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Alen Đurić. Quadratic normalisations and coherent presentations of monoids. Mathematics [math]. Université Paris Cité, 2023. English. NNT: . tel-04185552v1

HAL Id: tel-04185552

<https://hal.science/tel-04185552v1>

Submitted on 22 Aug 2023 (v1), last revised 25 May 2024 (v2)

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Quadratic normalisations and coherent presentations of monoids

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Thèse de doctorat de Mathématiques

Dirigée par Yves Guiraud et Pierre-Louis Curien

Présentée et soutenue publiquement le 13 avril 2023

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**Normalisations quadratiques et présentations
cohérentes de monoïdes**

Alen Đurić

Рафаилу и Михаилу

Titre : Normalisations quadratiques et présentations cohérentes de monoïdes

Résumé : Le but principal de cette thèse est d'utiliser le cadre des polygraphes pour une étude fine des formes des cellules de dimension supérieure (de dimension 3, et implicitement de dimension 4) provenant des normalisations quadratiques apparaissant en théorie combinatoire des groupes. La thèse propose une construction de présentations cohérentes pour une classe de monoïdes admettant une famille de Garside noethérienne à droite. Ainsi, elle présente une généralisation unificatrice de deux extensions distinctes de la construction originale de Deligne de présentations cohérentes de monoïdes d'Artin-Tits sphériques, données par Gaussent, Guiraud et Malbos : aux monoïdes d'Artin-Tits quelconques, et aux monoïdes de Garside.

En outre, des correspondances sont établies entre la notion de normalisation quadratique et la notion de structure de factorabilité. Les monoïdes factorisables sont caractérisés dans le cadre axiomatique des normalisations quadratiques. En outre, les normalisations quadratiques de classe $(4, 3)$ sont caractérisées en termes de structures de factorabilité et d'une condition garantissant la terminaison du système de réécriture associé.

De plus, on construit une présentation cohérente des monoïdes admettant une normalisation quadratique de classe $(4, 3)$, qui généralise la présentation cohérente construite par Hage et Malbos pour les monoïdes plaxiques de type A à l'aide de la présentation colonne.

Mots clefs : monoïde, présentation cohérente, réécriture, polygraphe, monoïde d'Artin-Tits, famille de Garside, factorabilité, monoïde plaxique, normalisation quadratique, réduction homotopique

Title: Quadratic normalisations and coherent presentations of monoids

Abstract: The principal goal of this thesis is to use the framework of polygraphs for a fine study of the shapes of higher-dimensional cells (of dimension 3, and implicitly of dimension 4) arising from quadratic normalisations appearing in combinatorial group theory. The thesis proposes a construction of coherent presentations for a class of monoids admitting a right-noetherian Garside family. Thereby, it presents a unifying generalisation of two distinct extensions of Deligne's original construction of coherent presentations of spherical Artin-Tits monoids, given by Gaussent, Guiraud and Malbos: to general Artin-Tits monoids and to Garside monoids.

Furthermore, correspondences are established between the notion of quadratic normalisation and the notion of factorability structure. Namely, factorable monoids are characterised in the axiomatic setting of quadratic normalisations. Additionally, quadratic normalisations of class $(4, 3)$ are characterised in terms of factorability structures and a condition ensuring the termination of the associated rewriting system.

Moreover, a coherent presentation of monoids admitting quadratic normalisation of class $(4, 3)$ is constructed and specialised to the already known column coherent presentation of plactic monoids of type A , constructed by Hage and Malbos.

Keywords: monoid, coherent presentation, rewriting, polygraph, Artin-Tits monoid, Garside family, factorability, plactic monoid, quadratic normalisation, homotopical reduction

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Introduction (français)

Les monoïdes figurent parmi les structures algébriques de base. Il s'agit d'ensembles non vides dotés d'une opération binaire associative et d'un élément identité bilatéral. Les monoïdes sont donc plus généraux que les groupes, car les éléments des monoïdes ne sont pas tenus d'être inversibles. Il existe une autre façon de considérer les monoïdes, à savoir comme des catégories ayant un seul objet, les éléments d'un monoïde correspondant aux flèches d'une catégorie.

Si S est un ensemble, le monoïde libre engendré par S est l'ensemble de toutes les séquences finies (ou mots) d'éléments de S , avec la concaténation comme opération du monoïde, et la séquence vide comme élément identité. Tous les monoïdes ne sont pas libres, car il peut exister des relations non triviales entre les éléments. Cependant, tout monoïde est une image homomorphe (ou un quotient) d'un monoïde libre. Cela permet de présenter un monoïde par un ensemble de générateurs et un ensemble de relations génératrices.

En remontant d'une dimension, on constate qu'il peut aussi y avoir des relations entre les relations, et ainsi de suite. Le cadre des polygraphes permet de garder une trace de toutes ces informations. Les polygraphes nous permettent également d'appliquer des transformations homotopiques à une présentation, afin d'obtenir une présentation du même monoïde mais possédant certaines propriétés.

L'un des attributs souhaitables d'une présentation est la convergence. Elle permet de résoudre le problème du mot (décider si deux représentants dans le quotient représentent bien le même élément du monoïde). Une propriété qui nous intéresse particulièrement est la cohérence.

Le but principal de cette thèse est d'utiliser le cadre des polygraphes pour une étude fine des formes des cellules de dimension supérieure (de dimension 3, et implicitement de dimension 4) provenant des normalisations quadratiques apparaissant en théorie combinatoire des groupes. La thèse propose une construction de présentations cohérentes pour une classe de monoïdes admettant une famille de Garside noethérienne à droite. Ainsi, elle présente une généralisation unificatrice de deux extensions distinctes de la construction originale de Deligne de présentations cohérentes de monoïdes d'Artin-Tits sphériques, données par Gaussent, Guiraud et Malbos : aux monoïdes d'Artin-Tits quelconques, et aux monoïdes de Garside. De plus, on construit une présentation cohérente des monoïdes admettant une normalisation quadratique de classe $(4, 3)$, qui généralise la présentation cohérente construite par Hage et Malbos pour les monoïdes plaxiques de type A à l'aide de la présentation colonne. En outre, des correspondances sont établies entre la notion de normalisation quadratique (développée en France, notamment par Dehornoy) et la notion de structure de factorabilité (développée en Allemagne, notamment par Bödigheimer, dans le but d'utiliser des formes normales appropriées pour comprendre l'homologie des groupes). Les monoïdes factorisables sont caractérisés dans le

cadre axiomatique des normalisations quadratiques. En outre, les normalisations quadratiques de classe $(4, 3)$ sont caractérisées en termes de structures de factorabilité et d'une condition garantissant la terminaison du système de réécriture associé.

Voici un aperçu des classes de monoïdes pertinentes, avec des flèches indiquant les inclusions, les flèches en pointillé « dash » étant notre contribution et les flèches en pointillé « dots » étant notre hypothèse :

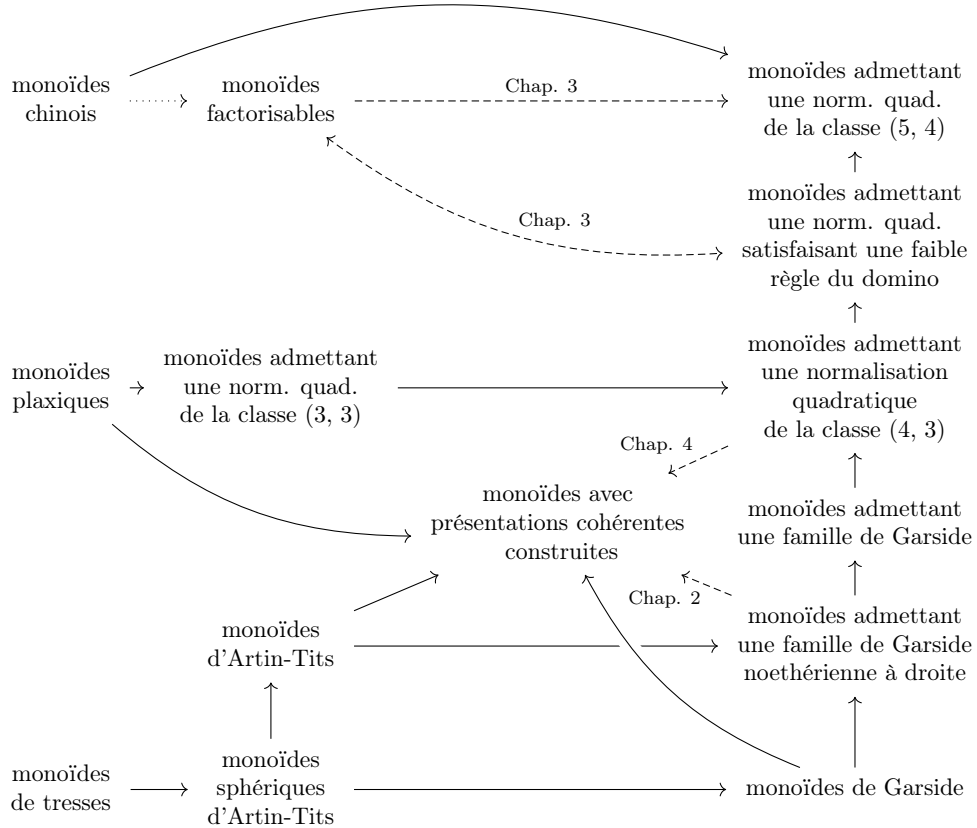


Schéma de la thèse

Brèves notes historiques. Nous préparons le terrain en donnant un très bref aperçu historique des sujets étudiés.

Ce que l'on appelle aujourd'hui la théorie de Garside a commencé avec les travaux de Garside sur les monoïdes de tresses dans les années 1960. Garside [25] a étudié les propriétés arithmétiques des groupes de tresses. Il a résolu le problème du mot et le problème de conjugaison pour les groupes de tresses en introduisant des monoïdes de tresses. Il a montré que la présentation d'Artin définit le monoïde des mots positifs dans le groupe des tresses. Il a notamment prouvé que le monoïde tressé B_n^+ est simplifiable à gauche, et que deux éléments quelconques de B_n^+ admettent un plus petit multiple commun. Il a également introduit l'élément de Garside (qu'il a appelé mot fondamental) d'un monoïde de tresses.

Les observations de Garside pour les monoïdes de tresses ont été généralisées aux monoïdes d'Artin-Tits (de type) sphériques par Brieskorn et Saito [5] et par Deligne [18].

La forme normale dite gloutonne a été introduite pour la première fois pour les monoïdes de tresses dans les années 1980, sur la base des observations de Garside (voir [13, Introduction] pour les références). Le développement s'est poursuivi en France, où Dehornoy et Paris [17] ont introduit les monoïdes de Garside afin d'abstraire les propriétés qui établissent l'existence de la forme normale gloutonne. Ensuite, Dehornoy, Digne et Michel [14] ont généralisé les monoïdes de Garside à des catégories admettant des familles de Garside. Par exemple, un groupe de Coxeter constitue une famille de Garside dans le monoïde d'Artin-Tits correspondant. De plus, Dehornoy, Dyer et Hohlweg [15] ont montré que tout monoïde Artin-Tits admet une famille de Garside finie (même si le groupe de Coxeter correspondant est infini). Un développement approfondi de la notion de famille de Garside se trouve dans la monographie [13].

Deligne [19] a étudié les actions faibles des monoïdes d'Artin-Tits sur les catégories. Il a prouvé que pour définir une action faible d'un monoïde d'Artin-Tits sphérique, il suffit de considérer des endofoncteurs pour chaque élément du groupe de Coxeter correspondant (c'est-à-dire qu'il n'est pas nécessaire de le faire pour chaque élément du monoïde). Cette définition alternative est basée sur ce qui est maintenant appelé la présentation de Garside des monoïdes d'Artin-Tits. Deligne a explicitement donné la présentation de Garside pour les monoïdes d'Artin-Tits sphériques. Michel [46] a étendu cette présentation à tous les monoïdes d'Artin-Tits. Gaussent, Guiraud et Malbos [26] ont prouvé que le résultat de Deligne est équivalent à l'affirmation que les monoïdes d'Artin-Tits sphériques admettent une présentation cohérente particulière, appelée présentation cohérente de Garside. Dans ce cadre, ils ont utilisé une complétion-réduction homotopique en s'appuyant sur les travaux de Squier [49] et de Brown [6], et ont étendu le résultat de Deligne à l'ensemble de tous les monoïdes d'Artin-Tits, ainsi qu'aux monoïdes de Garside.

La procédure de complétion-réduction homotopique a également été appliquée avec succès aux monoïdes plaxiques et chinois dans [32], ce qui donne des présentations cohérentes. Ceci suggère que les familles de Garside ne fournissent pas, en fin de compte, le cadre le plus général pour les formes normales gloutonnes. La notion de normalisation quadratique, introduite par Dehornoy et Guiraud [16] (influencés par Krammer [37]), semble remplir ce rôle car elle généralise, dans le même cadre axiomatique, deux classes bien connues de normalisations : celles issues des systèmes de réécriture quadratiques, telles qu'étudiées dans [26] pour les monoïdes Artin-Tits, et dans [4] et [7] pour les monoïdes plaxiques; et celles qui découlent de familles de Garside, comme cela a été étudié dans [13]. Nous renvoyons le lecteur à l'étude [12] pour une vue d'ensemble des extensions successives de la forme normale gloutonne, des monoïdes de tresses aux monoïdes admettant des normalisations quadratiques pondérées à gauche.

Pendant ce temps, en Allemagne, Bödigheimer et ses collaborateurs ont mis au point la notion de structure de factorabilité, introduite par Wang [51] et Heß [34] sur les monoïdes et les catégories comme une extension de la définition de structure de factorabilité sur les groupes introduite par Bödigheimer [3] et Visy [50]. Leur motivation était d'abstraire la structure, découverte dans les groupes symétriques, qui assure l'existence d'une forme normale permettant de réduire le

complexe « bar » associé à un complexe comportant beaucoup moins de cellules et permettant de calculer l’homologie de la structure en question. Cette réduction est obtenue grâce à la théorie des « schémas de contraction », qui a été introduite dans un contexte topologique par Brown [6], élaborée pour l’algèbre par Cohen [10], et redécouverte (sous le nom de théorie de Morse discrète) par Forman [23]. L’idée est d’établir, pour chaque entier positif n , une correspondance entre une classe de n -cellules dites redondantes et une classe de $(n + 1)$ -cellules dites « contractiles » d’une manière telle que l’élimination des cellules appariées préserve le type d’homotopie.

Structure des chapitres. La thèse se compose de quatre chapitres : Le chapitre 1 est une introduction, et les autres chapitres sont structurés comme suit.

Le chapitre 2 présente les résultats de [11], un travail réalisé conjointement avec Pierre-Louis Curien et Yves Guiraud. Par rapport à l’article, les préliminaires sont ici plus élaborés et, dans la mesure du possible, exprimés en termes de catégories plutôt que de monoïdes. De nouveaux détails sont ajoutés, par exemple l’exemple des monoïdes de tresses doubles. Le chapitre commence par fixer une terminologie issue du cadre des polygraphes; les transformations homotopiques sont l’objet d’une attention particulière. La notion clé de présentation cohérente est rappelée de [26] et illustrée par des exemples. La méthode principale, à savoir la procédure de complétion-réduction homotopique, est décrite sur les cas déjà connus des monoïdes d’Artin-Tits et des monoïdes de Garside. Une autre notion clé, celle de famille de Garside, est rappelée.

Enfin, le principal résultat de ce chapitre est donné. Nous montrons comment construire des présentations cohérentes de monoïdes simplifiables à gauche, admettant une famille de Garside noethérienne à droite ainsi que les multiples communs minimaux à droite, et ne contenant aucun élément inversible non trivial. Nous résolvons ainsi la question d’une généralisation commune aux deux extensions distinctes, données dans [26], du résultat original de Deligne pour les monoïdes Artin-Tits sphériques : aux monoïdes d’Artin-Tits généraux, et aux monoïdes de Garside. Nous appliquons nos résultats à certains monoïdes qui ne sont ni d’Artin-Tits ni de Garside. Un autre avantage de nos résultats est que nous pouvons prendre une famille de Garside finie pour ensemble de générateurs pour calculer une présentation cohérente des monoïdes d’Artin-Tits, alors qu’auparavant il fallait prendre le groupe de Coxeter correspondant, même s’il était infini.

Le chapitre 3 présente les résultats de [20], un travail supervisé par Viktoriya Ozornova. Par rapport à [20], nous expliquons ici les principales notions plus en détail et fournissons davantage d’exemples pour les illustrer. La notion de structure de factorabilité dans les monoïdes est rappelée, principalement à partir de [35]. La définition de la factorabilité locale est légèrement corrigée. Ensuite, des notions et des résultats sur les normalisations quadratiques dans les monoïdes sont rappelées de [16]. Après avoir posé les préliminaires, notre contribution est présentée.

Nous répondons à la question, laissée ouverte dans [13] et [16], de déterminer un lien entre les structures de factorisation et les normalisations quadratiques, malgré des origines et des motivations différentes pour ces deux notions. Plus concrètement, nous caractérisons les monoïdes factorisables dans le cadre axiomatique des normalisations quadratiques en tant que monoïdes admettant une normalisation quadratique satisfaisant une condition plus forte que la classe $(5, 4)$ mais plus faible

que la classe $(4, 4)$. Ici, la notion de classe est un paramètre évaluant la complexité de la normalisation de mots de longueur trois. Nous caractérisons également la classe $(4, 3)$ en termes de structures de factorabilité et d'une condition garantissant la terminaison du système de réécriture associé.

Le chapitre 4 commence par quelques préliminaires sur les monoïdes plaxiques en mettant l'accent sur leur présentation « colonne », introduite par Cain, Gray et Malheiro dans [7]. Nous donnons ensuite de nouveaux résultats, à savoir une présentation cohérente de monoïdes admettant une normalisation quadratique de classe $(4, 3)$, sans autres restrictions. Nous montrons également que la présentation cohérente construite se spécialise en la présentation cohérente construite par Hage et Malbos dans [33] pour les monoïdes plaxiques de type A à l'aide de la présentation colonne.

CHAPTER 1

Introduction

A monoid is one of the basic algebraic structures. It is a nonempty set equipped with an associative binary operation and a two-sided identity element. Monoids are more general than groups, which have an additional requirement that all elements are invertible. There is another way of looking at monoids, namely as categories having a single object, with elements of a monoid corresponding to arrows of a category.

For a set S , the free monoid generated by S is the set of all finite sequences (or words) of elements of S , with concatenation as the monoid operation, and the empty sequence as the identity element. Not every monoid is free as there can be some nontrivial relations among elements. However, every monoid is a homomorphic image (or a quotient) of a free monoid. This enables a presentation of a monoid by a generating set and a set of generating relations.

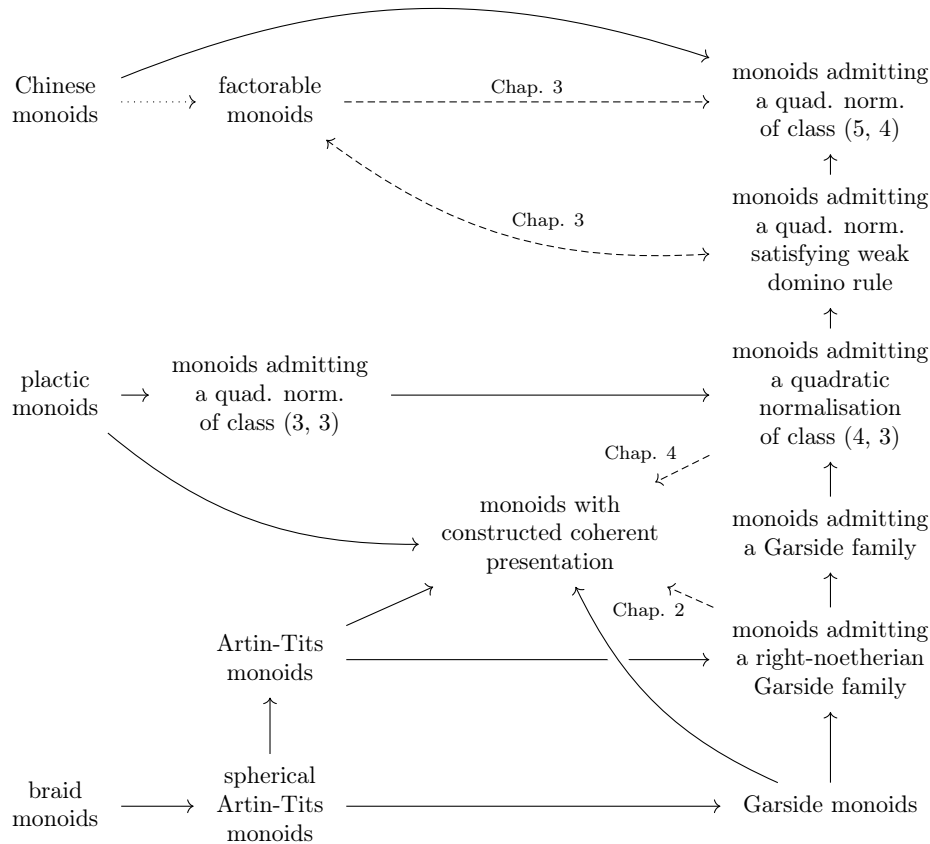
Climbing one dimension up, one observes that there can also be some relations among relations, and so on. A way to keep track of all this information is provided by the framework of polygraphs. Another benefit of using polygraphs is that they allow us to apply homotopical transformations to a presentation, in order to obtain a presentation of the same monoid but enjoying some preferable qualities.

One of the desirable attributes of a presentation is called convergence. It grants some very favourable computational properties, e.g. solvable word problem. A quality of a presentation, that we are particularly interested in, is a homotopical one called coherence.

The principal goal of this thesis is to use the framework of polygraphs for a fine study of the shapes of higher-dimensional cells (of dimension 3 and implicitly of dimension 4) arising from quadratic normalisations appearing in combinatorial group theory. The thesis proposes a construction of coherent presentations for a class of monoids admitting a right-noetherian Garside family. Thereby, it presents a unifying generalisation of two distinct extensions of Deligne's original construction of coherent presentations of spherical Artin-Tits monoids, given by Gaussent, Guiraud and Malbos: to general Artin-Tits monoids and to Garside monoids. In addition, a coherent presentation of monoids admitting quadratic normalisation of class $(4, 3)$ is constructed and specialised to the already known column coherent presentation of plactic monoids of type A , constructed by Hage and Malbos. Furthermore, correspondences are established between the notion of quadratic normalisation (developed in France, notably by Dehornoy) and the notion of factorability structure (developed by mathematicians in Germany led by Bödigheimer, with the goal of using suitable normal forms to understand group homology). Namely, factorable monoids are characterised in the axiomatic setting of quadratic normalisation. Additionally, quadratic normalisations of class $(4, 3)$ are characterised in

terms of factorability structures and a condition ensuring the termination of the associated rewriting system.

Here is an overview of relevant classes of monoids, with arrows denoting inclusions, the dashed ones being the present contribution with the dotted one being our hypothesis:



In this introductory chapter, we recall the main notions, but not necessarily in their full formality. Namely, technical definitions will not be given before chapters in which they are actually needed. Some new results are recollected, again not in a manner more technical than convenient.

The structure of the introduction is ‘homomorphic’ to the structure of the thesis. Namely, Section 1.1 provides a brief historical overview and a structural outline, whereas Sections 1.2, 1.3 and 1.4 respectively present Chapters 2, 3 and 4.

1.1. Thesis outline

1.1.1. Brief historical notes. We set up the stage by giving a very brief historical overview of the subjects investigated.

What is now called Garside theory has started with the work of Garside on braid monoids in the 1960s. Garside [25] investigated arithmetic properties of braid groups. He solved the word problem and the conjugacy problem for braid groups by introducing braid monoids. He showed that Artin’s presentation defines

the monoid of positive words in a braid group. Among other things, he proved that the braid monoid B_n^+ is left-cancellative and that each pair of elements of B_n^+ admits a least common multiple. He also introduced the Garside element (he called it the fundamental word) of a braid monoid.

Garside's observations for braid monoids were generalised to spherical Artin-Tits monoids by Brieskorn and Saito [5] and by Deligne [18].

The greedy normal form was first introduced for braid monoids in the 1980s, based on Garside's observations (see [13, Introduction] for references). The development continued in France, where Dehornoy and Paris [17] introduced Garside monoids in order to abstract properties which establish the existence of the greedy normal form. Then, Dehornoy, Digne and Michel [14] further generalised Garside monoids to categories admitting Garside families. For example, a Coxeter group constitutes a Garside family in the corresponding Artin-Tits monoid. Moreover, Dehornoy, Dyer and Hohlweg [15] showed that every Artin-Tits monoid admits a finite Garside family (even if the corresponding Coxeter group is infinite). A thorough development of the notion of Garside family can be found in the monograph [13].

Deligne [19] studied weak actions of Artin-Tits monoids on categories. He proved that to define a weak action of a spherical Artin-Tits monoid, it suffices to consider endofunctors for every element of the corresponding Coxeter group (i.e. it is not necessary to do it for every element of the monoid). This alternative definition is based on what is now called Garside's presentation of Artin-Tits monoids. Deligne explicitly gave Garside's presentation of spherical Artin-Tits monoids. Michel [46] extended this presentation to all Artin-Tits monoids. Gaussent, Guiraud and Malbos [26] proved that Deligne's result is equivalent to saying that spherical Artin-Tits monoids admit a particular coherent presentation, called Garside's coherent presentation. In this framework, they used the homotopical completion-reduction procedure, relying on the work of Squier [49] and Brown [6], to extend the result to all Artin-Tits monoids and also to Garside monoids.

The homotopical completion-reduction procedure was also successfully applied to plactic and Chinese monoids in [32], yielding coherent presentations. This suggested that Garside families did not provide, after all, an ultimate generalisation of the greedy normal form. The notion of quadratic normalisation, introduced by Dehornoy and Guiraud [16] (influenced by Krammer [37]), seems to succeed in this task, as it generalises, under the same axiomatic setting, two well-known classes of normalisations: those arising from quadratic rewriting systems, as studied in [26] for Artin-Tits monoids, and in [4] and [7] for plactic monoids; and those arising from Garside families, as investigated in [13]. We refer the reader to the survey [12] for an overview of the successive extensions of the greedy normal form from braid monoids to monoids admitting left-weighted quadratic normalisations of class $(4, 3)$.

Meanwhile in Germany, Bödigheimer and his collaborators developed the notion of factorability structure, introduced by Wang [51] and Heß [34] on monoids and categories as an extension of the definition of factorability structure on groups introduced by Bödigheimer [3] and Visy [50]. Their motivation was to abstract the structure, discovered in symmetric groups, which ensures the existence of a normal form allowing a reduction of the bar complex to a complex having considerably fewer cells, computing the homology of the structure in question. This reduction is

achieved using the theory of collapsing schemes, which was introduced in a topological flavour by Brown [6], elaborated for the algebraic setting by Cohen [10], and rediscovered (under the name of discrete Morse theory) by Forman [23]. The idea is to establish, for every nonnegative integer n , a matching between a class of n -cells called redundant and a class of $(n + 1)$ -cells called collapsible in such a way that collapsing matched pairs preserves the homotopy type.

1.1.2. Structure of chapters. The thesis consists of four chapters: the present Chapter 1 is introductory, and the other chapters are structured as follows.

Chapter 2 reports on the results of [11], a joint work with Pierre-Louis Curien and Yves Guiraud. Compared to the article, preliminaries here are elaborated and, where possible, expressed in terms of categories rather than monoids. New details are added, e.g. the example of dual braid monoids. The chapter starts by fixing some terminology from the framework of polygraphs, of which homotopical transformations are given a particular attention. The key notion of coherent presentation is recalled from [26] and illustrated by means of examples. The main method, namely, the homotopical completion-reduction procedure, is described and demonstrated using the known applications to Artin-Tits monoids and Garside monoids. Another key notion, that of Garside family, is recalled.

Finally, the principal result of the chapter is given. Namely, we show how to construct coherent presentations of left-cancellative monoids admitting a right-noetherian Garside family and minimal common right multiples, and containing no nontrivial invertible element. Thereby, we resolve the question of finding a common generalisation to the two distinct extensions, given in [26], of Deligne's original construction of coherent presentations of spherical Artin-Tits monoids: to general Artin-Tits monoids and to Garside monoids. The results are applied to some monoids that are neither Artin-Tits nor Garside. Another benefit of is that we can take a finite Garside family for a generating set in computing a coherent presentation of Artin-Tits monoids, whereas before one could only take the corresponding Coxeter group even if it was infinite.

Chapter 3 presents the results of [20], a work supervised by Viktoriya Ozornova. In comparison to [20], this thesis explains the main notions in greater detail and provides more examples to illustrate them. Definitions and results concerning factorability structures in monoids are recollected, mostly from [35]. Definition of local factorability is slightly corrected. Then, notions and results about quadratic normalisations for monoids are recalled from [16]. After the preliminaries are set, new results are presented.

We answer the question, left open in [13] and [16], of determining a connection between factorability structures and quadratic normalisations, despite different origins and motivations for these two notions. More concretely, factorable monoids are characterised in the axiomatic setting of quadratic normalisations as monoids admitting a quadratic normalisation satisfying a condition stronger than the class $(5, 4)$ yet weaker than the class $(4, 4)$. Here, the notion of class is a parameter evaluating the complexity of normalising length-three words. We also characterise the class $(4, 3)$ in terms of factorability structures and a condition ensuring the termination of the associated rewriting system, and we prove an equivalence between the two known such conditions.

Chapter 4 begins by giving some preliminaries on plactic monoids with a special emphasis on their column presentation, introduced by Cain, Gray and Malheiro in [7]. Then some new results are provided, namely, a coherent presentation of monoids admitting a quadratic normalisation of class $(4, 3)$, with no further restrictions. It is then shown that the constructed coherent presentation specialises to the already known column coherent presentation of plactic monoids of type A , constructed by Hage and Malbos in [33].

1.2. Coherent presentations and greedy normal form

Subsection 1.2.1 fixes basic terminology and notation to be used throughout the thesis. Subsections 1.2.2 and 1.2.4 recall coherent presentations and Garside families, respectively. In Subsection 1.2.3, notions concerning rewriting are recollected. Subsection 1.2.5 presents new results.

1.2.1. Basic terminology and notation. If S is a set, then S^* denotes the free monoid over S . Elements of S and S^* are called S -letters and S -words, respectively. So, an S -word is a finite sequence of S -letters, e.g. (s_1, \dots, s_n) . In the broader context of categories, words are generalised to paths of composable morphisms. The prefix S - is sometimes left out when the considered generating set is evident from the context.

The product in S^* of two words u and v is denoted by $u|v$. A letter is customarily identified with the single-letter word consisting of that letter. Accordingly, a word (s_1, \dots, s_n) can be written as the product (in S^*) of its letters: $s_1| \cdots |s_n$.

A monoid M is said to be generated by a set S , and elements of S are called generators of M , which is often written as (M, S) , if M is a homomorphic image of the free monoid S^* . Such a homomorphism is called an evaluation map and denoted by $\text{EV} : S^* \rightarrow M$. A **normal form** for M with respect to S is a set-theoretic section, denoted by NF , of the evaluation map. To rephrase it, a normal form maps elements of M to distinguished representative words in S^* . An S -word $s_1| \cdots |s_n$ is called a **decomposition** of an element g of M if the equality $s_1 \cdots s_n = g$ holds in M . An S -word u is said to be a **factor** of an S -word v if there exist S -words w and w' satisfying $wuw' = v$.

The length of u in S^* is denoted by $|u|$. For an element f of M , the minimal S -length of an S -word representing f is denoted by $|f|$. The same notation will be used for other length functions when they are introduced, but the proper interpretation will always be evident from the context (besides, all the length functions in this thesis concur with each other).

1.2.2. Coherent presentations of monoids. A monoid can be presented by a generating set and a set of generating relations between words over the generating set. A coherent presentation of a monoid consists of a generating set, a set of generating relations and, in addition, a set of generating relations among relations, by means of which each two parallel relations can be ‘transformed’ into each other.

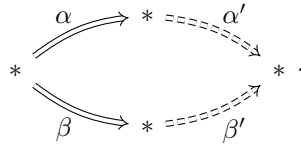
The notion of coherent presentation is closely related to other known notions: 2-syzygies for presentations of groups (see e.g. [43]); polygraphic resolutions of monoids, introduced by Métayer in [45], from which abelian resolutions can be deduced; cofibrant approximations in the canonical model structure on 2-categories, given by Lack in [39], [40]; weak actions of Artin-Tits monoids on categories, investigated by Deligne in [19]. Coherent presentations are also studied in [21],

under the name 3-presentations. Motivation and context for the notion of coherent presentation are briefly recalled in Subsection 2.1.5.

The construction of coherent presentations of spherical Artin-Tits monoids has been extended in [26] in two disjoint directions: to general Artin-Tits monoids and to Garside monoids. This has been done using methods from rewriting theory and Squier's work.

1.2.3. Rewriting methods. Generating relations, when considered directed from left to right (as ordered pairs), provide rewriting rules, which, together with the corresponding generating set, constitute a rewriting system. In the framework of higher rewriting, the notion of coherent presentation can be described topologically by saying that relations among relations fill all the spheres formed by two parallel rewriting paths.

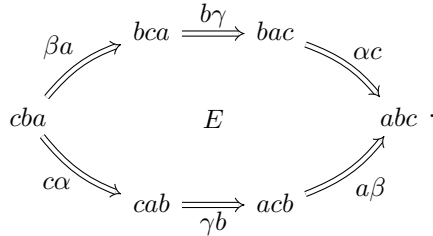
A presentation is called terminating if there is no infinite rewriting sequence. It is called confluent if each pair of distinct rewriting sequences starting from the same word can be completed in such a way that the sequences eventually lead to a common result:¹



A presentation is convergent if it is both terminating and confluent. The homotopical completion-reduction procedure enriches a terminating presentation to a coherent one. The main element is Squier's theorem (recalled here as Theorem 2.2.2.1), which allows one to simply compute generators of the relations among relations for a convergent presentation. Let us illustrate it by a simple example. Consider the following convergent presentation of the free abelian monoid \mathbb{N}^3 :

$$\langle a, b, c \mid ba \xrightarrow{\alpha} ab, cb \xrightarrow{\beta} bc, ca \xrightarrow{\gamma} ac \rangle.$$

Squier's theorem yields a generator of the relations among relations for each critical branching of the presentation, i.e. for each minimal overlap of the rewriting rules. In this example, there is exactly one such situation:

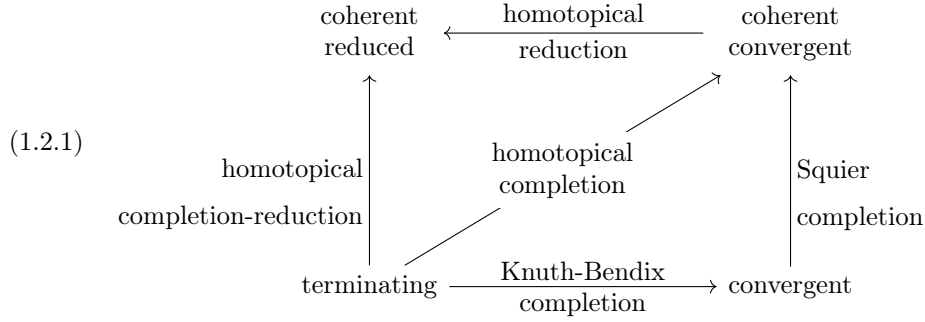


The generator E of the relations among relations is added.

The homotopical completion-reduction procedure has three stages. Firstly, the Knuth-Bendix completion procedure enriches a terminating presentation to a convergent one by adding a (not necessarily finite) number of generating relations in order to make every critical branching confluent. Secondly, the Squier completion

¹In a confluent presentation, one can safely follow the directions given by L. P. Berra, 'When you come to a fork in the road, take it.'

