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FACULTE DES SCIENCES
FORMATION DOCTORALE : MATHÉMATIQUES INFORMATIQUE ET APPLICATIONS

THÈSE DE DOCTORAT

Par

Mr/ Nizar El Idrissi

Sous le thème

Etude de quelques aspects de l'espace des repères, de l'indépendance et des ensembles quasi-invariants.

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Étude de quelques aspects de l'espace des repères, de l'indépendance et des ensembles quasi-invariants

Nizar El Idrissi

Une thèse présentée pour l'obtention du titre de
Docteur en mathématiques



Département de mathématiques
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Résumé

Cette thèse a pour objet l'étude de quelques aspects de l'espace des repères, de l'indépendance et des ensembles quasi-invariants. Dans la première partie, nous démontrons deux résultats sur la connexité par arcs et l'adhérence (respectivement) de quelques ensembles reliés à la variété de Stiefel non compacte (espace des repères), et nous illustrons le dernier résultat par quelques exemples. Dans la deuxième partie, nous prouvons quelques généralisations d'un résultat garantissant l'équivalence entre l'indépendance linéaire et la dimension infinie d'une suite infinie de vecteurs décalés par un opérateur linéaire. Dans la troisième partie, nous décrivons les fonctions admettant un nombre maximal d'ensembles ou de sur-ensembles finis invariants ou intérieurement-1-quasi-invariants.

Abstract

This thesis deals with some aspects of the frame space, of independence, and of quasi-invariant sets. In the first part, we prove two results on the path-connectedness and topological closure (respectively) of some sets related to the non-compact Stiefel manifold (frame space), and we illustrate the latter with some examples. In the second part, we prove some generalizations of a result guaranteeing the equivalence between the linear independence and infinite dimension of an infinite sequence of vectors shifted by a linear operator. In the third part, we describe the functions admitting a maximal number of finite invariant or internally-1-quasi-invariant sets or supersets.

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Chapitre 1

Introduction

Cette thèse est le fruit d'une recherche progressive dont le but initial était l'étude de quelques aspects des repères. Nous avons d'abord étudié quelques propriétés topologiques et d'existence de zéros de fonctions sur l'espace des repères, motivés par des articles récents comme [53, 27, 2, 21, 54]. Ensuite, nous avons approfondi un résultat reliant l'indépendance linéaire et la dimension infinie d'une suite infinie de vecteurs d'un espace vectoriel, motivés par l'article [22] portant sur l'échantillonnage dynamique en lien avec les repères et les systèmes indépendants. Enfin, désirant trouver un exemple à l'un de nos résultats du travail précédent, nous avons cherché les fonctions admettant un nombre maximal d'ensembles ou de sur-ensembles intérieurement-1-quasi-invariants, notion apparaissant dans le cadre des systèmes dynamiques [29, 25, 30] et dans celui des actions de groupes dans [13, 55, 14]. Cette thèse est constituée d'une introduction suivie de trois articles formant le corps de la thèse, et portant sur les thèmes suivants : repères et variétés de Stiefel, indépendance et dimension, fonctions et ensembles quasi-invariants. Dans cette introduction, nous présentons le contexte des thèmes traités dans cette thèse. Nous faisons d'abord la revue de la littérature sur le sujet, puis nous présentons brièvement nos contributions.

1.1 Repères et variétés de Stiefel

1.1.1 Littérature

Nous nous intéressons dans la première partie de la thèse à la notion de **repère** dans un espace de Hilbert et à l'**espace** des (X, Σ, μ) -repères dans un espace de Hilbert fixé (**variété de Stiefel non compacte**).

Notion de repère et espace des repères (variété de Stiefel)

Introduite par Duffin et Schaeffer en 1952 dans le cadre de l'espace $L^2((-\gamma, \gamma))$, et généralisée dans le même article aux espaces de Hilbert quelconques [26], la notion de repère permet la représentation des vecteurs par des coordonnées. A la différence d'autres représentations, cette dernière (réalisée concrètement comme une décomposition sommatoire) possède les qualités de la stabilité, la robustesse et la flexibilité. Après son introduction, cette notion n'a plus reçu d'attention de la part des mathématiciens, jusqu'à ce qu'en 1986, un article de Daubechies, Grossmann et Meyer [23], portant sur les repères de Gabor et d'ondelettes, ravive l'intérêt pour cette notion, dont l'étude continue jusqu'à maintenant.

D'un autre côté, l'espace des repères (variété de Stiefel non compacte) est un espace de configuration qui se retrouve dans de nombreux problèmes concrets. Introduit en 1935 [58] et étudié par des spécialistes de la topologie et de la géométrie bien avant et indépendamment de la notion de repère, son lien avec celle-ci fut néanmoins découvert par la suite.

Applications de la théorie des repères

La théorie des repères est devenue importante du fait des nombreuses applications qui utilisent ses résultats plus ou moins directement. Autres que les applications en mathématiques, où les développements par rapport aux repères sont utiles à la résolution de nombreux problèmes en analyse comme les équations aux dérivées partielles [64], il y a les très nombreuses applications aux théories du signal et de l'information. On peut citer le traitement d'image, du son, et de la vidéo, la reconstruction du signal, le débruitage, la transmission robuste, le codage et la compression. Dans le champ de la reconstruction du signal, on trouve les problématiques de l'acquisition comprimée et de la reconstruction de phase, qui ont des applications à la photographie, l'holographie, la microscopie électronique, la cristallographie aux rayons X, l'imagerie astronomique, la tomographie, la reconnaissance de la parole, etc.

Théorie des repères

Soit $\mathbb{F} = \mathbb{R}$ ou \mathbb{C} , K un espace de Hilbert sur \mathbb{F} et (X, Σ, μ) un espace mesuré.

Définition 1.1.1. [20] On dit qu'une famille faiblement mesurable $\Phi = (\varphi_x)_{x \in X}$ avec $\varphi_x \in K$ pour tout $x \in X$ est un **repère continu** dans K s'il existe deux constantes A et B strictement positives telles que

$$\forall v \in K : A\|v\|^2 \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq B\|v\|^2$$

Un repère est dit **serré** si on peut choisir $A = B$ comme bornes du repère. Un repère serré avec borne $A = B = 1$ est dit **de Parseval**. Une **famille de Bessel** continue est une famille satisfaisant seulement l'inégalité de droite. Un repère (ou une famille de Bessel) est **discret** si Σ est la tribu discrète et μ est la mesure de comptage. Si $(n, k) \in (\mathbb{N}^*)^2$, on note $\mathcal{F}_{(X, \Sigma, \mu), K}$, $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ et $\mathcal{F}_{k, n}^{\mathbb{F}}$ respectivement, l'ensemble des (X, Σ, μ) -repères continus à valeurs dans K , l'ensemble des (X, Σ, μ) -repères continus à valeurs dans \mathbb{F}^n , et l'ensemble des repères discrets indexés par $\llbracket 1, k \rrbracket$ et à valeurs dans \mathbb{F}^n .

Si $U = (u_x)_{x \in X}$ avec $u_x \in K$ pour tout $x \in X$ est une famille de Bessel continue dans K , on définit l'**opérateur d'analyse** $T_U : K \rightarrow L^2(X, \mu; \mathbb{F})$ par

$$\forall v \in K : T_U(v) := (\langle v, u_x \rangle)_{x \in X}.$$

L'adjoint de T_U est l'opérateur $T_U^* : L^2(X, \mu; \mathbb{F}) \rightarrow K$, appelé l'**opérateur de synthèse**, et donné par

$$\forall c \in L^2(X, \mu; \mathbb{F}) : T_U^*(c) = \int_X c(x) u_x d\mu(x).$$

La composée $S_U = T_U^* T_U : K \rightarrow K$ est donnée par

$$\forall v \in K : S_U(v) = \int_X \langle v, u_x \rangle u_x d\mu(x)$$

et s'appelle l'**opérateur repère** de U (utilisé surtout quand U est un repère, d'où le nom). Si U est une famille de Bessel, T_U , T_U^* et S_U sont des opérateurs bien définis et continus. Si Φ est un repère, alors S_Φ est un opérateur autoadjoint positif satisfaisant $0 < A \leq S_\Phi \leq B$, et il est donc inversible. Dans ce cas, les deux **formules de reconstruction** s'écrivent

$$\begin{aligned} \forall v \in K : v &= S_\Phi^{-1} S_\Phi v \\ &= \int_X \langle v, \varphi_x \rangle S_\Phi^{-1} \varphi_x d\mu(x) = \int_X \langle v, \varphi_x \rangle \tilde{\varphi}_x d\mu(x) \end{aligned}$$

et

$$\begin{aligned} \forall v \in K : v &= S_\Phi S_\Phi^{-1} v = \int_X \langle S_\Phi^{-1} v, \varphi_x \rangle \varphi_x d\mu(x) \\ &= \int_X \langle v, S_\Phi^{-1} \varphi_x \rangle \varphi_x d\mu(x) = \int_X \langle v, \tilde{\varphi}_x \rangle \varphi_x d\mu(x), \end{aligned}$$

et $\{\tilde{\varphi}_x\}_{x \in X} = \{S_\Phi^{-1} \varphi_x\}_{x \in X}$ s'appelle le **repère dual canonique** de Φ .

On a la caractérisation fondamentale suivante :

Theorème 1.1.1. [20] *Soit K un espace de Hilbert et (X, Σ, μ) un espace mesuré. Alors les propriétés suivantes sont équivalentes :*

1. $\Phi = (\varphi_x)_{x \in X}$ avec $\varphi_x \in K$ pour tout $x \in X$ est un repère continu dans K ,
2. T_Φ est bien défini, continu, injectif et d'image fermée,
3. T_Φ^* est bien défini, continu et surjectif,
4. S_Φ est bien défini, continu et inversible.

Supposons encore que l'espace de Hilbert K et l'espace mesuré (X, Σ, μ) soient fixés, et notons la remarque suivante : si $\Phi = (\varphi_x)_{x \in X}$ est un repère continu sur K , alors $\text{Range}(T_\Phi)$ est un **espace de Hilbert à noyau reproduisant** (RKHS) inclus dans $L^2(X, \mu; \mathbb{F})$, de noyau R donné par $\forall x, y \in X : R(x, y) = \langle \varphi_x, S_\Phi^{-1} \varphi_y \rangle$, et T_Φ détermine un isomorphisme de K dans $\text{Range}(T_\Phi)$.

Par conséquent, soit \mathbf{C}_1 l'ensemble formé par les repères continus $\Phi = (\varphi_x)_{x \in X}$ sur K .

Soit \mathbf{C}_2 l'ensemble formé par les isomorphismes $T : K \rightarrow H$ où H est un RKHS inclus dans $L^2(X, \mu; \mathbb{F})$.

Soit $S_1 : \mathbf{C}_1 \rightarrow \mathbf{C}_2$ la fonction définie par :

$$\Phi \mapsto T_\Phi \text{ corestreint à son image.}$$

Soit $S_2 : \mathbf{C}_2 \rightarrow \mathbf{C}_1$ la fonction définie par :

$$(T : K \rightarrow H \text{ isomorphisme où } H \text{ est un RKHS de noyau } R) \mapsto (T^*(R_x))_{x \in X}.$$

Alors S_1 et S_2 sont inverses l'une de l'autre, ce qui montre que les deux ensembles \mathbf{C}_1 et \mathbf{C}_2 sont en bijection. Parmi les références évoquant ce lien, on peut citer [6, 5]. De plus, dans S_2 , le repère $\Phi := (T^*(R_x))_{x \in X}$ est l'unique repère continu tel que $T_\Phi^H = T$ (ce dernier point découle du fait que deux suites de Bessel U et V sont égales ssi $T_U = T_V$). Enfin, il serait intéressant d'étendre les ensembles \mathbf{C}_1 et \mathbf{C}_2 en des catégories (en ajoutant des notions de morphismes dans \mathbf{C}_1 et \mathbf{C}_2) de sorte à ce que S_1 et S_2 deviennent des foncteurs, car cela signifierait que \mathbf{C}_1 et \mathbf{C}_2 sont isomorphes en tant que catégories.

Topologie et géométrie des espaces de repères (variétés de Stiefel)

Comme l'étude topologique pour la topologie L^2 est moins facile à entreprendre pour les repères à valeurs dans un espace de Hilbert de dimension infinie (car un tel repère a toujours une norme L^2 infinie), on s'intéresse habituellement à la topologie L^2 de l'espace des repères en dimension finie $n \in \mathbb{N}^*$. On sait que cet espace, noté dans le cas continu $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$, est isométrique à la variété de Stiefel $St(n, H)$, où $H := L^2(X, \mu; \mathbb{F})$ est un espace de Hilbert pouvant être de dimension infinie. Rappelons donc la définition de ces variétés et quelques résultats à leur propos.

Définition 1.1.2. La **variété de Stiefel non compacte des n -repères indépendants dans H** est définie par $St(n, H) := \{h = (h_1, \dots, h_n) \in H^n : \{h_1, \dots, h_n\} \text{ est un système libre}\}$. La **variété de Stiefel des n -repères orthonormés dans H** est définie par $St_o(n, H) := \{h = (h_1, \dots, h_n) \in H^n : \{h_1, \dots, h_n\} \text{ est un système orthonormé}\}$.

Si H est un espace de Hilbert, $St(n, H)$ est non vide exactement quand $\dim(H) \geq n$. Pour éviter les trivialités, nous allons constamment supposer que cette condition est satisfaite. L'isométrie que nous avons évoqué entre l'espace des repères à valeurs dans \mathbb{F}^n et la variété de Stiefel correspondante est concrètement réalisée par l'application transposée, et nous avons aussi par là même que l'espace des (X, Σ, μ) -repères de Parseval à valeurs dans \mathbb{F}^n est isométrique à $St_o(n, L^2(X, \mu; \mathbb{F}))$. En raison de cela, les résultats relatifs aux variétés de Stiefel que nous obtenons dans cette thèse s'appliquent aussi bien à l'espace des repères continus $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ pour chaque espace mesuré (X, Σ, μ) . Il est facile de voir que $St(n, H)$ est un ouvert non borné de H^n et $St_o(n, H)$ est un fermé borné de H^n . De plus, les propriétés différentielles suivantes sont connues.

Proposition 1.1.1. [46] *On a :*

1. $St(n, \mathbb{R}^k)$ est une variété réelle de dimension nk .
2. $St_o(n, \mathbb{R}^k)$ est une variété réelle de dimension $nk - \frac{n(n+1)}{2}$.
3. $St(n, \mathbb{C}^k)$ est une variété réelle de dimension $2nk$.
4. $St_o(n, \mathbb{C}^k)$ est une variété réelle de dimension $2nk - n^2$.
5. Si $\dim(H) = \infty$, alors $St(n, H)$ et $St_o(n, H)$ sont des variétés hilbertiennes de dimension infinie.

En joignant continûment un élément de $St(n, H)$ au système orthonormé qui lui correspond par le procédé de Gram-Schmidt, on sait aussi que $St_o(n, H)$ est un rétract par déformation de $St(n, H)$. En ce qui concerne les propriétés de connexité, nous avons :

Proposition 1.1.2. (voir pp. 382-383 of [36]) *On a :*

1. $St(n, \mathbb{R}^k)$ est $(k - n - 1)$ -connexe.
2. $St(n, \mathbb{C}^k)$ est $(2k - 2n)$ -connexe.
3. Si H est de dimension infinie, alors $St(n, H)$ est contractible,

où un espace topologique X est dit m -connexe si ses groupes d'homotopie $\pi_i(X, b)$ sont triviaux pour tout $i \in \llbracket 0, m \rrbracket$ et $b \in X$ (bien sûr, $\pi_0(X, b)$ n'est pas un groupe mais seulement un ensemble pointé).

En réalité, on en sait d'avantage, puisque tous les groupes d'homotopie et d'homologie des variétés de Stiefel ont été calculés. Enfin, on sait que $St(n, H)$ est dense dans H^n .

Les variétés de Stiefel sont étudiées en topologie différentielle et constituent un exemple fondamental dans ce domaine. Si la théorie des variétés de Stiefel de dimension finie est bien connue [46, 36, 42], celle des variétés de Stiefel de dimension infinie l'est beaucoup moins. D'un autre côté, plusieurs études topologiques et géométriques ont été réalisées directement sur l'espaces des repères. Des propriétés de connexité de quelques sous espaces importants de l'espace des repères $\mathcal{F}_{k,n}^{\mathbb{F}}$ ont été étudiées dans [16, 53]. Des propriétés différentielles et algébro-géométriques de ces sous-espaces ont été étudiées dans [27, 60, 59, 61] et dans le chapitre 4 de [19] respectivement. Une structure de fibré relativement aux normes L^1 et L^∞ fut établie pour les repères continus dans [1, 2]. Une notion de densité pour les repères généraux analogue à la densité de Beurling fut introduite et étudiée dans [8]. Finalement, des propriétés de connexité et de densité furent étudiées pour les repères de Gabor [9, 21, 37, 51] et d'ondelettes [12, 32, 34, 57].

Zéros de fonctions sur les variétés de Stiefel

Si H est un espace de Hilbert, $n \in \mathbb{N}^*$, E est un espace vectoriel et $f : H^n \rightarrow E$ est une fonction, une question importante est de savoir s'il existe des zéros de f appartenant à $St(n, H) \subset H^n$ ou $St_o(n, H) \subset H^n$, notamment lorsque f appartient à une classe de fonctions bien choisie. Cette question est d'un grand intérêt car les variétés de Stiefel sont des espaces de configuration intervenant dans de nombreux problèmes concrets, et on est amené dans ceux-ci à considérer la question de l'existence d'une configuration vérifiant une propriété $f(C) = 0$. D'après notre revue de la littérature, la question la plus souvent considérée est formulée plutôt comme un problème d'optimisation, concernant l'existence et la caractérisation des configurations optimales $C^* := \operatorname{argmin}_{C \in St(n, H) \text{ ou } St_o(n, H)} f(C)$ où f est une fonction positive et vérifiant diverses autres conditions selon les articles considérés [54, 65, 67] (sont des exemples d'articles récents traitant cette question).

1.1.2 Résultats obtenus

Connexité par arcs des intersections de translatés d'une variété de Stiefel non-compacte

Partant du fait que les propriétés de connexité des variétés de Stiefel (orthonormées ou non-compactes) sont connues depuis longtemps, nous nous sommes intéressés aux propriétés de connexité de quelques espaces similaires à celles-ci. Dans la littérature, nous avons observé que l'étude des propriétés de connexité

des espaces similaires semblait se concentrer sur certains sous-espaces importants des variétés de Stiefel de dimension finie (de manière équivalente, de l'espace des repères $\mathcal{F}_{k,n}^{\mathbb{F}}$) [16, 53]. Nous avons alors voulu sonder un espace, l'intersection de translatés d'une variété de Stiefel non-compacte, qui n'est pas inclus dans une variété de Stiefel mais dont la définition est naturelle dans la mesure où une variété de Stiefel peut être translatée dans son espace vectoriel sous-jacent. Nous avons dans ce cadre prouvé le résultat suivant :

Proposition 1.1.3. *Soit H un espace de Hilbert avec $\dim(H) \geq n$, J un ensemble d'indices et $(U(j))_{j \in J}$ une famille avec $U(j) \in H^n$ pour tout $j \in J$. Si $\text{codim}_H(\text{span}(\{u(j)_k : j \in J, k \in \llbracket 1, n \rrbracket\})) \geq 3n$, alors $\bigcap_{j \in J}(U(j) + \text{St}(n, H))$ est connexe par chemins polygonaux.*

Dans la preuve de cette proposition, la condition sur la codimension de $\text{span}(\{u(j)_k : j \in J, k \in \llbracket 1, n \rrbracket\})$ nous permet de garantir l'existence d'un point intermédiaire important du chemin reliant deux points X et Y de $\bigcap_{j \in J}(U(j) + \text{St}(n, H))$ que l'on cherche à construire. D'un autre côté, les arguments que nous utilisons ne permettent pas d'affirmer la connexité de $\bigcap_{j \in J}(U(j) + \text{St}_o(n, H))$ sous une condition similaire de codimension.

Adhérence de l'intersection d'un ensemble avec une variété de Stiefel non-compacte

Nous nous sommes demandés s'il est possible de généraliser le résultat connu de densité de la variété de Stiefel non-compacte dans H^n . En analysant sa preuve classique, nous avons constaté que l'idée clé qui est de considérer un chemin droit entre un élément de H^n et un élément quelconque de $\text{St}(n, H)$ et d'examiner les zéros de son déterminant Gramien (qui est alors polynomial en $t \in [0, 1]$), pouvait être généralisée en considérant plutôt un chemin polynomial dans H^n . Le déterminant Gramien d'un tel chemin est encore une fonction polynomiale en $t \in \mathbb{R}$, et le bénéfice de cette généralisation est qu'il est alors possible d'obtenir un résultat de densité relative de $\text{St}(n, H)$ dans des parties S de H^n , qui ne sont pas connexes par des chemins droits, mais le sont par des chemins polynomiaux. Nous obtenons donc le résultat suivant

Corollaire 1.1.1. *Soit $S \subseteq H^n$ tel que pour tout $U \in S$ il existe un chemin polynomial à reparamétrisation près joignant U à un $\Theta(U) \in \text{St}(n, H) \cap S$ et contenu dans S . Alors $\text{St}(n, H) \cap S$ est dense dans S .*

qui est un corollaire de notre résultat suivant, plus général et fondamental

Proposition 1.1.4. *Soit $S \subseteq H^n$ et*

$E := \{\gamma : [0, 1] \rightarrow S \text{ tel que } \gamma \text{ soit un chemin polynomial à reparamétrisation près et}$
 $\exists a_\gamma \in [0, 1] : \gamma(a_\gamma) \in St(n, H) \cap S\}.$

Alors $\bigcup_{\gamma \in E} \text{Range}(\gamma) \subseteq \overline{St(n, H) \cap S}$

Il en résulte que les n -uplets indépendants sont abondants non seulement dans H^n mais aussi dans plusieurs sous-ensembles S de H^n qui vérifient la condition du corollaire. Par exemple, la condition est satisfaite si S est une partie étoilée de H^n relativement à un $\Theta \in St(n, H) \cap S$.

Il est important de noter que pour qu'un sous-ensemble non vide S de H^n vérifie la condition du corollaire, il devrait exister au moins un élément $\Theta \in St(n, H) \cap S$. Cela nous a motivé à rechercher des sous-ensembles S de H^n qui contiennent un élément de $St(n, H)$, en supposant de plus que S est un ensemble "algébrique", formé par les solutions d'une équation $f(\mathbf{v}) = 0$, et que f est une fonction affine ou quadratique.

Zéros de fonctions affines et quadratiques sur une variété de Stiefel non-compacte

Si V et W sont deux espaces vectoriels, nous entendons par fonction affine toute fonction $u : V \rightarrow W$ du type $u(\theta) := A\theta + b$, où $A \in B(V; W)$ et $b \in W$. Par fonction quadratique, nous entendons une fonction $u : V \rightarrow W$ du type $u(\theta) := [\theta, \theta] + A\theta + b$, où $[\cdot, \cdot] : V \times V \rightarrow W$ est une application sesquilinéaire continue, $A \in B(V; W)$ et $b \in W$.

Nous démontrons tout d'abord en utilisant des arguments d'indépendance linéaire et de dimension que :

Proposition 1.1.5. *Soit $n \in \mathbb{N}^*$, V un \mathbb{F} -espace vectoriel de dimension $\geq n$ et $T : V^n \rightarrow \mathbb{F}$ une forme linéaire non nulle. Alors pour tout $d \neq 0$, il existe $(a_1, \dots, a_n) \in St(n, V)$ tel que $T(a_1, \dots, a_n) - d = 0$.*

Proposition 1.1.6. *Soit $n \in \mathbb{N}^*$, V un \mathbb{F} -espace vectoriel et $S : V \rightarrow \mathbb{F}$ une forme linéaire non nulle.*

On définit l'opérateur

$$T : \begin{cases} V^n & \rightarrow \mathbb{F}^n \\ (x_1, \dots, x_n) & \mapsto (S(x_1), \dots, S(x_n)) \end{cases}.$$

Pour tout $d \in \mathbb{F}^n$, supposons que

- si $d \neq 0$, alors $\dim(V) \geq n$,*

— si $d = 0$ alors $\dim(V) \geq n + 1$.

Alors il existe $(a_1, \dots, a_n) \in St(n, V)$ tel que $T(a_1, \dots, a_n) - d = 0$.

Dans la proposition précédente, les conditions sur la dimension de V sont nécessaires.

Dans les propositions qui suivent, nous utilisons un autre argument pour construire un zéro d'une fonction f sur une variété de Stiefel non-compacte $St(n, H)$, qui est de partir d'un élément de $St(n, K)$ où K est strictement contenu dans H , et de l'étendre à $St(n, H)$ de sorte à ce qu'il soit un zéro de la fonction f . De plus, nous nous concentrons sur le cas $H = L^2(X, \mu; \mathbb{F})$ et $K = L^2(Y, \mu; \mathbb{F})$ avec $Y \subset X$ convenable.

Proposition 1.1.7. Soit (X, Σ, μ) un espace mesuré, $H := L^2(X, \mu; \mathbb{F})$ et $h \in H$. On définit l'opérateur

$$T : \begin{cases} H^n & \rightarrow \mathbb{F}^n \\ (f_1, \dots, f_n) & \mapsto (\langle f_1, h \rangle, \dots, \langle f_n, h \rangle) \end{cases}.$$

S'il existe un sous-ensemble mesurable $Y \subset X$ tel que $\dim(L^2(Y, \mu; \mathbb{F})) \geq n$ et $\mu((X \setminus Y) \cap h^{-1}(\mathbb{F}^*)) > 0$, alors pour tout $d \in \mathbb{F}^n$, il existe $(f_1, \dots, f_n) \in St(n, H)$ tel que $T(f_1, \dots, f_n) - d = 0$.

Nous démontrons aussi

Proposition 1.1.8. Soit (X, Σ, μ) un espace mesuré, $l \in \mathbb{N}^*$, $(X_j)_{j \in [1, l]}$ une partition de X par des sous-ensembles mesurables, $H := L^2(X, \mu; \mathbb{F})$ et $h \in H$ tel qu'il existe une famille $(Y_j)_{j \in [1, l]}$ avec Y_j un sous-ensemble mesurable de X_j pour tout $j \in [1, l]$, $\mu((X_j \setminus Y_j) \cap h^{-1}(\mathbb{F}^*)) > 0$ pour tout $j \in [1, l]$, et $\sum_{j=1}^l \dim(L^2(Y_j, \mu; \mathbb{F})) \geq n$.

On définit l'opérateur

$$W : \begin{cases} H^n & \rightarrow \prod_{j \in [1, l]} \mathbb{F}^n \\ (f_1, \dots, f_n) & \mapsto ((\langle f_1, h.1_{X_j} \rangle, \dots, \langle f_n, h.1_{X_j} \rangle))_{j \in [1, l]} \end{cases}.$$

Alors pour tout $D = (d_j)_{j \in [1, l]} \in \prod_{j \in [1, l]} \mathbb{F}^n$, il existe $(f_1, \dots, f_n) \in St(n, H)$ tel que $W(f_1, \dots, f_n) - D = 0$.

Cette proposition peut être étendue à $l = +\infty$ ou à des partitions indexées par un ensemble quelconque J si nous nous limitons à $D = 0$ (à cause d'une difficulté de convergence). Notons que l'avant dernière proposition résulte de cette dernière en prenant $l = 1$.

Dans la proposition qui suit, nous considérons le cas d'une fonction quadratique. On a

Proposition 1.1.9. Soit (X, Σ, μ) un espace mesuré σ -fini, $b \in \mathbb{C}^n \setminus \{0\}$, $\epsilon > 0$ et $h \in L^\infty(X, \mu; \mathbb{C})$.

Considérons

$$q : \begin{cases} L^2(X, \mu; \mathbb{C}^n) & \rightarrow \mathbb{C}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) \langle b, f_x \rangle f_x d\mu(x) \end{cases}.$$

Alors pour tout $d \in \mathbb{C}^n$ tel que

- si $d \neq 0$, alors il existe un sous-ensemble mesurable $Y \subseteq X$, deux sous-ensembles mesurables $B_1 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(\langle b, d \rangle z) > \epsilon \text{ et } \operatorname{Im}(\langle b, d \rangle z) < -\epsilon\}$ et $B_2 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(\langle b, d \rangle z) > \epsilon \text{ et } \operatorname{Im}(\langle b, d \rangle z) > \epsilon\}$ tel que $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$ et $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$,
- si $d = 0$, alors il existe un sous-ensemble mesurable $Y \subseteq X$ tel que $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$ et $h(x) < 0$ μ -presque partout sur Y , et [(deux sous-ensembles mesurables $B_1 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ et } \operatorname{Im}(z) < 0\}$ et $B_2 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ et } \operatorname{Im}(z) > 0\}$ tel que $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$) ou (un sous-ensemble mesurable $B_3 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ et } \operatorname{Im}(z) = 0\}$ tel que $\mu((X \setminus Y) \cap h^{-1}(B_3)) > 0$)],

il existe $(f^{(1)}, \dots, f^{(n)}) \in St(n, H)$ tel que $q(f^{(1)}, \dots, f^{(n)}) - d = 0$ (où $H := L^2(X, \mu; \mathbb{C})$).

Exemples de sous-espaces relativement denses de l'espace des repères continus de dimension finie

Au vu du résultat de densité relative des variétés de Stiefel, et des résultats d'existence de zéros de fonctions affines et quadratiques sur une variété de Stiefel, nous avons pu proposer des exemples de parties S de $L^2(X, \mu; \mathbb{F}^n)$ dans lesquelles l'espace des repères $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ est relativement dense, dans le sens que $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}} \cap S$ est dense dans S .

Dans les trois corollaires suivants, les parties S sont de la forme $\mathbf{T}^{-1}(\{d\})$ où \mathbf{T} est un opérateur, et sont donc affines.

Corollaire 1.1.2. Soit $n \in \mathbb{N}^*$ et $T : L^2(X, \mu; \mathbb{F}^n) \rightarrow \mathbb{F}$ une forme linéaire non nulle.

Supposons que $\dim(L^2(X, \mu; \mathbb{F})) \geq n$.

Alors pour tout $d \neq 0$, $T^{-1}(\{d\})$ est relativement dense dans $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$.

Corollaire 1.1.3. Soit (X, Σ, μ) un espace mesuré, $d \in \mathbb{C}^n$, et $h \in L^2(X, \mu; \mathbb{F})$ tel que $T^{-1}(\{d\})$ contienne un repère continu $\Phi = (\varphi_x)_{x \in X}$ (voir par exemple le corollaire 2.5.2 et la proposition 2.5.3), où

$$T : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow \mathbb{F}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x) \end{cases}.$$

Alors $T^{-1}(\{d\})$ est relativement dense dans $\mathcal{F}_{(X,\Sigma,\mu),n}^{\mathbb{F}}$.

Corollaire 1.1.4. Soit (X, Σ, μ) un espace mesuré, $l \in \mathbb{N}^*$, $(X_j)_{j \in [1,l]}$ une partition de X par des sous-ensembles mesurables, $h \in L^2(X, \mu; \mathbb{F})$, et $D \in \mathbb{F}^n$ tel que $W^{-1}(\{D\})$ contienne un repère continu $\Phi = (\varphi_x)_{x \in X}$ (voir par exemple la proposition 2.5.4), où

$$W : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow \prod_{j \in [1,l]} \mathbb{F}^n \\ F = (f_x)_{x \in X} & \mapsto (\int_{X_j} h(x) f_x)_{j \in [1,l]} \end{cases}.$$

Alors $W^{-1}(\{d\})$ est relativement dense dans $\mathcal{F}_{(X,\Sigma,\mu),n}^{\mathbb{F}}$.

Dans le corollaire suivant, on peut montrer que la partie $S := q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ est une partie étoilée relativement à Φ .

Corollaire 1.1.5. Soit (X, Σ, μ) un espace mesuré σ -fini, $b \in \mathbb{C}^n \setminus \{0\}$, et $h \in L^\infty(X, \mu; \mathbb{C})$ tel que $q^{-1}(\{0\})$ contienne un repère continu $\Phi = (\varphi_x)_{x \in X}$ (voir par exemple la proposition 2.5.5), où

$$q : \begin{cases} L^2(X, \mu; \mathbb{C}^n) & \rightarrow \mathbb{C}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) \langle b, f_x \rangle f_x d\mu(x) \end{cases}.$$

Alors $q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ est relativement dense dans $\mathcal{F}_{(X,\Sigma,\mu),n}^{\mathbb{F}}$.

1.2 Indépendance et dimension

1.2.1 Littérature

Nous nous intéressons dans cette deuxième partie de la thèse aux notions d'**indépendance** et de **dimension**.

Indépendance linéaire et dimension

L'indépendance linéaire et la dimension sont des concepts bien connus et centraux de l'algèbre linéaire. Plusieurs conjectures sur l'indépendance linéaire de certaines familles de vecteurs sont ouvertes en mathématiques, l'une des plus proches peut-être de notre sujet de thèse étant la conjecture HRT [38] sur les translatées d'une fonction $g \in L^2(\mathbb{R})$ dans le domaine temps-fréquence. Les approches utilisées pour traiter ces problèmes diffèrent très largement.

Parmi les approches que nous avons rencontrées, une semblait faire exception en cela qu'elle établissait une implication entre la dimension infinie d'une suite infinie de vecteurs d'un espace vectoriel et son indépendance linéaire. Dans cette

proposition démontrée dans [22], les auteurs ont prouvé qu'il était suffisant de supposer l'existence d'un opérateur T qui décale les vecteurs de la suite, dans le sens que $\forall n \in \mathbb{N} : T(e_n) = e_{n+1}$.

σ -structure en algèbre universelle et théorie des modèles

Pour aborder les notions d'indépendance et de dimension sous un autre aspect, nous utilisons dans notre travail la notion de **σ -structure**. Rappelons donc qu'en algèbre universelle et en théorie des modèles, une σ -structure [40] consiste en un ensemble muni d'une collection d'opérations et de relations d'arité finie définies sur cet ensemble, et qui sont indiquées par la signature σ . Si la σ -structure ne contient pas de relations, on dit qu'elle est **algébrique**, et des exemples en sont les groupes, les anneaux, les corps et les espaces vectoriels.

La notion de σ -structure permet la formulation générale des concepts de sous-structures et d'homomorphismes entre σ -structures. De plus, du point de vue de la théorie des modèles, les σ -structures sont les objets utilisés pour définir la sémantique de la logique du premier ordre.

Théorie de l'ordre

La théorie de l'ordre [24] est une branche des mathématiques qui étudie la notion intuitive d'ordre en utilisant les relations binaires. La notion d'ordre est très générale et est utilisée partout en mathématiques.

Rappelons quelques définitions nécessaires à notre travail.

Définition 1.2.1. Considérons un ensemble P et une relation binaire \leq sur P . Alors \leq est un **préordre** s'il est réflexif et transitif; i.e., pour tout a, b et c dans P , on a :

- $a \leq a$ (réflexivité)
- si $a \leq b$ et $b \leq c$ alors $a \leq c$ (transitivité)

Un ensemble muni d'un préordre est appelé un **ensemble préordonné**.

Définition 1.2.2. Considérons un ensemble préordonné (P, \leq) et une application $p : P \rightarrow P$. Alors p est appelée une **projection** si pour tous a et b dans P , on a :

- $a \leq b$ implique $p(a) \leq p(b)$ (croissance)
- $p(p(a)) = p(a)$ (idempotence)

Définition 1.2.3. Considérons un ensemble P et une relation binaire \leq sur P . Alors \leq est un **ordre partiel** si c'est un préordre et pour tous a et b dans P , on a : $a \leq b \leq a$ implique $a = b$ (antisymétrie).

