{a_i+1}R$. We claim that f is

monomorphism. To do this, suppose that $\alpha x_i \in \sum_{j=1, j \neq i}^n x_j^{a_j}R + x_i^{a_i+1}R$ for some $\alpha \in x_i^{a_i-1}R$. Then we have the following equality

$$\alpha x_i = x_1^{a_1}r_1 + \dots + x_{i-1}^{a_{i-1}}r_{i-1} + x_i^{a_i+1}r_i + x_{i+1}^{a_{i+1}}r_{i+1} + \dots + x_n^{a_n}r_n,$$

for some $r_1, \dots, r_n \in R$. Since $\alpha \in x_i^{a_i-1}R$, there exists $r \in R$ such that $\alpha = x_i^{a_i-1}r$. Thus, $(r - r_i x_i)x_i^{a_i} \in \sum_{j=1, j \neq i}^n x_j^{a_j}R$. Since $x = x_1, \dots, x_n$ is a u.s.d.-sequence on R , we can conclude that

$$\left(\sum_{j=1, j \neq i}^n x_j^{a_j}R :_R x_i^{a_i} \right) = \left(\sum_{j=1, j \neq i}^n x_j^{a_j}R :_R x_i \right).$$

Hence, $rx_i \in \sum_{j=1, j \neq i}^n x_j^{a_j} R + x_i^2 R$, and so $\alpha \in \sum_{j=1}^n x_j^{a_j} R$. This implies that f is an R -monomorphism. Now, as mentioned in [85], consider the following exact sequence

$$0 \longrightarrow \frac{x_i^{a_i-1} R + \sum_{j=1}^n x_j^{a_j} R}{\sum_{j=1}^n x_j^{a_j} R} \xrightarrow{f} \frac{R}{\sum_{j=1, j \neq i}^n x_j^{a_j} R + x_i^{a_i+1} R} \xrightarrow{g} \frac{R}{\sum_{j=1}^n x_j^{a_j} R} \longrightarrow 0,$$

where g is the natural homomorphism. Thus, one obtains that

$$\begin{aligned} \text{Ass}_R\left(\frac{R}{\sum_{j=1, j \neq i}^n x_j^{a_j} R + x_i^{a_i+1} R}\right) &\subseteq \text{Ass}_R\left(\frac{x_i^{a_i-1} R + \sum_{j=1}^n x_j^{a_j} R}{\sum_{j=1}^n x_j^{a_j} R}\right) \\ &\cup \text{Ass}_R\left(\frac{R}{\sum_{j=1}^n x_j^{a_j} R}\right). \end{aligned}$$

In the light of

$$\text{Ass}_R\left(\frac{x_i^{a_i-1} R + \sum_{j=1}^n x_j^{a_j} R}{\sum_{j=1}^n x_j^{a_j} R}\right) \subseteq \text{Ass}_R\left(\frac{R}{\sum_{j=1}^n x_j^{a_j} R}\right),$$

we can derive that

$$\text{Ass}_R\left(\frac{R}{\sum_{j=1, j \neq i}^n x_j^{a_j} R + x_i^{a_i+1} R}\right) \subseteq \text{Ass}_R\left(\frac{R}{\sum_{j=1}^n x_j^{a_j} R}\right),$$

for all $(x_1^{a_1}, \dots, x_n^{a_n}) \in \Phi_x$ and $i \in \mathbb{N}$ with $1 \leq i \leq n$. This implies that for two arbitrary elements $\mathbf{a} = (x_1^{a_1}, \dots, x_n^{a_n})$ and $\mathbf{b} = (x_1^{b_1}, \dots, x_n^{b_n})$ of Φ_x with $\mathbf{a} \leq \mathbf{b}$, we have the inclusion $\text{Ass}_R(R/\mathbf{a}) \subseteq \text{Ass}_R(R/\mathbf{b})$, and also the set $\bigcup_{\mathbf{c} \in \Phi_x} \text{Ass}_R(R/\mathbf{c})$ is contained in the finite set $\text{Ass}_R(R/(x_1^2, \dots, x_n^2))$. We thus get $(\Phi_x, \leq)$ has the persistence property. This completes the proof. $\square$

Remark 2.4.5. Let $x_1, \dots, x_n$ be a permutable R -sequence. Then $x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ is also an R -sequence for all $\alpha_1, \dots, \alpha_n \in \mathbb{N}$. By virtue of $(x_1, \dots, x_n) \supseteq (x_1^{\alpha_1}, \dots, x_n^{\alpha_n})$, Theorem 2.4.2 implies that

$$\text{Ass}_R(R/(x_1, \dots, x_n)) \subseteq \text{Ass}_R(R/(x_1^{\alpha_1}, \dots, x_n^{\alpha_n})),$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{N}$. On the other hand, since every permutable R -sequence is a u.s.d.-sequence on R , it follows from Theorem 2.4.4 that

$$\text{Ass}_R(R/(x_1^{\alpha_1}, \dots, x_n^{\alpha_n})) \subseteq \text{Ass}_R(R/(x_1^2, \dots, x_n^2)),$$

for all $\alpha_1, \dots, \alpha_n \geq 2$.

Now, consider the following R -monomorphism with $0 \leq t \leq n - 1$,

$$f : R/(x_1^2, \dots, x_t^2, x_{t+1}, \dots, x_n) \rightarrow R/(x_1^2, \dots, x_n^2),$$

given by

$$f(r + (x_1^2, \dots, x_t^2, x_{t+1}, \dots, x_n)) = rx_{t+1} \cdots x_n + (x_1^2, \dots, x_n^2).$$

We therefore obtain

$$\text{Ass}_R(R/(x_1^{\alpha_1}, \dots, x_n^{\alpha_n})) = \text{Ass}_R(R/(x_1, \dots, x_n)).$$

Chapter 3

On the strong persistence property of monomial ideals

3.1 Associated primes of cover ideals under graph operations

In this chapter, we focus on the strong persistence property of monomial ideals. In particular, we argue on the cover ideals of graphs. It should be noted that the results of this section can be found in [104].

Throughout this section, G is an undirected finite simple graph, that is to say, G has no loops and no multiple edges. The symbol $V(G)$ (respectively, $E(G)$) is the set of vertices (respectively, the set of edges) of a graph G . Furthermore, $I(G)$ (respectively, $J(G)$) stands for the edge ideal (respectively, the cover ideal) of a graph G . In addition, $[n] := \{1, \dots, n\}$.

Next lemma is necessary for us which will be used in proving Theorem 3.1.2. Also, recall that an ideal I in a commutative Noetherian ring R has the *strong persistence property* if $(I^{k+1} :_R I) = I^k$ for all $k \geq 1$.

Lemma 3.1.1. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, v and h be two monomials in R such that $\gcd(v, h) = 1$, $h \in I$, and $\gcd(v, u_i) = 1$ for all $i = 1, \dots, m$. If I has the strong persistence property, then $L := vI + hR$ has so.*

Proof. Assume that I has the strong persistence property. Due to the definition, it is sufficient for us to show that $(L^{t+1} :_R L) = L^t$ for all positive integers t . Based on the binomial expansion, it follows that $L^{t+1} = \sum_{i=0}^{t+1} (hR)^{t+1-i} (vI)^i$, and according

to $\gcd(v, h) = 1$, we get the following equalities

$$\begin{aligned}
(L^{t+1} :_R L) &= \left(\sum_{i=0}^{t+1} (hR)^{t+1-i} (vI)^i :_R vI \right) \cap \left(\sum_{i=0}^{t+1} (hR)^{t+1-i} (vI)^i :_R hR \right) \\
&= (h^{t+1}R + \sum_{i=1}^{t+1} (hR)^{t+1-i} v^{i-1} I^i :_R I) \\
&\quad \cap ((v^{t+1} I^{t+1} :_R hR) + \sum_{i=0}^t (hR)^{t-i} (vI)^i).
\end{aligned}$$

Since $h \in I$, one can conclude that $(hR)^{t+1-i} I^i \subseteq I^{t+1}$ for all $i = 0, \dots, t+1$, and thus $(h^{t+1}R + \sum_{i=1}^{t+1} (hR)^{t+1-i} v^{i-1} I^i :_R I) \subseteq (I^{t+1} :_R I)$. Because I has the strong persistence property, we have $(I^{t+1} :_R I) = I^t$. Hence, one can deduce that

$$(h^{t+1}R + \sum_{i=1}^{t+1} (hR)^{t+1-i} v^{i-1} I^i :_R I) \subseteq I^t.$$

By virtue of $\sum_{i=0}^t (hR)^{t-i} (vI)^i = L^t$, this implies the containment below

$$(L^{t+1} :_R L) \subseteq I^t \cap (v^{t+1} I^{t+1} :_R hR) + I^t \cap L^t.$$

In the light of $\gcd(v, h) = 1$, we can assert that

$$I^t \cap (v^{t+1} I^{t+1} :_R hR) = I^t \cap v^{t+1} (I^{t+1} :_R hR),$$

and by $\gcd(v, u_i) = 1$ for all $i = 1, \dots, m$, one has $I^t \cap (v^{t+1} I^{t+1} :_R hR) \subseteq (vI)^t$. Accordingly, $I^t \cap (v^{t+1} I^{t+1} :_R hR) \subseteq L^t$, and so $(L^{t+1} :_R L) \subseteq L^t$. On account of $L^t \subseteq (L^{t+1} :_R L)$, one derives $(L^{t+1} :_R L) = L^t$, as required. $\square$

We now give one of the main results in the subsequent theorem. Indeed, we show that if we start with any graph G whose cover ideal has the strong persistence property, and if we then add a new vertex and join it to every vertex of G , then the cover ideal of the new graph will also have the strong persistence property.

Theorem 3.1.2. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and*

$$E(H) = E(G) \cup \{\{v, w\} : \text{for all } v \in V(G)\}.$$

If $J(G)$ has the strong persistence property, then $J(H)$ has so.

Proof. Suppose that $J(G)$ has the strong persistence property. Without loss of generality, one may assume that $V(G) = [n]$ and $V(H) = V(G) \cup \{n+1\}$. Also, let $R = K[x_1, \dots, x_{n+1}]$ be a polynomial ring over a field K . Let us simplify the notation by $L := J(H)$, $I := J(G)$, and $g := \prod_{i=1}^n x_i$.

In the following, our goal is to apply Lemma 3.1.1. Now, one can easily derive $L = I \cap \bigcap_{i=1}^n (x_i, x_{n+1})$. Due to $\bigcap_{i=1}^n (x_i, x_{n+1}) = (\prod_{i=1}^n x_i, x_{n+1})$, we deduce that $L = I \cap (\prod_{i=1}^n x_i, x_{n+1})$. By virtue of $g = \prod_{i=1}^n x_i$ and $\mathcal{G}(I) \subseteq K[x_1, \dots, x_n]$, we obtain $g \in I$. It is routine to check that $L = x_{n+1}I + gR$. Assume that $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. This leads to $\mathcal{G}(L) = \{x_{n+1}u_1, \dots, x_{n+1}u_m, g\}$. In the light of $L = x_{n+1}I + gR$, $\gcd(x_{n+1}, g) = 1$, $g \in I$, $\gcd(x_{n+1}, u_i) = 1$ for all $i = 1, \dots, m$, and because I has the strong persistence property, Lemma 3.1.1 yields that L has the strong persistence property, as claimed. $\square$

In order to prove Corollary 3.1.7 and Theorem 3.1.8, we need the following theorem. To do this, we recall some definitions that will be necessary for understanding Theorem 3.1.6.

Definition 3.1.3. ([45, Definition 2.7]) Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph. A d -coloring of $\mathcal{H}$ is any partition of $\mathcal{V} = C_1 \cup \dots \cup C_d$ into d disjoint sets such that for every $E \in \mathcal{E}$, we have $E \not\subseteq C_i$ for all $i = 1, \dots, d$. (In the case of a graph G , this simply means that any two vertices connected by an edge receive different colors.) The C_i s are called the color classes. Each color class C_i is an independent set, meaning that C_i does not contain any edge of the hypergraph. The chromatic number of $\mathcal{H}$, denoted $\chi(\mathcal{H})$, is the minimal d such that $\mathcal{H}$ has a d -coloring.

Definition 3.1.4. ([45, Definition 2.8]) A hypergraph $\mathcal{H}$ is called critically d -chromatic if $\chi(\mathcal{H}) = d$, but for every vertex $x \in \mathcal{V}$, $\chi(\mathcal{H} \setminus \{x\}) < d$, where $\mathcal{H} \setminus \{x\}$ denotes the hypergraph $\mathcal{H}$ with x and all edges containing x removed.

Definition 3.1.5. ([45, Definition 4.2]) Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a hypergraph with $V(\mathcal{H}) = \{x_1, \dots, x_n\}$. For each s , the s -th expansion of $\mathcal{H}$ is defined to be the hypergraph obtained by replacing each vertex $x_i \in V(\mathcal{H})$ by a collection $\{x_{ij} \mid j = 1, \dots, s\}$, and replacing $E(\mathcal{H})$ by the edge set that consists of edges $\{x_{i_1 l_1}, \dots, x_{i_r l_r}\}$ whenever $\{x_{i_1}, \dots, x_{i_r}\} \in E(\mathcal{H})$ and edges $\{x_{il}, x_{ik}\}$ for $l \neq k$. We denote this hypergraph by $\mathcal{H}^s$. The new variables x_{ij} are called the shadows of x_i . The process of setting x_{il} to equal to x_i for all i and l is called the depolarization.

Theorem 3.1.6. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and $E(H) = E(G) \cup \{\{v, w\}\}$ for some vertex $v \in V(G)$. Then $\text{Ass}_{R'}(R'/J(H)^s) = \text{Ass}_R(R/J(G)^s) \cup \{(x_v, x_w)\}$ for all s , where $R = K[x_\alpha : \alpha \in V(G)]$ and $R' = K[x_\alpha : \alpha \in V(H)]$.

Proof. To simplify the notation, set $I := J(G)$ and $J := J(H)$. We first show that $\text{Ass}_R(R/I^s) \cup \{(x_v, x_w)\} \subseteq \text{Ass}_{R'}(R'/J^s)$ for all s . To do this, fix $s \in \mathbb{N}$, and assume that $\mathbf{p} = (x_{i_1}, \dots, x_{i_r})$ is an arbitrary element of $\text{Ass}_R(R/I^s)$. It follows from [45, Lemma 2.11] that $\mathbf{p} \in \text{Ass}(K[\mathbf{p}]/J(G_{\mathbf{p}})^s)$, where $K[\mathbf{p}] = K[x_{i_1}, \dots, x_{i_r}]$ and $G_{\mathbf{p}}$ is the induced subgraph of G on the vertex set $\{i_1, \dots, i_r\} \subseteq V(G)$. As $G_{\mathbf{p}} = H_{\mathbf{p}}$,

we obtain $\mathfrak{p} \in \text{Ass}(K[\mathfrak{p}]/J(H_{\mathfrak{p}})^s)$. This implies that $\mathfrak{p} \in \text{Ass}_{R'}(R'/J^s)$. By virtue of $(x_v, x_w) \in \text{Ass}_{R'}(R'/J^s)$, one derives $\text{Ass}_R(R/I^s) \cup \{(x_v, x_w)\} \subseteq \text{Ass}_{R'}(R'/J^s)$. For completing the proof, it is sufficient to verify the reverse inclusion. To do this, assume that $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ is an arbitrary element of $\text{Ass}_{R'}(R'/J^s)$ with $\{i_1, \dots, i_r\} \subseteq V(H)$. If $\{i_1, \dots, i_r\} \subseteq V(G)$, then [45, Lemma 2.11] yields that $\mathfrak{p} \in \text{Ass}_R(R/I^s)$, and the proof is complete. We thus let $w \in \{i_1, \dots, i_r\}$. Without loss of generality, one may assume that $i_1 = v$ and $i_2 = w$. According to [45, Corollary 4.5], the associated primes of $J(H)^s$ will correspond to critical chromatic subgraphs of size $s+1$ in the s -th expansion of H . This means that one can take the induced subgraph on the vertex set $\{i_1, \dots, i_r\}$, and then form the s -th expansion on this induced subgraph, and within this new graph find a critical $(s+1)$ -chromatic graph. Due to w is only connected to v in the graph H , and since this induced subgraph is critical, if we remove the vertex w , we can color the resulting graph with s colors. This gives that w has to be adjacent to s vertices. But the only thing w is adjacent to is the shadows of w and the shadows of v , and hence one has a clique among these vertices. Consequently, w and its neighbors will form a clique of size $s+1$. By virtue of a clique is a critical graph, it follows that we do not need any element of $\{i_3, \dots, i_r\}$ or their shadows when making the critical $(s+1)$ -chromatic graph. This implies that $\mathfrak{p} = (x_v, x_w)$, as required. $\square$

In the following, we present the other main results in this section in Corollary 3.1.7 and Theorem 3.1.8. In fact, we prove that if we take any graph G which its cover ideal has the (strong) persistence property, and if we then add a leaf to G , then the cover ideal of the new graph satisfies the (strong) persistence property as well.

Corollary 3.1.7. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and $E(H) = E(G) \cup \{\{v, w\}\}$ for some vertex $v \in V(G)$. If $J(G)$ has the persistence property, then $J(H)$ has so.*

Proof. Let $J(G)$ have the persistence property, and fix a positive integer s . Based on Theorem 3.1.6, we gain $\text{Ass}_{R'}(R'/J(H)^s) = \text{Ass}_R(R/J(G)^s) \cup \{(x_v, x_w)\}$, and $\text{Ass}_{R'}(R'/J(H)^{s+1}) = \text{Ass}_R(R/J(G)^{s+1}) \cup \{(x_v, x_w)\}$, where $R = K[x_\alpha : \alpha \in V(G)]$ and $R' = K[x_\alpha : \alpha \in V(H)]$. Due to $\text{Ass}_R(R/J(G)^s) \subseteq \text{Ass}_R(R/J(G)^{s+1})$, one can conclude that $\text{Ass}_{R'}(R'/J(H)^s) \subseteq \text{Ass}_{R'}(R'/J(H)^{s+1})$. This means that $J(H)$ has the persistence property, and the desired result has been proved. $\square$

Theorem 3.1.8. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and $E(H) = E(G) \cup \{\{v, w\}\}$ for some vertex $v \in V(G)$. If $J(G)$ has the strong persistence property, then $J(H)$ has so.*

Proof. Suppose that $J(G)$ has the strong persistence property. Without loss of generality, one may assume that $V(G) = [n]$, $V(H) = V(G) \cup \{n+1\}$, and $E(H) = E(G) \cup \{\{1, n+1\}\}$. In addition, let $R = K[x_1, \dots, x_{n+1}]$ be a polynomial ring over a field K . In order to simplify our notation, put $I := J(G)$, $L := J(H)$, and $Q := (x_1, x_{n+1})$. By virtue of $L = I \cap Q$, we need only prove that $I \cap Q$ has the strong persistence property. To achieve this, one has to demonstrate

$$((I \cap Q)^{k+1} :_R I \cap Q) = (I \cap Q)^k,$$

for all k . Fix $k \in \mathbb{N}$. In the light of [89, Exercise 6.4], it is enough to verify that $(L_{\mathfrak{a}}^{k+1} :_{R_{\mathfrak{a}}} L_{\mathfrak{a}}) = L_{\mathfrak{a}}^k$ for every $\mathfrak{a} \in \text{Ass}_R(R/L^k)$. Let $\mathfrak{a}$ be an arbitrary element in $\text{Ass}_R(R/L^k)$. It follows also from Theorem 3.1.6 that $\text{Ass}_R(R/L^k) = \text{Ass}(I^k) \cup \{Q\}$. We thus can consider the following two cases:

Case 1. $\mathfrak{a} \in \text{Min}(L^k)$. Since $\text{Min}(L^k) = \text{Min}(L)$, one derives $\mathfrak{a} \in \text{Min}(L)$. In the following, we show $\mathfrak{a} \notin \text{Supp}_R(R/\mathfrak{b})$ for every $\mathfrak{b} \in \text{Ass}_R(R/L) \setminus \{\mathfrak{a}\}$. Suppose, on the contrary, that there exists $\mathfrak{b} \in \text{Ass}_R(R/L) \setminus \{\mathfrak{a}\}$ such that $\mathfrak{a} \in \text{Supp}_R(R/\mathfrak{b})$. Due to $\text{Supp}_R(R/\mathfrak{b}) = V(\text{Ann}_R(R/\mathfrak{b}))$, one has $\text{Ann}_R(R/\mathfrak{b}) \subseteq \mathfrak{a}$, and hence $\mathfrak{b} \subseteq \mathfrak{a}$. On account of $\mathfrak{a}$ is a minimal prime ideal of L , we get $\mathfrak{b} = \mathfrak{a}$, which is a contradiction. This implies that $\mathfrak{b}_{\mathfrak{a}} = R_{\mathfrak{a}}$ for every $\mathfrak{b} \in \text{Ass}_R(R/L) \setminus \{\mathfrak{a}\}$. Therefore, $L_{\mathfrak{a}} = \mathfrak{a}_{\mathfrak{a}}$, and hence $(L^k)_{\mathfrak{a}} = (\mathfrak{a}^k)_{\mathfrak{a}}$. It also arises from direct computation that $(\mathfrak{a}^{k+1} :_R \mathfrak{a}) = \mathfrak{a}^k$. We thus gain $(\mathfrak{a}^{k+1} :_{R_{\mathfrak{a}}} \mathfrak{a}_{\mathfrak{a}}) = \mathfrak{a}_{\mathfrak{a}}^k$, and so $(L_{\mathfrak{a}}^{k+1} :_{R_{\mathfrak{a}}} L_{\mathfrak{a}}) = L_{\mathfrak{a}}^k$, as required.

Case 2. $\mathfrak{a} \in \text{Ass}_R(R/L^k) \setminus \text{Min}(L^k)$. By $\text{Min}(L^k) = \text{Min}(L) = \text{Min}(I) \cup \{Q\}$, we deduce that $\mathfrak{a} \neq Q$, and thus $\mathfrak{a} \in \text{Ass}(I^k) \setminus \text{Min}(I)$. This implies that $x_{n+1} \notin \mathfrak{a}$. We next prove that $(I \cap Q)_{\mathfrak{a}} = I_{\mathfrak{a}}$. By virtue of $(I \cap Q)_{\mathfrak{a}} \subseteq I_{\mathfrak{a}}$, it is sufficient to show that $I_{\mathfrak{a}} \subseteq (I \cap Q)_{\mathfrak{a}}$. For this purpose, assume that $r/s \in I_{\mathfrak{a}}$. This gives that $r/s = \alpha/\beta$ for some $\alpha \in I$ and $\beta \notin \mathfrak{a}$. It is routine to check that $\alpha x_{n+1} \in I \cap Q$ and $\beta x_{n+1} \notin \mathfrak{a}$. On the other hand, on account of the following equality

$$\frac{\alpha}{\beta} = \frac{\alpha x_{n+1}}{\beta x_{n+1}},$$

one derives $r/s \in (I \cap Q)_{\mathfrak{a}}$, and so $(I \cap Q)_{\mathfrak{a}} = I_{\mathfrak{a}}$. By virtue of I has the strong persistence property, we get $(I^{k+1} :_R I) = I^k$, and hence $(I_{\mathfrak{a}}^{k+1} :_{R_{\mathfrak{a}}} I_{\mathfrak{a}}) = I_{\mathfrak{a}}^k$. Therefore, $(L_{\mathfrak{a}}^{k+1} :_{R_{\mathfrak{a}}} L_{\mathfrak{a}}) = L_{\mathfrak{a}}^k$, and the proof is complete. $\square$

From now onwards, we give some applications of the results which appeared in this section. Indeed, we show that the cover ideals of some imperfect graphs satisfy the strong persistence property. To achieve this, one should remember the definition of imperfect graphs, see Section 1.2.

3.1.1 Cover ideals of odd cycle graphs

In the first subsection, we investigate the cover ideals of cycle graphs of odd orders. In other words, Theorem 3.1.11 states that the cover ideal of every cycle graph of odd order satisfies the strong persistence property. For this purpose, we require Remark 3.1.9 and Lemma 3.1.10. We start with the following comment.

Let $R = K[x_1, \dots, x_{2n+1}]$ be a polynomial ring over a field K . Let $I = J(C_{2n+1}) \subset R$ be the *cover ideal* associated to the cycle graph C_{2n+1} with vertex set $[2n+1]$. That is,

$$I = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1}) = (x_1, x_2) \cap \dots \cap (x_{2n}, x_{2n+1}) \cap (x_{2n+1}, x_1),$$

where $2n+2=1$. Note that a monomial $m := x_1^{l_1} \dots x_{2n+1}^{l_{2n+1}}$ belongs to I if and only if $m \in (x_i, x_{i+1})$, if and only if $l_i + l_{i+1} \geq 1$ for every $i \in [2n+1]$. As R has an odd number of indeterminates, then $l_j + l_{j+1} \geq 2$ for some $j \in [2n+1]$. Therefore, we conclude the following remark.

Remark 3.1.9. A monomial $m = x_1^{l_1} \dots x_{2n+1}^{l_{2n+1}} \in I$ if and only if $l_i + l_{i+1} \geq 1$ for all $i \in [2n+1]$ and $\sum_{i=1}^{2n+1} [(l_i + l_{i+1}) - 1] \geq 1$.

According to the remark above, consider a special subset $\{\mu_1, \dots, \mu_{2n+1}\}$ of generators of I with (emphasizing that $2n+2=1$) such that

$$\mu_j := x_j x_{j+1} \prod_{k=1}^{n-1} x_{j+1+2k}.$$

Lemma 3.1.10. A monomial $f = x_1^{l_1} \dots x_{2n+1}^{l_{2n+1}} \in I^t$ if and only if $l_i + l_{i+1} \geq t$ for every $i \in [2n+1]$ and $\sum_{i=1}^{2n+1} [(l_i + l_{i+1}) - t] \geq t$.

Proof. The necessity is direct from the above remark. To prove the sufficiency, we proceed by induction on t . Based on Remark 3.1.9, the claim can be proved for the case $t = 1$. Assume that the statement is true for some $t > 1$. Let $f = x_1^{l_1} \dots x_{2n+1}^{l_{2n+1}}$ be with $l_i + l_{i+1} \geq t+1$ for every $i \in [2n+1]$ and $\sum_{i=1}^{2n+1} [(l_i + l_{i+1}) - (t+1)] \geq t+1$. Want to show $f \in I^{t+1}$. First, we claim that if $\mu_j \mid f$ for some j and if $l_j + l_{j+1} \geq t+2$, then $f/\mu_j \in I^t$, and hence $f \in I^{t+1}$. To prove the claim, write $f/\mu_j = x_1^{l'_1} \dots x_{2n+1}^{l'_{2n+1}}$ and note that $l'_j + l'_{j+1} = l_j + l_{j+1} - 2$ and $l'_i + l'_{i+1} = l_i + l_{i+1} - 1$ for $i \neq j$. Consider,

$$\begin{aligned} \sum_{i=1}^{2n+1} [(l'_i + l'_{i+1}) - t] &= (l_j + l_{j+1} - 2) - t + \sum_{i=1, i \neq j}^{2n+1} [(l_i + l_{i+1} - 1) - t] \\ &= (l_j + l_{j+1}) - (t+1) - 1 + \sum_{i=1, i \neq j}^{2n+1} [(l_i + l_{i+1}) - (t+1)] \\ &= \sum_{i=1}^{2n+1} [(l_i + l_{i+1}) - (t+1)] - 1 \geq t; \end{aligned}$$

thus, $f/\mu_j \in I^t$ by induction.

Define the set $A = \{j \mid l_j = 0\}$ and rewrite $A = \{j_1, \dots, j_s\}$ with $j_i < j_{i+1}$. Define a monomial N , and for every two consecutive elements j_i, j_{i+1} in A with $1 \leq i \leq s-1$ define a monomial $M_{i,i+1}$, as follows (with addition of indices is taken modulo $2n+1$ as of the cyclic behavior of the graph)

$$N = x_{j_1-1}^\sigma \prod_{k=0}^{(j_1+2n-1-\sigma-j_s)/2} x_{j_s+1+2k} \quad \text{and} \quad M_{i,i+1} = x_{j_i+1}^\varepsilon \prod_{k=0}^{(j_{i+1}-j_i-2-\varepsilon)/2} x_{j_i+1+\varepsilon+2k},$$

where $\sigma = 1$ or 0 according as $j_s - j_1$ is even or odd, and $\varepsilon = 0$ or 1 according as $j_{i+1} - j_i$ is even or odd. Roughly speaking, whenever j_i and j_{i+1} are both odd or both even, then $M_{i,i+1}$ is the product of every second variable between x_{j_i} and $x_{j_{i+1}}$ beginning with x_{j_i+1} and ending with $x_{j_{i+1}-1}$. Also, whenever one of j_i and j_{i+1} is odd and the other is even, then $M_{i,i+1}$ is x_{j_i+1} multiplied by the product of every second variable beginning with x_{j_i+2} and ending with $x_{j_{i+1}-1}$. Similarly, whenever one of j_1 and j_s is odd and the other is even, then N is the product of every second variable beginning with x_{j_s+1} and ending with x_{j_1-1} taken in a cyclic order. Also, whenever j_i and j_{i+1} are both odd or both even, then N is x_{j_1-1} multiplied by the product of every second variable beginning with x_{j_s+1} and ending with x_{j_1-2} , taken in a cyclic order.

Define $g = N \prod_{i=1}^{s-1} M_{i,i+1} = x_1^{h_1} \cdots x_{2n+1}^{h_{2n+1}}$. It is clear that $h_i + h_{i+1} = 1$ or 2 for all i . Precisely, $h_i + h_{i+1} = 2$ if and only if $i = j_1 - 2$ with $j_s - j_1$ is even or $i = j_r + 1$ with $j_{r+1} - j_r$ is odd. Hence, $g \in I$. Recall that $l_{i-2} + l_{i-1} \geq t + 2$ for any $i \in A$ with $i - 2 \notin A$, and also $l_{i+1} + l_{i+2} \geq t + 2$ for any $i \in A$ with $i + 2 \notin A$. Therefore, if $f/g = x_1^{l'_1} \cdots x_{2n+1}^{l'_{2n+1}}$ with $l'_i = l_i - h_i$, then $l'_i + l'_{i+1} \geq t$ for all i .

If A consists (or can be represented) entirely of odd (or even) integers, then $g = \mu_{j_1-2}$; thus, the proof is over by the above claim (note if $j_1 = 1$, then $j_1 - 2$ equals $2n$ because computations are taken modulo $2n+1$). Pick $j_r, j_{r+1} \in A$ such that $j_{r+1} - j_r$ is odd. Note that $l_{j_r+1} \geq t + 1$ since $j_r \in A$, and $l_{j_r+2} = a \geq 1$ since $j_r + 2 \notin A$. Also, since $l_i + l_{i+1} \geq t + 1$ for all i , then $l_{j_r+3} \geq t + 1 - a$, which in turn forces $l_{j_r+4} \geq a$. Continuing in this manner, and since $j_{r+1} - j_r$ is odd, we end up with $l_{j_{r+1}-2} \geq t + 1 - a$. But $l_{j_{r+1}-1} \geq t + 1$ since $j_{r+1} \in A$. Therefore, we have $(l_{j_r+1} + l_{j_r+2}) + (l_{j_{r+1}-2} + l_{j_{r+1}-1}) \geq 3(t + 1)$. Thus,

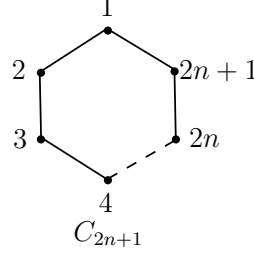
$$\begin{aligned} (l'_{j_r+1} + l'_{j_r+2}) + (l'_{j_{r+1}-2} + l'_{j_{r+1}-1}) &= (l_{j_r+1} + l_{j_r+2}) - (h_{j_r+1} + h_{j_r+2}) \\ &\quad + (l_{j_{r+1}-2} + l_{j_{r+1}-1}) - (h_{j_{r+1}-2} + h_{j_{r+1}-1}) \\ &\geq 3(t + 1) - 2 - 1 = 3t; \end{aligned}$$

therefore, $\sum_{i=1}^{2n+1} [(l'_i + l'_{i+1}) - t] \geq t$; thus, $f/g \in I^t$ by the inductive hypothesis. This completes the inductive step, and therefore the claim has been proved by induction. $\square$

We are now in a position to show Theorem 3.1.11.

Theorem 3.1.11. *Suppose that C_{2n+1} is a cycle graph with vertex set $[2n+1]$. Then the cover ideal associated to C_{2n+1} satisfies the strong persistence property.*

Proof. Label the vertices of C_{2n+1} in counterclockwise order with $1, \dots, 2n+1$, as shown in the figure below.



Let $I = J(C_{2n+1})$ be the cover ideal. Our aim is to show that $(I^{t+1} : I) = I^t$ for all $t \geq 1$. Fix $t \geq 1$. We only need to show that $(I^{t+1} : I) \subseteq I^t$. Suppose $f' = x_1^{l'_1} \cdots x_{2n+1}^{l'_{2n+1}} \in (I^{t+1} : I)$. For any $i \in [2n+1]$ choose $i_0 \neq i$ and consider $f' \mu_{i_0} := x_1^{l'_1} \cdots x_{2n+1}^{l'_{2n+1}} \in I^{t+1}$ since $\mu_{i_0} \in I$; hence, $l_i + l_{i+1} \geq t+1$ for every $i \in [2n+1]$; and therefore, $l'_i + l'_{i+1} \geq t$ for every $i \in [2n+1]$. Now, fix $j_0 \in [2n+1]$ and consider $f' \mu_{j_0} := x_1^{l'_1} \cdots x_{2n+1}^{l'_{2n+1}} \in I^{t+1}$; thus, $\sum_{i=1}^{2n+1} [(l'_i + l'_{i+1}) - (t+1)] \geq t+1$. Note $l_{j_0} + l_{j_0+1} = l'_{j_0} + l'_{j_0+1} + 2$ and $l_i + l_{i+1} = l'_i + l'_{i+1} + 1$ for $i \neq j_0$. Therefore, by the same process as the first part of the above proof, we obtain $\sum_{i=1}^{2n+1} [(l'_i + l'_{i+1}) - t] \geq t$; hence, $f' \in I^t$, which finishes the proof. $\square$

In order to establish Proposition 3.1.13, we require the following proposition. This proposition is a well-known result, but we re-prove it by a new proof.

Proposition 3.1.12. *Let I be an ideal in a commutative Noetherian ring S such that satisfies the strong persistence property. Then I has the persistence property.*

Proof. Fix $k \geq 1$, and choose an arbitrary element $\mathfrak{p} \in \text{Ass}_S(S/I^k)$. This implies that $\mathfrak{p} = (I^k :_S h)$ for some $h \in S$. Since I satisfies the strong persistence property, we have $(I^{k+1} :_S I) = I^k$, and so $\mathfrak{p} = ((I^{k+1} :_S I) :_S h)$. Let $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Hence, one obtains $\mathfrak{p} = (I^{k+1} :_S h \sum_{i=1}^m u_i S) = \bigcap_{i=1}^m (I^{k+1} :_S h u_i)$. Accordingly, we get $\mathfrak{p} = (I^{k+1} :_S h u_i)$ for some $1 \leq i \leq m$. Therefore, $\mathfrak{p} \in \text{Ass}_S(S/I^{k+1})$. This means that I has the persistence property, as claimed. $\square$

In the next proposition, our goal is to detect when the unique homogeneous maximal ideal appears. This result can also be deduced directly from [45, Corollary 4.7].

Proposition 3.1.13. *Suppose that C_{2n+1} is a cycle graph on the vertex set $[2n+1]$, $R = K[x_1, \dots, x_{2n+1}]$ is a polynomial ring over a field K , and $\mathfrak{m}$ is the unique homogeneous maximal ideal of R . Then $\mathfrak{m} \in \text{Ass}_R(R/(J(C_{2n+1}))^s)$ for all $s \geq 2$, and $\mathfrak{m} \notin \text{Ass}_R(R/(J(C_{2n+1})))$.*

Proof. Label the vertices of C_{2n+1} in counterclockwise order with $1, \dots, 2n+1$. In order to simplify the notation, we set $I := J(C_{2n+1})$. Due to Theorem 3.1.11 and Proposition 3.1.12, one obtains the ideal I has the persistence property, and by virtue of $\mathfrak{m}$ is the unique homogeneous maximal ideal of R , it is sufficient to verify that $\mathfrak{m} \subseteq (I^2 :_R \prod_{k=1}^{2n+1} x_k)$. To achieve this, we need to show that $x_i \prod_{k=1}^{2n+1} x_k \in I^2$ for all $i = 1, \dots, 2n+1$. To see this, let i be an arbitrary positive integer with $1 \leq i \leq 2n+1$, and set $g_i := x_i \prod_{k=1}^{2n+1} x_k$. Now, write

$$g_i = x_i \prod_{k=1}^{2n+1} x_k = \prod_{k=1}^{2n+1} x_k^{\ell_k}.$$

It is routine to check that $\ell_j + \ell_{j+1} \geq 2$ for all $j \in [2n+1]$, and also

$$\sum_{k=1}^{2n+1} [(\ell_k + \ell_{k+1}) - 2] = (3+3) - 2(2) = 2.$$

Now, Lemma 3.1.10 yields that $g_i \in I^2$, as required. Moreover, since the ideal I is a square-free monomial ideal and $2n+1 \geq 3$, one can see that $\mathfrak{m} \notin \text{Ass}_R(R/I)$. This gives $\mathfrak{m} \in \text{Ass}_R(R/(J(C_{2n+1}))^s)$ for all $s \geq 2$, and the desired conclusion has been obtained. $\square$

We now explore the associated prime ideals of powers of the cover ideals of cycle graphs of odd orders. To do this, we first recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal which has been stated in [65], and then use it in the proof of Proposition 3.1.15.

Definition 3.1.14. Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . We also denote by $V^*(I)$ the set of monomial prime ideals containing I . Let $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ be a monomial prime ideal. The monomial localization of I with respect to $\mathfrak{p}$, denoted by $I(\mathfrak{p})$, is the ideal in the polynomial ring $R(\mathfrak{p}) = K[x_{i_1}, \dots, x_{i_r}]$ which is obtained from I by applying the K -algebra homomorphism $R \rightarrow R(\mathfrak{p})$ with $x_j \mapsto 1$ for all $x_j \notin \{x_{i_1}, \dots, x_{i_r}\}$. It is well-known that $\mathfrak{p} \in \text{Ass}_R(R/I)$ if and only if $\mathfrak{m}_{\mathfrak{p}} \in \text{Ass}(I(\mathfrak{p}))$, where $\mathfrak{m}_{\mathfrak{p}}$ is the graded maximal ideal of $R(\mathfrak{p})$.

It should be noted that a special case of Proposition 3.1.15, when $s = 2$, has already appeared in [44].

Proposition 3.1.15. Suppose that C_{2n+1} is a cycle graph on the vertex set $[2n+1]$, $R = K[x_1, \dots, x_{2n+1}]$ is a polynomial ring over a field K , and $\mathfrak{m}$ is the unique homogeneous maximal ideal of R . Then

$$\text{Ass}_R(R/(J(C_{2n+1}))^s) = \text{Ass}_R(R/J(C_{2n+1})) \cup \{\mathfrak{m}\},$$

for all $s \geq 2$. In particular,

$$\text{Ass}^\infty(J(C_{2n+1})) = \{(x_i, x_{i+1}) : i = 1, \dots, 2n\} \cup \{(x_{2n+1}, x_1)\} \cup \{\mathfrak{m}\}.$$

Proof. Label the vertices of C_{2n+1} in counterclockwise order with $1, \dots, 2n+1$. To simplify our notation, we take $I := J(C_{2n+1})$. Fix $s \in \mathbb{N}$ with $s \geq 2$. It follows from Proposition 3.1.13 that $\text{Ass}_R(R/I) \cup \{\mathfrak{m}\} \subseteq \text{Ass}_R(R/I^s)$. Therefore, one has to demonstrate the reverse inclusion. For this purpose, let $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ be an arbitrary element in $\text{Ass}_R(R/I^s) \setminus \{\mathfrak{m}\}$. This implies that $(x_{i_1}, \dots, x_{i_r}) \in \text{Ass}(I^s(\mathfrak{p}))$. On account of above argument, one can assume that the induced subgraph of C_{2n+1} on the vertex set $\{i_1, \dots, i_r\}$ is a union of some disjoint path subgraphs. On the other hand, union of these disjoint path subgraphs can be viewed as a bipartite graph, say H . Accordingly, $I(\mathfrak{p})$ is the cover ideal of H . Due to [48, Corollary 2.6], it follows that $I(\mathfrak{p})$ is normally torsion-free, i.e., $\text{Ass}((I(\mathfrak{p}))^t) = \text{Ass}(I(\mathfrak{p}))$ for all t . Since $I^t(\mathfrak{p}) = (I(\mathfrak{p}))^t$ for all t , we get $(x_{i_1}, \dots, x_{i_r}) \in \text{Ass}(I(\mathfrak{p}))$. This leads to $\mathfrak{p} \in \text{Ass}_R(R/I)$, and the proof is complete. The last assertion follows readily from the fact that $\text{Ass}_R(R/J(C_{2n+1})) = \{(x_i, x_{i+1}) : i = 1, \dots, 2n\} \cup \{(x_{2n+1}, x_1)\}$. $\square$

3.1.2 Cover ideals of wheel graphs

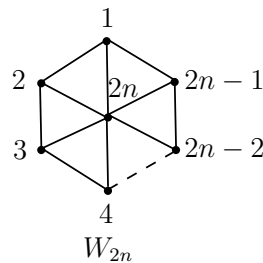
In the second subsection, we probe the cover ideals of wheel graphs. To do this, we first express the definition of wheel graphs.

Definition 3.1.16. *A wheel graph W_n of order n is a graph that contains a cycle of order $n-1$, and for which every vertex in the cycle is connected to one other vertex which is known as the hub. The edges of a wheel which include the hub are called spokes.*

In the following, we show that every cover ideal of wheel graphs of even order has the strong persistence property.

Theorem 3.1.17. *Suppose that W_{2n} is a wheel graph of order $2n$ on the vertex set $[2n]$. Then $J(W_{2n})$ satisfies the strong persistence property.*

Proof. Label the vertices of the cycle graph C_{2n-1} of order $2n-1$ in counterclockwise order with $1, \dots, 2n-1$, and the hub with $2n$, as shown in the figure below.



Now, the desired conclusion follows readily from Theorems 3.1.11 and 3.1.2. This finishes the proof. $\square$

Proposition below illuminates when the unique homogeneous maximal ideal appears.

Proposition 3.1.18. *Suppose that W_{2n} is a wheel graph of order $2n$ on the vertex set $[2n]$, $R = K[x_1, \dots, x_{2n}]$ is a polynomial ring over a field K , and $\mathfrak{m}$ is the unique homogeneous maximal ideal of R . Then $\mathfrak{m} \in \text{Ass}_R(R/(J(W_{2n}))^s)$ for all $s \geq 3$, and $\mathfrak{m} \notin \text{Ass}_R(R/(J(W_{2n}))^s)$ for $s = 1, 2$.*

Proof. Label the vertices of C_{2n-1} in counterclockwise order with $1, \dots, 2n-1$, and the hub with $2n$. Set $I := J(W_{2n})$. By virtue of Theorem 3.1.17 and Proposition 3.1.12, one can conclude that the ideal I has the persistence property. On the other hand, [78, Theorem 2.2] yields that $\mathfrak{m} \in \text{Ass}_R(R/I^3) \setminus \text{Ass}_R(R/I^2)$. According to $2n \geq 4$, this gives rise to $\mathfrak{m} \notin \text{Ass}_R(R/I)$. Therefore, we gain $\mathfrak{m} \in \text{Ass}_R(R/(J(W_{2n}))^s)$ for all $s \geq 3$, and $\mathfrak{m} \notin \text{Ass}_R(R/(J(W_{2n}))^s)$ for $s = 1, 2$, as claimed. $\square$

3.1.3 Cover ideals of helm graphs

The third subsection is devoted to the cover ideals of helm graphs. For this purpose, we begin with the definition of helm graphs.

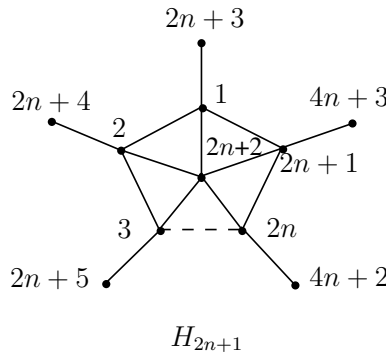
Definition 3.1.19. *The helm graph H_n , where has $2n + 1$ vertices, is the graph obtained from a wheel graph W_{n+1} of order $n + 1$ by adjoining a pendant edge at each node of the outer circuit of the wheel graph W_{n+1} .*

The following theorem illustrates every cover ideal of helm graphs of odd order with greater than or equal to 5, has the strong persistence property.

Theorem 3.1.20. *Suppose that H_{2n+1} with $n \geq 2$ is a helm graph of order $2n + 1$ on the vertex set $[4n + 3]$. Then $J(H_{2n+1})$ satisfies the strong persistence property.*

Proof. Label the vertices of C_{2n+1} in counterclockwise order with $1, \dots, 2n + 1$, and the hub with $2n + 2$, as shown in the figure below, such that we have

$$E(H_{2n+1}) = E(W_{2n+2}) \cup \{\{i, 2n + 2 + i\} : i = 1, \dots, 2n + 1\}.$$



We can now combine together Theorem 3.1.17 and the iteration of Theorem 3.1.8 to obtain the claim. $\square$

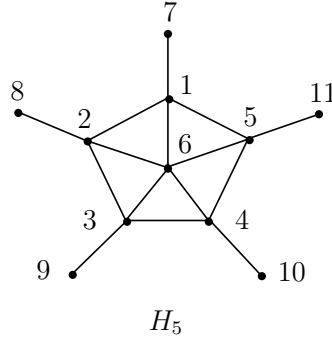
In the following proposition, one can see when appears the unique associated prime ideal which is maximal with respect to inclusion.

Proposition 3.1.21. *Suppose that H_{2n+1} with $n \geq 2$ is a helm graph of order $2n+1$ on the vertex set $[4n+3]$, $R = K[x_1, \dots, x_{4n+3}]$ is a polynomial ring over a field K , and $\mathfrak{m} = (x_1, \dots, x_{2n+2})$. Then $\mathfrak{m} \in \text{Ass}_R(R/(J(H_{2n+1}))^s)$ for all $s \geq 3$, and $\mathfrak{m} \notin \text{Ass}_R(R/(J(H_{2n+1}))^s)$ for $s = 1, 2$.*

Proof. The result follows immediately from Proposition 3.1.18, Theorem 3.1.20, and the iteration of Theorem 3.1.6. $\square$

We next give an example which explains Proposition 3.1.21.

Example 3.1.22. *Let H_5 be the helm graph, as shown in the figure below, on the vertex set $[11] = \{1, \dots, 11\}$, $R = K[x_1, \dots, x_{11}]$ be the polynomial ring over a field K , and $\mathfrak{m} = (x_1, \dots, x_6)$.*



It follows from Proposition 3.1.21 that for the unique associated prime ideal $\mathfrak{m}$ which is maximal with respect to inclusion, one derives $\mathfrak{m} \in \text{Ass}_R(R/(J(H_5))^s)$ for all $s \geq 3$, and $\mathfrak{m} \notin \text{Ass}_R(R/(J(H_5))^s)$ for $s = 1, 2$.

3.2 Superficial ideals in commutative Noetherian rings

The concept of superficial elements was first introduced by P. Samuel in [119]. Indeed, let I be an ideal in a commutative ring S , and let $k \in \mathbb{N}$. An element x in S is called a *superficial element of degree k for I* if $x \in I^k$ and there exists $c \in \mathbb{N}$ such that $(I^{n+k} :_S x) \cap I^c = I^n$ for all $n \geq c$. If $(I^{n+k} :_S x) = I^n$ for all $n \in \mathbb{N}$, we say that x is a *strongly superficial element of degree k for I* (see [90, 4.1.2]).

The motivation of this section comes back to the notion of strong persistence property. In fact, exploring the proof of Proposition 2.2.17 gives us that if an ideal I has a strongly superficial element of degree 1, then I has the strong persistence property. This result gives rise to a natural question in the context of the notion of

strong persistence property: May one consider a part of a minimal set of generators of I instead of all elements of the minimal set of generators in the definition of the strong persistence property? To answer this question, we provide the definition of superficial ideals.

Definition 3.2.1. *let I and J be two ideals in a commutative Noetherian ring S . We say that J is a superficial ideal for I if the following conditions are satisfied:*

- (i) $\mathcal{G}(J) \subseteq \mathcal{G}(I)$; and,
- (ii) $(I^{k+1} :_S J) = I^k$ for all positive integers k .

On the other hand, it is easy to see that an ideal I has the strong persistence property if and only if I has a superficial ideal. Therefore, one can replace the concept of the strong persistence property with superficial ideals. It should be noted that the results of this section can be found in [112].

We begin with the following proposition which will be used in the proof of Proposition 3.2.3.

Proposition 3.2.2. *Let I and J be two ideals of a commutative Noetherian ring S with $\mathcal{G}(J) \subseteq \mathcal{G}(I)$. Then J is a superficial ideal for I if and only if $(I^r :_S J^s) = I^{r-s}$ for all $r, s \in \mathbb{N}$ with $r \geq s$.*

Proof. First, assume that J is a superficial ideal for I . We use induction on s . Our assumption gives that the assertion is true in the case $s = 1$. Now, suppose, inductively, that $s > 1$ and that the result has been proved for $s - 1$. Due to J is a superficial ideal for I , and on account of $(I^r :_S J^s) = ((I^r :_S J) :_S J^{s-1})$, one derives $(I^r :_S J^s) = (I^{r-1} :_S J^{s-1})$. Now, the inductive hypothesis implies that $(I^{r-1} :_S J^{s-1}) = I^{r-s}$. Thus, we have $(I^r :_S J^s) = I^{r-s}$. This completes the inductive step, and so the claim has been proved by induction. The converse implication is straightforward. $\square$

The next result describes the relation between powers of ideals and powers of their superficial ideals.

Proposition 3.2.3. *Let I and J be two ideals of a commutative Noetherian ring S such that J is a superficial ideal for I . Then J^s is a superficial ideal for I^s for all $s \in \mathbb{N}$.*

Proof. Fix $s \in \mathbb{N}$. By virtue of J is a superficial ideal for I , Proposition 3.2.2 yields that $(I^{sk+s} :_S J^s) = I^{sk}$ for all positive integers k . Thus, one can deduce that $((I^s)^{k+1} :_S J^s) = (I^s)^k$ for all positive integers k . Moreover, one can easily verify that $\mathcal{G}(J^s) \subseteq \mathcal{G}(I^s)$. Accordingly, J^s is a superficial ideal for I^s , as required. $\square$

Along with these consequences, the following question arises:

Question 3.2.4. *Does every commutative ring have the property that any of its ideals has a superficial ideal?*

Proposition 3.2.5. *Every commutative Noetherian ring does not have this property that any of its ideals has a superficial ideal.*

Proof. We give a counterexample. To achieve this goal, one should use a graph which has been stated in [76]. Indeed, for a positive integer n , let $[n]$ denote the set $\{0, \dots, n-1\}$. Assume that P_n is a path with vertex set $[n]$, with vertices in the increasing order along P_n . Let also K_3 be the complete graph whose vertex set is the group $\mathbb{Z}_3$. For $n \geq 4$, we define H_n as the graph obtained from the Cartesian product $P_n \square K_3$ by adding the three edges joining $(0, j)$ to $(n-1, -j)$ for $j \in \mathbb{Z}_3$. Figure below is the graph of H_4 .

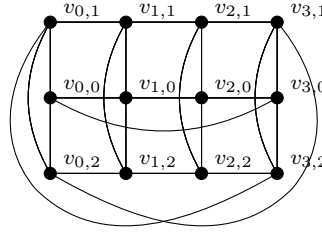


Figure 3.2.1: H_4

Assume that $I = J(H_4)$ denotes the cover ideal of H_4 in the polynomial ring $R = K[x_1, \dots, x_{12}]$ over a field K . Suppose, on the contrary, that there exists an ideal J with $\mathcal{G}(J) \subseteq \mathcal{G}(I)$ such that $(I^{k+1} :_R J) = I^k$ for all k . This implies that $(I^{k+1} :_R I) = I^k$ for all k , i.e., the ideal I has the strong persistence property, and so the ideal I has the persistence property, i.e., $\text{Ass}_R(R/I^k) \subseteq \text{Ass}_R(R/I^{k+1})$ for all k . By computing the set of associated primes of I^3 and I^4 , one derives $\mathfrak{m} = (x_1, \dots, x_{12}) \in \text{Ass}_R(R/I^3) \setminus \text{Ass}_R(R/I^4)$. This leads to a contradiction. In other words, there exists a commutative Noetherian ring R and an ideal I of R such that I has no superficial ideal. $\square$

The following proposition states that under a certain condition, one can always find a superficial ideal for some power of an ideal.

Proposition 3.2.6. *Let S be a commutative Noetherian ring and I be an ideal of S such that I has a non-zerodivisor element. Then there exists a positive integer s such that I^s is a superficial ideal for I^s .*

Proof. By the assumption, I has a non-zerodivisor element, say z . It follows from the Artin-Rees lemma that there exists a positive integer ℓ such that for all $n > \ell$, we have $I^n \cap zS = I^{n-\ell}(I^\ell \cap zS)$. In the light of $z(I^n :_S z) = I^n \cap zS$ and

$I^{n-\ell}(I^\ell \cap zS) \subseteq zI^{n-\ell}$, one can conclude that $(I^n :_S z) \subseteq I^{n-\ell}$ for all $n > \ell$. Now, assume that $R = \text{gr}_I(S) = S/I \oplus I/I^2 \oplus I^2/I^3 \oplus \cdots$. This implies that R is a Noetherian $\mathbb{N}$ -graded ring. The ideal $(0 :_R I/I^2)$ is finitely generated, in degrees at most some positive integer r . Accordingly, one derives $(0 :_R I/I^2) \cap I^n/I^{n+1} = 0$ for all $n > r$. This yields that $(I^{n+1} :_S I) \cap I^r = I^n$ for all $n > r$. Therefore, for all $n > \ell + r$, we get $(I^{n+1} :_S I) \subseteq (I^{n+1} :_S z) \subseteq I^{n+1-\ell}$. This gives that $(I^{n+1} :_S I) \subseteq I^r$ for all $n > \ell + r$, and so $(I^{n+1} :_S I) = I^n$ for all $n > \ell + r$. Here put $s := \ell + r + 1$ and fix $k \in \mathbb{N}$. It is routine to check that $((I^s)^{k+1} :_R I^s) = (I^s)^k$. This means that I^s is a superficial ideal for I^s , as claimed. $\square$

3.2.1 Some recursive theorems on the superficiality

Here, we give two recursive theorems which allow us to build new ideals that have superficial ideals. In order to demonstrate Theorem 3.2.8, one needs to prove the following lemma.

Lemma 3.2.7. *Let I be a monomial ideal of $R = K[x_1, \dots, x_n]$ such that $I = I_1R + I_2R$, where $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer $1 \leq m < n$. Then $I^s = \bigcap_{i=1}^s (I_1^i + I_2^{s+1-i})$ for all $s \in \mathbb{N}$.*

Proof. Fix $s \in \mathbb{N}$ and set $L_r := I_1^r + \sum_{i=0}^{r-1} I_1^i \cap I_2^{s-i}$ with $1 \leq r \leq s$. Our aim is to prove that $L_r = \bigcap_{i=1}^r (I_1^i + I_2^{s+1-i})$. To do this, we proceed by induction on r . One can easily see that the assertion is true for the case in which $r = 1$. Now, suppose, inductively, that $r > 1$ and that the result has been proved for all r less than t with $t \leq s$. It follows from the inductive hypothesis that $L_{t-1} = \bigcap_{i=1}^{t-1} (I_1^i + I_2^{s+1-i})$. On account of $I_1^t \cap \sum_{i=0}^{t-2} I_1^i \cap I_2^{s-i} \subseteq \sum_{i=0}^{t-2} I_1^i \cap I_2^{s-i}$ and $\sum_{i=0}^{t-2} I_1^i \cap I_2^{s-i} \subseteq I_2^{s+1-t}$, one can

derive the following equalities

$$\begin{aligned}
L_t &= I_1^t + \sum_{i=0}^{t-1} I_1^i \cap I_2^{s-i} \\
&= I_1^t + I_1^{t-1} \cap I_2^{s-t+1} + \sum_{i=0}^{t-2} I_1^i \cap I_2^{s-i} \\
&= I_1^t \cap I_1^{t-1} + I_1^t \cap \sum_{i=0}^{t-2} I_1^i \cap I_2^{s-i} + I_1^{t-1} \cap I_2^{s-t+1} + \sum_{i=0}^{t-2} I_1^i \cap I_2^{s-i} \\
&= (I_1^t + I_2^{s+1-t}) \cap (I_1^{t-1} + \sum_{i=0}^{t-2} I_1^i \cap I_2^{s-i}) \\
&= (I_1^t + I_2^{s+1-t}) \cap L_{t-1} \\
&= (I_1^t + I_2^{s+1-t}) \cap \bigcap_{i=1}^{t-1} (I_1^i + I_2^{s+1-i}) \\
&= \bigcap_{i=1}^t (I_1^i + I_2^{s+1-i}).
\end{aligned}$$

This completes the inductive step, and so the claim has been proved by induction. In particular, this implies that $L_s = \bigcap_{i=1}^s (I_1^i + I_2^{s+1-i})$, and hence

$$\bigcap_{i=1}^s (I_1^i + I_2^{s+1-i}) = I_1^s + \sum_{i=0}^{s-1} I_1^i \cap I_2^{s-i} = \sum_{i=0}^s I_1^i \cap I_2^{s-i}.$$

The assumption yields that $I_1^i \cap I_2^{s-i} = I_1^i I_2^{s-i}$ for each $i = 0, \dots, s$. This gives that $\bigcap_{i=1}^s (I_1^i + I_2^{s+1-i}) = \sum_{i=0}^s I_1^i I_2^{s-i}$. Finally, it follows from the binomial expansion that $\bigcap_{i=1}^s (I_1^i + I_2^{s+1-i}) = (I_1 + I_2)^s = I^s$, as required. $\square$

We are now ready to express Theorem 3.2.8, which is fundamental for proving Theorem 3.2.26.

Theorem 3.2.8. *Let I be a monomial ideal of $R = K[x_1, \dots, x_n]$ such that $I = I_1 R + I_2 R$, where $\mathcal{G}(I_1) \subseteq R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subseteq R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer m . If I_1 or I_2 has a superficial ideal, then I has a superficial ideal.*

Proof. Without loss of generality, assume that J_1 is a superficial ideal for I_1 with $\mathcal{G}(J_1) = \{v_1, \dots, v_t\}$. Because $\mathcal{G}(J_1) \subseteq \mathcal{G}(I)$, it is sufficient for us to verify that $(I^{k+1} :_R J_1) = I^k$ for all k . To do this, fix $k \in \mathbb{N}$. By virtue of the assumption, one has $(I_1^i :_R J_1) = I_1^{i-1}$ for each $i = 1, \dots, k+1$. In view of Lemma 3.2.7, we obtain

the following equalities

$$\begin{aligned}
(I^{k+1} :_R J_1) &= \left(\bigcap_{i=1}^{k+1} (I_1^i + I_2^{k+2-i}) :_R J_1 \right) \\
&= \bigcap_{i=1}^{k+1} \bigcap_{j=1}^t ((I_1^i :_R v_j) + (I_2^{k+2-i} :_R v_j)) \\
&= \bigcap_{i=1}^{k+1} \bigcap_{j=1}^t ((I_1^i :_R v_j) + I_2^{k+2-i}) \\
&= \bigcap_{i=1}^{k+1} (I_2^{k+2-i} + \cap_{j=1}^t (I_1^i :_R v_j)) \\
&= \bigcap_{i=1}^{k+1} (I_2^{k+2-i} + (I_1^i :_R J_1)) \\
&= \bigcap_{i=2}^{k+1} (I_2^{k+2-i} + I_1^{i-1}) \\
&= \bigcap_{\ell=1}^k (I_2^{k+1-\ell} + I_1^\ell) \\
&= I^k.
\end{aligned}$$

This yields that J_1 is a superficial ideal for I . A similar argument shows that if J_2 is a superficial ideal for I_2 , then one obtains $(I^{k+1} :_R J_2) = I^k$, i.e., J_2 is a superficial ideal for I . This completes the proof. $\square$

We state the second main result in the following theorem, which will be used in the proof of Theorem 3.2.29.

Theorem 3.2.9. *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, v and h be two monomials in R such that $\gcd(v, h) = 1$, $h \in I$, and $\gcd(v, u_i) = 1$ for all $i = 1, \dots, m$. If J is a superficial ideal for I , then $vJ + hR$ is a superficial ideal for $vI + hR$.*

Proof. In order to simplify our notation, set $L := vI + hR$. Let J be a superficial ideal for I . This implies that $(I^{t+1} :_R J) = I^t$ for all positive integers t . Fix $t \in \mathbb{N}$. Due to the binomial expansion, it follows that $L^{t+1} = \sum_{i=0}^{t+1} (hR)^{t+1-i} (vI)^i$. In the

light of $\gcd(v, h) = 1$, one obtains the following equalities

$$\begin{aligned}
(L^{t+1} :_R vJ + hR) &= \left(\sum_{i=0}^{t+1} (hR)^{t+1-i} (vI)^i :_R vJ \right) \cap \left(\sum_{i=0}^{t+1} (hR)^{t+1-i} (vI)^i :_R hR \right) \\
&= (h^{t+1}R + \sum_{i=1}^{t+1} (hR)^{t+1-i} v^{i-1} I^i :_R J) \\
&\quad \cap ((v^{t+1} I^{t+1} :_R hR) + \sum_{i=0}^t (hR)^{t-i} (vI)^i).
\end{aligned}$$

Because $h \in I$, one can conclude that $(hR)^{t+1-i} I^i \subseteq I^{t+1}$ for each $i = 0, \dots, t+1$, and so $(h^{t+1}R + \sum_{i=1}^{t+1} (hR)^{t+1-i} v^{i-1} I^i :_R J) \subseteq (I^{t+1} :_R J)$. Accordingly, one has

$$(h^{t+1}R + \sum_{i=1}^{t+1} (hR)^{t+1-i} v^{i-1} I^i :_R J) \subseteq I^t.$$

On account of $\sum_{i=0}^t (hR)^{t-i} (vI)^i = L^t$, one can deduce the containment below

$$(L^{t+1} :_R vJ + hR) \subseteq I^t \cap (v^{t+1} I^{t+1} :_R hR) + I^t \cap L^t.$$

By virtue of $\gcd(v, h) = 1$, one can yield the following equality

$$I^t \cap (v^{t+1} I^{t+1} :_R hR) = I^t \cap v^{t+1} (I^{t+1} :_R hR).$$

As $\gcd(v, u_i) = 1$ for each $i = 1, \dots, m$, we get $I^t \cap (v^{t+1} I^{t+1} :_R hR) \subseteq (vI)^t$. Consequently, $I^t \cap (v^{t+1} I^{t+1} :_R hR) \subseteq L^t$, and so $(L^{t+1} :_R vJ + hR) \subseteq L^t$. Based on $L^t \subseteq (L^{t+1} :_R vJ + hR)$, we gain $(L^{t+1} :_R vJ + hR) = L^t$. On the other hand, it follows from $\mathcal{G}(J) \subseteq \mathcal{G}(I)$ that $\mathcal{G}(vJ + hR) \subseteq \mathcal{G}(vI + hR)$. Therefore, $vJ + hR$ is a superficial ideal for $vI + hR$, and the proof is complete. $\square$

3.2.2 Superficiality under some monomial operators

In the following, we introduce several methods for constructing new classes of monomial ideals which have superficial ideals. To accomplish this purpose, we start with the first method. Thus, we need to consider the expansion operator on monomial ideals which has been stated in Definition 2.2.9, and utilize it as a criterion for the existence of superficial ideals.

The following theorem provides a powerful tool for producing new superficial ideals.

Theorem 3.2.10. *Let I and J be two monomial ideals of R . Then J is a superficial ideal for I if and only if J^* is a superficial ideal for I^* .*

Proof. For necessity, assume that J is a superficial ideal for I . Fix $k \in \mathbb{N}$. This implies that $(I^{k+1} :_R J) = I^k$. In the light of [13, Lemma 1.1(v)], we obtain $((I^{k+1})^* :_{R^*} J^*) = (I^k)^*$. Furthermore, [13, Lemma 1.1(iii)] yields that $(I^k)^* = (I^*)^k$ and $(I^{k+1})^* = (I^*)^{k+1}$. Accordingly, $((I^*)^{k+1} :_{R^*} J^*) = (I^*)^k$. It is clear that $\mathcal{G}(J^*) \subseteq \mathcal{G}(I^*)$. Therefore, one can deduce that J^* is a superficial ideal for I^* . Sufficiency follows similarly, and the proof is over. $\square$

We now state the second method for constructing new superficial ideals. To achieve this, one requires to recall the next definition.

Definition 3.2.11. *A weight over a polynomial ring $R = K[x_1, \dots, x_n]$ is a function $W : \{x_1, \dots, x_n\} \rightarrow \mathbb{N}$, and $w_i = W(x_i)$ is called the weight of the variable x_i . Given a monomial ideal I and a weight W , we define the weighted ideal, denoted by I_W , to be the ideal generated by $\{h(u) : u \in \mathcal{G}(I)\}$, where h is the unique homomorphism $h : R \rightarrow R$ given by $h(x_i) = x_i^{w_i}$. For a monomial $u \in R$, we denote $h(u) = u_W$.*

For instance, consider the monomial ideal $I = (x_1^2 x_2 x_3^6, x_2^3 x_4 x_5^4)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. Furthermore, let $W : \{x_1, x_2, x_3, x_4, x_5\} \rightarrow \mathbb{N}$ be a weight over R with $W(x_1) = 2$, $W(x_2) = 4$, $W(x_3) = 2$, $W(x_4) = 3$, and $W(x_5) = 1$. Therefore, the weighted ideal I_W is given by $I_W = (x_1^4 x_2^4 x_3^{12}, x_2^{12} x_4^3 x_5^4)$.

The next lemma is essential for us to prove Theorem 3.2.13.

Lemma 3.2.12. *Let I and J be two monomial ideals of R , and W a weight over R . Then the following statements hold.*

- (i) $(I + J)_W = I_W + J_W$;
- (ii) $(IJ)_W = I_W J_W$;
- (iii) $(I \cap J)_W = I_W \cap J_W$;
- (iv) $(I :_R J)_W = (I_W :_{R^*} J_W)$.

Proof. (i) It is routine to check that if $u \mid v$ (respectively, $v \mid u$) for some $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$, then $u_W \mid v_W$ (respectively, $v_W \mid u_W$). Now, the desired conclusion follows from the facts that $\mathcal{G}(I_W) = \{u_W : u \in \mathcal{G}(I)\}$, $\mathcal{G}(J_W) = \{v_W : v \in \mathcal{G}(J)\}$, and $\mathcal{G}((I + J)_W) \subseteq \mathcal{G}(I_W) \cup \mathcal{G}(J_W)$.

(ii) One can easily see that $(uv)_W = u_W v_W$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$. Here, the assertion follows directly from the fact that $\mathcal{G}((IJ)_W) \subseteq \mathcal{G}(I_W) \mathcal{G}(J_W)$.

(iii) It is straightforward to show that $(\text{lcm}(u, v))_W = \text{lcm}(u_W, v_W)$, and thus one can conclude the claim from the fact that $I_W \cap J_W$ is generated by the following set

$$\{\text{lcm}(u_W, v_W) : u_W \in \mathcal{G}(I_W), v_W \in \mathcal{G}(J_W)\}.$$

(iv) In the light of $(I :_R J) = \bigcap_{v \in \mathcal{G}(J)} (I :_R v)$, and by part (iii), one can derive $(I :_R J)_W = \bigcap_{v \in \mathcal{G}(J)} (I :_R v)_W$. Our aim is to show that $(I :_R v)_W = (I_W :_R v_W)$ for all $v \in \mathcal{G}(J)$. On account of $I_W = \sum_{u \in \mathcal{G}(I)} u_W R$, it follows that

$$(I_W :_R v_W) = \left(\sum_{u \in \mathcal{G}(I)} u_W R :_R v_W \right) = \sum_{u \in \mathcal{G}(I)} (u_W :_R v_W).$$

For completing the proof, one has to demonstrate $(u_W :_R v_W) = (u :_R v)_W$, and then apply part (i). This claim follows readily from the fact that $(\gcd(u, v))_W = \gcd(u_W, v_W)$. $\square$

We are now ready to prove Theorem 3.2.13 by using Lemma 3.2.12.

Theorem 3.2.13. *Let I and J be two monomial ideals of R , and W a weight over R . Then J is a superficial ideal for I if and only if J_W is a superficial ideal for I_W .*

Proof. For the forward implication, assume that J is a superficial ideal for I . Fix $k \in \mathbb{N}$. This gives that $(I^{k+1} :_R J) = I^k$. On account of Lemma 3.2.12(iv), one has $((I^{k+1})_W :_R J_W) = (I^k)_W$. It follows from Lemma 3.2.12(ii) that $(I^{k+1})_W = (I_W)^{k+1}$ and $(I^k)_W = (I_W)^k$. Hence, we get $((I_W)^{k+1} :_R J_W) = (I_W)^k$. Therefore, J_W is a superficial ideal for I_W . A similar argument shows the converse implication, and the proof is complete. $\square$

The subsequent lemma says that under certain conditions the monomial multiple of a monomial ideal which has a superficial ideal, always has a superficial ideal.

Lemma 3.2.14. *Let I and J be two monomial ideals of R with $\mathcal{G}(J) = \{v_1, \dots, v_t\}$. Let also h be a monomial in R such that $\gcd(h, v_i) = 1$ for each $i = 1, \dots, t$. Then J is a superficial ideal for I if and only if hJ is a superficial ideal for hI .*

Proof. To simplify our notation, we set $L := hI$. First, let J be a superficial ideal for I . Fix $k \in \mathbb{N}$. Thanks to $\mathcal{G}(hJ) \subseteq \mathcal{G}(L)$, for completing the proof, one has to demonstrate $(L^{k+1} :_R hJ) = L^k$. Due to $(h^{k+1} I^{k+1} :_R h) = h^k I^{k+1}$, one derives $(L^{k+1} :_R hJ) = (h^k I^{k+1} :_R J)$. By virtue of $\gcd(h, v_i) = 1$ for each $i = 1, \dots, t$, this gives that $(h^k I^{k+1} :_R J) = h^k (I^{k+1} :_R J)$. It follows from $(I^{k+1} :_R J) = I^k$ that $(L^{k+1} :_R hJ) = L^k$, as required. The converse implication is straightforward. This completes the proof. $\square$

In order to prove Theorem 3.2.16, we need to verify Lemma 3.2.15. To do this, we should recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal as has been stated in Definition 3.1.14.

Lemma 3.2.15. *Let I and J be two monomial ideals in $R = K[x_1, \dots, x_n]$, and $\mathfrak{p}$ be a monomial prime ideal of R . Then the following statements hold.*

- (i) $(I + J)(\mathfrak{p}) = I(\mathfrak{p}) + J(\mathfrak{p})$;
- (ii) $(IJ)(\mathfrak{p}) = I(\mathfrak{p})J(\mathfrak{p})$;
- (iii) $(I \cap J)(\mathfrak{p}) = I(\mathfrak{p}) \cap J(\mathfrak{p})$;
- (iv) $(I :_R J)(\mathfrak{p}) = (I(\mathfrak{p}) :_{R(\mathfrak{p})} J(\mathfrak{p}))$;
- (v) $\sqrt{I(\mathfrak{p})} = \sqrt{I}(\mathfrak{p})$;
- (vi) If Q is a $\mathfrak{q}$ -primary monomial ideal in R with $\mathfrak{q} \subseteq \mathfrak{p}$, then $Q(\mathfrak{p})$ is a $\mathfrak{q}$ -primary monomial ideal in $R(\mathfrak{p})$;
- (vii) If $\mathfrak{p} \in V^*(I)$, and $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of I , then

$$I(\mathfrak{p}) = \bigcap_{i=1, \dots, s, \sqrt{Q_i} \subseteq \mathfrak{p}} Q_i,$$

is a minimal primary decomposition of $I(\mathfrak{p})$ in $R(\mathfrak{p})$;

- (viii) If $\mathfrak{p} \in V^*(I)$, then

$$\text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/I(\mathfrak{p})) = \{\mathfrak{q} : \mathfrak{q} \in \text{Ass}_R(R/I) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\};$$

- (ix) If $\mathfrak{p} \in V^*(I)$, then $\text{Ass}^\infty(I(\mathfrak{p})) = \{\mathfrak{q} : \mathfrak{q} \in \text{Ass}^\infty(I) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$, where $\text{Ass}^\infty(I)$ denotes the stable set of associated prime ideals of I ;
- (x) If $\mathfrak{p} \in V^*(I)$, then $\text{astab}(I) \geq \text{astab}(I(\mathfrak{p}))$, where $\text{astab}(I)$ denotes the index of stability for the associated prime ideals of I .

Proof. (i) It is easy to see that if $u|v$ (respectively, $v|u$), then $u(\mathfrak{p})|v(\mathfrak{p})$ (respectively, $v(\mathfrak{p})|u(\mathfrak{p})$) for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$. Because $I(\mathfrak{p})$ (respectively, $J(\mathfrak{p})$) is generated by $\{u(\mathfrak{p}) : u \in \mathcal{G}(I)\}$ (respectively, $\{v(\mathfrak{p}) : v \in \mathcal{G}(J)\}$), and by virtue of $\mathcal{G}((I + J)(\mathfrak{p})) \subseteq \mathcal{G}(I(\mathfrak{p})) \cup \mathcal{G}(J(\mathfrak{p}))$, one can conclude that $(I + J)(\mathfrak{p}) = I(\mathfrak{p}) + J(\mathfrak{p})$.

(ii) It is straightforward to show that $(uv)(\mathfrak{p}) = u(\mathfrak{p})v(\mathfrak{p})$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$. Now, the desired conclusion follows immediately from the fact that $\mathcal{G}((IJ)(\mathfrak{p})) \subseteq \mathcal{G}(I(\mathfrak{p}))\mathcal{G}(J(\mathfrak{p}))$.

(iii) On account of $I(\mathfrak{p})$ (respectively, $J(\mathfrak{p})$) is generated by $\{u(\mathfrak{p}) : u \in \mathcal{G}(I)\}$ (respectively, $\{v(\mathfrak{p}) : v \in \mathcal{G}(J)\}$), this implies that $I(\mathfrak{p}) \cap J(\mathfrak{p})$ is generated by the set $\{\text{lcm}(u(\mathfrak{p}), v(\mathfrak{p})) : \text{for all } u \in \mathcal{G}(I) \text{ and } v \in \mathcal{G}(J)\}$. By virtue of $I \cap J$ is generated by $\{\text{lcm}(u, v) : \text{for all } u \in \mathcal{G}(I) \text{ and } v \in \mathcal{G}(J)\}$, one has $(I \cap J)(\mathfrak{p})$ is generated by $\{\text{lcm}(u, v)(\mathfrak{p}) : \text{for all } u \in \mathcal{G}(I) \text{ and } v \in \mathcal{G}(J)\}$. Now, the claim follows readily from the fact that $\text{lcm}(u, v)(\mathfrak{p}) = \text{lcm}(u(\mathfrak{p}), v(\mathfrak{p}))$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$.

(iv) Since $(I :_R J) = \bigcap_{v \in \mathcal{G}(J)} (I :_R v)$, and by virtue of statement (iii), one has $(I :_R J)(\mathfrak{p}) = \bigcap_{v \in \mathcal{G}(J)} (I :_R v)(\mathfrak{p})$. We prove that $(I :_R v)(\mathfrak{p}) = (I(\mathfrak{p}) :_{R(\mathfrak{p})} v(\mathfrak{p}))$ for all $v \in \mathcal{G}(J)$. Due to $I(\mathfrak{p}) = \sum_{u \in \mathcal{G}(I)} u(\mathfrak{p})R(\mathfrak{p})$, one derives the following equalities

$$(I(\mathfrak{p}) :_{R(\mathfrak{p})} v(\mathfrak{p})) = \left(\sum_{u \in \mathcal{G}(I)} u(\mathfrak{p})R(\mathfrak{p}) :_{R(\mathfrak{p})} v(\mathfrak{p}) \right) = \sum_{u \in \mathcal{G}(I)} (u(\mathfrak{p}) :_{R(\mathfrak{p})} v(\mathfrak{p})).$$

To conclude the argument, one has to demonstrate $(u(\mathfrak{p}) :_{R(\mathfrak{p})} v(\mathfrak{p})) = (u :_R v)(\mathfrak{p})$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$. Here, the assertion follows directly from the fact that $\gcd(u, v)(\mathfrak{p}) = \gcd(u(\mathfrak{p}), v(\mathfrak{p}))$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$.

(v) Suppose that $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Applying statement (i) implies the subsequent equalities

$$\begin{aligned} \sqrt{I(\mathfrak{p})} &= \sqrt{\left(\sum_{i=1}^m u_i R \right)(\mathfrak{p})} = \sqrt{\sum_{i=1}^m u_i(\mathfrak{p})R(\mathfrak{p})} = \sqrt{\sum_{i=1}^m \sqrt{u_i(\mathfrak{p})}R(\mathfrak{p})} \\ &= \sum_{i=1}^m \sqrt{u_i(\mathfrak{p})}R(\mathfrak{p}) = \sum_{i=1}^m \sqrt{u_i}(\mathfrak{p})R(\mathfrak{p}). \end{aligned}$$

On the other hand, since $\sqrt{I}$ is generated by $\{\sqrt{u_i} : i = 1, \dots, m\}$, using statement (i) yields the following equalities

$$\sqrt{I}(\mathfrak{p}) = \left(\sum_{i=1}^m \sqrt{u_i} R \right)(\mathfrak{p}) = \sum_{i=1}^m \sqrt{u_i}(\mathfrak{p})R(\mathfrak{p}).$$

Therefore, one can conclude that $\sqrt{I(\mathfrak{p})} = \sqrt{I}(\mathfrak{p})$.

(vi) Let Q be a $\mathfrak{q}$ -primary monomial ideal in R with $\mathfrak{q} \subseteq \mathfrak{p}$. Assume that $\mathfrak{q} = (x_{i_1}, \dots, x_{i_t})$. We thus get $Q = (x_{i_1}^{\alpha_1}, \dots, x_{i_t}^{\alpha_t})$ with $\alpha_1, \dots, \alpha_t$ are some positive integers. On account of $\mathfrak{q} \subseteq \mathfrak{p}$, one has $Q(\mathfrak{p}) = (x_{i_1}^{\alpha_1}, \dots, x_{i_t}^{\alpha_t})$, $Q(\mathfrak{p}) \neq R(\mathfrak{p})$, and $\mathfrak{q}(\mathfrak{p}) = (x_{i_1}, \dots, x_{i_t})$. Accordingly, one derives $Q(\mathfrak{p})$ is a $\mathfrak{q}$ -primary monomial ideal in $R(\mathfrak{p})$.

(vii) Let $\mathfrak{p} \in V^*(I)$, and $I = Q_1 \cap \dots \cap Q_s$ be a minimal primary decomposition of I . Since $\mathfrak{p} \in V^*(I)$, one has $\bigcap_{i=1}^s Q_i \subseteq \mathfrak{p}$. Hence, there exists a positive integer $1 \leq i \leq s$ such that $Q_i \subseteq \mathfrak{p}$, and so $\sqrt{Q_i} \subseteq \mathfrak{p}$. Without loss of generality, one may assume that $\sqrt{Q_i} \subseteq \mathfrak{p}$ for all $i = 1, \dots, t$ with $t \geq 1$, and $\sqrt{Q_i} \not\subseteq \mathfrak{p}$ for all $i = t+1, \dots, s$. By virtue of statement (vi), for each $i = 1, \dots, t$, Q_i is a primary monomial ideal in $R(\mathfrak{p})$. Moreover, for all $i = t+1, \dots, s$, we have $Q_i(\mathfrak{p}) = R(\mathfrak{p})$. It follows now from statements (iii) and (vi) that $I(\mathfrak{p}) = Q_1(\mathfrak{p}) \cap \dots \cap Q_t(\mathfrak{p})$ is a minimal primary decomposition of $I(\mathfrak{p})$.

(viii) This claim follows directly from statement (vii).

(ix) Let $\mathfrak{p} \in V^*(I)$. Thanks to statement (ii), one has $I^\ell(\mathfrak{p}) = (I(\mathfrak{p}))^\ell$ for all positive integers ℓ . Furthermore, statement (viii) gives the following equality

$$\text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/I^\ell(\mathfrak{p})) = \{\mathfrak{q} : \mathfrak{q} \in \text{Ass}_R(R/I^\ell) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

We thus gain

$$\text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/(I(\mathfrak{p}))^\ell) = \{\mathfrak{q} : \mathfrak{q} \in \text{Ass}_R(R/I^\ell) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}.$$

Therefore, one derives $\text{Ass}^\infty(I(\mathfrak{p})) = \{\mathfrak{q} : \mathfrak{q} \in \text{Ass}^\infty(I) \text{ and } \mathfrak{q} \subseteq \mathfrak{p}\}$.

(x) This assertion follows readily from statement (ix). $\square$

Theorem 3.2.16. *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$, and $\mathfrak{p} \in V^*(I)$. Then the following statements hold.*

(i) *If I has the strong persistence property, then $I(\mathfrak{p})$ has the strong persistence property. In particular, $I(\mathfrak{p})$ has a superficial ideal.*

(ii) *If I has the persistence property, then $I(\mathfrak{p})$ has the persistence property.*

Proof. (i) Assume that I has the strong persistence property. Hence, one can deduce that $(I^{k+1} :_R I) = I^k$ for all positive integers k . Fix $k \in \mathbb{N}$. It follows from $(I^{k+1} :_R I) = I^k$ that $(I^{k+1} :_R I)(\mathfrak{p}) = (I^k)(\mathfrak{p})$. Now, parts (ii) and (iv) of Lemma 3.2.15 yield that $((I(\mathfrak{p}))^{k+1} :_{R(\mathfrak{p})} I(\mathfrak{p})) = (I(\mathfrak{p}))^k$. This means that $I(\mathfrak{p})$ has the strong persistence property. In particular, $I(\mathfrak{p})$ has a superficial ideal, as claimed.

(ii) Let I have the persistence property, and k be an arbitrary positive integer. Consider $\mathfrak{a} \in \text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/(I(\mathfrak{p}))^k)$. Since $(I(\mathfrak{p}))^k = I^k(\mathfrak{p})$, it follows from part (viii) of Lemma 3.2.15 that $\mathfrak{a} = \mathfrak{q}(\mathfrak{p})$ with $\mathfrak{q} \in \text{Ass}_R(R/I^k)$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Due to I has the persistence property, one can conclude that $\mathfrak{q} \in \text{Ass}_R(R/I^{k+1})$. Once again, by $(I(\mathfrak{p}))^{k+1} = I^{k+1}(\mathfrak{p})$ and part (viii) of Lemma 3.2.15, we obtain $\mathfrak{q} \in \text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/(I(\mathfrak{p}))^{k+1})$. Accordingly, one derives $I(\mathfrak{p})$ has the persistence property, as desired. $\square$

It should be noted that Theorem 3.2.16 allows us to refute the (strong) persistence property for some monomial ideals, and helps us to construct new counterexamples inspired by some well-known counterexamples.

Corollary 3.2.17. *Let I be a monomial ideal in $R_1 = K[x_1, \dots, x_r]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, and $J_1, \dots, J_m$ be some monomial ideals in $R_2 = K[x_{r+1}, \dots, x_n]$. If I does not satisfy the persistence (respectively, strong persistence) property, then $L := u_1 J_1 R + \dots + u_m J_m R$ does not satisfy the persistence (respectively, strong persistence) property, where $R = K[x_1, \dots, x_n]$.*

Proof. Suppose, on the contrary, that L has the persistence (respectively, strong persistence) property. Put $\mathfrak{p} := (\bigcup_{i=1}^m \text{supp}(u_i))$. It is clear that $\mathfrak{p}$ is a monomial prime ideal in R , and $L \subseteq \mathfrak{p}$. Thanks to Theorem 3.2.16, one derives $L(\mathfrak{p})$ has the persistence (respectively, strong persistence) property. Since $L(\mathfrak{p}) = I$, we get a contradiction. Therefore, L does not satisfy the persistence (respectively, strong persistence) property, as claimed. $\square$

The following example illuminates what happens in Corollary 3.2.17.

Example 3.2.18. (1) Consider the monomial ideal I generated by $x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6$ in the polynomial ring $R_1 = K[x_1, x_2, x_3, x_4, x_5, x_6]$ over a field K , and the monomial ideal L generated by $x_1x_2x_3x_8, x_1x_2x_4, x_1x_3x_5x_7, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6x_7, x_2x_4x_5, x_2x_5x_6x_8, x_3x_4x_5, x_3x_4x_6x_7$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$. Now, set $\mathfrak{p} := (x_1, x_2, x_3, x_4, x_5, x_6)$. Certainly, $\mathfrak{p} \in V^*(L)$ and $L(\mathfrak{p}) = I$. By virtue of $(I^3 :_R I) \neq I^2$, one can conclude that the monomial ideal I does not satisfy the strong persistence property. Finally, based on Corollary 3.2.17, we deduce that the monomial ideal L does not satisfy the strong persistence property. On the other hand, direct computations show that $(L^3 :_R L) \neq L^2$ which confirm our result.

(2) Suppose that $I := (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1, x_5x_1^2x_2)$ and

$$L := (x_1x_2^2x_3x_7, x_2x_3^2x_4x_6, x_3x_4^2x_5x_6, x_4x_5^2x_1x_7, x_5x_1^2x_2x_8),$$

be two monomial ideals in $R = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ over a field K . Take $\mathfrak{p} := (x_1, x_2, x_3, x_4, x_5)$. It is easy to see that $\mathfrak{p} \in V^*(L)$ and $L(\mathfrak{p}) = I$. On account of $(x_1, x_2, x_3, x_4, x_5) \in \text{Ass}_R(R/I) \setminus \text{Ass}_R(R/I^2)$, one can conclude that I does not satisfy the persistence property. It follows now from Corollary 3.2.17 that L does not satisfy the persistence property. On the other hand, it is routine to check that $(x_1, x_2, x_3, x_4, x_5) \in \text{Ass}_R(R/L) \setminus \text{Ass}_R(R/L^2)$, as required.

To state the subsequent results, we need the definitions of the contraction and deletion operators as given in [50, P. 303].

Definition 3.2.19. Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. For some $1 \leq j \leq n$, the contraction I/x_j (respectively, deletion $I \setminus x_j$) is obtained by setting $x_j = 1$ in u_i (respectively, $x_j = 0$ in u_i) for each $i = 1, \dots, m$. It should be observed that the contraction I/x_j is exactly the monomial localization of I with respect to $\mathfrak{p} = \mathfrak{m} \setminus \{x_j\}$, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of $R = K[x_1, \dots, x_n]$.

Inspired by Lemma 3.2.15 and Theorem 3.2.16, one can conclude Lemma 3.2.20 and Theorem 3.2.21.

Lemma 3.2.20. Let I and J be two monomial ideals in $R = K[x_1, \dots, x_n]$ over a field K , $\mathfrak{m} = (x_1, \dots, x_n)$, and $1 \leq j \leq n$. Then the following statements hold.

- (i) $(I + J)/x_j = I/x_j + J/x_j$;
- (ii) $(IJ)/x_j = (I/x_j)(J/x_j)$;
- (iii) $(I \cap J)/x_j = (I/x_j) \cap (J/x_j)$;

(iv) $(I :_R J)/x_j = (I/x_j :_{R/x_j} J/x_j)$;

(v) $\sqrt{I/x_j} = \sqrt{I}/x_j$;

(vi) If Q is a $\mathfrak{q}$ -primary monomial ideal in R with $x_j \notin \mathfrak{q}$, then Q/x_j is a $\mathfrak{q}$ -primary monomial ideal in R/x_j ;

(vii) If $I \subseteq \mathfrak{m} \setminus \{x_j\}$, and $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of I , then

$$I/x_j = \bigcap_{i=1, \dots, s, x_j \notin \sqrt{Q_i}} Q_i,$$

is a minimal primary decomposition of I/x_j in R/x_j ;

(viii) If $I \subseteq \mathfrak{m} \setminus \{x_j\}$, then

$$\text{Ass}_{R/x_j}((R/x_j)/(I/x_j)) = \{\mathfrak{q} : \mathfrak{q} \in \text{Ass}_R(R/I) \text{ and } x_j \notin \mathfrak{q}\};$$

(ix) If $I \subseteq \mathfrak{m} \setminus \{x_j\}$, then $\text{Ass}^\infty(I/x_j) = \{\mathfrak{q} : \mathfrak{q} \in \text{Ass}^\infty(I) \text{ and } x_j \notin \mathfrak{q}\}$, where $\text{Ass}^\infty(I)$ denotes the stable set of associated prime ideals of I ;

(x) If $I \subseteq \mathfrak{m} \setminus \{x_j\}$, then $\text{astab}(I) \geq \text{astab}(I/x_j)$, where $\text{astab}(I)$ denotes the index of stability for the associated prime ideals of I .

Theorem 3.2.21. Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$, and $1 \leq j \leq n$. Then the following statements hold.

(i) If I has the strong persistence property, then I/x_j has the strong persistence property. In particular, I/x_j has a superficial ideal.

(ii) If I has the persistence property, then I/x_j has the persistence property.

To express Lemma 3.2.22 and Question 3.2.23, we must employ the definition of the deletion operator as given in Definition 3.2.19.

Lemma 3.2.22. Let I and J be two monomial ideals in $R = K[x_1, \dots, x_n]$ over a field K , and $1 \leq j \leq n$. Then the following statements hold.

(i) $(I + J) \setminus x_j = I \setminus x_j + J \setminus x_j$;

(ii) $(IJ) \setminus x_j = (I \setminus x_j)(J \setminus x_j)$;

(iii) $(I \cap J) \setminus x_j = (I \setminus x_j) \cap (J \setminus x_j)$;

(iv) $\sqrt{I \setminus x_j} = \sqrt{I} \setminus x_j$;

(v) If Q is a $\mathfrak{q}$ -primary monomial ideal in R , then $Q \setminus x_j$ is a $(\mathfrak{q} \setminus x_j)$ -primary monomial ideal in $R \setminus x_j$;

(vi) If $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of I in R , then $I \setminus x_j = (Q_1 \setminus x_j) \cap \cdots \cap (Q_s \setminus x_j)$ is a primary decomposition of $I \setminus x_j$ in $R \setminus x_j$;

(vii) $\text{Ass}_{R \setminus x_j}((R \setminus x_j)/(I \setminus x_j)) \subseteq \{\mathfrak{q} \setminus x_j : \mathfrak{q} \in \text{Ass}_R(R/I)\}$.

Proof. (i) It is routine to check that if $x_j|u$ (respectively, $x_j|v$) for some $u \in \mathcal{G}(I)$ (respectively, $v \in \mathcal{G}(J)$), then $u \notin \mathcal{G}(I \setminus x_j)$ (respectively, $v \notin \mathcal{G}(J \setminus x_j)$) and $u \notin \mathcal{G}((I+J) \setminus x_j)$ (respectively, $v \notin \mathcal{G}((I+J) \setminus x_j)$). Also, if $x_j \nmid uv$, then $u \setminus x_j = u$ and $v \setminus x_j = v$. Since $I \setminus x_j$ (respectively, $J \setminus x_j$) is generated by $\{u : u \in \mathcal{G}(I) \text{ and } x_j \nmid u\}$ (respectively, $\{v : v \in \mathcal{G}(J) \text{ and } x_j \nmid v\}$), and because $\mathcal{G}((I+J) \setminus x_j) \subseteq \mathcal{G}(I \setminus x_j) \cup \mathcal{G}(J \setminus x_j)$, we get $(I+J) \setminus x_j = I \setminus x_j + J \setminus x_j$.

(ii) One can easily see that $(uv) \setminus x_j = (u \setminus x_j)(v \setminus x_j)$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$. Since $\mathcal{G}((IJ) \setminus x_j) \subseteq \mathcal{G}(I \setminus x_j)\mathcal{G}(J \setminus x_j)$, we obtain $(IJ) \setminus x_j = (I \setminus x_j)(J \setminus x_j)$.

(iii) By using the convention that if $x_j|u$ or $x_j|v$, then $\text{lcm}(u \setminus x_j, v \setminus x_j) = 0$, one can conclude that $(I \setminus x_j) \cap (J \setminus x_j)$ is generated by

$$\{\text{lcm}(u \setminus x_j, v \setminus x_j) : \text{for all } u \in \mathcal{G}(I) \text{ and } v \in \mathcal{G}(J)\}.$$

Furthermore, we know that $(I \cap J) \setminus x_j$ is generated by

$$\{\text{lcm}(u, v) \setminus x_j : \text{for all } u \in \mathcal{G}(I) \text{ and } v \in \mathcal{G}(J)\}.$$

Based on $\text{lcm}(u, v) \setminus x_j = \text{lcm}(u \setminus x_j, v \setminus x_j)$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$, we gain $(I \cap J) \setminus x_j = (I \setminus x_j) \cap (J \setminus x_j)$.

(iv) Assume that $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Using statement (i) yields the subsequent equalities

$$\begin{aligned} \sqrt{I \setminus x_j} &= \sqrt{(\sum_{i=1}^m u_i R) \setminus x_j} = \sqrt{\sum_{i=1}^m (u_i \setminus x_j)(R \setminus x_j)} = \sqrt{\sum_{i=1}^m \sqrt{u_i \setminus x_j}(R \setminus x_j)} \\ &= \sum_{i=1}^m \sqrt{u_i \setminus x_j}(R \setminus x_j) = \sum_{i=1}^m (\sqrt{u_i} \setminus x_j)(R \setminus x_j). \end{aligned}$$

In addition, by virtue of $\sqrt{I}$ is generated by $\{\sqrt{u_i} : i = 1, \dots, m\}$, applying statement (i) gives the following equalities

$$\sqrt{I} \setminus x_j = (\sum_{i=1}^m \sqrt{u_i} R) \setminus x_j = \sum_{i=1}^m (\sqrt{u_i} \setminus x_j)(R \setminus x_j).$$

Consequently, we get $\sqrt{I \setminus x_j} = \sqrt{I} \setminus x_j$.

(v) Suppose that Q is a $\mathfrak{q}$ -primary monomial ideal in R . Let $\mathfrak{q} = (x_{i_1}, \dots, x_{i_t})$. This implies that $Q = (x_{i_1}^{\alpha_1}, \dots, x_{i_t}^{\alpha_t})$ with $\alpha_1, \dots, \alpha_t$ are some positive integers. If $x_j \notin \mathfrak{q}$, then $Q \setminus x_j = Q$ and $\mathfrak{q} \setminus x_j = \mathfrak{q}$. Accordingly, $Q \setminus x_j$ is a $(\mathfrak{q} \setminus x_j)$ -primary monomial ideal in $R \setminus x_j$. If $x_j \in \mathfrak{q}$, then there exists a positive integer $1 \leq s \leq t$

such that $i_s = j$. This yields that $Q \setminus x_j = (x_{i_1}^{\alpha_1}, \dots, x_{i_{s-1}}^{\alpha_{s-1}}, x_{i_{s+1}}^{\alpha_{s+1}}, \dots, x_{i_t}^{\alpha_t})$ and $\mathfrak{q} \setminus x_j = (x_{i_1}, \dots, x_{i_{s-1}}, x_{i_{s+1}}, \dots, x_{i_t})$. Therefore, one can conclude that $Q \setminus x_j$ is a $(\mathfrak{q} \setminus x_j)$ -primary monomial ideal in $R \setminus x_j$.

(vi) Suppose that $I = Q_1 \cap \dots \cap Q_s$ is a minimal primary decomposition of I . It follows from statements (iii) and (v) that $I \setminus x_j = (Q_1 \setminus x_j) \cap \dots \cap (Q_s \setminus x_j)$ is a primary decomposition of $I \setminus x_j$ in $R \setminus x_j$.

(vii) In the light of statement (vi), one can immediately deduce this assertion.

□

Here, we continue with the following question. We will argue on this question in Question 10.2.15.

Question 3.2.23. *Suppose that $I \subset R = K[x_1, \dots, x_n]$ is a square-free monomial ideal with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, $\bigcup_{i=1}^m \text{supp}(u_i) = \{x_1, \dots, x_n\}$, and $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of R . If there exists a positive integer $1 \leq j \leq n$ such that $\mathfrak{m} \setminus x_j \in \text{Ass}_{R \setminus x_j}((R \setminus x_j)/(I \setminus x_j)^k)$ for some positive integer k , then can we deduce that $\mathfrak{m} \in \text{Ass}_R(R/I^k)$?*

Remarks 3.2.24. (1) We note that the converse of Question 3.2.23 may be false. For example, suppose that C_5 denotes the cycle graph on the vertex set $V(C_5) = \{1, 2, 3, 4, 5\}$, and the edge set $E(C_5) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}\}$. Let

$$\mathfrak{a} := J(C_5) = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1),$$

be the cover ideal associated to C_5 in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$ over a field K . Hence, we get $\mathfrak{a} = (x_2x_4x_5, x_2x_3x_5, x_1x_3x_5, x_1x_3x_4, x_1x_2x_4)$. Now, one can easily see that $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5) \in \text{Ass}_R(R/\mathfrak{a}^2)$. Here, direct computations yield the subsequent table:

$\mathfrak{a} \setminus x_1 = (x_2x_4x_5, x_2x_3x_5)$	$\text{Ass}_{R \setminus x_1}((R \setminus x_1)/(\mathfrak{a} \setminus x_1)^2) = \{(x_2), (x_5), (x_3, x_4)\}$
$\mathfrak{a} \setminus x_2 = (x_1x_3x_5, x_1x_3x_4)$	$\text{Ass}_{R \setminus x_2}((R \setminus x_2)/(\mathfrak{a} \setminus x_2)^2) = \{(x_3), (x_1), (x_5, x_4)\}$
$\mathfrak{a} \setminus x_3 = (x_2x_4x_5, x_1x_2x_4)$	$\text{Ass}_{R \setminus x_3}((R \setminus x_3)/(\mathfrak{a} \setminus x_3)^2) = \{(x_4), (x_2), (x_5, x_1)\}$
$\mathfrak{a} \setminus x_4 = (x_2x_3x_5, x_1x_3x_5)$	$\text{Ass}_{R \setminus x_4}((R \setminus x_4)/(\mathfrak{a} \setminus x_4)^2) = \{(x_3), (x_5), (x_2, x_1)\}$
$\mathfrak{a} \setminus x_5 = (x_1x_3x_4, x_1x_2x_4)$	$\text{Ass}_{R \setminus x_5}((R \setminus x_5)/(\mathfrak{a} \setminus x_5)^2) = \{(x_4), (x_1), (x_3, x_2)\}$

For example, for $\mathfrak{a} \setminus x_1 = (x_2x_4x_5, x_2x_3x_5)$ in $R \setminus x_1 = K[x_2, x_3, x_4, x_5]$, one obtains $\mathfrak{a} \setminus x_1 = (x_2) \cap (x_5) \cap (x_4, x_3) = (x_2)(x_5)(x_4, x_3)$, and inspired by Proposition 2.1.8, we gain

$$\text{Ass}_{R \setminus x_1}((R \setminus x_1)/(\mathfrak{a} \setminus x_1)^t) = \text{Ass}_{R \setminus x_1}((R \setminus x_1)/(\mathfrak{a} \setminus x_1)),$$

for all positive integers t . A similar argument shows that

$$\text{Ass}_{R \setminus x_j}((R \setminus x_j)/(\mathfrak{a} \setminus x_j)^t) = \text{Ass}_{R \setminus x_j}((R \setminus x_j)/(\mathfrak{a} \setminus x_j)),$$

for all positive integers t and for all $j = 2, 3, 4, 5$. This implies that

$$\mathfrak{m} \setminus x_j \notin \text{Ass}_{R \setminus x_j}((R \setminus x_j)/(\mathfrak{a} \setminus x_j)^t),$$

for all positive integers t and for all $j = 1, 2, 3, 4, 5$.

(2) Question 3.2.23 can fail if we drop the word *square-free*. As an example, consider the monomial ideal $I = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1, x_5x_1^2x_2, x_1x_3^2x_6)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$ over a field K . It is routine to check that $\mathfrak{m} \setminus x_6 = (x_1, x_2, x_3, x_4, x_5) \in \text{Ass}_{R \setminus x_6}((R \setminus x_6)/(I^4 \setminus x_6))$, while $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5, x_6) \notin \text{Ass}_R(R/I^4)$.

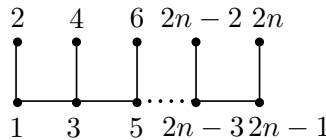
3.2.3 Examples of monomial superficial ideals

In the following, we introduce some classes of monomial ideals in which every ideal has a superficial ideal. To do this, one has to recall the definition of centipede graphs from Definition 2.2.18.

In view of [97], one can provide the following result which gives that the path ideals generated by paths of length two in centipede graphs have superficial ideals.

Theorem 3.2.25. *Let W_n , for some $n \in \mathbb{N}$ with $n \geq 2$, be a centipede graph with corresponding path ideal $I_2(W_n)$, i.e., the path ideal generated by paths of length two. Then $I_2(W_n)$ has a superficial ideal.*

Proof. To simplify our notation, one may consider the following centipede graph on the vertex set $V(W_n) = \{1, 2, 3, \dots, 2n-2, 2n-1, 2n\}$, and also set $L := I_2(W_n)$.



Put $g := x_1x_2x_3$. Certainly, $g \in \mathcal{G}(L)$. On account of the proof of Theorem 2.2.19, one derives $(L^{k+1} :_R g) = L^k$ for all $k \in \mathbb{N}$, where $R = K[x_1, \dots, x_{2n}]$. It follows directly that gR is a superficial ideal for L , as required. $\square$

Here, we focus on two classes of monomial ideals which were introduced in Section 2.3, see Definitions 2.3.1 and 2.3.3. In fact, our aim is to verify that every ideal in these two classes has a superficial ideal. We are now going to show that every unisplit monomial ideal has a superficial ideal.

Theorem 3.2.26. *Every unisplit monomial ideal has a superficial ideal.*

Proof. Let I be a unisplit monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Without loss of generality, assume that $\gcd(u_1, u_i) = 1$ for each $i = 2, \dots, m$. By virtue of $(u_1^{k+1} :_R u_1) = u_1^k$ for all k , one can conclude that $J := u_1R$ is a superficial ideal for

J . In view of the proof of Theorem 3.2.8, it follows that $J = u_1 R$ is a superficial ideal for I , i.e., $(I^{s+1} :_R u_1) = I^s$ for all s , as desired. $\square$

The next theorem illustrates every separable monomial ideal has a superficial ideal. In this theorem, we see an application of Theorem 3.2.26.

Theorem 3.2.27. *Every separable monomial ideal has a superficial ideal.*

Proof. Assume that I is a separable monomial ideal of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Without loss of generality, assume there are monomials f and w in R such that $w \neq 1$, $u_1 = wf$, $\gcd(w, f) = 1$, and $\gcd(u_1, u_i) = w$ for each $i = 2, \dots, m$. This gives that there exist monomials $h_2, \dots, h_m$ in R such that $u_i = wh_i$ for each $i = 2, \dots, m$. Accordingly, one can deduce that $I = w(fR + \sum_{i=2}^m h_i R)$. Put $L := fR + \sum_{i=2}^m h_i R$. This implies that L is a unisplit monomial ideal, and Theorem 3.2.26 yields that $(L^{k+1} :_R f) = L^k$ for all k . Here, our goal is to demonstrate $(I^{k+1} :_R wf) = I^k$ for all k . We thus have the following equalities

$$\begin{aligned} (I^{k+1} :_R wf) &= (w^{k+1} L^{k+1} :_R wf) \\ &= (w^k L^{k+1} :_R f) \\ &= w^k (L^{k+1} :_R f) \\ &= w^k L^k \\ &= I^k. \end{aligned}$$

Therefore, $J := u_1 R$ is a superficial ideal for I , as required. $\square$

As another application of Theorems 3.2.9 and 3.2.26, we provide the following theorem. To see this, one needs the proposition below.

Proposition 3.2.28. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K , v be a square-free monomial in R , and $u_1, \dots, u_m$ be some monomials in R . Then, for all positive integers $\alpha_1, \dots, \alpha_m$, the following statement holds*

$$(u_1^{\alpha_1} \cdots u_m^{\alpha_m} :_R v) = u_1^{\alpha_1-1} \cdots u_m^{\alpha_m-1} (u_1 \cdots u_m :_R v).$$

Proof. Let $\alpha_1, \dots, \alpha_m$ be some arbitrary positive integers. By virtue of v is a square-free monomial in R , one can easily see that

$$\gcd(u_1^{\alpha_1} \cdots u_m^{\alpha_m}, v) = \gcd(u_1 \cdots u_m, v).$$

We thus gain the following equalities

$$\begin{aligned}
(u_1^{\alpha_1} \cdots u_m^{\alpha_m} :_R v) &= \frac{u_1^{\alpha_1} \cdots u_m^{\alpha_m}}{\gcd(u_1^{\alpha_1} \cdots u_m^{\alpha_m}, v)} \\
&= \frac{u_1^{\alpha_1} \cdots u_m^{\alpha_m}}{\gcd(u_1 \cdots u_m, v)} \\
&= u_1^{\alpha_1-1} \cdots u_m^{\alpha_m-1} \frac{u_1 \cdots u_m}{\gcd(u_1 \cdots u_m, v)} \\
&= u_1^{\alpha_1-1} \cdots u_m^{\alpha_m-1} (u_1 \cdots u_m :_R v).
\end{aligned}$$

This completes the proof. $\square$

Theorem 3.2.29. *Suppose that I is a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ such that, for each $i = 2, \dots, m-1$, $\gcd(u_1, u_2, u_{i+1}) = 1$ and $(u_i :_R u_1)$ divides $(u_{i+1} :_R u_1)$. Then I has a superficial ideal.*

Proof. Fix $k \in \mathbb{N}$. If $m = 1$, then $\mathcal{G}(I) = \{u_1\}$. Since u_1 is a non-zerodivisor on R , one has $(u_1^{k+1}R :_R u_1R) = u_1^kR$, and so $(I^{k+1} :_R I) = I^k$. Hence, the claim is true for the case $m = 1$. If $m = 2$, then $\mathcal{G}(I) = \{u_1, u_2\}$. It follows from the binomial expansion that $I^{k+1} = \sum_{\ell=0}^{k+1} u_1^{k+1-\ell} u_2^\ell R$. This implies the subsequent equalities

$$\begin{aligned}
(I^{k+1} :_R u_1) &= \left(\sum_{\ell=0}^{k+1} u_1^{k+1-\ell} u_2^\ell R :_R u_1 \right) \\
&= \sum_{\ell=0}^{k+1} (u_1^{k+1-\ell} u_2^\ell R :_R u_1) \\
&= \sum_{\ell=0}^k (u_1^{k+1-\ell} u_2^\ell R :_R u_1) + (u_2^{k+1} :_R u_1) R \\
&= \sum_{\ell=0}^k u_1^{k-\ell} u_2^\ell R + (u_2^{k+1} :_R u_1) R.
\end{aligned}$$

Furthermore, Proposition 3.2.28 yields that $(u_2^{k+1} :_R u_1) = u_2^k(u_2 :_R u_1)$. Due to $u_1^{k-\ell} u_2^\ell \in I^k$ for all $\ell = 0, \dots, k$, and by virtue of $u_2^k(u_2 :_R u_1) \in I^k$, we deduce that $(I^{k+1} :_R u_1) \subseteq I^k$. As $I^k \subseteq (I^{k+1} :_R u_1)$, one has $(I^{k+1} :_R u_1) = I^k$. Thus, the assertion holds for the case $m = 2$. So let $m \geq 3$. To simplify our notation, we take $M_i := \gcd(u_{i+1}, u_1)$ and $N_i := (u_{i+1} :_R u_1)$ for all $i = 1, \dots, m-1$. This gives that $u_{i+1} = M_i N_i$ for all $i = 1, \dots, m-1$. On the other hand, the hypothesis implies that $N_i | N_{i+1}$ for all $i = 1, \dots, m-2$, and thus $N_1 | N_j$ for all $j = 2, \dots, m-1$. Hence, $N_j = N_1 H_j$ for some monomial H_j in R for all $j = 2, \dots, m-1$. Accordingly, one can write $I = (u_1, N_1 M_1, N_1 H_2 M_2, \dots, N_1 H_{m-1} M_{m-1})$. This gives rise to the equation below

$$I = N_1(M_1, H_2 M_2, \dots, H_{m-1} M_{m-1}) + u_1 R.$$

Here, set $L := (M_1, H_2M_2, \dots, H_{m-1}M_{m-1})$. Therefore, one has $I = N_1L + u_1R$. Because $M_1 = \gcd(u_2, u_1)$, we have $M_1|u_1$, and so $u_1 \in L$. Also, by virtue of u_i , for all $i = 1, \dots, m$, is a square-free monomial, one can conclude that $\gcd(N_1, u_1) = 1$, $\gcd(N_1, M_1) = 1$, and $\gcd(N_1, H_iM_i) = 1$ for all $i = 2, \dots, m-1$. On account of

$$\gcd(u_1, u_2, u_{i+1}) = \gcd(\gcd(u_1, u_2), u_{i+1}),$$

for all $i = 2, \dots, m-1$, we get $\gcd(M_1, u_{i+1}) = 1$ for all $i = 2, \dots, m-1$. This gives that $\gcd(M_1, M_iN_i) = 1$ for all $i = 2, \dots, m-1$, and hence $\gcd(M_1, M_iH_i) = 1$ for all $i = 2, \dots, m-1$. This yields that L is a unisplit monomial ideal. Thanks to Theorem 3.2.26, one derives that the monomial ideal M_1R is a superficial ideal for L . It follows now from Theorem 3.2.9 that $N_1M_1R + u_1R = (u_1, u_2)R$ is a superficial ideal for I , as required. $\square$

Corollary 3.2.30. *Suppose that I is a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ such that, for each $i = 2, \dots, m-1$, $\gcd(u_1, u_2, u_{i+1}) = 1$ and $\text{lcm}(u_1, u_i)$ divides $\text{lcm}(u_1, u_{i+1})$. Then I has a superficial ideal.*

Proof. We know that $(u_i :_R u_1) = u_i/\gcd(u_i, u_1)$ and $u_iu_1 = \text{lcm}(u_i, u_1)\gcd(u_i, u_1)$ for each $i = 1, \dots, m$. The assertion follows readily from Theorem 3.2.29. $\square$

We next give an example which illustrates Theorem 3.2.29.

Example 3.2.31. *Consider the following monomial ideal*

$$I = (x_1x_2x_3x_7x_8, x_1x_2x_4, x_3x_4x_5x_8, x_3x_4x_5x_6x_7, x_4x_5x_6x_7x_8),$$

in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ over a field K . Now, set $u_1 := x_1x_2x_3x_7x_8$, $u_2 := x_1x_2x_4$, $u_3 := x_3x_4x_5x_8$, $u_4 := x_3x_4x_5x_6x_7$, and $u_5 := x_4x_5x_6x_7x_8$. It is routine to check that, for each $i = 2, 3, 4$, $\gcd(u_1, u_2, u_{i+1}) = 1$ and $(u_i :_R u_1)$ divides $(u_{i+1} :_R u_1)$. Now, Theorem 3.2.29 yields that the monomial ideal $(u_1, u_2) = (x_1x_2x_3x_7x_8, x_1x_2x_4)$ is a superficial ideal for I .

We close this argument by giving two results which show that for an arbitrary monomial ideal I in a polynomial ring $R = K[x_1, \dots, x_n]$, several superficial ideals can be constructed.

Proposition 3.2.32. *Suppose that $G = (V(G), E(G))$ is a graph, $I = I(G)$ the edge ideal associated to G , v a pendant of G , and f is the unique neighbor of v . Then vfR is a superficial ideal for I , where $R = K[x_t : t \in V(G)]$.*

Proof. This assertion follows readily from [94, Lemma 2.10]. $\square$

Proposition 3.2.33. *Suppose $G = (V(G), E(G))$ is a bipartite graph, $J = J(G)$ the cover ideal associated to G , and $V(G) = U \cup W$ is a bipartition for the $V(G)$. Put $u := \prod_{\alpha \in U} x_\alpha$ and $w := \prod_{\beta \in W} x_\beta$. Then vR and wR are superficial ideals for J , where $R = K[x_t : t \in V(G)]$.*

Proof. In the light of [124, Lemma 3.2], one can immediately deduce this claim. $\square$

3.2.4 A criterion for detecting the non-normality

In this subsection, we present an elegant and useful criterion for detecting the non-normality of monomial ideals. We express it in the following theorem.

Theorem 3.2.34. *Every normal monomial ideal has the strong persistence property.*

Proof. Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. Without loss of generality, one may assume that $x_1 \in \bigcup_{i=1}^m \text{supp}(u_i)$. After relabeling, we have $u_i = x_1^{\alpha_i} g_i$ with $\alpha_i \in \mathbb{N}$ for all $i = 1, \dots, t$, and $\gcd(u_i, x_1) = 1$ for all $i = t+1, \dots, m$. To complete the proof, one has to prove that $(I^{k+1} :_R I) = I^k$ for all $k \in \mathbb{N}$. To achieve this, fix $k \in \mathbb{N}$, and set $L := (I :_R x_1)$. We thus gain the following consequences

$$\begin{aligned}
(I^{k+1} :_R I) &= (I^{k+1} :_R (u_1, \dots, u_m)) \\
&= (I^{k+1} :_R x_1(x_1^{\alpha_1-1} g_1, \dots, x_1^{\alpha_t-1} g_t) + (u_{t+1}, \dots, u_m)) \\
&\subseteq (I^{k+1} :_R x_1(x_1^{\alpha_1-1} g_1, \dots, x_1^{\alpha_t-1} g_t, u_{t+1}, \dots, u_m)) \\
&= ((I^{k+1} :_R x_1) :_R (x_1^{\alpha_1-1} g_1, \dots, x_1^{\alpha_t-1} g_t, u_{t+1}, \dots, u_m)) \\
&= (I^k(I :_R x_1) :_R (x_1^{\alpha_1-1} g_1, \dots, x_1^{\alpha_t-1} g_t, u_{t+1}, \dots, u_m)) \\
&= (I^k(I :_R x_1) :_R (I :_R x_1)) \\
&= (I^k L :_R L).
\end{aligned}$$

It follows now from [73, Corollary 6.8.7] that $(\overline{I^k L} :_R L) = \overline{I^k}$, and by virtue of $(I^k L :_R L) \subseteq (\overline{I^k L} :_R L)$, one has $(I^k L :_R L) \subseteq \overline{I^k}$. On account of I is normal, one derives $\overline{I^k} = I^k$, and so $(I^{k+1} :_R I) \subseteq I^k$. Due to $I^k \subseteq (I^{k+1} :_R I)$, one can conclude that $(I^{k+1} :_R I) = I^k$, as claimed. $\square$

The following corollary allows us to refute the normality of a monomial ideal based on the persistence property.

Corollary 3.2.35. *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$ over a field K which does not satisfy the persistence property. Then I is non-normal.*

Corollary 3.2.36. *Every normal monomial ideal has a superficial ideal.*

We now present the following question which is devoted to the superficiality of monomial ideals under polarization operator. To do this, we recall the definition of polarization.

Definition 3.2.37. ([50, Definition 4.1]) *The process of polarization replaces a power x_i^t by a product of t new variables $x_{(i,1)} \cdots x_{(i,t)}$. We call $x_{(i,j)}$ a shadow of x_i . We will use $\tilde{I}^t$ to denote the polarization of I^t , will use S_t for the new polynomial ring in this polarization, and will use $\tilde{w}$ to denote the polarization in S_t of a monomial w in $R = K[x_1, \dots, x_n]$. The depolarization of an ideal in S_t is formed by setting $x_{(i,j)} = x_i$ for all i, j .*

Question 3.2.38. *Let I and J be two monomial ideals in $R = K[x_1, \dots, x_n]$. If $\tilde{J}$ is a superficial ideal for $\tilde{I}$, then J is a superficial ideal for I ?*

As an application of Corollary 3.2.36, we show that Question 3.2.38 can fail in general. To do this, we need Lemma 3.2.39 and Theorem 3.2.40 to provide a counterexample.

Lemma 3.2.39. *Let $R = K[x_1, \dots, x_{2n}]$ be a polynomial ring over a field K , and I be a square-free monomial ideal in R with $\mathcal{G}(I) = \{x_{2i-3}x_{2i-1}x_{2i}x_{2i+1} : i = 1, \dots, n\}$, where $2n + 1 = 1$. Then, for all positive integers t , $x_1^{\ell_1} \cdots x_{2n}^{\ell_{2n}} \in \mathcal{G}(I^t)$ if and only if (1) $\sum_{i=1}^n \ell_{2i} = t$ and (2) $\ell_{2i+1} = \ell_{2i} + \ell_{2i+2} + \ell_{2i+4}$ for each $i = 1, \dots, n$.*

Proof. ($\implies$) Set $u_{2i} := x_{2i-3}x_{2i-1}x_{2i}x_{2i+1}$ for each $i = 1, \dots, n$. Clearly, our claim is true for the case $t = 1$, because of u_{2i} satisfies the statement for each $i = 1, \dots, n$. For the same reason, the proof follows directly by induction.

($\impliedby$) Any square-free monomial in R that satisfies the sufficient condition, must clearly be of the form u_{2i} for some $1 \leq i \leq n$. Hence, the statement is true for the case $t = 1$. Follow by induction. Assume that $\sum_{i=1}^n \ell_{2i} = t + 1$ and $\ell_{2i+1} = \ell_{2i} + \ell_{2i+2} + \ell_{2i+4}$ for each $i = 1, \dots, n$, where $2n + 1 = 1$. Choose i_0 such that $x_{2i_0+2} | x_1^{\ell_1} \cdots x_{2n}^{\ell_{2n}}$ for some nonnegative integers $\ell_1, \dots, \ell_{2n}$. Put $f := x_1^{\ell_1} \cdots x_{2n}^{\ell_{2n}}$. Then, by (2), we must have each of $x_{2i_0-1}, x_{2i_0+1}, x_{2i_0+3}$ divides f ; that is, $u_{2i_0+2} | f$. Write $f = u_{2i_0+2} f'$. Let $f' = x_1^{\ell'_1} \cdots x_{2n}^{\ell'_{2n}}$ for some nonnegative integers $\ell'_1, \dots, \ell'_{2n}$. Then it must be clear that (1) and (2) hold for the ℓ' 's, and so $f' \in \mathcal{G}(I^t)$. Therefore, $f \in \mathcal{G}(I^{t+1})$. This completes the inductive step, and so the claim has been proved by induction. $\square$

Theorem 3.2.40. *Let $R = K[x_1, \dots, x_{2n}]$ be a polynomial ring over a field K , and I be a square-free monomial ideal in R with $\mathcal{G}(I) = \{x_{2i-3}x_{2i-1}x_{2i}x_{2i+1} : i = 1, \dots, n\}$, where $2n + 1 = 1$. Then I is a normal ideal.*

Proof. Our strategy is to use [60, Theorem 1.4.2]. For this purpose, assume that $m \in \overline{I}^t$ for some positive integer t . Let $\alpha \in \mathbb{Z}^{2n}$ be the tuple of exponents of m . Also, let $\{\mu_i : i = 1, \dots, r\}$ be the tuples of exponents of the minimal generators of I^t . Suppose that μ_{ij} is the j -th component of μ_i . Since $m \in \overline{I}^t$, one can conclude that there exist $\lambda_1, \dots, \lambda_r \in \mathbb{Q}_{\geq 0}$ with $\sum_{i=1}^r \lambda_i = 1$ such that $\alpha = \lambda_1 \mu_1 + \dots + \lambda_r \mu_r$ (see [60, Corollary 1.4.3]). On account of $\{\mu_i : i = 1, \dots, r\}$ is the set of tuples of exponents of the minimal generators of I^t , Lemma 3.2.39 implies the following equalities

$$(*) \quad \sum_{i=1}^n \alpha_{2i} = \sum_{i=1}^n \sum_{j=1}^r \lambda_j \mu_{j,2i} = \sum_{j=1}^r \lambda_j t = t.$$

Furthermore, for each $i = 1, \dots, n$, applying Lemma 3.2.39 derives that

$$\begin{aligned} \alpha_{2i+1} &= \sum_{j=1}^r \lambda_j \mu_{j,2i+1} \\ &= \sum_{j=1}^r \lambda_j (\mu_{j,2i} + \mu_{j,2i+2} + \mu_{j,2i+4}) \\ (**) \quad &= \alpha_{2i} + \alpha_{2i+2} + \alpha_{2i+4}. \end{aligned}$$

We can now combine (*) and (**) to obtain $m \in I^t$, as claimed. $\square$

We are now in a position to state the following counterexample.