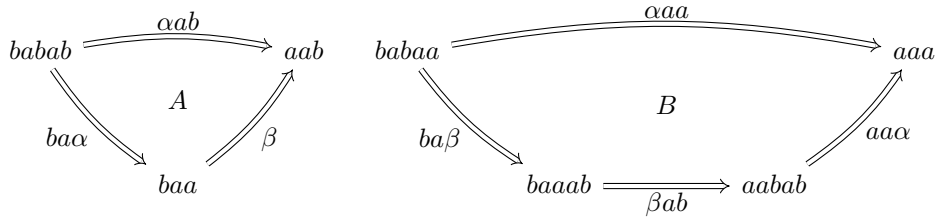
procedure adjoins generating relations among relations, thus providing a coherent presentation of the monoid admitting the original presentation. Thirdly, the homotopical reduction procedure removes extra relations (among relations). These homotopical transformations of presentations having certain properties are depicted in the following diagram (and recollected in Section 2.2):



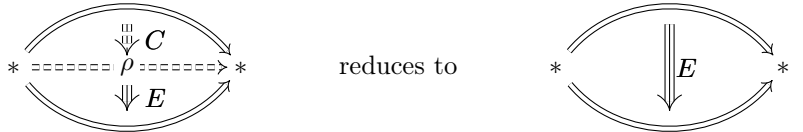
Let us illustrate the first two stages by giving a preview of Example 2.2.2.3. Consider the following presentation of the Klein bottle monoid:

$$\langle a, b \mid bab \xrightarrow{\alpha} a, baa \xrightarrow{\beta} aab \rangle.$$

There are exactly two critical branchings: $\{\alpha ab, ba\alpha\}$ and $\{\alpha aa, ba\beta\}$. Both branchings are confluent. The Squier completion procedure adjoins the generators A and B of the relations among relations. Here are the shapes of A and B :



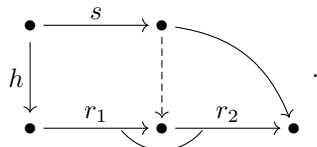
During the homotopical completion procedure, superfluous relations (among relations) may be adjoined. The homotopical reduction procedure provides a systematic way to eliminate them by recursively removing pairs (ρ, C) of redundant and collapsible generating cells, as illustrated by the following diagram (and explained in Subsection 2.2.4):



1.2.4. Garside families. A Garside family in a monoid is a generating family, not minimal in general, but ensuring the existence of the greedy normal form.

The greedy normal form is simple (the language of normal words is regular) and easily computed (there are simple incremental methods, thanks to the diagrammatic tool called the domino rule) as it has favourable locality properties (recalled in Subsection 1.3.2).

Assume that S is a subfamily S of a left-cancellative (meaning that $fg = fg'$ implies $g = g'$) monoid M having no nontrivial invertible element. A length-two S -word $r_1|r_2$ is said to be S -greedy if, for all s in S and h in M , commutativity of the following diagram without the dashed arrow implies the existence of a dashed arrow making the square (and thus also the triangle) commute:



A longer S -word is S -greedy if all of its length-two factors are S -greedy. A Garside family in M is a subfamily S such that, for every element of M , there exists an S -greedy word representing it. A precise definition is recalled in Section 2.4.

1.2.5. Attaining coherence. One of the main results presented in this thesis is a unification of the two above-mentioned results of [26] in the same generalisation. Namely, we apply the homotopical completion-reduction procedure to compute coherent presentations for a certain class of monoids admitting a Garside family. We take the following two main steps.

- (1) First, we use the fact that, for every left-cancellative monoid M with no nontrivial invertible element and every Garside family S in M , there is a presentation, here denoted by $\text{Gar}_2(S)$, having $S \setminus \{1\}$ as a generating set, with generating relations α of the form $s|t = st$, for s, t in $S \setminus \{1\}$ such that st lies in S (Proposition 2.4.4.4, adapted from [16]). We observe (Example 2.4.4.5) that Garside's presentation $\text{Gar}_2(W)$ of an Artin-Tits monoid $B^+(W)$ is a special case, with S being the Coxeter group W . Similarly (Example 2.4.4.6), Garside's presentation $\text{Gar}_2(M)$ of a Garside monoid M is another special case of $\text{Gar}_2(S)$.
- (2) Then, starting from $\text{Gar}_2(S)$, we embark on extending [26, Theorem 3.1.3] (which we recall in Example 2.3.3.2) to a wider class of monoids, including left-cancellative noetherian monoids having no nontrivial invertible element, admitting a Garside family. Working in a more general setting, we encounter additional critical branchings which cannot occur in the case of Artin-Tits or Garside monoids due to specific properties not shared by Garside families in general. Therefore, we construct new generating relations among relations. Conveniently, we then remove all the additional relations using the homotopical reduction procedure.

This results in Theorem 2.5.1.4, the main result of Chapter 2, of which we give here a weaker but simpler version (Corollary 2.5.5.1).

THEOREM. *Let M be a left-cancellative noetherian monoid containing no non-trivial invertible element. If S is a Garside family containing 1 in M , then M*

admits the coherent presentation $\text{Gar}_3(S)$ which extends $\text{Gar}_2(S)$ with the following set of generating relations among relations:

$$\begin{array}{ccccc}
 & & uv|w & & \\
 & \nearrow^{\alpha_{u,v}|w} & & \searrow_{\alpha_{uv,w}} & \\
 u|v|w & & A_{u,v,w} & & uvw \\
 & \searrow_{u|\alpha_{v,w}} & & \nearrow^{\alpha_{u,vw}} & \\
 & & u|vw & &
 \end{array}$$

for each triple (u, v, w) of elements of $S \setminus \{1\}$ such that uv , vw and uvw lie in S .

Note that $A_{u,v,w}$ can be read as a relation among relations ensuring associativity. We reach $\text{Gar}_3(S)$ by applying the homotopical completion-reduction procedure to the presentation $\text{Gar}_2(S)$.

In Section 2.6, the result is used to compute coherent presentations of some monoids which are neither Artin-Tits nor Garside and to construct a finite coherent presentation of the Artin-Tits monoid of type \tilde{A}_2 , taking a finite generating set. In some cases, homotopical reduction can be carried further. As a matter of fact, that is how we prove (in Subsection 2.6.3) that Artin's presentation of the Artin-Tits monoid of type \tilde{A}_2 is coherent (with the empty set of generating relations among relations).

We mainly consider monoids because that is where our present applications lie, but the approach presented here can be extended to categories, viewed as monoids with partial multiplication. To prepare the grounds for potential more general applications in future, we therefore prefer to state preliminaries in terms of categories.

1.3. Factorability and normalisation

The notions of factorability and normalisation are recalled in Subsections 1.3.1 and 1.3.2, respectively. Subsection 1.3.3 reports on new results.

1.3.1. The idea of factorability. For a monoid M and its generating set S , the idea of factorability is to determine a way to split off a generator from an element of the monoid. The goal is to ensure the existence of a normal form, obtained by recursively factorising, that would yield a favourable bijection in the bar complex, in order to collapse superfluous cells (as recalled in Subsection 1.1.1).

Before we go any further, let us recall the seminal example given by the symmetric group S_n and its generating set consisting of all transpositions, proved to be factorable in [50]. Let σ be a nontrivial permutation in S_n and j the largest value in $\{1, \dots, n\}$ that is not fixed by σ . Then the generator $(j, \sigma^{-1}(j))$ is split off. In the remainder of σ , the value j is fixed, which guarantees termination of the process. We illustrate this using a concrete permutation (note that we multiply permutations from left to right):

$$\begin{aligned}
 & \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 5 & 1 & 3 & 6 \end{pmatrix} = \\
 & \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix} =
 \end{aligned}$$

$$\begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix} = \\ \begin{pmatrix} 3 & 5 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}.$$

A desired factorisation is achieved by the notion of factorability structure, which consists of a factorisation map $\eta = (\eta', \bar{\eta}) : M \rightarrow M^2$ subject to several axioms ensuring, in particular, a compatibility with the multiplication $\mu : M^2 \rightarrow M$ in the monoid. In a manner of speaking, η acts on an element f of M by splitting off, in a suitable way, a generator $\eta'(f)$.

For every factorability structure, there is an associated rewriting system having rules of the form $(r, s) \rightarrow \eta \circ \mu(r, s)$. This rewriting system is confluent but it is not necessarily terminating. However, termination is obtained under the additional assumption that, for all s in S and f in M , the following equalities hold:

$$(1.3.1) \quad \eta'(sf) = \eta'(s \cdot \eta'(f)), \quad \bar{\eta}(sf) = \bar{\eta}(s \cdot \eta'(f)) \cdot \bar{\eta}(f).$$

1.3.2. The idea of quadratic normalisations. As hinted in Subsection 1.1.1, quadratic normalisations in monoids generalise, in the same axiomatic framework, two well-known classes of normalisations: those arising from quadratic rewriting systems (notably, those associated with plactic monoids); and those arising from Garside families, resulting from successive generalisations of the greedy normal form in braid monoids. So, quadratic normalisations provide an extension past Garside families of the greedy normal form. One of the advantages of this step further is that certain monoids that are not even left-cancellative, e.g. plactic monoids, become eligible.

Assume that a monoid is generated by a set S . By a normalisation, we mean a syntactic transformation of an arbitrary word over S to a ‘canonical’ one, called normal. Quadratic normalisations admit the following locality properties: a word is normal if, and only if, its length-two factors are normal; and the procedure of transforming a word into a normal form consists of a finite sequence of rewriting steps, each of which transforms a length-two factor.

The notion of class of a quadratic normalisation is defined in order to measure the complexity of normalising length-three words. The class (m, n) means that every length-three word admits at most m rewriting steps starting from the left and at most n rewriting steps starting from the right.

The class $(4, 3)$ is explored in great detail in [16] as it exhibits exceptionally favourable computational properties, mainly thanks to the following diagrammatic characterisation called the domino rule. If the solid arcs connect normal length-two words in the diagram

$$(1.3.2) \quad \begin{array}{c} \bullet \text{---} \xrightarrow{\quad} \bullet \text{---} \xrightarrow{\quad} \bullet \\ \downarrow \qquad \downarrow \qquad \downarrow \\ \bullet \text{---} \xrightarrow{\quad} \bullet \text{---} \xrightarrow{\quad} \bullet \end{array}$$

of elements of S , then so does the dashed arc. In particular, the rewriting system associated with a quadratic normalisation of class $(4, 3)$ is always terminating.

1.3.3. A correspondence between factorability and normalisation. The main result here answers the question, mentioned in [13] and [16], of determining the relation between the two approaches to combinatorial properties of a certain class of monoids: factorability and normalisation.

A correspondence is established between factorability structures and quadratic normalisations for monoids, despite the different origins and motivations for these two notions. By a correspondence, we mean maps in both directions between appropriate subclasses, which are inverse to each other up to technicalities, and moreover this bijection is compatible with the associated rewriting systems.

Since the rewriting system associated with a factorability structure is not necessarily terminating, whereas the rewriting system associated with a quadratic normalisation of class $(4, 3)$ is always terminating, it was known that a quadratic normalisation corresponding to a factorability structure is not necessarily of class $(4, 3)$. It is shown here that a quadratic normalisation corresponding to a factorability structure is always of class $(5, 4)$ and not smaller in general. A necessary and sufficient condition is given for a quadratic normalisation of class $(5, 4)$ to correspond to a factorability structure.

THEOREM. A monoid is factorable if, and only if, it admits a quadratic normalisation such that the domino rule holds for a diagram (1.3.2) whenever 1 does not occur in the upper or the right side (of the displayed rectangle).

Thereby, factorable monoids are characterised in terms of quadratic normalisations, thus adding another important family of monoids to those unified under the axiomatic framework of quadratic normalisations.

In particular, a quadratic normalisation of class $(4, 3)$ always yields a factorability structure, but not *vice versa*. However, the converse does hold under the stronger condition described as follows. Consider the map $F := \eta \circ \mu$ from the set M^2 to itself, with $\mu : M^2 \rightarrow M$ denoting the multiplication in M . Denote by F_1 (resp. F_2) the application of F to the first two elements (resp. the second and the third element) of a triple in M^3 . In general, a factorability structure ensures the equality

$$F_1 F_2 F_1 F_2(r, s, t) = F_2 F_1 F_2(r, s, t)$$

for each triple (r, s, t) in S^3 such that $F_2 F_1 F_2(r, s, t)$ contains no 1. The stronger condition states that this equality holds for every (r, s, t) in S^3 . Since this condition is implied by the class $(4, 3)$, quadratic normalisations of class $(4, 3)$ are characterised in terms of factorability structures.

Furthermore, it is shown that this stronger condition is equivalent to the aforementioned additional assumption (1.3.1) which is known to grant termination of the rewriting system associated with a factorability structure. Simply put, the class $(4, 3)$ equals factorability plus termination.

One of the benefits of the established correspondence between factorability structures and quadratic normalisations is that it provides a way of importing the results concerning homology, derived from the former (see e.g. [50], [51], [35]) to the framework of the latter, with the hope of generalising those results to higher classes. This is one of the potential further steps in the present direction of research.

1.4. Coherent presentations from class $(4, 3)$

Since a quadratic normalisation of class $(4, 3)$ always yields a convergent rewriting system, a Squier completion of the corresponding presentation can be computed. Conveniently, several important families of monoids do admit such a normalisation: monoids admitting a Garside family and plactic monoids, to name a few. Therefore, obtaining results about monoids admitting quadratic normalisation of class $(4, 3)$ provides *a fortiori* results about those specific classes of monoids. This is precisely what is done in Chapter 4.

Namely, we construct a coherent presentation of a general monoid admitting a quadratic normalisation of class $(4, 3)$. Then we show that this construction specialises to the column coherent presentation of plactic monoids of type A , constructed by Hage and Malbos [33].

Acknowledgements

I am very grateful to everyone who has helped me in this quest.

Especially, I would like to thank Pierre-Louis Curien and Yves Guiraud, who have been wonderful supervisors, for embarking upon this journey with me and for providing just the right amount of guidance along the way so I am free to find my own path and yet free from wandering off. I am also thankful for their enduring encouragement and enormous patience, despite vast external difficulties.

I warmly thank Viktoriya Ozornova for supervising a part of my thesis, for her kindness and support.

I am also grateful to Barbara Baumeister, António Malheiro, Ivan Marin, Victoria Lebed and Matthieu Picantin for agreeing to take part in my thesis defence committee, for their useful questions and remarks.

I also thank Nohra Hage for being kindly responsive to my questions about the column coherent presentations of plactic monoids.

I gratefully acknowledge the FSMP and especially Ariela Briani, as well as Inria and especially Anne Mathurin, for the practical support. Many thanks to IRIF and especially fellow doctoral students for making a fruitful working environment and a friendly community.

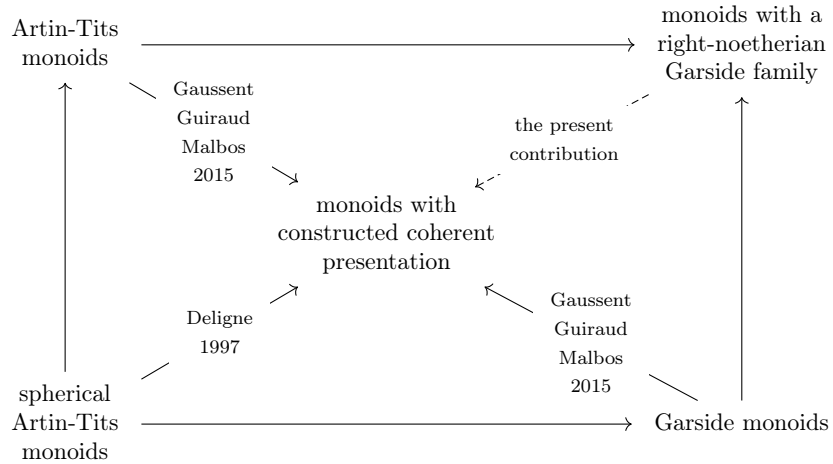
The document was prepared in $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$ (based on $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$). The diagrams were made using the package `tikz-cd` (with the aid of `quiver`, a graphical editor, for diagrams on page 95).

This programme has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 754362.

CHAPTER 2

Coherent presentations arising from Garside families

This chapter aims to motivate, present and apply a construction of coherent presentations for a class of monoids admitting a Garside family, after recalling preliminaries. The following diagram displays the context, with arrows denoting inclusions:



Section 2.1 recalls the framework of polygraphs, in which the notion of coherent presentation of a category is defined. In Section 2.2, the main method, namely the homotopical completion-reduction procedure, is recalled. Section 2.3 recollects some known applications of this method, which will later turn out to be special cases of our main result. Section 2.4 briefly recollects relevant facts about Garside families. Section 2.5, reports on new results. Finally, some applications are presented in Section 2.6.

2.1. Presentations of categories by polygraphs

In this section, we briefly recall the notions concerning polygraphic presentations of monoids (of which a technical elaboration can be found in [26]). Basic terminology is given in Subsection 2.1.1. Some basic notions of polygraphic rewriting theory are recollects in Subsection 2.1.2. The notion of coherent presentation of a monoid is recalled in Subsection 2.1.3 and technically elaborated in Subsection 2.1.4.

Throughout the present article, 2-categories and 3-categories are always assumed to be strict (see e.g. [31, Section 2]).

2.1.1. Presentations by 2-polygraphs. The framework of polygraphs encompasses (in the same globular object) words, rewriting rules, and homotopical properties of a rewriting system. Polygraphs provide a generalisation of a presentation of a monoid by generators and relations to the higher categories that are free up to codimension 1.

A polygraph is a higher-dimensional generalisation of a graph. Recall that a (directed) graph is a pair (X_0, X_1) of sets, together with two maps, called the source and target, from X_1 to X_0 . A 0-polygraph (X_0) is a set. A **1-polygraph** (X_0, X_1) is a graph. The free category generated by a 1-polygraph (X_0, X_1) , denoted by X_1^* , contains: X_0 as the set of objects, and paths (freely concatenated elements of X_1) between objects as morphisms. A **2-polygraph** is a triple $X = (X_0, X_1, X_2)$, where (X_0, X_1) is a 1-polygraph and X_2 is a set of **1-spheres**, i.e. pairs of parallel paths, in X_1^* . Elements of X_k are called **generating k -cells**. In diagrams, distinct arrows are used to denote generating k -cells for low k : \rightarrow , \Rightarrow , \Rrightarrow for k equal to 1, 2 and 3, respectively.

For a 2-polygraph X , the **category presented by X** , denoted by $\overline{X} = X_1^*/X_2$, is obtained by factoring out generating 2-cells, regarded as relations among 1-cells of X_1^* . For a category \mathcal{C} , a **presentation of \mathcal{C}** is a 2-polygraph X such that \mathcal{C} is isomorphic to \overline{X} . In the case where category is a monoid, which we are mainly interested in, X_0 is a singleton so each pair of paths in X_1^* forms a 1-sphere, and X_1^* is the free monoid generated by the set X_1 .

EXAMPLE 2.1.1.1 (The standard presentation). Let \mathcal{C} be a category. The **standard presentation** of \mathcal{C} is the 2-polygraph $\text{Std}_2(\mathcal{C})$ consisting of:

- one generating 0-cell x for every 0-cell x of \mathcal{C} ;
- one generating 1-cell $\widehat{u} : x \rightarrow y$ for every 1-cell $u : x \rightarrow y$ of \mathcal{C} ;
- one generating 2-cell $\gamma_{u,v} : \widehat{u}\widehat{v} \Rrightarrow \widehat{uv}$ for every pair of composable 1-cells u and v of \mathcal{C} ;
- one generating 2-cell $\iota_x : 1_x \Rrightarrow \widehat{1_x}$ for every 0-cell x of \mathcal{C} . Here, 1_x denotes the empty path (consisting of zero generating 1-cells) from x to x . By contrast, $\widehat{1_x}$ is the path of length 1, consisting of the generating 1-cell (of $\text{Std}_2(\mathcal{C})$) associated with the identity of x in \mathcal{C} .

2.1.2. Rewriting properties of 2-polygraphs. Let X be a 2-polygraph. Generating 2-cells of X are called **rewriting rules**. The **free 2-category over X** , denoted by $X_2^* = X_1^*[X_2]$, is obtained by adjoining to X_1^* all the formal compositions of elements of X_2 , treated as formal 2-cells. Standard notions from rewriting theory naturally translate into the framework of polygraphs. A **rewriting step** of a 2-polygraph X is a 2-cell of the free category X_2^* that contains a single generating 2-cell of X , here considered as a transformation of its source into its target. So, a rewriting step has a shape

$$\bullet \xrightarrow{w} \bullet \begin{array}{c} \xrightarrow{u} \\ \Downarrow \alpha \\ \xrightarrow{v} \end{array} \bullet \xrightarrow{w'} \bullet$$

where $\alpha : u \Rrightarrow v$ is a generating 2-cell of X , and w and w' are 1-cells of X_2^* , and every 0-cell involved is denoted by \bullet .

Let u and v be 1-cells of X_2^* . We say that u **rewrites** to v if there is a finite composable sequence of rewriting steps, such that the source of the first step of the

sequence is u and the target of the last step is v . A 1-cell v is **reduced** if there is no rewriting step whose source is v .

A **termination order** on X is a well-founded order relation \leq on parallel 1-cells of X_2^* having the following properties:

- the compositions by 1-cells of X_2^* are strictly monotone in both arguments, i.e. \leq is compatible with the composition of 1-cells;
- for every generating 2-cell α of X , the strict inequality $t(\alpha) < s(\alpha)$ holds.

A 2-polygraph X is **terminating** if it has no infinite sequence of rewriting steps. Admitting a termination order is equivalent to being terminating (in a terminating polygraph, a termination order is generated by putting $u > v$ for each pair of 1-cells u and v such that u rewrites to v).

A **branching** of a 2-polygraph X is a two-element multiset $\{\alpha, \beta\}$ of sequences of rewriting steps of X_2^* having the same source, called the source of the branching. If α and β are rewriting steps, then a branching $\{\alpha, \beta\}$ is called **local**. A local branching is **trivial** if it has one of the following two shapes: $\{\alpha, \alpha\}$, or $\{\alpha v, u\beta\}$ for $u = s(\alpha)$ and $v = s(\beta)$. Local branchings can be compared by the order \preceq generated by the relations $\{\alpha, \beta\} \preceq \{w\alpha w', w\beta w'\}$ given for every local branching $\{\alpha, \beta\}$ and all possible 1-cells w and w' of X_2^* . A minimal nontrivial local branching is called **critical**. A branching $\{\alpha, \beta\}$ is confluent if α and β can be expanded into sequences having the same target. A 2-polygraph X is **confluent** (resp. locally confluent, resp. critically confluent) if all its branchings (resp. local branchings, resp. critical branchings) are confluent. If a 2-polygraph is terminating and confluent, then it is said to be **convergent**. A convergent 2-polygraph X is called a **convergent presentation** of any category isomorphic to \overline{X} . In that case, for every 1-cell u of X^* , there is a unique reduced word, denoted by \hat{u} , to which u rewrites.

The following theorem states that two basic results of rewriting theory concerning confluence, called Newman's lemma [47, Theorem 3] and the critical branchings theorem respectively, are also valid for polygraphs.

THEOREM 2.1.2.1 ([31, Theorem 3.1.6]).

- (1) *If a 2-polygraph is terminating, then it is confluent if, and only if, it is locally confluent.*
- (2) *A 2-polygraph is locally confluent if, and only if, it is critically confluent.*

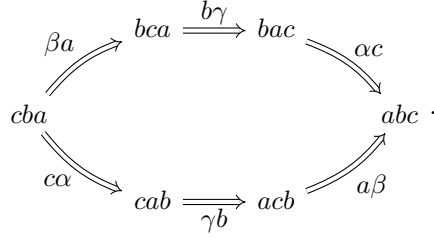
As a consequence of Theorem 2.1.2.1, a 2-polygraph is convergent if, and only if, it is terminating and its critical branchings are confluent.

EXAMPLE 2.1.2.2. Consider the free abelian monoid:

$$(2.1.1) \quad \mathbb{N}^3 = \langle a, b, c \mid ba \xrightarrow{\alpha} ab, cb \xrightarrow{\beta} bc, ca \xrightarrow{\gamma} ac \rangle.$$

This presentation admits the following termination order: comparing the lengths of words, then (i.e. if words are equal in length) applying, from left to right, the lexicographic order induced by $a < b < c$. Hence, it is terminating.

Let us illustrate confluence of the presentation (2.1.1) on the unique critical branching $\{\beta a, c\alpha\}$:



Thus, the presentation (2.1.1) is convergent, by Theorem 2.1.2.1.

2.1.3. Coherent presentations. A 2-category (resp. 3-category) is called a $(2, 1)$ -category (resp. a $(3, 1)$ -category) if all its 2-cells (resp. 2-cells and 3-cells) are invertible. For a 2-polygraph X , the **free $(2, 1)$ -category over X** , denoted by $X_2^\top = X_1^*(X_2)$, is constructed by adjoining to X_1^* all the formal compositions of elements of X_2 and formal inverses of elements of X_2 , and then factoring out the compositions of elements with their corresponding inverses. A **$(3, 1)$ -polygraph** is a quadruple $X = (X_0, X_1, X_2, X_3)$, where (X_0, X_1, X_2) is a 2-polygraph and X_3 is a set of **2-spheres**, i.e. pairs of parallel paths of 2-cells, in X_2^\top . For a $(3, 1)$ -polygraph X , the **free $(3, 1)$ -category over X** , denoted by $X_3^\top = X_2^\top(X_3)$, is constructed by adjoining to X_2^\top all the formal compositions of elements of X_3 and formal inverses of elements of X_3 , and then factoring out the compositions of elements with their corresponding inverses. A $(3, 1)$ -polygraph is called convergent if its underlying 2-polygraph is convergent. The **category presented by a $(3, 1)$ -polygraph X** is again \overline{X} , the category presented by its underlying 2-polygraph.

An **extended presentation of a monoid M** is a $(3, 1)$ -polygraph X such that M is isomorphic to \overline{X} .

DEFINITION 2.1.3.1. A **coherent presentation of a monoid M** is an extended presentation (X_0, X_1, X_2, X_3) of M such that factoring out elements of X_3 leaves only trivial 2-spheres (where the parallel paths are equal).

For example, presentations of the free abelian monoid \mathbb{N}^3 and the Klein bottle monoid in Subsection 1.2.3, obtained by Squier completion, are coherent. For simplicity, the single generating 0-cell is sometimes left out of a presentation of a monoid.

2.1.4. Cellular extensions of higher categories. An n -category is called an (n, p) -category if its k -cells are invertible for all $k > p$. For a k -cell f of an n -category \mathcal{C} , the identity $(k + 1)$ -cell and the i -source and i -target of f are respectively denoted by 1_f and $s_i(f)$ and $t_i(f)$; the subscript i is dropped if $i = k - 1$. If $t_i(f) = s_i(g)$, that is if f and g are i -composable k -cells, then their i -composite is denoted by $f \star_i g$; abbreviated to fg if $i = 0$. When 1_f (resp. the identity of 1_f , etc.) is composed with cells of dimension $k + 1$ (resp. $k + 2$, etc.), it is denoted by f , for simplicity.

For $k \geq 1$, two k -cells are called **parallel** if they have a common source and a common target. Any two 0-cells are considered parallel. An (ordered) pair of parallel k -cells in an n -category is called a **k -sphere**. The source and target of a k -sphere $\gamma = (f, g)$ are k -cells f and g , respectively, which is written as $s(\gamma) = f$

and $t(\gamma) = g$. For a k -cell γ of \mathcal{C} , the $(k-1)$ -sphere $(s(\gamma), t(\gamma))$ is called the **boundary** of γ .

For a set Γ and an n -category \mathcal{C} , a map from Γ to the set of n -spheres of \mathcal{C} is called a **cellular extension** of \mathcal{C} . Usually, we identify elements of Γ with their images and say that Γ is a cellular extension of \mathcal{C} . A cellular extension of \mathcal{C} can be interpreted as a set of directed relations between parallel n -cells. Identifying the n -cells $s(\gamma)$ and $t(\gamma)$ for every n -sphere γ of Γ , produces the quotient n -category \mathcal{C}/Γ . A cellular extension of \mathcal{C} can also be interpreted as a set of formal $(n+1)$ -cells, filling the n -spheres of \mathcal{C} . For an n -category \mathcal{C} and a cellular extension Γ , one constructs the **free $(n+1)$ -category generated by Γ over \mathcal{C}** , denoted by $\mathcal{C}[\Gamma]$, by adjoining to \mathcal{C} all the formal compositions of elements of Γ , treated as formal $(n+1)$ -cells. Similarly, for an $(n,1)$ -category \mathcal{C} and its cellular extension Γ , one constructs the **free $(n+1,1)$ -category generated by Γ over \mathcal{C}** , denoted by $\mathcal{C}(\Gamma)$, as the quotient $\mathcal{C}(\Gamma) = \mathcal{C}[\Gamma, \check{\Gamma}] / \text{Inv}(\Gamma)$, where $\check{\Gamma}$ denotes the cellular extension obtained by reversing directions (i.e. switching components) of spheres in Γ , and $\text{Inv}(\Gamma)$ denotes the cellular extension of $\mathcal{C}[\Gamma, \check{\Gamma}]$ consisting of $(n+2)$ -cells $\check{\chi} \star_n \chi \rightarrow 1_{t(\chi)}$ and $\chi \star_n \check{\chi} \rightarrow 1_{s(\chi)}$ for every $(n+1)$ -cell χ in Γ .

That is why, in Section 2.1.3, the free 2-category over a 2-polygraph X and the free $(2,1)$ -category over X are denoted respectively by $X_2^* = X_1^*[X_2]$ and $X_2^\top = X_1^\top(X_2)$. Technically, that is how they are, in fact, defined.

In formal terms, a $(3,1)$ -polygraph is a quadruple $X = (X_0, X_1, X_2, X_3)$, where (X_0, X_1, X_2) is a 2-polygraph and X_3 is a cellular extension of X_2^\top . Hence, the free $(3,1)$ -category X_3^\top over $(3,1)$ -polygraph X and the $(2,1)$ -category presented by X are denoted (technically, defined) respectively by $X_3^\top = X_2^\top(X_3)$ and $\tilde{X} = X_2^\top/X_3$. The category presented by a $(3,1)$ -polygraph X is again \tilde{X} , the category presented by its underlying 2-polygraph.

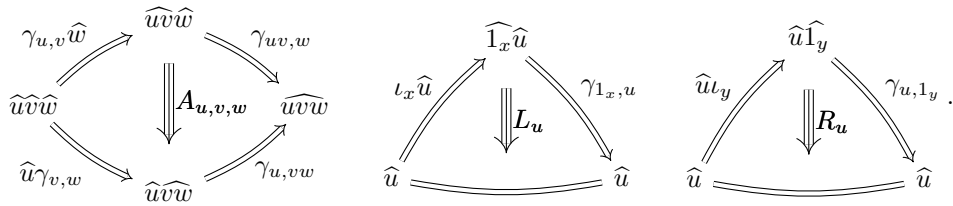
An **extended presentation** of \mathcal{C} is a $(3,1)$ -polygraph X such that \mathcal{C} is isomorphic to \tilde{X} .

An **acyclic cellular extension** of an n -category \mathcal{C} is a cellular extension Γ of \mathcal{C} such that all the n -spheres of the quotient n -category \mathcal{C}/Γ are of the form (f, f) or, equivalently, such that, for every n -sphere γ of \mathcal{C} , there is an $(n+1)$ -cell whose boundary is γ in the $(n+1)$ -category $\mathcal{C}(\Gamma)$.

DEFINITION 2.1.4.1. A **coherent presentation of a category \mathcal{C}** is an extended presentation (X_0, X_1, X_2, X_3) of \mathcal{C} such that X_3 is an acyclic cellular extension of X_2^\top .

Note that this definition agrees with the one for monoids (given in the previous subsection), seen as categories having a single object.

EXAMPLE 2.1.4.2 (The standard coherent presentation). Recalling the standard presentation in Example 2.1.1.1, let us extend it with the generating 3-cells



The generating 3-cell $A_{u,v,w}$ is adjoined for each triple (u, v, w) of 1-cells of \mathcal{C} such that the pairs (u, v) and (v, w) are composable; the generating 3-cell L_u and R_u are adjoined for each 1-cell u of \mathcal{C} . The resulting $(3, 1)$ -polygraph, denoted by $\text{Std}_3(\mathcal{C})$, is called the **standard coherent presentation** of \mathcal{C} (see [29, Subsection 3.3.3] for the explanation why $\text{Std}_3(\mathcal{C})$ is, indeed, a coherent presentation).

2.1.5. Motivation for coherent presentations. Coherent presentations generalise 2-szygies for presentations of groups. They also form the first dimensions of polygraphic resolutions of monoids. The notion of coherent presentation is also closely connected with other two known notions: weak actions of Artin-Tits monoids on categories, and cofibrant approximations of 2-categories. In what follows, we are going to briefly recall the last.

Assume that \mathcal{C} and \mathcal{D} are 2-categories. A 2-functor $G : \mathcal{D} \rightarrow \mathcal{C}$ is called a weak equivalence if there exists a pseudofunctor (i.e. a 2-functor preserving composition and identities of 1-cells only up to 2-isomorphism; see [26, 5.1.1]) $F : \mathcal{C} \rightarrow \mathcal{D}$ such that GF is isomorphic to $1_{\mathcal{C}}$ and FG is isomorphic to $1_{\mathcal{D}}$. A 2-category \mathcal{D} is called cofibrant if its underlying 1-category is free. A 2-category \mathcal{D} is called a cofibrant approximation of a category \mathcal{C} if \mathcal{D} is cofibrant and if there exists a weak equivalence $\mathcal{D} \rightarrow \mathcal{C}$. The next theorem displays the close relation between coherent presentations of categories and their cofibrant approximations. Recall that \tilde{X} denotes the $(2, 1)$ -category presented by X .

THEOREM ([26, Theorem 1.3.1]). *Let \mathcal{C} be a category, and let X be an extended presentation of \mathcal{C} . Then X is a coherent presentation of \mathcal{C} if, and only if, \tilde{X} is a cofibrant approximation of \mathcal{C} (viewed as a 2-category).*

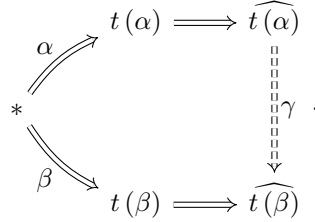
A strict action of a monoid M on a category \mathcal{C} is a morphism of monoids $F : M \rightarrow \text{End}(\mathcal{C})$. A weak action of M on \mathcal{C} replaces the morphism equalities $F(fg) = F(f)F(g)$ and $F(1_M) = 1_{\mathcal{C}}$ with natural isomorphisms satisfying coherence relations. Pseudofunctors generalise weak actions of monoids on categories. A consequence of the next theorem is that the Deligne's result [19, Theorem 1.5] is equivalent to saying that [26, Theorem 3.1.3] holds for Artin-Tits monoids of spherical type.