Un ensemble muni d'un ordre partiel est appelé un **ensemble partiellement ordonné**.

Définition 1.2.4. Un ensemble S partiellement ordonné par la relation binaire \leq est un **semi-treillis** si pour tous éléments x et y de S , la plus petite borne supérieure de l'ensemble $\{x, y\}$ existe dans S .

La plus petite borne supérieure de l'ensemble $\{x, y\}$ est appelée la **jointure** de x et y , notée $x \vee y$.

1.2.2 Résultats obtenus

Dans une σ -structure \mathcal{A} , " $\dim(e_n)_{n=0}^\infty = \infty(D\infty) \wedge (\exists f \in \text{End}(\mathcal{A}) : \forall n \in \mathbb{N} : T(e_n) = e_{n+1}) \Rightarrow$ indépendance de $(e_n)_{n=0}^\infty$ "

Considérons une σ -structure algébrique $\mathcal{A} = (A, \sigma)$. $(\mathcal{P}(A), \subseteq, \cup, \cap, \emptyset, A)$ est alors une algèbre de Boole et donc un semi-treillis $(\mathcal{P}(A), \subseteq, \cup)$. Un résultat sur les semi-treillis que nous prouvons nous permet alors de démontrer que

Proposition 1.2.1. *Soit $\mathcal{A} = (A, \sigma)$ une σ -structure algébrique.*

Soit $(e_n)_{n \in \mathbb{N}}$ une suite d'éléments de A .

Supposons qu'il existe $f \in \text{End}_\sigma(A)$ telle que $\forall n \in \mathbb{N} : f(e_n) = e_{n+1}$.

Alors $(\exists m \in \mathbb{N} : e_m \in \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}]) \Rightarrow (\exists m \in \mathbb{N} : \forall n \in \mathbb{N} : e_{m+n} \in \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}])$.

ou encore sa contraposée

Corollaire 1.2.1. *Soit $\mathcal{A} = (A, \sigma)$ une σ -structure algébrique.*

Soit $(e_n)_{n \in \mathbb{N}}$ une suite d'éléments de A .

Supposons qu'il existe $f \in \text{End}_\sigma(A)$ telle que $\forall n \in \mathbb{N} : f(e_n) = e_{n+1}$.

Alors $(\forall m \in \mathbb{N} : \exists \varphi(m) \geq m : e_{\varphi(m)} \notin \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}]) \Rightarrow (\forall m \in \mathbb{N} : e_m \notin \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}])$.

En effet, si $(e_n)_{n \in \mathbb{N}}$ est une suite d'éléments de A et qu'un endomorphisme de A décalant les éléments de la suite $(e_n)_{n \in \mathbb{N}}$ existe, on peut définir une projection p sur $\mathcal{P}(A)$ qui envoie $X \in \mathcal{P}(A)$ sur $\text{Terms}^{\mathcal{A}}[X]$, l'ensemble des termes de la σ -structure \mathcal{A} dont les variables appartiennent à X , poser $\forall n \in \mathbb{N} : b_n = \{e_0, \dots, e_{n-1}\}$, remarquer que l'endomorphisme f induit une application image directe par f sur $\mathcal{P}(A)$, notée f_\bullet , qui est croissante et qui commute avec p , et que $(e_n)_{n \in \mathbb{N}}$ induit une suite de singletons $(\{e_n\})_{n \in \mathbb{N}}$ dans $\mathcal{P}(A)$ qui sont de même décalés par f_\bullet . Il ne reste plus qu'à appliquer le dernier lemme sur les semi-treillis dans la suite de lemmes suivants (que nous démontrons dans notre travail).

Lemme 1.2.1. *Soit (P, \leq) un ensemble préordonné, $(a_n)_{n \in \mathbb{N}}$ une suite dans P , $b \in P$ et p une projection sur P .*

Supposons qu'il existe une application croissante $f : P \rightarrow P$ telle que $f(p(b)) \leq p(f(b))$, $\forall n \in \mathbb{N} : a_{n+1} \leq f(a_n)$, et $f(b) \leq a_0$.

Alors $(a_0 \leq p(b)) \Rightarrow (\forall n \in \mathbb{N} : a_n \leq p(b))$.

Lemme 1.2.2. Soit (S, \leq, \vee) un semi-treillis, p une projection sur S , $(a_{m,n})_{(m,n) \in \mathbb{N}^2}$ une suite double dans S , $(b_m)_{m \in \mathbb{N}}$ une suite croissante dans S telles que $\forall m \in \mathbb{N} : a_{m,0} \leq p(b_{m+1})$.

Supposons qu'il existe $m_\bullet \in \mathbb{N}$ et une application croissante $f : S \rightarrow S$ tel que $f(p(b_{m_\bullet})) \leq p(f(b_{m_\bullet}))$, $\forall (m,n) \in \mathbb{N}^2 : a_{m,n+1} \leq f(a_{m,n})$, et $f(b_{m_\bullet}) \leq \bigvee_{i=0}^{m_\bullet} a_{i,0}$.

Alors $(a_{m_\bullet,0} \leq p(b_{m_\bullet})) \Rightarrow (\forall n \in \mathbb{N} : a_{m_\bullet,n} \leq p(b_{m_\bullet}))$.

Lemme 1.2.3. Soit (S, \leq, \vee) un semi-treillis, p une projection sur S , $(e_n)_{n \in \mathbb{N}}$ et $(b_n)_{n \in \mathbb{N}}$ deux suites dans S telles que $(b_n)_{n \in \mathbb{N}}$ soit croissante et $\forall n \in \mathbb{N} : e_n \leq p(b_{n+1})$.

Supposons qu'il existe $m_\bullet \in \mathbb{N}$ et une application croissante $f : S \rightarrow S$ tel que $f(p(b_{m_\bullet})) \leq p(f(b_{m_\bullet}))$, $\forall n \in \mathbb{N} : e_{n+1} \leq f(e_n)$, et $f(b_{m_\bullet}) \leq \bigvee_{i=0}^{m_\bullet} e_i$.

Alors $(e_{m_\bullet} \leq p(b_{m_\bullet})) \Rightarrow (\forall n \in \mathbb{N} : e_{m_\bullet+n} \leq p(b_{m_\bullet}))$.

S'il est possible, dans une σ -structure (A, σ) , de définir une notion de dimension infinie d'une suite infinie $(e_n)_{n \in \mathbb{N}}$ d'éléments de A par « $\forall m \in \mathbb{N} : \exists \varphi(m) \geq m : e_{\varphi(m)} \notin \text{Terms}^A[e_0, \dots, e_{m-1}]$ », et une notion d'indépendance de $(e_n)_{n \in \mathbb{N}}$ par « $\forall m \in \mathbb{N} : e_m \notin \text{Terms}^A[e_0, \dots, e_{m-1}]$ », alors le corollaire 1.2.1 établit une implication entre ces conditions sous l'hypothèse première qu'il existe un endomorphisme de (A, σ) décalant les éléments de $(e_n)_{n \in \mathbb{N}}$. En considérant le cas d'un espace vectoriel sur un corps, avec sa signature classique décrite par exemple dans ([40] pp. 3-4), alors on retrouve le résultat de [22] auquel on a fait référence dans la sous-sous-section «Indépendance linéaire et dimension».

Dans un espace vectoriel E , « $(D_\infty) \wedge (\exists T \in \text{End}(E) : \forall n \in \mathbb{N} : T(e_n) = e_{\phi(n)}) \Rightarrow$ indépendance linéaire de $(e_n)_{n=0}^\infty$ » ssi ϕ est conjuguée à $n \mapsto n+1$

Nous nous plaçons dorénavant dans un espace vectoriel E pour retrouver plus de souplesse. À la question de savoir si la condition « $\forall n \in \mathbb{N} : T(e_n) = e_{n+1}$ » peut être affaiblie en remplaçant la fonction $\text{succ} : n \mapsto n+1$ par une fonction $\phi : I \rightarrow I$ (avec I dénombrable), nous avons répondu par la proposition suivante :

Proposition 1.2.2. Soit E un espace vectoriel de dimension infinie et I un ensemble infini dénombrable.

Pour tout $(e_i)_{i \in I} \in E^I$ et $\phi : I \rightarrow I$, soit $P((e_i)_{i \in I}, \phi)$ la proposition

$$\begin{aligned} (\exists T \in L(\text{span}(e_i)_{i \in I}, E) : \forall i \in I : T(e_i) = e_{\phi(i)} \text{ et } \dim \text{span}(e_i)_{i \in I} = +\infty) \\ \Rightarrow (e_i)_{i \in I} \text{ est libre.} \end{aligned}$$

Alors nous avons

$$\forall \phi : I \rightarrow I : \left[(\forall (e_i)_{i \in I} \in E^I : P((e_i)_{i \in I}, \phi)) \Leftrightarrow \exists a \in I : \text{Orb}_\phi(a) = I \right]$$

Pour cela, nous démontrons que toutes les orbites sont cofinies, puis nous raisonnons sur une orbite de complémentaire minimal pour montrer qu'elle couvre l'ensemble I tout entier. En utilisant par ailleurs le fait bien connu que tout fonction $\phi : I \rightarrow I$ vérifiant $\exists a \in I : \text{Orb}_\phi(a) = I$ est conjuguée à la fonction succ définie sur \mathbb{N} , on en conclut que le résultat de [22] auquel on a fait référence dans la sous-section «*Indépendance linéaire et dimension*» ne peut être généralisé davantage en remplaçant la fonction succ par d'autres fonctions $\phi : I \rightarrow I$ fondamentalement différentes de celle-ci (avec I dénombrable), et en supposant $\dim \text{span}(e_i)_{i \in I} = +\infty$ (et non pas par exemple $\dim \text{span}(e_i)_{i \in J} = +\infty$ pour J strictement inclus dans I).

Dans un espace vectoriel E , " $(D_\infty) \wedge$ condition généralisée \Rightarrow indépendance linéaire de $(e_n)_{n=0}^\infty$ "

Toujours dans un espace vectoriel E , nous généralisons maintenant vraiment la condition « $\forall n \in \mathbb{N} : T(e_n) = e_{n+1}$ » en supposant une condition plus forte que « $\dim \text{span}(e_i)_{i \in I} = +\infty$ ». Nous employons les opérateurs de manière plus élaborée pour généraliser la première condition et durcir la seconde, ce qui nous coûte l'introduction d'hypothèses supplémentaires impliquant des fonctions ensemblistes et une relation binaire. La proposition obtenue est la suivante :

Proposition 1.2.3. *Soit E un espace vectoriel de dimension infinie, V un sous-espace vectoriel de dimension finie de E , $(e_i)_{i \in I}$ une famille de vecteurs de E , et J un ensemble infini. Supposons que :*

1. *Il existe deux fonctions $u : \mathcal{P}_{\omega,*}(I) \rightarrow I$ et $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ telles que pour tout $I^* \in \mathcal{P}_{\omega,*}(I)$, $u(I^*) \in I^*$ et $I^* \subseteq G(I^*)$,*
2. *Il existe une fonction $T : \mathcal{P}_{\omega,*}(I) \times J \rightarrow L(*, E)$ telle que pour tout $(I^*, j) \in \mathcal{P}_{\omega,*}(I) \times J$, $T(I^*, j) \in L(\text{span}(e_i)_{i \in G(I^*)}, E)$,*
3. *Il existe un sous-ensemble fini J_0 de J et une relation binaire R sur J tel que $R[j] \subseteq J_0$ pour tout $j \in J_0$ et $(\forall j \in J)(\exists n_j \in \mathbb{N})R^{n_j}[j] \subseteq J_0$,*
4. *$\forall i \in I : \dim \text{span}\{T(I^*, j)e_i\}_{j \in J} = \infty$,*
5. *$\forall (I^*, j) \in \mathcal{P}_{\omega,*}(I) \times J : \forall i \in G(I^*) \setminus \{u(I^*)\} :$*

$$T(I^*, j)e_i \in \text{span}\{T(I^*, j')e_{i'}\}_{(j', i') \in R[j] \times G(I^*)} + V.$$

Alors la famille $(e_i)_{i \in I}$ est libre.

Ici, $\mathcal{P}_{\omega,*}(I)$ désigne l'ensemble des parties finies non vides de I , $L(*, E)$ la classe des opérateurs définis sur un espace vectoriel variable et à valeurs dans E et R^n la composition n -ème de la relation R avec elle-même (voir [56] pp. 12-13 pour ce dernier point).

Nous avons prouvé cette proposition par l'absurde en supposant que la famille

$(e_i)_{i \in I}$ est linéairement dépendante et en montrant par récurrence que la relation d'appartenance du point 5 s'étend à la relation binaire R^n , pour tout $n \in \mathbb{N}$, pour enfin arriver à la contradiction que $\dim \text{span}\{T(I^*, j)e_{u(I^*)}\}_{j \in J} < \infty$ (on utilise pour cela toutes les hypothèses de finitude qu'on a faites), ce qui contredit le point 4.

Un exemple d'application de cette proposition serait le cas d'une suite de vecteurs $(e_n)_{n \in \mathbb{Z}}$ indexée par \mathbb{Z} d'un espace vectoriel E , décalée par un opérateur T sur \mathbb{Z} tout entier, et vérifiant $\dim \text{span}\{e_n\}_{n \geq k} = \infty$ pour un et donc pour tout $k \in \mathbb{Z}$. On peut dans ce cas prendre $J = \mathbb{N}$, $J_0 = \{0\}$, $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{Z}) : u(I^*) = \max(I^*)$ et $G(I^*) = [\min(I^*), \max(I^*)]$, $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{Z}) : \forall j \in \mathbb{N} : T(I^*, j) = T^j$ et $R = (0, 0) \cup (\cup_{m \in \mathbb{N}^*}(m, m-1)) \subseteq \mathbb{N} \times \mathbb{N}$. Donc $(e_n)_{n \in \mathbb{Z}}$ est linéairement indépendante.

1.3 Fonctions et ensembles quasi-invariants

1.3.1 Littérature

Nous nous intéressons dans cette troisième partie de la thèse à la notion d'**ensemble quasi-invariant** pour une fonction. Cette notion apparait dans la théorie des systèmes dynamiques pour une fonction mesurable [29, 25] et pour un flot [30], ainsi qu'en dynamique des groupes pour une action de groupe [13, 55, 14]. Elle est une extension de la notion plus fondamentale d'ensemble invariant [48, 15], qui permet notamment de décrire les caractéristiques conservées de phénomènes naturels en évolution. Réduites au cas d'une seule fonction $f : I \rightarrow I$, les définitions d'un ensemble quasi-invariant $A \subseteq I$ fini selon [29] et [13] peuvent s'énoncer grossièrement comme $|f^{-1}(A) \cap A| \approx |A|$, où $|\cdot|$ est la fonction cardinal. Plus précisément, en transposant la définition [13] aux fonctions, un **ensemble k -quasi-invariant** par une fonction devrait être **ou bien**

1. **intérieurement- k -quasi-invariant** : vérifier $\exists P \subseteq I : f(A \setminus P) \subseteq A$ avec $|P| \leq k$, ou de manière équivalente $|f^{-1}(A) \cap A| \geq |A| - k$, **ou bien**
2. **extérieurement- k -quasi-invariant** : vérifier $\exists P \subseteq I : f(A) \subseteq A \cup P$ avec $|P| \leq k$,

ces deux conditions n'étant équivalentes que dans le cas d'une fonction $f : I \rightarrow I$ bijective.

Dans la dernière partie de notre travail, nous avons étudié les fonctions possédant un nombre maximal d'ensembles ou de sur-ensembles invariants ou intérieurement-1-quasi-invariants (un analogue de cette question dans le cas des actions de groupes a été étudié dans [55]). La motivation à notre recherche était la caractérisation des fonctions $\phi : I \rightarrow I$ d'orbites infinies tel qu'il existe deux fonctions $G : \mathcal{P}_{\omega,*}(I) \rightarrow$

$\mathcal{P}_{\omega,*}(I)$ et $u : \mathcal{P}_{\omega,*}(I) \rightarrow I$ satisfaisant

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in I^*, I^* \subseteq G(I^*) \text{ et } \phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*),$$

en vue de construire de nouveaux exemples à notre proposition 3.6.1 du deuxième article. Nous avons correctement caractérisé ces fonctions grâce à la proposition 4.7.5, que nous allons expliquer dans la prochaine sous-section.

Notons que la notion de fonction possédant un nombre maximal d'ensembles invariants ou quasi-invariants peut être vue comme l'antipode de la notion d'ergodicité.

1.3.2 Résultats obtenus

Fonctions avec un nombre maximal d'ensembles invariants finis

Notre première proposition dans cette troisième partie de la thèse ne fait que rappeler que la fonction identité $id : I \rightarrow I$ est la seule fonction dont toutes les parties finies sont invariantes (il suffit de considérer tous les singletons de I).

Fonctions avec un nombre maximal d'ensembles intérieurement-1-quasi-invariants finis

La proposition suivante décrit les fonctions admettant un nombre maximal d'ensembles intérieurement-1-quasi-invariants finis dans le cas général.

Proposition 1.3.1. *Soit I un ensemble tel que $|I| \geq 3$, $\beta : I \rightarrow I$ et $w : \mathcal{P}_{\omega,*}(I) \rightarrow I$ tels que :*

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : w(I^*) \in I^* \text{ et } \beta(I^* \setminus \{w(I^*)\}) \subseteq I^*.$$

Alors soit :

Cas (1).

$$\exists(a, b, c) \in I^3 : (\beta(a) = b \text{ et } \beta(b) = c) \text{ et } (\forall x \in I \setminus \{a, b\} : \beta(x) = x)$$

et

$$w = \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto b \text{ si } \{a, b\} \subseteq I^* \\ I^* & \mapsto a \text{ si } a \in I^* \text{ et } b \notin I^* \\ I^* & \mapsto b \text{ si } a \notin I^* \text{ et } b \in I^* \\ I^* & \mapsto n'importe \text{ quel } \text{élément de } I^* \text{ sinon} \end{cases}$$

ou

Cas (2).

$$\exists(a, b, c) \in I^3 \text{ avec } a \neq b, b \neq c, c \neq a, \beta(a) = b, \beta(b) = c, \beta(c) = a$$

et

$$\forall x \in I \setminus \{a, b, c\} : \beta(x) = x$$

et

$$w = \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto a \text{ si } a \in I^* \text{ et } b, c \notin I^* \\ I^* & \mapsto b \text{ si } b \in I^* \text{ et } a, c \notin I^* \\ I^* & \mapsto c \text{ si } c \in I^* \text{ et } a, b \notin I^* \\ I^* & \mapsto b \text{ si } \{a, b\} \subseteq I^* \text{ et } c \notin I^* \\ I^* & \mapsto a \text{ si } \{a, c\} \subseteq I^* \text{ et } b \notin I^* \\ I^* & \mapsto c \text{ si } \{b, c\} \subseteq I^* \text{ et } a \notin I^* \\ I^* & \mapsto \text{n'importe quel \u00e9l\u00e9ment de } I^* \text{ sinon} \end{cases} .$$

Nous prouvons la partie de cette proposition concernant la description de β en choisissant pour I^* diverses parties de I de cardinal 2 et en s\u00e9parant diff\u00e9rents cas. Nous en d\u00e9duisons apr\u00e8s assez facilement une description de la fonction w en observant de nouveau la condition $\forall I^* \in \mathcal{P}_{\omega,*}(I) : w(I^*) \in I^*$ et $\beta(I^* \setminus \{w(I^*)\}) \subseteq I^*$. Notons que nous ne pouvons pas demander que $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \beta(w(I^*)) \notin I^*$ puisque ces fonctions β poss\u00e8dent plusieurs points fixes (sauf si $|I| = 3$).

Nous nous pla\u00e7ons maintenant dans \mathbb{N} et nous consid\u00e9rons uniquement la pr\u00e9servation des intervalles \u00e0 un \u00e9l\u00e9ment pr\u00e8s. La prochaine proposition montre qu'il y a d'autres fonctions $\beta : \mathbb{N} \rightarrow \mathbb{N}$ telles qu'il existe une fonction $w : \text{Int}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ telle que pour tout $(a, b) \in \mathbb{N}^2$ avec $a \leq b$, $w(\llbracket a, b \rrbracket) \in \llbracket a, b \rrbracket$ et $\beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket$ (pas seulement la fonction successeur $\beta = \text{succ}$ avec $w : \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b \end{cases}$).

Proposition 1.3.2. *Soit $\beta : \mathbb{N} \rightarrow \mathbb{N}$ une fonction telle qu'il existe une autre fonction $w : \text{Int}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ telle que*

$$\forall \llbracket a, b \rrbracket \in \text{Int}_{\omega,*}(\mathbb{N}) : w(\llbracket a, b \rrbracket) \in \llbracket a, b \rrbracket \text{ et } \beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket.$$

Soit $\{b_n\}_{n \in [0, l]}$ une indexation strictement croissante du compl\u00e9mentaire des points fixes de β , o\u00f9 $l \in \mathbb{N} \cup \{-1, \infty\}$. Nous \u00e9tendons $\{b_n\}_{n \in [0, l]}$ et β en posant $b_{-1} = -1$ et $\beta(-1) = 0$, et si $l < \infty$, on pose $b_{l+1} = \infty$. Alors on a soit

Cas (1). $\forall n \in [0, l] : b_n < \beta(b_n) \leq b_{n+1}$ et

$$w = \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ si } \exists ! n \in [0, l] : b_n \in \llbracket a, b \rrbracket \text{ et } \beta(b_n) > b. \\ \llbracket a, b \rrbracket & \mapsto \text{n'importe quel \u00e9l\u00e9ment de } \llbracket a, b \rrbracket \text{ sinon} \end{cases}$$

Cas ②. Il existe $n^* \in \llbracket -1, l-1 \rrbracket$ tel que $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) < b_{n^*+1}$, $\forall m \in \llbracket n^*+2, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$, et

$$w = \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ si } \exists! n \in \llbracket 0, n^* \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ et } \beta(b_n) > b \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ si } \exists! n \in \llbracket n^*+1, l \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ et } \beta(b_n) < a \\ \llbracket a, b \rrbracket & \mapsto n' \text{ importe quel } \text{élément de } \llbracket a, b \rrbracket \text{ sinon} \end{cases}.$$

Cas ③. Il existe $n^* \in \llbracket -1, l-1 \rrbracket$ tel que $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) > b_{n^*+2}$, $\forall m \in \llbracket n^*+2, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$, et

$$w = \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ si } \exists! n \in \llbracket 0, n^* \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ et } \beta(b_n) > b \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ si } \exists! n \in \llbracket n^*+1, l \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ et } \beta(b_n) < a \\ \llbracket a, b \rrbracket & \mapsto n' \text{ importe quel } \text{élément de } \llbracket a, b \rrbracket \text{ sinon} \end{cases}.$$

De manière analogue, nous prouvons la partie de cette proposition concernant la description de β en choisissant pour $\llbracket a, b \rrbracket$ les intervalles $[b_n, b_{n+1}]$ et en séparant différents cas. Nous en déduisons après assez facilement une description de la fonction w en observant de nouveau la condition $\forall \llbracket a, b \rrbracket \in \text{Int}_{\omega,*}(\mathbb{N}) : w(\llbracket a, b \rrbracket) \in \llbracket a, b \rrbracket$ et $\beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket$. En demandant que pour de telles fonctions β , $\beta(w(\llbracket a, b \rrbracket)) \notin \llbracket a, b \rrbracket$ pour tous $a \leq b \in \mathbb{N}$, on obtient le corollaire :

Corollaire 1.3.1. Soit $\beta : \mathbb{N} \rightarrow \mathbb{N}$ et $w : \text{Int}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ deux fonctions telles que

$$\forall \llbracket a, b \rrbracket \in \text{Int}_{\omega,*}(\mathbb{N}) : \quad w(\llbracket a, b \rrbracket) \in \llbracket a, b \rrbracket \\ \text{et } \beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket \text{ et } \beta(w(\llbracket a, b \rrbracket)) \notin \llbracket a, b \rrbracket.$$

Alors soit

$$\mathbf{Cas} \text{ ①. } \beta : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n+1 \end{cases} \text{ et } w : \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b \end{cases}$$

$$\mathbf{Cas} \text{ ②. } \text{ Il existe } n^* \neq u \in \mathbb{N} \text{ tel que } \beta : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n+1 \text{ pour } n < n^* \\ n^* & \mapsto u \\ n & \mapsto n-1 \text{ pour } n > n^* \end{cases} \text{ et } w :$$

$$\begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b \text{ si } b \in \llbracket 0, n^* \rrbracket \\ \llbracket a, b \rrbracket & \mapsto a \text{ si } a \geq n^*+1 \end{cases}.$$

Pour prouver ce corollaire, on applique la proposition précédente après avoir noté que, pour tout $a \in \mathbb{N}$, $w([a, a]) \in [a, a]$ et $\beta(w([a, a])) \notin [a, a]$ impliquent que $\beta(a) \neq a$, et donc qu'une indexation strictement croissante du complémentaire des points fixes de β est $\{n\}_{n \in \mathbb{N}}$.

Fonctions avec un nombre maximal de sur-ensembles invariants finis

La proposition suivante décrit les fonctions admettant un nombre maximal de sur-ensembles invariants finis dans le cas général. Sa démonstration est assez simple.

Proposition 1.3.3. *Soit $\alpha : I \rightarrow I$ et $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$. Alors les conditions suivantes sont équivalentes :*

1. $\forall I^* \in \mathcal{P}_{\omega,*}(I) : I^* \subseteq G(I^*)$ et $\alpha(G(I^*)) \subseteq G(I^*)$.
2. Les orbites de α sont finies et il existe $H : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ tel que

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_{\alpha}^+(a).$$

Nous nous plaçons maintenant dans \mathbb{N} et nous considérons la question de décrire les fonctions $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ admettant un nombre maximal de sur-ensembles invariants finis ($G(I^*)$ pour chaque $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$) avec la condition supplémentaire que $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \max(G(I^*)) \in \alpha(I^*)$. Nous démontrons

Proposition 1.3.4. *Soit $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ et $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega,*}(\mathbb{N})$ tels que*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*), \alpha(G(I^*)) \subseteq G(I^*) \text{ et } \max(G(I^*)) \in \alpha(I^*).$$

Alors α admet un nombre infini de points fixes, et si $\{b_n\}_{n \in \mathbb{N}}$ en est une indexation strictement croissante, alors il existe une fonction $j : \mathbb{N} \rightarrow \mathbb{N}$ telle que

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)}$$

et

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) = n \text{ si } \exists n \in \mathbb{N} : a = b_n \\ j(a) \geq r(a) \text{ si } a \notin \{b_n\}_{n \in \mathbb{N}} \end{aligned}$$

où $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, et il existe une fonction $H : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega,*}(\mathbb{N})$ telle que $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \{a, \alpha(a)\}$ et $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in I^* : \forall x \in H(I^*) : j(x) \leq j(a)$. Réciproquement, toutes fonctions α et G de cette forme vérifient l'hypothèse initiale.

La preuve de cette proposition découle essentiellement de la formule $\forall n \in \mathbb{N} : \alpha(\alpha(n)) \geq \alpha(n)$, qu'on obtient en appliquant l'hypothèse aux singletons $\{n\}$. Dans la proposition suivante, nous supposons la condition plus restrictive que G est à valeurs dans $\text{Int}_{\omega,*}(\mathbb{N})$.

Proposition 1.3.5. *Soit $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ et $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \text{Int}_{\omega,*}(\mathbb{N})$ tel que*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*), \alpha(G(I^*)) \subseteq G(I^*) \text{ et } \max(G(I^*)) \in \alpha(I^*).$$

Alors α admet un nombre infini de points fixes, et si $\{b_n\}_{n \in \mathbb{N}}$ en est une indexation strictement croissante, il existe une fonction $j : \mathbb{N} \rightarrow \mathbb{N}$ telle que

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)},$$

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) = n \text{ si } \exists n \in \mathbb{N} : a = b_n \\ j(a) = r(a) \text{ si } a > b_0 \text{ et } a \notin \{b_n\}_{n \in \mathbb{N}} \end{aligned}$$

où $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, et

$$j(0) \geq j(1) \cdots \geq j(b_0 - 1),$$

et il existe une fonction $u : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ telle que $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \llbracket u(I^*), \max\{\alpha(a)\}_{a \in I^*} \rrbracket$, et

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : u^*(I^*) := \min\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\} \leq u(I^*) \leq \min(I^*),$$

Réciproquement, toutes fonctions α et G de cette forme vérifient l'hypothèse initiale.

Nous avons prouvé cette proposition en s'appuyant sur celle qui la précède et en appliquant l'hypothèse aux ensembles de la forme $\{b_n, b_{n+1}\}$ et aux singletons.

Fonctions avec un nombre maximal de sur-ensembles intérieurement-1-quasi-invariants

Nous commençons par donner des définitions sur lesquelles vont s'appuyer les résultats qui suivent.

Définition 1.3.1. On dit qu'un triplet $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ satisfait la propriété $P_1(\phi, G, u)$ si :

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in G(I^*), I^* \subseteq G(I^*) \text{ et } \phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*).$$

Définition 1.3.2. On dit qu'un triplet $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ satisfait la propriété $P_2(\phi, G, u)$ si

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in I^*, I^* \subseteq G(I^*) \text{ et } \phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*).$$

Définition 1.3.3. Soit I un ensemble infini et $\phi : I \rightarrow I$.

Soit $D_\phi = \{I^* \in \mathcal{P}_{\omega,*}(I) : \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \neq \emptyset\} \subseteq \mathcal{P}_{\omega,*}(I)$.

Pour tout $I^* \in D_\phi$, $z \in \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$ et $a \in I^*$, on définit m_a^z par $\min\{m \in \mathbb{N} : \phi^m(a) = z\}$.

On définit une application *noncanonique* $\xi_\phi : D_\phi \rightarrow I$ en choisissant pour tout $I^* \in D_\phi$, $\xi_\phi(I^*) \in \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$ avec $\sum_{a \in I^*} m_a^{\xi_\phi(I^*)}$ minimal.

Pour tout $I^* \in D_\phi$, on a $\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) = \mathcal{O}_\phi^+(\xi_\phi(I^*))$.

Définition 1.3.4. Soit I un ensemble et $\phi : I \rightarrow I$. Soit $\tilde{P}(\phi)$ la proposition

$$\forall a, b \in I : \left[(|\mathcal{O}_\phi^+(a)| = |\mathcal{O}_\phi^+(b)| = \infty) \Rightarrow |\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b)| = \infty \right].$$

Définition 1.3.5. Soit I un ensemble et $\phi : I \rightarrow I$.

On définit l'ordre partiel \preceq_ϕ sur I par : $a \preceq_\phi b \Leftrightarrow b \in \mathcal{O}_\phi^+(a)$.

Premièrement, nous avons considéré P_1 . Nous avons prouvé la caractérisation suivante :

Proposition 1.3.6. Soit I un ensemble et $\beta : I \rightarrow I$.

Soit $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ et $v : \mathcal{P}_{\omega,*}(I) \rightarrow I$. Alors

$P_1(\beta, G, v) \Leftrightarrow$ il existe trois fonctions $H, \overline{H}, \widetilde{H} : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_\omega(I)$ telles que pour tout $I^* \in \mathcal{P}_{\omega,*}(I)$, $I^* \subseteq H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$, $H(I^*) \subseteq \{a \in I : |\mathcal{O}_\beta^+(a)| = +\infty\}$,

$\overline{H}(I^*), \widetilde{H}(I^*) \subseteq \{a \in I : |\mathcal{O}_\beta^+(a)| < +\infty\}$, et

(a) $\exists z \in G(I^*) : |\mathcal{O}_\beta^+(z)| = +\infty$ et

$$\begin{cases} H(I^*) & \neq \emptyset \\ \overline{H}(I^*) & = \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_\beta^+(a) \right) \cup \left(\bigcup_{a \in H(I^*)} \{a, \beta(a), \beta^2(a), \dots, \xi_\beta(H(I^*))\} \right) \\ v(I^*) & = \xi_\beta(H(I^*)) \end{cases}$$

ou

(b) $\forall a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty$ et

$$\begin{cases} H(I^*) & = \emptyset \\ \overline{H}(I^*) & \neq \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_\beta^+(a) \right) \cup \left(\bigcup_{a \in \overline{H}(I^*)} \{a, \beta(a), \beta^2(a), \dots, v(I^*)\} \right) \end{cases}.$$

Puis, nous avons caractérisé le problème existentiel $\exists G \exists v : P_1(\beta, G, v)$ en fonction de $\tilde{P}(\beta)$.

Proposition 1.3.7. Soit I un ensemble infini et $\beta : I \rightarrow I$ une fonction. Alors :

$$(\exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : \exists v : \mathcal{P}_{\omega,*}(I) \rightarrow I : P_1(\beta, G, v)) \Leftrightarrow \tilde{P}(\beta).$$

Deuxièmement, nous avons considéré P_2 . Nous avons déduit trivialement de la caractérisation de P_1 la caractérisation suivante de P_2 :

Proposition 1.3.8. *Soit I un ensemble infini et $\beta : I \rightarrow I$.*

Soit $G : \mathcal{P}_{\omega,}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ et $v : \mathcal{P}_{\omega,*}(I) \rightarrow I$. Alors*

$P_2(\beta, G, v) \Leftrightarrow$ il existe trois fonctions $H, \bar{H}, \tilde{H} : \mathcal{P}_{\omega,}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ telles que pour tout $I^* \in \mathcal{P}_{\omega,*}(I)$, $I^* \subseteq H(I^*) \cup \bar{H}(I^*) \cup \tilde{H}(I^*)$, $H(I^*) \subseteq \{a \in I : |\mathcal{O}_\beta^+(a)| = +\infty\}$, $\bar{H}(I^*), \tilde{H}(I^*) \subseteq \{a \in I : |\mathcal{O}_\beta^+(a)| < +\infty\}$, et*

(a) $\exists z \in G(I^) : |\mathcal{O}_\beta^+(z)| = +\infty$ et*

$$\begin{cases} H(I^*) & \neq \emptyset \\ \bar{H}(I^*) & = \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \tilde{H}(I^*)} \mathcal{O}_\beta^+(a) \right) \cup \left(\bigcup_{a \in H(I^*)} \{a, \beta(a), \beta^2(a), \dots, \xi_\beta(H(I^*))\} \right) \\ v(I^*) & = \xi_\beta(H(I^*)) \in I^* \end{cases}$$

ou

(b) $\forall a \in G(I^) : |\mathcal{O}_\beta^+(a)| < +\infty$ et*

$$\begin{cases} H(I^*) & = \emptyset \\ \bar{H}(I^*) & \neq \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \tilde{H}(I^*)} \mathcal{O}_\beta^+(a) \right) \cup \left(\bigcup_{a \in \bar{H}(I^*)} \{a, \beta(a), \beta^2(a), \dots, v(I^*)\} \right) \\ v(I^*) & \in I^* \end{cases}$$

Puis, nous avons considéré le problème existentiel $\exists G \exists u : P_2(\beta, G, u)$ que nous avons essayé de caractériser en termes d'une condition sur ϕ . Sous la condition que toutes les orbites de ϕ sont infinies, la proposition suivante caractérise ce problème existentiel en termes de la structure orbitale de ϕ .