Example 3.2.41. Let $I = (x_1 x_2^2 x_3, x_2 x_3^2 x_4, x_3 x_4^2 x_5, x_4 x_5^2 x_1, x_5 x_1^2 x_2)$ be a monomial ideal in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$ over a field K . Suppose, on the contrary, that I has a superficial ideal. This implies that I satisfies the persistence property, while one can easily check that

$$\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5) \in \text{Ass}_R(R/I) \setminus \text{Ass}_R(R/I^2).$$

This gives rise to a contradiction. Consequently, the monomial ideal I has no superficial ideal. We now explore the superficiality of the polarization of I . It is routine to compute that

$$\tilde{I} = (x_{11} x_{21} x_{22} x_{31}, x_{21} x_{31} x_{32} x_{41}, x_{31} x_{41} x_{42} x_{51}, x_{41} x_{51} x_{52} x_{11}, x_{51} x_{11} x_{12} x_{21}),$$

in the polynomial ring $S = K[x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32}, x_{41}, x_{42}, x_{51}, x_{52}]$ over the field K . In order to simplify the notation, we set $L := \tilde{I}$, $z_{2k-1} := x_{k1}$ and $z_{2k} := x_{k2}$ for all $k = 1, \dots, 5$. Accordingly, one obtains

$$L = (z_1 z_3 z_4 z_5, z_3 z_5 z_6 z_7, z_5 z_7 z_8 z_9, z_1 z_7 z_9 z_{10}, z_1 z_2 z_3 z_9),$$

in the polynomial ring $S = K[z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9, z_{10}]$. Our aim is to verify that L has a superficial ideal. To establish this claim, we use Theorem 3.2.40

and Corollary 3.2.36. Indeed, it follows from Theorem 3.2.40 that L is a normal ideal, and Corollary 3.2.36 yields that L has a superficial ideal. Therefore, $\tilde{I}$ has a superficial ideal, while I has no superficial ideal, as desired.

In the next theorem, we probe the relation between normally torsion-freeness and superficiality.

Theorem 3.2.42. *Every normally torsion-free square-free monomial ideal has the strong persistence property.*

Proof. Assume that I is a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . It follows from the assumption that $\text{Min}(I) = \text{Ass}_R(R/I) = \text{Ass}_R(R/I^k)$ for all k . Since $I^t \subseteq (I^{t+1} :_R I)$ for all t , we need to show that $(I^{t+1} :_R I) \subseteq I^t$ for all t . Fix $t \in \mathbb{N}$. Due to [89, Exercise 6.4], it is sufficient to demonstrate that $(I_a^{t+1} :_{R_a} I_a) = I_a^t$ for every $\mathfrak{a} \in \text{Ass}_R(R/I^t)$. To do this, consider an arbitrary element $\mathfrak{a} \in \text{Ass}_R(R/I^t)$. Hence, $\mathfrak{a} \in \text{Min}(I)$. We next prove that, for every $\mathfrak{b} \in \text{Ass}_R(R/I) \setminus \{\mathfrak{a}\}$, one has $\mathfrak{a} \notin \text{Supp}_R(R/\mathfrak{b})$. On the contrary, assume that there exists $\mathfrak{b} \in \text{Ass}_R(R/I) \setminus \{\mathfrak{a}\}$ such that $\mathfrak{a} \in \text{Supp}_R(R/\mathfrak{b})$. As $\text{Supp}_R(R/\mathfrak{b}) = V(\text{Ann}_R(R/\mathfrak{b}))$, this implies that $\text{Ann}_R(R/\mathfrak{b}) \subseteq \mathfrak{a}$, and so $\mathfrak{b} \subseteq \mathfrak{a}$. By virtue of $\mathfrak{a} \in \text{Min}(I)$, one derives $\mathfrak{b} = \mathfrak{a}$. This leads to a contradiction. This yields that $\mathfrak{b}_a = R_a$ for every $\mathfrak{b} \in \text{Ass}_R(R/I) \setminus \{\mathfrak{a}\}$. Thus, $I_a = \mathfrak{a}_a$, and so $(I^t)_a = (\mathfrak{a}^t)_a$. In addition, we know that $(\mathfrak{a}^{t+1} :_R \mathfrak{a}) = \mathfrak{a}^t$. This implies that $(\mathfrak{a}_a^{t+1} :_{R_a} \mathfrak{a}_a) = \mathfrak{a}_a^t$, and hence $(I_a^{t+1} :_{R_a} I_a) = I_a^t$, as required. $\square$

Corollary 3.2.43. *Every normally torsion-free square-free monomial ideal has a superficial ideal.*

We conclude this subsection with showing that if I is a square-free monomial ideal of R , then we always have $(I^2 :_R I) = I$.

Proposition 3.2.44. *Let S be a Noetherian integral domain, and I be an integrally closed ideal of S . Then $(I^{k+1} :_S I^k) = I$ for all positive integers k .*

Proof. Fix $k \in \mathbb{N}$. The claim follows readily from [131, Proposition 1.58] by setting $L := I^k$. $\square$

Corollary 3.2.45. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K , and I be a square-free monomial ideal of R . Then $(I^{k+1} :_R I^k) = I$ for all positive integers k . In particular, $(I^2 :_R I) = I$.*

Proof. Since K is a field, one derives R is an integral domain. Furthermore, it is well-known that prime ideals are integrally closed, and also intersections of integrally closed ideals are integrally closed. This gives that I is integrally closed. Thus, the desired conclusion follows from Proposition 3.2.44. The last assertion is clear. $\square$

3.3 More results on the strong persistence property

In the following, we study the strong persistence property of monomial ideals with some suitable assumption on its support. It should be noted that the results of this section can be found in [105].

We begin with the following lemma which allows us to discuss the strong persistence property of the intersection and product of two monomial ideals which are generated by two disjoint sets of variables. Moreover, recall that for a monomial ideal I of R with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, we define $\text{supp}(I) := \bigcup_{i=1}^m \text{supp}(u_i)$.

Lemma 3.3.1. *Suppose that I_1 and I_2 are two monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that $\text{supp}(I_1) \cap \text{supp}(I_2) = \emptyset$. If I_1 and I_2 have the strong persistence property, then $I_1 \cap I_2$ has the strong persistence property.*

Proof. Assume that I_1 and I_2 have the strong persistence property. Since I_1 and I_2 are generated by disjoint sets of variables, [70, Lemma 1.1] yields that $I_1^a \cap I_2^b = I_1^a I_2^b$ for any positive integers a, b . Let $k \geq 1$. By observing Lemma 2.1.12, one can deduce that

$$(I_1^{k+1} I_2^{k+1} :_R I_1) = I_2^{k+1} (I_1^{k+1} :_R I_1) \quad \text{and} \quad (I_1^k I_2^{k+1} :_R I_2) = I_1^k (I_2^{k+1} :_R I_2).$$

To end the proof, it is enough to consider the following equalities,

$$\begin{aligned} ((I_1 \cap I_2)^{k+1} :_R I_1 \cap I_2) &= ((I_1 I_2)^{k+1} :_R I_1 I_2) \\ &= ((I_1^{k+1} I_2^{k+1} :_R I_1) :_R I_2) \\ &= (I_2^{k+1} (I_1^{k+1} :_R I_1) :_R I_2) \\ &= (I_1^k I_2^{k+1} :_R I_2) \\ &= I_1^k (I_2^{k+1} :_R I_2) \\ &= I_1^k I_2^k \\ &= (I_1 \cap I_2)^k. \end{aligned}$$

□

Lemma 3.3.2. *Suppose that I_1 and I_2 are two monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that $\text{supp}(I_1) \cap \text{supp}(I_2) = \emptyset$. Then, $I_1 + I_2$ has the strong persistence property if and only if I_1 or I_2 has the strong persistence property.*

Proof. The backward implication can be immediately deduced from Theorem 3.2.8. To establish the forward implication, suppose, on the contrary, that I_1 and I_2 do not satisfy the strong persistence property. This implies that there exist a positive integer k_1 (respectively, k_2) and a monomial m_1 (respectively, m_2) such that $m_1 \in \mathcal{G}(I_1^{k_1+1} :_R I_1) \setminus \mathcal{G}(I_1^{k_1})$ (respectively, $m_2 \in \mathcal{G}(I_2^{k_2+1} :_R I_2) \setminus \mathcal{G}(I_2^{k_2})$). Take the nonnegative integer a_1 (respectively, a_2) such that $m_1 \in I_1^{a_1} \setminus I_1^{a_1+1}$ (respectively, $m_2 \in I_2^{a_2} \setminus I_2^{a_2+1}$). This gives that $a_1 \leq k_1 - 1$ (respectively, $a_2 \leq k_2 - 1$). Put $I := I_1 + I_2$, $m := m_1 m_2$, and $b := a_1 + a_2$. Thus, one has $m \in I^b$. Note that $m \notin I_1^i \cap I_2^j$ for either $i > a_1$ or $j > a_2$, so, by [70, Lemma 1.1], $m \notin I^{b+1}$. To conclude the proof, it is enough to show that $m \in (I^{b+2} :_R I)$, this contradicts the assumption that I has the strong persistence property. So, take a monomial $u \in I = I_1 + I_2$. Without loss of generality, we can assume that $u \in I_1$. Thus, $um = (um_1)m_2 \in I_1^{k_1+1}I_2^{a_2} \subseteq (I_1 + I_2)^{k_1+1+a_2}$ and, since $k_1 > a_1$, we are done. $\square$

An application of Lemma 3.3.2 is the corollary below.

Corollary 3.3.3. *Every irreducible primary monomial ideal has the strong persistence property.*

Proof. Let $k \geq 1$ be an integer, and recall that, by [133, Proposition 6.1.7], a monomial ideal is a primary irreducible ideal if and only if it has the form $Q = (x_{i_1}^{\alpha_1}, \dots, x_{i_t}^{\alpha_t})$ in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\alpha_1, \dots, \alpha_t$ are positive integers and $\{x_{i_1}, \dots, x_{i_t}\} \subseteq \{x_1, \dots, x_n\}$. Set $I_1 := (x_{i_1}^{\alpha_1})$ and $I_2 = (x_{i_2}^{\alpha_2}, \dots, x_{i_t}^{\alpha_t})$. Since I_1 has the strong persistence property, Lemma 3.3.2 implies that Q has the strong persistence property, as desired. $\square$

By a repeated application of Lemma 3.1.1, we have the following result.

Lemma 3.3.4. *Let I and J be two monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, $\mathcal{G}(J) = \{v_1, \dots, v_s\}$, and h a monomial in R such that for each $i = 1, \dots, m$ and $j = 1, \dots, s$, $\gcd(h, v_j) = 1$, $\gcd(v_j, u_i) = 1$, and $h \in I$. If I has the strong persistence property, then $L := JI + hR$ has the strong persistence property.*

Remark 3.3.5. *It should be noted that Lemma 3.3.4 may be false if we consider the ideal L as $L = JI + H$ with H not a principal monomial ideal. To see this, assume that L is the Stanley-Reisner ideal that corresponds to the natural triangulation of the projective plane, that is, $L \subset R = K[x_1, x_2, x_3, x_4, x_5, x_6]$, one has*

$$L = (x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, \\ x_3x_4x_5, x_3x_4x_6).$$

In the following, put $J := (x_3)$, $I := (x_1x_2, x_1x_5, x_2x_6, x_4x_5, x_4x_6)$, and

$$H := (x_1x_2x_4, x_1x_4x_6, x_1x_5x_6, x_2x_4x_5, x_2x_5x_6).$$

It can be rapidly checked that $L = JI + H$, $H \subseteq I$, $\gcd(x_3, h) = 1$ for all $h \in \mathcal{G}(H)$, $\gcd(x_3, u) = 1$ for all $u \in \mathcal{G}(I)$. We prove that I has the strong persistence property. To do this, assume that G is the cycle graph with $V(G) = \{1, 2, 4, 5, 6\}$ and $E(G) = \{\{1, 2\}, \{2, 6\}, \{6, 4\}, \{4, 5\}, \{5, 1\}\}$. It is easy to detect that the edge ideal of G is $I(G) = (x_1x_2, x_1x_5, x_2x_6, x_4x_5, x_4x_6)$. This implies that $I(G) = I$. In addition, [133, Theorem 7.7.14] yields that $I(G)$ has the strong persistence property, and thus the monomial ideal I has the strong persistence property. On the other hand, by using Macaulay2 [49], one can detect that $(L^3 :_R L) \neq L^2$, that is, L does not satisfy the strong persistence property.

As an application of Lemma 3.1.1, we present Theorem 3.3.7. To understand the importance of this theorem, we first review some background.

Definition 3.3.6. ([133, Definition 10.5.4]) The cone $C(G)$, over the graph G , is obtained by adding a new vertex t to G and joining every vertex of G to t .

More generally, we can take any graph G and form a new graph H by adding new vertices, joining each to every vertex of G . Then $J(G)$ has the strong persistence property if and only if $J(H)$ has the strong persistence property. Explicitly,

Theorem 3.3.7. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w_1, \dots, w_r\}$ with $w_i \notin V(G)$ for all $i = 1, \dots, r$, and

$$E(H) = E(G) \cup \{\{v, w_i\} : \text{for all } v \in V(G) \text{ and for all } i = 1, \dots, r\}.$$

Then $J(G)$ has the strong persistence property if and only if $J(H)$ has the strong persistence property.

Proof. We give a sketch of the proof. Assume that $V(G) = [n]$ and $V(H) = V(G) \cup \{n+1, \dots, n+r\}$. In addition, let $R = K[x_1, \dots, x_{n+r}]$ be the polynomial ring over a field K . Put $L := J(H)$, $I := J(G)$, $h := \prod_{i=1}^n x_i$, and $g := \prod_{i=n+1}^{n+r} x_i$. It follows from [133, Exercise 6.1.23] that

$$\bigcap_{i=1}^n \bigcap_{j=n+1}^{n+r} (x_i, x_j) = \left(\prod_{i=1}^n x_i, \prod_{j=n+1}^{n+r} x_j \right).$$

Also, it is easy to see that $L = gIR + hR$. Suppose that I has the strong persistence property. Now, Lemma 3.1.1 yields that L has the strong persistence property, as claimed. Conversely, let $J(H)$ have the strong persistence property. Put $\mathfrak{p} := (x_1, \dots, x_n)$. Since $I = L(\mathfrak{p})$, where $L(\mathfrak{p})$ denotes the monomial localization of L with respect to $\mathfrak{p}$, the claim follows at once from Theorem 3.2.16. $\square$

Here, we concentrate on the notion of a whisker graph. Our aim is to explore the strong persistence property of the cover ideal of a whisker graph. To do this, we recollect the subsequent definition.

Definition 3.3.8. ([133, Definition 7.3.10]) Let G_0 be a graph on the vertex set $Y = \{y_1, \dots, y_n\}$ and take a new set of variables $X = \{x_1, \dots, x_n\}$. The whisker graph or suspension of G_0 , denoted by $G_0 \cup W(Y)$, is the graph obtained from G_0 by attaching to each vertex y_i a new vertex x_i and the edge $\{x_i, y_i\}$. The edge $\{x_i, y_i\}$ is called a whisker.

Theorem 3.1.2 says that if we take any graph G whose cover ideal has the strong persistence property, and if we then add a leaf to G , then the cover ideal of the new graph satisfies the strong persistence property as well.

Corollary 3.3.9. With the notation of Definition 3.3.8, if the cover ideal of a finite simple graph G_0 has the strong persistence property, then the cover ideal of the whisker graph of G_0 has the strong persistence property.

Proof. Proceed by using Theorem 3.1.2 repeatedly, n times. $\square$

3.3.1 Strong persistence property of primary monomial ideals

Several questions may be asked along our argument. In this subsection, we investigate the strong persistence property of primary monomial ideals as a case study. To accomplish this, we start with the following main question. In addition, the symbol $\mathbb{Q}$ will always denote the set of rational numbers.

Question 3.3.10. Which classes of monomial ideals have the strong persistence property?

We present a class of primary monomial ideals which do not satisfy the strong persistence property. To do this, consider the monomial ideals

$$I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n),$$

in the polynomial ring $R = K[x, y]$, $n \geq 4$ over a field K . The idea is to describe the monomials in $(I_n^d)_{nd+i}$ for $i \geq 0$, where subscripts denote the degree in R . We list all the monomials in $(I_n^d)_{nd+i}$ in lexicographic order with respect to $x < y$ (i.e., $x^{nd+i}, x^{nd+i-1}y, \dots, y^{nd+i}$). In order to simplify the notation, we will often describe a monomial in $(I_n^d)_{nd+i}$ with its power of y because we can identify such a monomial with its power of y .

It is clear that $(I_n^d)_j = 0$ for $j < nd$ and that the monomials in $(I_n^d)_{nd}$ are exactly the products of d generators of I_n (with repetition). First of all, we consider the monomials $\{(x^n)^\alpha (y^n)^\beta\}_{\alpha+\beta=d, \alpha \geq 0, \beta \geq 0}$ which we will refer to as “subdivision points”. Start with the subdivision point $(x^n)^\alpha (y^n)^\beta$. If $\alpha \geq 1$, then some of the x^n can be replaced by $x^{n-1}y$, and if $\beta \geq 1$, then some of the y^n can be replaced by xy^{n-1} . This leads to an “interval” of consecutive monomials $\{x^{n\alpha-j}y^{n\beta+j}\}_{-\beta \leq j \leq \alpha}$ about the subdivision point $(x^n)^\alpha (y^n)^\beta$. The first monomial in this interval is $\{x^{n\alpha+\beta}y^{n\beta-\beta}\}$ and the last is $\{x^{n\alpha-\alpha}y^{n\beta+\alpha}\}$. Describing this interval with the powers of y , we have an increasing sequence of consecutive powers of y , $\{y^j\}_{n\beta-\beta \leq j \leq n\beta+\alpha}$. The number of monomials in this interval is $(n\beta + \alpha) - (n\beta - \beta) + 1 = \alpha + \beta + 1 = d + 1$. The previous subdivision point (if $\beta \geq 1$) is $x^{n(\alpha+1)}y^{n(\beta-1)}$ and so to avoid overlap of intervals we require that $n(\beta - 1) + (\alpha + 1) < n\beta - \beta$ or equivalently $d + 1 < n$, so that we have $1 \leq d \leq n - 2$.

If $d = n - 2$, then there is no overlap of intervals, but also no gap. If $1 \leq d \leq n - 3$, between adjacent intervals

$$\{x^{n(\alpha+1)-j}y^{n(\beta-1)+j}\}_{-(\beta-1) \leq j \leq (\alpha+1)} \text{ and } \{x^{n\alpha-j}y^{n\beta+j}\}_{-\beta \leq j \leq \alpha},$$

there is a gap. For the left-hand side interval the largest exponent of y is $n(\beta - 1) + (\alpha + 1)$ and the smallest exponent of y in the right-hand side interval is $n\beta - \beta$, so the powers of y in the gap are $\{y^j\}_{n(\beta-1)+(\alpha+1) < j < n\beta-\beta}$, there being

$$(n\beta - \beta) - (n(\beta - 1) + (\alpha + 1)) - 1 = n - d - 2,$$

monomials in the gap. As a simple check for consistency, we have $d + 1$ intervals each containing $d + 1$ monomials, and d gaps each containing $n - d + 2$ monomials, and $(d + 1)^2 + d(n - d - 2) = nd + 1$, the total number of monomials in R_{nd} .

Now, consider the monomials in $(I_n^d)_{nd+i}$, $i \geq 0$. Then the interval

$$\{x^{n(\alpha+1)-j}y^{n(\beta-1)+j}\}_{-(\beta-1) \leq j \leq (\alpha+1)},$$

(of cardinality $\alpha + \beta + 1 = d + 1$) expands to include

$$\{x^{n(\alpha+1)-j+r}y^{n(\beta-1)+j+s}\}_{-(\beta-1) \leq j \leq (\alpha+1), r \geq 0, s \geq 0, r+s=i},$$

(of cardinality $\alpha + \beta + 1 + i = d + i + 1$) and $\{x^{n\alpha-j}y^{n\beta+j}\}_{-\beta \leq j \leq \alpha}$ expands to include $\{x^{n\alpha-j+r}y^{n\beta+j+s}\}_{-\beta \leq j \leq \alpha, r \geq 0, s \geq 0, r+s=i}$ (again respectively of cardinalities $d + 1$ and $d + i + 1$). As above we will describe the gap between these intervals by giving the exponents of y in the monomials in this gap.

Lemma 3.3.11. *[Gap Lemma]*

(1) If $i \geq n - d - 2$, then $(I_n^d)_{nd+i} = R_{nd+i}$.

(2) If $0 \leq i \leq n - d - 2$, then $(I_n^d)_{nd+i}$ has $d + 1$ subdivision intervals each of whose cardinality is $d + i + 1$, which is i more than that for $(I_n^d)_{nd}$.

(3) If $0 \leq i < n - d - 2$, then the list of y -exponents of monomials in the gap between consecutive intervals

$$\{x^{n(\alpha+1)-j+r}y^{n(\beta-1)+j+s}\}_{-(\beta-1) \leq j \leq (\alpha+1), r \geq 0, s \geq 0, r+s=i},$$

and

$$\{x^{n\alpha-j+r}y^{n\beta+j+s}\}_{-\beta \leq j \leq \alpha, r \geq 0, s \geq 0, r+s=i},$$

of $(I_n^d)_{nd+i}$, $i \geq 0$ contains $n - d - 2 - i$ consecutive values (that is, i less than for $(I_n^d)_{nd}$). Also, the largest y -exponent in the gap is $n\beta - \beta - 1$.

Proof. Part (2) was observed in the paragraph before the statement of the lemma. The largest exponent of y of a monomial in the first interval is $n(\beta - 1) + \alpha + 1 + i$ (coming from $j = \alpha + 1$ and $s = i$) and the smallest exponent of y in the right most interval is $n\beta - \beta$ (coming from $j = -\beta$ and $s = 0$). If

$$(n\beta - \beta) - (n(\beta - 1) + \alpha + 1 + i) - 1 = n - d - 2 - i \leq 0,$$

there are no gaps and part (1) follows. Otherwise we have (3). $\square$

Thus, we start with gap size $n - d - 2$ previously obtained for $i = 0$, and drop one exponent from the beginning of the list of exponents for each increase of 1 in the value of i , until $(I_n^d)_{nd+i}$ consists of all R_{nd+i} for $i = n - d - 2$.

To make the notation more manageable we will refer to

$$\{x^{n\alpha-j+r}y^{n\beta+j+s}\}_{-\beta \leq j \leq \alpha, r \geq 0, s \geq 0, r+s=i},$$

as the (α, β) subdivision interval of $(I_n^d)_{nd+i}$ ($0 \leq \beta \leq d, \alpha + \beta = d$). With this notation, Lemma 3.3.11(3) takes on the simpler appearance:

Lemma 3.3.12. *[simplified Gap Lemma] For $1 \leq \beta \leq d$, the list of y -exponents of monomials in the gap between consecutive subdivision intervals $(\alpha + 1, \beta - 1)$ and (α, β) of $(I_n^d)_{nd+i}$ with $0 \leq i < n - d - 2$ contains $n - d - 2 - i$ consecutive values, with the largest equal to $n\beta - \beta - 1$.*

Now, we calculate $I_n^{d+1} : I_n$ for a fixed $d \geq 1$. Note that $I_n^{d+1} : I_n$ is a monomial ideal and $I_n^d \subseteq (I_n^{d+1} : I_n)$ so we seek monomials in $I_n^{d+1} : I_n$ but not in I_n^d . If $d \geq n - 2$, then both I_n^d and I_n^{d+1} are all of R_j in all degrees j for which they are nonzero (namely $j \geq nd$ for I_n^d and $j \geq n(d + 1)$ for I_n^{d+1}) by Lemma 3.3.11(1). From this, it follows that $(I_n^{d+1} : I_n) = I_n^d$ for $d \geq n - 2$.

Now, consider $1 \leq d \leq n - 3$. Let $i = n - d - 3$. Then, by Lemma 3.3.11 part (3), $(I_n^d)_{nd+i}$ has gaps of size 1 and $(I_n^{d+1})_{n(d+1)+i}$ is equal to $R_{n(d+1)+i}$. The

monomials in these gaps of size 1 are mapped into $(I_n^{d+1})_{n(d+1)+i} = R_{n(d+1)+i}$ when multiplied by any minimal generator of I_n (and indeed, by any element of R_n), hence lie in $(I_n^{d+1} : I_n) \setminus I_n^d$. These monomials are $\{x^{nd+n-d-3-n\beta+\beta+1}y^{n\beta-\beta-1}\}_{1 \leq \beta \leq d} = \{x^{nd-d-1}y^{n-2}, x^{nd-n-d}y^{2n-3}, \dots, x^{n-2}y^{nd-d-1}\}$.

If $0 \leq i < n - d - 3$, then $(I_n^d)_{nd+i}$ has gaps of size $t > 1$, which are again mapped into $R_{n(d+1)+i}$ when multiplied by any minimal generator of I_n , in which $(I_n^{d+1})_{n(d+1)+i}$ has gaps of size $t - 1 > 0$. Suppose that one of our gaps of $(I_n^d)_{nd+i}$ lies between subdivision intervals $(\alpha + 1, \beta - 1)$ and (α, β) . Then, by Lemma 3.3.12, the monomials in the gap (described by giving only their exponent of y) consist of t consecutive powers of y with the largest being $n\beta - \beta - 1$. There is not a one-to-one correspondence between the gaps of $(I_n^{d+1})_{n(d+1)+i}$ and $(I_n^d)_{nd+i}$, the former having $d + 1$ gaps, and the latter having d gaps so we have to make a choice. Choose the gap of $(I_n^{d+1})_{n(d+1)+i}$ between subdivision intervals $(\alpha + 2, \beta - 1)$ and $(\alpha + 1, \beta)$. Then the monomials in the gap between these intervals consist of $t - 1$ consecutive powers of y with the largest again being $n\beta - \beta - 1$. The monomials in our gap for $(I_n^d)_{nd+i}$ are then represented by $\{y^{n\beta-\beta-t}, y^{n\beta-\beta-(t-1)}, \dots, y^{n\beta-\beta-1}\}$ and the monomials in our gap for $(I_n^{d+1})_{n(d+1)+i}$ are represented by $\{y^{n\beta-\beta-(t-1)}, \dots, y^{n\beta-\beta-1}\}$. Recall that x^n and $x^{n-1}y$ are minimal generators of I_n and multiplication by them raises the degree by n . Thus, if we multiply a monomial represented by y^a of the gap sequence for $(I_n^d)_{nd+i}$ by x^n we get a monomial of the gap sequence for $(I_n^{d+1})_{n(d+1)+i}$ that is also represented by y^a , except for the first power $y^{n\beta-\beta-t}$ where we can multiply instead by $x^{n-1}y$ and get the monomial represented by $y^{n\beta-\beta-(t-1)}$ which is in the gap sequence for $(I_n^{d+1})_{n(d+1)+i}$. This shows that if $0 \leq i < n - d - 3$, then the monomials in the gaps of $(I_n^d)_{nd+i}$ do not lie in $I_n^{d+1} : I_n$. Therefore, $(I_n^{d+1} : I_n) \setminus I_n^{d+1}$ consists only of the monomials $\{x^{nd-d-1}y^{n-2}, x^{nd-n-d}y^{2n-3}, \dots, x^{n-2}y^{nd-d-1}\}$ of degree $nd+i$ where $i = n - d - 3$ found in the previous paragraph.

In the expression $\{x^{nd+n-d-3-n\beta+\beta+1}y^{n\beta-\beta-1}\}_{1 \leq \beta \leq d}$ there is an annoying apparent lack of symmetry between x and y and between α and β . This can be removed by setting $i = \beta - 1, j = \alpha, a = d - 1$ so that this expression becomes $\{x^{n-2}y^{n-2}(x^{n-1})^j(y^{n-1})^i\}$ and our result becomes:

Proposition 3.3.13. *Let $I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n)$ in the polynomial ring $R = \mathbb{Q}[x, y]$, $n \geq 4$. Then $I_n^{a+2} : I_n \supsetneq I_n^{a+1}$ for $0 \leq a \leq n - 4$ and $I_n^{a+2} : I_n = I_n^{a+1}$ for $a \geq n - 3$. More precisely, if $0 \leq a \leq n - 4$, then $(I_n^{a+2} : I_n) / I_n^{a+1}$ is an $(a + 1)$ -dimensional vector space with basis $\{x^{n-2}y^{n-2}(x^{n-1})^j(y^{n-1})^i\}_{i \geq 0, j \geq 0, i+j=a}$.*

It should be noted that, in particular, if $a = 0$, then Proposition 3.3.13 yields the result that $x^{n-2}y^{n-2}$ is the only monomial in $I_n^2 : I_n$ but not in I_n .

To clarify our discussion, we provide the following example.

Example 3.3.14. *Let us now look at I_6^2 and $(I_6^3 : I_6) \setminus I_6^2$. This will illustrate the ideas in Lemma 3.3.11 and Proposition 3.3.13 and help ensure that we have gotten*

the notation straight. By direct computation one has

$$I_6^2 = (x^{12}, x^{11}y, x^{10}y^2, x^7y^5, x^6y^6, x^5y^7, x^2y^{10}, xy^{11}, y^{12}).$$

The monomials in $(I_6^2)_{12}$ are just the 9 minimal generators of I_6^2 and $(I_6^2)_j = 0$ for $j < 12$. Furthermore, the set of all monomials in $(I_6^2)_{13}$ is

$$\{x^{13}, x^{12}y, x^{11}y^2, x^{10}y^3, x^8y^5, x^7y^6, x^6y^7, x^5y^8, x^3y^{10}, x^2y^{11}, xy^{12}, y^{13}\},$$

and the set of all monomials in $(I_6^2)_{14}$ is

$$\begin{aligned} &\{x^{14}, x^{13}y, x^{12}y^2, x^{11}y^3, x^{10}y^4, x^9y^5, x^8y^6, x^7y^7, x^6y^8, x^5y^9, \\ &x^4y^{10}, x^3y^{11}, x^2y^{12}, xy^{13}, y^{14}\}, \end{aligned}$$

that is, all monomials of degree 14. Obviously, $(I_6^2)_j = R_j$ for $j > 14$. In the notation of Lemma 3.3.11, we have $n = 6$ and $d = 2$. Part (1) of this Lemma says that $(I_6^2)_{nd+i} = (I_6^2)_{12+i} = R_{12+i}$ for $i \geq 6 - 2 - 2 = 2$, or $(I_6^2)_j = R_j$ for $j \geq 14$, which is what we have observed. Furthermore, the $d + 1 = 3$ subdivision intervals of $(I_6^2)_{12}$ are each of cardinality $d + i + 1 = 2 + 0 + 1 = 3$ and the subdivision intervals of $(I_6^2)_{13}$ are each of cardinality $d + i + 1 = 2 + 1 + 1 = 4$, both in agreement with Lemma 3.3.11 part (2), increasing by 1 each time we increase i by 1. There are $d = 2$ gaps in $(I_6^2)_{12}$, namely $\{x^9y^3, x^8y^4\}$ and $\{x^4y^8, x^3y^9\}$, each of cardinality $n - d - 2 - i = 6 - 2 - 2 - 0 = 2$ and the largest exponent of y in these gaps is $n\beta - \beta - 1 = 4, 9$ for $\beta = 1, 2$, all in agreement with Lemma 3.3.11 part (3). The gaps in $(I_6^2)_{13}$, namely $\{x^9y^4\}$ and $\{x^4y^9\}$, each of cardinality $n - d - 2 - i = 6 - 2 - 2 - 1 = 1$ and the largest exponent of y in these gaps is again $n\beta - \beta - 1 = 4, 9$ for $\beta = 1, 2$, also in agreement with Lemma 3.3.11 part (3). We are interested in $I_6^3 : I_6$. This contains I_6^2 and potentially the gaps of $(I_6^2)_{12}$ and $(I_6^2)_{13}$. The minimal generators of I_6 are all of degree 6, and multiplication by one of these maps $(I_6^2)_{13}$ into R_{19} . But by Lemma 3.3.11 part (1), $(I_6^3)_{19} = R_{19}$, so the gaps of $(I_6^2)_{13}$ are in $(I_6^3 : I_6) \setminus (I_6)^2$. But the gaps of $(I_6^2)_{12}$ are not mapped into I_6^3 by all the minimal generators of I_6 . The gaps of $(I_6^3)_{18}$ are $\{x^{14}y^4\}$, $\{x^9y^9\}$ and $\{x^4y^{14}\}$, and for example $x^6(x^8y^4) = x^{14}y^4 \notin (I_6)^3$ so $x^8y^4 \notin I_6^3 : I_6$. Similarly, $x^5y(x^9y^3) = x^{14}y^4 \notin (I_6)^3$ so $x^9y^3 \notin I_6^3 : I_6$ either. Instead of the gap $\{x^{14}y^4\}$ we can use $\{x^9y^9\}$. Thus, $y^6(x^9y^3) = x^9y^9 \notin (I_6)^3$, again yielding that $x^9y^3 \notin I_6^3 : I_6$. We thus conclude that $(I_6^3 : I_6) \setminus (I_6)^2 = \{x^9y^4, x^4y^9\}$. (In the language of Proposition 3.3.13, $a = 1$.)

We are ready to present the second question.

Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and let I be a monomial ideal in R that has the strong persistence property. Also, let Q be an irreducible primary monomial ideal of R . Does the monomial ideal $I + Q$ satisfy the strong persistence property?

We provide a counterexample. For this purpose, assume that $R = K[x, y]$ is the polynomial ring over a field K , and put $I := (x^4, x^3y, xy^3, y^5)$, and $Q = (y^4)$. We thus have $I + Q = (x^4, x^3y, xy^3, y^4)$. It follows from Proposition 3.3.13 that $I + Q$ does not satisfy the strong persistence property. To conclude our argument, it remains to verify that the ideal I has the strong persistence property. To see this, fix $k \geq 1$. Since $I^k \subseteq (I^{k+1} :_R I) \subseteq (I^{k+1} :_R xy^3)$, it is enough for us to show that $(I^{k+1} :_R xy^3) \subseteq I^k$. It is well-known that

$$I^{k+1} = \sum_{\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = k+1} (x^4)^{\lambda_1} (x^3y)^{\lambda_2} (y^5)^{\lambda_3} (xy^3)^{\lambda_4} R.$$

If $\lambda_4 \geq 1$, then we have

$$((x^4)^{\lambda_1} (x^3y)^{\lambda_2} (y^5)^{\lambda_3} (xy^3)^{\lambda_4} :_R xy^3) = ((x^4)^{\lambda_1} (x^3y)^{\lambda_2} (y^5)^{\lambda_3} (xy^3)^{\lambda_4-1}) \subseteq I^k.$$

Thus, let $\lambda_4 = 0$. If $\lambda_2 \geq 3$, then one obtains

$$\begin{aligned} ((x^4)^{\lambda_1} (x^3y)^{\lambda_2} (y^5)^{\lambda_3} :_R xy^3) &= ((x^4)^{\lambda_1} (x^3y)^{\lambda_2-3} (y^5)^{\lambda_3} x^8) \\ &= ((x^4)^{\lambda_1+2} (x^3y)^{\lambda_2-3} (y^5)^{\lambda_3}) \subseteq I^k. \end{aligned}$$

Hence, let $\lambda_4 = 0$ and $0 \leq \lambda_2 \leq 2$. Accordingly, one may consider the following cases:

Case 1. $\lambda_4 = 0$ and $\lambda_2 = 0$. If $\lambda_1 = 0$, then $\lambda_3 = k + 1$. This leads to $((y^5)^{\lambda_3} :_R xy^3) = ((y^5)^{\lambda_3-1} y^2) \subseteq I^k$. Let $\lambda_1 \geq 1$. If $\lambda_3 = 0$, then $\lambda_1 = k + 1$, and thus $((x^4)^{\lambda_1} :_R xy^3) = ((x^4)^{\lambda_1-1} x^3) \subseteq I^k$. Hence, let $\lambda_3 \geq 1$. Due to $\lambda_1 \geq 1$ and $\lambda_3 \geq 1$, we have $((x^4)^{\lambda_1} (y^5)^{\lambda_3} :_R xy^3) = ((x^4)^{\lambda_1-1} (y^5)^{\lambda_3-1} (x^3y)y) \subseteq I^k$.

Case 2. $\lambda_4 = 0$ and $\lambda_2 = 1$. Let $\lambda_1 = 0$. Hence, $\lambda_3 = k \geq 1$, and so

$$((x^3y)(y^5)^{\lambda_3} :_R xy^3) = (x(xy^3)(y^5)^{\lambda_3-1}) \subseteq I^k.$$

Therefore, let $\lambda_1 \geq 1$. If $\lambda_3 = 0$, then $\lambda_1 = k$, and thus

$$((x^4)^{\lambda_1} (x^3y) :_R xy^3) = ((x^4)^{\lambda_1} x^2) \subseteq I^k.$$

Hence, let $\lambda_3 \geq 1$. Thanks to $\lambda_1 \geq 1$ and $\lambda_3 \geq 1$, one derives that

$$((x^4)^{\lambda_1} (x^3y)(y^5)^{\lambda_3} :_R xy^3) = ((x^4)^{\lambda_1-1} (x^3y)^2 (y^5)^{\lambda_3-1} y) \subseteq I^k.$$

Case 3. $\lambda_4 = 0$ and $\lambda_2 = 2$. If $\lambda_3 \geq 1$, then

$$\begin{aligned} ((x^4)^{\lambda_1} (x^3y)^2 (y^5)^{\lambda_3} :_R xy^3) &= ((x^4)^{\lambda_1} (x^5y^2) (y^5)^{\lambda_3-1} y^2) \\ &= ((x^4)^{\lambda_1} (x^3y)(y^5)^{\lambda_3-1} (xy^3)x) \subseteq I^k. \end{aligned}$$

Therefore, let $\lambda_3 = 0$. This gives that $\lambda_1 = k - 1$. Hence, we gain

$$((x^4)^{\lambda_1} (x^3y)^2 :_R xy^3) = ((x^4)^{\lambda_1} x^5) = ((x^4)^{\lambda_1+1} x) \subseteq I^k.$$

This terminates our argument.

3.3.2 Strong persistence property of the cover ideals

In this subsection, we focus on the strong persistence property of the cover ideals of the union of simple finite graphs. For this purpose, we consider the following lemma which examines the relation between associated primes of powers of the cover ideal of the union of a finite simple connected graph and a tree with the associated primes of powers of the cover ideals of each of them, under the condition that they have only one common vertex.

It should be noted that throughout this subsection, all trees are non-trivial, that is, they have at least two vertices.

A repeated application of Theorem 3.1.6 yields the following lemma:

Lemma 3.3.15. *Let $G = (V(G), E(G))$ be a finite simple connected graph and T be a tree such that $|V(G) \cap V(T)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(T)$ and $E(L) := E(G) \cup E(T)$. Then*

$$\text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(T)^s),$$

for all s , where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(T)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field K .

The next lemma explores the relation between associated primes of powers of the cover ideal of the union of a finite simple connected graph and a tree with the associated primes of powers of the cover ideals of each of them, under the condition that they have only a path in common. In fact, a repeated application of Lemma 3.3.15 gives the following lemma:

Lemma 3.3.16. *Let $G = (V(G), E(G))$ be a finite simple connected graph, $T_1, \dots, T_r$ be some trees with $V(G) \cap V(T_i) = \{v_i\}$ for each $i = 1, \dots, r$, $V(T_i) \cap V(T_j) = \emptyset$ for $i \neq j$, and $P = (V(P), E(P))$ be a path of G with*

$$V(P) = \{v_1, \dots, v_r, v_{r+1}, \dots, v_m\} \subseteq V(G),$$

and

$$E(P) = \{\{v_i, v_{i+1}\} : \text{for } i = 1, \dots, m-1\} \subseteq E(G).$$

Let $T = (V(T), E(T))$ be the tree with

$$V(T) = \left(\bigcup_{i=1}^r V(T_i) \right) \cup V(P) \text{ and } E(T) = \left(\bigcup_{i=1}^r E(T_i) \right) \cup E(P).$$

Also, let $L = (V(L), E(L))$ be the finite simple graph such that

$$V(L) := V(G) \cup V(T) \text{ and } E(L) := E(G) \cup E(T).$$

Then

$$\text{Ass}_R(R/J(L)^s) = \text{Ass}_{R'}(R'/J(G)^s) \cup \text{Ass}_{R''}(R''/J(T)^s),$$

for all s , where $R' = K[x_\alpha : \alpha \in V(G)]$, $R'' = K[x_\alpha : \alpha \in V(T)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field K .

To establish Theorem 3.3.19, one needs to know the following auxiliary propositions. Indeed, by considering the fact that localizing at a minimal prime ideal inverts everything outside of it, one can deduce the following proposition:

Proposition 3.3.17. *Let I be an ideal in a commutative Noetherian ring R . Also, let $I = Q_1 \cap \cdots \cap Q_t \cap Q_{t+1} \cap \cdots \cap Q_r$ be a minimal primary decomposition of I with $\mathfrak{p}_i = \sqrt{Q_i}$ for $i = 1, \dots, r$, and $\text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Then $I_{\mathfrak{p}_i} = (Q_i)_{\mathfrak{p}_i}$ for $i = 1, \dots, t$.*

Proposition 3.3.18. *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ and $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$. Then, the following statements hold.*

- (i) *If $x_i|u_t$ for some i with $1 \leq i \leq n$, and for some t with $1 \leq t \leq m$, then there exists j with $1 \leq j \leq s$, such that $x_i \in \mathfrak{p}_j$.*
- (ii) *If $x_i \in \mathfrak{p}_j$ for some i with $1 \leq i \leq n$, and for some j with $1 \leq j \leq s$, then there exists t with $1 \leq t \leq m$, such that $x_i|u_t$.*

In particular, $\bigcup_{j=1}^s \text{supp}(\mathfrak{p}_j) = \bigcup_{t=1}^m \text{supp}(u_t)$.

Proof. (i) Suppose that $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of I such that $\sqrt{Q_z} = \mathfrak{p}_z$ for all $z = 1, \dots, s$. Also, let $x_i|u_t$ for some $1 \leq i \leq n$ and $1 \leq t \leq m$. Since $u_t \in \mathcal{G}(I)$, one has $u_t \in Q_z$ for all $z = 1, \dots, s$, and so $u_t \in \mathfrak{p}_z$ for all $z = 1, \dots, s$. It follows also from $x_i|u_t$ that there exists some monomial v in R such that $u_t = x_i v$. Suppose, on the contrary, that $x_i \notin \mathfrak{p}_z$ for all $z = 1, \dots, s$. Then one can conclude that $v \in Q_z$ for all $z = 1, \dots, s$ because $x_i v \in Q_z$, $x_i \notin \mathfrak{p}_z$, and Q_z is primary. Therefore, $v \in \bigcap_{z=1}^s Q_z$, that is, $v \in I$. This contradicts the minimality of u_t . We thus have there exists some $1 \leq j \leq s$ such that $x_i \in \mathfrak{p}_j$.

(ii) Let $x_i \in \mathfrak{p}_j$ for some $1 \leq i \leq n$ and $1 \leq j \leq s$. Since $\mathfrak{p}_j \in \text{Ass}_R(R/I)$, there exists some monomial v in R such that $\mathfrak{p}_j = (I :_R v)$. In addition, the assumption $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ yields that $I = \sum_{r=1}^m u_r R$. By virtue of $\mathfrak{p}_j = (I :_R v) = \sum_{r=1}^m (u_r R :_R v)$ and $x_i \in \mathfrak{p}_j$, one can conclude that there exists some $1 \leq t \leq m$ such that $x_i \in (u_t R :_R v)$. We thus have $u_t h = x_i v$ for some monomial h in R . If $x_i|h$, then $v \in I$, which is a contradiction. Therefore, one can derive that $x_i|u_t$, as claimed.

The last assertion is an immediate consequence from parts (i) and (ii). $\square$

In the next theorem, we turn our attention to studying the strong persistence property of the cover ideal of the union of two finite simple connected graphs.

Theorem 3.3.19. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $J(G)$ and $J(H)$ have the strong persistence property. Also, let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$, $E(L) := E(G) \cup E(H)$. Assume that*

$$\text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s),$$

for all s , where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field K . Then, under each of the following cases, $J(L)$ has the strong persistence property.

$$(i) \ V(G) \cap V(H) = \{v\}.$$

$$(ii) \ V(G) \cap V(H) = \{v, w\} \text{ and } E(G) \cap E(H) = \{\{v, w\}\}.$$

$$(iii) \ V(G) \cap V(H) = \{v, w, z\} \text{ and } E(G) \cap E(H) = \{\{v, w\}, \{w, z\}\}.$$

Proof. To simplify our notation, set $I_1 := J(G)$, $I_2 := J(H)$, and $I := J(L)$. Note that $I = I_1 \cap I_2$. We require to show that $(I^{k+1} :_R I) = I^k$ for all $k \geq 1$. To achieve this, fix $k \geq 1$. In view of [89, Exercise 6.4], it is sufficient to prove that $(I^{k+1} :_{R_{\mathfrak{p}}} I_{\mathfrak{p}}) = I_{\mathfrak{p}}^k$ for every $\mathfrak{p} \in \text{Ass}_R(R/I^k)$. For this purpose, pick an arbitrary element $\mathfrak{p} \in \text{Ass}_R(R/I^k)$. We therefore can consider the following two cases:

Case (1). $\mathfrak{p} \in \text{Min}(I^k)$. Because $\text{Min}(I^k) = \text{Min}(I)$, one has $\mathfrak{p} \in \text{Min}(I)$. Now, Proposition 3.3.17 yields that $I_{\mathfrak{p}} = \mathfrak{p}_{\mathfrak{p}}$, and we thus have $I_{\mathfrak{p}}^k = \mathfrak{p}_{\mathfrak{p}}^k$ and $I_{\mathfrak{p}}^{k+1} = \mathfrak{p}_{\mathfrak{p}}^{k+1}$. On the other hand, Corollary 3.3.3 implies that $(\mathfrak{p}^{k+1} :_R \mathfrak{p}) = \mathfrak{p}^k$. Hence, one derives that $(\mathfrak{p}_{\mathfrak{p}}^{k+1} :_{R_{\mathfrak{p}}} \mathfrak{p}_{\mathfrak{p}}) = \mathfrak{p}_{\mathfrak{p}}^k$, and so $(I_{\mathfrak{p}}^{k+1} :_{R_{\mathfrak{p}}} I_{\mathfrak{p}}) = I_{\mathfrak{p}}^k$, as required.

Case (2). $\mathfrak{p} \in \text{Ass}_R(R/I^k) \setminus \text{Min}(I^k)$. By considering the assumption, one can deduce that $\mathfrak{p} \in \text{Ass}_{R_1}(R_1/I_1^k)$ or $\mathfrak{p} \in \text{Ass}_{R_2}(R_2/I_2^k)$. In view of the fact that $\text{Min}(I^k) = \text{Min}(I) = \text{Min}(I_1) \cup \text{Min}(I_2)$, we get

$$\mathfrak{p} \in \text{Ass}_R(R/I_1^k) \setminus \text{Min}(I_1) \quad \text{or} \quad \mathfrak{p} \in \text{Ass}_R(R/I_2^k) \setminus \text{Min}(I_2).$$

We only demonstrate the case $\mathfrak{p} \in \text{Ass}_R(R/I_1^k) \setminus \text{Min}(I_1)$, while another case is proved similarly. To finish the argument, we show that $I_{\mathfrak{p}} = (I_1)_{\mathfrak{p}}$. Thanks to $I_{\mathfrak{p}} \subseteq (I_1)_{\mathfrak{p}}$, it remains to establish $(I_1)_{\mathfrak{p}} \subseteq I_{\mathfrak{p}}$. To do this, take an arbitrary element r/s in $(I_1)_{\mathfrak{p}}$. This gives that $r/s = \alpha/\beta$ for some $\alpha \in I_1$ and $\beta \notin \mathfrak{p}$. We now have to consider the following cases:

Case (i). $V(G) \cap V(H) = \{v\}$. One can easily see that

$$\alpha \prod_{\ell \in V(H) \setminus \{v\}} x_{\ell} \in I_1 \cap I_2.$$

Furthermore, Proposition 3.3.18 implies that $x_\ell \notin \mathfrak{p}$ for any $\ell \in V(H) \setminus \{v\}$, and thus

$$\beta \prod_{\ell \in V(H) \setminus \{v\}} x_\ell \notin \mathfrak{p}.$$

Due to the following equality

$$\frac{\alpha}{\beta} = \frac{\alpha \prod_{\ell \in V(H) \setminus \{v\}} x_\ell}{\beta \prod_{\ell \in V(H) \setminus \{v\}} x_\ell},$$

we obtain $r/s \in I_{\mathfrak{p}}$.

Case (ii). $V(G) \cap V(H) = \{v, w\}$ and $E(G) \cap E(H) = \{\{v, w\}\}$. Since $\alpha \in I_1$, we get $\alpha \in (x_v, x_w)$, and so $x_v | \alpha$ or $x_w | \alpha$. Hence, one can readily deduce that

$$x_v | \alpha \prod_{\ell \in V(H) \setminus \{v, w\}} x_\ell \text{ or } x_w | \alpha \prod_{\ell \in V(H) \setminus \{v, w\}} x_\ell.$$

This leads to $\alpha \prod_{\ell \in V(H) \setminus \{v, w\}} x_\ell \in I_2$, and so $\alpha \prod_{\ell \in V(H) \setminus \{v, w\}} x_\ell \in I_1 \cap I_2$. Based on Proposition 3.3.18, we derive that $x_\ell \notin \mathfrak{p}$ for any $\ell \in V(H) \setminus \{v, w\}$, and hence $\beta \prod_{\ell \in V(H) \setminus \{v, w\}} x_\ell \notin \mathfrak{p}$. In the light of the following equality

$$\frac{\alpha}{\beta} = \frac{\alpha \prod_{\ell \in V(H) \setminus \{v, w\}} x_\ell}{\beta \prod_{\ell \in V(H) \setminus \{v, w\}} x_\ell},$$

one can conclude that $r/s \in I_{\mathfrak{p}}$.

Case (iii). $V(G) \cap V(H) = \{v, w, z\}$ and $E(G) \cap E(H) = \{\{v, w\}, \{w, z\}\}$. On account of $\alpha \in I_1$, one has $\alpha \in (x_v, x_w) \cap (x_w, x_z)$. If $x_w | \alpha$, then we obtain $x_w | \alpha \prod_{\ell \in V(H) \setminus \{v, w, z\}} x_\ell$, and so $\alpha \prod_{\ell \in V(H) \setminus \{v, w, z\}} x_\ell \in I_2$. Let $x_w \nmid \alpha$. Because I_1 is the cover ideal of the graph G , one must have $x_v | \alpha$ and $x_z | \alpha$. This gives rise to $\alpha \prod_{\ell \in V(H) \setminus \{v, w, z\}} x_\ell \in I_2$. It follows also from Proposition 3.3.18 that $x_\ell \notin \mathfrak{p}$ for any $\ell \in V(H) \setminus \{v, w, z\}$, and thus $\beta \prod_{\ell \in V(H) \setminus \{v, w, z\}} x_\ell \notin \mathfrak{p}$. Now, by observing the following equality

$$\frac{\alpha}{\beta} = \frac{\alpha \prod_{\ell \in V(H) \setminus \{v, w, z\}} x_\ell}{\beta \prod_{\ell \in V(H) \setminus \{v, w, z\}} x_\ell},$$

one derives that $r/s \in I_{\mathfrak{p}}$.

As the ideal I_1 has the strong persistence property, one gains $(I_1^{k+1} :_R I_1) = I_1^k$, and hence $((I_1)_{\mathfrak{p}}^{k+1} :_{R_{\mathfrak{p}}} (I_1)_{\mathfrak{p}}) = (I_1)_{\mathfrak{p}}^k$. We thus have $(I_{\mathfrak{p}}^{k+1} :_{R_{\mathfrak{p}}} I_{\mathfrak{p}}) = I_{\mathfrak{p}}^k$. This completes the proof. $\square$

Corollary 3.3.20. *Let $G = (V(G), E(G))$ be a finite simple connected graph such that $J(G)$ has the strong persistence property and T be a tree. Also, let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(T)$ and $E(L) := E(G) \cup E(T)$. Then, under each of the following cases, $J(L)$ has the strong persistence property.*

$$(i) \ V(G) \cap V(T) = \{v\}.$$

$$(ii) \ V(G) \cap V(T) = \{v, w\} \text{ and } E(G) \cap E(T) = \{\{v, w\}\}.$$

$$(iii) \ V(G) \cap V(T) = \{v, w, z\} \text{ and } E(G) \cap E(T) = \{\{v, w\}, \{w, z\}\}.$$

Proof. Since T is a tree, [48, Corollary 2.6] yields that $J(T)$ is a normally torsion-free square-free monomial ideal. It follows now from Theorem 3.2.42 that $J(T)$ has the strong persistence property. Hence, this claim is a direct consequence from Lemmas 3.3.15, 3.3.16, and Theorem 3.3.19. $\square$

The following remark says that, with the notation of Theorem 3.3.19, it is possible $J(G)$, $J(H)$, and $J(L)$ have the strong persistence property, while, for some s , we have

$$\text{Ass}_R(R/J(L)^s) \neq \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s).$$

Remark 3.3.21. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $J(G)$ and $J(H)$ have the strong persistence property, $|V(G) \cap V(H)| > 2$, and $|E(G) \cap E(H)| > 1$. Also, let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then, it is possible that $J(L)$ has the strong persistence property, while, for some s , we have

$$\text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s) \subsetneq \text{Ass}_R(R/J(L)^s),$$

where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ are polynomial rings over a field K . As an example, consider the graph $G = (V(G), E(G))$ with $V(G) = \{1, 2, 3, 4, 5\}$ and

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}\},$$

and also the graph $H = (V(H), E(H))$ with $V(H) = \{1, 4, 5, 6\}$ and

$$E(H) = \{\{1, 5\}, \{4, 5\}, \{1, 6\}, \{5, 6\}, \{4, 6\}\}.$$

Hence, as shown in the figure below, we have $V(L) = \{1, 2, 3, 4, 5, 6\}$, and

$$V(L) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}, \{1, 6\}, \{5, 6\}, \{4, 6\}\}.$$

It is easy to compute that

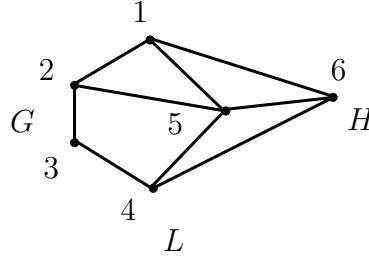
$$\begin{aligned} I_1 := J(G) &= (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1) \cap (x_5, x_2) \\ &= (x_2x_4x_5, x_2x_3x_5, x_1x_3x_5, x_1x_2x_4), \end{aligned}$$

and

$$\begin{aligned} I_2 &:= J(H) = (x_1, x_5) \cap (x_4, x_5) \cap (x_1, x_6) \cap (x_5, x_6) \cap (x_4, x_6) \\ &= (x_5x_6, x_1x_4x_6, x_1x_4x_5), \end{aligned}$$

and

$$\begin{aligned} I &:= J(L) = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_5, x_1) \cap (x_5, x_2) \\ &\quad \cap (x_4, x_5) \cap (x_1, x_6) \cap (x_5, x_6) \cap (x_4, x_6) \\ &= (x_2x_4x_5x_6, x_2x_3x_5x_6, x_1x_3x_5x_6, x_1x_2x_4x_6, x_1x_3x_4x_5, \\ &\quad x_1x_2x_4x_5). \end{aligned}$$



In the first step, we verify that I_1 and I_2 have the strong persistence property. To accomplish this, one can write $I_1 = x_5(x_2x_4, x_2x_3, x_1x_3) + x_1x_2x_4R_1$ and $I_2 = x_1x_4(x_5, x_6) + x_5x_6R_2$. Let P be the path graph with $V(P) = \{1, 2, 3, 4\}$ and $E(P) = \{\{1, 3\}, \{3, 2\}, \{2, 4\}\}$. Since the edge ideal of P is $I(P) = (x_2x_4, x_2x_3, x_1x_3)$, by virtue of [133, Theorem 7.7.14], we gain the monomial ideal (x_2x_4, x_2x_3, x_1x_3) has the strong persistence property. Moreover, it follows from Corollary 3.3.3 that the prime ideal (x_5, x_6) has the strong persistence property. Consequently, Lemma 3.3.4 gives that I_1 and I_2 have the strong persistence property. In the second step, we demonstrate that I has the strong persistence property. For this purpose, set $u_1 := x_1x_3x_4$, $u_2 := x_1x_3x_6$, $u_3 := x_2x_3x_6$, $u_4 := x_2x_4x_6$, $u_5 := x_1x_2x_4$, and $F := (u_1, u_2, u_3, u_4, u_5)R$. This gives rise to $I = x_5F + x_1x_2x_4x_6R$. Our strategy is to show that F has the strong persistence property. For this purpose, let $G = (V(G), E(G))$ be the graph with the vertex set $V(G) = \{1, 2, 3, 4, 6\}$ and the edge set $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 6\}, \{6, 1\}\}$. It is routine to check that G is the odd cycle graph of order 5. By using Macaulay2 [49], we can deduce that F is the cover ideal of G . Also, by virtue of Theorem 3.1.11, F has the strong persistence property. It follows now from Lemma 3.3.4 that I has the strong persistence property. Ultimately, by using Macaulay2 [49], we note that

$$(x_1, x_2, x_3, x_4, x_5, x_6) \in \text{Ass}_R(R/I^3) \setminus (\text{Ass}_{R_1}(R_1/I_1^3) \cup \text{Ass}_{R_2}(R_2/I_2^3)).$$

This completes our discussion.

We continue this argument with the following result, which examines the relation between associated primes of powers of cover ideal of a finite simple connected graph with the associated primes of powers of the cover ideals of each connected subgraph of that graph.

Proposition 3.3.22. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s) \subseteq \text{Ass}_R(R/J(L)^s).$$

Proof. To simplify our notation, put $I_1 := J(G)$, $I_2 := J(H)$, and $I := J(L)$. Take an arbitrary element $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ in $\text{Ass}_{R_1}(R_1/I_1^s) \cup \text{Ass}_{R_2}(R_2/I_2^s)$ with $\{i_1, \dots, i_r\} \subseteq V(L)$. Let $\mathfrak{p} \in \text{Ass}_{R_1}(R_1/I_1^s)$. In view of Proposition 3.3.18, one has $\text{supp}(\mathfrak{p}) \subseteq \bigcup_{u \in \mathcal{G}(I_1)} \text{supp}(u)$. This leads to $\{i_1, \dots, i_r\} \subseteq V(G)$. It follows from [45, Lemma 2.11] that $\mathfrak{p} \in \text{Ass}(K[\mathfrak{p}]/J(G_{\mathfrak{p}})^s)$, where $K[\mathfrak{p}] = K[x_{i_1}, \dots, x_{i_r}]$ and $G_{\mathfrak{p}}$ is the induced subgraph of G on the vertex set $\{i_1, \dots, i_r\} \subseteq V(G)$. Thanks to $G_{\mathfrak{p}} = L_{\mathfrak{p}}$, we get $\mathfrak{p} \in \text{Ass}(K[\mathfrak{p}]/J(L_{\mathfrak{p}})^s)$. Once again, using [45, Lemma 2.11] yields that $\mathfrak{p} \in \text{Ass}_R(R/I^s)$. A similar discussion shows that if $\mathfrak{p} \in \text{Ass}_{R_2}(R_2/I_2^s)$, then $\mathfrak{p} \in \text{Ass}_R(R/I^s)$. Therefore, one can conclude that

$$\text{Ass}_{R_1}(R_1/I_1^s) \cup \text{Ass}_{R_2}(R_2/I_2^s) \subseteq \text{Ass}_R(R/I^s).$$

This finishes the proof. $\square$

The following theorem examines the relation between associated primes of powers of the cover ideal of the union of two finite simple graphs with the associated primes of powers of the cover ideals of each of them, under the condition that they have only one common vertex.

Theorem 3.3.23. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $|V(G) \cap V(H)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s),$$

for all s , where $R_1 = K[x_{\alpha} : \alpha \in V(G)]$, $R_2 = K[x_{\alpha} : \alpha \in V(H)]$, and $R = K[x_{\alpha} : \alpha \in V(L)]$. In particular, if $J(G)$ and $J(H)$ have the strong persistence property, then $J(L)$ has the strong persistence property.

Proof. Fix $s \geq 1$, and let $V(G) \cap V(H) = \{v\}$. If G or H is a tree, then the claim follows rapidly from Lemma 3.3.15 and Theorem 3.3.19. Hence, we can assume that

neither G nor H is a tree. On account of Proposition 3.3.22, we gain the following containment

$$\text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s) \subseteq \text{Ass}_R(R/J(L)^s).$$

In order to complete the proof, it is sufficient for us to verify the reverse inclusion. To achieve this, pick an arbitrary element $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ in $\text{Ass}_R(R/J(L)^s)$. Based on Proposition 3.3.18, we have $\{i_1, \dots, i_r\} \subseteq V(L)$. Moreover, it follows from [45, Lemma 2.11] that $\mathfrak{p} \in \text{Ass}(K[\mathfrak{p}]/J(L_{\mathfrak{p}})^s)$, where $K[\mathfrak{p}] = K[x_{i_1}, \dots, x_{i_r}]$ and $L_{\mathfrak{p}}$ is the induced subgraph of L on the vertex set $\{i_1, \dots, i_r\}$. Hereafter, we assume that $\mathfrak{p}$ is the maximal ideal in the polynomial ring $R = K[\mathfrak{p}]$. To simplify the notation, put $\Gamma := \{x_c : c \in V(G) \setminus \{v\}\}$ and $\Lambda := \{x_c : c \in V(H) \setminus \{v\}\}$. Take an arbitrary element u in $\mathcal{G}(J(L))$. Because of neither G nor H is a tree, one can conclude that neither G nor H is a star graph or an edge, and so $\text{supp}(u) \cap \Gamma \neq \emptyset$ and $\text{supp}(u) \cap \Lambda \neq \emptyset$. Since $J(L) = J(G) \cap J(H)$, this implies that $u = ab/\text{gcd}(a, b)$ for some $a \in \mathcal{G}(J(G))$ and $b \in \mathcal{G}(J(H))$. Hence, one may consider the following cases:

Case 1. $a = f$ and $b = g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Hence, one has $u = fg$.

Case 2. $a = x_v f$ and $b = g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). We thus have $u = x_v fg$.

Case 3. $a = f$ and $b = x_v g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Therefore, one obtains $u = x_v fg$.

Case 4. $a = x_v f$ and $b = x_v g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Due to $\text{gcd}(a, b) = x_v$, one can conclude that $u = x_v fg$.

The argument above gives that if $u \in \mathcal{G}(J(L))$, then $u = Afg$, where $A|x_v$ and f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ) such that $Af \in J(G)$ (respectively, $Ag \in J(H)$). Since $\mathfrak{p} \in \text{Ass}_R(R/J(L)^s)$, we get there exists some monomial h in R such that $\mathfrak{p} = (J(L)^s :_R h)$. Assume that $h = h_1 h_2 x_v^\rho$ with h_1 (respectively, h_2) is a monomial in the variables Γ (respectively, Λ), and ρ is a nonnegative integer. It follows readily from Lemma 2.1.12 that $(J(G)^s :_R h_1 h_2 x_v^\rho) = (J(G)^s :_R h_1 x_v^\rho)$ and $(J(H)^s :_R h_1 h_2 x_v^\rho) = (J(H)^s :_R h_2 x_v^\rho)$. Since $(J(L)^s :_R h) \subseteq (J(G)^s :_R h)$ (respectively, $(J(L)^s :_R h) \subseteq (J(H)^s :_R h)$), one can conclude that $\mathfrak{p} \subseteq (J(G)^s :_R h_1 x_v^\rho)$ (respectively, $\mathfrak{p} \subseteq (J(H)^s :_R h_2 x_v^\rho)$). Our aim is to demonstrate that $h_1 x_v^\rho \notin J(G)$ or $h_2 x_v^\rho \notin J(H)$. Suppose, on the contrary, that $h_1 x_v^\rho \in J(G)$ and $h_2 x_v^\rho \in J(H)$. It follows from $h_1 x_v^\rho \in J(G)^s$ (respectively, $h_2 x_v^\rho \in J(H)^s$) that there exist square-free monomials $M_1, f_1, \dots, f_s$ (respectively, $M_2, g_1, \dots, g_s$) in the variables Γ (respectively, Λ), and $A_1, \dots, A_s$ (respectively, $A'_1, \dots, A'_s$) with $A_i|x_v$ (respectively, $A'_i|x_v$) and $A_i f_i \in J(G)$ (respectively, $A'_i g_i \in$

$J(H)$) for each $i = 1, \dots, s$, such that

$$h_1 x_v^\rho = \left(\prod_{i=1}^s A_i f_i \right) M_1 x_v^\theta \text{ (respectively, } h_2 x_v^\rho = \left(\prod_{i=1}^s A'_i g_i \right) M_2 x_v^\lambda),$$

for some nonnegative integer θ (respectively, λ). We thus have $h_1 = (\prod_{i=1}^s f_i) M_1$, $h_2 = (\prod_{i=1}^s g_i) M_2$, and $(\prod_{i=1}^s A_i) x_v^\theta = x_v^\rho = (\prod_{i=1}^s A'_i) x_v^\lambda$. Accordingly, one has

$$h_1 h_2 x_v^\rho = \left(\prod_{i=1}^s f_i \right) M_1 \left(\prod_{i=1}^s g_i \right) M_2 x_v^\rho = \left(\prod_{i=1}^s f_i g_i \right) M_1 M_2 x_v^\rho.$$

Since $(\prod_{i=1}^s A_i) x_v^\theta = x_v^\rho = (\prod_{i=1}^s A'_i) x_v^\lambda$, we get $h_1 h_2 x_v^\rho \in J(L)^s$, and so $h \in J(L)^s$; this contradicts the fact that $\mathfrak{p} = (J(L)^s :_R h)$. We therefore gain $h_1 x_v^\rho \notin J(G)$ or $h_2 x_v^\rho \notin J(H)$. As $\mathfrak{p}$ is the maximal ideal, one has $\mathfrak{p} \in \text{Ass}_{R_1}(R_1/J(G)^s)$ or $\mathfrak{p} \in \text{Ass}_{R_2}(R_2/J(H)^s)$, and so we have the following equality

$$(3.3.1) \quad \text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s),$$

as required. The last claim is an immediate consequence of Theorem 3.3.19 and (3.3.1). $\square$

To demonstrate Theorem 3.3.25, one needs to employ the following lemma.

Lemma 3.3.24. *Let $G = (V(G), E(G))$ be a finite simple connected graph and H be a triangle graph such that $|V(G) \cap V(H)| = 2$ and $|E(G) \cap E(H)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s),$$

for all s , where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field K . In particular, if $J(G)$ has the strong persistence property, then $J(L)$ has the strong persistence property.

Proof. Let $V(H) = \{v, w, z\}$, $V(G) \cap V(H) = \{v, w\}$, and $E(G) \cap E(H) = \{\{v, w\}\}$. Since the claim is true for the case $s = 1$, we only argue for all $s \geq 2$. Fix $s \geq 2$. It follows at once from Proposition 3.3.22 the containment below

$$\text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s) \subseteq \text{Ass}_R(R/J(L)^s).$$

For completing the proof, it is enough for us to prove the reverse inclusion. To accomplish this, select an arbitrary element $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ in $\text{Ass}_R(R/J(L)^s)$. One can conclude from Proposition 3.3.18 that $\{i_1, \dots, i_r\} \subseteq V(L)$. If $z \notin \{i_1, \dots, i_r\}$, then $\{i_1, \dots, i_r\} \subseteq V(G)$, and [45, Lemma 2.11] implies that $\mathfrak{p} \in \text{Ass}_{R_1}(R_1/J(G)^s)$. We thus let $z \in \{i_1, \dots, i_r\}$. In view of [17, Lemma 2.4], $L_{\mathfrak{p}}$ is a connected graph,

where L_p denotes the induced graph on $\{i_1, \dots, i_r\}$. This leads to the following cases:

Case 1. $v \in \{i_1, \dots, i_r\}$ but $w \notin \{i_1, \dots, i_r\}$ (or $w \in \{i_1, \dots, i_r\}$ but $v \notin \{i_1, \dots, i_r\}$). We only consider the case $v \in \{i_1, \dots, i_r\}$ but $w \notin \{i_1, \dots, i_r\}$, while the other case is proved similarly. Because $\mathbf{p} \in \text{Ass}_R(R/J(L)^s)$, [45, Lemma 2.11] concludes that $\mathbf{p} \in \text{Ass}_R(R/J(L_p)^s)$. Since $w \notin \{i_1, \dots, i_r\}$, one derives that $V(L_p) = V(G_p) \cup \{v, z\}$ and $E(L_p) = E(G_p) \cup \{\{v, z\}\}$. By Theorem 3.1.6, we have $\text{Ass}(J(L_p)^s) = \text{Ass}(J(G_p)^s) \cup \{(x_v, x_z)\}$. If $\mathbf{p} = (x_v, x_z)$, then $\mathbf{p} \in \text{Min}(J(H))$, and hence $\mathbf{p} \in \text{Ass}_{R_2}(R_2/J(H)^s)$. If $\mathbf{p} \in \text{Ass}(J(G_p)^s)$, then [45, Lemma 2.11] yields that $\mathbf{p} \in \text{Ass}_{R_1}(R_1/J(G)^s)$, and so $\mathbf{p} \in \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s)$.

Case 2. $v, w \in \{i_1, \dots, i_r\}$. Without loss of generality, one may assume that $i_1 = v$, $i_2 = w$, and $i_3 = z$. Based on [45, Corollary 4.5], the associated primes of $J(L)^s$ will correspond to critical chromatic subgraphs of size $s+1$ in the s -th expansion of L . This means that one can take the induced subgraph on the vertex set $\{i_1, \dots, i_r\}$, and then form the s -th expansion on this induced subgraph, and within this new graph find a critical $(s+1)$ -chromatic graph. Thanks to z is only connected to v and w in the graph L , and since this induced subgraph is critical, if we remove the vertex z , we can color the resulting graph with s colors. Also, by virtue of [18, Theorem 14.6], the vertex z has to be adjacent to at least s vertices. But the only things z is adjacent to are the shadows of z and the shadows of v and w , and hence one has a clique among these vertices. We thus gain that z and its neighbors will form a clique of size $s+1$. On account of a clique is a critical graph, this gives that we do not need any element of $\{i_4, \dots, i_r\}$ or their shadows when making the critical $(s+1)$ -chromatic graph. This implies that $\mathbf{p} = (x_v, x_w, x_z)$. Due to Proposition 3.1.15, one can conclude that $(x_v, x_w, x_z) \in \text{Ass}_{R_2}(R_2/J(H)^s)$.

Accordingly, we obtain the following equality

$$(3.3.2) \quad \text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s).$$

Since the cover ideal of any triangle has the strong persistence property, the last assertion is a straightforward consequence of Theorem 3.3.19 and (3.3.2). $\square$

The next theorem probes the relation between associated primes of powers of the cover ideal of the union of two finite simple connected graphs with the associated primes of powers of the cover ideals of each of them, under the condition that they have only one edge in common.

Theorem 3.3.25. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs such that $|V(G) \cap V(H)| = 2$ and $|E(G) \cap E(H)| = 1$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then*

$$\text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s),$$

for all s , where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, and $R = K[x_\alpha : \alpha \in V(L)]$. In particular, if $J(G)$ and $J(H)$ have the strong persistence property, then $J(L)$ has the strong persistence property.

Proof. Fix $s \geq 1$. Let $V(G) \cap V(H) = \{v, w\}$ and $E(G) \cap E(H) = \{\{v, w\}\}$. If G or H is a tree or triangle, then the assertion is true by virtue of Lemma 3.3.16 and Theorem 3.3.24. We therefore can assume that neither G nor H is a tree or triangle. As a direct consequence of Proposition 3.3.22, we have the following containment

$$\text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s) \subseteq \text{Ass}_R(R/J(L)^s).$$

To finish the argument, one has to establish the reverse inclusion. To do this, choose an arbitrary element $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ in $\text{Ass}_R(R/J(L)^s)$. By Proposition 3.3.18, one has $\{i_1, \dots, i_r\} \subseteq V(L)$. On account of [45, Lemma 2.11], one can derive that $\mathfrak{p} \in \text{Ass}(K[\mathfrak{p}]/J(L_{\mathfrak{p}})^s)$, where $K[\mathfrak{p}] = K[x_{i_1}, \dots, x_{i_r}]$ and $L_{\mathfrak{p}}$ is the induced subgraph of L on the vertex set $\{i_1, \dots, i_r\}$. We thus assume that $\mathfrak{p}$ is the maximal ideal in the polynomial ring $R = K[\mathfrak{p}]$. To simplify the notation, set

$$\Gamma := \{x_c : c \in V(G) \setminus \{v, w\}\} \text{ and } \Lambda := \{x_c : c \in V(H) \setminus \{v, w\}\}.$$

Pick an arbitrary element u in $\mathcal{G}(J(L))$. Due to neither G nor H is a tree or triangle, we can deduce that $\text{supp}(u) \cap \Gamma \neq \emptyset$ and $\text{supp}(u) \cap \Lambda \neq \emptyset$. It follows also from $J(L) = J(G) \cap J(H)$ that $u = ab/\text{gcd}(a, b)$ for some $a \in \mathcal{G}(J(G))$ and $b \in \mathcal{G}(J(H))$. Consequently, one of the following cases occurs:

Case 1. $a = x_v f$ and $b = x_v g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Because $\text{gcd}(a, b) = x_v$, we have $u = x_v f g$.

Case 2. $a = x_v f$ and $b = x_w g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Since $\text{gcd}(f, g) = 1$, this implies that $u = x_v x_w f g$.

Case 3. $a = x_v f$ and $b = x_v x_w g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). By $\text{gcd}(a, b) = x_v$, this yields that $u = x_v x_w f g$.

Case 4. $a = x_w f$ and $b = x_v g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Similarly to Case 2, one has $u = x_v x_w f g$.

Case 5. $a = x_w f$ and $b = x_w g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). It follows from $\text{gcd}(a, b) = x_w$ that $u = x_w f g$.

Case 6. $a = x_w f$ and $b = x_v x_w g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Since $\text{gcd}(a, b) = x_w$, we thus have $u = x_v x_w f g$.

Case 7. $a = x_v x_w f$ and $b = x_v g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). Similarly to Case 6, one can conclude that $u = x_v x_w f g$.

Case 8. $a = x_v x_w f$ and $b = x_w g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). On account of $\gcd(a, b) = x_w$, this gives that $u = x_v x_w f g$.

Case 9. $a = x_v x_w f$ and $b = x_v x_w g$, where f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ). According to $\gcd(a, b) = x_v x_w$, we get $u = x_v x_w f g$.

It follows from the discussion above that if $u \in \mathcal{G}(J(L))$, then $u = A f g$, where $A \neq 1$, $A | x_v x_w$, and f (respectively, g) is a square-free monomial in the variables Γ (respectively, Λ) such that $A f \in J(G)$ (respectively, $A g \in J(H)$). Since $\mathfrak{p} \in \text{Ass}_R(R/J(L)^s)$, we obtain $\mathfrak{p} = (J(L)^s :_R h)$ for some monomial h in R . Suppose that $h = h_1 h_2 x_v^\rho x_w^\delta$ with h_1 (respectively, h_2) is a monomial in the variables Γ (respectively, Λ), and ρ (respectively, δ) is a nonnegative integer. One can conclude from Lemma 2.1.12 that $(J(G)^s :_R h_1 h_2 x_v^\rho x_w^\delta) = (J(G)^s :_R h_1 x_v^\rho x_w^\delta)$ and also $(J(H)^s :_R h_1 h_2 x_v^\rho x_w^\delta) = (J(H)^s :_R h_2 x_v^\rho x_w^\delta)$. Thanks to

$$(J(L)^s :_R h) \subseteq (J(G)^s :_R h) \text{ (respectively, } (J(L)^s :_R h) \subseteq (J(H)^s :_R h)),$$

this implies that $\mathfrak{p} \subseteq (J(G)^s :_R h_1 x_v^\rho x_w^\delta)$ (respectively, $\mathfrak{p} \subseteq (J(H)^s :_R h_2 x_v^\rho x_w^\delta)$). Our purpose is to verify $h_1 x_v^\rho x_w^\delta \notin J(G)$ or $h_2 x_v^\rho x_w^\delta \notin J(H)$. Assume to the contrary that $h_1 x_v^\rho x_w^\delta \in J(G)$ and $h_2 x_v^\rho x_w^\delta \in J(H)$. Because of $h_1 x_v^\rho x_w^\delta \in J(G)^s$ (respectively, $h_2 x_v^\rho x_w^\delta \in J(H)^s$), this gives that there exist square-free monomials $M_1, f_1, \dots, f_s$ (respectively, $M_2, g_1, \dots, g_s$) in the variables Γ (respectively, Λ), and $A_1, \dots, A_s$ (respectively, $A'_1, \dots, A'_s$) with $A_i \neq 1$ (respectively, $A'_i \neq 1$), $A_i | x_v x_w$ (respectively, $A'_i | x_v x_w$), and $A_i f_i \in J(G)$ (respectively, $A'_i g_i \in J(H)$) for each $i = 1, \dots, s$, such that we have

$$h_1 x_v^\rho x_w^\delta = \left(\prod_{i=1}^s A_i f_i \right) M_1 x_v^{\theta_1} x_w^{\theta_2} \text{ (respectively, } h_2 x_v^\rho x_w^\delta = \left(\prod_{i=1}^s A'_i g_i \right) M_2 x_v^{\lambda_1} x_w^{\lambda_2}),$$

for some nonnegative integers θ_1 and θ_2 (respectively, λ_1 and λ_2). We thus get

$$\left(\prod_{i=1}^s A_i \right) x_v^{\theta_1} x_w^{\theta_2} = x_v^\rho x_w^\delta = \left(\prod_{i=1}^s A'_i \right) x_v^{\lambda_1} x_w^{\lambda_2},$$

$h_1 = (\prod_{i=1}^s f_i) M_1$, and $h_2 = (\prod_{i=1}^s g_i) M_2$. This gives rise to the following equalities

$$h_1 h_2 x_v^\rho x_w^\delta = \left(\prod_{i=1}^s f_i \right) M_1 \left(\prod_{i=1}^s g_i \right) M_2 x_v^\rho x_w^\delta = \left(\prod_{i=1}^s f_i g_i \right) M_1 M_2 x_v^\rho x_w^\delta.$$

Due to $(\prod_{i=1}^s A_i) x_v^{\theta_1} x_w^{\theta_2} = x_v^\rho x_w^\delta = (\prod_{i=1}^s A'_i) x_v^{\lambda_1} x_w^{\lambda_2}$, we obtain $h_1 h_2 x_v^\rho x_w^\delta \in J(L)^s$. Hence, $h \in J(L)^s$, which contradicts the fact that $\mathfrak{p} = (J(L)^s :_R h)$. Accordingly,

one can derive $h_1 x_v^\rho x_w^\delta \notin J(G)$ or $h_2 x_v^\rho x_w^\delta \notin J(H)$. On account of $\mathfrak{p}$ is the maximal ideal, this implies that $\mathfrak{p} \in \text{Ass}_{R_1}(R_1/J(G)^s)$ or $\mathfrak{p} \in \text{Ass}_{R_2}(R_2/J(H)^s)$, and so we get the following equality

$$(3.3.3) \quad \text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s).$$

We can now combine together Theorem 3.3.19 and (3.3.3) to obtain the last claim.

□

As an application of Theorem 3.3.25, we express the following proposition.

Proposition 3.3.26. (i) *The cover ideal of every cycle graph has the strong persistence property.*

(ii) *The cover ideal of every cycle graph with one chord has the strong persistence property.*

Proof. (i) Let C_n denote the cycle graph of order n . If n is an even number, then C_n is a bipartite graph, and by virtue of [48, Corollary 2.6], $J(C_n)$ is a normally torsion-free square-free monomial ideal, and Theorem 3.2.42 implies that $J(C_n)$ has the strong persistence property. If n is an odd number, then the claim follows immediately from Theorem 3.1.11.

(ii) The assertion can be deduced from Theorem 3.3.25 and part (i). □

The next question examines the relation between associated primes of powers of the cover ideal of the union of two finite simple connected graphs with the associated primes of powers of the cover ideals of each of them, in a general case.

Question 3.3.27. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple connected graphs. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. Then can we deduce that, for all s , one of the following statements holds?*

$$(i) \quad \text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s).$$

$$(ii) \quad \text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(H)^s) \cup \{\mathfrak{m}\},$$

where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(H)]$, $R = K[x_\alpha : \alpha \in V(L)]$ over a field K , and $\mathfrak{m} = (x_\alpha : \alpha \in V(L))$ is the unique homogeneous maximal ideal of R .

The answer is negative. We provide a counterexample. Let $G := \mathcal{K}_4$ be the complete graph with $V(G) = \{1, 2, 3, 4\}$ and

$$E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\},$$

and $H := S_4$ be the star graph with $V(H) = \{1, 2, 3, 4, 5\}$ and

$$E(H) = \{\{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}\}.$$

Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(H)$ and $E(L) := E(G) \cup E(H)$. By using *Macaulay2* [49], we obtain

$$(x_2, x_3, x_4, x_5) \in \text{Ass}(J(L)^3) \setminus (\text{Ass}(J(G)^3) \cup \text{Ass}(J(H)^3)),$$

and

$$(x_1, x_2, x_4, x_5) \in \text{Ass}(J(L)^4) \setminus (\text{Ass}(J(G)^4) \cup \text{Ass}(J(H)^4) \cup \mathfrak{m}),$$

where $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5)$. In view of [48, Corollary 2.6] and Theorem 3.2.42, we deduce that the cover ideal of every tree has the strong persistence property, and so $J(H)$ has the strong persistence property. On the other hand, note that the graph L is exactly the complete graph $\mathcal{K}_5$. It follows also from Corollary 7.1.6 that the cover ideal of every complete graph is normal, and by virtue of Theorem 3.2.34, one can conclude that $J(G)$ and $J(L)$ have the strong persistence property.

In the subsequent question, our aim is to investigate the relation between associated primes of powers of the cover ideal of the union of a finite simple connected graph and a complete graph with the associated primes of powers of the cover ideals of each of them, under the condition that they are common in a path with length 2.

Question 3.3.28. *Let $G = (V(G), E(G))$ be a finite simple connected graph and $\mathcal{K}_n$ the complete graph of order n such that $|V(G) \cap V(\mathcal{K}_n)| = 3$ and $|E(G) \cap E(\mathcal{K}_n)| = 2$. Let $L = (V(L), E(L))$ be the finite simple graph such that $V(L) := V(G) \cup V(\mathcal{K}_n)$ and $E(L) := E(G) \cup E(\mathcal{K}_n)$. Then can we conclude that*

$$\text{Ass}_R(R/J(L)^s) = \text{Ass}_{R_1}(R_1/J(G)^s) \cup \text{Ass}_{R_2}(R_2/J(\mathcal{K}_n)^s),$$

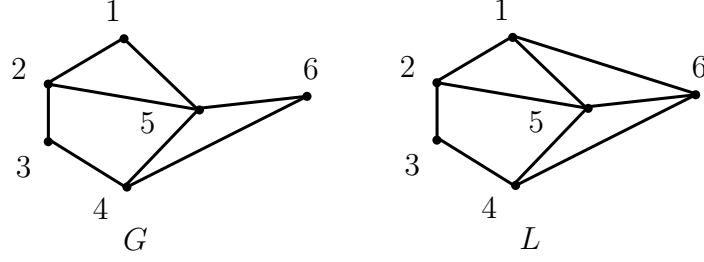
for all s , where $R_1 = K[x_\alpha : \alpha \in V(G)]$, $R_2 = K[x_\alpha : \alpha \in V(\mathcal{K}_n)]$, and $R = K[x_\alpha : \alpha \in V(L)]$ over a field K ? In particular, if $J(G)$ has the strong persistence property, then does $J(L)$ have the strong persistence property?

By giving a counterexample, we show that the answer is negative. To do this, consider the graph $G = (V(G), E(G))$, the left graph in the figure below, with $V(G) = \{1, 2, 3, 4, 5, 6\}$ and

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}, \{4, 6\}, \{5, 6\}\},$$

and also the graph $L = G \cup \mathcal{K}_3$, the right graph in the figure below, with $V(\mathcal{K}_3) = \{1, 6, 5\}$, $E(\mathcal{K}_3) = \{\{5, 6\}, \{1, 6\}, \{1, 5\}\}$, $V(L) = \{1, 2, 3, 4, 5, 6\}$, and

$$V(L) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{5, 2\}, \{1, 6\}, \{5, 6\}, \{4, 6\}\}.$$



It is routine to check that

$$\begin{aligned}
 J(G) &= (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1) \cap (x_5, x_2) \\
 &\quad \cap (x_4, x_6) \cap (x_5, x_6) \\
 &= (x_2x_4x_5, x_2x_3x_5x_6, x_1x_3x_5x_6, x_1x_2x_4x_6, x_1x_3x_4x_5),
 \end{aligned}$$

and

$$\begin{aligned}
 J(L) &= (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1) \cap (x_5, x_2) \\
 &\quad \cap (x_1, x_6) \cap (x_5, x_6) \cap (x_4, x_6) \\
 &= (x_2x_4x_5x_6, x_2x_3x_5x_6, x_1x_3x_5x_6, x_1x_2x_4x_6, x_1x_3x_4x_5, \\
 &\quad x_1x_2x_4x_5).
 \end{aligned}$$

Using *Macaulay2* [49] yields that

$$(x_1, x_2, x_3, x_4, x_5, x_6) \in \text{Ass}(J(L)^3) \setminus (\text{Ass}(J(G)^3) \cup \text{Ass}(J(\mathcal{K}_3)^3)).$$

On the other hand, since the graph G is the union of a cycle graph with one chord and a triangle, Proposition 3.3.26(ii) and Lemma 3.3.24 imply that $J(G)$ has the strong persistence property. Furthermore, one can write

$$J(L) = x_5(x_2x_4x_6, x_2x_3x_6, x_1x_3x_6, x_1x_3x_4, x_1x_2x_4) + (x_1x_2x_4x_6).$$

Because $F := (x_2x_4x_6, x_2x_3x_6, x_1x_3x_6, x_1x_3x_4, x_1x_2x_4)$ is the cover ideal of the odd cycle H with $V(H) = \{1, 2, 3, 4, 6\}$ and $E(H) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 6\}, \{6, 1\}\}$, Proposition 3.3.26(i) gives that F has the strong persistence property, and by virtue of $x_1x_2x_4x_6 \in F$, it follows from Lemma 3.1.1 that $J(L) = x_5F + (x_1x_2x_4x_6)$ has the strong persistence property. This finishes our discussion.

Chapter 4

On the symbolic strong persistence property of monomial ideals

4.1 Some results on the symbolic strong persistence property

In this section, our aim is to prove that any square-free monomial ideal satisfies the symbolic strong persistence property. To achieve this, we start with the definition of symbolic powers of an ideal. It should be noted that the results of this section can be found in [105].

Definition 4.1.1. ([133, Definition 4.3.22]) *Let I be an ideal of a ring R and $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ the minimal primes of I . Given an integer $n \geq 1$, the n -th symbolic power of I is defined to be the ideal*

$$I^{(n)} = \mathfrak{q}_1 \cap \dots \cap \mathfrak{q}_r,$$

where $\mathfrak{q}_i$ is the primary component of I^n corresponding to $\mathfrak{p}_i$.

Remark 4.1.2. *In much literature on symbolic powers a different definition is used based on all the primary decomposition, not just the minimal primes, that is,*

$$I^{(n)} = \bigcap_{\mathfrak{p} \in \text{Min}(I)} (I^n R_{\mathfrak{p}} \cap R) \quad \text{and} \quad I^{(n)} = \bigcap_{\mathfrak{p} \in \text{Ass}_R(R/I)} (I^n R_{\mathfrak{p}} \cap R).$$

See for example page 1 of [32] for more information. If all associated primes of I are minimal (as is the case with square-free monomial ideals), then the two definitions

of $I^{(n)}$ are the same. Otherwise, they are different. In general, both definitions have advantages. When we take $\mathfrak{p}$ ranging over $\text{Ass}_R(R/I)$, we get $I^{(1)} = I$, while taking $\mathfrak{p}$ ranging over $\text{Min}(I)$ means that $I^{(n)}$ coincides with the intersection of the primary components of I^n corresponding to its minimal primes.

Here, we list some properties of symbolic powers in the following remark.

Remark 4.1.3. *Let I be a radical ideal in a Noetherian ring R . Then the following basic properties hold:*

- (1) $I^{(1)} = I$;
- (2) For all $n \geq 1$, $I^n \subseteq I^{(n)}$;
- (3) $I^a \subseteq I^{(b)}$ if and only if $a \geq b$;
- (4) If $a \geq b$, then $I^{(a)} \subseteq I^{(b)}$;
- (5) For all $a, b \geq 1$, $I^{(a)}I^{(b)} \subseteq I^{(a+b)}$;
- (6) $I^n = I^{(n)}$ if and only if I^n has no embedded primes.

Now, we recollect the definition of symbolic strong persistence property of ideals.

Definition 4.1.4. ([118, Definition 13]) *Let I be an ideal in a commutative Noetherian ring R . Then, we say that I has the symbolic strong persistence property if $(I^{(i+1)} :_R I^{(1)}) = I^{(i)}$ for each i .*

The proposition below investigates the symbolic strong persistence property of powers of ideals.

Proposition 4.1.5. *Let R be a commutative Noetherian ring and I be an ideal of R such that I has a non-zerodivisor element. Then there exists a positive integer s such that I^s has the symbolic strong persistence property.*

Proof. According to Proposition 3.2.6, one can conclude that there exists a positive integer s such that I^s is a superficial ideal for I^s , that is, I^s has the strong persistence property. Now, the claim follows readily from [118, Theorem 11]. $\square$

The following proposition says that if an ideal has the symbolic strong persistence property, then any power of it has the symbolic strong persistence property as well.

Proposition 4.1.6. *Let I be an ideal in a commutative Noetherian ring R such that I has the symbolic strong persistence property. Then I^s has the symbolic strong persistence property for all positive integers s .*

Proof. Fix $s, k \geq 1$. It suffices to prove that $((I^s)^{(k+1)} :_R (I^s)^{(1)}) = (I^s)^{(k)}$. Let $\text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Set $S := R \setminus \bigcup_{i=1}^r \mathfrak{p}_i$. It follows from [133, Proposition 4.3.23] that $I^{(n)} = S^{-1}I^n \cap R$ for all $n \geq 1$. Since $\text{Min}(I^n) = \text{Min}(I)$ for all $n \geq 1$, this implies the following equalities

$$\begin{aligned} (I^s)^{(n)} &= S^{-1}(I^s)^n \cap R \\ &= S^{-1}I^{sn} \cap R \\ &= I^{(sn)}. \end{aligned}$$

Accordingly, one obtains $(I^s)^{(k+1)} = I^{(sk+s)}$, $(I^s)^{(k)} = I^{(sk)}$, and $(I^s)^{(1)} = I^{(s)}$. Therefore, we get $((I^s)^{(k+1)} :_R (I^s)^{(1)}) = (I^{(sk+s)} :_R I^{(s)})$. Because the ideal I has the symbolic strong persistence property, it follows from [118, Proposition 12] that $(I^{(sk+s)} :_R I^{(s)}) = I^{(sk)}$, and so $((I^s)^{(k+1)} :_R (I^s)^{(1)}) = (I^s)^{(k)}$, as claimed. $\square$

Here, we turn our attention to studying the symbolic strong persistence property of monomial ideals. In fact, one may ask the following question:

Question 4.1.7. *Does every monomial ideal satisfy the symbolic strong persistence property?*

The answer is negative. To see this, we come back to Proposition 3.3.13. Indeed, we proved that for the monomial ideal $I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n)$ in the polynomial ring $R = \mathbb{Q}[x, y]$, $n \geq 4$, one has $(I_n^{a+2} : I_n) \supsetneq I_n^{a+1}$ for $0 \leq a \leq n-4$. On the other hand, since $I_n = (x, y^n) \cap (x^{n-1}, y^{n-1}) \cap (x^n, y)$, we can deduce that I_n is a (x, y) -primary monomial ideal. This yields that we have $I_n^{(k)} = I_n^k$ for all $k \geq 1$. Therefore, we get $(I_n^{(a+2)} : I_n^{(1)}) \supsetneq I_n^{(a+1)}$ for $0 \leq a \leq n-4$. This means that I_n does not satisfy the symbolic strong persistence property.

Based on [118, Theorem 11], the strong persistence property implies the symbolic strong persistence property. Moreover, in view of Proposition 3.1.12, one can conclude that the strong persistence property implies the persistence property. Now, one may ask the following questions:

Questions 4.1.8. (i) *Does the persistence property imply the symbolic strong persistence property?*

(ii) *Does normally torsion-freeness imply the symbolic strong persistence property?*

Our answers are negative. To accomplish this, consider the monomial ideal $I_n = (x^n, x^{n-1}y, xy^{n-1}, y^n)$ in the polynomial ring $R = \mathbb{Q}[x, y]$, $n \geq 4$. Since $\text{Ass}_R(R/I_n^k) = \{(x, y)\}$ for all $k \geq 1$, one can conclude that I has the persistence property, and also is normally torsion-free. While, by the argument which has been

mentioned before, we obtain I does not satisfy the symbolic strong persistence property.

In the following, our intent is to show that every square-free monomial ideal satisfies the symbolic strong persistence property. To achieve this, we require Proposition 4.1.9 and Lemma 4.1.10.

Proposition 4.1.9. *Every power of a primary monomial ideal is primary.*

Proof. Assume that Q is a primary monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and fix $t \geq 1$. Let $x_i | u_1 \cdots u_t$, where $1 \leq i \leq n$ and $u_1, \dots, u_t \in \mathcal{G}(Q)$. This yields that $x_i | u_s$ for some $1 \leq s \leq t$. Since Q is a primary monomial ideal, it follows from [133, Proposition 6.1.7] that there exists a positive integer k such that $x_i^k \in \mathcal{G}(Q)$, and hence $x_i^{kt} \in \mathcal{G}(Q^t)$. By setting $\alpha := kt$, one has $x_i^\alpha \in \mathcal{G}(Q^t)$. Once again, [133, Proposition 6.1.7] implies that Q^t is a primary monomial ideal, as required. $\square$

Lemma 4.1.10. *Let $Q_1, \dots, Q_r$ be primary monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that $\sqrt{Q_i} \neq \sqrt{Q_j}$ for $1 \leq i \neq j \leq r$, and $\sqrt{Q_1}, \dots, \sqrt{Q_r}$ are incomparable with respect to inclusion. If Q_i has the strong persistence property for each $i = 1, \dots, r$, then, for all positive integers k ,*

$$\left(\bigcap_{i=1}^r Q_i^{k+1} :_R \bigcap_{i=1}^r Q_i \right) = \bigcap_{i=1}^r Q_i^k.$$

Proof. We give a sketch of the proof. Assume that Q_i has the strong persistence property for each $i = 1, \dots, r$. Fix $k \geq 1$, and pick u in $(\bigcap_{i=1}^r Q_i^{k+1} :_R \bigcap_{i=1}^r Q_i)$. Put $\mathfrak{p}_i := \sqrt{Q_i}$ for each $i = 1, \dots, r$. Fix $1 \leq j \leq r$. One can choose an element such as $v_i \in Q_i \setminus \mathfrak{p}_j$ for $1 \leq i \neq j \leq r$. Let λ be an arbitrary element in Q_j . Therefore, $u\lambda v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \in \bigcap_{i=1}^r Q_i^{k+1} \subseteq Q_j^{k+1}$, and also $v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \notin \mathfrak{p}_j$. Since Q_j is primary, Proposition 4.1.9 gives that Q_j^{k+1} is primary. Hence, $u\lambda \in Q_j^{k+1}$, and so $u \in (Q_j^{k+1} :_R Q_j) = Q_j^k$. Therefore, $(\bigcap_{i=1}^r Q_i^{k+1} :_R \bigcap_{i=1}^r Q_i) \subseteq \bigcap_{i=1}^r Q_i^k$. To prove the reverse inclusion, one should note that $(\bigcap_{i=1}^r Q_i)(\bigcap_{i=1}^r Q_i^k) \subseteq \bigcap_{i=1}^r Q_i^{k+1}$, and so $\bigcap_{i=1}^r Q_i^k \subseteq (\bigcap_{i=1}^r Q_i^{k+1} :_R \bigcap_{i=1}^r Q_i)$. This finishes our argument. $\square$

We are now in a position to express the main result of this subsection.

Theorem 4.1.11. *Every square-free monomial ideal has the symbolic strong persistence property.*

Proof. Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. On account of $\text{Ass}_R(R/I) = \text{Min}(I)$, it follows from Lemma 4.1.10 and Corollary 3.3.3 that $(\bigcap_{i=1}^r \mathfrak{p}_i^{k+1} :_R \bigcap_{i=1}^r \mathfrak{p}_i) =$

$\bigcap_{i=1}^r \mathfrak{p}_i^k$ for all $k \geq 1$. In addition, [60, Proposition 1.4.4] implies that $I^{(k)} = \bigcap_{i=1}^r \mathfrak{p}_i^k$ for all $k \geq 1$. Therefore, we have $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k \geq 1$, that is, I has the symbolic strong persistence property, as desired. $\square$

We can now ask the following question:

Question 4.1.12. *Is there an ideal satisfying the symbolic strong persistence property but does not have the strong persistence property?*

The answer is positive. We give such an ideal. Consider the following square-free monomial ideal $\mathcal{I}$ in the polynomial $R = K[x_1, \dots, x_{12}]$ over a field K ,

$$\begin{aligned} \mathcal{I} = (& x_1x_3x_6x_8x_9x_{10}x_{11}x_{12}, x_2x_4x_5x_7x_9x_{10}x_{11}x_{12}, x_1x_2x_4x_5x_7x_{10}x_{11}x_{12}, \\ & x_2x_3x_5x_6x_8x_9x_{11}x_{12}, x_1x_2x_3x_6x_8x_9x_{11}x_{12}, x_2x_4x_5x_6x_7x_9x_{11}x_{12}, \\ & x_1x_3x_6x_7x_8x_9x_{10}x_{12}, x_2x_3x_5x_7x_8x_9x_{10}x_{12}, x_2x_3x_4x_5x_7x_9x_{10}x_{12}, \\ & x_1x_3x_4x_5x_6x_7x_{10}x_{12}, x_1x_2x_3x_4x_5x_7x_{10}x_{12}, x_1x_3x_4x_6x_8x_9x_{10}x_{11}, \\ & x_1x_2x_4x_5x_7x_8x_{10}x_{11}, x_1x_3x_4x_5x_6x_8x_{10}x_{11}, x_1x_2x_4x_6x_7x_8x_9x_{11}, \\ & x_1x_2x_3x_4x_6x_8x_9x_{11}). \end{aligned}$$

Furthermore, using *Macaulay2* [49] implies that $(\mathcal{I}^4 :_R \mathcal{I}) \neq \mathcal{I}^3$. We thus get $\mathcal{I}$ does not satisfy the strong persistence property, whereas Theorem 4.1.11 shows that $\mathcal{I}$ satisfies the symbolic strong persistence property.

We terminate this argument with the following corollary. In fact, it has been established in Theorem 3.2.42, and we now re-prove it by using Theorem 4.1.11.

Corollary 4.1.13. *Every normally torsion-free square-free monomial ideal has the strong persistence property.*

Proof. Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. In the light of Theorem 4.1.11, one has $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all $k \geq 1$. On the other hand, [60, Theorem 1.4.6] implies that $I^{(k)} = I^k$ for all $k \geq 1$. Consequently, by Theorem 4.1.11, we get $(I^{k+1} :_R I) = I^k$ for all $k \geq 1$, that is, I has the strong persistence property. $\square$

4.2 Symbolic strong persistence property under monomial operations

The aim of this section is to state some methods for constructing new monomial ideals which have the symbolic strong persistence property based on the monomial

ideals which have the symbolic strong persistence property. It should be noted that the results of this section can be found in [84].

In what follows, we first show that a monomial ideal has the symbolic strong persistence property if and only if its weighted ideal has the symbolic strong persistence property. To see this, one requires to recall the definition of weighted ideals, see Definition 3.2.11.

The following proposition and lemma are necessary for us to prove the subsequent theorem.

Proposition 4.2.1. *Let I , J , and L be monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and w a weight over R . Then $(I :_R J) = L$ if and only if $(I_W :_R J_W) = L_W$.*

Proof. This result is a straightforward consequence of the fact that gcd and lcm of two monomials behave well with respect to taking weights. $\square$

Lemma 4.2.2. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and W a weight over R . Then $(I^{(k)})_W = (I_W)^{(k)}$ for all $k \in \mathbb{N}$.*

Proof. It is straightforward, and can be deduced by definitions. $\square$

We are ready to state one of the main results of this section in the following theorem.

Theorem 4.2.3. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and W a weight over R . Then I has the symbolic strong persistence property if and only if I_W has the symbolic strong persistence property.*

Proof. We can combine together Proposition 4.2.1 and Lemma 4.2.2 to obtain the claim. $\square$

In what follows, our goal is to establish that if a monomial ideal has the symbolic strong persistence property, then its contracted ideal has the symbolic strong persistence property as well, see Definition 3.2.19. To accomplish this, we first need to prove some auxiliary results as follows.

Proposition 4.2.4. *Let A be a finite set of irreducible monomial ideals whose radicals are mutually incomparable with respect to inclusion. Then $\bigcap_{Q \in A} Q$ is a minimal primary decomposition.*

Proof. Assume that $A := \{Q_1, \dots, Q_r\}$. Let $\mathfrak{p}_i = \sqrt{Q_i}$ for all i . If there exists j such that $\bigcap_{i \neq j} Q_i \subseteq Q_j$, then $\bigcap_{i \neq j} \mathfrak{p}_i \subseteq \mathfrak{p}_j$, which implies that $\mathfrak{p}_i \subseteq \mathfrak{p}_j$ for some $i \neq j$, a contradiction. $\square$

Lemma 4.2.5. *Let I and J be monomial ideals in $R = K[x_1, \dots, x_n]$ over a field K , and $1 \leq i \leq n$. Then*

(i) *If $I = Q_1 \cap \dots \cap Q_r$ is a minimal primary decomposition of I , then*

$$I/x_i = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j})} Q_j \text{ and } (I/x_i)^{(1)} = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j}), \sqrt{Q_j} \in \text{Min}(I)} Q_j.$$

(ii) $I^{(k)}/x_i = (I/x_i)^{(k)}$ for all $k \in \mathbb{N}$.

Proof. (i) If $I = Q_1 \cap \dots \cap Q_r$ is a minimal primary decomposition of I , then Lemma 3.2.20 implies that $I/x_i = Q_1/x_i \cap \dots \cap Q_r/x_i$. In addition, for each $j = 1, \dots, r$, we have

$$Q_j/x_i = \begin{cases} R & \text{if } x_i \in \mathcal{G}(\sqrt{Q_j}) \\ Q_j & \text{if } x_i \notin \mathcal{G}(\sqrt{Q_j}). \end{cases}$$

Accordingly, we get $I/x_i = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j})} Q_j$. Based on Proposition 4.2.4, one can conclude the minimal primary decomposition of I/x_i . Finally, since $\mathfrak{p} \in \text{Min}(I/x_i)$ if and only if $\mathfrak{p} \in \text{Min}(I)$ and $x_i \notin \mathcal{G}(\mathfrak{p})$, we gain the following equality

$$(I/x_i)^{(1)} = \bigcap_{x_i \notin \mathcal{G}(\sqrt{Q_j}), \sqrt{Q_j} \in \text{Min}(I)} Q_j.$$

(ii) Assume that $I^k = Q_1 \cap \dots \cap Q_r$ is a minimal primary decomposition of I^k , where $\sqrt{Q_j}$ is a minimal prime of I if and only if $j \leq r''$, and $x_i \notin \mathcal{G}(\sqrt{Q_j})$ if and only if $j \leq r' \leq r''$. We thus have $I^{(k)} = Q_1 \cap \dots \cap Q_{r''}$. By Lemma 3.2.20, one can conclude that $(I/x_i)^k = I^k/x_i = Q_1/x_i \cap \dots \cap Q_r/x_i$. Furthermore, part (i) and the fact that $L^{(k)} = (L^{(1)})^{(k)}$ for every monomial ideal L , yield that

$$(I/x_i)^{(k)} = Q_1 \cap \dots \cap Q_{r'} = I^{(k)}/x_i.$$

This terminates the proof. $\square$

We are now in a position to express another main result of this section in Theorem 4.2.6.

Theorem 4.2.6. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and $1 \leq i \leq n$. If I has the symbolic strong persistence property, then I/x_i has the symbolic strong persistence property.*

Proof. Suppose that I has the symbolic strong persistence property. Fix $k \geq 1$. In the light of $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$, it follows from Lemma 3.2.20(iv) that $(I^{(k+1)}/x_i :_{R/x_i} I^{(1)}/x_i) = I^{(k)}/x_i$. By virtue of Lemma 4.2.5(ii), one can deduce that $((I/x_i)^{(k+1)} :_{R/x_i} (I/x_i)^{(1)}) = (I/x_i)^{(k)}$. That is, I/x_i has the symbolic strong persistence property. This completes the proof. $\square$

Theorem 4.2.7. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that I has the symbolic strong persistence property. Then $I(\mathfrak{p})$ has the strong persistence property for all $\mathfrak{p} \in \text{Min}(I)$, and hence has the symbolic strong persistence property.*

Proof. Fix $k \geq 1$ and $\mathfrak{p} \in \text{Min}(I)$. The assumption implies that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$. This yields that $(I^{(k+1)} :_R I^{(1)})(\mathfrak{p}) = I^{(k)}(\mathfrak{p})$. It follows now from Lemma 3.2.15(iv) that $(I^{(k+1)}(\mathfrak{p}) :_{R(\mathfrak{p})} I^{(1)}(\mathfrak{p})) = I^{(k)}(\mathfrak{p})$. Since $\mathfrak{p} \in \text{Min}(I)$, by Lemma 3.2.15(vii), we obtain $I^{(s)}(\mathfrak{p}) = I^s(\mathfrak{p})$ for all s . Thus, we get $(I^{k+1}(\mathfrak{p}) :_{R(\mathfrak{p})} I(\mathfrak{p})) = I^k(\mathfrak{p})$. On account of Lemma 3.2.15(ii), we get $((I(\mathfrak{p}))^{k+1} :_{R(\mathfrak{p})} I(\mathfrak{p})) = (I(\mathfrak{p}))^k$. Therefore, $I(\mathfrak{p})$ has the strong persistence property, and hence has the symbolic strong persistence property, as required. $\square$

To see an application of Theorem 4.2.7, one can consider Question 4.2.10. To do this, we begin with the definition of monomial ideals of clutter type in the following definition.

Definition 4.2.8. *Let I be a non-square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\mathcal{G}(I) = \{u_1, \dots, u_r\}$. We say that I is of clutter type if $\sqrt{u_i} \nmid \sqrt{u_j}$ (or equivalently, $\text{supp}(u_i) \not\subseteq \text{supp}(u_j)$) for each $1 \leq i \neq j \leq r$.*

Example 4.2.9. *Let $I = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1, x_5x_1^2x_2)$ be a monomial ideal in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$ over a field K . Then one can rapidly see that I is of clutter type. Note that I does not satisfy both the persistence property and strong persistence property since $\mathfrak{m} = (x_1, x_2, x_3, x_4, x_5) \in \text{Ass}_R(R/I) \setminus \text{Ass}_R(R/I^2)$ and $(I^2 :_R I) \neq I$.*

Question 4.2.10. *Does every non-square-free monomial ideal of clutter type have the symbolic strong persistence property?*

The answer is negative. We provide a counterexample. To accomplish this, consider the monomial ideal $I = (x_1^4x_3, x_1^3x_2x_4, x_1x_2^3x_5, x_2^4x_6)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$ over a field K . It is easy to check that I is a non-square-free monomial ideal of clutter type. Furthermore, using *Macaulay2* [49] implies that $\mathfrak{p} = (x_1, x_2) \in \text{Min}(I)$. On account of Theorem 4.2.7, one can deduce that $I(\mathfrak{p}) = (x_1^4, x_1^3x_2, x_1x_2^3, x_2^4)$ has the symbolic strong persistence property, whereas the monomial ideal $I(\mathfrak{p})$ does not satisfy the symbolic strong persistence property since $(I(\mathfrak{p}))^{(2)} : I(\mathfrak{p})^{(1)} \neq I(\mathfrak{p})^{(1)}$. This finishes our argument.

Finally, we want to examine the symbolic strong persistence property under expansion operation, see Definition 2.2.9. Indeed, we show that a monomial ideal has the symbolic strong persistence property if and only if its expansion has the symbolic strong persistence property.

Let us provide the other main result of this section in the subsequent theorem.

Theorem 4.2.11. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . Then I has the symbolic strong persistence property if and only if I^* has the symbolic strong persistence property.*

Proof. We first note that $I = J$ if and only if $I^* = J^*$. To prove the forward implication, let I have the symbolic strong persistence property. Fix $k \geq 1$. The assumption yields that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$. In view of [13, Lemma 1.1 and Corollary 1.4], one has the following equalities

$$\begin{aligned} ((I^*)^{(k+1)} :_{R^*} (I^*)^{(1)}) &= ((I^{(k+1)})^* :_{R^*} (I^{(1)})^*) \\ &= (I^{(k+1)} :_R I^{(1)})^* \\ &= (I^{(k)})^* \\ &= (I^*)^{(k)}. \end{aligned}$$

This means that I^* has the symbolic strong persistence property. Conversely, assume that I^* has the symbolic strong persistence property. Accordingly, we have $((I^*)^{(k+1)} :_{R^*} (I^*)^{(1)}) = (I^*)^{(k)}$. On account of [13, Lemma 1.1 and Corollary 1.4], we get $((I^*)^{(k+1)} :_{R^*} (I^*)^{(1)}) = (I^{(k+1)} :_R I^{(1)})^*$ and $(I^*)^{(k)} = (I^{(k)})^*$. This implies that $(I^{(k+1)} :_R I^{(1)})^* = (I^{(k)})^*$, and so $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$. That is, I has the symbolic strong persistence property. $\square$

4.3 Some relations between SPP and SSPP

In this section, we give some relations between the strong persistence property (SPP for short) and symbolic strong persistence property (SSPP for short). It should be noted that the results of this section can be found in [84].

We first show Theorem 4.3.1, which will be used in the proof of Corollary 4.3.3.

Theorem 4.3.1. *Let I be an ideal in a commutative Noetherian ring S . Then I has the symbolic strong persistence property if and only if $I_{\mathfrak{p}}$ has the strong persistence property for all $\mathfrak{p} \in \text{Min}(I)$, where $I_{\mathfrak{p}}$ denotes the localization of I at $\mathfrak{p}$.*

Proof. ($\Rightarrow$) Assume that I has the symbolic strong persistence property. Suppose that $\text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Without loss of generality, it is enough to show that $(I_{\mathfrak{p}_1}^{k+1} :_{S_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}) = I_{\mathfrak{p}_1}^k$ for all $k \geq 1$. Fix $k \geq 1$. Also, for all s , suppose that

$$I^s = Q_{s,1} \cap \dots \cap Q_{s,t} \cap Q_{s,t+1} \cap \dots \cap Q_{s,r_s},$$

is a minimal primary decomposition of I^s with $\sqrt{Q_{s,i}} = \mathfrak{p}_i$ for each $i = 1, \dots, t$, and $\sqrt{Q_{s,i}}$ is not minimal for each $i = t+1, \dots, r_s$. We therefore have

$$I^{(1)} = \bigcap_{i=1}^t Q_{1,i}, \quad I^{(k)} = \bigcap_{i=1}^t Q_{k,i}, \quad \text{and} \quad I^{(k+1)} = \bigcap_{i=1}^t Q_{k+1,i}.$$

Since I has the symbolic strong persistence property, one has $(I^{(k+1)} :_S I^{(1)}) = I^{(k)}$, and hence $(I_{\mathfrak{p}_1}^{(k+1)} :_{S_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}^{(1)}) = I_{\mathfrak{p}_1}^{(k)}$. It follows also from Proposition 3.3.17 that, for all s , $(I^s)_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$ and $(I^{(s)})_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$. We thus gain $(I_{\mathfrak{p}_1}^{(k+1)} :_{S_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}) = I_{\mathfrak{p}_1}^k$, as required.

($\Leftarrow$) Assume that $I_{\mathfrak{p}}$ has the strong persistence property for all $\mathfrak{p} \in \text{Min}(I)$. Want to show that $(I^{(k+1)} :_S I^{(1)}) = I^{(k)}$ for all $k \geq 1$. To accomplish this, fix $k \geq 1$. Our strategy is to use [89, Exercise 6.4]. For this purpose, one has to prove that $(I_{\mathfrak{q}}^{(k+1)} :_{S_{\mathfrak{q}}} I_{\mathfrak{q}}^{(1)}) = I_{\mathfrak{q}}^{(k)}$ for all $\mathfrak{q} \in \text{Ass}_S(S/I^{(k)})$. With the notation which has been used in the proof of the forward implication, and by considering the fact that $\text{Ass}_S(S/I^{(s)}) = \text{Min}(I^s) = \text{Min}(I)$ for all s , one can deduce that $\text{Ass}_S(S/I^{(k)}) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Without loss of generality, we need only demonstrate that $(I_{\mathfrak{p}_1}^{(k+1)} :_{S_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}^{(1)}) = I_{\mathfrak{p}_1}^{(k)}$. Since $I_{\mathfrak{p}_1}$ has the strong persistence property, we obtain $(I_{\mathfrak{p}_1}^{(k+1)} :_{S_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}) = I_{\mathfrak{p}_1}^k$. As $(I^s)_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$ and $(I^{(s)})_{\mathfrak{p}_1} = (Q_{s,1})_{\mathfrak{p}_1}$ for all s , one derives that $(I_{\mathfrak{p}_1}^{(k+1)} :_{S_{\mathfrak{p}_1}} I_{\mathfrak{p}_1}^{(1)}) = I_{\mathfrak{p}_1}^{(k)}$. This terminates the proof. $\square$

To show the next corollary, one needs the following proposition. In fact, we showed this result in Corollary 3.3.3, but we re-prove it here by a different way.

Proposition 4.3.2. *Every irreducible primary monomial ideal has the strong persistence property. In particular, every monomial prime ideal has the strong persistence property.*

Proof. Fix $k \geq 1$, and assume that $Q = (x_{i_1}^{\alpha_1}, \dots, x_{i_t}^{\alpha_t})$ is an irreducible primary monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K with $\alpha_1, \dots, \alpha_t$ are positive integers and $\{x_{i_1}, \dots, x_{i_t}\} \subseteq \{x_1, \dots, x_n\}$. We have to prove that $(Q^{k+1} :_R Q) = Q^k$ for all positive integers k . Fix $k \geq 1$. Because $Q^k \subseteq (Q^{k+1} :_R Q)$, it remains to show that $(Q^{k+1} :_R Q) \subseteq Q^k$. To do this, one needs to verify that $(Q^{k+1} :_R (x_{i_1}^{\alpha_1})) \subseteq Q^k$. Here, inspired by Remark 2.3.5, one can deduce the following equalities,

$$\begin{aligned} Q^{k+1} :_R (x_{i_1}^{\alpha_1}) &= \left(\sum_{\theta_1 + \dots + \theta_t = k+1} ((x_{i_1}^{\alpha_1})^{\theta_1} \dots (x_{i_t}^{\alpha_t})^{\theta_t}) \right) :_R (x_{i_1}^{\alpha_1}) \\ &= \sum_{\theta_1 + \dots + \theta_t = k+1} ((x_{i_1}^{\alpha_1})^{\theta_1} \dots (x_{i_t}^{\alpha_t})^{\theta_t}) :_R (x_{i_1}^{\alpha_1}) \\ &= \sum_{\theta_1=0, \theta_2 + \dots + \theta_t = k+1} ((x_{i_2}^{\alpha_2})^{\theta_2} \dots (x_{i_t}^{\alpha_t})^{\theta_t}) \\ &\quad + \sum_{\theta_1 \geq 1, \theta_1 + \dots + \theta_t = k+1} ((x_{i_1}^{\alpha_1})^{\theta_1-1} \dots (x_{i_t}^{\alpha_t})^{\theta_t}). \end{aligned}$$

As $((x_{i_2}^{\alpha_2})^{\theta_2} \dots (x_{i_t}^{\alpha_t})^{\theta_t}) \subseteq Q^k$ with $\theta_2 + \dots + \theta_t = k+1$ and $((x_{i_1}^{\alpha_1})^{\theta_1-1} \dots (x_{i_t}^{\alpha_t})^{\theta_t}) \subseteq Q^k$ with $\theta_1 \geq 1$, $\theta_1 + \dots + \theta_t = k+1$, we conclude that $(Q^{k+1} :_R (x_{i_1}^{\alpha_1})) \subseteq Q^k$, as required.

$\square$

Corollary 4.3.3. *Let I be an ideal in a commutative Noetherian ring S . Let $I = Q_1 \cap \cdots \cap Q_m \cap Q_{m+1} \cap \cdots \cap Q_r$ be a minimal primary decomposition of I such that $\text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$, where $\mathfrak{p}_i = \sqrt{Q_i}$ for each $i = 1, \dots, m$. If Q_i has the strong persistence property for each $i = 1, \dots, m$, then I has the symbolic strong persistence property. In particular, every square-free monomial ideal has the symbolic strong persistence property.*

Proof. Fix $k \geq 1$. By virtue of Proposition Proposition 3.3.17, we get $I_{\mathfrak{p}_i} = (Q_i)_{\mathfrak{p}_i}$ for each $i = 1, \dots, m$. Since Q_i has the strong persistence property for each $i = 1, \dots, m$, one has $(Q_i^{k+1} :_S Q_i) = Q_i^k$ for each $i = 1, \dots, m$. Thus, $((Q_i)_{\mathfrak{p}_i}^{k+1} :_{S_{\mathfrak{p}_i}} (Q_i)_{\mathfrak{p}_i}) = (Q_i)_{\mathfrak{p}_i}^k$ for each $i = 1, \dots, m$. This implies that $I_{\mathfrak{p}_i}$ has the strong persistence property for each $i = 1, \dots, m$. Now, the claim follows from Theorem 4.3.1. The last assertion is a direct consequence from Proposition 4.3.2, which says that every monomial prime ideal has the strong persistence property. $\square$

We are ready to present the next result in the following theorem.

Theorem 4.3.4. *Suppose that I is a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . Also, for all $s \geq 2$, suppose that $\text{Ass}_R(R/I^s) = \text{Ass}_R(R/I) \cup \{\mathfrak{q}\}$ such that $\mathfrak{q}^{st}$ is the primary component of the embedded prime $\mathfrak{q}$, where*

$$t := \min\{|\{x_{j_1}, \dots, x_{j_\ell}\}| : x_{j_1} \cdots x_{j_\ell} \in I\}.$$

Then I has the strong persistence property.

Proof. To show the claim, it is sufficient to verify that $(I^{k+1} :_R I) \subseteq I^k$ for all k . Fix $k \geq 1$. Let $\text{Min}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$. Since $\text{Ass}_R(R/I^s) = \text{Ass}_R(R/I) \cup \{\mathfrak{q}\}$ and $\text{Min}(I) = \text{Ass}_R(R/I)$, we deduce from [133, Definition 4.3.22 and Proposition 4.3.25] and the assumption that $I^s = \mathfrak{p}_1^s \cap \cdots \cap \mathfrak{p}_r^s \cap \mathfrak{q}^{st}$ is a minimal primary decomposition of I^s for all $s \geq 2$. Pick an arbitrary monomial u in $(I^{k+1} :_R I)$. Fix $1 \leq j \leq r$. Since $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for $1 \leq i \neq j \leq r$, this implies that one can choose an element such as $v_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$. Let λ be an element in $\mathfrak{p}_j$. Because $u \in (I^{k+1} :_R I)$, we have $uI \subseteq I^{k+1}$. By virtue of $v_1 \cdots v_{j-1} \lambda v_{j+1} \cdots v_r \in \bigcap_{i=1}^r \mathfrak{p}_i$ and $I = \bigcap_{i=1}^r \mathfrak{p}_i$, one gains $u \lambda v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \in \bigcap_{i=1}^r \mathfrak{p}_i^{k+1} \cap \mathfrak{q}^{(k+1)t}$. This yields that $u \lambda v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \in \mathfrak{p}_j^{k+1}$. Due to $v_i \notin \mathfrak{p}_j$ for $1 \leq i \neq j \leq r$, we obtain $v_1 \cdots v_{j-1} v_{j+1} \cdots v_r \notin \mathfrak{p}_j$. Thanks to $\mathfrak{p}_j^{k+1}$ is primary, one has $u \lambda \in \mathfrak{p}_j^{k+1}$, and hence $u \in (\mathfrak{p}_j^{k+1} :_R \mathfrak{p}_j)$. It follows from Proposition 4.3.2 that $\mathfrak{p}_j$ has the strong persistence property, and so $u \in \mathfrak{p}_j^k$. Accordingly, we have $u \in \bigcap_{i=1}^r \mathfrak{p}_i^k$. To finish the proof, one requires to establish $u \in \mathfrak{q}^{kt}$. Without loss of generality, let $x_1 \cdots x_t \in \bigcap_{i=1}^r \mathfrak{p}_i$. This implies that $u x_1 \cdots x_t \in I^{k+1}$. We thus have $u x_1 \cdots x_t \in \mathfrak{q}^{(k+1)t}$, and therefore there exists some monomial $h \in \mathcal{G}(\mathfrak{q}^{(k+1)t})$ such that $h | u x_1 \cdots x_t$. Assume that $h = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $u = x_1^{\beta_1} \cdots x_n^{\beta_n}$ with $\alpha_i \geq 0$ and $\beta_i \geq 0$ for each i . Since $h \in$

$\mathcal{G}(\mathfrak{q}^{(k+1)t})$, this means that h is a minimal generator of $\mathfrak{q}^{(k+1)t}$, and by considering the assumption that $\mathfrak{q}$ is a monomial prime ideal, one can conclude that $\alpha_1 + \cdots + \alpha_n = (k+1)t$. On account of $h|ux_1 \cdots x_t$, this yields that $(k+1)t \leq \beta_1 + \cdots + \beta_n + t$, and hence $kt \leq \beta_1 + \cdots + \beta_n$. Consequently, one can conclude that $u \in \mathfrak{q}^{kt}$. This implies that $u \in I^k$, and the proof is done. $\square$

It has already been proved that the cover ideal of any odd cycle graph satisfies the strong persistence property, see Theorem 3.1.11. As an application of Theorem 4.3.4, we re-prove this fact in Theorem 4.3.6. To do this, we need the following auxiliary lemma. Indeed, in Lemma 4.3.5, we give a minimal primary decomposition of the powers of the cover ideal of any odd cycle graph.