THEOREM ([26, Theorem 5.1.6]). *Let \mathcal{C} be a category, and let X be an extended presentation of \mathcal{C} . Then X is a coherent presentation of \mathcal{C} if, and only if, for every 2-category \mathcal{D} , the category of pseudofunctors from \mathcal{C} to \mathcal{D} and the category of 2-functors from \tilde{X} to \mathcal{D} are equivalent, and this equivalence is natural in \mathcal{D} .*

2.2. Homotopical transformations of polygraphs

This section elaborates the diagram (1.2.1), by recalling the notion of homotopical completion-reduction, introduced in [26]. Subsection 2.2.1 recollects the Knuth-Bendix completion procedure, which transforms a terminating 2-polygraph into a convergent one. Subsection 2.2.2 describes the Squier completion procedure, which upgrades a convergent 2-polygraph to a convergent coherent $(3, 1)$ -polygraph. Subsection 2.2.3 recollects the basic transformations one can perform on a polygraph without changing the presented category. In Subsection 2.2.4, we recall the homotopical reduction procedure, which turns a coherent $(3, 1)$ -polygraph into a coherent one having fewer generating cells. Finally, Subsection 2.2.5 describes a particular method for obtaining a homotopical reduction in the case when the starting coherent $(3, 1)$ -polygraph is also convergent.

2.2.1. Knuth-Bendix completion. Starting with a terminating 2-polygraph X equipped with a total termination order \leq , the Knuth-Bendix completion procedure adjoins generating 2-cells aiming to produce a convergent 2-polygraph that presents the category presented by X . It works by iteratively examining all the critical branchings and adjoining a new generating 2-cell whenever the branching is not already confluent. Namely, for a critical branching $\{\alpha, \beta\}$, if $t(\widehat{\alpha}) > t(\widehat{\beta})$ (resp. $t(\widehat{\beta}) > t(\widehat{\alpha})$), then a generating 2-cell $\gamma : t(\widehat{\alpha}) \Rightarrow t(\widehat{\beta})$ (resp. $\gamma : t(\widehat{\beta}) \Rightarrow t(\widehat{\alpha})$) is adjoined, thus forcing the confluence of the branching:

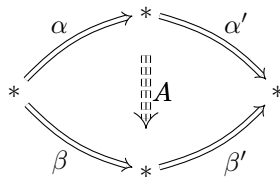


If new critical branchings are created by adjoining additional generating 2-cells, then confluence of such critical branchings is also examined. For details, see [31, 3.2.1]. This procedure is not guaranteed to terminate. In fact, its termination depends on the chosen termination order (see [22, Example 6.3.1]). If it does terminate, then the result is a convergent 2-polygraph. Otherwise, it produces an increasing sequence of 2-polygraphs, and the result is the union of all members of this sequence. Either way, the result is called a **Knuth-Bendix completion** of X . Note that different orders of examining critical branchings may result in different 2-polygraphs.

THEOREM 2.2.1.1 ([31, Theorem 3.2.2]). *If X is a 2-polygraph equipped with a total termination order, then every Knuth-Bendix completion of X is a convergent presentation of the category \bar{X} .*

REMARK 2.2.1.2. The Knuth-Bendix completion procedure, as described above, requires not only termination, but also a specification a total termination order, to be able to orient the generating 2-cells which are added and to be able to maintain the termination during the completion. There is an alternative approach. Namely, we can orient the newly added generating 2-cells ‘manually’, according to our inspiration, and verify after each addition in an *ad hoc* manner whether we maintain a terminating presentation, without having defined a total order beforehand. Therefore, we can invoke Theorem 2.2.1.1 even if we do not provide a total termination order in advance, as long as we are able to ensure termination after each addition of a generating 2-cell (we shall do this in the proof of Proposition 2.5.3.1).

2.2.2. Squier completion. A family of generating confluences of a convergent 2-polygraph X is a cellular extension of X_2^\top containing, for every critical branching $\{\alpha, \beta\}$ of X , exactly one 3-cell A :



where the rewriting sequences α' and β' are completing α and β , respectively, in such a way that they eventually reach the same target (such α' and β' exist by the assumption of confluence), with all the 1-cells involved denoted by $*$.

A **Squier completion** of a convergent 2-polygraph X is a $(3,1)$ -polygraph, with X as underlying 2-polygraph, whose generating 3-cells form a family of generating confluences of X . The following result is due to Squier ([49, Theorem 5.2]); we state a version in terms of polygraphs and higher-dimensional categories proved in [31].

THEOREM 2.2.2.1 ([31, Theorem 4.3.2]). *If X is a convergent 2-polygraph, then every Squier completion of X is an acyclic cellular extension of X_2^\top .*

In particular, if X is a convergent presentation of a monoid M , then every Squier completion of X is a convergent coherent presentation of M . Theorem 2.2.2.1 is extended to higher-dimensional polygraphs in [30, Proposition 4.3.4].

Let X be a terminating 2-polygraph equipped with a total termination order \leq . A homotopical completion of X is a Squier completion of a Knuth-Bendix completion of X . We have seen that the Knuth-Bendix completion procedure enriches a terminating 2-polygraph to a convergent one and that a Squier completion of a convergent 2-polygraph X is a coherent presentation of \overline{X} . These two transformations can be performed consecutively. They can also be performed simultaneously (see [26, 2.2.4]). Either way, the result is called a **homotopical completion** of X . Theorem 2.2.2.1 has the following consequence.

THEOREM 2.2.2.2. *Let \mathcal{C} be a category, and let X be a 2-polygraph. If X is a terminating presentation of \mathcal{C} , then every homotopical completion of X is a coherent convergent presentation of \mathcal{C} .*

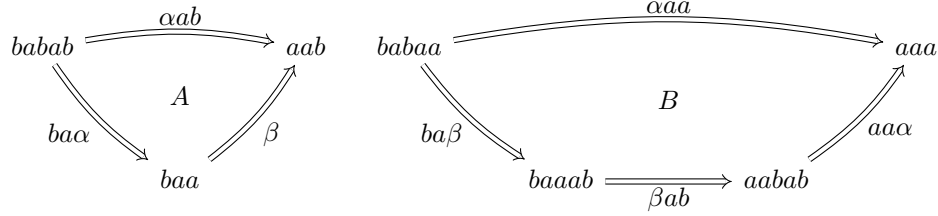
EXAMPLE 2.2.2.3 (Klein bottle monoid). Let us consider the Klein bottle monoid K^+ , as defined in [13, Subsection I.3.2]. It has the following presentation:

$$(2.2.1) \quad \langle a, b \mid bab = a \rangle.$$

The name comes from the fact that K^+ is the submonoid generated by a and b of the fundamental group of the Klein bottle generated by a and b , subject to the relation $bab = a$.

Let us apply the homotopical completion procedure to the presentation (2.2.1), which consists of the generating 1-cells a and b , and the single generating 2-cell $\alpha : bab \Rightarrow a$. Let us adopt the following termination order: comparing the lengths of words, then applying, from left to right, the lexicographic order induced by $a < b$ if words have the same length. For instance, $b < aa < ab$. The only critical branching is $\{\alpha ab, baa\}$, with source $babab$. The homotopical completion procedure adjoins the generating 2-cell $\beta : baa \Rightarrow ab$ and the generating 3-cell A , thereby granting convergence and coherence, respectively. The generating 2-cell β causes only one new critical branching, namely $\{\alpha aa, ba\beta\}$ with source $babaa$, which is confluent, hence only the generating 3-cell B is adjoined. Diagrammatically, the

generating 3-cells have the shapes as follows:



By Theorem 2.2.2.2, we have thus obtained a convergent coherent presentation of the Klein bottle monoid, having two generating 1-cells, two generating 2-cells, and two generating 3-cells:

$$\left(a, b \mid bab \xrightarrow{\alpha} a, baa \xrightarrow{\beta} aab \mid A, B \right).$$

CONVENTION 2.2.2.4. For simplicity, we mostly leave implicit the orientation of the 3-cells in the diagrams. We only label the corresponding area with the name of a 3-cell. The convention is that the source and target of a 3-cell are the upper and lower path, respectively, of the sphere bounding the area, unless explicitly stated otherwise.

2.2.3. Tietze transformations and Tietze equivalence. Polygraphs allow one to treat presentations of categories homotopically. This subsection recalls a framework for homotopical transformations of $(3, 1)$ -polygraphs, established in [26], formalising the idea of collapsing scheme introduced in [6]. We will use it to formally describe the homotopical reduction procedure in Subsection 2.2.4.

Let X be a $(3, 1)$ -polygraph. A generating 2-cell (resp. generating 3-cell, resp. 3-sphere) α of X is called **collapsible** if it meets the following two requirements:

- the target of α is a generating 1-cell (resp. 2-cell, resp. 3-cell) of X ,
- the source of α is a 1-cell (resp. 2-cell, resp. 3-cell) of the free $(3, 1)$ -category over $X \setminus \{t(\alpha)\}$.

The target of a collapsible cell is said to be **redundant**.

The following six operations are called **elementary Tietze transformations** of X :

- simultaneous adjunction or elimination of a redundant generating 1-cell v and a collapsible 2-cell $\alpha : u \Rightarrow v$,
- simultaneous adjunction or elimination of a redundant generating 2-cell β and a collapsible 3-cell $A : \alpha \Rightarrow \beta$,
- simultaneous adjunction or elimination of a redundant generating 3-cell B such that there is a collapsible 3-sphere whose target is B .

Formally, these transformations define 3-functors with domain X_3^\top (see [26, 2.1.1]). A **Tietze transformation** is a composition of elementary Tietze transformations.

Two $(3, 1)$ -polygraphs are said to be **Tietze-equivalent** if they satisfy the following two conditions:

- the 1-categories they present are isomorphic,
- the $(2, 1)$ -categories they present are equivalent.

Consequently, Tietze-equivalent $(3, 1)$ -polygraphs have the same 0-cells (up to a bijection), and two coherent presentations of the same category are Tietze-equivalent.

It is easily shown that two $(3, 1)$ -polygraphs related by a Tietze transformation are Tietze-equivalent. Namely, it suffices to verify that every one of the six elementary Tietze transformations preserves Tietze equivalence. Furthermore, Tietze transformations preserve the property of being coherent, which we formally state for future reference.

PROPOSITION 2.2.3.1 ([26, Theorem 2.1.3]). *If a $(3, 1)$ -polygraph X is a coherent presentation of a category \mathcal{C} , and if there exists a Tietze transformation from X to a $(3, 1)$ -polygraph Y , then Y is a coherent presentation of \mathcal{C} .*

An **elementary Nielsen transformation** on a $(3, 1)$ -polygraph X is any of the following operations:

- replacement of a 2-cell or a 3-cell with its formal inverse;
- replacement of a 3-cell $A : \alpha \Rightarrow \beta$ with

$$\begin{array}{ccccc}
 & & \alpha & & \\
 & \nearrow \chi & * \longrightarrow & * & \searrow \chi' \\
 & & \Downarrow \tilde{A} & & \\
 & \nwarrow \chi & * \longrightarrow & * & \swarrow \chi' \\
 & & \beta & &
 \end{array}$$

where χ and χ' are 2-cells of X_3^\top , and all the 1-cells involved are denoted by $*$.

Elementary Nielsen transformations are Tietze transformations (see [26, 2.1.4]). Therefore, they preserve presented 1-categories, equivalence of presented $(2, 1)$ -categories and homotopy type of $(3, 1)$ -polygraphs (see [26, 2.1.4]). In particular, they transform a coherent presentation of a category \mathcal{C} into another coherent presentation of \mathcal{C} . A **Nielsen transformation** is a composition of elementary ones. In the homotopical completion-reduction procedure, Nielsen transformations are often performed implicitly for convenience.

2.2.4. Homotopical reduction. A coherent presentation obtained by the homotopical completion procedure is not necessarily minimal, in the sense that it may contain superfluous cells. The homotopical reduction procedure aims to remove such superfluous cells by performing a series of elementary collapses, analogous to that used by Brown in [6]. In the present terminology, these are Tietze transformations (up to Nielsen transformations).

For a $(3, 1)$ -polygraph $X = (X_0, X_1, X_2, X_3)$, a **collapsible part** of X is a triple $\Gamma = (\Gamma_2, \Gamma_3, \Gamma_4)$ where $\Gamma_2, \Gamma_3, \Gamma_4$ respectively denote families of generating 2-cells of X , generating 3-cells of X , 3-spheres of X_3^\top , such that the following requirements are met:

- every γ of every Γ_k is collapsible (possibly up to a Nielsen transformation);
- no γ of Γ_k is the target of an element of Γ_{k+1} ;
- there exist well-founded order relations on X_1, X_2 and X_3 such that, for every γ in every Γ_k , the target of γ is strictly greater than every generating $(k - 1)$ -cell that occurs in the source of γ .

Here, we say that a generating 1-cell (resp. 2-cell, resp. 3-cell) x **occurs** in a 1-cell (resp. 2-cell, resp. 3-cell) γ of X_3^\top if it occurs in a minimal-length decomposition of γ into generating 1-cells (resp. 2-cells, resp. 3-cells).

The result of the **homotopical reduction of X with respect to Γ** is the $(3, 1)$ -polygraph

$$X/\Gamma = (X_0, X_1 \setminus t(\Gamma_2), X_2 \setminus t(\Gamma_3), X_3 \setminus t(\Gamma_4)).$$

Sources and targets are given by $\pi_\Gamma \circ s$ and $\pi_\Gamma \circ t$, where π_Γ is the 3-functor from X^\top to $(X/\Gamma)^\top$ given by the recursive formula

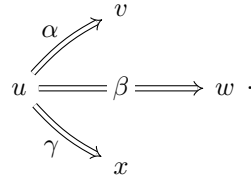
$$\pi_\Gamma(x) = \begin{cases} \pi_\Gamma(s(\gamma)) & \text{if } x = t(\gamma) \text{ for } \gamma \text{ in } \Gamma \\ 1_{\pi_\Gamma(s(x))} & \text{if } x \text{ in } \Gamma \\ x & \text{otherwise.} \end{cases}$$

Such a Tietze transformation of a $(3, 1)$ -polygraph is called the homotopical reduction procedure.

Let X be a terminating 2-polygraph, with a termination order \leq . A **homotopical completion-reduction** of X is a $(3, 1)$ -polygraph, obtained as a homotopical reduction, with respect to a collapsible part, of a homotopical completion of X . Proposition 2.2.3.1 and Theorem 2.2.2.1 imply the following result.

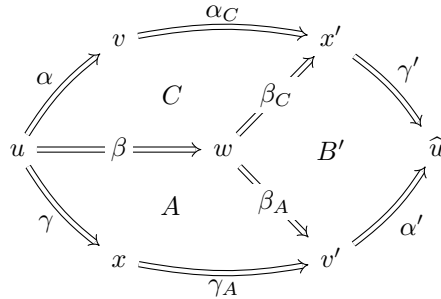
THEOREM 2.2.4.1. *If X is a terminating 2-polygraph presenting a category \mathcal{C} , then every homotopical completion-reduction of X is a coherent presentation of \mathcal{C} .*

2.2.5. A special case of reduction. We have just recalled the definition of a generic collapsible part of a $(3, 1)$ -polygraph X . For the applications considered here, however, it is practical to also recall a particular technique, given in [26, Subsection 3.2], to construct a collapsible part in the case where X is convergent and coherent. A **local triple branching** is a three-element multiset $\{\alpha, \beta, \gamma\}$ of rewriting steps having a common source:

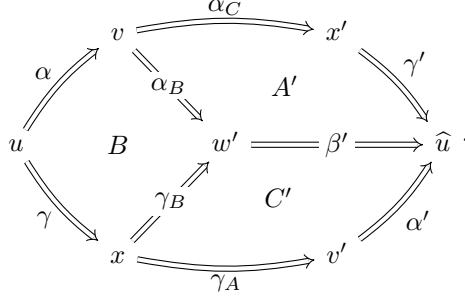


A local triple branching is **trivial** if two of its elements are equal or if one of its elements forms branchings of the type $\{\alpha v, u\beta\}$, for $u = s(\alpha)$ and $v = s(\beta)$, with the other two. In a manner analogous to the case of local branchings, local triple branchings can be ordered by ‘inclusion’, and a minimal nontrivial local triple branching is called **critical**.

Let X be a convergent and coherent $(3, 1)$ -polygraph. A **generating triple confluence** of X is a 3-sphere Φ whose boundary consists of the following two parts:



and



Here, $\{\alpha, \beta, \gamma\}$ is a critical triple branching, and the other cells are obtained in the following way. Consider the branching $\{\alpha, \beta\}$: use confluence to get the 2-cells α_C and β_C ; then use coherence to get the 3-cell C . Similarly, treat the branchings $\{\beta, \gamma\}$ and $\{\alpha, \gamma\}$. Then, consider the branching $\{\alpha_C, \alpha_B\}$: use convergence to get the 2-cells γ' and β' which have the 1-cell \hat{u} as their common target; then use coherence to get the 3-cell A' . Treat the branching $\{\gamma_B \star_1 \beta', \gamma_A\}$ similarly. Finally, use coherence to get the 3-cell B' .

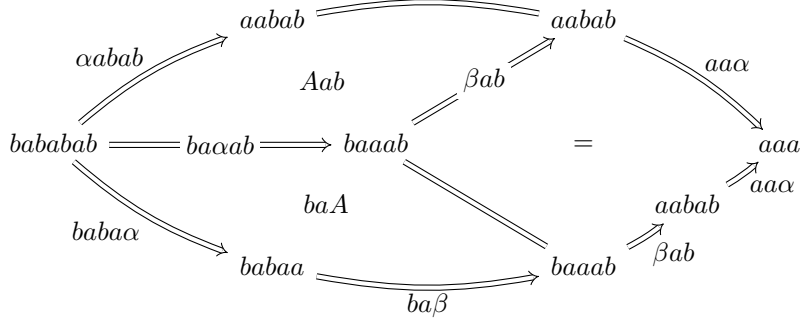
The boundary of the 3-sphere Φ consist of generating 3-cells of X in context. They have the shape zFz' where F is a generating 3-cell and z and z' are 1-cells. If one of the generating 3-cells, say F , occurs only once (in the boundary of Φ) and in an empty context (meaning $z = z' = 1$), then Φ provides a definition of F in terms of the other 3-cells involved. Thus, the 3-sphere Φ is collapsible, up to a Nielsen transformation (transforming Φ into a 3-sphere having F as target). The component Γ_4 of a collapsible part can be constructed by iteratively examining all the critical triple branchings and taking the collapsible 3-spheres, detected as described above.

After constructing Γ_4 , proceed to construct Γ_3 in the following way. For every 3-cell D of X_3^\top , consider its source and target, which are 1-composites of rewriting steps $z\delta z'$, where δ is a generating 2-cell and z and z' are 1-cells. If any such δ occurs only once and with $z = z' = 1$, then D is collapsible (up to a Nielsen transformation). Iteratively examining all the 3-cells and taking the ones such as D , the component Γ_3 of a collapsible part can be constructed. Finally, Γ_2 is constructed by iteratively taking 2-cells whose source or target consists of a single generating 1-cell occurring only once in the boundary.

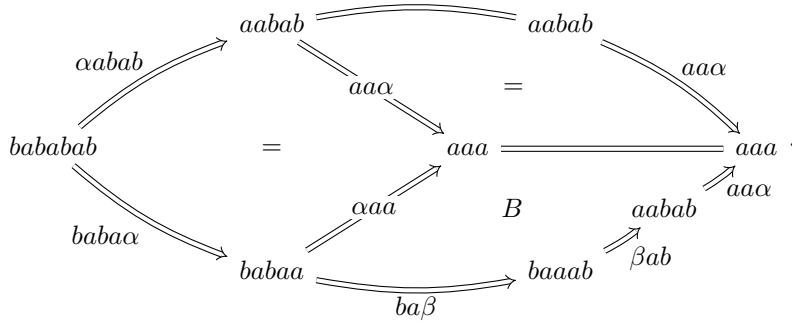
We illustrate this technique by means of an example.

EXAMPLE 2.2.5.1. Let us perform the homotopical reduction procedure on the homotopical completion of the Klein bottle monoid, computed in Example 2.2.2.3. We construct a collapsible part $\Gamma = (\Gamma_2, \Gamma_3, \Gamma_4)$. There is only one critical triple branching, namely $\{\alpha bab, ba\alpha ab, baba\alpha\}$. It yields the generating triple confluence Φ whose boundary consists of the following two parts (we display the generating 3-cells A and B differently now, to make the generating triple confluence more

evident):



and



Hence the component Γ_4 of the collapsible part contains the 3-sphere Φ which has the generating 3-cell B as target (recall that we implicitly perform a Nielsen transformation when needed). By the definition of collapsible part, we also need to provide a well-founded order relation on the set of generating 3-cells, such that, for every 3-sphere γ in Γ_4 , the target of γ is strictly greater than every generating 3-cell that occurs in the source of γ . So, we put $B > A$. Proceeding as described above, we examine the remaining 3-cells and construct the component Γ_3 out of those 3-cells whose boundary contains a generating 2-cell occurring only once. There is only one generating 3-cell left, namely A , and the generating 2-cell β appears only once in the boundary of A . So, Γ_3 contains A , and we order the set of generating 2-cells by setting $\beta > \alpha$. The component Γ_2 is empty because there is no 2-cell whose source or target consists of a single generating 1-cell appearing only once in the boundary.

Therefore, after performing the homotopical reduction procedure with respect to the collapsible part $(\emptyset, \Gamma_3, \Gamma_4)$, we are left with the presentation

$$(a, b \mid bab \xrightarrow{\alpha} a \mid \emptyset),$$

which is thus coherent by Theorem 2.2.4.1. Having a coherent presentation X with the empty set of generating 3-cells means that any two parallel rewriting paths represent the same 2-cell in X_3^\top .

Note that the homotopical reduction procedure, in general, does not preserve convergence because it can remove generating 2-cells that enabled confluence, as witnessed by the previous example.

2.3. Garside's coherent presentations of Artin-Tits and Garside monoids

In this section, we recollect some results of [26, Section 3], namely two instances of application of the homotopical completion-reduction procedure. We shall revisit these examples in Subsection 2.5.1 since the theorems of [26, Section 3] are special cases of the main result in the present chapter. Note that the homotopical completion-reduction procedure was also applied to other examples (see [32] for plactic and Chinese monoids).

2.3.1. Divisibility in categories. First, let us adopt a terminology concerning divisibility in categories. Since categories here are regarded as generalisations of monoids, morphisms will henceforth be called elements of a category (following [13, Subsection II.1.1]). A category \mathcal{C} is **left-cancellative** (resp. **right-cancellative**) if, for all f, g and g' in \mathcal{C} , the equality $fg = fg'$ (resp. $gf = g'f$) implies the equality $g = g'$. A category is **cancellative** if it is both left-cancellative and right-cancellative.

An element f of a category \mathcal{C} is said to be a **left divisor** of g in \mathcal{C} , and g is said to be a **right multiple** of f , denoted by $f \preceq g$, if there is an element f' in \mathcal{C} such that $ff' = g$. If, additionally, f' is not invertible, then divisibility is called **proper**, and written as $f \prec g$.

If \mathcal{C} is a left-cancellative category and $f \preceq g$ holds in \mathcal{C} , then there is a unique element f' in \mathcal{C} such that $ff' = g$. Here, f' is called **the right complement of f in g** .

For an element h of a left-cancellative category \mathcal{C} and a subfamily S of \mathcal{C} (i.e. family of elements of \mathcal{C}), we say that h is a greatest common left divisor, or **left-gcd**, (resp. least common right multiple, or **right-lcm**) of S if $h \preceq s$ (resp. $s \preceq h$) holds for all s in S and if every element of \mathcal{C} that is a left divisor (resp. right multiple) of all s in S is also a left divisor (resp. right multiple) of h . Left-gcds and right-lcms are unique up to multiplication by an invertible element.

A (proper) right divisor, a left-lcm and a right-gcd are defined similarly.

We say that a left-cancellative category \mathcal{C} **admits conditional right-lcms** if each pair of elements having a common right multiple has a right-lcm.

2.3.2. Coxeter groups and Artin-Tits monoids. For elements s and t of a set S , the element $\langle st \rangle^m$ of the free monoid S^* is defined inductively by $\langle st \rangle^0 = 1$ and $\langle st \rangle^{m+1} = s \langle ts \rangle^m$. A **Coxeter group** W is a group generated by a finite set S and subject to relations

$$(2.3.1) \quad \{s^2 = 1, \langle st \rangle^{m_{st}} = \langle ts \rangle^{m_{st}} \mid s, t \in S\}$$

with $(m_{st})_{s,t \in S}$ being a symmetric matrix whose entries are positive integers or ∞ (here, $m_{st} = \infty$ means that there is no given relation between s and t) and whose diagonal entries are all equal to 1. Note that a Coxeter group can admit several generating sets, but we always assume that such a set is fixed and totally ordered. One defines the length $|u|$ of an element u in W as the minimal length of an S -word needed to represent the element u , which is consistent with the definition of length (in monoids) given in Subsection 1.2.1 on page 11. The relations in (2.3.1), other than the involutions, are called **braid relations**. The **Artin-Tits monoid** $B^+(W)$ corresponding to a Coxeter group W is the monoid presented by

$$(2.3.2) \quad \langle S \mid \{\langle st \rangle^{m_{st}} = \langle ts \rangle^{m_{st}} \mid s, t \in S\} \rangle.$$

This presentation is called **Artin's presentation** of an Artin-Tits monoid. The Coxeter group W corresponding to an Artin-Tits monoid $B^+(W)$ is obtained by adding an involution $s^2 = 1$ for each generator s in (2.3.2). Since all the relations in $B^+(W)$ are length-preserving, one can naturally define the length of an element of $B^+(W)$ as the length of any S -word representing it. By Matsumoto's lemma ([13, Corollary IX.1.11] or [28, Theorem 1.2.2]), two minimal-length expressions (as products of elements of S) for an element in W can be obtained from one another by braid relations alone, hence they represent the same element in $B^+(W)$. Therefore, the canonical projection map $\pi : B^+(W) \rightarrow W$ has a canonical set-theoretic section $\sigma : W \rightarrow B^+(W)$. The image of σ consists of all the elements of the monoid $B^+(W)$ whose length is preserved by π . To simplify notation, we often identify elements of $\text{im}(\sigma)$ with their images under π . An Artin-Tits monoid is said to be of **spherical type**, or **spherical** for short, if its corresponding Coxeter group is finite.

Artin-Tits monoids are cancellative (see [5, Proposition 2.3]). Furthermore, if we regard w in W as an element of $B^+(W)$, by the canonical embedding provided by Matsumoto's lemma, then an equality $uw = w$ in $B^+(W)$ implies that both u and v are elements of W , which further implies that the condition $|uv| = |u| + |v|$ holds.

Artin-Tits monoids admit conditional right-lcms (see [5, Proposition 4.1]).

In a presentation of a category, a relation $u = v$ is called an ε -**relation** if one of the paths u and v is the empty path and the other one is not. By [13, Lemma II.1.42], if a presentation of a category \mathcal{C} contains no ε -relation, then \mathcal{C} contains no nontrivial invertible element. Note that a braid relation is not an ε -relation. Hence, an Artin-Tits monoid has no nontrivial invertible element.

EXAMPLE 2.3.2.1.

- Every free abelian monoid on a finite set is an Artin-Tits monoid because commutativity relations are braid relations. It is of spherical type because making generators involutive results in a finite Coxeter group.
- The **braid monoid** B_n^+ on n strands is the monoid presented by

$$(2.3.3) \quad \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j, |i - j| \geq 2; \sigma_i \sigma_j = \sigma_j \sigma_i, |i - j| = 1 \rangle.$$

By definition, braid monoids are Artin-Tits monoids. The Coxeter group corresponding to the braid monoid B_n^+ is the symmetric group S_n . So, braid monoids are Artin-Tits monoids of spherical type.

2.3.3. Garside's coherent presentation of Artin-Tits monoids. The monoid $B^+(W)$ admits another presentation (see [28, Proposition 4.1.3]). Garside's presentation (see [46, Section 1]) of an Artin-Tits monoid $B^+(W)$, seen as a 2-polygraph and denoted by $\text{Gar}_2(W)$, has a single generating 0-cell, elements of $W \setminus \{1\}$ as generating 1-cells, and a generating 2-cell

$$\alpha_{u,v} : u|v \Rightarrow uv$$

for each pair (u, v) of elements of $W \setminus \{1\}$ such that $|uv| = |u| + |v|$ holds. Here, $u|v$ denotes the product in W^* , whereas uv denotes the product in M . Note that the number of generating 1-cells in Garside's presentation of an Artin-Tits monoid is finite if, and only if, the Artin-Tits monoid is of spherical type.

REMARK 2.3.3.1. In this thesis, the letters u , v and w are commonly used to denote words over a generating set (i.e. 1-cells). However, in the present subsection,

these letters are actually used to denote only elements of a generating set (i.e. generating 1-cells). The reason for this is the fact that elements of a Coxeter group, which form a generating set for Garside's presentation, are themselves words over a generating set for Artin's presentation (2.3.2). For consistency, we shall extend this practice to the rest of the present section.

Now, let us recall the seminal example of application of the homotopical completion-reduction procedure.

EXAMPLE 2.3.3.2. Let $\text{Gar}_3(W)$ denote the extended presentation of $B^+(W)$ obtained by adjoining to $\text{Gar}_2(W)$ a generating 3-cell

$$\begin{array}{ccccc}
 & & uv|w & & \\
 \alpha_{u,v}|w & \nearrow & & \searrow & \alpha_{uv,w} \\
 u|v|w & & A_{u,v,w} & & uvw \\
 & \searrow & & \nearrow & \\
 u|\alpha_{v,w} & & u|vw & & \alpha_{u,vw}
 \end{array}$$

for each triple (u, v, w) of elements of $W \setminus \{1\}$ such that the equalities $|uv| = |u| + |v|$, $|vw| = |v| + |w|$ and $|uvw| = |u| + |v| + |w|$ hold. By [26, Theorem 3.1.3], the $(3, 1)$ -polygraph $\text{Gar}_3(W)$ is a homotopical completion-reduction of $\text{Gar}_2(W)$ so, by Theorem 2.2.4.1, it is a coherent presentation of $B^+(W)$.

The $(3, 1)$ -polygraph $\text{Gar}_3(W)$ is called Garside's coherent presentation of the Artin-Tits monoid $B^+(W)$.

2.3.4. Garside monoids. Recall that a **Garside monoid** (see [13, Definition I.2.1]) is a pair (M, Δ) such that the following conditions are satisfied:

- (1) M is a cancellative monoid;
- (2) there is a map $\lambda : M \rightarrow \mathbb{N}$ such that $\lambda(fg) \geq \lambda(f) + \lambda(g)$ holds for all f and g in M , and $\lambda(f) = 0$ implies $f = 1$;
- (3) each pair of elements has a left-gcd, a right-gcd, a left-lcm and a right-lcm;
- (4) there is an element Δ in M , called the Garside element, such that the left and the right divisors of Δ coincide and they generate M ;
- (5) the family of all divisors of Δ is finite.

Observe that the second condition implies that M has no nontrivial (here meaning different from 1) invertible element. Namely, $\lambda(1) = \lambda(1 \cdot 1) \geq \lambda(1) + \lambda(1)$ implies $\lambda(1) = 0$, whence $fg = 1$ implies $0 \geq \lambda(f) + \lambda(g)$ which further implies $f = g = 1$. Consequently, a left-gcd is unique in a Garside monoid. We write $f \wedge g$ for the left-gcd of f and g . For a (left) divisor f of Δ , we write $\partial(f)$ for the right complement of f in Δ .

EXAMPLE 2.3.4.1. The class of Garside monoids includes Artin-Tits monoids of spherical type (see [17]) and, consequently, braid monoids (see [5]).

- In the free abelian monoid $(\mathbb{N}^n, +)$, the Garside element is $\Delta_n = (1, \dots, 1)$.
- For the braid monoid B_n^+ , the Garside element Δ_n (called the half-turn braid) is defined inductively: $\Delta_1 = 1$, $\Delta_2 = \sigma_1$, $\Delta_n = \Delta_{n-1}\sigma_{n-1} \cdots \sigma_1$ for $n \geq 2$. Divisors of Δ_n are called simple braids.

Dual braid monoids (to be recalled in Subsection 2.6.4) make another important family of Garside monoids.

For more examples, see [13, Examples I.2.7 and I.2.8].

2.3.5. Greedy decomposition. As mentioned in Subsection 1.1.1, Garside monoids were introduced in order to axiomatise the greedy normal form that was observed in braid monoids. In a Garside monoid (M, Δ) , a word $u_1 | \cdots | u_m$, with u_1, \dots, u_m being divisors of Δ , is said to be Δ -**normal** if, for all $i < m$,

$$(2.3.4) \quad \text{for all } h \in M \setminus \{1\}, \quad h \preceq u_{i+1} \text{ implies } u_i h \not\preceq \Delta,$$

and it is said to be **strict** if, additionally, u_m is not equal to 1.

A (strict) Δ -normal decomposition of g in M is a (strict) Δ -normal word representing g with respect to the generating set consisting of all divisors of Δ . Note that the strict Δ -normal decomposition of 1 is the empty word. Let us examine how to compute a Δ -normal decomposition of g in M represented by a length-two word $u|v$ that is not Δ -normal. Notice that, in the length-two case, the condition (2.3.4) translates to $\partial(u) \wedge v = 1$. Indeed, $u|v$ is Δ -normal if, and only if, there is no nontrivial left divisor h of v such that uh is a divisor of Δ (or, equivalently, $h \preceq \partial(u)$). Hence, a procedure can be established to find a Δ -normal decomposition of g , as follows. If there is a nontrivial left divisor h of v such that $h \preceq \partial(u)$, then $\partial(u) \wedge v$ is nontrivial too. A Δ -normal form of g is obtained in a single step by replacing u and v respectively with $u' = u(\partial(u) \wedge v)$ and the right complement v' of $\partial(u) \wedge v$ in v . Thus we have rewritten $u|v$ to $u'|v'$, which is Δ -normal. This step is called **local sliding** in [27, Subsection 2.2] as $(\partial(u) \wedge v)$ ‘slides’ from v to u .

For a strict Δ -normal decomposition, one further distinguishes two cases:

- $\partial(u) \wedge v = v$, in which case v' is equal to 1, and $u' = uv$ is returned as a length-one strict Δ -normal decomposition of g ;
- $\partial(u) \wedge v$ properly divides v , in which case $u'|v'$ is a strict Δ -normal decomposition of g .

The condition (2.3.4) says that u_i is the maximal divisor of Δ that left-divides $u_i u_{i+1} \cdots u_m$ (hence this decomposition is called **greedy**, a term to be formally recalled in Subsection 2.4.3), as illustrated by the following simple example.

EXAMPLE 2.3.5.1. Let us compute a strict Δ_3 -normal decomposition of the element $g = (2, 1, 3)$ of the free abelian monoid \mathbb{N}^3 . The maximal divisor of Δ that left-divides g is Δ_3 . The right complement of Δ_3 in g is $(1, 0, 2)$. The maximal divisor of Δ that left-divides $(1, 0, 2)$ is $(1, 0, 1)$. The right complement of Δ_3 in g is $(0, 0, 1)$, which is a divisor of Δ . The decomposition, therefore, is complete. Denoting the generators $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ of \mathbb{N}^3 by a , b and c , respectively, gives $a^2 b c^3 = abc|ac|c$.

The main idea is that the finite lattice formed by divisors of Δ reveals the structure of the entire Garside monoid, as expressed by the following result.

PROPOSITION 2.3.5.2 ([12, Proposition 2.4]). *If (M, Δ) is a Garside monoid, then every g in M admits a unique strict Δ -normal decomposition.*

We refer the reader to [12, Subsection 2.2] for an overview of the computational properties of the Δ -normal form.

2.3.6. Garside’s coherent presentation of Garside monoids. Garside’s presentation of a Garside monoid M is the 2-polygraph $\text{Gar}_2(M)$, having divisors of Δ other than 1 as generating 1-cells and a generating 2-cell $\alpha_{u,v} : u|v \Rightarrow uv$ whenever the condition $\partial(u) \wedge v = v$ is satisfied.¹ Here, $u|v$ denotes the product in the free monoid over the set of proper divisors of Δ , whereas uv denotes the product in M . It is easily verified that $\text{Gar}_2(M)$ is, indeed, a presentation of M , because uv is the strict (we leave out the 1 from $uv|1$) Δ -normal form of $u|v$ if, and only if, the condition $\partial(u) \wedge v = v$ holds.