Proposition 1.3.9. *Soit I un ensemble infini, $\phi : I \rightarrow I$ tel que $(\forall a \in I) : \mathcal{O}_\phi^+(a)$ est infini.*

Alors les conditions suivantes sont équivalentes :

1. $\exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : \exists u : \mathcal{P}_{\omega,*}(I) \rightarrow I : P_2(\phi, G, u)$.
2. $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^* \neq \emptyset$,
3. $D_\phi = \mathcal{P}_{\omega,*}(I)$ et $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^* = \{\xi_\phi(I^*)\}$.
4. $D_\phi = \mathcal{P}_{\omega,*}(I)$ et $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \xi_\phi(I^*) \in I^*$,
5. $\forall n \in \mathbb{N}^* : \forall (a_1, \dots, a_n) \in I^n$ qui sont distincts : $\exists d \in \llbracket 1, n \rrbracket : \exists (m_1, \dots, m_n) \in \mathbb{N}^n$ avec $m_d = 0 : \forall i \in \llbracket 1, n \rrbracket : a_d = \phi^{m_i}(a_i)$.
6. $\forall (a, b) \in I^2$ avec $a \neq b$, $(\exists n \in \mathbb{N} : a = \phi^n(b))$ ou $(\exists n \in \mathbb{N} : b = \phi^n(a))$.
7. \preceq_ϕ est un ordre total sur I .

À partir de cette proposition, le corollaire suivant caractérisant le problème existentiel $\exists G \exists u : P_2(\phi, G, u)$ sans hypothèse supplémentaire peut être facilement déduit.

Corollaire 1.3.2. *Soit I un ensemble infini et $\phi : I \rightarrow I$. Notons I_ϕ^{inf} (resp. I_ϕ^{fin}) les sous-ensembles de I consistant en les éléments d'orbite infinie (resp. finie) sous ϕ . Remarquons que I_ϕ^{inf} et I_ϕ^{fin} sont tous les deux stables par ϕ et que $\{I_\phi^{inf}, I_\phi^{fin}\}$ est une partition de I . Alors les conditions suivantes sont équivalentes :*

1. $\exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : \exists u : \mathcal{P}_{\omega,*}(I) \rightarrow I : P_2(\phi, G, u)$.
2. $(\preceq_\phi)_{|I_\phi^{inf}}$ est un ordre total sur I_ϕ^{inf} .

La proposition suivante est un supplément à la proposition et au corollaire précédents, où nous supposons également l'existence d'une orbite cofinie.

Proposition 1.3.10. *Soit I un ensemble infini et $\phi : I \rightarrow I$ tels que $\forall a \in I : \mathcal{O}_\phi^+(a)$ est infinie. Alors nous avons :*

$$\begin{aligned}
& (\exists \tilde{a} \in I : \mathcal{O}_\phi^+(\tilde{a}) \text{ est cofinie}) \\
& \text{et } (\exists u : \mathcal{P}_{\omega,*}(I) \rightarrow I : \exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : P_2(\phi, G, u)) \\
& \Leftrightarrow (\exists \tilde{a} \in I : \mathcal{O}_\phi^+(\tilde{a}) \text{ est cofinie}) \text{ et } (\preceq_\phi \text{ est un ordre total sur } I) \\
& \Leftrightarrow \exists \alpha : (\mathbb{N}, \leq) \rightarrow (I, \preceq_\phi) \text{ telle que } \alpha \text{ est une bijection croissante} \\
& \Leftrightarrow \exists \tilde{a} \in I : \mathcal{O}_\phi^+(\tilde{a}) = I.
\end{aligned}$$

Enfin, nous avons démontré une propriété d'indivisibilité pour les triplets (ϕ, G, u) vérifiant $P_2(\phi, G, u)$. L'énoncé précis de notre proposition est le suivant

Proposition 1.3.11. *Soit I un ensemble et $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ un triplet vérifiant $P_2(\phi, G, u)$.*

Alors si $\phi = \beta \circ \alpha$ où $\beta : I \rightarrow I$ est une application vérifiant $\forall I^ \in \mathcal{P}_{\omega,*}(I) : \exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$ et $\alpha : I \rightarrow I$ est une application bijective telle que $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \alpha(G(I^*)) \subseteq G(I^*)$, alors $\alpha = id$ et $\beta = \phi$.*

Pour prouver cette proposition, nous analysons en particulier les orbites des éléments de I par α et par β .

Chapitre 2

Path-connectedness and topological closure of some sets related to the non-compact Stiefel manifold

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Abstract. If H is a Hilbert space, the non-compact Stiefel manifold $St(n, H)$ consists of independent n -tuples in H . In this article, we contribute to the topological study of non-compact Stiefel manifolds, mainly by proving two results on the path-connectedness and topological closure of some sets related to the non-compact Stiefel manifold. In the first part, after introducing and proving an essential lemma, we prove that $\bigcap_{j \in J} (U(j) + St(n, H))$ is path-connected by polygonal paths under a condition on the codimension of the span of the components of the translating J -family. Then, in the second part, we show that the topological closure of $St(n, H) \cap S$ contains all polynomial paths contained in S and passing through a point in $St(n, H)$. As a consequence, we prove that $St(n, H)$ is relatively dense in a certain class of subsets which we illustrate with many examples from frame theory coming from the study of the solutions of some linear and quadratic equations which are finite-dimensional continuous frames. Since $St(n, L^2(X, \mu; \mathbb{F}))$ is isometric to $\mathcal{F}_{(X, \mu, n)}^{\mathbb{F}}$, this article is also a contribution to the theory of finite-dimensional continuous Hilbert space frames.

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2.1 Introduction

Duffin and Shaeffer introduced in 1952 [26] the notion of a Hilbert space frame to study some deep problems in nonharmonic Fourier series. However, the general idea of signal decomposition in terms of elementary signals was known to Gabor [31] in 1946. The landmark paper of Daubechies, Grossmann, and Meyer [23] (1986) accelerated the development of the theory of frames which then became more widely known to the mathematical community. Nowadays, frames have a wide range of applications in both engineering science and mathematics : they have found applications in signal processing, image processing, data compression, and sampling theory. They are also used in Banach space theory. Intuitively, a frame in a Hilbert space K is an overcomplete basis allowing non-unique linear expansions, though technically, it must satisfy a double inequality called the frame inequality. There are many generalizations of frames in the literature, for instance frames in Banach space [18] or Hilbert C^* -modules [28]. A general introduction to frame theory can be found in [17, 20].

The space $\mathcal{F}_{(X,\Sigma,\mu),n}^{\mathbb{F}}$ of continuous frames indexed by (X, Σ, μ) and with values in \mathbb{F}^n is isometric to the Stiefel manifold $St(n, L^2(X, \mu; \mathbb{F}))$. If H is a Hilbert space, the non-compact Stiefel manifold $St(n, H)$ is the set of independent n -frames in H , where an independent n -frame simply denotes an independent n -tuple. Stiefel manifolds are studied in differential topology and are one of the fundamental examples in this area. Even though the theory of finite dimensional Stiefel manifolds is generally well-known [46, 36, 42], there are still some aspects under study [39, 47]. The theory of infinite dimensional Stiefel manifolds is less studied and some recent results can be found in [11, 35].

There have been also many studies directly devoted to the geometry of frames and their subsets. Connectivity properties of some important subsets of the frame space $\mathcal{F}_{k,n}^{\mathbb{F}}$ were studied in [16, 53]. Differential and algebro-geometric properties of these subsets were studied in [27, 60, 59, 61] and chapter 4 of [19] respectively. A fiber bundle structure with respect to the L^1 and L^∞ norms was established for continuous frames in [1, 2]. A notion of density for general frames analogous to Beurling density was introduced and studied in [8]. Finally, connectivity and density properties were studied for Gabor [9, 21, 37, 51] and wavelet [12, 32, 34, 57] frames.

Plan of the article. This article is organized as follows. In section 2.2, we set some notations, introduce the definition of continuous Bessel and frame families and their basic properties in \mathbb{F}^n , and present Stiefel manifolds with an emphasis on their topological aspects. In section 2.3, after introducing and proving an essential lemma, we prove that $\bigcap_{j \in J} (U(j) + St(n, H))$ is path-connected by polygonal paths under a condition on the codimension of the span of the

components of the translating J -family. Then, in section 2.4, we show that the topological closure of $St(n, H) \cap S$ contains all polynomial paths contained in S and passing through a point in $St(n, H)$. As a consequence, we prove that $St(n, H)$ is relatively dense in a certain class of subsets which we illustrate, in section 2.6, with many examples from frame theory coming from the study of the solutions of some linear and quadratic equations which are finite-dimensional continuous frames (section 2.5).

2.2 Preliminaries

2.2.1 Notations

The following notations are used throughout this article.

\mathbb{N} denotes the set of natural numbers including 0 and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

We denote by n an element of \mathbb{N}^* and by \mathbb{F} one of the fields \mathbb{R} or \mathbb{C} .

If K is a Hilbert space, we denote by $L(K)$ and $B(K)$ respectively the set of linear and bounded operators in K . Id_K is the identity operator of K .

If K is a Hilbert space, $m \in \mathbb{N}^*$, and $\theta_1, \dots, \theta_m \in H$, the Gram matrix of $(\theta_1, \dots, \theta_m)$ is the matrix $\text{Gram}(\theta_1, \dots, \theta_m)$ whose k, l -coefficient is $\text{Gram}(\theta_1, \dots, \theta_m)_{k,l} = \langle \theta_k, \theta_l \rangle$.

If $\sigma, \tau \in \mathbb{N}^*$, we denote by $M_{\sigma, \tau}(\mathbb{F})$ the algebra of matrices of size $\sigma \times \tau$ over the field \mathbb{F} . When $\sigma = \tau$, we denote this algebra $M_\sigma(\mathbb{F})$.

An element $x \in \mathbb{F}^n$ is a n -tuple (x^1, \dots, x^n) with $x^k \in \mathbb{F}$ for all $k \in \llbracket 1, n \rrbracket$.

If $S \in L(\mathbb{F}^n)$, we denote by $[S] \in M_n(\mathbb{F})$ the matrix of S in the standard basis of \mathbb{F}^n , and we write I_n as a shorthand for $[Id_{\mathbb{F}^n}]$.

If $U = (u_x)_{x \in X}$ is a family in \mathbb{F}^n indexed by X , then for each $k \in \llbracket 1, n \rrbracket$, we denote by U^k the family $(u_x^k)_{x \in X}$.

2.2.2 Continuous frames in \mathbb{F}^n

Let K be a Hilbert space over \mathbb{F} and (X, Σ, μ) a measure space.

Definition 2.2.1. [20] We say that a family $\Phi = (\varphi_x)_{x \in X}$ with $\varphi_x \in K$ for all $x \in X$ is a continuous frame in K if

$$\exists 0 < A \leq B : \forall v \in K : A\|v\|^2 \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq B\|v\|^2$$

A frame is tight if we can choose $A = B$ as frame bounds. A tight frame with bound $A = B = 1$ is called a Parseval frame. A Bessel family is a family satisfying only the upper inequality. A frame is discrete if Σ is the discrete σ -algebra and μ is the counting measure. If $(n, k) \in (\mathbb{N}^*)^2$, we denote by $\mathcal{F}_{(X, \Sigma, \mu), K}$, $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ and

$\mathcal{F}_{k,n}^{\mathbb{F}}$ respectively the set of (X, Σ, μ) -continuous frames with values in K , the set of (X, Σ, μ) -continuous frames with values in \mathbb{F}^n , and the set of discrete frames indexed by $\llbracket 1, k \rrbracket$ and with values in \mathbb{F}^n .

If $U = (u_x)_{x \in X}$ with $u_x \in K$ for all $x \in X$ is a continuous Bessel family in K , we define its analysis operator $T_U : K \rightarrow L^2(X, \mu; \mathbb{F})$ by

$$\forall v \in K : T_U(v) := (\langle v, u_x \rangle)_{x \in X}.$$

The adjoint of T_U is an operator $T_U^* : L^2(X, \mu; \mathbb{F}) \rightarrow K$ given by

$$\forall c \in L^2(X, \mu; \mathbb{F}) : T_U^*(c) = \int_X c(x) u_x d\mu(x).$$

The composition $S_U = T_U^* T_U : K \rightarrow K$ is given by

$$\forall v \in K : S_U(v) = \int_X \langle v, u_x \rangle u_x d\mu(x)$$

and called the frame operator of U . Since U is a Bessel family, T_U , T_U^* , and S_U are all well defined and continuous. If U is a frame in K , then S_U is a positive self-adjoint operator satisfying $0 < A \leq S_U \leq B$ and thus, it is invertible.

We now recall a proposition preventing that a frame belongs to $L^2(X, \mu; K)$ when $\dim(K) = \infty$. Here the set $L^2(X, \mu; K)$ refers to Bochner square integrable (classes) of functions in $\mathcal{M}(X; K)$, where the latter refers to the set of measurable functions from X to K . It explains why we only study the L^2 topology of frame subspaces in the finite dimensional case.

Proposition 2.2.1. *Let K be a Hilbert space with $\dim K = \infty$. Then $\mathcal{F}_{(X, \Sigma, \mu), K} \cap L^2(X, \mu; K) = \emptyset$*

Démonstration. Let $\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}_{(X, \Sigma, \mu), K} \cap L^2(X, \mu; K)$. Let $\{e_m\}_{m \in M}$ be an orthonormal basis of K . We have

$$\begin{aligned} \text{Tr}(S_\Phi) &= \text{Tr}(T_\Phi^* T_\Phi) = \sum_{m \in M} \|T_\Phi(e_m)\|^2 \\ &= \sum_{m \in M} \int_X |\langle e_m, \varphi_x \rangle|^2 d\mu(x) \\ &= \int_X \left(\sum_{m \in M} |\langle e_m, \varphi_x \rangle|^2 \right) d\mu(x) \\ &= \int_X \|\varphi_x\|^2 d\mu(x). \end{aligned}$$

Since $\Phi \in \mathcal{F}_{(X, \Sigma, \mu), K}$, there exists a constant $A > 0$ such that

$$S_\Phi \geq A \cdot Id,$$

so

$$\int_X \|\varphi_x\|^2 d\mu(x) = \text{Tr}(S_\Phi) = +\infty$$

since $\dim(K) = \infty$. Hence $\Phi \notin L^2(X, \mu; K)$. \square

From now on, we consider $K = \mathbb{F}^n$. In what follows, we will recall some elementary facts about Bessel sequences and frames in this setting.

Proposition 2.2.2. *A family $U = (u_x)_{x \in X}$ with $u_x \in \mathbb{F}^n$ for all $x \in X$ is a continuous Bessel family if and only if it belongs to $L^2(X, \mu, \mathbb{F}^n)$.*

Démonstration. (\Rightarrow) Suppose that $U = (u_x)_{x \in X}$ is a continuous Bessel family. For each $k \in \llbracket 1, n \rrbracket$, denote by e_k the k -th vector of the standard basis of \mathbb{F}^n .

Applying the definition to the vector e_k , we have for each $k \in \{1, \dots, n\}$: $\|U^k\|_{L^2(X, \mu, \mathbb{F})}^2 < \infty$, and so

$$\|U\|_{L^2(X, \mu, \mathbb{F}^n)}^2 = \sum_{k=1}^n \|U^k\|_{L^2(X, \mu, \mathbb{F})}^2 < \infty,$$

which implies $U \in L^2(X, \mu; \mathbb{F}^n)$.

(\Leftarrow) Suppose that $U = (u_x)_{x \in X} \in L^2(X, \mu; \mathbb{F}^n)$. We have

$$\forall v \in \mathbb{F}^n : \int_{x \in X} |\langle v, u_x \rangle|^2 d\mu(x) \leq \|U\|_{L^2(X, \mu; \mathbb{F}^n)}^2 \|v\|^2 < \infty$$

by the Cauchy-Schwarz inequality, which implies that $U = (u_x)_{x \in X}$ is a continuous Bessel family. \square

Lemma 2.2.1. *If $U \in L^2(X, \mu; \mathbb{F}^n)$, then $[S_U] = \text{Gram}(U^1, \dots, U^n)$.*

Démonstration. Let $(e_k)_{k \in \llbracket 1, n \rrbracket}$ the standard basis of \mathbb{F}^n . Let $i, j \in \llbracket 1, n \rrbracket$. Then

$$[S_U]_{i,j} = \langle S e_j, e_i \rangle = \int_X \langle e_j, u_x \rangle \langle u_x, e_i \rangle d\mu(x) = \int_X \overline{u_x^j} u_x^i d\mu(x) = \langle U^i, U^j \rangle.$$

\square

Proposition 2.2.3. *[20] Suppose $\Phi = (\varphi_x)_{x \in X}$ is a family in \mathbb{F}^n . Then*

$$\begin{aligned} \Phi \text{ is a continuous frame} &\Leftrightarrow \Phi \in L^2(X, \mu; \mathbb{F}^n) \text{ and } S_\Phi \text{ is invertible} \\ &\Leftrightarrow \Phi \in L^2(X, \mu; \mathbb{F}^n) \text{ and } \det(\text{Gram}(\Phi^1, \dots, \Phi^n)) > 0 \\ &\Leftrightarrow \Phi \in L^2(X, \mu; \mathbb{F}^n) \text{ and } \{\Phi^1, \dots, \Phi^n\} \text{ is free.} \end{aligned}$$

Proposition 2.2.4. [20] Suppose $\Phi = \{\varphi_x\}_{x \in X}$ is a family in \mathbb{F}^n and let $a > 0$. Then

$$\begin{aligned} \Phi \text{ is a measurable } a\text{-tight frame} &\Leftrightarrow \Phi \in L^2(X, \mu; \mathbb{F}^n) \text{ and } S_\Phi = aI_n \\ &\Leftrightarrow \Phi \in L^2(X, \mu; \mathbb{F}^n) \text{ and } \text{Gram}(\Phi^1, \dots, \Phi^n) = aI_n \\ &\Leftrightarrow \Phi \in L^2(X, \mu; \mathbb{F}^n) \text{ and } (\Phi^1, \dots, \Phi^n) \\ &\text{is an orthogonal family of } L^2(X, \mu; \mathbb{F}) \\ &\text{and } (\forall i \in \llbracket 1, n \rrbracket) : \|\Phi^i\| = \sqrt{a}. \end{aligned}$$

Example 2.2.1. Define $\varphi_m^1 = \frac{1}{m}e^{2\pi iam}$ and $\varphi_m^2 = \frac{1}{m}e^{2\pi ibm}$ with a, b two real numbers such that $a-b$ is not an integer. Then $\Phi^1 = (\varphi_m^1)_{m \in \mathbb{N}}$ and $\Phi^2 = (\varphi_m^2)_{m \in \mathbb{N}}$ are square summable with sum $\frac{\pi^2}{6}$. Since the sequences Φ^1 and Φ^2 are not proportional due to the constraint on a and b , it follows by 2.2.3 that Φ is a discrete frame in \mathbb{C}^2 . It is not however a tight frame since Φ^1 and Φ^2 are not orthogonal.

2.2.3 Basic topological properties of $St(n, H)$ and $St_o(n, H)$

In this subsection, we introduce $St(n, H)$ and $St_o(n, H)$ as well as some of their basic topological properties. We recall that n is a fixed element of \mathbb{N}^* . If H is a Hilbert space, then $St(n, H)$ is non-empty exactly when $\dim(H) \geq n$. In the following, we will always suppose this condition.

Definition 2.2.2. The *non-compact Stiefel manifold* of independent n -frames in H is defined by $St(n, H) := \{h = (h_1, \dots, h_n) \in H^n : \{h_1, \dots, h_n\} \text{ is free}\}$. The *Stiefel manifold of orthonormal n -frames in H* is defined by $St_o(n, H) := \{h = (h_1, \dots, h_n) \in H^n : \{h_1, \dots, h_n\} \text{ is an orthonormal system}\}$.

Proposition 2.2.5. We have

1. $St(n, L^2(X, \mu; \mathbb{F}))$ is isometric to $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$.
2. $St_o(n, L^2(X, \mu; \mathbb{F}))$ is isometric to the set of continuous (X, Σ, μ) -Parseval frames with values in \mathbb{F}^n .

Démonstration. Define

$$\mathbf{Transpose} : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow L^2(X, \mu; \mathbb{F})^n \\ F = (f_x)_{x \in X} & \mapsto ((f_x^1)_{x \in X}, \dots, (f_x^n)_{x \in X}) \end{cases}$$

Then **Transpose** is clearly an isometry, and it sends $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ to $St(n, L^2(X, \mu; \mathbb{F}))$ and $St_o(n, L^2(X, \mu; \mathbb{F}))$ to the set of continuous (X, Σ, μ) -Parseval frames with values in \mathbb{F}^n by propositions 2.2.3 and 2.2.4 respectively. \square

Remark 2.2.1. Because of proposition 2.2.5, the reader should keep in mind that the following topological properties and the new results of this article are also shared, for any measure space (X, Σ, μ) , by $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ or the set of continuous (X, Σ, μ) -Parseval frames with values in \mathbb{F}^n , depending on the context.

Proposition 2.2.6. *We have*

1. $St(n, H)$ is open in H^n .
2. $St_o(n, H)$ is closed in H^n .

Démonstration. 1. $St(n, H)$ is open because $St(n, H) = (\det \circ \text{Gram})^{-1}((0, \infty))$.

2. $St_o(n, H)$ is closed because $St_o(n, H) = \text{Gram}^{-1}(I_n)$.

□

By joining continuously each element of $St(n, H)$ to its corresponding Gram-Schmidt orthonormalized system in $St_o(n, H)$, we can prove

Proposition 2.2.7. *$St_o(n, H)$ is a deformation retract of $St(n, H)$.*

Concerning the connectedness properties of Stiefel manifolds, we have the following

Definition 2.2.3. Let X be a topological space and $m \in \mathbb{N}$. Then X is said to be m -connected if its homotopy groups $\pi_i(X, b)$ are trivial for all $i \in \llbracket 0, m \rrbracket$ and $b \in X$ (of course, $\pi_0(X, b)$ is not a group but merely a pointed set).

Proposition 2.2.8. *(see pp. 382-383 of [36]) We have*

1. $St(n, \mathbb{R}^k)$ is $(k - n - 1)$ -connected.
2. $St(n, \mathbb{C}^k)$ is $(2k - 2n)$ -connected.
3. If H is infinite dimensional, then $St(n, H)$ is contractible.

Moreover, we have

Proposition 2.2.9. *If H is infinite dimensional, then $St(n, H)$ is contractible.*

A proof can be found in [this math.stackexchange thread](#).

The following proposition asserts the density of $St(n, H)$ in H^n . A corollary of one of our results in this article (corollary 2.4.1) gives a generalization of this proposition.

Proposition 2.2.10. *$St(n, H)$ is dense in H^n .*

Démonstration. Consider $h = (h_1, \dots, h_n) \in H^n$. Pick some $\theta = (\theta_1, \dots, \theta_n) \in St(n, H)$. Let γ be the straight path connecting θ to h , i.e. for each $t \in [0, 1]$: $\gamma(t) = th + (1 - t)\theta \in H^n$. Let $\Gamma(t) = \det(\text{Gram}((\gamma(t)_1, \dots, \gamma(t)_n)))$. Clearly, $\Gamma(t)$ is a polynomial function in t which satisfies $\Gamma(0) \neq 0$ since $\theta \in St(n, H)$. Therefore

$$\Gamma(t) \neq 0 \text{ except for a finite number of } t's.$$

Moreover,

$$\|\gamma(t) - h\|_{H^n}^2 = \sum_{k=1}^n \|\gamma(t)_k - h_k\|_H^2 = \sum_{k=1}^n |1 - t|^2 \|\theta_k - h_k\|_H^2 \rightarrow 0 \text{ when } t \rightarrow 1$$

Hence, there exists $t \in [0, 1]$ such that $\gamma(t)$ is close to h and $\Gamma(t) \neq 0$, and so $\gamma(t) \in St(n, H)$. \square

We also include the following proposition on the differential structure of the Stiefel manifolds.

Proposition 2.2.11. [46] *We have*

1. $St(n, \mathbb{R}^k)$ is a real manifold of dimension nk .
2. $St_o(n, \mathbb{R}^k)$ is a real manifold of dimension $nk - \frac{n(n+1)}{2}$.
3. $St(n, \mathbb{C}^k)$ is a real manifold of dimension $2nk$.
4. $St_o(n, \mathbb{C}^k)$ is a real manifold of dimension $2nk - n^2$.
5. If $\dim(H) = \infty$, then $St(n, H)$ and $St_o(n, H)$ are Hilbert manifolds of infinite dimension.

We ask the reader to keep in mind remark 2.2.1 when reading the remaining parts of this article.

2.3 Path-connectedness of $\bigcap_{j \in J} (U(j) + St(n, H))$

Definition 2.3.1. Let E be a topological vector space and $\gamma : [0, 1] \rightarrow E$ be a continuous path. We say that γ is a *polygonal path* if there exist $q \in \mathbb{N}^*$, $(e_k)_{k \in [1, q+1]}$ a finite sequences with $e_k \in E$ for all $k \in [1, q+1]$, and $(\gamma_k)_{k \in [1, q]}$ a finite sequence

of (continuous) straight paths with $\gamma_k = \begin{cases} [0, 1] & \rightarrow E \\ t & \mapsto (1 - t)e_k + te_{k+1} \end{cases}$ such that

$\gamma = \gamma_1 * \dots * \gamma_q$, where $*$ is the path composition operation. We say that a subset $S \subseteq E$ is polygonally connected if every two points of S are connected by a polygonal path.

In the following, when we say that $S \subseteq E$ is polygonally connected, we mean that each two points of S are connected by a polygonal path of the type $\gamma_1 * \gamma_2$ where γ_1 and γ_2 are two straight paths.

Before we prove the main proposition of this section, let's prove a useful lemma.

Lemma 2.3.1. 1. Suppose we have a family $(a(j))_{j \in J}$ indexed by J where each $a(j)$ belongs to $St(n, H)$. Then if $(a(j)_k)_{j \in J, k \in \llbracket 1, n \rrbracket}$ is free, we have

$$\text{span}(\{a(j)\}_{j \in J}) \setminus \{0\} \subseteq St(n, H)$$

2. Suppose we have a family indexed by J where each $a(j)$ belongs to $St_o(n, H)$ for all $j \in J$. Then if $(a(j)_k)_{j \in J, k \in \llbracket 1, n \rrbracket}$ is an orthonormal system, we have

$$\{x \in \text{span}(\{a(j)\}_{j \in J}) : \|x\| = \sqrt{n}\} \subseteq St_o(n, H)$$

Démonstration. 1. Let $h = (h_1, \dots, h_n) \in \text{span}(\{a(j)\}_{j \in J}) \setminus \{0\}$. We can write $h = \sum_{u=1}^r \lambda_u a(j_u)$ with $\lambda_u \in \mathbb{F}$ for all $u \in \llbracket 1, r \rrbracket$ and the λ_u 's are not all zeros. We need to show that (h_1, \dots, h_n) is an independent system. Suppose otherwise $\sum_{k=1}^n c_k h_k = 0$. This means that $\sum_{k=1}^n c_k (\sum_{u=1}^r \lambda_u a(j_u)_k) = 0$, and so $\sum_{k=1}^n \sum_{u=1}^r (\lambda_u c_k) a(j_u)_k = 0$. Since $\bigcup_{j \in J} \bigcup_{k \in \llbracket 1, n \rrbracket} \{a(j)_k\}$ is free, we deduce that $\lambda_u c_k = 0$ for all $u \in \llbracket 1, r \rrbracket$ and $k \in \llbracket 1, n \rrbracket$, which implies that $c_k = 0$ for all $k \in \llbracket 1, n \rrbracket$ since the λ_u 's are not all zeros. Therefore, $h \in St(n, H)$.

2. Let $h = (h_1, \dots, h_n) \in \text{span}(\{a(j)\}_{j \in J})$ such that $\|h\| = \sqrt{n}$. We can write $h = \sum_{u=1}^r \lambda_u a(j_u)$ with $\lambda_u \in \mathbb{F}$ for all $u \in \llbracket 1, r \rrbracket$. We need to show that $\langle h_k, h_l \rangle = \delta_{k,l}$ for all $k, l \in \llbracket 1, n \rrbracket$. For $k \neq l$, we have : $\langle h_k, h_l \rangle = \langle \sum_{u=1}^r \lambda_u a(j_u)_k, \sum_{u=1}^r \lambda_u a(j_u)_l \rangle = \sum_{u=1}^r \sum_{v=1}^r \lambda_u \bar{\lambda}_v \langle a(j_u)_k, a(j_v)_l \rangle = 0$ since $\bigcup_{j \in J} \bigcup_{k \in \llbracket 1, n \rrbracket} \{a(j)_k\}$ is an orthogonal system. Moreover, $\|h_k\|^2 = \sum_{u=1}^r \sum_{v=1}^r \lambda_u \bar{\lambda}_v \langle a(j_u)_k, a(j_v)_k \rangle = \sum_{u=1}^r |\lambda_u|^2 \|a(j_u)_k\|^2 = \sum_{u=1}^r |\lambda_u|^2$ since $\bigcup_{j \in J} \bigcup_{k \in \llbracket 1, n \rrbracket} \{a(j)_k\}$ is an orthonormal system. By hypothesis, $n = \|h\|^2 = \sum_{k=1}^n \|h_k\|^2 = n(\sum_{u=1}^r |\lambda_u|^2)$, so $\|h_k\|^2 = \sum_{u=1}^r |\lambda_u|^2 = 1$ for all $k \in \llbracket 1, n \rrbracket$ as desired. □

Proposition 2.3.1. Let H be a Hilbert space with $\dim(H) \geq n$, J an index set and $(U(j))_{j \in J}$ a family with $U(j) \in H^n$ for all $j \in J$. If $\text{codim}_H(\text{span}(\{u(j)_k : j \in J, k \in \llbracket 1, n \rrbracket\})) \geq 3n$, then $\bigcap_{j \in J} (U(j) + St(n, H))$ is polygonally-connected.

Démonstration. Let $X = (x_1, \dots, x_n)$ and $Y = (y_1, \dots, y_n)$ in $\bigcap_{j \in J} (U(j) + St(n, H))$.

Let (z_1, \dots, z_n) be an independent family in H such that

$$\text{span}(\{z_k : k \in \llbracket 1, n \rrbracket\}) \cap \text{span}(\{x_k : k \in \llbracket 1, n \rrbracket\} \cup \{y_k : k \in \llbracket 1, n \rrbracket\} \cup \{u(j)_k : j \in J, k \in \llbracket 1, n \rrbracket\}) = \{0\}$$

This is possible since $\text{codim}_H(\text{span}(\{u(j)_k : j \in J, k \in \llbracket 1, n \rrbracket\})) \geq 3n$.

This ensures that we have for all $j \in J$

$$\text{span}(\{-u(j)_k + z_k : k \in \llbracket 1, n \rrbracket\}) \cap \text{span}(\{-u(j)_k + x_k : k \in \llbracket 1, n \rrbracket\}) = \{0\},$$

$$\text{span}(\{-u(j)_k + z_k : k \in \llbracket 1, n \rrbracket\}) \cap \text{span}(\{-u(j)_k + y_k : k \in \llbracket 1, n \rrbracket\}) = \{0\},$$

and

$$(-u(j)_k + z_k)_{k \in \llbracket 1, n \rrbracket} \text{ is independent.}$$

We define the straight paths

$$\begin{cases} \gamma_1 & : [0, 1] \rightarrow H^n \\ \gamma_2 & : [0, 1] \rightarrow H^n \end{cases}$$

by $\gamma_1(t) = tZ + (1-t)X$ and $\gamma_2(t) = tY + (1-t)Z$ respectively.

We have $\gamma_1(0) = X$, $\gamma_1(1) = \gamma_2(0) = Z$, and $\gamma_2(1) = Y$.

Since for all $j \in J$

$$\text{span}(\{-u(j)_k + z_k : k \in \llbracket 1, n \rrbracket\}) \cap \text{span}(\{-u(j)_k + x_k : k \in \llbracket 1, n \rrbracket\}) = \{0\},$$

we have for all $t \in [0, 1]$ and $j \in J$

$$-U(j) + tZ + (1-t)X = t(-U(j) + Z) + (1-t)(-U(j) + X) \in \text{St}(n, H)$$

by lemma 2.3.1, and so $\gamma_1(t) \in \bigcap_{j \in J} (U(j) + \text{St}(n, H))$.

Similarly, $\gamma_2(t) \in \bigcap_{j \in J} (U(j) + \text{St}(n, H))$ for all $t \in [0, 1]$.

Composing γ_1 with γ_2 , we see that $\bigcap_{j \in J} (U(j) + \text{St}(n, H))$ is polygonally connected. \square

2.4 Topological closure of $\text{St}(n, H) \cap S$

Before moving on, we need a definition and a small lemma.

Definition 2.4.1. Let V be a \mathbb{F} -vector space, $q \in \mathbb{N}$, and $v, v' \in V$. We say that $\gamma : [0, 1] \rightarrow V$ is a *polynomial path up to reparametrization joining v and v'* if there exist $q \in \mathbb{N}$, a finite sequence of vectors $(v^k)_{k \in \llbracket 0, q \rrbracket}$ with $v^k \in V$ for all $k \in \llbracket 0, q \rrbracket$ and a homeomorphism $\phi : [0, 1] \rightarrow [a, b] \subseteq \mathbb{R}$ such that $\forall t \in [a, b] : \gamma(\phi^{-1}(t)) = \sum_{k=0}^q t^k v^k$, $\gamma(0) = v$ and $\gamma(1) = v'$. If V is equipped with a topology, then we say that γ is a *continuous polynomial path* when it is continuous as a map from $[0, 1]$ to V .

Remark 2.4.1. Every expression of the form $\sum_{k=0}^q P_k(t)v^k$ where $v^k \in V$ and $P_k \in \mathbb{F}[X]$ for all $k \in \llbracket 0, q \rrbracket$ can be written in the form $\sum_{k=0}^{q'} t^k w^k$ where $q' \in \mathbb{N}$ and $w^k \in V$ for all $k \in \llbracket 0, q' \rrbracket$ (group by increasing powers of t). Therefore there is no difference whether we define polynomial paths using the first expression or the second.

Lemma 2.4.1. *Let E be a normed vector space, and $v, v' \in E$. Then each polynomial path $\gamma : [0, 1] \rightarrow E$ up to reparametrization joining v and v' is continuous.*

Démonstration. Let $\gamma : [0, 1] \rightarrow E$ be a polynomial path up to reparametrization joining v and v' . Hence we can write $\forall t \in [0, 1] : \gamma(t) = \sum_{k=0}^q \phi(t)^k v^k$. The continuity of γ follows from

$$\|\gamma(t) - \gamma(t')\| \leq \sum_{k=0}^q \|v^k\| |\phi(t)^k - \phi(t')^k|$$

which goes to 0 when t goes to t' by continuity of ϕ^k for all $k \in \llbracket 0, q \rrbracket$. \square

The following is our first original proposition in this section.