Lemma 4.3.5. *Let C_{2n+1} be an odd cycle graph with $V(C_{2n+1}) = \{1, \dots, 2n+1\}$ and $E(C) = \{\{i, i+1\} : i = 1, \dots, 2n+1\}$, where $2n+2 = 1$. Then, for all $s \geq 2$,*

$$J(C_{2n+1})^s = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})^s \cap \mathfrak{m}^{s(n+1)},$$

where $\mathfrak{m} = (x_1, \dots, x_{2n+1})$ is the unique homogeneous maximal ideal in the polynomial ring $R = K[x_1, \dots, x_{2n+1}]$ over a field K .

Proof. Since $J(C_{2n+1}) = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})$, it is sufficient for us to prove that

$$(4.3.1) \quad \left(\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}) \right)^s = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})^s \cap \mathfrak{m}^{s(n+1)}.$$

Let A (respectively, B) denote the ideal on the left-hand (respectively, right-hand) side of (4.3.1). Fix $s \geq 2$. We first show that $A \subseteq B$. As $\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}) \subseteq (x_i, x_{i+1})$ for each $i = 1, \dots, 2n+1$, this implies that $(\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}))^s \subseteq \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})^s$. To complete the argument, it suffices to prove that $(\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}))^s \subseteq \mathfrak{m}^{s(n+1)}$. To see this, consider a minimal generator u in $(\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}))^s$. We thus have $u = \prod_{i=1}^s g_i$, where each g_i is a minimal generator of $\bigcap_{i=1}^{2n+1} (x_i, x_{i+1})$. By virtue of $\bigcap_{i=1}^{2n+1} (x_i, x_{i+1})$ is exactly the cover ideal of the odd cycle C_{2n+1} and because of any minimal generator of $J(C_{2n+1})$ corresponds to a minimal vertex cover set of C_{2n+1} , and also by considering the fact that any minimal vertex cover set of C_{2n+1} has at least $n+1$ elements, we can deduce that $\deg u = \sum_{i=1}^s \deg g_i \geq s(n+1)$. This implies that $u \in \mathfrak{m}^{s(n+1)}$. Accordingly, one has $A \subseteq B$.

We now verify that $B \subseteq A$. For this purpose, select a minimal generator u in B . Let $u := x_1^{\ell_1} \cdots x_{2n+1}^{\ell_{2n+1}}$ with $\ell_i \geq 0$ for each $i = 1, \dots, 2n+1$. Our strategy is to use [104, Lemma 3.2]. To accomplish this, one has to demonstrate that $\ell_i + \ell_{i+1} \geq s$ for each $i = 1, \dots, 2n+1$ and $\sum_{i=1}^{2n+1} [(\ell_i + \ell_{i+1}) - s] \geq s$. Fix $1 \leq i \leq 2n+1$. It follows from $u \in B$ that $u \in (x_i, x_{i+1})^s$, and so $x_i^{\alpha_i} x_{i+1}^{s-\alpha_i} | u$ for some $0 \leq \alpha_i \leq s$. This implies that $\ell_i + \ell_{i+1} \geq s$. To finish the proof, we establish $\sum_{i=1}^{2n+1} [(\ell_i + \ell_{i+1}) - s] \geq s$. Since

$u \in B$, one has $u \in \mathfrak{m}^{s(n+1)}$, and so $x_1^{\theta_1} \cdots x_{2n+1}^{\theta_{2n+1}} | u$ with $\sum_{i=1}^{2n+1} \theta_i = s(n+1)$. By virtue of $\sum_{i=1}^{2n+1} \ell_i \geq \sum_{i=1}^{2n+1} \theta_i$, this leads to

$$\begin{aligned} \sum_{i=1}^{2n+1} [(\ell_i + \ell_{i+1}) - s] &= \sum_{i=1}^{2n+1} (\ell_i + \ell_{i+1}) - s(2n+1) \\ &= 2 \sum_{i=1}^{2n+1} \ell_i - s(2n+1) \\ &\geq 2 \sum_{i=1}^{2n+1} \theta_i - s(2n+1) \\ &= s. \end{aligned}$$

We therefore have $u \in A$, and thus $B \subseteq A$, as required. $\square$

Theorem 4.3.6. *The cover ideal of any odd cycle graph satisfies the strong persistence property.*

Proof. Let C_{2n+1} denote an odd cycle graph with $V(C_{2n+1}) = \{1, \dots, 2n+1\}$ and $E(C) = \{\{i, i+1\} : i = 1, \dots, 2n+1\}$, where $2n+2 = 1$. Fix $s \geq 2$. It follows from Lemma 4.3.5 that

$$(4.3.2) \quad J(C_{2n+1})^s = \left(\bigcap_{i=1}^{2n+1} (x_i, x_{i+1}) \right)^s = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1})^s \cap \mathfrak{m}^{s(n+1)},$$

where $\mathfrak{m} = (x_1, \dots, x_{2n+1})$ is the unique homogeneous maximal ideal in the polynomial ring $R = K[x_1, \dots, x_{2n+1}]$ over a field K . In other words, (4.3.2) is a minimal primary decomposition of $J(C_{2n+1})^s$. This gives rise to the following equality

$$\text{Ass}_R(R/J(C_{2n+1})^s) = \text{Ass}_R(R/J(C_{2n+1})) \cup \{\mathfrak{m}\}.$$

In the light of any minimal geneator of $J(C_{2n+1})$ corresponds to a minimal vertex cover set of C_{2n+1} , and by remembering the fact that any minimal vertex cover set of C_{2n+1} has at least $n+1$ elements, and also $J(C_{2n+1}) = \bigcap_{\mathfrak{p} \in \text{Ass}_R(R/J(C_{2n+1}))} \mathfrak{p}$, one can derive $t = \min\{|\{x_{j_1}, \dots, x_{j_\ell}\}| : x_{j_1} \cdots x_{j_\ell} \in J(C_{2n+1})\} = n+1$. Now, Theorem 4.3.4 gives that $J(C_{2n+1})$ satisfies the strong persistence property. $\square$

To establish Lemma 4.3.9, we require to know the following proposition.

Proposition 4.3.7. *Suppose that I_1 and I_2 are two monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer $m, 1 \leq m < n$. Then $(I_1 I_2)^{(k)} = I_1^{(k)} I_2^{(k)}$ for all $k \geq 1$.*

Proof. Fix $k \geq 1$. On account of I_1 and I_2 are generated by disjoint sets of variables, [70, Lemma 1.1] implies that $I_1 I_2 = I_1 \cap I_2$. It follows also from the assumption that $\text{Min}(I_1 \cap I_2) = \text{Min}(I_1) \cup \text{Min}(I_2)$. Hence, one can conclude the following equalities

$$\begin{aligned} (I_1 \cap I_2)^{(k)} &= \bigcap_{\mathfrak{p} \in \text{Min}(I_1 \cap I_2)} ((I_1 \cap I_2)^k R_{\mathfrak{p}} \cap R) \\ &= \bigcap_{\mathfrak{p} \in \text{Min}(I_1)} ((I_1)^k R_{\mathfrak{p}} \cap R) \cap \bigcap_{\mathfrak{p} \in \text{Min}(I_2)} ((I_2)^k R_{\mathfrak{p}} \cap R) \\ &= I_1^{(k)} \cap I_2^{(k)}. \end{aligned}$$

We therefore obtain $(I_1 I_2)^{(k)} = (I_1 \cap I_2)^{(k)} = I_1^{(k)} \cap I_2^{(k)} = I_1^{(k)} I_2^{(k)}$. $\square$

The theorem below is crucial for us to show Lemma 4.3.9.

Theorem 4.3.8. ([51, Theorem 3.4]) $(I + J)^{(n)} = \sum_{i+j=n} I^{(i)} J^{(j)}$.

In order to establish Theorem 4.3.10, one needs to prove the following lemma.

Lemma 4.3.9. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that $I = I_1 R + I_2 R$, where $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer $1 \leq m < n$. Then $I^{(s)} = \bigcap_{i=1}^s (I_1^{(i)} + I_2^{(s+1-i)})$ for all $s \in \mathbb{N}$.*

Proof. Fix $s \in \mathbb{N}$, and set $L_t := I_1^{(t)} + \sum_{i=0}^{t-1} I_1^{(i)} \cap I_2^{(s-i)}$ with $1 \leq t \leq s$. In what follows, want to prove that $L_t = \bigcap_{i=1}^t (I_1^{(i)} + I_2^{(s+1-i)})$. To achieve this, we proceed by induction on t . One can easily see that the assertion is true for the case in which $t = 1$. Now, suppose, inductively, that $t > 1$ and that the result has been proved for all r less than t with $t \leq s$. It follows also from the inductive hypothesis that $L_{t-1} = \bigcap_{i=1}^{t-1} (I_1^{(i)} + I_2^{(s+1-i)})$. In view of Remark 4.1.3, we get $I_1^{(t)} \subseteq I_1^{(t-1)}$, and so $I_1^{(t)} \cap I_1^{(t-1)} = I_1^{(t)}$. In addition, if $0 \leq i \leq t-2 \leq s$, then $s+1-t \leq s-i$, and thus $I_2^{(s-i)} \subseteq I_2^{(s+1-t)}$. This implies $\sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \subseteq I_2^{(s+1-t)}$, and so $I_2^{(s+1-t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} = \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}$. Now, by considering the fact that $I_1^{(t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \subseteq \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}$, we get

$$\begin{aligned} &I_1^{(t)} \cap I_1^{(t-1)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} + I_1^{(t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} + I_2^{(s+1-t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \\ &= I_1^{(t)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} + \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}. \end{aligned}$$

Hence, one can derive the following equalities

$$\begin{aligned}
L_t &= I_1^{(t)} + \sum_{i=0}^{t-1} I_1^{(i)} \cap I_2^{(s-i)} \\
&= I_1^{(t)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} + \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \\
&= I_1^{(t)} \cap I_1^{(t-1)} + I_1^{(t-1)} \cap I_2^{(s-t+1)} \\
&\quad + I_1^{(t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} + I_2^{(s+1-t)} \cap \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)} \\
&= (I_1^{(t)} + I_2^{(s+1-t)}) \cap (I_1^{(t-1)} + \sum_{i=0}^{t-2} I_1^{(i)} \cap I_2^{(s-i)}) \\
&= (I_1^{(t)} + I_2^{(s+1-t)}) \cap L_{t-1} \\
&= (I_1^{(t)} + I_2^{(s+1-t)}) \cap \bigcap_{i=1}^{t-1} (I_1^{(i)} + I_2^{(s+1-i)}) \\
&= \bigcap_{i=1}^t (I_1^{(i)} + I_2^{(s+1-i)}).
\end{aligned}$$

This completes the inductive step, and so the claim has been proved by induction. Especially, one can conclude that $L_s = \bigcap_{i=1}^s (I_1^{(i)} + I_2^{(s+1-i)})$, and hence

$$\bigcap_{i=1}^s (I_1^{(i)} + I_2^{(s+1-i)}) = I_1^{(s)} + \sum_{i=0}^{s-1} I_1^{(i)} \cap I_2^{(s-i)} = \sum_{i=0}^s I_1^{(i)} \cap I_2^{(s-i)}.$$

On account of $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \dots, x_n]$, Proposition 4.3.7 yields that $I_1^{(i)} \cap I_2^{(s-i)} = I_1^{(i)} I_2^{(s-i)}$ for each $i = 0, \dots, s$. Therefore, we get $\bigcap_{i=1}^s (I_1^{(i)} + I_2^{(s+1-i)}) = \sum_{i=0}^s I_1^{(i)} I_2^{(s-i)}$. Finally, it follows from Theorem 4.3.8 that $\bigcap_{i=1}^s (I_1^{(i)} + I_2^{(s+1-i)}) = (I_1 + I_2)^{(s)} = I^{(s)}$, as required. $\square$

We are now ready to express and prove Theorem 4.3.10.

Theorem 4.3.10. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K and $\mathcal{G}(I) = G_1 \cup \dots \cup G_r$ such that*

$$\{x_s : x_s | m \text{ for some } m \in G_i\} \cap \{x_t : x_t | m \text{ for some } m \in G_j\} = \emptyset,$$

for all $1 \leq i \neq j \leq r$. Then $(G_i)R$ has the symbolic strong persistence property for some $1 \leq i \leq r$ if and only if I has the symbolic strong persistence property.

Proof. It is enough to show our claim only for $r = 2$. To accomplish this, let I be a monomial ideal in $R = K[x_1, \dots, x_n]$ such that $I = I_1 R + I_2 R$, where $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer $1 \leq m <$

n . We first show the forward implication. Without loss of generality, assume that I_1 has the symbolic strong persistence property. Also, let $\mathcal{G}(I_1^{(1)}) = \{v_1, \dots, v_t\}$. Our main aim is to prove that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all k . Now, fix $k \in \mathbb{N}$. By virtue of the assumption, one has $(I_1^{(i)} :_R I_1^{(1)}) = I_1^{(i-1)}$ for all i . By applying Lemma 4.3.9, we have the following equalities

$$\begin{aligned}
(I^{(k+1)} :_R I_1^{(1)}) &= \left(\bigcap_{i=1}^{k+1} (I_1^{(i)} + I_2^{(k+2-i)}) \right) :_R I_1^{(1)} \\
&= \bigcap_{i=1}^{k+1} \bigcap_{j=1}^t ((I_1^{(i)} :_R v_j) + (I_2^{(k+2-i)} :_R v_j)) \\
&= \bigcap_{i=1}^{k+1} \bigcap_{j=1}^t ((I_1^{(i)} :_R v_j) + I_2^{(k+2-i)}) \\
&= \bigcap_{i=1}^{k+1} (I_2^{(k+2-i)} + \bigcap_{j=1}^t (I_1^{(i)} :_R v_j)) \\
&= \bigcap_{i=1}^{k+1} (I_2^{(k+2-i)} + (I_1^{(i)} :_R I_1^{(1)})) \\
&= \bigcap_{i=2}^{k+1} (I_2^{(k+2-i)} + I_1^{(i-1)}) \\
&= \bigcap_{\ell=1}^k (I_2^{(k+1-\ell)} + I_1^{(\ell)}) \\
&= I^{(k)}.
\end{aligned}$$

Thanks to Remark 4.1.3, one can derive $I^{(k)} \subseteq (I^{(k+1)} :_R I^{(1)})$, and by considering this fact that $(I^{(k+1)} :_R I^{(1)}) \subseteq (I^{(k+1)} :_R I_1^{(1)})$, one has $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$, as required.

To establish the converse implication, suppose, on the contrary, that I_1 and I_2 do not satisfy the symbolic strong persistence property. This implies that there exist a positive integer k_1 (respectively, k_2) and a monomial m_1 (respectively, m_2) such that $m_1 \in \mathcal{G}(I_1^{(k_1+1)} :_R I_1^{(1)}) \setminus \mathcal{G}(I_1^{(k_1)})$ (respectively, $m_2 \in \mathcal{G}(I_2^{(k_2+1)} :_R I_2^{(1)}) \setminus \mathcal{G}(I_2^{(k_2)})$). Take the nonnegative integer a_1 (respectively, a_2) such that $m_1 \in I_1^{(a_1)} \setminus I_1^{(a_1+1)}$ (respectively, $m_2 \in I_2^{(a_2)} \setminus I_2^{(a_2+1)}$). In the light of Remark 4.1.3, we deduce that $a_1 \leq k_1 - 1$ (respectively, $a_2 \leq k_2 - 1$). Put $m := m_1 m_2$ and $b := a_1 + a_2$. Thus, one has $m \in I^{(b)}$. We claim that $m \notin I^{(b+1)}$. Suppose, on the contrary, that $m \in I^{(b+1)}$. Since, by Theorem 4.3.8, $I^{(b+1)} = \sum_{\ell=0}^{b+1} I_1^{(\ell)} I_2^{(b+1-\ell)}$, one obtains $m \in I_1^{(\ell)} I_2^{(b+1-\ell)}$ for some $0 \leq \ell \leq b+1$. We thus have there exist $u_1 \in \mathcal{G}(I_1^{(\ell)})$ and $u_2 \in \mathcal{G}(I_2^{(b+1-\ell)})$ such that $u_1 u_2 | m_1 m_2$. Consequently, $u_1 | m_1 m_2$ and $u_2 | m_1 m_2$. Because $\gcd(u_1, m_2) = 1$ and $\gcd(u_2, m_1) = 1$, one has $u_1 | m_1$ and $u_2 | m_2$. This implies that $m_1 \in I_1^{(\ell)}$ and $m_2 \in I_2^{(b+1-\ell)}$. It follows from Remark 4.1.3, $m_1 \in I_1^{(a_1)} \setminus I_1^{(a_1+1)}$, and $m_2 \in I_2^{(a_2)} \setminus I_2^{(a_2+1)}$

that $a_1 \geq \ell$ and $a_2 \geq b + 1 - \ell$. This gives that $a_1 + a_2 \geq b + 1$, which contradicts the fact that $b = a_1 + a_2$. Accordingly, $m \notin I^{(b+1)}$. By setting $s_1 := k_1 + 1 + a_2$ and $s_2 := a_1 + k_2 + 1$, one can easily see that $s_i \geq a_1 + a_2 + 2$ for each $i = 1, 2$. Hence, $s := \min\{s_1, s_2\} \geq a_1 + a_2 + 2$. In addition, it is routine to check that if $u \in \mathcal{G}(I_1^{(1)})$ (respectively, $u \in \mathcal{G}(I_2^{(1)})$), then $mu \in I^{(s_1)}$ (respectively, $mu \in I^{(s_2)}$). Therefore, we get $m \in (I^{(s)} :_R I^{(1)}) \setminus I^{(s-1)}$, which contradicts the assumption that I has the symbolic strong persistence property, as desired. $\square$

The following lemma says that, under certain condition, a monomial ideal has the symbolic strong persistence property if and only if its monomial multiple has the symbolic strong persistence property.

Lemma 4.3.11. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and h be a monomial in R . Also, let $\gcd(h, u) = 1$ for all $u \in \mathcal{G}(I)$. Then I has the symbolic strong persistence property if and only if hI has the symbolic strong persistence property.*

Proof. To simplify our notation, set $L := hI$. We first assume that I has the symbolic strong persistence property. This means that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all k . In order to complete the argument, one has to show that $(L^{(k+1)} :_R L^{(1)}) = L^{(k)}$ for all k . Fix $k \geq 1$. It follows readily from Proposition 4.3.7 that $(hI)^{(k)} = h^k I^{(k)}$. In view of $(L^{(k+1)} :_R h) = (h^{k+1} I^{(k+1)} :_R h) = h^k I^{(k+1)}$, one can conclude that $(L^{(k+1)} :_R L^{(1)}) = (h^k I^{(k+1)} :_R I^{(1)})$. By virtue of $\gcd(h, u) = 1$ for all $u \in \mathcal{G}(I)$, we get $\gcd(h, v) = 1$ for all $v \in \mathcal{G}(I^{(1)})$. The assumption yields that $(L^{(k+1)} :_R L^{(1)}) = h^k I^{(k)} = (hI)^{(k)}$. Therefore, $(L^{(k+1)} :_R L^{(1)}) = L^{(k)}$, and the proof is over.

To establish the converse implication, suppose that L has the symbolic strong persistence property. Want to prove that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$ for all k . To do this, fix $k \geq 1$. Because of L has the symbolic strong persistence property, we get $(L^{(k+1)} :_R L^{(1)}) = L^{(k)}$, and hence $(h^{k+1} I^{(k+1)} :_R hI^{(1)}) = h^k I^{(k)}$. This yields that $(h^k I^{(k+1)} :_R I^{(1)}) = h^k I^{(k)}$. Moreover, a similar argument gives that $(h^k I^{(k+1)} :_R I^{(1)}) = h^k (I^{(k+1)} :_R I^{(1)})$, and so $h^k (I^{(k+1)} :_R I^{(1)}) = h^k I^{(k)}$. This implies that $(I^{(k+1)} :_R I^{(1)}) = I^{(k)}$, that is, I has the symbolic strong persistence property, as claimed. $\square$

Chapter 5

On the normally torsion-freeness of monomial ideals

5.1 Normally torsion-freeness of path ideals of directed trees

Let $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ be a simple hypergraph on the vertex set $V(\mathcal{H})$ and edge set $E(\mathcal{H})$. The edge ideal of $\mathcal{H}$, denoted by $I(\mathcal{H})$, is the ideal generated by the monomials corresponding to the edges of $\mathcal{H}$. A hypergraph $\mathcal{H}$ is called *Mengerian* if it satisfies a certain min-max equation, which is known as the Mengerian property in hypergraph theory or as the max-flow min-cut property in integer programming. Algebraically, it is equivalent to $I(\mathcal{H})$ being normally torsion-free, see [60, Corollary 10.3.15] and [133, Theorem 14.3.6]. Hence, the notion of normally torsion-freeness is important for both Commutative Algebra and Combinatorics.

Let R be a commutative Noetherian ring and I be an ideal of R . In addition, let $\text{Ass}_R(R/I)$ be the set of all prime ideals associated to I . An ideal I is called *normally torsion-free* if $\text{Ass}(R/I^k) \subseteq \text{Ass}(R/I)$ for all $k \geq 1$, see [60, Definition 1.4.5]. Normally torsion-free ideals have been a topic of several papers, however, a few classes of these ideals originate from graph theory. Simis, Vasconcelos, and Villarreal showed in [126] that a finite simple graph is bipartite if and only if its edge ideal is normally torsion-free. An analogue of bipartite graphs in higher dimensions is considered to be a hypergraph that avoids “special odd cycles”. Such hypergraphs are called *balanced*, and a well-known result of Fulkerson, Hoffman, and Oppenheim in [46] states that balanced hypergraphs are Mengerian. It follows immediately that the edge ideals of balanced hypergraphs are normally torsion-free. However, unlike the case of bipartite graphs, it should be noted that the converse of this statement

is not true. More information related to definitions in hypergraph theory can be found in [15].

The main motivation of this section is a conjecture which has been stated in [16]. In fact, in [16, Conjecture 6.1], made up the conjecture below.

Conjecture 5.1.1. *Let $\Gamma = (V_\Gamma, E_\Gamma)$ be a rooted tree with the corresponding Alexander dual J_2 . Then $\text{Ass}_R(R/J_2) = \text{Ass}_R(R/J_2^s)$ for all $s \geq 1$.*

Our goal in this section is to provide an affirmative answer to above conjecture (see Theorem 5.1.6). It should be noted that the results of this section can be found in [79].

Also, for a simple finite graph $G = (V(G), E(G))$, the symbol $N_G(v)$ stands for the neighbor set of a vertex $v \in V(G)$.

We first recall some definitions from [54] and [137] which will be needed in the following.

Definition 5.1.2. *A directed edge of a graph is an assignment of a direction to an edge of a graph. If $\{w, u\}$ is an edge, we write (w, u) to denote the directed edge, where the direction is from w to u . A graph is a directed graph if each edge has been assigned a direction. A path of length t in a directed graph is a sequence of vertices $i_1, \dots, i_{t+1}$ such that $e_j = (i_j, i_{j+1})$ is a directed edge for $j = 1, \dots, t$.*

Fix a positive integer t and a directed graph G , the path ideal of G of length t is the following monomial ideal

$$I_t(G) := (x_{i_1} \cdots x_{i_{t+1}} \mid i_1, \dots, i_{t+1} \text{ is a path of } G \text{ of length } t),$$

and also the Alexander dual of $I_t(G)$ is defined as follows:

$$I_t(G)^\vee := \bigcap_{i_1, \dots, i_{t+1} \text{ is a path of } G \text{ of length } t} (x_{i_1}, \dots, x_{i_{t+1}}).$$

A tree T can be considered as a directed graph by choosing any vertex of T to be the root of the tree, and assigning to each edge the direction “away” from the root. Since T is a tree, the assignment of a direction will always be possible. A rooted tree T is a tree with one vertex chosen as root. If tree T has no such root, then we say that T is unrooted.

To remember the definition and properties of hypergraphs, which will be needed in the rest of this section, one can refer to Section 1.3. In particular, see [43, 52] for more information.

To establish Theorem 5.1.6, one has to apply the notions of parameter ideals and corner-elements which were first introduced in [55]. We recall them in the subsequent definitions.

Definition 5.1.3. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring. A parameter ideal of R is an ideal of the form $(x_1^{a_1}, \dots, x_n^{a_n})$ with $a_1, \dots, a_n$ are positive integers. If $u = x_1^{a_1} \cdots x_n^{a_n}$ with nonnegative integers $a_1, \dots, a_n$ is a monomial in R , then we set $P_R(u) := (x_1^{a_1+1}, \dots, x_n^{a_n+1})$. It is well-known that for a monomial z in R and monomial ideal I of R , one has $z \notin I$ if and only if $I \subseteq P_R(z)$ (cf. [93, Exercise 6.1.12]).

Definition 5.1.4. Let $R = K[x_1, \dots, x_n]$ be a polynomial ring, and I be an ideal of R . A monomial z in R is called an I -corner-element if $z \notin I$ and $x_i z \in I$ for each $i = 1, \dots, n$.

We continue with the subsequent proposition which is essential for us to prove Theorem 5.1.6.

Proposition 5.1.5. Let T be a rooted tree with root z on the vertex set $\{z, 1, \dots, 2k\}$ for some positive integer k , and the edge set $E(T) = \{(z, i), (i, i+k) \mid i = 1, \dots, k\}$. Then $I_2(T)^\vee$ is normally torsion-free.

Proof. By virtue of $E(T) = \{(z, i), (i, i+k) \mid i = 1, \dots, k\}$, one can conclude that $I_2(T) = (x_z x_i x_{i+k} \mid i = 1, \dots, k)$, and so $I_2(T)^\vee = \bigcap_{i=1}^k (x_z, x_i, x_{i+k})$. This implies that $I_2(T)^\vee = x_z R' + \bigcap_{i=1}^k (x_i, x_{i+k})$, where $R' = K[x_z, x_1, \dots, x_{2k}]$ over a field K . Now, put $\mathfrak{p}_i := (x_i, x_{i+k})$ for each $i = 1, \dots, k$, and $L := \bigcap_{i=1}^k \mathfrak{p}_i$. Fix an arbitrary positive integer s . If $\mathfrak{p} \in \text{Ass}_{R'}(R'/(I_2(T)^\vee)^s)$, it follows from [50, Lemma 3.4] that $\mathfrak{p} = (x_z) + \mathfrak{q}$ for some $\mathfrak{q} \in \text{Ass}_R(R/L^s)$, where $R = K[x_1, \dots, x_{2k}]$. Thus, it is sufficient to show that L is normally torsion-free. As $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are generated by disjoint non-empty subsets of $\{x_1, \dots, x_{2k}\}$, according to [113, Remark 2.8], for all positive integers $r_1, \dots, r_k$, one has the following equalities

$$\text{Ass}_R(R/\mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_k^{r_k}) = \bigcup_{i=1}^k \text{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}.$$

Furthermore, [113, Theorem 2.7] implies that $\mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_k^{r_k} = \mathfrak{p}_1^{r_1} \cap \cdots \cap \mathfrak{p}_k^{r_k}$ for all positive integers $r_1, \dots, r_k$. Since $L = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$, we obtain

$$\text{Ass}_R(R/L^s) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\} = \text{Ass}_R(R/L).$$

This means that L is normally torsion-free, and so $I_2(T)^\vee$ is also normally torsion-free, as claimed. $\square$

We are now ready to express the main result of this section in the following theorem.

Theorem 5.1.6. Let T be a rooted tree. Then $I_2(T)^\vee$ is normally torsion-free.

Proof. Let $R = K[x_v \mid v \in V(T)]$ be the polynomial ring over a field K . To simplify our notation, we put $J_2(T) := I_2(T)^\vee$. We argue by induction on $n := |V(T)|$. If $n \leq 2$, then there is nothing to prove. So let $n = 3$, say $V(T) := \{z, 1, 2\}$ with z as the root. If $\deg_T z = 2$, then there is nothing to verify. If $\deg_T z = 1$, then we get $J_2(T) = (x_z, x_1, x_2)$. Since $\text{Ass}(J_2(T)) = \text{Ass}(J_2(T)^s)$ for all s , one can conclude that $J_2(T)$ is normally torsion-free. Therefore, the claim holds in the case in which $n = 3$. Now, suppose, inductively, that $n > 3$ and that the result has been proved for any rooted tree with the size of its vertex set is less than n . Assume that $V(T) = \{z, 1, \dots, n-1\}$, z is the root of T , and L_T denotes the set of leaves of T . If $L_T = V(T) \setminus \{z\}$, then there is nothing to show. Hence, one may assume that $L_T \neq V(T) \setminus \{z\}$. Furthermore, if $v' \in N_T(z) \cap L_T$, then the variable $x_{v'}$ does not appear in any of the variables of generators of $I_2(T)$, and so we can assume that $N_T(z) \cap L_T = \emptyset$. Therefore, one may consider the following two cases:

Case 1. The tree T has no subtree which is a claw or a path with length three. Then T is a rooted tree with root z on the vertex set $\{z, 1, \dots, 2k\}$ for some positive integer k , and the edge set $E(T) = \{(z, i), (i, i+k) \mid i = 1, \dots, k\}$. On account of Proposition 5.1.5, one can conclude that $I_2(T)^\vee$ is normally torsion-free, and the proof is over.

Case 2. The tree T has a subtree T_1 which is a claw with $V(T_1) = \{f, v, w, g\}$, $w \in L_T$, and $E(T_1) = \{(f, v), (v, w), (v, g)\}$, or a path with length three such that $V(T_1) = \{f, v, w, g\}$, $w \in L_T$, and $E(T_1) = \{(g, f), (f, v), (v, w)\}$. Set $G := T \setminus \{w\}$. This gives that $|V(G)| = |V(T)| - 1$. We now construct the finite simple hypergraph $\mathcal{H}$ corresponding to T , with $V(\mathcal{H}) = V(T)$ and

$$E(\mathcal{H}) = \{\{x_{i_1}, x_{i_2}, x_{i_3}\} : i_1, i_2, i_3 \text{ is a path of } T \text{ of length two}\}.$$

Also, the edge ideal $I(\mathcal{H})$ of the hypergraph $\mathcal{H}$ is equal to the path ideal $I_2(T)$. Therefore, Proposition 2.7 in [43] implies that

$$J_2(T) = I_2(T)^\vee = I(\mathcal{H})^\vee = J(\mathcal{H}),$$

where $J(\mathcal{H})$ denotes the cover ideal of the hypergraph $\mathcal{H}$. Once again, we can make the finite simple hypergraph $\mathcal{L}$ corresponding to G , with $V(\mathcal{L}) = V(G)$ and

$$E(\mathcal{L}) = \{\{x_{j_1}, x_{j_2}, x_{j_3}\} : j_1, j_2, j_3 \text{ is a path of } G \text{ of length two}\}.$$

Furthermore, the edge ideal $I(\mathcal{L})$ of the hypergraph $\mathcal{L}$ is equal to the path ideal $I_2(G)$. Thus, Proposition 2.7 in [43] yields that

$$J_2(G) = I_2(G)^\vee = I(\mathcal{L})^\vee = J(\mathcal{L}),$$

where $J(\mathcal{L})$ denotes the cover ideal of the hypergraph $\mathcal{L}$. In the following, our aim is to demonstrate that

$$\text{Ass}(J(\mathcal{H})^s) = \text{Ass}(J(\mathcal{L})^s) \cup \{(x_w, x_v, x_f)\},$$

for all s . Now, fix $s \in \mathbb{N}$. We first show that

$$\text{Ass}(J(\mathcal{L})^s) \cup \{(x_w, x_v, x_f)\} \subseteq \text{Ass}(J(\mathcal{H})^s).$$

To do this, let $\mathbf{p} = (x_{i_1}, \dots, x_{i_r})$ be an arbitrary element of $\text{Ass}(J(\mathcal{L})^s)$. It follows from [45, Lemma 2.11] that $\mathbf{p} \in \text{Ass}(K[\mathbf{p}]/J(\mathcal{L}_{\mathbf{p}})^s)$, where $K[\mathbf{p}] = K[x_{i_1}, \dots, x_{i_r}]$ and $\mathcal{L}_{\mathbf{p}}$ is the induced hypergraph of $\mathcal{L}$ on the vertex set $\{i_1, \dots, i_r\} \subseteq V(\mathcal{L})$. As $\mathcal{L}_{\mathbf{p}} = \mathcal{H}_{\mathbf{p}}$, we obtain $\mathbf{p} \in \text{Ass}(K[\mathbf{p}]/J(\mathcal{H}_{\mathbf{p}})^s)$. This implies that $\mathbf{p} \in \text{Ass}(J(\mathcal{H})^s)$. By virtue of $(x_w, x_v, x_f) \in \text{Min}(J(\mathcal{H}))$, one derives $(x_w, x_v, x_f) \in \text{Ass}(J(\mathcal{H})^s)$, and so $\text{Ass}(J(\mathcal{L})^s) \cup \{(x_w, x_v, x_f)\} \subseteq \text{Ass}(J(\mathcal{H})^s)$. For completing the argument, it is sufficient to verify the reverse inclusion. To achieve this, assume that $\mathbf{p} = (x_{i_1}, \dots, x_{i_r})$ is an arbitrary element of $\text{Ass}(J(\mathcal{H})^s)$ with $\{i_1, \dots, i_r\} \subseteq V(\mathcal{H})$. If $\{i_1, \dots, i_r\} \subseteq V(G)$, similar to discussion above, Lemma 2.11 in [45] yields that $\mathbf{p} \in \text{Ass}(J(\mathcal{L})^s)$, and the proof is complete. We thus let $w \in \{i_1, \dots, i_r\}$. We next show that $x_v \in \mathbf{p}$ and $x_f \in \mathbf{p}$. Suppose, on the contrary, that $x_v \notin \mathbf{p}$ or $x_f \notin \mathbf{p}$. Thanks to $\mathbf{p} \in \text{Ass}(J(\mathcal{H})^s)$, we get $\mathbf{m}_{\mathbf{p}} \in \text{Ass}(J(\mathcal{H})^s(\mathbf{p}))$, where $\mathbf{m}_{\mathbf{p}}$ is the graded maximal ideal of $R(\mathbf{p})$. Because $J(\mathcal{H})^s(\mathbf{p}) = (J(\mathcal{H})(\mathbf{p}))^s$, one derives $\mathbf{m}_{\mathbf{p}} \in \text{Ass}((J(\mathcal{H})(\mathbf{p}))^s)$. It follows from $x_v \notin \mathbf{p}$ or $x_f \notin \mathbf{p}$ that, in $R(\mathbf{p})$, we have $(x_w, x_v, x_f) = (1)$, and so $(x_w, x_v, x_f) \notin \text{Ass}(J(\mathcal{H})(\mathbf{p}))$. On account of the only minimal prime ideal which contains x_w is (x_w, x_v, x_f) , one can easily see that the variable x_w does not appear in any of the variables of generators of $(J(\mathcal{H})(\mathbf{p}))^s$, while $x_w \in \mathbf{m}_{\mathbf{p}}$. This leads to a contradiction, and hence one can conclude that $(x_w, x_v, x_f) \subseteq \mathbf{p}$. Let $r \leq n - 1$. Hence, Lemma 2.11 in [45] implies that $\mathbf{p} \in \text{Ass}(K[\mathbf{p}]/J(\mathcal{H}_{\mathbf{p}})^s)$, where $K[\mathbf{p}] = K[x_{i_1}, \dots, x_{i_r}]$. On the other hand, in view of the proof of Lemma 2.4 in [17], when $t = 2$, one derives $\mathcal{H}_{\mathbf{p}}$ is connected. This means that $\mathcal{H}_{\mathbf{p}}$ is a rooted subtree with the size of its vertex set is less than n . We see immediately from the inductive hypothesis that $J(\mathcal{H}_{\mathbf{p}})$ is normally torsion-free. Therefore, $\mathbf{p} = (x_w, x_v, x_f)$, and the proof is over. To conclude the proof, it is sufficient for us to demonstrate the case $r = n$ is impossible. On the contrary, assume that $\mathbf{p}$ is the unique homogeneous maximal ideal of R . It follows from $\mathbf{p} \in \text{Ass}(J(\mathcal{H})^s)$ that there exists a monomial $h \in R$ with $h \notin J(\mathcal{H})^s$ such that $\mathbf{p} = (J(\mathcal{H})^s : h)$, and so $\mathbf{p}h \subseteq J(\mathcal{H})^s$. This implies that $x_w h \in J(\mathcal{H})^s$, and hence there exist monomials $u_1, \dots, u_s \in \mathcal{G}(J(\mathcal{H}))$ such that $x_w h = u_1 \cdots u_s Q$ for some monomial $Q \in R$. If $x_w | Q$, then $h \in J(\mathcal{H})^s$, which is a contradiction. Therefore, $x_w | u_k$ for some $1 \leq k \leq s$, and thus $h \in J(\mathcal{H})^{s-1}$. Since $J(\mathcal{H})^s \subseteq J(\mathcal{L})^s$, one has $\mathbf{p} \subseteq (J(\mathcal{L})^s : h)$. If $\mathbf{p} = (J(\mathcal{L})^s : h)$, then $x_w h \in J(\mathcal{L})^s$, and so there exists an element $A \in \mathcal{G}(J(\mathcal{L})^s)$ such that $A | x_w h$. As $\gcd(A, x_w) = 1$, we obtain $A | h$, and thus $h \in J(\mathcal{L})^s$. This leads to a contradiction. Therefore, $(J(\mathcal{L})^s : h) = R$, and hence $h \in J(\mathcal{L})^s$. By virtue of $h \notin J(\mathcal{H})^s$ and $\mathbf{p} = (J(\mathcal{H})^s : h)$, we get h is a $J(\mathcal{H})^s$ -corner-element. It follows from [93, Corollary 6.3.3] that $J(\mathcal{H})^s \subseteq P_R(h)$. This yields that $(J(\mathcal{H})^s : h) \subseteq (P_R(h) : h)$, and hence $\mathbf{p} \subseteq (P_R(h) : h)$. Based on [93, Lemma 6.1.3], one has $h \notin P_R(h)$, and so $\mathbf{p} = (P_R(h) : h)$. Let $h = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$

with nonnegative integers $\alpha_1, \dots, \alpha_n$. Then $P_R(h) = (x_1^{\alpha_1+1}, \dots, x_n^{\alpha_n+1})$. We next prove that $h | (\prod_{i=1}^n x_i)^{s-1}$. Suppose, on the contrary, that $h \nmid (\prod_{i=1}^n x_i)^{s-1}$. This implies that there exists some x_i in h such that $\alpha_i \geq s$. Therefore, the exponent of x_i in hx_i is at least $s+1$. Due to $hx_i \in J(\mathcal{H})^s$, one can conclude that there exist monomials $u_1, \dots, u_s \in \mathcal{G}(J(\mathcal{H}))$ such that $hx_i = u_1 \cdots u_s Q$ for some monomial Q in R . Thanks to, for each $j = 1, \dots, s$, u_j is a square-free monomial, one has the exponent of x_i in each u_j is at most one. Accordingly, we get $x_i | Q$, and hence $h \in J(\mathcal{H})^s$, which is a contradiction. Therefore, $h | (\prod_{i=1}^n x_i)^{s-1}$. In the following, our aim is to verify that $h = (\prod_{i=1}^n x_i)^{s-1}$. Since $h \in J(\mathcal{H})^{s-1}$, [93, Exercise 6.1.12] derives $J(\mathcal{H})^{s-1} \not\subseteq P_R(h)$, and thus there exists an element $M \in \mathcal{G}(J(\mathcal{H})^{s-1})$ such that $M \notin P_R(h)$. It follows from $M \in \mathcal{G}(J(\mathcal{H})^{s-1})$ that there exist monomials $A_1, \dots, A_{s-1} \in \mathcal{G}(J(\mathcal{H}))$ such that $M = A_1 \cdots A_{s-1}$. This implies that $x_i^{\alpha_i+1} \nmid A_1 \cdots A_{s-1}$ for each $i = 1, \dots, n$. In the light of, for each $j = 1, \dots, s-1$, A_j is a square-free monomial, one obtains the exponent of x_i in A_j , for each $i = 1, \dots, n$, is at most one. This yields that $\alpha_i + 1 > s-1$ for each $i = 1, \dots, n$, and so $\alpha_i > s-2$ for each $i = 1, \dots, n$. It follows from $h | (\prod_{i=1}^n x_i)^{s-1}$ that $\alpha_i \leq s-1$ for each $i = 1, \dots, n$. Consequently, $\alpha_i = s-1$ for each $i = 1, \dots, n$, and thus $h = (\prod_{i=1}^n x_i)^{s-1}$. Because $h \in J(\mathcal{L})^s$, [93, Exercise 6.1.12] gives that $J(\mathcal{L})^s \not\subseteq (x_1^s, \dots, x_n^s)$, and hence there exist monomials $B_1, \dots, B_s \in \mathcal{G}(J(\mathcal{L}))$ such that $x_i^s \nmid B_1 \cdots B_s$ for each $i = 1, \dots, n$. On the other hand, by Proposition 1.2.1 in [60], it is routine to check that if $u \in \mathcal{G}(J(\mathcal{L}))$ with $x_v | u$ or $x_f | u$, then $u \in J(\mathcal{H})$, but if $x_v \nmid u$ and $x_f \nmid u$, then $ux_w \in J(\mathcal{H})$. Let $x_v | B_i$ for each $i = 1, \dots, \ell$, $x_f | B_i$ for each $i = \ell+1, \dots, t$, and $x_v \nmid B_i$ and $x_f \nmid B_i$ for each $i = t+1, \dots, s$. Therefore, we obtain $B_1, \dots, B_t \in J(\mathcal{H})$, and $B_{t+1}x_w, \dots, B_sx_w \in J(\mathcal{H})$. This implies that $B_1 \cdots B_s x_w^{s-t} \in J(\mathcal{H})^s$. Since $h \notin J(\mathcal{H})^s$, Exercise 6.1.12 in [93] yields that $J(\mathcal{H})^s \subseteq P_R(h)$, and hence $J(\mathcal{H})^s \subseteq (x_1^s, \dots, x_n^s)$. Accordingly, there exists a positive integer j with $1 \leq j \leq n$ such that $x_j^s | B_1 \cdots B_s x_w^{s-t}$. If $j \neq w$, then $x_j^s | B_1 \cdots B_s$, which contradicts that $x_i^s \nmid B_1 \cdots B_s$ for each $i = 1, \dots, n$. Hence, $j = w$, and so $t = 0$. Let $B_i = \prod_{\alpha \in W_i} x_\alpha$ for each $i = 1, \dots, s$. In view of the definition, we deduce that W_i is a minimal vertex cover of $\mathcal{L}$ for each $i = 1, \dots, s$. As $\{x_g, x_f, x_v\} \in E(\mathcal{L})$, one derives $\{x_g, x_f, x_v\} \cap \{x_\alpha \mid \alpha \in W_i\} \neq \emptyset$ for each $i = 1, \dots, s$. Due to, for each $i = 1, \dots, s$, $x_v \nmid B_i$ and $x_f \nmid B_i$, we get $x_g | B_i$ for each $i = 1, \dots, s$. This gives that $x_g^s | B_1 \cdots B_s$. This leads to a contradiction. So, the case $r = n$ is impossible. Hence, we gain

$$\text{Ass}(J(\mathcal{H})^s) = \text{Ass}(J(\mathcal{L})^s) \cup \{(x_w, x_v, x_f)\}.$$

Since $|V(G)| = |V(T)| - 1$, the inductive hypothesis implies that $J_2(G) = I_2(G)^\vee$ is normally torsion-free, and hence $J(\mathcal{L})$ is so. Accordingly, one can deduce that $\text{Ass}(J(\mathcal{L})^s) = \text{Ass}(J(\mathcal{L}))$ for all s . Due to $\text{Ass}(J(\mathcal{H})) = \text{Ass}(J(\mathcal{L})) \cup \{(x_w, x_v, x_f)\}$, we get $\text{Ass}(J(\mathcal{H})^s) = \text{Ass}(J(\mathcal{H}))$ for all s . Therefore, $J(\mathcal{H})$ is normally torsion-free,

and thus $J_2(T)$ is so. This completes the inductive step, and so the theorem has been proved by induction. $\square$

We end this section with an example that illustrates Theorem 5.1.6. To accomplish this, we consider an important class of rooted trees which are called broom graphs.

Example 5.1.7. ([19, Definition 4.3]) A broom graph Γ of type t consists of a handle, which is a directed path $x_0, \dots, x_s$, such that every edge in Γ (not on the handle) has the form (x_i, y_j) for some $i \geq s + 1 - t$ and $y_j \in V(\Gamma) \setminus \{x_0, \dots, x_s\}$.

Suppose that Γ is a broom graph with x_0 as the root and vertex set

$$V(\Gamma) = \{x_0, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, y_5, y_6, y_7\},$$

and the following edge set

$$\begin{aligned} E(\Gamma) = \{ & (x_0, x_1), (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_3, y_1), (x_3, y_2), \\ & (x_3, y_3), (x_4, y_4), (x_4, y_5), (x_4, y_6), (x_4, y_7) \}. \end{aligned}$$

Based on Theorem 5.1.6, one can easily see that the square-free monomial ideal

$$\begin{aligned} I_2(\Gamma)^\vee = & (x_0, x_1, x_2) \cap (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_2, x_3, y_1) \cap (x_2, x_3, y_2) \\ & \cap (x_2, x_3, y_3) \cap (x_3, x_4, y_4) \cap (x_3, x_4, y_5) \cap (x_3, x_4, y_6) \cap (x_3, x_4, y_7), \end{aligned}$$

is normally torsion-free.

5.2 Normally torsion-freeness and the Alexander dual

In this section, our aim is to show that the Alexander dual of the monomial ideal generated by the paths of maximal lengths in an unrooted starlike tree (respectively, a rooted starlike tree) has the persistence property (respectively, is normally torsion-free). It should be noted that the results of this section can be found in [102].

Throughout this section, for a simple finite graph $G = (V(G), E(G))$, the symbol L_G stands for the set of leaves of G (i.e., the set of vertices of degree one in G). We also denote the distance between two vertices u and v in $V(G)$ by $d(u, v)$.

We now explain one of the main results in this section in the following theorem.

Theorem 5.2.1. Let T be a rooted starlike tree on the vertex set $\{z, 1, \dots, n\}$ with root z . Let I be the monomial ideal corresponding to T which is generated by the paths of maximal lengths such that every path is naturally oriented away from z , and corresponding Alexander dual J . Then the ideal J is normally torsion-free.

Proof. Suppose that $\deg_T z = k$. Hence, $T \setminus \{z\}$ has exactly k connected components, say $L_1, \dots, L_k$, where each component is a path graph with $|V(L_i)| = h_i$ for any $i = 1, \dots, k$. Set $h_0 := 0$ and for all $i = 1, \dots, k$, let

$$V(L_i) := \{h_1 + \dots + h_{i-1} + j : j = 1, \dots, h_i\}.$$

This implies that $E(T)$ is given by

$$\{(z, \sum_{t=1}^{i-1} h_t + 1), (\sum_{t=1}^{i-1} h_t + j, \sum_{t=1}^{i-1} h_t + j + 1) : i = 1, \dots, k, j = 1, \dots, h_i - 1\}.$$

In the light of Lemma 2.2.5, one can conclude that $x_z \in \mathcal{G}(J)$. Assume that $\mathcal{G}(J) = \{x_z, u_1, \dots, u_r\}$. Fix an arbitrary positive integer s . If $\mathfrak{p} \in \text{Ass}_{R'}(R'/J^s)$, where $R' = K[x_z, x_1, \dots, x_n]$ over a field K , it follows from [50, Lemma 3.4] that $\mathfrak{p} = (x_z) + \mathfrak{q}$ for some $\mathfrak{q} \in \text{Ass}_R(R/I^s)$, where $R = K[x_1, \dots, x_n]$ and $I = (u_1, \dots, u_r)$. Thus, it is sufficient to show that I is normally torsion-free. Now, put

$$\mathfrak{p}_i := (x_{h_1 + \dots + h_{i-1} + j} : j = 1, \dots, h_i),$$

for all $i = 1, \dots, k$. As $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ are generated by disjoint non-empty subsets of $\{x_1, \dots, x_n\}$, according to [113, Remark 2.8], for all positive integers $r_1, \dots, r_k$, one has the following equalities

$$\text{Ass}_R(R/\mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_k^{r_k}) = \bigcup_{i=1}^k \text{Ass}_R(R/\mathfrak{p}_i) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}.$$

Furthermore, [113, Theorem 2.7] implies that $\mathfrak{p}_1^{r_1} \cdots \mathfrak{p}_k^{r_k} = \mathfrak{p}_1^{r_1} \cap \cdots \cap \mathfrak{p}_k^{r_k}$ for all positive integers $r_1, \dots, r_k$. Since $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_k$, we obtain

$$\text{Ass}_R(R/I^s) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\} = \text{Ass}_R(R/I).$$

Therefore, I is normally torsion-free, and so J is also normally torsion-free, as claimed. $\square$

Corollary 5.2.2. *Suppose that T is a rooted symmetric starlike tree on the vertex set $V(T) = \{z, 1, \dots, n\}$ with root z such that every path is naturally oriented away from z , and the following edge set*

$$E(T) = \{(z, i), (kj + i, kj + k + i) \mid i = 1, \dots, k \text{ and } j = 0, \dots, m - 1\},$$

such that $n = k(m + 1)$ for some $k \in \mathbb{N}$ and $m \in \mathbb{N}_0$. Suppose also that

$$J_{m+1} := \bigcap_{i=1}^k (x_z, x_i, x_{k+i}, \dots, x_{mk+i}).$$

Then J_{m+1} is normally torsion-free.

In order to verify another main result in this section, we need the following lemma. To recall the notion of the expansion operator on monomial ideals refer to Definition 2.2.9.

Lemma 5.2.3. *Let I be a monomial ideal of R . Then I is normally torsion-free if and only if I^* is.*

Proof. For necessity, consider $\mathfrak{p} \in \text{Ass}_R(R/I^n)$ for an arbitrary $n \in \mathbb{N}$. Due to Proposition 2.2.10, it follows that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/(I^n)^*)$. According to Lemma 2.2.11(iii), we have $(I^n)^* = (I^*)^n$. Hence, $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/(I^*)^n)$. By hypothesis, we deduce that $\mathfrak{p}^* \in \text{Ass}_{R^*}(R^*/I^*)$. In view of Proposition 2.2.10, one can conclude that $\mathfrak{p} \in \text{Ass}_R(R/I)$. Therefore, I is a normally torsion-free ideal of R . The sufficiency part follows from in a similar way, and the proof is complete. $\square$

In the following example we clarify the importance of Lemma 5.2.3.

Example 5.2.4. *Consider the following monomial ideal*

$$\begin{aligned} J := & (x_{1,1}x_{3,1}x_{4,1}, x_{1,1}x_{3,1}x_{4,2}, x_{1,1}x_{3,2}x_{4,1}, x_{1,1}x_{3,2}x_{4,2}, x_{1,2}x_{3,1}x_{4,1}, \\ & x_{1,2}x_{3,1}x_{4,2}, x_{1,2}x_{3,2}x_{4,1}, x_{1,2}x_{3,2}x_{4,2}, x_{1,1}x_{5,1}, x_{1,2}x_{5,1}, x_{1,1}x_{5,2}, \\ & x_{1,2}x_{5,2}, x_{1,1}x_{5,3}, x_{1,2}x_{5,3}, x_{2,1}x_{3,1}x_{4,1}, x_{2,1}x_{3,1}x_{4,2}, x_{2,1}x_{3,2}x_{4,1}, \\ & x_{2,1}x_{3,2}x_{4,2}), \end{aligned}$$

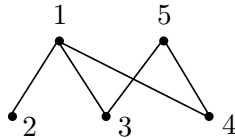
in the following polynomial ring, where K is a field,

$$R^* = K[x_{1,1}, x_{1,2}, x_{2,1}, x_{3,1}, x_{3,2}, x_{4,1}, x_{4,2}, x_{5,1}, x_{5,2}, x_{5,3}].$$

Now, set $\mathfrak{p}_1 := (x_{1,1}, x_{1,2})$, $\mathfrak{p}_2 := (x_{2,1})$, $\mathfrak{p}_3 := (x_{3,1}, x_{3,2})$, $\mathfrak{p}_4 := (x_{4,1}, x_{4,2})$, and $\mathfrak{p}_5 := (x_{5,1}, x_{5,2}, x_{5,3})$. Thus, it is routine to check that

$$J = \mathfrak{p}_1\mathfrak{p}_3\mathfrak{p}_4 + \mathfrak{p}_1\mathfrak{p}_5 + \mathfrak{p}_2\mathfrak{p}_3\mathfrak{p}_4.$$

Consider the square-free monomial ideal $I := (x_1x_3x_4, x_1x_5, x_2x_3x_4)$ in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. One can easily see that J is the expansion of I with respect to the 5-tuple $(2, 1, 2, 2, 3)$. Here and in the following, our aim is to show that I is normally torsion-free. To do this, consider the graph G , as shown in the figure below, on the vertex set $V(G) := [5] = \{1, 2, 3, 4, 5\}$ and the following edge set $E(G) := \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{3, 5\}, \{4, 5\}\}$.



It is clear that G is a bipartite graph with the following edge ideal

$$I(G) := (x_1x_2, x_1x_3, x_1x_4, x_3x_5, x_4x_5),$$

and the following cover ideal

$$J(G) := (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_4) \cap (x_3, x_5) \cap (x_4, x_5).$$

One can easily compute that $J(G) = I$. On the other hand, according to [48, Corollary 2.6], it follows that $J(G)$ is normally torsion-free. Hence, I is normally torsion-free and Lemma 5.2.3 implies that the ideal J is also normally torsion-free.

The subsequent definition is essential for us to provide Theorem 5.2.7.

Definition 5.2.5. An $(k_1, k_2, \dots, k_r)$ -banana tree is a graph obtained by connecting one leaf of each of an k_i -star graph, for all $i = 1, \dots, r$, with a single root vertex that is distinct from all the stars.

It should be observed that, for all $i = 1, \dots, r$, the number k_i in the definition of an $(k_1, k_2, \dots, k_r)$ -banana tree refers to the total number of vertices in the associated star graph. Furthermore, it is necessary to note that when $k_1 = \dots = k_r = k$, we get an (r, k) -banana tree, as defined by Chen et al. [26].

We next give two examples which illustrate our definitions.

Examples 5.2.6. (i) Suppose that T is the tree which is shown in Figure 5. One can easily see that T is a $(4, 5, 7)$ -banana tree.

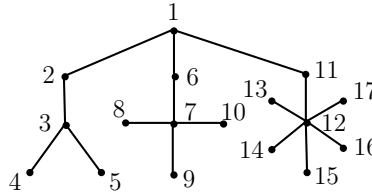


Figure 5

(ii) Assume that T is the tree which is shown in Figure 6. It is routine to check that T is a $(3, 5)$ -banana tree.

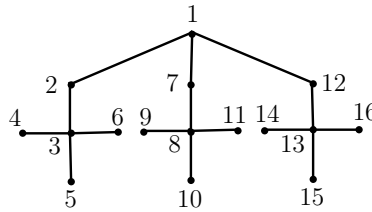


Figure 6

We are now in a position to prove another main result in this section.

Theorem 5.2.7. Suppose that T is a rooted $(k_1, k_2, \dots, k_r)$ -banana tree on the vertex set $V(T) = \{i \in \mathbb{N} : i = 1, 2, \dots, k_1 + k_2 + \dots + k_r + 1\}$ with vertex 1 chosen as root, $s_0 := 0$, $s_i := \sum_{t=1}^i k_t$, and the edge set $E(T)$ is given by

$$\{(1, s_i+2), (s_i+2, s_i+3), (s_i+3, s_i+j) : i = 0, 1, \dots, r-1 \text{ and } j = 4, 5, \dots, k_{i+1}+1\}.$$

Suppose also that

$$I_2 := (x_1 x_{s_i+2} x_{s_i+3}, x_{s_i+2} x_{s_i+3} x_{s_i+j} : i = 0, 1, \dots, r-1 \text{ and } j = 4, 5, \dots, k_{i+1}+1),$$

and J_2 is the Alexander dual of I_2 . Then the ideal J_2 is normally torsion-free.

Proof. It follows from the hypothesis that

$$J_2 = \bigcap_{i=0}^{r-1} \bigcap_{j=4}^{k_{i+1}+1} ((x_1, x_{s_i+2}, x_{s_i+3}) \cap (x_{s_i+2}, x_{s_i+3}, x_{s_i+j})).$$

Suppose that $R = K[x_i : 1 \leq i \leq \sum_{t=1}^r k_t + 1]$ and set $\mathbf{q}_i := (x_{s_i+2}, x_{s_i+3})$ for all $i = 0, 1, \dots, r-1$. Thus, $J_2 = \bigcap_{i=0}^{r-1} (\mathbf{q}_i + x_1 \prod_{j=4}^{k_{i+1}+1} x_{s_i+j} R)$. Now, put $\mathbf{p}_1 := x_1 R$, $\mathbf{p}_{s_i+2} := \mathbf{q}_i$ for all $i = 0, 1, \dots, r-1$, and $\mathbf{p}_{s_i+j} := x_{s_i+j} R$ for all $i = 0, 1, \dots, r-1$ and $j = 4, 5, \dots, k_{i+1}+1$. One can deduce that

$$J_2 = \bigcap_{i=0}^{r-1} (\mathbf{p}_{s_i+2} + \mathbf{p}_1 \prod_{j=4}^{k_{i+1}+1} \mathbf{p}_{s_i+j}).$$

Let F be the following monomial ideal with $k_1 + k_2 + \dots + k_r - r + 1$ variables

$$F := \bigcap_{i=0}^{r-1} (x_{s_i+2} R + x_1 \prod_{j=4}^{k_{i+1}+1} x_{s_i+j} R).$$

Accordingly, one can easily see that J_2 is the expansion of F . Our next aim is to show that F is normally torsion-free. To do this, consider the graph G on the following vertex set $V(G)$

$$\{s_i+2 : i = 0, 1, \dots, r-1\} \cup \{1, s_i+j : i = 0, 1, \dots, r-1 \text{ and } j = 4, 5, \dots, k_{i+1}+1\},$$

and the following edge set $E(G)$

$$\{\{x_{s_i+2}, x_1\}, \{x_{s_i+2}, x_{s_i+j}\} : i = 0, 1, \dots, r-1 \text{ and } j = 4, 5, \dots, k_{i+1}+1\}.$$

Since

$$\bigcap_{i=0}^{r-1} (x_{s_i+2} R + x_1 \prod_{j=4}^{k_{i+1}+1} x_{s_i+j} R) = \bigcap_{i=0}^{r-1} (x_{s_i+2}, x_1) \cap \bigcap_{i=0}^{r-1} \bigcap_{j=4}^{k_{i+1}+1} (x_{s_i+2}, x_{s_i+j}),$$

one has $G = (V(G), E(G))$ is a bipartite graph such that F is the cover ideal of G . On the other hand, according to [48, Corollary 2.6], it follows that F is normally torsion-free, and Lemma 5.2.3 implies that the ideal J_2 is also normally torsion-free, as claimed. $\square$

We conclude this section with the corollary below.

Corollary 5.2.8. *Suppose that T is a rooted (n, k) -banana tree on the vertex set $V(T) = \{i \in \mathbb{N} : i = 1, 2, \dots, nk + 1\}$ with vertex 1 chosen as root and the edge set $E(T)$ is given by*

$$\{(1, ik+2), (ik+2, ik+3), (ik+3, ik+j) : i = 0, 1, \dots, n-1 \text{ and } j = 4, 5, \dots, k+1\}.$$

Suppose also that

$$I_2 := (x_1x_{ik+2}x_{ik+3}, x_{ik+2}x_{ik+3}x_{ik+j} : i = 0, 1, \dots, n-1 \text{ and } j = 4, 5, \dots, k+1),$$

and J_2 is the Alexander dual of I_2 . Then the ideal J_2 is normally torsion-free.

5.3 Normally torsion-freeness under some monomial operators

A few examples of normally torsion-free monomial ideals appear from graph theory. Already, in [126], it has been proved that a finite simple graph G is bipartite if and only if its edge ideal is normally torsion-free. Moreover, by [48], it is well-known that the cover ideals of bipartite graphs are normally torsion-free. In addition, in [66], it has been verified that every transversal polymatroidal ideal is normally torsion-free. However, normally torsion-free square-free monomial ideals have been studied in [50, 126, 128], but little is known for the normally torsion-free monomial ideals which are not square-free, for example see [110]. In this section, we continue this argument and by using some monomial operators such as expansion, weighted, monomial multiple, monomial localization, contraction, and deletion, we introduce several methods for constructing new normally torsion-free monomial ideals based on the monomial ideals which have normally torsion-freeness. It should be noted that the results of this section can be found in [120].

We start our argument with the proposition below, which says that for an arbitrary ideal in a commutative Noetherian ring, there exists some power of it, which is normally torsion-free.

Proposition 5.3.1. *Let I be an ideal in a commutative Noetherian ring S . Then there exists a positive integer t such that I^t is a normally torsion-free ideal.*

Proof. Assume that t is the index of stability of I , i.e., $\text{Ass}_R(R/I^m) = \text{Ass}_R(R/I^t)$ for all $m \geq t$. Fix $k \in \mathbb{N}$, and consider $\mathfrak{p} \in \text{Ass}_R(R/(I^t)^k)$. The assumption yields that $\text{Ass}_R(R/I^{tk}) = \text{Ass}_R(R/I^t)$, and so $\mathfrak{p} \in \text{Ass}_R(R/I^t)$. This means that I^t is normally torsion-free, as required. $\square$

The following lemma guarantees that any power of a normally torsion-free square-free monomial ideal, is normally torsion-free as well.

Lemma 5.3.2. *Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$. If I is normally torsion-free, then, for all positive integers s , I^s is normally torsion-free.*

Proof. Assume that I is a normally torsion-free square-free monomial ideal of R . Fix positive integers k and s , and pick $\mathfrak{p} \in \text{Ass}_R(R/(I^s)^k)$. It follows from the assumption that $\text{Ass}_R(R/I^{sk}) \subseteq \text{Ass}_R(R/I)$, and so $\mathfrak{p} \in \text{Ass}_R(R/I)$. Since I is a square-free monomial ideal, one has $\text{Ass}_R(R/I) = \text{Ass}_R(R/I^s)$. This yields that $\mathfrak{p} \in \text{Ass}_R(R/I^s)$, and hence $\text{Ass}_R(R/(I^s)^k) \subseteq \text{Ass}_R(R/I^s)$, that is, I^s is normally torsion-free, as desired. $\square$

In the following, we focus on the normally torsion-freeness of the sum of two monomial ideals. For this purpose, we first recall the following well-known result.

Theorem 5.3.3. *([126, Theorem 5.4]) Let R_1 and R_2 be polynomial rings over a field k , and let $I_1 \subset R_1$ and $I_2 \subset R_2$ be reduced ideals. Suppose the total rings of fractions of R_1/I_1 and R_2/I_2 are separably generated over k . If I_1 and I_2 are normally torsion-free, then $I = (I_1, I_2)$ is a reduced, normally torsion-free ideal of $R = R_1 \otimes_k R_2$.*

We are now ready to express Theorem 5.3.4, which is related to the normally torsion-freeness of the sum of two normally torsion-free monomial ideals.

Theorem 5.3.4. *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$ such that $I = I_1R + I_2R$, where $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer $1 \leq m < n$. If I_1 and I_2 are normally torsion-free, then I is so.*

Proof. Assume that I_1 and I_2 are normally torsion-free monomial ideals. Fix $s \in \mathbb{N}$, and let $\mathfrak{p} \in \text{Ass}_R(R/I^s)$. This implies that $\mathfrak{p} = (I^s :_R c)$ for some monomial c in R . Certainly, one can write $c = c_1c_2$ with $c_1 \in R_1$ and $c_2 \in R_2$. By virtue of Lemma

3.2.7, one has $I^s = \bigcap_{i=1}^s (I_1^i + I_2^{s+1-i})$. Accordingly, we obtain the following equalities

$$\begin{aligned}
\mathfrak{p} &= (I^s :_R c) \\
&= \left(\bigcap_{i=1}^s (I_1^i + I_2^{s+1-i}) :_R c \right) \\
&= \bigcap_{i=1}^s (I_1^i + I_2^{s+1-i} :_R c) \\
&= \bigcap_{i=1}^s ((I_1^i :_R c_1 c_2) + (I_2^{s+1-i} :_R c_1 c_2)) \\
&= \bigcap_{i=1}^s ((I_1^i :_{R_1} c_1) + (I_2^{s+1-i} :_{R_2} c_2)).
\end{aligned}$$

Consequently, one can conclude that there exists a positive integer r with $1 \leq r \leq s$ such that $\mathfrak{p} = (I_1^r :_{R_1} c_1) + (I_2^{s+1-r} :_{R_2} c_2)$. On the other hand, one can write $\mathfrak{p} = \mathfrak{p}_1 R + \mathfrak{p}_2 R$ such that $\mathfrak{p}_1 \subset R_1$ and $\mathfrak{p}_2 \subset R_2$ are monomial prime ideals. This gives that $(I_1^r :_{R_1} c_1) = \mathfrak{p}_1$ and $(I_2^{s+1-r} :_{R_2} c_2) = \mathfrak{p}_2$. Therefore, $\mathfrak{p}_1 \in \text{Ass}_{R_1}(R_1/I_1^r)$ and $\mathfrak{p}_2 \in \text{Ass}_{R_2}(R_2/I_2^{s+1-r})$. Based on I_1 and I_2 are normally torsion-free, we have $\mathfrak{p}_1 \in \text{Ass}_{R_1}(R_1/I_1)$ and $\mathfrak{p}_2 \in \text{Ass}_{R_2}(R_2/I_2)$. Hence, one can deduce that there exist monomials $d_1 \in R_1$ and $d_2 \in R_2$ such that $\mathfrak{p}_1 = (I_1 :_{R_1} d_1)$ and $\mathfrak{p}_2 = (I_2 :_{R_2} d_2)$. Thus, one gains the following equalities

$$\begin{aligned}
(I :_R d_1 d_2) &= (I_1 R + I_2 R :_R d_1 d_2) \\
&= (I_1 :_R d_1 d_2) + (I_2 :_R d_1 d_2) \\
&= (I_1 :_{R_1} d_1) + (I_2 :_{R_2} d_2) \\
&= \mathfrak{p}_1 + \mathfrak{p}_2 \\
&= \mathfrak{p}.
\end{aligned}$$

Therefore, $\mathfrak{p} \in \text{Ass}_R(R/I)$, and so I is normally torsion-free, as required. $\square$

In what follows, we introduce several methods for constructing new classes of monomial ideals which have normally torsion-freeness. To accomplish this purpose, we begin with the first method. Thus, we need to recall the definition of the expansion operator on monomial ideals, which has been stated in Definition 2.2.9.

The subsequent theorem reconsiders a powerful tool for producing new normally torsion-free monomial ideals. Indeed, it says that a monomial ideal is normally torsion-free if and only if its expansion is normally torsion-free.

Theorem 5.3.5. (see Lemma 5.2.3) *Let I be a monomial ideal of R . Then I is normally torsion-free if and only if I^* is.*

Here, we state the second method for constructing new normally torsion-free monomial ideals. To achieve this, one requires to recall Definition 3.2.11. Some basic properties of the weighted operator are given in Lemma 3.2.12, which are essential for us to prove Theorem 5.3.10. To demonstrate Proposition 5.3.8, one needs to apply Proposition 5.3.6 and Corollary 5.3.7.

Proposition 5.3.6. *Let $Q = (x_{i_1}^{\ell_1}, \dots, x_{i_r}^{\ell_r})$ be an irreducible monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$. Then the following statements hold.*

- (i) *If I is a monomial ideal of R such that $Q \subsetneq I$, then there exist positive integers $\alpha_{r+1}, \dots, \alpha_n$ such that $x_{i_1}^{\ell_1-1} \dots x_{i_r}^{\ell_r-1} x_{i_{r+1}}^{\alpha_{r+1}} \dots x_{i_n}^{\alpha_n} \in I$, where $\{i_1, \dots, i_r, i_{r+1}, \dots, i_n\} = \{1, \dots, n\}$.*
- (ii) *If I and J are two monomial ideals of R such that $Q = I \cap J$, then $Q = I$ or $Q = J$.*

Proof. (i) It follows from $Q \subsetneq I$ that there exists a monomial $u \in R$ such that $u \in I \setminus Q$. Assume that $u = x_{i_1}^{\alpha_1} \dots x_{i_r}^{\alpha_r} x_{i_{r+1}}^{\alpha_{r+1}} \dots x_{i_n}^{\alpha_n}$ for some nonnegative integers $\alpha_1, \dots, \alpha_n$. Because $u \notin Q$, one can conclude that $x_{i_j}^{\ell_j} \nmid u$ for each $j = 1, \dots, r$. This implies that $\ell_i > \alpha_i$ for each $i = 1, \dots, r$. Hence, we get $\ell_i - 1 \geq \alpha_i$ for each $i = 1, \dots, r$, and so $x_{i_1}^{\ell_1-1} \dots x_{i_r}^{\ell_r-1} x_{i_{r+1}}^{\alpha_{r+1}} \dots x_{i_n}^{\alpha_n} = (x_{i_1}^{\ell_1-1-\alpha_1} \dots x_{i_r}^{\ell_r-1-\alpha_r}) \cdot u \in I$.

(ii) We prove the contrapositive of this claim, i.e., if $Q \subsetneq I$ and $Q \subsetneq J$, then $Q \subsetneq I \cap J$. Because $Q \subsetneq I$, part (i) implies that there exist positive integers $\lambda_{r+1}, \dots, \lambda_n$ such that $x_{i_1}^{\ell_1-1} \dots x_{i_r}^{\ell_r-1} x_{i_{r+1}}^{\lambda_{r+1}} \dots x_{i_n}^{\lambda_n} \in I$. Once again, by virtue of $Q \subsetneq J$, part (i) gives that there exist positive integers $\theta_{r+1}, \dots, \theta_n$ such that $x_{i_1}^{\ell_1-1} \dots x_{i_r}^{\ell_r-1} x_{i_{r+1}}^{\theta_{r+1}} \dots x_{i_n}^{\theta_n} \in J$. Now, set $v := x_{i_1}^{\ell_1-1} \dots x_{i_r}^{\ell_r-1} x_{i_{r+1}}^{\beta_{r+1}} \dots x_{i_n}^{\beta_n}$, where $\beta_i := \max\{\lambda_i, \theta_i\}$ for each $i = r+1, \dots, n$. It is routine to check that $v \in I \cap J \setminus Q$. Therefore, $Q \subsetneq I \cap J$, as required. $\square$

Corollary 5.3.7. *Let Q be an irreducible monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$, and $I_1, \dots, I_s$ be some monomial ideals of R . If $Q = I_1 \cap \dots \cap I_s$, then $Q = I_j$ for some j with $1 \leq j \leq s$.*

Proof. Follow by induction on s , and apply Proposition 5.3.6. $\square$

Proposition 5.3.8. *Let Q be an irreducible monomial ideal and $Q_1, \dots, Q_s$ be some primary monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$. If $Q_1 \cap \dots \cap Q_s \subseteq Q$, then $Q_j \subseteq Q$ for some j with $1 \leq j \leq s$.*

Proof. Due to $Q_1 \cap \dots \cap Q_s \subseteq Q$, one derives $Q_1 \cap \dots \cap Q_s + Q = Q$. Here, our aim is to show that

$$Q_1 \cap \dots \cap Q_s + Q = (Q_1 + Q) \cap \dots \cap (Q_s + Q).$$

Since $Q_1 \cap \cdots \cap Q_s \subseteq Q_i$ for each $i = 1, \dots, s$, one has $Q_1 \cap \cdots \cap Q_s + Q \subseteq Q_i + Q$ for each $i = 1, \dots, s$. This implies that $Q_1 \cap \cdots \cap Q_s + Q \subseteq (Q_1 + Q) \cap \cdots \cap (Q_s + Q)$. Conversely, we verify that $(Q_1 + Q) \cap \cdots \cap (Q_s + Q) \subseteq Q_1 \cap \cdots \cap Q_s + Q$. To do this, consider $u \in \mathcal{G}((Q_1 + Q) \cap \cdots \cap (Q_s + Q))$. This yields that $u \in Q_i + Q$ for each $i = 1, \dots, s$. Hence, one can conclude that $u \in Q_i$ or $u \in Q$ for each $i = 1, \dots, s$. If $u \in Q$, then $u \in Q_1 \cap \cdots \cap Q_s + Q$, and the argument is complete. Thus, let $u \notin Q$. This implies that $u \in Q_i$ for each $i = 1, \dots, s$, and so $u \in Q_1 \cap \cdots \cap Q_s$. Consequently, one has $u \in Q_1 \cap \cdots \cap Q_s + Q$. Therefore, we get

$$Q_1 \cap \cdots \cap Q_s + Q = (Q_1 + Q) \cap \cdots \cap (Q_s + Q).$$

Accordingly, one obtains $Q = (Q_1 + Q) \cap \cdots \cap (Q_s + Q)$. It follows now from Corollary 5.3.7 that there exists a positive integer j with $1 \leq j \leq s$ such that $Q_j + Q = Q$. This gives that $Q_j \subseteq Q$, as claimed. $\square$

The following lemma probes the relation between the set of associated primes of a monomial ideal and the set of associated primes of its weighted.

Lemma 5.3.9. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$, and W a weight over R . Then $\text{Ass}_R(R/I_W) = \text{Ass}_R(R/I)$.*

Proof. Suppose that the weight $W : \{x_1, \dots, x_n\} \rightarrow \mathbb{N}$ is given by $w_i = W(x_i)$ for each $i = 1, \dots, n$. Moreover, by virtue of [60, Theorem 1.3.1], assume that $I = Q_1 \cap \cdots \cap Q_s$ is the irredundant presentation of I . From now on, fix $1 \leq i \leq s$, and let $Q_i = (x_{i_1}^{\alpha_1}, \dots, x_{i_r}^{\alpha_r})$ with $i_1, \dots, i_r \in \{1, \dots, n\}$ and $\alpha_1, \dots, \alpha_r$ be positive integers. It follows from the definition of weighted ideal that $(Q_i)_W = (x_{i_1}^{\alpha_1 w_1}, \dots, x_{i_r}^{\alpha_r w_r})$. This concludes that $(Q_i)_W$ is a $(x_{i_1}, \dots, x_{i_r})$ -primary ideal of R . Accordingly, one can deduce that $I_W = (Q_1)_W \cap \cdots \cap (Q_s)_W$ is a presentation of I_W . To complete the proof, one has to demonstrate that this presentation is irredundant. On the contrary, assume that this presentation is not irredundant, that is, there exists a positive integer j with $1 \leq j \leq s$ such that

$$(Q_1)_W \cap \cdots \cap (Q_{j-1})_W \cap (Q_{j+1})_W \cap \cdots \cap (Q_s)_W \subseteq (Q_j)_W.$$

Now, Proposition 5.3.8 yields that there exists a positive integer $1 \leq t \leq s$ with $t \neq j$ such that $(Q_t)_W \subseteq (Q_j)_W$. Thus, we get $Q_t \subseteq Q_j$, which shows that Q_j is redundant in the presentation of I , a contradiction. Hence, one derives that $I_W = (Q_1)_W \cap \cdots \cap (Q_s)_W$ is the irredundant presentation of I_W . This leads to $\text{Ass}_R(R/I_W) = \text{Ass}_R(R/I)$, and the proof is over. $\square$

We are now ready to prove Theorem 5.3.10 by using Lemmas 3.2.12 and 5.3.9. In fact, it tells us that a monomial ideal is normally torsion-free if and only if its weighted is normally torsion-free.

Theorem 5.3.10. *Let I be a monomial ideal of R , and W a weight over R . Then I is normally torsion-free if and only if I_W is.*

Proof. For the forward implication, assume that I is a normally torsion-free monomial ideal of R . Fix $k \in \mathbb{N}$, and consider $\mathfrak{p} \in \text{Ass}_R(R/I_W^k)$. On account of Lemma 3.2.12(ii), one has $I_W^k = (I^k)_W$. This gives that $\mathfrak{p} \in \text{Ass}_R(R/(I^k)_W)$. Due to Lemma 5.3.9, we obtain $\mathfrak{p} \in \text{Ass}_R(R/I^k)$. Since I is normally torsion-free, one can conclude that $\mathfrak{p} \in \text{Ass}_R(R/I)$. Once again, Lemma 5.3.9 implies that $\mathfrak{p} \in \text{Ass}_R(R/I_W)$. This means that I_W is a normally torsion-free monomial ideal of R .

For the converse, assume that I_W is a normally torsion-free monomial ideal of R . Fix $k \in \mathbb{N}$, and pick $\mathfrak{p} \in \text{Ass}_R(R/I^k)$. In the light of Lemma 5.3.9, one derives $\mathfrak{p} \in \text{Ass}_R(R/(I^k)_W)$. It follows from Lemma 3.2.12(ii) that $I_W^k = (I^k)_W$. This yields that $\mathfrak{p} \in \text{Ass}_R(R/I_W^k)$. Because I_W is normally torsion-free, one can deduce that $\mathfrak{p} \in \text{Ass}_R(R/I_W)$. Once again, Lemma 5.3.9 concludes that $\mathfrak{p} \in \text{Ass}_R(R/I)$, and thus I is a normally torsion-free monomial ideal of R . $\square$

The subsequent lemma says that a monomial ideal is normally torsion-free if and only if its monomial multiple is normally torsion-free.

Lemma 5.3.11. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, and $h = x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ with $j_1, \dots, j_s \in \{1, \dots, n\}$ be a monomial in R . Then I is normally torsion-free if and only if hI is normally torsion-free.*

Proof. For necessity, assume that I is normally torsion-free, and fix $k \in \mathbb{N}$. Thanks to Theorem 2.1.27, we get $\text{Ass}_R(R/hI) = \text{Ass}_R(R/I) \cup \{(x_{j_1}), \dots, (x_{j_s})\}$, and also $\text{Ass}_R(R/(hI)^k) = \text{Ass}_R(R/I^k) \cup \{(x_{j_1}), \dots, (x_{j_s})\}$. Since I is normally torsion-free, one has $\text{Ass}_R(R/I^k) \subseteq \text{Ass}_R(R/I)$. Hence, we get $\text{Ass}_R(R/(hI)^k) \subseteq \text{Ass}_R(R/hI)$, that is, hI is normally torsion-free, as claimed.

To prove the sufficiency, let hI be normally torsion-free, and fix $k \in \mathbb{N}$. This yields that $\text{Ass}_R(R/(hI)^k) \subseteq \text{Ass}_R(R/hI)$. It follows from Theorem 2.1.27 that

$$\text{Ass}_R(R/I^k) \cup \{(x_{j_1}), \dots, (x_{j_s})\} \subseteq \text{Ass}_R(R/I) \cup \{(x_{j_1}), \dots, (x_{j_s})\}.$$

Pick $\mathfrak{p} \in \text{Ass}_R(R/I^k)$. Hence, we obtain $\mathfrak{p} \in \text{Ass}_R(R/I) \cup \{(x_{j_1}), \dots, (x_{j_s})\}$. If $\mathfrak{p} \in \text{Ass}_R(R/I)$, then the proof is complete. Thus, let $\mathfrak{p} = (x_{j_t})$ for some positive integer t with $1 \leq t \leq s$. This means that $\mathfrak{p} \in \text{Min}(I^k)$, and by virtue of $\text{Min}(I^k) = \text{Min}(I)$, one derives $\mathfrak{p} \in \text{Min}(I)$. Consequently, one has $\mathfrak{p} \in \text{Ass}_R(R/I)$. Therefore, I is normally torsion-free, as required. $\square$

In order to prove Theorem 5.3.12, we need to recall the notion of the monomial localization of a monomial ideal with respect to a monomial prime ideal as has been introduced in Definition 3.1.14.

Theorem 5.3.12. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$, and $\mathfrak{p} \in V^*(I)$. If I is normally torsion-free, then $I(\mathfrak{p})$ is so.*

Proof. Assume that I is a normally torsion-free monomial ideal of R . Fix $k \in \mathbb{N}$, and consider $\mathfrak{q} \in \text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/I(\mathfrak{p})^k)$. By Lemma 3.2.15(ii), one deduces that $I(\mathfrak{p})^k = I^k(\mathfrak{p})$, and so $\mathfrak{q} \in \text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/I^k(\mathfrak{p}))$. It follows now from Lemma 3.2.15(viii) that $\mathfrak{q} \in \text{Ass}_R(R/I^k)$ and $\mathfrak{q} \subseteq \mathfrak{p}$. Due to I is normally torsion-free, we gain $\mathfrak{q} \in \text{Ass}_R(R/I)$. In the light of $\mathfrak{q} \in \text{Ass}_R(R/I)$ and $\mathfrak{q} \subseteq \mathfrak{p}$, Lemma 3.2.15(viii) yields that $\mathfrak{q} \in \text{Ass}_{R(\mathfrak{p})}(R(\mathfrak{p})/I(\mathfrak{p}))$. Therefore, one can conclude that $I(\mathfrak{p})$ is normally torsion-free, as desired. $\square$

It should be noted that Theorem 5.3.12 helps us to construct new normally torsion-free monomial ideals inspired by some well-known normally torsion-free monomial ideals, as will be mentioned in Corollary 5.3.13.

Corollary 5.3.13. *Let I be a monomial ideal in $R_1 = K[x_1, \dots, x_r]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, and $J_1, \dots, J_m$ be some monomial ideals in $R_2 = K[x_{r+1}, \dots, x_n]$. If $L := u_1 J_1 R + \dots + u_m J_m R$ is normally torsion-free, where $R = K[x_1, \dots, x_n]$, then I is so.*

Proof. Put $\mathfrak{p} := (\bigcup_{i=1}^m \text{supp}(u_i))R$. It is obvious that $\mathfrak{p}$ is a monomial prime ideal of R , and $L \subseteq \mathfrak{p}$. By virtue of Theorem 5.3.12, one derives $L(\mathfrak{p})$ is normally torsion-free. Because $L(\mathfrak{p}) = I$, we get I is normally torsion-free, as claimed. $\square$

It is necessary to observe that the following corollary allows us to refute the normally torsion-freeness of some monomial ideals based on some non-normally torsion-free monomial ideals.

Corollary 5.3.14. *Let I be a monomial ideal in $R_1 = K[x_1, \dots, x_r]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, and $J_1, \dots, J_m$ be some monomial ideals in $R_2 = K[x_{r+1}, \dots, x_n]$. If I is not normally torsion-free, then $L := u_1 J_1 R + \dots + u_m J_m R$ is not normally torsion-free, where $R = K[x_1, \dots, x_n]$.*

To state the subsequent result, one has to recall the definition of the contraction operator, which has been stated in Definition 3.2.19.

Note that the contraction I/x_j is exactly the monomial localization of I with respect to $\mathfrak{p} = \mathfrak{m} \setminus \{x_j\}$, where $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of $R = K[x_1, \dots, x_n]$, then inspired by Lemma 3.2.20 and Theorem 5.3.12, one can easily conclude Theorem 5.3.15.

Theorem 5.3.15. *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$, and $1 \leq j \leq n$. If I is normally torsion-free, then I/x_j is so.*

To express Theorem 5.3.16, we require to recall the definition of the deletion operator, which has been stated in Definition 3.2.19.

Theorem 5.3.16. *Let I be a square-free monomial ideal in $R = K[x_1, \dots, x_n]$, and $1 \leq j \leq n$. If I is normally torsion-free, then $I \setminus x_j$ is so.*

Proof. Assume that I is normally torsion-free. Fix $k \in \mathbb{N}$, and suppose that $\text{Ass}_R(R/I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_r\}$, i.e., we have $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$. Since I is a square-free monomial ideal, [133, Proposition 3.3.26] yields that $I^k = I^{(k)}$, where $I^{(k)}$ denotes the k -th symbolic power of I . On the other hand, it follows from [133, Exercise 5.1.29] that $I^{(k)} = \mathfrak{p}_1^k \cap \dots \cap \mathfrak{p}_r^k$. Consequently, one has $I^k = \mathfrak{p}_1^k \cap \dots \cap \mathfrak{p}_r^k$. Thanks to parts (ii) and (iii) of Lemma 3.2.22, one can conclude the following equalities

$$\begin{aligned} (I \setminus x_j)^k &= I^k \setminus x_j \\ &= (\mathfrak{p}_1^k \cap \dots \cap \mathfrak{p}_r^k) \setminus x_j \\ &= (\mathfrak{p}_1^k \setminus x_j) \cap \dots \cap (\mathfrak{p}_r^k \setminus x_j) \\ &= (\mathfrak{p}_1 \setminus x_j)^k \cap \dots \cap (\mathfrak{p}_r \setminus x_j)^k. \end{aligned}$$

Furthermore, part (v) of Lemma 3.2.22 deduces that

$$(I \setminus x_j)^k = (\mathfrak{p}_1 \setminus x_j)^k \cap \dots \cap (\mathfrak{p}_r \setminus x_j)^k,$$

is a primary decomposition of $(I \setminus x_j)^k$. Suppose that

$$\text{Ass}_{R \setminus x_j}((R \setminus x_j)/(I \setminus x_j)) = \{\mathfrak{p}_{i_1} \setminus x_j, \dots, \mathfrak{p}_{i_t} \setminus x_j\},$$

where $\{i_1, \dots, i_t\} \subseteq \{1, \dots, r\}$. Accordingly, one derives that

$$I \setminus x_j = (\mathfrak{p}_{i_1} \setminus x_j) \cap \dots \cap (\mathfrak{p}_{i_t} \setminus x_j),$$

is the minimal primary decomposition of $I \setminus x_j$. Due to $\mathfrak{p}_{i_1} \setminus x_j, \dots, \mathfrak{p}_{i_t} \setminus x_j$ are minimal associated primes of $I \setminus x_j$, we get

$$(I \setminus x_j)^k = (\mathfrak{p}_{i_1} \setminus x_j)^k \cap \dots \cap (\mathfrak{p}_{i_t} \setminus x_j)^k.$$

Once again, by virtue of $I \setminus x_j = (\mathfrak{p}_{i_1} \setminus x_j) \cap \dots \cap (\mathfrak{p}_{i_t} \setminus x_j)$, [133, Exercise 5.1.29] yields the following equality

$$(I \setminus x_j)^{(k)} = (\mathfrak{p}_{i_1} \setminus x_j)^k \cap \dots \cap (\mathfrak{p}_{i_t} \setminus x_j)^k.$$

Therefore, we obtain $(I \setminus x_j)^{(k)} = (I \setminus x_j)^k$. Now, [133, Proposition 3.3.26] implies that $I \setminus x_j$ is normally torsion-free, as claimed. $\square$

5.4 Neighborhood hypergraphs and their transversal hypergraphs

The edge ideals and cover ideals of bipartite graphs are known to be normally torsion-free, see [126] and [48]. The bipartite graphs are characterized as the graphs that avoid any cycle of odd lengths. With the help of *Macaulay2* [49], one can verify that $NI(C_8)$ and $DI(C_8)$ are not normally torsion-free. This shows that one cannot directly extend the results of [126] and [48] in the case of closed neighborhood ideals and dominating ideals of bipartite graphs. However, if we consider the neighborhood hypergraph $\mathcal{N}(G)$ of a graph G such that there are no hypergraphic cycles in $\mathcal{N}(G)$, then one obtains a similar statement as in the case of edge ideals and cover ideals of bipartite graphs. To establish this, we recall some definitions and notions from hypergraph theory. For more details about these concepts, we refer the reader to [15]. It should be noted that the results of this section can be found in [106].

Let $\mathcal{H}$ be a simple hypergraph with n vertices and m edges, and let M be the incidence matrix of $\mathcal{H}$. A hypergraph $\mathcal{H}$ is called *Mengerian* if for all $\mathbf{c} \in \mathbb{Z}_+^n$, we have the following equality

$$\min\{\mathbf{a} \cdot \mathbf{c} : \mathbf{a} \in \mathbb{Z}_+^m, M \cdot \mathbf{a} \geq \mathbf{1}\} = \max\{\mathbf{b} \cdot \mathbf{1} : \mathbf{b} \in \mathbb{Z}_+^n, M^T \cdot \mathbf{b} \leq \mathbf{c}\},$$

where $\mathbb{Z}_+$ denotes the nonnegative integers. Algebraically, it is equivalent to $I(\mathcal{H})$ being normally torsion-free; it can be found in [60, Corollary 10.3.15] and [133, Theorem 14.3.6]. We next define some of the well-known classes of Mengerian hypergraphs.

Definition 5.4.1. A cycle of length $k \geq 2$ in a hypergraph $\mathcal{H}$ is a sequence

$$\mathcal{C} : x_1, E_1, x_2, E_2, \dots, x_k, E_k, x_1$$

such that

1. $E_1, \dots, E_k$ are distinct edges of $\mathcal{H}$,
2. $x_1, \dots, x_k$ are distinct vertices of $\mathcal{H}$,
3. $x_1, x_k \in E_k$ and $x_i, x_{i+1} \in E_i$ for all $i = 1, \dots, k-1$.

The vertices of $\mathcal{C}$ are $x_1, \dots, x_k$ and the edges of $\mathcal{C}$ are $E_1, \dots, E_k$. Following [60, Section 10.3.4], a cycle $\mathcal{C}$ of length $k \geq 3$ is said to be *special* if all the edges of $\mathcal{C}$ contain exactly two vertices of $\mathcal{C}$. Note that if we reduce to the case of graphs, then the special cycles of length $k \geq 3$ are just the chordless cycles.

A hypergraph $\mathcal{H}$ is called *balanced* if every cycle of $\mathcal{H}$ of odd length $k \geq 3$ has an edge containing at least three vertices of the cycle. Moreover, $\mathcal{H}$ is called *totally balanced* if every cycle of $\mathcal{H}$ of length $k \geq 3$ has an edge containing at least three vertices of the cycle, see [15, Chapter 5]. In other words, $\mathcal{H}$ is balanced if it does not have special odd cycles and $\mathcal{H}$ is totally balanced if it does not have any special cycles. It follows from the definition that every totally balanced hypergraph is also balanced.

Remark 5.4.2. *A well-known result of Fulkerson, Hoffman, and Oppenheim in [46] states that balanced hypergraphs are Mengerian. In particular, totally balanced hypergraphs are also Mengerian. Furthermore, [15, Theorem 19 on page 207] states that transversal hypergraphs of simple balanced hypergraphs are also Mengerian.*

In addition, following [133, Theorem 6.3.39], we can rephrase [122, Lemma 2.2] as follows:

Proposition 5.4.3. *Let G be a graph. Then $DI(G)$ is the edge ideal of the transversal hypergraph of $\mathcal{N}(G)$.*

It is known that the neighborhood hypergraphs of trees are totally balanced, for example see [15, Example 3 on page 174]. We can now combine together Remark 5.4.2 and Proposition 5.4.3 to obtain the following theorem. In particular, it is an affirmative answer to [107, Question 2.8(i)] and [107, Question 2.20(i)].

Theorem 5.4.4. *The closed neighborhood ideals and dominating ideals of trees are normally torsion-free.*

It is natural to ask that if there exists a complete characterization of graphs for which their associated neighborhood hypergraphs are balanced or totally balanced. Such graphs are characterized in [38] and [23]. To describe these graphs we need to recall some definitions from [38] and [23].

Definition 5.4.5. *Let $n \geq 3$. An incomplete trampoline is a chordal graph G on $2n$ vertices whose vertex set can be partitioned into two sets, $W = \{y_1, \dots, y_n\}$ and $U = \{u_1, \dots, u_n\}$ such that no vertices of W are adjacent to each other, and for each i and j , w_i is adjacent to u_i if and only if $i = j$ or $i \equiv j + 1 \pmod{n}$. In addition, an incomplete trampoline is called trampoline if the induced subgraph on U is a complete graph. If n is odd, then (incomplete) trampoline is called an odd (incomplete) trampoline.*

Trampolines on $2n$ vertices are sometimes also mentioned as *n-sun graphs*. It is shown in [23, Theorem 2] that for a given graph G , we have $\mathcal{N}(G)$ is balanced if and only if G is chordal with no induced odd incomplete trampoline. Once again, we can deduce from Remark 5.4.2 and Proposition 5.4.3 the next theorem.

Theorem 5.4.6. *Let G be a chordal graph with no induced odd incomplete trampoline. Then the neighborhood ideal and the dominating ideal of G is normally torsion-free.*

Remark 5.4.7. *It is crucial to mention that there are Mengerian neighborhood hypergraphs that are not balanced. To see this, suppose that $C_6 = (V(C_6), E(C_6))$ is the even cycle graph of order 6 on the vertex set $V(C_6) = \{x_1, \dots, x_6\}$ and the edge set $E(C_6) = \{\{x_i, x_{i+1}\} : i = 1, \dots, 5\} \cup \{\{x_6, x_1\}\}$. We can view the C_9 in a similar way. The cycle graphs C_6 and C_9 are not chordal, and hence [23, Theorem 2] tells us $\mathcal{N}(C_6)$ and $\mathcal{N}(C_9)$ are not balanced. Indeed, the edges of $\mathcal{N}(C_6)$ are*

$$N[x_1] = \{x_1, x_2, x_6\}, N[x_2] = \{x_1, x_2, x_3\}, N[x_3] = \{x_2, x_3, x_4\},$$

$$N[x_4] = \{x_3, x_4, x_5\}, N[x_5] = \{x_4, x_5, x_6\}, N[x_6] = \{x_5, x_6, x_1\},$$

and the sequence $\mathcal{C} : x_1, N[x_2], x_3, N[x_4], x_5, N[x_6], x_1$ gives a special odd cycle in the hypergraph $\mathcal{N}(C_6)$. One can construct a special odd cycle in $\mathcal{N}(C_9)$ in a similar fashion. In Subsection 5.4.2, we will show that not only $NI(C_6)$ and $NI(C_9)$ are normally torsion-free, but also their Alexander duals $DI(C_6)$ and $DI(C_9)$ are normally torsion-free. This highlights that the converse of Theorem 5.4.6 is not true.

Now, we turn our attention to the characterization of graphs for which the associated neighborhood hypergraphs are totally balanced. To accomplish this, one has to recall the following definitions.

A *strong elimination ordering* of G is an ordering $v_1, \dots, v_n$ of $V(G)$ with the property that for each i, j, k , and l , if $i < j$, $k < l$ with $v_k, v_l \in N_G[v_i]$ and $v_k \in N_G[v_j]$, then $v_l \in N_G[v_j]$. Graphs that admit a strong elimination ordering are called *strongly chordal graphs*. The strongly chordal graphs were introduced by Farber in [38]. They form a subclass of chordal graphs, because a strong elimination ordering is also a perfect elimination ordering, by letting $i = k$. Among many other characterizations of strongly chordal graphs, it is shown in [38] that a graph G is strongly chordal if and only if it is chordal and does not contain any induced trampoline. In [38, Theorem 3.3], a polynomial procedure to recognize strongly chordal graphs and to construct strong elimination orderings is given. Some of the special subclasses of strongly chordal graphs are trees and interval graphs. It is proved in [38, Theorem 5.2] and re-proved in [23, Theorem 2] that a graph G is strongly chordal if and only if $\mathcal{N}(G)$ is totally balanced. We provide a neat and straightforward proof of the forward implication in the next theorem.

Theorem 5.4.8. *Let G be a strongly chordal graph. Then $\mathcal{N}(G)$ is totally balanced.*

Proof. Let G be a strongly chordal graph. To simplify the notation in the following text, for any $p, q \in V(G)$ we write $p < q$ if p appears before q in the strong elimination ordering of G . Let

$$\mathcal{C} : x_1, E_1, x_2, E_2, \dots, x_k, E_k, x_1,$$

be a cycle in $\mathcal{N}(G)$ with $k \geq 3$. Let $v_1, \dots, v_k \in G$ be such that $E_i = N[v_i]$ for all $i = 1, \dots, k$. We can rearrange the edges in $\mathcal{C}$ such that $v_1 < v_i$ for all $i = 2, \dots, k$, and $x_2 < x_1$. Since $x_1, x_2 \in E_1 = N[v_1]$ and $x_2 \in E_2 = N[v_2]$, it follows that $x_1 \in E_2$. This shows that $\mathcal{C}$ is not a special cycle, and so $\mathcal{N}(G)$ is totally balanced, as claimed. $\square$

Totally balanced hypergraphs have further nice algebraic implications. It follows from [63, Theorem 2.3] that a simple hypergraph $\mathcal{H}$ is a simplicial forest if and only if $\mathcal{H}$ is totally balanced. Simplicial forests were introduced by Faridi in [41] in the language of simplicial complexes. The facet ideals of simplicial forests possess many nice properties as noted in [41] and [40]. For example, Faridi proved in [40, Corollaries 5.5 and 5.6] that if Δ is a simplicial forest, then its facet ideal $I(\Delta)$ is sequentially Cohen-Macaulay and $I(\Delta)^\vee$ is componentwise linear. More details and related definitions can be found in [41] and [40].

We summarize the discussion above together with Remark 5.4.2 and Proposition 5.4.3, in the next theorem.

Theorem 5.4.9. *Let G be a strongly chordal graph. Then the following statements hold:*

- (i) $NI(G)$ is normally torsion-free.
- (ii) $NI(G)$ is sequentially Cohen-Macaulay.
- (iii) $DI(G)$ is normally torsion-free.
- (iv) $DI(G)$ is componentwise linear.

5.4.1 The t -path hypergraphs of path graphs

Let P_n be a path on n vertices. It follows from [54, Theorem 2.7] that $\mathcal{H}_t(P_n)$ is a simplicial forest and using [63, Theorem 3.2] it can be seen that $\mathcal{H}_t(P_n)$ is totally balanced. Consequently, $I_t(P_n)$ is normally torsion-free. Furthermore, we conclude from Remark 5.4.2 that $I_t(P_n)^\vee$ is normally torsion-free. We provide another proof of these facts by using algebraic tools. To do this, we first introduce a technique to construct new normally torsion-free square-free monomial ideals based on the existing ones. This technique will be implemented in the subsequent result.

Theorem 5.4.10. *Let I be a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ and v be a square-free monomial in R . Let $\mathcal{G}(I) = \{u_1, \dots, u_m\}$ such that $\gcd(v, u_i)$ divides $\gcd(v, u_{i+1})$ for all $i = 1, \dots, m-1$. Then the following statements hold:*

- (i) $I + vR$ is normally torsion-free.
- (ii) $I + vR$ is nearly normally torsion-free.
- (iii) $I + vR$ is normal.
- (iv) $I + vR$ has the strong persistence property.
- (v) $I + vR$ has the persistence property.
- (vi) $I + vR$ has the symbolic strong persistence property.

Proof. (i) To simplify the notation, set $J := I + vR$. If $J \setminus x_t = \mathfrak{m} \setminus x_t$ for some $1 \leq t \leq n$, then we can write $J = x_t L + \mathfrak{m} \setminus x_t$. If $L = R$, then $J = \mathfrak{m}$, and the assertion holds. Assume that $L \neq R$. Let $h \in \mathcal{G}(L)$. If $x_\alpha \mid h$ for some $\alpha \in \{1, \dots, n\} \setminus \{t\}$, this gives that $h \in \mathfrak{m} \setminus x_t$, and hence $L \subseteq \mathfrak{m} \setminus x_t$. We thus get $J = \mathfrak{m} \setminus x_t$, and the proof is over. Thus, we assume that $J \setminus x_t \neq \mathfrak{m} \setminus x_t$ for all $t = 1, \dots, n$. In what follows, we first show that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \text{Min}(J)$. Pick an arbitrary element $\mathfrak{p} \in \text{Min}(J)$. Due to $v \in J$ and $J \subseteq \mathfrak{p}$, we have $v \in \mathfrak{p}$. On the contrary, assume that $v \in \mathfrak{p}^2$. Because v is a square-free monomial, this yields that $|\text{supp}(\mathfrak{p}) \cap \text{supp}(v)| \geq 2$. If $\gcd(v, u_m) = 1$, then $\gcd(v, u_i) = 1$ for all $i = 1, \dots, m-1$. In this case, the claim can be deduced directly from Theorem 5.3.4. Let $\text{supp}(\gcd(v, u_m)) = \{x_{i_1}, \dots, x_{i_k}\}$ with $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. Here, one may consider the following cases:

Case 1. $x_{i_r} \in \mathfrak{p}$ for some $x_{i_r} \in \{x_{i_1}, \dots, x_{i_k}\}$. Since $|\text{supp}(\mathfrak{p}) \cap \text{supp}(v)| \geq 2$, we can take $x_\lambda \in \text{supp}(\mathfrak{p}) \cap \text{supp}(v)$ with $x_\lambda \neq x_{i_r}$. We claim that $x_\lambda \notin \{x_{i_1}, \dots, x_{i_k}\}$. Suppose, on the contrary, that $x_\lambda \in \{x_{i_1}, \dots, x_{i_k}\}$. Let

$$c := \min\{j : x_{i_r} \mid u_j, \text{ where } j = 1, \dots, m\},$$

and

$$d := \min\{j : x_\lambda \mid u_j, \text{ where } j = 1, \dots, m\}.$$

If $c \leq d$ (respectively, $c > d$), then the assumption yields that $I \subseteq \mathfrak{p} \setminus \{x_\lambda\}$ (respectively, $I \subseteq \mathfrak{p} \setminus \{x_{i_r}\}$), and by virtue of $v \in \mathfrak{p} \setminus \{x_\lambda\}$ (respectively, $v \in \mathfrak{p} \setminus \{x_{i_r}\}$), this gives rise to $J \subseteq \mathfrak{p} \setminus \{x_\lambda\}$ (respectively, $J \subseteq \mathfrak{p} \setminus \{x_{i_r}\}$), which contradicts the minimality of $\mathfrak{p}$. Consequently, we conclude that $x_\lambda \notin \{x_{i_1}, \dots, x_{i_k}\}$, and thus $x_\lambda \notin \text{supp}(I)$. In the light of $v \in \mathfrak{p} \setminus \{x_\lambda\}$, this leads to $J \subseteq \mathfrak{p} \setminus \{x_\lambda\}$, a contradiction to the minimality of $\mathfrak{p}$.

Case 2. $x_{i_r} \notin \mathfrak{p}$ for any $x_{i_r} \in \{x_{i_1}, \dots, x_{i_k}\}$. Select two distinct variables $x_\alpha, x_\beta \in \text{supp}(\mathfrak{p}) \cap \text{supp}(v)$. Accordingly, we have $x_\alpha, x_\beta \notin \text{supp}(I)$. We thus get $I \subseteq \mathfrak{p} \setminus \{x_\alpha\}$. On account of $v \in \mathfrak{p} \setminus \{x_\alpha\}$, one can conclude that $J \subseteq \mathfrak{p} \setminus \{x_\alpha\}$. This contradicts the minimality of $\mathfrak{p}$.

Therefore, we deduce that $v \notin \mathfrak{p}^2$. In addition, note that $J \setminus x_t = I \setminus x_t$ for any $x_t \in \text{supp}(v)$. It follows also from Theorem 5.3.16 that $I \setminus x_t$ is normally torsion-free for any $x_t \in \text{supp}(v)$, and so $J \setminus x_t$ is normally torsion-free for any $x_t \in \text{supp}(v)$. Fix $s \geq 1$ and $x_t \in \text{supp}(v)$. Suppose, on the contrary, that $\mathfrak{m} \setminus x_t \in \text{Ass}(R/(J \setminus x_t)^s)$. In view of $\text{Ass}(R/(J \setminus x_t)^s) = \text{Min}(J \setminus x_t)$, we derive that $\mathfrak{m} \setminus x_t \in \text{Min}(J \setminus x_t)$, and thus $J \setminus x_t = \mathfrak{m} \setminus x_t$, which is a contradiction. We therefore have $\mathfrak{m} \setminus x_t \notin \text{Ass}(R/(J \setminus x_t)^s)$ for all s and $x_t \in \text{supp}(v)$. It follows now from Theorem 10.2.6 that J is normally torsion-free, and the proof is complete.

(ii)-(vi) can be shown similar to the proof of Corollary 6.2.3. $\square$

As an application of Theorem 5.4.10, we give the following result.

Proposition 5.4.11. *The t -path ideals of path graphs are normally torsion-free, for all $t \geq 0$.*

Proof. Let P_n be a path graph with the vertex set $V(P_n) = \{x_1, \dots, x_n\}$ and the edge set $E(P_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\}$. Then the t -path ideal of P_n is given by $I_t(P_n) = (x_i x_{i+1} \cdots x_{i+t-1} : i = 1, \dots, n-t+1)$. The proof is by induction on n . If $n = t$, then $I_t(P_n) = (x_1 x_2 \cdots x_t)$, and there is nothing to show. Let $n > t$ and that the claim has been proved for any value less than n . Now, put $I := (x_i x_{i+1} \cdots x_{i+t-1} : i = 1, \dots, n-t)$, $v := x_{n-t+1} x_{n-t+2} \cdots x_{n-1} x_n$, and $u_i := x_i x_{i+1} \cdots x_{i+t-1}$ for all $i = 1, \dots, n-t$. It is easily seen that $\gcd(v, u_i) = 1$ for all $i = 1, \dots, n-2t+1$, and $\gcd(v, u_i) = x_{n-t+1} x_{n-t+2} \cdots x_{t+i-1}$ for all $i = n-2t+2, \dots, n-t$. In particular, this implies that $\gcd(v, u_i)$ divides $\gcd(v, u_{i+1})$ for all $i = 1, \dots, n-t-1$. It follows also from the inductive hypothesis that I is normally torsion-free. Thanks to $I_t(P_n) = I + vR$, where $R = K[x_1, \dots, x_n]$, we deduce immediately the assertion from Theorem 5.4.10(i). $\square$

In what follows, our aim is to study the normally torsion-freeness of the Alexander duals of 3-path ideals of path graphs. For this, we will need the following proposition.

Proposition 5.4.12. *Suppose that I and J are two normally torsion-free square-free monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that $\text{supp}(I) \cap \text{supp}(J) = \emptyset$. Then $I \cap J = IJ$ is normally torsion-free.*

Proof. Due to $\text{supp}(I) \cap \text{supp}(J) = \emptyset$, we deduce from [60, Proposition 1.2.1] that $IJ = I \cap J$. In particular, we have $\text{Min}(I \cap J) = \text{Min}(I) \cup \text{Min}(J)$. On the other

hand, we know from [60, Theorem 1.4.6] that a square-free monomial ideal L is normally torsion-free if and only if $L^k = L^{(k)}$ for all $k \geq 1$, where $L^{(k)}$ denotes the k -th symbolic power of L . Hence, it is enough for us to show that $(I \cap J)^k = (I \cap J)^{(k)}$ for all $k \geq 1$. To achieve this, fix $k \geq 1$. Since I and J are normally torsion-free, one can conclude that $I^k = I^{(k)}$ and $J^k = J^{(k)}$. Thus, we get the following equalities

$$\begin{aligned}
(I \cap J)^{(k)} &= \bigcap_{\mathfrak{p} \in \text{Min}(I \cap J)} ((I \cap J)^k R_{\mathfrak{p}} \cap R) \\
&= \bigcap_{\mathfrak{p} \in \text{Min}(I)} (I^k R_{\mathfrak{p}} \cap R) \cap \bigcap_{\mathfrak{p} \in \text{Min}(J)} (J^k R_{\mathfrak{p}} \cap R) \\
&= I^{(k)} \cap J^{(k)} \\
&= I^k \cap J^k \\
&= (I \cap J)^k.
\end{aligned}$$

This shows that $I \cap J = IJ$ is normally torsion-free, as claimed. $\square$

We are now ready to prove the following proposition.

Lemma 5.4.13. *The Alexander duals of 3-path ideals of path graphs are normally torsion-free square-free monomial ideals.*

Proof. Let P_n be a path graph with the vertex set $V(P_n) = \{x_1, \dots, x_n\}$ and the edge set $E(P_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\}$. Then the Alexander dual of the 3-path ideal of the path graph P_n , denoted by $I_3(P_n)^\vee$, is given by $I_3(P_n)^\vee = \bigcap_{i=1}^{n-2} (x_i, x_{i+1}, x_{i+2})$. We show the claim by induction on n . Since every monomial prime ideal is normally torsion-free, this implies that the assertion is true for the case in which $n = 3$. Let $n > 3$ and that the claim has been shown for any value less than n . Our strategy is to use Theorem 10.2.6. For this purpose, set $\mathfrak{p}_i := (x_i, x_{i+1}, x_{i+2})$ for all $i = 1, \dots, n-2$, and

$$v := \prod_{k=0}^{\lfloor \frac{n-1}{3} \rfloor} x_{1+3k}.$$

It is easy to check that $|\text{supp}(v) \cap \text{supp}(\mathfrak{p}_i)| = 1$ for all $i = 1, \dots, n-2$. This yields that $v \in \mathfrak{p}_i \setminus \mathfrak{p}_i^2$ for all $i = 1, \dots, n-2$. To complete the argument, one has to verify that $\mathfrak{m} \setminus x_i \notin \text{Ass}(R/(I_3(P_n)^\vee \setminus x_i)^s)$ for all s and $x_i \in \text{supp}(v)$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Fix $s \geq 1$ and $x_i \in \text{supp}(v)$. One can check that $I_3(P_n)^\vee \setminus x_i$, after deleting the redundant components, can be viewed as $J_1 \cap J_2 \cap \Gamma$, where $J_1 = \bigcap_{r=1}^{\ell_1} \mathfrak{q}_r$ (respectively, $J_2 = \bigcap_{t=1}^{\ell_2} \mathfrak{q}'_t$) is the Alexander dual of a 3-path ideal of a path graph such as P_1 (respectively, P_2) with $V(P_1) < n$ (respectively, $V(P_2) < n$), $V(P_1) \cap V(P_2) = \emptyset$ and $\Gamma = \bigcap_{c=1}^{\lambda} (x_{\alpha_c}, x_{\beta_c})$ such that for any $1 \leq r \leq \ell_1$, $1 \leq t \leq \ell_2$, and $1 \leq c \neq d \leq \lambda$, we have $|\text{supp}(\mathfrak{q}_r) \cap \text{supp}(x_{\alpha_c}, x_{\beta_c})| \leq 1$, $|\text{supp}(\mathfrak{q}'_t) \cap \text{supp}(x_{\alpha_c}, x_{\beta_c})| \leq$

1, and $|\text{supp}(x_{\alpha_c}, x_{\beta_c}) \cap \text{supp}(x_{\alpha_d}, x_{\beta_d})| = 1$. Furthermore, the inductive hypothesis gives that J_1 and J_2 are normally torsion-free, and Proposition 5.4.12 implies that $J_1 \cap J_2$ is normally torsion-free. It follows now by repeated use of Lemma 6.4.9 that $J_1 \cap J_2 \cap \Gamma$ is normally torsion-free, and so $I_3(P_n)^\vee \setminus x_i$ is normally torsion-free. This means that $\mathfrak{m} \setminus x_i \notin \text{Ass}(R/(I_3(P_n)^\vee \setminus x_i)^s)$, as required. We conclude from Theorem 10.2.6 that $I_3(P_n)^\vee$ is normally torsion-free. This completes the inductive step, and therefore the claim has been shown by induction. $\square$

5.4.2 Associated primes of dominating ideals of cycles

In this subsection, we will mainly discuss the normally torsion-freeness of $DI(C_n)$ and the stable sets of associated primes of $DI(C_n)$. In [8, Theorem 3.9], Alilooee and Banerjee showed that the 3-path ideal of a graph G is normally torsion-free if and only if G is a path graph P_k for some $k \geq 3$ or G is a cycle C_{3k} when $k = 1, 2, 3$. Observe that the 3-path ideal of a cycle C_n with $n \geq 3$ coincides with $NI(C_n)$. With this observation, we can rephrase [8, Theorem 3.9] in the following way.

Proposition 5.4.14. *Let C_n be a cycle graph with $n \geq 3$. Then $NI(C_n)$ is normally torsion-free if and only if $n \in \{3, 6, 9\}$.*

As the first main result of this subsection, in Theorem 5.4.16, we will prove that the statement above also holds for $DI(C_n)$. To show this, we first consider the case of $DI(C_6)$ and $DI(C_9)$ and verify that they are normally torsion-free.

Lemma 5.4.15. *The dominating ideals of C_6 and C_9 are normally torsion-free.*

Proof. (i) Suppose that $E(C_6) = \{\{x_i, x_{i+1}\} : i = 1, \dots, 5\} \cup \{\{x_6, x_1\}\}$. Then the dominating ideal of C_6 is given by

$$\begin{aligned} DI(C_6) &= (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5) \cap (x_4, x_5, x_6) \cap (x_5, x_6, x_1) \\ &\quad \cap (x_6, x_1, x_2) \\ &= (x_3x_6, x_2x_5, x_1x_4, x_2x_4x_6, x_1x_3x_5). \end{aligned}$$

We want to demonstrate that $DI(C_6)$ is normally torsion-free by using Theorem 10.2.6. For this purpose, set $v := x_3x_6$ and $L := DI(C_6)$. One can easily see that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \text{Min}(L)$. Here, we must show that $\mathfrak{m} \setminus x_i \notin \text{Ass}(R/(L \setminus x_i)^s)$ for all s and $x_i \in \{x_3, x_6\}$. Fix $s \geq 1$. First, we establish $\mathfrak{m} \setminus x_3 \notin \text{Ass}(R/(L \setminus x_3)^s)$, where $\mathfrak{m} = (x_1, \dots, x_6)$. According to the definition of deletion, we obtain

$$\begin{aligned} L \setminus x_3 &= (x_1, x_2) \cap (x_2, x_4) \cap (x_4, x_5) \cap (x_5, x_6, x_1) \\ &= (x_2x_5, x_1x_4, x_2x_4x_6). \end{aligned}$$

It follows from Theorem 5.3.4 and Lemma 5.3.11 that $x_4(x_1, x_2x_6)$ is normally torsion-free. Because $\gcd(x_2x_5, x_1x_4) = 1$ and $\gcd(x_2x_5, x_2x_4x_6) = x_2$, one can derive from Theorem 5.4.10 that $L \setminus x_3$ is normally torsion-free, and so $\mathfrak{m} \setminus x_3 = (x_1, x_2, x_4, x_5, x_6) \notin \text{Ass}(R/(L \setminus x_3)^s)$. Now, we show that $\mathfrak{m} \setminus x_6 \notin \text{Ass}(R/(L \setminus x_3)^s)$. A similar computation gives that

$$\begin{aligned} L \setminus x_6 &= (x_1, x_2) \cap (x_2, x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1) \\ &= (x_2x_5, x_1x_4, x_1x_3x_5). \end{aligned}$$

On account on Theorem 5.3.4 and Lemma 5.3.11, we get $x_1(x_4, x_3x_5)$ is normally torsion-free. As $\gcd(x_2x_5, x_1x_4) = 1$ and $\gcd(x_2x_5, x_1x_3x_5) = x_5$, Theorem 5.4.10 implies that $L \setminus x_6$ is normally torsion-free, and hence $\mathfrak{m} \setminus x_6 = (x_1, x_2, x_3, x_4, x_5) \notin \text{Ass}(R/(L \setminus x_6)^s)$. It follows now from Theorem 10.2.6 that $DI(C_6)$ is normally torsion-free, as claimed.

(ii) Let $E(C_9) = \{\{x_i, x_{i+1}\} : i = 1, \dots, 8\} \cup \{\{x_9, x_1\}\}$. Then the dominating ideal of C_9 is given by

$$\begin{aligned} DI(C_9) &= (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5) \cap (x_4, x_5, x_6) \cap (x_5, x_6, x_7) \\ &\quad \cap (x_6, x_7, x_8) \cap (x_7, x_8, x_9) \cap (x_8, x_9, x_1) \cap (x_9, x_1, x_2). \end{aligned}$$

Our strategy is to use Theorem 10.2.6. To do this, put $v := x_3x_6x_9$ and $L := DI(C_9)$. It is routine to investigate that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \text{Min}(L)$. To complete the argument, one has to verify that $\mathfrak{m} \setminus x_i \notin \text{Ass}(R/(L \setminus x_i)^s)$ for all s and $x_i \in \{x_3, x_6, x_9\}$. Fix $s \geq 1$. We first prove that $\mathfrak{m} \setminus x_3 \notin \text{Ass}(R/(L \setminus x_3)^s)$, where $\mathfrak{m} = (x_1, \dots, x_9)$. It follows from the definition of deletion that

$$\begin{aligned} L \setminus x_3 &= (x_1, x_2) \cap (x_2, x_4) \cap (x_4, x_5) \cap (x_5, x_6, x_7) \cap (x_6, x_7, x_8) \cap (x_7, x_8, x_9) \\ &\quad \cap (x_8, x_9, x_1). \end{aligned}$$

Set $\alpha := x_1x_4x_7$ and $A := L \setminus x_3$. It is easy to see that $\alpha \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \text{Min}(A)$. We show that $Q \setminus x_i \notin \text{Ass}(R/(A \setminus x_i)^s)$ for all s and $x_i \in \{x_1, x_4, x_7\}$, where $Q = \mathfrak{m} \setminus x_3$. Fix $s \geq 1$. We first prove that $Q \setminus x_1 \notin \text{Ass}(R/(A \setminus x_1)^s)$. Direct computation gives that

$$A \setminus x_1 = (x_6, x_7, x_8) \cap (x_5, x_6, x_7) \cap (x_8, x_9) \cap (x_4, x_5) \cap (x_2).$$

It follows from Lemma 5.4.13 that $(x_6, x_7, x_8) \cap (x_5, x_6, x_7)$ is normally torsion-free, and by virtue of Lemma 6.4.9, we get $(x_6, x_7, x_8) \cap (x_5, x_6, x_7) \cap (x_8, x_9) \cap (x_4, x_5)$ is normally torsion-free. According to Proposition 5.4.12, one has $A \setminus x_1$ is normally torsion-free, and so $Q \setminus x_1 \notin \text{Ass}(R/(A \setminus x_1)^s)$. By a similar argument, one can show that

$$A \setminus x_4 = (x_2) \cap (x_5) \cap (x_6, x_7, x_8) \cap (x_7, x_8, x_9) \cap (x_8, x_9, x_1),$$

and

$$A \setminus x_7 = (x_1, x_2) \cap (x_2, x_4) \cap (x_4, x_5) \cap (x_5, x_6) \cap (x_6, x_8) \cap (x_8, x_9),$$

are normally torsion-free. Hence, we get $Q \setminus x_4 \notin \text{Ass}(R/(A \setminus x_4)^s)$ and $Q \setminus x_7 \notin \text{Ass}(R/(A \setminus x_7)^s)$. We deduce from Theorem 10.2.6 that $A = L \setminus x_3$ is normally torsion-free, and so $\mathfrak{m} \setminus x_3 \notin \text{Ass}(R/(L \setminus x_3)^s)$. In what follows, we give the sketch of the proof of normally torsion-freeness of $B := L \setminus x_6$ and $C := L \setminus x_9$. We can rapidly conclude the following

$$\begin{aligned} L \setminus x_6 &= (x_4, x_5) \cap (x_5, x_7) \cap (x_7, x_8) \cap (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_1, x_8, x_9) \\ &\quad \cap (x_1, x_2, x_9), \end{aligned}$$

and

$$\begin{aligned} L \setminus x_9 &= (x_7, x_8) \cap (x_8, x_1) \cap (x_1, x_2) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5) \cap (x_4, x_5, x_6) \\ &\quad \cap (x_5, x_6, x_7). \end{aligned}$$

It is routine to check that $\alpha = x_1 x_4 x_7 \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \text{Min}(B)$ (respectively, for any $\mathfrak{p} \in \text{Min}(C)$). Furthermore, we have the following

$$\begin{aligned} B \setminus x_1 &= (x_4, x_5) \cap (x_5, x_7) \cap (x_7, x_8) \cap (x_8, x_9) \cap (x_9, x_2) \cap (x_2, x_3), \\ B \setminus x_4 &= (x_5) \cap (x_7, x_8) \cap (x_2, x_3) \cap (x_1, x_8, x_9) \cap (x_1, x_2, x_9), \\ B \setminus x_7 &= (x_5) \cap (x_8) \cap (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_1, x_2, x_9), \\ C \setminus x_1 &= (x_8) \cap (x_2) \cap (x_3, x_4, x_5) \cap (x_4, x_5, x_6) \cap (x_5, x_6, x_7), \\ C \setminus x_4 &= (x_7, x_8) \cap (x_8, x_1) \cap (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_5) \cap (x_5, x_6), \end{aligned}$$

and

$$C \setminus x_7 = (x_8) \cap (x_1, x_2) \cap (x_5, x_6) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5).$$

By using Lemma 5.4.13, Theorem 5.3.4, and Lemma 6.4.9, we can conclude that $B \setminus x_1$, $B \setminus x_4$, and $B \setminus x_7$ (respectively, $C \setminus x_1$, $C \setminus x_4$, and $C \setminus x_7$) are normally torsion-free. This gives rise to $\mathfrak{m} \setminus x_6 \notin \text{Ass}(R/(I \setminus x_6)^s)$ (respectively, $\mathfrak{m} \setminus x_9 \notin \text{Ass}(R/(I \setminus x_9)^s)$). Finally, Theorem 10.2.6 gives that $DI(C_9)$ is normally torsion-free. $\square$

We are now ready to establish the first main result of this subsection in the following theorem.