Having noticed in the previous subsection that, in the case of a length-two word $u|v$, the condition (2.3.4) of being Δ -normal amounts to $\partial(u) \wedge v = 1$, we can take the condition $\partial(u) \wedge v = v$ here to mean ‘as far as possible from being Δ -normal’.

EXAMPLE 2.3.6.1. We want to describe the extended presentation $\text{Gar}_3(M)$, which is introduced in [26, 3.3.2].² To that end, we are going to reformulate the condition $\partial(u) \wedge v = v$ in a suitable way, in order to generalise it to three elements. Let us first notice that the condition $\partial(u) \wedge v = v$ is equivalent to saying that v is a left divisor of $\partial(u)$. In other words, there is an element w in M such that $vw = \partial(u)$. By definition of $\partial(u)$, this means that $uvw = \Delta$, so uv and w are divisors of Δ . This reformulation allows us to extend the given condition to a greater number of elements. Let $\text{Gar}_3(M)$ denote the extended presentation of M obtained by adjoining to $\text{Gar}_2(M)$ a generating 3-cell

$$\begin{array}{ccccc}
 & & uv|w & & \\
 & \nearrow \alpha_{u,v|w} & & \searrow \alpha_{uv,w} & \\
 u|v|w & & A_{u,v,w} & & uvw \\
 & \searrow u|\alpha_{v,w} & & \nearrow \alpha_{u,vw} & \\
 & & u|vw & &
 \end{array}$$

for each triple (u, v, w) of divisors of Δ , not equal to 1, such that uv , vw and uvw are divisors of Δ . By [26, Theorem 3.3.3], $\text{Gar}_3(M)$ is a homotopical completion-reduction of $\text{Gar}_2(M)$ so, by Theorem 2.2.4.1, it is a coherent presentation of M .

2.4. Garside families

The present section briefly recollects the basic notions and results concerning Garside families (for technical elaboration, see the monograph [13]).

2.4.1. Motivation for Garside family. Introducing Garside monoids generalised spherical Artin-Tits monoids (and *a fortiori* braid monoids) and axiomatised properties allowing the greedy decomposition. Shortly afterwards, a further extension was invited by the examples that were not Garside monoids but whose elements nevertheless admitted decompositions similar to the Δ -normal decomposition (provided by Proposition 2.3.5.2).

Free abelian monoids over infinite basis, infinite braids, the Klein bottle group, ribbon categories, among others, fail to fit in the frame of Garside monoids, yet they do grant decompositions structurally identical to the greedy decomposition. Such examples suggested seeking more general requirements which a subfamily of

¹There is a typo in [26, 3.3.2], where it is written $\partial(u) \wedge v = 1$ instead of $\partial(u) \wedge v = v$.

²Conditions for the generating 3-cells of $\text{Gar}_3(M)$ are omitted in [26, 3.3.2].

a category has to meet in order to guarantee the existence of a decomposition such as the Δ -normal decomposition.

Some of these examples will be recollected in Section 2.6 and more can be found in [13, Section I.3].

2.4.2. Right-mcms. Let \mathcal{C} be a left-cancellative category and, let S be a subfamily of \mathcal{C} . The left-divisibility relation \preceq is a preorder of elements; it is an order if, and only if, \mathcal{C} has no nontrivial invertible element.

A subfamily S of a left-cancellative category \mathcal{C} is said to be **closed under right comultiple** if every common right multiple of two elements s and t of S (if there is any) is a right multiple of a common right multiple of s and t that lies in S .

For f and g in a category \mathcal{C} , a minimal common right multiple, or **right-mcm**, of f and g is a right multiple h of f and g , such that no proper left divisor of h is a common right multiple of f and g . A category \mathcal{C} **admits right-mcms** if, for all f and g of \mathcal{C} , every common right multiple of f and g is a right multiple of some right-mcm of f and g . Observe that, in a left-cancellative category admitting conditional right-lcms, the notions of a right-mcm and right-lcm coincide. Let us state a rather basic observation about right-mcm in a left-cancellative monoid, which we will use in one step of the main proof in Subsection 2.5.3. The following lemma is similar to [35, Lemma 11.24], which deals with lcms, whereas here it suffices to consider mcms, under weaker assumptions.

LEMMA 2.4.2.1. *Let M be a left-cancellative monoid. If v' is a right-mcm of v_1 and v_2 in M , then uv' is a right-mcm of uv_1 and uv_2 for every u in M .*

PROOF. Let x_1 and x_2 denote the right complements in v' of v_1 and v_2 , respectively. Note that uv' is a common right multiple of uv_1 and uv_2 , namely, $uv_1x_1 = uv' = uv_2x_2$. Assume that there is a common right multiple f of uv_1 and uv_2 such that f is a proper left divisor of uv' . Then there exist elements f_1, f_2 and a non-invertible element g in M satisfying $uv_1f_1 = f = uv_2f_2$ and $fg = uv'$. By the left cancellation property, the equations $uv_kf_kg = uv'$ yield $v_kf_kg = v'$ for k in $\{1, 2\}$. Thus we obtain the common right multiple $v_1f_1 = v_2f_2$ of v_1 and v_2 , which is a proper left divisor of v' . This contradicts the fact that v' is a right-mcm of v_1 and v_2 . Therefore, uv' is a right-mcm of uv_1 and uv_2 . \square

Relying on [13, Propositions II.2.28 and II.2.29], we say that a left-cancellative category \mathcal{C} is **left-noetherian** (resp. **right-noetherian**) if for every g in \mathcal{C} , every increasing sequence of right (resp. left) divisors of g with respect to proper right divisibility (resp. left divisibility) is finite. A left-cancellative category \mathcal{C} is **noetherian** if it is both left-noetherian and right-noetherian.

EXAMPLE 2.4.2.2.

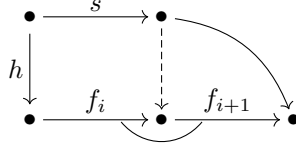
- Proper division, left or right, strictly reduces the length of an element of an Artin-Tits monoid. Therefore, no element admits an infinite number of divisors, so Artin-Tits monoids are noetherian.
- Garside monoids are noetherian by definition (thanks to the map $\lambda : M \rightarrow \mathbb{N}$).
- For an example of a monoid that is right-noetherian but not noetherian, we refer to reader to the lifted omega monoid given in [13, Exercises II.14 and IV.46].

2.4.3. Garside families. In this subsection, we recollect the definition and some basic properties of the all-important notion of Garside family, which provides a way of extending the notion of greedy decomposition beyond Garside monoids.

For a subfamily S of a left-cancellative small category \mathcal{C} , a \mathcal{C} -path $f_1|\cdots|f_m$ is said to be S -greedy if, for all $i < m$,

$$(2.4.1) \quad \text{for all } s \in S \text{ and } h \in \mathcal{C}, \quad s \preceq hf_i f_{i+1} \text{ implies } s \preceq hf_i.$$

This property can be expressed diagrammatically as follows. Commutativity of the diagram



without the dashed arrow implies the existence of a dashed arrow making the square (and thus also the triangle) commute. The arc joining f_i and f_{i+1} denotes greediness.

By definition, a path of length 0 or 1 is S -greedy for any subfamily S . The following result observes an immediate but important consequence of the definition of S -greedy \mathcal{C} -path.

LEMMA 2.4.3.1 ([14, Lemma 2.12]). *Let S be a subfamily of a left-cancellative category \mathcal{C} . If a \mathcal{C} -path $f_1|\cdots|f_m$ is S -greedy, then the length-two \mathcal{C} -path $f_1|f_2\cdots f_m$ is S -greedy, as well.*

PROOF. Suppose that $s \preceq hf_1(f_2\cdots f_m)$ for some s in S and h in \mathcal{C} . The idea is to use associativity to group together all factors except the last two, then to use the fact that the remaining pair is S -greedy. Namely, $s \preceq (hf_1f_2\cdots f_{m-2})f_{m-1}f_m$ implies $s \preceq (hf_1f_2\cdots f_{m-2})f_{m-1}$. We repeat this process until we get $s \preceq (h)f_1f_2$ which implies $s \preceq hf_1$. Therefore, $f_1|f_2\cdots f_m$ is S -greedy. \square

Let \mathcal{C}^\times denote the family of all invertible elements of \mathcal{C} . For a subfamily S of \mathcal{C} , a \mathcal{C} -path $r_1|\cdots|r_m$ is said to be S -normal if it is S -greedy and if, moreover, r_1, \dots, r_m lie in $S\mathcal{C}^\times \cup \mathcal{C}^\times$. An S -normal path $r_1|\cdots|r_m$ is called **strict** if, additionally, r_1, \dots, r_{m-1} lie in $S \setminus \mathcal{C}^\times$ and r_m lies in $S\mathcal{C}^\times \setminus \mathcal{C}^\times$. In (other) words, an S -normal path is strict if all its components are non-invertible and if all, except possibly the last one, lie in S .

An S -normal decomposition of f in \mathcal{C} is an S -normal path $r_1|\cdots|r_m$ such that $f = r_1\cdots r_m$ holds in \mathcal{C} . In general, an S -normal decomposition of an element is not unique. Nevertheless, the number of non-invertible letters is the same in all S -normal decompositions of f (see [14, Proposition 2.11] or [13, Proposition III.1.25] for exposition). This motivates the following notion. For a subfamily S of a left-cancellative category \mathcal{C} and an element f of \mathcal{C} admitting at least one S -normal decomposition, the S -length of f is defined as the common number of non-invertible letters in all S -normal decompositions of f . The next subsection will show that the S -length concurs with the length in monoids, as given in Subsection 1.2.1 on page 11 (and, in particular, with the lengths in Coxeter groups and Artin-Tits monoids, mentioned in Subsection 2.3.2).

DEFINITION 2.4.3.2. A subfamily S of a left-cancellative small category \mathcal{C} is called a **Garside family** in \mathcal{C} if every element of \mathcal{C} admits an S -normal decomposition.

Since every left-cancellative category \mathcal{C} is a Garside family in itself (for every f in \mathcal{C} , simply take the length-one word f as a \mathcal{C} -normal decomposition of f), we are interested only in proper (meaning other than \mathcal{C} itself) Garside families.

EXAMPLE 2.4.3.3.

- The Coxeter group W corresponding to a Artin-Tits monoid $B^+(W)$ is a, possibly infinite, Garside family in $B^+(W)$.
- Every Artin-Tits monoid admits a finite Garside family (see [15]). In the case of an Artin-Tits monoid of spherical type, a finite Garside family is given by the corresponding Coxeter group. In the particular case of a braid monoid, the family of all simple braids is a Garside family.
- Every Garside monoid (M, Δ) has a finite Garside family given by the family of all divisors of Δ (see [14, Proposition 2.18] or [13, Proposition III.1.43]).

More examples will be presented in Section 2.6. The following proposition gives a simple characterisation of a Garside family.

PROPOSITION 2.4.3.4 ([14, Proposition 3.1] or [13, Proposition III.1.39]). *A subfamily S of a left-cancellative category \mathcal{C} is a Garside family if, and only if, the following conjunction holds: $S\mathcal{C}^\times \cup \mathcal{C}^\times$ generates \mathcal{C} and every length-two word over $S\mathcal{C}^\times \cup \mathcal{C}^\times$ admits an S -normal decomposition.*

We recall another characterisation of Garside family, which we will invoke in Subsection 2.5.3. More characterisations of Garside families can be found in [14, Subsection 3.2] or [13, Subsection IV.1.2].

PROPOSITION 2.4.3.5 ([14, Proposition 3.9]). *Let \mathcal{C} be a left-cancellative category. A subfamily S of \mathcal{C} is a Garside family if, and only if, the following conditions are satisfied:*

- $S\mathcal{C}^\times \cup \mathcal{C}^\times$ generates \mathcal{C} ,
- it is closed under right comultiple and right divisor,
- and evaluation of every non-invertible length-two word over $S\mathcal{C}^\times \cup \mathcal{C}^\times$ admits a \prec -maximal left divisor in S .

Let us recall another result to be used in Subsection 2.5.3.

LEMMA 2.4.3.6 ([13, Lemma IV.2.24]). *Let \mathcal{C} be a left-cancellative category. If \mathcal{C} admits right-mcms, then, for every subfamily S of \mathcal{C} , the following are equivalent.*

- The family S is closed under right comultiple.
- The family $S\mathcal{C}^\times \cup \mathcal{C}^\times$ is **closed under right-mcm**, i.e. if f and g lie in $S\mathcal{C}^\times \cup \mathcal{C}^\times$, then so does every right-mcm of f and g .

2.4.4. Special case of left-cancellative monoids with no nontrivial invertible element. In the case of a left-cancellative monoid M having no nontrivial invertible element, the normality conditions simplify to the following. For a subfamily S of M , an M -word $r_1|\cdots|r_m$ is called S -normal if it is S -greedy and if, moreover, all r_1, \dots, r_m lie in S . An S -normal word $r_1|\cdots|r_m$ is strict if, additionally, $r_m \neq 1$. Therefore, under the given restrictions, the existence of an S -normal

form implies the existence of a strict one. Furthermore, by [12, Lemma 3.5], every element admits at most one strict S -normal decomposition.

PROPOSITION 2.4.4.1 ([12, Proposition 3.10]). *Let M be a left-cancellative monoid containing no nontrivial invertible element, and let S be a Garside family in M . Then, for every S -word v , there exists a unique S -normal word $N^S(v)$ that represents the same element as v and has the same length as v .*

Observe that, consequently, the S -length agrees with the length (for monoids) given in Subsection 1.2.1 on page 11.

For a Garside family S in a left-cancellative monoid M with no nontrivial invertible element, the **normalisation map derived from the Garside family** S is the length-preserving map $N^S : S^* \rightarrow S^*$ which assigns to each v in $S^* \setminus \{1\}$ the S -normal decomposition of the element of M represented by v ; and $N^S(1) = 1$. It is well defined due to Proposition 2.4.4.1. The following result provides an important property of S -normal decomposition. It is a straightforward extension of [16, Lemma 6.9], which deals with length-two S -words, to all S -words.

LEMMA 2.4.4.2. *Let M be a left-cancellative monoid having no nontrivial invertible element, and let S be a Garside family in M . Then, for every word v in S^* , the leftmost letter of v is a left divisor of the leftmost letter of $N^S(v)$.*

PROOF. Let $r_1|r_2 \cdots |r_m$ be the word $N^S(v)$. Since v and $r_1|r_2 \cdots |r_m$ evaluate to the same element of M , the leftmost letter of v is a left divisor of $r_1(r_2 \cdots r_m)$. By Lemma 2.4.3.1, the length-two M -word $r_1|r_2 \cdots |r_m$ is S -greedy. Hence, the desired conclusion follows. \square

A property granted by Lemma 2.4.4.2, but limited to S -words of length 2, is called **left-weightedness** and will be revisited in a more general framework in Subsection 3.3.5.

A Garside family yields a presentation in the following sense.

PROPOSITION 2.4.4.3 ([16, Proposition 6.17] or [29, Corollary 6.6.4]). *Let M be a left-cancellative monoid containing no nontrivial invertible element, and let S be a Garside family in M . Then M admits, as a convergent presentation, the 2-polygraph that contains a single generating 0-cell, one generating 1-cell for each element of $S \setminus \{1\}$, and one generating 2-cell of the form*

$$(2.4.2) \quad s|t \Rightarrow \text{the strict } S\text{-normal decomposition of } s|t$$

for each pair s, t in $S \setminus \{1\}$ such that $s|t$ is not S -normal. In particular, every Artin-Tits monoid admits a finite convergent presentation.

A Garside family also induces a ‘smaller’ presentation, beside the one provided by Proposition 2.4.4.3, which will be instrumental in deriving a main result in the next section. The following proposition is an adaptation of [16, Proposition 6.15] using [16, Proposition 6.10].

PROPOSITION 2.4.4.4. *Let M be a left-cancellative monoid containing no nontrivial invertible element, and let S be a Garside family containing 1 in M . Then M admits, as a presentation, the 2-polygraph $\text{Gar}_2(S)$ that contains a single generating 0-cell, one generating 1-cell for each element of $S \setminus \{1\}$, and one generating 2-cell of the form*

$$(2.4.3) \quad s|t \Rightarrow st$$

for each pair s, t in $S \setminus \{1\}$ such that the product st in M lies in S .

PROOF. Proposition 2.4.4.3 grants a presentation of M in terms of S by the relations (2.4.2). In this paragraph, we want to show that the relations (2.4.3) are included in the relations (2.4.2). If s and t in $S \setminus \{1\}$ are such that st lies in $S \setminus \{1\}$, then the strict S -normal decomposition of $s|t$ is st . Otherwise, $st = 1$ holds and the strict S -normal decomposition of $s|t$ is 1. In both cases, the strict S -normal decomposition of $s|t$ is st . Hence, the relations (2.4.3) are included in the relations (2.4.2).

Conversely, let us show that each relation (2.4.2), with s and t in $S \setminus \{1\}$, follows from a finite number of relations (2.4.3). Assume that s and t lie in $S \setminus \{1\}$ and let $s'|t' := N^S(s|t)$. If $t' = 1$ holds, then $s' = st$, which is a (2.4.3) relation, so the result is true in this case. Otherwise, Lemma 2.4.4.2 implies that there exists r in M , satisfying $sr = s'$, which is a (2.4.3) relation. Indeed, being a right divisor of s' in S , the element r also lies in S by Proposition 2.4.3.5. Multiplying the equality $sr = s'$ by t' on the right yields $srt' = s't' = st$. Then the left cancellation property of M implies $rt' = t$, which is a (2.4.3) relation. Therefore, the relation $s|t = s|r|t' = s'|t'$ follows from the relations $s|r = s'$ and $r|t' = t$. \square

The 2-polygraph $\text{Gar}_2(S)$ is called **Garside's presentation** of M , with respect to Garside family S . We study it in the next section. Here, let us just observe that it extends Garside's presentation of Artin-Tits monoids, recalled in Subsection 2.3.3.

EXAMPLE 2.4.4.5. Garside's presentation $\text{Gar}_2(W)$ of an Artin-Tits monoid $B^+(W)$ is an instance of a Garside's presentation $\text{Gar}_2(S)$ with respect to a Garside family S . Indeed, the Artin-Tits monoid $B^+(W)$ is a cancellative monoid (see [5]) with no nontrivial invertible element, and the Coxeter group W is a Garside family containing 1. Hence, $B^+(W)$ meets all the requirements of Proposition 2.4.4.4, which, for this particular Garside family as input for S , produces precisely Garside's presentation $\text{Gar}_2(W)$.

EXAMPLE 2.4.4.6. Garside's presentation $\text{Gar}_2(M)$ of a Garside monoid M is another instance of a Garside's presentation with respect to a Garside family S . Namely, a Garside monoid is cancellative, by definition. Note that the property (2) of Garside monoid implies that it has no nontrivial invertible element. All the divisors of Δ form a Garside family containing 1. Hence, M meets all the requirements of Proposition 2.4.4.4, which, for this particular Garside family as input for S , produces precisely Garside's presentation $\text{Gar}_2(M)$.

2.5. Coherent presentations from Garside families

Having recalled necessary notions and results in previous sections, in this section we present new results. Subsection 2.5.1 gives the statement and sketches a proof of Theorem 2.5.1.4, a main result in the present chapter, which gives a construction of coherent presentations for a class of monoids admitting a Garside family, thus providing a unifying generalisation of theorems recalled in Examples 2.3.3.2 and 2.3.6.1 concerning Artin-Tits and Garside monoids, respectively. Subsections 2.5.2, 2.5.3 and 2.5.4 aim to prove Theorem 2.5.1.4 by respectively providing termination, homotopical completion and homotopical reduction, starting with Garside's presentation $\text{Gar}_2(S)$. In Subsection 2.5.5, the necessity of noetherianity is illustrated by means of example.

For the rest of this chapter, we shall resume the practice observed in Remark 2.3.3.1 of using the letters u, v, w to denote generators rather than words.

2.5.1. Main statement and sketch of proof. In this subsection, we adapt some notation from [26] and set a suitable noetherianity condition. Then we state Theorem 2.5.1.4.

Let M be a monoid generated by a set S containing 1. We define the notations $u \frown v$ and $u \times v$, as follows. Given two elements u and v of $S \setminus \{1\}$, write:

$$\begin{aligned} u \frown v &\iff uv \in S, \\ u \times v &\iff uv \notin S. \end{aligned}$$

The notation extends to a greater number of elements. For three elements u, v, w in S , we write $u \frown v \frown w$ if both conditions $u \frown v$ and $v \frown w$ hold. The case $u \frown v \frown w$ splits into two mutually exclusive subcases:

$$\begin{aligned} u \frown v \frown w &\iff (u \frown v \frown w \text{ and } uvw \in S), \\ u \times v \frown w &\iff (u \frown v \frown w \text{ and } uvw \notin S). \end{aligned}$$

Let us formally redefine symbols Gar_2 and Gar_3 in a more general context, as follows. The 2-polygraph $\text{Gar}_2(S)$ contains: a single generating 0-cell; one generating 1-cell for every element of $S \setminus \{1\}$; one generating 2-cell of the form

$$\alpha_{u,v} : u|v \Rightarrow uv$$

for each pair (u, v) of elements of $S \setminus \{1\}$ such that $u \frown v$ holds. Here, $u|v$ denotes the product in S^* , whereas uv denotes the product in M . The $(3, 1)$ -polygraph $\text{Gar}_3(S)$ is consisting of the 2-polygraph $\text{Gar}_2(S)$ and the generating 3-cells of the form

$$\begin{array}{ccc} & uv|w & \\ \alpha_{u,v}|w \nearrow & & \searrow \alpha_{uv,w} \\ u|v|w & A_{u,v,w} & uvw \\ u|\alpha_{v,w} \searrow & & \nearrow \alpha_{u,vw} \\ & u|vw & \end{array}$$

for each triple (u, v, w) of elements of $S \setminus \{1\}$ such that $u \frown v \frown w$.

REMARK 2.5.1.1. Note that the 2-polygraph $\text{Gar}_2(S)$ is not a presentation of M , in general. Consequently, since $\text{Gar}_3(S)$ is an extended presentation of a monoid presented by $\text{Gar}_2(S)$, it is not necessarily an extended presentation of M . Proposition 2.4.4.4 gives sufficient conditions for $\text{Gar}_2(S)$ to be a presentation of M , thus making $\text{Gar}_3(S)$ an extended presentation of M .

To formulate the main result, we need a restriction of right noetherianity to a Garside family.

DEFINITION 2.5.1.2. A Garside family S in a left-cancellative monoid M is said to be **right-noetherian** if for every s in S , every increasing sequence of proper left divisors in S of s with respect to proper left divisibility is finite.³

EXAMPLE 2.5.1.3. Every Garside family in a right-noetherian left-cancellative monoid M is right-noetherian. In Subsection 2.5.5, we shall see an example of a Garside family that is not right-noetherian.

Now, we state the main result of this chapter.

THEOREM 2.5.1.4. *Let M be a left-cancellative monoid containing no nontrivial invertible element, and let S be a Garside family containing 1 in M . If S is right-noetherian, and if M admits right-mcms, then M admits the $(3, 1)$ -polygraph $\text{Gar}_3(S)$ as a coherent presentation.*

REMARK 2.5.1.5. Results of [13, Subsection IV.2.2] enable us to rephrase Theorem 2.5.1.4 by referring to a right-noetherian monoid rather than just a right-noetherian Garside family. However, we prefer the above formulation since it agrees better with the rather formal construction of a homotopical completion which we will give in Subsection 2.5.3.

Before we proceed to prove the theorem, let us show that it gives a common generalisation of the two distinct directions of extension, given in [26], of Deligne's result [19, Theorem 1.5].

COROLLARY 2.5.1.6 ([26, Theorem 3.1.3]). *For every Coxeter group W , the Artin-Tits monoid $B^+(W)$ admits $\text{Gar}_3(W)$ as a coherent presentation.*

PROOF. Let us restrict the conditions $u \widehat{v}$ and $u \widehat{v} \widehat{w}$, defined in the beginning of the current subsection, to the case of the Artin-Tits monoid $B^+(W)$, with the Coxeter group W as a Garside family S . Observe that, for u, v in $W \setminus \{1\}$, the condition $u \widehat{v}$ amounts to the condition $|uv| = |u| + |v|$ given in Example 2.3.3.2 (see Matsumoto's lemma, e.g. [13, Corollary IX.1.11]). Accordingly, the condition $u \widehat{v} \widehat{w}$ becomes the conjunction of $u \widehat{v}$ and $v \widehat{w}$ and $|uvw| = |u| + |v| + |w|$. Recall from Section 2.3 that Artin-Tits monoids are cancellative and noetherian (Example 2.4.2.2), and that they contain no nontrivial invertible element. Note that they also admit right-mcms: in fact, they admit conditional right-lcms ([5, Proposition 4.1]). Consequently, Theorem 2.5.1.4 specialises to [26, Theorem 3.1.3] when a monoid considered is Artin-Tits with Coxeter group as a Garside family. \square

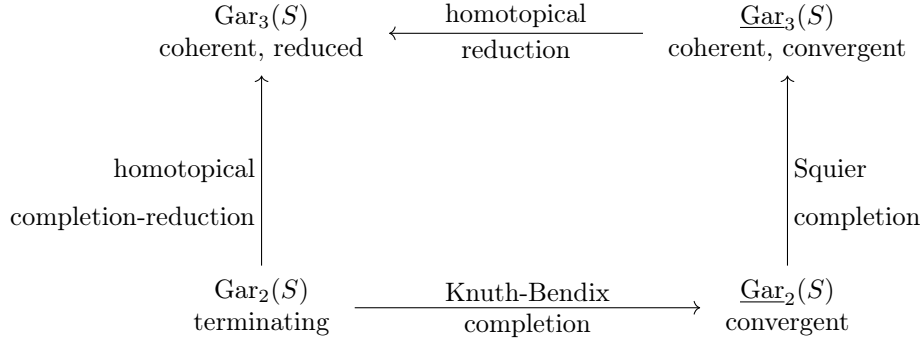
Similarly, one shows that Theorem 2.5.1.4 specialises to [26, Theorem 3.3.3] when a monoid considered is Garside with S being the set of divisors of the Garside element.

COROLLARY 2.5.1.7 ([26, Theorem 3.3.3]). *Every Garside monoid M admits $\text{Gar}_3(M)$ as a coherent presentation.*

³Technically, we can drop the condition that the family S is Garside and relax the finiteness condition by considering only such sequences with every quotient (of a member of the sequence by its predecessor) also lying in S . However, for simplicity, we already assume that S is Garside family (since that will be the case in applications, anyway), so the quotients automatically lie in S due to the closure under right divisor.

PROOF. If we restrict the conditions $u \frown v$ and $u \frown v \frown w$ to the case of a Garside monoid (M, Δ) , with divisors of Δ making a Garside family S , then we get precisely our equivalent reformulation, given in Example 2.3.6.1, of the conditions stated in [26, Subsection 3.3]. Literally, the condition $u \frown v \frown w$ then says that uv is a divisor of Δ . Garside monoids are cancellative by definition. Note that the property (2) of a Garside monoid (see Subsection 2.3.4) implies noetherianity as well as the fact that there is no nontrivial invertible element. They also admit right-mcms: in fact, they admit right-lcms (by definition). \square

The following diagram summarises key steps of the proof of Theorem 2.5.1.4 and thus motivates the next three subsections which together contain the proof.



In Subsection 2.5.2, Garside's presentation $\text{Gar}_2(S)$ of M is extended with the generating 2-cells β , which results in a terminating presentation $\underline{\text{Gar}}_2(S)$. In fact, $\underline{\text{Gar}}_2(S)$ is a convergent presentation, namely a Knuth-Bendix completion of $\text{Gar}_2(S)$. The proof of confluence of $\underline{\text{Gar}}_2(S)$ is postponed until Subsection 2.5.3, but looking ahead prompts us to begin Subsection 2.5.2 with a formal definition of the 2-polygraph $\underline{\text{Gar}}_2(S)$.

In Subsection 2.5.3, a Squier completion of the polygraph $\underline{\text{Gar}}_2(S)$ is computed formally, under certain assumptions on the monoid. The resulting $(3, 1)$ -polygraph is denoted by $\underline{\text{Gar}}_3(S)$. It is then shown that this construction applies to a terminating presentation of the monoid M given in the statement of Theorem 2.5.1.4 and produces a coherent convergent presentation $\underline{\text{Gar}}_3(S)$.

Finally, in Subsection 2.5.4, a homotopical reduction of $\underline{\text{Gar}}_3(S)$ is computed resulting in the $(3, 1)$ -polygraph $\text{Gar}_3(S)$ as a coherent presentation of M .

2.5.2. Attaining termination. In this subsection, we ensure that a certain presentation, denoted by $\underline{\text{Gar}}_2(S)$, is terminating. This presentation will arise naturally (in the next subsection) as a result of applying the Knuth-Bendix completion procedure to Garside's presentation $\text{Gar}_2(S)$. Hence the motivation for the formal definition of the 2-polygraph $\underline{\text{Gar}}_2(S)$ here.

Let M be a monoid generated by a subset S containing 1. Observe that the 2-polygraph $\text{Gar}_2(S)$ has exactly one critical branching for all u, v and w of $S \setminus \{1\}$

such that $u \frown v \frown w$ holds:

$$\begin{array}{ccc} & & uv|w \\ & \nearrow^{\alpha_{u,v}|w} & \\ u|v|w & & \\ & \searrow_{u|\alpha_{v,w}} & \\ & & u|vw \end{array} .$$

If the subcase $u \frown v \frown w$ holds, then the branching is already confluent. Otherwise $u \frown^{\times} v \frown w$ holds, and the branching requires a new generating 2-cell to reach confluence, so the generating 2-cell $\beta_{u,v,w} : u|vw \Rightarrow uv|w$ is adjoined. We write $\underline{\text{Gar}}_2(S)$ for the 2-polygraph that contains a single generating 0-cell, one generating 1-cell for every element of $S \setminus \{1\}$, and the generating 2-cells

$$\begin{aligned} \alpha_{u,v} : u|v &\Rightarrow uv, & u, v &\in S \setminus \{1\}, \quad u \frown v, \\ \beta_{u,v,w} : u|vw &\Rightarrow uv|w, & u, v, w &\in S \setminus \{1\}, \quad u \frown^{\times} v \frown w. \end{aligned}$$

To show that the 2-polygraph $\underline{\text{Gar}}_2(S)$, under certain conditions, is a Knuth-Bendix completion of the 2-polygraph $\text{Gar}_2(S)$, we need to ensure two things: a way to maintain a terminating presentation in the sense of Remark 2.2.1.2, and a demonstration that all new critical branchings caused by the generating 2-cells β are confluent. These are respectively given by Proposition 2.5.2.1, and the proof of Proposition 2.5.3.1.

PROPOSITION 2.5.2.1. *Let M be a left-cancellative monoid containing no non-trivial invertible element, and let S be a Garside family containing 1 in M . If S is right-noetherian, then the 2-polygraph $\underline{\text{Gar}}_2(S)$ is terminating.*

PROOF. Let us first adopt some notation, to facilitate exposition. For an S -word x , we write $h(x)$ for the leftmost letter of x , and $t(x)$ for the word obtained from x by removing the letter $h(x)$. For a generating 2-cell χ , a χ -step is a rewriting step in which the generating 2-cell is χ , and χ_i is a χ -step

$$\bullet \xrightarrow{z} \bullet \begin{array}{c} \xrightarrow{x} \\ \Downarrow \chi \\ \xrightarrow{y} \end{array} \bullet \xrightarrow{z'} \bullet ,$$

where z is an S -word of length $i - 1$. For an infinite sequence of positive integers $i_1|i_2|\dots$, we denote the path $\dots \circ \chi_{i_2} \circ \chi_{i_1}$ by $\chi_{i_1|i_2|\dots}$.

Having introduced the needed notation, we proceed to prove the proposition. Suppose that there is an infinite rewriting path. Note that an α -step strictly reduces the $(S \setminus \{1\})$ -length of a word. So, in every rewriting path, there can be only finitely many of the generating 2-cells α . Hence, there is no loss in generality if we consider only β -steps. Namely, we can simply consider an infinite path after the last α -step is applied and we are left with an infinite path composed out of β -steps only. Suppose, then, that there is an infinite rewriting path consisting of β -steps. Let $\beta_{i_1|i_2|\dots}$ be such a path having source x of minimal $(S \setminus \{1\})$ -length. Note that $|x|$, the length of x , is at least 2.

The minimality assumption about $|x|$ implies that the number 1 occurs infinitely many times in the sequence $i_1|i_2|\cdots$. In other words, infinitely many generating 2-cells β of the path modify letters at the leftmost position. Indeed, if the number 1 occurred only finitely many times in $i_1|i_2|\cdots$, then, denoting the last occurrence of 1 by i_q , we would have an infinite path of β -steps $\beta_{i_{q+1}-1|i_{q+2}-1|\cdots}$, starting at the word $t(\beta_{i_1|i_2|\dots|i_q}(x))$, whose $(S \setminus \{1\})$ -length would equal $|x| - 1$. But that would contradict the minimality assumption about $|x|$. Therefore, the number 1 occurs infinitely many times in the sequence $i_1|i_2|\cdots$.

We write $i_{c_1}|i_{c_2}|\cdots$ for the constant subsequence of the sequence $i_1|i_2|\cdots$ taking all the members whose value equals 1. To be specific, denote by c_1 the least d such that $i_d = 1$; and, for all p in \mathbb{N} , denote by c_{p+1} the least d such that the conditions $d > c_p$ and $i_d = 1$ hold.

Let $x^{(m)}$ denote the m th word in the path $\beta_{i_1|i_2|\dots}$, i.e. the source of the step β_{i_m} . Notice that the leftmost letter of a word is modified by the step β_{i_m} if, and only if, i_m equals 1. In this case, the modification is such that the current leftmost letter $h(x^{(m)})$ is a proper left divisor of the next leftmost letter $h(x^{(m+1)})$, and the corresponding complement lies in $S \setminus \{1\}$ by the definition of the generating 2-cell β . In formal terms,

$$(2.5.1) \quad h(x^{(m+1)}) = \begin{cases} h(x^{(m)}) & \text{if } i_m \neq 1, \\ h(x^{(m)})u_m \text{ for some } u_m \in S \setminus \{1\} & \text{if } i_m = 1. \end{cases}$$

Let r denote the leftmost letter of the S -normal form of x . Notice that all the words in the path $\beta_{i_1|i_2|\dots}$ have the same evaluation in M and that, consequently, the equality $N^S(x) = N^S(x^{(m)})$ holds for all m by the definition of N^S . Hence, Lemma 2.4.4.2 implies that $h(x^{(m)})$ is a left divisor of r for all m .

Consider the sequence

$$(2.5.2) \quad \left(h(x^{(c_p)}) \right)_{p=1}^{\infty}$$

of elements of S that divide r . Observe that, by (2.5.1), we have $h(x^{(c_{p+1})}) = h(x^{(c_p)})u_{c_p}$. Since u_{c_p} lies in $S \setminus \{1\}$ and there is no nontrivial invertible element, $h(x^{(c_p)})$ is a proper left divisor of $h(x^{(c_{p+1})})$. Therefore, the existence of the sequence (2.5.2) contradicts the fact that the Garside family S is right-noetherian. We conclude that the 2-polygraph $\underline{\text{Gar}}_2(S)$ is terminating. \square

2.5.3. Homotopical completion of Garside's presentation. In this subsection, we enrich Garside's presentation to reach a coherent convergent presentation. First we compute purely formally, in Proposition 2.5.3.1, a homotopical completion of a terminating presentation of a monoid satisfying certain conditions, but not presumed to have a proper Garside family. Then we show, in Corollary 2.5.3.5, that this provides a coherent convergent presentation of a left-cancellative monoid containing no nontrivial invertible element, admitting right-mcms and a right-noetherian Garside family containing 1.