Proposition 2.4.1. *Let $S \subseteq H^n$ and*

$E := \{\gamma : [0, 1] \rightarrow S \text{ such that } \gamma \text{ is a polynomial path up to reparametrization and}$
 $\exists a_\gamma \in [0, 1] : \gamma(a_\gamma) \in St(n, H) \cap S\}.$

Then $\bigcup_{\gamma \in E} \text{Range}(\gamma) \subseteq \overline{St(n, H) \cap S}$

Démonstration. Let $\gamma \in E$. We have $\forall t \in [a, b] : \gamma(\phi^{-1}(t)) = \sum_{k=0}^q t^k V^k$.
Let $\Gamma(t) := \det(\text{Gram}((\gamma(\phi^{-1}(t)))_1, \dots, \gamma(\phi^{-1}(t)))_n))$ for all $t \in [a, b]$.
Since for all $i, j \in \llbracket 1, n \rrbracket$ and $t \in [a, b]$

$$\begin{aligned} \langle \gamma(\phi^{-1}(t))_i, \gamma(\phi^{-1}(t))_j \rangle &= \left\langle \sum_{k=0}^q t^k v_i^k, \sum_{k=0}^q t^k v_j^k \right\rangle \\ &= \sum_{k, k'=0}^q \langle v_i^k, v_j^{k'} \rangle t^{k+k'} \end{aligned}$$

is a polynomial function in $t \in [a, b]$, and the determinant of a matrix in $M_{n,n}(\mathbb{F})$ is a polynomial function in its coefficients, $\Gamma(t)$ is a polynomial function in $t \in [a, b]$ which satisfies $\Gamma(\phi(a_\gamma)) \neq 0$ since $\gamma(a_\gamma) \in St(n, H)$. Therefore

$$\Gamma(t) \neq 0 \text{ for } t \text{ in a cofinite set } L \subseteq [a, b].$$

Hence $\text{Range}(\gamma) \setminus \{\gamma(\phi^{-1}(t))\}_{t \in [a, b] \setminus L} \subseteq St(n, H) \cap S \subseteq \overline{St(n, H) \cap S}$. Since $[a, b] \setminus L$ is finite, the continuity of γ (see lemma 2.4.1) at $\{\phi^{-1}(t)\}_{t \in [a, b] \setminus L}$ implies

$$\text{Range}(\gamma) = \overline{\text{Range}(\gamma)} = \overline{\text{Range}(\gamma) \setminus \{\gamma(\phi^{-1}(t))\}_{t \in [a, b] \setminus L}} \subseteq \overline{St(n, H) \cap S}.$$

This being true for all $\gamma \in E$, the result follows. \square

The following is a corollary.

Corollary 2.4.1. *Let $S \subseteq H^n$ such that for all $U \in S$ there exists a polynomial path up to reparametrization connecting U to some $\Theta(U) \in St(n, H) \cap S$ and contained in S . Then $St(n, H) \cap S$ is dense in S .*

Démonstration. For all $U \in S$, there exists by the hypothesis $\gamma \in E$ such that $\gamma(0) = U$. By proposition 2.4.1, we have $Range(\gamma) \subseteq \overline{St(n, H) \cap S}$. It follows that $U \in \overline{St(n, H) \cap S}$ since $U \in Range(\gamma)$. \square

This result shows the abundance of independent n -frames not only in H^n but also in many subsets S of the form of corollary 2.4.1. Importantly, notice that for $S \neq \emptyset$, there should exist at least one $\Theta \in St(n, H) \cap S$ (i.e. $St(n, H) \cap S \neq \emptyset$) for the result to follow.

As an example, this is true when S is a star domain of H^n with respect to some $\Theta \in St(n, H) \cap S$; if it is a convex subset of H^n and contains some $\Theta \in St(n, H) \cap S$; and if it is in particular an affine subspace containing some $\Theta \in St(n, H) \cap S$.

In the next section, we will find sufficient conditions under which some sets of the form $f^{-1}(\{d\}) \subseteq L^2(X, \mu; \mathbb{F}^n)$ where f is some linear or quadratic function contain a continuous frame. This will allow us to apply corollary 2.4.1 to these examples, which will be done in section 2.6.

2.5 Existence of solutions of some linear and quadratic equations which are finite-dimensional continuous frames

In this section, we show how to construct continuous finite-dimensional frames that are mapped to a given element by a linear operator or a quadratic function. In other words, we show the existence of frames in the inverse image of singletons by these functions.

2.5.1 Linear equations

Proposition 2.5.1. *Let $n \in \mathbb{N}^*$, V an \mathbb{F} -vector field of dimension $\geq n$ and $T : V^n \rightarrow \mathbb{F}$ a non-zero linear form. Then for all $d \neq 0$, there exists $(a_1, \dots, a_n) \in V^n$ such that the system (a_1, \dots, a_n) is free and $T(a_1, \dots, a_n) = d$.*

The proof relies on the following lemma, which may be of independent interest.

Lemma 2.5.1. *Let $n \in \mathbb{N}^*$ and V be a vector space of dimension $\geq n$. Let $(x_1, \dots, x_n) \in V^n \setminus \{(0, \dots, 0)\}$. Then there exist $e \in \mathbb{N}^*$ with $e =$*

$\begin{cases} 1 & \text{if } k = n \\ 2 & \text{if } k \in \llbracket 1, n-1 \rrbracket \end{cases}$
 where $k = \dim(\text{span}\{x_1, \dots, x_n\})$, and e independent systems (a_1^u, \dots, a_n^u) in V for $u \in [1, e]$ such that $(x_1, \dots, x_n) = \sum_{u=1}^e (a_1^u, \dots, a_n^u)$.

Démonstration. Without loss of generality, suppose that (x_1, \dots, x_k) is free where $k = \dim(\text{span}\{x_1, \dots, x_n\}) \geq 1$. Let $(b_i)_{i \in \llbracket 1, n-k \rrbracket}$ be an $(n-k)$ -tuple of vectors of V such that the system $(x_1, \dots, x_k, b_1, \dots, b_{n-k})$ is free.

- If $k = n$, then (x_1, \dots, x_n) is already free and so we can choose $e = 1$ and $a_i^1 = x_i$ for all $i \in \llbracket 1, n \rrbracket$.
- If $k \in \llbracket 1, n-1 \rrbracket$, we can write

$$(x_1, \dots, x_n) = \left(\frac{x_1}{2}, \dots, \frac{x_k}{2}, b_1, \dots, b_{n-k}\right) + \left(\frac{x_1}{2}, \dots, \frac{x_k}{2}, x_{k+1}-b_1, \dots, x_n-b_{n-k}\right).$$

$\left(\frac{x_1}{2}, \dots, \frac{x_k}{2}, b_1, \dots, b_{n-k}\right)$ is free by assumption, and we can easily show that $\left(\frac{x_1}{2}, \dots, \frac{x_k}{2}, x_{k+1}-b_1, \dots, x_n-b_{n-k}\right)$ is free by expressing x_{k+1}, \dots, x_n in terms of x_1, \dots, x_k . Hence we can choose $e = 2$ and $a_i^1 = a_i^2 = \frac{x_i}{2}$ for all $i \in \llbracket 1, k \rrbracket$, $a_i^1 = b_{i-k}$, and $a_i^2 = x_i - b_{i-k}$ for all $i \in \llbracket k+1, n \rrbracket$.

□

Démonstration. (of proposition 2.5.1)

Let's show that there exists a free system (a_1, \dots, a_n) such that $T(a_1, \dots, a_n) \neq 0$. Suppose to the contrary that $T(a_1, \dots, a_n) = 0$ for all free systems (a_1, \dots, a_n) in V . Let $(x_1, \dots, x_n) \in V^n \setminus \{(0, \dots, 0)\}$. By lemma 2.5.1, (x_1, \dots, x_n) decomposes as a finite sum of free systems. By linearity of T , we thus have $T(x_1, \dots, x_n) = 0$. Hence T is the zero form, a contradiction. Hence there exists a free system (a_1, \dots, a_n) such that $T(a_1, \dots, a_n) \neq 0$. Then $\frac{d}{T(a_1, \dots, a_n)}(a_1, \dots, a_n)$ satisfies the requirement. □

Corollary 2.5.1. Let $n \in \mathbb{N}^*$ and $T : L^2(X, \mu; \mathbb{F}^n) \rightarrow \mathbb{F}$ be a non-zero linear form. Suppose that $\dim(L^2(X, \mu; \mathbb{F})) \geq n$.

Then for all $d \neq 0$, there exists a continuous frame $\Phi = (\varphi_x)_{x \in X}$ such that $T(\Phi) = d$.

Proposition 2.5.2. Let $n \in \mathbb{N}^*$, V be a vector space over \mathbb{F} , and $S : V \rightarrow \mathbb{F}$ be a non-zero linear form.

Define the linear operator

$$T : \begin{cases} V^n & \rightarrow \mathbb{F}^n \\ (x_1, \dots, x_n) & \mapsto (S(x_1), \dots, S(x_n)) \end{cases}.$$

For all $d \in \mathbb{F}^n$, suppose that

- if $d \neq 0$, then $\dim(V) \geq n$,

— if $d = 0$ then $\dim(V) \geq n + 1$.

Then $T^{-1}(\{d\})$ contains an independent n -tuple (a_1, \dots, a_n) .

Démonstration. — Suppose that $d \neq 0$ and $\dim(V) \geq n$. Let $h \in V$ such that $S(h) = 1$, $(h_{(2)}, \dots, h_{(n)})$ be an independent system in $\text{Ker}(S)$ (this is possible since $\text{codim}(\text{Ker}(S)) = 1$), and $(d, d_{(2)}, \dots, d_{(n)})$ be an independent

system in \mathbb{F}^n . Consider the V -valued column matrix $\mathbf{H} = \begin{bmatrix} h \\ h_{(2)} \\ \vdots \\ h_{(n)} \end{bmatrix}$ and

the \mathbb{F} -valued square matrix $\mathbf{D} = \begin{bmatrix} d^1 & d_{(2)}^1 & \cdots & d_{(n)}^1 \\ \vdots & \vdots & \vdots & \vdots \\ d^n & d_{(2)}^n & \cdots & d_{(n)}^n \end{bmatrix}$. Moreover, we set

$$\mathbf{A} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \mathbf{D}\mathbf{H}.$$

We have $T(a_1, \dots, a_n) = (S(a_1), \dots, S(a_n)) = (S(d^i h + \sum_{k=2}^n d_{(k)}^i h_{(k)}))_{i \in [1, n]} = (d^i)_{i \in [1, n]} = d$ since $h_{(2)}, \dots, h_{(n)}$ belong to $\text{Ker}(S)$.

Let's show that (a_1, \dots, a_n) is free. Let $\lambda = (\lambda^1, \dots, \lambda^n) \in \mathbb{F}^n$

such that $\sum_{i=1}^n \lambda^i a_i = 0$. Consider $\mathbf{\Lambda} = \begin{bmatrix} \lambda^1 \\ \vdots \\ \lambda^n \end{bmatrix}$. Therefore we have

$0 = \mathbf{\Lambda}^\top \mathbf{A} = \mathbf{\Lambda}^\top \mathbf{D}\mathbf{H} \in V$. At this point, we can complete $(h, h_{(2)}, \dots, h_{(n)})$ into a Hamel basis of V , and denote by $(h^*, h_{(2)}^*, \dots, h_{(n)}^*)$ the first n linear forms of its dual basis. Applying these linear forms to $\mathbf{\Lambda}^\top \mathbf{D}\mathbf{H}$, we see that $\mathbf{\Lambda}^\top \mathbf{D} = [0 \mid \cdots \mid 0]$, therefore $\lambda = 0$ as \mathbf{D}^\top is obviously invertible since $(h, d_{(2)}, \dots, d_{(n)})$ is an independent system in \mathbb{F}^n .

— Suppose that $d = 0$ and $\dim(V) \geq n + 1$. Let $h \in V$ such that $S(h) = 1$ and (a_1, \dots, a_n) be an independent system in $\text{Ker}(S)$.

We have $T(a_1, \dots, a_n) = (S(a_1), \dots, S(a_n)) = 0 \in \mathbb{F}^n$.

Moreover, we have by construction that (a_1, \dots, a_n) is free. □

Remark 2.5.1. In the previous proposition, if $d \neq 0$, the condition $\dim(V) \geq n$ is necessary for the existence of an independent n -tuple because the existence of an independent n -tuple implies that $\dim(V) \geq n$, and if $d = 0$, the condition $\dim(V) \geq n + 1$ is also necessary for the existence of an independent n -tuple in $T^{-1}(\{0\})$ because the existence of an independent n -tuple (a_1, \dots, a_n) in $T^{-1}(\{0\})$

implies that $S(a_i) = 0$ for all $i \in \llbracket 1, n \rrbracket$, and since there exists $h \in V$ such that $S(h) = 1$ because S is non-zero, then (a_1, \dots, a_n, h) is free which implies $\dim(V) \geq n + 1$.

Corollary 2.5.2. *Let $h \in L^2(X, \mu; \mathbb{F}) \neq 0$. Define the linear operator*

$$T : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow \mathbb{F}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x) \end{cases}.$$

For all $d \in \mathbb{F}^n$, suppose that

- if $d \neq 0$, then $\dim(L^2(X, \mu; \mathbb{F})) \geq n$,
- if $d = 0$ then $\dim(L^2(X, \mu; \mathbb{F})) \geq n + 1$.

Then $T^{-1}(\{d\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$.

Proposition 2.5.3. *Let $h \in L^2(X, \mu; \mathbb{F}) \neq 0$. Define the linear operator*

$$T : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow \mathbb{F}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x) \end{cases}.$$

If there exists a measurable subset $Y \subset X$ such that $\dim(L^2(Y, \mu; \mathbb{F})) \geq n$ and $\mu((X \setminus Y) \cap h^{-1}(\mathbb{F}^*)) > 0$, then for all $d \in \mathbb{F}^n$, $T^{-1}(\{d\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$.

Démonstration. Since $\dim(L^2(Y, \mu; \mathbb{F})) \geq n$, there exists a continuous frame $(\varphi_y)_{y \in Y} \in \mathcal{F}_{(Y, \mu), n}^{\mathbb{F}}$. We extend $(\varphi_y)_{y \in Y}$ by setting

$$\varphi_x := \frac{\overline{h(x)}}{\|h\|_{L^2(X \setminus Y, \mu; \mathbb{F})}^2} \left(d - \int_Y h(y) \varphi_y d\mu(y) \right) \quad \text{for all } x \in X \setminus Y$$

Let $\Phi = (\varphi_x)_{x \in X}$. We have

$$\begin{aligned} T(\Phi) &= \int_X h(x) \varphi_x d\mu(x) \\ &= \int_Y h(y) \varphi_y d\mu(y) + \left(\int_{X \setminus Y} h(x) \frac{\overline{h(x)}}{\|h\|_{L^2(X \setminus Y, \mu; \mathbb{F})}^2} d\mu(x) \right) \left(d - \int_Y h(y) \varphi_y d\mu(y) \right) \\ &= d. \end{aligned}$$

Moreover, $\Phi \in \mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ since we have only completed $(\varphi_y)_{y \in Y}$ by a function in $L^2(X \setminus Y, \mu; \mathbb{F}^n)$. \square

Proposition 2.5.4. Let (X, Σ, μ) be a measure space, $l \in \mathbb{N}^*$, $(X_j)_{j \in \llbracket 1, l \rrbracket}$ a partition of X by measurable subsets, and $h \in L^2(X, \mu; \mathbb{F})$ such that there exist a family $(Y_j)_{j \in \llbracket 1, l \rrbracket}$ with Y_j a measurable subset of X_j for all $j \in \llbracket 1, l \rrbracket$, $\mu((X_j \setminus Y_j) \cap h^{-1}(\mathbb{F}^*)) > 0$ for all $j \in \llbracket 1, l \rrbracket$, and $\sum_{j=1}^l \dim(L^2(Y_j, \mu; \mathbb{F})) \geq n$.

Define the operator

$$W : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow \prod_{j \in \llbracket 1, l \rrbracket} \mathbb{F}^n \\ F = (f_x)_{x \in X} & \mapsto (f_{X_j} h(x) f_x d\mu(x))_{j \in \llbracket 1, l \rrbracket} \end{cases}.$$

Then for all $D = (d_j)_{j \in \llbracket 1, l \rrbracket} \in \prod_{j \in \llbracket 1, l \rrbracket} \mathbb{F}^n$, $W^{-1}(\{D\})$ contains at least one continuous frame $\Phi \in \mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$.

Remark 2.5.2. Proposition 2.5.3 results from proposition 2.5.4 by taking $l = 1$.

Remark 2.5.3. Proposition 2.5.4 can be generalized to $l = +\infty$ or to partitions indexed by a general index set J if we restrict to $D = 0$ (due to convergence issues).

Démonstration. For each $i \in \llbracket 1, n \rrbracket$, let e_i be the i -th vector of the standard basis of \mathbb{F}^n . Since $\sum_{j=1}^l \dim(L^2(Y_j, \mu; \mathbb{F})) \geq n$, we can find distinct $j_1, \dots, j_r \in \llbracket 1, l \rrbracket$ such that for each $u \in \llbracket 1, r \rrbracket$, $\dim(L^2(Y_{j_u}, \mu; \mathbb{F})) \geq 1$ and $\sum_{u=1}^r \dim(L^2(Y_{j_u}, \mu; \mathbb{F})) \geq n$. Take a partition P_1, \dots, P_r of $\{e_1, \dots, e_n\}$ with $|P_u| \leq \dim(L^2(Y_{j_u}, \mu; \mathbb{F}))$ for all $u \in \llbracket 1, r \rrbracket$.

For all $u \in \llbracket 1, r \rrbracket$, let $(g_u^p)_{p \in P_u}$ be an orthonormal family in $L^2(Y_{j_u}, \mu; \mathbb{F})$ and define $(\varphi_x)_{x \in X_{j_u}}$ by

$$\varphi_y = \sum_{p \in P_u} g_u^p(y) p \quad \text{for all } y \in Y_{j_u}$$

and

$$\varphi_x := \frac{\overline{h(x)}}{\|h\|_{L^2(X_{j_u} \setminus Y_{j_u}, \mu; \mathbb{F})}^2} \left(d_{j_u} - \int_Y h(y) \varphi_y d\mu(y) \right) \quad \text{for all } x \in X_{j_u} \setminus Y_{j_u}.$$

For all $j \notin \{j_u : u \in \llbracket 1, r \rrbracket\}$, define $(\varphi_x)_{x \in X_j}$ by

$$\varphi_y = 0 \quad \text{for all } y \in Y_j$$

and

$$\varphi_x := \frac{\overline{h(x)}}{\|h\|_{L^2(X_j \setminus Y_j, \mu; \mathbb{F})}^2} d_j \quad \text{for all } x \in X_j \setminus Y_j.$$

Let $\Phi = (\varphi_x)_{x \in X}$. We have

$$\begin{aligned}
W(\Phi) &= \left(\int_{X_j} h(x) \varphi_x d\mu(x) \right)_{j \in \llbracket 1, l \rrbracket} \\
&= \left(\int_{Y_j} h(y) \varphi_y d\mu(y) \right. \\
&\quad \left. + \left(\int_{X_j \setminus Y_j} h(x) \frac{\overline{h(x)}}{\|h\|_{L^2(X_j \setminus Y_j, \mu; \mathbb{F})}^2} d\mu(x) \right) \left(d_j - \int_{Y_j} h(y) \varphi_y d\mu(y) \right) \right)_{j \in \llbracket 1, l \rrbracket} \\
&= (d_j)_{j \in \llbracket 1, l \rrbracket} = D.
\end{aligned}$$

Moreover, $\Phi \in \mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}}$ since

$$\forall v \in \mathbb{F}^n : \|v\|^2 = \sum_{u=1}^r \int_{Y_{j_u}} |\langle v, \varphi_x \rangle|^2 d\mu(x) \leq \int_X |\langle v, \varphi_x \rangle|^2 d\mu(x)$$

and

$$\begin{aligned}
\int_X |\langle v, \varphi_x \rangle|^2 d\mu(x) &\leq \|v\|^2 + \left(\sum_{u=1}^r \frac{\|d_{j_u} - \int_{Y_{j_u}} h(y) \varphi_y d\mu(y)\|^2}{\|h\|_{L^2(X_{j_u} \setminus Y_{j_u}, \mu; \mathbb{F})}^2} \right) \|v\|^2 \\
&\quad + \left(\sum_{j \notin \{j_u : u \in \llbracket 1, r \rrbracket\}} \frac{\|d_j\|^2}{\|h\|_{L^2(X_j \setminus Y_j, \mu; \mathbb{F})}^2} \right) \|v\|^2
\end{aligned}$$

□

2.5.2 A quadratic equation

Proposition 2.5.5. *Let (X, Σ, μ) be a σ -finite measure space, $b \in \mathbb{C}^n \setminus \{0\}$, $\epsilon > 0$ and $h \in L^\infty(X, \mu; \mathbb{C})$.*

Consider

$$q : \begin{cases} L^2(X, \mu; \mathbb{C}^n) & \rightarrow \mathbb{C}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) \langle b, f_x \rangle f_x d\mu(x) \end{cases}.$$

Then for all $d \in \mathbb{C}^n$ such that

- *if $d \neq 0$, then there exists a measurable subset $Y \subseteq X$, two measurable subsets $B_1 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(\langle b, d \rangle z) > \epsilon\}$ and $B_2 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(\langle b, d \rangle z) > \epsilon \text{ and } \operatorname{Im}(\langle b, d \rangle z) > \epsilon\}$ such that $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$ and $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$,*

- if $d = 0$, then there exist a measurable subset $Y \subseteq X$ such that $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$ and $h(x) < 0$ μ -almost everywhere on Y , and [(two measurable subsets $B_1 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) < 0\}$ and $B_2 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0\}$ such that $\mu((X \setminus Y) \cap h^{-1}(B_1)), \mu((X \setminus Y) \cap h^{-1}(B_2)) > 0$) or (a measurable subset $B_3 \subseteq \{z \in \mathbb{C} : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) = 0\}$ such that $\mu((X \setminus Y) \cap h^{-1}(B_3)) > 0$),

there exists a continuous frame $\Phi = (\varphi_x)_{x \in X} \in q^{-1}(\{d\})$.

Démonstration. — Suppose $d \neq 0$. Let $\widetilde{B}_1 = (X \setminus Y) \cap h^{-1}(B_1)$ and $\widetilde{B}_2 = (X \setminus Y) \cap h^{-1}(B_2)$. There is no loss in generality in assuming that $\mu(\widetilde{B}_1)$ and $\mu(\widetilde{B}_2)$ are finite since μ is σ -finite. Let $0 < a < \frac{\epsilon}{\|h\|_\infty^2 \|b\|^2}$. Since $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$, we can pick an a -tight frame $(\varphi_y)_{y \in Y} \in \mathcal{F}_{(Y, \mu, n)}^{\mathbb{C}}$. Let $\widetilde{h}(x) = (\langle b, d \rangle - \int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y)) h(x)$ for all $x \in X$. Notice that we have

$$\begin{aligned} \operatorname{Re}(\widetilde{h}(x)) &> 0 && \mu - \text{almost everywhere on } \widetilde{B}_1, \\ \operatorname{Im}(\widetilde{h}(x)) &< 0 && \mu - \text{almost everywhere on } \widetilde{B}_1, \\ \operatorname{Re}(\widetilde{h}(x)) &> 0 && \mu - \text{almost everywhere on } \widetilde{B}_2, \end{aligned}$$

and

$$\operatorname{Im}(\widetilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_2.$$

Let

$$A = \frac{1}{\frac{\langle -\operatorname{Im}(\widetilde{h}), \operatorname{Re}(\widetilde{h}) \rangle_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}}{\|\operatorname{Im}(\widetilde{h})\|_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}^2} + \frac{\langle \operatorname{Im}(\widetilde{h}), \operatorname{Re}(\widetilde{h}) \rangle_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}}{\|\operatorname{Im}(\widetilde{h})\|_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}^2}} > 0$$

and

$$g(x) = \sqrt{A} \frac{\sqrt{-\operatorname{Im}(\widetilde{h}(x))}}{\|\operatorname{Im}(\widetilde{h})\|_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}} 1_{\widetilde{B}_1}(x) + \sqrt{A} \frac{\sqrt{\operatorname{Im}(\widetilde{h}(x))}}{\|\operatorname{Im}(\widetilde{h})\|_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}} 1_{\widetilde{B}_2}(x)$$

for all $x \in X \setminus Y$.

Then it is easily seen that

$$\left(\int_{X \setminus Y} h(x) |g(x)|^2 d\mu(x) \right) \left(-\langle b, d \rangle + \int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y) \right) = -1. \quad (2.1)$$

Consider $(\varphi_x)_{x \in X \setminus Y}$ defined by $\varphi_x = g(x) (-d + \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y))$ for all $x \in X \setminus Y$. Then $\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{C}}$ since we have only completed

$(\varphi_y)_{y \in Y}$ by a function in $L^2(X \setminus Y, \mu; \mathbb{C}^n)$. Moreover

$$\begin{aligned}
q(\Phi) &= \int_X h(x) \langle b, \varphi_x \rangle \varphi_x d\mu(x) \\
&= \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \\
&\quad + \int_{X \setminus Y} h(x) \langle b, g(x) \left(-d + \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right) \rangle \\
&\quad \cdot g(x) \left(-d + \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right) d\mu(x) \\
&= \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \\
&\quad - \underbrace{\left[\int_{X \setminus Y} h(x) |g(x)|^2 \left(-\langle b, d \rangle + \int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y) \right) d\mu(x) \right]}_{=-1} d \\
&\quad + \underbrace{\left[\int_{X \setminus Y} h(x) |g(x)|^2 \left(-\langle b, d \rangle + \int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y) \right) d\mu(x) \right]}_{=-1} \\
&\quad \cdot \left(\int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right) \\
&= d
\end{aligned}$$

using equality 2.1.

- Suppose $d = 0$. Let $\widetilde{B}_1 = (X \setminus Y) \cap h^{-1}(B_1)$ and $\widetilde{B}_2 = (X \setminus Y) \cap h^{-1}(B_2)$. Suppose first that $\mu(\widetilde{B}_1), \mu(\widetilde{B}_2) > 0$. There is no loss in generality in assuming that $\mu(\widetilde{B}_1)$ and $\mu(\widetilde{B}_2)$ are finite since μ is σ -finite. Since $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$, we can pick a frame $(\varphi_y)_{y \in Y} \in \mathcal{F}_{(Y, \mu), n}^{\mathbb{C}}$. Let $\widetilde{h}(x) = -(\int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y)) h(x)$ for all $x \in X$. Notice that we have

$$\operatorname{Re}(\widetilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_1,$$

$$\operatorname{Im}(\widetilde{h}(x)) < 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_1,$$

$$\operatorname{Re}(\widetilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_2,$$

and

$$\operatorname{Im}(\widetilde{h}(x)) > 0 \quad \mu - \text{almost everywhere on } \widetilde{B}_2.$$

Let

$$A = \frac{1}{\frac{\langle -\operatorname{Im}(\widetilde{h}), \operatorname{Re}(\widetilde{h}) \rangle_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}}{\|\operatorname{Im}(\widetilde{h})\|_{L^2(\widetilde{B}_1, \mu; \mathbb{C})}^2} + \frac{\langle \operatorname{Im}(\widetilde{h}), \operatorname{Re}(\widetilde{h}) \rangle_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}}{\|\operatorname{Im}(\widetilde{h})\|_{L^2(\widetilde{B}_2, \mu; \mathbb{C})}^2}} > 0$$

and

$$g(x) = \sqrt{A} \frac{\sqrt{-\operatorname{Im}(\tilde{h}(x))}}{\|\operatorname{Im}(\tilde{h})\|_{L^2(\tilde{B}_1, \mu; \mathbb{C})}} 1_{\tilde{B}_1}(x) + \sqrt{A} \frac{\sqrt{\operatorname{Im}(\tilde{h}(x))}}{\|\operatorname{Im}(\tilde{h})\|_{L^2(\tilde{B}_2, \mu; \mathbb{C})}} 1_{\tilde{B}_2}(x)$$

for all $x \in X \setminus Y$.

Then it is easily seen that

$$\left(\int_{X \setminus Y} h(x) |g(x)|^2 d\mu(x) \right) \left(\int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y) \right) = -1. \quad (2.2)$$

Consider $(\varphi_x)_{x \in X \setminus Y}$ defined by $\varphi_x = g(x) \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y)$ for all $x \in X \setminus Y$. Then $\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{C}}$ since we have only completed $(\varphi_y)_{y \in Y}$ by a function in $L^2(X \setminus Y, \mu; \mathbb{C}^n)$. Moreover

$$\begin{aligned} q(\Phi) &= \int_X h(x) \langle b, \varphi_x \rangle \varphi_x d\mu(x) \\ &= \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \\ &\quad + \int_{X \setminus Y} h(x) \langle b, g(x) \left(\int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right) \rangle \\ &\quad \cdot g(x) \left(\int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right) d\mu(x) \\ &= \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \\ &\quad + \underbrace{\left[\int_{X \setminus Y} h(x) |g(x)|^2 \left(\int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y) \right) d\mu(x) \right]}_{=-1} \\ &\quad \cdot \left(\int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y) \right) \\ &= 0 \end{aligned}$$

using equality 2.2.

Now let $\tilde{B}_3 = (X \setminus Y) \cap h^{-1}(B_3)$ and suppose instead that $\mu(\tilde{B}_3) > 0$. There is no loss in generality in assuming that $\mu(\tilde{B}_3)$ is finite since μ is σ -finite. Since $\dim(L^2(Y, \mu; \mathbb{C})) \geq n$, we can pick a frame $(\varphi_y)_{y \in Y} \in \mathcal{F}_{(Y, \mu), n}^{\mathbb{C}}$. Let $\tilde{h}(x) = -(\int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y)) h(x)$ for all $x \in X$. Notice that we have

$$\tilde{h}(x) > 0 \quad \mu - \text{almost everywhere on } \tilde{B}_3.$$

Let

$$g(x) = \frac{\sqrt{\tilde{h}(x)}}{\|\tilde{h}\|_{L^2(\tilde{B}_3, \mu; \mathbb{C})}} 1_{\tilde{B}_3}(x) \text{ for all } x \in X \setminus Y$$

Then it is easily seen that

$$\left(\int_{X \setminus Y} h(x) |g(x)|^2 d\mu(x) \right) \left(\int_Y h(y) |\langle b, \varphi_y \rangle|^2 d\mu(y) \right) = -1. \quad (2.3)$$

Consider $(\varphi_x)_{x \in X \setminus Y}$ defined by $\varphi_x = g(x) \int_Y h(y) \langle b, \varphi_y \rangle \varphi_y d\mu(y)$ for all $x \in X \setminus Y$. Then $\Phi = (\varphi_x)_{x \in X} \in \mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{C}}$ since we have only completed $(\varphi_y)_{y \in Y}$ by a function in $L^2(X \setminus Y, \mu; \mathbb{C}^n)$. Moreover we can prove that $q(\Phi) = 0$ as before using equality 2.3. □

2.6 Examples of subspaces in which the space of continuous frames is relatively dense

Corollary 2.6.1. *Let $n \in \mathbb{N}^*$ and $T : L^2(X, \mu; \mathbb{F}^n) \rightarrow \mathbb{F}$ be a non-zero linear form. Suppose that $\dim(L^2(X, \mu; \mathbb{F})) \geq n$. Then since $T^{-1}(\{d\})$ is affine and by corollaries 2.4.1 and 2.5.1, for all $d \neq 0$, $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}} \cap T^{-1}(\{d\})$ is dense in $T^{-1}(\{d\})$*

Corollary 2.6.2. *Let (X, Σ, μ) be a measure space, $d \in \mathbb{C}^n$, and $h \in L^2(X, \mu; \mathbb{F})$ such that $T^{-1}(\{d\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$ (see for instance corollary 2.5.2 and proposition 2.5.3), where*

$$T : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow \mathbb{F}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) f_x d\mu(x) \end{cases}.$$

Then since $T^{-1}(\{d\})$ is affine, by corollary 2.4.1, $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}} \cap T^{-1}(\{d\})$ is dense in $T^{-1}(\{d\})$.

Corollary 2.6.3. *Let (X, Σ, μ) be a measure space, $l \in \mathbb{N}^*$, $(X_j)_{j \in \llbracket 1, l \rrbracket}$ a partition of X by measurable subsets, $h \in L^2(X, \mu; \mathbb{F})$, and $D \in \mathbb{F}^n$ such that $W^{-1}(\{D\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$ (see for instance proposition 2.5.4), where*

$$W : \begin{cases} L^2(X, \mu; \mathbb{F}^n) & \rightarrow \prod_{j \in \llbracket 1, l \rrbracket} \mathbb{F}^n \\ F = (f_x)_{x \in X} & \mapsto (\int_{X_j} h(x) f_x)_{j \in \llbracket 1, l \rrbracket} \end{cases}.$$

Then since $W^{-1}(\{D\})$ is affine, and by corollary 2.4.1, $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{F}} \cap W^{-1}(\{D\})$ is dense in $W^{-1}(\{D\})$.

Remark 2.6.1. Consider the function q of proposition 2.5.5. If $\Phi = (\varphi_x)_{x \in X} \in q^{-1}(\{0\})$, then we also have $\Phi \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ since $q(2\Phi) = 0$.

Proposition 2.6.1. Consider the function q of proposition 2.5.5. Let $\Phi = (\varphi_x)_{x \in X} \in q^{-1}(\{0\})$ and $U = (u_x)_{x \in X} \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$. Then for all $\lambda, \mu \in \mathbb{R}$, $\lambda\Phi + \mu U \in q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$.

In particular, $q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ is a star domain relatively to Φ .

Démonstration. Let $\lambda, \mu \in \mathbb{R}$. Let s be the sesquilinear form $\begin{cases} L^2(X, \mu; \mathbb{C}^n) & \rightarrow \mathbb{C}^n \\ (F, G) & \mapsto \int_X \langle b, g_x \rangle f_x d\mu(x) \end{cases}$. We have

$$\begin{aligned} q(\lambda\Phi + \mu U) &= \lambda^2 \underbrace{q(\Phi)}_0 + \lambda\mu(s(\Phi, U) + s(U, \Phi)) + \mu^2 \underbrace{q(U)}_0 \\ &= \lambda\mu \left(\underbrace{q(\Phi + U)}_0 - \underbrace{q(\Phi)}_0 - \underbrace{q(U)}_0 \right) \\ &= 0. \end{aligned}$$

We can show similarly that $q((\lambda + 1)\Phi + \mu U) = 0$. □

Corollary 2.6.4. Let (X, Σ, μ) be a σ -finite measure space, $b \in \mathbb{C}^n \setminus \{0\}$, and $h \in L^\infty(X, \mu; \mathbb{C})$ such that $q^{-1}(\{0\})$ contains a continuous frame $\Phi = (\varphi_x)_{x \in X}$ (see for instance proposition 2.5.5), where

$$q : \begin{cases} L^2(X, \mu; \mathbb{C}^n) & \rightarrow \mathbb{C}^n \\ F = (f_x)_{x \in X} & \mapsto \int_X h(x) \langle b, f_x \rangle f_x d\mu(x) \end{cases}.$$

Then by proposition 2.6.1 and corollary 2.4.1, $\mathcal{F}_{(X, \Sigma, \mu), n}^{\mathbb{C}} \cap q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$ is dense in $q^{-1}(\{0\}) \cap (q^{-1}(\{0\}) - \Phi)$.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Chapitre 3

Independence, infinite dimension, and operators

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Abstract. In [Appl. Comput. Harmon. Anal., 46(3) :664–673, 2019] O. Christensen and M. Hasannasab observed that assuming the existence of an operator T sending e_n to e_{n+1} for all $n \in \mathbb{N}$ (where $(e_n)_{n \in \mathbb{N}}$ is a sequence of vectors) guarantees that $(e_n)_{n \in \mathbb{N}}$ is linearly independent if and only if $\dim\{e_n\}_{n \in \mathbb{N}} = \infty$. In this article, we recover this result as a particular case of a general order-theory-based model-theoretic result. We then return to the context of vector spaces to show that, if we want to use a condition like $T(e_i) = e_{\phi(i)}$ for all $i \in I$ where I is countable as a replacement of the previous one, the conclusion will only stay true if $\phi : I \rightarrow I$ is conjugate to the successor function $\text{succ} : n \mapsto n + 1$ defined on \mathbb{N} . We finally prove a tentative generalization of the result, where we replace the condition $T(e_i) = e_{\phi(i)}$ for all $i \in I$ where ϕ is conjugate to the successor function with a more sophisticated one, and to which we have not managed to find a new application yet.

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3.1 Introduction

Linear algebra is an entrenched subject of mathematics that started with the introduction of coordinates in geometry by René Descartes. Its modern theory

emerged in the late nineteenth century after Peano gave the definition of a vector space. This theory makes heavy use of the concepts of linear independence and dimension, which often allow to state important theorems and conjectures.