Theorem 5.4.16. *Let $C_n = (V(C_n), E(C_n))$ be a cycle graph of order $n \geq 3$. Then $DI(C_n)^t = DI(C_n)^{(t)}$ for all $t \geq 1$ if and only if $n \in \{3, 6, 9\}$. In particular, $DI(C_n)$ is normally torsion-free if and only if $n \in \{3, 6, 9\}$.*

Proof. We first assume that $n \in \{3, 6, 9\}$. Let $V(C_n) = \{x_1, \dots, x_n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\} \cup \{\{x_n, x_1\}\}$. It is easy to see that $DI(C_3) = (x_1, x_2, x_3)$ is normally torsion-free. According to Lemma 5.4.15, it follows that $DI(C_6)$ and $DI(C_9)$ are also normally torsion-free. To complete the proof, we need to show that $DI(C_n)^t \neq DI(C_n)^{(t)}$ for some $t \geq 1$ when $n \notin \{3, 6, 9\}$. To do this, we divide the proof into three cases. We will repeatedly use the following facts in our proof.

- (i) The domination number of C_{3k} is k , and the domination number of C_{3k+1} and C_{3k+2} is $k+1$, see [56] for more information.
- (ii) The associated primes of $DI(C_n)$ correspond to the closed neighborhoods of the vertices of C_n .

Now, we proceed with the proof.

Case 1. Let $n = 3k$ for some $k \geq 4$. It follows from [56, Lemma 4] that there are exactly three minimal dominating sets of C_n of cardinality k , given by

$$A = \{x_1, x_4, x_7, \dots, x_{3k-2}\}, B = \{x_2, x_5, x_8, \dots, x_{3k-1}\}, \text{ and}$$

$$C = \{x_3, x_6, x_9, \dots, x_{3k}\}.$$

Our aim is to construct an element f with $\deg f = 2k+1$ such that none of A or B or C is in $\text{supp}(f)$. To do this, set $h := x_4x_8 \prod_{j=4}^k x_{3j}$ and $g := \prod_{i=1}^{3k} x_i$. We claim that $f = g/h \notin DI(C_{3k})^2$ but $f \in DI(C_{3k})^{(2)}$. It is routine to check that $\deg h = k-1$, and hence $\deg f = 3k - k + 1 = 2k + 1$. In addition, from the constructions of A , B , and C , we know that the only elements of degree k in $\mathcal{G}(DI(C_{3k}))$ are

$$\left\{ \prod_{x_i \in A} x_i, \prod_{x_i \in B} x_i, \prod_{x_i \in C} x_i \right\}.$$

As $x_4, x_8, x_{12} \notin \text{supp}(f)$, we deduce that $\prod_{x_i \in A} x_i \nmid f$, $\prod_{x_i \in B} x_i \nmid f$, and $\prod_{x_i \in C} x_i \nmid f$. Therefore, none of the elements of degree k in $DI(C_{3k})$ divides f , and consequently $f \notin DI(C_{3k})^2$. In the light of [60, Proposition 1.4.4], we obtain

$$DI(C_{3k})^{(2)} = \bigcap_{\mathfrak{p} \in \text{Ass}(DI(C_{3k}))} \mathfrak{p}^2.$$

To see $f \in DI(C_{3k})^{(2)}$, note that the closed neighborhood set of each vertex of C_{3k} is covered twice by $\text{supp}(f)$. Indeed, for each $x_i \in C_{3k}$, we have either $x_i x_{i+1} \in \text{supp}(f)$, or $x_{i-1} x_i \in \text{supp}(f)$, or $x_{i-1} x_{i+1} \in \text{supp}(f)$.

Case 2. It follows from $x_1 x_2 x_3 \in DI(C_4)^{(2)} \setminus DI(C_4)^2$ that $DI(C_4)^2 \neq DI(C_4)^{(2)}$, and so the assertion is true for C_4 . Let $n = 3k+1$ for some $k \geq 2$. Let $f' := x_{3k+1} \prod_{i=0}^{k-1} x_{3i+1} x_{3i+2}$. Then $\deg f' = 2k+1$. By virtue of the minimal degree of any generator in $DI(C_{3k+1})^2$ is $2k+2$, this shows that $f' \notin DI(C_{3k+1})^2$. Moreover, the

closed neighborhood set of each vertex of C_{3k+1} is covered twice by $\text{supp}(f')$, which yields that $f' \in DI(C_{3k+1})^{(2)}$.

Case 3. Let $n = 3k + 2$ for some $k \geq 1$ and put $f'' := \prod_{i=1}^{3k+2} x_i$. Then $\deg f'' = 3k + 2$. In view of the minimal degree of any generator in $DI(C_{3k+2})^3$ is $3k + 3$, we get $f'' \notin DI(C_{3k+2})^3$. Furthermore, the closed neighborhood set of each vertex of C_{3k+2} is covered three times by $\text{supp}(f'')$, which gives that $f'' \in DI(C_{3k+2})^{(3)}$.

Accordingly, the discussion above shows that if $n \notin \{3, 6, 9\}$, then $DI(C_n)^t \neq DI(C_n)^{(t)}$ for some $t \geq 1$. This finishes the proof. $\square$

In general, it is well-known that the cover ideals of even cycle graphs are always normally torsion-free (cf. [48, Corollary 2.6]). It has already been shown in Theorem 6.2.1 that the cover ideals of odd cycle graphs are nearly normally torsion-free (to recall the definition of nearly normally torsion-freeness refer to Section 1.1). Furthermore, the associated primes of powers of cover ideals of odd cycle graphs have already been studied in [104], refer to Proposition 3.1.15.

In what follows, our aim is to establish that a similar result as above holds for the associated primes of powers of dominating ideals of cycle graphs of any order. Before stating the next theorem, one has to recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal in Definition 3.1.14.

We are now in a position to state our next result related to the associated primes of powers of dominating ideals of cycle graphs.

Theorem 5.4.17. *Let C_n be a cycle graph of order n with $V(C_n) = \{x_1, \dots, x_n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\} \cup \{\{x_n, x_1\}\}$. Let $DI(C_n)$ be the dominating ideal of C_n in the polynomial ring $R = K[x_1, \dots, x_n]$. Then the following statements hold.*

(i) *The monomial ideal $DI(C_n)$ satisfies the persistence property.*

(ii) *For all $s \geq 1$, we have*

$$\text{Ass}(DI(C_n)^s) \subseteq \{(x_i, x_{i+1}, x_{i+2}) : i = 1, \dots, n\} \cup \{\mathbf{m}\},$$

where $\mathbf{m} = (x_1, \dots, x_n)$ and x_{n+1} (respectively, x_{n+2}) represents x_1 (respectively, x_2).

(iii) $\text{Ass}^\infty(DI(C_n)) \subseteq \{(x_i, x_{i+1}, x_{i+2}) : i = 1, \dots, n\} \cup \{\mathbf{m}\}$.

Proof. For convenience of notation, put $L := DI(C_n)$. Since all claims are true for $DI(C_3) = (x_1, x_2, x_3)$, we assume that $n \geq 4$. We first show that $\text{Ass}(L^s) \subseteq \text{Min}(L) \cup \{\mathbf{m}\}$ for all $s \geq 1$. To see this, fix $s \geq 1$, and pick an arbitrary element $\mathfrak{p} \in \text{Ass}(L^s)$. If $\mathfrak{p} = \mathbf{m}$, then there is nothing to show. We thus assume that

$\mathfrak{p} \neq \mathfrak{m}$. Without loss of generality, one may assume that $\mathfrak{p} = (x_i : i = 1, \dots, r)$, where $3 \leq r < n$. Then we get $L(\mathfrak{p}) = \bigcap_{i=1}^{\ell} I_3^{\vee}(P_i)$, where $P_1, \dots, P_{\ell}$ are some disjoint paths in C_n . By virtue of Lemma 5.4.13, one can immediately conclude that $I_3^{\vee}(P_i)$ is normally torsion-free for all $i = 1, \dots, \ell$. In the light of Proposition 5.4.12, we get $L(\mathfrak{p})$ is normally torsion-free, and hence $\mathfrak{p} \in \text{Min}(L(\mathfrak{p}))$. On the other hand, based on Lemma 3.2.15(viii), we know that $\mathfrak{p} \in \text{Ass}(L^s)$ if and only if $\mathfrak{p} \in \text{Ass}(R(\mathfrak{p})/L(\mathfrak{p})^s)$. Consequently, one obtains $\mathfrak{p} \in \text{Ass}(R(\mathfrak{p})/L(\mathfrak{p})^s)$. We therefore get $\mathfrak{p} = (x_{\alpha}, x_{\alpha+1}, x_{\alpha+2})$ for some $1 \leq \alpha \leq r-2$, and thus $\mathfrak{p} \in \text{Min}(L)$. Accordingly, we derive that $\text{Ass}(L^s) \subseteq \text{Min}(L) \cup \{\mathfrak{m}\}$ for all $s \geq 1$. This proves (ii).

We now show that L has the persistence property. Note that, in view of Theorem 3.2.34 and Proposition 3.1.12, every normal monomial ideal has the persistence property. Hence, it is sufficient for us to demonstrate that L is normal. Our strategy is to use Theorem 6.5.1. To accomplish this, set

$$A := \bigcap_{i=1}^{n-3} (x_i, x_{i+1}, x_{i+2}) \text{ and } B := (x_{n-1}, x_{n-2}) \cap (x_{n-1}, x_1) \cap (x_1, x_2).$$

Then we have $L = A \cap (x_n R + B) = x_n A + A \cap B$. Due to A is the Alexander dual of the 3-path ideal of a path graph $P = (V(P), E(P))$ with the vertex set $V(P) = \{x_1, \dots, x_{n-1}\}$ and the following edge set

$$E(P) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-2\},$$

Lemma 5.4.13 gives that A is normally torsion-free, and by [60, Theorem 1.4.6], we deduce that A is normal. Also, note that $A + A \cap B = A$ is normal. In what follows, our aim is to establish $A \cap B$ is normal. It is easy to check that

$$\begin{aligned} A \cap B &= \bigcap_{i=2}^{n-4} (x_i, x_{i+1}, x_{i+2}) \cap (x_{n-1}, x_{n-2}) \cap (x_{n-1}, x_1) \cap (x_1, x_2) \\ &= x_1 \left(\bigcap_{i=2}^{n-4} (x_i, x_{i+1}, x_{i+2}) \cap (x_{n-1}, x_{n-2}) \right) \\ &\quad + x_2 R \cap \bigcap_{i=3}^{n-4} (x_i, x_{i+1}, x_{i+2}) \cap (x_{n-1}, x_{n-2}) \cap (x_{n-1}, x_1). \end{aligned}$$

To simplify the notation, put $C := \bigcap_{i=2}^{n-4} (x_i, x_{i+1}, x_{i+2}) \cap (x_{n-1}, x_{n-2})$ and

$$D := x_2 R \cap \bigcap_{i=3}^{n-4} (x_i, x_{i+1}, x_{i+2}) \cap (x_{n-1}, x_{n-2}) \cap (x_{n-1}, x_1).$$

This gives rise to $A \cap B = x_1 C + D$. As the monomial ideal $\bigcap_{i=2}^{n-4} (x_i, x_{i+1}, x_{i+2})$ can be viewed as the Alexander dual of the 3-path ideal of a path graph, by Lemma 5.4.13, is normally torsion-free, and on account of Lemma 6.4.9, we can conclude that C is

normally torsion-free, and based on [60, Theorem 1.4.6], is normal. Moreover, since $D \subseteq C$, one has $C + D = C$ is normal as well. Now, a similar argument shows that the monomial ideal $\bigcap_{i=3}^{n-4} (x_i, x_{i+1}, x_{i+2})$ is normally torsion-free, and after two times using Lemma 6.4.9, we can derive that $\bigcap_{i=3}^{n-4} (x_i, x_{i+1}, x_{i+2}) \cap (x_{n-1}, x_{n-2}) \cap (x_{n-1}, x_1)$ is normally torsion-free. It follows now from Lemma 7.4.5 that D is normal. Thanks to Theorem 6.5.1, we get $A \cap B = x_1 C + D$ is normal. Due to A and $A \cap B$ are normal, Theorem 6.5.1 yields that $L = x_n A + A \cap B$ is normal, and so has the persistence property, as we claimed in (i).

Finally, combining (i) and (ii) leads us to

$$\text{Ass}^\infty(DI(C_n)) \subseteq \{(x_i, x_{i+1}, x_{i+2}) : i = 1, \dots, n\} \cup \{\mathfrak{m}\}.$$

This demonstrates (iii), and the proof is complete. $\square$

Chapter 6

On the nearly normally torsion-freeness of monomial ideals

In this chapter, we deal with the concept of nearly normally torsion-free monomial ideals. Recall that a monomial ideal I in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K is called *nearly normally torsion-free* if there exist a positive integer k and a monomial prime ideal $\mathfrak{p}$ such that $\text{Ass}_R(R/I^m) = \text{Min}(I)$ for all $1 \leq m \leq k$, and $\text{Ass}_R(R/I^m) \subseteq \text{Min}(I) \cup \{\mathfrak{p}\}$ for all $m \geq k + 1$, see [9, Definition 2.1]. This notion generalizes normally torsion-freeness such that every normally torsion-free monomial ideal is nearly normally torsion-free.

More recently, in [9, Theorem 2.3], the author characterized all connected graphs whose edge ideals are nearly normally torsion-free. The main purpose of this chapter is to continue this argument. In particular, we present some new classes of nearly normally torsion-free monomial ideals.

6.1 Some classes of nearly normally torsion-free ideals

In this section, our aim is to give additional results related to nearly normally torsion-free monomial ideals. To achieve this, one has to recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal, see Definition 3.1.14. It should be noted that the results of this section can be found in [108].

Lemma 6.1.1. *Let I be a nearly normally torsion-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . Then there exists a monomial prime ideal $\mathfrak{p} \in V^*(I)$ such that $I(\mathfrak{p} \setminus \{x_i\})$ is normally torsion-free for all $x_i \in \mathfrak{p}$.*

Proof. Since I is nearly normally torsion-free, there exists a positive integer k and a monomial prime ideal $\mathfrak{p}$ such that $\text{Ass}_R(R/I^m) = \text{Min}(I)$ for all $1 \leq m \leq k$, and $\text{Ass}_R(R/I^m) \subseteq \text{Min}(I) \cup \{\mathfrak{p}\}$ for all $m \geq k+1$. We claim that $I(\mathfrak{p} \setminus \{x_i\})$ is normally torsion-free for all $x_i \in \mathfrak{p}$. To do this, fix $x_i \in \mathfrak{p}$, and set $\mathfrak{q} := \mathfrak{p} \setminus \{x_i\}$. We need to show that $\text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/(I(\mathfrak{q}))^\ell) \subseteq \text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/I(\mathfrak{q}))$ for all ℓ . Fix $\ell \geq 1$. It follows from Lemma 3.2.15 that

$$\text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/(I(\mathfrak{q}))^\ell) = \text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/I^\ell(\mathfrak{q})) = \{Q : Q \in \text{Ass}_R(R/I^\ell) \text{ and } Q \subseteq \mathfrak{q}\}.$$

Pick an arbitrary element $Q \in \text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/(I(\mathfrak{q}))^\ell)$. Thus, $Q \in \text{Ass}_R(R/I^\ell)$ and $Q \subseteq \mathfrak{q}$. Since $\mathfrak{q} = \mathfrak{p} \setminus \{x_i\}$, this yields that $Q \neq \mathfrak{p}$; thus, one must have $Q \in \text{Min}(I)$. Hence, $Q \in \text{Ass}_R(R/I)$, and so $Q \in \text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/I(\mathfrak{q}))$. Therefore, $I(\mathfrak{p} \setminus \{x_i\})$ is normally torsion-free. $\square$

Lemma 6.1.2. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K such that $\text{Ass}_R(R/I) = \text{Min}(I)$. Let $I(\mathfrak{m} \setminus \{x_i\})$ be normally torsion-free for all $i = 1, \dots, n$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Then I is nearly normally torsion-free.*

Proof. In the light of $\text{Ass}_R(R/I) = \text{Min}(I)$, it is enough for us to show that $\text{Ass}_R(R/I^k) \subseteq \text{Min}(I) \cup \{\mathfrak{m}\}$ for all $k \geq 2$. To achieve this, fix $k \geq 2$, and take an arbitrary element $Q \in \text{Ass}_R(R/I^k)$. If $Q = \mathfrak{m}$, then the proof is over. Hence, let $Q \neq \mathfrak{m}$. On account of Q is a monomial prime ideal, this implies that $Q \subseteq \mathfrak{m} \setminus \{x_j\}$ for some $x_j \in \mathfrak{m}$. Put $\mathfrak{q} := \mathfrak{m} \setminus \{x_j\}$. Because $I(\mathfrak{q})$ is normally torsion-free, we thus have $\text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/(I(\mathfrak{q}))^k) \subseteq \text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/I(\mathfrak{q}))$. Also, one can deduce from Lemma 3.2.15 that $Q \in \text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/(I(\mathfrak{q}))^k)$, and so $Q \in \text{Ass}_{R(\mathfrak{q})}(R(\mathfrak{q})/I(\mathfrak{q}))$. This yields that $Q \in \text{Ass}_R(R/I)$, and hence $Q \in \text{Min}(I)$. This completes the proof. $\square$

As an immediate consequence of Lemma 6.1.2, we obtain the following corollary:

Corollary 6.1.3. *Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . Let $I(\mathfrak{m} \setminus \{x_i\})$ be normally torsion-free for all $i = 1, \dots, n$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Then I is nearly normally torsion-free.*

As an application of Lemma 6.1.2, we give a class of nearly normally torsion-free monomial ideals in the subsequent theorem. To see this, one needs to recall from [68] that a monomial ideal is said to be a *monomial ideal of intersection type* when it can be presented as an intersection of powers of monomial prime ideals.

Proposition 6.1.4. *Let $I = \bigcap_{\mathfrak{p} \in \text{Ass}_R(R/I)} \mathfrak{p}^{d_{\mathfrak{p}}} \subset R = K[x_1, \dots, x_n]$ be a monomial ideal of intersection type such that for any $1 \leq i \leq n$, there exists unique $\mathfrak{p} \in \text{Ass}_R(R/I)$ with $I(\mathfrak{m} \setminus \{x_i\}) = \mathfrak{p}^{d_{\mathfrak{p}}}$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Then I is nearly normally torsion-free.*

Proof. We first note that the assumption concludes that the monomial ideal I must have the following form

$$I = (\mathfrak{m} \setminus \{x_1\})^{d_1} \cap (\mathfrak{m} \setminus \{x_2\})^{d_2} \cap \cdots \cap (\mathfrak{m} \setminus \{x_n\})^{d_n},$$

for some positive integers $d_1, \dots, d_n$. This gives rise to $\text{Ass}_R(R/I) = \text{Min}(I)$. In addition, since $(\mathfrak{m} \setminus \{x_i\})^{d_i}$, for each $i = 1, \dots, n$, is normally torsion-free, the claim can be promptly deduced from Lemma 6.1.2. $\square$

The next lemma says that a monomial ideal is nearly normally torsion-free if and only if its monomial multiple is nearly normally torsion-free under certain conditions. It is an updated version of [64, Lemma 3.5] and Lemma 5.3.11.

Lemma 6.1.5. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$, and h be a monomial in R such that $\gcd(h, u) = 1$ for all $u \in \mathcal{G}(I)$. Then I is nearly normally torsion-free if and only if hI is nearly normally torsion-free.*

Proof. ($\Rightarrow$) Assume that I is nearly normally torsion-free. Let $h = x_{j_1}^{b_1} \cdots x_{j_s}^{b_s}$ with $j_1, \dots, j_s \in \{1, \dots, n\}$. On account of Theorem 2.1.27, we obtain, for all ℓ ,

$$(6.1.1) \quad \text{Ass}_R(R/(hI)^\ell) = \text{Ass}_R(R/I^\ell) \cup \{(x_{j_1}), \dots, (x_{j_s})\}.$$

Due to $\gcd(h, u) = 1$ for all $u \in \mathcal{G}(I)$, it is routine to check that

$$(6.1.2) \quad \text{Min}(hI) = \text{Min}(I) \cup \{(x_{j_1}), \dots, (x_{j_s})\}.$$

Since I is nearly normally torsion-free, there exist a positive integer k and a monomial prime ideal $\mathfrak{p}$ such that $\text{Ass}_R(R/I^m) = \text{Min}(I)$ for all $1 \leq m \leq k$, and $\text{Ass}_R(R/I^m) \subseteq \text{Min}(I) \cup \{\mathfrak{p}\}$ for all $m \geq k+1$. Select an arbitrary element $\mathfrak{q} \in \text{Ass}_R(R/(hI)^m)$. Then (6.1.1) implies that $\mathfrak{q} \in \text{Ass}_R(R/I^m) \cup \{(x_{j_1}), \dots, (x_{j_s})\}$. If $1 \leq m \leq k$, then (6.1.2) yields that $\mathfrak{q} \in \text{Min}(hI)$. Hence, let $m \geq k+1$. The claim follows readily from (6.1.1), (6.1.2), and $\text{Ass}_R(R/I^m) \subseteq \text{Min}(I) \cup \{\mathfrak{p}\}$.

($\Leftarrow$) Let hI be nearly normally torsion-free. This means that there exist a positive integer k and a monomial prime ideal $\mathfrak{p}$ such that $\text{Ass}_R(R/(hI)^m) = \text{Min}(hI)$ for all $1 \leq m \leq k$, and $\text{Ass}_R(R/(hI)^m) \subseteq \text{Min}(hI) \cup \{\mathfrak{p}\}$ for all $m \geq k+1$. Take an arbitrary element $\mathfrak{q} \in \text{Ass}_R(R/I^m)$. Because $\mathfrak{q} \in \text{Ass}_R(R/I^m)$, we get $\mathfrak{q} \notin \{(x_{j_1}), \dots, (x_{j_s})\}$. If $1 \leq m \leq k$, then (6.1.1) gives that $\mathfrak{q} \in \text{Ass}_R(R/(hI)^m)$, and so $\mathfrak{q} \in \text{Min}(hI)$. As $\mathfrak{q} \notin \{(x_{j_1}), \dots, (x_{j_s})\}$, we gain $\mathfrak{q} \in \text{Min}(I)$. We thus assume that $m \geq k+1$. One can derive the assertion according to the facts $\mathfrak{q} \notin \{(x_{j_1}), \dots, (x_{j_s})\}$, (6.1.1), (6.1.2), and $\text{Ass}_R(R/(hI)^m) \subseteq \text{Min}(hI) \cup \{\mathfrak{p}\}$. $\square$

We conclude this section by observing that nearly normally torsion-freeness does not imply the strong persistence property. It is not known if nearly normally torsion-freeness implies the persistence property.

Example 6.1.6. Let $R = K[x_1, x_2, x_3]$ be the polynomial ring over a field K and $I := (x_2^4, x_1x_2^3, x_1^3x_2, x_1^4x_3)$ be a monomial ideal of R . On account of

$$I = (x_1, x_2^4) \cap (x_1^3, x_2^3) \cap (x_1^4, x_2) \cap (x_2, x_3),$$

one can conclude that $\text{Ass}_R(R/I) = \text{Min}(I) = \{(x_1, x_2), (x_2, x_3)\}$. We claim that $\text{Ass}_R(R/I^m) = \text{Min}(I) \cup \{(x_1, x_2, x_3)\}$ for all $m \geq 2$. To prove this claim, fix $m \geq 2$. In what follows, we verify that $(x_1, x_2, x_3) \in \text{Ass}_R(R/I^m)$. To establish this, we show that $(I^m :_R v) = (x_1, x_2, x_3)$, where $v := x_1^{3m-1}x_2^{m+1}$. To see this, one may consider the following statements:

- (i) Since $vx_1 = x_1^{3m}x_2^{m+1} = (x_1^3x_2)^mx_2$ and $x_1^3x_2 \in I$, we get $vx_1 \in I^m$, and so $x_1 \in (I^m :_R v)$;
- (ii) Due to $vx_2 = x_1^{3m-1}x_2^{m+2} = (x_1^3x_2)^{m-1}(x_1^2x_2^3)$ and $x_1^2x_2^3 \in I$, one has $vx_2 \in I^m$, and hence $x_2 \in (I^m :_R v)$;
- (iii) As $vx_3 = x_1^{3m-1}x_2^{m+1}x_3 = (x_1^3x_2)^{m-2}(x_1^5x_2^3x_3)$ and $x_1^5x_2^3x_3 \in I^2$, we obtain $vx_3 \in I^m$, and thus $x_3 \in (I^m :_R v)$.

Consequently, $(x_1, x_2, x_3) \subseteq (I^m :_R v)$. For the converse, let $v \in I^m$. This leads to there exist monomials $h_1, \dots, h_m \in \mathcal{G}(I)$ such that $x_3 \nmid h_i$ for each $i = 1, \dots, m$, and $h_1 \cdots h_m | x_1^{3m-1}x_2^{m+1}$. Hence, we have the following equality

$$(6.1.3) \quad h_1 \cdots h_m = (x_2^4)^{\alpha_1} (x_1x_2^3)^{\alpha_2} (x_1^3x_2)^{\alpha_3},$$

for some nonnegative integers $\alpha_1, \alpha_2, \alpha_3$ with $\alpha_1 + \alpha_2 + \alpha_3 = m$. Particularly, one can derive from (6.1.3) that

$$(6.1.4) \quad 4\alpha_1 + 3\alpha_2 + \alpha_3 \leq m + 1,$$

and

$$(6.1.5) \quad \alpha_2 + 3\alpha_3 \leq 3m - 1.$$

Since $\sum_{i=1}^3 \alpha_i = m$, we obtain from (6.1.4) that $m + 3\alpha_1 + 2\alpha_2 \leq m + 1$, and so $3\alpha_1 + 2\alpha_2 \leq 1$. We thus have $\alpha_1 = \alpha_2 = 0$, and so $\alpha_3 = m$. Moreover, it follows from (6.1.5) that $3m \leq 3m - 1$, which is a contradiction. Therefore, $v \notin I^m$, and so $(I^m :_R v) = (x_1, x_2, x_3)$. This gives rise to $(x_1, x_2, x_3) \in \text{Ass}_R(R/I^m)$ for all $m \geq 2$. Hence, $\text{Ass}_R(R/I^m) = \text{Min}(I) \cup \{(x_1, x_2, x_3)\}$ for all $m \geq 2$. This means that I has the persistence property and is a nearly normally torsion-free ideal. On the other hand, using Macaulay2 [49] shows that $(I^2 :_R I) \neq I$. We therefore deduce that I does not satisfy the strong persistence property.

6.2 Nearly normally torsion-freeness and cover ideals

In the following, our goal is to characterize all finite simple connected graphs such that their cover ideals are nearly normally torsion-free, see Theorem 6.2.1. It should be noted that the results of this section can be found in [108].

To understand the proof of Theorem 6.2.1, we first review some notations from [75] as follows:

Let $G = (V(G), E(G))$ be a finite simple connected graph. For any $x, y \in V(G)$, an (x, y) -path is simply a path between the vertices x and y in G . Also, for a vertex subset W of a graph G , $\langle W \rangle$ will denote the subgraph of G induced by W .

Let G be an *almost bipartite* graph, that is to say, G has only one induced odd cycle subgraph, say C_{2k+1} . For each $i \in V(C_{2k+1})$, let

$$A_i = \{x \in V(G) \mid x \neq i \text{ and for all } j \in V(C_{2k+1}), i \text{ appears on every } (x, j)\text{-path}\}.$$

Based on [75, Page 540], it should be noted that the set A_i may be empty for some i , and it follows from [75, Lemma 2.3] that if $A_i \neq \emptyset$, then the induced subgraph $\langle A_i \rangle$ is bipartite in its own right. Also, for every edge $e = \{i, j\} \in E(C_{2k+1})$, let

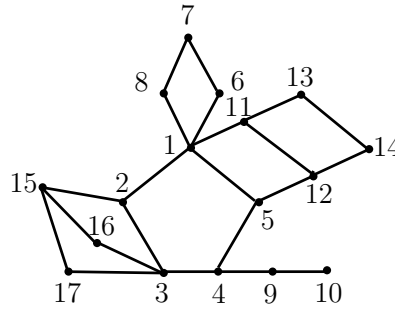
$$B_e = \{x \in V(G) \setminus V(C_{2k+1}) \mid \text{for every } m \in V(C_{2k+1}) \setminus \{i, j\}, \text{ there}$$

is an (x, m) -path in which i appears but j does not, and an

(x, m) -path in which j appears but i does not\}.

Once again, according to [75, Page 541], one must note that the set B_e may be empty for some e , and it can be deduced from [75, Lemma 2.3] that if $B_e \neq \emptyset$, then the induced subgraph $\langle B_e \rangle$ is bipartite in its own right. Furthermore, the A_i 's and B_e 's are all mutually disjoint. In other words, each vertex of G is in exactly one of the following sets: (i) $V(C_{2k+1})$, (ii) A_i for some i , and (iii) B_e for some e . Moreover, for each $i, j \in V(C_{2k+1})$ with $i \neq j$, and $e, e' \in E(C_{2k+1})$ with $e \neq e'$, one can easily derive from the definitions that there is no path between any two vertices of $\langle A_i \rangle$ and $\langle B_e \rangle$, or $\langle A_i \rangle$ and $\langle A_j \rangle$, or $\langle B_e \rangle$ and $\langle B_{e'} \rangle$.

As an example, consider the following graph G from [75].



G

Direct computations show that $A_1 = \{6, 7, 8\}$, $A_4 = \{9, 10\}$, $A_2 = A_3 = A_5 = \emptyset$, and $B_{\{1,5\}} = \{11, 12, 13, 14\}$, $B_{\{2,3\}} = \{15, 16, 17\}$, $B_{\{1,2\}} = B_{\{3,4\}} = B_{\{4,5\}} = \emptyset$.

The following theorem is the first main result of this section.

Theorem 6.2.1. *Assume that $G = (V(G), E(G))$ is a finite simple connected graph, and $J(G)$ denotes the cover ideal of G . Then $J(G)$ is nearly normally torsion-free if and only if G is either a bipartite graph or an almost bipartite graph.*

Proof. To show the forward implication, let $J(G)$ be nearly normally torsion-free. Suppose, on the contrary, that G is neither bipartite nor almost bipartite. This gives that G has at least two induced odd cycle subgraphs, say C and C' . It follows from Proposition 3.1.15 that $\mathbf{p} = (x_j : j \in V(C)) \in \text{Ass}(J(C)^s)$ and $\mathbf{p}' = (x_j : j \in V(C')) \in \text{Ass}(J(C')^s)$ for all $s \geq 2$. Since $G_{\mathbf{p}} = C$ and $G_{\mathbf{p}'} = C'$, we can deduce from [45, Lemma 2.11] that $\mathbf{p}, \mathbf{p}' \in \text{Ass}_R(R/J(G)^s)$ for all $s \geq 2$. This contradicts the assumption that $J(G)$ is nearly normally torsion-free.

Conversely, if G is bipartite, then on account of [48, Corollary 2.6], one has $J(G)$ is normally torsion-free, and so $J(G)$ is nearly normally torsion-free. Next, we assume that G is an almost bipartite graph, and let C be its unique induced odd cycle subgraph. Put $\mathbf{p} = (x_j : j \in V(C))$. We claim that $\text{Ass}(J(G)^s) = \text{Min}(J(G)) \cup \{\mathbf{p}\}$ for all $s \geq 2$. Fix $s \geq 2$. For any $i \in V(C)$ and $e \in E(C)$, assume that A_i and B_e are the vertex subsets of G as defined in the discussion above. Without loss of generality, suppose that $A_i \neq \emptyset$ for all $i = 1, \dots, r$ and $B_{e_j} \neq \emptyset$ for all $j = 1, \dots, t$. Set $H_i := \langle A_i \cup \{i\} \rangle$ for all $i = 1, \dots, r$, and $L_j := \langle B_{e_j} \cup \{\alpha_j, \beta_j\} \rangle$ for all $j = 1, \dots, t$, where $e_j = \{\alpha_j, \beta_j\}$. Accordingly, we get $|V(C) \cap V(H_i)| = 1$ for all $i = 1, \dots, r$, $|V(C) \cap V(L_j)| = 2$ and $|E(C) \cap E(L_j)| = 1$ for all $j = 1, \dots, t$. On the other hand, it should be noted that all H_i and L_j are bipartite, and so [48, Corollary 2.6] yields that $\text{Ass}(J(H_i)^s) = \text{Min}(J(H_i))$ and $\text{Ass}(J(L_j)^s) = \text{Min}(J(L_j))$ for all $i = 1, \dots, r$ and $j = 1, \dots, t$. Now, repeated applications of Theorems 3.3.23 and 3.3.25 give that $\text{Ass}(J(G)^s) = \text{Ass}(J(C)^s) \cup \text{Min}(J(H_i)) \cup \text{Min}(J(L_j))$ for all $i = 1, \dots, r$ and $j = 1, \dots, t$. By virtue of Proposition 3.1.15, one obtains $\text{Ass}(J(C)^s) = \{\mathbf{p}\} \cup \text{Min}(J(C))$. We thus have $\text{Ass}(J(G)^s) = \text{Min}(J(G)) \cup \{\mathbf{p}\}$, as claimed. This shows that $J(G)$ is nearly normally torsion-free, and the proof is done. $\square$

Now, we focus on cover ideals of hypergraphs. For this purpose, we state the following theorem. To do this, one has to recollect Definitions 3.1.3, 3.1.4, and 3.1.5, which will be necessary for understanding Theorem 6.2.2.

Theorem 6.2.2. *Assume that $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ and $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ are two finite simple hypergraphs such that $V(\mathcal{H}) = V(\mathcal{G}) \cup \{w\}$ with $w \notin V(\mathcal{G})$, and $E(\mathcal{H}) = E(\mathcal{G}) \cup \{\{v, w\}\}$ for some vertex $v \in V(\mathcal{G})$. Then*

$$\text{Ass}_{R'}(R'/J(\mathcal{H})^s) = \text{Ass}_R(R/J(\mathcal{G})^s) \cup \{(x_v, x_w)\},$$

for all s , where $R = K[x_\alpha : \alpha \in V(\mathcal{G})]$ and $R' = K[x_\alpha : \alpha \in V(\mathcal{H})]$.

Proof. For convenience of notation, set $I := J(\mathcal{G})$ and $J := J(\mathcal{H})$. We first prove that $\text{Ass}_R(R/I^s) \cup \{(x_v, x_w)\} \subseteq \text{Ass}_{R'}(R'/J^s)$ for all s . Fix $s \geq 1$, and assume that $\mathbf{p} = (x_{i_1}, \dots, x_{i_r})$ is an arbitrary element of $\text{Ass}_R(R/I^s)$. According to [45, Lemma 2.11], we get $\mathbf{p} \in \text{Ass}(K[\mathbf{p}]/J(\mathcal{G}_{\mathbf{p}})^s)$, where $K[\mathbf{p}] = K[x_{i_1}, \dots, x_{i_r}]$ and $\mathcal{G}_{\mathbf{p}}$ is the induced subhypergraph of $\mathcal{G}$ on the vertex set $\{i_1, \dots, i_r\} \subseteq V(\mathcal{G})$. Since $\mathcal{G}_{\mathbf{p}} = \mathcal{H}_{\mathbf{p}}$, we have $\mathbf{p} \in \text{Ass}(K[\mathbf{p}]/J(\mathcal{H}_{\mathbf{p}})^s)$. This yields that $\mathbf{p} \in \text{Ass}_{R'}(R'/J^s)$. On account of $(x_v, x_w) \in \text{Ass}_{R'}(R'/J^s)$, one derives $\text{Ass}_R(R/I^s) \cup \{(x_v, x_w)\} \subseteq \text{Ass}_{R'}(R'/J^s)$. To complete the proof, it is enough for us to show the reverse inclusion. Assume that $\mathbf{p} = (x_{i_1}, \dots, x_{i_r})$ is an arbitrary element of $\text{Ass}_{R'}(R'/J^s)$ with $\{i_1, \dots, i_r\} \subseteq V(\mathcal{H})$. If $\{i_1, \dots, i_r\} \subseteq V(\mathcal{G})$, then [45, Lemma 2.11] implies that $\mathbf{p} \in \text{Ass}_R(R/I^s)$, and the proof is done. Thus, let $w \in \{i_1, \dots, i_r\}$. It follows from [45, Corollary 4.5] that the associated primes of $J(\mathcal{H})^s$ will correspond to critical chromatic subhypergraphs of size $s + 1$ in the s -th expansion of $\mathcal{H}$. This means that one can take the induced subhypergraph on the vertex set $\{i_1, \dots, i_r\}$, and then form the s -th expansion on this induced subhypergraph, and within this new hypergraph find a critical $(s + 1)$ -chromatic hypergraph. Notice that since this expansion cannot have any critical chromatic subgraphs, this implies that $\mathcal{H}_{\mathbf{p}}$ must be connected. Hence, $v \in \{i_1, \dots, i_r\}$. Without loss of generality, one may assume that $i_1 = v$ and $i_2 = w$. Thanks to w is only connected to v in the hypergraph $\mathcal{H}$, and because this induced subhypergraph is critical, if we remove the vertex w , one can color the resulting hypergraph with at least s colors. This leads to that w has to be adjacent to at least s vertices. But the only thing w is adjacent to is the shadows of w and the shadows of v , and so one has a clique among these vertices. Accordingly, w and its neighbors will form a clique of size $s + 1$. Since a clique is a critical graph, it follows that we do not need any element of $\{i_3, \dots, i_r\}$ or their shadows when making the critical $(s + 1)$ -chromatic hypergraph. Consequently, we obtain $\mathbf{p} = (x_v, x_w)$, as required. $\square$

Before stating the next result, it should be noted that one can always view a square-free monomial ideal as the cover ideal of a simple hypergraph. In fact, assume that I is a square-free monomial ideal, and $I^\vee$ denotes its Alexander dual. Also, let $\mathcal{H}$ denote the hypergraph corresponding to $I^\vee$. Then, we have $I = J(\mathcal{H})$, where $J(\mathcal{H})$ denotes the cover ideal of the hypergraph $\mathcal{H}$. Consult [43] for further details and information.

For instance, consider the following square-free monomial ideal in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$ over a field K ,

$$I = (x_1x_2x_3, x_2x_3x_4, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2).$$

Then the Alexander dual of I is given by

$$\begin{aligned} I^\vee &= (x_1, x_2, x_3) \cap (x_2, x_3, x_4) \cap (x_3, x_4, x_5) \cap (x_4, x_5, x_1) \cap (x_5, x_1, x_2) \\ &= (x_3x_5, x_2x_5, x_2x_4, x_1x_4, x_1x_3). \end{aligned}$$

Now, define the hypergraph $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ with $\mathcal{X} = \{x_1, x_2, x_3, x_4, x_5\}$ and

$$\mathcal{E} = \{\{x_3, x_5\}, \{x_2, x_5\}, \{x_2, x_4\}, \{x_1, x_4\}, \{x_1, x_3\}\}.$$

Then the edge ideal and cover ideal of the hypergraph $\mathcal{H}$ are given by

$$I(\mathcal{H}) = (x_3x_5, x_2x_5, x_2x_4, x_1x_4, x_1x_3),$$

and

$$J(\mathcal{H}) = I(\mathcal{H})^\vee = (x_3, x_5) \cap (x_2, x_5) \cap (x_2, x_4) \cap (x_1, x_4) \cap (x_1, x_3).$$

It is easy to see that $J(\mathcal{H}) = I$, as claimed.

We are in a position to provide the second main result of this section in the following corollary.

Corollary 6.2.3. *Let I be a normally torsion-free square-free monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then the ideal $L := IS \cap (x_n, x_{n+1}) \subset S = R[x_{n+1}]$ satisfies the following statements:*

- (i) L is normally torsion-free.
- (ii) L is nearly normally torsion-free.
- (iii) L is normal.
- (iv) L has the strong persistence property.
- (v) L has the persistence property.
- (vi) L has the symbolic strong persistence property.

Proof. (i) In the light of the argument which has been stated above, we can assume that $I = J(\mathcal{H})$ such that the hypergraph $\mathcal{H}$ corresponds to $I^\vee$, where $I^\vee$ denotes the Alexander dual of I . Fix $t \geq 1$. It follows now from Theorem 6.2.2 that

$$\text{Ass}_S(S/L^t) = \text{Ass}_R(R/J(\mathcal{H})^t) \cup \{(x_n, x_{n+1})\}.$$

Since I is normally torsion-free, one can deduce that $\text{Ass}_R(R/J(\mathcal{H})^t) = \text{Min}(J(\mathcal{H}))$, and so $\text{Ass}_S(S/L^t) = \text{Min}(J(\mathcal{H})) \cup \{(x_n, x_{n+1})\}$. Therefore, $\text{Ass}_S(S/L^t) = \text{Min}(L)$. This means that L is normally torsion-free, as desired.

- (ii) It is obvious from the definition of nearly normally torsion-freeness and (i).

(iii) By virtue of [60, Theorem 1.4.6], every normally torsion-free square-free monomial ideal is normal. Now, the assertion can be deduced from (i).

(iv) Due to Theorem 3.2.34, every normal monomial ideal has the strong persistence property, and hence the claim follows readily from (iii).

(v) It is shown in [65] or Proposition 3.1.12 that the strong persistence property implies the persistence property. Therefore, one can derive the assertion from (iv).

(vi) Thanks to [118, Theorem 11], the strong persistence property implies the symbolic strong persistence property, and thus the claim is an immediate consequence of (iv). $\square$

6.3 The case of t -spread principal Borel ideals

In this section, we focus on normally torsion-free and nearly normally torsion-free t -spread principal Borel ideals. It should be noted that the results of this section can be found in [108].

We first recall the definition of t -spread monomial ideals from [37].

Definition 6.3.1. Let $t \geq 0$ and $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K . A monomial $x_{i_1}x_{i_2} \cdots x_{i_d} \in R$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ is called t -spread if $i_j - i_{j-1} \geq t$ for all $j = 2, \dots, d$. A monomial ideal in R is called a t -spread monomial ideal if it is generated by t -spread monomials. A 0-spread monomial ideal is just an ordinary monomial ideal, while a 1-spread monomial ideal is just a square-free monomial ideal.

Throughout this section, we will assume that $t \geq 1$.

Definition 6.3.2. Let $I \subset R = K[x_1, \dots, x_n]$ be a t -spread monomial ideal. Then I is called a t -spread strongly stable ideal if for all t -spread monomials $u \in \mathcal{G}(I)$, all $j \in \text{supp}(u)$ and all $1 \leq i < j$ such that $x_i(u/x_j)$ is a t -spread monomial, it follows that $x_i(u/x_j) \in I$.

Definition 6.3.3. A monomial ideal $I \subset R = K[x_1, \dots, x_n]$ is called a t -spread principal Borel if there exists a t -spread monomial $u \in \mathcal{G}(I)$ such that I is the smallest t -spread strongly stable ideal which contains u . We denote it as $I = B_t(u)$. It should be noted that for a t -spread monomial $u = x_{i_1}x_{i_2} \cdots x_{i_d} \in R$, we have $x_{j_1}x_{j_2} \cdots x_{j_d} \in \mathcal{G}(B_t(u))$ if and only if $j_1 \leq i_1, \dots, j_d \leq i_d$ and $j_k - j_{k-1} \geq t$ for $k \in \{2, \dots, d\}$. More information can be found in [37].

To see an application of Corollary 6.1.3, we re-prove Proposition 4.4 in [9].

Proposition 6.3.4. Let $u = x_i x_n$ be a t -spread monomial in $R = K[x_1, \dots, x_n]$ with $i \geq t$. Then $I = B_t(u)$ is nearly normally torsion-free.

Proof. We first assume that $i = t$. In this case, it follows from the definition that

$$I = (x_1x_{t+1}, x_1x_{t+2}, \dots, x_1x_n, x_2x_{t+2}, \dots, x_2x_n, \dots, x_tx_{2t}, x_tx_{2t+1}, \dots, x_tx_n).$$

It is routine to check that I is the edge ideal of a bipartite graph with the vertex set $\{1, 2, \dots, t\} \cup \{t+1, t+2, \dots, n\}$. In addition, [133, Corollary 14.3.15] implies that I is normally torsion-free. Now, let $i > t$. One can conclude from the definition that the minimal generators of I are as follows:

$$\begin{aligned} & x_ix_n, x_ix_{n-1}, \dots, x_ix_{i+t}, x_{i-1}x_n, x_{i-1}x_{n-1}, \dots, x_{i-1}x_{i+t-1}, \\ & x_{i-2}x_n, x_{i-2}x_{n-1}, \dots, x_{i-2}x_{i+t-2}, \dots, x_1x_n, x_1x_{n-1}, \dots, x_1x_{t+1}. \end{aligned}$$

Our strategy is to use Corollary 6.1.3. To do this, one has to show that $I(\mathfrak{m} \setminus \{x_z\})$ is normally torsion-free for all $z = 1, \dots, n$, where $\mathfrak{m} = (x_1, \dots, x_n)$. First, let $1 \leq z \leq i-1$. Direct computation gives that

$$\begin{aligned} I(\mathfrak{m} \setminus \{x_z\}) &= (x_\alpha x_\beta : \alpha = 1, \dots, z-1, \beta = t+1, \dots, z+t-1, \beta - \alpha \geq t) \\ &\quad + (x_n, x_{n-1}, \dots, x_{z+t}). \end{aligned}$$

One can deduce that $(x_\alpha x_\beta : \alpha = 1, \dots, z-1, \beta = t+1, \dots, z+t-1, \beta - \alpha \geq t)$ is the edge ideal of a bipartite graph with the following vertex set

$$\{1, \dots, z-1\} \cup \{t+1, \dots, z+t-1\},$$

and so is normally torsion-free. Also, we know that every monomial prime ideal is normally torsion-free. Moreover, on account of the two ideals $(x_n, x_{n-1}, \dots, x_{z+t})$ and $(x_\alpha x_\beta : \alpha = 1, \dots, z-1, \beta = t+1, \dots, z+t-1, \beta - \alpha \geq t)$ do not have any common variables, we conclude from Theorem 5.3.4 that $I(\mathfrak{m} \setminus \{x_z\})$ is normally torsion-free as well.

Here, we consider the case $z = i$. Then we have

$$\begin{aligned} I(\mathfrak{m} \setminus \{x_i\}) &= (x_\alpha x_\beta : \alpha = i-t+1, \dots, i-1, \beta = i+1, \dots, i+t-1, \beta - \alpha \geq t) \\ &\quad + (x_n, x_{n-1}, \dots, x_{i+t}) + (x_1, x_2, \dots, x_{i-t}). \end{aligned}$$

A similar technique shows that $I(\mathfrak{m} \setminus \{x_i\})$ is again normally torsion-free.

Now, let $i+1 \leq z \leq n$. It is routine to check that

$$\begin{aligned} I(\mathfrak{m} \setminus \{x_z\}) &= (x_\alpha x_\beta : \alpha = z-t+1, \dots, i, \beta = z+1, \dots, n, \beta - \alpha \geq t) \\ &\quad + (x_1, x_2, \dots, x_{z-t}). \end{aligned}$$

It is easy to see that $(x_\alpha x_\beta : \alpha = z-t+1, \dots, i, \beta = z+1, \dots, n, \beta - \alpha \geq t)$ is the edge ideal of a bipartite graph with the following vertex set

$$\{z-t+1, \dots, i\} \cup \{z+1, \dots, n\},$$

and hence is normally torsion-free. In addition, by a similar argument, the ideal $(x_1, x_2, \dots, x_{z-t})$ is normally torsion-free. Furthermore, since $(x_1, x_2, \dots, x_{z-t})$ and $(x_\alpha x_\beta : \alpha = z-t+1, \dots, i, \beta = z+1, \dots, n, \beta - \alpha \geq t)$ have no common variables, we are able to derive from Theorem 5.3.4 that $I(\mathfrak{m} \setminus \{x_z\})$ is normally torsion-free too. Inspired by Corollary 6.1.3, we conclude that I is nearly normally torsion-free, as claimed. $\square$

We illustrate the statement of Proposition 6.3.4 through the following example.

Example 6.3.5. Let $u = x_4x_7$ and $t = 3$. If $v = x_ax_b \in \mathcal{G}(B_3(u))$, then $a \in [1, 4]$ and $b \in [4, 7]$. The complete list of minimal generators of $B_3(x_4x_7)$ is given below:

$$\begin{array}{cccc} x_1x_4 & & & \\ x_1x_5 & x_2x_5 & & \\ x_1x_6 & x_2x_6 & x_3x_6 & \\ x_1x_7 & x_2x_7 & x_3x_7 & x_4x_7 \end{array}$$

The monomial localizations of I at $\mathfrak{m} \setminus \{x_k\}$, for each $k = 1, \dots, 7$, are listed below:

$$\begin{aligned} I(\mathfrak{m} \setminus \{x_1\}) &= (x_4, x_5, x_6, x_7); \\ I(\mathfrak{m} \setminus \{x_2\}) &= (x_5, x_6, x_7) + (x_1x_4); \\ I(\mathfrak{m} \setminus \{x_3\}) &= (x_6, x_7) + (x_1x_4, x_1x_5, x_2x_5); \\ I(\mathfrak{m} \setminus \{x_4\}) &= (x_7, x_1) + (x_2x_5, x_2x_6, x_3x_6); \\ I(\mathfrak{m} \setminus \{x_5\}) &= (x_1, x_2) + (x_3x_6, x_3x_7, x_4x_7); \\ I(\mathfrak{m} \setminus \{x_6\}) &= (x_1, x_2, x_3) + (x_4x_7); \\ I(\mathfrak{m} \setminus \{x_7\}) &= (x_1, x_2, x_3, x_4). \end{aligned}$$

Therefore, for each $k = 1, \dots, 7$, the monomial localization $I(\mathfrak{m} \setminus \{x_k\})$ is either a monomial prime ideal, or sum of a monomial prime ideal and an edge ideal of a bipartite graph, which have no common variables. In each of the above cases, $I(\mathfrak{m} \setminus \{x_k\})$ is a normally torsion-free ideal. Hence, it follows from Corollary 6.1.3 that I is nearly normally torsion-free.

We further investigate the class of ideals of type $B_t(u)$ in the context of normally torsion-freeness and nearly normally torsion-freeness. Given $a, b \in \mathbb{N}$, we set $[a, b] := \{c \in \mathbb{N} : a \leq c \leq b\}$. Let $u := x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}$ and

$$(6.3.1) \quad A_k := [(k-1)t+1, i_k], \text{ for each } k = 1, \dots, d.$$

We can describe the minimal generators of $I = B_t(u)$ in the following way: if $x_{j_1} \cdots x_{j_d} \in \mathcal{G}(I)$, then for each $k = 1, \dots, d$, we have $j_k \in A_k$. For example, a complete list of minimal generators of $B_2(x_3x_5x_7)$ is given below:

$$\begin{array}{lll}
x_1x_3x_5 & & \\
x_1x_3x_6 & & \\
x_1x_3x_7 & & \\
x_1x_4x_6 & x_2x_4x_6 & \\
x_1x_4x_7 & x_2x_4x_7 & \\
x_1x_5x_7 & x_2x_5x_7 & x_3x_5x_7
\end{array}$$

Here, $A_1 = [1, 3]$, $A_2 = [3, 5]$, and $A_3 = [5, 7]$.

Let $u = x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}$ with $i_{d-1} \leq (d-1)t$, then it forces us to have $i_k \leq kt$, for each $k = 1, \dots, d-2$ because u is a t -spread monomial. In this case, $A_i \cap A_j = \emptyset$. For example, a complete list of minimal generators of $B_3(x_3x_6x_9)$ is given below:

$$\begin{array}{lll}
x_1x_4x_7 & & \\
x_1x_4x_8 & & \\
x_1x_4x_9 & & \\
x_1x_5x_8 & x_2x_5x_8 & \\
x_1x_5x_9 & x_2x_5x_9 & \\
x_1x_6x_9 & x_2x_6x_9 & x_3x_6x_9
\end{array}$$

Here, $A_1 = [1, 3]$, $A_2 = [4, 6]$, and $A_3 = [7, 9]$.

Remark 6.3.6. If $u = x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}$ with $i_{d-1} \leq (d-1)t$, then $B_t(u)$ is an edge ideal of a d -uniform d -partite hypergraph whose vertex partition is given by $A_1, \dots, A_d$. Recall that $\mathcal{H}$ is a d -partite hypergraph if its vertex set $V(\mathcal{H})$ is a disjoint union of sets $V_1, \dots, V_d$ such that if E is an edge of $\mathcal{H}$, then $|E \cap V_i| \leq 1$. Moreover, a hypergraph is d -uniform if each edge of $\mathcal{H}$ has size d . In particular, if $\mathcal{H}$ is a d -uniform d -partite hypergraph with vertex partition $V_1, \dots, V_d$, then $|E| = d$ and $|E \cap V_i| = 1$ for each $E \in E(\mathcal{H})$.

Let $\mathcal{A}_t$ denote the family of all ideals of the form $B_t(u)$ such that $B_t(u)$ is an edge ideal of a d -partite d -uniform hypergraph for some d . From the above discussion it follows that $B_t(u) \in \mathcal{A}_t$ if and only if there exists some d such that $u = x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}$ with $i_{d-1} \leq (d-1)t$.

Let $\mathcal{H}$ be a hypergraph. A sequence $v_1, E_1, v_2, E_2, \dots, v_s, E_s, v_{s+1} = v_1$ of distinct edges and vertices of $\mathcal{H}$ is called a *cycle* in $\mathcal{H}$ if $v_i, v_{i+1} \in E_i$ for all $i = 1, \dots, s$. Such a cycle is called *special* if no edge contain more than two vertices of the cycle. After translating the language of simplicial complexes to hypergraphs, it can be seen from [60, Theorem 10.3.16] that if a hypergraph does not have any special odd cycles, then its edge ideal is normally torsion-free. The following example shows that hypergraphs with edge ideals in $\mathcal{A}_t$, may contain special odd cycles.

Example 6.3.7. Let $\mathcal{H}$ be the hypergraph whose edge ideal is $I = B_3(x_3x_6x_9x_{12})$. Then the following sequence of the vertices and the edges of $\mathcal{H}$ gives a special odd cycle.

$$1, \{1, 4, 9, 12\}, 9, \{2, 5, 9, 12\}, 5, \{1, 5, 8, 11\}, 1$$

Here, remember the definition of the deletion operator, which has been stated in Definition 3.2.19.

Remark 6.3.8. Let $u = x_{i_1}x_{i_2}\cdots x_{i_{d-1}}x_{i_d}$ with $i_{d-1} \leq (d-1)t$, and $I = B_t(u) \subset R$. Note that we have $I \in \mathcal{A}_t$.

1. If $i_k < kt$, for some $1 \leq k \leq d-1$, then the variables $x_{i_k+1}, \dots, x_{kt}$ do not appear in $\text{supp}(I)$. For example, a complete list of minimal generators of $B_3(x_2x_5x_9)$ is given below:

$$x_1x_4x_7$$

$$x_1x_4x_8$$

$$x_1x_4x_9$$

$$x_1x_5x_8$$

$$x_2x_5x_8$$

$$x_1x_5x_9$$

$$x_2x_5x_9$$

Here, $A_1 = [1, 2]$, $A_2 = [4, 5]$, and $A_3 = [7, 9]$, and $x_3, x_6 \notin \text{supp}(B_3(x_2x_5x_9))$.

2. If $i_d < n$, then $x_{i_d+1}, \dots, x_n \notin \text{supp}(I)$. Hence, we can always assume that $i_d = n$.
3. If $i_k > (k-1)t+1$, that is, $|A_k| > 1$ for some $k = 1, \dots, d$, then $I \setminus x_{i_k} = B_t(v)$ where v is chosen with the following property: $v = x_{j_1}\cdots x_{j_d} \in \mathcal{G}(I \setminus x_{i_k})$ and for any other $w = x_{l_1}\cdots x_{l_d} \in \mathcal{G}(I \setminus x_{i_k})$, we have $l_k \leq j_k$. For example, $B_3(x_2x_5x_9) \setminus x_5 = B_3(x_1x_4x_9)$. Therefore, we conclude that $I \setminus x_{i_k} \in \mathcal{A}_t$ for all x_{i_k} with $k = 1, \dots, d$.
4. The definition of A_k immediately implies that $|A_k| = 1$ if and only if $i_k = (k-1)t+1$. Moreover, if $i_k = (k-1)t+1$ for some $1 \leq k \leq d$, then $i_j = (j-1)t+1$ for all $j \leq k$ because u is a t -spread monomial. In this case, $x_1x_{t+1}\cdots x_{(k-1)t+1}$ divides every minimal generator of $B_t(u)$, and hence $B_t(u) = x_1x_{t+1}\cdots x_{(k-1)t+1}J$, where J can be identified as a t -spread principal Borel ideal generated in degree $d-k$ in its ambient polynomial ring. Hence, $J = B_t(v) \in \mathcal{A}_t$ with $v = u/x_1x_{t+1}\cdots x_{(k-1)t+1}$.

For example, if $u = x_1x_4x_9x_{12}$, then $B_3(u) = x_1x_4J$ with $J = B_t(v) \subset K[x_7, \dots, x_{12}]$ and $v = x_9x_{12}$. In fact, by substituting $y_{i-6} = x_i$ for $i = 7, \dots, 12$, J can be identified with $B_3(y_3y_6) \subset K[y_1, \dots, y_6]$.

5. If $|A_k| \geq 2$, then $i_k > (k-1)t + 1$. This forces $|A_j| \geq 2$ for all $j = k, \dots, d$ because u is a t -spread monomial.

In what follows, our aim is to show that for any fixed t , all ideals in $\mathcal{A}_t$ are normally torsion-free. It is known from [10, Corollary 2.6] that $B_t(u)$ satisfies the persistence property and the Rees algebra $\mathcal{R}(B_t(u))$ is a normal Cohen-Macaulay domain. It is a well-known fact that for any nonzero graded ideal $I \subset R = K[x_1, \dots, x_n]$, if $\mathcal{R}(I)$ is Cohen-Macaulay, then $\lim_{k \rightarrow \infty} \text{depth}(R/I^k) = n - \ell(I)$, for example see [60, Proposition 10.3.2], where $\ell(I)$ denotes the analytic spread of I , that is, the Krull dimension of the fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. This leads us to the following corollary which will be used in the subsequent results. Note that due to Remark 6.3.8(1), one needs to pay attention to the ambient ring of $B_t(u)$. From now on, by the ambient ring, we mean the polynomial ring R containing $B_t(u)$ such that all variables in R appear in $\text{supp}(B_t(u))$.

Corollary 6.3.9. *Let $I = B_t(u) \subset R = K[x_i : x_i \in \text{supp}(I)]$. If $\ell(I) < n$, then $\lim_{k \rightarrow \infty} \text{depth}(R/I^k) \neq 0$. In particular, if $\ell(I) < n$, then $\mathfrak{m} \notin \text{Ass}(R/I^k)$, where $\mathfrak{m}$ is the unique graded maximal ideal of R .*

Next, we compute $\ell(I)$ for all $I \in \mathcal{A}_t$. To do this, we first recollect the definition of linear relation graph from [65, Definition 3.1].

Definition 6.3.10. *Let $I \subset R$ be a monomial ideal with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. The linear relation graph Γ of I is the graph with the edge set*

$$E(\Gamma) = \{\{i, j\} : \text{there exist } u_k, u_l \in \mathcal{G}(I) \text{ such that } x_i u_k = x_j u_l\},$$

and the vertex set $V(\Gamma) = \bigcup_{\{i, j\} \in E(\Gamma)} \{i, j\}$.

It is known from [34, Lemma 5.2] that if I is a monomial ideal generated in degree d and the first syzygy of I is generated in degree $d + 1$, then

$$(6.3.2) \quad \ell(I) = r - s + 1,$$

where r is the number of vertices and s is the number of connected components of the linear relation graph of I . If u is a t -spread monomial of degree d , then it can be concluded from [10, Theorem 2.3] that the first syzygy of $B_t(u)$ is generated in degree $d + 1$. Hence, [34, Lemma 5.2] gives a way to compute the analytic spread of $B_t(u)$. Before proving the other main result of this section, we first analyze the linear relation graph of $B_t(u)$.

Lemma 6.3.11. *Let $u = x_{i_1} x_{i_2} \cdots x_{i_{d-1}} x_{i_d}$ and $A_k = [(k-1)t + 1, i_k]$ for each $k = 1, \dots, d$. Moreover, let Γ be the linear relation graph of $I = B_t(u)$. Then the following statements hold:*

(i) For some $1 \leq k \leq d$, $|A_k| \geq 2$ if and only if $A_k \subseteq V(\Gamma)$. Moreover, if $|A_k| \geq 2$, then the induced subgraph of Γ on A_k is a complete graph.

(ii) Let $i_k < kt + 1$ for some $1 \leq k \leq d - 1$. Then Γ does not contain any edge with one endpoint lies in A_r and the other endpoint lies in A_s for any $1 \leq r < s \leq k$.

Proof. (i) We first show that if $A_k \subseteq V(\Gamma)$, then $|A_k| \geq 2$. Assume that $|A_k| = 1$ for some k . Then from Remark 6.3.8(4), we see that $i_k = (k - 1)t + 1$. In this case, each generator of $B_t(u)$ is a multiple of x_{i_k} , and according to the definition of Γ , we deduce that $A_k = \{i_k\} \not\subseteq V(\Gamma)$.

Conversely, we verify that if $|A_k| \geq 2$, for some k , then $A_k \subseteq V(\Gamma)$ and the induced subgraph on A_k is a complete graph. Take any $f, h \in A_k$ with $f < h$. If $k = 1$, then $f < h \leq i_1$ and the monomials $v = x_h(u/x_{i_1})$ and $v' = x_f(u/x_{i_1})$ belong to $\mathcal{G}(I)$ because I is t -spread strongly stable. Hence, $\{f, h\} \in \Gamma$, as required. A similar argument shows that if $|A_d| \geq 2$, then $A_d \subseteq V(\Gamma)$ and the induced subgraph on A_d is also a complete graph. Now, assume that $1 < k < d$. Due to $|A_k| \geq 2$, we deduce that $i_k > (k - 1)t + 1$ and $(k - 1)t + 1 \leq f < h \leq i_k$. By using the fact that I is t -spread strongly stable, we see that $v = x_1 \cdots x_{(k-2)t+1} x_f x_{kt+1} \cdots x_{i_d} \in \mathcal{G}(I)$ and $v' = x_1 \cdots x_{(k-2)t+1} x_h x_{kt+1} \cdots x_{i_d} \in \mathcal{G}(I)$. Moreover, we have $v = (v'/x_h)x_f$, and so $\{f, h\} \in E(\Gamma)$, as required.

(ii) If $|A_r| = 1$ or $|A_s| = 1$, then by using (i), we see that the statement holds trivially. Assume that $|A_r| \geq 2$ and $|A_s| \geq 2$. Since $i_k < kt + 1$, we have $i_j < jt + 1$ for all $j = 1, \dots, k$ because u is a t -spread monomial. In this case, $A_r \cap A_s = \emptyset$ for all $1 \leq r < s \leq k$. Moreover, $|A_r| \leq t$ for all $r = 1, \dots, k$. Take $f \in A_r$ and $h \in A_s$ for some $1 < r < s \leq k$. Then for any $u \in \mathcal{G}(I)$ with $x_h|u$, we have $(u/x_h)x_f \notin \mathcal{G}(I)$. This can be easily verified because $(u/x_h)x_f$ is not a t -spread monomial as it contains two variables with indices in A_r and $|A_r| \leq t$. Hence, we do not have any edge in Γ of the form $\{f, h\}$ where $f \in A_r$ and $h \in A_s$. $\square$

Proposition 6.3.12. Let $I = B_t(x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}) \subset R = K[x_i : x_i \in \text{supp}(I)]$ with $i_{d-1} \leq (d - 1)t$. Then $\ell(I) < n = |\text{supp}(I)|$. In particular, $\mathfrak{m} \notin \text{Ass}(R/I^k)$ for all $k \geq 1$, where $\mathfrak{m}$ is the unique graded maximal ideal of R .

Proof. If I is a principal ideal, then the assertion holds trivially. Therefore, we may assume that I is not a principal ideal. To show that $\ell(I) < n$, from the equality in (6.3.2), it is enough to prove that the linear relation graph Γ of I has more than one connected components. Let $u = x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}$ and $A_k = [(k - 1)t + 1, i_k]$ for $k = 1, \dots, d$. Since I is not principal, it follows from Remark 6.3.8(4) that $|A_k| \geq 2$ for some $1 \leq k \leq d$.

Since $i_{d-1} \leq (d - 1)t$, one can deduce from Lemma 6.3.11 that we do not have any edge in Γ of the form $\{f, h\}$ with f and h lie in different A_k 's and if $|A_k| \geq 2$,

for some k , then $A_k \subseteq V(\Gamma)$ and the induced subgraph on A_k is a complete graph. This shows that $V(\Gamma)$ is the union of all A_i 's for which $|A_i| \geq 2$. Moreover, each such A_i determines a connected component in Γ . Hence, Γ has only one connected components if and only if $|A_d| \geq 2$, and $|A_i| = 1$ for all $i = 1, \dots, d-1$. In this case, $\ell(I) = |A_d| - 1 + 1 = |A_d| < n$. Otherwise, Γ has at least two connected components and again we obtain $\ell(I) < n$, as required. Then the assertion $\mathfrak{m} \notin \text{Ass}(R/I^k)$, for all $k \geq 1$, follows from Corollary 6.3.9. $\square$

We now state the second main result of this section in the next theorem.

Theorem 6.3.13. *Let $I = B_t(x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}) \subset R = K[x_i : x_i \in \text{supp}(I)]$ with $i_{d-1} \leq (d-1)t$. Then I is normally torsion-free.*

Proof. We may assume that $i_k \neq (k-1)t + 1$ for all $1 \leq k \leq d$. Otherwise, from Remark 6.3.8(4), it follows that $I = wB_t(v)$, where w is the product of all variables for which $i_k = (k-1)t + 1$, $v = u/w$, and $B_t(v)$ is a t -spread principal Borel ideal in its ambient ring. Then from Lemma 5.3.11, it follows that I is normally torsion-free if and only if $B_t(v)$ is normally torsion-free. Therefore, one may reduce the discussion to $B_t(v)$ whose generators are not a multiple of a fixed monomial.

Let $u = x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}$ with $i_1 > 1$. To show that $I = B_t(u)$ is normally torsion-free, we will use Theorem 10.2.6. It can be seen from [9, Theorem 4.2] that $u \in \mathfrak{p} \setminus \mathfrak{p}^2$ for all $\mathfrak{p} \in \text{Min}(I)$. We proceed by induction on $n := |\text{supp}(I)|$. Let $n = 2$. Then $d \leq 2$. Hence, $i_1 \leq t$, and so we can view I as the edge ideal of a bipartite graph, and so I is normally torsion-free. Let $n > 2$. Recall that the family of ideals $\mathcal{A}_t$ is defined by: $B_t(u) \in \mathcal{A}_t$ if and only if there exists some d such that $u = x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}$ with $i_{d-1} \leq (d-1)t$. From Remark 6.3.8(3), it follows that $I \setminus x_{i_k} \in \mathcal{A}_t$ for all $k = 1, \dots, d$. It follows now from the inductive hypothesis that $I \setminus x_{i_k}$ is normally torsion-free for all $k = 1, \dots, d$. This completes the inductive step, and the proof is over. $\square$

In the subsequent theorem, we prove the converse of Theorem 6.3.13 to obtain the complete characterization of normally torsion-free t -spread principal Borel ideals.

Theorem 6.3.14. *Let $I = B_t(x_{i_1}x_{i_2} \cdots x_{i_{d-1}}x_{i_d}) \subset R = K[x_i : x_i \in \text{supp}(I)]$. Then I is normally torsion-free if and only if $i_{d-1} \leq (d-1)t$. In other words, I is normally torsion-free if and only if I can be viewed as an edge ideal of a d -uniform d -partite hypergraph.*

Proof. On account of Theorem 6.3.13, it is enough to show that if $i_{d-1} > (d-1)t$, then I is not normally torsion-free. Let k be the smallest integer for which $i_k > kt$. Then by remembering this fact that $i_{k+1} - i_k \geq t$, we get $i_{k+1} > (k+1)t$. Continuing this process, we obtain $i_j > jt$ for all $j = k, \dots, d$, and as we explained

in Remark 6.3.8(5), one has $|A_j| \geq 2$ for all $j = k, \dots, d$. The sets A_j 's are defined in (6.3.1). It follows from Lemma 6.3.11 that $A = A_k \cup A_{k+1} \cup \dots \cup A_d \subset V(\Gamma)$, where Γ is linear relation graph of I and for each $j = k, \dots, d$, the induced subgraph of Γ on A_j is a complete graph. Moreover, $i_j > jt$ for each $j = k, \dots, d$ gives that $i_j \in A_j \cap A_{j+1} \neq \emptyset$ for each $j = k, \dots, d-1$. Therefore, we conclude that the induced subgraph on A is connected.

Set $\mathfrak{p} := (x_{(k-1)t+1}, \dots, x_{i_d})$. Here, the crucial observation is that if we take the monomial localization of I at $\mathfrak{p}$ (that is, we map all variables x_i to 1, where $i \in A_1 \cup \dots \cup A_{k-1}$), then we are reducing the degree of each generator of $B_t(u)$ by $k-1$. It is because $A \cap B = \emptyset$, where $B = A_1 \cup \dots \cup A_{k-1}$. Hence, $I(\mathfrak{p})$ can be viewed as a t -spread principal Borel ideal by a shift of indices of variables. More precisely, each $j \in A$ is shifted to $j - (k-1)t$. Therefore, $I(\mathfrak{p}) = B_t(x_{j_1} \cdots x_{j_k})$, where $t < j_1 < \dots < j_k$. The linear relation graph of $I(\mathfrak{p})$ is isomorphic to the induced subgraph of Γ on the vertex set A . Then we can conclude that

$$\ell(I(\mathfrak{p})) = |\text{supp}(B_t(x_{j_1} \cdots x_{j_k}))| = \dim(R(\mathfrak{p})), \text{ and } \lim_{k \rightarrow \infty} \text{depth}(R(\mathfrak{p})/I(\mathfrak{p})^k) = 0.$$

This shows that $\mathfrak{p} \in \text{Ass}(R/I^k)$ for some $k > 1$. Moreover, $\mathfrak{p} \notin \text{Ass}(R/I)$ because of [9, Theorem 4.2]. Hence, we conclude that I is not normally torsion-free. $\square$

In the next example, we illustrate the construction of $\mathfrak{p}$ as in the proof of Theorem 6.3.14.

Example 6.3.15. Let $u = x_2x_7x_{10}x_{13}$ be a 3-spread monomial and $I = B_3(u)$. Here, $i_1 = 2, i_2 = 7, i_3 = 10$, and $i_4 = 13$. Moreover, $i_1 = 2 < t = 3$, but $i_2 = 7 > 2 \cdot 3$. Set $\mathfrak{p} := (x_4, x_5, \dots, x_{13})$ as in the proof of Theorem 6.3.14. Then the minimal generators of $I(\mathfrak{p})$ are listed below. In the following table, $u \rightarrow v$ indicates that $u \in \mathcal{G}(I(\mathfrak{p}))$ and v is the monomial obtained by shifting each $j \in [4, 13]$ to $j - 3$.

$x_4x_7x_{10} \rightarrow x_1x_4x_7$			
$x_4x_7x_{11} \rightarrow x_1x_4x_8$			
$x_4x_7x_{12} \rightarrow x_1x_4x_9$			
$x_4x_7x_{13} \rightarrow x_1x_4x_{10}$			
$x_4x_8x_{11} \rightarrow x_1x_5x_8$	$x_5x_8x_{11} \rightarrow x_2x_5x_8$		
$x_4x_8x_{12} \rightarrow x_1x_5x_9$	$x_5x_8x_{12} \rightarrow x_2x_5x_9$		
$x_4x_8x_{13} \rightarrow x_1x_5x_{10}$	$x_5x_8x_{13} \rightarrow x_2x_5x_{10}$		
$x_4x_9x_{12} \rightarrow x_1x_6x_9$	$x_5x_9x_{12} \rightarrow x_2x_6x_9$	$x_6x_9x_{12} \rightarrow x_3x_6x_9$	
$x_4x_9x_{13} \rightarrow x_1x_6x_{10}$	$x_5x_9x_{13} \rightarrow x_2x_6x_{10}$	$x_6x_9x_{13} \rightarrow x_3x_6x_{10}$	
$x_4x_{10}x_{13} \rightarrow x_1x_7x_{10}$	$x_5x_{10}x_{13} \rightarrow x_2x_7x_{10}$	$x_6x_{10}x_{13} \rightarrow x_3x_7x_{10}$	$x_7x_{10}x_{13} \rightarrow x_4x_7x_{10}$

Therefore, $I(\mathfrak{p})$ can be viewed as a 3-spread principal Borel ideal $B_3(x_4x_7x_{10})$. A direct computation with Macaulay2 [49] shows that $(x_1, \dots, x_{10})$ is an associated prime of the third power of $B_3(x_4x_7x_{10})$. Consequently, $\mathfrak{p}$ is also an associated prime of the third power of I .

When u is a monomial of degree 3, then we have the following outcome.

Lemma 6.3.16. *Let $u = x_a x_b x_n$ be a t -spread monomial and $I = B_t(u) \subset R = K[x_i : x_i \in \text{supp}(I)]$. Then the following statements hold.*

- (i) *if $b < 2t + 1$, then I is normally torsion-free.*
- (ii) *if $a = 1$ and $b \geq 2t + 1$, then I is nearly normally torsion-free.*
- (iii) *if $a > 1$ and $b \geq 2t + 1$, then I is not nearly normally torsion-free.*

Proof. (i) This claim can be shown by virtue of Theorem 6.3.13.

(ii) Let $u \in \mathcal{G}(I)$. Then $u = x_1 x_{i_2} x_{i_3}$, where $i_2 \in \{t + 1, \dots, b\}$ and $i_3 \in \{2t + 1, \dots, n\}$. It can be easily seen that $I = x_1 J$, where $J = B_t(x_b x_n)$. It follows from Proposition 6.3.4 that $B_t(x_b x_n)$ is nearly normally torsion-free. Then we get the required result by using Lemma 6.1.5.

(iii) We will prove the assertion by constructing two monomial prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ that belong to $\text{Ass}(R/I^k)$, for some $k > 1$, but $\mathfrak{p}_1, \mathfrak{p}_2 \notin \text{Ass}(R/I)$. Recall from (6.3.1) that if $v \in \mathcal{G}(I)$, then $v = x_{a'} x_{b'} x_{c'}$ with $a' \in A_1 = [1, a]$, $b' \in A_2 = [t + 1, b]$, and $c' \in A_3 = [2t + 1, n]$. Since $a > 1$, we have $|A_1| > 1$ and since $b \geq 2t + 1$, we have $|A_2| > 1$ and $|A_3| > 1$.

Let $\mathfrak{p}_1 = (x_{t+1}, \dots, x_{i_d})$. As shown in the last part of the proof of Theorem 6.3.14, $\mathfrak{p}_1 \in \text{Ass}(R/I^k)$, for some $k > 1$, but $\mathfrak{p}_1 \notin \text{Ass}(R/I)$. Let $\mathfrak{p}_2 = (x_1, x_{t+2}, x_{t+3}, \dots, x_n)$; then from [9, Theorem 4.2], we have $\mathfrak{p}_2 \notin \text{Ass}(R/I)$. We claim that $I(\mathfrak{p}_2)$ can be viewed as the t -spread principal Borel ideal $B_t(x_r x_n)$, where $r > t$. Indeed, by substituting $x_{t+1} = 1$, all t -spread monomials of the form $x_1 x_{t+1} x_{c'}$, with $c' \in A_3$ are reduced to $x_1 x_{c'}$. Therefore, the monomials $x_1 x_{b'} x_{c'} \in \mathcal{G}(I)$ with $b' \in [t + 2, b]$ and $c' \in [2t + 2, n]$ do not appear in $\mathcal{G}(I(\mathfrak{p}_2))$.

Since $a > 1$, we have $2 \in A_1$. Moreover, if $v = x_{a'} x_{b'} x_{c'} \in \mathcal{G}(I)$ with $a' \geq 2$, then $b' \geq t + 2$ and $c' \geq 2t + 2$ because v is a t -spread monomial.

By substituting $x_i = 1$, for all $i \in [2, a]$, all monomials of the form $x_i x_{b'} x_{c'}$ with $b' \in [t + 2, b]$ and $c' \in [2t + 2, n]$ are reduced to $x_{b'} x_{c'}$. This shows that $I(\mathfrak{p}_2)$ is generated in degree 2 by t -spread monomials of the following form:

$$\begin{aligned} & x_1 x_{c'} \text{ with } c' \in A_3 = [2t + 1, n]; \\ & x_{b'} x_{c'} \text{ with } b' \in [t + 2, b] \text{ and } c' \in [2t + 2, n] \text{ and } c' - b' \geq t. \end{aligned}$$

Note that $\text{supp}(I(\mathfrak{p}_2)) = \{x_1, x_{t+2}, x_{t+3}, \dots, x_n\}$. Moreover, if we shift the indices of variables in $\{x_{t+2}, x_{t+3}, \dots, x_n\}$ by $k \rightarrow k - t$, then $I(\mathfrak{p}_2)$ can be viewed as a t -spread principal Borel ideal $B_t(x_r x_s)$, where $s = n - t$ and $r = b - t > t$ because $b \geq 2t + 1$. Moreover, the linear relation graph of $B_t(x_r x_s)$ is a connected with $|\text{supp}(I(\mathfrak{p}_2))|$ vertices. Hence, $\lim_{k \rightarrow \infty} \text{depth}(R(\mathfrak{p}_2)/I(\mathfrak{p}_2)^k) = 0$, and consequently $\mathfrak{p}_2 \in \text{Ass}(R/I^k)$ for some $k > 1$. This finishes our argument. $\square$

6.4 Closed neighborhood and dominating ideals of classes of trees

Let G be a simple graph. The closed neighborhood ideal $NI(G)$ of G is generated by square-free monomials that correspond to the closed neighborhoods of the vertices of G , whereas, the dominating ideal $DI(G)$ of G is generated by the monomials that correspond to the dominating sets of G (see Section 1.2 for the formal definitions). As shown in [122], $NI(G)$ and $DI(G)$ are the Alexander dual of each other, a similar relation that exists between edge ideals and cover ideals of G .

Domination in graphs was mathematically formulated by Berge and Ore in 1960's and has been widely studied by many researchers due to its enormous and growing applications in various fields including computer sciences, operations research, linear algebra and optimization. Let G be a simple graph with the vertex set $V(G)$. A set $S \subseteq V(G)$ is known as dominating set of G if every vertex in $V(G) \setminus S$ is adjacent to at least one vertex in S . We refer to [53] for further concepts related to the domination in graphs.

In this section, our main goal is to further extend the study on closed neighborhood ideals and dominating ideals. In particular, we focus on normally torsion-freeness and certain stability property of these ideals. Moreover, we establish that the closed neighborhood ideals and the dominating ideals of star graphs are normally torsion-free. To do this, we will first prove several results of general nature. It should be noted that the results of this section can be found in [107].

The next theorem will be used frequently to formulate proofs of some main results of this section. It provides a way to create new normally torsion-free ideals based on the existing ones.

Theorem 6.4.1. *Let I be a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ and h be a square-free monomial in R . Let there exist two variables x_r and x_s with $1 \leq r \neq s \leq n$ such that $\gcd(h, u) = 1$ or $\gcd(h, u) = x_r$ or $\gcd(h, u) = x_r x_s$ for all $u \in \mathcal{G}(I)$. Then the following statements hold:*

- (i) $I + hR$ is normally torsion-free.
- (ii) $I + hR$ is nearly normally torsion-free.
- (iii) $I + hR$ is normal.
- (iv) $I + hR$ has the strong persistence property.
- (v) $I + hR$ has the persistence property.
- (vi) $I + hR$ has the symbolic strong persistence property.

Proof. (i) For convenience of notation, put $L := I + hR$. If $L \setminus x_k = \mathfrak{m} \setminus x_k$ for some $1 \leq k \leq n$, then one can write $L = x_k J + \mathfrak{m} \setminus x_k$. If $J = R$, then $L = \mathfrak{m}$, and there is nothing to prove. Let $J \neq R$, and take an arbitrary element $v \in \mathcal{G}(J)$. If $x_\ell \mid v$ for some $\ell \in \{1, \dots, n\} \setminus \{k\}$, then $v \in \mathfrak{m} \setminus x_k$, and so $J \subseteq \mathfrak{m} \setminus x_k$. This implies that $L = \mathfrak{m} \setminus x_k$, and hence the assertion holds. We thus assume that $L \setminus x_k \neq \mathfrak{m} \setminus x_k$ for all $k = 1, \dots, n$. We claim that $h \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \text{Min}(L)$. Take an arbitrary element $\mathfrak{p} \in \text{Min}(L)$. Since $h \in L$ and $L \subseteq \mathfrak{p}$, one has $h \in \mathfrak{p}$. Suppose, on the contrary, that $h \in \mathfrak{p}^2$. Due to h is square-free, this gives that $|\text{supp}(h) \cap \text{supp}(\mathfrak{p})| \geq 2$. We observe the following:

- (i) If $x_s \in \text{supp}(u)$ for some $u \in \mathcal{G}(I)$, then $x_r \in \text{supp}(u)$ as well. It is due to the assumption on $\gcd(h, u)$ with $u \in \mathcal{G}(I)$.
- (ii) At most one of x_r and x_s can be in $\text{supp}(\mathfrak{p})$. Indeed, if both $x_r, x_s \in \text{supp}(\mathfrak{p})$, then $x_r, x_s \in \text{supp}(h) \cap \text{supp}(\mathfrak{p})$. From (i), we see that $u \in \mathfrak{p} \setminus \{x_s\}$ for all $u \in \mathcal{G}(I)$. Also, $h \in \mathfrak{p} \setminus \{x_s\}$. Hence, $L \subset \mathfrak{p} \setminus \{x_s\}$, a contradiction to the minimality of $\mathfrak{p}$.

In order to establish our claim, we have the following cases to discuss:

Case 1. $x_r \in \mathfrak{p}$. Take any $x_i \in \text{supp}(h) \cap \text{supp}(\mathfrak{p})$ such that $x_r \neq x_i$. Then $x_s \neq x_i$ due to (ii). From the assumption on $\gcd(h, u)$ with $u \in \mathcal{G}(I)$ it follows that $x_i \notin \text{supp}(I)$. Therefore, $I \subset \mathfrak{p} \setminus \{x_i\}$. Since $h \in \mathfrak{p} \setminus \{x_i\}$, we conclude that $L \subset \mathfrak{p} \setminus \{x_i\}$, a contradiction to the minimality of $\mathfrak{p}$.

Case 2. $x_s \in \mathfrak{p}$. By mimicking the same argument as in Case 1, we again obtain a contradiction to the minimality of $\mathfrak{p}$.

Case 3. $x_r \notin \mathfrak{p}$ and $x_s \notin \mathfrak{p}$. Take any $x_i, x_j \in \text{supp}(h) \cap \text{supp}(\mathfrak{p})$. Then $x_i, x_j \notin \text{supp}(I)$, due to the assumption on $\gcd(h, u)$ with $u \in \mathcal{G}(I)$. It yields that $I \subset \mathfrak{p} \setminus \{x_i\}$. Since $h \in \mathfrak{p} \setminus \{x_i\}$, we conclude that $L \subset \mathfrak{p} \setminus \{x_i\}$, again a contradiction to the minimality of $\mathfrak{p}$.

This shows that our claim holds true. To complete the proof, note that for all $x_k \in \text{supp}(h)$, one has $L \setminus x_k = I \setminus x_k$. Based on Theorem 5.3.16, we gain $I \setminus x_k$ is normally torsion-free as well. This leads to $L \setminus x_k$ is normally torsion-free. Fix $s \geq 1$. Suppose, on the contrary, that $\mathfrak{m} \setminus x_k \in \text{Ass}(R/(L \setminus x_k)^s)$ for some k . Because $\text{Ass}(R/(L \setminus x_k)^s) = \text{Min}(L \setminus x_k)$, we get $\mathfrak{m} \setminus x_k \in \text{Min}(L \setminus x_k)$, and so

$L \setminus x_k = \mathfrak{m} \setminus x_k$, which is a contradiction. Therefore, $\mathfrak{m} \setminus x_i \notin \text{Ass}(R/(I \setminus x_i)^s)$ for all s and $x_i \in \text{supp}(h)$. Consequently, the assertion can be readily concluded from Theorem 10.2.6.

(ii) - (vi) can be shown similar to the proof of Corollary 6.2.3. $\square$

As an immediate consequence of Theorem 6.4.1, we give the following corollary.

Corollary 6.4.2. *The path ideals corresponding to path graphs of length two are normally torsion-free.*

Proof. Let $P = (V(P), E(P))$ denote a path graph with the vertex set $V(P) = \{x_1, \dots, x_n\}$ and the edge set $E(P) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\} \cup \{\{x_n, x_1\}\}$. Hence, the path ideal corresponding to the path graph P of length two is given by

$$L := (x_i x_{i+1} x_{i+2} : i = 1, \dots, n-2).$$

We proceed by induction on n . If $n = 3$, then $L = (x_1 x_2 x_3)$, and there is nothing to prove. Let $n > 3$ and that the claim has been proved for $n-1$. Set $h := x_{n-2} x_{n-1} x_n$ and $I := (x_i x_{i+1} x_{i+2} : i = 1, \dots, n-3)$. One can easily check that, for each $u \in \mathcal{G}(I)$, we have $\gcd(h, u) = 1$ or $\gcd(h, u) = x_{n-2}$ or $\gcd(h, u) = x_{n-2} x_{n-1}$. It follows now from the inductive hypothesis that I is normally torsion-free. Since $L = I + hR$, where $R = K[x_1, \dots, x_n]$, we can derive the assertion from Theorem 6.4.1. $\square$

To see another application of Theorem 6.4.1, we give the following lemma.

Lemma 6.4.3. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and $E(H) = E(G) \cup \{\{v, w\}\}$ for some vertex $v \in V(G)$. If $NI(G)$ is normally torsion-free and $\prod_{j \in N_G[v]} x_j \notin \mathcal{G}(NI(G))$, then $NI(H)$ is normally torsion-free.*

Proof. Let $NI(G)$ be a normally torsion-free monomial ideal. It is routine to check that $NI(H) = NI(G) + (x_v x_w)R$, where $R = K[x_\alpha : \alpha \in V(H)]$. Also, one can easily see that either $\gcd(x_v x_w, u) = 1$ or $\gcd(x_v x_w, u) = x_v$ for all $u \in \mathcal{G}(NI(G))$. Therefore, the claim follows immediately from Theorem 6.4.1. $\square$

We are ready to state the first main result as an immediate corollary of Theorem 6.4.1 and Lemma 6.4.3.

Corollary 6.4.4. *The closed neighborhood ideals of star graphs are normally torsion-free.*

Proof. Proceed by induction on the number of vertices and use Lemma 6.4.3. $\square$

In what follows, we investigate the closed neighborhood ideals related to the cone of a graph.

Definition 6.4.5. ([133, Definition 10.5.4]) The cone $C(G)$, over the graph G , is obtained by adding a new vertex t to G and joining every vertex of G to t .

Lemma 6.4.6. Let G be a graph and let $H := C(G)$ be its cone. Then the following statements hold:

- (i) $NI(G)$ is normally torsion-free if and only if $NI(H)$ is normally torsion-free.
- (ii) $NI(G)$ is nearly normally torsion-free if and only if $NI(H)$ is nearly normally torsion-free.
- (iii) $NI(G)$ is normal if and only if $NI(H)$ is normal.
- (iv) $NI(G)$ has the strong persistence property if and only if $NI(H)$ has the strong persistence property.
- (v) $NI(G)$ has the persistence property if and only if $NI(H)$ has the persistence property.
- (vi) Both $NI(G)$ and $NI(H)$ have the symbolic strong persistence property.

Proof. Assume that the cone $H = C(G)$ is obtained by adding the new vertex w to G and joining every vertex of G to w . Then one can easily see that

$$NI(H) = x_w NI(G) + (x_w \prod_{i \in V(G)} x_i).$$

Since $\prod_{i \in V(G)} x_i \in NI(G)$, this implies that $NI(H) = x_w NI(G)$.

- (i) This claim can be deduced from Lemma 5.3.11.
- (ii) On account of Lemma 6.1.5, one can derive this claim.
- (iii) We can conclude this assertion by virtue of Remark 7.1.1.
- (iv) This claim is an immediate consequence of Lemma 3.2.14.
- (v) Due to Theorem 2.1.27, we can deduce this assertion.
- (vi) This claim follows readily from Theorem 4.1.11. □

In order to establish the next lemma, one has to remember the definition of unisplit monomial ideals, see Definition 2.3.1.

Lemma 6.4.7. Let G be a graph and let $H := C(G)$ be its cone. Then the following statements hold:

- (i) $DI(G)$ is normally torsion-free if and only if $DI(H)$ is normally torsion-free.
- (ii) $DI(G)$ is nearly normally torsion-free if and only if $DI(H)$ is nearly normally torsion-free.
- (iii) $DI(G)$ is normal if and only if $DI(H)$ is normal.

(iv) $DI(H)$ has the strong persistence property.

(v) $DI(H)$ has the persistence property.

(vi) Both $DI(G)$ and $DI(H)$ have the symbolic strong persistence property.

Proof. Suppose that the cone $H = C(G)$ is obtained by adding the new vertex w to G and joining every vertex of G to w . Using [122, Lemma 2.2] yields that

$$DI(H) = DI(G) + (x_w).$$

It follows now from [50, Lemma 3.4] that, for all s ,

$$(6.4.1) \quad \text{Ass}(DI(H)^s) = \{(\mathfrak{p}, x_w) : \mathfrak{p} \in \text{Ass}(DI(G)^s)\}.$$

(i) Let $DI(G)$ be normally torsion-free. Then the claim can be deduced from Theorem 5.3.4. Conversely, let $DI(H)$ be normally torsion-free. By using Theorem 5.3.16, we obtain $DI(G)$ is normally torsion-free.

(ii) Based on (6.4.1), and by considering the fact that

$$\text{Min}(DI(H)) = \{(\mathfrak{p}, x_w) : \mathfrak{p} \in \text{Min}(DI(G))\},$$

one can easily show this claim.

(iii) One concludes this assertion by [2, Theorem 3.12].

(iv) By Definition 2.3.1, $DI(H)$ is a unisplit monomial ideal. Hence, Theorems 2.3.10 and 2.3.17 imply that $DI(H)$ has the strong persistence property.

(v) Proposition 3.1.12 together with (iv) yield that $DI(H)$ has the persistence property.

(vi) This assertion follows promptly from Theorem 4.1.11. $\square$

Our next goal is to show that the dominating ideals of star graphs are normally torsion-free. To do this, we first prove some results of general nature. The following result is a generalized form of Theorem 6.2.2.

Theorem 6.4.8. *Assume that $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ and $\mathcal{H} = (V(\mathcal{H}), E(\mathcal{H}))$ are two finite simple hypergraphs such that $V(\mathcal{H}) = V(\mathcal{G}) \cup \{w_1, \dots, w_t\}$ with $w_i \notin V(\mathcal{G})$ for each $i = 1, \dots, t$, and $E(\mathcal{H}) = E(\mathcal{G}) \cup \{\{v, w_1, \dots, w_t\}\}$ for some vertex $v \in V(\mathcal{G})$. Then*

$$\text{Ass}_{R'}(R'/J(\mathcal{H})^s) = \text{Ass}_R(R/J(\mathcal{G})^s) \cup \{(x_v, x_{w_1}, \dots, x_{w_t})\},$$

for all s , where $R = K[x_\alpha : \alpha \in V(\mathcal{G})]$ and $R' = K[x_\alpha : \alpha \in V(\mathcal{H})]$.

Proof. For convenience of notation, set $I := J(\mathcal{G})$ and $J := J(\mathcal{H})$. We first prove that $\text{Ass}_R(R/I^s) \cup \{(x_v, x_{w_1}, \dots, x_{w_t})\} \subseteq \text{Ass}_{R'}(R'/J^s)$ for all s . Fix $s \geq 1$, and assume that $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ is an arbitrary element of $\text{Ass}_R(R/I^s)$. According to

[45, Lemma 2.11], we get $\mathfrak{p} \in \text{Ass}(K[\mathfrak{p}]/J(\mathcal{G}_{\mathfrak{p}})^s)$, where $K[\mathfrak{p}] = K[x_{i_1}, \dots, x_{i_r}]$ and $\mathcal{G}_{\mathfrak{p}}$ is the induced subhypergraph of $\mathcal{G}$ on the vertex set $\{i_1, \dots, i_r\} \subseteq V(\mathcal{G})$. Since $\mathcal{G}_{\mathfrak{p}} = \mathcal{H}_{\mathfrak{p}}$, we have $\mathfrak{p} \in \text{Ass}(K[\mathfrak{p}]/J(\mathcal{H}_{\mathfrak{p}})^s)$. This yields that $\mathfrak{p} \in \text{Ass}_{R'}(R'/J^s)$. On account of $(x_v, x_{w_1}, \dots, x_{w_t}) \in \text{Ass}_{R'}(R'/J^s)$, one derives

$$\text{Ass}_R(R/I^s) \cup \{(x_v, x_{w_1}, \dots, x_{w_t})\} \subseteq \text{Ass}_{R'}(R'/J^s).$$

To complete the proof, it is enough for us to show the reverse inclusion. Assume that $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ is an arbitrary element of $\text{Ass}_{R'}(R'/J^s)$ with $\{i_1, \dots, i_r\} \subseteq V(\mathcal{H})$. If $\{i_1, \dots, i_r\} \subseteq V(\mathcal{G})$, then [45, Lemma 2.11] implies that $\mathfrak{p} \in \text{Ass}_R(R/I^s)$, and the proof is done. Thus, let $\{w_1, \dots, w_t\} \cap \{i_1, \dots, i_r\} \neq \emptyset$. It follows from [45, Corollary 4.5] that the associated primes of $J(\mathcal{H})^s$ will correspond to critical chromatic subhypergraphs of size $s+1$ in the s -th expansion of $\mathcal{H}$. This means that one can take the induced subhypergraph on the vertex set $\{i_1, \dots, i_r\}$, and then form the s -th expansion on this induced subhypergraph, and within this new hypergraph find a critical $(s+1)$ -chromatic hypergraph. Notice that since this expansion cannot have any critical chromatic subgraphs, this implies that $\mathcal{H}_{\mathfrak{p}}$ must be connected. Without loss of generality, one may assume that $i_1 = v$ and $i_2 = w_1, i_3 = w_2, \dots, i_{t+1} = w_t$. Thanks to $w_1, \dots, w_t$ are connected to v in the hypergraph $\mathcal{H}$, and because this induced subhypergraph is critical, if we remove any vertex w_k for some $1 \leq k \leq t$, one can color the resulting hypergraph with at least s colors. This leads to that w_k has to be adjacent to at least s vertices. But the only things w_k is adjacent to are the shadows of w_i for each $i = 1, \dots, t$, and the shadows of v , and so one has a clique among these vertices. Accordingly, w_k and its neighbors will form a clique of size $s+1$. Since a clique is a critical graph, it follows that we do not need any element of $\{i_{t+2}, \dots, i_r\}$ or their shadows when making the critical $(s+1)$ -chromatic hypergraph. Hence, we obtain $\mathfrak{p} = (x_v, x_{w_1}, \dots, x_{w_t})$. This finishes the proof. $\square$

Lemma 6.4.9. *Let I be a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then the ideal*

$$L := IS \cap (x_n, x_{n+1}, x_{n+2}, \dots, x_m) \subset S = R[x_{n+1}, x_{n+2}, \dots, x_m],$$

is normally torsion-free.

Proof. It is well-known that one can view the square-free monomial ideal I as the cover ideal of a simple hypergraph $\mathcal{H}$ such that the hypergraph $\mathcal{H}$ corresponds to $I^\vee$, where $I^\vee$ denotes the Alexander dual of I . Then we have $I = J(\mathcal{H})$, where $J(\mathcal{H})$ denotes the cover ideal of the hypergraph $\mathcal{H}$. Fix $k \geq 1$. On account of Theorem 6.4.8, we get the following equality

$$\text{Ass}_S(S/L^k) = \text{Ass}_R(R/J(\mathcal{H})^k) \cup \{(x_n, x_{n+1}, x_{n+2}, \dots, x_m)\}.$$

Because I is normally torsion-free, one derives that $\text{Ass}_R(R/J(\mathcal{H})^k) = \text{Min}(J(\mathcal{H}))$, and hence $\text{Ass}_S(S/L^k) = \text{Min}(J(\mathcal{H})) \cup \{(x_n, x_{n+1}, x_{n+2}, \dots, x_m)\}$. This gives rise to $\text{Ass}_S(S/L^k) = \text{Min}(L)$. Therefore, L is normally torsion-free, as claimed. $\square$

Lemma 6.4.10. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and $E(H) = E(G) \cup \{v, w\}$ for some vertex $v \in V(G)$ and $\prod_{j \in N_G[v]} x_j \notin \mathcal{G}(NI(G))$. If $DI(G)$ is normally torsion-free, then $DI(H)$ is normally torsion-free.*

Proof. Let $DI(G)$ be normally torsion-free. It follows from [122, Lemma 2.2] that $DI(H) = DI(G) \cap (x_v, x_w)R$, where $R = K[x_\alpha : \alpha \in V(H)]$. Now, we can conclude the assertion from Lemma 6.4.9. $\square$

We are in a position to give the second main result of this section in the following corollary, which is related to dominating ideals of star graphs.

Corollary 6.4.11. *The dominating ideals of star graphs are normally torsion-free.*

Proof. We use the induction on the number of vertices together with Lemma 6.4.10. $\square$

6.5 Closed neighborhood ideals and dominating ideals of cycles

As we know, the edge ideals and the cover ideals of bipartite graphs are known to be normally torsion-free, see [48, 126]. In particular, the edge ideals and the cover ideals of even cycles are normally torsion-free. However, this behavior changes when we consider the odd cycles. The cover ideals of odd cycles happen to be nearly normally torsion-free, see [104], but edge ideals of odd cycles do not admit such tamed behavior for the set of their associated primes. Given these facts, it is natural to expect some irregularities for the closed neighborhood ideals and dominating ideals of even and odd cycles. It can be verified by using *Macaulay2* [49] that in general, the closed neighborhood ideals of cycles, regardless of the parity of their lengths, are neither normally torsion-free nor nearly normally torsion-free. However, in this section, we will show that the closed neighborhood ideals of cycles admit the strong persistence property. On the other side, as another main result of this section, we will show that the dominating ideals of cycles are nearly normally torsion-free. It should be noted that the results of this section can be found in [107].

To establish above-mentioned results, we begin by proving the following theorem which gives an inductive way to study the normality of a square-free monomial ideal.

Theorem 6.5.1. *Let I and H be two normal square-free monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ such that $I + H$ is normal. Let $x_c \in \{x_1, \dots, x_n\}$ be a variable with $\gcd(v, x_c) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$. Then $L := I + x_c H$ is normal.*

Proof. Let $\mathcal{G}(I) = \{u_1, \dots, u_s\}$ and $\mathcal{G}(H) = \{h_1, \dots, h_r\}$. Since $\gcd(v, x_c) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$, without loss of generality, one may assume that $x_c = x_1 \in K[x_1]$ and

$$\mathcal{G}(I) \cup \mathcal{G}(H) = \{u_1, \dots, u_s, h_1, \dots, h_r\} \subseteq K[x_2, \dots, x_n].$$

We must show that $\overline{L^t} = L^t$ for all integers $t \geq 1$. For this purpose, it is enough to prove that $\overline{L^t} \subseteq L^t$. Let α be a monomial in $\overline{L^t}$ and write $\alpha = x_1^b \delta$ with $x_1 \nmid \delta$ and $\delta \in R$. On account of [60, Theorem 1.4.2], $\alpha^k \in L^{tk}$ for some integer $k \geq 1$. Write

$$(6.5.1) \quad \alpha^k = x_1^{bk} \delta^k = \prod_{i=1}^s u_i^{p_i} x_1^{q+\varepsilon} \prod_{j=1}^r h_j^{q_j} \beta,$$

with $\sum_{i=1}^s p_i = p$, $\sum_{j=1}^r q_j = q$, $p + q = tk$, $\varepsilon \geq 0$, and β is some monomial in R such that $x_1 \nmid \beta$. Because $x_1 \nmid \beta$, $x_1 \nmid \delta$, and $\gcd(v, x_1) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$, one can conclude that $bk = q + \varepsilon$. Accordingly, by virtue of (6.5.1), we obtain

$$\delta^k = \prod_{i=1}^s u_i^{p_i} \prod_{j=1}^r h_j^{q_j} \beta \in (I + H)^{tk}.$$

This leads to $\delta \in \overline{(I + H)^t}$. Thanks to $I + H$ is normal, we can conclude that $\overline{(I + H)^t} = (I + H)^t$, and so $\delta \in (I + H)^t$. Therefore, one can write

$$(6.5.2) \quad \delta = \prod_{i=1}^s u_i^{l_i} \prod_{j=1}^r h_j^{z_j} \gamma,$$

with $\sum_{i=1}^s l_i = l$, $\sum_{j=1}^r z_j = z$, $l + z = t$, and γ is some monomial in R . Note that $x_1 \nmid \gamma$ as $x_1 \nmid \delta$. Due to $x_1^{bk} \delta^k \in L^{tk}$, it follows immediately from (6.5.2) that

$$\prod_{i=1}^s u_i^{l_i k} x_1^{bk} \prod_{j=1}^r h_j^{z_j k} \gamma^k \in L^{tk} = (I + x_1 H)^{tk}.$$

Consequently, we conclude that $bk \geq zk$, that is, $b \geq z$. This gives rise to

$$x_1^b \delta = \prod_{i=1}^s u_i^{l_i} x_1^b \prod_{j=1}^r h_j^{z_j} \gamma \in (I + x_1 H)^t.$$

We thus get $\alpha \in L^t$, and the proof is over. $\square$

We here state the third main result in the next theorem, which is related to the closed neighborhood ideals of cycles.

Theorem 6.5.2. *Let C_n be a cycle graph of order n . Then the following statements hold:*

(i) $NI(C_n)$ is normal.

(ii) $NI(C_n)$ has the strong persistence property.

(iii) $NI(C_n)$ has the persistence property.

Proof. (i) Let $C_n = (V(C_n), E(C_n))$ be a cycle graph of order n with $V(C_n) = \{x_1, \dots, x_n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\} \cup \{\{x_n, x_1\}\}$. Then the closed neighborhood ideal of C_n is given by

$$NI(C_n) = (x_i x_{i+1} x_{i+2} : i = 1, \dots, n) \subset R = K[x_1, \dots, x_n],$$

where x_{n+1} (respectively, x_{n+2}) represents x_1 (respectively, x_2). If $n = 3$, then $NI(C_3) = (x_1 x_2 x_3)$, and so there is nothing to prove. Thus, let $n \geq 4$. Put $H := (x_2 x_3, x_{n-1} x_n, x_2 x_n)$ and $I := (x_i x_{i+1} x_{i+2} : i = 2, \dots, n-2)$. One can easily see that $NI(C_n) = I + x_1 H$. Our strategy is to use Theorem 6.5.1 to complete the proof. To do this, we first show that I , H , and $I + H$ are normal. Assume that G is a path graph with $V(G) = \{x_2, x_3, x_{n-1}, x_n\}$ and $E(G) = \{\{x_2, x_3\}, \{x_{n-1}, x_n\}, \{x_2, x_n\}\}$. It is routine to check that $I(G) = H$, where $I(G)$ denotes the edge ideal of G . Since, by [48, Corollary 2.6], the edge ideal of any path graph is normally torsion-free, and by remembering this fact that every normally torsion-free square-free monomial ideal is normal, we deduce that H is a normal square-free monomial ideal. Now, assume that P is a path graph with $V(P) = \{x_2, x_3, \dots, x_{n-1}, x_n\}$ and $E(P) = \{\{x_i, x_{i+1}\} : i = 2, \dots, n-1\}$. It is not hard to check that $I = I_3(P)$, where $I_3(P)$ denotes the path ideal of length 2 of P . It follows readily from Corollary 6.4.2 that $I = I_3(P)$ is normally torsion-free, and so is normal. To complete the proof, we show that $I + H$ is normal. To accomplish this, we note that

$$I + H = (x_2 x_3, x_{n-1} x_n, x_2 x_n, x_i x_{i+1} x_{i+2} : i = 3, \dots, n-3).$$

Set $A := (x_3, x_n)$ and $B := (x_{n-1} x_n, x_i x_{i+1} x_{i+2} : i = 3, \dots, n-3)$. Notice that $I + H = B + x_2 A$. It is clear that A is a normal ideal. Furthermore, it follows from Corollary 6.4.2 and Theorem 6.4.1 that B is normally torsion-free, and so is normal. In addition, we have

$$B + A = (x_3, x_n, x_i x_{i+1} x_{i+2} : i = 4, \dots, n-3).$$

One can easily conclude from Corollary 6.4.2 and Theorem 6.4.1 that $B + A$ is normally torsion-free, and hence is normal. By virtue of Theorem 6.5.1, we deduce that $B + x_2 A$ is normal, and so $I + H$ is normal as well. Finally, note that $\gcd(v, x_1) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$. This finishes the proof.

The claims (ii) and (iii) can be shown similar to the proof of Corollary 6.2.3.

□

The neighborhood ideals of cycles are particularly nice because they are generated by monomial of the same degree. This fact together with Theorem 6.5.2 enables us to study the depth of powers of $NI(C_n)$. For this purpose, we need to recall the definition of linear relation graph, see Definition 6.3.10, and also the following result from [65].

Theorem 6.5.3. ([65, Theorem 3.3]) *Let $I \subset R = K[x_1, \dots, x_n]$ be a monomial ideal generated in a single degree whose linear relation graph has r vertices and s connected components. Then*

$$\text{depth}(R/I^t) \leq n - t - 1 \text{ for } t = 1, \dots, r - s.$$

In order to employ the above theorem, we first analyze the linear relation graph of $NI(C_n)$. Let $V(C_n) = [n]$ and $E(C_n) = \{\{1, 2\}, \{2, 3\}, \dots, \{n-1, n\}, \{n, 1\}\}$. We set the following notations.

1. $u_i = \prod_{j \in N[i]} x_j$. In simple words, u_i is the monomial that corresponds to the closed neighborhood of the vertex i for all $i = 1, \dots, n$
2. Note that $u_i = x_{i-1}x_i x_{i+1}$, where $i = 2, \dots, n-1$ and $u_1 = x_n x_1 x_2$, $u_n = x_{n-1} x_n x_1$. To synchronize this notation for all i with $i > n$, we read i as $i \pmod{n}$. In this way, we can write $u_i = x_{i-1}x_i x_{i+1}$ for all $i = 1, \dots, n$.

Remark 6.5.4. *Let $i \neq j$. Note that each variable x_i appears in exactly three monomials in $\mathcal{G}(NI(C_n))$, and these monomials are $u_{i-1} = x_{i-2}x_{i-1}x_i$, $u_i = x_{i-1}x_i x_{i+1}$ and $u_{i+1} = x_i x_{i+1} x_{i+2}$. From this observation, we conclude that $\{x_i, x_j\} \in E(\Gamma)$ if and only if there exists a path of length three from i to j in C_n . Note that here a path P of length n is defined on $n+1$ vertices and n edges.*

Remark 6.5.5. *Let $n \geq 4$, and set $I_n := NI(C_n)$. Remark 6.5.4 leads us to the following:*

1. $|V(\Gamma_{I_n})| = n$. This can be easily verified because for every i , we can find another vertex j such that there is a path of length three from i to j in C_n .
2. Γ_{I_n} has one connected component if $n \neq 3k$, for all $k \geq 2$. Indeed, if $n = 1 \pmod{3}$, that is, $n = 3k + 1$ for some $k \geq 1$, then we have

$$\begin{aligned} E(\Gamma_{I_n}) = & \{\{x_1, x_4\}, \{x_4, x_7\}, \dots, \{x_{3k-2}, x_{3k+1}\}, \\ & \{x_{3k+1}, x_3\}, \{x_3, x_6\}, \dots, \{x_{3k-3}, x_{3k}\}, \\ & \{x_{3k}, x_2\}, \{x_2, x_5\}, \dots, \{x_{3k-2}, x_1\}\}. \end{aligned}$$

If $n = 2(\bmod 3)$, that is, $n = 3k + 2$ for some $k \geq 1$, then we have

$$\begin{aligned} E(\Gamma_{I_n}) = & \{\{x_1, x_4\}, \{x_4, x_7\}, \dots, \{x_{3k-2}, x_{3k+1}\}, \\ & \{x_{3k+1}, x_2\}, \{x_2, x_5\}, \dots, \{x_{3k-1}, x_{3k+2}\}, \\ & \{x_{3k+2}, x_3\}, \{x_3, x_6\}, \dots, \{x_{3k}, x_1\}\}. \end{aligned}$$

3. Γ_{I_n} has three connected components if $n = 3k$, for some $k \geq 2$. In fact, set $V(\Gamma_1) = \{x_1, x_4, \dots, x_{3k-2}\}$, and

$$E(\Gamma_1) = \{\{x_1, x_4\}, \{x_4, x_7\}, \dots, \{x_{3k-2}, x_1\}\}.$$

Set $V(\Gamma_2) = \{x_2, x_5, \dots, x_{3k-1}\}$, and

$$E(\Gamma_2) = \{\{x_2, x_5\}, \{x_5, x_8\}, \dots, \{x_{3k-1}, x_2\}\}.$$

Set $V(\Gamma_3) = \{x_3, x_6, \dots, x_{3k}\}$, and

$$E(\Gamma_3) = \{\{x_3, x_6\}, \{x_6, x_9\}, \dots, \{x_{3k}, x_3\}\}.$$

It can be easily verified that Γ_{I_n} is the disjoint union of Γ_1 , Γ_2 , and Γ_3 .

Now, Theorem 6.5.3 together with Remark 6.5.5 leads us to the following corollary.

Corollary 6.5.6. *Let $n \neq 0(\bmod 3)$. Set $I_n := NI(C_n) \subset R = K[x_1, \dots, x_n]$. Then $\text{depth}(R/I_n^{n-1}) = 0$. In particular, $\mathfrak{m} \in \text{Ass}(R/I_n^{n-1})$ and $\lim_{k \rightarrow \infty} \text{depth} R/I_n^k = 0$.*

We close this section by providing the fourth main result in the subsequent theorem, which is related to the dominating ideals of cycles.

Theorem 6.5.7. *The dominating ideals of cycles are nearly normally torsion-free.*

Proof. Let C_n denote a cycle graph of order n with $V(C_n) = \{x_1, \dots, x_n\}$ and $E(C_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n-1\} \cup \{\{x_n, x_1\}\}$. In the light of [122, Lemma 2.2], the dominating ideal of C_n is given by

$$DI(C_n) = \bigcap_{i=1}^n (x_i, x_{i+1}, x_{i+2}) \subset R = [x_1, \dots, x_n],$$

where x_{n+1} (respectively, x_{n+2}) represents x_1 (respectively, x_2). Set $I := DI(C_n)$. Our strategy is to use Corollary 6.1.3. To do this, we must show that $I(\mathfrak{m} \setminus \{x_i\})$ is normally torsion-free for all $i = 1, \dots, n$, where $\mathfrak{m} = (x_1, \dots, x_n)$. Without loss of generality, it is sufficient for us to prove that $I(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. To simplify our notation, set $F := \bigcap_{i=2}^{n-2} (x_i, x_{i+1}, x_{i+2})$. By virtue of Corollary

6.1.3, one has to show that the ideal $F = I(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. To do this, let $T = (V(T), E(T))$ be the rooted tree with the root 2, the vertex set $V(T) = \{x_2, \dots, x_n\}$, and the edge set $E(T) = \{(x_i, x_{i+1}) : i = 2, \dots, n-1\}$, where (x_i, x_{i+1}) denotes the directed edge from the vertex x_i to the vertex x_{i+1} for all $i = 2, \dots, n-1$. It is not hard to check that F is the Alexander dual of the path ideal generated by all paths of length 2 in the rooted tree T . Now, one can deduce from Theorem 5.1.6 that $F = I(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. This finishes the proof. $\square$

Chapter 7

On the normality of monomial ideals

Let I be an ideal in a commutative Noetherian ring R . An element $f \in R$ is *integral* over I if there exists an equation

$$f^k + c_1 f^{k-1} + \cdots + c_{k-1} f + c_k = 0 \quad \text{with } c_i \in I^i.$$

The set of elements $\bar{I}$ in R which are integral over I is the *integral closure* of I . The ideal I is *integrally closed*, if $I = \bar{I}$, and I is *normal* if all powers of I are integrally closed. If I is a monomial ideal, then the notion integrality becomes simpler, namely, an element $u \in R$ is integral over I if and only if there exists an integer k such that $u^k \in I^k$, see Section 1.1. The normality of square-free monomial ideals has been of interest for many authors, for instance see [39, 117, 126, 135]. Such normality has connections to combinatorial commutative algebra as in [35, 88, 134]. One aspect of these connections is that the square-free monomial ideals come out as the edge and cover ideals of graphs. For example, it is shown in [126] that the edge ideals of bipartite graphs are normal. Also, it has been shown in [133, Corollary 14.6.25] that the cover ideals of perfect graphs are normal. However, little is known for the cover ideals of imperfect graphs.

In this chapter, we first argue on the normality of monomial ideals under some monomial operations. After that, we employ these consequences in cover ideals of some imperfect graphs. Next, we turn our attention to exploring the normality and associated primes of powers of the cover ideals of theta graphs and Jahangir's graphs. Moreover, we handle the normality of closed neighborhood and dominating ideals of complete bipartite graphs. We finally deal with the normality of dominating ideals of h -wheel graphs.

7.1 Normality and Cover Ideals

In this section, we are interested in proving the normality of cover ideals of graphs. In Subsection 7.1.1, we develop a technique for proving such normality. In Subsection 7.1.2, we prove that the cover ideals of the cycle graphs of odd orders are normal, and thus, using the results of Subsection 7.1.1, we prove that the cover ideal of the wheel graph is normal as well. Motivated by the method of the proof of Theorem 7.1.3, we tailor this section with Subsection 7.1.3 that contains some results on the integral closedness. It should be noted that the results of this section can be found in [4].

Given a monomial ideal I in the polynomial ring $R = K[x_1, \dots, x_n]$, where K is a field. Let $\Gamma(I)$ be the set of exponents of all monomials in I . It is well-known that finding the integral closure of the monomial ideal I is the same as finding all the integer lattice points in the convex hull in $\mathbb{R}^n$ of $\Gamma(I)$, see [129, Proposition 1.4.6] or [60, Corollary 1.4.3]. According to the discussion before Proposition 1.4.2 of [129], the problem of finding an equation of integral dependence of a monomial m over a monomial ideal I is reduced to finding an integer k and monomials $u_1, \dots, u_k$ in I such that $m^k = u_1 \cdots u_k$. Thus, $m \in \bar{I}$ if and only if $m^k \in I^k$ for some k .

The following remark is crucial for us to provide our results and can be directly proved by using [60, Theorem 1.4.2].

Remark 7.1.1. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ and let $h \in R$ be a monomial. Then $h\bar{I} = \overline{hI}$, and also that I is normal if and only if hI is normal.*

Studying the behavior of the integral closedness of powers of monomial ideals has been of interest for many researchers. Before we proceed to our results of this subsection, we would like to demonstrate some of these behaviors through certain monomial ideals. We do this in the following remark.

Remark 7.1.2. (i) *We give an ideal I which is not integrally closed but I^k is integrally closed for every $k \geq 2$. Let $I = (x^6, x^4y^2, x^3y^3, x^2y^4, y^6) \subset K[x, y]$. Then $\bar{I} = (x^{6-i}y^i)_{i=0}^6$, and thus I is not integrally closed. However, simple computations show that $I^2 = (x^{12-i}y^i)_{i=0}^{12}$ and $I^3 = JI^2$, where $J = (x^6, y^6)$. Hence, J is a reduction of I with reduction number 2. That is, $I^{k+1} = JI^k$ for $k \geq 2$. Thus, I^k is integrally closed for every $k \geq 2$.*

(ii) *We give an ideal I which is integrally closed but I^k is not integrally closed for every $k \geq 2$. In an unpublished work of Jockusch and Swanson (see Exercise 1.14 of [129]) it is pointed out that if $I = \overline{(x^2, y^3, z^7)} \subset K[x, y, z]$, then I^2 is not integrally closed. Here, we show that $\alpha := xy^2z^6z^{7(k-2)} \in \bar{I^k} \setminus I^k$ for any positive integer $k \geq 2$; hence, I^k is not integrally closed for every $k \geq 2$. To show $\alpha \in \bar{I^k}$ it*

suffices to show that $(1, 2, 6 + 7(k - 2))$ is in the convex hull of the set of exponents of the monomials x^2, y^3 , and z^7 . But this is equivalent to finding nonnegative rational numbers λ_1, λ_2 , and λ_3 with $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and with $2k\lambda_1 \leq 1$, $3k\lambda_2 \leq 2$, and $7k\lambda_3 \leq 6 + 7(k - 2)$. It is routine to see that this is achieved if we choose $\lambda_1 = 1/2k$, $\lambda_2 = 1/3k$, and $\lambda_3 = 1 - (\lambda_1 + \lambda_2)$. Now, we show that $\alpha \notin I^k$ for any $k \geq 2$. Using Normaliz [24] we find that the set of generators of I is given by $\mathcal{G}(I) = \{x^2, y^3, z^7\} \cup \{y^2z^3, yz^5, xy^2, xyz^2, xz^4\}$. If $\alpha \in I^k$ for some $k \geq 2$, then $\alpha = z^{7(k-(i_1+\dots+i_5))} (y^2z^3)^{i_1} (yz^5)^{i_2} (xy^2)^{i_3} (xyz^2)^{i_4} (xz^4)^{i_5}$ with i_j nonnegative. Now, the x -deg(α) forces that $i_3 + i_4 + i_5 = 1$. If $i_3 = 1$, then $i_4 = i_5 = 0$, also $i_1 = i_2 = 0$ by virtue of the y -deg(α); hence, z -deg(α) = $7(k - 1) > 6 + 7(k - 2)$, a contradiction. The cases $i_4 = 1$ and $i_5 = 1$ are manipulated similarly.

(iii) We give an ideal L such that L^2 is not integrally closed but L^3 is integrally closed. This ideal is inspired to us while working with cover ideals of graphs.

Given the ideals $I = (x_1x_4, x_1x_5, x_2x_3, x_2x_5, x_3x_4)$ and

$$J = (x_1x_2x_4, x_1x_3x_5, x_1x_2x_3, x_2x_4x_5, x_3x_4x_5).$$

Let $L = x_6I + x_7J$. Then L is integrally closed since it is square-free (see Theorem 7.1.9). Note that $\alpha := x_1x_2x_3x_4x_5x_6x_7 \notin L^2$ and

$$\alpha^2 = (x_2x_5x_6)(x_3x_4x_6)(x_1x_2x_4x_7)(x_1x_3x_5x_7) \in L^4,$$

hence $\alpha \in \overline{L^2}$ by virtue of [60, Theorem 1.4.2]; therefore, L^2 is not integrally closed. On the other hand, computations with Normaliz [24] verify that L^3 is integrally closed. In fact, our computations also show that L^k is integrally closed for $k = 4, 5, 6, 7$. The authors wonder whether L^k is integrally closed for all $k \geq 3$.

7.1.1 Constructing Normal Ideals with Application to Cover Ideals

In the following theorem, we introduce a procedure for obtaining normal monomial ideals from other ideals that are assumed to be normal. This theorem enables us to prove the normality of the cover ideals of graphs upon the normality of cover ideals of other graphs. This is illustrated in Theorem 7.1.5 which in turn is used to prove that the cover ideals of complete graphs are normal (see Corollary 7.1.6), and also to prove that the cover ideals of the wheel graph is normal (see Theorem 7.1.8).

Theorem 7.1.3. *Let I be a monomial ideal in the polynomial ring $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) = \{u_1, \dots, u_r\}$, v a square-free monomial in R such that $\gcd(v, u_i) = 1$ for all i , and $h \in I$ a monomial. If I is normal, then $vI + hR$ is normal.*

Proof. We first note that one may assume $\gcd(v, h) = 1$. To see this, let $d := \gcd(v, h)$. This yields that $v = dv_1$ and $h = dh_1$ for some monomials v_1 and h_1 . Consequently, one derives $vI + hR = d(v_1I + h_1R)$ with $\gcd(v_1, h_1) = 1$. Also, it follows from $h \in I$ that there exists an element $u \in \mathcal{G}(I)$ such that $u|h$, and so $u|h_1d$. By virtue of $\gcd(u, v) = 1$, one has $\gcd(u, d) = 1$, and hence $u|h_1$, that is, $h_1 \in I$. Furthermore, since $\gcd(v, u) = 1$ for every $u \in \mathcal{G}(I)$, one can conclude that $\gcd(v_1, u) = 1$ for every $u \in \mathcal{G}(I)$. Accordingly, if we show that $v_1I + h_1R$ is normal, then Remark 7.1.1 implies that $vI + hR$ is also normal. Thus, one may assume that $\gcd(v, h) = 1$. Write $v = x_1 \cdots x_m \in K[x_1, \dots, x_m]$ and also let $\mathcal{G}(I) \subset K[x_{m+1}, \dots, x_n]$. Assume α is a monomial in $(vI + hR)^t$, thus $\alpha^k \in (vI + hR)^{tk}$ for some k . Hence, $\alpha^k \in (vI)^p (hR)^q$ for some p and q with $p + q = tk$. Choose q to be the minimal according to this membership. If $q = 0$, then $\alpha^k \in (vI)^{tk}$; thus, $\alpha \in \overline{(vI)^t}$. Since I is normal, then vI is also normal by Remark 7.1.1. Therefore, $\alpha \in (vI)^t \subseteq (vI + hR)^t$, and the proof is done. Therefore, assume $q > 0$. As $\mathcal{G}(I) = \{u_1, \dots, u_r\}$ and $\alpha^k \in (vI)^p (hR)^q$, we thus get

$$(7.1.1) \quad \alpha^k = v^p u_1^{p_1} \cdots u_r^{p_r} h^q \beta,$$

with $\sum_{i=1}^r p_i = p$ and β is some monomial in $K[x_1, \dots, x_n]$. As $h \in I$ and I is a monomial ideal, then by [60, Proposition 1.1.5], h is divisible by some element of $\mathcal{G}(I)$, say u_r . Thus, we obtain $h = u_r h'$ for some monomial h' . Now, if v divides β , say $\beta = v\beta'$, then (7.1.1) can be rewritten as

$$\alpha^k = v^{p+1} u_1^{p_1} \cdots u_r^{p_r+1} h^{q-1} h' \beta' \in (vI)^{p+1} (hR)^{q-1},$$

contradicting the minimality of q . Therefore, we may assume in (7.1.1) that $v \nmid \beta$, say $x_1 \nmid \beta$. Write $\alpha = x_1^{b_1} \cdots x_m^{b_m} \delta$ with each b_i nonnegative and $\delta \in K[x_{m+1}, \dots, x_n]$. Since v is square-free, $\gcd(v, h) = 1$, and $\gcd(v, u_i) = 1$ for all i , we then obtain $b_1 k = p$, and so $q = (t - b_1)k$. Consequently, $\alpha^k \in (vI)^{b_1 k} (hR)^{(t-b_1)k}$, and hence $\alpha \in \overline{(vI)^{b_1} h^{t-b_1}}$. Due to I is normal, it follows from Remark 7.1.1 that vI is normal as well, that is, $\overline{(vI)^{b_1}} = (vI)^{b_1}$. Again, by Remark 7.1.1, we conclude $\alpha \in (vI)^{b_1} h^{t-b_1} \subseteq (vI + hR)^t$. This finishes the proof. $\square$

Remark 7.1.4. We show that the hypotheses in the above theorem are best possible.