PROPOSITION 2.5.3.1. *Let M be a left-cancellative monoid admitting right-mcms, and let S be a subfamily of M . Assume that S is closed under right-mcm and right divisor, and assume that the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a terminating presentation of M . Then M admits, as a coherent convergent presentation, the $(3, 1)$ -polygraph $\underline{\text{Gar}}_3(S)$ which extends $\underline{\text{Gar}}_2(S)$ with the following twelve families of*

generating 3-cells, indexed by all possible elements of $S \setminus \{1\}$ (i.e. for which all the generating 2-cells involved are defined):

$$\begin{array}{ccc}
\begin{array}{ccc}
\alpha_{u,v}|w & \xrightarrow{\quad} & uv|w \\
u|v|w & \xrightarrow{A_{u,v,w}} & uvw \\
u|\alpha_{v,w} & \xrightarrow{\quad} & u|vw \\
& & \alpha_{u,vw}
\end{array}
& &
\begin{array}{ccc}
& \xrightarrow{\alpha_{u,v}|w} & \\
u|v|w & \xrightarrow{B_{u,v,w}} & uv|w \\
& \xrightarrow{\beta_{u,v,w}} & \\
u|\alpha_{v,w} & \xrightarrow{\quad} & u|vw
\end{array}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{u,v}|wx & \xrightarrow{\quad} & uv|wx \\
u|v|wx & \xrightarrow{C_{u,v,w,x}} & uvw|x \\
u|\beta_{v,w,x} & \xrightarrow{\quad} & u|vw|x \\
& & \alpha_{u,vw}|x
\end{array}
\quad
\begin{array}{ccc}
& \xrightarrow{\alpha_{u,v}|wx} & \\
u|v|wx & \xrightarrow{D_{u,v,w,x}} & uv|wx \\
& \xrightarrow{\beta_{u,v,w}|x} & \\
u|\beta_{v,w,x} & \xrightarrow{\beta_{u,v,w}|x} & uv|wx \\
& & \alpha_{u,vw}|x
\end{array}$$

$$\begin{array}{ccc}
\beta_{u,v,w}|x & \xrightarrow{\quad} & uv|w|x \\
u|vw|x & \xrightarrow{E_{u,v,w,x}} & uv|wx \\
u|\alpha_{vw,x} & \xrightarrow{\quad} & u|vw|x \\
& & \beta_{u,v,w|x}
\end{array}
\quad
\begin{array}{ccc}
\beta_{u,v,w}|x & \xrightarrow{\quad} & uv|w|x \\
u|vw|x & \xrightarrow{E'_{u,v,w,x}} & uv|wx \\
u|\alpha_{vw,x} & \xrightarrow{\quad} & u|vw|x \\
& & \alpha_{uv,w|x} \\
& & \alpha_{u,vwx}
\end{array}$$

$$\begin{array}{ccc}
& \xrightarrow{\beta_{u,v,w}|xy} & uv|w|xy \\
u|vw|xy & \xrightarrow{F_{u,v,w,x,y}} & uv|wx|y \\
u|\beta_{vw,x,y} & \xrightarrow{\quad} & u|vw|x|y \\
& & \beta_{u,v,wx}|y \\
& & \alpha_{uv,wx}|y
\end{array}$$

$$\begin{array}{ccc}
\beta_{u,v,w}|xy & \xrightarrow{\quad} & uv|w|xy \\
u|vw|xy & \xrightarrow{F'_{u,v,w,x,y}} & uv|wx|y \\
u|\beta_{vw,x,y} & \xrightarrow{\quad} & u|vw|x|y \\
& & \alpha_{u,vwx}|y
\end{array}$$

$$\begin{array}{ccc}
\beta_{u,v,w}|xy & \xrightarrow{\quad} & uv|w|xy \\
u|vw|xy & \xrightarrow{G_{u,v,w,x,y}} & uv|wx|y \\
u|\beta_{vw,x,y} & \xrightarrow{\quad} & u|vw|x|y \\
& & \beta_{u,v,wx}|y \\
& & \alpha_{uv,wx}|y
\end{array}
\quad
\begin{array}{ccc}
\beta_{u,v,w}|xy & \xrightarrow{\quad} & uv|w|xy \\
u|vw|xy & \xrightarrow{G'_{u,v,w,x,y}} & uv|wx|y \\
u|\beta_{vw,x,y} & \xrightarrow{\quad} & u|vw|x|y \\
& & \alpha_{u,vwx}|y
\end{array}$$

$$\begin{array}{ccc}
\beta_{u,v,xy} & \xrightarrow{\quad} & uv|xy \\
u|vxy & \xrightarrow{H_{u,v,x,y}} & uvx|y \\
& & \beta_{u,vx,y}
\end{array}$$

$$\begin{array}{ccc}
\beta_{u,v_1,w_1} & \xrightarrow{\quad} & uv_1|w_1 = uv_1|x_1y \\
u|v_1w_1 & \xrightarrow{I_{u,v_1,w_1,v_2,w_2}} & uv_1x_1|y \\
= & & = \\
u|v_2w_2 & \xrightarrow{\quad} & uv_2|x_2y \\
\beta_{u,v_2,w_2} & \xrightarrow{\quad} & uv_2|w_2 = uv_2|x_2y \\
& & \beta_{uv_2,x_2,y}
\end{array}$$

The meanings of the 1-cells (i.e. words) x_1 , x_2 , y and x , y which appear respectively in the definitions of the generating 3-cells I and H , are as follows. Since v_1 and v_2 have the common right multiple $v_1w_1 = v_2w_2$, they also have a right-mcm. The

words x_1 and x_2 are the right complements of v_1 and v_2 , respectively, in their right-mcm. The word y is the right complement of $v_1x_1 = v_2x_2$ in $v_1w_1 = v_2w_2$. If either x_1 or x_2 is equal to 1, then the other one is simply denoted by x (in the generating 3-cell H).

The structure of the following proof closely resembles that of the proof of [26, Proposition 3.2.1], but we need to devise more general arguments to assure favourable properties in a more general context.

PROOF. Termination of the 2-polygraph $\text{Gar}_2(S)$ is assumed, so we can perform a relaxed version of the Knuth-Bendix completion procedure, as described in Remark 2.2.1.2, simultaneously with the Squier completion procedure. It will turn out that all critical branchings are confluent. Accordingly, only a Squier completion will be actually computed, i.e. no further 2-generating cells will be added.

Let us first consider critical branchings consisting only of the generating 2-cells α . There is only one such critical branching for each triple (u, v, w) of elements of $S \setminus \{1\}$ satisfying $u \widehat{v} \widehat{w}$:

$$\begin{array}{ccc} & & uv|w \\ & \nearrow^{\alpha_{u,v}|w} & \\ u|v|w & & \\ & \searrow_{u|\alpha_{v,w}} & \\ & & u|vw \end{array} .$$

If the subcase $u \widehat{v} \widehat{w}$ holds, then the branching is already confluent, so the homotopical completion procedure adjoins only the generating 3-cell $A_{u,v,w}$. If $u \widehat{v}^x \widehat{w}$ holds, then the branching is again confluent, so the generating 3-cell $B_{u,v,w}$ is adjoined.

Let us now consider critical branchings containing the generating 2-cell β . The sources of 2-cells forming such a branching can either overlap on one element of $S \setminus \{1\}$ or be equal, as the lengths in $(S \setminus \{1\})^*$ of the sources of the generating 2-cells α and β equal 2. We consider the two cases accordingly.

For the first case (when the overlap is of length 1, i.e. the source of branching is of length 3), the proof of [26, Proposition 3.2.1] applies here to a certain extent. The source of a branching has length 3, as a word in $(S \setminus \{1\})^*$. One of the 2-cells which form a branching, changes the leftmost two generating 1-cells of the source, and the other one changes the rightmost two. There are three distinct forms of such branchings, which we discuss in the following three paragraphs.

The form

$$\begin{array}{ccc} & & uv|wx \\ & \nearrow^{\alpha_{u,v}|wx} & \\ u|v|wx & & \\ & \searrow_{u|\beta_{v,w,x}} & \\ & & u|vw|x \end{array}$$

is defined under the condition $u \frown v \overset{x}{\frown} w \frown x$, which splits into two mutually exclusive possibilities

$$u \frown v \overset{x}{\frown} w \frown x \quad \text{and} \quad u \overset{x}{\frown} v \overset{x}{\frown} w \frown x$$

yielding respectively the generating 3-cells $C_{u,v,w,x}$ and $D_{u,v,w,x}$ by the homotopical completion procedure.

The branching

$$\begin{array}{ccc} & & uv|w|x \\ & \nearrow \beta_{u,v,w|x} & \\ u|vw|x & & \\ & \searrow u|\alpha_{vw,x} & \\ & & u|vwx \end{array}$$

is defined when $u \overset{x}{\frown} v \frown w \frown x$. This conjunction splits into two disjoint possibilities

$$u \overset{x}{\frown} v \frown w \frown x \quad \text{and} \quad u \overset{x}{\frown} v \overset{x}{\frown} w \frown x$$

producing respectively the generating 3-cells $E_{u,v,w,x}$ and $E'_{u,v,w,x}$.

The form

$$\begin{array}{ccc} & & uv|w|xy \\ & \nearrow \beta_{u,v,w|xy} & \\ u|vw|xy & & \\ & \searrow u|\beta_{vw,x,y} & \\ & & u|vwx|y \end{array}$$

is defined under the condition $u \overset{x}{\frown} v \overset{x}{\frown} w \overset{x}{\frown} x \frown y$. This conjunction splits into two mutually exclusive possibilities

$$u \overset{x}{\frown} v \overset{x}{\frown} w \overset{x}{\frown} x \frown y \quad \text{and} \quad u \overset{x}{\frown} v \overset{x}{\frown} w \overset{x}{\frown} x \overset{x}{\frown} y$$

The former possibility further splits into

$$u \overset{x}{\frown} v \overset{x}{\frown} w \overset{x}{\frown} x \overset{x}{\frown} y \quad \text{and} \quad u \overset{x}{\frown} v \overset{x}{\frown} w \overset{x}{\frown} x \overset{x}{\frown} y$$

yielding the generating 3-cells $F_{u,v,w,x,y}$ and $F'_{u,v,w,x,y}$, respectively; the latter splits into

$$u \overset{x}{\frown} v \overset{x}{\frown} w \overset{x}{\frown} x \overset{x}{\frown} y \quad \text{and} \quad u \overset{x}{\frown} v \overset{x}{\frown} w \overset{x}{\frown} x \overset{x}{\frown} y$$

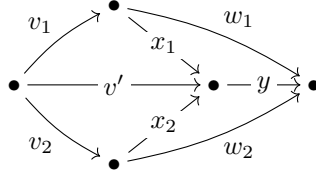
yielding the generating 3-cells $G_{u,v,w,x,y}$ and $G'_{u,v,w,x,y}$, respectively.

We have thus considered the first case. The second case (when the source of branching is of length 2), is going to be considered in greater detail because this is where new justifications are needed. Assume that the two generating 2-cells which generate a critical branching have the same source. One of these two generating 2-cells has to be β (otherwise, the branching is trivial). Therefore, the source has to

have a form $u|v_1w_1$ satisfying the condition $u\widehat{v_1^x}w_1$. Since the generating 2-cells α are not defined under this condition, the other generating 2-cell also has to be β . The only way for the generating 2-cells β with the same source $u|v_1w_1$ to form a critical branching is for v_1w_1 to have another decomposition $v_1w_1 = v_2w_2$ such that $u\widehat{v_2^x}w_2$. Then the branching is as follows:

$$\begin{array}{ccc}
 & & uv_1|w_1 \\
 & \nearrow \beta_{u,v_1,w_1} & \\
 u|v_1w_1 & & \\
 = & & \\
 u|v_2w_2 & & \\
 & \searrow \beta_{u,v_2,w_2} & \\
 & & uv_2|w_2
 \end{array}$$

Let us invoke the assumed property of M admitting right-mcms. Since v_1 and v_2 have a common right multiple, namely $v_1w_1 = v_2w_2$, they also have a right-mcm, say v' . Since S is closed under right-mcm by assumption, v' lies in S . By the left cancellation property which grants the uniqueness of right complements, we define x_1 and x_2 as the right complements in v' of v_1 and v_2 , respectively. Since S is closed under right divisor, x_1 and x_2 are elements of S . We also define y as the right complement of v' in $v_1w_1 = v_2w_2$. Note that y lies in S as a right divisor of v_1w_1 which lies in S . Uniqueness of the right complements of v_1 and v_2 in v_1w_1 and v_2w_2 , respectively, yields $w_1 = x_1y$ and $w_2 = x_2y$. To sum up, the diagram



commutes (with \bullet denoting the unique 0-cell). Furthermore, the equality $w_k = x_k y$, the fact that v' lies in S , and the condition $v_k\widehat{w_k}$ together imply $v_k\widehat{x_k}y$ for k in $\{1, 2\}$.

We have only shown that x_1 , x_2 and y are elements of S . In order to construct the generating 3-cell I_{u,v_1,w_1,v_2,w_2} , let us verify that all the generating 1-cells involved are, indeed, elements of $S \setminus \{1\}$. First we demonstrate that y cannot be equal to 1. Suppose the opposite. Then the condition $u\widehat{v_1^x}w_1$ reduces to $u\widehat{v_1^x}x_1$. On the other hand, uv' is a right-mcm of uv_1 and uv_2 by Lemma 2.4.2.1. Since S is closed under right-mcm, uv' lies in S , which contradicts the condition $u\widehat{v_1^x}x_1$. Thus, we deduce that y is not equal to 1.

Note that if x_1 and x_2 were both equal to 1, then the branching $\{\beta_{u,v_1,w_1}, \beta_{u,v_2,w_2}\}$ would be trivial. So, at most one of the 1-cells x_1 and x_2 can be equal to 1. If x_2 is equal to 1, then the generating 3-cell $H_{u,v,x,y}$ is constructed, with $v := v_1$ and $x := x_1$. Similarly, if x_1 is equal to 1, the generating 3-cell $H_{u,v,x,y}$ is constructed, with $v := v_2$ and $x := x_2$. Finally, if neither x_1 nor x_2 is equal to 1, then the generating 3-cell I_{u,v_1,w_1,v_2,w_2} is adjoined.

By Theorem 2.2.2.2, the constructed $(3, 1)$ -polygraph $\underline{\text{Gar}}_3(S)$ is a coherent convergent presentation of M . \square

The rest of this subsection is made up of remarks and corollaries of Proposition 2.5.3.1.

REMARK 2.5.3.2. Notice that Proposition 2.5.3.1 gives three new families of generating 3-cells (namely, E' , F' and G') that were not a part of the [26, Proposition 3.2.1], an analogous result for Artin-Tits monoids. The reason for this is that the Garside families considered in [26] for Artin-Tits monoids and Garside monoids are closed under left and right divisors, whereas a family S in Proposition 2.5.3.1 is only closed under right divisor (like a Garside family in general). Consequently, certain conjunctions of conditions, discussed in the proof of Proposition 2.5.3.1, could not be satisfied in the setting of Artin-Tits monoids. For instance, here we consider the possibility of uvw lying in S under the condition that uvw is not an element of S , to construct the generating 3-cell E' . In an Artin-Tits monoid, on the other hand, if uvw lies in S then so does uvw due to closure under left divisor.

COROLLARY 2.5.3.3. *Let M be a left-cancellative monoid containing no non-trivial invertible element, and let S be a Garside family containing 1 in M . If S is right-noetherian, then the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a convergent presentation of M .*

PROOF. Proposition 2.4.4.4 grants that the 2-polygraph $\text{Gar}_2(S)$ is a presentation of M . Since the generating 2-cells α strictly decrease the S -length, the 2-polygraph $\text{Gar}_2(S)$ is terminating. Thanks to Proposition 2.5.2.1, we can compute its Knuth-Bendix completion in a manner described in Remark 2.2.1.2. As shown in Subsection 2.5.2, the generating 2-cells β are added.

Note that Proposition 2.4.3.5 and Lemma 2.4.3.6, together with the assumptions that S contains 1 and that there is no nontrivial invertible element in M , yield the property of S being closed under right-mcm. By Proposition 2.4.3.5, we know that S is closed under right divisor. With all these conditions satisfied, the proof of Proposition 2.5.3.1 applies in a straightforward fashion. In particular, it shows that all new critical branchings caused by the generating 2-cells β are confluent. Thus, the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a Knuth-Bendix completion of Garside's presentation $\text{Gar}_2(S)$, which yields the desired conclusion by Theorem 2.2.1.1 and Remark 2.2.1.2. \square

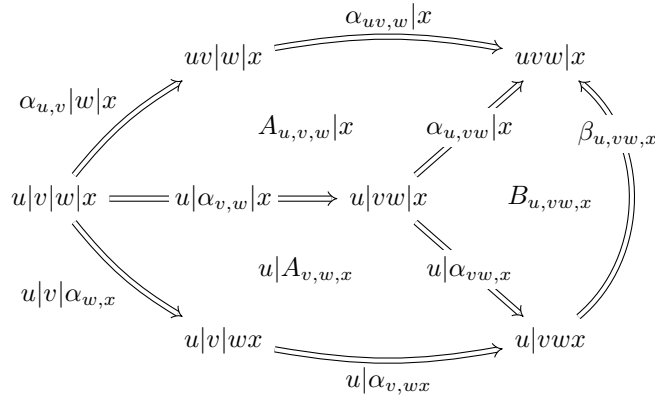
REMARK 2.5.3.4. Observe that Proposition 2.5.2.1, together with Proposition 2.4.4.4, immediately implies that the 2-polygraph $\underline{\text{Gar}}_2(S)$ is a terminating presentation of M . On the other hand, the fact that $\underline{\text{Gar}}_2(S)$ is also a convergent presentation of M has been reachable only after Proposition 2.5.3.1, which showed that no additional generating 2-cells were required to obtain confluence.

COROLLARY 2.5.3.5. *Let M be a left-cancellative monoid containing no non-trivial invertible element, and let S be a Garside family containing 1 in M . If S is right-noetherian, and if M admits right-mcms, then M admits the $(3, 1)$ -polygraph $\underline{\text{Gar}}_3(S)$ as a coherent convergent presentation.*

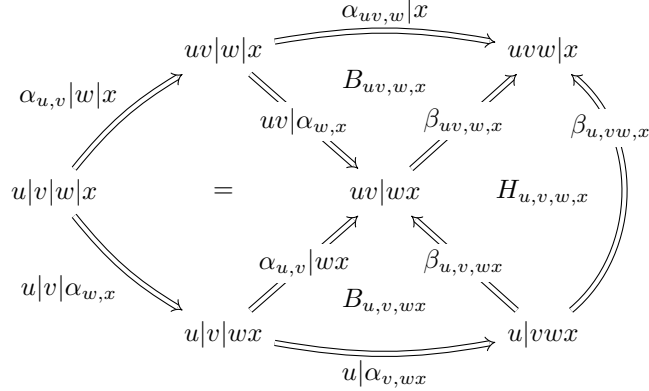
PROOF. Corollary 2.5.3.3 grants that $\underline{\text{Gar}}_2(S)$ is a terminating presentation of M . As shown in the proof of Corollary 2.5.3.3, all the requirements are met for applying Proposition 2.5.3.1, which completes the proof. \square

2.5.4. Homotopical reduction of Garside's presentation. The homotopical reduction procedure described in [26, 3.2.2] applies verbatim to the coherent convergent presentation provided by Proposition 2.5.3.1 (and echoed by Corollary 2.5.3.5), with respect to the collapsible part Γ which is obtained as follows. The component Γ_4 of Γ contains seven generating triple confluences whose targets are the families C, \dots, I of generating 3-cells, with the order $I > H > \dots > C$. For the sake of illustration, we recall one such generating triple confluence in the case

$u \begin{array}{c} \diagup \quad \diagdown \\ v \quad w \\ \diagdown \quad \diagup \\ x \end{array}$ (we refer the reader to [26, 3.2.2] for the other six generating triple confluences). Its boundary consists of the following two parts:



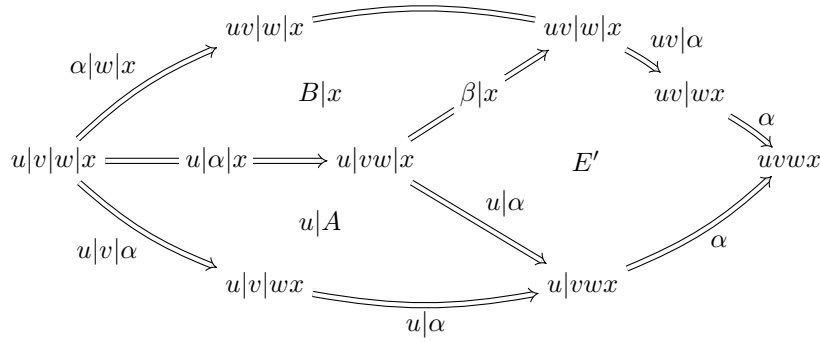
and



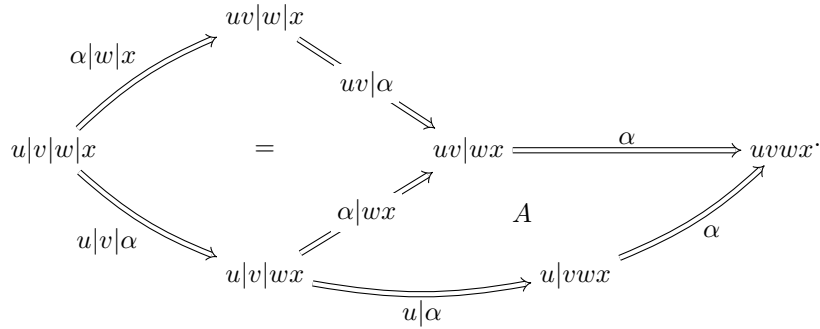
The target of this particular generating triple confluence is the generating 3-cell $H_{u,v,w,x}$.

Note, however, that applying the homotopical reduction procedure from [26, 3.2.2] does not suffice to eliminate any of the generating 3-cells $E'_{u,v,w,x}$, $F'_{u,v,w,x,y}$ and $G'_{u,v,w,x,y}$ since these particular families of generating 3-cells did not even occur in [26, Section 3], as observed in Remark 2.5.3.2. So, we have yet to eliminate these cells here. To this end, we consider the following generating triple confluences in the $(3, 1)$ -polygraph $\text{Gar}_3(S)$.

The boundary of our first 3-sphere of interest consists of

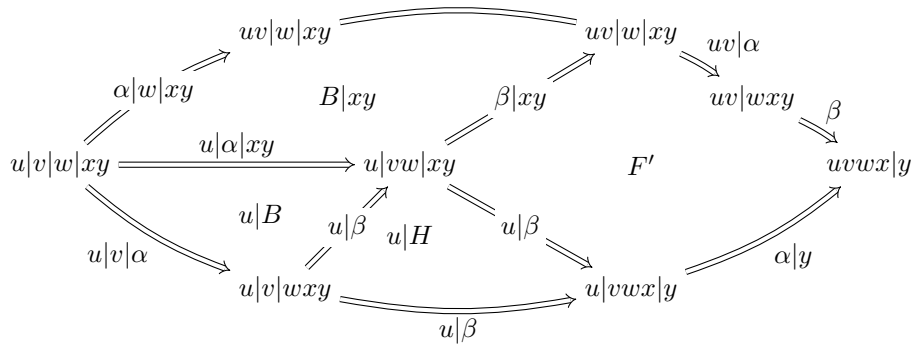


and



The target is the generating 3-cell $E'_{u,v,w,x}$.

Second generating triple confluence which we are going to use has the boundary consisting of



2.5.5. Noetherianity. Let us state an immediate corollary of Theorem 2.5.1.4, having somewhat simpler, although more restrictive, requirements.

COROLLARY 2.5.5.1. *Let M be a left-cancellative monoid containing no non-trivial invertible element, and let S be a Garside family containing 1 in M . If M is noetherian, then M admits the $(3, 1)$ -polygraph $\text{Gar}_3(S)$ as a coherent presentation.*

PROOF. Since M is right-noetherian, so is S . By [13, Proposition II.2.40], every left-cancellative left-noetherian monoid admits right-mcms, so M admits right-mcms. Hence, all the conditions of Theorem 2.5.1.4 are satisfied. \square

The next section demonstrates advantages of using our results in applications. The following example, however, shows that taking a Garside family as a generating set is not always the most practical way to get a coherent presentation.

EXAMPLE 2.5.5.2. We revisit the Klein bottle monoid K^+ , considered in Examples 2.2.2.3 and 2.2.5.1. One of the infinitely many Garside families in K^+ , none of which is finite (see [13, Example IV.2.35]), is the set of left divisors of a^2 , which we denote by S . Let us check if the conditions of Theorem 2.5.1.4 are satisfied. Note that K^+ is cancellative as it is embeddable in a group. The presentation (2.2.1) contains no ε -relation, hence K^+ has no nontrivial invertible element. Note that the left divisibility relation of K^+ is a total order ([13, Figure I.6]), which is more than necessary for admitting conditional right-lcms (and *a fortiori* right-mcms). However, the sequence $(ab^n)_{n=1}^\infty$ shows that S is not right-noetherian. Even worse, S contains an infinite path of the generating 2-cells β , as defined in Proposition 2.5.2.1:

$$b|a^2 \mapsto b^2|aba \mapsto b^3|ab^2a \mapsto \dots \mapsto b^n|ab^{n-1}a \mapsto \dots$$

Even if we took another Garside family, we would not be successful, as witnessed by [13, Example IV.2.35]. Therefore, neither Theorem 2.5.1.4 nor its proof is applicable to K^+ .

If one found a way to use a Garside family as a generating set, then they would have an infinite number of generating 1-cells. On the other hand, by directly performing the homotopical completion-reduction procedure in Examples 2.2.2.3 and 2.2.5.1, we have demonstrated that the presentation (2.2.1), which has two generating 1-cells and one generating 2-cell, is coherent. Therefore, for this particular example, the direct application of the homotopical completion-reduction procedure is a preferable way of reaching a coherent presentation.

2.6. Applications of Theorem 2.5.1.4

In this section, we consider applications of Theorem 2.5.1.4 to certain monoids. In Subsections 2.6.1 and 2.6.2, we apply the theorem to monoids which are neither Artin-Tits nor Garside. In Subsection 2.6.3, we compute a finite coherent presentation of an Artin-Tits monoid that is not of spherical type, taking a finite Garside family (hence, different from the corresponding Coxeter group) as a generating set. Finally, we construct a coherent presentation of a dual braid monoid in Subsection 2.6.4.

2.6.1. The free abelian monoid over an arbitrary basis. Consider the free abelian monoid $\mathbb{N}^{(J)}$ of all J -indexed sequences of nonnegative integers with

finite support. Define

$$S_J = \left\{ g \in \mathbb{N}^{(J)} \mid \forall j \in J, g(j) \in \{0, 1\} \right\}.$$

If $\mathbb{N}^{(J)}$ is finitely generated by S_J , that is for $J = \{1, 2, \dots, n\}$, then the set S_J coincides with the set of divisors of the Garside element $\Delta_n = (1, \dots, 1)$. Otherwise, the monoid $\mathbb{N}^{(J)}$ is neither Artin-Tits nor Garside. Nevertheless, by [13, Subsection I.3.1], the Δ -normal decomposition results established for the free abelian monoid \mathbb{N}^n (which is a Garside monoid and a spherical Artin-Tits monoid) remain valid after replacing the set of divisors of Δ with S_J . Namely, every element of $\mathbb{N}^{(J)}$ admits a unique decomposition of the form $u_1 | \dots | u_m$ with u_1, \dots, u_m in S_J and $u_m \neq 1$ such that

$$\text{for all } h \in \mathbb{N}^{(J)} \setminus \{1\}, \quad h \preceq u_{i+1} \text{ implies } u_i h \notin S_J.$$

Observe that S_J is a Garside family in $\mathbb{N}^{(J)}$ (say, by applying Proposition 2.4.3.4, or by recognising that the above implication defines greediness). The following properties follow from the fact that the definition of the product in $\mathbb{N}^{(J)}$ is based on the pointwise addition of nonnegative integers: $\mathbb{N}^{(J)}$ is a cancellative monoid, it has no nontrivial invertible element, and it admits conditional right-lcms. Since every element of $\mathbb{N}^{(J)}$ has only finitely many divisors, $\mathbb{N}^{(J)}$ is noetherian. So, all the conditions of Theorem 2.5.1.4 are satisfied.

Let us describe the cells of the coherent presentation of $\mathbb{N}^{(J)}$ granted by Theorem 2.5.1.4. The generating 2-cells are relations $\alpha_{u,v} : u|v \Rightarrow uv$ for u, v in $S_J \setminus \{1\}$ such that uv lies in S , which in this particular context means that u and v have disjoint supports. A generating 3-cell $A_{u,v,w}$ is adjoined for each triple (u, v, w) of elements of $S_J \setminus \{1\}$ having pairwise disjoint supports.

As expected, for $J = \{1, 2, \dots, n\}$, we recover Garside's coherent presentation of the Artin-Tits monoid \mathbb{N}^n recalled in Example 2.3.3.2, as well as Garside's coherent presentation of the Garside monoid \mathbb{N}^n recalled in Example 2.3.6.1.

2.6.2. Infinite braids. Denote by B_∞^+ the monoid of all positive braids on infinitely many strands indexed by positive integers, as defined in [13, Subsection I.3.1]. It is shown that B_∞^+ does not admit a finite generating set. Therefore, it is neither Artin-Tits nor Garside. However, its elements do admit decompositions similar to the Δ -normal decomposition. Namely, put

$$S_\infty = \bigcup_{n \geq 1} \{ \text{the family of all divisors of } \Delta_n \}$$

where Δ_n denotes the half-turn braid on n strands. In other words, S_∞ consists of all simple braids on n strands for all $n \geq 1$. This is made precise in [13, Subsection I.3.1]. Basically, B_n^+ is identified with its image in B_{n+1}^+ under the homomorphism induced by the identity map on $\{\sigma_1, \dots, \sigma_n\}$. In that sense, B_∞^+ is seen as the union of all braid monoids B_n^+ . Every element of B_∞^+ admits a unique decomposition of the form $u_1 | \dots | u_m$ with u_1, \dots, u_m in S_∞ and $u_m \neq 1$ satisfying

$$\text{for all } h \in B_\infty^+ \setminus \{1\}, \quad h \preceq u_{i+1} \text{ implies } u_i h \notin S_\infty.$$

Observe that the family S_∞ is a Garside family in B_∞^+ (like in the previous subsection). Cancellation and divisibility properties, as well as the property of having no nontrivial invertible element, are preserved from braid monoids because the respective properties do not depend on the number of strands. The monoid is

noetherian for the same reason as Artin-Tits monoids (Example 2.4.2.2). So, we can apply Theorem 2.5.1.4 to construct a coherent presentation.

The generating 2-cells are relations $\alpha_{u,v} : u|v \Rightarrow uv$ for u, v in $S_\infty \setminus \{1\}$ such that uv lies in S_∞ , which in this example means that uv is a simple braid. A generating 3-cell $A_{u,v,w}$ is adjoined for each triple (u, v, w) of elements of $S_\infty \setminus \{1\}$ with uv, vw, uvw lying in S_∞ , which here means that uv, vw and uvw are simple braids. So, formally, each cell is constructed exactly like in the coherent presentation provided by [26, Theorem 3.1.3] for a (finite) braid monoid, regarded as an Artin-Tits monoid, which comes as no surprise since Theorem 2.5.1.4 is a formal generalisation of [26, Theorem 3.1.3].

REMARK 2.6.2.1. Observe the similarities between the free abelian monoid over an infinite basis and the monoid of all positive braids on infinitely many strands. Even though the monoids involved are quite different (for starters, the former is abelian unlike the latter), there is a strong analogy between the normal decompositions their elements admit: \mathbb{N}^n , $\mathbb{N}^{(J)}$, the set of divisors of Δ_n , and S_J behave like B_n^+ , B_∞^+ , simple braids, and S_∞ , respectively. Such observations invited the notion of Garside family, as hinted in Subsection 2.4.1.

2.6.3. Artin-Tits monoids that are not of spherical type. For an Artin-Tits monoid $B^+(W)$ of spherical type, [26, Theorem 3.1.3], recalled in Subsection 2.3.3, provides a finite coherent presentation having $W \setminus \{1\}$ as a generating set. On the other hand, if a Coxeter group W is infinite, then [26, Theorem 3.1.3] still provides a coherent presentation but an infinite one. Recall that every Artin-Tits monoid admits a finite Garside family (we refer the reader to [15] for elaboration), regardless of whether the monoid is of spherical type or not. An advantage of having Theorem 2.5.1.4 at our disposal is that we can take a finite Garside family for a generating set in computing a coherent presentation (whereas with [26, Theorem 3.1.3], one has to take the corresponding Coxeter group even if it is infinite).

Let us consider the Artin-Tits monoid of type \tilde{A}_2 , i.e. the monoid presented by

$$(2.6.1) \quad \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_3\sigma_1\sigma_3 = \sigma_1\sigma_3\sigma_1 \rangle^+.$$

By [15, Table 1 and Proposition 5.1], the smallest Garside family F in this monoid consists of the sixteen right divisors of the elements $\sigma_3\sigma_1\sigma_2\sigma_1$, $\sigma_1\sigma_2\sigma_3\sigma_2$, and $\sigma_2\sigma_3\sigma_1\sigma_3$. Namely,

$$F = \{1, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_3, \sigma_3\sigma_2, \sigma_3\sigma_1, \sigma_1\sigma_3, \\ \sigma_1\sigma_2\sigma_1, \sigma_2\sigma_3\sigma_2, \sigma_3\sigma_1\sigma_3, \sigma_3\sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_3\sigma_2, \sigma_2\sigma_3\sigma_1\sigma_3\}.$$

The Cayley graph of F can be seen in [15, Figure 1].

As observed in Corollary 2.5.1.6, all the conditions of Theorem 2.5.1.4 are satisfied. Following Theorem 2.5.1.4, we construct a generating 2-cell $u|v \Rightarrow uv$ for u, v in $F \setminus \{1\}$ with uv in F . This yields the following family of generating 2-cells: three pairs of generating 2-cells of the form

$$\alpha_{\sigma_i, \sigma_j} : \sigma_i|\sigma_j \Rightarrow \sigma_i\sigma_j, \quad \alpha_{\sigma_j, \sigma_i} : \sigma_j|\sigma_i \Rightarrow \sigma_j\sigma_i,$$

three pairs of generating 2-cells of the form

$$\alpha_{\sigma_i, \sigma_j\sigma_i} : \sigma_i|\sigma_j\sigma_i \Rightarrow \sigma_i\sigma_j\sigma_i, \quad \alpha_{\sigma_j, \sigma_i\sigma_j} : \sigma_j|\sigma_i\sigma_j \Rightarrow \sigma_i\sigma_j\sigma_i,$$

three pairs of generating 2-cells of the form

$$\alpha_{\sigma_i\sigma_j,\sigma_i} : \sigma_i\sigma_j|\sigma_i \Rightarrow \sigma_i\sigma_j\sigma_i, \quad \alpha_{\sigma_j\sigma_i,\sigma_j} : \sigma_j\sigma_i|\sigma_j \Rightarrow \sigma_i\sigma_j\sigma_i,$$

three generating 2-cells of the form

$$\alpha_{\sigma_k,\sigma_i\sigma_j\sigma_i} : \sigma_k|\sigma_i\sigma_j\sigma_i \Rightarrow \sigma_k\sigma_i\sigma_j\sigma_i,$$

and three pairs of generating 2-cells of the form

$$\alpha_{\sigma_k\sigma_i,\sigma_j\sigma_i} : \sigma_k\sigma_i|\sigma_j\sigma_i \Rightarrow \sigma_k\sigma_i\sigma_j\sigma_i, \quad \alpha_{\sigma_k\sigma_j,\sigma_i\sigma_j} : \sigma_k\sigma_j|\sigma_i\sigma_j \Rightarrow \sigma_k\sigma_i\sigma_j\sigma_i,$$

with i, j, k in $\{1, 2, 3\}$, and $j = i + 1$ and $k = j + 1$ modulo 3.