Generally, linear independence of an infinite sequence implies that it spans an infinite-dimensional space, but not the opposite. As a result, it is interesting to consider the conditions of a reverse statement. Such reverse statements may allow to solve standing problems on linear independence.

In [22], O. Christensen and M. Hasannasab observed that assuming the existence of an operator T sending e_n to e_{n+1} for all $n \in \mathbb{N}$ (where $(e_n)_{n \in \mathbb{N}}$ is a sequence of vectors) guarantees that $(e_n)_{n \in \mathbb{N}}$ is linearly independent if and only if $\dim\{e_n\}_{n \in \mathbb{N}} = \infty$. To wit :

Proposition 3.1.1. (*O. Christensen and M. Hasannasab*)

Let E be a vector space and $(e_n)_{n \in \mathbb{N}}$ a family in E indexed by \mathbb{N} . Then

$$(\exists T \in L(\text{span}\{e_n\}_{n \in \mathbb{N}}, E) : \forall n \in \mathbb{N} : T(e_n) = e_{n+1}) \text{ and } \dim \text{span}\{e_n\}_{n \in \mathbb{N}} = +\infty \Rightarrow \\ (e_n)_{n \in \mathbb{N}} \text{ is free.}$$

In this article, we prove some additional results related to proposition 3.1.1. First, we recover proposition 3.1.1 as a particular case of a general order-theory-based model-theoretic result. We then return to the context of vector spaces to show that, if we want to use a condition like $T(e_i) = e_{\phi(i)}$ for all $i \in I$ where I is countable as a replacement of the previous one, the conclusion will only stay true if $\phi : I \rightarrow I$ is conjugate to the successor function $\text{succ} : n \mapsto n + 1$ defined on \mathbb{N} . We finally prove a tentative generalization of the result, where we replace the condition $T(e_i) = e_{\phi(i)}$ for all $i \in I$ where ϕ is conjugate to the successor function with a more sophisticated one, and to which we have not managed to find a new application yet.

Plan of the article. We dedicate section 3.2 to the notations. We then prove in section 3.3 some order-theoretic lemmas. These lemmas will allow us to prove in the next section 3.4 a model-theoretic result and recover proposition 3.1.1 as a particular case. We then return in section 3.5 to the context of vector spaces and show that proposition 3.1.1 can at most be generalized in the countable case to families $(e_i)_{i \in I}$ indexed by a countable set I and maps $\phi : I \rightarrow I$ that are conjugate to the successor function $\text{succ} : n \mapsto n + 1$ defined on \mathbb{N} , at least if we want to preserve a condition like $T(e_i) = e_{\phi(i)}$ for all $i \in I$. We finally prove in section 3.6 a tentative generalization of the result, where we replace the condition $T(e_i) = e_{\phi(i)}$ for all $i \in I$ where ϕ is conjugate to the successor function with a more sophisticated one.

3.2 Notations

In the sequel, \mathbb{N} denotes the set $\{0, 1, 2, \dots\}$ of natural numbers including 0. \mathbb{N}^* denotes $\mathbb{N} \setminus \{0\}$.

If A is a set, we denote by $|A|$ the cardinality of A , $\mathcal{P}(A)$ the powerset of A , $\mathcal{P}_\omega(A)$ the set $\{B \subseteq A : |B| < \infty\}$, and $\mathcal{P}_{\omega,*}(A)$ the set $\{B \subseteq A : 0 < |B| < \infty\}$.

If A is a set, $\phi : A \rightarrow A$ a self map and $n \in \mathbb{N}^*$, we denote by ϕ^n the composition of ϕ with itself n times : $\phi \circ \dots \circ \phi : A \rightarrow A$. In addition, we define ϕ^0 to be the identity function on A . Moreover, if $a \in A$, we denote by $Orb_\phi(a)$ the forward orbit of a under the iterates of ϕ : $\{\phi^n(a) : n \in \mathbb{N}\}$.

If E and F are two vector spaces, $L(E, F)$ denotes the set of linear operators from E to F . When $E = F$, we simply write $L(E)$. If E is a vector space, we denote by $L(*, E)$ the **class** $\{T \in L(E^*, E) : E^* \text{ is a vector space}\}$.

3.3 Lemmas in order theory

Definition 3.3.1. Consider some set P and a binary relation \leq on P . Then \leq is a *preorder* if it is reflexive and transitive; i.e., for all a, b and c in P , we have that :

- $a \leq a$ (reflexivity)
- if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)

A set that is equipped with a preorder is called a *preordered set*.

Definition 3.3.2. Consider a preordered set (P, \leq) and a map $p : P \rightarrow P$. Then p is called a *projection* if for all a and b in P , we have that :

- $a \leq b$ implies $p(a) \leq p(b)$ (p is monotone/increasing/order-preserving/isotone)
- $p(p(a)) = p(a)$ (idempotence)

Lemma 3.3.1. Let (P, \leq) be a preordered set, $(a_n)_{n \in \mathbb{N}}$ a sequence in P , $b \in P$ and p a projection on P .

Suppose there exists an increasing map $f : P \rightarrow P$ such that $f(p(b)) \leq p(f(b))$, $\forall n \in \mathbb{N} : a_{n+1} \leq f(a_n)$, and $f(b) \leq a_0$.

Then : $(a_0 \leq p(b)) \Rightarrow (\forall n \in \mathbb{N} : a_n \leq p(b))$.

Démonstration. Indeed, suppose that $a_0 \leq p(b)$. Let's show by induction that $\forall n \in \mathbb{N} : a_n \leq p(b)$. The base case is the hypothesis $a_0 \leq p(b)$. Suppose we have that $a_n \leq p(b)$ for some $n \in \mathbb{N}$. Then $a_{n+1} \leq f(a_n) \leq f(p(b)) \leq p(f(b)) \leq p(a_0) \leq p(p(b)) = p(b)$. Hence $a_n \leq p(b)$ holds for all $n \in \mathbb{N}$ and the lemma is proved. \square

Definition 3.3.3. Consider some set P and a binary relation \leq on P . Then \leq is a *partial order* if it is a preorder and for all a and b in P , we have $a \leq b \leq a$ implies $a = b$ (antisymmetry).

A set that is equipped with a partial order is called a *partially ordered set* or *poset*.

Definition 3.3.4. A set S partially ordered by the binary relation \leq is a *join-semilattice* if for all elements x and y of S , the smallest upper bound of the set $\{x, y\}$ exists in S .

The smallest upper bound of the set $\{x, y\}$ is called the join of x and y , denoted $x \vee y$.

Lemma 3.3.2. Let (S, \leq, \vee) be a join-semilattice, p a projection on S , $(a_{m,n})_{(m,n) \in \mathbb{N}^2}$ a double sequence in S , $(b_m)_{m \in \mathbb{N}}$ an increasing sequence in S such that $\forall m \in \mathbb{N} : a_{m,0} \leq p(b_{m+1})$.

Suppose there exists $m_\bullet \in \mathbb{N}$ and an increasing map $f : S \rightarrow S$ such that $f(p(b_{m_\bullet})) \leq p(f(b_{m_\bullet}))$, $\forall (m, n) \in \mathbb{N}^2 : a_{m,n+1} \leq f(a_{m,n})$, and $f(b_{m_\bullet}) \leq \bigvee_{i=0}^{m_\bullet} a_{i,0}$. Then : $(a_{m_\bullet,0} \leq p(b_{m_\bullet})) \Rightarrow (\forall n \in \mathbb{N} : a_{m_\bullet,n} \leq p(b_{m_\bullet}))$.

Démonstration. For all $(m, n) \in \mathbb{N}^2$, let $\widetilde{a_{m,n}} = \bigvee_{i=0}^m a_{i,n}$. Suppose that $a_{m_\bullet,0} \leq p(b_{m_\bullet})$. Then $\widetilde{a_{m_\bullet,0}} = \bigvee_{i=0}^{m_\bullet} a_{i,0} \leq p(b_{m_\bullet})$ since $\forall i \in \llbracket 0, m_\bullet - 1 \rrbracket : a_{i,0} \leq p(b_{i+1})$, $a_{m_\bullet,0} \leq p(b_{m_\bullet})$, and $(p(b_i))_{i \in \mathbb{N}}$ is increasing. Moreover, f being increasing implies $\forall a, b \in S : f(a) \vee f(b) \leq f(a \vee b)$, and so $\forall n \in \mathbb{N} : \widetilde{a_{m_\bullet,n+1}} = \bigvee_{i=0}^{m_\bullet} a_{i,n+1} \leq \bigvee_{i=0}^{m_\bullet} f(a_{i,n}) \leq f(\bigvee_{i=0}^{m_\bullet} a_{i,n}) = f(\widetilde{a_{m_\bullet,n}})$. Therefore, applying lemma 3.3.1 to $(\widetilde{a_{m_\bullet,n}})_{n \in \mathbb{N}}$, b_{m_\bullet} , and f , we have that $\forall n \in \mathbb{N} : \widetilde{a_{m_\bullet,n}} \leq p(b_{m_\bullet})$ which implies $\forall n \in \mathbb{N} : a_{m_\bullet,n} \leq p(b_{m_\bullet})$. \square

Lemma 3.3.3. Let (S, \leq, \vee) be a join-semilattice, p a projection on S , $(e_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ two sequences in S such that $(b_n)_{n \in \mathbb{N}}$ is increasing and $\forall n \in \mathbb{N} : e_n \leq p(b_{n+1})$.

Suppose there exists $m_\bullet \in \mathbb{N}$ and an increasing map $f : S \rightarrow S$ such that $f(p(b_{m_\bullet})) \leq p(f(b_{m_\bullet}))$, $\forall n \in \mathbb{N} : e_{n+1} \leq f(e_n)$, and $f(b_{m_\bullet}) \leq \bigvee_{i=0}^{m_\bullet} e_i$.

Then : $(e_{m_\bullet} \leq p(b_{m_\bullet})) \Rightarrow (\forall n \in \mathbb{N} : e_{m_\bullet+n} \leq p(b_{m_\bullet}))$.

Démonstration. Define $a_{m,n}$ as e_{m+n} for all $(m, n) \in \mathbb{N}^2$. Then $((a_{m,n})_{(m,n) \in \mathbb{N}^2}, (b_n)_{n \in \mathbb{N}}, f)$ is a triple satisfying the conditions of lemma 3.3.2 and so we deduce the result. \square

3.4 Application to model theory

In this section, we will use lemma 3.3.3 to prove a model-theoretic result. For a quick reference on model theory, see the book [40]. The model-theoretic result allows to recover proposition 3.1.1 when we choose the σ -structure to be a vector space over a field. The idea of the following model-theoretic proposition is to consider an unsorted algebraic σ -structure $\mathcal{A} = (A, \sigma)$ and regard the map sending a set $X \subseteq A$ to the set $\text{Terms}^{\mathcal{A}}[X]$ of interpreted terms with variables taken from X as a special projection map on the boolean algebra $\mathcal{P}(A)$.

Proposition 3.4.1. *Let σ be an algebraic signature and $\mathcal{A} = (A, \sigma)$ a σ -structure. Let $(e_n)_{n \in \mathbb{N}}$ be a sequence of elements in A . Suppose there exists a map $f \in \text{End}_\sigma(A)$ such that $\forall n \in \mathbb{N} : f(e_n) = e_{n+1}$. Then $(\exists m \in \mathbb{N} : e_m \in \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}]) \Rightarrow (\exists m \in \mathbb{N} : \forall n \in \mathbb{N} : e_{m+n} \in \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}])$.*

Démonstration. Notice that $(\mathcal{P}(A), \subseteq, \cup, \cap, \emptyset, A)$ is a boolean algebra and so a join-semilattice $(\mathcal{P}(A), \subseteq, \cup)$. Define p as the map sending a set $X \subseteq A$ to the set $\text{Terms}^{\mathcal{A}}[X]$ of interpreted terms with variables taken from X , and $b_n = \{e_0, \dots, e_{n-1}\}$. The sequence $(e_n)_{n \in \mathbb{N}}$ in A induces a sequence of singletons $(\{e_n\})_{n \in \mathbb{N}}$ in $\mathcal{P}(A)$. Also, the endomorphism f induces the direct image map $f_\bullet = \begin{cases} \mathcal{P}(A) & \rightarrow \mathcal{P}(A) \\ X & \mapsto f(X) \end{cases}$ which is increasing, satisfies $\forall X \in \mathcal{P}(A) : f_\bullet(p(X)) = p(f_\bullet(X))$, $\forall i \in \mathbb{N} : \{e_{i+1}\} = f_\bullet(\{e_i\})$, and $f_\bullet(\{e_0, \dots, e_{m-1}\}) = \{f(e_0), \dots, f(e_{m-1})\} = \{e_1, \dots, e_m\} \subseteq \bigcup_{i=0}^m \{e_i\}$. Moreover $(b_n)_{n \in \mathbb{N}}$ is increasing and $\forall n \in \mathbb{N} : \{e_n\} \subseteq \text{Terms}^{\mathcal{A}}[e_0, \dots, e_n]$. Suppose $\exists m_\bullet \in \mathbb{N} : e_{m_\bullet} \in \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m_\bullet-1}]$. Then the implication follows from lemma 3.3.3 applied to $(S, \leq, \vee) := (\mathcal{P}(A), \subseteq, \cup)$, p , $(\{e_n\})_{n \in \mathbb{N}}$, $(b_n)_{n \in \mathbb{N}}$, f_\bullet , and m_\bullet . \square

The previous proposition can be better appreciated after considering its contrapositive :

Corollary 3.4.1. *Let σ be an algebraic signature and $\mathcal{A} = (A, \sigma)$ a σ -structure. Let $(e_n)_{n \in \mathbb{N}}$ be a sequence of elements in A . Suppose there exists a map $f \in \text{End}_\sigma(A)$ such that $\forall n \in \mathbb{N} : f(e_n) = e_{n+1}$. Then $(\forall m \in \mathbb{N} : \exists \varphi(m) \geq m : e_{\varphi(m)} \notin \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}]) \Rightarrow (\forall m \in \mathbb{N} : e_m \notin \text{Terms}^{\mathcal{A}}[e_0, \dots, e_{m-1}])$.*

Example 3.4.1. Consider the case of a vector space over a field, with its classical signature described for instance in [40] pp. 3-4. Then the corollary means that if we have an infinite sequence $(e_n)_{n \in \mathbb{N}}$ of vectors and a linear map with the property $f(e_n) = e_{n+1}$ for all $n \in \mathbb{N}$, then in order to show the linear independence of the sequence $(e_n)_{n \in \mathbb{N}}$, it is sufficient to prove that it spans an infinite-dimensional space. This is the result appearing in the paper [22] by O. Christensen and M. Hasannasab (proposition 3.1.1 of the present article).

Corollary 3.4.2. *Let E be an infinite-dimensional vector space, $e \in E$, and S a linear operator in E . Define the infinite sequence $(e_n)_{n \in \mathbb{N}}$ in E as $e_n = S^n(e)$ for all $n \in \mathbb{N}$. Suppose that $\dim \text{span}(e_n)_{n \in \mathbb{N}} = \infty$. Then $(e_n)_{n \in \mathbb{N}}$ is free.*

Démonstration. We have $\forall n \in \mathbb{N} : S(e_n) = e_{n+1}$ and we conclude by proposition 3.1.1. \square

We do not know if it is possible to formulate a definition of infinite dimension or independence in the context of general algebraic σ -structures as in corollary 3.4.1. If possible, corollary 3.4.1 may be advantageously applied to other algebraic σ -structures like groups, rings, algebras, etc., and have the same intuitive meaning of establishing a link between infinite dimension and independence.

3.5 The only possible countable extensions

Before establishing the main proposition 3.5.1, we need to recall some well-known lemmas.

Lemma 3.5.1. *Let I be an infinite set, $a \in I$ and $\phi : I \rightarrow I$.*

Suppose that $Orb_\phi(a)$ is infinite.

Then $a, \phi(a), \phi(\phi(a)), \dots$ are distinct.

Démonstration. We use euclidean division. Suppose that $\phi^n(a) = \phi^m(a)$ for $n < m$. By induction, we have $\phi^{n+j}(a) = \phi^{m+j}(a)$ for all $j \in \mathbb{N}$. Let $e \geq n$. Let $e - n = q(m - n) + r$ be the division with remainder of $e - n \in \mathbb{N}$ by $m - n \in \mathbb{N}^*$. If $q \geq 1$, we have $\phi^e(a) = \phi^{n+q(m-n)+r}(a) = \phi^{m+(q-1)(m-n)+r}(a) = \phi^{n+(q-1)(m-n)+r}(a)$. By immediate induction, we have that $\phi^e(a) = \phi^{n+r}(a)$, where $0 \leq r < m - n$. So $Orb_\phi(a) = \{a, \phi(a), \dots, \phi^{m-1}(a)\}$ is finite, contradiction. \square

Lemma 3.5.2. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \rightarrow I$.*

Suppose that $Orb_\phi(a)$ is infinite and $Orb_\phi(b)$ is cofinite.

Then $(\exists(m, n) \in \mathbb{N}^2) : \phi^m(a) = \phi^n(b)$.

Démonstration. Assume by way of contradiction that $(\forall(m, n) \in \mathbb{N}^2 : \phi^m(a) \neq \phi^n(b))$. Then $Orb_\phi(a) \subseteq I \setminus Orb_\phi(b)$. But this is impossible since $Orb_\phi(a)$ is infinite and $I \setminus Orb_\phi(b)$ is finite. Hence the result. \square

Lemma 3.5.3. *Let I be a countably infinite set and $\phi : I \rightarrow I$. Then*

$$(\exists a \in I : Orb_\phi(a) = I) \Leftrightarrow (\phi \text{ is conjugate to } succ : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n + 1 \end{cases} \text{ in the sense that } \\ \exists \alpha : \mathbb{N} \rightarrow I \text{ such that } \alpha \text{ is bijective and } \alpha \circ succ = \phi \circ \alpha).$$

Démonstration. (\Rightarrow) Suppose that $\exists a \in I : Orb_\phi(a) = I$. Define $\alpha : \begin{cases} \mathbb{N} & \rightarrow I \\ n & \mapsto \phi^n(a) \end{cases}$.

By lemma 4.3.1, α is bijective. Moreover, we have clearly $\alpha \circ succ = \phi \circ \alpha$.

(\Leftarrow) Suppose ϕ is conjugated to $succ : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n + 1 \end{cases}$ by α . Then we have $Orb_\phi(\alpha(0)) = \{\phi^n(\alpha(0))\}_{n \in \mathbb{N}} = \{\alpha(succ^n(0))\}_{n \in \mathbb{N}} = \{\alpha(n)\}_{n \in \mathbb{N}} = I$. \square

The following proposition is the main result of this section.

Proposition 3.5.1. *Let E be an infinite dimensional vector space and I a countably infinite set.*

For all $(e_i)_{i \in I} \in E^I$ and $\phi : I \rightarrow I$, let $P((e_i)_{i \in I}, \phi)$ be the proposition

$$\begin{aligned} (\exists T \in L(\text{span}(e_i)_{i \in I}, E) : \forall i \in I : T(e_i) = e_{\phi(i)} \text{ and } \dim \text{span}(e_i)_{i \in I} = +\infty) \\ \Rightarrow (e_i)_{i \in I} \text{ is free.} \end{aligned}$$

Then we have

$$\forall \phi : I \rightarrow I : \left[\left(\forall (e_i)_{i \in I} \in E^I : P((e_i)_{i \in I}, \phi) \right) \Leftrightarrow \exists a \in I : \text{Orb}_\phi(a) = I \right]$$

Démonstration. Let $\phi : I \rightarrow I$.

(\Rightarrow) : Suppose that $\forall (e_i)_{i \in I} : P((e_i)_{i \in I}, \phi)$.

Let's show first that all the orbits of ϕ are cofinite. Suppose by way of contradiction that $\exists a \in I : |I \setminus \text{Orb}_\phi(a)| = +\infty$. Let $(a_n)_{n \in \mathbb{N}}$ be a free family in E . We let

$$u : I \setminus \text{Orb}_\phi(a) \rightarrow \mathbb{N} \text{ be a bijection, and set for all } i \in I, e_i = \begin{cases} 0 & \text{if } i \in \text{Orb}_\phi(a) \\ a_{u(i)} & \text{otherwise} \end{cases}.$$

We define an operator on $\text{span}(e_i)_{i \in I} = \text{span}(a_n)_{n \in \mathbb{N}}$ by setting $T(a_n) = e_{\phi(u^{-1}(n))}$ for all $n \in \mathbb{N}$ and extending linearly. Then we have $\forall i \in I : T(e_i) = e_{\phi(i)}$. Indeed, if $i \in \text{Orb}_\phi(a)$, then $\phi(i) \in \text{Orb}_\phi(a)$ and so $e_i = e_{\phi(i)} = 0$ by definition which makes the relation true. Otherwise, we have $e_i = a_{u(i)}$ and so $T(e_i) = T(a_{u(i)}) = e_{\phi(u^{-1}(u(i)))} = e_{\phi(i)}$. Besides, we have $\dim \text{span}(e_i)_{i \in I} = +\infty$ since $\text{span}(e_i)_{i \in I} = \text{span}(a_n)_{n \in \mathbb{N}}$ and $\dim \text{span}(a_n)_{n \in \mathbb{N}} = +\infty$. Since $P((e_i)_{i \in I}, \phi)$ is true, it follows that $(e_i)_{i \in I}$ is free which is impossible since $e_a = 0$.

Now take $a \in I$ such that $|I \setminus \text{Orb}_\phi(a)|$ is minimal. From lemma 3.5.2, $\forall b \in I : \exists (m(b), n(b)) \in \mathbb{N}^2 : \phi^{m(b)}(b) = \phi^{n(b)}(a)$. If we choose $m(b)$ and $n(b)$ such that $m(b) + n(b)$ is minimal, then $n(b) \geq m(b)$ by minimality of $I \setminus \text{Orb}_\phi(a)$. Let $(a_n)_{n \in \mathbb{N}}$ be a free family in E . For all $i \in I$, we set $e_i = a_{n(i)-m(i)}$. We define an operator on $\text{span}(e_i)_{i \in I} = \text{span}(a_n)_{n \in \mathbb{N}}$ by setting $T(a_n) = a_{n+1}$ for all $n \in \mathbb{N}$ and extending linearly. Then we have $\forall i \in I : T(e_i) = e_{\phi(i)}$. Indeed, if $i \in \text{Orb}_\phi(a)$, then $m(i) = m(\phi(i)) = 0$ and $n(\phi(i)) = n(i) + 1$ which implies $T(e_i) = T(a_{n(i)}) = a_{n(i)+1} = a_{n(\phi(i))} = e_{\phi(i)}$. Otherwise, $m(i) \geq 1$, $m(\phi(i)) = m(i) - 1$ and $n(\phi(i)) = n(i)$ since $\phi^{m(i)}(i) = \phi^{n(i)}(a)$ with $m(i) + n(i)$ minimal and $m(i) \geq 1$ implies $\phi^{m(i)-1}(\phi(i)) = \phi^{n(i)}(a)$ with $(m(i) - 1) + n(i)$ minimal. So $T(e_i) = a_{n(i)-m(i)+1} = a_{n(i)-(m(i)-1)} = a_{n(\phi(i))-m(\phi(i))} = e_{\phi(i)}$. Besides, we have $\dim \text{span}(e_i)_{i \in I} = +\infty$ since $\text{span}(e_i)_{i \in I} = \text{span}(a_n)_{n \in \mathbb{N}}$ and $\dim \text{span}(a_n)_{n \in \mathbb{N}} = +\infty$. Since $P((e_i)_{i \in I}, \phi)$ is true, it follows that $(e_i)_{i \in I}$ is free which implies $\text{Orb}_\phi(a) = I$ (otherwise, $\exists b \in I \setminus \text{Orb}_\phi(a)$, and so $e_b = a_{n(b)-m(b)} = e_{\phi^{n(b)}(a)}$ with $b \neq \phi^{n(b)}(a)$, contradicting the independence of $(e_i)_{i \in I}$).

(\Leftarrow) : Suppose that $\exists a \in I : Orb_\phi(a) = I$. By lemma 3.5.3, ϕ is conjugated to $succ : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n + 1 \end{cases}$ by α . Let $(e_i)_{i \in I} \in E^I$ and suppose there exists $T \in L(\text{span}(e_i)_{i \in I}, E)$ such that for all $i \in I : T(e_i) = e_{\phi(i)}$ and $\dim \text{span}(e_i)_{i \in I} = +\infty$. Define $(f_n)_{n \in \mathbb{N}}$ by $f_n = e_{\alpha(n)}$ for all $n \in \mathbb{N}$. Then $(f_n)_{n \in \mathbb{N}}$ satisfies $\forall n \in \mathbb{N} : T(f_n) = f_{n+1}$ and $\dim \text{span}(f_n)_{n \in \mathbb{N}} = +\infty$ which by proposition 3.1.1 implies that $(f_n)_{n \in \mathbb{N}}$ is free. Since α is a bijection, it follows that $(e_i)_{i \in I}$ is free. \square

3.6 A tentative generalization

Definition 3.6.1. Consider two sets X and Y . A *binary relation* R on X and Y is a subset of the cartesian product $X \times Y$. The *direct image* of a subset $S \subseteq X$ under a binary relation R on X and Y is written $R[S]$ and refers to $\{y \in Y : \exists x \in S : (x, y) \in R\}$. If $X = Y$, a binary relation on X and Y is simply called a relation on X . The relation $\Delta_X := \{(x, x) : x \in X\}$ is called the *identity relation* on X . The *composition* of two binary relations R_1 and R_2 over X and Y , and Y and Z (respectively), is the binary relation over X and Z , denoted $R_1 \circ R_2$, and given by the subset $\{(x, z) \in X \times Z : \exists y \in Y : (x, y) \in R_1 \wedge (y, z) \in R_2\}$. If R is a binary relation on X , we define R^0 as Δ_X and for all $n \geq 1$, R^n as $R \circ R^{n-1}$.

The following proposition which uses the language of functions with set-valued inputs or/and outputs is the main result of this section.

Proposition 3.6.1. *Let E be an infinite dimensional vector space, V a finite dimensional subspace of E , $(e_i)_{i \in I}$ a family of vectors in E , and J an infinite set. Suppose that*

1. *There exist two functions $u : \mathcal{P}_{\omega,*}(I) \rightarrow I$ and $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ such that for all $I^* \in \mathcal{P}_{\omega,*}(I)$, $u(I^*) \in I^*$ and $I^* \subseteq G(I^*)$,*
2. *There exists a function $T : \mathcal{P}_{\omega,*}(I) \times J \rightarrow L(*, E)$ such that for all $(I^*, j) \in \mathcal{P}_{\omega,*}(I) \times J$, $T(I^*, j) \in L(\text{span}(e_i)_{i \in G(I^*)}, E)$,*
3. *There exists a finite subset J_0 of J and a relation R on J such that $R[j] \subseteq J_0$ for all $j \in J_0$ and $(\forall j \in J)(\exists n_j \in \mathbb{N})R^{n_j}[j] \subseteq J_0$,*
4. $\forall i \in I : \dim \text{span}\{T(I^*, j)e_i\}_{j \in J} = \infty$,
5. $\forall (I^*, j) \in \mathcal{P}_{\omega,*}(I) \times J : \forall i \in G(I^*) \setminus \{u(I^*)\} :$

$$T(I^*, j)e_i \in \text{span}\{T(I^*, j')e_{i'}\}_{(j', i') \in R[j] \times G(I^*)} + V.$$

Then $(e_i)_{i \in I}$ is free.

Démonstration. (of proposition 3.6.1) Assume by way of contradiction that $(e_i)_{i \in I}$ is dependent. Then, there exists $I^* \in \mathcal{P}_{\omega,*}(I)$ and coefficients $\{c_i\}_{i \in I^*}$ in \mathbb{C}^* such that

$$\sum_{i \in I^*} c_i e_i = 0,$$

which implies

$$e_{u(I^*)} = \sum_{i \in I^* \setminus \{u(I^*)\}} \frac{-c_i}{c_{u(I^*)}} e_i.$$

By linearity of $T(I^*, j)$, we then have for all $j \in J$

$$T(I^*, j)e_{u(I^*)} = \sum_{i \in I^* \setminus \{u(I^*)\}} \frac{-c_i}{c_{u(I^*)}} T(I^*, j)e_i. \quad (*)$$

We will prove by induction on $n \in \mathbb{N}$ that

$$\forall (n, j, i) \in \mathbb{N} \times J \times (G(I^*) \setminus \{u(I^*)\}) : T(I^*, j)e_i \in \text{span}\{T(I^*, j')e_{i'}\}_{(j', i') \in R^n[j] \times G(I^*)} + V. \quad (**)$$

- For $n = 0$, the relation is true since $R^0(j) = \Delta_J[j] = \{j\}$ and $i \in G(I^*)$ (take $(j', i') := (j, i)$ and $v := 0$).
- Suppose the induction hypothesis is true at the order $n \in \mathbb{N}$. So for all $(j, i) \in J \times (G(I^*) \setminus \{u(I^*)\})$, there exist complex coefficients $\{\alpha_{j', i'}^{j, i}\}_{(j', i') \in R^n[j] \times G(I^*)}$ and $\{\beta_{j', i'}^{j, i}\}_{(j', i') \in R^n[j] \times G(I^*)}$, and vectors $v^{j, i}$ and $w^{j, i}$ in V such that

$$T(I^*, j)e_i = \sum_{(j', i') \in R^n[j] \times G(I^*)} \alpha_{j', i'}^{j, i} T(I^*, j')e_{i'} + v^{j, i}, \quad (***)$$

and

$$T(I^*, j)e_i = \sum_{(j', i') \in R^n[j] \times G(I^*)} \beta_{j', i'}^{j, i} T(I^*, j')e_{i'} + w^{j, i}, \quad (***)$$

because of condition 5 and the induction hypothesis.

Let $(j, i) \in J \times (G(I^*) \setminus \{u(I^*)\})$. We have

$$\begin{aligned} T(I^*, j)e_i &= \sum_{(j', i') \in R^n[j] \times G(I^*)} \beta_{j', i'}^{j, i} T(I^*, j')e_{i'} + w^{j, i} \\ &= \sum_{(j', i') \in R^n[j] \times (G(I^*) \setminus \{u(I^*)\})} \beta_{j', i'}^{j, i} T(I^*, j')e_{i'} \\ &\quad + \sum_{j' \in R^n[j]} \beta_{j', u(I^*)}^{j, i} T(I^*, j')e_{u(I^*)} + w^{j, i} \\ &= \sum_{(j', i') \in R^n[j] \times (G(I^*) \setminus \{u(I^*)\})} \beta_{j', i'}^{j, i} T(I^*, j')e_{i'} \\ &\quad + \sum_{(j', i') \in R^n[j] \times (G(I^*) \setminus \{u(I^*)\})} \beta_{j', u(I^*)}^{j, i} \frac{-c_{i'}}{c_{u(I^*)}} T(I^*, j')e_{i'} + w^{j, i}, \end{aligned}$$

where we have used equation $(*)$ and extended $\{c_i\}_{i \in I^*}$ to $\{c_i\}_{i \in G(I^*)}$ by setting $\forall i' \in G(I^*) \setminus I^* : c_{i'} = 0$. Simplifying, we have

$$T(I^*, j)e_i = \sum_{(j', i') \in R^n[j] \times (G(I^*) \setminus \{u(I^*)\})} (\beta_{j', i'}^{j, i} - \beta_{j', u(I^*)}^{j, i} \frac{c_{i'}}{c_{u(I^*)}}) T(I^*, j')e_{i'} + w^{j, i}.$$

Using equality $(***)$ for each term $T(I^*, j')e_{i'}$ and then rearranging the resulting sum, we have that $T(I^*, j)e_i$ is equal to

$$\begin{aligned} & \sum_{(j'', i'') \in R^{n+1}[j] \times G(I^*)} \left[\sum_{(j', i') \in F(j'', j, n) \times G(I^*) \setminus \{u(I^*)\}} \alpha_{j'', i''}^{j', i'} \left(\beta_{j', i'}^{j, i} - \beta_{j', u(I^*)}^{j, i} \frac{c_{i'}}{c_{u(I^*)}} \right) \right] \\ & \cdot T(I^*, j'')e_{i''} + \left[\sum_{(j', i') \in R^n[j] \times (G(I^*) \setminus \{u(I^*)\})} (\beta_{j', i'}^{j, i} - \beta_{j', u(I^*)}^{j, i} \frac{c_{i'}}{c_{u(I^*)}}) w^{j', i'} + w^{j, i} \right] \\ & \in \text{span}\{T(I^*, j')e_{i'}\}_{(j', i') \in R^{n+1}[j] \times G(I^*)} + V, \end{aligned}$$

where $F(j'', j, n) := \{j' \in R^n[j] : j'' \in R[j']\}$.

Hence the claim is proved. Now letting $j \in J$ and using condition 3, there exists $n_j \in \mathbb{N}$ such that $R^{n_j}[j] \subseteq J_0$, which implies by $(*)$ and the claim $(**)$ that

$$T(I^*, j)e_{u(I^*)} \in \text{span}\{T(I^*, j')e_{i'}\}_{(j', i') \in J_0 \times G(I^*)} + V.$$

Therefore

$$\text{span}\{T(I^*, j)e_{u(I^*)}\}_{j \in J} \subseteq \text{span}\{T(I^*, j')e_{i'}\}_{(j', i') \in J_0 \times G(I^*)} + V.$$

Since $\text{span}\{T(I^*, j')e_{i'}\}_{(j', i') \in J_0 \times G(I^*)} + V$ is finite-dimensional, this contradicts condition 4. \square

Example 3.6.1. (from O. Christensen and M. Hasannab [22])

Let E be an infinite-dimensional vector space and $(e_n)_{n \in \mathbb{N}}$ a sequence in E .

Suppose that $\dim \text{span}(e_n)_{n \in \mathbb{N}} = \infty$ and there exists an operator $S : \text{span}(e_n)_{n \in \mathbb{N}} \rightarrow E$ such that $S(e_n) = e_{n+1}$ for all $n \in \mathbb{N}$. Then $(e_n)_{n \in \mathbb{N}}$ is free.

Indeed, define $u : \mathcal{P}_{\omega, *}(\mathbb{N}) \rightarrow \mathbb{N}$ and $G : \mathcal{P}_{\omega, *}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega, *}(\mathbb{N})$ by $u(I^*) = \max(I^*)$ and $G(I^*) = \llbracket 0, \max(I^*) \rrbracket$ for all $I^* \in \mathcal{P}_{\omega, *}(\mathbb{N})$. Moreover, define $T : \mathcal{P}_{\omega, *}(\mathbb{N}) \times \mathbb{N} \rightarrow L(*, E)$ by $T(I^*, m) = (S^m)|_{\text{span}(e_n)_{n \in \llbracket 0, \max(I^*) \rrbracket}}$ for all $(I^*, m) \in \mathcal{P}_{\omega, *}(\mathbb{N}) \times \mathbb{N}$. Take $J_0 = \{0\}$ and define $R = (0, 0) \cup (\bigcup_{m \in \mathbb{N}^*} (m, m-1)) \subseteq \mathbb{N} \times \mathbb{N}$. Examining the conditions of proposition 3.6.1, we see that conditions 1-2 are valid, 3 is true because $R^m[m] = \{0\} = J_0$ for all $m \in \mathbb{N}$, 4 is valid because $\{T(I^*, m)e_{u(I^*)}\}_{m \in \mathbb{N}} = \{e_{u(I^*)+m}\}_{m \in \mathbb{N}}$ spans an infinite dimensional space since $(e_m)_{m \in \mathbb{N}}$ does, and finally condition 5 is also valid since for all $(I^*, m) \in \mathcal{P}_{\omega, *}(\mathbb{N}) \times \mathbb{N}^*$ and for all $n \in \llbracket 0, \max(I^*) - 1 \rrbracket$ we have $T(I^*, m)e_n = e_{n+m} = T(I^*, m-1)e_{n+1}$ (notice that $m-1 \in R[m]$ and $n+1 \in \llbracket 0, \max(I^*) \rrbracket$), and for all $I^* \in \mathcal{P}_{\omega, *}(\mathbb{N})$ and $n \in \llbracket 0, \max(I^*) - 1 \rrbracket$ we have $T(I^*, 0)e_n = e_n = T(I^*, 0)e_n$ (notice that $0 \in R[0]$ and $n \in \llbracket 0, \max(I^*) \rrbracket$). Hence the result.