(i) The square-freeness of the monomial v in the above theorem is necessary. For instance, let $I = (x_1, x_2) \subset R = K[x_1, x_2, x_3]$ and $h = x_1^2 x_2^2 \in I$ and $v = x_3^2$. Then $vI + hR = (x_1 x_3^2, x_2 x_3^2, x_1^2 x_2^2)$. Note that $x_1^2 x_2 x_3 \notin vI + hR$ but $(x_1^2 x_2 x_3)^2 \in (vI + hR)^2$, and therefore, the ideal $vI + hR$ is not integrally closed, hence not normal.

(ii) The condition that $\gcd(u, v) = 1$ for every $u \in \mathcal{G}(I)$ is necessary. For instance, let $I = \overline{(x_1^8, x_2^8)} \subset R = K[x_1, x_2, x_3]$ and $h = x_1^4 x_2^4 \in I$. If $v = x_1 x_2 x_3$, then the ideal $L := vI + hR = x_1 x_2 x_3 \overline{(x_1^8, x_2^8)} + (x_1^4 x_2^4)R$ is not integrally closed as $\alpha := x_1^3 x_2^6 x_3 \notin L$, while $\alpha^2 = (x_1^2 x_2^8 x_3^2)(x_1^4 x_2^4) \in L^2$; thus, $\alpha \in \overline{L} \setminus L$.

A direct application of Theorem 7.1.3 to graphs is the following theorem which is needed to conclude Corollary 7.1.6 and also to prove Theorem 7.1.8.

Theorem 7.1.5. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and*

$$E(H) = E(G) \cup \{\{v, w\} : \text{for all } v \in V(G)\}.$$

Let $J(G)$ (respectively, $J(H)$) denote the cover ideal of the graph G (respectively, graph H). If $J(G)$ is normal, then $J(H)$ is normal.

Proof. Suppose that $J(G)$ is normal. Without loss of generality, one may assume that $V(G) = [n]$ and $V(H) = V(G) \cup \{n+1\}$. Also, let $R = K[x_1, \dots, x_{n+1}]$ be a polynomial ring over a field K . Let us simplify the notation by $L := J(H)$, $I := J(G)$, and $g := \prod_{i=1}^n x_i$. In the following, our goal is to apply Theorem 7.1.3. By definition $L = IR \cap \bigcap_{i=1}^n (x_i, x_{n+1})$. It follows from [133, Exercises 6.1.22 and 6.1.23] that $\bigcap_{i=1}^n (x_i, x_{n+1}) = (\prod_{i=1}^n x_i, x_{n+1})$, and so one has $L = IR \cap (\prod_{i=1}^n x_i, x_{n+1})$. By virtue of $g = \prod_{i=1}^n x_i$ and $\mathcal{G}(I) \subseteq K[x_1, \dots, x_n]$, one obtains $g \in I$. We next prove that $L = x_{n+1}IR + gR$. Let $M \in R$ be a monomial. The condition that M will be in L is that for each edge $\{i, j\}$ of H , we have at least one of $x_i|M$, $x_j|M$ (in which case we will say that M covers $\{i, j\}$). First consider the case $x_{n+1} \nmid M$. In order to cover all the edges $\{i, n+1\}, 1 \leq i \leq n$ of H we must have $x_i|M, 1 \leq i \leq n$. Thus, in this case $M \in L \iff g|M$. Now, consider the case $x_{n+1}|M$. Then the edges $\{i, n+1\}, 1 \leq i \leq n$ are all covered, so $M \in L$ if and only if M covers all the edges $\{i, j\} \in E(G)$. This will be the case $\iff M = x_{n+1}M'$ with $M' \in IR$, from which it follows that $L = x_{n+1}IR + gR$. In the following, we show that IR is normal in $R = K[x_1, \dots, x_{n+1}]$. To see this, let F be any monomial ideal in $R' = K[x_1, \dots, x_n]$ with minimal generating set $\mathcal{G}(F) = \{\alpha_1, \dots, \alpha_r\}$. Then by definition FR has minimal generating set $\mathcal{G}(F)$, now regarded as a subset of R by the canonical inclusion $R' \subset R$. Similarly $F^t R = (FR)^t$ because both have generating set all products of t elements of $\mathcal{G}(F)$. We claim that the same sort of thing holds for integral closure, namely that $\overline{FR} = \overline{F}R$ (where each integral closure operation takes place in the ring of the ideal). The inclusion $\overline{F}R \subseteq \overline{FR}$ is obvious. Conversely, suppose that M is a monomial of R in $\overline{FR}$. This means that there is a positive integer d such that $M^d \in (FR)^d$. Write $M = M'x_{n+1}^a$ with a a nonnegative integer and M' a monomial in R' . We now have $M'^d x_{n+1}^{ad} = \prod_{i=1}^r \alpha_i^{a_i} N$, where $\sum_{i=1}^r a_i = d$ and N is a monomial in R . Setting $x_{n+1} = 1$ in this expression we obtain $M'^d \in F^d$, so that $M' \in \overline{F}$ and finally that $M \in \overline{F}R$, proving the reverse inclusion. Since I is normal, applying the previous observations to I^d and noting that $\overline{I^d} = I^d$ for all d we have $(IR)^d = I^d R = \overline{I^d} R = \overline{I^d R} = (\overline{IR})^d$, so $(IR)^d$ is integrally closed for all d , hence IR is normal. Now, assume that $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. This leads to

$$\mathcal{G}(L) = \{x_{n+1}u_1, \dots, x_{n+1}u_m, g\}.$$

In the light of $L = x_{n+1}IR + gR$, $g \in I$, $\gcd(x_{n+1}, u_i) = 1$ for all $i = 1, \dots, m$, and because IR is normal, Theorem 7.1.3 yields that L is normal, as claimed. $\square$

The following corollary, which is an immediate consequence of Theorem 7.1.5, tells us that the cover ideals of complete graphs are normal.

Corollary 7.1.6. *Let K_n be the complete graph on n vertices with $n \geq 2$. Let $J(K_n)$ denote the cover ideal of the graph K_n . Then $J(K_n)$ is a normal square-free monomial ideal.*

Proof. Let $V(K_n) = \{v_1, \dots, v_n\}$. We claim that one can inductively construct the graph K_n . Since $E(K_n) = \{\{v_i, v_j\} : \text{for all } 1 \leq i \neq j \leq n\}$, we get $\{v_1, v_2\} \in E(K_n)$. Consider a graph H_1 with $V(H_1) = \{v_1, v_2\}$ and $E(H_1) = \{\{v_1, v_2\}\}$. Now, for each $2 \leq k \leq n-1$, define a graph H_k with vertices $V(H_k) = \{v_1, \dots, v_{k+1}\}$ and with edges $E(H_k) = E(H_{k-1}) \cup \{\{v_{k+1}, w\} : \text{for all } w \in V(H_{k-1})\}$. It is routine to check that $H_{n-1} = K_n$. Because $J(H_1) = (x_{v_1}, x_{v_2})$ as a monomial prime ideal is normal, it follows from the iteration of Theorem 7.1.5 that, for each $i = 2, \dots, n-1$, $J(H_i)$ is normal as well. In particular, we deduce that $J(K_n)$ is normal, and the proof is complete. $\square$

7.1.2 Normality of Cover Ideals of Some Imperfect Graphs

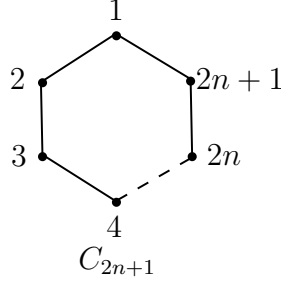
We now explore the normality of the cover ideals of some imperfect graphs. It is well-known that the cover ideals of perfect graphs are normal, see [133, Corollary 14.6.25]. For example, cycle graphs of even orders are perfect, and so their cover ideals will be normal. However, little is known for the cover ideals of imperfect graphs. In what follows, we consider some classes of imperfect graphs, i.e., cycle graphs of odd orders and wheel graphs of even orders. Our aim is to show that the cover ideals of these graphs are normal.

Let $R = K[x_1, \dots, x_{2n+1}]$ be a polynomial ring over a field K . Let $I = J(C_{2n+1}) \subset R$ be the *cover ideal* associated to the cycle graph C_{2n+1} with the vertex set $[2n+1]$. That is, $I = \bigcap_{i=1}^{2n+1} (x_i, x_{i+1}) = (x_1, x_2) \cap \dots \cap (x_{2n}, x_{2n+1}) \cap (x_{2n+1}, x_1)$, where subscripts are taken mod $2n+1$, for instance x_{2n+1+i} represents x_i .

Now, we are in a position to demonstrate that the cover ideals of cycle graphs of odd orders are normal.

Theorem 7.1.7. *Let $R = K[x_1, \dots, x_{2n+1}]$ be a polynomial ring over a field K , C_{2n+1} be the cycle graph on the vertex set $\{1, \dots, 2n+1\}$, and $I = J(C_{2n+1})$ be the cover ideal associated to C_{2n+1} . Then I is a normal square-free monomial ideal.*

Proof. First proof. Label the vertices of C_{2n+1} in counterclockwise order with $1, \dots, 2n+1$, as shown in figure below.



Fix $t \in \mathbb{N}$, and assume that $m = x_1^{\alpha_1} \cdots x_{2n+1}^{\alpha_{2n+1}} \in \overline{I}^t$ with $\alpha = (\alpha_1, \dots, \alpha_{2n+1}) \in \mathbb{N}_0^{2n+1}$. Let $\{\mu_j\}_{j=1}^r \subset \mathbb{N}_0^{2n+1}$ be the set of exponents of the minimal generators of I^t and let $\mu_{j,i}$ be the i -th component of μ_j . Since $m \in \overline{I}^t$, it follows by [60, Corollary 1.4.3] that α belongs to the Newton polyhedron of I . That is, there exist $\lambda_1, \dots, \lambda_r \in \mathbb{Q}_{\geq 0}$ with $\sum_{j=1}^r \lambda_j = 1$ such that $\alpha_i \geq \sum_{j=1}^r \lambda_j \mu_{j,i}$ for each $i = 1, \dots, 2n+1$. If $i \in [2n+1]$, then (with $2n+2 = 1$)

$$\begin{aligned}
 \alpha_i + \alpha_{i+1} &\geq \sum_{j=1}^r \lambda_j \mu_{j,i} + \sum_{j=1}^r \lambda_j \mu_{j,i+1} \\
 &= \sum_{j=1}^r \lambda_j (\mu_{j,i} + \mu_{j,i+1}) \\
 &\geq t \sum_{j=1}^r \lambda_j \quad (\text{by Lemma 3.1.10}) \\
 (7.1.2) \quad &= t.
 \end{aligned}$$

Furthermore, we have

$$\begin{aligned}
 \sum_{i=1}^{2n+1} (\alpha_i + \alpha_{i+1}) - t(2n+1) &\geq \sum_{i=1}^{2n+1} \left[\sum_{j=1}^r \lambda_j (\mu_{j,i} + \mu_{j,i+1}) \right] - t(2n+1) \\
 &= \sum_{j=1}^r \lambda_j \left[\sum_{i=1}^{2n+1} (\mu_{j,i} + \mu_{j,i+1}) - t(2n+1) \right] \\
 &\geq t \sum_{j=1}^r \lambda_j \quad (\text{by Lemma 3.1.10}) \\
 (7.1.3) \quad &= t.
 \end{aligned}$$

By virtue of (7.1.2) and (7.1.3), along with Lemma 3.1.10, we obtain $m \in I^t$. Hence, $\overline{I}^t = I^t$, and therefore I is normal.

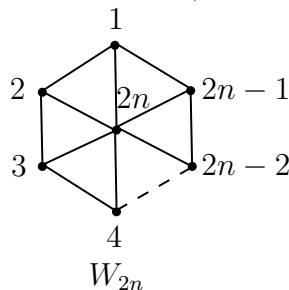
Second proof. Similar to the first proof, label the vertices of C_{2n+1} in counter-clockwise order with $1, \dots, 2n+1$, as shown in figure above. Let X be the convex set in $\mathbb{R}^{2n+1}$ defined by the inequalities $l_i + l_{i+1} \geq d$ and $\sum_{i=1}^{2n+1} [(l_i + l_{i+1} - d)] \geq d$ from Lemma 3.1.10, together with $l_i \geq 0$ for all i . Let $\Gamma(I^d)$ be the exponent set of I^d . Then Lemma 3.1.10 says that $\Gamma(I^d) = \mathbb{N}_0^{2n+1} \cap X$. It is known [60, Corollary 1.4.3] that $\Gamma(\overline{I}^d) = \mathbb{N}_0^{2n+1} \cap NP(I^d)$, where NP denotes the Newton polyhedron of

an ideal. By definition, the Newton polyhedron of an ideal is the smallest convex set that contains the exponent set of the ideal. Therefore, $NP(I^d) \subseteq X$ and we have $\Gamma(\overline{I^d}) = \mathbb{N}_0^{2n+1} \cap NP(I^d) \subset \mathbb{N}_0^{2n+1} \cap X = \Gamma(I^d)$, which implies that $\overline{I^d} = I^d$. We now obtain that I^d is integrally closed for all d . This gives that I is normal, as required. $\square$

We are now going to investigate the normality of the cover ideals of wheel graphs of even orders. To remember the definition of wheel graphs, refer to Definition 3.1.16.

Theorem 7.1.8. *Suppose that W_{2n} is a wheel graph of order $2n$ on the vertex set $[2n]$. Then $J(W_{2n})$ is normal.*

Proof. Label the vertices of the cycle graph C_{2n-1} of order $2n-1$ in counterclockwise order with $1, \dots, 2n-1$, and the hub with $2n$, as shown in the figure below.



Now, the desired conclusion follows readily from Theorems 7.1.7 and 7.1.5, as claimed. $\square$

7.1.3 Some Related Results in Normality

Motivated by the methods of proving the integral closedness in the previous subsections, we finish this section with some results on the integral closedness of monomial ideals.

The result below is well-known, but we re-prove it by a different method from the proof which has been introduced in Corollary 3.2.45.

Theorem 7.1.9. *Every square-free monomial ideal is integrally closed.*

Proof. Let $I \subset R = K[x_1, \dots, x_n]$ be a square-free monomial ideal with $\mathcal{G}(I) = \{u_1, \dots, u_r\}$. Write $u_i = x_1^{a_{i1}} \cdots x_n^{a_{in}}$ with $a_{ij} \in \{0, 1\}$. Let $\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \overline{I}$. According to Proposition 1.4.6 of [129] and the discussion before it, there exist $\lambda_1, \dots, \lambda_r \in \mathbb{Q}_{\geq 0}$ with $\sum_{j=1}^r \lambda_j = 1$ such that $\alpha_k \geq \sum_{j=1}^r \lambda_j a_{jk}$ for all k . As $\sum_{j=1}^r \lambda_j = 1$, this yields that $\lambda_l > 0$ for some $l \in \{1, \dots, r\}$, and thus $\alpha_k \geq \lambda_l a_{lk}$ for all k . Since a_{lk} equals 0 or 1 and since α_k is a nonnegative integer, we derive that $\alpha_k \geq a_{lk}$ for all k . But this gives that α is a multiple of u_l , that is, $\alpha \in I$; hence I is integrally closed. $\square$

In the following lemma we give the best possible hypotheses so that the integral closure commutes under addition.

Lemma 7.1.10. *Let $J \subseteq I \subseteq R = K[x_1, \dots, x_n]$ be monomial ideals. Let v be a square-free monomial in R such that $\gcd(v, f) = 1$ and $\gcd(v, g) = 1$ for every $f \in \mathcal{G}(I)$ and $g \in \mathcal{G}(J)$. Let w be any monomial in R . Then $\overline{vI + wJ} = v\overline{I} + w\overline{J}$.*

Proof. By Remark 7.1.1, we may assume that $\gcd(v, w) = 1$. It is known that the integral closure operation on ideals preserves inclusion. Thus, and by Remark 7.1.1, $v\overline{I} + w\overline{J} \subseteq \overline{vI + wJ}$. It remains to prove $\overline{vI + wJ} \subseteq v\overline{I} + w\overline{J}$. Write $v = x_1 \cdots x_m \in K[x_1, \dots, x_m]$ and let $\mathcal{G}(I) \subseteq K[x_{m+1}, \dots, x_n]$ and $\mathcal{G}(J) \subseteq K[x_{m+1}, \dots, x_n]$. Assume α is a monomial in $\overline{vI + wJ}$, thus $\alpha^k \in (vI + wJ)^k$ for some k . Hence, $\alpha^k \in (vI)^p (wJ)^q$ for some p and q with $p + q = k$. Choose q to be the minimal according to this membership. If $q = 0$, then $\alpha^k \in (vI)^k$; thus, $\alpha \in \overline{vI} = v\overline{I}$. Accordingly, assume $q > 0$. Let $\mathcal{G}(I) = \{f_1, \dots, f_r\}$ and $\mathcal{G}(J) = \{g_1, \dots, g_s\}$. On account of $\alpha^k \in (vI + wJ)^k$, we get the following equality

$$(7.1.4) \quad \alpha^k = \prod_{i=1}^r f_i^{p_i} v^p \prod_{j=1}^s g_j^{q_j} w^q \beta,$$

with $\sum_{i=1}^r p_i = p$, $\sum_{j=1}^s q_j = q$, and β is some monomial in $K[x_1, \dots, x_n]$. By our hypothesis, every $g_j \in \mathcal{G}(J)$ is divisible by some $f_{i_j} \in \mathcal{G}(I)$, say $g_j = f_{i_j} g'_j$. Therefore, if v divides β , say $\beta = v\beta'$, then (7.1.4) can be rewritten as (after some reordering)

$$\alpha^k = \prod_{i=1}^{r-1} f_i^{p_i} f_r^{p_r+1} v^{p+1} \prod_{j=1}^{s-1} g_j^{q_j} g_s^{q_s-1} w^{q-1} g'_s w \beta',$$

and hence $\alpha^k \in (vI)^{p+1} (wJ)^{q-1}$, contradicting the minimality of q . Hence, we may assume in (7.1.4) that $v \nmid \beta$, say $x_1 \nmid \beta$. Write $\alpha = x_1^{b_1} \cdots x_m^{b_m} \delta$ with each b_i nonnegative and $\delta \in K[x_{m+1}, \dots, x_n]$. Now, note that $b_1 k = p$ because v is square-free with $\gcd(v, w) = 1$, $\gcd(v, f_i) = 1$, and $\gcd(v, g_j) = 1$ for all i and j . As $p + q = k$, we obtain $q = (1 - b_1)k$. But $q > 0$, hence $b_1 = 0$; thus, $p = 0$ and $q = k$. Therefore, $\alpha^k \in (wJ)^k$, and thus $\alpha \in \overline{wJ} = w\overline{J}$. $\square$

Remark 7.1.11. *We show that the hypotheses in the above lemma are best possible.*

(i) *The square-freeness of the monomial v is necessary. For instance, let $I = (x_1, x_2)$ and $J = (x_1^2)$ in the polynomial ring $K[x_1, x_2, x_3, x_4]$ and let $v = x_3^2$ and $w = x_4$. Then we have $v\overline{I} + w\overline{J} = vI + wJ = (x_1 x_3^2, x_2 x_3^2, x_1^2 x_4)$. Consider $\alpha := x_1 x_2 x_3 x_4$ and also notice $\alpha^2 \in (vI + wJ)^2$, hence $\alpha \in \overline{vI + wJ} \setminus (v\overline{I} + w\overline{J})$.*

(ii) *The containment $J \subseteq I$ is necessary. For instance, let $I = (x_1^2, x_1 x_2, x_2^2)$ and $J = (x_3^2)$ in the polynomial ring $K[x_1, \dots, x_5]$ and let $v = x_4$ and $w = x_5$. Then we have $v\overline{I} + w\overline{J} = vI + wJ = (x_1^2 x_4, x_1 x_2 x_4, x_2^2 x_4, x_3^2 x_5)$. Consider $\alpha := x_1 x_3 x_4 x_5$ and notice $\alpha^2 \in (vI + wJ)^2$, hence $\alpha \in \overline{vI + wJ} \setminus (v\overline{I} + w\overline{J})$.*

Remark 7.1.12. By Theorem 7.1.9, any square-free monomial ideal is integrally closed. Therefore, if I and J are square-free monomial ideals, and if v and w are square-free monomials with $\gcd(v, f) = 1$ and $\gcd(w, g) = 1$ for every $f \in \mathcal{G}(I)$ and $g \in \mathcal{G}(J)$, then $vI + wJ$ is always integrally closed since it is square-free as well, in particular, $\overline{vI + wJ} = v\overline{I} + w\overline{J}$, and so the containment $J \subseteq I$ and the condition $\gcd(v, g) = 1$ for every $g \in \mathcal{G}(J)$ are not necessary for the conclusion of Lemma 7.1.10 as long as the ideal $vI + wJ$ is square-free. Though, to guarantee the normality of the ideal $vI + wJ$, even though it is square-free and the condition $\gcd(v, g) = 1$ for every $g \in \mathcal{G}(J)$ holds, one cannot give up the containment condition of $J \subseteq I$, as the following example shows.

Example 7.1.13. Let $I = (x_1x_2, x_2x_3, x_3x_4)$ and $J = (x_1x_3, x_2x_3, x_2x_4)$ in the polynomial ring $K[x_1, \dots, x_6]$ and let $v = x_6$ and $w = x_5$. Then we have

$$vI + wJ = (x_1x_2x_6, x_2x_3x_6, x_3x_4x_6, x_1x_3x_5, x_2x_3x_5, x_2x_4x_5).$$

Consider $\alpha := x_1x_2x_3x_4x_5x_6 \notin (vI + wJ)^2$. Direct computations show that

$$\alpha^2 = (x_1x_2x_6)(x_2x_4x_5)(x_1x_3x_5)(x_3x_4x_6) \in (vI + wJ)^4,$$

so $\alpha \in \overline{(vI + wJ)^2}$. Thus, $(vI + wJ)^2$ is not integrally closed; hence, $vI + wJ$ is not normal.

7.2 Normality under some Operations

In this section, we investigate the normality under expansion, weighting, and polarization. In Subsection 7.2.1, we prove some results concerning the normality under expansion and then apply these results to obtain a generalization of the whole work of [1]. In Subsection 7.2.2, we show that if a weighted ideal is normal, then the ideal itself is normal. It should be noted that the results of this section can be found in [4].

7.2.1 Normality under Expansion

First, one should recall the definition of the expansion of a monomial ideal as has been stated in Definition 2.2.9.

The following theorem states that the normality of a monomial ideal is equivalent to the normality of its expansion. As an application of this theorem, we show in Proposition 7.2.2 that the integral closure commutes with expansion.

Theorem 7.2.1. *Let I be a monomial ideal of R . Then I is normal if and only if I^* is normal.*

Proof. First assume that I is a normal monomial ideal of R . Also, define the K -algebra homomorphism $\pi : R^* \rightarrow R$ given by $\pi(x_{ij}) = x_i$ for all i and j . One can easily observe that if u is a monomial in R^* , then $\pi(u) \in I$ if and only if $u \in I^*$. Fix $t \in \mathbb{N}$, and pick an arbitrary monomial u in $\overline{(I^*)^t}$. Hence, [60, Theorem 1.4.2] implies that there exists a positive integer k such that $u^k \in ((I^*)^t)^k$, and so $u^k \in (I^*)^{tk}$. On account of [13, Lemma 1.1(iii)], one has $(I^*)^{tk} = (I^{tk})^*$. Consequently, we can derive $u^k \in (I^{tk})^*$, and thus $\pi(u^k) \in I^{tk}$. Because $\pi(u^k) = (\pi(u))^k$, one gains that $(\pi(u))^k \in (I^t)^k$. This yields that $\pi(u) \in \overline{I^t}$. Thanks to I is normal, one can conclude that $\pi(u) \in I^t$, and so $u \in (I^t)^*$. Once again, [13, Lemma 1.1(iii)] gives that $(I^t)^* = (I^*)^t$. Therefore, $u \in (I^*)^t$; hence, I^* is a normal monomial ideal in R^* .

Conversely, let I^* be normal. Fix $t \in \mathbb{N}$, and consider a monomial $v \in \overline{I^t}$. Then $v^k \in (I^t)^k$ for some $k \in \mathbb{N}$. Let u be a monomial in R^* such that $\pi(u) = v$. Then $(\pi(u))^k \in I^{tk}$, and so $u^k \in (I^{tk})^* = (I^*)^{tk}$. This gives that $u \in \overline{(I^*)^t}$. Since I^* is normal, one has $u \in (I^*)^t = (I^t)^*$. Therefore, $v \in I^t$, and hence I is a normal monomial ideal. $\square$

Proposition 7.2.2. *Let I be a monomial ideal of R . Then $(\bar{I})^* = \overline{(I^*)}$.*

Proof. It follows from $I \subseteq \bar{I}$ that $I^* \subseteq (\bar{I})^*$. Since $\bar{I}$ is integrally closed, the proof of Theorem 7.2.1 yields that $(\bar{I})^*$ is also integrally closed, and hence $\overline{(I^*)} \subseteq (\bar{I})^*$. Conversely, suppose that $u \in (\bar{I})^*$ for some monomial u in R^* . Then $\pi(u) \in \bar{I}$, and so $(\pi(u))^k \in I^k$ for some positive integer k . This implies that $u^k \in (I^k)^* = (I^*)^k$, and thus $u \in \overline{(I^*)}$. Therefore, we get $(\bar{I})^* \subseteq \overline{(I^*)}$, and the proof is done. $\square$

Let R be an arbitrary Noetherian ring and I, J, K, L be ideals of R . By Remark 1.3.2(4) in [129], $\bar{I} \bar{J} \subseteq \overline{IJ}$, hence $\bar{I} \bar{J} \subseteq \overline{IJ + K\bar{L}}$. Similarly, $\bar{K} \bar{L} \subseteq \overline{IJ + K\bar{L}}$. Since the latter is integrally closed, we have $\overline{\bar{I} \bar{J} + \bar{K} \bar{L}} \subseteq \overline{IJ + K\bar{L}}$. The reverse inclusion is obvious, thus we obtain $\overline{\bar{I} \bar{J} + \bar{K} \bar{L}} = \overline{IJ + K\bar{L}}$. This last equality helps us to prove a general relation between expansions and the integral closure as in the following lemma.

Lemma 7.2.3. *Let $L = (u_1, \dots, u_s) \subset K[x_1, \dots, x_n]$ be a monomial ideal. Write $u_j = x_1^{a_j} u'_j$ with $a_j \geq 0$ and $\gcd(x_1, u'_j) = 1$. Let L^* be the expansion of L with respect to the n -tuple $(m, 1, \dots, 1)$, that is,*

$$L^* = \sum_{j=1}^s (x_{11}, \dots, x_{1m})^{a_j} u'_j.$$

Define the ideal N in the ring $K[x_{11}, \dots, x_{1m}, x_2, \dots, x_n]$ as follows

$$N = \sum_{j=1}^s (x_{11}^{a_j}, \dots, x_{1m}^{a_j}) u'_j.$$

Then L^* and N have the same integral closure, that is, $\overline{L^*} = \overline{N}$.

Proof. Note $\overline{(x_{11}^{a_j}, \dots, x_{1m}^{a_j})} = (x_{11}, \dots, x_{1m})^{a_j}$. Remark 7.1.1 and the discussion above give

$$\begin{aligned}\overline{L^*} &= \overline{\sum_{j=1}^s (x_{11}, \dots, x_{1m})^{a_j} u'_j} \\ &= \overline{\sum_{j=1}^s \overline{(x_{11}^{a_j}, \dots, x_{1m}^{a_j})} u'_j} = \overline{\sum_{j=1}^s (x_{11}^{a_j}, \dots, x_{1m}^{a_j}) u'_j} = \overline{N},\end{aligned}$$

as required. $\square$

Let $I = (x_1^{a_1}, \dots, x_n^{a_n}) \subset R = K[x_1, \dots, x_n]$ and let $J := \overline{I}$. The normality of the ideal J has been of interest for many authors, see [1, 2, 33, 116]. A result of Reid, Roberts, and Vitulli [116] relates the normality of both of the ideals J and L , where $L = \overline{(x_1^{a_1}, \dots, x_n^{a_n+l})}$ and $l = \text{lcm}(a_1, \dots, a_{n-1})$. Yet, it is still a challenging problem to give necessary and sufficient conditions, on the integers a_i , for the normality of the ideal J . Al-Ayyoub [1] proved that the integral closure of the ideal $I = (x_1^{a_1}, \dots, x_n^{a_n})$ in R is normal provided that $a_i \in \{s, l\}$, where s and l are positive integers. Generalizing the whole work of [1], it is proved in [2] that the integral closure of $(x_1^{a_1}, \dots, x_{i-1}^{a_{i-1}}, x_{i1}^{a_i}, x_{i2}^{a_i}, x_{i+1}^{a_{i+1}}, \dots, x_n^{a_n})$ is normal in $R[x_{i1}, x_{i2}]/(x_i)$ if and only if the integral closure of $(x_1^{a_1}, \dots, x_n^{a_n})$ is normal in R , where $1 \leq i \leq n$. The proof of [2] is long since its accomplished after developing an algorithm and some consequences. As an application of Theorem 7.2.1, we give a direct proof of this result.

Theorem 7.2.4. *Let I be the integral closure of $L_1 := (x_1^{a_1}, \dots, x_n^{a_n})$ in the polynomial ring $R = K[x_1, \dots, x_n]$. Let J be the integral closure of the following monomial ideal*

$$L_2 := (x_1^{a_1}, \dots, x_{i-1}^{a_{i-1}}, x_{i1}^{a_i}, x_{i2}^{a_i}, x_{i+1}^{a_{i+1}}, \dots, x_n^{a_n}),$$

in $R[x_{i1}, x_{i2}]/(x_i)$. Then I is normal if and only if J is normal.

Proof. By virtue of Theorem 7.2.1, it suffices to show that $J = I^*$, the expansion of I with respect to the n -tuple $(1, \dots, 1, 2, 1, \dots, 1)$, where 2 is in the i -th place. By Lemma 7.2.3, $\overline{L_1^*} = \overline{L_2}$. But $J = \overline{L_2}$ and $\overline{L_1^*} = (\overline{L_1})^* = I^*$ by Proposition 7.2.2. This leads to $J = I^*$, as desired. $\square$

7.2.2 Normality under Weighting

To state Theorem 7.2.5, one first should recall the definition and properties of weighted ideals, see Definition 3.2.11 and Lemma 3.2.12.

Theorem 7.2.1 gives that the ideal $M = (x_1, x_2, \dots, x_n)$ is normal in $R = K[x_1, \dots, x_n]$. But if we consider the weight $W := \{x_1, \dots, x_n\} \rightarrow \mathbb{N}$ over R with $W(x_i) \geq 2$ for all i , then it is obvious that $x_i x_j \in \overline{M_W} \setminus M_W$; hence, M_W is not integrally closed. This shows that if the original monomial ideal is normal, then it

does not necessarily imply that the weighted ideal is also normal. But the converse is always true. The following theorem asserts that if the weighted ideal of a monomial ideal is normal, then the original ideal is also normal. An application of this theorem will be appeared in Chapter 11.

Theorem 7.2.5. *Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$, and W a weight over R . If I_W is normal, then I is normal.*

Proof. We use the notations that have been introduced in Definition 3.2.11, in particular, for a monomial $u \in R$, we denote $h(u) = u_W$. Let I_W be normal. Fix $t \in \mathbb{N}$, and pick a monomial $u \in \overline{I^t}$. Hence, [60, Theorem 1.4.2] implies that there exists a positive integer k such that $u^k \in (I^t)^k$. We thus gain $(u^k)_W \in (I^{tk})_W$. It follows from Lemma 3.2.12(ii) that $(I^{tk})_W = (I_W)^{tk}$. Consequently, one can conclude that $u_W^k \in (I_W^t)^k$. Now, [60, Theorem 1.4.2] gives that $u_W \in \overline{(I_W)^t}$. The assumption deduces that $\overline{(I_W)^t} = (I_W)^t$, and so $u_W \in (I_W)^t$. By virtue of $(I_W)^t = (I^t)_W$, we get $u_W \in (I^t)_W$. Hence, there exists a monomial $v \in \mathcal{G}(I^t)$ such that $v_W | u_W$. Accordingly, one has $v | u$, and so $u \in I^t$. This means that I is a normal ideal, and the proof is complete. $\square$

7.3 Normality under More Operations

In this section, we investigate the normality under localization, deletion, and contraction. In Subsection 7.3.1, we show that monomial localization of a normally torsion-free square-free monomial ideal I , in a commutative Noetherian ring S , with respect to $\mathfrak{p}$ is normal, where $\mathfrak{p} \in \text{Ass}_S(S/I)$, see Theorem 7.3.2. We also show that the converse of this theorem is not true, see Remark 7.3.4. Upon our results on normality of cover ideals of cycle graphs of odd order, and also upon [133, Corollary 10.5.9], we construct a square-free monomial ideal $L \subset R = K[x_1, \dots, x_n]$ which is not normal, while the contraction $L/x_j = (L :_R x_i)$ is normal for some $1 \leq i \leq n$, see Subsection 7.3.2. In Subsection 7.3.3, we show that if I is a normally torsion-free square-free monomial ideal, then every deletion $I \setminus x_j$ is normal, see Theorem 7.3.9. We also show that the converse of this theorem is not true, see Example 7.3.11 in which proving normality depends on Theorem 7.1.3. It should be noted that the results of this section can be found in [3].

The subsequent theorem expresses that under certain conditions, the sum of two square-free monomial ideals is normal, see also [130, Corollary 5.5.13]. This theorem is needed (along with Theorem 7.1.3 and Remark 7.1.1) to produce an interesting example about the normality under deletion, see Example 7.3.11.

Theorem 7.3.1. *Suppose that I is a square-free monomial ideal in $R = K[x_1, \dots, x_n]$ such that $I = I_1R + I_2R$, where $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 =$*

$K[x_{m+1}, \dots, x_n]$ for some positive integer $1 \leq m < n$. If I_1 and I_2 are normally torsion-free square-free monomial ideals, then I is a square-free normally torsion-free monomial ideal; hence, normal.

Proof. Theorem 5.3.4 together with [60, Theorem 1.4.6] yield the claim. $\square$

7.3.1 Normality under Localization

First, we need to recall the definition of the monomial localization of a monomial ideal with respect to a monomial prime ideal as has been stated in Definition 3.1.14.

In the following theorem, we prove that if I is a normally torsion-free square-free monomial ideal, then every monomial localization of I is normal. However, one should note that if I is a normal (not necessarily square-free) monomial ideal, then $I(\mathfrak{p})$ is normal by [133, Proposition 12.2.3].

Theorem 7.3.2. *Let I be a square-free monomial ideal in $R = K[x_1, \dots, x_n]$ and $\mathfrak{p} \in V^*(I)$. If I is normally torsion-free, then $I(\mathfrak{p})$ is square-free normally torsion-free; hence, normal.*

Proof. It is enough to combine Theorem 5.3.12 and [60, Theorem 1.4.6]. $\square$

The subsequent corollary is an immediate consequence of Theorem 7.3.2.

Corollary 7.3.3. *Let I be a square-free monomial ideal in $R_1 = K[x_1, \dots, x_r]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, $\bigcup_{i=1}^m \text{supp}(u_i) = \{x_1, \dots, x_r\}$, and $J_1, \dots, J_m$ be some nonzero square-free monomial ideals in $R_2 = K[x_{r+1}, \dots, x_n]$. If $L := u_1 J_1 R + \dots + u_m J_m R$ is normally torsion-free, where $R = K[x_1, \dots, x_n]$, then I is square-free normally torsion-free, hence normal.*

Proof. Set $\mathfrak{p} := (\bigcup_{i=1}^m \text{supp}(u_i))R = (x_1, \dots, x_r)R$. It is routine to check that $\mathfrak{p}$ is a monomial prime ideal of R , and $L \subseteq \mathfrak{p}$. It follows now from Theorem 7.3.2 that $L(\mathfrak{p})$ is normally torsion-free and normal as well. Since $L(\mathfrak{p}) = I$, one can conclude that I is normally torsion-free and normal as well, as required. $\square$

The opposite of Theorem 7.3.2 is not true as we show in the following remark. Recall from Theorem 3.2.34 that any normal monomial ideal has the strong persistence property. In particular, it has been proved in [133, Theorem 7.7.3] that every normal monomial ideal has the persistence property.

Remark 7.3.4. *Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ such that $I(\mathfrak{p})$ is normal for all $\mathfrak{p} \in \text{Ass}_R(R/I)$. Then we show that, in general, we cannot deduce that I is normal. To see this, consider the graph $G = H_4$, which has been presented in Proposition 3.2.5.*

To simplify the notation, one may assume that $V(G) = [12]$. Let I be the cover ideal associated to the graph G in the polynomial ring $R = K[x_1, \dots, x_{12}]$, that is, $I = \bigcap_{\{i,j\} \in E(G)} (x_i, x_j)$, and so $\text{Ass}_R(R/I) = \{(x_i, x_j) \mid \{i, j\} \in E(G)\}$. Since $I(\mathfrak{p}) = \mathfrak{p}$ for each $\mathfrak{p} \in \text{Ass}_R(R/I)$, one can readily conclude that $I(\mathfrak{p})$ is normal for each $\mathfrak{p} \in \text{Ass}_R(R/I)$. Now, suppose, on the contrary, that I is normal. It follows now from [133, Theorem 7.7.3] that I has the persistence property. On the other hand, using the commutative algebra program Macaulay2 [49], we get $\mathfrak{m} = (x_1, \dots, x_{12}) \in \text{Ass}_R(R/I^3) \setminus \text{Ass}_R(R/I^4)$. Consequently, I does not satisfy the persistence property. This leads to a contradiction. Therefore, I is not normal, as required.

As we have mentioned before the above remark, it has been proved in Theorem 3.2.34 that any normal monomial ideal has the strong persistence property. Therefore, in the light of Theorems 7.1.7 and 7.1.8, the following two corollaries follow immediately.

Corollary 7.3.5. *Let $R = K[x_1, \dots, x_{2n+1}]$ be a polynomial ring over a field K , C_{2n+1} be the cycle graph on the vertex set $\{1, \dots, 2n+1\}$, and $I = J(C_{2n+1}) \subset R$ be the cover ideal of C_{2n+1} . Then I has the strong persistence property, and hence persistence property.*

Corollary 7.3.6. *Suppose that W_{2n} is a wheel graph of order $2n$ on the vertex set $[2n]$. Then $J(W_{2n})$ has the strong persistence property, and hence persistence property.*

7.3.2 Normality under Contraction

To state the subsequent result, we require to remember the definition of the contraction operator as has been given in Definition 3.2.19.

By virtue of Theorem 7.3.2, one can conclude the following result.

Corollary 7.3.7. *Suppose that I is a square-free monomial ideal in the polynomial ring $R = K[x_1, \dots, x_n]$. If I is normally torsion-free, then I/x_j is normally torsion-free square-free; hence, normal.*

It is well-known that if I is a square-free monomial ideal in a polynomial ring $K[x_1, \dots, x_n]$, then $I/x_j = (I :_R x_j)$ for each $j = 1, \dots, n$. Hence, one should be aware that it is possible a square-free monomial ideal such as $I \subset R = K[x_1, \dots, x_n]$ is not normal, while $(I :_R x_i)$ is normal for some $1 \leq i \leq n$. To see this, consider the following example.

Example 7.3.8. *Consider two monomial ideals $I := (x_1x_4, x_1x_5, x_2x_3, x_2x_5, x_3x_4)$ and $J := (x_1x_2x_4, x_1x_3x_5, x_1x_2x_3, x_2x_4x_5, x_3x_4x_5)$ in $K[x_1, \dots, x_7]$. Set $L := x_6I +$*

x_7J . One can rapidly conclude that I and J have the following minimal primary decomposition

$$I = (x_3, x_2, x_1) \cap (x_5, x_4, x_3) \cap (x_5, x_3, x_1) \cap (x_5, x_4, x_2) \cap (x_4, x_2, x_1), \text{ and}$$

$$J = (x_1, x_4) \cap (x_4, x_3) \cap (x_3, x_2) \cap (x_2, x_5) \cap (x_5, x_1).$$

Now, let C_5 be the odd cycle graph on the vertex set $V(G) = \{1, 2, 3, 4, 5\}$ and the edge set $E(C_5) = \{\{1, 4\}, \{4, 3\}, \{3, 2\}, \{2, 5\}, \{5, 1\}\}$. It is easy to see that I is the edge ideal associated to C_5 , and J is the cover ideal associated to C_5 . Based on Theorem 7.1.7, the ideal J is normal. By virtue of [133, Corollary 10.5.9], one derives I is normal. Moreover, it follows from $J \subseteq I$ that $(L :_R x_6) = I$, and hence $(L :_R x_6)$ is normal. Now, put $\alpha := x_1x_2x_3x_4x_5x_6x_7$. Direct computation gives that

$$\alpha^2 = (x_1x_2x_3x_7)(x_1x_5x_6)(x_2x_4x_5x_7)(x_3x_4x_6) \in (L^2)^2,$$

and so $\alpha \in \overline{L^2}$, while $\alpha \notin L^2$. This yields that L is not normal.

7.3.3 Normality under Deletion

To express Theorem 7.3.9, we must recollect the definition of the deletion operator as has been stated in Definition 3.2.19.

In the next theorem, we show that if I is a normally torsion-free square-free monomial ideal, then every deletion $I \setminus x_j$ is normal.

Theorem 7.3.9. *Suppose that I is a square-free monomial ideal in the ring $R = K[x_1, \dots, x_n]$. If I is normally torsion-free, then $I \setminus x_j$ is normally torsion-free square-free; hence, normal.*

Proof. It is enough to combine Theorem 5.3.16 and [60, Theorem 1.4.6]. $\square$

Remark 7.3.10. *We finally emphasize that if I is a normal (not necessarily square-free) monomial ideal, then both I/x_j and $I \setminus x_j$ are normal by [133, Proposition 12.2.3].*

As an application of Theorem 7.3.1, Remark 7.1.1, and Theorem 7.1.3, we show that if I is a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ such that $I \setminus x_j$ is normal for all $i = 1, \dots, n$, then we cannot, in general, conclude that I is normal. This is illustrated in the following example.

Example 7.3.11. *Consider the following monomial ideal*

$$I = (x_3x_5x_6, x_3x_4x_6, x_2x_5x_6, x_2x_4x_5, x_1x_3x_5, x_1x_2x_6),$$

in the polynomial ring $R = K[x_1, \dots, x_6]$. We demonstrate that $I \setminus x_i$ is normal for all $i = 1, \dots, 6$. One can rapidly gain the following results:

- (i) $I \setminus x_1 = x_6(x_3(x_5, x_4) + (x_2x_5)) + (x_2x_4x_5)$.
- (ii) $I \setminus x_2 = x_3(x_5(x_1, x_6) + (x_4x_6))$.
- (iii) $I \setminus x_3 = x_2(x_6(x_5, x_1) + (x_4x_5))$.
- (iv) $I \setminus x_4 = x_6(x_5(x_3, x_2) + (x_1x_2)) + (x_1x_3x_5)$.
- (v) $I \setminus x_5 = x_6(x_1x_2, x_3x_4)$.
- (vi) $I \setminus x_6 = x_5(x_1x_3, x_2x_4)$.

Using Theorem 7.3.1 and Remark 7.1.1, we get $I \setminus x_5$ and $I \setminus x_6$ are normal. Moreover, applying Theorem 7.1.3 verifies that $I \setminus x_1$ and $I \setminus x_4$ are normal. Finally, combining Theorem 7.1.3 and Remark 7.1.1 yields that $I \setminus x_2$ and $I \setminus x_3$ are normal. We claim that I is not normal. To show this, put $\alpha := x_1x_2x_3x_4x_5x_6$. Since

$$\alpha^2 = (x_1x_3x_5)(x_1x_2x_6)(x_2x_4x_5)(x_3x_4x_6) \in (I^2)^2,$$

we get $\alpha \in \overline{I^2}$. On the other hand, it is easy to see that $\alpha \notin I^2$, and so $\alpha \in \overline{I^2} \setminus I^2$. Accordingly, one derives that I is not normal.

7.4 More results on the normality of square-free monomial ideals

In this section, we express several outcomes on the normality of square-free monomial ideals. It should be noted that the results of this section can be found in [3].

We begin with the following theorem, which is essential for us to show Corollary 7.4.2.

Theorem 7.4.1. *Let I be a normal monomial ideal in R and $h \in R$ a monomial. Assume $v \in R$ is a square-free monomial with $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I) \cup \{h\}$. Then $L := I + vhR$ is normal if and only if $J := I + hR$ is normal.*

Proof. Because $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I) \cup \{h\}$, one may assume that $v = x_1 \cdots x_m \in K[x_1, \dots, x_m]$ and $\mathcal{G}(I + hR) := \{u_1, \dots, u_s, h\} \subseteq K[x_{m+1}, \dots, x_n]$ for some positive integer $1 \leq m < n$. The ideal J is obtained from L by making every variable x_i (for $i = 1, \dots, m$) is equal to 1. Proposition 12.2.3 of Villarreal [133] asserts that a normal monomial ideal stays normal if we make any variable equals 1. This proves the necessary part. Conversely, we prove that $\overline{L^t} = L^t$ for all integers $t \geq 1$, provided that J is normal. It suffices to show that $\overline{L^t} \subseteq L^t$. Let α be a

monomial in $\overline{L}^t$ and write $\alpha = v^b \delta$ with $v \nmid \delta$ and $\delta \in R$. By [60, Theorem 1.4.2], $\alpha^k \in L^{tk}$ for some integer $k \geq 1$. Write

$$(7.4.1) \quad v^{bk} \delta^k = \prod_{i=1}^s u_i^{p_i} v^{q+\varepsilon} h^q \beta,$$

with $\sum_{i=1}^s p_i + q = tk$, $\varepsilon \geq 0$, and β is some monomial in R such that $v \nmid \beta$. Let $x_l \in \text{supp}(v) \setminus \text{supp}(\beta)$ for some $l \in \{1, \dots, m\}$. Now, if $x_l \in \text{supp}(\delta)$, then, since v is square-free, the x_l -degree in both sides of (7.4.1) gives that $bk + jk = q + \varepsilon$, where $j = x_l\text{-deg}(\delta)$. Hence, cancelling v^{bk} from both sides of (7.4.1) gives that $\delta^k = \prod_{i=1}^s u_i^{p_i} v^{q+\varepsilon-bk} h^q \beta$. Since v is square-free, this yields that v must divide δ , a contradiction. Therefore, we proceed with the assumption that there exists $l \in \{1, \dots, m\}$ with $x_l \in \text{supp}(v) \setminus (\text{supp}(\beta) \cup \text{supp}(\delta))$. This assumption, along with the hypothesis $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I) \cup \{h\}$, forces that $bk = q + \varepsilon$. Therefore, by (7.4.1) we obtain $\delta^k = \prod_{i=1}^s u_i^{p_i} h^q \beta \in J^{tk}$. This implies that $\delta \in \overline{J}^t$. Since J is normal, we obtain $\overline{J}^t = J^t$, and thus, $\delta \in J^t$; hence, write

$$(7.4.2) \quad \delta = \prod_{i=1}^s u_i^{l_i} h^z \gamma,$$

with $\sum_{i=1}^s l_i + z = t$, $z \geq 0$, and γ is some monomial in R . Note $v \nmid \gamma$ as $v \nmid \delta$. Now, since $v^{bk} \delta^k \in L^{tk}$, by virtue of (7.4.2), we get $\prod_{i=1}^s u_i^{l_i k} v^{bk} h^{zk} \gamma^k \in L^{tk} = (I + v h R)^{tk}$. We therefore have $bk \geq zk$, that is, $b \geq z$. Thus, $v^b \delta = \prod_{i=1}^s u_i^{l_i} v^b h^z \gamma \in (I + v h R)^t$, and the proof is done. $\square$

Setting $h = 1$ in Theorem 7.4.1, we obtain the following corollary.

Corollary 7.4.2. *Let I be a normal monomial ideal in R and $v \in R$ a square-free monomial with $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I)$. Then $I + vR$ is normal.*

Inductively and in view of [133, Proposition 12.2.3], the following proposition follows directly from Corollary 7.4.2.

Proposition 7.4.3. *Any square-free monomial ideal of R with a set of pairwise relatively prime generators is normal. In particular, assume $\mathcal{G}(I) \subset K[x_1, \dots, x_m]$ and J is square-free with $\mathcal{G}(J) \subset K[x_{m+1}, \dots, x_n]$, where $1 \leq m < n$, and the generators of J are pairwise relatively prime. Then $I + J$ is normal if and only if I is normal.*

Remark 7.4.4. *It should be noted that Proposition 7.4.3 may be false if we drop the condition that the generators of J are pairwise relatively prime. To see a counterexample, let $I := (x_1 x_2 x_4, x_1 x_3 x_4, x_1 x_3 x_5, x_2 x_3 x_5, x_2 x_4 x_5) \subset K[x_1, \dots, x_5]$ and $J := (y_1 y_2 y_4, y_1 y_3 y_4, y_1 y_3 y_5, y_2 y_3 y_5, y_2 y_4 y_5) \subset K[y_1, \dots, y_5]$ be the cover ideals of the odd cycle graphs G and H , respectively, where $V(G) = \{x_1, x_2, x_3, x_4, x_5\}$ with*

$E(G) = \{\{x_i, x_{i+1}\}\}_{i=1}^5$, and $V(H) = \{y_1, y_2, y_3, y_4, y_5\}$ with $E(H) = \{\{y_i, y_{i+1}\}\}_{i=1}^5$, where x_6 represents x_1 , and y_6 represents y_1 . It follows from Theorem 7.1.7 that I and J are normal monomial ideals. Set $\alpha := \prod_{i=1}^5 x_i \prod_{i=1}^5 y_i$ in the monomial ideal

$$Q := IK[x_1, \dots, x_5, y_1, \dots, y_5] + JK[x_1, \dots, x_5, y_1, \dots, y_5].$$

Because $\alpha^2 = (x_1x_2x_4)(x_1x_3x_5)(x_2x_3x_5)x_4(y_1y_2y_4)(y_1y_3y_5)(y_2y_3y_5)y_4 \in Q^6$, one can deduce that $\alpha \in \overline{Q^3}$. On the other hand, it is easy to check that $\alpha \notin Q^3$. This means that Q is non-normal.

Assume a graph H is obtained from a graph G by connecting all the vertices of G with a new vertex. In Theorem 7.1.5 it is proved that the cover ideal of the graph H is normal provided that the cover ideal of the graph G is normal. The proof relies on Theorem 7.1.3 in which it is proved that $vI + hR$ is normal provided that I is a normal monomial ideal, $h \in I$, and v is a square-free monomial which is relatively prime to every generator of I . In Theorem 7.4.6, we give a generalization of Theorem 7.1.3. The following lemma is needed in the proofs of Theorems 7.4.6 and 7.4.7. It is a straightforward application of [60, Theorem 1.4.2].

Lemma 7.4.5. *Suppose that I and J are two normal monomial ideals in R such that $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$. Then $I \cap J = IJ$ is normal.*

Theorem 7.4.6. *Let I be a normal monomial ideal of R , and J be a normal square-free monomial ideal of R whose generators are pairwise relatively prime. Let $h \in I$ a monomial and suppose $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I) \cup \{h\}$ and $v \in \mathcal{G}(J)$. Then $L := JI + hR$ is normal.*

Proof. Assume that $\mathcal{G}(I) := \{u_1, \dots, u_r\}$ and $\mathcal{G}(J) := \{v_1, \dots, v_s\}$. As before, it suffices to show that $\overline{L^t} \subseteq L^t$ for all $t \geq 1$. Let α be a monomial in $\overline{L^t}$. By [60, Theorem 1.4.2], we have $\alpha^k \in (JI + hR)^{tk}$ for some positive integer k . The binomial expansion implies that $\alpha^k \in (JI)^p(hR)^q$ for some nonnegative integers p and q with $p + q = tk$. Choose q to be the minimal according to this membership. If $q = 0$, then $p = tk$; hence, $\alpha^k \in (JI)^{tk}$. Accordingly, $\alpha \in \overline{(JI)^t}$. In the light of I and J being normal and $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I)$ and $v \in \mathcal{G}(J)$, Lemma 7.4.5 yields that JJ is normal, and so $\alpha \in (JI)^t$; thus, we have $\alpha \in L^t$ and the proof is over. Therefore, let $q \geq 1$. Since $\alpha^k \in (JI)^p(hR)^q$, we get the following equality

$$(7.4.3) \quad \alpha^k = v_1^{z_1} \cdots v_s^{z_s} u_1^{p_1} \cdots u_r^{p_r} h^q \beta,$$

with $\sum_{i=1}^s z_i = p$, $\sum_{i=1}^r p_i = p$, and β some monomial in R . Since $h \in I$, this implies that $h = h' u_\lambda$ for some $1 \leq \lambda \leq r$ and monomial h' in R . If $\beta \in J$, then $\beta = v_\theta \beta'$ for some $1 \leq \theta \leq s$ and monomial β' in R . Hence, (7.4.3) can be rewritten as follows

$$\alpha^k = v_1^{z_1} \cdots v_s^{z_s} v_\theta u_1^{p_1} \cdots u_r^{p_r} u_\lambda h^{q-1} h' \beta'.$$

This leads to $\alpha^k \in (JI)^{p+1}(hR)^{q-1}$, which contradicts the minimality of q . Accordingly, one can assume in (7.4.3) that $v_i \nmid \beta$ for each $i = 1, \dots, s$. Let $x_{j_i} | v_i$ but $x_{j_i} \nmid \beta$ for each $i = 1, \dots, s$. Note that $x_{j_1}, \dots, x_{j_s}$ are distinct since $\gcd(v_i, v_j) = 1$ for any $1 \leq i \neq j \leq s$. Write $\alpha = x_{j_1}^{b_1} \cdots x_{j_s}^{b_s} \delta$ with $x_{j_d} \nmid \delta$ for each $d = 1, \dots, s$. This gives rise to the following equality

$$(7.4.4) \quad x_{j_1}^{b_1 k} \cdots x_{j_s}^{b_s k} \delta^k = v_1^{z_1} \cdots v_s^{z_s} u_1^{p_1} \cdots u_r^{p_r} h^q \beta.$$

In view of (7.4.4), we have $b_i k = z_i$ for each $i = 1, \dots, s$. Set $b := b_1 + \cdots + b_s$. Thanks to $\sum_{i=1}^s z_i = p$, one has $b k = p$, and so $q = (t-b)k$. We therefore obtain $\alpha^k \in (JI)^{bk}(hR)^{(t-b)k}$. Consequently, $\alpha \in \overline{(JI)^b(hR)^{t-b}}$. It follows from Lemma 7.4.5 that $\overline{(JI)^b(hR)^{t-b}} = (JI)^b(hR)^{t-b}$. This yields that $\alpha \in (JI)^b(hR)^{t-b} \subseteq (JI+hR)^t = L^t$, as required. $\square$

The subsequent theorem is one of the main results in this section which is used in proving Theorem 7.4.12.

Theorem 7.4.7. *Let I be a normal square-free monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then the ideal $L := IS \cap (x_n, x_{n+1}) \subset S = R[x_{n+1}]$ is normal.*

Proof. Since $I \cap (x_n) = x_n(I :_S x_n)$ and $I \cap (x_{n+1}) = x_{n+1}I$, one can conclude that $L = x_n(I :_S x_n) + x_{n+1}I$. To simplify our notation, set $F := (I :_S x_n)$ and $L = x_n F + x_{n+1}I$. Since I is square-free, then the ideal F is obtained from I by making the variable x_n is equal to 1. Therefore, F is normal by virtue of Proposition 12.2.3 in [133], and hence $x_n F$ is also normal. Let $\mathcal{G}(I) = \{g_1, \dots, g_d, g_{d+1}, \dots, g_s\}$ with $x_n | g_j$ for $j = 1, \dots, d$ and $x_n \nmid g_j$ for $j = d+1, \dots, s$. Then a generating set (not necessarily minimal) of F is given by $\{f_1, \dots, f_d, f_{d+1}, \dots, f_s\}$ with $f_i x_n = g_i$ for $i = 1, \dots, d$ and $f_i = g_i$ for $i = d+1, \dots, s$. Note that $x_n F + I = I$; therefore, $x_n F + I$ is normal, that is, $\overline{(x_n F + I)^t} = (x_n F + I)^t$ for all $t \geq 1$. Our goal is to show that $\overline{L^t} = L^t$ for all $t \geq 1$.

Let α be a monomial in $\overline{L^t}$ and write $\alpha = x_{n+1}^b \delta$ for some integer b and some monomial $\delta \in R$ with $x_{n+1} \nmid \delta$. As $\alpha \in \overline{L^t}$, [60, Theorem 1.4.2] implies that $\alpha^k \in L^{tk} = (x_n F + x_{n+1}I)^{tk}$ for some integer k ; therefore, $\alpha^k \in (x_n F)^p (x_{n+1}I)^q$ for some integers p and q with $p+q = tk$. Assume q is maximal according to this membership. Note that if $p = 0$, then $\alpha^k \in (x_{n+1}I)^{tk}$, and hence $\alpha \in \overline{(x_{n+1}I)^t} = (x_{n+1}I)^t \subset L^t$. Henceforth, assume $p > 0$. Similarly, and since $x_n F$ is normal, we may also assume $q > 0$. Write

$$(7.4.5) \quad \alpha^k = x_{n+1}^{bk} \delta^k = \prod_{i=1}^s f_i^{p_i} x_n^p \prod_{j=1}^s g_j^{q_j} x_{n+1}^q \beta,$$

with $\sum_{i=1}^s p_i = p$, $\sum_{j=1}^s q_j = q$, and β some monomial in S . If $x_{n+1} | \beta$, then we get a contradiction to the maximality of q since either $f_i x_n = g_i$ or $f_i = g_i$. Therefore,

we may assume in (7.4.5) that $x_{n+1} \nmid \beta$, and thus we can conclude that $q = bk$ and also that

$$(7.4.6) \quad \delta^k = \prod_{i=1}^s f_i^{p_i} x_n^p \prod_{j=1}^s g_j^{q_j} \beta \in (x_n F + I)^{tk}.$$

Therefore, $\delta \in \overline{(x_n F + I)^t}$, and hence $\delta \in (x_n F + I)^t$; thus, let $\delta \in (x_n F)^l I^h$ with $l + h = t$ and l being maximal with respect to this membership. Note that if $h = 0$, then $\delta \in (x_n F)^t \subset L^t$. Henceforth, assume $h > 0$. Write

$$\delta = \prod_{i=1}^s f_i^{l_i} x_n^l \prod_{j=1}^s g_j^{h_j} \gamma,$$

with $\sum_{i=1}^s l_i = l$, $\sum_{j=1}^s h_j = h$, and γ is some monomial in S . Note that if $d = 0$, then $L = I \cap (x_n, x_{n+1})$ is normal by Lemma 7.4.5. In addition, note that if $d = s$, then $L = I$, which is normal. Henceforth, assume that $s > d > 0$. The maximality of l implies that $h_j = 0$ for $j = 1, \dots, d$ and also that $x_n \nmid \gamma$ since $g_j = f_j$ for $j = d + 1, \dots, s$. Hence, $x_n\text{-deg}(\delta) = l$. Since $p = (t - b)k$, then (7.4.6) gives that $x_n\text{-deg}(\delta) \geq t - b$, that is, $l \geq t - b$; thus, $b \geq h$. Therefore, since $\delta \in (x_n F)^l I^h$, one can deduce that

$$x_{n+1}^b \delta \in (x_n F)^l (x_{n+1} I)^h \subseteq (x_n F + x_{n+1} I)^{l+h} = L^t,$$

which finishes the proof. $\square$

Remark 7.4.8. As the reader may notice, in the proof of Theorem 7.4.7 we proved that the linear combination $x_n F + x_{n+1} I$ is normal, where $I \subseteq F$. Initial endeavors of the authors were to prove a more general result, that is, investigating the normality of square-free monomial ideals resulting from linear combinations $vF + wI$, where F and I are normal ideals, v and w are square-free monomials with $\gcd(v, f) = 1$ for all $f \in \mathcal{G}(F)$ and $\gcd(w, g) = 1$ for all $g \in \mathcal{G}(I)$. Since $vF + wI$ is square-free, one gains it is integrally closed. However, we found that one cannot guarantee the normality of $vF + wI$, even though one has $I \subseteq F$ or $\mathcal{G}(I) \subset \mathcal{G}(F)$. In this remark we demonstrate an example supporting this assertion. Let $F = (x_1 x_2 x_4, x_1 x_3 x_5, x_2 x_3, x_2 x_5, x_3 x_4)$ and $I = (x_1 x_2 x_4, x_1 x_3 x_5)$ in $R = K[x_1, \dots, x_7]$, and let $v = x_6$ and $w = x_7$. Then we get

$$T := vF + wI = (x_1 x_2 x_4 x_6, x_1 x_3 x_5 x_6, x_2 x_3 x_6, x_2 x_5 x_6, x_3 x_4 x_6, x_1 x_2 x_4 x_7, x_1 x_3 x_5 x_7).$$

The normality of I can be deduced from Proposition 7.4.3. In addition, using Normaliz [24] yields that F is a normal monomial ideal. Now, put $\alpha := x_1 x_2 x_3 x_4 x_5 x_6 x_7$. Direct computations show that $\alpha \notin T^2$. Since

$$\alpha^2 = (x_2 x_5 x_6) (x_3 x_4 x_6) (x_1 x_2 x_4 x_7) (x_1 x_3 x_5 x_7) \in T^4,$$

we conclude that $\alpha \in \overline{T^2} \setminus T^2$, that is, $vF + wI$ is not normal.

Corollary 7.4.9. *Let I be a normal square-free monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then the ideal $L := IS \cap (x_n, x_{n+1} \cdots x_m) \subset S = R[x_{n+1}, \dots, x_m]$ is normal.*

Proof. Since $I \cap (x_n, x_{n+1} \cdots x_m) = I \cap (x_n, x_{n+1}) \cap (x_n, x_{n+2}) \cap \cdots \cap (x_n, x_m)$, this claim follows at once from Theorem 7.4.7. $\square$

The above corollary motivates for the following two questions.

Question 7.4.10. *Let I be a normal square-free monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$, and $\{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$ with $r > 1$. Then, in general, can we deduce that the ideal $L := IS \cap (x_{i_1} \cdots x_{i_r}, x_{n+1}) \subset S = R[x_{n+1}]$ is normal?*

Question 7.4.11. *Let I be a normal square-free monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then, in general, can one conclude $L := IS \cap (x_n, x_{n+1}^\ell) \subset S = R[x_{n+1}]$ with $\ell > 1$, is normal?*

We show that Question 7.4.10 has a negative answer, while we will show that Question 7.4.11 has an affirmative answer, see Proposition 7.7.6. For this purpose, we provide a counterexample. Firstly, one should recall that, given a graph G , if v is a vertex of G , we may obtain a graph on $n - 1$ vertices by deleting from G the vertex v together with all the edges incident with v . The resulting graph is denoted by $G \setminus v$. In the following, one should consider a graph which has been stated in [76]. In fact, for a positive integer n , let $[n]$ denote the set $\{0, \dots, n - 1\}$. Assume that P_n is a path with vertex set $[n]$, with vertices in the increasing order along P_n . Let also K_3 be the complete graph whose vertex set is the group $\mathbb{Z}_3$. For $n \geq 4$, we define H_n as the graph obtained from the Cartesian product $P_n \square K_3$ by adding the three edges joining $(0, j)$ to $(n - 1, -j)$ for $j \in \mathbb{Z}_3$. Figure below is the graph of H_4 in [76].

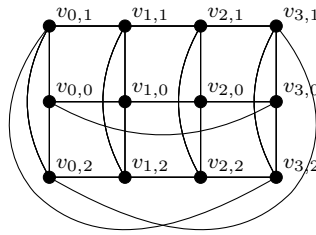


Figure 7.4.1: H_4

Assume that $J(H_4)$ denotes the cover ideal of H_4 in the polynomial ring $S = K[x_\alpha : \alpha \in V(H_4)]$ over a field K . Now, put $G := H_4 \setminus v_{0,1}$, where the graph G is obtained from H_4 by deleting from H_4 the vertex $v_{0,1}$ together with all the edges

incident with $v_{0,1}$. Let $J(G)$ denote the cover ideal of the graph G . It should be noted that

$$\begin{aligned} J(H_4) &= J(G) \cap (x_{v_{0,1}}, x_{v_{1,1}}) \cap (x_{v_{0,1}}, x_{v_{0,0}}) \cap ((x_{v_{0,1}}, x_{v_{0,2}}) \cap (x_{v_{0,1}}, x_{v_{3,2}})) \\ &= J(G) \cap (x_{v_{0,1}}, x_{v_{1,1}} x_{v_{0,0}} x_{v_{0,2}} x_{v_{3,2}}). \end{aligned}$$

It follows from Normaliz [24] that $J(G)$ is a normal monomial ideal, while based on Remark 7.3.4, $J(H_4)$ is non-normal.

7.4.1 An application to the cover ideals of imperfect graphs

The point of this subsection is to hire some of the consequences of the previous section to cover ideals of imperfect graphs. Indeed, we first prove Theorem 7.4.12, which says that if we take any graph G with normal cover ideal, and add a leaf to G , then the cover ideal of the new graph is also normal. Note that, in this section, $[n] = \{1, \dots, n\}$.

Theorem 7.4.12. *Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be finite simple graphs such that $V(H) = V(G) \cup \{w\}$ with $w \notin V(G)$, and $E(H) = E(G) \cup \{v, w\}$ for some vertex $v \in V(G)$. Let $J(G)$ and $J(H)$ be the cover ideals of the graphs G and H , respectively. If $J(G)$ is normal, then $J(H)$ is normal.*

Proof. Suppose that $J(G)$ is normal. Without loss of generality, one may assume that $V(G) = [n]$, $V(H) = V(G) \cup \{n+1\}$, and $E(H) = E(G) \cup \{n, n+1\}$. Since $J(H) = J(G) \cap (x_n, x_{n+1})$, the claim is a straightforward consequence of Theorem 7.4.7, that is, $J(H)$ is normal. $\square$

Here, we want to explore the normality of the cover ideals of helm graphs H_n for $n \geq 5$ odd. For this purpose, one requires to recall the definition of helm graphs, see Definition 3.1.19.

As an application of Theorem 7.4.12, we illustrate that every cover ideal of helm graphs H_n for $n \geq 5$ odd, is normal. Since H_{2n+1} with $n \geq 2$, contains an induced odd cycle of length $n \geq 5$, this graph is imperfect, and so the normality of its cover ideal is of special interest.

Theorem 7.4.13. *Suppose that H_{2n+1} with $n \geq 2$ is a helm graph on the vertex set $[4n+3]$. Then $J(H_{2n+1})$ is normal. Therefore, it has the strong persistence property, and hence the persistence property.*

Proof. Label the vertices of C_{2n+1} in counterclockwise order with $1, \dots, 2n+1$, and the hub with $2n+2$, as shown in the figure below, such that we have

$$E(H_{2n+1}) = E(W_{2n+2}) \cup \{\{i, 2n+2+i\} : i = 1, \dots, 2n+1\}.$$

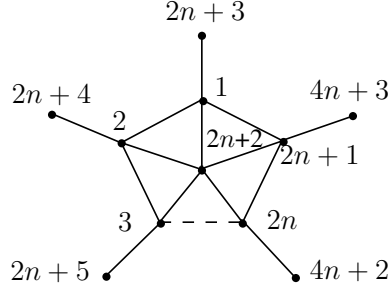


Figure 7.4.2: H_{2n+1}

We can now combine together Theorem 7.1.8 and the iteration of Theorem 7.4.12 to obtain the normality of $J(H_{2n+1})$. The last assertion follows readily from the normality of $J(H_{2n+1})$, Theorem 3.2.34, and Proposition 3.1.12. $\square$

7.4.2 On the normality of linear combinations of normal ideals

Let $R = \mathbb{Q}[x_1, \dots, x_{n+2}]$, let I and J be two square-free monomial ideals in the ring $\mathbb{Q}[x_1, \dots, x_n]$, and let $L = x_{n+1}IR + x_{n+2}JR$. As we have mentioned in Remark 7.4.8, this construction is a fruitful source of interesting square-free monomial ideals L which are not normal.

In what follows, we investigate the case where I is the edge ideal

$$I_n = (x_1x_2, x_2x_3, \dots, x_{n-1}x_n, x_nx_1),$$

of an n -cycle C_n and J is the cover ideal $J_n = (x_n, x_1) \cap \bigcap_{i=1}^{n-1} (x_i, x_{i+1})$ of I_n . We will write L_n instead of L , so that $L_n = x_{n+1}I_nR + x_{n+2}J_nR$. By abuse of notation, we will write more simply $L_n = x_{n+1}I_n + x_{n+2}J_n$. Also, all statements about L_n, I_n, J_n will take place in the ring $R = \mathbb{Q}[x_1, \dots, x_{n+2}]$, which for simplicity of notation, we might not explicitly mention. In a similar vein, $\overline{L_n^i} \setminus L_n^i$ literally means the complement of L_n^i in $\overline{L_n^i}$, i.e., those elements of $\overline{L_n^i}$ not in L_n^i . But computationally we like to think of it as all the monomials in $\overline{L_n^i}$ that are not in L_n^i . We need $n \geq 3$ for cycles to exist at all (as simple undirected graphs). *Macaulay2* calculation shows that L_3 is normal. In the following, we will always have $n \geq 4$.

Our first conclusion is that L_n is not normal for $n \geq 4$. More specifically, L_n^2 is not integrally closed for any $n \geq 4$ (Theorem 7.4.14). If $i \geq 3$, then L_n^i is not integrally closed for even $n \geq 4$, and for odd $n > 5$ (Theorem 7.4.16). If $n = 5$, then L_5^i is integrally closed for $i \geq 3$ (Theorem 7.4.17). We used the Hilbert basis of the Rees cone to prove the last result. Thus, we characterize all cases when L_n^i is integrally closed, i.e., when $\overline{L_n^i}/L_n^i = 0$.

We finally show that $\overline{L_n^2}/L_n^2$ is not a finite dimensional vector space for all odd n , $n \geq 5$ (Theorem 7.4.18).

The case of L_n^2

We start by proving our first claim in the following theorem.

Theorem 7.4.14. *If $n \geq 4$, then L_n^2 is not integrally closed.*

Proof. Let $f = x_1x_2 \cdots x_nx_{n+1}x_{n+2}$. Then we claim that $f \notin L_n^2$. For if it were, then f would be the product of one minimal generator of $x_{n+1}I_n$ and one minimal generator of $x_{n+2}J_n$. By cyclic symmetry of C_n , we can assume that the element of $x_{n+1}I_n$ is $x_1x_2x_{n+1}$. What is left is $x_3x_4 \cdots x_nx_{n+2}$ which is not in $x_{n+2}J_n$ because $x_3x_4 \cdots x_n$ does not contain one of the variables $\{x_1, x_2\}$ (according to the definition of the cover ideal). But now consider $f^2 = x_1^2x_2^2 \cdots x_n^2x_{n+1}^2x_{n+2}^2$. If n is even, this can be written as

$$\begin{aligned} f^2 &= (x_1x_3x_5 \cdots x_{n-1}x_{n+2})(x_2x_4x_6 \cdots x_nx_{n+2})(x_1x_2x_{n+1})(x_3x_4x_{n+1})(x_5 \cdots x_n) \\ &\in L_n^4, \end{aligned}$$

and so $f \in \overline{L_n^2} \setminus L_n^2$. If n is odd, then we have

$$\begin{aligned} f^2 &= (x_1x_2x_{n+1})(x_1x_3 \cdots x_nx_{n+2})(x_2x_4 \cdots x_{n-1}x_nx_{n+2})(x_3x_4x_{n+1})(x_5 \cdots x_{n-1}) \\ &\in L_n^4, \end{aligned}$$

and so again $f \in \overline{L_n^2} \setminus L_n^2$. This finishes our proof. $\square$

In view of Theorem 7.4.14 and remembering the fact that a monomial ideal is normal if and only if all its powers are integrally closed, we can deduce the following corollary.

Corollary 7.4.15. *If $n \geq 4$, then L_n is not normal.*

The case of $L_n^i, i \geq 3$

We are in a position to establish our second claim in the following theorem.

Theorem 7.4.16. *If $n \geq 4$, n even, or $n > 5$, n odd, then $L_n^i, i \geq 3$ is not integrally closed.*

Proof. We produce an explicit element in $\overline{L_n^i} \setminus L_n^i$ for $i \geq 3$ in these ranges of n .

For n even, $n \geq 4$, consider $h_i = x_1x_2^{i-1}x_3x_4^{i-1} \cdots x_{n-1}x_n^{i-1}x_{n+1}x_{n+2}^{i-1}$. Because of the $x_{n+1}x_{n+2}^{i-1}$ portion, h_i is potentially in $(x_{n+1}I_n)(x_{n+2}^{i-1}J_n^{i-1}) \subset L_n^i$. By cyclic

symmetry, we can assume that the generator in $x_{n+1}I_n$ is $x_1x_2x_{n+1}$. Dividing by this, we get $x_2^{i-2}x_3x_4^{i-1}\cdots x_{n-1}x_n^{i-1}x_{n+2}^{i-1}$. We now need $i-1$ minimal generators of $x_{n+2}J_n$ each of which has to be divisible by a variable from $\{x_1, x_2\}$, which must be x_2 since x_1 is now absent. But there are only $i-2$ x_2 factors left, so this is not possible. Thus, $h_i \notin L_n^i$. But we can write

$$h_i^2 = x_1^2x_2^{2i-2}x_3^2x_4^{2i-2}\cdots x_{n-1}^2x_n^{2i-2}x_{n+1}^2x_{n+2}^{2i-2} = (x_2x_4\cdots x_nx_{n+2})^{2i-4}f^2 \in L_n^{2i},$$

since we saw in the proof of Theorem 7.4.14 that $f^2 \in L_n^4$. Thus, $h_i \in \overline{L_n^i} \setminus L_n^i$, as desired.

For n odd, $n > 5$, let $g_i = x_2x_3x_4\cdots x_{n-2}x_{n-1}^{i-1}x_n^{i-1}x_{n+1}^{i-1}x_{n+2}$. Because of the $x_{n+1}^{i-1}x_{n+2}$ portion, if $g_i \in L_n^i$, then $g_i \in (x_{n+1}I_n)^{i-1}(x_{n+2}J_n)$. If p is any minimal generator of $x_{n+2}J_n$, then p must be divisible by at least one generator from each of the pairs $\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \{x_n, x_1\}$. Because x_1 is absent from g_i , p must be divisible by x_2 and x_n . Furthermore, g_i/p cannot be divisible by any of the products $x_1x_2, x_2x_3, \dots, x_{n-3}x_{n-2}$. The only part of g_i/p left to give an element of $(x_{n+1}I_n)^{i-1}$ is possibly $x_{n-2}x_{n-1}^{i-1}x_n^{i-2}x_{n+1}^{i-1}$. Here, the exponents of x_n and x_{n+1} are certain but those of x_{n-2} and x_{n-1} are possibly one lower. Thus, g_i/p is always divisible by $(x_{n-1}x_n)^{i-2}$. Dividing by this, we are left at most with $x_{n-2}x_{n-1}$. But p has to be divisible by at least one of these variables, so g_i/p cannot be in $(x_{n+1}I_n)^{i-1}$, and g_i cannot be in $(x_{n+1}I_n)^{i-1}(x_{n+2}J_n)$, and hence is not in L_n^i .

Note that g_i is of degree $n-2+3(i-1)$ and that $x_{n+2}J_n$ has minimal generators of degree $2+(n-1)/2$ (e.g., $x_1x_3x_5x_7x_9$ for C_7). Thus, potentially g_i^2 is the product of 2 minimal generators of $x_{n+2}J_n$ and $6(i-1)$ minimal generators of $x_{i+1}I_n$, with $2n-4+6(i-1)-(4+(n-1)+6(i-1))=n-7$ extra variables. Indeed

$$g_i^2 = (x_2x_3x_5x_7\cdots x_nx_{n+2})(x_2x_4x_6\cdots x_{n-1}x_nx_{n+2})(x_3x_4x_{n+1})(x_5x_6x_{n+1}) \\ (x_{n-1}x_nx_{n+1})^{2i-4}(x_7\cdots x_{n-1}).$$

Notice that the last bit $x_7\cdots x_{n-1}$ being the “extra variables”. Thus, $g_i^2 \in L_n^{2i}$, and so $g_i \in \overline{L_n^i}$. The final conclusion is that $g_i \in \overline{L_n^i} \setminus L_n^i$. $\square$

It should be noted that the proof of Theorem 7.4.16 fails for $n=5$ because $5-7$ is negative. The part of the proof showing that $g_i \notin L_5^i$ is valid, but the expression for $g_i^2 \in L_n^2$ does not work for $n=5$. In order to tackle $n=5$, we found the Rees cone helpful. This is reviewed in the next subsection.

The Rees Cone, illustrated by application to L_5

The Hilbert basis of the Rees cone can be computed with Normaliz [24] (as in Example 2.16 of version 3.8.4). A definition of the Rees cone and its properties can

be found in [133, Chapters 13 and 14]. Let I be a monomial ideal in a polynomial ring $\mathbb{Q}[x_1, \dots, x_{n+2}]$ whose minimal generators have exponent vectors $\{v_1, \dots, v_m\}$. Let $\{e_i, 1 \leq i \leq n+3\}$ be the unit vectors in $\mathbb{Q}^{n+3}$ (dimension 1 more than the number of variables). Then the Rees cone $C(I)$ is the rational cone in $\mathbb{Q}^{n+3}$ spanned by the vectors $(v_i, 1), 1 \leq i \leq m$ and $e_i, 1 \leq i \leq n+2$ (leave out e_{n+3}). This has the property that $(v, d) \in C(I)$ if and only if $\mathbf{x}^v \in \overline{I^d}$. The next example is motivated by Remark 7.1.2(iii).

Using Normaliz directly we find that the Hilbert basis of the Rees cone of I_5 is the rows of the matrix

$$\begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \end{array}$$

The rows of this matrix are the $e_i, 1 \leq i \leq 7, (v_i, 1)$ where v_i is the exponent vector of a minimal generator of $x_6 I_5$ (e.g., $(0, 0, 0, 1, 1, 1, 0, 1)$ corresponds to $x_4 x_5 x_6$), there are five of these, and $(0, 1, 0, 1, 1, 0, 1, 1)$ corresponds to $x_2 x_4 x_5 x_7$, a minimal generator of $x_7 J_5$ (there are also five of these). The final row corresponds to $x_1 x_2 x_3 x_4 x_5 x_6 x_7$, which according to the theory of the Rees cone (and Theorem 7.4.14 above) is in $\overline{L_5^2} \setminus L_5^2$.

Theorem 7.4.17. *If $i \geq 3$, then L_5^i is integrally closed.*

Proof. What we must show is that if $(v, i), i \geq 3$ is in the Rees cone of I_5 , then (v, i) is the sum of i rows of the above matrix that end in 1 (possibly with some rows ending in 0). If there is no row $(1, 1, 1, 1, 1, 1, 1, 2)$, then we are done. At

most one $(1, 1, 1, 1, 1, 1, 1, 2)$ is required in this summation, by the proof of Theorem 7.4.14. Thus, it suffices to show that $(1, 1, 1, 1, 1, 1, 1, 2)$ plus any row ending in 1 can be rewritten without the $(1, 1, 1, 1, 1, 1, 1, 2)$. By cyclic symmetry in the variables $\{x_1, \dots, x_5\}$, it suffices to consider one minimal generator of x_6I_5 and one of x_7J_5 . Thus, we have the following equalities

$$\begin{aligned} (1, 1, 1, 1, 1, 1, 1, 2) + (0, 0, 0, 1, 1, 1, 0, 1) &= (1, 1, 1, 2, 2, 2, 1, 3) \\ &= (0, 1, 0, 1, 1, 0, 1, 1) + (1, 0, 0, 0, 1, 1, 0, 1) \\ &\quad + (0, 0, 1, 1, 0, 1, 0, 1), \end{aligned}$$

and

$$\begin{aligned} (1, 1, 1, 1, 1, 1, 1, 2) + (0, 1, 0, 1, 1, 0, 1, 1) &= (1, 2, 1, 2, 2, 1, 2, 3) \\ &= (1, 1, 0, 1, 0, 0, 1, 1) + (0, 1, 1, 0, 1, 0, 1, 1) \\ &\quad + (0, 0, 0, 1, 1, 1, 0, 1). \end{aligned}$$

□

We have already seen that $f = x_1x_2x_3x_4x_5x_6x_7$ is not in L_5^2 . In fact, we have the following stronger result. The proof works for any odd $n \geq 5$, so we state it in that generality.

Theorem 7.4.18. *Let $f = x_1x_2 \cdots x_nx_{n+1}x_{n+2}$, where n is an odd integer ≥ 5 . For any $i \geq 1$, $x_{n+2}^i f \notin L_n^2$, so that $\overline{L_n^2}/L_n^2$ is an infinite dimensional vector space.*

Proof. We have $x_{n+2}^i f = x_1x_2 \cdots x_{n+1}x_{n+2}^{i+1}$. If this is in L_n^2 , then from the $x_{n+1}x_{n+2}^{i+1}$ portion, it must be in either $(x_{n+2}J_n)^2$ or $x_{n+1}I_nx_{n+2}J_n$. It cannot be in the former because all minimal generators of J_n are of degree $\geq (n+1)/2$, and $x_{n+2}^i f$ is only of degree n in $\{x_1, \dots, x_n\}$. It cannot be in $x_{n+1}I_nx_{n+2}J_n$ by the same argument that $f \notin L_5^2$ (based on the proof of Theorem 7.4.14). In the proof of Theorem 7.4.14, it was seen that $f \in \overline{L_n^2}$ and of course $x_{n+2}^i f$ is still in $\overline{L_n^2}$. Finally, note that all ideals involved here are monomial ideals, and those monomials in $\overline{L_n^2}$, but not in L_n^2 form a $\mathbb{Q}$ -basis of $\overline{L_n^2}/L_n^2$. We have found an infinite number of these, and so the conclusion follows. □

We are ready to verify our last outcome in the subsequent theorem.

Theorem 7.4.19. $L_5^2 : \overline{L_5^2} = (x_1, \dots, x_6)$.

Proof. We have $x_1f = x_1^2x_2x_3x_4x_5x_6x_7 = (x_1x_2x_4x_7)(x_1x_5x_6)x_3 \in L_5^2$. By cyclic symmetry in $\{x_1, \dots, x_5\}$, we also obtain $x_2f, x_3f, x_4f, x_5f \in L_5^2$. We also deduce that

$$x_6f = x_1x_2x_3x_4x_5x_6^2x_7 = (x_1x_5x_6)(x_2x_3x_6)x_4x_7 \in L_5^2.$$

As in the proof of Theorem 7.4.17, any monomial in L_5^2 can be written with at most one factor f . It follows now that $(x_1, \dots, x_6) \in L_5^2 : \overline{L_5^2}$. The only monomial ideals between $(x_1, \dots, x_6)$ and (1) are of the form $(x_1, \dots, x_6, x_7^i), i \geq 1$. By virtue of Theorem 7.4.18, we get $x_7^i \notin L_5^2 : \overline{L_5^2}$. This finishes the proof. $\square$

It should be noted that, by using similar techniques, we have been able to prove that $L_4^i : \overline{L_4^i} = (x_1, \dots, x_6)$ for all $i \geq 2$ and $\overline{L_4^i} \setminus L_4^i$ is a finite dimensional vector space for all $i \geq 1$.

7.5 Powers of cover ideals of Jahangir's graphs

The Jahangir's graph $\mathcal{J}_{n,m}$, for $m \geq 3$, is a graph on $nm + 1$ vertices, i.e., a graph consisting of a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm} . More explicitly, it consists of m consecutive cycles of equal length $n+2$ such that all these cycles have one vertex in common and every pair of consecutive cycles has exactly one edge in common. Also, a vertex v in the vertex set of C_{nm} is called *radial* if $\{v, w\}$ is an edge of $\mathcal{J}_{n,m}$, where w denotes the additional vertex of $\mathcal{J}_{n,m}$. Moreover, each edge $\{v, w\}$ is called a *spoke*. In fact, $\mathcal{J}_{n,m}$ has exactly m spokes.

As an example, consider the Figure below. It presents the Jahangir's graphs $\mathcal{J}_{2,8}$, $\mathcal{J}_{5,4}$, and $\mathcal{J}_{3,5}$. The Jahangir's graph $\mathcal{J}_{2,8}$ appears on Jahangir's tomb in his mausoleum, which lies 5 kilometers north-west of Lahore, Pakistan across the River Ravi ([7]).

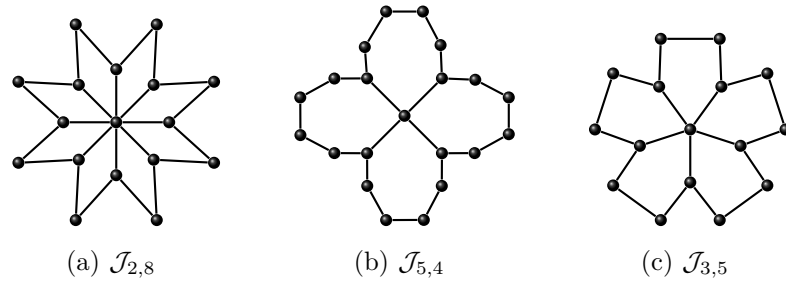


Figure 7.5.1: Jahangir's graphs

In general, Jahangir's graphs have been explored by many people. In [74], algebraic and combinatorial properties and a computation of the number of the spanning trees are developed for a Jahangir's graph. Moreover, some algebraic and combinatorial characterizations of the spanning simplicial complex $\Delta_s(\mathcal{J}_{n,m})$ of the Jahangir's graph $\mathcal{J}_{n,m}$ are examined by Raza et al. in [115]. In addition, it follows from [133, Theorem 7.7.14] that the edge ideals of Jahangir's graphs satisfy the strong persis-

tence property. It should be noted that, based on [133, Corollary 10.5.9], the edge ideals of Jahangir's graphs are normal.

In this section, we concentrate on the cover ideals of Jahangir's graphs. Our first main aim is to study the normality of the cover ideals of Jahangir's graphs. In fact, we show that the cover ideals of Jahangir's graphs are normal, see Theorem 7.5.2. The second main goal of us is to study the associated primes of the powers of cover ideals of Jahangir's graphs. Indeed, we demonstrate that the cover ideals of Jahangir's graphs satisfy the (symbolic) (strong) persistence property, see Corollary 7.5.3. In the third main result of this section, we determine when the unique homogeneous maximal ideal appears in the set of associated primes of powers of the cover ideals of Jahangir's graphs, refer to Theorem 7.5.7. It should be noted that the results of this section can be found in [5].

We continue with the following auxiliary result, which we require it to establish Theorem 7.5.2. In addition, throughout the following text, if G is a finite simple graph, then $J(G)$ stands for the cover ideal of G .

Proposition 7.5.1. *The cover ideals of cycle graphs are normal.*

Proof. Let C_n denote a cycle graph of order n . If n is even, then C_n is bipartite. It follows from [48, Corollary 2.6] that $J(C_n)$ is normally torsion-free. Now, [60, Theorem 1.4.6] yields that $J(C_n)$ is normal. If n is odd, then C_n is an odd cycle graph, and the claim can be deduce immediately from Theorem 7.1.7. $\square$

We are now in a position to state the first main result of this section in the following theorem.

Theorem 7.5.2. *The cover ideals of the Jahangir's graphs are normal.*

Proof. Assume that $\mathcal{J}_{n,m}$ is the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \dots, nm\}$, 0 is the additional vertex, and the following edge set

$$E(\mathcal{J}_{n,m}) = \{\{i, i+1\} : i = 1, \dots, nm-1\} \cup \{\{nm, 1\}\} \\ \cup \{\{0, 1+rn\} : r = 0, 1, \dots, m-1\}.$$

Let $R = K[x_1, \dots, x_{nm}]$, $u := \prod_{r=0}^{m-1} x_{1+rn}$, and $S = R[x_0]$. Let $J(\mathcal{J}_{n,m})$ (respectively, $J(C_{nm})$) denote the cover ideal of $\mathcal{J}_{n,m}$ (respectively, C_{nm}). To simplify our notation, set $L := J(\mathcal{J}_{n,m})$, $I := J(C_{nm})$, and $F := (I :_S u)$. Clearly, $L = I \cap (u, x_0) \subset S$. Because $I \cap (u) = u(I :_S u)$ and $I \cap (x_0) = x_0 I$, we derive that $L = u(I :_S u) + x_0 I$; thus, $L = uF + x_0 I$. Moreover, based on Proposition 7.5.1, the ideal I is normal. On account of I is square-free, then the ideal F is obtained from I by putting $x_{1+rn} = 1$ for all $r = 0, 1, \dots, m-1$. Hence, F is normal by [133, Proposition 12.2.3], and so uF is normal according to Remark 7.1.1. Due to $F = (I :_S u)$,

one gains $uF + I = I$; therefore, $uF + I$ is normal. Since $\gcd(v, x_0) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(uF)$, it follows from Theorem 6.5.1 that L is normal. This completes the proof. $\square$

As an application of Theorem 7.5.2, we can present the second main result in the next corollary.

Corollary 7.5.3. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph, and $J(\mathcal{J}_{n,m})$ its cover ideal. Then the following statements hold.*

- (i) $J(\mathcal{J}_{n,m})$ has the strong persistence property.
- (ii) $J(\mathcal{J}_{n,m})$ has the persistence property.
- (iii) $J(\mathcal{J}_{n,m})$ has the symbolic strong persistence property.

Proof. (i) - (iii) can be shown similar to the proof of Corollary 6.2.3. $\square$

Here, we give the following proposition, which helps us to demonstrate Lemma 7.5.5.

Proposition 7.5.4. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph, n odd, and W a vertex cover set of $\mathcal{J}_{n,m}$. If W does not contain the additional vertex, then $|W| \geq m(n+1)/2$. If W contains the additional vertex, then $|W| \geq (nm+3)/2$ if m is odd, or $|W| \geq (nm+2)/2$ if m is even.*

Proof. Let w denote the additional vertex of $\mathcal{J}_{n,m}$. First, assume that $w \notin W$. Let $v_1, \dots, v_m$ be the radial vertices of $\mathcal{J}_{n,m}$. It follows from $w \notin W$ that $v_i \in W$ for $i = 1, \dots, m$. Since $W \cap V(C_{nm})$ is a vertex cover of C_{nm} , and $v_i, v_{i+1} \in W \cap V(C_{nm})$, we deduce that in the path from v_i to v_{i+1} , one needs at least $(n-1)/2$ vertices for $i = 1, \dots, m$, where v_{m+1} represents v_1 . We therefore gain $|W| \geq m + m(n-1)/2$, and so $|W| \geq m(n+1)/2$. Now, let $w \in W$. Then $W \cap V(C_{nm})$ must be a vertex cover of C_{nm} . If m is odd, then C_{nm} is an odd cycle, and so $W \cap V(C_{nm})$ has at least $(nm+1)/2$ elements; thus, we get $|W| \geq 1 + (nm+1)/2$, and hence $|W| \geq (nm+3)/2$. If m is even, then C_{nm} is an even cycle. Accordingly, $W \cap V(C_{nm})$ has at least $nm/2$ elements, and so $|W| \geq (nm+2)/2$. This completes the proof. $\square$

In order to prove Theorem 7.5.7, we require the subsequent lemma.

Lemma 7.5.5. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \dots, nm\}$, 0 be the additional vertex, and the following edge set*

$$E(\mathcal{J}_{n,m}) = \{\{i, i+1\} : i = 1, \dots, nm-1\} \cup \{\{nm, 1\}\} \\ \cup \{\{0, 1+kn\} : k = 0, 1, \dots, m-1\}.$$

Let $C_{2e_1+1}, \dots, C_{2e_z+1}$ be all induced odd cycle subgraphs of $\mathcal{J}_{n,m}$. Then, for each positive integer s , if $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in J(\mathcal{J}_{n,m})^s$, then the following conditions hold:

- (i) $\ell_\alpha + \ell_\beta \geq s$ for each $\{\alpha, \beta\} \in E(\mathcal{J}_{n,m})$;
- (ii) $\sum_{\{\alpha, \beta\} \in E(C_{2e_j+1})} ((\ell_\alpha + \ell_\beta) - s) \geq s$ for each $j = 1, \dots, z$;
- (iii) If n and m are odd, then $((m-1)/2)\ell_0 + \sum_{i=1}^{nm} \ell_i \geq sm(n+1)/2$.

Proof. Fix $s \geq 1$. To verify the claim, consider a monomial $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in J(\mathcal{J}_{n,m})^s$. Take an arbitrary edge $\{\alpha, \beta\}$ in $E(\mathcal{J}_{n,m})$. Since $J(\mathcal{J}_{n,m}) \subseteq (x_\alpha, x_\beta)$, this implies that $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in (x_\alpha, x_\beta)^s$, and so $\ell_\alpha + \ell_\beta \geq s$. This proves condition (i). By virtue of $J(\mathcal{J}_{n,m})^s \subseteq J(C_{2e_j+1})^s$ for all j , we have $\prod_{i \in V(C_{2e_j+1})} x_i^{\ell_i} \in J(C_{2e_j+1})^s$ for all j , where $J(C_{2e_j+1})$ is the cover ideal of C_{2e_j+1} . Therefore, condition (ii) follows directly from Lemma 3.1.10. It follows from $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} \in J(\mathcal{J}_{n,m})^s$ that there exist monomials $g_1, \dots, g_s \in \mathcal{G}(J(\mathcal{J}_{n,m}))$ and some monomial h in $K[x_0, x_1, \dots, x_{nm}]$ such that $x_0^{\ell_0} x_1^{\ell_1} \cdots x_{nm}^{\ell_{nm}} = g_1 \cdots g_s h$. Note that each g_i corresponds to a minimal vertex cover set of $\mathcal{J}_{n,m}$. After reordering, there exists some integer $\lambda \leq \ell_0$ such that $x_0 \mid g_i$ for $i = 1, \dots, \lambda$, and $x_0 \nmid g_i$ for $i = \lambda + 1, \dots, s$. If n and m are odd, then one can derive from Proposition 7.5.4 that $\deg(g_i) \geq (nm+3)/2$ for $i = 1, \dots, \lambda$, and also $\deg(g_i) \geq m(n+1)/2$ for $i = \lambda + 1, \dots, s$. Note that $m \geq 3$ and $\lambda \leq \ell_0$. Hence, we get the following

$$\begin{aligned} 2 \sum_{i=0}^{nm} \ell_i &\geq \lambda(nm+3) + (s-\lambda)m(n+1) \\ &= smn + sm - \lambda(m-3) \\ &\geq smn + sm - \ell_0(m-3). \end{aligned}$$

Consequently, we deduce that $((m-1)/2)\ell_0 + \sum_{i=1}^{nm} \ell_i \geq sm(n+1)/2$. Therefore, one concludes that condition (iii) holds. This finishes the proof. $\square$

Remark 7.5.6. Assume that $\mathcal{J}_{n,m}$ is the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \dots, nm\}$ with labeling counterclockwise, 0 is the additional vertex, $\mathfrak{m} = (x_0, x_1, \dots, x_{nm})$ the unique homogeneous maximal ideal, and $J(\mathcal{J}_{n,m})$ its cover ideal. If n is an even number, then $\mathcal{J}_{n,m}$ has no any odd cycle, and so $\mathcal{J}_{n,m}$ is a bipartite graph. This implies that $J(\mathcal{J}_{n,m})$ is normally torsion-free, and so the unique homogeneous maximal ideal does not appear in the set of associated primes of powers of the cover ideal of the Jahangir's graph, that is, $\mathfrak{m} \notin \text{Ass}(J(\mathcal{J}_{n,m})^s)$ for all $s \geq 1$.

Now, assume that n is an odd number and m is an even number. Our computations by Macaulay2 [49] show that the unique homogeneous maximal ideal $\mathfrak{m}$ does not appear in the set of associated primes of powers of the cover ideal of the Jahangir's graph, that is, $\mathfrak{m} \notin \text{Ass}(J(\mathcal{J}_{n,m})^s)$ for all $s \geq 1$. In particular, it should be noted that

in this case, $J(\mathcal{J}_{n,m})/\{x_0\}$ is exactly the cover ideal of the rim, and since m is even, this yields that the rim is an even cycle graph, and hence $J(\mathcal{J}_{n,m})/\{x_0\}$ is normally torsion-free. Nevertheless, we need to discuss the case in which n is an odd number and m is an even number separately. In what follows, we explore the case in which m, n are odd numbers.

We are ready to give the third main result of this section in the next theorem. Indeed, we determine when the unique homogeneous maximal ideal appears in the set of associated primes of powers of the cover ideals of Jahangir's graphs.

Theorem 7.5.7. *Let $\mathcal{J}_{n,m}$ be the Jahangir's graph, $J(\mathcal{J}_{n,m})$ its cover ideal, and m, n odd numbers. Then, for all $s \geq 3$,*

$$(x_i : i \in V(\mathcal{J}_{n,m})) \in \text{Ass}(J(\mathcal{J}_{n,m})^s) \setminus (\text{Ass}(J(\mathcal{J}_{n,m})) \cup \text{Ass}(J(\mathcal{J}_{n,m})^2)).$$

Proof. Without loss of generality, assume that $\mathcal{J}_{n,m}$ is the Jahangir's graph with the vertex set $V(\mathcal{J}_{n,m}) = \{0, 1, 2, \dots, nm\}$ with labeling counterclockwise, 0 is the additional vertex, $\mathbf{m} = (x_0, x_1, \dots, x_{nm})$, and the following edge set

$$E(\mathcal{J}_{n,m}) = \{\{i, i+1\} : i = 1, \dots, nm-1\} \cup \{\{nm, 1\}\} \\ \cup \{\{0, 1+kn\} : k = 0, 1, \dots, m-1\}.$$

Set $L := J(\mathcal{J}_{n,m})$. Since L is a square-free monomial ideal, we have $\text{Ass}(L) = \text{Min}(L)$, and so $\mathbf{m} \notin \text{Ass}(L)$. In addition, it follows from [44, Corollary 3.4] that $\mathbf{m} \notin \text{Ass}(L^2)$. According to Corollary 7.5.3, the ideal L satisfies the persistence property. Hence, it is sufficient for us to show that $\mathbf{m} \in \text{Ass}(J(\mathcal{J}_{n,m})^3)$. To accomplish this, set

$$h := x_0^2 \prod_{k=0}^{m-1} \prod_{j=1}^{n-1} x_{1+kn}^\lambda x_{1+kn+j}^\lambda, \text{ where } \lambda = \begin{cases} 1 & \text{if } j \text{ is odd,} \\ 2 & \text{if } j \text{ is even.} \end{cases}$$

In what follows, our aim is to show that $h \notin L^3$ and $x_i h \in L^3$ for each $i = 0, 1, \dots, nm$. Suppose, on the contrary, that $h \in L^3$. One can promptly deduce from Lemma 7.5.5(iii) that

$$(7.5.1) \quad ((m-1)/2)(2) + \sum_{i=1}^{nm} \deg_{x_i} h \geq 3m(n+1)/2.$$

Note that there exist exactly $n-1$ numbers between any two consecutive radial vertices, and by virtue of n is odd, this implies that there exist $(n-1)/2$ odd numbers and also $(n-1)/2$ even numbers between any two consecutive radial vertices. We thus get

$$\begin{aligned} ((m-1)/2)(2) + \sum_{i=1}^{nm} \deg_{x_i} h &= m-1 + 2m + 2m(n-1)/2 + m(n-1)/2 \\ &= (3m(n+1) - 2)/2, \end{aligned}$$

which contradicts (7.5.1). Accordingly, we obtain $h \notin L^3$. To complete the proof, one has to verify that $x_i h \in L^3$ for each $i = 0, 1, \dots, nm$. To achieve this, fix $1 \leq t \leq nm$. We relabel the vertices of the odd cycle C_{nm} starting from $t = y_1$ counterclockwise $y_2, y_3, \dots, y_{nm}$. Now, set

$$u_1 := x_0 \prod_{i \text{ is odd}} y_i, \quad u_2 := x_0 y_1 \prod_{i \text{ is even}} y_i, \quad u_3 := \prod_{k=0}^{m-1} \prod_{\substack{j=1, \dots, n-1, \\ \text{where } j \text{ is even}}} x_{1+kn} x_{1+kn+j}.$$

It is not hard to check that $x_t h = u_1 u_2 u_3$, and also u_1, u_2 , and u_3 correspond to vertex cover sets of $\mathcal{J}_{n,m}$. This yields that $u_1, u_2, u_3 \in L$, and so $x_t h \in L^3$. It remains to show that $x_0 h \in L^3$. To do this, put

$$v_1 := x_0 \prod_{i \text{ is odd}} x_i, \quad v_2 := x_0 \prod_{\substack{i=n+1, \dots, nm, \\ \text{where } i \text{ is even}}} x_i \prod_{\substack{i=1, \dots, n+1, \\ \text{where } i \text{ is odd}}} x_i, \quad \text{and} \\ v_3 := x_0 \prod_{\substack{i=1, \dots, n, \\ \text{where } i \text{ is even}}} x_i \prod_{k=1}^{m-1} \prod_{\substack{j=1, \dots, n-1, \\ \text{where } j \text{ is even}}} x_{1+kn} x_{1+kn+j}.$$

It is routine to see that $x_0 h = v_1 v_2 v_3$, and also v_1, v_2 , and v_3 correspond to vertex cover sets of $\mathcal{J}_{n,m}$. We thus gain $v_1, v_2, v_3 \in L$, and hence $x_0 h \in L^3$. This terminates the argument. $\square$

As an immediate consequence of Theorem 7.5.7, we give the following result:

Corollary 7.5.8. *Let $G = (V(G), E(G))$ be a finite simple graph with the vertex set $V(G) = \{x_1, \dots, x_n\}$, and let G have a subgraph $\mathcal{J}_{n,m}$ as the Jahangir's graph, where m, n be odd numbers. Let $R = K[x_1, \dots, x_n]$ and J denote the cover ideal of G . Then, for all $s \geq 3$,*

$$(x_i : i \in V(\mathcal{J}_{n,m})) \in \text{Ass}(R/J^s) \setminus (\text{Ass}(R/J) \cup \text{Ass}(R/J^2)).$$

Proof. This claim can be deduced from [45, Lemma 2.11] and Theorem 7.5.7. $\square$

7.6 Powers of cover ideals of theta graphs

Studying the strong persistence property and normality of the cover ideals of theta graphs are the major points of this section. It should be noted that the results of this section can be found in [6].

In order to formulate the outcomes of this section, one has to recall the definition of simple graphs of the form $\theta_{n_1, \dots, n_k}$ which were introduced in [92].

Definition 7.6.1. Let $k > 1$ be an integer, and $n_1, \dots, n_k$ be a sequence of positive integers. Then $\theta_{n_1, \dots, n_k}$ is the graph constructed by k paths of length $n_1, \dots, n_k$ such that only their endpoints being in common. By length of a path, we mean the number of edges in the path. We call $\theta_{n_1, \dots, n_k}$ the theta graph of order $n_1, \dots, n_k$.

As an example, consider the figure below. It presents the theta graph $\theta_{3,1,4,2}$ of order 3, 1, 4, 2, the left-hand side graph in Figure 1, and the theta graph $\theta_{3,4,2,3}$ of order 3, 4, 2, 3, the right-hand side graph in Figure 1.

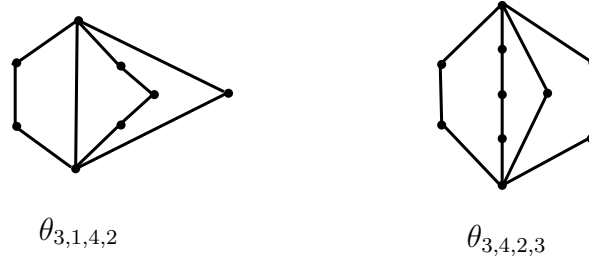


Figure 1

Remark 7.6.2. If $k = 2$, then $\theta_{n_1, \dots, n_k}$ is a cycle of length $n_1 + n_2$; thus, theta graphs are a generalization of cycle graphs. Furthermore, it should be noted that since we assume $\theta_{n_1, \dots, n_k}$ is a simple graph, there exists at most one i with $1 \leq i \leq k$ such that $n_i = 1$.

7.6.1 Strong persistence property of cover ideals of theta graphs

In this subsection, our aim is to explore the strong persistence property of the cover ideals of theta graphs. To prove Theorem 7.6.4, we have to employ the following auxiliary theorem.

Theorem 7.6.3. With the notation of Definition 7.6.1, let $\theta_{n_1, \dots, n_k}$ be a theta graph of order $n_1, \dots, n_k$ with $V(\theta_{n_1, \dots, n_k}) = \{1, \dots, m\}$ and cover ideal $J(\theta_{n_1, \dots, n_k}) \subset R = K[x_1, \dots, x_m]$ over a field K . Let $C_{2e_1+1}, \dots, C_{2e_z+1}$ be all induced odd cycle subgraphs of $\theta_{n_1, \dots, n_k}$. Then, for each positive integer s , $x_1^{\ell_1} \cdots x_m^{\ell_m} \in (J(\theta_{n_1, \dots, n_k}))^s$ if and only if the following conditions hold:

- (i) $\ell_\alpha + \ell_\beta \geq s$ for each $\{\alpha, \beta\} \in E(\theta_{n_1, \dots, n_k})$;
- (ii) $\sum_{\{\alpha, \beta\} \in E(C_{2e_j+1})} [(\ell_\alpha + \ell_\beta) - s] \geq s$ for each $j = 1, \dots, z$.

Proof. To simplify the notation, set $G := \theta_{n_1, \dots, n_k}$. Fix $s \geq 1$. The forward implication can be deduced from Lemma 3.1.10. Conversely, assume that a monomial $x_1^{\ell_1} \cdots x_m^{\ell_m}$ satisfies the conditions (i) and (ii). We show that $x_1^{\ell_1} \cdots x_m^{\ell_m} \in J(G)^s$. Let $z = 0$. This means that there is no induced odd cycle subgraph, and hence G is bipartite. By virtue of [48, Corollary 2.6], $J(G)$ is normally torsion-free; thus, [60, Theorem 1.4.6] implies that $J(G)^s = J(G)^{(s)}$, where $J(G)^{(s)}$ denotes the s -th symbolic power of $J(G)$. Because $J(G)^{(s)} = \bigcap_{\mathfrak{p} \in \text{Min}(J(G))} \mathfrak{p}^s$, this yields that $J(G)^s = \bigcap_{\mathfrak{p} \in \text{Min}(J(G))} \mathfrak{p}^s$. Thanks to the monomial $x_1^{\ell_1} \cdots x_m^{\ell_m}$ satisfies the condition (i), one obtains $\ell_\alpha + \ell_\beta \geq s$ for each $\{\alpha, \beta\} \in E(G)$, and so $x_1^{\ell_1} \cdots x_m^{\ell_m} \in (x_\alpha, x_\beta)^s$ for each $\{\alpha, \beta\} \in E(G)$. In the light of

$$\text{Min}(J(G)) = \{(x_\alpha, x_\beta) : \text{ for all } \{\alpha, \beta\} \in E(G)\},$$

this gives rise to $x_1^{\ell_1} \cdots x_m^{\ell_m} \in \bigcap_{\mathfrak{p} \in \text{Min}(J(G))} \mathfrak{p}^s$, and hence $x_1^{\ell_1} \cdots x_m^{\ell_m} \in J(G)^s$. We therefore assume that $z \geq 1$. Let C be an induced odd cycle subgraph of G . Without loss of generality, one may assume that $V(C) \cap \{1, \dots, m\} = \{1, \dots, t\}$. Since $x_1^{\ell_1} \cdots x_m^{\ell_m}$ satisfies the conditions (i) and (ii), Lemma 3.1.10 implies that $x_1^{\ell_1} \cdots x_t^{\ell_t} \in J(C)^s$. Consequently, there exist monomials $g_1, \dots, g_s \in \mathcal{G}(J(C))$ and some monomial h in R such that $x_1^{\ell_1} \cdots x_t^{\ell_t} = g_1 \cdots g_s h$. Note that any minimal generator of $J(G)$ (respectively, $J(C)$) corresponds to a minimal vertex cover set of G (respectively, C). Hence, to finish the proof, one has to verify that $g_1 \cdots g_s h x_{t+1}^{\ell_{t+1}} \cdots x_m^{\ell_m}$ can be written as $f_1 \cdots f_s h'$ such that each f_i corresponds to a vertex cover set of G . To accomplish this, let $e = \{\alpha, \beta\}$ be an edge in $E(G) \setminus E(C)$. By the condition (i), we have $\ell_\alpha + \ell_\beta \geq s$. This enables us to cover the edge $e = \{\alpha, \beta\}$ at least s times. Accordingly, one can derive that $x_1^{\ell_1} \cdots x_m^{\ell_m} \in J(G)^s$, and the proof is done. $\square$

We are now in a position to express and prove Theorem 7.6.4.

Theorem 7.6.4. *With the notation of Definition 7.6.1, let $\theta_{n_1, \dots, n_k}$ be a theta graph of order $n_1, \dots, n_k$ with $V(\theta_{n_1, \dots, n_k}) = \{1, \dots, m\}$ and the cover ideal $J(\theta_{n_1, \dots, n_k})$ in $R = K[x_1, \dots, x_m]$ over a field K . Let $C_{2e_1+1}, \dots, C_{2e_z+1}$ be all induced odd cycle subgraphs of $\theta_{n_1, \dots, n_k}$. Then, for all $s \geq 2$,*

$$J(\theta_{n_1, \dots, n_k})^s = \bigcap_{\{\alpha, \beta\} \in E(\theta_{n_1, \dots, n_k})} (x_\alpha, x_\beta)^s \cap \bigcap_{j=1}^z (x_i : i \in V(C_{2e_j+1}))^{s(e_j+1)},$$

is a minimal primary decomposition of $J(\theta_{n_1, \dots, n_k})^s$.

Proof. Fix $s \geq 2$, and put $G := \theta_{n_1, \dots, n_k}$. Because $J(G) = \bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta)$, we need only prove the following equality

$$(7.6.1) \quad \left(\bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta) \right)^s = \bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta)^s \cap \bigcap_{j=1}^z (x_i : i \in V(C_{2e_j+1}))^{s(e_j+1)}.$$

Assume A (respectively, B) denotes the ideal on the left-hand (respectively, right-hand) side of (7.6.1). We first establish $A \subseteq B$. Since, for any $\{\alpha, \beta\} \in E(G)$, $\bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta) \subseteq (x_\alpha, x_\beta)$, this yields the following containment

$$\left(\bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta) \right)^s \subseteq \bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta)^s.$$

Fix $1 \leq j \leq z$. In order to complete the proof, we have to verify that

$$\left(\bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta) \right)^s \subseteq (x_i : i \in V(C_{2e_j+1}))^{s(e_j+1)}.$$

Thanks to $\bigcap_{\{\alpha, \beta\} \in E(G)} (x_\alpha, x_\beta) \subseteq \bigcap_{\{\alpha, \beta\} \in E(C_{2e_j+1})} (x_\alpha, x_\beta)$, it is enough for us to show that $(\bigcap_{\{\alpha, \beta\} \in E(C_{2e_j+1})} (x_\alpha, x_\beta))^s \subseteq (x_i : i \in V(C_{2e_j+1}))^{s(e_j+1)}$. To do this, pick a minimal generator u in $(\bigcap_{\{\alpha, \beta\} \in E(C_{2e_j+1})} (x_\alpha, x_\beta))^s$. This implies that $u = \prod_{i=1}^s g_i$, where each g_i is a minimal generator of $\bigcap_{\{\alpha, \beta\} \in E(C_{2e_j+1})} (x_\alpha, x_\beta)$. Due to $\bigcap_{\{\alpha, \beta\} \in E(C_{2e_j+1})} (x_\alpha, x_\beta)$ is exactly the cover ideal of the odd cycle C_{2e_j+1} and since any minimal generator of $J(C_{2e_j+1})$ corresponds to a minimal vertex cover of C_{2e_j+1} , and also by observing the fact that any minimal vertex cover of C_{2e_j+1} has at least $e_j + 1$ elements, one derives that $\deg u = \sum_{i=1}^s \deg g_i \geq s(e_j + 1)$. This gives that $u \in (x_i : i \in V(C_{2e_j+1}))^{s(e_j+1)}$. Accordingly, we get $A \subseteq B$.

We now demonstrate that $B \subseteq A$. Take a minimal generator u in B . Let $u := x_1^{\ell_1} \cdots x_m^{\ell_m}$ with $\ell_i \geq 0$ for each $i = 1, \dots, m$. Our strategy for completing the proof is to use Theorem 7.6.3. It follows at once from $u \in B$ that $u \in (x_\alpha, x_\beta)^s$ for each $(\alpha, \beta) \in E(G)$. We thus have $x_\alpha^\lambda x_\beta^{s-\lambda} | u$ for some $0 \leq \lambda \leq s$. This leads to $\ell_\alpha + \ell_\beta \geq s$. To finish the argument, we must establish that, for each $j = 1, \dots, z$,

$$\sum_{\{\alpha, \beta\} \in E(C_{2e_j+1})} [(\ell_\alpha + \ell_\beta) - s] \geq s.$$

To achieve this, fix $1 \leq j \leq z$. In the light of $u \in B$, we can deduce that $u \in (x_i : i \in V(C_{2e_j+1}))^{s(e_j+1)}$. Without loss of generality, one may assume that $V(C_{2e_j+1}) = \{1, \dots, 2e_j + 1\}$ and $E(C_{2e_j+1}) = \{\{i, i+1\} : i = 1, \dots, 2e_j + 1\}$ with $2e_j + 2 = 1$. Consequently, there exist nonnegative integers $\theta_1, \dots, \theta_{2e_j+1}$ with $\sum_{i=1}^{2e_j+1} \theta_i = s(e_j + 1)$ such that $x_1^{\theta_1} \cdots x_{2e_j+1}^{\theta_{2e_j+1}} | u$. This gives rise to

$$\begin{aligned} \sum_{\{\alpha, \beta\} \in E(C_{2e_j+1})} [(\ell_\alpha + \ell_\beta) - s] &= \sum_{i=1}^{2e_j+1} [(\ell_i + \ell_{i+1})] - s(2e_j + 1) \\ &= 2 \sum_{i=1}^{2e_j+1} \ell_i - s(2e_j + 1) \\ &\geq 2 \sum_{i=1}^{2e_j+1} \theta_i - s(2e_j + 1) \\ &= s. \end{aligned}$$

One can deduce from Theorem 7.6.3 that $u \in A$; thus, $B \subseteq A$, as required. $\square$

The corollaries below are some immediate consequences of Theorem 7.6.4.

Corollary 7.6.5. *With the notation of Definition 7.6.1, let $\theta_{n_1, \dots, n_k}$ be a theta graph of order $n_1, \dots, n_k$. Then, for all positive integers $s \geq 2$, we have*

$$\begin{aligned} \text{Ass}(J(\theta_{n_1, \dots, n_k})^s) = & \text{Min}(J(\theta_{n_1, \dots, n_k})) \\ & \cup \{(x_{j_1}, \dots, x_{j_c}) : \text{induced graph on } \{j_1, \dots, j_c\} \text{ is an odd cycle}\}. \end{aligned}$$

Corollary 7.6.6. *With the notation of Definition 7.6.1, let $\theta_{n_1, \dots, n_k}$ be a theta graph of order $n_1, \dots, n_k$, $k > 2$, and let $\mathfrak{m}$ be the unique homogeneous maximal ideal of $R = K[x_\alpha : \alpha \in V(\theta_{n_1, \dots, n_k})]$. Then $\mathfrak{m} \notin \text{Ass}_R(R/J(\theta_{n_1, \dots, n_k})^s)$ for all $s \geq 1$.*

Corollary 7.6.7. *Let C be a cycle graph with a path between two vertices of the cycle C . Then, for all positive integers $s \geq 2$,*

$$\begin{aligned} \text{Ass}(J(C)^s) = & \text{Min}(J(C)) \\ & \cup \{(x_{i_1}, \dots, x_{i_r}) : \text{induced graph on } \{i_1, \dots, i_r\} \text{ is an odd cycle}\}. \end{aligned}$$

To illustrate Corollary 7.6.5, we give the following example.

Example 7.6.8. *We now number the vertices of the graphs in Figure 1, as are illustrated by Figure 2 below.*

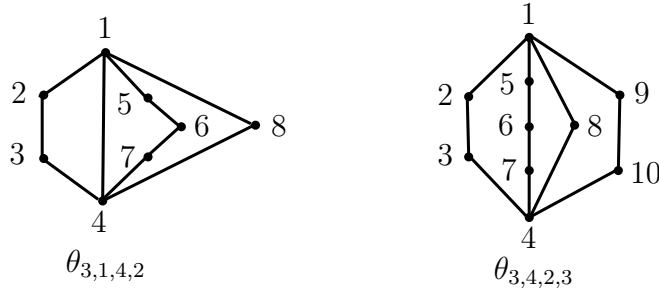


Figure 2

It is routine to check that there are only two induced odd cycle subgraphs of $\theta_{3,1,4,2}$. In fact, the first odd cycle is 1, 5, 6, 7, 4, 1 and the second odd cycle is 1, 8, 4, 1. Note that the induced subgraph of $\theta_{3,1,4,2}$ on $\{1, 2, 3, 4, 5, 6, 7\}$ (respectively, $\{1, 2, 3, 4, 8\}$) is not an odd cycle since it has the chord $\{1, 4\}$, while any cycle graph has no chord. In the light of Corollary 7.6.5, for all $s \geq 2$, one can derive that

$$\begin{aligned} \text{Ass}(J(\theta_{3,1,4,2})^s) = & \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_1, x_4), (x_1, x_5), (x_5, x_6), \\ & (x_6, x_7), (x_7, x_4), (x_1, x_8), (x_4, x_8), (x_1, x_8, x_4), \\ & (x_1, x_5, x_6, x_7, x_4)\}. \end{aligned}$$

In addition, one can easily see that there are exactly four induced odd cycle subgraphs C_1, C_2, C_3 , and C_4 such that $V(C_1) = \{1, 2, 3, 4, 5, 6, 7\}$, $V(C_2) = \{1, 2, 3, 4, 8\}$, $V(C_3) = \{1, 5, 6, 7, 4, 9, 10\}$, and $V(C_4) = \{1, 8, 4, 9, 10\}$. In view of Corollary 7.6.5, for all $s \geq 2$, we deduce that

$$\begin{aligned} \text{Ass}(J(\theta_{3,4,2,3})^s) = & \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_1, x_5), (x_5, x_6), (x_6, x_7), \\ & (x_7, x_4), (x_1, x_8), (x_8, x_4), (x_1, x_9), (x_9, x_{10}), (x_{10}, x_4), \\ & (x_1, x_2, x_3, x_4, x_8), (x_1, x_5, x_6, x_7, x_4, x_9, x_{10}), \\ & (x_1, x_8, x_4, x_9, x_{10}), (x_1, x_2, x_3, x_4, x_5, x_6, x_7)\}. \end{aligned}$$

We are ready to state and prove Theorem 7.6.9 as the main theorem of this section.