We proceed to construct the generating 3-cells $A_{u,v,w}$ for u, v, w in $F \setminus \{1\}$ with uv, vw, uvw lying in F . We obtain pairs of generating 3-cells of the form

$$\begin{array}{ccc} \alpha_{\sigma_i,\sigma_j}|\sigma_i & \xrightarrow{\sigma_i\sigma_j|\sigma_i} & \sigma_i\sigma_j|\sigma_i \\ \sigma_i|\sigma_j|\sigma_i & \xrightarrow{A_{\sigma_i,\sigma_j,\sigma_i}} & \sigma_i\sigma_j\sigma_i \\ \sigma_i|\alpha_{\sigma_j,\sigma_i} & \xrightarrow{\sigma_i|\sigma_j\sigma_i} & \sigma_i|\sigma_j\sigma_i \end{array} \quad \begin{array}{ccc} \alpha_{\sigma_j,\sigma_i}|\sigma_j & \xrightarrow{\sigma_j\sigma_i|\sigma_j} & \sigma_j\sigma_i|\sigma_j \\ \sigma_j|\sigma_i|\sigma_j & \xrightarrow{A_{\sigma_j,\sigma_i,\sigma_j}} & \sigma_i\sigma_j\sigma_i \\ \sigma_j|\alpha_{\sigma_i,\sigma_j} & \xrightarrow{\sigma_j|\sigma_i\sigma_j} & \sigma_j|\sigma_i\sigma_j \end{array}$$

and pairs of generating 3-cells of the form

$$\begin{array}{ccc} \alpha_{\sigma_k,\sigma_i}|\sigma_j\sigma_i & \xrightarrow{\sigma_k\sigma_i|\sigma_j\sigma_i} & \sigma_k\sigma_i|\sigma_j\sigma_i \\ \sigma_k|\sigma_i|\sigma_j\sigma_i & \xrightarrow{A_{\sigma_k,\sigma_i,\sigma_j\sigma_i}} & \sigma_k\sigma_i\sigma_j\sigma_i \\ \sigma_k|\alpha_{\sigma_i,\sigma_j\sigma_i} & \xrightarrow{\sigma_k|\sigma_i\sigma_j\sigma_i} & \sigma_k|\sigma_i\sigma_j\sigma_i \end{array} \quad \begin{array}{ccc} \alpha_{\sigma_k,\sigma_j}|\sigma_i\sigma_j & \xrightarrow{\sigma_k\sigma_j|\sigma_i\sigma_j} & \sigma_k\sigma_j|\sigma_i\sigma_j \\ \sigma_k|\sigma_j|\sigma_i\sigma_j & \xrightarrow{A_{\sigma_k,\sigma_j,\sigma_i\sigma_j}} & \sigma_k\sigma_i\sigma_j\sigma_i \\ \sigma_k|\alpha_{\sigma_j,\sigma_i\sigma_j} & \xrightarrow{\sigma_k|\sigma_i\sigma_j\sigma_i} & \sigma_k|\sigma_i\sigma_j\sigma_i \end{array}$$

with i, j and k as above.

We have thus computed the finite coherent presentation of the Artin-Tits monoid of type A_2 , which consists of fifteen generating 1-cells, twenty-seven generating 2-cells, and twelve generating 3-cells.

Note that some of the generating 3-cells are superfluous, so let us further perform the homotopical reduction procedure as described in Subsection 2.2.4. Let us construct a collapsible part Γ . There are no critical triple branchings, so we take the empty set for Γ_4 . For the component Γ_3 , we take the Nielsen transformations of the generating 3-cells

$$A_{\sigma_3,\sigma_1,\sigma_2\sigma_1}, A_{\sigma_3,\sigma_2,\sigma_1\sigma_2}, A_{\sigma_1,\sigma_2,\sigma_3\sigma_2}, A_{\sigma_1,\sigma_3,\sigma_2\sigma_3}, A_{\sigma_2,\sigma_3,\sigma_1\sigma_3}, A_{\sigma_2,\sigma_1,\sigma_3\sigma_1}$$

whose respective targets are the following generating 2-cells:

$$\alpha_{\sigma_3\sigma_1,\sigma_2\sigma_1}, \alpha_{\sigma_3\sigma_2,\sigma_1\sigma_2}, \alpha_{\sigma_1\sigma_2,\sigma_3\sigma_2}, \alpha_{\sigma_1\sigma_3,\sigma_2\sigma_3}, \alpha_{\sigma_2\sigma_3,\sigma_1\sigma_3}, \alpha_{\sigma_2\sigma_1,\sigma_3\sigma_1}.$$

We order the generating 2-cells as follows:

$$\begin{aligned} \alpha_{\sigma_3\sigma_1,\sigma_2\sigma_1} &> \alpha_{\sigma_3\sigma_2,\sigma_1\sigma_2} > \alpha_{\sigma_1\sigma_2,\sigma_3\sigma_2} > \alpha_{\sigma_1\sigma_3,\sigma_2\sigma_3} > \alpha_{\sigma_2\sigma_3,\sigma_1\sigma_3} > \alpha_{\sigma_2\sigma_1,\sigma_3\sigma_1} > \\ &> \text{other generating 2-cells in any order (as they are not targets in } \Gamma_3). \end{aligned}$$

For the component Γ_2 , we take the generating 2-cells

$$\begin{aligned} \alpha_{\sigma_1,\sigma_2}, \alpha_{\sigma_2,\sigma_1}, \alpha_{\sigma_2,\sigma_3}, \alpha_{\sigma_3,\sigma_2}, \alpha_{\sigma_3,\sigma_1}, \alpha_{\sigma_1,\sigma_3}, \\ \alpha_{\sigma_1,\sigma_2\sigma_1}, \alpha_{\sigma_2,\sigma_3\sigma_2}, \alpha_{\sigma_3,\sigma_1\sigma_3}, \alpha_{\sigma_3,\sigma_1\sigma_2\sigma_1}, \alpha_{\sigma_1,\sigma_2\sigma_3\sigma_2}, \alpha_{\sigma_2,\sigma_3\sigma_1\sigma_3}. \end{aligned}$$

Any ordering of the generating 1-cells which respects the length in F^* will do.

The homotopical reduction of $\text{Gar}_3(F)$ with respect to the collapsible part $(\Gamma_2, \Gamma_3, \emptyset)$ contains: a single generating 0-cell; the generating 1-cells $\sigma_1, \sigma_2, \sigma_3$; the

generating 2-cells $\alpha_{\sigma_2, \sigma_1 \sigma_2}$, $\alpha_{\sigma_3, \sigma_2 \sigma_3}$, $\alpha_{\sigma_1, \sigma_3 \sigma_1}$; and no generating 3-cells. As a side result, we have thus shown that Artin's presentation ('extended' with the empty set of generating 3-cells) of the Artin-Tits monoid of type \tilde{A}_2 is coherent.

2.6.4. Dual braid monoids. Let us consider the dual braid monoid B_n^{+*} , as defined in [13, Subsection I.1.3]. Its generating set consists of braids

$$b_{i,j} = \sigma_i \cdots \sigma_{j-2} \sigma_{j-1} \sigma_{j-2}^{-1} \cdots \sigma_i^{-1}$$

for $1 \leq i < j \leq n$. They obey the following relations:

$$\begin{aligned} b_{i,j} b_{i',j'} &= b_{i',j'} b_{i,j} && \text{for } [i,j] \text{ and } [i',j'] \text{ disjoint or nested,} \\ b_{i,j} b_{j,k} &= b_{j,k} b_{i,k} = b_{i,k} b_{i,j} && \text{for } 1 \leq i < j < k \leq n. \end{aligned}$$

The pair (B_n^{+*}, Δ_n^*) , with $\Delta_n^* = b_{1,2} \cdots b_{n-1,n}$, is a Garside monoid, as essentially shown in [2] for B_3^{+*} (although without using the terminology of Garside monoids) and in [1] for the general case. Therefore, computing a coherent presentation of a dual braid monoid does not really require Theorem 2.5.1.4 because [26, Theorem 3.3.3], recalled in Example 2.3.6.1, would suffice. To the best of our knowledge, however, it has not been done yet, so we do it here. Let us take B_4^{+*} for example. For simplicity, we leave out the comma which separates the two indices of b .

The Garside family we consider is the set D of all (left) divisors of Δ_4^* . Namely,

$$D = \{1, b_{12}, b_{23}, b_{24}, b_{13}, b_{34}, b_{14}, b_{12}b_{23}, b_{12}b_{34}, b_{23}b_{34}, b_{12}b_{24}, b_{23}b_{14}, b_{13}b_{34}, b_{12}b_{23}b_{34}\}.$$

The lattice (D, \preceq) can be seen in [13, Figure I.4].

The requirements of Theorem 2.5.1.4 are met, as observed in Corollary 2.5.1.6. By Theorem 2.5.1.4, we take generating 2-cells of the form $u|v \Rightarrow uv$ for u, v in $D \setminus \{1\}$ such that uv divides Δ_4^* . This yields the following family of generating 2-cells:

$$\begin{aligned} \alpha_{b_{12}, b_{23}} : b_{12}|b_{23} &\Rightarrow b_{12}b_{23}, & \alpha_{b_{23}, b_{34}} : b_{23}|b_{34} &\Rightarrow b_{23}b_{34}, \\ \alpha_{b_{23}, b_{13}} : b_{23}|b_{13} &\Rightarrow b_{12}b_{23}, & \alpha_{b_{34}, b_{24}} : b_{34}|b_{24} &\Rightarrow b_{23}b_{34}, \\ \alpha_{b_{13}, b_{12}} : b_{13}|b_{12} &\Rightarrow b_{12}b_{23}, & \alpha_{b_{24}, b_{23}} : b_{24}|b_{23} &\Rightarrow b_{23}b_{34}, \\ \\ \alpha_{b_{12}, b_{24}} : b_{12}|b_{24} &\Rightarrow b_{12}b_{24}, & \alpha_{b_{13}, b_{34}} : b_{13}|b_{34} &\Rightarrow b_{13}b_{34}, \\ \alpha_{b_{24}, b_{14}} : b_{24}|b_{14} &\Rightarrow b_{12}b_{24}, & \alpha_{b_{34}, b_{14}} : b_{34}|b_{14} &\Rightarrow b_{13}b_{34}, \\ \alpha_{b_{14}, b_{12}} : b_{14}|b_{12} &\Rightarrow b_{12}b_{2,4}, & \alpha_{b_{14}, b_{13}} : b_{14}|b_{13} &\Rightarrow b_{13}b_{34}, \\ \\ \alpha_{b_{12}, b_{34}} : b_{12}|b_{34} &\Rightarrow b_{12}b_{34}, & \alpha_{b_{23}, b_{14}} : b_{23}|b_{14} &\Rightarrow b_{23}b_{14}, \\ \alpha_{b_{34}, b_{12}} : b_{34}|b_{12} &\Rightarrow b_{12}b_{34}, & \alpha_{b_{14}, b_{23}} : b_{14}|b_{23} &\Rightarrow b_{23}b_{14}, \end{aligned}$$

$$\begin{array}{ll}
\alpha_{b_{12}, b_{23} b_{34}} : b_{12} | b_{23} b_{34} \Rightarrow b_{12} b_{23} b_{34}, & \alpha_{b_{12} b_{23}, b_{34}} : b_{12} b_{23} | b_{34} \Rightarrow b_{12} b_{23} b_{34}, \\
\alpha_{b_{23}, b_{13} b_{34}} : b_{23} | b_{13} b_{34} \Rightarrow b_{12} b_{23} b_{34}, & \alpha_{b_{12} b_{34}, b_{24}} : b_{12} b_{34} | b_{24} \Rightarrow b_{12} b_{23} b_{34}, \\
\alpha_{b_{24}, b_{23} b_{14}} : b_{24} | b_{23} b_{14} \Rightarrow b_{12} b_{23} b_{34}, & \alpha_{b_{23} b_{34}, b_{14}} : b_{23} b_{34} | b_{14} \Rightarrow b_{12} b_{23} b_{34}, \\
\alpha_{b_{13}, b_{12} b_{34}} : b_{13} | b_{12} b_{34} \Rightarrow b_{12} b_{23} b_{34}, & \alpha_{b_{12} b_{24}, b_{23}} : b_{12} b_{24} | b_{23} \Rightarrow b_{12} b_{23} b_{34}, \\
\alpha_{b_{34}, b_{12} b_{24}} : b_{34} | b_{12} b_{24} \Rightarrow b_{12} b_{23} b_{34}, & \alpha_{b_{23} b_{14}, b_{13}} : b_{23} b_{14} | b_{13} \Rightarrow b_{12} b_{23} b_{34}, \\
\alpha_{b_{14}, b_{12} b_{23}} : b_{14} | b_{12} b_{23} \Rightarrow b_{12} b_{23} b_{34}, & \alpha_{b_{13} b_{34}, b_{12}} : b_{13} b_{34} | b_{12} \Rightarrow b_{12} b_{23} b_{34}.
\end{array}$$

Now we construct the generating 3-cells $A_{u,v,w}$ for u, v, w in $D \setminus \{1\}$ with uw, vw , and uvw being divisors of Δ_4^* :

$$\begin{array}{ll}
\begin{array}{ccc}
\alpha_{b_{12}, b_{23} | b_{34}} & \xrightarrow{\quad} & b_{12} b_{23} | b_{34} \\
& \searrow & \nearrow \\
b_{12} | b_{23} | b_{34} & A_{b_{12}, b_{23}, b_{34}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{12} | \alpha_{b_{23}, b_{34}} & \xrightarrow{\quad} & b_{12} | b_{23} b_{34} \\
& \nearrow & \searrow \\
& \alpha_{b_{12}, b_{23} b_{34}} &
\end{array} &
\begin{array}{ccc}
\alpha_{b_{23}, b_{34} | b_{14}} & \xrightarrow{\quad} & b_{23} b_{34} | b_{14} \\
& \searrow & \nearrow \\
b_{23} | b_{34} | b_{14} & A_{b_{23}, b_{34}, b_{14}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{23} | \alpha_{b_{34}, b_{14}} & \xrightarrow{\quad} & b_{23} | b_{13} b_{34} \\
& \nearrow & \searrow \\
& \alpha_{b_{23}, b_{13} b_{34}} &
\end{array} \\
\begin{array}{ccc}
\alpha_{b_{23}, b_{13} | b_{34}} & \xrightarrow{\quad} & b_{12} b_{23} | b_{34} \\
& \searrow & \nearrow \\
b_{23} | b_{13} | b_{34} & A_{b_{23}, b_{13}, b_{34}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{23} | \alpha_{b_{13}, b_{34}} & \xrightarrow{\quad} & b_{23} | b_{13} b_{34} \\
& \nearrow & \searrow \\
& \alpha_{b_{23}, b_{13} b_{34}} &
\end{array} &
\begin{array}{ccc}
\alpha_{b_{34}, b_{24} | b_{14}} & \xrightarrow{\quad} & b_{23} b_{34} | b_{14} \\
& \searrow & \nearrow \\
b_{34} | b_{24} | b_{14} & A_{b_{34}, b_{24}, b_{14}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{34} | \alpha_{b_{24}, b_{14}} & \xrightarrow{\quad} & b_{34} | b_{12} b_{24} \\
& \nearrow & \searrow \\
& \alpha_{b_{34}, b_{12} b_{24}} &
\end{array} \\
\begin{array}{ccc}
\alpha_{b_{13}, b_{12} | b_{34}} & \xrightarrow{\quad} & b_{12} b_{23} | b_{34} \\
& \searrow & \nearrow \\
b_{13} | b_{12} | b_{34} & A_{b_{13}, b_{12}, b_{34}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{13} | \alpha_{b_{12}, b_{34}} & \xrightarrow{\quad} & b_{13} | b_{12} b_{34} \\
& \nearrow & \searrow \\
& \alpha_{b_{13}, b_{12} b_{34}} &
\end{array} &
\begin{array}{ccc}
\alpha_{b_{24}, b_{23} | b_{14}} & \xrightarrow{\quad} & b_{23} b_{34} | b_{14} \\
& \searrow & \nearrow \\
b_{24} | b_{23} | b_{14} & A_{b_{24}, b_{23}, b_{14}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{24} | \alpha_{b_{23}, b_{14}} & \xrightarrow{\quad} & b_{24} | b_{23} b_{14} \\
& \nearrow & \searrow \\
& \alpha_{b_{24}, b_{23} b_{14}} &
\end{array} \\
\begin{array}{ccc}
\alpha_{b_{12}, b_{24} | b_{23}} & \xrightarrow{\quad} & b_{12} b_{24} | b_{23} \\
& \searrow & \nearrow \\
b_{12} | b_{24} | b_{23} & A_{b_{12}, b_{24}, b_{23}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{12} | \alpha_{b_{24}, b_{23}} & \xrightarrow{\quad} & b_{12} | b_{23} b_{34} \\
& \nearrow & \searrow \\
& \alpha_{b_{12}, b_{23} b_{34}} &
\end{array} &
\begin{array}{ccc}
\alpha_{b_{13}, b_{34} | b_{12}} & \xrightarrow{\quad} & b_{13} b_{34} | b_{12} \\
& \searrow & \nearrow \\
b_{13} | b_{34} | b_{12} & A_{b_{13}, b_{34}, b_{12}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{13} | \alpha_{b_{34}, b_{12}} & \xrightarrow{\quad} & b_{13} | b_{12} b_{34} \\
& \nearrow & \searrow \\
& \alpha_{b_{13}, b_{12} b_{34}} &
\end{array} \\
\begin{array}{ccc}
\alpha_{b_{24}, b_{14} | b_{23}} & \xrightarrow{\quad} & b_{12} b_{24} | b_{23} \\
& \searrow & \nearrow \\
b_{24} | b_{14} | b_{23} & A_{b_{24}, b_{14}, b_{23}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{24} | \alpha_{b_{14}, b_{23}} & \xrightarrow{\quad} & b_{24} | b_{23} b_{14} \\
& \nearrow & \searrow \\
& \alpha_{b_{24}, b_{23} b_{14}} &
\end{array} &
\begin{array}{ccc}
\alpha_{b_{34}, b_{14} | b_{12}} & \xrightarrow{\quad} & b_{13} b_{34} | b_{12} \\
& \searrow & \nearrow \\
b_{34} | b_{14} | b_{12} & A_{b_{34}, b_{14}, b_{12}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{34} | \alpha_{b_{14}, b_{12}} & \xrightarrow{\quad} & b_{34} | b_{12} b_{24} \\
& \nearrow & \searrow \\
& \alpha_{b_{34}, b_{12} b_{24}} &
\end{array} \\
\begin{array}{ccc}
\alpha_{b_{14}, b_{12} | b_{23}} & \xrightarrow{\quad} & b_{12} b_{24} | b_{23} \\
& \searrow & \nearrow \\
b_{14} | b_{12} | b_{23} & A_{b_{14}, b_{12}, b_{23}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{14} | \alpha_{b_{12}, b_{23}} & \xrightarrow{\quad} & b_{14} | b_{12} b_{23} \\
& \nearrow & \searrow \\
& \alpha_{b_{14}, b_{12} b_{23}} &
\end{array} &
\begin{array}{ccc}
\alpha_{b_{14}, b_{13} | b_{12}} & \xrightarrow{\quad} & b_{13} b_{34} | b_{12} \\
& \searrow & \nearrow \\
b_{14} | b_{13} | b_{12} & A_{b_{14}, b_{13}, b_{12}} & b_{12} b_{23} b_{34} \\
& \swarrow & \nwarrow \\
b_{14} | \alpha_{b_{13}, b_{12}} & \xrightarrow{\quad} & b_{14} | b_{12} b_{23} \\
& \nearrow & \searrow \\
& \alpha_{b_{14}, b_{12} b_{23}} &
\end{array}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{b_{12}, b_{34}} | b_{24} & \xrightarrow{\quad} & b_{12} b_{34} | b_{24} & \xrightarrow{\quad} & \alpha_{b_{12} b_{34}, b_{24}} \\
b_{12} | b_{34} | b_{24} & \xrightarrow{\quad} & A_{b_{12}, b_{34}, b_{24}} & \xrightarrow{\quad} & b_{12} b_{23} b_{34} \\
b_{12} | \alpha_{b_{34}, b_{24}} & \xrightarrow{\quad} & b_{12} | b_{23} b_{34} & \xrightarrow{\quad} & \alpha_{b_{12}, b_{23} b_{34}}
\end{array}
\quad
\begin{array}{ccc}
\alpha_{b_{23}, b_{14}} | b_{13} & \xrightarrow{\quad} & b_{23} b_{14} | b_{13} & \xrightarrow{\quad} & \alpha_{b_{23} b_{14}, b_{13}} \\
b_{23} | b_{14} | b_{13} & \xrightarrow{\quad} & A_{b_{23}, b_{14}, b_{13}} & \xrightarrow{\quad} & b_{12} b_{23} b_{34} \\
b_{23} | \alpha_{b_{14}, b_{13}} & \xrightarrow{\quad} & b_{23} | b_{13} b_{34} & \xrightarrow{\quad} & \alpha_{b_{23}, b_{13} b_{34}}
\end{array}$$

$$\begin{array}{ccc}
\alpha_{b_{34}, b_{12}} | b_{24} & \xrightarrow{\quad} & b_{12} b_{34} | b_{24} & \xrightarrow{\quad} & \alpha_{b_{12} b_{34}, b_{24}} \\
b_{34} | b_{12} | b_{24} & \xrightarrow{\quad} & A_{b_{34}, b_{12}, b_{24}} & \xrightarrow{\quad} & b_{12} b_{23} b_{34} \\
b_{34} | \alpha_{b_{12}, b_{24}} & \xrightarrow{\quad} & b_{34} | b_{12} b_{24} & \xrightarrow{\quad} & \alpha_{b_{34}, b_{12} b_{24}}
\end{array}
\quad
\begin{array}{ccc}
\alpha_{b_{14}, b_{23}} | b_{13} & \xrightarrow{\quad} & b_{23} b_{14} | b_{13} & \xrightarrow{\quad} & \alpha_{b_{23} b_{14}, b_{13}} \\
b_{14} | b_{23} | b_{13} & \xrightarrow{\quad} & A_{b_{14}, b_{23}, b_{13}} & \xrightarrow{\quad} & b_{12} b_{23} b_{34} \\
b_{14} | \alpha_{b_{23}, b_{13}} & \xrightarrow{\quad} & b_{14} | b_{12} b_{23} & \xrightarrow{\quad} & \alpha_{b_{14}, b_{12} b_{23}}
\end{array}$$

We have thus computed the coherent presentation $\text{Gar}_3(D)$ of the dual braid monoid B_4^{+*} , which consists of thirteen generating 1-cells, twenty-eight generating 2-cells, and sixteen generating 3-cells.

As in Subsection 2.6.3, we can further perform the homotopical reduction procedure. A quick inspection reveals that there are no critical triple branchings, so we take the component Γ_4 of the collapsible part to be the empty set. For the component Γ_3 , we take the Nielsen transformations of the generating 3-cells

$$\begin{aligned}
& A_{b_{24}, b_{23}, b_{14}}, A_{b_{13}, b_{34}, b_{12}}, A_{b_{23}, b_{14}, b_{13}}, A_{b_{23}, b_{34}, b_{14}}, A_{b_{14}, b_{12}, b_{23}}, \\
& A_{b_{34}, b_{12}, b_{24}}, A_{b_{13}, b_{12}, b_{34}}, A_{b_{23}, b_{13}, b_{34}}, A_{b_{12}, b_{24}, b_{23}}, A_{b_{12}, b_{34}, b_{24}}, A_{b_{12}, b_{23}, b_{34}}
\end{aligned}$$

whose respectively selected targets are the following generating 2-cells:

$$\begin{aligned}
& \alpha_{b_{24}, b_{23} b_{14}}, \alpha_{b_{13} b_{34}, b_{12}}, \alpha_{b_{23} b_{14}, b_{13}}, \alpha_{b_{23} b_{34}, b_{14}}, \alpha_{b_{14}, b_{12} b_{23}}, \\
& \alpha_{b_{34}, b_{12} b_{24}}, \alpha_{b_{13}, b_{12} b_{34}}, \alpha_{b_{23}, b_{13} b_{34}}, \alpha_{b_{12} b_{24}, b_{23}}, \alpha_{b_{12} b_{34}, b_{24}}, \alpha_{b_{12} b_{23}, b_{34}}.
\end{aligned}$$

We order the generating 2-cells as follows:

$$\begin{aligned}
& \alpha_{b_{24}, b_{23} b_{14}} > \alpha_{b_{13}, b_{34}, b_{12}} > \alpha_{b_{23} b_{14}, b_{13}} > \alpha_{b_{23} b_{34}, b_{14}} > \\
& > \alpha_{b_{14}, b_{12} b_{23}} > \alpha_{b_{34}, b_{12} b_{24}} > \alpha_{b_{13}, b_{12} b_{34}} > \alpha_{b_{23}, b_{13} b_{34}} > \\
& > \alpha_{b_{12} b_{24}, b_{23}} > \alpha_{b_{12} b_{34}, b_{24}} > \alpha_{b_{12} b_{23}, b_{34}} > \alpha_{b_{12}, b_{23} b_{34}} > \\
& > \text{other generating 2-cells in any order (as they are not targets in } \Gamma_3).
\end{aligned}$$

For the component Γ_2 , we take the generating 2-cells

$$\alpha_{b_{12}, b_{23}}, \alpha_{b_{23}, b_{34}}, \alpha_{b_{12}, b_{24}}, \alpha_{b_{13}, b_{34}}, \alpha_{b_{12}, b_{34}}, \alpha_{b_{23}, b_{14}}, \alpha_{b_{12}, b_{23} b_{34}}.$$

Any ordering of the generating 1-cells which respects the length in D will do.

Recall that, when cells are removed from a polygraph, the boundaries of the remaining cells are not necessarily defined anymore, in a manner of speaking. More precisely, if a boundary contained a removed cell, then it would be ‘updated’ according to the recursive assignment given in Subsection 2.2.4. For convenience, let us write A' to denote the new version of the generating 3-cell A which shares the same index. Then, the homotopical reduction of $\text{Gar}_3(D)$ with respect to the collapsible part $(\Gamma_2, \Gamma_3, \emptyset)$ contains: a single generating 0-cell; the generating 1-cells $b_{12}, b_{23}, b_{24}, b_{13}, b_{34}, b_{14}$; the generating 2-cells

$$\begin{array}{ll}
\alpha_{b_{23}, b_{13}} : b_{23}|b_{13} \Rightarrow b_{12}|b_{23}, & \alpha_{b_{34}, b_{24}} : b_{34}|b_{24} \Rightarrow b_{23}|b_{34}, \\
\alpha_{b_{13}, b_{12}} : b_{13}|b_{12} \Rightarrow b_{12}|b_{23}, & \alpha_{b_{24}, b_{23}} : b_{24}|b_{23} \Rightarrow b_{23}|b_{34}, \\
\alpha_{b_{24}, b_{14}} : b_{24}|b_{14} \Rightarrow b_{12}|b_{24}, & \alpha_{b_{34}, b_{14}} : b_{34}|b_{14} \Rightarrow b_{13}|b_{34}, \\
\alpha_{b_{14}, b_{1,2}} : b_{14}|b_{12} \Rightarrow b_{12}|b_{24}, & \alpha_{b_{14}, b_{13}} : b_{14}|b_{13} \Rightarrow b_{13}|b_{34}, \\
\alpha_{b_{34}, b_{12}} : b_{34}|b_{12} \Rightarrow b_{12}|b_{34}, & \alpha_{b_{14}, b_{23}} : b_{14}|b_{23} \Rightarrow b_{23}|b_{14};
\end{array}$$

and the generating 3-cells

$$\begin{array}{ccccc}
\alpha_{b_{34}, b_{24}}|b_{14} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{23}|b_{34}|b_{14} & \xrightarrow{b_{23}|\alpha_{b_{34}, b_{14}}} & b_{23}|b_{13}|b_{34} & \begin{array}{c} \searrow \\ \nearrow \end{array} & \alpha_{b_{23}, b_{13}}|b_{34} \\
b_{34}|b_{24}|b_{14} & & & \xrightarrow{A'_{b_{34}, b_{24}, b_{14}}} & & & b_{12}|b_{23}|b_{34} \\
b_{34}|\alpha_{b_{24}, b_{14}} & \begin{array}{c} \searrow \\ \nearrow \end{array} & b_{34}|b_{12}|b_{24} & \xrightarrow{\alpha_{b_{34}, b_{12}}|b_{24}} & b_{12}|b_{34}|b_{24} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{12}|\alpha_{b_{34}, b_{2,4}}
\end{array}$$

$$\begin{array}{ccccc}
\alpha_{b_{24}, b_{14}}|b_{23} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{12}|b_{24}|b_{23} & \xrightarrow{b_{12}|\alpha_{b_{24}, b_{23}}} & b_{12}|b_{23}|b_{34} & \begin{array}{c} \searrow \\ \nearrow \end{array} & \alpha_{b_{23}, b_{13}}|b_{34} \\
b_{24}|b_{14}|b_{23} & & & \xrightarrow{A'_{b_{24}, b_{14}, b_{23}}} & & & b_{23}|b_{13}|b_{34} \\
b_{24}|\alpha_{b_{14}, b_{23}} & \begin{array}{c} \searrow \\ \nearrow \end{array} & b_{24}|b_{23}|b_{14} & \xrightarrow{\alpha_{b_{24}, b_{23}}|b_{14}} & b_{23}|b_{34}|b_{14} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{23}|\alpha_{b_{34}, b_{14}}
\end{array}$$

$$\begin{array}{ccccc}
\alpha_{b_{34}, b_{14}}|b_{12} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{13}|b_{34}|b_{12} & \xrightarrow{b_{13}|\alpha_{b_{34}, b_{12}}} & b_{13}|b_{12}|b_{34} & \begin{array}{c} \searrow \\ \nearrow \end{array} & \alpha_{b_{13}, b_{12}}|b_{34} \\
b_{34}|b_{14}|b_{12} & & & \xrightarrow{A'_{b_{34}, b_{14}, b_{12}}} & & & b_{12}|b_{23}|b_{34} \\
b_{34}|\alpha_{b_{14}, b_{12}} & \begin{array}{c} \searrow \\ \nearrow \end{array} & b_{34}|b_{12}|b_{24} & \xrightarrow{\alpha_{b_{34}, b_{12}}|b_{24}} & b_{12}|b_{34}|b_{24} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{12}|\alpha_{b_{34}, b_{2,4}}
\end{array}$$

$$\begin{array}{ccccc}
\alpha_{b_{14}, b_{13}}|b_{12} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{13}|b_{34}|b_{12} & \xrightarrow{b_{13}|\alpha_{b_{34}, b_{12}}} & b_{13}|b_{12}|b_{34} & \begin{array}{c} \searrow \\ \nearrow \end{array} & \alpha_{b_{13}, b_{34}, b_{12}} \\
b_{14}|b_{13}|b_{12} & & & \xrightarrow{A'_{b_{14}, b_{13}, b_{12}}} & & & b_{12}|b_{23}|b_{34} \\
b_{14}|\alpha_{b_{13}, b_{12}} & \begin{array}{c} \searrow \\ \nearrow \end{array} & b_{14}|b_{12}|b_{23} & \xrightarrow{\alpha_{b_{14}, b_{12}}|b_{23}} & b_{12}|b_{24}|b_{23} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{12}|\alpha_{b_{24}, b_{23}}
\end{array}$$

$$\begin{array}{ccccc}
\alpha_{b_{14}, b_{23}}|b_{13} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{23}|b_{14}|b_{13} & \xrightarrow{b_{23}|\alpha_{b_{14}, b_{13}}} & b_{23}|b_{13}|b_{34} & \begin{array}{c} \searrow \\ \nearrow \end{array} & \alpha_{b_{23}, b_{13}}|b_{34} \\
b_{14}|b_{23}|b_{13} & & & \xrightarrow{A'_{b_{14}, b_{23}, b_{13}}} & & & b_{12}|b_{23}|b_{34} \\
b_{14}|\alpha_{b_{23}, b_{13}} & \begin{array}{c} \searrow \\ \nearrow \end{array} & b_{14}|b_{12}|b_{23} & \xrightarrow{\alpha_{b_{14}, b_{12}}|b_{23}} & b_{12}|b_{24}|b_{23} & \begin{array}{c} \nearrow \\ \searrow \end{array} & b_{12}|\alpha_{b_{24}, b_{23}}
\end{array}$$

A view towards further applications. A coherent presentation provides the first dimensions of a polygraphic resolution. We would like to prove Guiraud's conjecture that Theorem 2.5.1.4 generalises to all dimensions: a monoid M having a Garside family S , under certain reasonable conditions, admits the polygraphic resolution

$$\text{Gar}(S) = \left\{ \gamma_{s_0 \cdots s_n} \mid s_0 \cdots s_n \in S \setminus \{1\}, s_0 \widehat{\cdot} \cdots \widehat{\cdot} s_n \right\}$$

where $\gamma_{s_0 \cdots s_n}$ is an n -cell. A challenge in this direction is determining boundary maps.

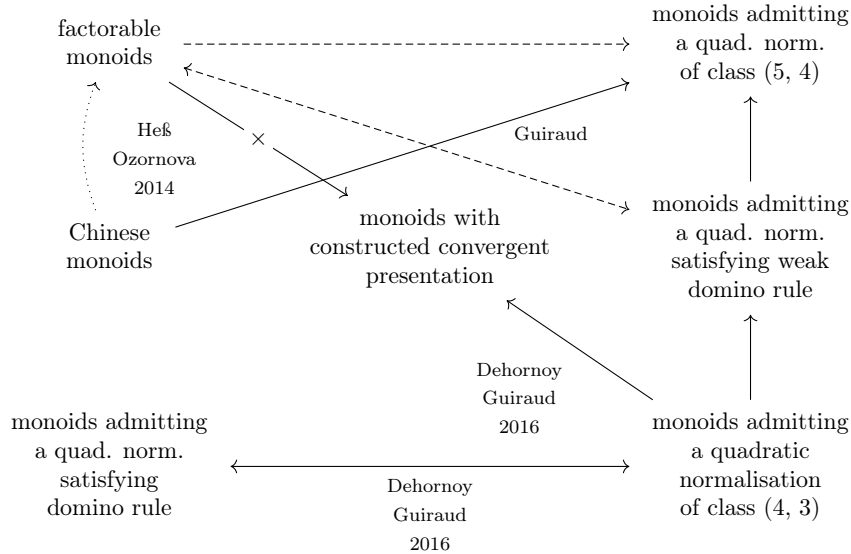
Now that we have a mechanism for constructing a coherent presentation of monoids admitting a Garside family, other examples of application are readily feasible. Here, we mention two such examples.

Consider braid ribbons as defined in [13, Subsection I.3.4]. One easily checks that all the requirements of the Theorem 2.5.1.4 would be met if the theorem was extended from monoids to categories, so a coherent presentation of the category of braid ribbons could be constructed. Then, a task remaining to be done would be computing the binary products of elements of the given Garside family.