Before we move on to the next example, we need to prove two additional lemmas.

Lemma 3.6.1. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \rightarrow I$. Suppose that $\text{Orb}_\phi(a)$ is cofinite. Then*

1. *If $\exists n \in \mathbb{N} : b = \phi^n(a)$, then $\text{Orb}_\phi(b)$ is cofinite,*
2. *If $\exists n \in \mathbb{N} : a = \phi^n(b)$, then $\text{Orb}_\phi(b)$ is cofinite.*

Démonstration. First, notice that by lemma 4.3.1, $a, \phi(a), \phi^2(a), \dots$ are distinct.
(1) : Suppose $\exists n \in \mathbb{N} : b = \phi^n(a)$. Then $I \setminus \text{Orb}_\phi(b) = I \setminus \{\phi^m(a)\}_{m \geq n} = (I \setminus \text{Orb}_\phi(a)) \cup \{\phi^m(a)\}_{m < n}$ is finite.
(2) : Suppose $\exists n \in \mathbb{N} : a = \phi^n(b)$. Then $I \setminus \text{Orb}_\phi(b) = (I \setminus \{\phi^m(b)\}_{m \geq n}) \setminus \{\phi^m(b)\}_{m < n} = (I \setminus \text{Orb}_\phi(a)) \setminus \{\phi^m(b)\}_{m < n}$ is finite. \square

Corollary 3.6.1. *Let I be an infinite set, $a \in I$ and $\phi : I \rightarrow I$. Suppose that $\text{Orb}_\phi(a) = I$. Then $\forall b \in I : \text{Orb}_\phi(b)$ is cofinite*

Lemma 3.6.2. *Let I be an infinite set and $\phi : I \rightarrow I$ such that $\forall a \in I : \text{Orb}_\phi(a)$ is infinite. Then we have*

$$(\exists a \in I : \text{Orb}_\phi(a) = I) \Rightarrow$$

(there exist two functions $u : \mathcal{P}_{\omega,}(I) \rightarrow I$ and $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in I^*, I^* \subseteq G(I^*),$ and $\phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*)$).*

Démonstration. Define

$$n : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathbb{N} \\ I^* & \mapsto \max\{n \in \mathbb{N} : \phi^n(a) \in I^*\} \end{cases}$$

which is well-defined because $\{a, \phi(a), \phi^2(a), \dots\}$ are distinct (lemma 4.3.1) and let

$$u : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto \phi^{n(I^*)}(a) \end{cases}$$

and

$$G : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_{\omega,*}(I) \\ I^* & \mapsto \{a, \phi(a), \dots, \phi^{n(I^*)}(a)\} \end{cases}.$$

With this choice, since $\text{Orb}_\phi(a) = I$, u and G do satisfy the requirements. \square

Example 3.6.2. Let E be an infinite dimension vector space, I a countably infinite set, $(e_i)_{i \in I}$ a family in E and $\phi : I \rightarrow I$ such that $\exists a \in I : \text{Orb}_\phi(a) = I$. Suppose that $\dim \text{span}\{e_i\}_{i \in I} = \infty$ and there exists an operator $S : \text{span}\{e_i\}_{i \in I} \rightarrow$

E such that $S(e_i) = e_{\phi(i)}$ for all $i \in I$. Then $\{e_i\}_{i \in I}$ is free. Indeed, from lemma 3.6.2, there exist two functions $u : \mathcal{P}_{\omega,*}(I) \rightarrow I$ and $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ such that for all $I^* \in \mathcal{P}_{\omega,*}(I)$, $u(I^*) \in I^*$, $I^* \subseteq G(I^*)$ and $\phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*)$. Define $T : \mathcal{P}_{\omega,*}(I) \times \mathbb{N} \rightarrow L(*, E)$ by $T(I^*, m) = (S^m)|_{\text{span}\{e_i\}_{i \in G(I^*)}}$ for all $(I^*, m) \in \mathcal{P}_{\omega,*}(I) \times \mathbb{N}$. Take $J_0 = \{0\}$ and define $R = (0, 0) \cup (\bigcup_{m \in \mathbb{N}^*} (m, m-1)) \subseteq \mathbb{N} \times \mathbb{N}$. Examining the conditions of proposition 3.6.1, we see that conditions 1-2 are valid, 3 is true because $R^m[m] = \{0\} = J_0$ for all $m \in \mathbb{N}$, 4 is valid because $\{T(I^*, m)e_{u(I^*)}\}_{m \in \mathbb{N}} = \{e_{\phi^m(u(I^*))}\}_{m \in \mathbb{N}}$ spans an infinite-dimensional space since $(e_i)_{i \in I}$ does and $|I \setminus \{\phi^m(u(I^*))\}_{m \in \mathbb{N}}| < \infty$ (corollary 4.3.3), and finally condition 5 is also valid since for all $(I^*, m) \in \mathcal{P}_{\omega,*}(I) \times \mathbb{N}^*$ and for all $i \in G(I^*) \setminus \{u(I^*)\}$ we have $T(I^*, m)e_i = e_{\phi^m(i)} = T(I^*, m-1)e_{\phi(i)}$ (notice that $m-1 \in R[m]$ and $\phi(i) \in G(I^*)$), and for all $I^* \in \mathcal{P}_{\omega,*}(I)$ and $i \in G(I^*) \setminus \{u(I^*)\}$ we have $T(I^*, 0)e_i = e_i = T(I^*, 0)e_i$ (notice that $0 \in R[0]$ and $i \in G(I^*)$). Hence the result.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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Chapitre 4

Functions with a maximal number of finite invariant or internally-1-quasi-invariant sets or supersets

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Abstract. A relaxation of the notion of invariant set, known as k -quasi-invariant set, has appeared several times in the literature in relation to group dynamics. The results obtained in this context depend on the fact that the dynamic is generated by a group. In our work, we consider the notions of invariant and 1-internally-quasi-invariant sets as applied to an action of a function f on a set I . We answer several questions of the following type, where $k \in \{0, 1\}$: what are the functions f for which every finite subset of I is internally- k -quasi-invariant? More restrictively, if $I = \mathbb{N}$, what are the functions f for which every finite interval of I is internally- k -quasi-invariant? Last, what are the functions f for which every finite subset of I admits a finite internally- k -quasi-invariant superset? This parallels a similar investigation undertaken by C. E. Praeger in the context of group actions.

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4.1 Introduction

Invariant sets play an essential role in the qualitative study of dynamical systems [48, 15]. In group dynamics, a relaxation of this notion, known as *k-quasi-invariant sets*, or *k-almost-invariant sets* (where $k \in \mathbb{N}$), appears in the works [13, 55, 14, 49, 50, 4, 3] and the references therein. Most of the results contained in these articles crucially depend on the fact that G is a group.

In our work, we depart from the group setting by extending the notion of a *k-quasi-invariant set* under a group action to the context of an action of a set A on another set I (this simply consists of a function $\rho : A \times I \rightarrow I$ with no additional requirement). The lack of bijectivity of the functions $\{\rho(a, \cdot)\}_{a \in A}$ forces us to make the following definitions. If we denote by $|\cdot|$ the cardinality function and $\Lambda^a := \{\rho(a, x) : x \in \Lambda\}$ for $a \in A$ and $\Lambda \subseteq I$, then Λ is *externally-k-quasi-invariant* under the action ρ if

$$\forall a \in A : |\Lambda^a \setminus \Lambda| \leq k,$$

or equivalently

$$\forall a \in A : \exists P \subseteq I : |P| \leq k \text{ and } \Lambda^a \subseteq \Lambda \cup P,$$

and is *internally-k-quasi-invariant* under the action ρ if

$$\forall a \in A : \exists P \subseteq I : |P| \leq k \text{ and } (\Lambda \setminus P)^a \subseteq \Lambda.$$

We focus in this article on internally-*k*-quasi-invariant sets when A is a singleton and $k \in \{0, 1\}$. In this case, the action $\rho : A \times I \rightarrow I$ becomes just a function $f : I \rightarrow I$, and therefore a set Λ is internally-0-quasi-invariant under the action of f if it is a (forward)-invariant set of f , i.e.

$$f(\Lambda) \subseteq \Lambda \tag{4.1}$$

and is internally-1-quasi-invariant under the action of f if

$$\exists a \in \Lambda : f(\Lambda \setminus \{a\}) \subseteq \Lambda. \tag{4.2}$$

Note that the notions of external or internal *k*-quasi-invariance under the action of f that we have just defined, differ from external or internal *k*-quasi-invariance under the action of the monoid \mathbb{N} (by iterates of a function f), which we will not be concerned about in this article.

Most of the findings in [13, 55, 14, 49, 50, 4, 3] about *k*-quasi-invariant sets under a group action tend to be wrong for internally-*k*-quasi-invariant sets under the action of $f : I \rightarrow I$. In [55], the author shows that if every finite subset of I is

k -quasi-invariant under a group action with no fixed points, then I must be finite. The analogous statement for internally-1-quasi-invariant sets under the action of a single function f is wrong, as can be attested by the function $succ : \mathbb{N} \rightarrow \mathbb{N}$. However, the problem considered by the author in [55] is interesting, and can be adapted to our setting : what are the functions $f : I \rightarrow I$ for which every finite subset of I is internally- k -quasi-invariant ? More restrictively, if $I = \mathbb{N}$, what are the functions $f : I \rightarrow I$ for which every finite interval of I is internally- k -quasi-invariant ? Last, what are the functions $f : I \rightarrow I$ for which every finite subset of I admits a finite internally- k -quasi-invariant superset ? In this article, we investigate these types of questions in the $k \in \{0, 1\}$ cases. Besides, we inform the reader that the idea of this article emerged when we were looking for an example to a proposition related to linear independence in ([44], Proposition 6.1 and Example 6.2).

4.2 Notations

In the sequel, \mathbb{N} denotes the set $\{0, 1, 2, \dots\}$ of natural numbers including 0. \mathbb{N}^* denotes $\mathbb{N} \setminus \{0\}$.

If A is a set, we denote by $|A|$ the cardinality of A , $\mathcal{P}(A)$ the powerset of A , $\mathcal{P}_\omega(A)$ the set $\{B \subseteq A : |B| < \infty\}$, $\mathcal{P}_{\omega,*}(A)$ the set $\{B \subseteq A : 0 < |B| < \infty\}$, and $\mathcal{P}_{\omega,n^+}(A)$ the set $\{B \subseteq A : n \leq |B| < \infty\}$.

If (A, \leq) is a non-empty totally ordered set, we denote by $Int_{\omega,*}(A)$ the set $\{[a, b] : a \leq b, |[a, b]| < \infty\}$, and by $Int_{\omega,n^+}(A)$ the set $\{[a, b] : a \leq b, n \leq |[a, b]| < \infty\}$.

If A is a set, $\phi : A \rightarrow A$ a self map and $n \in \mathbb{N}^*$, we denote by ϕ^n the composition of ϕ with itself n times : $\phi \circ \dots \circ \phi : A \rightarrow A$. In addition, we define ϕ^0 to be the identity function on A . Moreover, if $a \in A$, we denote by $\mathcal{O}_\phi^+(a)$ the forward orbit of a under the iterates of ϕ : $\{\phi^n(a) : n \in \mathbb{N}\}$.

4.3 Preliminaries on discrete time forward orbits

4.3.1 Elementary lemmas

The following well-known lemmas are easy to prove and are only reminded for convenience.

Lemma 4.3.1. *Let I be an infinite set, $a \in I$ and $\phi : I \rightarrow I$. Then*

$$\mathcal{O}_\phi^+(a) \text{ is infinite} \Leftrightarrow a, \phi(a), \phi(\phi(a)), \dots \text{ are distinct}$$

or equivalently

$$\mathcal{O}_\phi^+(a) \text{ is finite} \Leftrightarrow \text{the sequence } a, \phi(a), \phi(\phi(a)), \dots \text{ is eventually periodic.}$$

Démonstration. (\Rightarrow) Suppose that $\mathcal{O}_\phi^+(a)$ is infinite. Suppose that $\phi^n(a) = \phi^m(a)$ for some $n < m$. By induction, we have $\phi^{n+j}(a) = \phi^{m+j}(a)$ for all $j \in \mathbb{N}$. Let $e \geq n$. Let $e-n = q(m-n)+r$ be the division with remainder of $e-n \in \mathbb{N}$ by $m-n \in \mathbb{N}^*$. If $q \geq 1$, we have $\phi^e(a) = \phi^{n+q(m-n)+r}(a) = \phi^{m+(q-1)(m-n)+r}(a) = \phi^{n+(q-1)(m-n)+r}(a)$. By immediate induction, we have that $\phi^e(a) = \phi^{n+r}(a)$, where $0 \leq r < m-n$. So $\mathcal{O}_\phi^+(a) = \{a, \phi(a), \dots, \phi^{m-1}(a)\}$ is finite, contradiction.
(\Leftarrow) This is clear. □

Therefore, for any map $\phi : I \rightarrow I$ and $a \in I$, the following simple conjugation result for $\phi|_{\mathcal{O}_\phi^+(a)}$ follows.

Lemma 4.3.2. *Let I be an infinite set, $a \in I$ and $\phi : I \rightarrow I$.*

1. *Suppose that $\mathcal{O}_\phi^+(a)$ is infinite. Then $\phi|_{\mathcal{O}_\phi^+(a)}$ is conjugate to succ :*

$$\begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n+1 \end{cases}.$$

2. *Suppose that $\mathcal{O}_\phi^+(a)$ is finite. Then $\exists(u, v) \in \mathbb{N} \times \mathbb{N}^*$ and a bijection $\alpha : [0, u+v-1] \rightarrow \mathcal{O}_\phi^+(a)$ such that*

$$\mathcal{O}_\phi^+(a) = \{a, \phi(a), \dots, \phi^{u-1}(a)\} \sqcup \{\phi^u(a), \dots, \phi^{u+v-1}(a)\},$$

$$\phi|_{\{a, \phi(a), \dots, \phi^{u-1}(a)\}} \text{ is conjugated to } \text{succ}_{[u]} : \begin{cases} [0, u-1] & \rightarrow [1, u] \\ n & \mapsto n+1 \end{cases} \text{ by } \alpha|_{[0, u-1]},$$

and

$\phi|_{\{\phi^u(a), \dots, \phi^{u+v-1}(a)\}}$ is conjugated to

$$\text{cycle}_{[v]} : \begin{cases} [u, u+v-1] & \rightarrow [u, u+v-1] \\ n & \mapsto n+1 \text{ if } n \in [u, u+v-2] \\ u+v-1 & \mapsto u \end{cases} \text{ by } \alpha|_{[u, u+v-1]}.$$

Démonstration. The map α that realizes the conjugation is, in both cases, the one that sends n to $\phi^n(a)$. □

Thus, we can deduce

Corollary 4.3.1. *Let I be an infinite set, $a \in I$ and $\phi : I \rightarrow I$.*

Suppose that $\mathcal{O}_\phi^+(a)$ is infinite.

Then $\phi|_{\mathcal{O}_\phi^+(a)} : \mathcal{O}_\phi^+(a) \rightarrow \mathcal{O}_\phi^+(a) \setminus \{a\}$ is well-defined and bijective.

Lemma 4.3.3. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \rightarrow I$.*

Suppose that $\mathcal{O}_\phi^+(a)$ is infinite and $\mathcal{O}_\phi^+(b)$ is finite.

Then $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ don't have an intersection point.

Démonstration. Assume by way of contradiction that $(\exists u, v \in \mathbb{N}) : \phi^u(a) = \phi^v(b)$. Then $\mathcal{O}_\phi^+(a) = \{a, \phi(a), \dots, \phi^{u-1}(a)\} \cup \mathcal{O}_\phi^+(\phi^v(b)) \subseteq \{a, \phi(a), \dots, \phi^{u-1}(a)\} \cup \mathcal{O}_\phi^+(b)$, which is finite, a contradiction. \square

Lemma 4.3.4. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \rightarrow I$. Suppose that $\mathcal{O}_\phi^+(a)$ is cofinite and $\mathcal{O}_\phi^+(b)$ is infinite. Then $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ have at least one intersection point.*

Démonstration. Assume by way of contradiction that $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ don't intersect. Then $\mathcal{O}_\phi^+(b) \subseteq I \setminus \mathcal{O}_\phi^+(a)$. But this is impossible since $\mathcal{O}_\phi^+(b)$ is infinite and $I \setminus \mathcal{O}_\phi^+(a)$ is finite. Hence the result. \square

Corollary 4.3.2. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \rightarrow I$. Suppose that $\mathcal{O}_\phi^+(a)$ is cofinite. Then $\mathcal{O}_\phi^+(a)$ intersects $\mathcal{O}_\phi^+(b)$ if and only if $\mathcal{O}_\phi^+(b)$ is infinite.*

Démonstration. Combine lemmas 4.3.3 and 4.3.4. \square

Lemma 4.3.5. *Let I be an infinite set, $(a, b) \in I^2$ and $\phi : I \rightarrow I$. Suppose that $\mathcal{O}_\phi^+(a)$ is cofinite and $\mathcal{O}_\phi^+(b)$ is infinite. Then $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b)$, and consequently $\mathcal{O}_\phi^+(b)$, are cofinite.*

Démonstration. First, notice that by lemma 4.3.1, $a, \phi(a), \phi^2(a), \dots$ are distinct. From the hypothesis, there exist $u, v \in \mathbb{N}$ such that $\phi^u(a) = \phi^v(b)$. So

$$\begin{aligned} I \setminus (\mathcal{O}_\phi^+(b) \cap \mathcal{O}_\phi^+(b)) &= (I \setminus \mathcal{O}_\phi^+(a)) \cup \left[(I \setminus \mathcal{O}_\phi^+(\phi^v(b))) \setminus \{b, \dots, \phi^{v-1}(b)\} \right] \\ &= (I \setminus \mathcal{O}_\phi^+(a)) \cup \left[(I \setminus \mathcal{O}_\phi^+(\phi^u(a))) \setminus \{b, \dots, \phi^{v-1}(b)\} \right] \\ &= (I \setminus \mathcal{O}_\phi^+(a)) \\ &\quad \cup \left[((I \setminus \mathcal{O}_\phi^+(a)) \cup \{a, \dots, \phi^{u-1}(a)\}) \setminus \{b, \dots, \phi^{v-1}(b)\} \right] \\ &\subseteq (I \setminus \mathcal{O}_\phi^+(a)) \cup \{a, \dots, \phi^{u-1}(a)\} \end{aligned}$$

is finite. The fact that $\mathcal{O}_\phi^+(b)$ is also cofinite is due to the inclusion $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b) \subseteq \mathcal{O}_\phi^+(b)$. \square

Corollary 4.3.3. *Let I be an infinite set, $a \in I$ and $\phi : I \rightarrow I$. Suppose that $\mathcal{O}_\phi^+(a) = I$. Then $\forall b \in I : \mathcal{O}_\phi^+(b)$ is cofinite*

Démonstration. This can be seen directly or as a consequence of corollary 4.3.2 and lemma 4.3.5 (since $\forall b \in I : b \in \mathcal{O}_\phi^+(a)$). \square

Remark 4.3.1. The converse statement is false. Indeed, let $\phi : \mathbb{N} \cup \{\bullet\} \rightarrow \mathbb{N} \cup \{\bullet\}$:
 $\begin{cases} n & \mapsto n+1 \\ \bullet & \mapsto 1 \end{cases}$. Then $\forall a \in \mathbb{N} \cup \{\bullet\} : (\mathcal{O}_\phi^+(a))$ is cofinite and $\mathcal{O}_\phi^+(a) \neq \mathbb{N} \cup \{\bullet\}$.

4.3.2 Nearest element to a finite set lying in the intersection of orbits of elements of that set

We now associate to each self-map $\phi : I \rightarrow I$ on an infinite set I , a particular *noncanonical* map $\xi_\phi : D_\phi \rightarrow I$, which gives for each $I^* \in D_\phi$ one of the nearest elements to I^* that lies in the intersection of orbits of elements of I^* . The value of $\xi_\phi(I^*)$, for $I^* \in D_\phi$, is uniquely determined if all the orbits of elements in I^* are infinite (use lemmas 4.3.6 and 4.3.1), which arises if and only if $\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$ is infinite by corollary 4.3.4. However, if $\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$ is finite, which arises if and only if all the orbits of elements in I^* are finite by corollary 4.3.8, then $\xi_\phi(I^*)$ may be non-uniquely determined as in the case where I^* is precisely a finite cycle of a map $\phi : I \rightarrow I$. This definition will be used in the rest of this article.

Definition 4.3.1. Let I be an infinite set and $\phi : I \rightarrow I$.

Let $D_\phi = \{I^* \in \mathcal{P}_{\omega,*}(I) : \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \neq \emptyset\} \subseteq \mathcal{P}_{\omega,*}(I)$.

For all $I^* \in D_\phi$, $z \in \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$ and $a \in I^*$, we define m_a^z as $\min\{m \in \mathbb{N} : \phi^m(a) = z\}$.

We define a *noncanonical* map $\xi_\phi : D_\phi \rightarrow I$ by selecting for all $I^* \in D_\phi$, $\xi_\phi(I^*) \in \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$ with $\sum_{a \in I^*} m_a^{\xi_\phi(I^*)}$ minimal.

Example 4.3.1. If I is infinite and $I^* \in \mathcal{P}_{\omega,*}(I)$ is such that $(\forall a \in I^*) : \mathcal{O}_\phi^+(a)$ is cofinite, then $I^* \in D_\phi$, because a finite intersection of cofinite sets is cofinite and hence non-empty.

Lemma 4.3.6. Let I be an infinite set, $\phi : I \rightarrow I$ and $I^* \in \mathcal{P}_{\omega,*}(I)$ such that $\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \neq \emptyset$. Then $\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) = \mathcal{O}_\phi^+(\xi_\phi(I^*))$.

Démonstration. Suppose that $z \in \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$. So $(\forall a \in I^*) : z = \phi^{m_a^z}(a)$. Also, we have $(\forall a \in I^*) : \xi_\phi(I^*) = \phi^{m_a^{\xi_\phi(I^*)}}(a)$ where the $m_a^{\xi_\phi(I^*)}$'s satisfy a minimality property.

Hence $\sum_{a \in I^*} m_a^z \geq \sum_{a \in I^*} m_a^{\xi_\phi(I^*)}$, and so $(\exists a \in I^*) : m_a^z \geq m_a^{\xi_\phi(I^*)}$. This implies that

$$z = \phi^{m_a^z}(a) = \phi^{m_a^z - m_a^{\xi_\phi(I^*)}}(\phi^{m_a^{\xi_\phi(I^*)}}(a)) = \phi^{m_a^z - m_a^{\xi_\phi(I^*)}}(\xi_\phi(I^*)) \in \mathcal{O}_\phi^+(\xi_\phi(I^*)).$$

Conversely, suppose that $z \in \mathcal{O}_\phi^+(\xi_\phi(I^*))$. So $(\exists n \in \mathbb{N}) : z = \phi^n(\xi_\phi(I^*))$. Also, we have $(\forall a \in I^*) : \xi_\phi(I^*) = \phi^{m_a^{\xi_\phi(I^*)}}(a)$. Therefore $(\forall a \in I^*) : z = \phi^{n+m_a^{\xi_\phi(I^*)}}(a) \in \mathcal{O}_\phi^+(a)$. Hence the two sets are equal. \square

Lemma 4.3.7. Let I be an infinite set, $(a, b, c) \in I^3$ and $\phi : I \rightarrow I$.

Suppose that $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ are infinite and have at least one intersection point, and that $c \in \mathcal{O}_\phi^+(b)$

Then $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(c)$ is infinite.

Démonstration. Since $\mathcal{O}_\phi^+(a)$ and $\mathcal{O}_\phi^+(b)$ intersect, there exist $(m, n) \in \mathbb{N}^2$ such that $\phi^n(a) = \phi^m(b)$. Also, since $c \in \mathcal{O}_\phi^+(b)$, there exists $p \in \mathbb{N}$ such that $c = \phi^p(b)$. Therefore $\phi^{n+p}(a) = \phi^{m+p}(b) = \phi^m(c)$, and so this common element belongs to the orbits of a and c . Moreover, since $\mathcal{O}_\phi^+(a)$ is infinite, and this common element and its iterates by ϕ belong to it, we have by lemma 4.3.1 that the orbit of this element is infinite. This orbit is included in the orbits of a and c , and so in their intersection, which proves that $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(c)$ is infinite. \square

Corollary 4.3.4. *Let I be an infinite set and $\phi : I \rightarrow I$.*

Let $I^ = \{a_1, \dots, a_n\} \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_\phi^+(a)| = +\infty\})$ such that the orbits $\mathcal{O}_\phi^+(a_i)$ intersect jointly or in pairs. Then $\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$ is infinite.*

Démonstration. If the orbits intersect jointly, then they intersect in pairs. Therefore it suffices to consider the latter case. Now, we have $\forall i \in \llbracket 1, n \rrbracket : |\mathcal{O}_\phi^+(a_i)| = +\infty$, so using lemma 4.3.7 (in conjunction with lemma 4.3.6) repeatedly, we see that $|\bigcap_{i \in \llbracket 1, n \rrbracket} \mathcal{O}_\phi^+(a_i)| = \infty$. \square

The following lemma asserts that we cannot find a subset $I^* \in D_\phi$ containing both an element of infinite orbit and another element of finite orbit.

Lemma 4.3.8. *Let I be an infinite set and $\phi : I \rightarrow I$. Then we have*

$$D_\phi \subseteq \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_\phi^+(a)| = +\infty\}) \cup \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_\phi^+(a)| < \infty\})$$

Démonstration. Let $I^* \in D_\phi$. Suppose by way of contradiction that $\{a, b\} \subseteq I^*$ with $\mathcal{O}_\phi^+(a) = +\infty$ and $\mathcal{O}_\phi^+(b) < +\infty$. Let $c \in \mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b)$. If $\mathcal{O}_\phi^+(c)$ were finite, we would have $\phi^n(c) = \phi^m(c)$ for some $m \neq n$ by lemma 4.3.1, and so $\phi^{n+u}(a) = \phi^{m+u}(a)$ for some $u \in \mathbb{N}$ since $c \in \mathcal{O}_\phi^+(a)$, which contradicts lemma 4.3.1 since $\mathcal{O}_\phi^+(a)$ is infinite. So $\mathcal{O}_\phi^+(c)$ must be infinite. But $c \in \mathcal{O}_\phi^+(b)$ implies $\mathcal{O}_\phi^+(c) \subseteq \mathcal{O}_\phi^+(b)$ which is finite, a contradiction. So either $I^* \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_\phi^+(a)| = +\infty\})$ or $I^* \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_\phi^+(a)| < \infty\})$. \square

4.3.3 Maps with pairwise intersecting infinite orbits

The next definition will be helpful in the next section.

Definition 4.3.2. Let I be a set and $\phi : I \rightarrow I$. We let $\tilde{P}(\phi)$ be the proposition

$$\forall a, b \in I : [(|\mathcal{O}_\phi^+(a)| = |\mathcal{O}_\phi^+(b)| = \infty) \Rightarrow |\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b)| = \infty] .$$

Remark 4.3.2. Using lemmas 4.3.7 and 4.3.3, $\tilde{P}(\phi)$ is easily seen to be equivalent to

$$\forall a, b \in I : [(|\mathcal{O}_\phi^+(a)| = |\mathcal{O}_\phi^+(b)| = \infty) \Rightarrow \mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b) \neq \emptyset] .$$

Example 4.3.2. 1. $\tilde{P}(\phi)$ is vacuously true when I is finite.

2. $\tilde{P}(\phi)$ is true when $(\exists a \in I) : \mathcal{O}_\phi^+(a)$ is cofinite, see lemma 4.3.10.

Lemma 4.3.9. *Let I be an infinite set and $\phi : I \rightarrow I$. Then we have $\tilde{P}(\phi) \Leftrightarrow D_\phi \supseteq \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_\phi^+(a)| = +\infty\})$.*

Démonstration. (\Rightarrow) Suppose that $\tilde{P}(\phi)$ is true. Let $I^* = \{a_1, \dots, a_n\} \in \mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_\phi^+(a)| = +\infty\})$. Then it follows from corollary 4.3.4 that $I^* \in D_\phi$.

(\Leftarrow) Suppose that $D_\phi \supseteq \mathcal{P}_{\omega,*}(\{x \in I : |\mathcal{O}_\phi^+(x)| = +\infty\})$. Let $a, b \in I$ such that $|\mathcal{O}_\phi^+(a)| = |\mathcal{O}_\phi^+(b)| = +\infty$. So $\{a, b\} \in \mathcal{P}_{\omega,*}(\{x \in I : |\mathcal{O}_\phi^+(x)| = +\infty\}) \subseteq D_\phi$. Therefore $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b) \neq \emptyset$ by definition of D_ϕ . Hence $\tilde{P}(\phi)$ is true by remark 4.3.2. \square

Lemma 4.3.10. *Let I be an infinite set, $\phi : I \rightarrow I$, and $a \in I$ such that $\mathcal{O}_\phi^+(a)$ is cofinite. Then $\tilde{P}(\phi)$ holds.*

Démonstration. Let $b_1, b_2 \in I$ such that $|\mathcal{O}_\phi^+(b_1)| = |\mathcal{O}_\phi^+(b_2)| = \infty$. Then by lemma 4.3.5, $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b_1)$ is non-empty and cofinite. By lemma 4.3.6, $\mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b_1) = \mathcal{O}_\phi^+(\xi_\phi(\{a, b_1\}))$. Again, by lemma 4.3.5, $\mathcal{O}_\phi^+(\xi_\phi(\{a, b_1\})) \cap \mathcal{O}_\phi^+(b_2) = \mathcal{O}_\phi^+(a) \cap \mathcal{O}_\phi^+(b_1) \cap \mathcal{O}_\phi^+(b_2)$ is non-empty and cofinite. Since this set is included in $\mathcal{O}_\phi^+(b_1) \cap \mathcal{O}_\phi^+(b_2)$, we have that $\mathcal{O}_\phi^+(b_1) \cap \mathcal{O}_\phi^+(b_2)$ is also non-empty and cofinite. Therefore $\tilde{P}(\phi)$ holds. \square

4.4 Functions with a maximal number of finite invariant sets

The results of this section are obvious and only proved for completeness. They express the fact that, as long as k is strictly less than the cardinality of I , the identity function is the only function with a maximal number of finite invariant sets in $\mathcal{P}_{\omega,k+}(I)$ or $\text{Int}_{\omega,k+}(\mathbb{N})$.

4.4.1 Preservation of all finite sets

Proposition 4.4.1. *Let I be a set, $\alpha : I \rightarrow I$, and k a natural number such that $1 \leq k \leq |I|$. Suppose that*

$$\forall I^* \in \mathcal{P}_{\omega,k+}(I) : \alpha(I^*) \subseteq I^*.$$

Then we have

- (a) *if $1 \leq k < |I|$, then $\alpha = \text{Id}$.*
- (b) *if $k = |I|$, then α can be arbitrary.*

Démonstration. Suppose $1 \leq k < |I|$. If $\alpha(a) \neq a$ for some $a \in I$, one can choose $I^* \in \mathcal{P}_{\omega, k^+}(I)$ such that $a \in I^*$ but $\alpha(a) \notin I^*$ to obtain a contradiction. Conversely the identity function satisfies the requirement.

Suppose now that $k = |I|$. Then I is a finite set and $\mathcal{P}_{\omega, k^+}(I) = \{I\}$, and every function $\alpha : I \rightarrow I$ obviously preserves I . \square

4.4.2 Preservation of all finite intervals in \mathbb{N}

Proposition 4.4.2. *Let $\alpha : \mathbb{N} \rightarrow \mathbb{N}$, and k a natural number such that $k \geq 1$. Suppose that*

$$\forall [a, b] \in \text{Int}_{\omega, k^+}(\mathbb{N}) : \alpha([a, b]) \subseteq [a, b].$$

Then $\alpha = \text{Id}$.

Démonstration. Let $a \in \mathbb{N}$. If $\alpha(a) > a$, one can choose $[a - k + 1, a]$ to obtain a contradiction. If $\alpha(a) < a$, one can choose $[a, a + k - 1]$ to obtain a contradiction. Conversely the identity function satisfies the requirement. \square

4.5 Functions with a maximal number of finite internally-1-quasi-invariant sets

4.5.1 Preservation of all finite sets up to one element

If I is a set such that $|I| \geq 3$, the next proposition shows that there are two 3-parameter families of functions $\beta : I \rightarrow I$ such that there exists a function $w : \mathcal{P}_{\omega, *}(I) \rightarrow I$ such that

$$\forall I^* \in \mathcal{P}_{\omega, *}(I) : w(I^*) \in I^* \text{ and } \beta(I^* \setminus \{w(I^*)\}) \subseteq I^*.$$

Note that we cannot require that $\forall I^* \in \mathcal{P}_{\omega, *}(I) : \beta(w(I^*)) \notin I^*$ since these functions have many fixed points (except if $|I| = 3$).

Proposition 4.5.1. *Let I be a set such that $|I| \geq 3$, $\beta : I \rightarrow I$ and $w : \mathcal{P}_{\omega, *}(I) \rightarrow I$ such that*

$$\forall I^* \in \mathcal{P}_{\omega, *}(I) : w(I^*) \in I^* \text{ and } \beta(I^* \setminus \{w(I^*)\}) \subseteq I^*.$$

Then either

Case (1).

$$\exists (a, b, c) \in I^3 : (\beta(a) = b \text{ and } \beta(b) = c) \text{ and } (\forall x \in I \setminus \{a, b\} : \beta(x) = x)$$

and

$$w = \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto b \text{ if } \{a, b\} \subseteq I^* \\ I^* & \mapsto a \text{ if } a \in I^* \text{ and } b \notin I^* \\ I^* & \mapsto b \text{ if } a \notin I^* \text{ and } b \in I^* \\ I^* & \mapsto \text{any element of } I^* \text{ otherwise} \end{cases}$$

or

Case (2).

$$\exists(a, b, c) \in I^3 \text{ with } a \neq b, b \neq c, c \neq a, \beta(a) = b, \beta(b) = c, \beta(c) = a$$

and

$$\forall x \in I \setminus \{a, b, c\} : \beta(x) = x$$

and

$$w = \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto a \text{ if } a \in I^* \text{ and } b, c \notin I^* \\ I^* & \mapsto b \text{ if } b \in I^* \text{ and } a, c \notin I^* \\ I^* & \mapsto c \text{ if } c \in I^* \text{ and } a, b \notin I^* \\ I^* & \mapsto b \text{ if } \{a, b\} \subseteq I^* \text{ and } c \notin I^* \\ I^* & \mapsto a \text{ if } \{a, c\} \subseteq I^* \text{ and } b \notin I^* \\ I^* & \mapsto c \text{ if } \{b, c\} \subseteq I^* \text{ and } a \notin I^* \\ I^* & \mapsto \text{any element of } I^* \text{ otherwise} \end{cases}.$$

Démonstration. If β is the identity function, then β is clearly of type (1).

Otherwise, there exists $p \in I : \beta(p) \neq p$.