Theorem 7.6.9. *With the notation of Definition 7.6.1, let $\theta_{n_1, \dots, n_k}$ be a theta graph of order $n_1, \dots, n_k$ with the cover ideal $J(\theta_{n_1, \dots, n_k})$. Then $J(\theta_{n_1, \dots, n_k})$ has the strong persistence property.*

Proof. For convenience of notation, put $L := J(\theta_{n_1, \dots, n_k})$. In addition, let a and b be the endpoint vertices of the path graphs. Want to show that $(L^{m+1} :_R L) = L^m$ for all positive integers m , where $R = K[x_\alpha : \alpha \in V(\theta_{n_1, \dots, n_k})]$. Fix $m \geq 1$. By [89, Exercise 6.4], it is enough to prove that $(L_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} L_{\mathfrak{p}}) = L_{\mathfrak{p}}^m$ for every $\mathfrak{p} \in \text{Ass}_R(R/L^m)$. In view of Corollary 7.6.5, we may consider the following cases:

Case 1. $\mathfrak{p} \in \text{Min}(L^m)$. Because $\text{Min}(L^m) = \text{Min}(L)$, one can conclude from Proposition 3.3.17 that $L_{\mathfrak{p}}^m = \mathfrak{p}_{\mathfrak{p}}^m$ and $L_{\mathfrak{p}}^{m+1} = \mathfrak{p}_{\mathfrak{p}}^{m+1}$, and so $(L_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} L_{\mathfrak{p}}) = (\mathfrak{p}_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} \mathfrak{p}_{\mathfrak{p}})$. In view of Proposition 4.3.2, $\mathfrak{p}$ has the strong persistence property, that is, $(\mathfrak{p}^{m+1} :_R \mathfrak{p}) = \mathfrak{p}^m$, and hence $(\mathfrak{p}_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} \mathfrak{p}_{\mathfrak{p}}) = \mathfrak{p}_{\mathfrak{p}}^m$. This yields that $(L_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} L_{\mathfrak{p}}) = L_{\mathfrak{p}}^m$.

Case 2. $\mathfrak{p} \in \text{Ass}_R(R/L^m) \setminus \text{Min}(L^m)$. It follows readily from Corollary 7.6.5 that $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r})$ such that the induced subgraph on $\{i_1, \dots, i_r\}$ is an odd cycle, say C_r . Now, set $x_e := (x_\lambda, x_\rho)$ for each edge $e = \{\lambda, \rho\} \in E(\theta_{n_1, \dots, n_k})$. We thus have the following equalities

$$L_{\mathfrak{p}} = \left(\bigcap_{e \in E(\theta_{n_1, \dots, n_k})} x_e \right)_{\mathfrak{p}} = \bigcap_{e \in E(\theta_{n_1, \dots, n_k})} (x_e)_{\mathfrak{p}}.$$

Here, one can consider the following subcases:

Subcase 2.1. $\{a, b\} \in E(C_r)$. Pick an edge $e = \{\alpha, \beta\} \in E(\theta_{n_1, \dots, n_k}) \setminus E(C_r)$. Want to demonstrate that $(x_e)_{\mathfrak{p}} = R_{\mathfrak{p}}$. To see this, it is sufficient for us to verify that $\mathfrak{p} \notin \text{Supp}(R/x_e)$. Suppose, on the contrary, that $\mathfrak{p} \in \text{Supp}(R/x_e)$. Because $\text{Supp}(R/x_e) = V(\text{Ann}(R/x_e))$, this yields $\mathfrak{p} \in V(\text{Ann}(R/x_e))$, and so $\mathfrak{p} \in V(x_e)$.

Hence, $x_e \subseteq \mathfrak{p}$. By virtue of $e \notin E(C_r)$ and $\{a, b\} \in E(C_r)$, this gives rise to a contradiction. We therefore get $\mathfrak{p} \notin \text{Supp}(R/x_e)$, and thus $(x_e)_{\mathfrak{p}} = R_{\mathfrak{p}}$. Accordingly, one can conclude the following equalities

$$L_{\mathfrak{p}} = \bigcap_{e \in E(C_r)} (x_e)_{\mathfrak{p}} = \left(\bigcap_{e \in E(C_r)} x_e \right)_{\mathfrak{p}}.$$

Note that $\bigcap_{e \in E(C_r)} x_e$ is equal to the cover ideal of the induced graph on $\{i_1, \dots, i_r\}$, that is, $\bigcap_{e \in E(C_r)} x_e = J(C_r)$. It follows from Proposition 3.3.26(i) that $J(C_r)$ has the strong persistence property, that is, $((J(C_r))^{m+1} :_R J(C_r)) = (J(C_r))^m$. This implies that $((J(C_r))_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} J(C_r)_{\mathfrak{p}}) = (J(C_r)_{\mathfrak{p}})^m$, and so $(L_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} L_{\mathfrak{p}}) = L_{\mathfrak{p}}^m$. We therefore deduce that $L_{\mathfrak{p}}$ has the strong persistence property, as required.

Subcase 2.2. $\{a, b\} \in E(\theta_{n_1, \dots, n_k}) \setminus E(C_r)$. A similar argument gives the following equality

$$L_{\mathfrak{p}} = ((x_a, x_b) \cap \bigcap_{e \in E(C_r)} x_e)_{\mathfrak{p}}.$$

Notice that $\{a, b\} \cup E(C_r)$ can be viewed as the edge set of the cycle graph C_r with one chord $\{a, b\}$, say C . This yields that $(x_a, x_b) \cap \bigcap_{e \in E(C_r)} x_e$ equals the cover ideal of C , that is, $J(C) = (x_a, x_b) \cap \bigcap_{e \in E(C_r)} x_e$. Furthermore, one can obtain from Proposition 3.3.26(ii) that $J(C)$ has the strong persistence property, that is, $((J(C))^{m+1} :_R J(C)) = (J(C))^m$. Thus, $((J(C))_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} J(C)_{\mathfrak{p}}) = (J(C)_{\mathfrak{p}})^m$, and so $(L_{\mathfrak{p}}^{m+1} :_{R_{\mathfrak{p}}} L_{\mathfrak{p}}) = L_{\mathfrak{p}}^m$. Therefore, $L_{\mathfrak{p}}$ has the strong persistence property, as desired.

This completes our discussion. $\square$

7.6.2 On the normality of the cover ideals of theta graphs

In general, there are well-known relations among the persistence property, strong persistence property, and normality. In fact, every normal monomial ideal has the strong persistence property (see Theorem 3.2.34), and the strong persistence property implies the persistence property (see Proposition 3.1.12). However, there exists some square-free monomial ideal which does not satisfy the persistence property. To see such an example, consider the following square-free monomial ideal in the polynomial ring $R = K[x_1, \dots, x_{12}]$,

$$\begin{aligned} I = & (x_2x_4x_5x_7, x_1x_3x_6x_8, x_3x_6x_8x_9, x_1x_4x_7x_{10}, x_4x_5x_7x_{10}, x_1x_3x_8x_{10}, \\ & x_2x_4x_5x_{11}, x_1x_4x_6x_{11}, x_1x_6x_8x_{11}, x_2x_8x_9x_{11}, x_6x_8x_9x_{11}, x_2x_5x_7x_{12}, \\ & x_3x_6x_9x_{12}, x_2x_7x_9x_{12}, x_3x_5x_{10}x_{12}, x_5x_7x_{10}x_{12}). \end{aligned}$$

By using *Macaulay2* [49], we obtain $\mathfrak{m} = (x_1, \dots, x_{12}) \in \text{Ass}_R(R/I^3) \setminus \text{Ass}_R(R/I^4)$, that is, I does not satisfy the persistence property. It should be noted that this

counterexample is different from the counterexample which has been presented in Proposition 3.2.5.

In what follows, our target is to probe the normality of the cover ideals of the theta graphs. Indeed, we found that proving the normality of odd cycle graphs, theta graphs, and bipartite graphs follow the same scheme. This is illustrated in the following theorem which introduces a class of normal monomial ideals. Note that $R = K[x_1, \dots, x_m]$ is a polynomial ring over a field K and $[m] = \{1, \dots, m\}$.

Theorem 7.6.10. *Let I be a monomial ideal in R . Let $A_h \subseteq [m] \times [m]$ for all $h = 1, \dots, s$ with $s \in \mathbb{N}$. For each $t \geq 1$, let c_t and d_t be arbitrary positive integers. Assume that the following statement is true*

$$x_1^{l_1} \cdots x_m^{l_m} \in I^t \iff \begin{cases} (i) & l_p + l_q \geq c_t \text{ for each } \{p, q\} \in A_h \text{ and each } h \in [s]; \\ (ii) & \sum_{\{p, q\} \in A_h} [(l_p + l_q) - c_t] \geq d_t \text{ for some } h \in \Omega \subseteq [s], \end{cases}$$

with Ω possibly empty. Then I is normal.

Proof. Let $\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in \overline{I^t}$. Our goal is to prove that $\alpha \in I^t$. Assume that $\{\omega_j\}_{j=1}^r \subset \mathbb{N}_0^m$ is the set of exponents of the minimal generators of I^t , and let $\omega_{j,i}$ be the i -th component of ω_j . Corollary 1.4.3 of [60] implies that $(\alpha_1, \dots, \alpha_m)$ belongs to the Newton polyhedron of I . This means that there exist $\lambda_1, \dots, \lambda_r \in \mathbb{Q}_{\geq 0}$ with $\sum_{j=1}^r \lambda_j = 1$ such that $\alpha_i \geq \sum_{j=1}^r \lambda_j \omega_{j,i}$ for each $i \in [m]$. If $\{p, q\} \in A_h$ and $h \in [s]$, then we obtain

$$\begin{aligned} \alpha_p + \alpha_q &\geq \sum_{j=1}^r \lambda_j (\omega_{j,p} + \omega_{j,q}) \\ (7.6.2) \quad &\geq \sum_{j=1}^r \lambda_j c_t = c_t. \end{aligned}$$

In addition, if Ω is not empty, then for every $h \in \Omega$, we have

$$\begin{aligned} \sum_{\{p, q\} \in A_h} [(\alpha_p + \alpha_q) - c_t] &\geq \sum_{\{p, q\} \in A_h} \left[\sum_{j=1}^r \lambda_j (\omega_{j,p} + \omega_{j,q}) - c_t \right] \\ &= \sum_{j=1}^r \lambda_j \left[\sum_{\{p, q\} \in A_h} [(\omega_{j,p} + \omega_{j,q}) - c_t] \right] \\ (7.6.3) \quad &\geq \sum_{j=1}^r \lambda_j d_t = d_t. \end{aligned}$$

It follows from (7.6.2) and (7.6.3) that $\alpha \in I^t$, and hence I is normal. $\square$

As an immediate consequence of Theorem 7.6.10, we get the corollary below.

Corollary 7.6.11. *Let $G = (V(G), E(G))$ be a finite simple graph with the cover ideal $J \subset R$, and $V(G) = [m] = \bigcup_{h=1}^s V_h$ with $\phi \subseteq V_h \subseteq V(G)$. For each $t \geq 1$, let c_t and d_t be arbitrary positive integers. Assume that the following statement is true*

$$x_1^{l_1} \cdots x_m^{l_m} \in J^t \iff \begin{cases} (i) & l_p + l_q \geq c_t \text{ for each } \{p, q\} \in E(G) \text{ with } p, q \in V_h \\ & \text{and each } h \in [s]; \\ (ii) & \sum_{\substack{\{p, q\} \in E(G) \\ \text{with } p, q \in V_h}} [(l_p + l_q) - c_t] \geq d_t \text{ for some } h \in \Omega \subseteq [s], \end{cases}$$

with Ω possibly empty. Then J is normal.

Proof. The claim can be proved by direct application of Theorem 7.6.10 with $A_h := \{\{p, q\} \in E(G) : p, q \in V_h\}$ for each $h \in [s]$. $\square$

In Lemma 3.1.10 it has been proved that if J is the cover ideal of the odd cycle graph C of order $2n + 1$, then $x_1^{l_1} \cdots x_{2n+1}^{l_{2n+1}} \in J^t$ if and only if $l_i + l_{i+1} \geq t$ for every $i \in [2n + 1]$ and $\sum_{i=1}^{2n+1} [(l_i + l_{i+1}) - t] \geq t$. In Theorem 7.1.7, it has been shown that J is normal. The proof in [4] is essentially the same as the proof of the above corollary but with $s = 1$, $V_1 = V(C) = [2n + 1]$, and $c_t = d_t = 1$. As an application of Corollary 7.6.11 above, we give in the following theorem a generalization such that we prove that cover ideals of theta graphs are normal.

Theorem 7.6.12. *Let J be the cover ideal of the theta graph $\theta_{n_1, \dots, n_k}$. Then J is normal.*

Proof. Assume that $C_{2e_1+1}, \dots, C_{2e_z+1}$ are all induced odd cycle subgraphs of $\theta_{n_1, \dots, n_k}$. Set $c_t = d_t = t$ for all $t \geq 1$, $V_h := V(C_{2e_h+1})$ for each $h = 1, \dots, z$, and $V_{z+1} := V(G) \setminus \bigcup_{h=1}^z V(C_{2e_h+1})$. Now, the assertion can be deduced from Theorem 7.6.3 and Corollary 7.6.11. $\square$

To give another application of Corollary 7.6.11, we can re-prove a well-known result which is related to the cover ideals of bipartite graphs, see [133, Corollary 14.6.25].

Proposition 7.6.13. *Let $G = (V(G), E(G))$ be a bipartite graph with the cover ideal J , and $V(G) = [m]$. Then J is normal.*

Proof. Since G is bipartite, it follows from [48, Corollary 2.6] that J is normally torsion-free, and [60, Theorem 1.4.6] implies that $J^t = J^{(t)}$ for all t , where $J^{(t)}$ denotes the t -th symbolic power of J . By virtue of $J^{(t)} = \bigcap_{\mathfrak{p} \in \text{Min}(J)} \mathfrak{p}^t$, one obtains $J^t = \bigcap_{\mathfrak{p} \in \text{Min}(J)} \mathfrak{p}^t$. This yields that $x_1^{\ell_1} \cdots x_m^{\ell_m} \in J^t$ if and only if $\ell_p + \ell_q \geq t$ for each $\{p, q\} \in E(G)$. In view of Corollary 7.6.11, we can derive that J is a normal square-free monomial ideal, as desired. $\square$

We terminate this argument by stating two theorems which introduce some classes of normal monomial ideals.

Theorem 7.6.14. *Let I be a monomial ideal in $R = K[x_1, \dots, x_m]$ with $m+1 = 1$. For each $t \geq 1$, let c_t be an arbitrary positive integer. Assume there are some fixed subsets $S \subseteq [m]$ and $W \subseteq [m]$ such that the following statement is true*

$$x_1^{l_1} \cdots x_m^{l_m} \in I^t \iff \begin{cases} (i) & \sum_{i \in S} l_{2i} \geq c_t; \\ (ii) & \text{for each } z \in W \text{ there is } T_z \subseteq S \text{ with } \sum_{i \in T_z} l_{2i} \leq l_{2z+1}. \end{cases}$$

Then I is normal.

Proof. Let $\alpha = x_1^{\alpha_1} \cdots x_m^{\alpha_m} \in \overline{I^t}$. Assume that $\{\omega_j\}_{j=1}^r \subset \mathbb{N}_0^m$ is the set of exponents of the minimal generators of I^t , and also assume that $\omega_{j,i}$ is the i -th component of ω_j . Due to $\alpha \in \overline{I^t}$, one can deduce from [60, Corollary 1.4.3] that $(\alpha_1, \dots, \alpha_m)$ belongs to the Newton polyhedron of I . This gives that there exist $\lambda_1, \dots, \lambda_r \in \mathbb{Q}_{\geq 0}$ with $\sum_{j=1}^r \lambda_j = 1$ such that $\alpha_i = \sum_{j=1}^r \lambda_j \omega_{j,i}$ for each $i \in [m]$. Note that

$$\begin{aligned} \sum_{i \in S} \alpha_{2i} &= \sum_{i \in S} \sum_{j=1}^r \lambda_j \omega_{j,2i} \\ &= \sum_{j=1}^r \lambda_j \sum_{i \in S} \omega_{j,2i} \\ &\geq \sum_{j=1}^r \lambda_j c_t = c_t. \end{aligned} \tag{7.6.4}$$

Since $\omega_j \in I^t$, then for each $z \in W$ there is $T_z \subseteq S$ with $\sum_{i \in T_z} \omega_{j,2i} \leq \omega_{j,2z+1}$. We thus get

$$\begin{aligned} \sum_{i \in T_z} \alpha_{2i} &= \sum_{i \in T_z} \sum_{j=1}^r \lambda_j \omega_{j,2i} \\ &= \sum_{j=1}^r \lambda_j \sum_{i \in T_z} \omega_{j,2i} \\ &\leq \sum_{j=1}^r \lambda_j \omega_{j,2z+1} = \alpha_{2z+1}. \end{aligned} \tag{7.6.5}$$

We can now combine (7.6.4) and (7.6.5) to conclude $\alpha \in I^t$, and hence I is normal. This finishes the proof. $\square$

To clarify Theorem 7.6.14, we present the example below.

Example 7.6.15. Let $R = K[x_1, \dots, x_{2n}]$ be the polynomial ring over a field K , and I be the square-free monomial ideal in R with

$$\mathcal{G}(I) = \{x_{2i-3}x_{2i-1}x_{2i}x_{2i+1} : i = 1, \dots, n\},$$

where x_{2n+1} represents x_1 . For any $t \geq 1$, it has already been shown in Lemma 3.2.39 that $x_1^{\ell_1} \cdots x_{2n}^{\ell_{2n}} \in \mathcal{G}(I^t)$ if and only if (i) $\sum_{i=1}^n \ell_{2i} = t$ and (ii) $\ell_{2i+1} = \ell_{2i} + \ell_{2i+2} + \ell_{2i+4}$ for each $i = 1, \dots, n$. By Theorem 3.2.40 the ideal I is normal. But this can now be deduced directly from Theorem 7.6.14 by setting $c_t = t$, $S = \{1, \dots, n\}$, $W = \{1, \dots, n\}$, and $T_z = \{z, z+1, z+2\}$ for each $z \in W$.

More hypotheses can be made on the powers of square-free monomial ideals so that normality is deduced. For instance, we have the following theorem whose proof mimics the proof of Theorem 7.6.14.

Theorem 7.6.16. Let I be a monomial ideal in $R = K[x_1, \dots, x_m]$ with x_{m+1} represents x_1 . For each $t \geq 1$, let c_t be an arbitrary positive integer. Assume there are some fixed subsets $S \subseteq [m]$, $W \subseteq [m]$, and $T \subseteq S$ such that the following statement is true

$$x_1^{l_1} \cdots x_m^{l_m} \in I^t \iff \begin{cases} (i) & \sum_{i \in S} l_{2i} \geq c_t; \\ (ii) & \sum_{i \in T} l_{2i} \leq \sum_{z \in W} l_{2z+1}. \end{cases}$$

Then I is normal.

7.7 Powers of closed neighborhood and dominating ideals of graphs

Sharifan and Moradi, in [122], introduced the notion of closed neighborhood ideals and dominating ideals of graphs, refer to Section 1.2 for recalling the definitions. The authors calculated some algebraic invariants of these ideals such as regularity and projective dimension in terms of the information from the underlying graph. Next, Honeycutt and Sather-Wagstaff, in [71], investigated the Cohen-Macaulay, unmixed, and complete intersection properties of closed neighborhood ideals.

In this section, we concentrate on the normality, strong persistence property, persistence property, and symbolic strong persistence property of closed neighborhood ideals and dominating ideals of some classes of graphs. It should be noted that the results of this section can be found in [101].

Throughout this section, if G is a finite simple graph, then $NI(G)$ (respectively, $DI(G)$) stands for the closed neighborhood ideal (respectively, dominating ideal) of G .

7.7.1 Some criteria for normality of monomial ideals

This subsection is devoted to presenting some criteria which help us to detect the normality of monomial ideals. In particular, our results will be used in studying closed neighborhood ideals and dominating ideals of some classes of graphs.

The following theorem is essential for us to verify Theorem 7.7.3 and Proposition 7.7.6. It is necessary to observe that Theorem 7.7.1 is an updated version of Theorem 6.5.1. In fact, in Theorem 7.7.1, we focus on the monomial ideals, while in Theorem 6.5.1, the authors argued on square-free monomial ideals.

Theorem 7.7.1. *Let I and H be two normal monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ such that $I + H$ is normal. Let $x_d \in \{x_1, \dots, x_n\}$ be a variable with $\gcd(v, x_d) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$, and c be a positive integer number. Then the following statements hold:*

- (i) $L := I + x_d^c H$ is normal.
- (ii) $L := I + x_d^c H$ has the strong persistence property.
- (iii) $L := I + x_d^c H$ has the persistence property.
- (iv) $L := I + x_d^c H$ has the symbolic persistence property.

Proof. (i) Assume that $\mathcal{G}(I) = \{u_1, \dots, u_s\}$ and $\mathcal{G}(H) = \{h_1, \dots, h_r\}$. In the light of $\gcd(v, x_d) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$, without loss of generality, one can assume that $x_d = x_1 \in K[x_1]$ and

$$\mathcal{G}(I) \cup \mathcal{G}(H) = \{u_1, \dots, u_s, h_1, \dots, h_r\} \subseteq K[x_2, \dots, x_n].$$

To establish the normality of L , one has to prove that $\overline{L^t} = L^t$ for all integers $t \geq 1$. Obviously, $L^t \subseteq \overline{L^t}$, so it is enough to show that $\overline{L^t} \subseteq L^t$. For this it suffices to show that an arbitrary monomial $\alpha \in \overline{L^t}$ is in L^t . Write $\alpha = x_1^b \delta$ with $x_1 \nmid \delta$ and $\delta \in R$. In view of [60, Theorem 1.4.2], one has $\alpha^k \in L^{tk}$ for some integer $k \geq 1$. Hence, one can conclude that $\alpha^k \in I^p (x_1^c H)^q$ for some p and q with $p + q = tk$. If $p = 0$ (respectively, $q = 0$), then $x_1^{bk} \delta^k \in (x_1^c H)^{tk}$ (respectively, $x_1^{bk} \delta^k \in I^{tk}$), and so $x_1^b \delta \in \overline{(x_1^c H)^t}$ (respectively, $x_1^b \delta \in \overline{I^t}$). Because I and H are normal, this implies that $x_1^b \delta \in (x_1^c H)^t$ (respectively, $x_1^b \delta \in I^t$), and the proof is complete. Accordingly, one can assume that $p > 0$ and $q > 0$. Then, we can write

$$(7.7.1) \quad \alpha^k = x_1^{bk} \delta^k = \prod_{i=1}^s u_i^{p_i} x_1^{cq+\varepsilon} \prod_{j=1}^r h_j^{q_j} \beta,$$

with $\sum_{i=1}^s p_i = p$, $\sum_{j=1}^r q_j = q$, $p + q = tk$, $\varepsilon \geq 0$, and β is some monomial in R such that $x_1 \nmid \beta$. As $x_1 \nmid \beta$, $x_1 \nmid \delta$, and $\gcd(v, x_1) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$, we obtain $bk = cq + \varepsilon$, in particular, $bk \geq cq$. In addition, suppose that p is minimal according to the membership $\delta^k \in I^p H^q$. Now, one can deduce from (7.7.1) that

$$(7.7.2) \quad \delta^k = \prod_{i=1}^s u_i^{p_i} \prod_{j=1}^r h_j^{q_j} \beta \in (I + H)^{tk}.$$

It follows from (7.7.2) that $\delta \in \overline{(I + H)^t}$. Thanks to $I + H$ is normal, one has $\overline{(I + H)^t} = (I + H)^t$, and so $\delta \in (I + H)^t$. Thus, we get $\delta \in I^l H^z$ for some l and z with $l + z = t$, in particular, $\delta^k \in I^{lk} H^{zk}$. One can derive from the minimality of p that $p \leq lk$. On account of $\delta \in I^l H^z$, we can write

$$(7.7.3) \quad \delta = \prod_{i=1}^s u_i^{l_i} \prod_{j=1}^r h_j^{z_j} \gamma,$$

with $\sum_{i=1}^s l_i = l$, $\sum_{j=1}^r z_j = z$, $l + z = t$, and γ is some monomial in R . Observe that $x_1 \nmid \gamma$ due to $x_1 \nmid \delta$.

We have shown above that $p + q = tk$, $lk + zk = tk$, and $p \leq lk$. It follows from this that $q \geq zk$. We have also shown above that $bk \geq cq$. Thus, $bk \geq czk$, and since $k \geq 1$, we have $b \geq cz$. Multiplying (7.7.3) by x_1^b yields

$$x_1^b \delta = \prod_{i=1}^s u_i^{l_i} x_1^b \prod_{j=1}^r h_j^{z_j} \gamma.$$

The latter can be rewritten as

$$\prod_{i=1}^s u_i^{l_i} \prod_{j=1}^r (x_1^c h_j)^{z_j} (x_1^{b-cz} \gamma) \in I^l (x_1^c H)^z \subseteq L^t.$$

Therefore, $\alpha = x_1^b \delta \in L^t$, which makes L normal, completing the proof.

(ii)-(iv) can be established similar to the proof of Corollary 6.2.3. $\square$

To formulate Theorem 7.7.3, one requires the next auxiliary result. We first should recall the definition of linear relation graph, see Definition 6.3.10. As an immediate consequence of Theorem 6.5.3, we get the following corollary.

Corollary 7.7.2. *Let $I \subset R = K[x_1, \dots, x_n]$ be a monomial ideal generated in a single degree whose linear relation graph has n vertices and one connected component. Then*

- (i) $\text{depth}(R/I^{n-1}) = 0$. In particular, $\mathfrak{m} = (x_1, \dots, x_n) \in \text{Ass}(R/I^{n-1})$.
- (ii) If I satisfies the persistence property, then $\lim_{k \rightarrow \infty} \text{depth} R/I^k = 0$. In particular, $\mathfrak{m} \in \text{Ass}(R/I^k)$ for all $k \geq n - 1$.

The following theorem is crucial for us to prove Theorem 7.8.1 and Lemma 7.9.3.

Theorem 7.7.3. *Let I and H be two normal monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ such that $I + H$ is normal. Let h be a monomial in R with $\gcd(v, h) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$. Then the following statements hold:*

- (i) $L := I + hH$ is normal.
- (ii) $L := I + hH$ has the strong persistence property.
- (iii) $L := I + hH$ has the persistence property.
- (iv) $L := I + hH$ has the symbolic persistence property.
- (v) If $L := I + hH$ is generated in a single degree whose linear relation graph has n vertices and one connected component, then $\lim_{k \rightarrow \infty} \text{depth} R/L^k = 0$. In particular, $\mathfrak{m} = (x_1, \dots, x_n) \in \text{Ass}(R/L^k)$ for all $k \geq n - 1$.

Proof. (i) Let $h = x_1^{c_1} \cdots x_m^{c_m}$. We use induction on m . It follows from Theorem 7.7.1 that the claim is true for the case in which $m = 1$. Suppose now that $m > 1$ and also the assertion is true for $m - 1$, that is, $I + x_1^{c_1} \cdots x_{m-1}^{c_{m-1}} H$ is normal. Since I , $I + x_1^{c_1} \cdots x_{m-1}^{c_{m-1}} H$, and $I + (I + x_1^{c_1} \cdots x_{m-1}^{c_{m-1}} H)$ are normal, one can rapidly conclude from Theorem 7.7.1 that $I + x_m^{c_m}(I + x_1^{c_1} \cdots x_{m-1}^{c_{m-1}} H)$ is normal as well. Due to $I + x_m^{c_m}(I + x_1^{c_1} \cdots x_{m-1}^{c_{m-1}} H) = I + x_1^{c_1} \cdots x_m^{c_m} H$, this completes the inductive step, and hence the claim has been shown by induction. Therefore, $L = I + hH$ is normal, as claimed.

(ii)-(iv) can be established similar to the proof of Corollary 6.2.3.

(v) Corollary 7.7.2 together with (iii) yield the assertion. $\square$

To show Theorem 7.9.7, we have to utilize the next proposition.

Proposition 7.7.4. *Let $I \subseteq H$ be two normal monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$. Let J be a monomial ideal of R such that $\gcd(u, v) = 1$ for all $v, u \in \mathcal{G}(J)$ with $u \neq v$. Also, let $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I) \cup \mathcal{G}(H)$ and $v \in \mathcal{G}(J)$. Then the following statements hold:*

- (i) $L := I + JH$ is normal.
- (ii) $L := I + JH$ has the strong persistence property.
- (iii) $L := I + JH$ has the persistence property.

Proof. (i) Let $\mathcal{G}(J) = \{u_1, \dots, u_s\}$. We proceed by induction on s . The case in which $s = 1$ is true according to Theorem 7.7.3(i). Now, assume that the claim holds for $s - 1$. Set $F := I + (u_1, \dots, u_{s-1})H$. This implies that $L = F + u_s H$.

Based on the inductive hypothesis, we obtain F is normal. In addition, by virtue of $F + H = I + H = H$ is normal and also $\gcd(v, u_s) = 1$ for all $v \in \mathcal{G}(F) \cup \mathcal{G}(H)$, it follows readily from Theorem 7.7.3(i) that L is normal. This completes the inductive step, and so the claim has been shown by induction.

(ii) and (iii) can be shown similar to the proof of Corollary 6.2.3. $\square$

It has already been remained the following open question:

Question 7.7.5. (see Question 7.4.11) Let I be a normal square-free monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then, in general, can one conclude that $L := IS \cap (x_n, x_{n+1}^\ell) \subset S = R[x_{n+1}]$ with $\ell > 1$, is normal?

As an application of Theorem 7.7.1, we provide an affirmative answer to above open question in the following proposition:

Proposition 7.7.6. Let $I \subset R = K[x_1, \dots, x_n]$ be a normal square-free monomial ideal with $\mathcal{G}(I) \subset R$. Then $L := IS \cap (x_n, x_{n+1}^\ell) \subset S = R[x_{n+1}]$ with $\ell \geq 1$, is normal.

Proof. It is routine to check that

$$L = I \cap (x_n) + I \cap (x_{n+1}^\ell) = x_n(I :_S x_n) + x_{n+1}^\ell I.$$

Since I is normal, [133, Proposition 12.2.3] implies that $(I :_S x_n)$ is normal, and hence $x_n(I :_S x_n)$ is normal as well. In addition, $x_n(I :_S x_n) + I = I$ is normal. Now, one can rapidly deduce from Theorem 7.7.1 that L is normal, as claimed. $\square$

To see an application of Proposition 7.7.6, we give the subsequent corollary.

Corollary 7.7.7. Let I be a normal square-free monomial ideal in $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then the square-free monomial ideal

$$L := IS \cap (x_n, x_{n+1}, \dots, x_{n+m}) \subset S = R[x_{n+1}, \dots, x_{n+m}],$$

is normal.

Proof. We demonstrate the claim by using induction on m . In view of Proposition 7.7.6, we deduce that the claim is true for the case in which $m = 1$. Suppose that the claim holds for $m - 1$. Put $J := I \cap (x_n, x_{n+1}, \dots, x_{n+m-1})$. We thus have $L = J + x_{n+m}I$. The inductive hypothesis implies that J is normal. In the light of Theorem 7.7.3, one can conclude that L is normal. This completes the inductive step, and so the claim has been shown by induction. $\square$

The next theorem will be used to establish Lemma 7.9.3.

Theorem 7.7.8. *Let $I \subset R = K[x_1, \dots, x_n, x_{n+1}]$ be a normal square-free monomial ideal such that $I \cap (x_n) + (I :_R x_{n+1})$ is normal. Then $L := I \cap (x_n, x_{n+1})$ is normal.*

Proof. It is easy to check $I \cap (x_n) = x_n (I :_R x_n)$ and $I \cap (x_{n+1}) = x_{n+1} (I :_R x_{n+1})$. Hence, in view of [133, Exercise 6.1.23], we get $L = x_n (I :_R x_n) + x_{n+1} (I :_R x_{n+1})$. To simplify our notation, put $F := (I :_R x_n)$ and $G := (I :_R x_{n+1})$. Thus, $L = x_n F + x_{n+1} G$. Let $\mathcal{G}(I) = \{u_1, \dots, u_s\}$, and define $f_i = u_i$ if $x_n \nmid u_i$, and $f_i = u_i/x_n$ if $x_n \mid u_i$. By the square-free assumption on I , none of the f_i is divisible by x_n . Similarly, define a set of elements g_i by considering divisibility by x_{n+1} , i.e., $g_i = u_i$ if $x_{n+1} \nmid u_i$ and $g_i = u_i/x_{n+1}$ if $x_{n+1} \mid u_i$. Again, by the square-free assumption, none of the g_i is divisible by x_{n+1} . One can readily see that $F = (f_1, \dots, f_s)$ and $G = (g_1, \dots, g_s)$. By [133, Proposition 12.2.3], F and G are normal, and so $x_n F$ and $x_{n+1} G$ are also normal. By hypothesis, $x_n F + G$ is normal. Accordingly, we obtain $\overline{(x_n F + G)^t} = (x_n F + G)^t$ for all $t \geq 1$. One has to prove that $\overline{L^t} = L^t$ for all $t \geq 1$. To accomplish this, it is enough to show that $\overline{L^t} \subseteq L^t$ for all $t \geq 1$. To do this, pick an arbitrary monomial α in $\overline{L^t}$ and write $\alpha = x_{n+1}^b \delta$ for some integer b and some monomial $\delta \in R$ with $x_{n+1} \nmid \delta$. Since $\alpha \in \overline{L^t}$, [60, Theorem 1.4.2] gives that $\alpha^k \in L^{tk} = (x_n F + x_{n+1} G)^{tk}$ for some positive integer k . This implies that $\alpha^k \in (x_n F)^p (x_{n+1} G)^q$ for some integers p and q with $p + q = tk$. Assume that q is maximal according to this membership. This means that p is minimal according to this membership. Observe that if $p = 0$, then $\alpha^k \in (x_{n+1} G)^{tk}$, and hence $\alpha \in \overline{(x_{n+1} G)^t} = (x_{n+1} G)^t \subset L^t$. Henceforth, let $p > 0$. Thanks to $x_n F$ is normal, a similar argument shows that one may assume $q > 0$ too. Write

$$(7.7.4) \quad \alpha^k = x_{n+1}^{bk} \delta^k = \prod_{i=1}^s f_i^{p_i} x_n^p \prod_{j=1}^s g_j^{q_j} x_{n+1}^q \beta,$$

with $\sum_{i=1}^s p_i = p$, $\sum_{j=1}^s q_j = q$, and β some monomial in S . Let $x_{n+1} \mid \beta$. On account of either $f_j x_n = u_j$ or $f_j = u_j$, this contradicts the maximality of q . In addition, if there exists some f_j with $p_j > 0$ such that $x_{n+1} \mid f_j$, then once again this leads to a contradiction to the maximality of q . Therefore, we can assume in (7.7.4) that $x_{n+1} \nmid \beta$, and also $x_{n+1} \nmid f_j$ with $p_j > 0$. Consequently, one can derive that $q = bk$, in particular, $\delta^k \in (x_n F)^p G^q$. Now, we get the following equality

$$\delta^k = \prod_{i=1}^s f_i^{p_i} x_n^p \prod_{j=1}^s g_j^{q_j} \beta \in (x_n F + G)^{tk}.$$

Thus, $\delta \in \overline{(x_n F + G)^t}$, and so $\delta \in (x_n F + G)^t$. Hence, one obtains $\delta \in (x_n F)^l G^h$ for some l and h with $l + h = t$, in particular, $\delta^k \in (x_n F)^{lk} G^{hk}$. It follows from the minimality of p that $lk \geq p$. Because $lk \geq p$ and $p + q = lk + hk = tk$, we gain $q \geq hk$. Since $k \geq 1$, one can deduce from $q = bk$ and $q \geq hk$ that $b \geq h$. It follows now from $\delta \in (x_n F)^l G^h$ and $b \geq h$ that $\alpha = x_{n+1}^b \delta \in (x_n F + x_{n+1} G)^t = L^t$, and the argument is over. $\square$

7.8 Closed neighborhood and dominating ideals of complete bipartite graphs

Here, we are going to provide some applications of the previous section. In fact, we study the closed neighborhood ideals and dominating ideals of complete bipartite graphs. We begin with the following theorem which says that closed neighborhood ideals of complete bipartite graphs are normal, and so satisfy the (strong) persistence property.

Theorem 7.8.1. *Let $K_{r,s}$ be a complete bipartite graph. Then the following statements hold:*

- (i) $NI(K_{r,s})$ is normal.
- (ii) $NI(K_{r,s})$ has the strong persistence property.
- (iii) $NI(K_{r,s})$ has the persistence property.

Proof. (i) For convenience of notation, put $L := NI(K_{r,s})$. Let $V(K_{r,s}) = V_1 \cup V_2$, where $V_1 = \{x_1, \dots, x_r\}$ and $V_2 = \{x_{r+1}, \dots, x_{r+s}\}$. If $r = 1$, then

$$L = (x_1 \prod_{i=2}^{s+1} x_i, x_1 x_2, \dots, x_1 x_{s+1}) = x_1(x_2, \dots, x_{s+1}).$$

It is well-known that any monomial prime ideal is normal, and by virtue of Remark 7.1.1, one has L is normal. Similarly, if $s = 1$, then L is normal too. Accordingly, we assume that $r, s > 1$. Set $g := \prod_{i=2}^r x_i$ and $f := \prod_{i=r+1}^{r+s} x_i$. It is easy to check that $\mathcal{G}(L) = \{x_1 f, \dots, x_r f, x_1 x_{r+1} g, \dots, x_1 x_{r+s} g\}$. Hence, one can write $L = I + x_1 H$, where $H := (f, x_{r+1} g, \dots, x_{r+s} g)$ and $I := (x_2, \dots, x_r) f$. A similar argument shows that I is normal. Since $H = g(x_{r+1}, \dots, x_{r+s}) + f R$, where $R = K[x_1, \dots, x_r, x_{r+1}, \dots, x_{r+s}]$, it follows from Theorem 7.1.3 that H is normal. It is not hard to check that $I + H = H$, and so $I + H$ is normal as well. Note that $\gcd(x_1, v) = 1$ for all $v \in \mathcal{G}(I) \cup \mathcal{G}(H)$. Now, the claim can be deduced immediately from Theorem 7.7.3.

(ii) and (iii) can be proved similar to the proof of Corollary 6.2.3. This finishes the proof. $\square$

In the following proposition, we investigate when the unique homogeneous maximal ideal appears in the associated primes set of powers of closed neighborhood ideals of complete bipartite graphs.

Proposition 7.8.2. *Let $K_{r,s}$ with $r, s > 1$ be a complete bipartite graph, and*

$$\mathfrak{m} = (x_1, \dots, x_r, x_{r+1}, \dots, x_{r+s}) \subset R = K[x_1, \dots, x_r, x_{r+1}, \dots, x_{r+s}],$$

be the unique homogeneous maximal ideal. Then, for all $s \geq 3$,

$$\mathfrak{m} \in \text{Ass}(NI(K_{r,s})^s) \setminus (\text{Ass}(NI(K_{r,s})) \cup \text{Ass}(NI(K_{r,s})^2)).$$

In particular, $\text{depth}(R/NI(K_{r,s})^s) = 0$ for all $s \geq 3$, and

$$\lim_{k \rightarrow \infty} \text{depth}(R/NI(K_{r,s})^k) = 0.$$

Proof. Set $L := NI(K_{r,s})$. Let $V(K_{r,s}) = V_1 \cup V_2$, where $V_1 = \{x_1, \dots, x_r\}$ and $V_2 = \{x_{r+1}, \dots, x_{r+s}\}$. Put $g := \prod_{i=1}^r x_i$ and $f := \prod_{i=r+1}^{r+s} x_i$. It is not hard to investigate that $\mathcal{G}(L) = \{x_1 f, \dots, x_r f, x_{r+1} g, \dots, x_{r+s} g\}$. Due to L is a square-free monomial ideal, this implies that $\text{Ass}(L) = \text{Min}(L)$, and hence $\mathfrak{m} \notin \text{Ass}(L)$. We show that $\mathfrak{m} \notin \text{Ass}(L^2)$. Suppose, on the contrary, that $\mathfrak{m} \in \text{Ass}(L^2)$. This gives that $\mathfrak{m} = (L^2 : v)$ for some monomial $v \in R$. By virtue of Proposition 10.2.16, one has $\deg_{x_i}(v) \leq 1$ for all $i = 1, \dots, r+s$. Let $x_d \in \mathfrak{m}$ for some $1 \leq d \leq r+s$. Hence, $x_d v \in L^2$, and so there exists some monomial $w \in \mathcal{G}(L^2)$ such that $w \mid x_d v$. In addition, it follows from $\mathcal{G}(L) = \{x_1 f, \dots, x_r f, x_{r+1} g, \dots, x_{r+s} g\}$ that there exist i and j with $1 \leq i \neq j \leq r+s$ such that $\deg_{x_i}(w) = \deg_{x_j}(w) = 2$. Since $\deg_{x_i}(v) \leq 1$ and $\deg_{x_j}(v) \leq 1$, this leads to a contradiction. Consequently, $\mathfrak{m} \notin \text{Ass}(L^2)$. Based on Theorem 7.8.1(iii), the ideal L satisfies the persistence property. Hence, it is enough for us to show that $\mathfrak{m} \in \text{Ass}(L^3)$. To accomplish this, set $h := \prod_{i=1}^{r+s} x_i^2$. We claim that $h \notin L^3$ and $\mathfrak{m} \subseteq (L^3 : h)$. We show that $x_k h \in L^3$ for all $1 \leq k \leq r+s$. Let $1 \leq k \leq r$. Set $\alpha := (x_{r+1} g)(x_{r+2} g)(x_k f)$. Because $\alpha \in L^3$ and $\alpha \mid x_k h$, one can derive that $x_k h \in L^3$. Now, let $r+1 \leq k \leq r+s$. Put $\beta := (x_1 f)(x_2 f)(x_k g)$. Since $\beta \in L^3$ and $\beta \mid x_k h$, this implies that $x_k h \in L^3$. Therefore, we deduce that $\mathfrak{m} \subseteq (L^3 : h)$. On the contrary assume that $h \in L^3$. Thus, there exist monomials $u_1, u_2, u_3 \in \mathcal{G}(L)$ such that $u_1 u_2 u_3 \mid h$. If $u_1, u_2, u_3 \in \{x_1 f, \dots, x_r f\}$ (respectively, $u_1, u_2, u_3 \in \{x_{r+1} g, \dots, x_{r+s} g\}$), then $f^3 \mid h$ (respectively, $g^3 \mid h$), which contradicts the fact that $\deg_{x_i}(h) = 2$ for all i . Let $u_1, u_2 \in \{x_{r+1} g, \dots, x_{r+s} g\}$, and $u_3 \in \{x_1 f, \dots, x_r f\}$, say $u_3 = x_1 f$. Then, we get $x_1^3 \mid u_1 u_2 u_3$, and so $x_1^3 \mid h$, which is a contradiction. Finally, let $u_1, u_2 \in \{x_1 f, \dots, x_r f\}$, and $u_3 \in \{x_{r+1} g, \dots, x_{r+s} g\}$, say $u_3 = x_{r+1} g$. This gives rise to $x_{r+1}^3 \mid u_1 u_2 u_3$, and so $x_{r+1}^3 \mid h$, which is a contradiction. Accordingly, $h \notin L^3$, and thus $\mathfrak{m} = (L^3 : h)$. This terminates the proof. $\square$

We are in a position to state another main result of this section in the subsequent theorem.

Theorem 7.8.3. *The dominating ideals of complete bipartite graphs are nearly normally torsion-free.*

Proof. Let $K_{r,s}$ be a complete bipartite graph, $R = K[x_1, \dots, x_r, x_{r+1}, \dots, x_{r+s}]$, and $\mathfrak{m} = (x_1, \dots, x_r, x_{r+1}, \dots, x_{r+s})$ be the unique homogeneous maximal ideal. Let $V(K_{r,s}) = V_1 \cup V_2$, where $V_1 = \{x_1, \dots, x_r\}$ and $V_2 = \{x_{r+1}, \dots, x_{r+s}\}$. Put

$L := DI(K_{r,s})$, $I := (x_i : i = 1, \dots, r)$, $J := (x_i : i = r+1, \dots, r+s)$, $g := \prod_{i=1}^r x_i$, and $f := \prod_{i=r+1}^{r+s} x_i$. According to [122, Lemma 2.2] and [133, Exercise 6.1.23], we get the following equalities:

$$\begin{aligned}
L &= \bigcap_{i=1}^r (J + x_i R) \cap \bigcap_{i=r+1}^{r+s} (I + x_i R) \\
&= (J + \bigcap_{i=1}^r x_i R) \cap (I + \bigcap_{i=r+1}^{r+s} x_i R) \\
&= J \cap I + \bigcap_{i=1}^r x_i R + \bigcap_{i=r+1}^{r+s} x_i R \\
&= JI + gR + fR.
\end{aligned}$$

In what follows, our aim is to use Corollary 6.1.3. To do this, without loss of generality, it is enough to show that $L(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. Since $I(\mathfrak{m} \setminus \{x_1\}) = R$, $J(\mathfrak{m} \setminus \{x_1\}) = J$, $fR(\mathfrak{m} \setminus \{x_1\}) = fR$, and $gR(\mathfrak{m} \setminus \{x_1\}) = \prod_{i=2}^r x_i R$, we obtain

$$L(\mathfrak{m} \setminus \{x_1\}) = J + \prod_{i=2}^r x_i R + fR = J + \prod_{i=2}^r x_i R.$$

It follows now from Theorem 5.3.4 that $L(\mathfrak{m} \setminus \{x_1\})$ is normally torsion-free. Therefore, the claim can be deduced from Corollary 6.1.3, and the proof is done. $\square$

7.9 Dominating ideals of h -wheel graphs

The main point of this section is to explore dominating ideals of a class of graphs which are called h -wheel graphs. For this purpose, we start by stating the definition of h -wheel graphs.

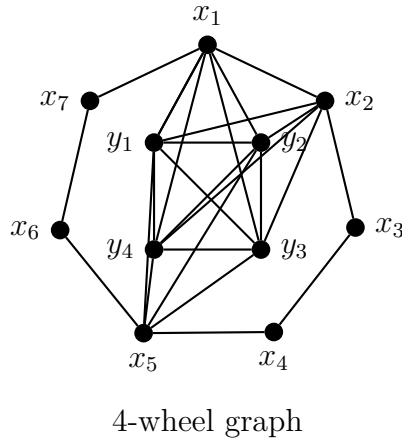
Definition 7.9.1. ([22, Definition 1.6]) *A graph G with the vertex set $V(G)$ is called an h -wheel if $V(G)$ can be written as the union of two disjoint sets, the set of rim vertices R^G and the set of center vertices C^G , such that the following conditions hold:*

- (1) *The subgraph induced by C^G is the complete graph on h vertices.*
- (2) *The subgraph induced by R^G is an odd cycle.*
- (3) *There exist $x_{i_1}, \dots, x_{i_k} \in R^G$ with $k \geq 3$ such that $N_{R^G}(y) = \{x_{i_1}, \dots, x_{i_k}\}$ for all $y \in C^G$.*
- (4) *For every $y \in C^G$, the vertex y belongs to at least two odd cycles in the subgraph induced by y and $N_{R^G}(y)$.*

The k is called the *radial number* for G . Also, for each $j = 1, \dots, k-1$, set ℓ_j as the length of the path along the subgraph induced by R^G from x_{i_j} to $x_{i_{j+1}}$, and set ℓ_k as the length from x_{i_k} to x_{i_1} . The positive integers $\ell_1, \dots, \ell_k$ are called the *radial lengths*.

It should be noted that, in [78, Page 265], the authors studied the 1-wheel, which we call a wheel for simplicity. In fact, given an h -wheel G and a vertex $y \in C^G$, the subgraph induced by y and R^G is a wheel.

Example 7.9.2. We give a 4-wheel graph G in the following figure.



Note that $C^G = \{y_1, y_2, y_3, y_4\}$, $R^G = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, and

$$N_{R^G}(y_1) = N_{R^G}(y_2) = N_{R^G}(y_3) = N_{R^G}(y_4) = \{x_1, x_2, x_5\}.$$

Lemma 7.9.3. Let C_n be a cycle graph with the vertex set $V(C_n) = \{1, \dots, n\}$ and the edge set $E(C_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n\}$, where x_0 (respectively, x_{n+1}) represents x_n (respectively, x_1). Let

$$H := \bigcap_{j \in \{1, \dots, n\} \setminus \{\ell_1, \dots, \ell_k\}} (x_{j-1}, x_j, x_{j+1}).$$

Then the following statements hold:

- (i) H is normal.
- (ii) H has the strong persistence property.
- (iii) H has the persistence property.

Proof. (i) Clearly, $n \geq 3$. Let $\{1, \dots, n\} \setminus \{\ell_1, \dots, \ell_k\} = \{i_1, \dots, i_{n-k}\}$. To simplify the notation, set $u_j := (x_{i_{j-1}}, x_{i_j}, x_{i_{j+1}}) \subset R$ for each $j = 1, \dots, n-k$, where $R = K[x_1, \dots, x_n]$. We use induction on $s := n-k$. If $s = 0$, then there is nothing to prove. Let $s = 1$. Then $H = u_1 = (x_{i_1-1}, x_{i_1}, x_{i_1+1})$, which is certainly

normal. Now, let $s > 1$, and the claim has been shown for all integers less than s . Put $J := \bigcap_{j=1}^{s-1} u_j$. Note that the inductive hypothesis implies that J is normal. Moreover, $H = J \cap u_s$. Here, one may consider the following cases:

Case 1. $\text{supp}(J) \cap \text{supp}(u_s) = \emptyset$. Then Lemma 7.4.5 yields that H is normal.

Case 2. $|\text{supp}(J) \cap \text{supp}(u_s)| = 1$. According to Corollary 7.7.7, one can derive that H is normal.

Case 3. $|\text{supp}(J) \cap \text{supp}(u_s)| = 2$, say $\text{supp}(J) \cap \text{supp}(u_s) = \{x_a, x_b\}$. In this case, one may consider the following subcases:

Subcase 3.1. $|a - b| = 1$, say $\text{supp}(J) \cap \text{supp}(u_s) = \{x_{i_s-1}, x_{i_s}\}$. This implies that $H = J \cap (x_{i_s-1}, x_{i_s}) + x_{i_s+1}J$. It is routine to check that one can write

$$J \cap (x_{i_s-1}, x_{i_s}) = \bigcap_{j=1, \text{ where } x_{i_s-1}x_{i_s} \notin u_j}^{s-1} u_j \cap (x_{i_s-1}, x_{i_s}).$$

The inductive hypothesis yields that $\bigcap_{j=1, \text{ where } x_{i_s-1}x_{i_s} \notin u_j}^{s-1} u_j$ is normal. Notice that any two consecutive integers can be appeared at most in two u_i 's. Also, recall that any integer can be appeared at most in three u_i 's. It follows now from Corollary 7.7.7 that $\bigcap_{j=1, \text{ where } x_{i_s-1}x_{i_s} \notin u_j}^{s-1} u_j \cap (x_{i_s-1}, x_{i_s})$ is normal too. On account of Theorem 7.7.3(i), we deduce that H is normal.

Subcase 3.2. $|a - b| = 2$, that is, $\text{supp}(J) \cap \text{supp}(u_s) = \{x_{i_s-1}, x_{i_s+1}\}$. We thus have $H = J \cap (x_{i_s-1}, x_{i_s+1}) + x_{i_s}J$. Want to show that $J \cap (x_{i_s-1}, x_{i_s+1})$ is normal. Our strategy is to use Theorem 7.7.8. For this purpose, it is sufficient to prove that $J \cap (x_{i_s-1}) + (J : x_{i_s+1})$ is normal. If $(x_{i_s-3}, x_{i_s-2}, x_{i_s-1})$ or $(x_{i_s+1}, x_{i_s+2}, x_{i_s+3})$ or both of them do not appear in J , then Corollary 7.7.7 yields that $J \cap (x_{i_s-1}, x_{i_s+1})$ is normal. Hence, assume that $(x_{i_s-3}, x_{i_s-2}, x_{i_s-1})$ and $(x_{i_s+1}, x_{i_s+2}, x_{i_s+3})$ appear in J . This means that we can write

$$J = J_1 \cap (x_{i_s-3}, x_{i_s-2}, x_{i_s-1}) \cap (x_{i_s+1}, x_{i_s+2}, x_{i_s+3}).$$

Note that the inductive hypothesis gives that J_1 is normal. Now, we obtain

$$\begin{aligned} J \cap (x_{i_s-1}) + (J : x_{i_s+1}) &= J_1 \cap (x_{i_s+1}, x_{i_s+2}, x_{i_s+3}) \cap (x_{i_s-1}) \\ &\quad + J_1 \cap (x_{i_s-3}, x_{i_s-2}, x_{i_s-1}) \\ &= J_1 \cap (x_{i_s-3}, x_{i_s-2}, x_{i_s-1}), \end{aligned}$$

where $\text{supp}(J_1) \cap \{x_{i_s-1}, x_{i_s+1}\} = \emptyset$. Since $x_{i_s-1} \notin J_1$, this implies that the normality of $J_1 \cap (x_{i_s-3}, x_{i_s-2}, x_{i_s-1})$ can be concluded from Case 1 or Case 2 or Subcase 3.1. Therefore, $J \cap (x_{i_s-1}) + (J : x_{i_s+1})$ is normal. It follows from Theorem 7.7.8 that $J \cap (x_{i_s-1}, x_{i_s+1})$ is normal. On account of Theorem 7.7.3(i), we deduce that H is normal.

Case 4. $\text{supp}(J) \cap \text{supp}(u_s) = \{x_{i_s-1}, x_{i_s}, x_{i_s+1}\}$. In this case, we can consider the following subcases:

Subcase 4.1. There exists exactly one integer $\alpha \in \{1, \dots, s-1\}$ such that $|\text{supp}(u_\alpha) \cap \text{supp}(u_s)| = 2$, say $\text{supp}(u_\alpha) \cap \text{supp}(u_s) = \{x_{i_s-1}, x_{i_s}\}$. Also, there exists unique $\beta \in \{1, \dots, s-1\} \setminus \{\alpha\}$ such that $x_{i_s+1} \in u_\beta$. Assume that $u_\beta = (x_{i_s+1}, x_r, x_t)$. Set $A := \bigcap_{j \in \{1, \dots, s-1\} \setminus \{\beta\}} u_j$. We thus get the following equalities

$$\begin{aligned} H &= A \cap ((x_{i_s+1}) + (x_{i_s-1}, x_{i_s}) \cap (x_r, x_t)) \\ &= A \cap (x_{i_s-1}, x_{i_s}) \cap (x_r, x_t) + x_{i_s+1}A. \end{aligned}$$

It follows from the inductive hypothesis that A is normal. Furthermore, by mimicking the argument in Case 3, one can derive that $A \cap (x_{i_s-1}, x_{i_s}) \cap (x_r, x_t)$ is normal. We therefore conclude from Theorem 7.7.3(i) that H is normal.

Subcase 4.2. There exist exactly two distinct integers $\alpha, \beta \in \{1, \dots, s-1\}$ such that $|\text{supp}(u_\alpha) \cap \text{supp}(u_s) \cap \text{supp}(u_\beta)| = 1$, say $\text{supp}(u_\alpha) \cap \text{supp}(u_s) \cap \text{supp}(u_\beta) = x_{i_s}$. Let $u_\alpha = (x_{i_s}, x_{\theta_1}, x_{\theta_2})$ and $u_\beta = (x_{i_s}, x_{\lambda_1}, x_{\lambda_2})$. Put $B := \bigcap_{j \in \{1, \dots, s-1\} \setminus \{\alpha, \beta\}} u_j$. This yields the following equalities

$$\begin{aligned} H &= B \cap ((x_{i_s}) + (x_{i_s-1}, x_{i_s+1}) \cap (x_{\theta_1}, x_{\theta_2}) \cap (x_{\lambda_1}, x_{\lambda_2})) \\ &= B \cap (x_{i_s-1}, x_{i_s+1}) \cap (x_{\theta_1}, x_{\theta_2}) \cap (x_{\lambda_1}, x_{\lambda_2}) + x_{i_s}B. \end{aligned}$$

One can deduce from the inductive hypothesis that B is normal. By repeating the argument in Case 3, we can deduce that $B \cap (x_{i_s-1}, x_{i_s+1}) \cap (x_{\theta_1}, x_{\theta_2}) \cap (x_{\lambda_1}, x_{\lambda_2})$ is normal as well. Thanks to Theorem 7.7.3(i), we obtain H is normal.

This completes the inductive step, and hence the claim has been proved by induction.

(ii) and (iii) can be shown similar to the proof of Corollary 6.2.3. □

The next corollary which is an immediate consequence of Lemma 7.9.3, says that the dominating ideals of cycle graphs are normal and also satisfy the (strong) persistence property. Moreover, this corollary will be used in proving Theorem 7.9.7.

Corollary 7.9.4. *Let C_n be a cycle graph with the vertex set $V(C_n) = \{1, \dots, n\}$ and the edge set $E(C_n) = \{\{x_i, x_{i+1}\} : i = 1, \dots, n\}$, where x_{n+1} represents x_1 , and $DI(C_n)$ be its dominating ideal. Then the following statements hold:*

- (i) $DI(C_n)$ is normal.
- (ii) $DI(C_n)$ has the strong persistence property.
- (iii) $DI(C_n)$ has the persistence property.

It has already been introduced, in [17], the notion of partial t -cover ideals of finite simple graphs. We first recall the definition of partial t -cover ideals in the following definition.

Definition 7.9.5. ([17, Definition 1.1]) Suppose that G is a finite simple graph on the vertex set $V(G) = \{x_1, x_2, \dots, x_n\}$ with the edge set $E(G)$. Also, for any $x \in V(G)$, let $N(x) = \{y : \{x, y\} \in E(G)\}$ denote the set of neighbors of x . Fix an integer $t \geq 1$. The partial t -cover ideal of G is the monomial ideal

$$J_t(G) = \bigcap_{x \in V(G)} \left(\bigcap_{\{x_{i_1}, \dots, x_{i_t}\} \subseteq N(x)} (x, x_{i_1}, \dots, x_{i_t}) \right).$$

When $t = 1$, above construction is simply the cover ideal of a finite simple graph G . It has already been shown in [17, Theorem 1.2] that if T is a tree, then, for any $t \geq 1$, $J_t(T)$ satisfies the persistence property. The next corollary states that the partial 2-cover ideal of any cycle graph satisfies the persistence property as well.

Corollary 7.9.6. Let C_n be a cycle graph and $J_2(C_n)$ be its partial 2-cover ideal. Then the following statements hold:

- (i) $J_2(C_n)$ is normal.
- (ii) $J_2(C_n)$ has the strong persistence property.
- (iii) $J_2(C_n)$ has the persistence property.

We are ready to express the main result of this section in the following theorem.

Theorem 7.9.7. Let G be an h -wheel graph with rim R^G and center C^G such that $V(C^G) = \{y_1, \dots, y_h\}$ and $V(R^G) = \{x_1, \dots, x_{2m+1}\}$, where $m \geq 2$. Also, let $x_{\ell_1}, \dots, x_{\ell_k}$ be the radial vertices such that there exist at least three consecutive numbers among them. Let $DI(G)$ denote the dominating ideal of G . Then the following statements hold:

- (i) $DI(G)$ is normal.
- (ii) $DI(G)$ has the strong persistence property.
- (iii) $DI(G)$ has the persistence property.

Proof. (i) To simplify the notation, put $F := (x_{\ell_1}, \dots, x_{\ell_k})S$ and $J := (y_1, \dots, y_h)S$, where $S = K[x_1, \dots, x_{2m+1}, y_1, \dots, y_h]$. Since there exist at least three consecutive numbers among $\ell_1, \dots, \ell_k$, this gives rise to $\bigcap_{i=1}^k (x_{\ell_i-1}, x_{\ell_i}, x_{\ell_i+1}) \subseteq F$. Hence, by

remembering this fact that x_0 (respectively, x_{2m+2}) represents x_{2m+1} (respectively, x_1), one can deduce the following equalities

$$\begin{aligned}
DI(G) &= (F + J) \cap \bigcap_{i=1}^k ((x_{\ell_i-1}, x_{\ell_i}, x_{\ell_i+1}) + J) \\
&\cap \bigcap_{j \in \{1, \dots, 2m+1\} \setminus \{\ell_1, \dots, \ell_k\}} (x_{j-1}, x_j, x_{j+1}) \\
&= \left(\bigcap_{i=1}^k (x_{\ell_i-1}, x_{\ell_i}, x_{\ell_i+1}) + J \right) \cap \bigcap_{j \in \{1, \dots, 2m+1\} \setminus \{\ell_1, \dots, \ell_k\}} (x_{j-1}, x_j, x_{j+1}) \\
&= \bigcap_{j=1}^{2m+1} (x_{j-1}, x_j, x_{j+1}) + J \bigcap_{j \in \{1, \dots, 2m+1\} \setminus \{\ell_1, \dots, \ell_k\}} (x_{j-1}, x_j, x_{j+1}) \\
&= DI(R^G) + J \bigcap_{j \in \{1, \dots, 2m+1\} \setminus \{\ell_1, \dots, \ell_k\}} (x_{j-1}, x_j, x_{j+1}).
\end{aligned}$$

Set $I := DI(R^G)$ and $H := \bigcap_{j \in \{1, \dots, 2m+1\} \setminus \{\ell_1, \dots, \ell_k\}} (x_{j-1}, x_j, x_{j+1})$. Hence, one has $DI(G) = I + JH$. By virtue of Lemma 7.9.3, we obtain H is normal. Moreover, Corollary 7.9.4 yields that I is normal. Because $I \subseteq H$, $\gcd(u, v) = 1$ for all $v, u \in \mathcal{G}(J)$ with $u \neq v$, and also $\gcd(u, v) = 1$ for all $u \in \mathcal{G}(I) \cup \mathcal{G}(H)$ and $v \in \mathcal{G}(J)$, one can conclude immediately from Proposition 7.7.4(i) that $DI(G)$ is normal, as desired.

(ii) and (iii) can be verified similar to the proof of Corollary 6.2.3. $\square$

Chapter 8

On the set defining minimal vertex covers and dominating sets

8.1 Sets defining minimal vertex covers of uniform hypergraphs

One motivation for this argument comes from a nice result in [87] which is devoted to finite simple graphs. In fact, Theorem 2.1 in [87] says that if V is a finite non-empty set of n elements, r a positive integer, and $\mathcal{A} = \{A_1, \dots, A_r\}$ a non-empty family of r subsets of V , then under certain conditions, there exists a unique simple graph G with vertex set V whose minimal vertex cover sets are the elements of $\mathcal{A}$. In what follows, we generalize the above mentioned result to d -uniform hypergraphs (see Theorem 8.1.3). To review the definition of hypergraphs, refer to Subsection 1.3. It should be noted that the results of this section can be found in [103].

Before stating the following definition, remember that an $m \times n$ matrix $M = (a_{ij})$ is called *binary* if $a_{ij} \in \{0, 1\}$ for each $i = 1, \dots, m$ and $j = 1, \dots, n$.

Definition 8.1.1. Let $M = (a_{ij})$ be an $m \times n$ binary matrix, and d be a positive integer with $2 \leq d \leq n$. We say that M satisfies the condition:

- (P1) when, for each $1 \leq i_1, i_2 \leq m$ with $i_1 \neq i_2$, there exists a positive integer $1 \leq j \leq n$ such that $a_{i_1, j} > a_{i_2, j}$.
- (P2)(d) when, for each $d-1$ distinct positive integers $1 \leq j_1, \dots, j_{d-1} \leq n$, there exists a positive integer $1 \leq i \leq m$ such that $a_{i, j_1} + \dots + a_{i, j_{d-1}} = 0$.
- (P3)(d) when, for each $\ell \geq d$ distinct positive integers $1 \leq j_1, \dots, j_\ell \leq n$ such that $a_{i, j_1} + \dots + a_{i, j_\ell} \geq 1$ for all $i = 1, \dots, m$, then there exist at least d distinct

integers $j_{\alpha_1}, \dots, j_{\alpha_d} \in \{j_1, \dots, j_\ell\}$ such that $a_{i,j_{\alpha_1}} + \dots + a_{i,j_{\alpha_d}} \geq 1$ for all $i = 1, \dots, m$.

When $d = 2$, the above definition coincides with conditions (P1), (P2) and (P3) in [87].

In the next definition, we correspond a matrix to a non-empty family of some subsets of a non-empty set.

Definition 8.1.2. Suppose that $V = \{1, \dots, n\}$ is a non-empty set of n elements, and let $\mathcal{A} = \{A_1, \dots, A_m\}$ be a non-empty family of m subsets of V . We define the incidence matrix associated to $\mathcal{A}$, denoted by $M(\mathcal{A})$, as the binary matrix $M(\mathcal{A}) = (a_{ij})$ with m rows and n columns such that $a_{ij} = 0$ if $j \notin A_i$, and $a_{ij} = 1$ if $j \in A_i$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

We are now ready to present the main theorem of this section.

Theorem 8.1.3. Suppose that $V = \{1, \dots, n\}$ is a non-empty set of n elements, $\mathcal{A} = \{A_1, \dots, A_m\}$ a non-empty family of m non-empty subsets of V , and $2 \leq d \leq n$ a positive integer. Then there exists a d -uniform hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = V$ where $\mathcal{A}$ is the set whose elements are the minimal vertex cover sets of $\mathcal{H}$ if and only if the matrix $M(\mathcal{A}) = (a_{ij})$ satisfies conditions (P1), (P2)(d), and (P3)(d).

If there exists such a hypergraph $\mathcal{H}$, then it is unique and its edge set is given by

$$E(\mathcal{H}) = \{\{j_1, \dots, j_d\} : a_{i,j_1} + \dots + a_{i,j_d} \geq 1 \text{ for all } i = 1, \dots, m, \\ \text{with } j_1, \dots, j_d \text{ are distinct positive integers}\}.$$

Proof. For necessity, assume that there exists a d -uniform hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = V$ and $\mathcal{A}$ is the set whose elements are the minimal vertex cover sets of $\mathcal{H}$. We first prove that $M(\mathcal{A})$ satisfies the condition (P1). To do this, let $1 \leq i_1, i_2 \leq m$ with $i_1 \neq i_2$. Suppose, on the contrary, that (P1) does not hold. This implies that, for each $j = 1, \dots, n$, we have $a_{i_1,j} \leq a_{i_2,j}$. Thus, one derives $A_{i_1} \subseteq A_{i_2}$, a contradiction to the minimality of A_{i_2} . Consequently, $M(\mathcal{A})$ satisfies the condition (P1).

We now show that $M(\mathcal{A})$ satisfies the condition (P2)(d). To achieve this, suppose that $1 \leq j_1, \dots, j_{d-1} \leq n$ are $d-1$ distinct positive integers. On the contrary, assume that for each $1 \leq i \leq m$, one has $a_{i,j_1} + \dots + a_{i,j_{d-1}} \neq 0$. Set $\mathbf{p}_i := (x_w : w \in A_i)$ for each $i = 1, \dots, m$. Thus, one can show that $I(\mathcal{H}) = \bigcap_{i=1}^m \mathbf{p}_i$, where $I(\mathcal{H})$ denotes the edge ideal of the hypergraph $\mathcal{H}$. It follows from the assumption that $\{j_1, \dots, j_{d-1}\} \cap A_i \neq \emptyset$ for each $i = 1, \dots, m$. This implies that $\{x_{j_1}, \dots, x_{j_{d-1}}\} \cap \mathbf{p}_i \neq \emptyset$ for each $i = 1, \dots, m$. Accordingly, $x_{j_1} \cdots x_{j_{d-1}} \in \mathbf{p}_i$ for each $i = 1, \dots, m$, and hence $x_{j_1} \cdots x_{j_{d-1}} \in \bigcap_{i=1}^m \mathbf{p}_i$. This implies that there exists an element $u \in \mathcal{G}(I(\mathcal{H}))$ such

that $u|x_{j_1} \cdots x_{j_{d-1}}$. Consequently, we deduce that $\deg u \leq d-1$, which contradicts the fact that $\deg u = d$. Therefore, $M(\mathcal{A})$ satisfies the condition $(\mathcal{P}2)(d)$.

To complete the argument, one has to demonstrate that $M(\mathcal{A})$ satisfies the condition $(\mathcal{P}3)(d)$. Suppose that for each $\ell \geq d$ distinct positive integers $1 \leq j_1, \dots, j_\ell \leq n$, we have $a_{i,j_1} + \cdots + a_{i,j_\ell} \geq 1$ for all $i = 1, \dots, m$. Our aim is to verify that there are at least d distinct integers $j_{\alpha_1}, \dots, j_{\alpha_d} \in \{j_1, \dots, j_\ell\}$ such that $a_{i,j_{\alpha_1}} + \cdots + a_{i,j_{\alpha_d}} \geq 1$ for all $i = 1, \dots, m$.

Set $\mathbf{p}_i := (x_w : w \in A_i)$ for each $i = 1, \dots, m$. It is routine to check that $a_{i,j_1} + \cdots + a_{i,j_\ell} \geq 1$ for all $i = 1, \dots, m$, if and only if $\{j_1, \dots, j_\ell\} \cap A_i \neq \emptyset$ for all $i = 1, \dots, m$, if and only if $\{x_{j_1}, \dots, x_{j_\ell}\} \cap \mathbf{p}_i \neq \emptyset$ for all $i = 1, \dots, m$, if and only if $x_{j_1} \cdots x_{j_\ell} \in \mathbf{p}_i$ for all $i = 1, \dots, m$. This gives $x_{j_1} \cdots x_{j_\ell} \in \bigcap_{i=1}^m \mathbf{p}_i$, and hence $x_{j_1} \cdots x_{j_\ell} \in I(\mathcal{H})$. Because $\mathcal{H}$ is a d -uniform hypergraph, we deduce that there exists an element $u \in \mathcal{G}(I(\mathcal{H}))$ with $\deg u = d$ such that $u|x_{j_1} \cdots x_{j_\ell}$. Assume that $u = x_{j_{\alpha_1}} \cdots x_{j_{\alpha_d}}$. Thus, $j_{\alpha_1}, \dots, j_{\alpha_d} \in \{j_1, \dots, j_\ell\}$. Also, $u \in \mathcal{G}(I(\mathcal{H}))$ yields that $x_{j_{\alpha_1}} \cdots x_{j_{\alpha_d}} \in \mathbf{p}_i$ for all $i = 1, \dots, m$. Accordingly, $a_{i,j_{\alpha_1}} + \cdots + a_{i,j_{\alpha_d}} \geq 1$ for all $i = 1, \dots, m$, as required.

To conclude the proof, we need to verify sufficiency. To do this, suppose that the matrix $M(\mathcal{A})$ satisfies conditions $(\mathcal{P}1)$, $(\mathcal{P}2)(d)$, and $(\mathcal{P}3)(d)$. We define a hypergraph $\mathcal{H}$ with the vertex set $V(\mathcal{H}) = V$, and the following edge set

$$E(\mathcal{H}) = \{\{j_1, \dots, j_d\} : a_{i,j_1} + \cdots + a_{i,j_d} \geq 1 \text{ for all } i = 1, \dots, m, \\ \text{with } j_1, \dots, j_d \text{ are distinct positive integers}\}.$$

Here, we need to demonstrate $E(\mathcal{H}) \neq \emptyset$. Since $A_i \neq \emptyset$ for all $i = 1, \dots, m$, it follows that for each row i , there exists an integer j such that $a_{ij} = 1$. Hence, $a_{i,1} + \cdots + a_{i,n} \geq 1$ for all $i = 1, \dots, m$. Now, the condition $(\mathcal{P}3)(d)$ yields that there exist at least d distinct integers $j_1, \dots, j_d \in \{1, \dots, n\}$ such that $a_{i,j_1} + \cdots + a_{i,j_d} \geq 1$ for all $i = 1, \dots, m$. Accordingly, $\{j_1, \dots, j_d\} \in E(\mathcal{H})$, and so $E(\mathcal{H}) \neq \emptyset$. Certainly, $\mathcal{H}$ is a d -uniform hypergraph.

Hence, our goal is to show that every element of $\mathcal{A}$ is a minimal vertex cover set of $\mathcal{H}$. To accomplish this, we first prove that $A_1, \dots, A_m$ are vertex cover sets of $\mathcal{H}$. To do this, consider $\{j_1, \dots, j_d\} \in E(\mathcal{H})$. This implies that $a_{i,j_1} + \cdots + a_{i,j_d} \geq 1$ for all $i = 1, \dots, m$, and so $\{j_1, \dots, j_d\} \cap A_i \neq \emptyset$ for all $i = 1, \dots, m$. Therefore, A_i is a vertex cover set of $\mathcal{H}$ for all $i = 1, \dots, m$.

In the following, we claim that if C is a vertex cover set of $\mathcal{H}$, then $A_i \subseteq C$ for some $1 \leq i \leq m$. Suppose, on the contrary, that $A_i \not\subseteq C$ for each $i = 1, \dots, m$. This implies that there exists an element $f_s \in A_s \setminus C$ for all $s = 1, \dots, m$. Assume that $\{f_1, \dots, f_m\} = \{y_1, \dots, y_r\}$ such that $y_1, \dots, y_r$ are r distinct integers. Due to $\{f_1, \dots, f_m\} = \{y_1, \dots, y_r\}$, for each $i = 1, \dots, m$, we get there exists a positive integer k_i with $1 \leq k_i \leq r$ such that $y_{k_i} \in A_i$, and hence $a_{i,y_1} + \cdots + a_{i,y_r} \geq 1$ for

each $i = 1, \dots, m$. We claim that $r \geq d$. On the contrary, let $r \leq d - 1$. It follows that one can pick $d - 1 - r$ distinct integers $z_1, \dots, z_{d-1-r} \in \{1, \dots, n\} \setminus \{y_1, \dots, y_r\}$. Since $a_{i,y_1} + \dots + a_{i,y_r} \geq 1$ for each $i = 1, \dots, m$, we obtain

$$a_{i,y_1} + \dots + a_{i,y_r} + a_{i,z_1} + \dots + a_{i,z_{d-1-r}} \geq 1,$$

for each $i = 1, \dots, m$. This contradicts the condition $(\mathcal{P}2)(d)$, and so one derives $r \geq d$. Now, the condition $(\mathcal{P}3)(d)$ follows that there exist at least d distinct integers $y_{\alpha_1}, \dots, y_{\alpha_d} \in \{y_1, \dots, y_r\}$ such that $a_{i,y_{\alpha_1}} + \dots + a_{i,y_{\alpha_d}} \geq 1$ for all $i = 1, \dots, m$. We thus gain $\{y_{\alpha_1}, \dots, y_{\alpha_d}\} \in E(\mathcal{H})$. This implies that $\{y_{\alpha_1}, \dots, y_{\alpha_d}\} \cap C \neq \emptyset$, and so $\{y_1, \dots, y_r\} \cap C \neq \emptyset$. Therefore, one has $\{f_1, \dots, f_m\} \cap C \neq \emptyset$, which is a contradiction. Consequently, one derives that if C is a vertex cover set of $\mathcal{H}$, then $A_i \subseteq C$ for some $1 \leq i \leq m$.

Moreover, the condition $(\mathcal{P}1)$ gives that $A_i \not\subseteq A_j$ for each $1 \leq i, j \leq m$ with $i \neq j$. Accordingly, we deduce that $A_1, \dots, A_m$ are the minimal vertex cover sets of $\mathcal{H}$, and thus one obtains $I(\mathcal{H}) = \bigcap_{i=1}^m \mathfrak{p}_i$, where $\mathfrak{p}_i := (x_w : w \in A_i)$ for all $i = 1, \dots, m$. By virtue of [60, Corollary 1.3.6], there exists only one square-free monomial ideal such that the set of its associated prime ideals is $\{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$. This establishes that the hypergraph $\mathcal{H}$ is unique, and the proof is over. $\square$

Remark 8.1.4. *Since there is a natural bijection between the set of simple hypergraphs $\mathcal{H} = (\mathcal{X}, \mathcal{E})$ with the vertex set $\mathcal{X} = \{1, \dots, n\}$, and the set of square-free monomial ideals $I \subset S = K[x_1, \dots, x_n]$, the algebraic interpretation of Theorem 8.1.3 is characterizing the possible prime ideals in the minimal primary decomposition of a square-free monomial ideal generated in degree d .*

The following examples will illuminate Theorem 8.1.3.

Example 8.1.5. *Let $V = \{1, 2, 3, 4, 5, 6\}$, and $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$, where $A_1 := \{3, 5\}$, $A_2 := \{2, 4, 5\}$, $A_3 := \{1, 3, 6\}$, $A_4 := \{2, 5\}$, and $A_5 := \{1, 4, 6\}$. This yields that the incidence matrix associated to $\mathcal{A}$, i.e., matrix $M(\mathcal{A})$:*

$$M(\mathcal{A}) = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Since $a_{2,j} \geq a_{4,j}$, for each $1 \leq j \leq 6$, one derives the condition $(\mathcal{P}1)$ does not hold, and so there is no uniform hypergraph such as $\mathcal{H}$ with $V(\mathcal{H}) = V$, and with $\mathcal{A}$ is the set whose elements are the minimal vertex cover sets of $\mathcal{H}$.

Example 8.1.6. Let $V = \{1, 2, 3, 4, 5, 6\}$, $d = 3$, and $\mathcal{A} = \{A_i \mid i = 1, \dots, 8\}$, where $A_1 := \{3, 4, 5, 6\}$, $A_2 := \{1, 4, 5, 6\}$, $A_3 := \{2, 3, 4, 6\}$, $A_4 := \{1, 2, 3, 6\}$, $A_5 := \{1, 2, 3, 4\}$, $A_6 := \{1, 2, 4, 5\}$, $A_7 := \{1, 2, 3, 5\}$, and $A_8 := \{1, 3, 5, 6\}$. Hence, one can conclude that the incidence matrix associated to $\mathcal{A}$ is the binary matrix below.

$$M(\mathcal{A}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

Because $a_{i,1} + a_{i,4} \neq 0$, for all $i = 1, \dots, 8$, this yields that the condition $(\mathcal{P}2)(d)$ for $d = 3$, does not hold. Thus, we cannot find any 3-uniform hypergraph like $\mathcal{H}$ such that $V(\mathcal{H}) = V$, and $\mathcal{A}$ is the set whose elements are the minimal vertex cover sets of $\mathcal{H}$.

Example 8.1.7. Let $V = \{1, 2, 3, 4, 5, 6, 7\}$, $d = 3$, and $\mathcal{A} = \{A_1, A_2, A_3, A_4, A_5\}$, where $A_1 := \{1\}$, $A_2 := \{2, 5, 7\}$, $A_3 := \{3, 4, 7\}$, $A_4 := \{3, 5, 6\}$, and $A_5 := \{2, 3, 4\}$. This gives that the incidence matrix associated to $\mathcal{A}$, i.e., matrix $M(\mathcal{A})$:

$$M(\mathcal{A}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

It is routine to check that the condition $(\mathcal{P}1)$ holds. Moreover, one can see that for any $d - 1 = 2$ distinct integers $1 \leq j_1, j_2 \leq 7$, there exists an integer $1 \leq i \leq 5$ such that $a_{i,j_1} + a_{i,j_2} = 0$. So the matrix $M(\mathcal{A})$ satisfies the condition $(\mathcal{P}2)(d)$ for $d = 3$. Furthermore, we have $a_{i,1} + a_{i,4} + a_{i,6} + a_{i,7} \geq 1$ for all $i = 1, 2, 3, 4, 5$, while there are no distinct integers $j_1, j_2, j_3 \in \{1, 4, 6, 7\}$ such that $a_{i,j_1} + a_{i,j_2} + a_{i,j_3} \geq 1$ for all $i = 1, 2, 3, 4, 5$. This means that the condition $(\mathcal{P}3)(d)$ for $d = 3$, does not hold. Therefore, we are not able to build any 3-uniform hypergraph such as $\mathcal{H}$ with

$V(\mathcal{H}) = V$, and with $\mathcal{A}$ is the set whose elements are the minimal vertex cover sets of $\mathcal{H}$.

Example 8.1.8. Let $V = \{1, 2, 3, 4, 5, 6\}$, $d = 3$, and $\mathcal{A} = \{A_i \mid i = 1, \dots, 7\}$, where $A_1 := \{3, 4\}$, $A_2 := \{1, 5, 6\}$, $A_3 := \{1, 2, 6\}$, $A_4 := \{2, 4\}$, $A_5 := \{1, 3, 5\}$, $A_6 := \{2, 3\}$, and $A_7 := \{1, 4\}$. The incidence matrix associated to $\mathcal{A}$ is the matrix:

$$M(\mathcal{A}) = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

One can easily see that the condition (P1) holds. In addition, for investigating the condition (P2)(d) for $d = 3$, one has to check that for any two distinct integers $1 \leq j_1, j_2 \leq 6$, there exists a positive integer $1 \leq i \leq 7$ such that $a_{i,j_1} + a_{i,j_2} = 0$. To do this, we only need to examine it $\binom{6}{2} = 15$ times. This shows that the matrix $M(\mathcal{A})$ satisfies the condition (P2)(d) for $d = 3$. Also, for checking the condition (P3)(d) for $d = 3$, we must check that for each $6 \geq \ell \geq 3$ and distinct integers $1 \leq j_1, \dots, j_\ell \leq 6$ such that $a_{i,j_1} + \dots + a_{i,j_\ell} \geq 1$ for all $i = 1, \dots, 7$, then there exist at least three distinct integers $j_{\alpha_1}, j_{\alpha_2}, j_{\alpha_3} \in \{j_1, \dots, j_\ell\}$ such that $a_{i,j_{\alpha_1}} + a_{i,j_{\alpha_2}} + a_{i,j_{\alpha_3}} \geq 1$ for all $i = 1, \dots, 7$. When $\ell = k$, where $3 \leq k \leq 6$, one is required to investigate the condition (P3)(d) for $d = 3$, exactly $\binom{6}{\ell}$ times. This implies that we have to check the condition (P3)(d) for $d = 3$, $\binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 42$ times. After checking them, we deduce that the condition (P3)(d) for $d = 3$, holds. Now, based on Theorem 8.1.3, it follows that the hypergraph $\mathcal{H}$ with $V(\mathcal{H}) = \{1, 2, 3, 4, 5, 6\}$, and the following edge set

$$E(\mathcal{H}) = \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 4, 5\}, \{3, 4, 6\}\},$$

is the unique 3-uniform hypergraph such that $\mathcal{A} = \{A_i \mid i = 1, \dots, 7\}$ is the set whose elements are the minimal vertex cover sets of $\mathcal{H}$.

8.2 Sets defining minimal dominating sets of regular graphs

Suppose that $V = \{1, \dots, n\}$ is a non-empty set of n elements, $\mathcal{S} = \{S_1, \dots, S_m\}$ a non-empty set of m non-empty subsets of V . In this section, by using some algebraic notions in commutative algebra, we investigate the question arises whether there exists an undirected finite simple graph G with $V(G) = V$, where $\mathcal{S}$ is the set whose elements are the minimal dominating sets of G .

It should be noted that the results of this section can be found in [96].

Before asserting the main result of this section, one needs to review the following definitions. Remember that an $m \times n$ matrix $M = (a_{ij})$ is called *binary* if $a_{ij} \in \{0, 1\}$ for each $i = 1, \dots, m$ and $j = 1, \dots, n$.

Definition 8.2.1. Let $M = (a_{ij})$ be an $m \times n$ binary matrix, and r be a positive integer with $2 \leq r \leq n$. We say that M satisfies the condition:

- (i) when, for any $1 \leq i_1, i_2 \leq m$ with $i_1 \neq i_2$, there exists a positive integer $1 \leq j \leq n$ such that $a_{i_1, j} > a_{i_2, j}$.
- (ii) when, for any $r - 1$ distinct positive integers $1 \leq j_1, \dots, j_{r-1} \leq n$, there exists a positive integer $1 \leq i \leq m$ such that $a_{i, j_1} + \dots + a_{i, j_{r-1}} = 0$.
- (iii) when, for any $\ell \geq r$ distinct positive integers $1 \leq j_1, \dots, j_\ell \leq n$ such that $a_{i, j_1} + \dots + a_{i, j_\ell} \geq 1$ for all $i = 1, \dots, m$, then there exist at least r distinct integers $j_{\alpha_1}, \dots, j_{\alpha_r} \in \{j_1, \dots, j_\ell\}$ such that $a_{i, j_{\alpha_1}} + \dots + a_{i, j_{\alpha_r}} \geq 1$ for all $i = 1, \dots, m$.
- (iv) when, there exist n subsets (possibly some of them are the same) $\Gamma_t := \{\theta_{t,1}, \dots, \theta_{t,r}\} \subseteq \{1, \dots, n\}$ for any integer $1 \leq t \leq n$ such that we have
 - (a) $a_{i, \theta_{t,1}} + \dots + a_{i, \theta_{t,r}} \geq 1$ for $i = 1, \dots, m$ and $t = 1, \dots, n$. Furthermore, if $z_1, \dots, z_r$ are r distinct integers such that $a_{i, z_1} + \dots + a_{i, z_r} \geq 1$ for $i = 1, \dots, m$, then $\{z_1, \dots, z_r\} = \Gamma_s$ for some $1 \leq s \leq n$;
 - (b) for any integer $1 \leq t \leq n$ there exist r sets $\Gamma_{i_1}, \dots, \Gamma_{i_r}$ with $t \in \Gamma_{i_k}$ for all $k = 1, \dots, r$;
 - (c) $t \in \Gamma_t$ for $t = 1, \dots, n$;
 - (d) $\theta_{t,i} \in \Gamma_t$ if and only if $t \in \Gamma_{\theta_{t,i}}$ for $i = 1, \dots, r$ and $t = 1, \dots, n$, which ensures us that the resulting graph is not directed.

Definition 8.2.2. Suppose that $V = \{1, \dots, n\}$ is a non-empty set of n elements, and let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a non-empty set of m subsets of V . We define the

incidence matrix associated to $\mathcal{S}$, denoted by $M(\mathcal{S})$, as the binary matrix $M(\mathcal{S}) = (a_{ij})$ with m rows and n columns such that $a_{ij} = 0$ if $j \notin S_i$, and $a_{ij} = 1$ if $j \in S_i$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.

We are ready to state the main result of this paper in the following theorem.

Theorem 8.2.3. *Suppose that $V = \{1, \dots, n\}$ is a non-empty set of n elements, $\mathcal{S} = \{S_1, \dots, S_m\}$ a non-empty set of m non-empty subsets of V , and $2 \leq r \leq n$ a positive integer. Then there exists an undirected $(r-1)$ -regular graph G with $V(G) = V$, where $\mathcal{S}$ is the set whose elements are the minimal dominating sets of G if and only if the matrix $M(\mathcal{S}) = (a_{ij})$ satisfies conditions (i)-(iv).*

If there exists such $(r-1)$ -regular graph G , then the closed neighborhood ideal of G is given by

$$NI(G) = (x_{j_1} \cdots x_{j_r} : a_{i,j_1} + \cdots + a_{i,j_r} \geq 1 \text{ for all } i = 1, \dots, m, \\ \text{with } j_1, \dots, j_r \text{ are distinct positive integers}).$$

Proof. To establish the forward implication, assume there exists an $(r-1)$ -regular graph G with $V(G) = V$, where $\mathcal{S}$ is the set whose elements are the minimal dominating sets of G . We show that $M(\mathcal{S})$ satisfies condition (i). On the contrary, assume that condition (i) does not hold. So there exist $1 \leq i_1, i_2 \leq m$ with $i_1 \neq i_2$ such that $a_{i_1,j} \leq a_{i_2,j}$ for each $j = 1, \dots, n$. Hence, we get $S_{i_1} \subseteq S_{i_2}$, which contradicts the minimality of S_{i_2} . Accordingly, we conclude that $M(\mathcal{S})$ satisfies condition (i).

We prove that $M(\mathcal{S})$ satisfies condition (ii). On the contrary, assume that there exist $r-1$ distinct positive integers $1 \leq j_1, \dots, j_{r-1} \leq n$, such that $a_{i,j_1} + \cdots + a_{i,j_{r-1}} \neq 0$ for each $1 \leq i \leq m$. To simplify the notation, put $\mathbf{p}_i := (x_w : w \in S_i)$ for each $i = 1, \dots, m$. Due to $S_1, \dots, S_m$ are the minimal dominating sets of G , [122, Lemma 2.2] implies that $NI(G) = \cap_{i=1}^m \mathbf{p}_i$, where $NI(G)$ denotes the closed neighborhood ideal of G . The assumption gives that $\{j_1, \dots, j_{r-1}\} \cap S_i \neq \emptyset$ for each $i = 1, \dots, m$. Hence, one can conclude that $\{x_{j_1}, \dots, x_{j_{r-1}}\} \cap \mathbf{p}_i \neq \emptyset$ for each $i = 1, \dots, m$. This yields that $x_{j_1} \cdots x_{j_{r-1}} \in \mathbf{p}_i$ for each $i = 1, \dots, m$, and so $x_{j_1} \cdots x_{j_{r-1}} \in \cap_{i=1}^m \mathbf{p}_i$. Thus, there exists an element $u \in \mathcal{G}(NI(G))$ such that $u \mid x_{j_1} \cdots x_{j_{r-1}}$. So $\deg u \leq r-1$. On the other hand, since G is an $(r-1)$ -regular graph, one obtains $\deg u = r$, which is a contradiction.

Here, we demonstrate that $M(\mathcal{S})$ satisfies condition (iii). Suppose that for each $\ell \geq r$ distinct positive integers $1 \leq j_1, \dots, j_\ell \leq n$, we have $a_{i,j_1} + \cdots + a_{i,j_\ell} \geq 1$ for all $i = 1, \dots, m$. Want to show that there are at least r distinct integers $j_{\alpha_1}, \dots, j_{\alpha_r} \in \{j_1, \dots, j_\ell\}$ such that $a_{i,j_{\alpha_1}} + \cdots + a_{i,j_{\alpha_r}} \geq 1$ for all $i = 1, \dots, m$. Set $\mathbf{p}_i := (x_w : w \in S_i)$ for each $i = 1, \dots, m$. It is not hard to see that $a_{i,j_1} + \cdots + a_{i,j_\ell} \geq 1$ for all $i = 1, \dots, m$, if and only if $\{j_1, \dots, j_\ell\} \cap S_i \neq \emptyset$ for all $i = 1, \dots, m$, if and only

if $\{x_{j_1}, \dots, x_{j_\ell}\} \cap \mathfrak{p}_i \neq \emptyset$ for all $i = 1, \dots, m$, if and only if $x_{j_1} \cdots x_{j_\ell} \in \mathfrak{p}_i$ for all $i = 1, \dots, m$. This leads to $x_{j_1} \cdots x_{j_\ell} \in \cap_{i=1}^m \mathfrak{p}_i$. On account of $NI(G) = \cap_{i=1}^m \mathfrak{p}_i$ and G is an $(r-1)$ -regular graph, this implies that there exists an element $u \in \mathcal{G}(NI(G))$ with $\deg u = r$ such that $u \mid x_{j_1} \cdots x_{j_\ell}$. Let $u = x_{j_{\alpha_1}} \cdots x_{j_{\alpha_r}}$. Accordingly, we get $j_{\alpha_1}, \dots, j_{\alpha_r} \in \{j_1, \dots, j_\ell\}$. One can derive from $u \in \mathcal{G}(NI(G))$ that $x_{j_{\alpha_1}} \cdots x_{j_{\alpha_r}} \in \mathfrak{p}_i$ for all $i = 1, \dots, m$. Therefore, $a_{i,j_{\alpha_1}} + \cdots + a_{i,j_{\alpha_r}} \geq 1$ for all $i = 1, \dots, m$, as claimed.

To finish the argument, we show that $M(\mathcal{S})$ satisfies condition (iv). By virtue of G is an $(r-1)$ -regular graph, this implies that there exist n subsets $N_G[1], \dots, N_G[n]$ such that, for each $t = 1, \dots, n$, $N_G[t]$ as the closed neighborhood of the vertex $t \in V(G)$ has r elements. Set $N_G[t] := \{\theta_{t,1}, \dots, \theta_{t,r}\}$ for each $t = 1, \dots, n$. It follows from the definition of closed neighborhood ideal of G that $NI(G) = (x_{\theta_{t,1}} \cdots x_{\theta_{t,r}} : t = 1, \dots, n)$. Since $NI(G) = \cap_{i=1}^m \mathfrak{p}_i$, where $\mathfrak{p}_i = (x_w : w \in S_i)$ for each $i = 1, \dots, m$, this gives that $x_{\theta_{t,1}} \cdots x_{\theta_{t,r}} \in \mathfrak{p}_i$ for each $i = 1, \dots, m$ and $t = 1, \dots, n$. Fix $1 \leq t \leq n$. Hence, we get $\{\theta_{t,1}, \dots, \theta_{t,r}\} \cap S_i \neq \emptyset$ for each $i = 1, \dots, m$, and so $a_{i,\theta_{t,1}} + \cdots + a_{i,\theta_{t,r}} \geq 1$ for each $i = 1, \dots, m$. Now, assume that $z_1, \dots, z_r$ are r distinct integers such that $a_{i,z_1} + \cdots + a_{i,z_r} \geq 1$ for $i = 1, \dots, m$. This implies that $\{z_1, \dots, z_r\} \cap S_i \neq \emptyset$ for each $i = 1, \dots, m$, and so $\{x_{z_1}, \dots, x_{z_r}\} \cap \mathfrak{p}_i \neq \emptyset$ for all $i = 1, \dots, m$. We can deduce that $x_{z_1} \cdots x_{z_r} \in \mathfrak{p}_i$ for all $i = 1, \dots, m$, and hence $x_{z_1} \cdots x_{z_r} \in \cap_{i=1}^m \mathfrak{p}_i$. As $NI(G) = \cap_{i=1}^m \mathfrak{p}_i$ and $NI(G) = (x_{\theta_{t,1}} \cdots x_{\theta_{t,r}} : t = 1, \dots, n)$, we get there exists an element $x_{\theta_{s,1}} \cdots x_{\theta_{s,r}} \in \mathcal{G}(NI(G))$ for some $1 \leq s \leq n$ such that $x_{\theta_{s,1}} \cdots x_{\theta_{s,r}} \mid x_{z_1} \cdots x_{z_r}$. Since both of $x_{\theta_{s,1}} \cdots x_{\theta_{s,r}}$ and $x_{z_1} \cdots x_{z_r}$ are square-free and in the same degree, we thus obtain that $x_{\theta_{s,1}} \cdots x_{\theta_{s,r}} = x_{z_1} \cdots x_{z_r}$, and therefore $\{z_1, \dots, z_r\} = \Gamma_s$, as required.

Moreover, since G is an $(r-1)$ -regular graph, we derive there exist r sets $N_G[i_1], \dots, N_G[i_r]$ with $t \in N_G[i_k]$ for all $k = 1, \dots, r$. Since G is undirected, this is equivalent to this statement that $\theta_{t,i} \in N_G[t]$ if and only if $t \in N_G[\theta_{t,i}]$ for $i = 1, \dots, r$ and $t = 1, \dots, n$. This yields that $M(\mathcal{S})$ satisfies condition (iv).

Conversely, assume that the matrix $M(\mathcal{S})$ satisfies conditions (i)-(iv). Our aim is to verify the existence of an $(r-1)$ -regular graph G with $V(G) = V$ where $\mathcal{S}$ is the set whose elements are the minimal dominating sets of G . Condition (iv) gives that there exist n subsets $\Gamma_t = \{\theta_{t,1}, \dots, \theta_{t,r}\} \subseteq \{1, \dots, n\}$ for any integer $1 \leq t \leq n$ such that $a_{i,\theta_{t,1}} + \cdots + a_{i,\theta_{t,r}} \geq 1$ for $i = 1, \dots, m$ and $t = 1, \dots, n$, and also for any integer $1 \leq t \leq n$ there exist r sets $\Gamma_{i_1}, \dots, \Gamma_{i_r}$ with $t \in \Gamma_{i_k}$ for all $k = 1, \dots, r$, $t \in \Gamma_t$, and $\theta_{t,i} \in \Gamma_t$ if and only if $t \in \Gamma_{\theta_{t,i}}$ for all $i = 1, \dots, r$. Hence, we define an $(r-1)$ -regular graph G with $N_G[t] := \Gamma_t = \{\theta_{t,1}, \dots, \theta_{t,r}\}$ for any integer $1 \leq t \leq n$. In other words, conditions (iv)(a)-(d) guarantee the existence of such a graph G . In particular, one can conclude from the definition of closed neighborhood ideal of G that $NI(G) = (x_{\theta_{t,1}} \cdots x_{\theta_{t,r}} : t = 1, \dots, n)$.

In what follows, we prove that every element of $\mathcal{S}$ is a minimal dominating set

of G . To see this, we first show that $S_1, \dots, S_m$ are dominating sets of G . To do this, assume that $N_G[t] = \{\theta_{t,1}, \dots, \theta_{t,r}\}$ is the closed neighborhood of the vertex t with $1 \leq t \leq n$. By condition (iv)(a), we have that $a_{i,\theta_{t,1}} + \dots + a_{i,\theta_{t,r}} \geq 1$ for all $i = 1, \dots, m$. Since, for all $i = 1, \dots, m$, the i -th row of the matrix $M(\mathcal{S})$ is associated to S_i and $a_{i,\theta_{t,1}} + \dots + a_{i,\theta_{t,r}} \geq 1$, we obtain $\{\theta_{t,1}, \dots, \theta_{t,r}\} \cap S_i \neq \emptyset$ for all $i = 1, \dots, m$. Consequently, $N_G[t] \cap S_i \neq \emptyset$, and so S_i is a dominating set of G for all $i = 1, \dots, m$.

Here, we claim that if D is an arbitrary dominating set of G , then $S_i \subseteq D$ for some $1 \leq i \leq m$. On the contrary, assume that $S_i \not\subseteq D$ for each $i = 1, \dots, m$. Thus there exists an element $f_i \in S_i \setminus D$ for all $i = 1, \dots, m$. Suppose that $\cup_{i=1}^m \{f_i\} = \{y_1, \dots, y_\lambda\}$ such that $y_1 < \dots < y_\lambda$ are λ distinct integers. Because $\cup_{i=1}^m \{f_i\} = \{y_1, \dots, y_\lambda\}$, we get $f_i \in \{y_1, \dots, y_\lambda\}$ for all $i = 1, \dots, m$. Thus, we deduce that there exists a positive integer k_i with $1 \leq k_i \leq \lambda$ such that $f_i = y_{k_i}$ for all $i = 1, \dots, m$. Due to $f_i \in S_i$ for all $i = 1, \dots, m$, this implies that $y_{k_i} \in S_i$ for all $i = 1, \dots, m$. Since, for all $i = 1, \dots, m$, the i -th row of the matrix $M(\mathcal{S})$ is associated to S_i , $y_{k_i} \in S_i$, and $1 \leq k_i \leq \lambda$, this yields that $a_{i,y_1} + \dots + a_{i,y_\lambda} \geq 1$ for each $i = 1, \dots, m$. We prove that $\lambda \geq r$. Suppose, on the contrary, that $\lambda \leq r - 1$. If $\lambda = r - 1$, then $a_{i,y_1} + \dots + a_{i,y_{r-1}} \geq 1$ for each $i = 1, \dots, m$, which is a contradiction with condition (ii). Now, let $\lambda < r - 1$, then this gives that one can choose $r - 1 - \lambda$ distinct integers $z_1, \dots, z_{r-1-\lambda} \in \{1, \dots, n\} \setminus \{y_1, \dots, y_\lambda\}$. On account of $a_{i,y_1} + \dots + a_{i,y_\lambda} \geq 1$ for each $i = 1, \dots, m$, one obtains $a_{i,y_1} + \dots + a_{i,y_\lambda} + a_{i,z_1} + \dots + a_{i,z_{r-1-\lambda}} \geq 1$ for each $i = 1, \dots, m$. This contradicts condition (ii), and hence we derive $\lambda \geq r$. It follows from condition (iii) that there exist at least r distinct integers $y_{\alpha_1}, \dots, y_{\alpha_r} \in \{y_1, \dots, y_\lambda\}$ such that $a_{i,y_{\alpha_1}} + \dots + a_{i,y_{\alpha_r}} \geq 1$ for all $i = 1, \dots, m$. In view of condition (iv)(a), this implies that $\{y_{\alpha_1}, \dots, y_{\alpha_r}\} = \Gamma_s = N_G[s]$ for some $1 \leq s \leq n$. Thus, $\{y_{\alpha_1}, \dots, y_{\alpha_r}\} \cap D \neq \emptyset$, and so $\{y_1, \dots, y_\lambda\} \cap D \neq \emptyset$. This yields that $(\cup_{i=1}^m \{f_i\}) \cap D \neq \emptyset$, which is a contradiction. Hence, one can conclude that if D is an arbitrary dominating set of G , then $S_i \subseteq D$ for some $1 \leq i \leq m$. On the other hand, condition (i) gives that $S_i \not\subseteq S_j$ for each $1 \leq i, j \leq m$ with $i \neq j$. Consequently, we derive that $S_1, \dots, S_m$ are the minimal dominating sets of G , and so the proof is over. $\square$

The next example illustrates how we can employ Theorem 8.2.3.

Example 8.2.4. Assume that $V = \{x_1, \dots, x_6\}$ and the S_i 's are a non-empty set of V as follows:

$$S_1 = \{x_1, x_2\}, S_2 = \{x_1, x_4\}, S_3 = \{x_1, x_6\}, S_4 = \{x_2, x_3\}, S_5 = \{x_2, x_5\},$$

$$S_6 = \{x_3, x_4\}, S_7 = \{x_3, x_6\}, S_8 = \{x_4, x_5\}, S_9 = \{x_5, x_6\}, S_{10} = \{x_1, x_3, x_5\},$$

$$S_{11} = \{x_2, x_4, x_6\}.$$

Conditions (ii) and (iii) force us to seek a 3-regular graph G . It is not hard to observe that the matrix $M(\mathcal{S})$ associated to $\mathcal{S}$ is the following matrix:

$$M(\mathcal{S}) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

It is easy to investigate that condition (i) holds. In order to show condition (ii), we should check that for each $r - 1 = 3$ distinct integers $1 \leq j_1, j_2, j_3 \leq 6$, there exists a positive integer $1 \leq i \leq 11$ such that $a_{i,j_1} + a_{i,j_2} + a_{i,j_3} = 0$. To do this, we have to examine it $\binom{6}{3} = 20$ times. This proves that $M(\mathcal{S})$ satisfies condition (ii). Moreover, for verifying condition (iii), one must check that for each $4 \leq \ell \leq 6$ and distinct integers $1 \leq j_1, \dots, j_\ell \leq 6$ such that $a_{i,j_1} + \dots + a_{i,j_\ell} \geq 1$ for all $i = 1, \dots, 11$, then there exist at least four distinct integers $j_{\alpha_1}, j_{\alpha_2}, j_{\alpha_3}, j_{\alpha_4} \in \{j_1, \dots, j_\ell\}$ such that $a_{i,j_{\alpha_1}} + a_{i,j_{\alpha_2}} + a_{i,j_{\alpha_3}} + a_{i,j_{\alpha_4}} \geq 1$ for all $i = 1, \dots, 11$. Hence, we have to check condition (iii) exactly $\binom{6}{4} + \binom{6}{5} + \binom{6}{6} = 22$ times. After checking them, we derive that condition (iii) holds. Finally, direct computations show that after checking $\binom{6}{4} = 15$ cases, there exist exactly six subsets $\Gamma_1 := \{1, 2, 4, 6\}$, $\Gamma_2 := \{1, 2, 3, 5\}$, $\Gamma_3 := \{2, 3, 4, 6\}$, $\Gamma_4 := \{1, 3, 4, 5\}$, $\Gamma_5 := \{2, 4, 5, 6\}$, and $\Gamma_6 := \{1, 3, 5, 6\}$ which satisfy condition (iv).

Since conditions (i)-(iv) hold, according to Theorem 8.2.3, one can deduce that the closed neighborhood ideal of G is given by

$$NI(G) = (x_1x_2x_4x_6, x_1x_2x_3x_5, x_2x_3x_4x_6, x_1x_3x_4x_5, x_2x_4x_5x_6, x_1x_3x_5x_6).$$

Since $x_1x_2x_4x_6 \in NI(G)$, one may consider the following cases:

Case 1. $N_G(1) = \{2, 4, 6\}$. Because $x_1x_2x_3x_5 \in NI(G)$, one can conclude that $N_G(2) = \{1, 3, 5\}$. Furthermore, thanks to $N_G(2) = \{1, 3, 5\}$ and $x_2x_3x_4x_6 \in NI(G)$, this implies that $N_G(3) = \{2, 4, 6\}$. Due to $N_G(3) = \{2, 4, 6\}$ and $x_1x_3x_4x_5 \in NI(G)$, we get $N_G(4) = \{1, 3, 5\}$. It follows from $N_G(4) = \{1, 3, 5\}$ and $x_2x_4x_5x_6 \in NI(G)$

that $N_G(5) = \{2, 4, 6\}$. Finally, since $N_G(5) = \{2, 4, 6\}$ and $x_1x_3x_5x_6 \in NI(G)$, this gives that $N_G(6) = \{1, 3, 5\}$. Therefore, we deduce that G is the graph G_1 which has been shown in the figure below.

Case 2. $N_G(2) = \{1, 4, 6\}$ and $N_G(4) = \{2, 3, 6\}$. A similar argument yields that $N_G(1) = \{2, 3, 5\}$, $N_G(3) = \{1, 4, 5\}$, $N_G(5) = \{1, 3, 6\}$, and $N_G(6) = \{2, 4, 5\}$. We thus gain that G is isomorphic to G_2 which has been shown in the figure below.

Case 3. $N_G(2) = \{1, 4, 6\}$ and $N_G(6) = \{2, 3, 4\}$. Following a similar discussion, we get $N_G(1) = \{2, 3, 5\}$, $N_G(3) = \{1, 5, 6\}$, $N_G(4) = \{2, 5, 6\}$, and $N_G(5) = \{1, 3, 4\}$. Consequently, one has G is isomorphic to G_2 .

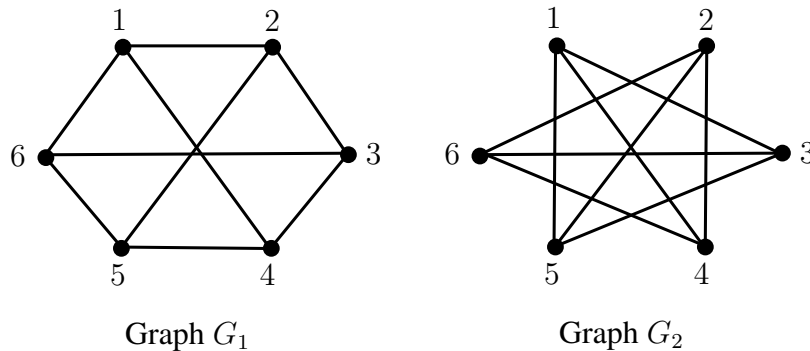
Case 4. $N_G(4) = \{1, 2, 6\}$ and $N_G(3) = \{1, 2, 5\}$. A similar discussion gives rise to $N_G(1) = \{3, 4, 5\}$, $N_G(2) = \{3, 4, 6\}$, $N_G(5) = \{1, 3, 6\}$, and $N_G(6) = \{2, 4, 5\}$. One can easily check that G is isomorphic to G_2 .

Case 5. $N_G(4) = \{1, 2, 6\}$ and $N_G(5) = \{1, 2, 3\}$. A similar argument implies that $N_G(1) = \{3, 4, 5\}$, $N_G(2) = \{4, 5, 6\}$, $N_G(3) = \{1, 5, 6\}$, and $N_G(6) = \{2, 3, 4\}$. It is not hard to investigate that G is isomorphic to G_2 .

Case 6. $N_G(6) = \{1, 2, 4\}$ and $N_G(5) = \{1, 2, 3\}$. By a similar discussion, we obtain that $N_G(1) = \{3, 5, 6\}$, $N_G(2) = \{4, 5, 6\}$, $N_G(3) = \{1, 4, 5\}$, and $N_G(4) = \{2, 3, 6\}$. One can rapidly check that G is isomorphic to G_2 .

Case 7. $N_G(6) = \{1, 2, 4\}$ and $N_G(3) = \{1, 2, 5\}$. A similar argument yields that $N_G(1) = \{3, 5, 6\}$, $N_G(2) = \{3, 4, 6\}$, $N_G(4) = \{2, 5, 6\}$, and $N_G(5) = \{1, 3, 4\}$. We can easily see that G is isomorphic to G_2 .

Since G_1 has no induced odd cycle, this implies that G_1 is bipartite, while G_2 has an induced odd cycle, and so is non-bipartite. This yields that G_1 and G_2 are non-isomorphic.



Chapter 9

On the max-path ideals and matroidal path ideals

9.1 Maximal paths and path cover sets

Monomial ideals play a fundamental role in studying the connection between commutative algebra and combinatorics. The relation between these two subjects allows us to use techniques and methods in commutative algebra to solve combinatorial problems, and vice versa. So, commutative algebraists have started exploring the properties of finite simple graphs through monomial ideals, see [91] for more information.

In the following text, our major goal is to introduce two new notions which are related to paths in a finite simple graph. In fact, similar to edge ideals and cover ideals, we define the notions max-path ideals and path cover ideals associated to simple finite graphs. It should be noted that the results of this section can be found in [123].

Let G be a graph on the vertex set $[n] = \{1, \dots, n\}$. Then the sequence of distinct vertices $i_1, \dots, i_r$ is called a *maximal path* when $i_1, \dots, i_r$ is a path and furthermore, for all $j \in \{1, \dots, n\} \setminus \{i_1, \dots, i_r\}$, none of $j, i_1, \dots, i_r$ and $i_1, \dots, i_r, j$ is a path.

For instance, let $V(G) := [4]$ and $E(G) := \{\{1, 2\}, \{1, 3\}, \{2, 4\}, \{3, 4\}\}$. Then the sequences $1, 2, 4, 3$ and $1, 3, 4, 2$ are two maximal paths in G .

Next, we define path cover sets of a graph.

Definition 9.1.1. *Let G be a graph on the vertex set $[n]$. Then $W \subseteq [n]$ is called a path cover set when, for any maximal path $i_1, \dots, i_t$ in G , we have*

$$W \cap \{i_1, \dots, i_t\} \neq \emptyset.$$

Example 9.1.2. If G is a cycle or a complete graph or a path graph, then for each $v \in V(G)$, the set $\{v\}$ is a path cover set.

Example 9.1.3. Let G be a graph on the vertex set $[6] = \{1, 2, 3, 4, 5, 6\}$ such that

$$E(G) := \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 5\}, \{4, 5\}, \{5, 6\}\}.$$

Then $W_1 := \{3, 5\}$ and $W_2 := \{1, 4\}$ are some path cover sets of G .

Definition 9.1.4. Suppose that G is a graph and set

$$\mathcal{I}_G := \min\{|W| : W \text{ is a path cover set of } G\}.$$

We call $\mathcal{I}_G$ the index of covering of G .

Clearly, if G is a cycle or a complete graph or a path graph, then $\mathcal{I}_G = 1$.

Example 9.1.5. Assume that G is the graph of Example 9.1.3. Then we can easily see that $\mathcal{I}_G = 2$.

In this section, we focus on the case that G is a tree. Furthermore, from now on, we consider the graphs K_n, C_n , and P_n , as the complete, the cycle, and the path graphs on the vertex set $[n] = \{1, \dots, n\}$, respectively.

In the next proposition we present some elementary properties of maximal paths in a tree.

Proposition 9.1.6. Suppose that T is a tree. Then

- (i) The path $i_1, \dots, i_t$ is a maximal path if and only if both i_1 and i_t belong to L_T .
- (ii) If $|L_T| = d$, then T has exactly $\binom{d}{2}$ maximal paths.
- (iii) If $i_1, \dots, i_t$ and $j_1, \dots, j_s$ are two distinct maximal paths, then we have

$$\{i_1, \dots, i_t\} \not\subseteq \{j_1, \dots, j_s\} \text{ and } \{j_1, \dots, j_s\} \not\subseteq \{i_1, \dots, i_t\}.$$

The following proposition is an immediate consequence of Proposition 9.1.6(i) which presents an upper bound for the index of covering of a tree.

Proposition 9.1.7. Suppose that T is a tree, $L_T = \{r_1, \dots, r_d\}$ and $W \subseteq L_T$. Then W is a minimal path cover set of T if and only if $|W| = d - 1$. In particular, $\mathcal{I}_T \leq d - 1$.

In the next theorem we present an upper bound for the number of elements of any minimal path cover set of a tree.

Theorem 9.1.8. Suppose that T is a tree and $|L_T| = d$. If W is a minimal path cover set of T , then $|W| \leq d - 1$.

Proof. We proceed by induction on $n := |V(T)|$. The case $n = 1$ is clear. Now, suppose, inductively, that $n > 1$ and that the result has been proved for all trees which have vertex sets with less than n elements. Let W be a minimal path cover set of a tree T and $V(T) = [n]$. If $W \subseteq L_T$ and $|L_T| = d$, then, in view of Proposition 9.1.7, it follows that $|W| = d - 1$. Hence, assume that $W \not\subseteq L_T$. Consider $v \in W \setminus L_T$ and set $\ell := \deg_T v$. Then $T \setminus \{v\}$ has exactly ℓ connected components such as $T_1, \dots, T_\ell$. Put $W_i := W \cap V(T_i)$ for all $i = 1, \dots, \ell$. It follows that there exists a unique vertex $w_i \in V(T_i)$ such that $\{v, w_i\} \in E(T)$ for all $i = 1, \dots, \ell$. Hence, for every $i = 1, \dots, \ell$, we may consider one of the following four cases:

Case 1. $\deg_T w_i \geq 3$. We show that W_i is a minimal path cover set of T_i . Let $r_1, \dots, r_s$ be a maximal path in T_i . It is easy to see that $L_{T_i} = L_T \cap V(T_i)$. Hence, $r_1, \dots, r_s$ is a maximal path in T . It follows that $\{r_1, \dots, r_s\} \cap W_i \neq \emptyset$, and so W_i is a path cover set of T_i . Now, let $V \subsetneq W_i$ be a minimal path cover set of T_i . Put

$$V' := \{v\} \cup \left(\bigcup_{j=1, j \neq i}^{\ell} W_j \right) \cup V.$$

It is easy to see that V' is a path cover set of T and $V' \subsetneq W$, which is the required contradiction. This shows that W_i is a minimal path cover set of T_i .

Case 2. $\deg_T w_i = 2$ and T_i is a path graph. By minimality of W , it follows that $W_i = \emptyset$.

Case 3. $\deg_T w_i = 2$ and T_i is not a path graph. Then T_i is the union of a path P_i and a tree T'_i such that if v'_i is the cut-point of this union, then $\deg_T v'_i \geq 3$. Hence, we can assume that $V(P_i) = \{v'_i, v_1, \dots, v_t, w_i\}$. By minimality of W , we have $W \cap (V(P_i) \setminus \{v'_i\}) = \emptyset$, and so $W_i = W \cap V(T'_i)$. Since $L_{T'_i} = L_T \cap V(T_i)$ and $\deg_T v'_i \geq 3$, similar to Case 1, one can conclude that W_i is a minimal path cover set of T'_i .

Case 4. $\deg_T w_i = 1$. Then $V(T_i) = \{w_i\}$, and so $W_i = \emptyset$.

Set $d_i := |L_T \cap V(T_i)|$ for all $i = 1, \dots, \ell$ and suppose that Case 1 occurs for $T_1, \dots, T_t$, Case 2 occurs for $T_{t+1}, \dots, T_k$, Case 3 occurs for $T_{k+1}, \dots, T_m$ and finally Case 4 occurs for $T_{m+1}, \dots, T_\ell$. Since $W = (\bigcup_{i=1}^{\ell} W_i) \cup \{v\}$, we get $|W| = \sum_{i=1}^{\ell} |W_i| + 1$. This implies that

$$\begin{aligned} |W| &= \sum_{i=1}^t |W_i| + \sum_{i=t+1}^k |W_i| + \sum_{i=k+1}^m |W_i| + \sum_{i=m+1}^{\ell} |W_i| + 1 \\ &\leq \sum_{i=1}^t d_i - t + 0 + \sum_{i=k+1}^m d_i - m + k + 0 + 1. \end{aligned}$$

On the other hand, we have the following equalities

$$\begin{aligned} d &= \sum_{i=1}^{\ell} d_i \\ &= \sum_{i=1}^t d_i + k - t + \sum_{i=k+1}^m d_i + \ell - m. \end{aligned}$$

This implies that $|W| \leq d - \ell + 1$. Since $v \notin L_T$, one has $\ell \geq 2$. Therefore, $|W| \leq d - 1$, and the proof is complete. $\square$

Next, we are going to find a better bound for the index of covering of a tree T which is not a path graph. For this purpose, we require the following lemma.

Lemma 9.1.9. *Let T be a tree, P be a maximal path, and P' be a path in T . If $V(P') \not\subseteq V(P)$ and $V(P) \cap V(P') \neq \emptyset$, then there exists $v \in V(P) \cap V(P')$ with $\deg_T v \geq 3$.*

Proof. The proof is straightforward. $\square$

Proposition 9.1.10. *Suppose that T is a tree which is not a path graph. Also, let S_T be the set of all vertices with degree greater than two in T . Then S_T is a path cover set of T . In particular, $\mathcal{I}_T \leq |S_T| \leq |L_T| - 2$.*

Proof. Let T be an arbitrary tree. It is easy to see that $|S_T| \leq |L_T| - 2$. So, it is enough to show that S_T is a path cover set of T when T is not a path graph. Suppose, on the contrary, that there exists a maximal path $P : v_1, \dots, v_s$ in T such that $V(P) \cap S_T = \emptyset$. Hence, $\deg_T v_j = 2$ for all $j = 2, \dots, s-1$. Choose an arbitrary vertex v_j in $V(P)$. Since T is not a path graph, it follows that there is a vertex such as w in $V(T)$ such that $\deg_T w \geq 3$, and so $w \notin V(P)$. By connectivity of T , one can conclude that there exists unique path P' in T between w and v_j . Clearly, $V(P') \cap V(P) \neq \emptyset$. Due to $V(P') \not\subseteq V(P)$, by virtue of Lemma 9.1.9, there exists $v \in V(P) \cap V(P')$ with $\deg_T v \geq 3$, which is a contradiction. Therefore, S_T is a path cover set of T , as claimed. $\square$

9.1.1 Max-path and path cover ideals

In this subsection, we turn to the study of monomial ideals arising from maximal paths of a graph G .

Definition 9.1.11. *Let G be a graph on the vertex set $[n]$. Then the ideal*

$$PI(G) := (x_{i_1} \cdots x_{i_t} : i_1, \dots, i_t \text{ is a maximal path in } G) \subset R = K[x_1, \dots, x_n],$$

is called the max-path ideal of G , and also

$PC(G) := (x_{j_1} \cdots x_{j_s} : \{j_1, \dots, j_s\} \text{ is a path cover set of } G) \subset R = K[x_1, \dots, x_n]$,

is called the path cover ideal of G .

We now focus on max-path ideals and path cover ideals of trees. Note that two different graphs on the vertex set $[n]$ may have the same max-path ideal. For instance, consider the graphs K_n , C_n , and P_n with vertex set $[n]$. Clearly, $PI(K_n) = PI(C_n) = PI(P_n) = (x_1 \cdots x_n)$ and $PC(K_n) = PC(C_n) = PC(P_n) = (x_1, \dots, x_n)$. In fact, in the graph K_n we have $n!/2$ different maximal paths that all of them are corresponded to the generator $x_1 \cdots x_n$ in $PI(K_n)$.

As another example, let G be a graph on $[5] = \{1, 2, 3, 4, 5\}$ with

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{2, 4\}, \{4, 5\}\}.$$

Then

$$\{L_1 = 1, 2, 4, 5, L_2 = 1, 2, 3, 4, 5, L_3 = 3, 4, 2, 1, L_4 = 3, 2, 4, 5\}$$

is the set of all maximal paths in G ,

$$PI(G) = (x_1x_2x_4x_5, x_1x_2x_3x_4, x_2x_3x_4x_5),$$

and

$$PC(G) = (x_2, x_4, x_1x_5, x_1x_3, x_3x_5).$$

In the following proposition we study some properties of max-path and path cover ideals.

Proposition 9.1.12. *Let G be a graph on the vertex set $[n]$. Then*

(i)

$$PI(G) = \bigcap_{\{j_1, \dots, j_\ell\} \text{ is a minimal path cover set of } G} (x_{j_1}, \dots, x_{j_\ell}),$$

is the minimal primary decomposition of $PI(G)$.

(ii) $PC(G) = PI(G)^\vee$, where $I^\vee$ means the Alexander dual ideal of I .

(iii) $\text{ht}(PI(G)) = \mathcal{I}_G$.

Proof. (i) By using the definition of the max-path ideal of G , we only need to show that

$$PI(G) \supseteq \bigcap_{\{j_1, \dots, j_\ell\} \text{ is a minimal path cover set of } G} (x_{j_1}, \dots, x_{j_\ell}).$$

Let $\mathfrak{p} = (x_{j_1}, \dots, x_{j_\ell})$ be a minimal prime ideal of $PI(G)$. We prove that $\{j_1, \dots, j_\ell\}$ is a minimal path cover set of G . To do this, let $i_1, \dots, i_t$ be an arbitrary maximal path in G . Then $x_{i_1} \cdots x_{i_t} \in PI(G)$. Since $PI(G) \subseteq \mathfrak{p}$, it follows that there exists an integer $1 \leq r \leq \ell$ such that $x_{j_r} | x_{i_1} \cdots x_{i_t}$. Thus, $x_{j_r} \in \{x_{i_1}, \dots, x_{i_t}\}$, and so $j_r \in \{i_1, \dots, i_t\}$. This implies that

$$\{j_1, \dots, j_\ell\} \cap \{i_1, \dots, i_t\} \neq \emptyset.$$

Therefore, $\{j_1, \dots, j_\ell\}$ is a path cover set of G . Now, suppose, on the contrary, that $\{j_1, \dots, j_\ell\}$ is not a minimal path cover set. Hence, there exists a minimal path cover set W such that $W \subsetneq \{j_1, \dots, j_\ell\}$. It is easy to see that

$$PI(G) \subseteq (x_\ell : \ell \in W) \subsetneq \mathfrak{p},$$

which is a contradiction with the minimality of $\mathfrak{p}$.