In Subsection 2.6.4, we have constructed a coherent presentation of the dual braid monoid B_4^{+*} . Our approach could be generalised to B_n^{+*} . To this end, one needs to compute the binary products of divisors of Δ_n^* . Non-crossing partitions ordered by inclusion form a lattice whose size is the n th Catalan number. This lattice is isomorphic to the Garside family of left divisors of the Garside element of B_n^{+*} . Determining the corresponding relations amounts to a combinatorial task of sorting out certain relations among partitions.

Correspondence between factorability and normalisation

This chapter investigates combinatorial properties of a certain class of monoids, seen from two different viewpoints, with a goal of unifying the two. The main contribution is providing answers to the question, left open in [13] and [16], of determining the relation between these two approaches. The following diagram puts it in a nutshell, with arrows denoting inclusions (and \times over an arrow negating inclusion), the dashed ones being new results with the dotted one being our hypothesis:



Section 3.1 fixes basic terminology to be used throughout the chapter. Sections 3.2 and 3.3 present the main characters of the play, namely, factorability structures and quadratic normalisations, respectively. In Section 3.4, the close interplay between the two is investigated.

3.1. Remarks on terminology

In the present chapter, the setting of polygraphs is not needed since no coherent presentations are constructed here. Therefore, we are going to use a simpler terminology, that of rewriting systems. A (word) **rewriting system** is a pair (S, R) consisting of a set S and a binary relation R on S^* , whose elements are called **rewriting rules**. An element (u, v) of R is also written as $u \rightarrow v$ to stress the fact

that it is directed. Seeing relations between words not as equalities but as rewriting rules is a key concept of rewriting theory.

For a rewriting rule (u, v) , and w, w' in S^* , a pair $(w|u|w', w|v|w')$ is called a **rewriting step**. For u and v in S^* , we say that u **rewrites** to v if there is a finite composable sequence of rewriting steps, such that the source of the first step of the sequence is u and the target of the last step is v . A word v is called **irreducible** with respect to R if there is no rewriting step whose source is v .

A rewriting system (S, R) is called:

- **confluent** if each pair of rewriting sequences starting with the same word can be completed in such a way that the sequences eventually reach a common result;
- **normalising** if every u in S^* rewrites to at least one irreducible word;
- **terminating** if it admits no infinite rewriting sequence;
- **convergent** if it is both confluent and terminating;
- **reduced** (or minimal) if for every rewriting rule $u \rightarrow v$, the word v is irreducible with respect to R , and the word u is irreducible with respect to $R \setminus \{(u, v)\}$;
- **strongly reduced** if it is reduced and, in addition, every element of S is irreducible;
- **quadratic** if the source and target of each element of R are of length 2.

The monoid **presented by** a rewriting system (S, R) is the quotient M of the free monoid S^* by the congruence relation generated by R . In a confluent rewriting system, if a word u rewrites to an irreducible word (which is necessarily unique), then the latter is denoted by \hat{u} . If (S, R) is convergent, then the map $M \rightarrow S^*$ defined by $f \mapsto \hat{u}$ for any u such that $\text{EV}(u) = f$, is called the **normal form** associated with the rewriting system (S, R) . We remind the reader of terminology and notation introduced in Subsection 1.2.1 on page 11.

A normal form NF for a monoid M with respect to a generating set S is called **geodesic** if, for every f in M , the inequality $|\text{NF}(f)| \leq |u|$ holds for every S -word u representing f , i.e. such that $\text{EV}(u) = f$.

REMARK 3.1.0.1. If a generating set S of a monoid M is a subset of M , then elements of S can be regarded in two ways: as length-one words in S^* , and as elements of M . When a rewriting system presenting M with respect to S is strongly reduced, this makes no essential difference, so elements of S are denoted in the same way, regardless of the viewpoint, relying on the context to provide the proper interpretation. In particular, one can say that a generating set contains (or that it does not contain) 1, the identity element of a monoid. This phrasing is the custom in the context of factorability (see [35], [34], [48]), but not in the context of normalisation in general (where a generating set is commonly distinguished from its image under the evaluation map, as can be seen in [16]). So, we will emphasise such situation by calling S a generating subset, not just a generating set, of M . When we characterise factorable monoids in terms of quadratic normalisations (Subsection 3.4.1), the corresponding normalisations will be eligible to share this custom so there will be no need to emphasise it.

For technical reasons, in the rest of this chapter, the letter S will be reserved for the following purpose.

CONVENTION 3.1.0.2. When the letter S is used to denote a generating set of a monoid, it is understood that S contains no letter representing the identity element of the monoid (i.e. no generator evaluated to 1).

On the other hand, if a generating set S is enriched, via disjoint union, with a letter representing the identity of the considered monoid, then the resulting pointed set (i.e. set equipped with a distinguished element, called basepoint, enjoying a special treatment in the given context) is denoted by S^e . In accordance with Remark 3.1.0.1, the basepoint of S^e is denoted by 1 if S is a generating subset of the considered monoid.

To simplify the presentation, we are considering exclusively monoids, but results stated here (recalled ones as well as new ones) mostly extend to categories, considered as monoids with partial multiplication. As a reminder that a monoid is thought of as a monoid of endomorphisms (of an object), we tend to use letters f and g for elements of a monoid.

3.2. Factorability structures

This section recalls the notion of factorability structure. For elaboration, the reader is referred to [35]. Subsection 3.2.1 recollects the basic terminology. In Subsection 3.2.2, we recall an alternative approach to factorability through the notion of local factorability. Certain notions are redefined in order to overcome the issues arising from the original definition, which are pointed out in Subsection 3.2.3. Subsection 3.2.4 recalls the rewriting system associated with a factorability structure.

3.2.1. The definition of factorability structure.

CONVENTION 3.2.1.1. Let us adopt the convention that elements of a finite sequence are indexed starting from the leftmost one, as in (s_1, s_2, \dots, s_n) , thereby not following the convention used in [35] where elements are indexed starting from the rightmost one. The purpose is to make the notions that concern factorability more easily comparable (in Section 3.4) with those concerning normalisation.

A pair (f, g) in M^2 is called **geodesic** if $|fg| = |f| + |g|$.

Let M be a monoid, and let S be a generating subset of M . A **factorisation map** for (M, S) is a map $\eta = (\eta', \bar{\eta}) : M \rightarrow M^2$ satisfying the following conditions:

- for f in $M \setminus \{1\}$, the element $\eta'(f)$ of S is a left divisor of f , and the element $\bar{\eta}(f)$ is a right complement of $\eta'(f)$ in f ;
- the pair $(\eta'(f), \bar{\eta}(f))$ is geodesic;
- η maps 1, the identity element of M , to $\eta(1) = (1, 1)$.

Whenever confusion is unlikely, $\eta'(f)$ and $\bar{\eta}(f)$ are abbreviated to f' and \bar{f} , respectively.

EXAMPLE 3.2.1.2. Assume that M is a free abelian monoid generated by a nonempty finite totally ordered set. Define $\eta = (\eta', \bar{\eta}) : M \rightarrow M^2$ by setting $\eta'(f)$ to be the least left divisor of f lying in the generating set. Note that η here is well-defined since the left cancellation property of M implies uniqueness of right complements, so knowing $\eta'(f)$ determines $\bar{\eta}(f)$.

NOTATION 3.2.1.3. Let A be a set, and let F be a map from A^l to A^m . Then the (partial) map $F_i : A^* \rightarrow A^*$ consists of applying F to l consecutive elements starting from position i , i.e. to the elements at positions $i, i+1, \dots, i+l-1$.

EXAMPLE 3.2.1.4. For the sake of illustration, take the set $A = \{a, b, c\}$ totally ordered by $a < b < c$. We write $<^*$ for the lexicographic extension of $<$ to A^* . Let $F : A^2 \rightarrow A^2$ map each length-two word to the $<^*$ -minimal word obtained by simply permuting the letters of the argument if needed. Then, we have:

$$c|b|a \xrightarrow{F_2} c|a|b \xrightarrow{F_1} a|c|b \xrightarrow{F_2} a|b|c.$$

The multiplication in M is denoted by $\mu : M^2 \rightarrow M$, and $\mu(f, g)$ is often abbreviated to $f \cdot g$ or fg .

DEFINITION 3.2.1.5 ([35, Definition 2.1]). Let M be a monoid, and let S be a generating subset of M . A **factorability structure** on (M, S) is a factorisation map $\eta : M \rightarrow M^2$ such that, denoting the map $\eta \circ \mu : M^2 \rightarrow M^2$ by F , for every triple in M^3 , the three maps

$$F_1 F_2 F_1 F_2, \quad F_2 F_1 F_2, \quad F_2 F_1 F_2 F_1$$

coincide or each map reduces the sum of the lengths of the elements of the triple. If $\eta : M \rightarrow M^2$ is a factorability structure on (M, S) , then the triple (M, S, η) is called a **factorable monoid**.

EXAMPLE 3.2.1.6. An element of an Artin-Tits monoid $B^+(W)$ presented by (2.3.2) on page 32 is said to be square-free if it cannot be represented by a word containing a square of a generator. For an element f of M , set $\eta'(f)$ to be a maximal square-free left divisor of f . This defines a factorability structure on $B^+(W)$ with respect to the set of square-free elements, as demonstrated in [48, Section 4.3].

EXAMPLE 3.2.1.7 ([34, Example 2.2.8]). Let (M, Δ) be a Garside monoid (re-called in Subsection 2.3.4 on page 34). For an element f of M , setting $\eta'(f) = f \wedge \Delta$ defines a factorability structure, as demonstrated in [48, Section 4.2] where a larger class of monoids is equipped with a factorability structure.

Assume that (M, S, η) is a factorable monoid. The **normal form associated with the factorability structure** η , or the η -normal form, for short, is the map $\text{NF}_\eta : M \rightarrow S^*$ defined as

$$f \mapsto \eta_{|f|-1} \cdots \eta_2 \eta_1(f).$$

EXAMPLE 3.2.1.8. The map $F : A^2 \rightarrow A^2$ in Example 3.2.1.4 can be regarded as a composition $\eta \circ \mu$ of the multiplication in A^* and a factorability structure splitting off the least letter. For $f = bacabc$, we get

$$\text{NF}_\eta(f) = \eta_5 \cdots \eta_2 \eta_1(f) = (a, a, b, b, c, c).$$

For a factorable monoid (M, S, η) , an M -word x is said to be **stable at the i th position** if $F_i(x) = x$; it is **everywhere stable** if it is stable at the i th position for every i in $\{1, \dots, |x| - 1\}$. The normal form NF_η admits the following locality property.

LEMMA 3.2.1.9 ([35, Lemma 3.2]). *If (M, S, η) is a factorable monoid, then, for every f in M , the η -normal form of f is everywhere stable.*

3.2.2. Local factorability structure. There is an alternative definition of factorability, due to Moritz Rodenhausen. In order to resolve an issue detected in the original definition (to be addressed in the next subsection), we introduce the following notation.

NOTATION 3.2.2.1. Let A be a set, and let φ be a map from A^2 to itself. The composite map $\underline{\varphi} : A^* \rightarrow A^*$ is defined as

$$u \mapsto \varphi_{|u|-1} \cdots \varphi_1(u).$$

Note that, in the above composition, $\varphi_{|u|-1}$ is applied last; its argument is the rightmost length-two factor of $\varphi_{|u|-2} \cdots \varphi_1(u)$. In particular, if u has length 1, then $\underline{\varphi}(u) = u$.

DEFINITION 3.2.2.2. Let S^e be a pointed set with basepoint 1, and let φ be a map from $(S^e)^2$ to itself. The **normalisation map associated with the map φ** is a map N_φ from $(S^e)^*$ to itself, defined as follows:

- (1) N_φ of the empty word is the empty word;
- (2) N_φ of a word containing 1 equals N_φ of the same word with 1 removed;
- (3) and

$$N_\varphi(s_1, \dots, s_n) := \begin{cases} \underline{\varphi}(s_1 | N_\varphi(s_2, \dots, s_n)) & \text{if it contains no 1} \\ N_\varphi\left(\underline{\varphi}(s_1 | N_\varphi(s_2, \dots, s_n))\right) & \text{otherwise.} \end{cases}$$

REMARK 3.2.2.3. Note that, by recursion, the computation of $N_\varphi(s_1, \dots, s_n)$ terminates and its length is bounded by n . Namely, all the recursive calls for N_φ are made on words of smaller length: directly in (2) and in the first case of (3); and indirectly in the second case of (3) which calls for (2). The length of $N_\varphi(s_1, \dots, s_n)$ is less than or equal to the length of $\underline{\varphi}(s_1 | N_\varphi(s_2, \dots, s_n))$, which is less than or equal to the length of $N_\varphi(s_2, \dots, s_n)$ plus 1.

We state anew the definition of local factorability structure on (M, S) , using Definition 3.2.2.2.

DEFINITION 3.2.2.4. Let M be a monoid, and let S be a generating subset of M . A **local factorability structure** is a map φ from $(S^e)^2$ to itself, having the following properties:

- (1) M admits the presentation

$$\langle S | \{(s, t) = \varphi(s, t) | s, t \in S\} \rangle;$$

- (2) φ is idempotent;
- (3) $\varphi(1, s)$ equals $(s, 1)$ for every s in S ;
- (4) for every (r, s, t) in S^3 , the equality

$$\varphi_1 \varphi_2 \varphi_1 \varphi_2(r, s, t) = \varphi_2 \varphi_1 \varphi_2(r, s, t)$$

holds or $\varphi_2 \varphi_1 \varphi_2(r, s, t)$ contains 1;

- (5) the normalisation map associated with φ satisfies

$$N_\varphi(r, s, t) = N_\varphi(\varphi_1(r, s, t))$$

for every (r, s, t) in S^3 .

By Definition 3.2.2.2, for every S^e -word v , the word $N_\varphi(v)$ contains no 1. If we add to $N_\varphi(v)$ a string of 1's on the right, then the result is called an **extended form** of $N_\varphi(v)$. We recall a result to be used in Section 3.4.

LEMMA 3.2.2.5 ([48, Lemma 2.3.5]). *Let S^e be a pointed set with basepoint 1, and let φ be a map from $(S^e)^2$ to itself. If φ formally satisfies the second, the third and the fourth condition of Definition 3.2.2.4, then $\varphi_1\varphi_2\varphi_1\varphi_2(r, s, t)$ is an extended form of $N_\varphi(r, s, t)$ for every length-three S^e -word (r, s, t) .*

A factorability structure is equivalent to a local factorability structure, in the following sense.

THEOREM 3.2.2.6 ([35, Theorem 3.4]).

- (1) *If (M, S, η) is a factorable monoid, then the restriction of the map $\eta\mu$ to $(S^e)^2$ defines a local factorability structure on M .*
- (2) *Conversely, one can construct a factorability structure out of a local factorability structure by setting: $\eta(1) = (1, 1)$; and*

$$\eta(f) = (r_1, \text{EV}(r_2, \dots, r_m)) \text{ for } N_\varphi(u) = (r_1, r_2, \dots, r_m)$$
with u being any S^e -word representing f .
- (3) *These constructions are inverse to each other.*
- (4) *By this correspondence, for f in M , the η -normal form $\text{NF}_\eta(f)$ equals $N_\varphi(u)$ for any S^e -word u representing f .*

The proof of Theorem 3.2.2.6 can be found in [48, Section 2.3].

REMARK 3.2.2.7. Here are some observations about local factorability structures that will be used implicitly from now on.

- The property (3) of Theorem 3.2.2.6 implies $N_\varphi(s) = s$ for every s in S .
- For every length-two S^e -word (s, t) , the first element of $\varphi(s, t)$ cannot be equal to 1 unless the second element is equal to 1. Namely, assume the opposite: $\varphi(s, t) = (1, t')$ for $t' \neq 1$. Then the idempotency of φ gives $\varphi(1, t') = (1, t')$ which contradicts the property (3).
- Note that, by Theorem 3.2.2.6, the equality $\varphi(s, t) = (1, 1)$ holds if, and only if, $st = 1$ in M .

3.2.3. Deviation from the original definition. As Convention 3.2.1.1 hints, the original definition of a factorisation map, which separates a right divisor, is reformulated in this thesis to separate a left divisor, instead. The definition of local factorability structure is also modified.

Let us recall the original definition of a local factorability structure, in order to justify its present modification (Definition 3.2.2.4). For simplicity, we still assume Convention 3.1.0.2, so we do not actually copy the original *verbatim*, but we do preserve its essence (as well as the convention of indexing from the right).

Here is a recollection of [35, Definition 3.3]. Let M be a monoid, and let S be a generating subset of M . A local factorability structure on (M, S) is a map φ from $(S^e)^2$ to itself, having the following properties:

- (1) M admits the presentation

$$\langle S \mid \{(t, s) = \varphi(t, s) \mid s, t \in S\} \rangle;$$

- (2) φ is idempotent;
- (3) $\varphi(s, 1)$ equals $(1, s)$ for every s in S ;
- (4) for every (t, s, r) in S^3 , applying any φ_i to the triple $\varphi_2\varphi_1\varphi_2(t, s, r)$ leaves it unchanged, or $\varphi_2\varphi_1\varphi_2(t, s, r)$ contains 1;
- (5) $\text{NF}(t, s, r)$ equals $\text{NF}(\varphi_1(t, s, r))$ for all (t, s, r) in S^3 .

Here, the map NF from $(S^e)^*$ to itself is defined inductively on the length of the word, as follows:

- NF of the empty word is the empty word;
- $\text{NF}(s)$ equals s for all s in S ;
- NF of a word containing 1 equals NF of the same word with 1 removed;
- and

$$\text{NF}(s_n, \dots, s_1) := \begin{cases} \varphi_{n-1} \cdots \varphi_1(\text{NF}(s_n, \dots, s_2), s_1) & \text{if it contains no 1} \\ \text{NF}(\varphi_{n-1} \cdots \varphi_1(\text{NF}(s_n, \dots, s_2), s_1)) & \text{otherwise.} \end{cases}$$

REMARK 3.2.3.1. Notice that there is a flaw in the above definition. Since φ is defined on the domain $(S^e)^2$, it is not clear what should one do when one gets the expression of the form $\varphi(s)$, where s is a single element of S . However, such an expression may indeed occur if φ is applied just after NF has eliminated 1 and thus shortened the word in question. For instance, take $s_2 = 1$. Then

$$\text{NF}(s_2, s_1) = \varphi_1(\text{NF}(s_2), s_1) = \varphi_1(s_1),$$

which is not defined. This issue has been resolved by introducing Notation 3.2.2.1.

REMARK 3.2.3.2. Note that, in [35] and [48], one the maps NF is called the normal form. In this thesis (following [16]), however, a normal form is a map from the presented monoid, not from the free monoid over a generating set (recall Subsection 1.2.1 on page 11). Accordingly, we do not call the map N_φ , which is an analogue of NF , a normal form. Instead, we call such a syntactic transformation (of arbitrary words to normal ones) a normalisation map (as in Definition 3.2.2.2).

3.2.4. Rewriting system associated with factorability. A rewriting system is associated with a factorable monoid in a canonical way, as follows.

LEMMA 3.2.4.1 ([35, Lemma 5.1]). *Let (M, S, η) be a factorable monoid. If R is the set of rewriting rules of the form*

$$(s, t) \rightarrow \eta\mu(s, t)$$

for all s and t in S such that (s, t) is not stable, then (S, R) is a confluent, strongly reduced rewriting system presenting M . Here, if $\bar{\eta}(st) = 1$, then the rewriting rule is interpreted as $(s, t) \rightarrow st$.

REMARK 3.2.4.2. Observe that, strictly speaking, the presentation given by the property (1) of Definition 3.2.2.4 (of local factorability structure) is not the same as the one obtained by turning rewriting rules into equations in Lemma 3.2.4.1. Namely, if st is in S (resp. equal to 1), then the former contains the relation $(s, t) = (st, 1)$ (resp. $(s, t) = (1, 1)$), whereas the latter has $(s, t) = st$ (resp. $(s, t) = 1$). However, one can obtain the latter from the former simply by removing the rightmost occurrence of 1, and *vice versa*.

The associated rewriting system in Lemma 3.2.4.1 is not necessarily terminating, even if S is finite and M is left-cancellative. The reader is referred to [35, Appendix] for an example of a factorable monoid whose associated rewriting system is not terminating. The following result gives sufficient conditions for the rewriting system associated with a factorable monoid to be terminating.

THEOREM 3.2.4.3 ([35, Theorem 7.3]). *Let (M, S, η) be a factorable monoid. If the equalities*

$$\eta'(sf) = \eta'(s \cdot \eta'(f)), \quad \bar{\eta}(sf) = \bar{\eta}(s \cdot \eta'(f)) \cdot \bar{\eta}(f)$$

hold for all s in S and f in M , then the associated rewriting system is terminating.

3.3. Quadratic normalisations

This section recalls the notion of quadratic normalisation. After presenting basic notions concerning normalisation in Subsection 3.3.1, we recollect the notion of quadratic normalisation in Subsection 3.3.2, then we focus on a particular class of quadratic normalisations in Subsection 3.3.3. Subsection 3.3.4 recalls the rewriting system associated with a quadratic normalisation. Subsection 3.3.5 recalls the notion of left-weighted normalisation. For technical elaboration, see [16].

3.3.1. Normalisation and normal form. Having already hinted in the Introduction that a normalisation is a syntactic transformation of an arbitrary word to a normal one, here we recall a formal definition.

DEFINITION 3.3.1.1. Let A be a set, and let N be a map from A^* to A^* . A **normalisation** is a pair (A, N) if

- (1) N is length-preserving,
- (2) restriction of N to A is the identity map,
- (3) the equality

$$N(u|v|w) = N(u|N(v)|w)$$

holds for all A -words u, v, w .

The map N is called a **normalisation map**. An A -word v such that $N(v) = v$ is called **N -normal**.

A **normalisation for a monoid** M is a normalisation (S, N) such that M admits the presentation

$$\langle S \mid \{v = N(v) \mid v \in S^*\} \rangle.$$

REMARK 3.3.1.2. Recall that the usage of S , as well as the superscript e in S^e (or the absence of it), is explained by Convention 3.1.0.2. The letter S is not used in Definition 3.3.1.1, in order to point out that normalisation itself is a purely syntactic notion, as opposed to the notion of normalisation for a monoid.

EXAMPLE 3.3.1.3 ([16, Example 2.2]). Assume that $(A, <)$ is a totally ordered nonempty finite set. Let $<^*$ denote the lexicographic extension of $<$ to A^* . The image under N of an A -word is defined as the $<^*$ -minimal word obtained by permuting letters of the argument. Then (A, N) is a normalisation for the free abelian monoid over A .

REMARK 3.3.1.4. Note that the definition of normalisation for a monoid M implies that there is a nontrivial monoid homomorphism, also known as grading, from M to the multiplicative monoid of nonnegative integers, such that the degree, i.e. image under grading, of every s in S equals 1. Namely, set the degree of g in M to be the common length of all the S -words representing g (which is well-defined due to the property (1) of normalisation). Such monoids (M, S) are called **graded**. In other words, M is graded with respect to a generating set S if all S -words representing the same element of M are equal in length.

REMARK 3.3.1.5 ([16, Proposition 2.6]). In a graded monoid, a normalisation and a normal form are just different aspects of looking at the same notion. Indeed, given a normalisation (S, N) for a monoid M , a normal form NF for M with respect to S is obtained by setting $\text{NF}(g) = N(v)$ for any S -word v representing g . Conversely, if NF is a normal form for a graded monoid M with respect to a generating set S , then one obtains a normalisation map by setting $N(v) = \text{NF}(\text{EV}(v))$. These two correspondences are inverse to each other.

On the other hand, if a monoid (M, S) is not graded, i.e. if an element of M may be represented by S -words of different lengths, then a new letter representing 1 can be introduced to formally preserve length. For a normalisation (A, N) , we say that an element e of A is N -**neutral** if the equalities

$$(3.3.1) \quad N(v|e) = N(e|v) = N(v)|e$$

hold for every A -word v .

Let S^e be a pointed set with basepoint e . We say that (S^e, N) is a **normalisation mod e for a monoid M** if e is an N -neutral element of S^e and M admits the presentation

$$(3.3.2) \quad \langle S^e \mid \{v = N(v) \mid v \in (S^e)^*\} \cup \{e = 1\} \rangle.$$

Note that there can be at most one N -neutral element. We write π_e for the canonical projection from $(S^e)^*$ onto S^* , which removes all the occurrences of e . This extends the equivalence between normalisation and normal form from graded monoids to monoids in general.

PROPOSITION 3.3.1.6 ([16, Proposition 2.9]).

- (1) If (S^e, N) is a normalisation mod e for a monoid M , then a geodesic normal form NF for M with respect to S^e is obtained by setting $\text{NF}(g) = \pi_e(N(v))$ for any S^e -word v representing g .
- (2) Conversely, let NF be a geodesic normal form for a monoid M with respect to a generating set S . Write EV^e for the extension of the evaluation map $\text{EV} : S^* \rightarrow M$ to $(S^e)^*$ by putting $\text{EV}^e(e) = 1$. Then a normalisation (S^e, N) mod e for M is provided by the map

$$N(v) = \text{NF}(\text{EV}^e(v))|e^q$$

with q denoting the number of letters e to be added in order to formally preserve length, namely $q = |v| - |\text{NF}(\text{EV}^e(v))|$.

- (3) These two correspondences are inverse to each other.

EXAMPLE 3.3.1.7. For a left-cancellative monoid M with no nontrivial invertible element and a Garside family S^e in M , Proposition 2.4.4.1 on page 40 yields a geodesic normal form on (M, S) . Then, applying Proposition 3.3.1.6(2) yields a normalisation (S^e, N^S) where, formally, S^e denotes S enriched with an N -neutral letter. This normalisation is said to be **derived from the Garside family S^e** . The monoid M then admits the presentation (3.3.2). Note that the definition of the map N^S here agrees with the one given in Subsection 2.4.4 on page 39.

REMARK 3.3.1.8 ([16, Remark 2.10]). If a monoid (M, S) is graded and NF is a normal form on (M, S) , then there are two normalisations arising from NF , provided by Remark 3.3.1.5 and Proposition 3.3.1.6(2), respectively. To make a clear distinction, let us temporarily write N^e for the normalisation map of the

latter. The normalisations (S, N) and (S^e, N^e) are closely related as the map $\pi_e \circ N^e$ is identically equal to the map $N \circ \pi_e$. Therefore, it suffices to formally consider normalisations mod e for monoids.

3.3.2. Quadratic normalisations. Let us extend Notation 3.2.1.3.

NOTATION 3.3.2.1. For a finite sequence of positive integers $u = (i_1, \dots, i_q)$, we denote the composite map $F_{i_q} \circ \dots \circ F_{i_1}$ (note that F_{i_1} is applied first) by F_u , often omitting commas in u when all its components are single-digit numbers (so there is no risk of ambiguity).

If (A, N) is a normalisation, let \bar{N} denote the restriction of N to the set of all length-two A -words.

DEFINITION 3.3.2.2. A normalisation (A, N) is **quadratic** if the following two requirements are met.

- (1) An A -word v is N -normal (meaning $N(v) = v$) if, and only if, every length-two factor of v is N -normal.
- (2) For every A -word v , there exists a finite sequence of positions u , depending on v , such that $N(v) = \bar{N}_u(v)$.

EXAMPLE 3.3.2.3. The normalisation given in Example 3.3.1.3 is quadratic. Indeed, a word is $<^*$ -minimal if, and only if, its every length-two factor is $<^*$ -minimal; and each word can be ordered by switching pairs of adjacent letters that are not ordered as expected.

An advantage of a quadratic normalisation is that it is completely determined by the restriction \bar{N} .

PROPOSITION 3.3.2.4 ([16, Proposition 3.6]).

- (1) If (S, N) is a quadratic normalisation for a monoid M , then \bar{N} is idempotent and M admits the presentation

$$\langle S \mid \{s|t = \bar{N}(s|t) \mid s, t \in S\} \rangle.$$

- (2) If (A, N) is a quadratic normalisation, then an element e of A is N -neutral if, and only if, the following equalities hold for every s in A :

$$\bar{N}(s|e) = \bar{N}(e|s) = \bar{N}(s)|e.$$

- (3) If (S^e, N) is a quadratic normalisation mod e for a monoid M , then \bar{N} is idempotent and M admits the presentation

$$\langle S \mid \{s|t = \pi_e(\bar{N}(s|t)) \mid s, t \in S\} \rangle.$$

If (A, N) is a quadratic normalisation, then the image under N of an A -word is computed by sequentially applying \bar{N} to length-two factors at various positions. For length-three words, there are only two such positions. Since \bar{N} is idempotent, it suffices to consider alternating sequences of positions. This motivates the notion of class, which measures the complexity of normalising length-three words. For $m \geq 0$, we write $12[m]$ (resp. $21[m]$) for the alternating sequence $121\dots$ (resp. $212\dots$) of length m . A quadratic normalisation (A, N) is said to be of **left-class** m (respectively, **right-class** n) if, on the set of all length-three A -words, the map N coincides with the map $\bar{N}_{12[m]}$ (respectively, $\bar{N}_{21[n]}$). A quadratic normalisation (A, N) is of **class** (m, n) if it is of left-class m and of right-class n .

EXAMPLE 3.3.2.5. Let us consider the quadratic normalisation given in Example 3.3.1.3. If A has only one element, then there is only one A -word of length 3 and it is N -normal. So, (A, N) is of class $(0, 0)$ in this case.

If A has at least two elements, then each of the maps \overline{N}_{121} and \overline{N}_{212} produces N -normal words. Namely, assuming that the word $a|b|c$ is normal, compute \overline{N}_{121} and \overline{N}_{212} of the word $c|b|a$ (which provides the worst-case scenario):

$$c|b|a \xrightarrow{\overline{N}_1^1} b|c|a \xrightarrow{\overline{N}_2^2} b|a|c \xrightarrow{\overline{N}_1^1} a|b|c, \quad c|b|a \xrightarrow{\overline{N}_2^2} c|a|b \xrightarrow{\overline{N}_1^1} a|c|b \xrightarrow{\overline{N}_2^2} a|b|c.$$

Therefore, (A, N) is of class $(3, 3)$ in this case. A smaller class cannot be obtained in general, as witnessed by $\overline{N}_{12}(b|b|a) = b|a|b$ and $\overline{N}_{21}(b|a|a) = a|b|a$, with a and b being any two distinct elements of A such that $a < b$.

A class of a quadratic normalisation (A, N) can be characterised by relations involving only the restriction \overline{N} , as follows.

PROPOSITION 3.3.2.6 ([16, Proposition 3.14]). *A quadratic normalisation (A, N) is*

- (1) *of left-class m if, and only if, the following three maps coincide on A^3 :*

$$\overline{N}_{12[m]}, \quad \overline{N}_{12[m+1]}, \quad \overline{N}_{21[m+1]}.$$

- (2) *of right-class n if, and only if, the following three maps coincide on A^3 :*

$$\overline{N}_{21[n]}, \quad \overline{N}_{21[n+1]}, \quad \overline{N}_{12[n+1]}.$$

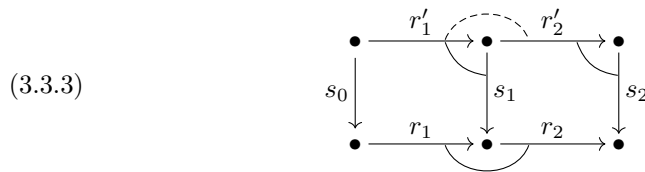
- (3) *of class (m, m) if, and only if, the map $\overline{N}_{12[m]}$ coincides with the map $\overline{N}_{21[m]}$ on A^3 .*

The **minimal left-class** of (A, N) is the smallest natural number m such that (A, N) is of left-class m if such m exists, and ∞ otherwise. The minimal right-class n of (A, N) is defined analogously. Then the minimal class of (A, N) is the pair (m, n) .

LEMMA 3.3.2.7 ([16, Lemma 3.13]). *The minimal class of a quadratic normalisation is either of the form (m, n) with $|m - n| \leq 1$, or (∞, ∞) .*

For example, the minimal class of the quadratic normalisation considered in Example 3.3.2.5 is $(3, 3)$ if the generating set has at least two elements.

3.3.3. Quadratic normalisations of class $(4, 3)$. The class $(4, 3)$ has particularly nice computational properties (see [16, Section 4]), not shared by higher classes (see [16, Example 3.23]), thanks to the diagrammatic tool called the domino rule. Let A be a set, and let F be a map from A^2 to itself. We say that the **domino rule**¹ is valid for F if, for all $r_1, r_2, r'_1, r'_2, s_0, s_1, s_2$ in A such that $F(s_0|r_1) = r'_1|s_1$ and $F(s_1|r_2) = r'_2|s_2$, the following implication holds: if $r_1|r_2$ and $r'_1|s_1$ and $r'_2|s_2$ are fixed points of F , then so is $r'_1|r'_2$. The domino rule is expressed by the commutative diagram



¹A more precise name would be the left domino rule (see e.g. [12]).

where arcs denote fixed points of F : the solid ones are assumptions, and the dashed one is the expected conclusion.

PROPOSITION 3.3.3.1 ([**16**, Lemma 4.2]). *A quadratic normalisation (A, N) is of class $(4, 3)$ if, and only if, the domino rule is valid for \overline{N} .*

The domino rule allows one to devise a simple universal recipe for computing the images under a normalisation map. We recall the notation needed to express it.

NOTATION 3.3.3.2. For a positive integer n , the finite sequence δ_n of positive integers is inductively defined as follows:

$$\delta_n := \begin{cases} \text{the empty sequence} & \text{if } n = 1 \\ \text{sh}(\delta_{n-1}) |1|2| \cdots |n-1 & \text{if } n > 1, \end{cases}$$

where sh denotes the shifting map which increases every entry by 1. So, we have $\delta_2 = 1$, $\delta_3 = 2|1|2$, $\delta_4 = 3|2|3|1|2|3$, $\delta_5 = 4|3|4|2|3|4|1|2|3|4$, etc.

For the sake of illustration, let us combine Notations 3.2.1.3 and 3.3.3.2.

EXAMPLE 3.3.3.3. Let us consider an instance of the class of monoids given in Example 3.3.1.3. Take $A = \{a, b, c\}$ and $a < b < c$. Then, $c|b|a \mapsto \overline{N}_{\delta_3}(c|b|a)$ consists of the following steps:

$$c|b|a \xrightarrow{\overline{N}_2} c|a|b \xrightarrow{\overline{N}_1} a|c|b \xrightarrow{\overline{N}_2} a|b|c.$$

We observe that $\overline{N}_{\delta_3}(c|b|a)$ is equal to $N(c|b|a)$, meaning that the corresponding quadratic normalisation is of right-class 3 (which agrees with Example 3.3.2.5).

Now we can formulate the aforementioned recipe for normalising arbitrarily long words.

PROPOSITION 3.3.3.4 ([**16**, Proposition 4.4]). *If (A, N) is a quadratic normalisation of class $(4, 3)$, then, for every positive integer n and every A -word v of length n , we have*

$$N(v) = \overline{N}_{\delta_n}(v).$$

The key step in proving this proposition is the following lemma.

LEMMA 3.3.3.5 ([**16**, Lemma 4.5]). *If (A, N) is a quadratic normalisation of class $(4, 3)$, then, for every s in A and every N -normal A -word $r_1| \cdots |r_m$, we have*

$$N(s|r_1| \cdots |r_m) = \overline{N}_{1|2| \cdots |m-1|m}(s|r_1| \cdots |r_m).$$

REMARK 3.3.3.6. Note, in particular, that Lemma 3.3.3.5 implies that the leftmost letter of $N(s|r_1| \cdots |r_m)$ does not depend on r_2, \dots, r_m but only on s and r_1 . We will use this observation in Subsection 3.4.2.