If β is the identity function on $I \setminus \{p\}$, then β is clearly of type (1).

Otherwise, let $q \in I \setminus \{p\}$ such that $\beta(q) \neq q$.

If for all $(m, n) \in I \setminus \{p\}$ with $m \neq n$ we had $w(\{p, m\}) = m$ and $w(\{p, n\}) = n$, then $\beta(p) \in \{p, m\} \cap \{p, n\} = \{p\}$, a contradiction. So for all $(m, n) \in I \setminus \{p\}$ with $m \neq n$, we have $w(\{p, m\}) = p$ or $w(\{p, n\}) = p$ which implies $\beta(m) \in \{p, m\}$ or $\beta(n) \in \{p, n\}$. Similarly, for all $(m, n) \in I \setminus \{q\}$ with $m \neq n$, $\beta(m) \in \{q, m\}$ or $\beta(n) \in \{q, n\}$.

Let $n \in I \setminus \{p, q, \beta(q)\}$. If $w(\{q, n\}) = n$, then $\beta(q) \in \{q, n\}$ so $\beta(q) = n$, a contradiction. So $w(\{q, n\}) = q$ and therefore $\beta(n) \in \{q, n\}$.

If $\exists r \in I \setminus \{p, q, \beta(q)\}$ such that $\beta(r) \neq r$, then $\beta(r) = q \notin \{p, r\}$, which implies $\forall m \in I \setminus \{p, r\} : \beta(m) \in \{p, m\}$. In particular $\beta(q) \in \{p, q\}$ and so $\beta(q) = p$. Since $\beta(r) \neq r$, we can prove in the same way as before that $\forall m, n \in I \setminus \{r\}$ with $m \neq n$, $\beta(m) \in \{r, m\}$ or $\beta(n) \in \{r, n\}$. So either β is the identity function on $I \setminus \{p, r\}$ or there exists $s \in I \setminus \{p, r\}$ such that $\forall n \in I \setminus \{p, r, s\} : \beta(n) \in \{q, n\} \cap \{r, n\} = \{n\}$

and so β is the identity function on $I \setminus \{p, r, s\}$. The first case is included in the second, and we have necessarily $s = q$ since $q \notin \{p, r\}$ and $\beta(q) = p \neq q$. Since $p \neq q$ and $p, q \in I \setminus \{r\}$, we have $\beta(p) \in \{r, p\}$ or $\beta(q) \in \{r, q\}$. The second case is impossible so $\beta(p) \in \{r, p\}$ which implies $\beta(p) = r$. So β is of type $\textcircled{2}$ ($a = p, b = r, c = q$).

Otherwise, we have $\forall n \in I \setminus \{p, q, \beta(q)\} : \beta(n) = n$.

If $\beta(q) = p$, then β is clearly of type $\textcircled{1}$.

Suppose then that $\beta(q) \neq p$.

If $\beta(p) \neq q$, then $\beta(\{p, q\} \setminus \{w(\{p, q\})\}) \subseteq \{p, q\}$ leads to a contradiction independently of $w(\{p, q\}) \in \{p, q\}$. So $\beta(p) = q$.

Since $q \neq \beta(q)$ and $q, \beta(q) \in I \setminus \{p\}$, we have $\beta(q) \in \{p, q\}$ or $\beta(\beta(q)) \in \{p, \beta(q)\}$. The first case is impossible. So $\beta(\beta(q)) \in \{p, \beta(q)\}$.

If $\beta(\beta(q)) = p$ then β is of type $\textcircled{2}$ ($a = p, b = q, c = \beta(q)$).

Otherwise $\beta(\beta(q)) = \beta(q)$ and so β is of type $\textcircled{1}$ ($a = p, b = q, c = \beta(q)$).

The statements about w are easy to prove. \square

4.5.2 Preservation of all finite intervals up to one element in \mathbb{N}

Apart from the successor function $\beta = \text{succ}$ with $w : \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b \end{cases}$, there exist other functions $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that there exists a function $w : \text{Int}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ such that

$$\forall \llbracket a, b \rrbracket \in \text{Int}_{\omega,*}(\mathbb{N}) : w(\llbracket a, b \rrbracket) \in \llbracket a, b \rrbracket \text{ and } \beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket,$$

as shown by the next proposition.

Proposition 4.5.2. *Let $\beta : \mathbb{N} \rightarrow \mathbb{N}$ such that there exists a function $w : \text{Int}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ such that*

$$\forall \llbracket a, b \rrbracket \in \text{Int}_{\omega,*}(\mathbb{N}) : w(\llbracket a, b \rrbracket) \in \llbracket a, b \rrbracket \text{ and } \beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket.$$

Let $\{b_n\}_{n \in [0, l]}$ the sequence of non-fixed points of β in strictly increasing order, where $l \in \mathbb{N} \cup \{-1, \infty\}$. We extend $\{b_n\}_{n \in [0, l]}$ and β by setting $b_{-1} = -1$ and $\beta(-1) = 0$, and if $l < \infty$, we set $b_{l+1} = \infty$. Then we have either

Case $\textcircled{1}$. $\forall n \in [0, l] : b_n < \beta(b_n) \leq b_{n+1}$ and

$$w = \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ if } \exists ! n \in [0, l] : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) > b_n. \\ \llbracket a, b \rrbracket & \mapsto \text{any element of } \llbracket a, b \rrbracket \text{ otherwise} \end{cases}$$

Case ②. There exists $n^* \in \llbracket -1, l-1 \rrbracket$ such that $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) < b_{n^*+1}$, $\forall m \in \llbracket n^*+2, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$, and

$$w = \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ if } \exists! n \in \llbracket 0, n^* \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) > b \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ if } \exists! n \in \llbracket n^*+1, l \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) < a \\ \llbracket a, b \rrbracket & \mapsto \text{any element of } \llbracket a, b \rrbracket \text{ otherwise} \end{cases}.$$

Case ③. There exists $n^* \in \llbracket -1, l-1 \rrbracket$ such that $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) > b_{n^*+2}$, $\forall m \in \llbracket n^*+2, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$, and

$$w = \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ if } \exists! n \in \llbracket 0, n^* \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) > b \\ \llbracket a, b \rrbracket & \mapsto b_n \text{ if } \exists! n \in \llbracket n^*+1, l \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) < a \\ \llbracket a, b \rrbracket & \mapsto \text{any element of } \llbracket a, b \rrbracket \text{ otherwise} \end{cases}.$$

Démonstration. Let's show that

$$\boxed{\forall n \in \llbracket 0, l \rrbracket : (\beta(b_n) < b_n \Rightarrow \forall m \in \llbracket n+1, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m)} \quad (4.3)$$

and

$$\boxed{\forall n \in \llbracket 0, l \rrbracket : (\beta(b_n) > b_{n+1} \Rightarrow \forall m \in \llbracket n+1, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m)}. \quad (4.4)$$

Suppose that $\beta(b_n) < b_n$ for a certain $n \in \mathbb{N}$ and $n+1 \leq l$. We have $\beta(\llbracket b_n, b_{n+1} \rrbracket \setminus \{w(\llbracket b_n, b_{n+1} \rrbracket)\}) \subseteq \llbracket b_n, b_{n+1} \rrbracket$. If $w(\llbracket b_n, b_{n+1} \rrbracket) = b_{n+1}$, then $b_n \leq \beta(b_n)$ which is a contradiction. Therefore $w(\llbracket b_n, b_{n+1} \rrbracket) \neq b_{n+1}$ and so $b_n \leq \beta(b_{n+1}) < b_{n+1}$ since b_{n+1} is not a fixed point. By immediate induction, we have $\forall m \in \llbracket n+1, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$.

Therefore assertion 4.3 holds.

Suppose that $\beta(b_n) > b_{n+1}$ for a certain $n \in \mathbb{N}$ and $n+1 \leq l$. We have $\beta(\llbracket b_n, b_{n+1} \rrbracket \setminus \{w(\llbracket b_n, b_{n+1} \rrbracket)\}) \subseteq \llbracket b_n, b_{n+1} \rrbracket$. If $w(\llbracket b_n, b_{n+1} \rrbracket) = b_{n+1}$, then $\beta(b_n) \leq b_{n+1}$ which is a contradiction. Therefore $w(\llbracket b_n, b_{n+1} \rrbracket) \neq b_{n+1}$ and so $b_n \leq \beta(b_{n+1}) < b_{n+1}$ since b_{n+1} is not a fixed point. By immediate induction, we have $\forall m \in \llbracket n+1, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$.

Therefore assertion 4.4 holds.

With these assertions in hand, we can now start the proof. We treat the $l = \infty$ and $l < \infty$ cases simultaneously.

Case ①. Suppose $\forall n \in \llbracket 0, l \rrbracket : b_n < \beta(b_n) \leq b_{n+1}$.

Let $a \leq b \in \mathbb{N}$.

Subcase (1.1). Suppose $\exists n \in \llbracket 0, l \rrbracket : b_n \in \llbracket a, b \rrbracket$ and $\beta(b_n) > b$. Let's show that there is a unique such n . Suppose that $n_1 > n_2$ both satisfy the condition. Since, $n_2 > n_1$, we have $n_2 \geq n_1 + 1$, and since $(b_n)_{n \in \mathbb{N}}$ is strictly increasing, we have

$$b \geq b_{n_2} \geq b_{n_1+1} \geq \beta(b_{n_1}) > b,$$

which is a contradiction. Therefore there exists a unique $n(a, b) \in \llbracket 0, l \rrbracket$ such that $b_{n(a,b)} \in \llbracket a, b \rrbracket$ and $\beta(b_{n(a,b)}) > b$. If we had $w(\llbracket a, b \rrbracket) \neq b_{n(a,b)}$, then since $\beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket$, we would have $\beta(b_{n(a,b)}) \in \llbracket a, b \rrbracket$, which contradicts $\beta(b_{n(a,b)}) > b$. Therefore $w(\llbracket a, b \rrbracket) = b_{n(a,b)}$.

Subcase (1.2). Suppose $\forall n \in \llbracket 0, l \rrbracket : (b_n \in \llbracket a, b \rrbracket \Rightarrow \beta(b_n) \leq b)$. There is no additional constraint in this subcase, we can choose $w(\llbracket a, b \rrbracket)$ arbitrarily from $\llbracket a, b \rrbracket$.

Conversely. If $l \in \mathbb{N} \cup \{-1, \infty\}$, $\{b_n\}_{n \in \llbracket 0, l \rrbracket}$ is a strictly increasing sequence, $\beta : \mathbb{N} \rightarrow \mathbb{N}$ is a function such that $\mathbb{N} \setminus \{b_n\}_{n \in \llbracket 0, l \rrbracket}$ are the fixed points of β , $\forall n \in \llbracket 0, l \rrbracket : b_n < \beta(b_n) \leq b_{n+1}$, and

$$w = \begin{cases} \text{Int}_{\omega, *}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_{n(a,b)} \text{ if } \exists n \in \llbracket 0, l \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) > b, \\ \llbracket a, b \rrbracket & \mapsto \text{any element of } \llbracket a, b \rrbracket \text{ otherwise} \end{cases}$$

then $\forall a \leq b \in \llbracket a, b \rrbracket : \beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket$. Indeed, let $a \leq b \in \mathbb{N}$. Because of the uniqueness property of $b_{n(a,b)}$ (when it exists), any $x \in \llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}$ is either a fixed point (in which case $\beta(x) = x \in \llbracket a, b \rrbracket$), or satisfies $x = b_n$ with $\beta(b_n) \leq b$, in which case, since $b_n < \beta(b_n)$, we have $a \leq x = b_n < \beta(b_n) = \beta(x) \leq b$.

Suppose now $\neg (\forall n \in \llbracket 0, l \rrbracket : b_n < \beta(b_n) \leq b_{n+1})$. Necessarily, $l \geq 0$. Let $n^* = \max\{n \in \llbracket -1, l \rrbracket : \forall m \leq n : b_m < \beta(b_m) \leq b_{m+1}\}$. Necessarily, $n^* \in \llbracket -1, l - 1 \rrbracket$. Notice that if $\beta(b_{n^*+1}) \leq b_{n^*+2}$, then by maximality of n^* , we have $\beta(b_{n^*+1}) < b_{n^*+1}$ since b_{n^*+1} is not a fixed point of β . Therefore we have either $\beta(b_{n^*+1}) < b_{n^*+1}$ or $\beta(b_{n^*+1}) > b_{n^*+2}$.

Case (2). Suppose $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$ and $\beta(b_{n^*+1}) < b_{n^*+1}$. By assertion 4.3, we have $\forall m \in \llbracket n^* + 2, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$.

Let $a \leq b \in \mathbb{N}$.

Subcase (2.1). Suppose $\exists n \in \llbracket 0, n^* \rrbracket : b_n \in \llbracket a, b \rrbracket$ and $\beta(b_n) > b$. Let's show that there is a unique such n . Suppose that $n_1 > n_2$ both satisfy the condition. Since, $n_2 > n_1$, we have $n_2 \geq n_1 + 1$, and since $(b_n)_{n \in \mathbb{N}}$ is strictly increasing, we have

$$b \geq b_{n_2} \geq b_{n_1+1} \geq \beta(b_{n_1}) > b,$$

which is a contradiction. Therefore there exists a unique $n(a, b) \in \llbracket 0, n^* \rrbracket$ such that $b_{n(a,b)} \in \llbracket a, b \rrbracket$ and $\beta(b_{n(a,b)}) > b$. If we had $w(\llbracket a, b \rrbracket) \neq b_{n(a,b)}$, then since $\beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket$, we would have $\beta(b_{n(a,b)}) \in \llbracket a, b \rrbracket$, which contradicts $\beta(b_{n(a,b)}) > b$. Therefore $w(\llbracket a, b \rrbracket) = b_{n(a,b)}$.

Subcase (2.2). Suppose $\exists n \in \llbracket n^* + 1, l \rrbracket : b_n \in \llbracket a, b \rrbracket$ and $\beta(b_n) < a$. Let's show that there is a unique such n . Suppose that $n_1 > n_2$ both satisfy the condition. Since, $n_2 > n_1$, we have $n_2 - 1 \geq n_1$ and $n_2 \geq n^* + 2$, and since $(b_n)_{n \in \mathbb{N}}$ is strictly increasing, we have

$$a > \beta(b_{n_2}) \geq b_{n_2-1} \geq b_{n_1} \geq a,$$

which is a contradiction. Therefore there exists a unique $n'(a, b) \in \llbracket n^* + 1, l \rrbracket$ such that $b_{n'(a,b)} \in \llbracket a, b \rrbracket$ and $\beta(b_{n'(a,b)}) > b$. If we had $w(\llbracket a, b \rrbracket) \neq b_{n'(a,b)}$, then since $\beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket$, we would have $\beta(b_{n'(a,b)}) \in \llbracket a, b \rrbracket$, which contradicts $\beta(b_{n'(a,b)}) > b$. Therefore $w(\llbracket a, b \rrbracket) = b_{n'(a,b)}$.

Let's show that subcases 2.1 and 2.1 are disjoint. Suppose the contrary. Then

$$\begin{aligned} a > \beta(b_{n'(a,b)}) &\geq b_{n'(a,b)-1} > \beta(b_{n'(a,b)-1}) \geq b_{n'(a,b)-2} > \cdots \\ &> \beta(b_{n^*+2}) \geq b_{n^*+1} \geq \beta(b_{n^*}) > b_{n^*} \geq \beta(b_{n^*-1}) > \cdots > \beta(b_{n(a,b)}) > b, \end{aligned}$$

which is a contradiction.

Subcase (2.3). Suppose $\forall n \in \llbracket 0, n^* \rrbracket : (b_n \in \llbracket a, b \rrbracket \Rightarrow \beta(b_n) \leq b)$ and $\forall n \in \llbracket n^* + 1, l \rrbracket : (b_n \in \llbracket a, b \rrbracket \Rightarrow \beta(b_n) \geq a)$. There is no additional constraint in this subcase, we can choose $w(\llbracket a, b \rrbracket)$ arbitrarily from $\llbracket a, b \rrbracket$.

Conversely. If $l \geq 0$, $\{b_n\}_{n \in \llbracket 0, l \rrbracket}$ is a strictly increasing sequence, $\beta : \mathbb{N} \rightarrow \mathbb{N}$ is a function such that $\mathbb{N} \setminus \{b_n\}_{n \in \llbracket 0, l \rrbracket}$ are the fixed points of β , $n^* \in \llbracket -1, l - 1 \rrbracket$, $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$, $\beta(b_{n^*+1}) < b_{n^*+1}$, $\forall m \in \llbracket n^* + 2, l \rrbracket : b_{m-1} \leq \beta(b_m) < b_m$, and

$$w = \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b_{n(a,b)} \text{ if } \exists n \in \llbracket 0, n^* \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) > b \\ \llbracket a, b \rrbracket & \mapsto b_{n'(a,b)} \text{ if } \exists n \in \llbracket n^* + 1, l \rrbracket : b_n \in \llbracket a, b \rrbracket \text{ and } \beta(b_n) < a \\ \llbracket a, b \rrbracket & \mapsto \text{any element of } \llbracket a, b \rrbracket \text{ otherwise} \end{cases}$$

then $\forall a \leq b \in \llbracket a, b \rrbracket : \beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket$. Indeed, let $a \leq b \in \mathbb{N}$. Because of the uniqueness properties of $b_{n(a,b)}$ and $b_{n'(a,b)}$ (when they exist) and the disjointness of subcases 2.1 and 2.2, any $x \in \llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}$ is either a fixed point (in which case $\beta(x) = x \in \llbracket a, b \rrbracket$), or satisfies $x = b_n$ with $n \in \llbracket 0, n^* \rrbracket$ and $\beta(b_n) \leq b$, in which case, since $b_n < \beta(b_n)$, we have $a \leq x = b_n < \beta(b_n) = \beta(x) \leq b$, or satisfies $x = b_n$ with $n \geq n^* + 1$ and $\beta(b_n) \geq a$, in which case, since $\beta(b_n) < b_n$, we have $a \leq \beta(b_n) = \beta(x) < b_n = x \leq b$.

Case ③. Suppose $\forall m \leq n^* : b_m < \beta(b_m) \leq b_{m+1}$ and $\beta(b_{n^*+1}) > b_{n^*+2}$. By assertion 4.4, we have $\forall m \geq n^* + 2 : b_{m-1} \leq \beta(b_m) < b_m$. Then the treatment from now on is analogous to case 2 (since we haven't used the conditions $\beta(b_{n^*+1}) < b_{n^*+1}$ or $\beta(b_{n^*+1}) > b_{n^*+2}$) with the same necessary and sufficient form of the function w . \square

Requiring that for such functions, $\beta(w(\llbracket a, b \rrbracket)) \notin \llbracket a, b \rrbracket$ for all $a \leq b \in \mathbb{N}$, leads to

Corollary 4.5.1. *Let $\beta : \mathbb{N} \rightarrow \mathbb{N}$ and $w : \text{Int}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ be two functions such that*

$$\forall \llbracket a, b \rrbracket \in \text{Int}_{\omega,*}(\mathbb{N}) : \quad w(\llbracket a, b \rrbracket) \in \llbracket a, b \rrbracket \\ \text{and } \beta(\llbracket a, b \rrbracket \setminus \{w(\llbracket a, b \rrbracket)\}) \subseteq \llbracket a, b \rrbracket \text{ and } \beta(w(\llbracket a, b \rrbracket)) \notin \llbracket a, b \rrbracket.$$

Then either

$$\text{Case ①. } \beta : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n + 1 \end{cases} \text{ and } w : \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b \end{cases}$$

$$\text{Case ②. } \text{There exists } n^* \neq u \in \mathbb{N} \text{ such that } \beta : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n + 1 \text{ for } n < n^* \\ n^* & \mapsto u \\ n & \mapsto n - 1 \text{ for } n > n^* \end{cases} \text{ and}$$

$$w : \begin{cases} \text{Int}_{\omega,*}(\mathbb{N}) & \rightarrow \mathbb{N} \\ \llbracket a, b \rrbracket & \mapsto b \text{ if } b \in \llbracket 0, n^* \rrbracket \\ \llbracket a, b \rrbracket & \mapsto a \text{ if } a \geq n^* + 1 \end{cases}.$$

Démonstration. Notice that for all $a \in \mathbb{N}$, $\beta(w(\llbracket a, a \rrbracket)) \notin \llbracket a, a \rrbracket$ implies that $\beta(a) \neq a$, and so the sequence of non-fixed points of β in strictly increasing order is $\{b_n\}_{n \in \mathbb{N}} = \{n\}_{n \in \mathbb{N}}$. We can then apply proposition 4.5.2. \square

4.6 Functions with a maximal number of finite invariant supersets

4.6.1 Solution to the general problem

Proposition 4.6.1. *Let $\alpha : I \rightarrow I$ and $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$. Then the following conditions are equivalent*

1. $\forall I^* \in \mathcal{P}_{\omega,*}(I) : I^* \subseteq G(I^*) \text{ and } \alpha(G(I^*)) \subseteq G(I^*)$.

2. The orbits of α are finite and there exists $H : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega}(I)$ such that

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_{\alpha}^+(a).$$

Démonstration. (1) \Rightarrow (2). Let $a \in I$. We have $\alpha(a) \in G(\{a\})$ since $a \in G(\{a\})$ and $\alpha(G(\{a\})) \subseteq G(\{a\})$. By immediate induction, we have $\mathcal{O}_{\alpha}^+(a) \subseteq G(\{a\})$. Since $G(\{a\})$ is finite, this implies that $\mathcal{O}_{\alpha}^+(a)$ is finite. Therefore the orbits of α are finite. Furthermore, let $H(I^*) = G(I^*)$ for all $I^* \in \mathcal{P}_{\omega,*}(I)$. Let $I^* \in \mathcal{P}_{\omega,*}(I)$. We have $I^* \cup H(I^*) = I^* \cup G(I^*) = G(I^*)$. Let $a \in G(I^*)$. We have $\alpha(a) \in G(I^*)$ since $a \in G(I^*)$ and $\alpha(G(I^*)) \subseteq G(I^*)$. By immediate induction, we have $\mathcal{O}_{\alpha}^+(a) \subseteq G(I^*)$. Therefore $\bigcup_{a \in G(I^*)} \mathcal{O}_{\alpha}^+(a) \subseteq G(I^*)$. The reverse inclusion is obvious since $a \in \mathcal{O}_{\alpha}^+(a)$ for all $a \in G(I^*)$.

(2) \Rightarrow (1). Let $I^* \in \mathcal{P}_{\omega,*}(I)$. To show that $I^* \subseteq G(I^*)$, let $a \in I^*$. We have $a \in \mathcal{O}_{\alpha}^+(a) \subseteq G(I^*)$. Therefore $I^* \subseteq G(I^*)$. To show that $\alpha(G(I^*)) \subseteq G(I^*)$, let $a \in G(I^*)$. We have

$$\alpha(a) \in \alpha \left(\bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_{\alpha}^+(a) \right) = \bigcup_{a \in I^* \cup H(I^*)} \alpha(\mathcal{O}_{\alpha}^+(a)) \subseteq \bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_{\alpha}^+(a).$$

Therefore $\alpha(G(I^*)) \subseteq G(I^*)$. □

4.6.2 Solution to a more constrained problem in \mathbb{N}

The following proposition gives a characterization of the functions $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega,*}(\mathbb{N})$ that satisfy the conditions of proposition 4.6.1, plus the additional hypothesis that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \max(G(I^*)) \in \alpha(I^*)$.

Proposition 4.6.2. *Let $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega,*}(\mathbb{N})$ such that*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*), \alpha(G(I^*)) \subseteq G(I^*) \text{ and } \max(G(I^*)) \in \alpha(I^*).$$

Then α admits an infinite number of fixed points, and letting this sequence be $\{b_n\}_{n \in \mathbb{N}}$ in strictly increasing order, there exists a function $j : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)}$$

and

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) = n \text{ if } \exists n \in \mathbb{N} : a = b_n \\ j(a) \geq r(a) \text{ if } a \notin \{b_n\}_{n \in \mathbb{N}} \end{aligned}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, and there exists a function $H : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega}(\mathbb{N})$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \{a, \alpha(a)\}$ and

$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in I^* : \forall x \in H(I^*) : j(x) \leq j(a)$. Conversely, any such α and G are valid solutions.

Démonstration. **Direct implication.** First, let's show that

$$\boxed{\forall n \in \mathbb{N} : \alpha(\alpha(n)) = \alpha(n) \geq n} \quad (4.5)$$

We have $\forall n \in \mathbb{N} : \max(G(\{n\})) \in \alpha(\{n\})$, so $\alpha(n) = \max(G(\{n\})) \in G(\{n\})$. Let $n \in \mathbb{N}$. We have $\alpha(\alpha(n)) \in G(\{n\})$ since $\alpha(n) \in G(\{n\})$ (we use the hypothesis $\alpha(G(I^*)) \subseteq G(I^*)$ for $I^* = \{n\}$), and so $\alpha(\alpha(n)) \leq \alpha(n)$ since $\alpha(n) = \max(G(\{n\}))$. Moreover, we have $n \in \{n\} \subseteq G(\{n\})$, and so $n \leq \max(G(\{n\})) = \alpha(n)$. Replacing n by $\alpha(n)$, we have $\forall n \in \mathbb{N} : \alpha(\alpha(n)) \geq \alpha(n)$.

Therefore $\forall n \in \mathbb{N} : \alpha(\alpha(n)) = \alpha(n) \geq n$, and so assertion 4.5 holds.

Notice that if α is injective, then by assertion 4.5 we have $\forall n \in \mathbb{N} : \alpha(n) = n$, and α is the identity function. Otherwise, α admits a sequence of fixed points $\{\alpha(n)\}_{n \in \mathbb{N}}$ tending to infinity since $\forall n \in \mathbb{N} : \alpha(n) \geq n$. Let $\{b_n\}_{n \in \mathbb{N}}$ be the sequence of all fixed points of α in strictly increasing order. Let $b_{-1} = -1$ and extend α by setting $\alpha(-1) = -1$. Then $\forall n \in \mathbb{N} \cup \{-1\} : \forall m \in [b_n + 1, b_{n+1} - 1] : \exists k \geq n + 1 : \alpha(m) = b_k$ since $\alpha(m)$ is a fixed point and $\alpha(m) \geq m > b_n$. Moreover, we have $r(m) \leq n + 1$ since $b_{n+1} > m$, and so $r(m) \leq k$. Therefore we can set $j(m) = n$ if $m = b_n$ and $j(m) = k$ if $m \in [b_n + 1, b_{n+1} - 1]$ for some $n \in \mathbb{N} \cup \{-1\}$.

Moreover, from proposition 4.6.1, there exists a function $H : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega}(\mathbb{N})$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \mathcal{O}_{\alpha}^+(a) = \bigcup_{a \in I^* \cup H(I^*)} \{a, \alpha(a)\}$. It's easy to show that the condition $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \max(G(I^*)) \in \alpha(I^*)$ means that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in I^* : \forall x \in H(I^*) : j(x) \leq j(a)$.

Conversely. Given a function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ which admits an infinite number of fixed points $\{b_n\}_{n \in \mathbb{N}}$ in strictly increasing order, such that there exists a function $j : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)}$$

and

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) = n \text{ if } \exists n \in \mathbb{N} : a = b_n \\ j(a) \geq r(a) \text{ if } a \notin \{b_n\}_{n \in \mathbb{N}} \end{aligned}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, and such that there exists a function $H : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega}(\mathbb{N})$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \bigcup_{a \in I^* \cup H(I^*)} \{a, \alpha(a)\}$ and $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in I^* : \forall x \in H(I^*) : j(x) \leq j(a)$, then it's easy to check that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \alpha(G(I^*)) \subseteq G(I^*)$ and $\max(G(I^*)) \in \alpha(I^*)$. \square

Remark 4.6.1. If α is assumed to be bijective, then we can also use proposition 4.7.7 to deduce $\alpha = Id$. Indeed, suppose that $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathcal{P}_{\omega,*}(\mathbb{N})$ are such that

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*), \alpha(G(I^*)) \subseteq G(I^*) \text{ and } \max(G(I^*)) \in \alpha(I^*),$$

and let $\beta : \begin{cases} \mathbb{N} & \rightarrow \mathbb{N} \\ n & \mapsto n + 1 \end{cases}$, $\phi = \beta \circ \alpha$, and for all $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$, $u(I^*) \in I^*$ such that $\alpha(u(I^*)) = \max(G(I^*))$.

Then since $P_2(\phi, G, u)$ is true and $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : \exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$, we have $\alpha = Id$.

In the following proposition, we require furthermore that $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \text{Int}_{\omega,*}(\mathbb{N})$.

Proposition 4.6.3. *Let $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ and $G : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \text{Int}_{\omega,*}(\mathbb{N})$ such that*

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : I^* \subseteq G(I^*), \alpha(G(I^*)) \subseteq G(I^*) \text{ and } \max(G(I^*)) \in \alpha(I^*).$$

Then α admits an infinite number of fixed points, and if this sequence is $\{b_n\}_{n \in \mathbb{N}}$ in strictly increasing order, there exists a function $j : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)},$$

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) &= n \text{ if } \exists n \in \mathbb{N} : a = b_n \\ j(a) &= r(a) \text{ if } a > b_0 \text{ and } a \notin \{b_n\}_{n \in \mathbb{N}} \end{aligned}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, and

$$j(0) \geq j(1) \cdots \geq j(b_0 - 1),$$

and there exists a function $u : \mathcal{P}_{\omega,*}(\mathbb{N}) \rightarrow \mathbb{N}$ such that $\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : G(I^*) = \llbracket u(I^*), \max\{\alpha(a)\}_{a \in I^*} \rrbracket$, and

$$\forall I^* \in \mathcal{P}_{\omega,*}(\mathbb{N}) : u^*(I^*) := \min\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\} \leq u(I^*) \leq \min(I^*),$$

Conversely, any such α and G are valid solutions.

Démonstration. Direct implication. α and G satisfy of course all the previous properties of proposition 4.6.2.

For all $I^* \in \mathcal{P}_{\omega,*}(\mathbb{N})$, we let $\llbracket u(I^*), v(I^*) \rrbracket := G(I^*)$. Assume by way of contradiction that $v(I^*) \neq \max\{\alpha(a)\}_{a \in I^*}$. We have $\forall a \in I^* \subseteq G(I^*) : \alpha(a) \leq v(I^*)$, so $\max\{\alpha(a)\}_{a \in I^*} \leq v(I^*)$. Since $v(I^*) \neq \max\{\alpha(a)\}_{a \in I^*}$, we have

$\max\{\alpha(a)\}_{a \in I^*} < v(I^*)$ which is a contradiction since $v(I^*) = \max(G(I^*)) \in \alpha(I^*)$. So $v(I^*) = \max\{\alpha(a)\}_{a \in I^*}$.

Applying this to $I^* = \{b_n, b_{n+1}\}$ for $n \in \mathbb{N}$ gives $G(\{b_n, b_{n+1}\}) = [u(\{b_n, b_{n+1}\}), b_{n+1}]$ where $u(\{b_n, b_{n+1}\}) \leq b_n$, and so $\forall x \in \llbracket b_n, b_{n+1} \rrbracket : \alpha(x) \leq b_{n+1}$ since $G(I^*)$ is preserved by α . Hence $\forall x \in \llbracket b_n, b_{n+1} \rrbracket : \alpha(x) = b_{n+1}$ (since $j(x) \geq n+1$ by proposition 4.6.2).

Applying it for $I^* = \{x\}$ for $x \in [0, b_0[$ gives $G(I^*) = \llbracket u(\{x\}), \alpha(x) \rrbracket$ where $u(\{x\}) \leq x$. Since $\alpha(x)$ is a fixed point, we have $\alpha(x) \geq b_0$. Therefore, $\forall y \in [a, b_0[: \alpha(y) \leq \alpha(x)$ since $G(I^*)$ is preserved by α . So $\alpha(0) \geq \alpha(1) \cdots \geq \alpha(b_0 - 1)$, which is equivalent to $j(0) \geq j(1) \geq \cdots \geq j(b_0 - 1)$. Moreover, for all $I^* \in \mathcal{P}_{\omega, *}(\mathbb{N})$, we have $u(I^*) \leq \min(I^*)$ since $I^* \subseteq G(I^*)$.

Furthermore, for all $I^* \in \mathcal{P}_{\omega, *}(\mathbb{N})$, the set $\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\}$ is non-empty (it contains b_0), and letting $u^*(I^*) := \min\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\}$, we have necessarily $u(I^*) \geq u^*(I^*)$. Indeed, if $u(I^*) > b_0$, the inequality is obvious. Otherwise, it is true since $\alpha(u(I^*)) \in G(I^*)$ (since $\alpha(G(I^*)) \subseteq G(I^*)$ and $u(I^*) \in G(I^*)$), and therefore $\alpha(u(I^*)) \leq \max(G(I^*)) = \max\{\alpha(a)\}_{a \in I^*}$.

Conversely. Given a function $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ which admits an infinite number of fixed points $\{b_n\}_{n \in \mathbb{N}}$ in strictly increasing order, such that there exists a function $j : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$\forall a \in \mathbb{N} : \alpha(a) = b_{j(a)}$$

and

$$\begin{aligned} \forall a \in \mathbb{N} : j(a) = n \text{ if } \exists n \in \mathbb{N} : a = b_n \\ j(a) = r(a) \text{ if } a > b_0 \text{ and } a \notin \{b_n\}_{n \in \mathbb{N}} \end{aligned}$$

where $r(a) := \min\{n \in \mathbb{N} : b_n > a\}$, and

$$j(0) \geq j(1) \cdots \geq j(b_0 - 1),$$

and such that $\forall I^* \in \mathcal{P}_{\omega, *}(\mathbb{N}) : G(I^*) = \llbracket u(I^*), \max\{\alpha(a)\}_{a \in I^*} \rrbracket$, where $u : \mathcal{P}_{\omega, *}(\mathbb{N}) \rightarrow \mathbb{N}$ is a function such that

$$\forall I^* \in \mathcal{P}_{\omega, *}(\mathbb{N}) : u^*(I^*) := \min\{n \leq b_0 : \max\{\alpha(a)\}_{a \in I^*} \geq \alpha(n)\} \leq u(I^*) \leq \min(I^*),$$

then it's clear that $\forall I^* \in \mathcal{P}_{\omega, *}(\mathbb{N}) : \max(G(I^*)) \in \alpha(I^*)$. To check that $\forall I^* \in \mathcal{P}_{\omega, *}(\mathbb{N}) : \alpha(G(I^*)) \subseteq G(I^*)$, we take $I^* \in \mathcal{P}_{\omega, *}(\mathbb{N})$ and $x \in G(I^*)$, and distinguish between the cases $x \geq b_0$ and $x < b_0$. In the first case, since $x \leq \max\{\alpha(a)\}_{a \in I^*}$, $\max\{\alpha(a)\}_{a \in I^*}$ is a fixed point, and $x \geq b_0$, then it follows that $u(I^*) \leq x \leq \alpha(x) \leq \max\{\alpha(a)\}_{a \in I^*}$. In the second case, we have $u^*(I^*) \leq u(I^*) \leq x < b_0$, therefore $\alpha(x) \leq \alpha(u^*(I^*)) \leq \max\{\alpha(a)\}_{a \in I^*}$ since α is non-increasing on $\llbracket 0, b_0 - 1 \rrbracket$ and by the definition of $u^*(I^*)$. Moreover, since $\alpha(x)$ is a fixed point, it is obvious that $\alpha(x) \geq b_0 > x \geq u(I^*)$. Therefore we are done in both cases and it follows that $\alpha(G(I^*)) \subseteq G(I^*)$. \square

4.7 Functions with a maximal number of finite internally-1-quasi-invariant supersets

Below, we define two similar predicates (in the sense of propositions depending on mathematical objects) that model the property of direct image preservation by $\phi : I \rightarrow I$ of finite supersets of finite subsets of I , up to one element, denoted by $P_1(\phi, G, u)$ and $P_2(\phi, G, u)$. We will be concerned with these predicates in the following three subsections.