(ii) It is clear by [60, Corollary 1.5.5].

(iii) It follows that from part (i).

Proposition 9.1.13. *Suppose that T is a tree which has exactly d leaves. Then we have*

(i) $\text{ht}(PC(T)) = \min\{d(i, j) + 1 : i, j \in L_T\}$, where by $d(i, j)$ we mean the distance between two vertices i and j .

(ii) If $i_1, \dots, i_r$ is a maximal path in T , then $x_{i_1} \cdots x_{i_r} \in \mathcal{G}(PI(T))$.

(iii) If T is not a path graph, then $\text{ht}(PI(T)) \leq d - 2$.

(iv) $\nu(PI(T)) = \binom{d}{2}$, where by $\nu(I)$ we mean the minimal number of homogeneous generators of a graded ideal I .

(v) $\text{ht} \mathfrak{p} \leq d - 1$ for all $\mathfrak{p} \in \text{Ass}(PI(T))$.

(vi) If $L_T = \{i_1, \dots, i_d\}$, then, for all $j = 1, \dots, d$, we have

$$\mathfrak{p}_j = (x_{i_1}, \dots, \widehat{x_{i_j}}, \dots, x_{i_d}) \in \text{Ass}(PI(T)).$$

Proof. It is enough to apply Propositions 9.1.12, 9.1.6, 9.1.7, 9.1.10, and Theorem 9.1.8. $\square$

In the following corollary, we characterize trees in which the max-path ideals are unmixed (or Cohen-Macaulay).

Corollary 9.1.14. *Let T be a tree. Then the following conditions are equivalent.*

(i) T is the path graph.

(ii) $PI(T)$ is Cohen-Macaulay.

(iii) $PI(T)$ is unmixed.

Proof. Suppose that T has the vertex set $[n]$ and also has exactly d leaves. Parts (i) $\implies$ (ii) and (ii) $\implies$ (iii) are clear. Hence, the only non-trivial part is (iii) $\implies$ (i). To do this, assume that $PI(T)$ is an unmixed ideal. If $d = 2$, then T is a path graph, and so there is nothing to prove. Hence, assume that $d \geq 3$, and thus T is not the path graph. By Proposition 9.1.13(iii), we have $\text{ht}(PI(T)) \leq d - 2$ and by Proposition 9.1.13(vi), $PI(T)$ has some associated primes of height $d - 1$, which is a contradiction. $\square$

Next corollary is another application of Propositions 9.1.12 and 9.1.13.

Corollary 9.1.15. *Let T be a tree on the vertex set $[n]$ and $v \in V(T)$. Then there exists a minimal path cover set of T which contains v .*

Proof. If $v \in V(T)$, then it is clear that we can find at least one maximal path in T such as $i_1, \dots, i_r$ that passes through v . So $x_v | x_{i_1} \cdots x_{i_r}$. Due to Proposition 9.1.13, it follows that $x_{i_1} \cdots x_{i_r} \in \mathcal{G}(PI(T))$. On the other hand, according to Proposition 9.1.12, one can conclude that

$$PI(T) = \bigcap_{\{j_1, \dots, j_\ell\} \text{ is a minimal path cover set of } T} (x_{j_1}, \dots, x_{j_\ell}),$$

is a minimal primary decomposition. Thus, there exists a minimal path cover set of T which contains v , as claimed. $\square$

Definition 9.1.16. ([20, P. 1190]) *Let G be a graph and let k be a positive integer. By the order of a path P we mean the number of vertices on P . A subset of vertices $S \subseteq V(G)$ is called a k -path vertex cover if every path of order k in G contains at least one vertex from S . Also, we denote by $\psi_k(G)$ the minimum cardinality of a k -path vertex cover in G .*

Let G be a graph on the vertex set $[n]$ and $t \leq n$ is a positive integer. We recall that $P_t(G)$, the path ideal of length t , is the monomial ideal generated by all monomials $x_{i_1} \cdots x_{i_{t+1}}$, where $i_1, \dots, i_{t+1}$ is a path of length t in G .

In a similar way, we define the ideals $PI_t(G)$ and $PC_t(G)$ in the following

$$PI_t(G) = (x_{i_1} \cdots x_{i_{t+1}} : i_1, \dots, i_{t+1} \text{ is a path of length } t \text{ in } G),$$

and

$$PC_t(G) = (x_{j_1} \cdots x_{j_\ell} : \{j_1, \dots, j_\ell\} \text{ is a } (t+1)\text{-path vertex cover of } G).$$

It is necessary to note that $PI_t(G) = P_t(G)$ for all $t \in \mathbb{N}$.

Proposition 9.1.17. *Let G be a graph on the vertex set $[n]$. Then*

(i)

$$PI_t(G) = \bigcap_{\{j_1, \dots, j_\ell\} \text{ is a minimal } (t+1)\text{-path vertex cover of } G} (x_{j_1}, \dots, x_{j_\ell})$$

is the minimal primary decomposition of $PI_t(G)$.

(ii) *If $i_1, \dots, i_{t+1}$ is a path of length t , then $x_{i_1} \cdots x_{i_{t+1}} \in \mathcal{G}(PI_t(G))$.*

(iii) $PC_t(G) = PI_t(G)^\vee$, *where $I^\vee$ means the Alexander dual ideal of I .*

(iv) $\text{ht}(PI_t(G)) = \psi_{t+1}(G)$.

(v) $\text{ht}(PC_t(G)) = t + 1$.

(vi) *If G is a tree, then $\text{ht}(PI_t(G)) \leq n/(t + 1)$.*

Proof. (i) Let $\mathfrak{p} = (x_{j_1}, \dots, x_{j_\ell})$ be a minimal prime ideal of $PI_t(G)$. We prove that $\{j_1, \dots, j_\ell\}$ is a minimal $(t + 1)$ -path vertex cover of G . For this purpose, assume that $i_1, \dots, i_{t+1}$ is an arbitrary path of order $t + 1$ in G . Then $x_{i_1} \cdots x_{i_{t+1}} \in PI_t(G)$. Since $PI_t(G) \subseteq \mathfrak{p}$, this implies that there exists an integer $1 \leq r \leq \ell$ such that $x_{j_r} | x_{i_1} \cdots x_{i_{t+1}}$. Hence, $x_{j_r} \in \{x_{i_1}, \dots, x_{i_{t+1}}\}$, and so $j_r \in \{i_1, \dots, i_{t+1}\}$. It follows that $\{j_1, \dots, j_\ell\} \cap \{i_1, \dots, i_{t+1}\} \neq \emptyset$. Therefore, $\{j_1, \dots, j_\ell\}$ is a $(t + 1)$ -path vertex cover of G . Now, suppose, on the contrary, that $\{j_1, \dots, j_\ell\}$ is not a minimal $(t + 1)$ -path vertex cover of G . Hence, there exists a minimal $(t + 1)$ -path vertex cover W such that $W \subsetneq \{j_1, \dots, j_\ell\}$. It is easy to see that $PI_t(G) \subseteq (x_\ell : \ell \in W) \subsetneq \mathfrak{p}$, which is a contradiction with the minimality of $\mathfrak{p}$. So, we get

$$PI_t(G) \supseteq \bigcap_{\{j_1, \dots, j_\ell\} \text{ is a minimal } (t+1)\text{-path vertex cover of } G} (x_{j_1}, \dots, x_{j_\ell}).$$

The reverse inclusion is clear by the definition of $PI_t(G)$.

(ii) Suppose, on the contrary, that $x_{i_1} \cdots x_{i_{t+1}} \in PI_t(G)$ but $x_{i_1} \cdots x_{i_{t+1}} \notin \mathcal{G}(PI_t(G))$. Hence, there exists a path of length t , say $j_1, \dots, j_{t+1}$, in G such that we have $x_{j_1} \cdots x_{j_{t+1}} | x_{i_1} \cdots x_{i_{t+1}}$. Then $\{x_{j_1}, \dots, x_{j_{t+1}}\} \subseteq \{x_{i_1}, \dots, x_{i_{t+1}}\}$, and so $\{x_{j_1}, \dots, x_{j_{t+1}}\} = \{x_{i_1}, \dots, x_{i_{t+1}}\}$, this is a contradiction.

(iii) It is straightforward by virtue of the definition of $PC_t(G)$, part (i) and [60, Corollary 1.5.5].

(iv) It follows from part (i).

(v) Since $PC_t(G) = PI_t(G)^\vee$, it follows that

$$PC_t(G) = \bigcap_{i_1, \dots, i_{t+1} \text{ is a path of length } t \text{ in } G} (x_{i_1}, \dots, x_{i_{t+1}}).$$

This implies that $\text{ht}(PC_t(G)) = t + 1$.

(vi) According to [20, Theorem 2], it follows that $\psi_{t+1}(G) \leq \frac{|V(G)|}{t+1}$. Now, by part (iv), one can conclude that $\text{ht}(PI_t(G)) \leq n/(t + 1)$. $\square$

To understand the next result, one has to recollect the definitions of *starlike trees* and *symmetric starlike trees*, see Definition 2.2.4.

In the following corollary, we investigate the relation between max-path ideals and path ideals.

Corollary 9.1.18. *Let T be a tree on the vertex set $[n]$ which is not a path graph. Then the following conditions are equivalent.*

- (i) $PI(T) = P_t(T)$ for some integer t .
- (ii) $PC(T)$ is unmixed.
- (iii) $\max\{d(u, v) : u, v \in L_T\} = \min\{d(u, v) : u, v \in L_T\}$.
- (iv) T is a symmetric starlike tree.

Proof. (i) $\implies$ (ii). Let $PI(T) = P_t(T)$ for some integer t . By Proposition 9.1.17(iii), we have the following equalities

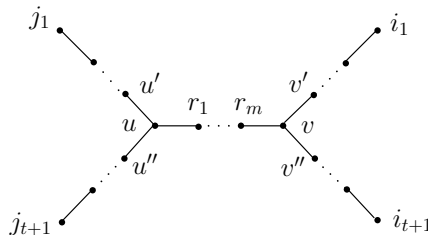
$$\begin{aligned} PC(T) &= PI(T)^\vee \\ &= PI_t(T)^\vee \\ &= PC_t(T). \end{aligned}$$

Now, in view of Proposition 9.1.17(v), it follows that $PC(T)$ is unmixed.

(ii) $\implies$ (iii). Let $u, v \in L_T$. Then there exists a path between them in T such as $u, i_1, \dots, i_r, v$. Hence, $x_u x_{i_1} \cdots x_{i_r} x_v \in \mathcal{G}(PI(T))$. Since $PC(T) = PI(T)^\vee$, we get $(x_u, x_{i_1}, \dots, x_{i_r}, x_v) \in \text{Ass}(PC(T))$. Suppose that $\text{ht}(PC(T)) = \alpha$. Hence, $\text{ht}(x_u, x_{i_1}, \dots, x_{i_r}, x_v) = \alpha$, and so $d(u, v) = \alpha - 1$. Therefore,

$$\max\{d(u, v) : u, v \in L_T\} = \min\{d(u, v) : u, v \in L_T\} = \alpha - 1.$$

(iii) $\implies$ (iv). Let $\max\{d(u, v) : u, v \in L_T\} = \min\{d(u, v) : u, v \in L_T\} = t$. According to (iii), it is enough to prove that T is a starlike tree. By contrary assume that there exist $u, v \in V(T)$ such that $u \neq v$ and $\deg_T u \geq \deg_T v \geq 3$. Suppose that $u, r_1, \dots, r_m, v$ is the path between u and v . Consider $u', u'' \in N_T(u) \setminus \{r_1\}$, $v', v'' \in N_T(v) \setminus \{r_m\}$ and extend the path v', v, v'' to a maximal path $i_1, \dots, v', v, v'', \dots, i_{t+1}$ and the path u', u, u'' to a maximal path $j_1, \dots, u', u, u'', \dots, j_{t+1}$ as the following figure.



On account of the following equalities

$$d(j_1, j_{t+1}) = d(j_1, i_1) = d(j_1, i_{t+1}) = d(j_{t+1}, i_1) = d(j_{t+1}, i_{t+1}) = d(i_1, i_{t+1}),$$

one can conclude that $d(u, v) = 0$, which is a contradiction. This implies that T has at most one vertex of degree greater than 2 and because T is not a path graph, it has exactly one vertex of degree greater than 2. This gives that T is starlike.

(iv) $\implies$ (i). The proof is based on the definition of path ideals. $\square$

Next theorem is the basic step to characterize trees with the maximal path ideal of height 1.

Theorem 9.1.19. *Let T be a tree on the vertex set $[n]$ and $|L_T| = d$ with $d \geq 3$. Then the following conditions are equivalent.*

(i) T has a vertex of degree d .

(ii) $\text{ht}(PI(T)) = 1$.

(iii) T is starlike.

(iv) If $\mathfrak{p} \in \text{Ass}(PI(T))$ and $\text{ht} \mathfrak{p} > 1$, then $\text{ht} \mathfrak{p} = d - 1$.

Proof. (i) $\implies$ (ii). Let $v \in V(T)$ be a vertex of degree d and $N_T(v) = \{v_1, \dots, v_d\}$. Then any vertex v_i is connected to a tree such as T_i for all $i = 1, \dots, d$. We show that T_i is a path graph for all $i = 1, \dots, d$. Suppose, on the contrary, that there exists a positive integer $1 \leq i \leq d$ such that T_i is not a path graph. Hence, the number of leaves of T exceed d , which is a contradiction. This implies that $\{v\}$ is a minimal path cover set, and so $\mathcal{I}_T = 1$. By Proposition 9.1.12(iii), one can conclude that $\text{ht}(PI(T)) = 1$, as claimed.

(ii) $\implies$ (iii). Assume that $\text{ht}(PI(T)) = 1$. Then by Proposition 9.1.12(iii), there exists a minimal path cover set such as $W = \{v\}$ with $v \in V(T)$. We first prove that $\deg_T v \geq 3$. To do this, let $i, j, k \in L_T$, $P : i = i_1, i_2, \dots, i_r = j$ be the maximal path between i and j , and $P' : i = j_1, j_2, \dots, j_s = k$ be the maximal path between i and k . Then by Lemma 9.1.9, put $\ell = \max\{m : i_m \in V(P) \cap V(P') \text{ and } \deg_T i_m \geq 3\}$. It is clear that $V(P) \cap V(P') = \{i_1, \dots, i_\ell\}$. So, $v \in \{i_1, \dots, i_\ell\}$. Suppose that P'' is the maximal path between j and k . It is easy to check that $V(P'') \cap \{i_1, \dots, i_\ell\} = \{i_\ell\}$. This implies that $v = i_\ell$, and hence $\deg_T v \geq 3$. Now, assume that $u \in V(T)$ with $u \neq v$ and $\deg_T u \geq 3$. Let P''' be the path between u and v , $u', u'' \in N_T(u) \setminus V(P''')$, $v', v'' \in N_T(v) \setminus V(P''')$. Then, similar to the proof of Corollary 9.1.18 (iii) $\implies$ (iv), if we extend the path u', u, u'' to a maximal path Q , it follows that $V(Q) \cap \{v\} = \emptyset$, which is a contradiction. This implies that v is the only vertex of degree greater than 2, and thus T is starlike.

(iii) $\implies$ (iv). Suppose that T is starlike. Then there exists a vertex $v \in V(T)$ such that $\deg_T v = d$. If $d = 3$, then $\text{htp} = 2$ and the proof is complete. We therefore suppose that $d \geq 4$. Let $P_1, \dots, P_d$ be the path graphs which are connected to the vertex v . Assume that $\mathbf{p} \in \text{Ass}(PI(T))$ and $\text{htp} > 1$. By Proposition 9.1.13(v), we have $\text{htp} \leq d - 1$. Suppose that $\text{htp} \leq d - 2$, and seek a contradiction. Then, by Proposition 9.1.12(i), it follows that $\mathbf{p} = (x_{j_1}, \dots, x_{j_s})$ such that $W = \{j_1, \dots, j_s\}$ is a minimal path cover set with $2 \leq s \leq d - 2$. By minimality of W , it follows that $|W \cap V(P_i)| \leq 1$ for all $i = 1, \dots, d$. Since $\text{htp} > 1$, we obtain $v \notin W$. Without loss of generality, suppose that $j_k \in V(P_k)$ for $k = 1, \dots, s$. This implies that $W \cap (\{v\} \cup V(P_{s+1}) \cup V(P_{s+2})) = \emptyset$, a contradiction.

(iv) $\implies$ (i). If there exists a minimal path cover set such as W such that $|W| = 1$, then $\text{ht}(PI(T)) = 1$, and so T is starlike. Since $|L_T| = d$, we get T has a vertex of degree d and the proof is complete. Suppose that, for each minimal path cover set W , we have $|W| > 1$. Then, according to our hypothesis, we conclude that $|W| = d - 1$ for all minimal path cover set of T . So, $\text{ht}(PI(T)) = d - 1$ which is a contradiction with Proposition 9.1.13(iii). $\square$

In the following corollary, we characterize trees in which $\text{ht}(PI(T)) = 1$.

Corollary 9.1.20. *Let T be a tree on the vertex set $[n]$ and $|L_T| = d$. Then $\text{ht}(PI(T)) = 1$ if and only if T is a path graph or starlike.*

Proof. If $d = 2$, then T is a path graph and there is nothing to prove. We thus suppose that $d \geq 3$. The claim is therefore an immediate consequence of Theorem 9.1.19. $\square$

Here, we characterize trees in which the max-path ideals have height 2.

Theorem 9.1.21. *Let T be a tree on the vertex set $[n]$ and $|L_T| = d$. Then $\text{ht}(PI(T)) = 2$ if and only if there exist two starlike trees S_1 and S_2 such that $T = S_1 \cup E \cup S_2$, where E satisfies one of the following conditions.*

(i) E is a path graph between the centers of S_1 and S_2 .

(ii) E is the union of two path graphs P and P' such that $|V(P) \cap V(S_i)| = 1$ for $i = 1, 2$, and $V(P) \cap V(P') = \{v\}$ with $\deg_T v = 3$.

Proof. First, suppose that T is as described in the theorem and z_i is the center of the starlike tree S_i for $i = 1, 2$. For sufficiency, the set $W = \{z_1, z_2\}$ is a minimal path cover set, and so $\text{ht}(PI(T)) = 2$.

For necessity, assume that $\text{ht}(PI(T)) = 2$. Then by Proposition 9.1.13(iii), we have $d \geq 4$ and also, there exists a minimal path cover set such as $W = \{z_1, z_2\}$. Let $\ell_1 := \deg_T z_1$ and $\ell_2 := \deg_T z_2$. If $\ell_1 \leq 2$ and $\ell_2 \leq 2$, then by considering all possible

shapes of maximal paths we get a contradiction. Hence, without loss of generality, assume that $\ell_1 \geq 3$.

Then $T \setminus \{z_1\}$ has ℓ_1 connected components such as $T_1, \dots, T_{\ell_1}$ and for each $1 \leq i \leq \ell_1$, there exists a unique vertex $w_i \in V(T_i)$ such that $\{z_1, w_i\} \in E(T)$.

Coming back to the proof of theorem 9.1.8, without loss of generality, we can assume that for each $\deg_T w_i$ ($2 \leq i \leq \ell_1$), Case 2 or Case 4 of the proof of Theorem 9.1.8 happens and for $\deg_T w_1$, Case 1 or Case 3 happens. Hence, z_1 is the center of a starlike subgraph S_1 of T and for T_1 one of the following two cases happens:

Case 1. $\deg_T w_1 \geq 3$. In this case, since $\{z_2\}$ is a minimal path cover set of T_1 , T_1 is a path graph or a starlike graph. Hence, we get $T = S_1 \cup E \cup S_2$, where S_2 is also a starlike graph whose center belongs to $V(T_1)$ and E satisfies one of the conditions (i) or (ii).

Case 2. $\deg_T w_1 = 2$ and T_1 is not a path graph. Then T_1 is the union of a path P_1 and a tree T'_1 such that if v'_1 is the cut point of this union, then $\deg_T v'_1 \geq 3$ and $\{z_2\}$ is a minimal path cover set of T'_1 . Therefore, T'_1 is a path graph or a starlike graph and again we conclude that $T = S_1 \cup E \cup S_2$, where S_2 is also a starlike graph whose center belongs to $V(T'_1)$ and E satisfies one of the conditions (i) or (ii). $\square$

In the subsequent corollary, we find $\text{ht}(PI(T))$ for a class of trees which can be considered as a generalization of the class of trees introduced in Theorem 9.1.21.

Corollary 9.1.22. *Let T be a tree on the vertex set $[n]$. Suppose also that*

$$T = S_1 \cup E_1 \cup S_2 \cup E_2 \cup \dots \cup E_{m-1} \cup S_m,$$

such that $S_1, \dots, S_m$ are starlike trees and E_i , for all $i = 1, \dots, m-1$, satisfies one of the following conditions.

- (i) E_i is a path graph between the centers of S_i and S_{i+1} .
- (ii) E_i is the union of two path graphs P_i and P'_i such that $|V(P_i) \cap V(S_k)| = 1$ for $k = i, i+1$, and $V(P_i) \cap V(P'_i) = \{v_i\}$ with $\deg_T v_i = 3$.

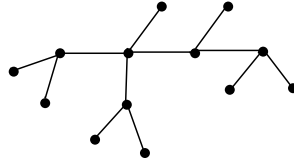
Then $\text{ht}(PI(T)) = m$.

Proof. Suppose that z_i is the center of the starlike tree S_i for all $i = 1, \dots, m$. Then the set $W = \{z_1, \dots, z_m\}$ is a minimal path cover set of T , and so $\text{ht}(PI(T)) \leq m$. Also, note that we can find some maximal paths in T like $Q_1, \dots, Q_m$ such that

$$V(Q_1) \subseteq V(S_1), V(Q_2) \subseteq V(E_1) \cup V(S_2), \dots, V(Q_m) \subseteq V(E_{m-1}) \cup V(S_m).$$

Since each of the Q_i s is a maximal path, the $V(Q_i)$ s ($1 \leq i \leq m$) are pairwise disjoint. Let W' be an arbitrary path cover set of T . Then for each $1 \leq i \leq m$, $W' \cap V(Q_i) \neq \emptyset$. This shows that $|W'| \geq m$. Therefore, $\text{ht}(PI(T)) = m$. $\square$

It is necessary to note that the converse of Corollary 9.1.22 is not valid. To observe this, consider the following tree T on a vertex set with thirteen elements.



It is routine to check that $\text{ht}(PI(T)) = 4$, while we cannot write T based on the format which has been stated in Corollary 9.1.22.

9.2 Path cover sets among some graphs

In this section, we are going to discuss the path cover sets among graphs. We begin with the following theorem. In fact, the next theorem links an arbitrary minimal path cover set of the union of a cycle and a connected graph with the minimal path cover set of the connected graph. It should be noted that the result of this section can be found in [80].

Recall that a *separating set* in a connected graph G is a set of vertices whose deletion disconnects G . If a separating set contains only one vertex v , we call v a *cut-point*.

Theorem 9.2.1. *Let G be the union of a cycle C and a connected graph H such that v is the cut-point of this union. Suppose that W_G is an arbitrary minimal path cover set of G and $V(C) \cap W_G = \{v'\}$. Then one of the following statements holds.*

- (i) W_G is a minimal path cover set of H .
- (ii) $W_G \setminus \{v'\}$ is a minimal path cover set of H .
- (iii) $(W_G \cup \{v\}) \setminus \{v'\}$ is a minimal path cover set of H .

Proof. If W_G is a minimal path cover set of H , then (i) holds. So, suppose that W_G is not a minimal path cover set of H . Thus, one can consider the following two cases:

Case 1. W_G is a path cover set of H , but it is not minimal. It follows that there exists an element such as w in W_G such that $W_G \setminus \{w\}$ is a path cover set of H . We show that $w = v'$. Suppose, on the contrary, that $w \neq v'$. We prove that $W_G \setminus \{w\}$ is a path cover set of G . Suppose that there exists a maximal path such as P in G such that $V(P) \cap (W_G \setminus \{w\}) = \emptyset$, and seek a contradiction. Hence, $V(P) \cap W_G \subseteq \{w\}$. Since $V(P) \cap W_G \neq \emptyset$, we have $V(P) \cap W_G = \{w\}$. If $w \in V(C)$, then $w \in V(C) \cap W_G$, and so $w = v'$, which is a contradiction. This implies that $w \notin V(C)$, and hence $w \neq v$. If $v \notin V(P)$, then P is a maximal path in H , and so

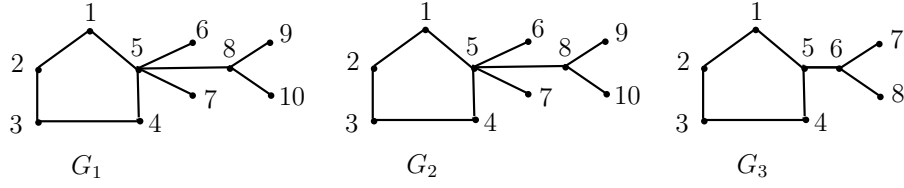
$V(P) \cap (W_G \setminus \{w\}) \neq \emptyset$, which is a contradiction. Therefore, $v \in V(P)$. If $v \in W_G$, then $v \in V(P) \cap W_G$, and hence $v = w$, which is a contradiction. This implies that $v \notin W_G$, and so $v \neq v'$. If $v' \in V(P)$, then $v' \in V(P) \cap W_G$, and hence $v' = w$, which is a contradiction. So $v' \notin V(P)$, and one has $V(P) \cap V(C) = \{v\}$. Thus, P is a maximal path in H . It follows that $V(P) \cap (W_G \setminus \{w\}) \neq \emptyset$, which is a contradiction. This implies that $w = v'$, and so $W_G \setminus \{v'\}$ is a minimal path cover set of H . Therefore, (ii) holds.

Case 2. W_G is not a path cover set of H . Set $W' := W_G \cup \{v\}$. We first show that W' is a path cover set of H . Suppose, on the contrary, that there exists a maximal path such as P in H such that $V(P) \cap W' = \emptyset$. Thus, $v \notin V(P)$ and $V(P) \cap W_G = \emptyset$. Since $v \notin V(P)$, one has $V(P) \cap V(C) = \emptyset$, and so P is a maximal path in G . Accordingly, $V(P) \cap W_G \neq \emptyset$, which is a contradiction. It follows that W' is a path cover set of H . If $v \in W_G$, then $W' = W_G$ is a path cover set of H , which is a contradiction. This implies that $v \notin W_G$, and thus $v \neq v'$. We next prove that $W' \setminus \{v'\}$ is a path cover set of H . Suppose, on the contrary, that there exists a maximal path such as P in H such that $V(P) \cap (W' \setminus \{v'\}) = \emptyset$. Therefore, $v \notin V(P)$ and $V(P) \cap W_G \subseteq \{v'\}$. Due to $v \notin V(P)$, one has $V(P) \cap V(C) = \emptyset$, and so P is a maximal path in G . It follows that $V(P) \cap W_G \neq \emptyset$, and hence $V(P) \cap W_G = \{v'\}$. This implies that $v' \in V(P)$, and so $V(P) \cap V(C) \neq \emptyset$, which is a contradiction. Therefore, $W' \setminus \{v'\}$ is a path cover set of H . Since W_G is not a path cover set of H , it follows that there exists a maximal path such as Q in H such that $V(Q) \cap W_G = \emptyset$. Due to $W' \setminus \{v'\}$ is a path cover set of H , one has $V(Q) \cap (W' \setminus \{v'\}) \neq \emptyset$. Hence, $v \in V(Q)$ and $v' \notin V(Q)$. If $\deg_H v \geq 2$, then Q is a maximal path in G , and so $V(Q) \cap W_G \neq \emptyset$, which is a contradiction. This implies that $\deg_H v = 1$. If $W' \setminus \{v'\}$ is a minimal path cover set of H , then the proof is complete. So, we assume that $W' \setminus \{v'\}$ is not minimal. It follows that there exists an element such as w in $W' \setminus \{v'\}$ such that $W' \setminus \{v', w\}$ is a path cover set of H with $w \neq v'$. If $w = v$, then $W' \setminus \{v', w\} = W_G \setminus \{v'\}$ is a path cover set of H , which is a contradiction. This implies that $w \neq v$, and so $w \in W_G$. In the following, we verify that $W_G \setminus \{w\}$ is a path cover set of G . Suppose, on the contrary, that there exists a maximal path such as P in G such that $V(P) \cap (W_G \setminus \{w\}) = \emptyset$. Hence, $V(P) \cap W_G = \{w\}$. If $v \notin V(P)$, then P is a maximal path in H , and thus $V(P) \cap (W' \setminus \{v', w\}) \neq \emptyset$. Since $V(P) \cap W' = \{w\}$, we get $V(P) \cap (W' \setminus \{v', w\}) = \emptyset$, which is a contradiction. This implies that $v \in V(P)$. By virtue of $v' \notin V(P)$ and $\deg_H v = 1$, one can conclude that P is not a maximal path in G , which is a contradiction. So $W_G \setminus \{w\}$ is a path cover set of G , which is a contradiction. Accordingly, $W' \setminus \{v'\}$ is a minimal path cover set of H . Therefore, (iii) holds, and the proof is complete. $\square$

The example below illuminates what happens in Theorem 9.2.1.

Example 9.2.2. Assume that G_1, G_2 , and G_3 are the graphs as shown in the fol-

lowing figures.



It is routine to check that $W_{G_1} = \{5, 9\}$ (respectively, $W_{G_2} = \{2, 7, 9, 10\}$, $W_{G_3} = \{2, 7\}$) is a minimal path cover set of G_1 (respectively, G_2 , G_3). On the other hand, one can easily see that $W_{G_1} = \{5, 9\}$ (respectively, $W_{G_2} \setminus \{2\} = \{7, 9, 10\}$, $(W_{G_3} \cup \{5\}) \setminus \{2\} = \{5, 7\}$) is a minimal path cover set of the tree which the vertex $v = 5$ is the cut-point of the union of the tree and cycle.

In the light of Theorem 9.2.1, we provide the following proposition.

Proposition 9.2.3. *Let G be the union of a cycle C and a tree T such that v is the cut-point of this union. Suppose that W_G is an arbitrary minimal path cover set of G . Then $|W_G| \leq |L_G| + 1$, where L_G is the set of leaves of G .*

Proof. Let $V(C) \cap W_G = \{v'\}$. Due to Theorem 9.2.1, we have the following two cases:

Case 1. W_G (or $(W_G \cup \{v\}) \setminus \{v'\}$) is a minimal path cover set of T . By virtue of Theorem 9.1.8, it follows that $|W_G| \leq |L_T| - 1$. If $v \in L_T$, then $|L_G| = |L_T| - 1$, and so $|W_G| \leq |L_G|$. If $v \notin L_T$, then $|L_G| = |L_T|$, and hence $|W_G| \leq |L_G| - 1$. Therefore, in this case, we get $|W_G| \leq |L_G| + 1$.

Case 2. $W_G \setminus \{v'\}$ is a minimal path cover set of T . If $v \notin L_T$, then $|L_G| = |L_T|$. According to Theorem 9.1.8, one can conclude that $|W_G| - 1 \leq |L_T| - 1$. Hence, $|W_G| \leq |L_T|$, and thus $|W_G| \leq |L_G|$. Now, let $v \in L_T$. Then $|L_G| = |L_T| - 1$. Also, Theorem 9.1.8 implies that $|W_G| - 1 \leq |L_T| - 1$. Consequently, $|W_G| - 1 \leq |L_G|$, and so $|W_G| \leq |L_G| + 1$. Therefore, in this case, we obtain $|W_G| \leq |L_G| + 1$.

We finally get $|W_G| \leq |L_G| + 1$, as desired. $\square$

Let G be the union of a path graph L and a connected graph H such that v is the cut-point of this union. Also, let W_G be an arbitrary minimal path cover set of G and $V(L) \cap W_G = S$. It follows now from the minimality of W_G that, if S is non-empty, then S is singleton. The following theorem discusses the minimal path cover set of H with respect to W_G .

Theorem 9.2.4. *Let G be the union of a path graph $L : v, r_1, \dots, r_s, w$ with $w \in L_G$ and a connected graph H such that v is the cut-point of this union. Suppose that W_G is an arbitrary minimal path cover set of G and $V(L) \cap W_G = \{t\}$. Then one of the following statements holds.*

- (i) W_G is a minimal path cover set of H .

(ii) $W_G \setminus \{t\}$ is a minimal path cover set of H .

(iii) $(W_G \setminus \{t\}) \cup \{v\}$ is a minimal path cover set of H .

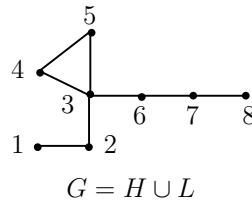
Proof. If $v = w$, then $G = H$. So, W_G is a minimal path cover set of H , and hence (i) holds. Thus, we may assume that $v \neq w$. If W_G is a minimal path cover set of H , then (i) holds. Therefore, assume that W_G is not a minimal path cover set of H . Hence, we can consider the following two cases:

Case 1. W_G is a path cover set of H , but it is not minimal. This implies that there exists an element z in W_G such that $W_G \setminus \{z\}$ is a path cover set of H . In the following, our goal is to prove that $z = t$. Suppose, on the contrary, that $z \neq t$. We show that $W_G \setminus \{z\}$ is a path cover set of G . Suppose, on the contrary, that there exists a maximal path such as P in G such that $V(P) \cap (W_G \setminus \{z\}) = \emptyset$. Thus, $V(P) \cap W_G \subseteq \{z\}$. By $V(P) \cap W_G \neq \emptyset$, one has $V(P) \cap W_G = \{z\}$. If $v \notin V(P)$, then P is a maximal path in H , and we obtain $V(P) \cap (W_G \setminus \{z\}) \neq \emptyset$, which is a contradiction. It follows that $v \in V(P)$. Let $\deg_H v = 1$. Then one can assume that $P : i_1, \dots, i_k, v, r_1, \dots, r_s, w$ with $\{i_1, \dots, i_k\} \subseteq V(H)$. Since $\deg_H v = 1$, it follows that $Q : i_1, \dots, i_k, v$ is a maximal path in H . Due to $V(P) \cap (W_G \setminus \{z\}) = \emptyset$ and $V(Q) \subseteq V(P)$, we get $V(Q) \cap (W_G \setminus \{z\}) = \emptyset$, which is a contradiction. Therefore, $\deg_H v \geq 2$. Since $z \neq t$ and $z \in W_G$, one has $z \notin V(L)$. If $w \notin V(P)$, by virtue of $\deg_H v \geq 2$, we have P is a maximal path in H . Thus, $V(P) \cap (W_G \setminus \{z\}) \neq \emptyset$, which is a contradiction. Therefore, $w \in V(P)$. According to $z \notin V(L)$, one can conclude that $z \neq v$ and $z \neq w$. This yields that $w \notin W_G$ and $v \notin W_G$. Since $t \in V(L)$, $v \in V(P)$, and $w \in V(P)$, we have $t \in V(P)$. According to $t \in W_G$, one has $t \in V(P) \cap W_G$. This implies that $t = z$. This leads to a contradiction. Therefore, $W_G \setminus \{z\}$ is a path cover set of G , which is a contradiction. Accordingly, one has $z = t$, and hence $W_G \setminus \{t\}$ is a minimal path cover set of H . Therefore, (ii) holds.

Case 2. W_G is not a path cover set of H . Put $W' := W_G \cup \{v\}$. We first show that W' is a path cover set of H . Suppose, on the contrary, that there exists a maximal path such as P in H such that $V(P) \cap W' = \emptyset$. Hence, $v \notin V(P)$ and $V(P) \cap W_G = \emptyset$. Since $v \notin V(P)$, it follows that P is a maximal path in G . Accordingly, $V(P) \cap W_G \neq \emptyset$, which is a contradiction. Therefore, W' is a path cover set of H . If $v \in W_G$, then $W' = W_G$ is a path cover set of H , which is a contradiction. This implies that $v \notin W_G$. In the following, our aim is to prove that $W' \setminus \{t\}$ is a path cover set of H . Suppose, on the contrary, that there exists a maximal path such as P in H such that $V(P) \cap (W' \setminus \{t\}) = \emptyset$. Since $v \notin W_G$, we have $v \neq t$. It follows that $v \notin V(P)$ and $V(P) \cap W_G \subseteq \{t\}$. Due to $v \notin V(P)$, one has P is a maximal path in G , and thus $V(P) \cap W_G \neq \emptyset$. This implies that $V(P) \cap W_G = \{t\}$, and so $t \in V(P)$. Since P is a maximal path in G and $t \in V(L)$, we get $v \in V(P)$,

which is a contradiction. Therefore, $W' \setminus \{t\}$ is a path cover set of H . If $W' \setminus \{t\}$ is a minimal path cover set of H , then the proof is complete. So, suppose that $W' \setminus \{t\}$ is not minimal. This implies that there exists an element like z in $W' \setminus \{t\}$ such that $W' \setminus \{z, t\}$ is a path cover set of H . If $z = v$, then $W' \setminus \{z, t\} = W_G \setminus \{t\}$ is a path cover set of H , which is a contradiction. It follows that $z \neq v$. Since $z \in W' \setminus \{t\}$ and $z \neq v$, one has $z \in W_G$. According to W_G is not a path cover set of H , one can conclude that there exists a maximal path like Q in H such that $V(Q) \cap W_G = \emptyset$. By $W' \setminus \{t\}$ is a path cover set of H , it follows that $V(Q) \cap (W' \setminus \{t\}) \neq \emptyset$. Hence, $v \in V(Q)$. If $\deg_H v \geq 2$, then Q is a maximal path in G , and so $V(Q) \cap W_G \neq \emptyset$, which is a contradiction. This implies that $\deg_H v = 1$. We now verify that $W_G \setminus \{z\}$ is a path cover set of G . Suppose, on the contrary, that there exists a maximal path such as Γ in G such that $V(\Gamma) \cap (W_G \setminus \{z\}) = \emptyset$. Due to $V(\Gamma) \cap W_G \neq \emptyset$, one has $V(\Gamma) \cap W_G = \{z\}$. If $v \notin V(\Gamma)$, then Γ is a maximal path in H , and so $V(\Gamma) \cap (W' \setminus \{z, t\}) \neq \emptyset$. Since $V(\Gamma) \cap W' = \{z\}$, we have $V(\Gamma) \cap (W' \setminus \{z, t\}) = \emptyset$, which is a contradiction. This yields that $v \in V(\Gamma)$. Since $\deg_H v = 1$, one can conclude that $V(L) \subseteq V(\Gamma)$. According to $t \in V(L)$, one has $t \in V(\Gamma)$, and hence $t \in V(\Gamma) \cap W_G$. Since $z \neq t$, this leads to a contradiction. Therefore, $W_G \setminus \{z\}$ is a path cover set of G , which is a contradiction. Thus, $W' \setminus \{t\}$ is a minimal path cover set of H . According to $v \neq t$, one can conclude that $W' \setminus \{t\} = (W_G \setminus \{t\}) \cup \{v\}$. Therefore, (iii) holds, and the proof is complete. $\square$

It is necessary to observe that Theorem 9.2.4 may be false if we omit the assumption that $V(L) \cap W_G$ must be non-empty. As a counterexample, consider the graph below, where $V(G) := [8]$, $V(H) := [5]$, $V(L) := \{3, 6, 7, 8\}$, and $v := 3$ is the cut-point. It is easy to see that $W_G := \{2, 4\}$ is a minimal path cover set of G , but neither W_G nor $W_G \cup \{3\}$ is a minimal path cover set of H .



Theorem 9.2.5. *Let G be the union of a tree T and a connected graph H such that v is the cut-point of this union with $\deg_T v = 1$. Let W_G be an arbitrary minimal path cover set of G and $V(T) \cap W_G = S$. Suppose that, for each $w \neq v$ in L_T , there exist a path $L' : v, \dots, z$ and a path graph $L : z, \dots, w$ with $\deg_T z \geq 3$ ($\deg_T z \leq 2$ when T is a path graph) in T such that $V(L) \cap W_G \neq \emptyset$. Then one of the following statements holds.*

- (i) W_G is a minimal path cover set of H .
- (ii) $W_G \setminus S$ is a minimal path cover set of H .

(iii) $(W_G \setminus S) \cup \{v\}$ is a minimal path cover set of H .

Proof. We use induction on $n := |L_T|$. Since T is a tree and $\deg_T v = 1$, one has $|L_T| \geq 2$. If $n = 2$, then T is a path graph, and the assertion follows from Theorem 9.2.4. Now, suppose, inductively, that $n \geq 3$ and that the result has been proved for any graph which satisfies the hypothesis of our theorem with the size of the set of leaves of the tree is less than n . Let G be the union of a tree T and a connected graph H such that v is the cut-point of this union with $\deg_T v = 1$ and $|L_T| = n$. Let also W_G be an arbitrary minimal path cover set of G and $V(T) \cap W_G = S$. Suppose that, for each $w \neq v$ in L_T , there exist a path $L' : v, \dots, z$ and a path graph $L : z, \dots, w$ with $\deg_T z \geq 3$ in T such that $V(L) \cap W_G \neq \emptyset$. Then one can consider the graph G as the union of a connected graph G' and the path graph $L : z, \dots, w$ with $\deg_T z \geq 3$ such that z is the cut-point of this union. Set $S' := V(L) \cap W_G$. By our hypothesis, we get $S' \neq \emptyset$. Now, according to Theorem 9.2.4, we can deduce that one of the following statements holds.

- (A) W_G is a minimal path cover set of G' .
- (B) $W_G \setminus S'$ is a minimal path cover set of G' .
- (C) $(W_G \setminus S') \cup \{z\}$ is a minimal path cover set of G' .

Put $V(T') := (V(T) \setminus V(L)) \cup \{z\}$. This implies that G' is the union of the tree T' and graph H such that v is the cut-point of this union. Set $S_1 := V(T') \cap W_G$, $S_2 := V(T') \cap (W_G \setminus S')$, and $S_3 := V(T') \cap ((W_G \setminus S') \cup \{z\})$. Since $|L_{T'}| < n$, according to the inductive hypothesis, we can conclude that one of the following statements holds.

- (1) W_G is a minimal path cover set of H .
- (2) $W_G \setminus S_1$ is a minimal path cover set of H .
- (3) $(W_G \setminus S_1) \cup \{v\}$ is a minimal path cover set of H .
- (4) $W_G \setminus S'$ is a minimal path cover set of H .
- (5) $(W_G \setminus S') \setminus S_2$ is a minimal path cover set of H .
- (6) $((W_G \setminus S') \setminus S_2) \setminus \{v\}$ is a minimal path cover set of H .
- (7) $(W_G \setminus S') \cup \{z\}$ is a minimal path cover set of H .
- (8) $((W_G \setminus S') \cup \{z\}) \setminus S_3$ is a minimal path cover set of H .
- (9) $((W_G \setminus S') \cup \{z\}) \setminus S_3 \cup \{v\}$ is a minimal path cover set of H .

Since $V(T) = V(L) \cup V(T')$, one can easily check that $S' \cup S_1 = S$, $S' \cup S_2 = S$, and $S' \cup S_3 = S \cup \{z\}$. If (1) holds, then we get (i).

Let (2) hold. It follows that $W_G \setminus S_1 = (W_G \setminus S) \cup ((W_G \setminus S_1) \cap (S' \setminus S_1))$, and so one has $W_G \setminus S_1 = (W_G \setminus S) \cup (S' \setminus S_1)$. Since $W_G \setminus S_1$ is a minimal path cover set of H , we can deduce that $W_G \setminus S_1 \subseteq V(H)$, and hence $S' \setminus S_1 \subseteq V(H)$. By $S' \setminus S_1 \subseteq V(L)$ and $V(H) \cap V(L) \subseteq \{v\}$, one can conclude that $S' \setminus S_1 \subseteq \{v\}$. It is easy to see that $S' \setminus S_1 = \emptyset$. This implies that $W_G \setminus S_1 = W_G \setminus S$, and so we obtain (ii).

In the statement (3), by using a similar argument which has been stated in statement (2), we have $W_G \setminus S_1 = (W_G \setminus S) \cup (S' \setminus S_1)$, and thus

$$(W_G \setminus S_1) \cup \{v\} = (W_G \setminus S) \cup (S' \setminus S_1) \cup \{v\}.$$

By virtue of $(W_G \setminus S_1) \cup \{v\}$ is a minimal path cover set of H , one can conclude that $(W_G \setminus S_1) \cup \{v\} \subseteq V(H)$, and so $S' \setminus S_1 \subseteq V(H)$. Since $S' \setminus S_1 \subseteq V(L)$ and $V(H) \cap V(L) \subseteq \{v\}$, it follows that $S' \setminus S_1 \subseteq \{v\}$. If $S' \setminus S_1 = \{v\}$, then $v \in S'$ and $v \notin S_1$. Since $v \in S'$, it follows that $v \in W_G$, and so $v \in S_1$, which is a contradiction. Accordingly, $S' \setminus S_1 = \emptyset$. Hence, $(W_G \setminus S_1) \cup \{v\} = (W_G \setminus S) \cup \{v\}$, and so we have (iii).

Let (4) hold. One can easily check that $W_G \setminus S' = (W_G \setminus S) \cup (S_1 \setminus S')$. Since $W_G \setminus S'$ is a minimal path cover set of H , one can conclude that $W_G \setminus S' \subseteq V(H)$, and hence $S_1 \setminus S' \subseteq V(H)$. According to $S_1 \setminus S' \subseteq V(T')$ and $V(H) \cap V(T') = \{v\}$, it follows that $S_1 \setminus S' \subseteq \{v\}$. If $S_1 \setminus S' = \emptyset$, then one has (ii), but if $S_1 \setminus S' = \{v\}$, then we get (iii).

Statement (5) implies that $(W_G \setminus S') \setminus S_2 = W_G \setminus S$, and so we have (ii).

It follows from (6) that $((W_G \setminus S') \setminus S_2) \cup \{v\} = (W_G \setminus S) \cup \{v\}$, and hence we obtain (iii).

In the statement (7), according to the argument has been stated in (4), one has $W_G \setminus S' = (W_G \setminus S) \cup (S_1 \setminus S')$, and so $(W_G \setminus S') \cup \{z\} = (W_G \setminus S) \cup (S_1 \setminus S') \cup \{z\}$. Since $(W_G \setminus S') \cup \{z\}$ is a minimal path cover set of H , we have $(W_G \setminus S') \cup \{z\} \subseteq V(H)$, and hence $z \in V(H)$. This implies that $z = v$. Due to $S_1 \setminus S' \subseteq V(H)$, $S_1 \setminus S' \subseteq V(T')$, and $V(H) \cap V(T') = \{v\}$, one has $S_1 \setminus S' \subseteq \{v\}$. Thus, we can conclude that $(W_G \setminus S') \cup \{z\} = (W_G \setminus S) \cup \{v\}$. Therefore, we get (iii).

Let the statement (8) hold. Since $S' \cup S_3 = S \cup \{z\}$, one can conclude that

$$((W_G \setminus S') \cup \{z\}) \setminus S_3 = (W_G \setminus (S \cup \{z\})) \cup (\{z\} \setminus S_3).$$

Since $z \in S_3$, it follows that $\{z\} \setminus S_3 = \emptyset$, and so $((W_G \setminus S') \cup \{z\}) \setminus S_3 = W_G \setminus (S \cup \{z\})$. If $z \in S$, then $W_G \setminus (S \cup \{z\}) = W_G \setminus S$. If $z \notin S$, then $z \notin W_G$, and hence $W_G \setminus (S \cup \{z\}) = W_G \setminus S$. This implies that $((W_G \setminus S') \cup \{z\}) \setminus S_3 = W_G \setminus S$. Accordingly, we have (ii).

Due to (9), by virtue of $\{z\} \setminus S_3 = \emptyset$, one can conclude that

$$(((W_G \setminus S') \cup \{z\}) \setminus S_3) \cup \{v\} = (W_G \setminus (S \cup \{z\})) \cup \{v\}.$$

If $z \notin S$, then $z \notin W_G$, and thus $(((W_G \setminus S') \cup \{z\}) \setminus S_3) \cup \{v\} = (W_G \setminus S) \cup \{v\}$. If $z \in S$, then $(((W_G \setminus S') \cup \{z\}) \setminus S_3) \cup \{v\} = (W_G \setminus S) \cup \{v\}$. Hence, one has (iii). This completes the inductive step, and therefore the theorem has been proved by induction. $\square$

We are now ready to present the main result of this section in the following theorem.

Theorem 9.2.6. *Let G be the union of trees $T_1, \dots, T_n$ and a connected graph H such that v is the cut-point of this union with $\deg_{T_i} v = 1$ for each $i = 1, \dots, n$. Let also W_G be an arbitrary minimal path cover set of G , and for each $w \neq v$ in L_{T_i} , there exist a path $L'_i : v, \dots, z$ and a path graph $L_i : z, \dots, w$ with $\deg_{T_i} z \geq 3$ ($\deg_{T_i} z \leq 2$ when T_i is a path graph) in T_i such that $V(L_i) \cap W_G \neq \emptyset$, for all $i = 1, \dots, n$. If $S = (\bigcup_{i=1}^n V(T_i)) \cap W_G$, then one of the following statements holds.*

- (i) W_G is a minimal path cover set of H .
- (ii) $W_G \setminus S$ is a minimal path cover set of H .
- (iii) $(W_G \setminus S) \cup \{v\}$ is a minimal path cover set of H .

Proof. We proceed by induction on n . By Theorem 9.2.5, the assertion is true for $n = 1$. Now, suppose, inductively, that $n > 1$ and that the result has been proved for all graphs which are the union of some trees and a connected graph with a cut-point, such that the number of trees is less than n and satisfy the condition has been stated in our theorem. Let G be the union of trees $T_1, \dots, T_n$ and a connected graph H such that v is the cut-point of this union. Let also W_G be an arbitrary minimal path cover set of G , and the tree T_i satisfies the condition has been stated in our theorem for all $i = 1, \dots, n$. Set $S := (\bigcup_{i=1}^n V(T_i)) \cap W_G$. Assume that T is the union of the trees $T_1, \dots, T_{n-1}$ with v is the cut-point of this union. Now, suppose that G' is the union of the tree T and the connected graph H such that v is the cut-point of this union. Therefore, G is the union of the graph G' and the tree T_n such that v is the cut-point of this union. Put $S' := V(T_n) \cap W_G$. By the inductive hypothesis, it follows that one of the following statements holds.

- (A) W_G is a minimal path cover set of G' .
- (B) $W_G \setminus S'$ is a minimal path cover set of G' .
- (C) $(W_G \setminus S') \cup \{v\}$ is a minimal path cover set of G' .

Set $S_1 := V(T) \cap W_G$, $S_2 := V(T) \cap (W_G \setminus S')$, and $S_3 := V(T) \cap ((W_G \setminus S') \cup \{v\})$. Once again, according to the inductive hypothesis, we can deduce that one of the following statements holds.

- (1) W_G is a minimal path cover set of H .
- (2) $W_G \setminus S_1$ is a minimal path cover set of H .
- (3) $(W_G \setminus S_1) \cup \{v\}$ is a minimal path cover set of H .
- (4) $W_G \setminus S'$ is a minimal path cover set of H .
- (5) $(W_G \setminus S') \setminus S_2$ is a minimal path cover set of H .
- (6) $((W_G \setminus S') \setminus S_2) \setminus \{v\}$ is a minimal path cover set of H .
- (7) $(W_G \setminus S') \cup \{v\}$ is a minimal path cover set of H .
- (8) $((W_G \setminus S') \cup \{v\}) \setminus S_3$ is a minimal path cover set of H .
- (9) $((W_G \setminus S') \cup \{v\}) \setminus S_3 \cup \{v\}$ is a minimal path cover set of H .

Since $\bigcup_{i=1}^n V(T_i) = V(T_n) \cup V(T)$, one can easily check that $S' \cup S_1 = S$, $S' \cup S_2 = S$ and $S' \cup S_3 = S \cup \{v\}$. If (1) holds, then we get (i).

Assume that statement (2) holds. It follows that $W_G \setminus S_1 = (W_G \setminus S) \cup ((W_G \setminus S_1) \cap (S' \setminus S_1))$, and one has $W_G \setminus S_1 = (W_G \setminus S) \cup (S' \setminus S_1)$. According to $W_G \setminus S_1$ is a minimal path cover set of H , we deduce that $W_G \setminus S_1 \subseteq V(H)$, and hence $S' \setminus S_1 \subseteq V(H)$. By $S' \setminus S_1 \subseteq V(T_n)$ and $V(H) \cap V(T_n) = \{v\}$, one can conclude that $S' \setminus S_1 \subseteq \{v\}$. If $S' \setminus S_1 = \{v\}$, then $v \in S'$ and $v \notin S_1$. Since $v \in S'$, it follows that $v \in W_G$, and so $v \in S_1$, which is a contradiction. Thus, $S' \setminus S_1 = \emptyset$. This implies that $W_G \setminus S_1 = W_G \setminus S$, and so we obtain (ii).

In the statement (3), based on the argument has been stated in (2), we have $W_G \setminus S_1 = (W_G \setminus S) \cup (S' \setminus S_1)$, and thus $(W_G \setminus S_1) \cup \{v\} = (W_G \setminus S) \cup (S' \setminus S_1) \cup \{v\}$. By virtue of $(W_G \setminus S_1) \cup \{v\}$ is a minimal path cover set of H , one can conclude that $(W_G \setminus S_1) \cup \{v\} \subseteq V(H)$, and so $S' \setminus S_1 \subseteq V(H)$. Since $S' \setminus S_1 \subseteq V(T_n)$ and $V(H) \cap V(T_n) = \{v\}$, it follows that $S' \setminus S_1 \subseteq \{v\}$. If $S' \setminus S_1 = \{v\}$, then $v \in S'$ and $v \notin S_1$. Since $v \in S'$, it follows that $v \in W_G$, and so $v \in S_1$, which is a contradiction. Accordingly, $S' \setminus S_1 = \emptyset$. Hence, $(W_G \setminus S_1) \cup \{v\} = (W_G \setminus S) \cup \{v\}$, and so we have (iii).

Assume that statement (4) holds. One can easily check that $W_G \setminus S' = (W_G \setminus S) \cup (S_1 \setminus S')$. Since $W_G \setminus S'$ is a minimal path cover set of H , one can conclude that $W_G \setminus S' \subseteq V(H)$, and hence $S_1 \setminus S' \subseteq V(H)$. According to $S_1 \setminus S' \subseteq V(T)$ and $V(H) \cap V(T) = \{v\}$, it follows that $S_1 \setminus S' \subseteq \{v\}$. If $S_1 \setminus S' = \emptyset$, then we gain (ii), but if $S_1 \setminus S' = \{v\}$, then one has (iii).

It follows from (5) that $(W_G \setminus S') \setminus S_2 = W_G \setminus S$, and so we have (ii).

Statement (6) implies that $((W_G \setminus S') \setminus S_2) \cup \{v\} = (W_G \setminus S) \cup \{v\}$, and thus we obtain (iii).

In the statement (7), by virtue of the argument has been stated in (4), we have $W_G \setminus S' = (W_G \setminus S) \cup (S_1 \setminus S')$, and hence $(W_G \setminus S') \cup \{v\} = (W_G \setminus S) \cup (S_1 \setminus S') \cup \{v\}$. Since $(W_G \setminus S') \cup \{v\}$ is a minimal path cover set of H , it follows that $(W_G \setminus S') \cup \{v\} \subseteq V(H)$, and so $S_1 \setminus S' \subseteq V(H)$. Due to $S_1 \setminus S' \subseteq V(T)$ and $V(H) \cap V(T) = \{v\}$, one has $S_1 \setminus S' \subseteq \{v\}$. If $S_1 \setminus S' = \{v\}$, then $v \in S_1$ and $v \notin S'$. Since $v \in S_1$, it follows that $v \in W_G$, and so $v \in S'$, which is a contradiction. Thus, $S_1 \setminus S' = \emptyset$, and hence $(W_G \setminus S') \cup \{v\} = (W_G \setminus S) \cup \{v\}$. Therefore, we get (iii).

Let statement (8) hold. Since $S' \cup S_3 = S \cup \{v\}$, one can conclude that

$$((W_G \setminus S') \cup \{v\}) \setminus S_3 = (W_G \setminus (S \cup \{v\})) \cup (\{v\} \setminus S_3).$$

Since $v \in S_3$, it follows that $\{v\} \setminus S_3 = \emptyset$, and so $((W_G \setminus S') \cup \{v\}) \setminus S_3 = W_G \setminus (S \cup \{v\})$. If $v \in S$, then $W_G \setminus (S \cup \{v\}) = W_G \setminus S$. If $v \notin S$, then $v \notin W_G$, and hence $W_G \setminus (S \cup \{v\}) = W_G \setminus S$. This implies that $((W_G \setminus S') \cup \{v\}) \setminus S_3 = W_G \setminus S$. Therefore, one has (ii).

In the statement (9), based on $\{v\} \setminus S_3 = \emptyset$, one can conclude that

$$(((W_G \setminus S') \cup \{v\}) \setminus S_3) \cup \{v\} = (W_G \setminus (S \cup \{v\})) \cup \{v\}.$$

If $v \notin S$, then $v \notin W_G$, and thus $((W_G \setminus S') \cup \{v\}) \setminus S_3 = (W_G \setminus S) \cup \{v\}$. If $v \in S$, then $v \in W_G$. Accordingly, $((W_G \setminus S') \cup \{v\}) \setminus S_3 = (W_G \setminus S) \cup \{v\}$. So, we have (iii). This is enough to complete the inductive step. $\square$

The following proposition says that the size of an arbitrary minimal path cover set of the union of two cycles is at most two.

Proposition 9.2.7. *Suppose that G is the union of two cycles C and C' such that $V(C) \cap V(C')$ is a path and W_G is an arbitrary minimal path cover set of G . Then either $|W_G| = 1$ or $|W_G| = 2$.*

Proof. Let Q be the path such that $V(Q) = V(C) \cap V(C')$ with starting point v and ending point v' . Now, pick an arbitrary maximal path P in G . By maximality of P , one can consider the following four cases:

Case 1. $V(P) = V(C)$.

Case 2. $V(P) = V(C')$.

Case 3. $V(P) = V(G)$.

Case 4. $V(P) = (V(G) \setminus V(Q)) \cup \{v, v'\}$.

It is straightforward to check that W_G may be one of the following four types:

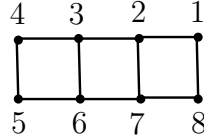
(A) $W_G = \{w, w'\}$, where $w \in V(C) \setminus V(Q)$ and $w' \in V(C') \setminus V(Q)$.

- (B) $W_G = \{w, w'\}$, where $w \in V(C) \setminus V(Q)$ and $w' \in V(Q) \setminus \{v, v'\}$.
- (C) $W_G = \{w, w'\}$, where $w \in V(C') \setminus V(Q)$ and $w' \in V(Q) \setminus \{v, v'\}$.
- (D) $W_G = \{w\}$, where $w \in \{v, v'\}$.

It follows that either $|W_G| = 1$ or $|W_G| = 2$, as claimed. $\square$

9.2.1 Max-path ideals with certain heights

One of the problems arising in this context is to explore the graphs which have a minimal path cover set with size one. As mentioned in Example 9.1.2, if G is a cycle or a complete graph or a path graph, then for each $v \in V(G)$, the set $\{v\}$ is a minimal path cover set of G . Nevertheless, there exist some graphs such that none of their vertices is a minimal path cover set, even for Hamiltonian graphs. For instance, consider the following Hamiltonian graph.



Since $P_1 : 1, 2, 7, 8$ and $P_2 : 4, 3, 6, 5$ are two maximal paths, we cannot find a singleton minimal path cover. In the following, our aim is to introduce a large class of finite simple graphs which have a minimal path cover set with size one. To achieve this, we need the following two propositions.

Proposition 9.2.8. *Let $G_1, \dots, G_m$ be some complete subgraphs of a finite simple graph G , and $\bigcap_{i=1}^m V(G_i)$ be a path. If $v \in \bigcap_{i=1}^m V(G_i)$, then $\{v\}$ is a minimal path cover set of the union of $G_1, \dots, G_m$.*

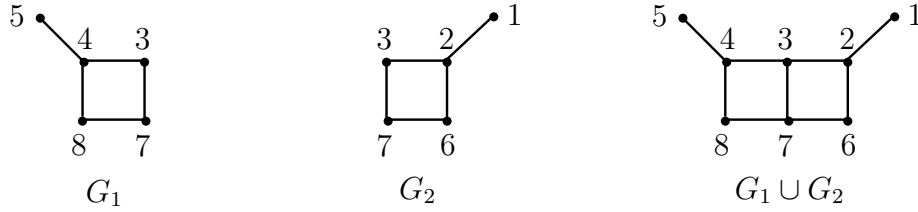
Proof. Let v be an arbitrary element of $\bigcap_{i=1}^m V(G_i)$. Suppose, on the contrary, that there exists a maximal path such as $P : v_1, \dots, v_k$ in the union of $G_1, \dots, G_m$ such that $v \notin V(P)$. Since $v_k \in \bigcup_{i=1}^m V(G_i)$, there exists a positive integer j with $1 \leq j \leq m$ such that $v_k \in G_j$. Due to G_j is a complete graph and $v \in G_j$, one can conclude that $\{v, v_k\} \in E(G_j)$. This implies that P is not a maximal path in the union of $G_1, \dots, G_m$, which is a contradiction. $\square$

Proposition 9.2.9. *Let $G_1, \dots, G_m$ be some subgraphs of a finite simple graph G , such that $\bigcap_{i=1}^m V(G_i) = \{v\}$ and the set $\{v\}$ is a minimal path cover set of the graphs $G_1, \dots, G_m$. Then $\{v\}$ is a minimal path cover set of the union of $G_1, \dots, G_m$.*

Proof. Let P be an arbitrary maximal path in the union of $G_1, \dots, G_m$. If there exists a positive integer j with $1 \leq j \leq m$ such that $V(P) \subseteq V(G_j)$, then P is a maximal path in the graph G_j , and so $v \in V(P)$. So, assume that there are two

positive integers r and s with $1 \leq r, s \leq m$ and $r \neq s$ such that $V(P) \cap V(G_r) \neq \emptyset$ and $V(P) \cap V(G_s) \neq \emptyset$. Suppose that $w \in V(P) \cap V(G_r)$ and $w' \in V(P) \cap V(G_s)$. Let $Q : w, j_1, \dots, j_k, w'$ be the path between w and w' with $V(Q) \subseteq V(P)$. By virtue of $\bigcap_{i=1}^m V(G_i) = \{v\}$, one has $v \in V(Q)$, and so $v \in V(P)$. This completes the proof. $\square$

The conclusion of the above proposition does not hold true if $\bigcap_{i=1}^m V(G_i)$ is a path and $v \in \bigcap_{i=1}^m V(G_i)$ such that the set $\{v\}$ is a minimal path cover set of graphs $G_1, \dots, G_m$. As a counterexample, consider graphs below. One can see that $V(G_1) \cap V(G_2)$ is a path.

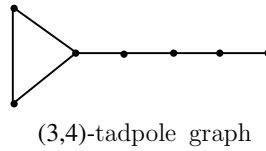


It is straightforward to check that $W := \{3\}$ is a minimal path cover set of both graphs G_1 and G_2 . Now, consider the maximal path $P : 5, 4, 8, 7, 6, 2, 1$. Due to $W \cap V(P) = \emptyset$, one can conclude that W is not a minimal path cover set of the union of G_1 and G_2 .

We next state the definition of tadpole graphs.

Definition 9.2.10. *The (m, n) -tadpole graph is the graph obtained by union a cycle C_m to a path graph P_n with a bridge.*

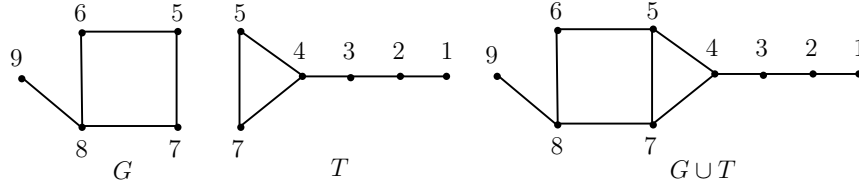
For example, the following figure is a $(3, 4)$ -tadpole graph.



The following proposition can be easily proved based on the definition of minimal path cover sets.

Proposition 9.2.11. *Let G be a cycle graph and let $T = C_3 \cup P_n$ be a $(3, n)$ -tadpole graph for some natural number n such that $E(G) \cap E(C_3) = \{u, v\}$ with $\deg_T u = 2$ and $\deg_T v = 2$. Then W is a minimal path cover set of G if and only if W is a minimal path cover set of $G \cup T$.*

It is important to note that Proposition 9.2.11 is not valid if we drop the assumption that the graph G is cycle. To see this, one just has to check that $W = \{5\}$ is a minimal path cover set of G , as shown below, but it is not a minimal path cover set of the union of G and the $(3, 3)$ -tadpole graph T , by virtue of the maximal path $9, 8, 7, 4, 3, 2, 1$.



Theorem 9.2.12. *Suppose that $G_1, \dots, G_k$ are path graphs, $G_{k+1}, \dots, G_t$ are cycle graphs, $G_{t+1}, \dots, G_r$ are complete graphs, and $G_{r+1}, \dots, G_s$ are graphs which each of them is the union of a cycle graph and a $(3, n)$ -tadpole graph for some natural number n such that satisfies the condition of Proposition 9.2.11. Suppose that $G = \bigcup_{i=1}^s G_i$ is the union of all the G_i and there is a vertex $v \in \bigcap_{i=1}^s V(G_i)$ such that the vertex v is the cut-point of this union and the set $\{v\}$ is a minimal path cover set of G_i for all $i = 1, \dots, s$. Then $\text{ht}(PI(G)) = 1$.*

Proof. We can combine together Propositions 9.2.8, 9.2.9, and 9.2.11 to obtain the set $\{v\}$ is a minimal path cover set of G , and so $\text{ht}(PI(G)) = 1$. This finishes the proof. $\square$

Along the argument has been stated in the proof of Theorem 9.1.19, we provide the proposition below.

Proposition 9.2.13. *Suppose that T is a tree which is not a path graph. Also, suppose that $\text{ht}(PI(T)) = k$. Then T has at least k vertices with degree greater than or equal to 3.*

Proof. Let S_T be the set of all vertices with degree greater than 2 in T . Due to Proposition 9.1.10, one has $\mathcal{I}_T \leq |S_T| \leq |L_T| - 2$, where $\mathcal{I}_T$ is the index of covering of T . On the other hand, it follows from Proposition 9.1.12 that $\text{ht}(PI(T)) = \mathcal{I}_T$. This implies that $|S_T| \geq k$, and the proof is complete. $\square$

Assume that T is a tree which is not a path graph, $V(T) = [n]$, $L_T = \{r_1, \dots, r_d\}$, and $\text{ht}(PI(T)) = k$. We thus can conclude that there exists a minimal path cover set of T with size k , say $\{j_1, \dots, j_k\} \subseteq [n]$. Put $\mathfrak{p} = (x_{j_1}, \dots, x_{j_k})$. Therefore, $\mathfrak{p}$ is a minimal prime ideal of $PI(T)$ such that $\text{ht}\mathfrak{p} = k$. Now, consider $W \subseteq L_T$ with $|W| = d - 1$, and set $\mathfrak{p} = (x_j : j \in W)$. Due to Proposition 9.1.7, one has $\mathfrak{p}$ is a minimal prime ideal of $PI(T)$ such that $\text{ht}\mathfrak{p} = d - 1$.

Another question that can be asked in this context is that does there exist a monomial prime ideal $\mathfrak{p} \in \text{Ass}(PI(T))$ with $\text{ht}\mathfrak{p} = r$ for all $r = k + 1, \dots, d - 2$? We provide an example that refutes this assertion. To do this, we consider a tree T on the vertex set $V(T) := [8]$ with the following edge set

$$E(T) := \{\{1, 2\}, \{2, 3\}, \{2, 4\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{4, 8\}\}.$$

It is easy to see that

$$PI(T) = (x_1x_2x_4x_5, x_1x_2x_4x_6, x_1x_2x_4x_7, x_1x_2x_4x_8, x_3x_2x_4x_5, x_3x_2x_4x_6, x_4x_7x_8, \\ x_3x_2x_4x_7, x_3x_2x_4x_8, x_1x_2x_3, x_4x_5x_6, x_4x_5x_7, x_4x_5x_8, x_4x_6x_7, x_4x_6x_8).$$

Hence, one can deduce that $\text{Ass}(PI(T))$ is given by

$$\{(x_4, x_2), (x_4, x_1), (x_8, x_6, x_5, x_2), (x_8, x_7, x_5, x_2), (x_8, x_6, x_5, x_3, x_1), \\ (x_8, x_7, x_6, x_2), (x_7, x_6, x_5, x_2), (x_7, x_6, x_5, x_3, x_1), (x_8, x_7, x_5, x_3, x_1), \\ (x_8, x_7, x_6, x_5, x_3), (x_8, x_7, x_6, x_3, x_1), (x_8, x_7, x_6, x_5, x_1), (x_4, x_3)\}.$$

This implies that there is no prime monomial ideal $\mathfrak{p}$ in $\text{Ass}(PI(T))$ with $\text{ht}\mathfrak{p} = 3$.

In the following, our goal is to find the whole of the minimal prime ideals of $PI(T)$ such as $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ such that $\mathcal{I}_T = \text{ht}\mathfrak{p}_i$ for all $i = 1, \dots, k$, where $\mathcal{I}_T$ is the index of covering of tree T .

Proposition 9.2.14. *Let T be a tree which is not a path graph. Then T has a vertex $u \in V(T)$ with $\deg_T u \geq 3$, and $w_1, w_2 \in L_T$ with $w_1 \neq w_2$, and*

$$v_1, \dots, v_r, v_{r+1}, \dots, v_s \in V(T),$$

(possibly $s = 0$) with $\deg_T v_i = 2$ for each $i = 1, \dots, s$ such that

$$Q : w_1, v_1, \dots, v_r, u, v_{r+1}, \dots, v_s, w_2,$$

is a maximal path in T .

Proof. Since the tree T is not a path graph, one can conclude that $|V(T)| \geq 4$. We argue by induction on $n := |V(T)|$. If $n = 4$, then T is the claw graph. The claim is true in this case. Now, suppose, inductively, that $n > 4$ and that the result has been proved for any tree which is not a path graph and the size of its set of vertices is less than n . Let T be a tree which is not a path graph with $n = |V(T)| > 4$. If T has no vertex with degree two, then T is the union of some star graphs such as $S_1, \dots, S_t$ such that, for each $i = 1, \dots, t-1$, $E(S_i) \cap E(S_{i+1})$ is an edge. Therefore, one can easily find a maximal path $Q : w_1, u, w_2$ in T such that $w_1, w_2 \in L_T$ with $w_1 \neq w_2$ and $\deg_T u \geq 3$. Thus, we assume that there exists a vertex in T with degree two. Here, we claim that there exists a leaf such as w_1 in T such that $N_T(w_1) = \{z\}$ with $\deg_T z = 2$, where $N_T(v)$ denotes the neighbor set of v in T . On the contrary, assume that for any leaf w in T , one has $N_T(w) = \{z_w\}$ with $\deg_T z_w \geq 3$. Let $L_T = \{w_1, \dots, w_m\}$, and $N_T(w_i) = \{z_i\}$ for each $i = 1, \dots, m$. If there exist positive integers i and j with $1 \leq i \neq j \leq m$ such that $N_T(w_i) = N_T(w_j) = p$, then the maximal path $Q : w_i, p, w_j$ is the required

path, and the proof is over. Thus, one may assume that $z_i \neq z_j$ for all positive integers i and j with $1 \leq i \neq j \leq m$, that is, all z_i are distinct. This implies that $\deg_T z_i \geq 3$ for each $i = 1, \dots, m$, and so $\sum_{i=1}^m \deg_T z_i \geq 3m$. It is clear that $\sum_{i=1}^m \deg_T w_i = m$. Note that $V(T) \setminus \{w_1, \dots, w_m, z_1, \dots, z_m\} \neq \emptyset$, and also for any $v \in V(T) \setminus \{w_1, \dots, w_m, z_1, \dots, z_m\}$, we have $\deg_T v \geq 2$. We therefore deduce that

$$\sum_{i=1}^m \deg_T w_i + \sum_{i=1}^m \deg_T z_i + \sum_{v \in V(T) \setminus \{w_1, \dots, w_m, z_1, \dots, z_m\}} \deg_T v \geq m + 3m + 2(n - 2m).$$

Thus, we get $\sum_{v \in V(T)} \deg_T v \geq 2n$. On the other hand, it is well-known that $\sum_{v \in V(T)} \deg_T v = 2n - 2$. Accordingly, one has $2n - 2 \geq 2n$, which is a contradiction. This shows that there exists a leaf such as w_1 in T such that $N_T(w_1) = \{z\}$ with $\deg_T z = 2$. Now, set $T_1 := T \setminus \{w_1\}$. Note that $E(T_1) = E(T) \setminus \{w_1, z\}$. Due to $|V(T_1)| < n$, it follows from the inductive hypothesis that the tree T_1 has a vertex $u \in V(T_1)$ with $\deg_{T_1} u \geq 3$, and $w', w_2 \in L_{T_1}$ with $w' \neq w_2$, and $v_1, \dots, v_r, v_{r+1}, \dots, v_s \in V(T_1)$ (possibly $s = 0$) with $\deg_{T_1} v_i = 2$ for each $i = 1, \dots, s$ such that $Q : w', v_1, \dots, v_r, u, v_{r+1}, \dots, v_s, w_2$ is a maximal path in T_1 . Note $L_{T_1} = (L_T \setminus \{w_1\}) \cup \{z\}$ since $\deg_{T_1} z = 1$. If $z \notin \{w', w_2\}$, then Q is the required path in T . Let $z \in \{w', w_2\}$. If $z = w'$, then $Q : w_1, w', v_1, \dots, v_r, u, v_{r+1}, \dots, v_s, w_2$ is a maximal path in T . If $z = w_2$, then since T has no cycle, we have $z \neq w'$, and so $\deg_T w' = 1$; this yields that $Q : w', v_1, \dots, v_r, u, v_{r+1}, \dots, v_s, w_2, w_1$ is a maximal path in T . This completes the inductive step, and so the claim has been proved by induction. $\square$

Theorem 9.2.15. *Suppose that T is a tree which is not a path graph. Then there is exactly one minimal path cover set of T such that the degree of its vertices is greater than or equal to 3.*

Proof. According to Proposition 9.1.10, the set of all vertices in T with degree greater than 2, denoted by S_T , is a path cover set. We proceed by induction on $n := |S_T|$. If $n = 1$, then T is a starlike graph with center z , and so the set $\{z\}$ is the unique minimal path cover set of T which the degree of its vertex is greater than or equal to 3. Hence, the result in the case where $n = 1$ holds. Now, suppose, inductively, that $n > 1$ and that the result has been proved for all trees which are not path graphs and the size of the set of all vertices with degree greater than 2 is less than n . Assume that T is a tree such that is not a path graph and $n = |S_T| > 1$. It therefore becomes of interest to seek a contradiction. Suppose that H_1 and H_2 are two distinct minimal path cover sets of tree T such that the degree of their vertices are greater than or equal to 3. In the following, our aim is to prove that $H_1 \cap H_2 \neq \emptyset$. Since T is a tree which is not a path graph, Proposition 9.2.14 yields that T has a vertex $u \in V(T)$ with $\deg_T u \geq 3$, and $w_1, w_2 \in L_T$ with $w_1 \neq w_2$, and $v_1, \dots, v_r, v_{r+1}, \dots, v_s \in V(T)$ (possibly $s = 0$) with $\deg_T v_i = 2$ for each $i = 1, \dots, s$

such that $Q : w_1, v_1, \dots, v_r, u, v_{r+1}, \dots, v_s, w_2$ is a maximal path in T . Due to the degree of each vertex of H_i is greater than or equal to 3 and $H_i \cap V(Q) \neq \emptyset$ for all $i = 1, 2$, it follows that $u \in H_i$ for all $i = 1, 2$, and thus $H_1 \cap H_2 \neq \emptyset$. Put $\alpha := \deg_T u$. Hence, $T \setminus \{u\}$ has exactly α connected components like $T_1, \dots, T_\alpha$. Put $B_{i,k} := H_i \cap V(T_k)$ for each $k = 1, \dots, \alpha$, and $i = 1, 2$. Furthermore, for each $k = 1, \dots, \alpha$, there is unique vertex $f_k \in V(T_k)$ such that $\{u, f_k\} \in E(T)$. Accordingly, for each $k = 1, \dots, \alpha$, we may consider one of the following four cases:

Case 1. $\deg_T f_k \geq 3$. In view of the proof of Theorem 9.1.8, $B_{1,k}$ and $B_{2,k}$ are two minimal path cover sets of T_k . Due to $|S_{T_k}|$ is less than n , it follows from the inductive hypothesis that $B_{1,k} = B_{2,k}$.

Case 2. $\deg_T f_k = 2$ and T_k is not a path graph. Hence, T_k is the union of a path Q and a tree T' such that if z is the cut-point of this union, then $\deg_T z \geq 3$. In view of the proof of Theorem 9.1.8, we get $B_{1,k}$ and $B_{2,k}$ are two minimal path cover sets of T' . According to $|S_{T'}|$ is less than n , the inductive hypothesis yields that $B_{1,k} = B_{2,k}$.

Case 3. $\deg_T f_k = 2$ and T_k is a path graph. These imply that $B_{1,k} = B_{2,k} = \emptyset$.

Case 4. $\deg_T f_k = 1$. Then $B_{1,k} = B_{2,k} = \emptyset$.

Based on $V(T) = \bigcup_{k=1}^{\alpha} V(T_k) \cup \{u\}$, $u \in H_1 \cap H_2$, and $B_{1,k} = B_{2,k}$ for each $k = 1, \dots, \alpha$, one can conclude that $H_1 = H_2$, which is a contradiction. This completes the inductive step, and so the theorem has been proved by induction.

□

Lemma 9.2.16. *Suppose that T is a tree which is not a path graph and L is the set of all vertices of T with degree smaller than or equal to 2. Then $(PI(T) : \prod_{v \in L} x_v)$ is a minimal prime ideal of $PI(T)$.*

Proof. In order to simplify the notation, we set $J := (PI(T) : \prod_{v \in L} x_v)$. Let $L = \{j_1, \dots, j_d\}$. We first prove that $\prod_{v \in L} x_v \notin PI(T)$. Suppose, on the contrary, that $\prod_{v \in L} x_v \in PI(T)$. Hence, there exists an element u in the unique minimal set of monomial generators of $PI(T)$ such that u divides $\prod_{v \in L} x_v$. By the definition of the max-path ideal, it follows that $u = x_{i_1} \cdots x_{i_k}$ and $Q : i_1, \dots, i_k$ is a maximal path in T . Since u divides $\prod_{v \in L} x_v$, one has $\deg_T i_s \leq 2$ for all $s = 1, \dots, k$. Choose an arbitrary vertex i_s in $V(Q)$. Since tree T is not a path graph, we obtain there exists a vertex v in $V(T)$ such that $\deg_T v \geq 3$, and thus $v \notin V(Q)$. It follows from connectivity of the tree T that there is unique path Q' in T which connects v and i_s . According to $i_s \in V(Q) \cap V(Q')$ and $v \notin V(Q)$, we have $V(Q) \cap V(Q') \neq \emptyset$ and $V(Q') \not\subseteq V(Q)$. In view of Lemma 9.1.9, there exists $v' \in V(Q) \cap V(Q')$ with $\deg_T v' \geq 3$, which is a contradiction. This implies that $\prod_{v \in L} x_v \notin PI(T)$. By virtue of Proposition 9.1.12, one can conclude that

$$PI(T) = \bigcap_{W \text{ is a minimal path cover set of } T} (x_k : k \in W),$$

is the minimal primary decomposition of $PI(T)$. Let W be an arbitrary minimal path cover set of T . It is easy to see that if $W \cap L \neq \emptyset$, then

$$((x_k : k \in W) : \prod_{v \in L} x_v) = (1).$$

This implies that J has the minimal primary decomposition $J = \bigcap_{i=1}^{\ell} \mathfrak{p}_i$ such that, for all $i = 1, \dots, \ell$, we have $\mathfrak{p}_i = (x_{t_{i,1}}, \dots, x_{t_{i,s}})$ which the set $\{t_{i,1}, \dots, t_{i,s}\}$ is a minimal path cover set of T and $\deg_T t_{i,k} \geq 3$ for each $k = 1, \dots, s$. On the other hand, Theorem 9.2.15 yields that $\ell = 1$. Therefore, we get $(PI(T) : \prod_{v \in L} x_v)$ is a prime monomial ideal, as desired. $\square$

Theorem 9.2.17. *Suppose that T is a tree which has a vertex u with $\deg_T u \geq 4$ and $w_1, \dots, w_k \in L_T$, where $k = \deg_T u - 1$, are such that, for all $i = 1, \dots, k$, if $u, r_1, \dots, r_s, w_i$ is the path between u and w_i , then $\deg_T r_j = 2$ for all $j = 1, \dots, s$. Also, suppose that L is the set of all vertices of T with degree smaller than or equal to 2. Then $\mathcal{I}_T = \text{ht} \mathfrak{p}$, where $\mathfrak{p} = (PI(T) : \prod_{v \in L} x_v)$.*

Proof. Suppose that $V(T) = \{1, \dots, \lambda\}$ and $R = K[x_1, \dots, x_\lambda]$ is the polynomial ring over a field K . We proceed by induction on $n := |L_T|$. According to the hypothesis, one has $|L_T| \geq 4$. If $|L_T| = 4$, then T is a starlike graph which u is the center of the starlike with $\deg_T u = 4$. We thus can consider the vertex set of T as $V(T) = \{i_1, i_2, i_3, i_4, i_5, \dots, i_s, u\}$ with $\deg_T i_k = 1$ for all $k = 1, 2, 3, 4$, and $\deg_T i_k = 2$ for all $k = 5, \dots, s$, and also $\deg_T u = 4$. This yields that $PI(T)$ is given by $PI(T) = (x_v \cdots x_u \cdots x_{v'} : v, v' \in L_T)$. It follows from Theorem 2.1.27 that $(x_u) \in \text{Ass}_R(R/PI(T))$, and so $\text{ht}(PI(T)) = 1$. Also, since $L = \{i_k : k = 1, \dots, s\}$, we get $(PI(T) : \prod_{v \in L} x_v) = (x_u)$. Therefore, the assertion holds for $n = 4$. Now, suppose, inductively, that $n > 4$ and that the result has been proved for any tree which satisfies the hypothesis of theorem and that the number of leaves is less than n . Let T be a tree that $|L_T| = n$ and it has a vertex u with $\deg_T u \geq 4$ and $w_1, \dots, w_k \in L_T$, where $k = \deg_T u - 1$, are such that, for all $i = 1, \dots, k$, if $u, r_1, \dots, r_s, w_i$ is the path between u and w_i , then $\deg_T r_j = 2$ for all $j = 1, \dots, s$. Also, suppose that L is the set of all vertices of T with degree smaller than or equal to two. Pick an arbitrary vertex in L_T , say w , such that if $u, v_1, \dots, v_s, w$ is the path between u and w , then $\deg_T v_i = 2$ for all $i = 1, \dots, s$. Set $T_1 := T \setminus \{w\}$ and $L_1 := \{x_v : \deg_{T_1} v \leq 2\}$. Let $PI(T_1) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_\ell$ be the minimal primary decomposition of $PI(T_1)$, and $PI(T) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_t$ be the minimal primary decomposition of $PI(T)$ such that $\mathfrak{p}_1 = (x_{j_1}, \dots, x_{j_k})$ with $\deg_T j_i \geq 3$ for all $i = 1, \dots, k$. Let $N_T(w) = \{w_1\}$ with $\deg_T w_1 = 2$. In the following, our aim is to prove that $\text{ht}(PI(T)) = \text{ht}(PI(T_1))$. According to $\deg_T w_1 = 2$, one can conclude that $(PI(T) :_R x_w) = PI(T_1)$, and so $\bigcap_{j=1}^t (\mathfrak{p}_j :_R x_w) = \bigcap_{i=1}^{\ell} \mathfrak{q}_i$. Without loss of generality, we may assume that $x_w \notin \mathfrak{p}_j$ only for $j = 1, \dots, \alpha$ with $1 \leq \alpha \leq t$. Hence,

$\bigcap_{j=1}^{\alpha} \mathfrak{p}_j = \bigcap_{i=1}^{\ell} \mathfrak{q}_i$. It follows immediately from the Second Uniqueness Theorem for Primary Decomposition that $\alpha = \ell$ and, after possibly reordering, $\mathfrak{p}_i = \mathfrak{q}_i$ for each $i = 1, \dots, \ell$. Also, it is clear that $PI(T) = \bigcap_{i=1}^{\ell} \mathfrak{q}_i \cap \bigcap_{j=\alpha+1}^t \mathfrak{p}_j$ is the minimal primary decomposition of $PI(T)$. Now, let $\mathfrak{p}_j = (x_{i_1}, \dots, x_{i_r})$ with $\alpha + 1 \leq j \leq t$. Therefore, $\Gamma = \{i_1, \dots, i_r\}$ is a minimal path cover set of T . On the other hand, we can consider T as the union of T_1 and a path graph L' with $V(L') = \{w_1, w\}$ such that w_1 is the cut-point of this union. According to $V(L') \cap \Gamma = \{w\}$ is a non-empty set, applying Theorem 9.2.4 yields the following three cases:

Case 1. Γ is a minimal path cover set of T_1 . Since $w \notin V(T_1)$, this case is impossible.

Case 2. $\Gamma \setminus \{w\}$ is a minimal path cover set of T_1 . Without loss of generality, assume that $i_1 = w$, and put $\Gamma_1 := \Gamma \setminus \{w\}$. Then $\Gamma_1 = \{i_2, i_3, \dots, i_r\}$, and so $(x_{i_2}, x_{i_3}, \dots, x_{i_r}) \in \text{Ass}(PI(T_1))$. By $\text{Ass}(PI(T_1)) = \{\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_{\ell}\}$, it follows that there exists a positive integer k with $1 \leq k \leq \ell$ such that $\mathfrak{q}_k = (x_{i_2}, x_{i_3}, \dots, x_{i_r})$. Since $\text{Ass}(PI(T_1)) \subseteq \text{Ass}(PI(T))$, we get $\mathfrak{q}_k \in \text{Ass}(PI(T))$, which contradicts the minimality of $(x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_r})$. Therefore, this case is impossible.

Case 3. $(\Gamma \setminus \{w\}) \cup \{w_1\}$ is a minimal path cover set of T_1 . Without loss of generality, we may assume that $i_1 = w$. Set $\Gamma_2 := (\Gamma \setminus \{w\}) \cup \{w_1\}$. This implies that $\Gamma_2 = \{w_1, i_2, i_3, \dots, i_r\}$, and so $(x_{w_1}, x_{i_2}, x_{i_3}, \dots, x_{i_r}) \in \text{Ass}(PI(T_1))$. Due to $\text{Ass}(PI(T_1)) = \{\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_{\ell}\}$, one has there exists a positive integer k with $1 \leq k \leq \ell$ such that $\mathfrak{q}_k = (x_{w_1}, x_{i_2}, x_{i_3}, \dots, x_{i_r})$. Therefore, $\text{ht} \mathfrak{p}_j = \text{ht} \mathfrak{q}_k$. Since $\text{ht}(PI(T)) = \min\{\text{ht} \mathfrak{q}_i, \text{ht} \mathfrak{p}_j : i = 1, \dots, \ell, j = \alpha + 1, \dots, t\}$, one can conclude that $\text{ht}(PI(T)) = \text{ht}(PI(T_1))$. According to $L_1 = L \setminus \{w\}$ and $(PI(T) :_R x_w) = PI(T_1)$, it follows that $(PI(T) :_R \prod_{v \in L} x_v) = (PI(T_1) :_R \prod_{v \in L_1} x_v)$.

We therefore shall reduce to the situation where $N_T(u) = \{v_1, v_2, \dots, v_{m+1}\}$ with $m + 1 = \deg_T u \geq 4$ and $\deg_T v_i = 1$ for any $i = 2, \dots, m + 1$. Let $w \in N_T(u)$ with $\deg_T w = 1$. Once again, set $T_1 := T \setminus \{w\}$, and thus $|L_{T_1}| = |L_T| - 1$. Let $PI(T_1) = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \dots \cap \mathfrak{q}_{\ell}$ be the minimal primary decomposition of $PI(T_1)$, and $PI(T) = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_t$ be the minimal primary decomposition of $PI(T)$ such that $\mathfrak{p}_1 = (x_{j_1}, \dots, x_{j_k})$ with $\deg_T j_i \geq 3$ for all $i = 1, \dots, k$. By virtue of the inductive hypothesis, we get $\text{ht}(PI(T_1)) = \text{ht} \mathfrak{q}_1$, where $\mathfrak{q}_1 = (PI(T_1) :_R \prod_{v \in L_1} x_v)$ with $L_1 = \{v \in V(T_1) : \deg_{T_1} v \leq 2\}$. Without loss of generality, we may assume that $x_w \notin \mathfrak{p}_j$ only for $j = 1, \dots, \alpha$ with $1 \leq \alpha \leq t$. Furthermore, it is routine to check that $(PI(T), x_w) = (PI(T_1), x_w)$. This implies that

$$\bigcap_{j=1}^{\alpha} (\mathfrak{p}_j, x_w) \cap \bigcap_{j=\alpha+1}^t \mathfrak{p}_j = \bigcap_{i=1}^{\ell} (\mathfrak{q}_i, x_w).$$

Pick a positive integer j with $1 \leq j \leq \alpha$. Then $\bigcap_{i=1}^{\ell} (\mathfrak{q}_i, x_w) \subseteq (\mathfrak{p}_j, x_w)$, and so there exists a positive integer i_1 with $1 \leq i_1 \leq \ell$ such that $(\mathfrak{q}_{i_1}, x_w) \subseteq (\mathfrak{p}_j, x_w)$.

Hence, $\mathbf{q}_{i_1} \subseteq \mathbf{p}_j$. It follows from the inductive hypothesis that $\text{ht}\mathbf{q}_1 \leq \text{ht}\mathbf{q}_{i_1}$, and thus $\text{ht}\mathbf{q}_1 \leq \text{ht}\mathbf{p}_j$. Now, pick a positive integer j with $\alpha + 1 \leq j \leq t$. Then $\bigcap_{i=1}^{\ell} (\mathbf{q}_i, x_w) \subseteq \mathbf{p}_j$, and hence there exists a positive integer i_2 with $1 \leq i_2 \leq \ell$ such that $(\mathbf{q}_{i_2}, x_w) \subseteq \mathbf{p}_j$. So $\mathbf{q}_{i_2} \subseteq \mathbf{p}_j$. The inductive hypothesis yields that $\text{ht}\mathbf{q}_1 \leq \text{ht}\mathbf{q}_{i_2}$, and one has $\text{ht}\mathbf{q}_1 \leq \text{ht}\mathbf{p}_j$. Therefore, we have $\text{ht}\mathbf{q}_1 \leq \text{ht}\mathbf{p}_j$ for any $j = 1, \dots, t$. In the following, our aim is to show that $\text{ht}\mathbf{q}_1 = \text{ht}\mathbf{p}_1$. According to $\mathbf{p}_1 \in \text{Ass}(PI(T))$, one can conclude that $\Gamma := \{j_1, \dots, j_k\}$ is a minimal path cover set of T . Also, we can consider T as the union of T_1 and a path graph L' with $V(L') = \{u, w\}$ such that u is the cut-point of this union. Since $\deg_T j_i \geq 3$ for all $i = 1, \dots, k$, we have the following two cases:

(A) $u \in \Gamma$. Due to $V(L') \cap \Gamma = \{u\}$ is a non-empty set, it follows from Theorem 9.2.4 that one has the following three subcases:

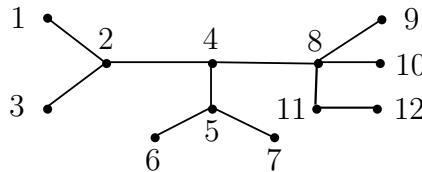
- (i) Γ is a minimal path cover set of T_1 . Hence, $\mathbf{p}_1 \in \text{Ass}(PI(T_1))$. Since $\deg_T j_i \geq 3$ for all $i = 1, \dots, k$, we obtain $\mathbf{p}_1 = \mathbf{q}_1$, and so $\text{ht}\mathbf{p}_1 = \text{ht}\mathbf{q}_1$.
- (ii) $\Gamma \setminus \{u\}$ is a minimal path cover set of T_1 . We now verify that this case is impossible. Without loss of generality, we may assume $j_1 = u$. Set $\Gamma_1 := \Gamma \setminus \{u\}$. Then $\Gamma_1 = \{j_2, j_3, \dots, j_k\}$. Since $\deg_T u = m + 1 \geq 4$, it follows that $\deg_{T_1} u = \deg_T u - 1 \geq 3$, and so there exist at least two distinct vertices such as $v, v' \in N_T(u)$ with $\deg_T v = \deg_T v' = 1$. Therefore, Γ_1 does not cover the maximal path v, u, v' in T_1 , this yields a contradiction.
- (iii) $(\Gamma \setminus \{u\}) \cup \{u\}$ is a minimal path cover set of T_1 . Due to the argument has been stated in (i), one can conclude that $\mathbf{p}_1 = \mathbf{q}_1$, and hence $\text{ht}\mathbf{p}_1 = \text{ht}\mathbf{q}_1$.

(B) $u \notin \Gamma$. Because of $\deg_T u = m + 1 \geq 4$, one has $\deg_{T_1} u = \deg_T u - 1 \geq 3$, and so there exist at least two distinct vertices such as $v, v' \in N_T(u)$ with $\deg_T v = \deg_T v' = 1$, and hence Γ has to cover the maximal path v, u, v' in T . This implies that Γ must contain a vertex with degree one, which is a contradiction.

In order to complete the proof, we note that $\mathbf{p}_1 = (PI(T) : \prod_{v \in L} x_v)$, by virtue of Lemma 9.2.16. $\square$

We conclude this section with an example which illuminates Theorem 9.2.17.

Example 9.2.18. Let T be a tree, as shown in the following figure, on the vertex set $[12]$.



It is routine to check that, in the polynomial ring $R = K[x_1, \dots, x_{12}]$ over a field K , we have

$$\begin{aligned} PI(T) = & (x_1x_2x_3, x_5x_6x_7, x_1x_2x_4x_5x_7, x_2x_3x_4x_5x_6, x_2x_3x_4x_5x_7, \\ & x_1x_2x_4x_8x_9, x_1x_2x_4x_8x_{10}, x_1x_2x_4x_8x_{11}x_{12}, x_2x_3x_4x_8x_9, \\ & x_2x_3x_4x_8x_{10}, x_2x_3x_4x_8x_{11}x_{12}, x_4x_5x_6x_8x_9, x_4x_5x_6x_8x_{10}, \\ & x_4x_5x_6x_8x_{11}x_{12}, x_4x_5x_7x_8x_9, x_4x_5x_7x_8x_{10}, x_4x_5x_7x_8x_{11}x_{12}, \\ & x_1x_2x_4x_5x_6, x_8x_9x_{10}, x_8x_9x_{11}x_{12}, x_8x_{10}x_{11}x_{12}). \end{aligned}$$

Direct computation shows that $(x_8, x_5, x_2) \in \text{Ass}(PI(T))$. On the other hand, Theorem 9.2.17 yields that $\text{ht}(PI(T)) = \text{ht}(x_8, x_5, x_2) = 3$, where

$$(x_8, x_5, x_2) = (PI(T) :_R x_1x_3x_6x_7x_9x_{10}x_{11}x_{12}).$$

9.3 On a class of polymatroidal ideals

Matroid theory is one of the most attractive areas in combinatorics and is deeply rooted in graph theory and linear algebra. The well-known examples of matroids include uniform matroids, graphical matroids, and linear matroids. We refer to [136], [111], and [59] for basic definitions and notions in matroid theory. The polymatroids originated in [36] and the *discrete polymatroids* appeared in [59] as a multiset analogue of matroids. In commutative algebra, the matroidal ideals as the square-free version of polymatroidal ideals, hold a very special place due to their nice algebraic and homological properties. A monomial ideal $I \subset K[x_1, \dots, x_n]$ is called *polymatroidal*, if the set of exponent vectors of the minimal generating set of I corresponds to the set of bases of a discrete polymatroid. In particular, a square-free polymatroidal ideal is simply referred to as a *matroidal* ideal, because in this case, the set of exponent vectors of the minimal generating set of I corresponds to the set of bases of a matroid. The algebraic and homological properties of polymatroidal ideals have been studied by many authors, for example, see [67], [66], [58], [12], [65]. It is known from [31] and [67] that the product of polymatroidal ideals is again polymatroidal. Moreover, polymatroidal ideals have linear quotients, and hence linear resolutions, see [31]. In particular, all powers of a polymatroidal ideal have linear resolutions. If I is a polymatroidal ideal, then the Rees algebra $\mathcal{R}(I)$ is normal, see [135, Proposition 3.11], [66, Theorem 3.4]. The stability indices related to associated primes and depth of powers of polymatroidal ideals are of particular interest and have been discussed in several papers, for example, see [66] and [65]. In fact, polymatroidal ideals are known to have the persistence property, see [66, Proposition 3.3], and they even have the strong persistence property, see [65, Proposition 2.4].

In this section, we introduce another class of matroidal ideals, namely, the class of the t -path ideals of complete multipartite graphs. Let G be a finite simple graph with n vertices. We refer to a path of length $t - 1$ in G as a t -path. Let S be a polynomial ring over a field K in n variables. To simplify the notation, throughout this text, we identify the vertices of G with the variables in S . The t -path ideal $I_t(G) \subset S = K[x_1, \dots, x_n]$ is generated by those monomials $x_{i_1} \cdots x_{i_t}$ such that $x_{i_1}, \dots, x_{i_t}$ is a t -path in G . The t -path ideals of graphs were introduced by Conca and De Negri in [30] and later on discussed by several authors for different classes of graphs, in particular for directed trees, trees and, cycles, see [54], [19], [25]. It should be noted that the results of this section can be found in [83].

In what follows, we first recall some basis definition and notion related to matroids. Let $[n] = \{1, \dots, n\}$ and $P([n])$ denote the set of all subsets of $[n]$. For any $S \subseteq [n]$, the cardinality of S is denoted by $|S|$. A *matroid* $\mathcal{M}$ on the ground set $[n]$ is a non-empty collection of subsets of $[n]$ satisfying the following properties:

1. if $A \in \mathcal{M}$ and $B \subset A$, then $B \in \mathcal{M}$;
2. if $A, B \in \mathcal{M}$ and $|B| \leq |A|$, then there exists $a \in A \setminus B$ such that $B \cup \{a\} \in \mathcal{M}$.

A maximal element (with respect to inclusion) in $\mathcal{M}$ is called a *base*. Condition (2) can be used to show that all bases of $\mathcal{M}$ have the same cardinalities. If $\mathcal{B}$ be the set containing all the bases of $\mathcal{M}$, then $\mathcal{B}$ is distinguished by the following “exchange property”.

(EP) For any $A, B \in \mathcal{B}$, if $a \in A \setminus B$, then there exists $b \in B \setminus A$ such that $(A \setminus \{a\}) \cup \{b\} \in \mathcal{B}$.

Given any subset $\mathcal{C}$ of $P([n])$, there exists a matroid on $[n]$ with $\mathcal{C}$ as its sets of bases if and only if $\mathcal{C}$ satisfies the exchange property.

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and I be a monomial ideal in S . For a given $\mathbf{a} = (a_1, \dots, a_n)$ with nonnegative entries, $\mathbf{x}^{\mathbf{a}}$ denotes the monomial $x_1^{a_1} \cdots x_n^{a_n}$ in S . A monomial ideal $I \subset S$ generated in a single degree is called *polymatroidal* if for all monomials $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}} \in \mathcal{G}(I)$ with $a_i > b_i$, there exists j with $a_j < b_j$ such that $x_j(\mathbf{x}^{\mathbf{a}}/x_i) \in \mathcal{G}(I)$. A square-free polymatroidal ideal is simply referred to as a *matroidal* ideal. In other words, a monomial ideal I in S is matroidal, if $\mathcal{G}(I)$ can be identified as a set of bases of a matroid.

Here, we recall some basic definitions and notions from graph theory. Throughout this section, all considered graphs will be simple, undirected, and finite. Let G be a graph with the vertex set $V(G)$ and the edge set $E(G)$. A subset $A \subseteq V(G)$ is called *independent* if no vertices in A are adjacent in G . A graph G is called *complete r -partite*, if $V(G)$ can be partitioned into r independent sets $V_1, \dots, V_r$ such that a and b are adjacent for all $a \in V_i$ and all $b \in V_j$ with $1 \leq i \neq j \leq r$. Such

a partition is called an r -partition of G . Moreover, if $|V_i| = n_i$ for all $i = 1, \dots, r$, then the complete r -partite graph is denoted by $K_{n_1, \dots, n_r}$. In this section, we focus on $I(K_{n_1, \dots, n_r})$.

Let G be a graph with $|V(G)| = n$. From now on, we identify the vertices of the graph G with the variables in the polynomial ring $S = K[x_1, \dots, x_n]$.

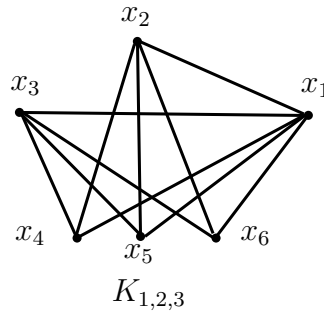
Definition 9.3.1. The t -path ideal of G , denoted by $I_t(G)$, is given by

$$I_t(G) := (x_{i_1} \cdots x_{i_t} : x_{i_1}, \dots, x_{i_t} \text{ is a } t\text{-path in } G) \subset S.$$

When $t = 2$, then $I_2(G)$ is simply the edge ideal $I(G)$ of G . If there is no t -path in G , then we set $I_t(G) = 0$.

The following example illustrates the definitions that have been stated above.

Example 9.3.2. Let $t = 4$ and $K_{1,2,3}$ be as shown in figure below with partition sets $\{x_1\}$, $\{x_2, x_3\}$ and $\{x_4, x_5, x_6\}$.



Then $I_4(K_{1,2,3})$ has the following minimal monomial generators:

$$x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_3x_6, x_1x_2x_4x_5, x_1x_2x_4x_6, x_1x_2x_5x_6,$$

$$x_1x_3x_4x_5, x_1x_3x_4x_6, x_1x_3x_5x_6, x_2x_3x_4x_5, x_2x_3x_4x_6, x_2x_3x_5x_6.$$

Remember that for a monomial ideal $I \subset S$, if $\text{supp}(I) = \{x_1, \dots, x_n\}$, then I is said to be *fully supported*. Furthermore, if $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, then $\text{gcd}(I) = \text{gcd}(u_1, \dots, u_m)$.

Below we give a list of remarks that will be used frequently throughout this text. For any integer a , the notations $\lceil a \rceil$ and $\lfloor a \rfloor$ denote the ceiling and floor functions of a , respectively. Let $G = K_{n_1, \dots, n_r}$ and $V_1, \dots, V_r$ be the r -partition of G . Then we have the following remarks.

Remark 9.3.3. Let $P : x_{i_1}, x_{i_2}, \dots, x_{i_t}$ be a t -path in G , and $u = x_{i_1}x_{i_2} \cdots x_{i_t} \in \mathcal{G}(I_t(G))$. If $x_{i_j} \in V_s$ for some $1 \leq s \leq r$ and $2 \leq j \leq t-1$, then $x_{i_{j-1}}, x_{i_{j+1}} \notin V_s$. This leads to the following conclusions:

- (i) A t -path in G can have at most $\lceil t/2 \rceil$ vertices in V_s .

- (ii) Let $x_{i_j} \in V_s$ for some $1 \leq j \leq t$. If there exists $x_k \in V_s$ such that $x_k \notin V(P)$, then x_{i_j} can be replaced by x_k in P to obtain a new t -path P' in G . In particular, $(u/x_{i_j})x_k \in \mathcal{G}(I_t(G))$.
- (iii) If there exists some V_s such that $V(P) \cap V_s = \emptyset$, then any $x_{i_j} \in V(P)$ can be replaced in P by any $x_k \in V_s$, to obtain a new path P' in G . In particular, $(u/x_{i_j})x_k \in \mathcal{G}(I_t(G))$, for all $x_k \in V_s$ and for each $j = 1, \dots, t$.
- (iv) If $I_t(G) \neq 0$, then $I_t(G)$ is fully supported. This can be easily seen due to statements (ii) and (iii).
- (v) Let $I_t(G) \neq 0$. Then $I_t(G)$ is a principal ideal if and only if $t = n_1 + \dots + n_r$. This also follows from statements (ii) and (iii).

9.3.1 Matroidal path ideals of graphs

It is proved in [12, Theorem 1.1] that a monomial ideal $I \subset S = K[x_1, \dots, x_n]$ is polymatroidal if and only if $I : u$ is polymatroidal for all monomials u in S . In particular, $(I : x_i)$ is polymatroidal if I is polymatroidal. The following theorem gives further information about $(I : x_i)$.

Theorem 9.3.4. *Let $I \subset S = K[x_1, \dots, x_n]$ be a polymatroidal ideal and $1 \leq i \leq n$. Assume that $I = \sum_{j=0}^d I_j x_i^j$, where $d = \max\{\deg_{x_i} u : u \in \mathcal{G}(I)\}$ and*

$$I_j = (u/x_i^j : u \in \mathcal{G}(I), \deg_{x_i} u = j),$$

for each $j = 0, \dots, d$. Then $I_0 \subseteq I_1 \subseteq \dots \subseteq I_d$.

Proof. If $d = 0$, then $I = I_0$ and there is nothing to prove. Let $d \geq 1$ and set $J_1 := \sum_{j=1}^d I_j x_i^{j-1}$. It is easy to see that $I = I_0 + x_i J_1$ and $(I : x_i) = I_0 + J_1$. Following the proof of [12, Theorem 1.1], one can deduce that $I_0 \subseteq J_1$, and hence $(I : x_i) = J_1$ is a polymatroidal ideal. Now, on account of $I_0 = (u : u \in \mathcal{G}(I), \deg_{x_i} u = 0)$ and $I_1 = (u/x_i : u \in \mathcal{G}(I), \deg_{x_i} u = 1)$, we obtain $I_0 \subseteq I_1$. If $d = 1$, then we are done. Otherwise, if $d \geq 2$, we set $J_2 := \sum_{j=2}^d I_j x_i^{j-2}$. Thus, $J_1 = I_1 + x_i J_2$. Once again, from the proof of [12, Theorem 1.1] we obtain $I_1 \subseteq J_2$, and thus $I_1 \subseteq I_2$. Therefore, $I_0 \subseteq I_1 \subseteq I_2$. Now, set $J_k = \sum_{j=k}^d I_j x_i^{j-k}$, where $1 \leq k \leq d$. By continuing this procedure, we can conclude that $I_0 \subseteq I_1 \subseteq \dots \subseteq I_d$. This finishes the proof. $\square$

If $I \subset K[x_1, \dots, x_n]$ is a matroidal ideal, then $\max\{\deg_{x_i} u : u \in \mathcal{G}(I)\}$ is either 0 or 1. For each variable $x_i \in \{x_1, \dots, x_n\}$, we set $I_{0,i} := (u : x_i \notin \text{supp}(u))$ and $I_{1,i} := (u/x_i : x_i \in \text{supp}(u))$. Then, we get $I = I_{0,i} + x_i I_{1,i}$, for all $i = 1, \dots, n$. With this notation, we obtain the following specialization of Theorem 9.3.4.

Proposition 9.3.5. *Let $I \subset K[x_1, \dots, x_n]$ be a fully supported matroidal ideal generated in degree d with $\gcd(I) = 1$. Then the following statements hold.*

- (i) *for each $i = 1, \dots, n$, we have $\mathcal{G}(I_{1,i}) \subseteq \bigcup_{t=1, t \neq i}^n \mathcal{G}(I_{1,t})$.*
- (ii) *if $I_{1,i} \subseteq I_{1,j}$ for some $1 \leq i, j \leq n$ with $i \neq j$, then $I_{1,i} = I_{1,j}$.*

Proof. (i) The assumption that I is fully supported implies that $I_{1,i} \neq 0$ for each $i = 1, \dots, n$. Moreover, $\gcd(I) = 1$ gives that $I_{0,i} \neq 0$ for each $i = 1, \dots, n$. Let $u' \in \mathcal{G}(I_{1,i})$ and $v \in \mathcal{G}(I_{0,i})$. Then it follows from the definition of $I_{0,i}$ and $I_{1,i}$ that $v \in \mathcal{G}(I)$ and $u := u'x_i \in \mathcal{G}(I)$. Since $\deg_{x_i} u > \deg_{x_i} v$, one can conclude from the exchange property that there exists some s with $\deg_{x_s} u < \deg_{x_s} v$ such that $u'x_s = x_s(u/x_i) \in \mathcal{G}(I)$. Thus, $u' \in \mathcal{G}(I_{1,s})$. We therefore get $\mathcal{G}(I_{1,i}) \subseteq \bigcup_{t=1, t \neq i}^n \mathcal{G}(I_{1,t})$, as claimed.

(ii) Suppose, on the contrary, that there exists some monomial $u \in \mathcal{G}(I_{1,j}) \setminus \mathcal{G}(I_{1,i})$. Thus, $x_j u \in \mathcal{G}(I)$. If $x_i \in \text{supp}(u)$, then $(x_j u)/x_i \in \mathcal{G}(I_{1,i})$. Using the assumption $I_{1,i} \subseteq I_{1,j}$, we obtain $(x_j u)/x_i \in I_{1,j}$ which gives $(x_j^2 u)/x_i \in I$. Since I is matroidal, $(x_j u)/x_i \in I$. This yields a contradiction to $x_j u \in \mathcal{G}(I)$. Hence, $x_i \notin \text{supp}(u)$. Since $x_j u \in \mathcal{G}(I)$, we conclude that $x_j u \in \mathcal{G}(I_{0,i})$. From Theorem 9.3.4, we know that $I_{0,i} \subseteq I_{1,i}$. Then there exists some monomial $w \in \mathcal{G}(I_{1,i})$ such that w divides $x_j u$. Again, by using the assumption, $I_{1,i} \subseteq I_{1,j}$, we have $x_j \notin \text{supp}(w)$, and therefore $w|u$. Since both w and u are monomials of degree $d-1$, we conclude that $u = w$, as required. $\square$

It follows from [28, Theorem 3.2] that if $I(G)$ is a matroidal ideal, then G is a complete multipartite graph. The converse follows as a corollary of [11, Theorem 2.2]. We give another straightforward proof of this fact in the language of edge ideals of complete multipartite graphs.

Theorem 9.3.6. *Let G be a graph without isolated vertices. The edge ideal $I(G)$ is a matroidal ideal if and only if G is a complete r -partite graph for some $r \geq 2$.*

Proof. We first assume that G is a complete r -partite graph, that is, $G = K_{n_1, \dots, n_r}$ with a vertex partition $V_1, \dots, V_r$ and $|V_i| = n_i$ for all $i = 1, \dots, r$. Let $u := x_k x_l \in I(G)$ and $v := x_i x_j \in I(G)$ such that x_k does not divide v . We need to show that either $x_i(u/x_k) = x_i x_l \in I(G)$ or $x_j(u/x_k) = x_j x_l \in I(G)$. Let $x_l \in V_t$ for some $1 \leq t \leq r$. The graph G is complete r -partite and $\{x_i, x_j\} \in E(G)$, hence it follows that at least one of the vertices between x_i and x_j does not belong to V_t . If $x_i \notin V_t$, then $x_i x_l \in I(G)$ and similarly if $x_j \notin V_t$, then $x_j x_l \in I(G)$, as required.

Conversely, let $I(G)$ be a matroidal ideal and $|V(G)| = n$. We can assume that G does not contain any isolated vertices, and hence $I(G)$ is fully supported. If $I(G)$ is a principal ideal, then there is nothing to prove and the assertion holds trivially.

Let $I(G)$ be not a principal ideal. If $\gcd(I(G)) = x_i$, for some variable x_i , then G is a complete bipartite graph with r -partition $V_1 = \{x_i\}$ and $V_2 = V(G) \setminus \{x_i\}$, and again the assertion holds in this case.

Finally, we assume that $\gcd(I) = 1$. Note that for each $i = 1, \dots, n$, we have $I(G)_{1,i} = (x_q : x_q \in N_G(x_i))$, where $N_G(x_i)$ denotes the neighborhood of x_i in G . Then from Proposition 9.3.5(ii), we conclude that if $I(G)_{1,i} \subseteq I(G)_{1,j}$ for some $i \neq j$, then $I(G)_{1,i} = I(G)_{1,j}$; equivalently, if $N_G(x_i) \subseteq N_G(x_j)$, then $N_G(x_i) = N_G(x_j)$. We can partition vertices of G into disjoint sets, say $V_1, \dots, V_r$ such that for all $k = 1, \dots, r$, the vertices in V_k have the same neighborhood. Since G is a simple graph, it is clear that each of the V_k is an independent set of G . We claim that $G = K_{n_1, \dots, n_r}$ with r -partition $V_1, \dots, V_r$. To prove our claim, it only remains to verify the following: if $x_i \in V_k$ and $x_j \in V_\ell$ with $k \neq \ell$, then $\{x_i, x_j\} \in E(G)$. By virtue of Proposition 9.3.5(ii), we have $N_G(x_i) \not\subseteq N_G(x_j)$ and $N_G(x_j) \not\subseteq N_G(x_i)$. Let $x_k \in N_G(x_i) \setminus N_G(x_j)$ and $x_\ell \in N_G(x_j) \setminus N_G(x_i)$. Then $\{x_i, x_k\}$ and $\{x_j, x_\ell\}$ are disjoint edges in G . Since $I(G)$ is matroidal, then by employing the exchange property on $x_i x_k$ and $x_j x_\ell$, we conclude that $\{x_i, x_j\} \in E(G)$, as claimed. This completes the proof. $\square$

To establish the main result of this section, we first need the following auxiliary lemma.

Lemma 9.3.7. *Let $G = K_{n_1, \dots, n_r}$ and $t \geq 3$. Furthermore, let $P_1 : x_{i_1}, \dots, x_{i_{t-1}}$ be a $(t-1)$ -path in G and $P_2 : x_{j_1}, \dots, x_{j_t}$ be a t -path in G . Then there exists $x_{j_k} \in V(P_2) \setminus V(P_1)$ such that one of the following statements is satisfied.*

- (1) $x_{j_k}, x_{i_1}, \dots, x_{i_{t-1}}$ is a t -path in G .
- (2) $x_{i_1}, \dots, x_{i_{t-1}}, x_{j_k}$ is a t -path in G .
- (3) $t \geq 4$ and there exists $2 \leq p \leq t-2$ such that $x_{i_1}, \dots, x_{i_p}, x_{j_k}, x_{i_{p+1}}, \dots, x_{i_{t-1}}$ is a t -path in G .

Proof. Let $V_1, \dots, V_r$ be the r -partition of $G = K_{n_1, \dots, n_r}$. We may assume that $x_{i_1} \in V_1$. If there exists $x_{j_k} \in V(P_2) \setminus V(P_1)$ such that $x_{j_k} \notin V_1$, then $\{x_{j_k}, x_{i_1}\} \in E(G)$, and $P : x_{j_k}, x_{i_1}, \dots, x_{i_{t-1}}$ is a t -path in G , as given in (1).

If (1) is not true, then every $x_{j_k} \in V(P_2) \setminus V(P_1)$ is such that $x_{j_k} \in V_1$. If $x_{i_{t-1}} \notin V_1$, then for any $x_{j_k} \in V(P_2) \setminus V(P_1)$, we have $\{x_{i_{t-1}}, x_{j_k}\} \in E(G)$, and hence we obtain a t -path $P : x_{i_1}, \dots, x_{i_{t-1}}, x_{j_k}$ in G , as given in (2). Note that if $t = 3$, then in $P_1 : x_{i_1}, x_{i_2}$, both vertices cannot be in G_1 because G is a complete r -partite graphs. Hence, either (1) or (2) must be true for $t = 3$.

If both (1) and (2) do not hold, then $t \geq 4$. In this case, we try to construct a path as given in (3). The negation of (1) and (2) gives us that $x_{i_1}, x_{i_{t-1}} \in V_1$

and every $x_{j_k} \in V(P_2) \setminus V(P_1)$ is such that $x_{j_k} \in V_1$. If there exists some $2 \leq p \leq t-2$ such that $x_{i_p}, x_{i_{p+1}} \notin V_1$, then for any $x_{j_k} \in V(P_2) \setminus V(P_1)$, we have $\{x_{i_p}, x_{j_k}\}, \{x_{j_k}, x_{i_{p+1}}\} \in E(G)$, and $P : x_{i_2}, \dots, x_{i_p}, x_{j_k}, x_{i_{p+1}}, \dots, x_{i_t}$ is a t -path in G as given in (3). Finally, suppose that each of (1), (2), and (3) do not hold. Then we are in the following situation:

- (i) $x_{i_1}, x_{i_{t-1}} \in V_1$;
- (ii) every $x_{j_k} \in V(P_2) \setminus V(P_1)$ is such that $x_{j_k} \in V_1$. This also gives that for every $x_{j_k} \in V(P_2) \setminus V_1$, we have $x_{j_k} \in V(P_1)$;
- (iii) for every $2 \leq p \leq t-2$ either $x_{i_p} \in V_1$ or $x_{i_{p+1}} \in V_1$.

Since $x_{i_1} \in V_1$ and $\{x_{i_1}, x_{i_2}\} \in E(G)$, this implies that $x_{i_2} \notin V_1$. It follows now from (iii) that $x_{i_3} \in V_1$. By continuing this process, we obtain from (iii) that $x_{i_{t-2}} \notin V_1$ since $x_{i_{t-1}} \in V_1$ and $\{x_{i_{t-2}}, x_{i_{t-1}}\} \in E(G)$. This shows that P_1 starts and ends at vertices in V_1 , and all vertices in P_1 with odd indices are also in V_1 . Furthermore, all vertices in $V(P_1)$ with even indices do not belong to V_1 . Hence, $t-1$ must be an odd integer, and in $V(P_1)$, there are $t/2-1$ vertices that do not belong to V_1 , that is, $|V(P_1) \setminus V_1| = t/2-1$. Moreover, it follows from (ii) that every vertex in $V(P_2)$ which is outside of V_1 must belong to $V(P_1)$, that is, $V(P_2) \setminus V_1 \subseteq V(P_1) \setminus V_1$. Since P_2 is a t -path with even number of vertices in a complete r -partite graph G , this gives that at least half of its vertices are not in V_1 , that is, $|V(P_2) \setminus V_1| \geq t/2$. This contradicts $|V(P_2) \setminus V_1| \leq |V(P_1) \setminus V_1| = t/2-1$. This yields that at least one of the statements in (1), (2), or (3) must hold. $\square$

We are now ready to state the main result of this section.

Theorem 9.3.8. *Let $t \geq 2$. If $I_t(K_{n_1, \dots, n_r}) \neq 0$, then it is a matroidal ideal.*

Proof. We set $G := K_{n_1, \dots, n_r}$ and let $V_1, \dots, V_r$ be the r -partition of G . The assertion is true for $I_2(G)$, as shown in Theorem 9.3.6. Let $t \geq 3$ and $I_t(G) \neq 0$. If $I_t(G)$ is a principal ideal, then there is nothing to prove and the assertion holds trivially. Let $|\mathcal{G}(I_t(G))| \geq 2$. To prove that $I_t(G)$ is matroidal, we must show that if $P_1 : x_{i_1}, \dots, x_{i_t}$ and $P_2 : x_{j_1}, \dots, x_{j_t}$ are two t -paths with $x_{i_l} \notin V(P_2)$ for some $1 \leq l \leq t$, then there exists some $x_{j_k} \in V(P_2) \setminus V(P_1)$, such that one obtains a new path by removing x_{i_l} from P_1 and inserting x_{j_k} in an appropriate position in P_1 .

We may assume that $x_{i_l} \in V_1$. If $x_{i_1}, \dots, x_{i_{l-1}}, x_{i_{l+1}}, \dots, x_{i_t}$ is a $(t-1)$ -path in G , then we obtain the desired conclusion by using Lemma 9.3.7. In particular, if $l = 1$ or $l = t$, then removing x_{i_l} from P_1 gives a $(t-1)$ -path in G . Now, assume that $x_{i_1}, \dots, x_{i_{l-1}}, x_{i_{l+1}}, \dots, x_{i_t}$ is not a $(t-1)$ -path in G . This is possible only if $\{x_{i_{l-1}}, x_{i_{l+1}}\} \notin E(G)$, that is, $x_{i_{l-1}}, x_{i_{l+1}}$ belong to V_a for some $2 \leq a \leq r$. If there

exists some $x_{j_k} \in V(P_2) \setminus V(P_1)$ such that $x_{j_k} \notin V_a$, then

$$x_{i_1}, \dots, x_{i_{l-1}}, x_{j_k}, x_{i_{l+1}}, \dots, x_{i_t},$$

is a t -path in G , and the proof is complete. Otherwise, every $x_{j_k} \in V(P_2) \setminus V(P_1)$ is such that $x_{j_k} \in V_a$. If $x_{i_1} \notin V_a$, then $\{x_{i_1}, x_{i_{l+1}}\} \in E(G)$, and we get $x_{i_{l-1}}, x_{i_{l-2}}, \dots, x_{i_1}, x_{i_{l+1}}, \dots, x_{i_t}$ as a $(t-1)$ -path in G . We again use Lemma 9.3.7 to obtain the desired conclusion. A similar argument gives us the desired conclusion when $x_{i_t} \notin V_a$. Otherwise, we are in the following situation:

- (1) $x_{i_{l-1}}, x_{i_{l+1}}, x_{i_1}, x_{i_t} \in V_a$;
- (2) every $x_{j_k} \in V(P_2) \setminus V(P_1)$ is such that $x_{j_k} \in V_a$. This implies that if $x_{j_k} \in V(P_2) \setminus V_a$, then $x_{j_k} \in V(P_1)$.