Definition 4.7.1. We say that a 3-tuple $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ satisfies property $P_1(\phi, G, u)$ if

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in G(I^*), I^* \subseteq G(I^*) \text{ and } \phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*).$$

Definition 4.7.2. We say that a 3-tuple $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ satisfies property $P_2(\phi, G, u)$ if

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : u(I^*) \in I^*, I^* \subseteq G(I^*) \text{ and } \phi(G(I^*) \setminus \{u(I^*)\}) \subseteq G(I^*).$$

The following simple lemma is often used (without mention) in the paper.

Lemma 4.7.1. *Let I be an infinite set and $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ a 3-tuple satisfying $P_1(\phi, G, u)$. Then we have*

$$\forall I^* \in \mathcal{P}_{\omega,*}(I) : (|\mathcal{O}_\phi^+(u(I^*))| = \infty \Rightarrow \phi(u(I^*)) \notin G(I^*) \text{ and } \phi(u(I^*)) \notin I^*).$$

Démonstration. Let $I^* \in \mathcal{P}_{\omega,*}(I)$ such that $|\mathcal{O}_\phi^+(u(I^*))| = \infty$. From lemma 4.3.1, $u(I^*), \phi(u(I^*)), \phi^2(u(I^*)), \dots$ must be distinct. In particular, $\forall n \geq 1, \phi^n(u(I^*)) \neq u(I^*)$. Thus we have $\phi(u(I^*)) \notin G(I^*)$ (otherwise we would have $\mathcal{O}_\phi^+(u(I^*)) \subseteq G(I^*)$ which is impossible since $G(I^*)$ is finite and $|\mathcal{O}_\phi^+(u(I^*))| = \infty$). Since $I^* \subseteq G(I^*)$, this implies that $\phi(u(I^*)) \notin I^*$. \square

We also have

Lemma 4.7.2. *Let I be an infinite set and $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ a 3-tuple satisfying $P_2(\phi, G, u)$. Then we have*

$$(\forall I^* \in \mathcal{P}_{\omega,*}(I) : |\mathcal{O}_\phi^+(u(I^*))| = \infty) \Rightarrow (\forall b \in I : \forall m \in \mathbb{N} : u(\{b, \dots, \phi^m(b)\}) = \phi^m(b)).$$

Démonstration. Suppose that $\forall I^* \in \mathcal{P}_{\omega,*}(I) : |\mathcal{O}_\phi^+(u(I^*))| = \infty$. Let $b \in I$ and $m \in \mathbb{N}$. Since we should have $u(\{b, \dots, \phi^m(b)\}) \in \{b, \dots, \phi^m(b)\}$ and $\phi(u(\{b, \dots, \phi^m(b)\})) \notin \{b, \dots, \phi^m(b)\}$ by lemma 4.7.1, we have necessarily $u(\{b, \dots, \phi^m(b)\}) = \phi^m(b)$. \square

4.7.1 Solution to problem $P_1(\phi, G, u)$

We first prove the following structural proposition.

Proposition 4.7.1. *Let I be an infinite set and $\beta : I \rightarrow I$.*

Let $G : \mathcal{P}_{\omega,}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ and $v : \mathcal{P}_{\omega,*}(I) \rightarrow I$. Then*

$P_1(\beta, G, v) \Leftrightarrow$ there exist three functions $H, \overline{H}, \widetilde{H} : \mathcal{P}_{\omega,}(I) \rightarrow \mathcal{P}_{\omega}(I)$ such that for all $I^* \in \mathcal{P}_{\omega,*}(I)$, $I^* \subseteq H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$, $H(I^*) \subseteq \{a \in I : |\mathcal{O}_{\beta}^+(a)| = +\infty\}$,*

$\overline{H}(I^), \widetilde{H}(I^*) \subseteq \{a \in I : |\mathcal{O}_{\beta}^+(a)| < +\infty\}$, and*

(a) $\exists z \in G(I^) : |\mathcal{O}_{\beta}^+(z)| = +\infty$ and*

$$\begin{cases} H(I^*) & \neq \emptyset \\ \overline{H}(I^*) & = \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_{\beta}^+(a) \right) \cup \left(\bigcup_{a \in H(I^*)} \{a, \beta(a), \beta^2(a), \dots, \xi_{\beta}(H(I^*))\} \right) \\ v(I^*) & = \xi_{\beta}(H(I^*)) \end{cases}$$

or

(b) $\forall a \in G(I^*) : |\mathcal{O}_{\beta}^+(a)| < +\infty$ and

$$\begin{cases} H(I^*) & = \emptyset \\ \overline{H}(I^*) & \neq \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_{\beta}^+(a) \right) \cup \left(\bigcup_{a \in \overline{H}(I^*)} \{a, \beta(a), \beta^2(a), \dots, v(I^*)\} \right) \end{cases}.$$

Démonstration. (\Rightarrow) : Suppose $P_1(\beta, G, v)$.

Let

$$H : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_{\omega}(I) \\ I^* & \mapsto \{a \in G(I^*) : |\mathcal{O}_{\beta}^+(a)| = +\infty\} \end{cases},$$

$$\overline{H} : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_{\omega}(I) \\ I^* & \mapsto \{a \in G(I^*) : |\mathcal{O}_{\beta}^+(a)| < +\infty \text{ and } v(I^*) \in \mathcal{O}_{\beta}^+(a)\} \end{cases}$$

and

$$\widetilde{H} : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_{\omega}(I) \\ I^* & \mapsto \{a \in G(I^*) : |\mathcal{O}_{\beta}^+(a)| < +\infty \text{ and } v(I^*) \notin \mathcal{O}_{\beta}^+(a)\} \end{cases}.$$

Let $I^* \in \mathcal{P}_{\omega,*}(I)$. We have $G(I^*) = H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$. Therefore $I^* \subseteq H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$.

(a) Suppose that $\exists z \in G(I^*) : |\mathcal{O}_{\beta}^+(z)| = +\infty$. So $H(I^*) \neq \emptyset$.

Let's show that $v(I^*) = \xi_{\beta}(H(I^*))$. Assume by way of contradiction that $\exists b \in H(I^*) : v(I^*) \notin \mathcal{O}_{\beta}^+(b)$. Then we must have $\mathcal{O}_{\beta}^+(b) \subseteq G(I^*)$,

which is impossible since $|\mathcal{O}_\beta^+(b)| = +\infty$ and $G(I^*)$ is finite. So $v(I^*) \in \bigcap_{b \in H(I^*)} \mathcal{O}_\beta^+(b) = \mathcal{O}_\beta^+(\xi_\beta(H(I^*)))$. In particular, $|\mathcal{O}_\beta^+(v(I^*))| = +\infty$ and so $v(I^*) \in H(I^*)$. So there exist $m, n \in \mathbb{N}$ such that $v(I^*) = \beta^m(\xi_\beta(H(I^*)))$ and $\xi_\beta(H(I^*)) = \beta^n(v(I^*))$. This can only work if $m = n = 0$ due to lemma 4.3.1 and therefore $v(I^*) = \xi_\beta(H(I^*))$.

Now we must have $\overline{H}(I^*) = \emptyset$ (otherwise there exists $a' \in G(I^*)$ such that $|\mathcal{O}_\beta^+(a')| < +\infty$ and $v(I^*) = \xi_\beta(H(I^*)) \in \mathcal{O}_\beta^+(z) \cap \mathcal{O}_\beta^+(a') = \emptyset$ by lemma 4.3.8, a contradiction).

Let's show that $G(I^*) = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_\beta^+(a) \right) \cup \left(\bigcup_{a \in H(I^*)} \{a, \beta(a), \beta^2(a), \dots, \xi_\beta(H(I^*))\} \right)$. $P_1(\beta, G, v)$ shows that the inclusion \supseteq is true. Conversely, let $a \in G(I^*)$. We have either $a \in H(I^*)$ or $a \in \widetilde{H}(I^*)$. Therefore the inclusion \subseteq is also true.

(b) Suppose that $\forall a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty$. So $H(I^*) = \emptyset$ and $\overline{H}(I^*) \neq \emptyset$.

Let's show that $G(I^*) = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_\beta^+(a) \right) \cup \left(\bigcup_{a \in \overline{H}(I^*)} \{a, \beta(a), \beta^2(a), \dots, v(I^*)\} \right)$. $P_1(\beta, G, v)$ shows that the inclusion \supseteq is true. Conversely, let $a \in G(I^*)$. We have $a \in \widetilde{H}(I^*)$ or $a \in \overline{H}(I^*)$ and so the inclusion \subseteq is true.

(\Leftarrow) : Suppose that H, \overline{H} and \widetilde{H} exist with the desired properties. Let $I^* \in \mathcal{P}_{\omega,*}(I)$.

(a) If $\exists z \in G(I^*) : |\mathcal{O}_\beta^+(z)| = +\infty$, then $\overline{H}(I^*) = \emptyset$ and so $G(I^*) \supseteq \widetilde{H}(I^*) \cup H(I^*) \supseteq I^*$ using the hypotheses. Moreover, we have clearly $v(I^*) \in G(I^*)$ and $\beta(G(I^*) \setminus \{v(I^*)\}) \subseteq G(I^*)$.

(b) If $\forall a \in G(I^*) : |\mathcal{O}_\beta^+(a)| < +\infty$, we have $H(I^*) = \emptyset$ and we can check similarly that $P_1(\beta, G, v)$ is true. \square

The following proposition characterizes the existential problem $\exists G \exists v : P_1(\beta, G, v)$ to $\widetilde{P}(\beta)$ in terms of $\widetilde{P}(\beta)$.

Proposition 4.7.2. *Let I be an infinite set and $\beta : I \rightarrow I$. Then*

$$(\exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : \exists v : \mathcal{P}_{\omega,*}(I) \rightarrow I : P_1(\beta, G, v)) \Leftrightarrow \widetilde{P}(\beta).$$

Démonstration. (\Rightarrow) : Let $a, b \in I$ such that $|\mathcal{O}_\beta^+(a)| = |\mathcal{O}_\beta^+(b)| = +\infty$. Assume by way of contradiction that $\mathcal{O}_\beta^+(a) \cap \mathcal{O}_\beta^+(b) = \emptyset$. So $a \neq b$. Let $I^* = \{a, b\}$. Let $G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ and $v : \mathcal{P}_{\omega,*}(I) \rightarrow I$ such that $P_1(\beta, G, v)$. Suppose by way of contradiction that $v(I^*) \in \mathcal{O}_\beta^+(a)$. Then $v(I^*) \notin \mathcal{O}_\beta^+(b)$ since $\mathcal{O}_\beta^+(a) \cap \mathcal{O}_\beta^+(b) = \emptyset$. Using the hypothesis, we obtain $\mathcal{O}_\beta^+(b) \subseteq G(I^*)$ which is impossible since $G(I^*)$ is finite and $\mathcal{O}_\beta^+(b)$ is infinite. Therefore $v(I^*) \notin \mathcal{O}_\beta^+(a)$. But this is also impossible for the same reason. So we must have $\mathcal{O}_\beta^+(a) \cap \mathcal{O}_\beta^+(b) \neq \emptyset$.

(\Leftarrow) : Suppose that $\widetilde{P}(\beta)$ is true. Let

$$H : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_\omega(I) \\ I^* & \mapsto \{a \in I^* : |\mathcal{O}_\beta^+(a)| = +\infty\} \end{cases},$$

$$\begin{aligned} \widetilde{H} &: \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_{\omega}(I) \\ I^* & \mapsto \{a \in I^* : |\mathcal{O}_{\beta}^+(a)| < +\infty\} \end{cases}, \\ G &: \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_{\omega,*}(I) \\ I^* & \mapsto \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_{\beta}^+(a) \right) \cup \left(\bigcup_{a \in H(I^*)} \{a, \beta(a), \beta^2(a), \dots, \xi_{\beta}(H(I^*))\} \right) \end{cases}, \end{aligned}$$

and

$$v : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto \xi_{\beta}(H(I^*)) \text{ if } H(I^*) \neq \emptyset \\ I^* & \mapsto \text{an arbitrary element of } I^* \text{ otherwise} \end{cases},$$

where we used the fact $\mathcal{P}_{\omega,*}(\{a \in I : |\mathcal{O}_{\beta}^+(a)| = +\infty\}) \subseteq D_{\beta}$ (see lemma 4.3.9). Then $P_1(\beta, G, v)$ holds. \square

4.7.2 Solution to problem $P_2(\phi, G, u)$

Proposition 4.7.1 implies, trivially, the following characterization of $P_2(\phi, G, u)$. It is not known to the authors if $P_2(\phi, G, u)$ admits a simpler description.

Proposition 4.7.3. *Let I be an infinite set and $\beta : I \rightarrow I$.*

Let $G : \mathcal{P}_{\omega,}(I) \rightarrow \mathcal{P}_{\omega,*}(I)$ and $v : \mathcal{P}_{\omega,*}(I) \rightarrow I$. Then*

$P_2(\beta, G, v) \Leftrightarrow$ there exist three functions $H, \overline{H}, \widetilde{H} : \mathcal{P}_{\omega,}(I) \rightarrow \mathcal{P}_{\omega}(I)$ such that for all $I^* \in \mathcal{P}_{\omega,*}(I)$, $I^* \subseteq H(I^*) \cup \overline{H}(I^*) \cup \widetilde{H}(I^*)$, $H(I^*) \subseteq \{a \in I : |\mathcal{O}_{\beta}^+(a)| = +\infty\}$, $\overline{H}(I^*), \widetilde{H}(I^*) \subseteq \{a \in I : |\mathcal{O}_{\beta}^+(a)| < +\infty\}$, and*

(a) $\exists z \in G(I^) : |\mathcal{O}_{\beta}^+(z)| = +\infty$ and*

$$\begin{cases} H(I^*) & \neq \emptyset \\ \overline{H}(I^*) & = \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_{\beta}^+(a) \right) \cup \left(\bigcup_{a \in H(I^*)} \{a, \beta(a), \beta^2(a), \dots, \xi_{\beta}(H(I^*))\} \right) \\ v(I^*) & = \xi_{\beta}(H(I^*)) \in I^* \end{cases}$$

or

(b) $\forall a \in G(I^) : |\mathcal{O}_{\beta}^+(a)| < +\infty$ and*

$$\begin{cases} H(I^*) & = \emptyset \\ \overline{H}(I^*) & \neq \emptyset \\ G(I^*) & = \left(\bigcup_{a \in \widetilde{H}(I^*)} \mathcal{O}_{\beta}^+(a) \right) \cup \left(\bigcup_{a \in \overline{H}(I^*)} \{a, \beta(a), \beta^2(a), \dots, v(I^*)\} \right) \\ v(I^*) & \in I^* \end{cases}.$$

In what follows, we direct our attention to the existential problem $\exists G \exists u : P_2(\phi, G, u)$, which we try to characterize in terms of a condition on ϕ . To achieve that, let's first prove

Proposition 4.7.4. *Let I be an infinite set and $\beta : I \rightarrow I$. Then, if $D_\beta = \mathcal{P}_{\omega,*}(I)$ and $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \xi_\beta(I^*) \in I^*$, then $(\exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : \exists v : \mathcal{P}_{\omega,*}(I) \rightarrow I : P_2(\beta, G, v))$.*

Démonstration. Suppose that $D_\beta = \mathcal{P}_{\omega,*}(I)$ and $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \xi_\beta(I^*) \in I^*$. Then, by lemma 4.3.8, and using the previous notations of proposition 4.7.2, we have $\forall I^* \in \mathcal{P}_{\omega,*}(I) : H(I^*) \in \{I^*, \emptyset\}$ and so $v(I^*) \in I^*$. Therefore, there exist G and v with the desired properties. \square

Definition 4.7.3. Let I be a set and $\phi : I \rightarrow I$.

We define the partial order \preceq_ϕ on I by : $a \preceq_\phi b \Leftrightarrow b \in \mathcal{O}_\phi^+(a)$.

The following proposition characterizes, under the condition that all the orbits of ϕ are infinite, the existential problem $\exists G \exists u : P_2(\phi, G, u)$ in terms of the orbital structure of ϕ .

Proposition 4.7.5. *Let I be an infinite set, $\phi : I \rightarrow I$ such that $(\forall a \in I) : \mathcal{O}_\phi^+(a)$ is infinite.*

Then the following conditions are equivalent :

1. $\exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : \exists u : \mathcal{P}_{\omega,*}(I) \rightarrow I : P_2(\phi, G, u)$.
2. $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^* \neq \emptyset$,
3. $D_\phi = \mathcal{P}_{\omega,*}(I)$ and $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^* = \{\xi_\phi(I^*)\}$.
4. $D_\phi = \mathcal{P}_{\omega,*}(I)$ and $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \xi_\phi(I^*) \in I^*$,
5. $\forall n \in \mathbb{N}^* : \forall (a_1, \dots, a_n) \in I^n$ which are distinct : $\exists d \in \llbracket 1, n \rrbracket : \exists (m_1, \dots, m_n) \in \mathbb{N}^n$ with $m_d = 0 : \forall i \in \llbracket 1, n \rrbracket : a_d = \phi^{m_i}(a_i)$.
6. $\forall (a, b) \in I^2$ with $a \neq b$, $(\exists n \in \mathbb{N} : a = \phi^n(b))$ or $(\exists n \in \mathbb{N} : b = \phi^n(a))$.
7. \preceq_ϕ is a total order on I .

Démonstration. (1) \Rightarrow (2) : Let $I^* \in \mathcal{P}_{\omega,*}(I)$. We have from proposition 4.7.1 that $u(I^*) = \xi_\phi(H(I^*)) \in \bigcap_{a \in H(I^*)} \mathcal{O}_\phi^+(a) = \bigcap_{a \in G(I^*)} \mathcal{O}_\phi^+(a) \subseteq \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)$, where we used the fact that here we have necessarily $H(I^*) = G(I^*) \supseteq I^*$. Moreover, we have $u(I^*) \in I^*$. So $\left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^* \neq \emptyset$.

(2) \Rightarrow (3) : Let $I^* \in \mathcal{P}_{\omega,*}(I)$. Since $\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \neq \emptyset$, then $I^* \in D_\phi$. Let $x \in \left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^*$. Since $x \in \bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) = \mathcal{O}_\phi^+(\xi_\phi(I^*))$ by lemma 4.3.6 and $\xi_\phi(I^*) = \phi^{m_x^{\xi_\phi(I^*)}}(x)$ because $x \in I^*$ (see definition 4.3.1), then $\exists n \geq m_x^{\xi_\phi(I^*)} : x = \phi^n(x)$. Since $\{x, \phi(x), \phi^2(x), \dots\}$ are distinct (see lemma 4.3.1), we have necessarily $n = 0$ and so $m_x^{\xi_\phi(I^*)} = 0$. This implies $\xi_\phi(I^*) = \phi^{m_x^{\xi_\phi(I^*)}}(x) = x \in I^*$. Since this is true for all $x \in \left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^*$, we have $\left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a) \right) \cap I^* = \{\xi_\phi(I^*)\}$.

(3) \Rightarrow (4) : This is clear.

(4) \Rightarrow (1) : This is proposition 4.7.4.

(4) \Rightarrow (5) : Let $n \in \mathbb{N}^*$ and let a_1, \dots, a_n be distinct elements of I . Applying (4) to $I^* = \{a_1, \dots, a_n\}$ gives $\xi_\phi(I^*) \in I^*$ which is the content of (5).

(5) \Rightarrow (2) : Let $I^* \in \mathcal{P}_{\omega,*}(I)$. Write I^* as $\{a_1, \dots, a_n\}$ where n is the cardinality of I^* . From (5), we have $a_d \in \left(\bigcap_{a \in I^*} \mathcal{O}_\phi^+(a)\right) \cap I^*$, and so this last set is not empty.

(5) \Rightarrow (6) : This is clear.

(6) \Rightarrow (5) : Let $n \in \mathbb{N}^*$ and let a_1, \dots, a_n be distinct elements of I . Assume by way of contradiction that $\forall d \in \llbracket 1, n \rrbracket : \exists l(d) \in \llbracket 1, n \rrbracket \setminus \{d\} : a_d \notin \mathcal{O}_\phi^+(a_{l(d)})$. By (6), this implies that $\forall d \in \llbracket 1, n \rrbracket : a_{l(d)} \in \mathcal{O}_\phi^+(a_d)$. Besides, we have necessarily $\exists d \in \llbracket 1, n \rrbracket : \exists e \in \mathbb{N}^* : l^e(d) = d$. Otherwise, we would have $\forall d \in \llbracket 1, n \rrbracket : \forall e \in \mathbb{N}^* : l^e(d) \neq d$, which implies that $\forall \tilde{d} \in \llbracket 1, n \rrbracket : \{\tilde{d}, l(\tilde{d}), l^2(\tilde{d}), \dots\}$ are distinct. This is impossible since this infinite set is included in $\llbracket 1, n \rrbracket$. Therefore, $\exists(n_1, \dots, n_e) \in \mathbb{N}^e : a_d = a_{l^e(d)} = \phi^{n_e}(a_{l^{e-1}(d)}) = \phi^{n_e}(\phi^{n_{e-1}}(a_{l^{e-2}(d)})) = \dots = \phi^{\sum_{i=1}^e n_i}(a_d)$. Since $\mathcal{O}_\phi^+(a_d)$ is infinite, we have by lemma 4.3.1 that $\forall i \in [1, e] : n_i = 0$, which implies in particular that $a_{l(d)} = a_d$, a contradiction. Therefore we have the result.

(6) \Leftrightarrow (7) : This is just the definition of a total order. \square

From this proposition, the following corollary characterizing the existential problem $\exists G \exists u : P_2(\phi, G, u)$ with no additional assumption can be easily derived.

Corollary 4.7.1. *Let I be an infinite set and $\phi : I \rightarrow I$. Denote by I_ϕ^{inf} (resp. I_ϕ^{fin}) the subsets of I consisting of the elements which have an infinite (resp. finite) orbit under ϕ . Notice that I_ϕ^{inf} and I_ϕ^{fin} are both invariant under ϕ and that $\{I_\phi^{inf}, I_\phi^{fin}\}$ is a partition of I . Then the following conditions are equivalent :*

1. $\exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : \exists u : \mathcal{P}_{\omega,*}(I) \rightarrow I : P_2(\phi, G, u)$.
2. $(\preceq_\phi)_{I_\phi^{inf}}$ is a total order on I_ϕ^{inf} .

Lemma 4.7.3. *Let I be a set and $\phi : I \rightarrow I$.*

Suppose that $\exists a \in I : \mathcal{O}_\phi^+(a) = I$.

Then \preceq_ϕ is a total order on I .

Démonstration. Let $x, y \in I$. We can write $x = \phi^n(a)$ and $y = \phi^m(a)$ for $m, n \in \mathbb{N}$. If $m \leq n$, we have $y \preceq_\phi x$. Otherwise we have $x \preceq_\phi y$. Hence \preceq_ϕ is a total order. \square

The following proposition is a supplement to proposition 4.7.5 where we also suppose the existence of a cofinite orbit.

Proposition 4.7.6. *Let I be an infinite set and $\phi : I \rightarrow I$ such that $\forall a \in I : \mathcal{O}_\phi^+(a)$ is infinite. Then we have*

$$\begin{aligned}
& (\exists \tilde{a} \in I : \mathcal{O}_\phi^+(\tilde{a}) \text{ is cofinite}) \\
& \text{and } (\exists u : \mathcal{P}_{\omega,*}(I) \rightarrow I : \exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : P_2(\phi, G, u)) \\
& \Leftrightarrow (\exists \tilde{a} \in I : \mathcal{O}_\phi^+(\tilde{a}) \text{ is cofinite}) \text{ and } (\preceq_\phi \text{ is a total order on } I) \\
& \Leftrightarrow \exists \alpha : (\mathbb{N}, \leq) \rightarrow (I, \preceq_\phi) \text{ such that } \alpha \text{ is an increasing bijection} \\
& \Leftrightarrow \exists \tilde{a} \in I : \mathcal{O}_\phi^+(\tilde{a}) = I.
\end{aligned}$$

Démonstration. Suppose that $\exists \tilde{a} \in I : \mathcal{O}_\phi^+(\tilde{a}) = I$. Let $\alpha = \begin{cases} \mathbb{N} & \rightarrow I \\ n & \mapsto \phi^n(\tilde{a}) \end{cases}$. Clearly,

α is increasing. It is injective by lemma 4.3.1. It is surjective by the hypothesis. Conversely, suppose that there is such an increasing bijection α from \mathbb{N} to I . For all $n \in \mathbb{N}$, we have $\alpha(n+1) = \phi(\alpha(n))$. Indeed, since α is increasing, we have $\exists l(n) \in \mathbb{N} : \alpha(n+1) = \phi^{l(n)}(\alpha(n))$. Suppose by way of contradiction that $l(n) \neq 1$. If $l(n) = 0$, we would have $\alpha(n+1) = \alpha(n)$, contradicting the injectivity of α . If $l(n) \geq 2$, let $m \in \mathbb{N}$ such that $\alpha(m) = \phi(\alpha(n))$. If $m \geq n+1$, then $\alpha(n+1) \preceq_\phi \alpha(m)$, and so $\phi(\alpha(n)) = \alpha(m) \in \mathcal{O}_\phi^+(\alpha(n+1)) = \mathcal{O}_\phi^+(\phi^{l(n)}(\alpha(n)))$. This implies $\exists u \in \mathbb{N} : \phi(\alpha(n)) = \phi^{l(n)-1+u}(\phi(\alpha(n)))$ where $l(n)-1+u \geq 1$, contradicting lemma 4.3.1. If $m \leq n$, then $\alpha(m) \preceq_\phi \alpha(n)$, and so $\alpha(n) \in \mathcal{O}_\phi^+(\alpha(m)) = \mathcal{O}_\phi^+(\phi(\alpha(n)))$. This implies $\exists u \in \mathbb{N} : \alpha(n) = \phi^{u+1}(\alpha(n))$ where $u+1 \geq 1$, contradicting again lemma 4.3.1. Hence $l(n) = 1$ and the relation $\alpha(n+1) = \phi(\alpha(n))$ is proved. Now consider $\tilde{a} = \alpha(0)$. Then clearly $\mathcal{O}_\phi^+(\tilde{a}) = \alpha(\mathbb{N}) = I$. This establishes the equivalence of the last two statements.

Using lemma 4.7.3, the last statement clearly implies the second. Let's prove that the second one implies the last one. Using lemma 4.3.5, we have that all the orbits of ϕ are cofinite. Let $b \in I$ such that $|I \setminus \mathcal{O}_\phi^+(b)|$ is minimal. Suppose by way of contradiction that $\mathcal{O}_\phi^+(b) \neq I$ and let $c \in I \setminus \mathcal{O}_\phi^+(b)$. Since \preceq_ϕ is a total order, we have necessarily $b \in \mathcal{O}_\phi^+(c)$. Specifically, we have $\exists n \geq 1 : b = \phi^n(c)$. Since $c, \phi(c), \phi^2(c), \dots$ are distinct by lemma 4.3.1, we have, thus, $|I \setminus \mathcal{O}_\phi^+(c)| < |I \setminus \mathcal{O}_\phi^+(b)|$ which is a contradiction. Hence $\mathcal{O}_\phi^+(b) = I$.

The first and second statements are equivalent by corollary 4.3.3 and proposition 4.7.5. \square

Note that we can also prove explicitly that

$$\exists a \in I : \mathcal{O}_\phi^+(a) = I \Rightarrow (\exists u : \mathcal{P}_{\omega,*}(I) \rightarrow I : \exists G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I) : P_2(\phi, G, u)).$$

Indeed, suppose $\exists a \in I : \mathcal{O}_\phi^+(a) = I$ and define

$$n : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathbb{N} \\ I^* & \mapsto \max\{n \in \mathbb{N} : \phi^n(a) \in I^*\} \end{cases}$$

which is well-defined because $\{a, \phi(a), \phi^2(a), \dots\}$ are distinct (lemma 4.3.1) and let

$$u : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto \phi^{n(I^*)}(a) \end{cases}$$

and

$$G : \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow \mathcal{P}_{\omega,*}(I) \\ I^* & \mapsto \{a, \phi(a), \dots, \phi^{n(I^*)}(a)\} \end{cases}.$$

Then u and G clearly satisfy the requirements.

Remark 4.7.1. There are many maps $\phi : \mathbb{N} \rightarrow \mathbb{N}$ such that $\exists a \in \mathbb{N} : \mathcal{O}_\phi^+(a) = \mathbb{N}$ (not just the successor function). They are conjugated to the successor function and can be found by choosing a bijection $\alpha : \mathbb{N} \rightarrow \mathbb{N}$ and setting $\phi = \alpha \circ \text{succ} \circ \alpha^{-1}$. For example, there is the function

$$\begin{cases} \mathbb{N} & \rightarrow \mathbb{N}^* \\ 0 & \mapsto 2 \\ 2i + 1 & \mapsto 2i + 4 \text{ for all } i \in \mathbb{N} \\ 2i + 2 & \mapsto 2i + 1 \text{ for all } i \in \mathbb{N} \end{cases}$$

4.7.3 An indivisibility property for 3-tuples (ϕ, G, u) satisfying $P_2(\phi, G, u)$

3-tuples $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ satisfying $P_2(\phi, G, u)$ enjoy the following indivisibility property with respect to composition of maps $\beta : I \rightarrow I$ satisfying $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$ with bijective maps $\alpha : I \rightarrow I$ satisfying $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \alpha(G(I^*)) \subseteq G(I^*)$.

Proposition 4.7.7. *Let I be a set and $(\phi : I \rightarrow I, G : \mathcal{P}_{\omega,*}(I) \rightarrow \mathcal{P}_{\omega,*}(I), u : \mathcal{P}_{\omega,*}(I) \rightarrow I)$ a 3-tuple satisfying $P_2(\phi, G, u)$.*

Then if $\phi = \beta \circ \alpha$ where $\beta : I \rightarrow I$ is a map satisfying $\forall I^ \in \mathcal{P}_{\omega,*}(I) : \exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$ and $\alpha : I \rightarrow I$ is a bijective map such that $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \alpha(G(I^*)) \subseteq G(I^*)$, then $\alpha = \text{id}$ and $\beta = \phi$.*

Démonstration. First, notice that for all $I^* \in \mathcal{P}_{\omega,*}(I)$, α bijective and $\alpha(G(I^*)) \subseteq G(I^*)$ implies that α induces a bijection on $G(I^*)$.

Let $v = \begin{cases} \mathcal{P}_{\omega,*}(I) & \rightarrow I \\ I^* & \mapsto \alpha(u(I^*)) \in G(I^*) \end{cases}$. Let $I^* \in \mathcal{P}_{\omega,*}(I)$. Since α induces a

bijection on $G(I^*)$, it is easily seen that $\beta(G(I^*) \setminus \{v(I^*)\}) \subseteq G(I^*)$. Since $\exists a \in G(I^*) : |\mathcal{O}_\beta^+(a)| = +\infty$, it follows from proposition 4.7.1 that $v(I^*) = \xi_\beta(H(I^*))$, where we have used the notation of that proposition. Specifying this relation for $I^* = \{b\}$, one has $\alpha(b) = \alpha(u(\{b\})) = v(\{b\}) = \xi_\beta(H(\{b\}))$ for all $b \in I$.

Since $\forall I^* \in \mathcal{P}_{\omega,*}(I) : \alpha(G(I^*)) \subseteq G(I^*)$, we have $\forall b \in I : \alpha(b) \in \alpha(G(\{b\})) \subseteq G(\{b\})$, and by induction $\mathcal{O}_\alpha^+(b) \subseteq G(\{b\})$. Since $G(\{b\})$ is finite and α is bijective, we have from lemma 4.3.1 $\exists n_b \geq 1 : \alpha^{n_b}(b) = b$. This implies in particular that $\mathcal{O}_\beta^+(b) = \mathcal{O}_\beta^+(\alpha^{n_b}(b)) = \mathcal{O}_\beta^+(\xi_\beta(H(\{\alpha^{n_b-1}(b)\}))) = \bigcap_{c \in H(\{\alpha^{n_b-1}(b)\})} \mathcal{O}_\beta^+(c)$ is an orbit arising as a finite intersection of infinite orbits. Thus it is infinite, which is true for all $b \in I$.

Now let $b \in I$. We have $|\mathcal{O}_\beta^+(b)| = |\mathcal{O}_\beta^+(\alpha(b))| = +\infty$. Let $I^* = H(\{\alpha(b)\})$. Since $\alpha(b) \in I^*$, we have $\xi_\beta(I^*) \in \mathcal{O}_\beta^+(\alpha(b))$. Hence $\alpha(\alpha(b)) = \xi_\beta(I^*) \in \mathcal{O}_\beta^+(\alpha(b))$. So $\forall b \in I : \exists k_b \in \mathbb{N} : \alpha(\alpha(b)) = \beta^{k_b}(\alpha(b))$. We have $\forall b \in I : \alpha(\alpha(\alpha(b))) = \beta^{k_{\alpha(b)}}(\alpha(\alpha(b))) = \beta^{k_{\alpha(b)} + k_b}(\alpha(b))$. In general, $\forall b \in I : \forall n \geq 1 : \alpha^n(b) \in \mathcal{O}_\beta^+(\alpha(b))$. Let $b \in I$ and choose $n = n_{\alpha(b)} \geq 1$. Then $\alpha(b) = \alpha^{n_{\alpha(b)}}(\alpha(b)) \in \mathcal{O}_\beta^+(\alpha(\alpha(b)))$, and so $\exists m \in \mathbb{N} : \alpha(b) = \beta^m(\alpha(\alpha(b)))$. We saw previously that $\alpha(\alpha(b)) \in \mathcal{O}_\beta^+(\alpha(b))$, so $\exists l \in \mathbb{N} : \alpha(\alpha(b)) = \beta^l(\alpha(b))$. Combining the two results, we have $\alpha(b) = \beta^{m+l}(\alpha(b))$ and so $m = l = 0$ since $|\mathcal{O}_\beta^+(\alpha(b))| = +\infty$. In turn, this implies that $\alpha(\alpha(b)) = \alpha(b)$ and so $\alpha(b) = b$ since α is a bijection.

Hence $\alpha = id$ and $\beta = \phi$. □

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Résumé (250 mots Max) :

Cette thèse a pour objet l'étude de quelques aspects de l'espace des repères, de l'indépendance et des ensembles quasi-invariants. Dans la première partie, nous démontrons deux résultats sur la connexité par arcs et l'adhérence (respectivement) de quelques ensembles reliés à la variété de Stiefel non compacte (espace des repères), et nous illustrons le dernier résultat par quelques exemples. Dans la deuxième partie, nous prouvons quelques généralisations d'un résultat garantissant l'équivalence entre l'indépendance linéaire et la dimension infinie d'une suite infinie de vecteurs décalés par un opérateur linéaire. Dans la troisième partie, nous décrivons les fonctions admettant un nombre maximal d'ensembles ou de sur-ensembles finis invariants ou intérieurement-1-quasi-invariants.

Abstract (250 words Max) :

This thesis deals with some aspects of the frame space, of independence, and of quasi-invariant sets. In the first part, we prove two results on the path-connectedness and topological closure (respectively) of some sets related to the non-compact Stiefel manifold (frame space), and we illustrate the latter with some examples. In the second part, we prove some generalizations of a result guaranteeing the equivalence between the linear independence and infinite dimension of an infinite sequence of vectors shifted by a linear operator. In the third part, we describe the functions admitting a maximal number of finite invariant or internally-1-quasi-invariant sets or supersets.