We claim that there must exist some p with either $2 \leq p \leq l-3$ or $l+2 \leq p \leq t-2$ such that $x_{i_p}, x_{i_{p+1}} \notin V_a$. Indeed, if our claim is true and we can find such p with $2 \leq p \leq l-3$, then $x_{i_1}, \dots, x_{i_p}, x_{i_{l-1}}, x_{i_{l-2}}, \dots, x_{i_{p+1}}, x_{i_{l+1}}, \dots, x_{i_t}$ is a $(t-1)$ -path in G , and the desired result can be deduced from Lemma 9.3.7. Similarly, if we can find such p with $l+2 \leq p \leq t-2$, then

$$x_{i_1}, \dots, x_{i_{l-1}}, x_{i_p}, x_{i_{p-1}}, \dots, x_{i_{l+1}}, x_{i_{p+1}}, \dots, x_{i_t}$$

is a $(t-1)$ -path in G , and one can conclude the desired result from Lemma 9.3.7.

To show our claim, suppose, on the contrary, that for every p with $2 \leq p \leq l-3$ and $l+2 \leq p \leq t-2$, we have either $x_{i_p} \in V_a$ or $x_{i_{p+1}} \in V_a$. Since $x_{i_1} \in V_a$ and $\{x_{i_1}, x_{i_2}\} \in E(G)$, this yields that $x_{i_2} \notin V_a$. It follows now from the assumption that $x_{i_3} \in V_a$. Moreover, one can conclude from (1) that $x_{i_{l-2}} \notin V_a$ since $x_{i_{l-1}} \in V_a$ and $\{x_{i_{l-2}}, x_{i_{l-1}}\} \in E(G)$. This forces $l-1$ to be an odd integer. Similarly, since $x_{i_{l+1}}, x_{i_t} \in V_a$, we derive that the number of vertices in the path $x_{i_{l+1}}, \dots, x_{i_t}$ is odd as well. Collectively, we obtain t itself must be odd. This shows that $V(P_1)$ contains $(t+1)/2$ vertices from V_a and the other $(t-1)/2$ vertices are not from V_a . Note that these $(t-1)/2$ vertices that are not from V_a include x_{i_l} as well. Considering that, by (2), every vertex in $V(P_2) \setminus V(P_1)$ belongs to V_a and $x_{i_l} \notin V(P_2)$, this yields that in $V(P_2)$ there are at most $(t-1)/2 - 1$ vertices that are not from V_a , which is impossible by Remark 9.3.3(i). Hence, the claim holds, and the proof is over. $\square$

9.3.2 Cohen–Macaulay property of $I_t(K_{n_1, \dots, n_r})$

We now investigate the Cohen–Macaulay property for $I_t(K_{n_1, \dots, n_r})$. Let I be a monomial ideal in the polynomial ring $S = K[x_1, \dots, x_n]$. The ideal I is called the *Veronese ideal* of degree d in S if I is generated by all monomials of degree d in S .

Similarly, one defines the square-free counterpart of a Veronese ideal. The monomial ideal I is called *square-free Veronese ideal* of degree d in S if I is generated by all square-free monomials of degree d in S . It follows from [58, Theorem 4.2] that a polymatroidal ideal I is Cohen-Macaulay if and only if I is

- (i) a principal ideal, or
- (ii) a Veronese ideal, or
- (iii) a square-free Veronese ideal.

In particular, if I is a matroidal ideal, then it is Cohen-Macaulay if and only if it is a principal ideal or square-free Veronese ideal. From Remark 9.3.3(v) it follows that $I_t(K_{n_1, \dots, n_r})$ is a principal ideal if and only if $t = n_1 + \dots + n_r$, and hence in this case it can be viewed as the square-free Veronese ideal of degree $t = n$ in S . Therefore, for a given t , to be able to characterize $n_1, \dots, n_r$ such that $I_t(K_{n_1, \dots, n_r})$ is Cohen-Macaulay, one only needs to check that when $I_t(K_{n_1, \dots, n_r})$ is square-free Veronese of degree t in S . More precisely, one needs to check that for which $n_1, \dots, n_r$ with $n_1 + \dots + n_r = n$, every subset of $[n]$ of size t can be viewed as a path in $K_{n_1, \dots, n_r}$. In the case of edge ideal of $K_{n_1, \dots, n_r}$, the answer follows immediately from [58, Theorem 4.2].

Proposition 9.3.9. *Let G be a graph such that $I(G)$ is matroidal. Then $I(G)$ is Cohen-Macaulay if and only if G is a complete graph.*

Proof. Let $|V(G)| = n$. From [58, Theorem 4.2], it follows that $I(G)$ is Cohen-Macaulay if and only if $I(G)$ is square-free Veronese of degree 2 in $S = K[x_1, \dots, x_n]$, that is, for every $1 \leq i \neq j \leq n$, we have $x_i x_j \in I(G)$. Moreover, $x_i x_j \in I(G)$ for every $1 \leq i \neq j \leq n$, if and only if G is a complete graph. $\square$

By using Theorem 9.3.6, the above proposition can be rephrased as follows:

Corollary 9.3.10. *The edge ideal $I(K_{n_1, \dots, n_r})$ is Cohen-Macaulay if and only if $n_1 = \dots = n_r = 1$.*

We are in a position to state the main result of this section in which we discuss the general case when $t \geq 2$.

Theorem 9.3.11. *The t -path ideal $I_t(K_{n_1, \dots, n_r}) \neq 0$ is Cohen-Macaulay if and only if $n_i \leq \lceil t/2 \rceil$ for each $i = 1, \dots, r$.*

Proof. Let $V_1, \dots, V_r$ be the r -partition of $G = K_{n_1, \dots, n_r}$ with $|V_i| = n_i$ for each $i = 1, \dots, r$. Let $n = n_1 + \dots + n_r$. From [58, Theorem 4.2] and Remark 9.3.3(v), one can conclude that if $I_t(G) \subset S = K[x_1, \dots, x_n]$ is Cohen-Macaulay, then it is a square-free Veronese ideal of degree t , and hence $x_{i_1} \dots x_{i_t} \in I_t(G)$ for every

$1 \leq i_1 < \cdots < i_t \leq n$. Moreover, from Remark 9.3.3(i), it follows that if there exists some V_i with $n_i > \lceil t/2 \rceil$, then for any subset T of V_i with $|T| = \lceil t/2 \rceil + 1$, there does not exist any t -path in G that contains all the elements of T . Hence, if there exists some V_i of G with $n_i > \lceil t/2 \rceil$, then $I(G)$ is not square-free Veronese, and hence not Cohen-Macaulay.

Now, assume that $n_i \leq \lceil t/2 \rceil$ for each $i = 1, \dots, r$. Let $u = x_{i_1} \cdots x_{i_t}$ be a square-free monomial in S . We have to show that the elements in $A := \text{supp}(u)$ can be interpreted as a t -path in G . Let $A_i = A \cap V_i$ and $a_i = |A_i|$. Since $n_i \leq \lceil t/2 \rceil$, we get $a_i \leq \lceil t/2 \rceil$. Without loss of generality, we may assume that the elements in $\{x_{i_1}, \dots, x_{i_t}\}$ are arranged such that the first a_1 elements lie in V_1 , the next a_2 elements lie in V_2 and so on. If t is odd, then

$$x_1, x_{\frac{t+1}{2}+1}, x_2, x_{\frac{t+1}{2}+2}, \dots, x_{\frac{t-1}{2}}, x_t, x_{\frac{t+1}{2}},$$

is a t -path in G , by virtue of consecutive vertices in the above sequence belong to different V_i 's. Similarly, if t is even, then $x_1, x_{\frac{t}{2}+1}, x_2, x_{\frac{t}{2}+2}, \dots, x_{\frac{t}{2}}, x_t$ is a t -path in G . This completes the proof. $\square$

9.3.3 limit depth of $I_t(K_{n_1, \dots, n_r})$

Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring or a standard graded algebra over a field K with the graded maximal ideal $\mathfrak{m}$. Let $I \subset R$ be an ideal, which is graded if R is a standard graded K -algebra. It is known from Brodmann [21] that $\text{depth}(R/I^k)$ stabilizes for large k , that is, $\text{depth}(R/I^k)$ is constant for $k \gg 0$. The smallest $t > 0$ such that $\text{depth}(R/I^t) = \text{depth}(R/I^k)$ for all $k \geq t$ is called the *index of depth stability* of I and is denoted by $\text{dstab}(I)$, as defined in [66]. Moreover, $\text{depth}(I^{\text{dstab}(I)})$ is called the *limit depth* of I and is denoted by $\lim_{k \rightarrow \infty} \text{depth}(R/I^k)$, see [61]. In this section, we study the $\text{dstab}(S/I_t(K_{n_1, \dots, n_r}))$ and $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(K_{n_1, \dots, n_r})^k)$. In addition, the linear relation graphs of monomial ideals were introduced to help in understanding their analytic spreads, see Definition 6.3.10. It is known from [65, Lemma 4.2 and Theorem 4.1] that if the linear relation graph of a matroidal ideal I has m vertices and s connected components, then

$$(9.3.1) \quad \text{dstab}(I) < \ell(I) = m - s + 1,$$

where $\ell(I)$ denotes the analytic spread of I , that is, the Krull dimension of the fiber ring $\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Combining this with [66, Corollary 3.5], one can deduce that

$$(9.3.2) \quad \lim_{k \rightarrow \infty} \text{depth}(S/I^k) = n - (m - s + 1).$$

Therefore, to compute the limit depth of t -path ideals of complete r -partite graphs, it is enough to compute the number of vertices and the number of connected components in their linear relation graphs.

Remark 9.3.12. In [61, Proposition 2.1], it is shown that for any graded ideal I , the $\text{depth}(S/I^k)$ is a non-increasing function of k if all powers of I have a linear resolution. This is indeed the case for matroidal ideals. Therefore, one concludes the following: if limit depth of a matroidal ideal is s and the k is the smallest integer for which $\text{depth}(S/I^k) = s$, then $\text{dstab}(I) = k$.

Remark 9.3.13. It follows from [28, Theorem 2.5] that if I is a fully supported matroidal ideal generated in degree d , then $\text{depth}(I) = d - 1$. It is shown in Remark 9.3.3(iv) that if $I_t(K_{n_1, \dots, n_r}) \neq 0$, then it is a fully supported ideal. This shows that whenever $I_t(K_{n_1, \dots, n_r}) \neq 0$, we have $\text{depth}(I_t(K_{n_1, \dots, n_r})) = t - 1$.

At first, we will compute limit depth of t -path ideals of complete bipartite graphs.

Theorem 9.3.14. Let $G = K_{p,q}$ with $n = p + q$ and $t \geq 2$. Then the following statements hold.

- (i) if $p = \lfloor t/2 \rfloor$ and $q = \lceil t/2 \rceil$, then $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = n - 1$ and $\text{dstab}(I_t(G)) = 1$.
- (ii) if $p = \lfloor t/2 \rfloor$ and $q > \lceil t/2 \rceil$, then $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = \lfloor t/2 \rfloor$ and $\text{dstab}(I_t(G)) = \lceil (q - 1)/(q - \lceil t/2 \rceil) \rceil$.
- (iii) if $p > \lfloor t/2 \rfloor$ and $q > \lceil t/2 \rceil$, then $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = 0$ and $1 < \text{dstab}(I_t(G)) < n$ if t is odd, and $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = 1$ and $1 \leq \text{dstab}(I_t(G)) < n - 1$ if t is even.

Proof. Let $V_1 = \{x_1, \dots, x_p\}$ and $V_2 = \{y_1, \dots, y_q\}$ be the 2-partition of G .

(i) In this case $n = p + q = t$ and $I_t(G)$ is a principal ideal and the assertion follows trivially.

(ii) If $p = \lfloor t/2 \rfloor$ and $q > \lceil t/2 \rceil$, then it is evident from Remark 9.3.3(i) that all vertices of V_1 appear in every t -path in G . Then, we deduce from the definition of Γ that $V_1 \cap V(\Gamma) = \emptyset$. It follows immediately from Remark 9.3.3(ii) that $\{y_i, y_j\} \in E(\Gamma)$ for all $1 \leq i \neq j \leq q$. Therefore, $V(\Gamma) = V_2$ and Γ is a complete graph on q vertices, and has only one connected component. We conclude $\ell(I) = q$ and use the equality in (9.3.2) to compute $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = n - q = p = \lfloor t/2 \rfloor$.

To prove the assertion about $\text{dstab}(I_t(G))$, we first observe that $I_t(G) = JL$, where $J = (x_1 \cdots x_p)$ and L is generated by all monomials of degree $\lceil t/2 \rceil$ in q variables. Indeed, as discussed before, all vertices of V_1 appear in every t -path in G , equivalently, $x_1 \cdots x_p$ divides every generator of $I_t(G)$. Moreover, for every subset $\{y_{i_1}, \dots, y_{i_m}\}$ of V_2 of size $m = \lceil t/2 \rceil$, there is a t -path P in G with $V(P) = \{y_{i_1}, \dots, y_{i_m}\} \cup V_1$, equivalently, $y_{i_1} \cdots y_{i_m} \in L$. Hence, J is a principal ideal and L is the square-free Veronese ideal of degree $\lceil t/2 \rceil$ in variables $y_1, \dots, y_q$. Let $S_1 = K[x_1, \dots, x_p]$ and $S_2 = K[y_1, \dots, y_q]$. Then $\text{depth}(S_1/J^k) = p - 1$ and it follows

from [66, Corollary 5.7] that $\text{depth}(S_2/L^k) = \max\{0, k(\lceil t/2 \rceil - q) + q - 1\}$ for all k and $\lim_{k \rightarrow \infty} \text{depth}(S_2/L^k) = 0$ with $\text{dstab}(L) = \lceil (q - 1)/(q - \lceil t/2 \rceil) \rceil$.

The fact that the ideals J and L are generated in different sets of variables facilitates to conclude that $(I_t(G))^k = J^k L^k$, and $\text{depth}(S/I_t(G)^k) = \text{depth}(S_1/J^k) + \text{depth}(S_2/L^k) + 1$, for all k , for example, see [70, Lemma 2.2]. Therefore,

$$\text{depth}(S/I_t(G)^k) = p + \max\{0, k(\lceil t/2 \rceil - q) + q - 1\}.$$

Since $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = p$ and $\text{dstab}(L) = \lceil (q - 1)/(q - \lceil t/2 \rceil) \rceil$, it follows that $\text{dstab}(I_t(G)) = \lceil (q - 1)/(q - \lceil t/2 \rceil) \rceil$.

(iii) Let $p > \lceil t/2 \rceil$ and $q > \lfloor t/2 \rfloor$. Then it follows from Remark 9.3.3(ii) that $\{x_i, x_j\} \in E(\Gamma)$ for all $1 \leq i \neq j \leq p$, and $\{y_i, y_j\} \in E(\Gamma)$ for all $1 \leq i \neq j \leq q$. If t is even, then every path in G contains half variables from V_1 and the other half from V_2 . Consequently, given any t -path in G , we cannot replace any vertex from V_1 with any vertex from V_2 . This shows that $\{x_i, y_j\} \notin E(\Gamma)$. Therefore, Γ has exactly two connected components, and $\ell(I) = p + q - 2 + 1 = n - 1$. By using the equality in (9.3.2), we obtain $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = n - (n - 1) = 1$ and we conclude from (9.3.1) that $\text{dstab}(I_t(G)) < n - 1$. Remark 9.3.13 gives that $\text{depth}(I_t(G)) = t - 1$. Hence, $\text{dstab}(I_t(G)) \geq 1$ and the equality holds only if $t = 2$ due to Remark 9.3.12.

If t is odd, then set $m = \lfloor t/2 \rfloor$. The sequence $x_1, y_1, x_2, y_2, \dots, x_m, y_m$ is a $(t - 1)$ -path in G . It can be extended to a t -path by either joining the edge $\{y_{m+1}, x_1\}$, or the edge $\{y_m, x_{m+1}\}$. Hence, for $u = y_{m+1}x_1y_1x_2y_2 \cdots x_my_m$ and $v = x_1y_1x_2y_2 \cdots x_my_mx_{m+1}$, we have $x_{m+1}u = y_{m+1}v$, which gives $\{x_{m+1}, y_{m+1}\} \in E(\Gamma)$. This shows that Γ is connected. In fact, by repeating the same argument as above, one can show that Γ is a complete graph on n vertices. Therefore, $\ell(I) = n - 1 + 1 = n$ and by using the equality in (9.3.2), we obtain $\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = 0$ and one can conclude from (9.3.1) that $\text{dstab}(I_t(G)) < n$. Again from Remark 9.3.13 we have $\text{depth}(I_t(G)) = t - 1 > 0$. Hence, $\text{dstab}(I_t(G)) > 1$. This finishes the proof. $\square$

Now, we consider the case in which G is an r -partite graph with $r \geq 3$. Let $d, a_1, \dots, a_n$ be positive integers with $\sum_{i=1}^n a_i \geq d$. Then the ideal generated by all monomials $x_1^{b_1} \cdots x_n^{b_n}$ of degree d with $b_i \leq a_i$, for all $i = 1, \dots, n$ is called the ideal of *Veronese type*, and is denoted by $I_{d;a_1, \dots, a_n}$. In [66], the index of depth stability of Veronese type ideals is discussed in detail. Note that if $a_1 = \cdots = a_n = 1$, then $I_{d;a_1, \dots, a_n}$ is square-free Veronese ideal in n variables. According to Theorem 9.3.11 and results obtained in [66], we conclude the following proposition.

Proposition 9.3.15. *Let $t \geq 2$ and $G = K_{n_1, \dots, n_r}$ with $|V(G)| = n$ and $n_i \leq \lceil t/2 \rceil$ for all $1 \leq i \leq r$. Then*

$$\lim_{k \rightarrow \infty} \text{depth}(S/I_t(G)^k) = 0 \quad \text{and} \quad \text{dstab}(I_t(G)) = \lceil (n - 1)/(n - t) \rceil.$$

In view of Proposition 9.3.15, it is enough to consider those cases in which at least one of the n_i 's is greater than $\lceil t/2 \rceil$. From Theorem 9.3.14, it is clear that limit depth of $I_t(G)$ depends on t and how vertices are distributed among the partition sets. We see this behavior in subsequent results too. Now, we will describe limit depth and dstab for $I_3(K_{n_1, \dots, n_r})$ when $r \geq 3$. Note that since $r \geq 3$, we must have $3 \leq n = n_1 + \dots + n_r$. First, we discuss the cases when $n \in \{3, 4\}$. The following lemma is a consequence of Proposition 9.3.15 and [133, Lemma 2.3.10].

Lemma 9.3.16. *Let $G = K_{n_1, \dots, n_r}$ with $r \geq 3$. Then the following statements hold.*

- (i) *If $|V(G)| = 3$, then $I_3(G)$ is a principal ideal and $\lim_{k \rightarrow \infty} \text{depth}(S/I_3(G)^k) = 2$ and $\text{dstab}(I_3(G)) = 1$.*
- (ii) *If $|V(G)| = 4$, then G is either a complete graph on four vertices or isomorphic to $K_{2,1,1}$. Moreover, $\lim_{k \rightarrow \infty} \text{depth}(S/I_3(G)^k) = 0$ and $\text{dstab}(I_3(G)) = 3$.*

In the rest of this section, $\mathfrak{m}$ denotes the unique graded maximal ideal in the polynomial ring S whose variables correspond to the vertices of the graph G .

Proposition 9.3.17. *Let $G = K_{n_1, \dots, n_r}$ with $r \geq 3$. Then $\lim_{k \rightarrow \infty} \text{depth}(I(G)^k) = 0$ and $\text{dstab}(I(G)) = 2$.*

Proof. From Remark 9.3.13, we know that $\text{depth}(I(G)) = 1$. Then by following Remark 9.3.12, it is enough to show that $\text{depth}(I(G)^2) = 0$. Given a graph H , in [57, Theorem 3.1], the equivalent condition for $\text{depth}(I(H)^2) = 0$ is described as follows: $\text{depth}(I(H)^2) = 0$ if and only if H contains a cycle of length 3 and every other vertex of G has a neighbor in this cycle. It can be easily seen that G satisfies this condition. Let $V_1, \dots, V_r$ be the r -partition of G and since $r \geq 3$, we can choose $x_1 \in V_1$, $x_2 \in V_2$, and $x_3 \in V_3$. Then the subgraph induced by x_1, x_2 , and x_3 is the cycle of length 3 in G . Moreover, it follows from the definition of G that any $x_j \in V(G)$ is adjacent to at least one of the vertices in $\{x_1, x_2, x_3\}$. Hence, $\text{depth}(I(G)^2) = 0$, as required. $\square$

Theorem 9.3.18. *Let $G = K_{n_1, \dots, n_r}$ with $r \geq 3$ and $|V(G)| \geq 5$. Then $\lim_{k \rightarrow \infty} \text{depth}(I_3(G)^k) = 0$ and $\text{dstab}(I_3(G)) = 2$.*

Proof. From Remark 9.3.13, we know that $\text{depth}(I_3(G)) = 2$, and then by following Remark 9.3.12, it is enough to show that $\text{depth}(I_3(G)^2) = 0$. Let $V_1, \dots, V_r$ be the r -partition of G .

First, assume that $|V_1| = \dots = |V_{r-1}| = 1$. If $|V_r| = 1$ or $|V_r| = 2$, then the assertion follows from Proposition 9.3.15. More precisely, If $|V_r| = 1$, then $|V(G)| = r$, and if $|V_r| = 2$, then $|V(G)| = r + 1$. Since $r \geq 5$, in both cases, we have $\text{dstab}(I_3(G)) = \lceil (s-1)/(s-3) \rceil = 2$ for $s = r, r + 1$. Finally, if $|V_r| > 2$, then take

$x_1 \in V_1$, $x_2 \in V_2$, and $x_3, x_4, x_5 \in V_r$, and set $u := x_1x_2x_3x_4x_5$. Then for any $x_i \in V_r$, we have $ux_i = (x_ix_2x_3)(x_4x_1x_5) \in I_3(G)^2$ because x_i, x_2, x_3 and x_4, x_1, x_5 are 3-paths in G . Furthermore, for any $x_i \in V(G) \setminus V_r$, we have $ux_i = (x_1x_3x_2)(x_4x_ix_5) \in I_3(G)^2$ because x_1, x_3, x_2 and x_4, x_i, x_5 are 3-paths in G . Therefore, $I_3(G)^2 : u = \mathfrak{m}$, and hence $\text{depth}(I_3(G)^2) = 0$, as required.

Now, assume that $|V_i|, |V_j| \geq 2$, for some $1 \leq i \neq j \leq r$. Take $x_1, x_2 \in V_i$, $x_3, x_4 \in V_j$, and $x_5 \in V_k$. Set $u := x_1x_2x_3x_4x_5$; then $u \notin I_3(G)^2$. We claim that $I_3(G)^2 : u = \mathfrak{m}$. Indeed, for any $x_m \in V_i$, we have $ux_m \in I_3(G)^2$ because x_1, x_5, x_2 and x_3, x_m, x_4 are 3-paths in G . Similarly, for any $x_m \in V_j$, we have $ux_m \in I_3(G)^2$. Finally, if $x_m \in V(G) \setminus (V_i \cup V_j)$, then again $ux_m \in I_3(G)^2$ because x_1, x_m, x_2 and x_3, x_5, x_4 are 3-paths in G . This shows that $\text{depth}(I_3(G)^2) = 0$, as claimed. $\square$

We are ready to give the final result of this subsection which shows that in the case of $r \geq 3$ and $t \geq 3$, if the number of vertices in each partition set is big enough, then limit depth of $I_t(G)$ is 0.

Theorem 9.3.19. *Let $G = K_{n_1, \dots, n_r}$ with $r \geq 3$ and $n_i \geq \lceil t/2 \rceil$ for all $1 \leq i \leq r$. Then $\lim_{k \rightarrow \infty} \text{depth}(I_t(G)^k) = 0$ and $1 < \text{dstab}(I_t(G)) < n$ for $t \geq 3$.*

Proof. From Remark 9.3.13, we conclude that $\text{depth}(I_t(G)) = t - 1 > 0$. Let $|V(G)| = n$. Due to (9.3.1) and (9.3.2), it is enough to prove that the linear relation graph Γ of $I_t(G)$ has one connected component and $|V(\Gamma)| = n$. To prove this, we will show that Γ is a complete graph with $V(\Gamma) = V(G)$. Let $V_1, \dots, V_r$ be the r -partition of G . Let $x_i, x_j \in V(G)$, for some $i \neq j$. We can choose $A \subset V(G)$ such that $|A| = t$, $x_i \in A$, $x_j \notin A$, and $|A \cap V_k| < \lceil t/2 \rceil$ for all $k = 1, \dots, r$. Such a choice of A is possible because $n_i \geq \lceil t/2 \rceil$ and $r \geq 3$. Take $B = (A \setminus \{x_i\}) \cup \{x_j\}$. Then $|B| = t$ and $|B \cap V_k| \leq \lceil t/2 \rceil$, for all $k = 1, \dots, r$. Then by following the similar construction of the path at the end of the proof of Theorem 9.3.11, we obtain a t -path P in G with vertices in A and a t -path P' in G with vertices in B . Let u be the monomial in $\mathcal{G}(I_t(G))$ that corresponds to the t -path P and v be the monomial in $\mathcal{G}(I_t(G))$ that corresponds to the t -path P' . Then $x_iv = x_ju$ and $\{x_i, x_j\} \in E(\Gamma)$, as required. This completes the proof. $\square$

Chapter 10

On the maximal associated prime ideal of monomial ideals

10.1 A criterion for existence the maximal ideal

Suppose that I is a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of R , and $x_1, \dots, x_n$ are indeterminates. One of the most important questions is the existence of the graded maximal ideal in the set of associated primes, see [27], [50], [102] for more details. Unfortunately, little is known on this subject in literature. In this section, our aim is to detect that whether $\mathfrak{m} \in \text{Ass}_R(R/I)$ solely by using the elements of the unique minimal set of monomial generators of I , and without computing any minimal primary decomposition of I . It should be noted that the results of this section can be found in [109].

The following lemma investigates the relation between the existence of the graded maximal ideal and the number of minimal generators.

Lemma 10.1.1. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. If $\mathfrak{m} = (x_1, \dots, x_n) \in \text{Ass}_R(R/I)$, then $m \geq n$.*

Proof. Suppose, on the contrary, that $m < n$. It follows from $\mathfrak{m} \in \text{Ass}_R(R/I)$ that there exists a monomial h in R such that $\mathfrak{m} = (I :_R h)$. Hence, one has $x_i h \in I$ for each $i = 1, \dots, n$. Note that $n \geq 2$ since $m \geq 1$. Because $m < n$, the Pigeonhole Principle implies that there exist positive integers $1 \leq i, j \leq n$ with $i \neq j$ and $1 \leq k \leq m$ such that $u_k | x_i h$ and $u_k | x_j h$. Accordingly, $x_i h = u_k w$ and $x_j h = u_k v$ for some monomials $v, w \in R$. This yields that $x_i u_k v = x_i x_j h = x_j u_k w$, and so $x_i v = x_j w$. Therefore, we get $x_i | w$, and thus $w = x_i g$ for some monomial g in R . Consequently, one derives $x_i v = x_j x_i g$, and hence $v = x_j g$. This gives that

$x_j h = u_k x_j g$, and so $h = u_k g$. That is, $h \in I$, which is a contradiction since $\mathfrak{m} = (I :_R h)$. Therefore, we obtain $m \geq n$, as claimed. $\square$

The proposition below says that any monomial ideal in a polynomial ring in two variables, which is generated by at least two minimal generators has the graded maximal ideal as an associated prime.

Proposition 10.1.2. *Let I be a monomial ideal in a polynomial ring $R = K[x, y]$ with $\mathcal{G}(I) = \{u_1, \dots, u_m\}$. If $m \geq 2$, then $\mathfrak{m} = (x, y) \in \text{Ass}_R(R/I)$.*

Proof. Suppose that $u_i = x^{\alpha_i} y^{\beta_i}$ with nonnegative integers α_i and β_i for each $i = 1, \dots, m$. Lexicographically order the u_i 's such that $u_i < u_j$ if either $\alpha_i < \alpha_j$ or $\alpha_i = \alpha_j$ and $\beta_i < \beta_j$. Then one may assume that $u_1 < u_2 < \dots < u_m$. On account of the u_i 's are minimal generators, one can conclude that $\alpha_1 < \alpha_2 < \dots < \alpha_m$ and $\beta_1 > \beta_2 > \dots > \beta_m$. Note that $\alpha_2 \geq 1$ and $\beta_1 \geq 1$. Set $h := x^{\alpha_2-1} y^{\beta_1-1}$. Want to show that $h \notin I$. On the contrary, assume that $h \in I$. This implies that there exists a positive integer $1 \leq i \leq m$ such that $u_i | h$. Hence, $\alpha_i \leq \alpha_2 - 1$ and $\beta_i \leq \beta_1 - 1$, and so $\alpha_i < \alpha_2$ and $\beta_i < \beta_1$. Accordingly, one must have $i = 1$ and $i \geq 2$, which is a contradiction. Therefore, we get $h \notin I$. We now demonstrate that $xh \in I$ and $yh \in I$. By virtue of $xh = x^{\alpha_2} y^{\beta_1-1} = x^{\alpha_2} y^{\beta_2} y^{\beta_1-\beta_2-1}$ and $yh = x^{\alpha_2-1} y^{\beta_1} = x^{\alpha_2-\alpha_1-1} x^{\alpha_1} y^{\beta_1}$, we obtain $xh \in I$ and $yh \in I$. This yields that $\mathfrak{m} = (I :_R h)$, and thus $\mathfrak{m} \in \text{Ass}_R(R/I)$, as desired. $\square$

The next theorem gives us a subtle tool for checking whether the set of associated primes contains the graded maximal ideal.

Theorem 10.1.3. *Suppose that I is a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$, $\mathfrak{m} = (x_1, \dots, x_n)$, and*

$$\mathcal{G}(I) = \{x_1^{r_{1,1}} \cdots x_n^{r_{1,n}}, \dots, x_1^{r_{k,1}} \cdots x_n^{r_{k,n}}\},$$

with $k \geq n$. Then $\mathfrak{m} \in \text{Ass}_R(R/I)$ if and only if there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, k\}$ such that the following conditions hold:

- (i) $r_{i_t, j} < r_{i_j, j}$ for each $t \neq j$, where $j = 1, \dots, n$;
- (ii) for each $i \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}$, there exists a positive integer $1 \leq t \leq n$ such that $r_{i, t} \geq r_{i_t, t}$.

Proof. We first assume that $\mathfrak{m} \in \text{Ass}_R(R/I)$. This implies that there exists a monomial h in R such that $\mathfrak{m} = (I :_R h)$. Let $h = x_1^{s_1} \cdots x_n^{s_n}$ with nonnegative integers $s_1, \dots, s_n$. It follows from $\mathfrak{m} = (I :_R h)$ that $h \notin I$, and also $hx_i \in I$ for each $i = 1, \dots, n$. Fix $1 \leq t \leq n$. Since $hx_t \in I$, one can conclude that there exists a positive integer $1 \leq i_t \leq k$ such that $x_1^{r_{i_t, 1}} \cdots x_n^{r_{i_t, n}} | x_1^{s_1} \cdots x_{t-1}^{s_{t-1}} x_t^{s_t+1} x_{t+1}^{s_{t+1}} \cdots x_n^{s_n}$.

Thus, $r_{i_t,j} \leq s_j$ for each $j \in \{1, \dots, t-1, t+1, \dots, n\}$, and also $r_{i_t,t} \leq s_t + 1$. On the other hand, $h \notin I$ implies that $x_1^{r_{i_t,1}} \cdots x_n^{r_{i_t,n}} \nmid x_1^{s_1} \cdots x_n^{s_n}$ for each $i = 1, \dots, k$. Therefore, we gain $x_1^{r_{i_t,1}} \cdots x_n^{r_{i_t,n}} \nmid x_1^{s_1} \cdots x_n^{s_n}$. Hence, one has to derive $r_{i_t,t} > s_t$, and so $r_{i_t,t} = s_t + 1$. Accordingly, we get $s_t = r_{i_t,t} - 1$ for each $t = 1, \dots, n$. Furthermore, by virtue of $r_{i_t,j} \leq s_j$ for each $j \in \{1, \dots, t-1, t+1, \dots, n\}$, and also $s_j = r_{i_j,j} - 1$, one has $r_{i_t,j} < r_{i_j,j}$. Consequently, the condition (i) is proved. This gives rise to that $i_1, \dots, i_n$ are distinct. To establish the condition (ii), consider $i \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}$. Due to $h \notin I$, we deduce that $x_1^{r_{i,1}} \cdots x_n^{r_{i,n}} \nmid x_1^{s_1} \cdots x_n^{s_n}$. This yields that there exists a positive integer $1 \leq t \leq n$ such that $r_{i,t} > s_t$. As $s_t = r_{i_t,t} - 1$, we obtain $r_{i,t} > r_{i_t,t} - 1$, and thus $r_{i,t} \geq r_{i_t,t}$. Therefore, the condition (ii) holds.

Conversely, assume that there are distinct integers $i_1, \dots, i_n \in \{1, \dots, k\}$ such that the conditions (i) and (ii) hold. Set $h := x_1^{s_1} \cdots x_n^{s_n}$, where $s_t = r_{i_t,t} - 1$ for each $t = 1, \dots, n$. Our aim is to show that $\mathfrak{m} = (I :_R h)$. To do this, we first prove that $h \notin I$. Clearly, for each $t = 1, \dots, n$, one has $s_t = r_{i_t,t} - 1 < r_{i_t,t}$. Moreover, the condition (ii) gives that, for each $i \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}$, there exists a positive integer $1 \leq \alpha \leq n$ such that $r_{i,\alpha} \geq r_{i_\alpha,\alpha}$, and hence $r_{i,\alpha} > s_\alpha$. Therefore, for each $i = 1, \dots, k$, we get $x_1^{r_{i,1}} \cdots x_n^{r_{i,n}} \nmid h$, and so $h \notin I$. In order to complete the proof, one needs only prove that $hx_t \in I$ for each $t = 1, \dots, n$. To accomplish this, fix $1 \leq t \leq n$. Remembering the assumption that $s_j = r_{i_j,j} - 1$ for each $j = 1, \dots, n$, one has the following equalities

$$\begin{aligned} hx_t &= x_1^{s_1} \cdots x_{t-1}^{s_{t-1}} x_t^{s_t+1} x_{t+1}^{s_{t+1}} \cdots x_n^{s_n} \\ &= x_1^{r_{i_1,1}-1} \cdots x_{t-1}^{r_{i_{t-1},t-1}-1} x_t^{r_{i_t,t}} x_{t+1}^{r_{i_{t+1},t+1}-1} \cdots x_n^{r_{i_n,n}-1}. \end{aligned}$$

Moreover, based on the condition (i), for each $j = 1, \dots, n$, we have $r_{i_t,j} < r_{i_j,j}$ for each $t \neq j$, and so $r_{i_t,j} \leq r_{i_j,j} - 1$. This yields that $x_1^{r_{i_t,1}} \cdots x_n^{r_{i_t,n}} \mid hx_t$, and thus $hx_t \in I$. Therefore, one can conclude that $\mathfrak{m} \in \text{Ass}_R(R/I)$, as required. $\square$

We now give the following example which clarifies the main aim of Theorem 10.1.3.

Example 10.1.4. Let $R = K[x_1, x_2, x_3]$ be the polynomial ring over a field K , and $I = (x_1x_2x_3^5, x_1^2x_2^5x_3^3, x_1^3x_2^2x_3^4, x_1^4x_2^4x_3, x_1^5x_2^3x_3^2)$ be a monomial ideal of R . Want to show that $\mathfrak{m} = (x_1, x_2, x_3) \in \text{Ass}_R(R/I)$. To do this, put $u_1 := x_1x_2x_3^5$, $u_2 := x_1^2x_2^5x_3^3$, $u_3 := x_1^3x_2^2x_3^4$, $u_4 := x_1^4x_2^4x_3$, and $u_5 := x_1^5x_2^3x_3^2$. After selecting $i_1 := 4$, $i_2 := 2$, and $i_3 := 3$, one can easily check that $r_{i_t,1} < r_{i_1,1}$ for all $t \in \{2, 3\}$, $r_{i_t,2} < r_{i_2,2}$ for all $t \in \{1, 3\}$, and $r_{i_t,3} < r_{i_3,3}$ for all $t \in \{1, 2\}$. That is, the condition (i) in Theorem 10.1.3 holds. On the other hand, since $r_{1,3} \geq r_{i_3,3}$ and $r_{5,1} \geq r_{i_1,1}$, we conclude that the condition (ii) in Theorem 10.1.3 holds. This implies that $\mathfrak{m} = (x_1, x_2, x_3) \in \text{Ass}_R(R/I)$.

In the subsequent lemma, we assume that the number of minimal generators and the number of variables are the same, and provide a criterion for the graded maximal ideal to be an associate prime.

Lemma 10.1.5. *Suppose that $I \subset R = K[x_1, \dots, x_n]$ is a monomial ideal, $\mathfrak{m} = (x_1, \dots, x_n)$, and $\mathcal{G}(I) = \{x_1^{r_{1,1}} \cdots x_n^{r_{1,n}}, \dots, x_1^{r_{n,1}} \cdots x_n^{r_{n,n}}\}$. Then $\mathfrak{m} \in \text{Ass}_R(R/I)$ if and only if there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, n\}$ such that $r_{i_t,j} < r_{i_j,j}$ for each $t \neq j$, where $j = 1, \dots, n$.*

Proof. If $\mathfrak{m} \in \text{Ass}_R(R/I)$, in view of Theorem 10.1.3, one derives that there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, n\}$ such that $r_{i_t,j} < r_{i_j,j}$ for each $t \neq j$, where $j = 1, \dots, n$. And the argument is done.

Conversely, assume that there are distinct integers $i_1, \dots, i_n \in \{1, \dots, n\}$ such that $r_{i_t,j} < r_{i_j,j}$ for each $t \neq j$, where $j = 1, \dots, n$. Since $i_1, \dots, i_n$ are distinct, one has $\{i_1, \dots, i_n\} = \{1, \dots, n\}$, and hence the condition (ii) in Theorem 10.1.3 is superfluous. Therefore, $\mathfrak{m} \in \text{Ass}_R(R/I)$, and the proof is over. $\square$

In order to verify the main result of this section, one relies on the following corollary which is an immediate consequence of Lemma 10.1.5.

Corollary 10.1.6. *Suppose that I is a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) = \{x_1^{r_{1,1}} \cdots x_n^{r_{1,n}}, \dots, x_1^{r_{n,1}} \cdots x_n^{r_{n,n}}\}$, and $\mathfrak{m} = (x_1, \dots, x_n)$. Also, let $C_j = \{i \mid r_{i,j} = \max\{r_{1,j}, \dots, r_{n,j}\}\}$ for each $j = 1, \dots, n$. Then $\mathfrak{m} \in \text{Ass}_R(R/I)$ if and only if the following conditions hold:*

(i) $|C_j| = 1$ for each $j = 1, \dots, n$;

(ii) $C_i \cap C_j = \emptyset$ for all $i \neq j$.

Proof. We first assume that $\mathfrak{m} \in \text{Ass}_R(R/I)$. Inspired by Lemma 10.1.5, we get there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, n\}$ such that $r_{i_t,j} < r_{i_j,j}$ for each $t \neq j$, where $j = 1, \dots, n$. Because $i_1, \dots, i_n$ are distinct, this gives that i_j is the unique number among $\{i_1, \dots, i_n\}$ such that $r_{i_j,j} = \max\{r_{i_1,j}, \dots, r_{i_n,j}\}$ for each $j = 1, \dots, n$. Also, by virtue of $\{i_1, \dots, i_n\} = \{1, \dots, n\}$, one derives that $\{r_{i_1,j}, \dots, r_{i_n,j}\} = \{r_{1,j}, \dots, r_{n,j}\}$. Consequently, the conditions (i) and (ii) are proved.

To prove the converse, suppose that the conditions (i) and (ii) hold. It follows that there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, n\}$ with $i_j \in C_j$ for all $j = 1, \dots, n$, such that $r_{i_t,j} < r_{i_j,j}$ for each $t \neq j$, where $j = 1, \dots, n$. Now, Lemma 10.1.5 implies that $\mathfrak{m} \in \text{Ass}_R(R/I)$, as required. $\square$

We are now in a position to prove the main result of this paper in the next theorem. In fact, it yields a necessary and sufficient condition whether the graded maximal ideal appears in the set of associated prime ideals.

Theorem 10.1.7. Suppose that I is a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$, $\mathbf{m} = (x_1, \dots, x_n)$, and $\mathcal{G}(I) = \{x_1^{r_{1,1}} \cdots x_n^{r_{1,n}}, \dots, x_1^{r_{k,1}} \cdots x_n^{r_{k,n}}\}$ with $k \geq n$. Then $\mathbf{m} \in \text{Ass}_R(R/I)$ if and only if there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, k\}$ such that the following conditions hold:

- (i) $|C_j| = 1$, where $C_j = \{i_t \mid r_{i_t,j} = \max\{r_{i_1,j}, \dots, r_{i_n,j}\}\}$ for all $j = 1, \dots, n$;
- (ii) $C_i \cap C_j = \emptyset$ for all $i \neq j$;
- (iii) $x_1^{r_{i,1}} \cdots x_n^{r_{i,n}} \nmid x_1^{r_{i_1,1}-1} \cdots x_n^{r_{i_n,n}-1}$ for each $i \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}$.

Proof. To prove the forward implication, let $\mathbf{m} \in \text{Ass}_R(R/I)$. It follows from Theorem 10.1.3 that there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, k\}$ such that $r_{i_t,j} < r_{i_j,j}$ for each $t \neq j$, where $j = 1, \dots, n$. Now, Lemma 10.1.5 yields that $\mathbf{m} \in \text{Ass}_R(R/J)$, where $J = (x_1^{r_{i_1,1}} \cdots x_n^{r_{i_1,n}}, \dots, x_1^{r_{i_n,1}} \cdots x_n^{r_{i_n,n}})$. Accordingly, Corollary 10.1.6 implies that the conditions (i) and (ii) hold. Moreover, Theorem 10.1.3 gives that for each $i \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}$, there exists a positive integer $1 \leq t \leq n$ such that $r_{i,t} \geq r_{i_t,t}$, and so $r_{i,t} > r_{i_t,t} - 1$. Thus, $x_1^{r_{i,1}} \cdots x_n^{r_{i,n}} \nmid x_1^{r_{i_1,1}-1} \cdots x_n^{r_{i_n,n}-1}$ for each $i \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}$. This means that the condition (iii) is proved.

Conversely, assume that there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, k\}$ such that the conditions (i)-(iii) hold. Put $J := (x_1^{r_{i_1,1}} \cdots x_n^{r_{i_1,n}}, \dots, x_1^{r_{i_n,1}} \cdots x_n^{r_{i_n,n}})$. On account of Corollary 10.1.6, one can conclude that $\mathbf{m} \in \text{Ass}_R(R/J)$. Now, Lemma 10.1.5 implies that there exist distinct integers $s_1, \dots, s_n \in \{i_1, \dots, i_n\}$ such that $r_{s_t,j} < r_{s_j,j}$ for each $t \neq j$, where $j = 1, \dots, n$. Since $\{s_1, \dots, s_n\} = \{i_1, \dots, i_n\}$, after rearranging, one may assume that $i_j = s_j$ for each $j = 1, \dots, n$. This shows that the condition (i) in Theorem 10.1.3 holds. In addition, it follows from the condition (iii) that for each $i \in \{1, \dots, k\} \setminus \{i_1, \dots, i_n\}$, there exists a positive integer $1 \leq t \leq n$ such that $r_{i,t} > r_{i_t,t} - 1$, and hence $r_{i,t} \geq r_{i_t,t}$. Therefore, the condition (ii) in Theorem 10.1.3 is proved. Consequently, we get $\mathbf{m} \in \text{Ass}_R(R/I)$, as desired. $\square$

We are now ready to present an example which illustrates Theorem 10.1.7.

Example 10.1.8. Consider the following monomial ideal

$$I = (x_1^2 x_2^4 x_3, x_1^3 x_2^3 x_3, x_1^4 x_2^2 x_3, x_1 x_2^3 x_3^4, x_1^3 x_2^2 x_3^3),$$

in the polynomial ring $R = K[x_1, x_2, x_3]$. Our purpose is to demonstrate that $\mathbf{m} = (x_1, x_2, x_3) \in \text{Ass}_R(R/I)$. To achieve this, it is sufficient to apply Theorem 10.1.7. To see this, set $u_1 := x_1^2 x_2^4 x_3$, $u_2 := x_1^3 x_2^3 x_3$, $u_3 := x_1^4 x_2^2 x_3$, $u_4 := x_1 x_2^3 x_3^4$, and $u_5 := x_1^3 x_2^2 x_3^3$. By choosing $i_1 = 1$, $i_2 = 4$, and $i_3 = 5$, one has $C_1 = \{5\}$, $C_2 = \{1\}$, and $C_3 = \{4\}$. Thus, $|C_1| = |C_2| = |C_3| = 1$, and also $C_i \cap C_j = \emptyset$ for all $i \neq j$, that is, the conditions (i) and (ii) are proved. Moreover, since $r_{5,1} = 3$, $r_{1,2} = 4$, and

$r_{4,3} = 4$, and by virtue of $u_i \nmid x_1^2 x_2^3 x_3^3$ for each $i \in \{1, 2, 3, 4, 5\} \setminus \{1, 4, 5\}$, one can conclude that the condition (iii) holds. Therefore, $\mathfrak{m} = (x_1, x_2, x_3) \in \text{Ass}_R(R/I)$. In general, there are $\binom{5}{3} = 10$ cases for choosing i_1 , i_2 , and i_3 . Direct computations show that we can also consider the following cases:

(1) $i_1 = 1$, $i_2 = 2$, and $i_3 = 4$.

(2) $i_1 = 2$, $i_2 = 3$, and $i_3 = 5$.

The following corollary is a well-known result, which can be shown by the minimal primary decomposition technique. But we re-prove it as an application of Theorem 10.1.7.

Corollary 10.1.9. *Suppose that $I \subset R = K[x_1, \dots, x_n]$ is a square-free monomial ideal, $\mathfrak{m} = (x_1, \dots, x_n)$, and $\mathcal{G}(I) = \{x_1^{r_{1,1}} \cdots x_n^{r_{1,n}}, \dots, x_1^{r_{k,1}} \cdots x_n^{r_{k,n}}\}$ with $k \geq n$. Then $\mathfrak{m} \in \text{Ass}_R(R/I)$ if and only if $I = \mathfrak{m}$.*

Proof. We first assume that $\mathfrak{m} \in \text{Ass}_R(R/I)$. In addition, for each $j = 1, \dots, n$, let $C_j = \{i_t \mid r_{i_t,j} = \max\{r_{i_1,j}, \dots, r_{i_n,j}\}\}$. Thanks to Theorem 10.1.7, one can derive that there exist distinct integers $i_1, \dots, i_n \in \{1, \dots, k\}$ such that $|C_j| = 1$ for all $j = 1, \dots, n$, and $C_i \cap C_j = \emptyset$ for all $i \neq j$. Since I is a square-free monomial ideal, one has $r_{i,j} \in \{0, 1\}$ for all i and j . This yields that $r_{i_j,j} = 1$ and $r_{i_t,j} = 0$ for all $t \neq j$, where $j = 1, \dots, n$. Accordingly, one must have $n = k$, and so $\{x_{i_1}, \dots, x_{i_n}\} = \{x_1, \dots, x_n\}$. Due to $x_{i_1}, \dots, x_{i_n} \in I$, one gains $\mathfrak{m} \subseteq I$, and thus $I = \mathfrak{m}$. The converse is obvious based on the fact that $\text{Ass}_R(R/\mathfrak{m}) = \{\mathfrak{m}\}$. $\square$

10.2 On the embedded associated primes and corner-elements

In this section, our first motivation is to give a large class of square-free monomial ideals which satisfy normality, normally torsion-freeness, and the (symbolic) (strong) persistence property. Moreover, our second aim is to present some results on the corner-elements of monomial ideals. It should be noted that the results of this section can be found in [121].

Here, we introduce some necessary definitions which will be used in the following subsections.

Let I be a square-free monomial ideal and $\Gamma \subseteq \mathcal{G}(I)$. We say that Γ is an *independent* set in I if $\gcd(f, g) = 1$ for each $f, g \in \Gamma$ with $f \neq g$. We denote the maximum cardinality of an independent set in I by $\beta_1(I)$.

Also, a *König* ideal is a square-free monomial ideal I with $\mathcal{G}(I) = \{u_1, \dots, u_r\}$ such that the maximum number of pairwise disjoint monomials $u_1, \dots, u_r$ is equal

to the height of I , that is, $\beta_1(I) = \text{ht}(I)$. Note that edge ideals of bipartite graphs form a large class of König ideals.

10.2.1 Results on the embedded associated prime ideals

In what follows, we focus on the embedded associated primes of monomial ideals. The main results of this section are Corollary 10.2.4 and Theorem 10.2.6. To achieve this goal, we start with the following proposition, which will be used in proving Theorem 10.2.2.

Proposition 10.2.1. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K and $\mathfrak{p} \in \text{Ass}(R/I)$. Then there exists a monomial v in R with $\text{supp}(v) \subseteq \bigcup_{u \in \mathcal{G}(I)} \text{supp}(u)$ such that $\mathfrak{p} = (I :_R v)$.*

Proof. By virtue of [60, Corollary 1.3.10], there exists a monomial w in R such that $\mathfrak{p} = (I :_R w)$. Without loss of generality, one may assume that $\bigcup_{u \in \mathcal{G}(I)} \text{supp}(u) = \{x_1, \dots, x_m\}$. Certainly, one can write $w = w_1 w_2$, where $\text{supp}(w_1) \subseteq \{x_1, \dots, x_m\}$ and $\text{supp}(w_2) \subseteq \{x_{m+1}, \dots, x_n\}$. It follows from Lemma 2.1.12 that $(I :_R w_2) = I$, and so we get the following equalities

$$\mathfrak{p} = (I :_R w) = ((I :_R w_2) :_R w_1) = (I :_R w_1).$$

This completes our argument. $\square$

The next theorem enables us to argue on the associated prime ideals of a monomial ideal which can be divided into two monomial ideals generated in disjoint sets of variables.

Theorem 10.2.2. *Let $I_1 \subset R_1 = K[x_1, \dots, x_n]$ and $I_2 \subset R_2 = K[y_1, \dots, y_m]$ be two monomial ideals in disjoint sets of variables. Let*

$$I = I_1 R + I_2 R \subset R = K[x_1, \dots, x_n, y_1, \dots, y_m].$$

Then $\mathfrak{p} \in \text{Ass}(R/I)$ if and only if $\mathfrak{p} = \mathfrak{p}_1 R + \mathfrak{p}_2 R$, where $\mathfrak{p}_1 \in \text{Ass}(R_1/I_1)$ and $\mathfrak{p}_2 \in \text{Ass}(R_2/I_2)$.

Proof. Let $I_1 = Q_1 \cap \dots \cap Q_s$ be a minimal primary decomposition of I_1 such that $\sqrt{Q_i} = \mathfrak{p}_i$ for all $i = 1, \dots, s$. In view of [133, Exercise 6.1.23], we have

$$I = \bigcap_{i=1}^s Q_i + I_2 = \bigcap_{i=1}^s (Q_i + I_2).$$

In addition, it is routine to check that there exists the following R -monomorphism

$$\theta : R/I \rightarrow \bigoplus_{i=1}^s R/(Q_i + I_2),$$

given by $\theta(r + I) = (r + Q_1 + I_2, \dots, r + Q_s + I_2)$ for all $r \in R$. This implies that

$$\text{Ass}(R/I) \subseteq \bigcup_{i=1}^s \text{Ass}(R/(Q_i + I_2)).$$

Assume that $I_2 = Q'_1 \cap \dots \cap Q'_t$ is a minimal primary decomposition of I_2 such that $\sqrt{Q'_j} = \mathfrak{p}'_j$ for all $j = 1, \dots, t$. Now, fix $1 \leq i \leq s$. Applying [133, Exercise 6.1.23] yields

$$Q_i + I_2 = Q_i + \bigcap_{j=1}^t Q'_j = \bigcap_{j=1}^t (Q_i + Q'_j).$$

Consequently, there exists the following R -monomorphism

$$\phi : R/(Q_i + I_2) \rightarrow \bigoplus_{j=1}^t R/(Q_i + Q'_j),$$

given by $\phi(r + Q_i + I_2) = (r + Q_i + Q'_1, \dots, r + Q_i + Q'_t)$ for all $r \in R$. We thus get

$$\text{Ass}(R/(Q_i + I_2)) \subseteq \bigcup_{j=1}^t \text{Ass}(R/(Q_i + Q'_j)).$$

Since $Q_i + Q'_j$ is a $(\mathfrak{p}_i + \mathfrak{p}'_j)$ -primary monomial ideal, for all $j = 1, \dots, t$, one has $\text{Ass}(R/(Q_i + Q'_j)) = \mathfrak{p}_i + \mathfrak{p}'_j$ for all $i = 1, \dots, s$ and $j = 1, \dots, t$. Therefore,

$$\text{Ass}(R/I) \subseteq \{\mathfrak{p} \mid \mathfrak{p} = \mathfrak{p}_i + \mathfrak{p}'_j \text{ for all } i = 1, \dots, s \text{ and } j = 1, \dots, t\}.$$

To establish the reverse inclusion, pick arbitrary elements $\mathfrak{p} \in \text{Ass}(R/I_1)$ and $\mathfrak{p}' \in \text{Ass}(R/I_2)$. Then there exist monomials v and v' in R such that $\mathfrak{p} = (I_1 :_R v)$ and $\mathfrak{p}' = (I_2 :_R v')$. Based on Proposition 10.2.1, we may assume that

$$(10.2.1) \quad \text{supp}(v) \subseteq \bigcup_{f \in \mathcal{G}(I_1)} \text{supp}(f) \text{ and } \text{supp}(v') \subseteq \bigcup_{g \in \mathcal{G}(I_2)} \text{supp}(g).$$

In the light of (10.2.1) and Lemma 2.1.12, one can deduce that $(I_1 :_R vv') = (I_1 :_R v)$ and $(I_2 :_R vv') = (I_2 :_R v')$. Accordingly, we obtain the following equalities

$$\begin{aligned} \mathfrak{p} + \mathfrak{p}' &= (I_1 :_R v) + (I_2 :_R v') \\ &= (I_1 :_R vv') + (I_2 :_R vv') \\ &= (I_1 + I_2 :_R vv') \\ &= (I :_R vv'). \end{aligned}$$

This implies that $\mathfrak{p} + \mathfrak{p}' \in \text{Ass}_R(R/I)$. Thus, the reverse inclusion holds. $\square$

To prove Corollary 10.2.4, we have to employ the following crucial theorem.

Theorem 10.2.3. *Let $I \subset R = K[x_1, \dots, x_n]$ be a monomial ideal, $\mathfrak{m} = (x_1, \dots, x_n)$, t a positive integer, and $y_1, \dots, y_s$ be distinct variables in R such that, for each $i = 1, \dots, s$, $\mathfrak{m} \setminus y_i \notin \text{Ass}(R/(I \setminus y_i)^t)$, where $I \setminus y_i$ denotes the deletion of I at y_i . Then $\mathfrak{m} \in \text{Ass}(R/I^t)$ if and only if $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^s y_i))$.*

Proof. We first assume that $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^s y_i))$. Based on [133, Exercise 2.1.62], we get the following short exact sequence

$$0 \longrightarrow R/(I^t : \prod_{i=1}^s y_i) \xrightarrow{\psi} R/I^t \xrightarrow{\phi} R/(I^t, \prod_{i=1}^s y_i) \longrightarrow 0,$$

where $\psi(\bar{r}) = \bar{r} \prod_{i=1}^s y_i$ and $\phi(\bar{r}) = \bar{r}$. In addition, by virtue of [125, Exercise 9.42], one can derive that

(10.2.2)

$$\text{Ass}(R/(I^t : \prod_{i=1}^s y_i)) \subseteq \text{Ass}(R/I^t) \subseteq \text{Ass}(R/(I^t : \prod_{i=1}^s y_i)) \cup \text{Ass}(R/(I^t, \prod_{i=1}^s y_i)).$$

Since $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^s y_i))$, we get $\mathfrak{m} \in \text{Ass}(R/I^t)$. To establish the converse implication, let $\mathfrak{m} \in \text{Ass}(R/I^t)$. Our aim is to prove that $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^s y_i))$. Proceed by induction on s . Let $J := I \setminus y_1$. Because the generators of J^t are precisely the generators of I^t that are not divisible by y_1 , this implies that $(I^t, y_1) = (J^t, y_1)$. By assumption, one has $\mathfrak{m} \setminus y_1 \notin \text{Ass}(R/J^t)$. Also, it follows from Theorem 10.2.2 that $\mathfrak{p} \in \text{Ass}(R/(J^t, y_1))$ if and only if $\mathfrak{p} = (\mathfrak{p}_1, y_1)$, where $\mathfrak{p}_1 \in \text{Ass}(R/J^t)$. We thus have $\mathfrak{m} \notin \text{Ass}(R/(J^t, y_1))$, and so $\mathfrak{m} \notin \text{Ass}(R/(I^t, y_1))$. Also, (10.2.2) yields that $\text{Ass}(R/I^t) \subseteq \text{Ass}(R/(I^t : y_1)) \cup \text{Ass}(R/(I^t, y_1))$. Accordingly, we obtain $\mathfrak{m} \in \text{Ass}(R/(I^t : y_1))$. Hence, the claim is true for the case in which $s = 1$. Now, suppose that the assertion has been shown for a product of $s - 1$ variables, and that $\mathfrak{m} \in \text{Ass}(R/I^t)$. Set $M := \prod_{i=1}^{s-1} y_i$. One can deduce from the induction that $\mathfrak{m} \in \text{Ass}(R/(I^t : M))$. Moreover,

$$(10.2.3) \quad \text{Ass}(R/(I^t : M)) \subseteq \text{Ass}(R/((I^t : M) : y_s)) \cup \text{Ass}(R/((I^t : M), y_s)).$$

Let $K := I \setminus y_s$. It remains to check that $((I^t : M), y_s) = ((K^t : M), y_s)$. Since $K \subseteq I$, one can easily conclude that $((K^t : M), y_s) \subseteq ((I^t : M), y_s)$. To prove the reverse inclusion, take a monomial u in $((I^t : M), y_s)$. If $y_s \mid u$, then $u \in ((K^t : M), y_s)$, and the argument is done. If $y_s \nmid u$, then $u \in (I^t : M)$, and hence $uM \in I^t$. This yields that there exists a monomial $f \in \mathcal{G}(I^t)$ such that $f \mid uM$. Because $y_s \nmid u$ and $y_s \nmid M$, this implies that $y_s \nmid f$, and so $f \in K^t$. We therefore get $u \in (K^t : M)$, and thus $((I^t : M), y_s) \subseteq ((K^t : M), y_s)$. It also follows from our assumption that $\mathfrak{m} \setminus y_s \notin \text{Ass}(R/K^t)$. As $\text{Ass}(R/(K^t : M)) \subseteq \text{Ass}(R/K^t)$, one can conclude that $\mathfrak{m} \setminus y_s \notin \text{Ass}(R/(K^t : M))$. Due to Theorem 10.2.2, we obtain $\mathfrak{p} \in \text{Ass}(R/((K^t : M), y_s))$ if and only if $\mathfrak{p} = (\mathfrak{p}_1, y_s)$, where $\mathfrak{p}_1 \in \text{Ass}(R/(K^t : M))$.

This gives rise to $\mathfrak{m} \notin \text{Ass}(R/((K^t : M), y_s))$, and so $\mathfrak{m} \notin \text{Ass}(R/((I^t : M), y_s))$. In the light of (10.2.3), one has $\mathfrak{m} \in \text{Ass}(R/((I^t : M) : y_s)) = \text{Ass}(R/(I^t : \prod_{i=1}^s y_i))$. This completes the inductive step, and therefore the claim has been proved by induction. $\square$

We are ready to express one of the main results of this section in the subsequent corollary.

Corollary 10.2.4. *Let $I \subset R = K[x_1, \dots, x_n]$ be a square-free monomial ideal, $\mathfrak{m} = (x_1, \dots, x_n)$, and $\{u_1, \dots, u_{\beta_1(I)}\}$ be a maximal independent set of minimal generators of I such that $\mathfrak{m} \setminus x_i \notin \text{Ass}(R/(I \setminus x_i)^t)$ for all $x_i \mid \prod_{i=1}^{\beta_1(I)} u_i$ and some positive integer t , where $I \setminus x_i$ denotes the deletion of I at x_i . If $\mathfrak{m} \in \text{Ass}(R/I^t)$, then $t \geq \beta_1(I) + 1$.*

Proof. To simplify the notation, set $\ell := \beta_1(I)$. Let $\mathfrak{m} \in \text{Ass}(R/I^t)$. Without loss of generality, one may assume that $\prod_{i=1}^{\ell} u_i = \prod_{i=1}^r x_i$. In view of Theorem 10.2.3, we can conclude that $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^r x_i))$. Since $\prod_{i=1}^{\ell} u_i \in I^{\ell}$, we get $\prod_{i=1}^r x_i \in I^{\ell}$. If $t \leq \ell$, then $(I^t : \prod_{i=1}^r x_i) = R$, which contradicts the fact that $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^r x_i))$. This implies that $t \geq \ell + 1$, as required. $\square$

Applying Corollary 10.2.4 enables us to study König ideals in the next theorem.

Theorem 10.2.5. *Let I be an unmixed König ideal in the polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , $\mathfrak{m} = (x_1, \dots, x_n)$, and $\{u_1, \dots, u_{\beta_1(I)}\}$ be a maximal independent set of minimal generators of I such that $I \setminus x_i$ is normally torsion-free for all $x_i \mid \prod_{i=1}^{\beta_1(I)} u_i$. Then the following statements hold:*

- (i) *I is normally torsion-free.*
- (ii) *I is normal.*
- (iii) *I has the strong persistence proeprty.*
- (iv) *I has the persistence property.*
- (v) *I has the symbolic strong persistence property.*

Proof. (i) For convenience of notation, put $\ell := \beta_1(I)$. Suppose, on the contrary, that I is not normally torsion-free. Thus, there exists a positive integer t such that I^t has embedded primes. We consider t minimal with respect to this property. Let $\mathfrak{p}$ be an embedded prime of I^t . In view of [125, Lemma 9.38], we get $\mathfrak{p} \in \text{Ass}(R/I^t)$ if and only if $\mathfrak{p}R_{\mathfrak{p}} \in \text{Ass}(R_{\mathfrak{p}}/(IR_{\mathfrak{p}})^t)$. This enables us to assume that $\mathfrak{p} = \mathfrak{m}$. Without loss of generality, one may assume that $\prod_{i=1}^{\ell} u_i = \prod_{i=1}^r x_i$. In the light of Theorem 10.2.3, one has $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^r x_i))$, and so $\mathfrak{m} \in \text{Ass}(R/(I^t : \prod_{i=1}^{\ell} u_i))$. On the other hand, since $\mathfrak{m} \setminus x_i \notin \text{Ass}(R/(I \setminus x_i)^t)$ for each $i = 1, \dots, r$, we deduce

from Corollary 10.2.4 that $t \geq \ell + 1$. Because t is minimal, $I^{t-\ell}$ has no embedded primes, and hence $I^{t-\ell} = I^{(t-\ell)}$. It follows now from [50, Proposition 3.8] that $(I^t : \prod_{i=1}^{\ell} u_i) = I^{t-\ell}$. We therefore obtain $\mathfrak{m} \in \text{Ass}(R/I^{t-\ell})$, which contradicts the fact that $I^{t-\ell}$ has no embedded primes. Consequently, I is a normally torsion-free square-free monomial ideal.

(ii)-(v) can be shown similar to the proof of Corollary 6.2.3. □

We are in a position to state the other main result of this section in the following theorem.

Theorem 10.2.6. *Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K and $\mathfrak{m} = (x_1, \dots, x_n)$. If there exists a square-free monomial $v \in I$ such that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for any $\mathfrak{p} \in \text{Min}(I)$, and $I \setminus x_i$ is normally torsion-free for all $x_i \in \text{supp}(v)$, then the following statements hold:*

- (i) *I is normally torsion-free.*
- (ii) *I is normal.*
- (iii) *I has the strong persistence property.*
- (iv) *I has the persistence property.*
- (v) *I has the symbolic strong persistence property.*

Proof. (i) Suppose, on the contrary, that I is not normally torsion-free. Let t be minimal such that I^t has embedded prime ideals. Since I is a square-free monomial ideal, this implies that $\text{Ass}(R/I) = \text{Min}(I)$, and hence $t \geq 2$. Take an arbitrary $\mathfrak{p} \in \text{Min}(I)$. In the following, we show that $(I^t : v) = I^{t-1}$. Because $v \in I$, one must have $I^{t-1} \subseteq (I^t : v)$. To prove the reverse inclusion, select a monomial f in $(I^t : v)$. Thus, $fv \in I^t$, and so there exist some monomials $g_1, \dots, g_t \in \mathcal{G}(I)$ and some monomial h in R such that $fv = g_1 \cdots g_t h$. Due to $\mathfrak{p} \in \text{Min}(I)$, this yields that $fv \in \mathfrak{p}^t$. Since $v \in \mathfrak{p} \setminus \mathfrak{p}^2$, one can conclude that $\mathfrak{p}$ contains exactly one variable that divides v . This gives that $f \in \mathfrak{p}^{t-1}$. Because $\mathfrak{p}$ is arbitrary, one obtains $f \in \bigcap_{\mathfrak{p} \in \text{Min}(I)} \mathfrak{p}^{t-1}$. Applying [133, Proposition 4.3.25] leads to $f \in I^{(t-1)}$. Since t is minimal such that I^t has embedded prime ideals, this yields that I^{t-1} has no embedded prime ideals, and so $I^{(t-1)} = I^{t-1}$. We thus gain $f \in I^{t-1}$. Consequently, $(I^t : v) \subseteq I^{t-1}$, and hence $(I^t : v) = I^{t-1}$. Let $\mathfrak{q}$ be an embedded prime ideal of I^t . It follows from [125, Lemma 9.38] that $\mathfrak{q} \in \text{Ass}(R/I^t)$ if and only if $\mathfrak{q}R_{\mathfrak{q}} \in \text{Ass}(R_{\mathfrak{q}}/(IR_{\mathfrak{q}})^t)$. We therefore may assume that $\mathfrak{q} = \mathfrak{m}$. Hence, $\mathfrak{m} \in \text{Ass}(R/I^t)$. In view of Theorem 10.2.3, one derives that $\mathfrak{m} \in \text{Ass}(R/(I^t : v))$. We thus get $\mathfrak{m} \in \text{Ass}(R/I^{t-1})$. This contradicts our assumption. Accordingly, the square-free monomial ideal I is normally torsion-free.

(ii)-(v) can be shown similar to the proof of Corollary 6.2.3. □

To illustrate Theorem 10.2.6, one has to review the definition of t -spread monomial ideals, see Definition 6.3.1.

Example 10.2.7. Assume that $I \subset R = K[x_1, \dots, x_7]$ is a square-free monomial ideal with K is a field and $I = (x_1x_3x_6, x_1x_3x_7, x_1x_4x_6, x_1x_4x_7, x_1x_5x_7, x_2x_4x_7, x_2x_5x_7)$. Direct computations show that I is a 2-spread monomial ideal. Using Macaulay2 [49], one can deduce that

$$\text{Min}(I) = \{(x_1, x_2), (x_1, x_7), (x_6, x_7), (x_1, x_4, x_5), (x_3, x_4, x_7), (x_3, x_4, x_5)\}.$$

Also, note that I is not unmixed. Set $v := x_1x_3x_6$. It is straightforward to check that $v \in \mathfrak{p} \setminus \mathfrak{p}^2$ for each $\mathfrak{p} \in \text{Min}(I)$. To complete our argument, one needs to demonstrate that $I \setminus x_i$ is normally torsion-free for all $i \in \{1, 3, 6\}$. Since $I \setminus x_1 = (x_2x_4x_7, x_2x_5x_7) = x_2x_7(x_4, x_5)$, it follows from Lemma 5.3.11 that $I \setminus x_1$ is normally torsion-free. Note that $F := I \setminus x_3 = (x_1x_4x_6, x_1x_4x_7, x_1x_5x_7, x_2x_4x_7, x_2x_5x_7)$. In addition, one has $\text{Ass}(R/(I \setminus x_3)) = \{(x_1, x_2), (x_4, x_5), (x_4, x_7), (x_6, x_7), (x_1, x_7)\}$. Let $G = (V(G), E(G))$ be the graph with the vertex set $V(G) = \{1, 2, 4, 5, 6, 7\}$ and the edge set $E(G) = \{\{1, 2\}, \{4, 5\}, \{4, 7\}, \{6, 7\}, \{1, 7\}\}$. It is routine to check that G is a tree, and F is its cover ideal. By virtue of [48, Corollary 2.6], F is normally torsion-free; thus, $I \setminus x_3$ is normally torsion-free. Finally, notice that

$$I \setminus x_6 = (x_1x_3x_7, x_1x_4x_7, x_1x_5x_7, x_2x_4x_7, x_2x_5x_7).$$

Put $L := (x_1x_3, x_1x_4, x_1x_5, x_2x_4, x_2x_5)$. Assume that $H = (V(H), E(H))$ is the graph with the vertex set $V(H) = \{1, 2, 3, 4, 5\}$ and the following edge set

$$E(H) = \{\{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}\}.$$

Thanks to H has no odd cycle subgraph, we get H is bipartite. In addition, L is the edge ideal of H , and so [126, Theorem 5.9] yields that L is normally torsion-free. Due to Lemma 5.3.11, one has $I \setminus x_6 = x_7L$ is normally torsion-free as well. It follows now from Theorem 10.2.6 that I is normally torsion-free, as desired.

We terminate this section by giving an application of Theorem 10.2.6. In fact, we re-prove the fact that every square-free transversal polymatroidal ideal is normally torsion-free, see [66, Corollary 3.6]. To accomplish this, one has to recall the following definition from [66, Page 298].

Definition 10.2.8. Let F be a non-empty subset of $[n] = \{1, \dots, n\}$. We denote by $\mathfrak{p}_F$ the monomial prime ideal $(x_j : j \in F)$. A transversal polymatroidal ideal is an ideal I of the form $I = \mathfrak{p}_{F_1} \cdots \mathfrak{p}_{F_r}$, where $F_1, \dots, F_r$ is a collection of non-empty subsets of $[n]$ with $r \geq 1$.

Theorem 10.2.9. Every square-free transversal polymatroidal ideal is normally torsion-free.

Proof. Let $I \subset R = K[x_1, \dots, x_n]$ be a square-free transversal polymatroidal ideal. Thus, we can write $I = \mathfrak{p}_{F_1} \cdots \mathfrak{p}_{F_r}$, where $F_1, \dots, F_r$ is a collection of non-empty subsets of $[n]$ with $r \geq 1$ and $F_i \cap F_j = \emptyset$ for all $1 \leq i \neq j \leq r$. Without loss of generality, one may assume that $\bigcup_{u \in \mathcal{G}(I)} \text{supp}(u) = \{x_1, \dots, x_n\}$. We proceed by induction on n . If $n = 1$, then $I = (x_1)$ which is normally torsion-free. Now, suppose that $n > 1$, and that the claim has been proved for $n - 1$. Without loss of generality, we may assume that $x_i \in \mathfrak{p}_{F_i}$ for each $i = 1, \dots, r$. Set $v := x_1 \cdots x_r$. Note that $\text{Min}(I) = \{\mathfrak{p}_{F_1}, \dots, \mathfrak{p}_{F_r}\}$. It is routine to check that $v \in \mathfrak{p}_{F_i} \setminus \mathfrak{p}_{F_i}^2$ for each $i = 1, \dots, r$. Fix $1 \leq i \leq r$. Since $I \setminus x_i = \mathfrak{p}_{F_1} \cdots \mathfrak{p}_{F_{i-1}}(\mathfrak{p}_{F_i} \setminus \{x_i\})\mathfrak{p}_{F_{i+1}} \cdots \mathfrak{p}_{F_r}$, it follows from the inductive hypothesis that $I \setminus x_i$ is normally torsion-free. By virtue of Theorem 10.2.6, we deduce that I is normally torsion-free. $\square$

10.2.2 Some results on the corner-elements of monomial ideals

Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K and $\mathfrak{p} \in \text{Ass}(R/I)$. It is well-known by [60, Corollary 1.3.10] that $\mathfrak{p} = (I : h)$ for some monomial h in R . Moreover, it has been proved that if $I \subset R = K[x_1, \dots, x_n]$ is a square-free monomial ideal and $\mathfrak{p} \in \text{Ass}(R/I)$, then there exists a square-free monomial h in R with $\text{supp}(h) \subseteq \bigcup_{u \in \mathcal{G}(I)} \text{supp}(u)$ such that $\mathfrak{p} = (I :_R h)$, see Theorem 2.1.15. In this subsection, we are going to detect which variables must divide h . To accomplish this, we begin with the following proposition.

Proposition 10.2.10. *Let $I \subset R = K[x_1, \dots, x_n]$ be a monomial ideal. Let $\mathfrak{p}, \mathfrak{q} \in \text{Ass}(R/I^t)$ with $\mathfrak{p} \neq \mathfrak{q}$ such that $\mathfrak{p} = (I^t : f)$ and $\mathfrak{q} = (I^t : g)$ for some positive integer t and some monomials f and g in R . Then $f \nmid g$ and $g \nmid f$.*

Proof. We need only show that $f \nmid g$, since the other claim can be proved by a similar argument. Suppose, on the contrary, that $f \mid g$. That is, $g = fv$ for some monomial v in R . This gives that $\mathfrak{q} = (I^t : fv)$. Since $(I^t : fv) = ((I^t : f) : v)$, one gains that $\mathfrak{q} = (\mathfrak{p} : v)$. It is easy to see that either $(\mathfrak{p} : v) = R$ or $(\mathfrak{p} : v) = \mathfrak{p}$. We thus have either $\mathfrak{q} = R$ or $\mathfrak{q} = \mathfrak{p}$. This leads to a contradiction. Therefore, we get $f \nmid g$, as claimed. $\square$

To establish Proposition 10.2.11 and Corollary 10.2.14, one has to apply the notion of corner-elements which has been introduced in Definition 5.1.4.

Proposition 10.2.11. *Let $I \subset R = K[x_1, \dots, x_n]$ be a monomial ideal and $\mathfrak{m} = (x_1, \dots, x_n)$. Let f and g be two I^t -corner-elements for some positive integer t . Then $f \nmid g$ and $g \nmid f$.*

Proof. Since f and g are two I^t -corner-elements for some positive integer t , this implies that $f \notin I^t$ (respectively, $g \notin I^t$) and $x_i f \in I^t$ (respectively, $x_i g \in I^t$) for each $i = 1, \dots, n$. We thus have $x_i \in (I^t : f)$ (respectively, $x_i \in (I^t : g)$) for each $i = 1, \dots, n$. Accordingly, $\mathfrak{m} \subseteq (I^t : f)$ (respectively, $\mathfrak{m} \subseteq (I^t : g)$). Because $f \notin I^t$ (respectively, $g \notin I^t$), this yields that $\mathfrak{m} = (I^t : f)$ (respectively, $\mathfrak{m} = (I^t : g)$). On the contrary, assume that $f \mid g$. This gives that $g = fh$ for some monomial h in R . We thus get $\mathfrak{m} = ((I^t : f) : h)$, and so $\mathfrak{m} = R$, a contradiction. A similar discussion shows that $g \nmid f$, and the proof is complete. $\square$

The following lemma will be useful in the proof of Corollary 10.2.14.

Lemma 10.2.12. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . Let $\mathfrak{p} = (I^t : h)$ for some positive integer t and some monomial h in R such that $x_i \nmid h$ for some $1 \leq i \leq n$. Then $\mathfrak{p} \setminus x_i = ((I \setminus x_i)^t : h)$, and so $\mathfrak{p} \setminus x_i \in \text{Ass}(I \setminus x_i)^t$.*

Proof. We show that $\mathfrak{p} \setminus x_i = ((I \setminus x_i)^t : h)$. To do this, we first prove that $\mathfrak{p} \setminus x_i \subseteq ((I \setminus x_i)^t : h)$. To see this, take $x_j \in \mathfrak{p} \setminus x_i$. Since $\mathfrak{p} \setminus x_i \subseteq \mathfrak{p}$, this implies that $x_j \in \mathfrak{p}$, and so $x_j h \in I^t$. Because $x_j \neq x_i$ and $x_i \nmid h$, one can conclude that $x_i \nmid x_j h$. Hence, $x_j h \in I^t \setminus x_i$. Since $I^t \setminus x_i = (I \setminus x_i)^t$, we get $x_j h \in (I \setminus x_i)^t$, and thus $x_j \in ((I \setminus x_i)^t : h)$. We therefore have $\mathfrak{p} \setminus x_i \subseteq ((I \setminus x_i)^t : h)$. Conversely, let $v \notin \mathfrak{p} \setminus x_i$. It suffices to show that $v \notin ((I \setminus x_i)^t : h)$. Suppose, on the contrary, that $v \in ((I \setminus x_i)^t : h)$. Then $vh \in (I \setminus x_i)^t$. Since $v \in \mathfrak{p}$ and $v \notin \mathfrak{p} \setminus x_i$, one must have $v = x_i^\ell a$ for some positive integer ℓ and some monomial $a \notin \mathfrak{p}$. Hence, $x_i^\ell ah \in (I \setminus x_i)^t$, and so $ah \in ((I \setminus x_i)^t : x_i^\ell)$. It follows readily from Lemma 2.1.12 that $ah \in (I \setminus x_i)^t$. Since $I \setminus x_i \subseteq I$, we obtain $ah \in I^t$, and so $a \in (I^t : h)$. Accordingly, one has $a \in \mathfrak{p}$, which is a contradiction. We thus have $v \notin ((I \setminus x_i)^t : h)$, and so $((I \setminus x_i)^t : h) \subseteq \mathfrak{p} \setminus x_i$. Consequently, $\mathfrak{p} \setminus x_i = ((I \setminus x_i)^t : h)$, and hence $\mathfrak{p} \setminus x_i \in \text{Ass}(I \setminus x_i)^t$. $\square$

We now express two main results of this subsection in the subsequent corollaries.

Corollary 10.2.13. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . If $\mathfrak{p} = (I^t : h)$ for some positive integer t and some monomial h in R such that $\mathfrak{p} \setminus x_i \notin \text{Ass}(I \setminus x_i)^t$ for some $1 \leq i \leq n$, then $x_i \mid h$.*

Corollary 10.2.14. *Let I be a monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K and z be an I^t -corner-element for some positive integer t such that $\mathfrak{m} \setminus x_i \notin \text{Ass}(I \setminus x_i)^t$ for some $1 \leq i \leq n$. Then $x_i \mid z$.*

The following question, which posed in [112, Question 4.13], asks if Lemma 10.2.12 can be strengthened further.

Question 10.2.15. (see Question 3.2.23) Suppose that I is a square-free monomial ideal in $R = K[x_1, \dots, x_n]$, $\mathcal{G}(I) = \{u_1, \dots, u_m\}$, $\bigcup_{i=1}^m \text{supp}(u_i) = \{x_1, \dots, x_n\}$, and $\mathfrak{m} = (x_1, \dots, x_n)$ is the graded maximal ideal of R . If there exists a positive integer $1 \leq j \leq n$ such that $\mathfrak{m} \setminus x_j \in \text{Ass}(R/(I \setminus x_j)^k)$ for some positive integer k , then can we deduce that $\mathfrak{m} \in \text{Ass}_R(R/I^k)$?

We provide a counterexample. To do this, let $G = (V(G), E(G))$ be the graph with the vertex set $V(G) = \{1, 2, 3, 4, 5, 6\}$, and the following edge set

$$E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 1\}, \{1, 6\}, \{5, 6\}, \{4, 6\}\}.$$

Assume that I is the cover ideal of G , that is,

$$I = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1) \cap (x_1, x_6) \cap (x_5, x_6) \cap (x_4, x_6).$$

Using *Macaulay2* [49], one can derive that $(x_1, x_2, x_3, x_4, x_5) \in \text{Ass}(R/(I \setminus x_6)^2)$, while $(x_1, x_2, x_3, x_4, x_5, x_6) \notin \text{Ass}(R/I^2)$.

The proposition below says that if $\mathfrak{p} = (I^s : h) \in \text{Ass}(R/I^s)$ for some positive integer s , then we can always find an upper bound for $\deg_{x_i} h$ for each i .

Proposition 10.2.16. Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . Let $\mathfrak{p} = (I^s : h) \in \text{Ass}(R/I^s)$ for some positive integer s and some monomial h in R . Then $\deg_{x_i} h \leq s - 1$ for each $i = 1, \dots, n$.

Proof. To verify the claim, it is enough for us to show that $h | (\prod_{i=1}^n x_i)^{s-1}$. On the contrary, assume that $h \nmid (\prod_{i=1}^n x_i)^{s-1}$. Thus, there exists some $1 \leq i \leq n$ such that $\deg_{x_i} h \geq s$. This implies that the exponent of x_i in hx_i is at least $s + 1$. By $\mathfrak{p} = (I^s : h)$, one has $hx_i \in I^s$, and so there exist monomials $u_1, \dots, u_s \in \mathcal{G}(I)$ such that $hx_i = u_1 \cdots u_s f$ for some monomial f in R . Since I is a square-free monomial ideal, then for each $j = 1, \dots, s$, u_j is a square-free monomial, one has the exponent of x_i in each u_j is at most one. Accordingly, we get $x_i | f$, and hence $h \in I^s$, which is a contradiction. We therefore have $h | (\prod_{i=1}^n x_i)^{s-1}$, as desired. $\square$

We close our argument with the subsequent proposition which focuses on the minimal prime ideals associated to square-free monomial ideals.

Proposition 10.2.17. Let I be a square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . If $\mathfrak{p} \in \text{Min}(I)$, then $\mathfrak{p} = (I : h)$, where h is the product of the variables that are not in $\mathfrak{p}$.

Proof. Let $\mathfrak{p} \in \text{Min}(I)$. We first demonstrate that $(I : h) \subseteq \mathfrak{p}$. To see this, pick a monomial u in $(I : h)$. Hence, $uh \in I$, and so $uh \in \mathfrak{p}$. We thus have $x_j | uh$ for some $x_j \in \mathfrak{p}$. Since $\gcd(x_j, h) = 1$, one can conclude that $x_j | u$, and so $u \in \mathfrak{p}$. Therefore,

$(I : h) \subseteq \mathfrak{p}$. To complete our discussion, we show that $\mathfrak{p} \subseteq (I : h)$. To achieve this, take an arbitrary element $x_j \in \mathfrak{p}$. Because $\mathfrak{p} \in \text{Min}(I)$, this yields that $I \not\subseteq \mathfrak{p} \setminus \{x_j\}$, and so there exists some $u \in \mathcal{G}(I)$ such that $u \notin \mathfrak{p} \setminus \{x_j\}$. As I is square-free, this forces $u \mid x_j h$, and hence $x_j h \in I$. Accordingly, one must have $x_j \in (I : h)$. We therefore get $\mathfrak{p} \subseteq (I : h)$, as required. $\square$

Chapter 11

Some perspectives for future works

The major point of this chapter, which has been divided into seven parts, is to give some open questions and ideas for future work. To do this, we will recall and repeat any needed notions and definitions from commutative algebra or combinatorics in each part.

11.1 Relations among NTF, NNTF, and SPP

Let R be a commutative Noetherian ring and I be an ideal of R . The ideal I is said to have the *persistence property* if $\text{Ass}(R/I^k) \subseteq \text{Ass}(R/I^{k+1})$ for all positive integers k . Moreover, the ideal I satisfies the *strong persistence property* (*SPP* for short) if $(I^{k+1} : I) = I^k$ for all positive integers k . Furthermore, the ideal I is called *normally torsion-free* (*NTF* for short) if $\text{Ass}_R(R/I^k) \subseteq \text{Ass}_R(R/I)$ for all positive integers k . In addition, a monomial ideal I in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K is called *nearly normally torsion-free* (*NNTF* for short) if there exist a positive integer k and a monomial prime ideal $\mathfrak{p}$ such that $\text{Ass}_R(R/I^m) = \text{Min}(I)$ for all $1 \leq m \leq k$, and $\text{Ass}_R(R/I^m) \subseteq \text{Min}(I) \cup \{\mathfrak{p}\}$ for all $m \geq k + 1$.

Based on Example 6.1.6, we know that, in general, nearly normally torsion-freeness does not imply the strong persistence property. In fact, let $R = K[x_1, x_2, x_3]$ be the polynomial ring over a field K and $I := (x_2^4, x_1x_2^3, x_1^3x_2, x_1^4x_3)$ be a monomial ideal of R . On account of $I = (x_1, x_2^4) \cap (x_1^3, x_2^3) \cap (x_1^4, x_2) \cap (x_2, x_3)$, one can conclude that $\text{Ass}_R(R/I) = \text{Min}(I) = \{(x_1, x_2), (x_2, x_3)\}$. We proved that $\text{Ass}_R(R/I^m) = \text{Min}(I) \cup \{(x_1, x_2, x_3)\}$ for all $m \geq 2$. Hence, $\text{Ass}_R(R/I^m) = \text{Min}(I) \cup \{(x_1, x_2, x_3)\}$ for all $m \geq 2$. This means that I has the persistence property and is also a nearly normally torsion-free ideal. On the other hand, using *Macaulay2* [49] shows that

$(I^2 :_R I) \neq I$. We therefore deduce that I does not satisfy the strong persistence property.

In this direction, one can ask the following question. In fact, up to now, the author could not find any counterexample for it.

Question 11.1.1. *Let I be a nearly normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K . Then can we conclude that I has the strong persistence property?*

To see a large class in this theme, we know that all cover ideals of odd cycle graphs are nearly normally torsion-free, and also satisfy the strong persistence property.

We finish this section with the following open questions.

Question 11.1.2. *Let I be a square-free monomial ideal in $R = K[x_1, \dots, x_n]$. Let I be minimally not normally torsion-free, i.e., I is not normally torsion-free but all its minors are. Find the least power $t \geq 1$ such that $\text{Ass}_R(R/I^t) \neq \text{Ass}_R(R/I)$.*

To see an example, consider the odd cycle graph $C_5 = (V(C_5), E(C_5))$ with $V(C_5) = \{1, 2, 3, 4, 5\}$ and $E(C_5) = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_1\}\}$. Then the cover ideal associated to C_5 is given by

$$L := J(C_5) = (x_1, x_2) \cap (x_2, x_3) \cap (x_3, x_4) \cap (x_4, x_5) \cap (x_5, x_1).$$

Since the cover ideal of any bipartite graph is normally torsion-free and also by remembering this fact that the intersection of two normally torsion-free square-free monomial ideals which have no common variable is again normally torsion-free, we can deduce that any minor of L is normally torsion-free. On the other hand, we know that $\text{Ass}(L^s) = \text{Min}(L) \cup \{(x_1, x_2, x_3, x_4, x_5)\}$ for all $s \geq 2$. Therefore, L is not normally torsion-free, but all its minors are normally torsion-free. In other words, L is minimally not normally torsion-free. Note that in this example, we have $t = 2$.

Question 11.1.3. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a field K and $\text{SPP}_+(n)$ (respectively, $\text{SPP}_-(n)$) be the set of square-free monomial ideals in R such that satisfy (respectively, do not satisfy) the strong persistence property. Then, does the following limit exist? can be 0?*

$$\lim_{n \rightarrow \infty} \frac{|\text{SPP}_-(n)|}{|\text{SPP}_+(n)|},$$

where $|A|$ denotes the cardinality of A .

11.2 Normally torsion-freeness and normality under polarization

We continue our argument with two open questions which are devoted to the normally torsion-freeness and normality of monomial ideals under the polarization operator. We should recall the definition of polarization operator as follows.

Definition 11.2.1. (see Definition 3.2.37) *The process of polarization replaces a power x_i^t by a product of t new variables $x_{(i,1)} \cdots x_{(i,t)}$. We call $x_{(i,j)}$ a shadow of x_i . We will use $\tilde{I}^t$ to denote the polarization of I^t , will use S_t for the new polynomial ring in this polarization, and will use $\tilde{w}$ to denote the polarization in S_t of a monomial w in $R = K[x_1, \dots, x_n]$. The depolarization of an ideal in S_t is formed by setting $x_{(i,j)} = x_i$ for all i, j .*

In general, the polarization operator is more complicated in comparison with the other monomial operators. Let I be a non-square-free monomial ideal in the polynomial ring $K[x_1, \dots, x_n]$. A natural question is about whether there is some relation between the normality (respectively, normally torsion-freeness) of a monomial ideal I and the normality (respectively, normally torsion-freeness) of its polarization $\tilde{I}$?

A partial answer to this question is that the normality of $\tilde{I}$ does not necessarily imply that I is normal. This assertion is supported in the following example which we give a monomial ideal I which is not normal while its polarization $\tilde{I}$ is normal.

Example 11.2.2. *Let $I = (x_1x_2^2x_3, x_2x_3^2x_4, x_3x_4^2x_5, x_4x_5^2x_1, x_5x_1^2x_2)$ be a monomial ideal in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5]$. In view of Example 3.2.41, one can conclude that $\tilde{I}$ is normal. On the other hand, we show that I is not integrally closed, and a fortiori is not normal. To see this, let $\alpha := x_1x_2x_3x_4x_5$; one derives $\alpha^2 \in I^2$ but $\alpha \notin I$; thus, $\alpha \in \bar{I} \setminus I$.*

The above example shows that the normality of I and the normality of $\tilde{I}$ are not equivalent. As an application of Theorem 7.2.5, we give in the following example a monomial ideal I such that both I and $\tilde{I}$ are not normal.

Example 11.2.3. *Consider the following monomial ideal*

$$I = (x_1^2x_2x_3, x_1^4x_2x_4, x_1^2x_3x_5, x_1^2x_4x_6, x_1^2x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6),$$

in the polynomial ring $R = K[x_1, \dots, x_6]$. To illustrate the non-normality of I , our strategy is to use Theorem 7.2.5. To do this, take the monomial ideal $L \subset R$ as follows

$$L = (x_1x_2x_3, x_1^2x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6).$$

Also, suppose that $W : \{x_1, \dots, x_6\} \rightarrow \mathbb{N}$ is the weight over R with $W(x_1) = 2$, and $W(x_i) = 1$ for all $i = 2, \dots, 6$. This implies that $I = L_W$. By Theorem 7.2.5, it suffices to verify that L is not normal. To see this, set $\alpha := x_1x_2x_3x_4x_5x_6$. Due to

$$\alpha^2 = (x_1x_2x_3)(x_1x_5x_6)(x_2x_4x_5)(x_3x_4x_6) \in (L^2)^2,$$

we get $\alpha \in \overline{L^2}$. Moreover, it is routine to check that $\alpha \notin L^2$, and hence $\alpha \in \overline{L^2} \setminus L^2$. This yields that L is not normal, and so I is not normal. To complete our argument, we must demonstrate that $\tilde{I}$ is not normal. Without loss of generality, one can write

$$\begin{aligned} \tilde{I} = & (x_1x_2x_3x_7, x_1x_2x_4x_7x_8x_9, x_1x_3x_5x_7, x_1x_4x_6x_7, x_1x_5x_6x_7, \\ & x_2x_3x_6, x_2x_4x_5, x_2x_5x_6, x_3x_4x_5, x_3x_4x_6), \end{aligned}$$

in the polynomial ring $S = K[x_1, \dots, x_9]$. Let $\alpha := x_1x_2x_3x_4x_5x_6x_7$. It is routine to see that $\alpha \notin (\tilde{I})^2$, while $\alpha^2 \in (\tilde{I})^4$; hence, $\alpha \in \overline{(\tilde{I})^2} \setminus (\tilde{I})^2$. Thus, $(\tilde{I})^2$ is not integrally closed, and so $\tilde{I}$ is not normal.

We now turn our attention to the relation between the persistence property (respectively, the strong persistence property) of a non-square-free monomial ideal I and its polarization, that is, I^{Pol} . This counterexample can be found in [100].

Question 11.2.4. *Let I be a non-square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K , and I^{Pol} denote its polarization. Let I satisfy the persistence property (respectively, strong persistence property). Then can we deduce that I^{Pol} satisfies the persistence property (respectively, strong persistence property)?*

Solution. Suppose the following monomial ideal

$$\begin{aligned} I = & (x_4^2x_5x_1, x_2x_3x_6, x_2x_3x_1, x_2x_4x_5, x_2x_4x_1, x_2x_5x_6, x_3x_4x_5, \\ & x_3x_4x_6, x_3x_5x_1, x_4x_6x_1, x_5^2x_6x_1), \end{aligned}$$

in the polynomial ring $R = K[x_1, x_2, x_3, x_4, x_5, x_6]$ over a field K . Then the polarization of I is the following square-free monomial ideal,

$$\begin{aligned} I^{\text{Pol}} = & (x_7x_4x_5x_1, x_2x_3x_6, x_2x_3x_1, x_2x_4x_5, x_2x_4x_1, x_2x_5x_6, x_3x_4x_5, \\ & x_3x_4x_6, x_3x_5x_1, x_4x_6x_1, x_8x_5x_6x_1), \end{aligned}$$

in the polynomial ring $S = K[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]$ over a field K . By using Macaulay2 [49], we detect that

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \text{Ass}(S/(I^{\text{Pol}})^2) \setminus \text{Ass}(S/(I^{\text{Pol}})^3),$$

and also $((I^{\text{Pol}})^3 :_S I^{\text{Pol}}) \neq (I^{\text{Pol}})^2$. This shows that I^{Pol} does not satisfy both of the persistence property and strong persistence property. To finish the argument,

we have to verify that I has both of the persistence property and strong persistence property. Our strategy is to use [112, Corollary 6.4], which says that every normal monomial ideal has a superficial ideal. Note that a monomial ideal has the strong persistence property if and only if it has a superficial ideal. A fast approach is to refer to *Normaliz* software, but we utilize a recursive method to demonstrate the normality of I . We establish this claim in four steps:

Step 1. Put $A_1 := (x_4, x_5^2)$ and $B_1 := (x_4x_5, x_2)$. Since A_1 is integrally closed, we can deduce from [132, Theorem 12.1.10] that A_1 is normal. It follows also from [4, Theorem 3.4 and Remark 1.2] that x_4B_1 is normal. Furthermore, note that $A_1 + x_4B_1 = A_1$, and so $A_1 + x_4B_1$ is normal. Now, Theorem 7.7.3 gives that $B_2 := x_6A_1 + x_4B_1$ is normal.

Step 2. Let $A_2 := (x_2, x_5)$. Clearly, A_2 is normal. Due to $A_2 + B_2 = (x_2, x_5, x_4x_6)$, we conclude from [4, Theorem 3.4] that $A_2 + B_2$ is normal. Also, based on Step 1, the ideal B_2 is normal. Hence, Theorem 7.7.3 yields that $H := x_3A_2 + B_2$ is normal.

Step 3. Put $A_3 := x_4x_5(x_2, x_3)$ and $B_3 := (x_2x_3, x_2x_5, x_3x_4)$. It follows from [4, Theorem 3.4 and Remark 1.2] that A_3 is normal. To show the normality of B_3 , consider a simple finite graph $G = (V(G), E(G))$ with $V(G) = \{x_2, x_3, x_4, x_5\}$ and $E(G) = \{\{x_2, x_3\}, \{x_2, x_5\}, \{x_3, x_4\}\}$. Since B_3 is the edge ideal of G and G is bipartite, [48, Corollary 2.6] yields that B_3 is normally torsion-free, and by [60, Theorem 1.4.6], we deduce that B_3 is normal. Also, note that $A_3 + B_3 = B_3$ is normal. Here, Theorem 7.7.3 implies $L := A_3 + x_6B_3$ is normal.

Step 4. Thanks to $I = L + x_1H$, and according to Steps 2 and 3, H and L are normal, it is enough for us to prove the normality of

$$H + L = (x_4^2x_5, x_2x_3, x_2x_4, x_3x_5, x_4x_6, x_5^2x_6, x_2x_5x_6).$$

Set $C := (x_4^2x_5, x_2x_3, x_2x_4, x_3x_5)$ and $D := (x_4, x_5^2, x_2x_5)$. Clearly, $H + L = C + x_6D$. Since $C = (x_4^2, x_3)x_5 + (x_3, x_4)x_2$, $D = (x_4, x_5^2) + x_2(x_5)$, and

$$C + D = (x_4, x_5^2) + x_3(x_5) + x_2(x_3, x_5),$$

similar to the argument which has been presented in Steps 2 and 3, we conclude that C , D , and $C + D$ are normal, and so $H + L$ is normal. In view of Theorem 7.7.3, I has the (strong) persistence property. This completes the discussion.

Along with this argument, we are ready to provide the following open questions.

Question 11.2.5. Let I be a normal non-square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$. Can we deduce that $\tilde{I}$ is normal?

Question 11.2.6. Let I be a normally torsion-free non-square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$. Then can we deduce that $\tilde{I}$ is a normally torsion-free monomial ideal?

11.3 Closed neighborhood ideals and dominating ideals

In this section, we present three open questions, which are related to closed neighborhood ideals and dominating ideals of graphs.

Let G be a simple finite graph. We are first going to give an argument on the relation between monomial prime ideals associated to $\text{Ass}(DI(G)^s)$ for some $s \geq 1$, and monomial prime ideals associated to $\text{Ass}(DI(G_{\mathfrak{p}})^s)$, where $G_{\mathfrak{p}}$ stands for the induced graph on $\mathfrak{p}$, i.e., the graph with vertex set $\mathfrak{p}$, and the following edge set

$$E(G_{\mathfrak{p}}) = \{e \in E(G) \mid e \subseteq \mathfrak{p}\}.$$

Francisco, Hà, and Van Tuyl, in [45, Lemma 2.11], showed that if $\mathcal{H}$ is a finite simple hypergraph on $V = \{x_1, \dots, x_n\}$ with cover ideal $J(\mathcal{H}) \subseteq R = k[x_1, \dots, x_n]$, then for all $d \geq 1$,

$$P = (x_{i_1}, \dots, x_{i_r}) \in \text{Ass}(R/J(\mathcal{H})^d) \Leftrightarrow P = (x_{i_1}, \dots, x_{i_r}) \in \text{Ass}(k[P]/J(\mathcal{H}_P)^d),$$

where $k[P] = k[x_{i_1}, \dots, x_{i_r}]$, and $\mathcal{H}_P$ is the induced hypergraph of $\mathcal{H}$ on the vertex set $P = \{x_{i_1}, \dots, x_{i_r}\} \subseteq V$. Here, we provide a counterexample which shows that one cannot mimic this result for dominating ideals of simple finite graphs. To do this, let $G = (V(G), E(G))$ be a simple finite graph with the vertex set $V(G) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ and the following edge set

$$E(G) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_6\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_4\}, \\ \{x_3, x_5\}, \{x_4, x_6\}\}.$$

One can easily see that the dominating ideal of G is given by

$$DI(G) = (x_1, x_2, x_3, x_6) \cap (x_1, x_2, x_4, x_5) \cap (x_1, x_3, x_4, x_5) \cap (x_2, x_3, x_4, x_6) \\ \cap (x_2, x_3, x_5) \cap (x_1, x_4, x_6).$$

Take $\mathfrak{p} := (x_1, x_2, x_3, x_6) \in \text{Ass}(DI(G)^2)$. It follows from the definition that $V(G_{\mathfrak{p}}) = \{x_1, x_2, x_3, x_6\}$ and $E(G_{\mathfrak{p}}) = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_6\}\}$. In particular, the dominating ideal of $G_{\mathfrak{p}}$ is given by

$$DI(G_{\mathfrak{p}}) = (x_1, x_2) \cap (x_1, x_3) \cap (x_1, x_6) = (x_1, x_2x_3x_6).$$

We can conclude from Theorem 5.3.4 that $DI(G_{\mathfrak{p}})$ is normally torsion-free, and so $\mathfrak{p} \notin \text{Ass}(DI(G_{\mathfrak{p}})^2)$. On the other hand, $\mathfrak{q} := (x_1, x_6) \in \text{Ass}(DI(G_{\mathfrak{p}})^2)$, while $\mathfrak{q} \notin \text{Ass}(DI(G)^2)$.

We now discuss the nature of $G_{\mathfrak{p}}$. To accomplish this, we first recall the definition of partial t -cover ideals which has already been introduced in [17].

Definition 11.3.1. ([17, Definition 1.1]) Suppose that G is a finite simple graph on the vertex set $V(G) = \{x_1, x_2, \dots, x_n\}$ with the edge set $E(G)$. Also, for any $x \in V(G)$, let $N(x) = \{y : \{x, y\} \in E(G)\}$ denote the set of neighbors of x . Fix an integer $t \geq 1$. The partial t -cover ideal of G is the monomial ideal

$$J_t(G) = \bigcap_{x \in V(G)} \left(\bigcap_{\{x_{i_1}, \dots, x_{i_t}\} \subseteq N(x)} (x, x_{i_1}, \dots, x_{i_t}) \right).$$

It should be noted that when $t = 1$, the construction above is simply the cover ideal of a finite simple graph G . In fact, Bhat, Biermann, and Van Tuyl, in [17, Lemma 2.4], proved the following lemma:

Lemma 11.3.2. ([17, Lemma 2.4]) Let G be a graph on the vertex set $\{x_1, \dots, x_n\}$, and $J_t(G)$ be the partial t -cover ideal of G . If $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r}) \in \text{Ass}(J_t(G)^s)$, then $G_{\mathfrak{p}}$ is connected.

In this direction, we leave the following open question:

Question 11.3.3. Let G be a connected simple finite graph on the vertex set $V(G) = \{x_1, \dots, x_n\}$, and let $DI(G)$ be the dominating ideal of G . If $\mathfrak{p} = (x_{i_1}, \dots, x_{i_r}) \in \text{Ass}(DI(G)^s)$ for some $s \geq 1$, then can we deduce that $G_{\mathfrak{p}}$ is connected?

In what follows, we investigate the closed neighborhood ideals related to the whisker graph of a graph.

Definition 11.3.4. ([133, Definition 7.3.10]) Let G_0 be a graph on the vertex set $Y = \{y_1, \dots, y_n\}$ and take a new set of variables $X = \{x_1, \dots, x_n\}$. The whisker graph or suspension of G_0 , denoted by $G_0 \cup W(Y)$, is the graph obtained from G_0 by attaching to each vertex y_i a new vertex x_i and the edge $\{x_i, y_i\}$. The edge $\{x_i, y_i\}$ is called a whisker.

Question 11.3.5. Let G_0 be a graph and let $H := G_0 \cup W(Y)$ be its whisker graph. If $NI(G_0)$ is normally torsion-free, then can we deduce that $NI(H)$ is normally torsion-free?

Question 11.3.6. Let G_0 be a graph and let $H := G_0 \cup W(Y)$ be its whisker graph. Then if $DI(G_0)$ is normally torsion-free, then can we deduce that $DI(H)$ is normally torsion-free?

11.4 t -spread principal Borel ideals

Here, we turn our attention to *vector-spread monomials of type $\mathbf{t}$* . In fact, in [108, Theorem 5.13 and Lemma 5.15], the authors characterized all normally torsion-free t -spread principal Borel ideals. We review the needed definitions as follows.

Definition 11.4.1. ([42, Definition 2.1]) Let $u = x_{j_1}x_{j_2}\cdots x_{j_\ell} \in R = K[x_1, \dots, x_n]$ be a monomial of degree $\ell \leq d$, with $1 \leq j_1 \leq j_2 \leq \cdots \leq j_\ell \leq n$. We say that u is a *vector-spread monomial of type $\mathbf{t}$* , or simply a *$\mathbf{t}$ -spread monomial*, if $j_{i+1} - j_i \geq t_i$ for all $i = 1, \dots, \ell - 1$.

Note that any variable x_j is $\mathbf{t}$ -spread, and we also assume that $u = 1$ is $\mathbf{t}$ -spread. In addition, we say that a monomial ideal $I \subset R = K[x_1, \dots, x_n]$ is a *vector-spread monomial ideal of type $\mathbf{t}$* , or simply a *$\mathbf{t}$ -spread monomial ideal*, if all monomials $u \in \mathcal{G}(I)$ are $\mathbf{t}$ -spread.

Definition 11.4.2. ([42, Definition 2.2]) With the notation above, if $t_i = t \geq 0$ for all $i = 1, \dots, d - 1$, then we say that a $\mathbf{t}$ -spread monomial (ideal) $u \in \text{Mon}(R)$, ($I \subseteq R$), is called an *ordinary or uniform t -spread monomial (ideal)*, which has been introduced in [37].

Definition 11.4.3. A $\mathbf{t}$ -spread monomial ideal I of R is called a *vector-spread Borel ideal of type $\mathbf{t}$* or simply a *$\mathbf{t}$ -spread strongly stable ideal* if for any $\mathbf{t}$ -spread monomial $u \in I$, and all $j < i$ such that x_i divides u and $x_j(u/x_i)$ is $\mathbf{t}$ -spread, then $x_j(u/x_i) \in I$. For $\mathbf{t} = \mathbf{0} = (0, 0, \dots, 0)$ ($\mathbf{t} = \mathbf{1} = (1, 1, \dots, 1)$), one obtains the classical notion of *strongly stable (square-free strongly stable) ideal*, see [60, Definition 4.2.2] (Problem 6.9 on page 112 in [60]).

Definition 11.4.4. A $\mathbf{t}$ -spread monomial ideal I of R is called a *$\mathbf{t}$ -spread principal Borel ideal* if there exists a $\mathbf{t}$ -spread monomial $u \in \mathcal{G}(I)$ such that I is the smallest $\mathbf{t}$ -spread strongly stable ideal which contains u . We denote it as $I = B_{\mathbf{t}}(u)$.

It should be noted that for a $\mathbf{t}$ -spread monomial $u = x_{i_1}\cdots x_{i_\ell} \in R$, we have $x_{j_1}\cdots x_{j_\ell} \in \mathcal{G}(B_{\mathbf{t}}(u))$ if and only if $j_1 \leq i_1, \dots, j_\ell \leq i_\ell$ and $j_{i+1} - j_i \geq t_i$ for all $i = 1, \dots, \ell - 1$.

Now, one can ask the following question:

Problem 11.4.5. Which classes of $\mathbf{t}$ -spread principal Borel ideals are normally torsion-freeness?

11.5 A comparison between clutters and complement clutters

To formulate the following question, we need to recall the notion of a clutter. A *clutter* (or *simple hypergraph*) $\mathcal{C}$ with vertex set $X = \{x_1, \dots, x_n\}$ is a family of subsets of X , called *edges*, none of which is included in another, see [133, Definition 6.3.33] for more details. The edge ideal of a clutter is defined in [133, Definition

6.3.35], and the cover ideal of a clutter can be defined as the ideal of all monomials M such that given any edge e of $\mathcal{C}$ there is some variable x_i such that $x_i \in e$ and $x_i \nmid M$. Note that the vertices of these clutters become the variables of the ring in which the edge ideal and cover ideal are allocated. Given a clutter $\mathcal{C}$ on $\{x_1, \dots, x_n\}$ with edges $e_1, \dots, e_r$, we define the *complement clutter*, denoted by $\mathcal{C}^c$, as the clutter whose edges are $\{x_1, \dots, x_n\} \setminus e_i$ for each $i = 1, \dots, r$. In particular, we have $(\mathcal{C}^c)^c = \mathcal{C}$. We denote the edge ideal of a clutter $\mathcal{C}$ by $I(\mathcal{C})$.

It is quite natural to want to compare the algebraic properties of clutters and complement clutters. The next question pursues this goal.

Question 11.5.1. (i) Does $I(\mathcal{C})$ have the strong persistence property if and only if $I(\mathcal{C}^c)$ have the strong persistence property?

(ii) Does $I(\mathcal{C})$ have the persistence property if and only if $I(\mathcal{C}^c)$ have the persistence property?

(iii) Is $I(\mathcal{C})$ normal if and only if $I(\mathcal{C}^c)$ is normal?

(iv) Is $I(\mathcal{C})$ normally torsion-free if and only if $I(\mathcal{C}^c)$ is normally torsion-free?

To illuminate the purpose of the above question, we give an example of a clutter $\mathcal{C}$ where both $I(\mathcal{C})$ and $I(\mathcal{C}^c)$ do not have the strong persistence property.

To do this, consider the graph which has been presented in Proposition 3.2.5. Set $F := J(H_4)$. By using *Macaulay2* [49], F is given by

$$\begin{aligned} F = & (x_1x_3x_6x_8x_9x_{10}x_{11}x_{12}, x_2x_4x_5x_7x_9x_{10}x_{11}x_{12}, x_1x_2x_4x_5x_7x_{10}x_{11}x_{12}, \\ & x_2x_3x_5x_6x_8x_9x_{11}x_{12}, x_1x_2x_3x_6x_8x_9x_{11}x_{12}, x_2x_4x_5x_6x_7x_9x_{11}x_{12}, \\ & x_1x_3x_6x_7x_8x_9x_{10}x_{12}, x_2x_3x_5x_7x_8x_9x_{10}x_{12}, x_2x_3x_4x_5x_7x_9x_{10}x_{12}, \\ & x_1x_3x_4x_5x_6x_7x_{10}x_{12}, x_1x_2x_3x_4x_5x_7x_{10}x_{12}, x_1x_3x_4x_6x_8x_9x_{10}x_{11}, \\ & x_1x_2x_4x_5x_7x_8x_{10}x_{11}, x_1x_3x_4x_5x_6x_8x_{10}x_{11}, x_1x_2x_4x_6x_7x_8x_9x_{11}, \\ & x_1x_2x_3x_4x_6x_8x_9x_{11}). \end{aligned}$$

It has already been shown in [76] that F does not satisfy the persistence property (and hence does not satisfy the strong persistence property either). One can show, by using *Macaulay2* [49], that in the polynomial ring $R = K[x_1, \dots, x_{12}]$ over a field K , $\mathfrak{m} = (x_1, \dots, x_{12}) \in \text{Ass}_R(R/F^3) \setminus \text{Ass}_R(R/F^4)$ and $(F^4 :_R F) \neq F^3$. Now, we construct the clutter $\mathcal{C}$ on $\{x_1, \dots, x_{12}\}$ whose edge ideal is F , that is, $F = I(\mathcal{C})$, as

follows

$$\begin{aligned}\mathcal{C} := & \{\{x_1, x_3, x_6, x_8, x_9, x_{10}, x_{11}, x_{12}\}, \{x_2, x_4, x_5, x_7, x_9, x_{10}, x_{11}, x_{12}\}, \\ & \{x_1, x_2, x_4, x_5, x_7, x_{10}, x_{11}, x_{12}\}, \{x_2, x_3, x_5, x_6, x_8, x_9, x_{11}, x_{12}\}, \\ & \{x_1, x_2, x_3, x_6, x_8, x_9, x_{11}, x_{12}\}, \{x_2, x_4, x_5, x_6, x_7, x_9, x_{11}, x_{12}\}, \\ & \{x_1, x_3, x_6, x_7, x_8, x_9, x_{10}, x_{12}\}, \{x_2, x_3, x_5, x_7, x_8, x_9, x_{10}, x_{12}\}, \\ & \{x_2, x_3, x_4, x_5, x_7, x_9, x_{10}, x_{12}\}, \{x_1, x_3, x_4, x_5, x_6, x_7, x_{10}, x_{12}\}, \\ & \{x_1, x_2, x_3, x_4, x_5, x_7, x_{10}, x_{12}\}, \{x_1, x_3, x_4, x_6, x_8, x_9, x_{10}, x_{11}\}, \\ & \{x_1, x_2, x_4, x_5, x_7, x_8, x_{10}, x_{11}\}, \{x_1, x_3, x_4, x_5, x_6, x_8, x_{10}, x_{11}\}, \\ & \{x_1, x_2, x_4, x_6, x_7, x_8, x_9, x_{11}\}, \{x_1, x_2, x_3, x_4, x_6, x_8, x_9, x_{11}\}\}.\end{aligned}$$

Then, $F = I(\mathcal{C})$ does not satisfy the strong persistence property. On the other hand, one can deduce from the definition that the complement clutter of $\mathcal{C}$, that is $\mathcal{C}^c$, is as follows

$$\begin{aligned}\mathcal{C}^c := & \{\{x_2, x_4, x_5, x_7\}, \{x_1, x_3, x_6, x_8\}, \{x_3, x_6, x_8, x_9\}, \{x_1, x_4, x_7, x_{10}\}, \\ & \{x_4, x_5, x_7, x_{10}\}, \{x_1, x_3, x_8, x_{10}\}, \{x_2, x_4, x_5, x_{11}\}, \{x_1, x_4, x_6, x_{11}\}, \\ & \{x_1, x_6, x_8, x_{11}\}, \{x_2, x_8, x_9, x_{11}\}, \{x_6, x_8, x_9, x_{11}\}, \{x_2, x_5, x_7, x_{12}\}, \\ & \{x_3, x_6, x_9, x_{12}\}, \{x_2, x_7, x_9, x_{12}\}, \{x_3, x_5, x_{10}, x_{12}\}, \{x_5, x_7, x_{10}, x_{12}\}\}.\end{aligned}$$

This implies that the edge ideal of $I(\mathcal{C}^c)$ is given by

$$\begin{aligned}I(\mathcal{C}^c) = & (x_2x_4x_5x_7, x_1x_3x_6x_8, x_3x_6x_8x_9, x_1x_4x_7x_{10}, x_4x_5x_7x_{10}, x_1x_3x_8x_{10}, \\ & x_2x_4x_5x_{11}, x_1x_4x_6x_{11}, x_1x_6x_8x_{11}, x_2x_8x_9x_{11}, x_6x_8x_9x_{11}, x_2x_5x_7x_{12}, \\ & x_3x_6x_9x_{12}, x_2x_7x_9x_{12}, x_3x_5x_{10}x_{12}, x_5x_7x_{10}x_{12}).\end{aligned}$$

By using *Macaulay2* [49], one can check that $(I(\mathcal{C}^c)^4 :_R I(\mathcal{C}^c)) \neq I(\mathcal{C}^c)^3$ and also $\mathfrak{m} = (x_1, \dots, x_{12}) \in \text{Ass}_R(R/I(\mathcal{C}^c)^3) \setminus \text{Ass}_R(R/I(\mathcal{C}^c)^4)$, that is, $I(\mathcal{C}^c)$ does not satisfy the strong persistence property and the persistence property.

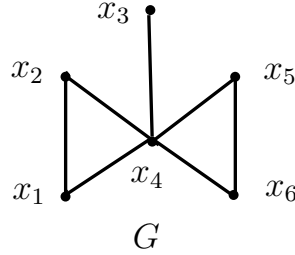
11.6 Matroidal path ideals of graphs

It is known from [135, Proposition 3.11] that polymatroidal ideals are normal. In [117, Corollary 2.11], it is shown that the t -path ideals of complete bipartite graphs are normal. In general, for a given $t \geq 3$, one can find graphs that are not complete

r -partite, but their t -path ideal $I_t(G)$ is matroidal. As a very simple example, let G itself be a path on t vertices. Then $I_t(G)$ is matroidal because it is a principal ideal, while G is not complete r -partite.

Theorems 9.3.6 and 9.3.8 together show that if $I_2(G)$ is matroidal, then $I_t(G)$ is also matroidal for all $t \geq 2$. Therefore, it is natural to ask the following question: If $I_t(G)$ is matroidal for some t , then is it true that $I_k(G)$ is also matroidal for all $k \geq t$? The following example shows that it is not true in general.

Example 11.6.1. *Let G be the graph as shown in the following figure.*



Then direct computation gives that

$$I_3(G) = x_4 J = (x_1 x_2 x_4, x_1 x_3 x_4, x_1 x_4 x_5, x_1 x_4 x_6, x_2 x_3 x_4, x_2 x_4 x_5, x_2 x_4 x_6, x_3 x_4 x_5, x_3 x_4 x_6, x_4 x_5 x_6),$$

where $J = (x_1 x_2, x_1 x_3, x_1 x_5, x_1 x_6, x_2 x_3, x_2 x_5, x_2 x_6, x_3 x_5, x_3 x_6, x_5 x_6)$. It is straight-forward to check that the ideal J is indeed the edge ideal of the complete graph on the vertex set $\{x_1, x_2, x_3, x_5, x_6\}$, and by Theorem 9.3.6, J is matroidal. This shows that $I_3(G)$ is also matroidal because it is the product of two matroidal ideals, namely (x_4) and J . Now, set $u := x_1 x_2 x_3 x_4$ and $v := x_3 x_4 x_5 x_6$. Then $u, v \in I_4(G)$, but both $x_5(u/x_1)$ and $x_6(u/x_1)$ do not belong to G . Hence, $I_4(G)$ is not matroidal.

In view of Theorem 9.3.8 and Example 11.6.1, we close this argument with the following questions.

Question 11.6.2. *For a given $t > 2$, characterize the graphs for which $I_t(G)$ is matroidal.*

Question 11.6.3. *For which graphs it is true that if $I_t(G)$ is matroidal for some t , then $I_k(G)$ is also matroidal for all $k \geq t$?*

11.7 A minimal counterexample to the Conforti-Cornuéjols conjecture

We terminate this chapter with the following questions and conjecture which has been stated in [43]. To accomplish this, we recall the notion of minor of a square-free

monomial ideal. Observe that, for a hypergraph $\mathcal{H}$, the *deletion* $\mathcal{H} \setminus x$ at a vertex $x \in V(\mathcal{H})$ has the effect of setting $x = 0$ in the edge ideal $I(\mathcal{H})$, and the *contraction* $\mathcal{H}/x$ has the effect of setting $x = 1$ in the edge ideal $I(\mathcal{H})$. We call an ideal I' a *minor* of a square-free monomial ideal I if I' can be obtained from I by a sequence of taking quotients (=deletion) and localizations (=contraction) at the variables. In this direction, one can ask the following questions.

Question 11.7.1. *Let I be a square-free monomial ideal in $R = K[x_1, \dots, x_n]$. Also, assume that I is minimally not normally torsion-free, that is, I is not normally torsion-free, but all its minors are normally torsion-free. Then find the least power t such that $\text{Ass}_R(R/I^t) \neq \text{Ass}_R(R/I)$.*

Question 11.7.2. *Let I be a square-free monomial ideal in $R = K[x_1, \dots, x_n]$. Then find the least integer ℓ (may depend on I) such that if $I^{(t)} = I^t$ for $0 \leq t \leq \ell$, then $I^{(t)} = I^t$ for all $t \geq 1$, that is, I is normally torsion-free.*

Let $\mathcal{H}$ denote a hypergraph over d vertices $\{x_1, \dots, x_d\}$ and $R = K[x_1, \dots, x_d]$ will be the corresponding polynomial ring. A vertex $x \in V(\mathcal{H})$ is called an *isolated vertex* if $\{x\} \in E(\mathcal{H})$. It follows from the definition that if x is an isolated vertex of $\mathcal{H}$, then $\{x\}$ is the only edge in $\mathcal{H}$ that contains x . A vertex cover of $\mathcal{H}$ is a set of vertices that has non-empty intersection with all of the edges of $\mathcal{H}$. The minimum cardinality of a vertex cover of $\mathcal{H}$ is denoted by $\alpha_0(\mathcal{H})$. It should be noted that by the correspondence between minimal primes and minimal vertex covers, $\alpha_0(I)$ is also the height of I . We say the generators of a square-free monomial ideal I as being *independent* if the corresponding edges of the associated hypergraph are pairwise disjoint; that is, the generators have disjoint support. We denote the maximum cardinality of an independent set in I by $\beta_1(I)$. Furthermore, note that a subset of the monomial generators of I is independent if and only if it forms a regular sequence. Thus, $\beta_1(I)$ is equal to the monomial grade of I , denoted by $\mathbf{m}\text{-grade}(I)$, where the monomial grade of an ideal I is the maximum length of a regular sequence of monomials in I . It not hard to see that $\alpha_0(I) \geq \beta_1(I)$.

Definition 11.7.3. *A square-free monomial ideal I is said to satisfy the König property if $\alpha_0(I) = \beta_1(I)$. Consequently, I satisfies the König property if and only if $\text{grade}(I) = \text{ht}(I) = \mathbf{m}\text{-grade}(I)$.*

Definition 11.7.4. *A square-free monomial ideal I has the packing property if I and all of its minors satisfy the König property.*

Definition 11.7.5. *A hypergraph $\mathcal{H}$ is called Mengerian if its edge ideal $I(\mathcal{H})$ is normally torsion-free.*

In 1993, M. Conforti and G. Cornuéjols [29] made up the following conjecture, and it is still open after 30 years.

Conjecture 11.7.6. (*Conforti-Cornuéjols conjecture*) *A hypergraph has the packing property if and only if it is Mengerian.*

In connection with combinatorial applications, we express an algebraic reformulation of the Conforti-Cornuéjols conjecture as follows.

Conjecture 11.7.7. *Let I be a square-free monomial ideal such that I and all of its minors satisfy the property that their heights are the same as their $\mathfrak{m}$ -grades. Then $\mathrm{gr}_I := \bigoplus_{q \geq 0} I^q / I^{q+1}$ is reduced; or equivalently, $I^k = I^{(k)}$ for all $k \geq 1$; or equivalently, I is normally torsion-free.*

In broadly speaking, finding a counterexample for this conjecture is too hard, and the famous result in this theme is due to Hà and Morey, which says that

Corollary 11.7.8. (*[50, Corollary 3.10]*) *A minimal counterexample to the Conforti-Cornuéjols conjecture cannot be unmixed.*

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