By Proposition 3.3.2.6(2), if (A, N) is a quadratic normalisation of class $(4, 3)$, then the maps \overline{N}_{1212} , \overline{N}_{212} and \overline{N}_{2121} coincide on A^3 . One of the major results of [**16**] is the converse: every idempotent map satisfying such condition arises from a quadratic normalisation of class $(4, 3)$, in the following sense.

PROPOSITION 3.3.3.7 ([**16**, Proposition 4.7]). *Let A be a set, and let F be a map from A^2 to itself. If F is idempotent, and if the maps F_{1212} , F_{212} and F_{2121} coincide on A^3 , then there is a quadratic normalisation (A, N) of class $(4, 3)$ such that the map \overline{N} is identically equal to the map F .*

Thus, quadratic normalisations of class $(4, 3)$ are fully axiomatised.

3.3.4. The rewriting system associated with a quadratic normalisation. There is a simple correspondence between quadratic normalisations and quadratic rewriting systems.

PROPOSITION 3.3.4.1 ([16, Proposition 3.7]).

- (1) *If (S, N) is a quadratic normalisation for a monoid M , then a quadratic, reduced, normalising and confluent rewriting system (S, R) presenting M , is obtained by defining R as the set of rewriting rules of the form*

$$s|t \rightarrow \overline{N}(s|t)$$

for all s and t in S such that $s|t$ is not N -normal.

- (2) *Conversely, if (S, R) is a quadratic, reduced, normalising and confluent rewriting system presenting a monoid M , then (S, N) is a quadratic normalisation for M , with*

$$N(v) = \widehat{v}$$

for every S -word v .

- (3) *These two correspondences are inverse to each other.*

Proposition 3.3.4.1(1) can be adapted to a case when there is an N -neutral element. In fact, an N -neutral element does not affect termination of the associated rewriting system.

PROPOSITION 3.3.4.2 ([16, Proposition 3.9]).

- (1) *If (S^e, N) is a quadratic normalisation mod e for a monoid M , then a reduced, normalising and confluent rewriting system (S, R) presenting M , is obtained by defining R as the set of rewriting rules of the form*

$$s|t \rightarrow \pi_e(\overline{N}(s|t))$$

for all s and t in S such that $s|t$ is not N -normal.

- (2) *If the rewriting system in Proposition 3.3.4.1(1) terminates, then so does the one in (1).*

The rewriting system associated with a quadratic normalisation in Proposition 3.3.4.1 need not be terminating (see [16, Proposition 5.7]). However, it is terminating for quadratic normalisations of class (4, 3).

PROPOSITION 3.3.4.3 ([16, Proposition 5.4]). *A rewriting system associated to a quadratic normalisation of class (4, 3) is convergent. More precisely, every rewriting sequence starting from an element of length n has length at most $2^n - n - 1$.*

3.3.5. Left-weighted normalisations. Before closing this section, let us recall (from [16, Subsection 6.2]) a notion that we glimpsed in Lemma 2.4.4.2. A normalisation (A, N) for a monoid M is called **left-weighted** if, for all s, t, s', t' in A , the equality $s'|t' = \overline{N}(s|t)$ implies the left divisibility $s \preceq s'$ in M . Reminding the reader of the notion of normalisation derived from a Garside family, defined in Example 3.3.1.7, we recall a characterisation of such normalisations among quadratic normalisations of class (4, 3).

PROPOSITION 3.3.5.1 ([16, Proposition 6.10]). *Let M be a left-cancellative monoid M containing no nontrivial invertible element. If (S^e, N) is a quadratic normalisation mod 1 for M , then the following are equivalent.*

- (1) The family S^e is a Garside family in M , and the map N is identically equal to the map N^s .
- (2) The normalisation (S^e, N) is of class $(4, 3)$ and is left-weighted.

REMARK 3.3.5.2. In particular, Proposition 3.3.5.1(1) means that, under given conditions, every N -normal word $r_1 | \cdots | r_m$ has the following property for all $i < m$,

$$\text{for all } s \in S \text{ and } h \in M, \quad s \preceq hr_i r_{i+1} \text{ implies } s \preceq hr_i.$$

It is the property of being S -greedy, expressed diagrammatically *mutatis mutandis* by (2.4.1) on page 38.

3.4. A correspondence between factorability structures and quadratic normalisations in monoids

In this section, a correspondence between factorability structures and quadratic normalisations is established. Subsection 3.4.1 gives a characterisation of factorable monoids in terms of quadratic normalisations. Subsection 3.4.2 shows that, although a quadratic normalisation corresponding to a factorable monoid is not of class $(4, 3)$ in general, it is so if a defining condition of local factorability structure is strengthened in a suitable way. Finally, this strengthened definition is shown to imply the additional assumption introduced in [35] (and recalled in Theorem 3.2.4.3) in order to reach termination of the associated rewriting system.

3.4.1. Characterisation of factorable monoids. In this subsection, a necessary and sufficient condition is given, in terms of quadratic normalisations, for a monoid to be factorable. This is achieved through a syntactic correspondence between a local factorability structure and the restriction of a quadratic normalisation map to length-two words. The property (4) of Definition 3.2.2.4 (of local factorability structure) is going to be essential in deriving main result of the present chapter so, for convenience, we are going to express it compactly using the domino rule.

DEFINITION 3.4.1.1. Let (A, N) be a quadratic normalisation with the N -neutral element e . The **weak domino rule** is valid for \overline{N} if the domino rule is valid for \overline{N} whenever none of the elements r'_1, r'_2, s_2 of the diagram (3.3.3) equals e .

Now, we can state the main result of the current chapter.

THEOREM 3.4.1.2. *A monoid (M, S) is factorable if, and only if, it admits a quadratic normalisation (N, S^e) mod 1 such that the weak domino rule is valid for \overline{N} .*

The rest of this subsection presents a (quite straightforward) proof, relying on the relation between factorability structure and local factorability structure, as stated in Theorem 3.2.2.6. First we verify that a factorability structure yields a quadratic normalisation, rather canonically. Assume that (M, S, η) is a factorable monoid. Since N_φ is not length-preserving in general, it does not make a suitable candidate for a normalisation map in the sense of Subsection 3.3.1. To repair this, let us introduce the following notation.

NOTATION 3.4.1.3. Let (M, S, η) be a factorable monoid. Denote by N'_φ the pointwise length-preserving extended form of N_φ , defined as follows:

$$v \mapsto N_\varphi(v) | 1^q, \text{ with } q = |v| - |N_\varphi(v)|.$$

LEMMA 3.4.1.4. *If (M, S, η) is a factorable monoid, then (S^e, N'_φ) is a quadratic normalisation mod 1 for M .*

PROOF. Let us check that (S^e, N'_φ) has, *mutatis mutandis*, the three properties of Definition 3.3.1.1 (of normalisation). Properties (1) and (2) are satisfied by construction. Indeed, N'_φ is length-preserving and $N'_\varphi(s) = s$ for every s in S^e , by definition. Lemma 3.2.1.9 implies the property (3), namely

$$N'_\varphi(u|v|w) = N'_\varphi(u|N'_\varphi(v)|w).$$

We verify that the obtained normalisation (S^e, N'_φ) is quadratic. The property (3) of Definition 3.2.2.2 (of the normalisation map N'_φ) implies the property (2) of Definition (3.3.2.2) (of quadratic normalisation) and, consequently, also the right-to-left implication of the property (1) of Definition (3.3.2.2). The other direction of the property (1) of Definition (3.3.2.2) follows from Lemma 3.2.1.9.

Finally, Lemma 3.2.4.1, together with Remark 3.2.4.2, provides a presentation showing that (S^e, N'_φ) is a normalisation mod 1 for M . \square

We say that (S^e, N'_φ) is the **quadratic normalisation corresponding to the factorability structure η** .

REMARK 3.4.1.5. A note on formality is in order before we continue. As announced by Remark 3.1.0.1, we have tried to respect the two original conventions: of taking a generating set S^e to be a subset of a factorable monoid; and of distinguishing S^e from its image under an evaluation map arising from a normalisation. Now, however, that we consider those quadratic normalisations (S^e, N) which correspond to a factorability structure, we interchangeably write e and 1 for an N -neutral S^e -letter, depending on whether we take the viewpoint of normalisation or factorability.

REMARK 3.4.1.6. Note that, by construction, the restriction $\overline{N'_\varphi}$ of N'_φ to length-two words is identically equal to the local factorability structure of (M, S, η) . This fact will be often used implicitly in the rest of the present chapter.

Having obtained a quadratic normalisation, we want to determine its class.

LEMMA 3.4.1.7. *If (M, S, η) is a factorable monoid, then the quadratic normalisation corresponding to η is of class $(5, 4)$.*

PROOF. Denote by φ the local factorability structure corresponding to η in the sense of Theorem 3.2.2.6. If (r, s, t) is a length-three S^e -word, then Lemma 3.2.2.5 says that the word $N'_\varphi(r, s, t)$ equals the word $\varphi_{2121}(r, s, t)$. We conclude that quadratic normalisation (S^e, N'_φ) is of right-class 4. Then Lemma 3.3.2.7 grants the left-class 5. \square

The following example demonstrates that the minimal right-class of a quadratic normalisation corresponding to a factorability structure is not smaller than 4, in general.

EXAMPLE 3.4.1.8 ([34, Example 2.1.13]). Consider the monoid $(\mathbb{Z}, +)$ with respect to the generating set $\{-1, +1\}$. The factorisation map is defined by $g \mapsto (\text{sgn}(g), g - \text{sgn}(g))$, where $\text{sgn} : \mathbb{Z} \rightarrow \{-1, 0, +1\}$ denotes the sign function. One

can check that this is a factorable monoid.² Note that $\varphi_{212}(1, -1, -1)$ equals $(0, -1, 0)$, whereas $\varphi_{2121}(1, -1, -1)$ equals $(-1, 0, 0)$. Therefore, the minimal right-class of the corresponding quadratic normalisation is at least 4; then it is exactly 4, by Lemma 3.4.1.7.

The next example (for which the reader is referred to [35, Proposition .7]) shows that the minimal left-class of a quadratic normalisation corresponding to a factorability structure is not smaller than 5 in general.

EXAMPLE 3.4.1.9. Consider the factorable monoid M given in [35, Proposition .7]. Computing $\varphi_{1212}(c_1, b_2, a_2)$ produces

$$(c_1, b_1, a_1) \xrightarrow{\varphi_1^1} (c_1, b_1, a_1) \xrightarrow{\varphi_2^2} (c_1, b_2, a_2) \xrightarrow{\varphi_1^1} (c_2, b_3, a_2) \xrightarrow{\varphi_2^2} (c_2, e_2, 1),$$

whereas

$$\varphi_{12121}(c_1, b_2, a_2) = \varphi_1(c_2, e_2, 1) = (g_2, f_2, 1).$$

Hence the minimal left-class of the corresponding quadratic normalisation is at least 5.

The above computation also gives

$$\varphi_{212}(c_1, b_1, a_1) = (c_2, e_2, 1),$$

whereas

$$\varphi_{2121}(c_1, b_1, a_1) = \varphi_1(c_2, e_2, 1) = (g_2, f_2, 1).$$

Thus, the minimal right-class is at least 4, as expected according to Lemma 3.3.2.7.

The previous two examples witness that the estimate of class in Lemma 3.4.1.7 is as good as one can hope for. On the other hand, observe that not every quadratic normalisation of class $(5, 4)$ is corresponding to a factorability structure, as demonstrated by the following example (adapted from [16, Example 3.15]).

EXAMPLE 3.4.1.10. Let $A = \{a, b_1, b_2, b_3, b_4, b_5\}$, and let R consist of the rules $ab_i \rightarrow ab_{i+1}$ for $i < 5$ even and $b_i a \rightarrow b_{i+1} a$ for $i < 5$ odd. The rewriting system (A, R) is clearly quadratic and reduced. Notice that it is also terminating because each rewriting rule only increases the index of a letter b in a word. Furthermore, it is confluent, as illustrated by the following diagram (recall that, by Theorem 2.1.2.1 on page 21, it suffices to investigate the critical branchings; or see [41, Theorem 0] for the same fact expressed without using the polygraphic setting):

$$\begin{array}{ccc}
 & b_{i+1}ab_j & \\
 \nearrow & & \searrow \\
 b_iab_j & & b_{i+1}ab_{j+1} \\
 \searrow & & \nearrow \\
 & b_iab_{j+1} &
 \end{array}$$

Denote by (A, N) the quadratic normalisation associated with (A, R) by Proposition 3.3.4.1. Let us determine the minimal class of (A, N) . If a length-three word does not begin and end with the letter a , then it is either N -normal or it becomes

²Technically speaking, this is a factorable group by [50, Example 3.2.2], and a weakly factorable monoid. Hence, it is also a factorable monoid by [34, Proposition 2.1.28].

N -normal in a single step. On the other hand, normalising $a|b_1|a$ takes four steps starting from the right:

$$a|b_1|a \begin{pmatrix} \xrightarrow{\overline{N}_1} a|b_1|a \\ \xrightarrow{\overline{N}_2} a|b_2|a \\ \xrightarrow{\overline{N}_1} a|b_3|a \\ \xrightarrow{N_2} a|b_4|a \\ \xrightarrow{\overline{N}_1} a|b_5|a \end{pmatrix}$$

Thus, the minimal class of (A, N) is $(5, 4)$.

However, the normalisation (A, N) is not corresponding to any factorability structure. To see this, observe that

$$\overline{N}_{212}(a, b_1, a) = (a, b_4, a),$$

whereas

$$\overline{N}_{2121}(a, b_1, a) = (a, b_5, a).$$

Thus, \overline{N} fails to admit the property (4) of Definition 3.2.2.4 (of local factorability structure). We conclude (by Remark 3.4.1.6) that \overline{N} does not correspond to a factorability structure.

A natural question to ask is: among quadratic normalisations of class $(5, 4)$, what distinguishes those which do correspond to a factorability structure? Example 3.4.1.10 suggests a candidate (by what it is lacking): simply impose the weak domino rule upon quadratic normalisation of class $(5, 4)$. Before testing sufficiency of these two properties combined, let us notice that they are not independent from each other.

LEMMA 3.4.1.11. *Let (A, N) be a quadratic normalisation having an N -neutral element e . If the weak domino rule is valid for \overline{N} , then (A, N) is of class $(5, 4)$.*

PROOF. Let (r, s, t) be an A -word. If

$$\overline{N}_{2121}(r, s, t) = \overline{N}_{212}(r, s, t),$$

then

$$N(r, s, t) = \overline{N}_{212}(r, s, t),$$

so it takes at most three steps starting with \overline{N}_2 to normalise (r, s, t) .

Otherwise, $\overline{N}_{212}(r, s, t)$ contains e by assumption. Denote

$$(3.4.1) \quad (a, b, c) := \overline{N}_{212}(r, s, t).$$

Case 1. If e occurs exactly once in the triple (a, b, c) , then it cannot be at the leftmost position after \overline{N}_1 is applied in (3.4.1). So, it has to be at the rightmost position after the last \overline{N}_2 is applied in (3.4.1). In other words, c has to be equal to e . Consequently,

$$(3.4.2) \quad N(a, b, c) = \overline{N}_1(a, b, c) = \overline{N}_{2121}(r, s, t).$$

Case 2. If e occurs exactly twice in the triple (a, b, c) , then either $a \neq e$ or $b \neq e$. In either case, the equalities (3.4.2) hold.

Case 3. If e occurs three times in the triple (a, b, c) , then clearly the equalities (3.4.2) hold.

Therefore, (A, N) is of right class 4, hence of class $(5, 4)$, by Lemma 3.3.2.7. \square

The following proposition shows that adding the weak domino rule to the properties of quadratic normalisation (and thus granting class $(5, 4)$ by Lemma 3.4.1.11), suffices to yield factorability.

PROPOSITION 3.4.1.12. *Let (S^e, N) be a quadratic normalisation mod e for a monoid M . If the weak domino rule is valid for \bar{N} , then \bar{N} is a local factorability structure.*

PROOF. The following list shows that the map \bar{N} has, *mutatis mutandis*, all the properties of Definition 3.2.2.4 (of local factorability structure).

- (1) By Proposition 3.3.2.4(3), M admits the presentation

$$\langle S \mid \{s|t = \pi_e(\bar{N}(s|t)) \mid s, t \in S\} \rangle.$$

Although this presentation is not the same as the one in Definition 3.2.2.4, the two are equivalent (see Remark 3.2.4.2).

- (2) By the property (3) of Definition 3.3.1.1 (of normalisation), \bar{N} is idempotent.
- (3) By Proposition 3.3.2.4(2), the equality $\bar{N}(e|s) = s|e$ holds for all $s \in S^e$.
- (4) This property is explicitly assumed.
- (5) By Lemma 3.4.1.11, the normalisation (S^e, N) is of class (5, 4). By the definition of right-class 4, the N -normal word $N(r, s, t)$ equals $\bar{N}_{2121}(r, s, t)$ which further equals $\bar{N}_{12121}(r, s, t)$ by the definition of left-class 5. Hence, $N(r, s, t)$ equals $(N \circ \bar{N}_1)(r, s, t)$ for all (r, s, t) in S^3 . □

We have, thus, proved one direction of Theorem 3.4.1.2. The other one follows from Remark 3.4.1.6, as the restriction to length-two words of a quadratic normalisation corresponding to a factorability structure has all the properties of the corresponding local factorability structure and, in particular, the weak domino rule is valid. Thereby, we have completed the proof of Theorem 3.4.1.2.

The following corollary is an immediate consequence.

COROLLARY 3.4.1.13.

- (1) *Associating a factorability structure to a quadratic normalisation such that the weak domino rule is valid, and associating a quadratic normalisation (the weak domino rule is valid automatically) to a factorability structure, as given above, are inverse transformations.*
- (2) *Normal forms with respect to a factorability structure and to the corresponding quadratic normalisation are the same.*
- (3) *The rewriting systems associated with a factorability structure and the corresponding quadratic normalisation are equivalent, the only difference being dummy letters to preserve length in the latter.*

3.4.2. Factorability in relation to quadratic normalisation of class (4, 3). Having established a general correspondence between a factorability structure and a quadratic normalisation in the previous subsection, we are now going to further elaborate these links in the case when the quadratic normalisation involved is of class (4, 3). First we emphasise a particular, yet important, consequence of Proposition 3.4.1.12.

COROLLARY 3.4.2.1. *If (S^e, N) is a quadratic normalisation of class (4, 3) mod 1 for a monoid M , then \bar{N} is a local factorability structure.*

Although the converse does not hold in general, it does so in the case of graded monoids (as defined in Subsection 3.3.1).

LEMMA 3.4.2.2. *If (M, S, η) is a graded factorable monoid, then (S, N_φ) is a quadratic normalisation of class $(4, 3)$.*

PROOF. In addition to the conclusion of Lemma 3.4.1.4, observe that the property of being of right-class 3 follows from the property (4) of Definition 3.2.2.4 (of local factorability structure) and the assumption that M is graded. Then the property of being of left-class 4 follows from Lemma 3.3.2.7. \square

Let us point out precisely which defining properties of quadratic normalisation of class $(4, 3)$ mod 1 for M do not necessarily arise from a factorability structure on M . To put it another way, we are looking for a (not too restrictive) property that would complement a factorability structure to a quadratic normalisation of class $(4, 3)$. The property (4) of Definition 3.2.2.4 (of local factorability structure) grants the equality

$$(3.4.3) \quad \varphi_{2121}(r, s, t) = \varphi_{212}(r, s, t)$$

only for length-three S -words (r, s, t) such that $\varphi_{212}(r, s, t)$ contains no 1. On the other hand, the right-class 3 (i.e. the domino rule) requires the equality (3.4.3) to hold for every length-three S^e -word (r, s, t) , regardless of whether $\varphi_{212}(r, s, t)$ contains 1 or not. Therefore, in order for a factorability structure to induce a quadratic normalisation of class $(4, 3)$, it suffices to strengthen the condition (4) of Definition 3.2.2.4, as follows.

PROPOSITION 3.4.2.3. *Let (M, S, η) be a factorable monoid. If the equality (3.4.3) holds for each (r, s, t) in S^3 such that $\varphi_{212}(r, s, t)$ contains 1, then (S^e, N'_φ) is a quadratic normalisation of class $(4, 3)$ mod 1 for M .*

PROOF. Lemma 3.4.1.4 says that (S^e, N'_φ) is a quadratic normalisation mod 1 for M .

To obtain the class $(4, 3)$, what remains to be shown is that the equality (3.4.3) holds for all (r, s, t) in $(S^e)^3 \setminus S^3$. If t equals 1, then $\varphi_1(r, s, t)$ equals $\varphi(r, s) | 1$, which is everywhere stable. If s equals 1, then $\varphi_{21}(r, s, t)$ equals $\varphi(r, t) | 1$, which is everywhere stable. If r equals 1, then $\varphi_{121}(r, s, t)$ equals $\varphi_{212}(r, s, t)$ which equals $\varphi(s, t) | 1$, which is, again, everywhere stable. \square

We have shown that a quadratic normalisation of class $(4, 3)$ yields a factorability structure (Corollary 3.4.2.1), but not *vice versa* (Examples 3.4.1.8 and 3.4.1.9, or Theorem 3.4.1.2). However, a factorability structure does yield a quadratic normalisation of class $(4, 3)$ under a stronger condition on local factorability (Proposition 3.4.2.3). Therefore, under the same condition, the rewriting system associated with a factorable monoid is terminating, by Proposition 3.3.4.3 and Corollary 3.4.1.13(3).

From another point of view, recall that Theorem 3.2.4.3 ensures termination of the rewriting system associated with a factorable monoid, under an additional assumption on the factorisation map. It is then natural to ask what is the relation between the additional condition of Proposition 3.4.2.3 and the additional assumption of Theorem 3.2.4.3, which are both known to ensure termination of the associated rewriting system.

In the rest of the present subsection, we investigate the relation between these two optional properties of a factorable monoid (M, S, η) : for each (r, s, t) in S^3 such

that $\varphi_{212}(r, s, t)$ contains 1, the condition

$$(3.4.4) \quad (\eta\mu)_{2121}(r, s, t) = (\eta\mu)_{212}(r, s, t);$$

and, for all s in S and f in M , the assumption

$$(3.4.5) \quad (sf)' = (sf')', \quad \overline{sf} = \overline{sf'} \cdot \overline{f}.$$

REMARK 3.4.2.4. Note that the assumption (3.4.5) is trivially valid in the case where f lies in S^e or in the case where (s, f) is stable.

It is already known that the assumption (3.4.5) implies the condition (3.4.4), as follows.

LEMMA 3.4.2.5 ([35, Lemma 7.1]). *Let (M, S, η) be a factorable monoid. If the additional assumption (3.4.5) is valid for all s in S and f in M , then the maps $(\eta\mu)_{212}$ and $(\eta\mu)_{2121}$ coincide on M^3 .*

COROLLARY 3.4.2.6. *Let (M, S, η) be a factorable monoid. If the additional assumption (3.4.5) is valid for all s in S and f in M , then the condition (3.4.4) is satisfied for all (r, s, t) in S^3 .*

For the reader's convenience, we adapt the proof of Lemma 3.4.2.5 here, to prove Corollary 3.4.2.6.

PROOF. If the assumption (3.4.5) is valid for all s in S and f in M , then the maps $\eta_1\mu_1$ and $\mu_2\eta_1\mu_1\eta_2$ coincide on $S \times M$. Composing each of these two maps with μ_2 and then composing η_2 with the obtained composite map, we see that the maps $\eta_2\eta_1\mu_1\mu_2$ and $\eta_2\mu_2\eta_1\mu_1\eta_2\mu_2$ coincide on $S \times M \times M$. Note that the map $\eta_2\eta_1$ produces the η -normal form by definition; and that the restriction of the map $\eta_2\mu_2\eta_1\mu_1\eta_2\mu_2$ to $S \times S \times S$ coincides with the map $\varphi_2\varphi_1\varphi_2$. Therefore, every image under $\varphi_2\varphi_1\varphi_2$ is everywhere stable by Lemma 3.2.1.9. Thus, the condition (3.4.4) is satisfied for all (r, s, t) in S^3 . \square

In the opposite direction, a partial result is already known. Namely, (3.4.4) implies (3.4.5) under certain (quite restrictive) additional requirements imposed on both the monoid and the normalisation. The next lemma is a straightforward adaptation of [13, Proposition IV.1.49]; it is also hinted in the proof of [48, Corollary 7.4.5].

REMARK 3.4.2.7. Relying on Theorem 3.4.1.2 and Corollary 3.4.1.13(2) in particular, we are going to abuse terminology by saying ‘the normal form’ without specifying whether it arises from factorability or normalisation.

LEMMA 3.4.2.8. *Let M be a left-cancellative monoid containing no nontrivial invertible element. If (S^e, N) is a left-weighted quadratic normalisation of class (4, 3) mod 1 for M , then the assumption (3.4.5) is valid for all s in S and f in M .*

PROOF. We need to show that the equalities $(sf)' = (sf')'$ and $\overline{sf} = \overline{sf'} \cdot \overline{f}$ hold for all s in S and f in M .

By the definition of factorability structure, we have $(sf')' \preceq sf'$, and $f' \preceq f$ hence also $sf' \preceq sf$. The transitivity gives $(sf')' \preceq sf'$, and the assumption that (S^e, N) is left-weighted then yields $(sf')' \preceq (sf)'$.

By Corollary 3.4.1.13(2), the normal form of f has the form $f'|r_2|\cdots|r_m$. Hence, $(sf)' \preceq sf = sf'r_2\cdots r_m$. Since $(sf)'$ lies in S^e and $f'|r_2|\cdots|r_m$ is normal,

we obtain $(sf)' \preceq sf'$ by Proposition 3.3.5.1 and Remark 3.3.5.2 (which deem this normal form S -greedy).

Since M is a left-cancellative monoid having no nontrivial invertible element, we conclude that $(sf)' = (sf')'$. Then $\overline{sf} = \overline{sf'} \cdot \overline{f}$ follows from the left cancellation property. \square

We want to find out whether the condition (3.4.4) is as strong as the assumption (3.4.5) in general (i.e. without all the additional requirements of the previous lemma) or is the latter strictly stronger than the former. Let us start by considering length-two words.

LEMMA 3.4.2.9. *Let (M, S, η) be a factorable monoid. If the condition (3.4.4) is satisfied for each (r, s, t) in S^3 such that $\varphi_{212}(r, s, t)$ contains 1, then the assumption (3.4.5) is valid for all s in S and length-two f in M .*

PROOF. The idea is to equate two different expressions of the normal form of sf , in order to obtain (3.4.5). Fix arbitrary s in S and a length-two element f of M .

First compute the η -normal form of sf , by definition:

$$sf \xrightarrow{\eta_1} ((sf)', \overline{sf}) \xrightarrow{\eta_2} ((sf)', \overline{sf'}, \overline{\overline{sf}}).$$

Since the length of f equals 2, we know that \overline{f} lies in S . Hence, (s, f', \overline{f}) lies in S^3 . So we have another length-three S -word evaluating to sf . By the property (4) of Definition 3.2.2.4 (of local factorability structure) and the condition (3.4.4) of the statement, the word $(\eta\mu)_{212}(s, f', \overline{f})$ is everywhere stable, hence also normal by Lemma 3.4.1.4 and the property (1) of Definition 3.3.2.2 (of quadratic normalisation). Therefore, $(\eta\mu)_{212}(s, f', \overline{f})$ is the N'_φ -normal form of the evaluation of (s, f', \overline{f}) , i.e. of sf . Since computing $(\eta\mu)_{212}(s, f', \overline{f})$ depends on whether any 1 occurs in the process, we consider the following cases.

Case 1. If $sf' \notin S^e$, then

$$(s, f', \overline{f}) \xrightarrow{(\eta\mu)_2} (s, f', \overline{f}) \xrightarrow{(\eta\mu)_1} ((sf')', \overline{sf'}, \overline{f}) \xrightarrow{(\eta\mu)_2} ((sf')', (\overline{sf'} \cdot \overline{f})', \overline{\overline{sf'} \cdot \overline{f}}).$$

Comparing the result to $\text{NF}_\eta(sf) = ((sf)', \overline{sf'}, \overline{\overline{sf}})$ yields (3.4.5), by Corollary 3.4.1.13(2). Indeed, equating the first components of the obtained normal forms gives $(sf)' = (sf')'$. Equating the second and the third components gives $\overline{sf} = \overline{sf'} \cdot \overline{f}$.

Case 2. If sf' is an element of S , then

$$(3.4.6) \quad (s, f', \overline{f}) \xrightarrow{(\eta\mu)_2} (s, f', \overline{f}) \xrightarrow{(\eta\mu)_1} ((sf')', 1, \overline{f}) \xrightarrow{(\eta\mu)_2} ((sf')', \overline{f}, 1).$$

Again, equating the first components of the result and $\text{NF}_\eta(sf)$ gives $(sf)' = (sf')'$. Equating the second and the third components now gives $\overline{sf} = \overline{f}$, but note that, in the present case, this equality is equivalent to $\overline{sf} = \overline{sf'} \cdot \overline{f}$.

Finally, observe that the case $sf' = 1$ cannot occur. Namely, if sf' were equal to 1, then applying $(\eta\mu)_1$ after (3.4.6) would yield $(\overline{f}, 1, 1)$ since \overline{f} is in S , and that would contradict the condition (3.4.4). \square

Lemma 3.4.2.9 suggests itself as the base case for an induction, which we are going to achieve in Proposition 3.4.2.11. First we introduce some notation, in order to simplify exposition.

NOTATION 3.4.2.10. If (r_1, r_2, \dots, r_m) is the normal form of f in M , then the product $r_1 r_2 \cdots r_{m-1}$ in M is denoted by \underline{f} .

PROPOSITION 3.4.2.11. *Let (M, S, η) be a factorable monoid. If the condition (3.4.4) is satisfied for each (r, s, t) in S^3 such that $\varphi_{212}(r, s, t)$ contains 1, then the assumption (3.4.5) is valid for all s in S and f in M .*

PROOF. By Proposition 3.4.2.3, the quadratic normalisation corresponding to the given factorability structure is of class (4, 3). Then, Remark 3.3.3.6 (on Lemma 3.3.3.5) implies the equality $(sf)' = (sf')'$ for all s in S and f in M .

We need to show that the equality $\overline{sf} = \overline{sf'} \cdot \underline{f}$ also holds for all s in S and f in M . We prove this using induction on the length of f . Let $P(n)$ be the statement: the equality $(sf)' = (sf')'$ holds for all s in S and f of length n in M . The statement $P(2)$ (resp. $P(1)$) holds by Lemma 3.4.2.9 (resp. trivially).

Let m be an integer greater than 2, and suppose that the statement $P(m-1)$ holds. Fix an arbitrary s in S and f of length m in M . Since the length of \underline{f} equals $m-1$, the equality $\overline{sf} = \overline{sf'} \cdot \underline{f}$ holds by inductive hypothesis.³ Denote the normal form of f by (r_1, r_2, \dots, r_m) . Multiplying the equality $\overline{sf} = \overline{sf'} \cdot \underline{f}$ by r_m on the right yields

$$(3.4.7) \quad \overline{sf} \cdot r_m = \overline{sf'} \cdot \underline{f}.$$

Hence it suffices to show that the equality $\overline{sf} \cdot r_m = \overline{sf}$ holds.

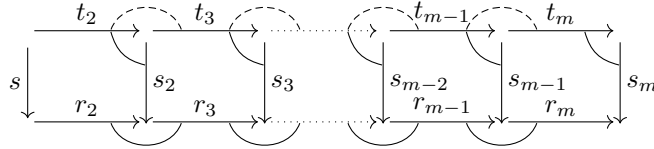
Let us compute the normal form of \overline{sf} and \overline{sf} using Lemma 3.3.3.5. Denoting $t_i | s_i := \overline{N}(s_{i-1} | r_i)$ for i in $\{2, \dots, m-1\}$, we obtain

$$\text{NF}(\overline{sf}) = t_2 | t_3 | \cdots | t_{m-1} | s_{m-1}$$

and

$$\text{NF}(\overline{sf}) = t_2 | t_3 | \cdots | t_m | s_m,$$

as displayed by the diagram



with arcs having the same meaning as in the diagram (3.3.3). Therefore, $\overline{sf} = t_2 t_3 \cdots t_{m-1} s_{m-1}$. Multiplying by r_m on the right yields

$$\overline{sf} \cdot r_m = t_2 t_3 \cdots t_{m-1} s_{m-1} r_m = t_2 t_3 \cdots t_{m-1} t_m = \overline{sf},$$

which, together with the equality (3.4.7), implies $P(m)$. \square

The results of the present subsection enable the following characterisation.

PROPOSITION 3.4.2.12. *Let (M, S, η) be a factorable monoid. Then the following properties are equivalent.*

³Statement $P(1)$ could not serve as the base case because \underline{f} in the inductive hypothesis would not be defined, which is why we needed Lemma 3.4.2.9.

(1) For all s in S and f in M , the equalities

$$(sf)' = (sf')', \quad \bar{s}f = \overline{sf'} \cdot \bar{f}$$

hold.

(2) For every (f, g, h) in M^3 , the equality

$$(\eta\mu)_{2121}(f, g, h) = (\eta\mu)_{212}(f, g, h)$$

holds.

(3) For each (r, s, t) in S^3 such that $\varphi_{212}(r, s, t)$ contains 1, the equality

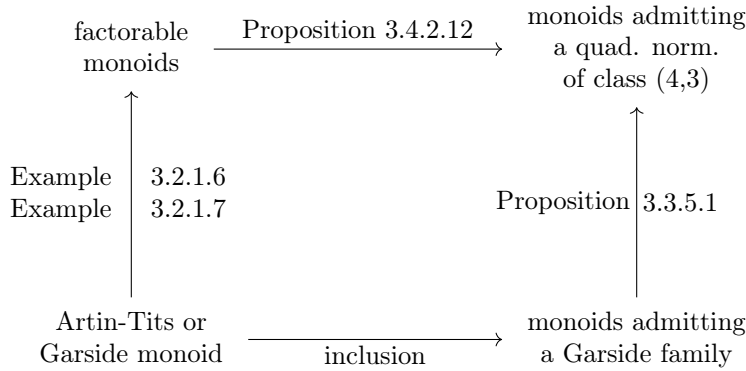
$$(\eta\mu)_{2121}(r, s, t) = (\eta\mu)_{212}(r, s, t)$$

holds.

(4) The quadratic normalisation (S^e, N'_φ) mod 1 for M is of class (4, 3). In other words, the domino rule is valid for $\overline{N'_\varphi}$.

PROOF. It was already known (Lemma 3.4.2.5) that (1) implies (2), which, in turn, clearly implies (3). The properties (3) and (4) are equivalent by the definitions of the notions concerned. Finally, Proposition 3.4.2.11 says that (3) implies (1). \square

REMARK 3.4.2.13. Reminding the reader of Subsection 2.3, we point out that applying Proposition 3.4.2.12 to the factorable monoids given in Examples 3.2.1.6 and 3.2.1.7, both having property (3) yields special cases of a normalisation provided by Proposition 3.3.5.1 when the Garside family involved is a Coxeter group corresponding to an Artin-Tits monoid and a set of divisors of a Garside element in a Garside monoid, respectively. The commutative diagram



summarises the point.

Before leaving this chapter, let us observe that, thanks to Theorem 3.4.1.2, Proposition 3.4.2.12 can also be read another way, as a characterisation of monoids admitting a quadratic normalisation of class (4, 3) among factorable monoids.

PROPOSITION 3.4.2.14. *A monoid (M, S) admits a quadratic normalisation of class (4, 3) if, and only if, it is factorable and has any of the properties (1), (2) and (3) of Proposition 3.4.2.12.*

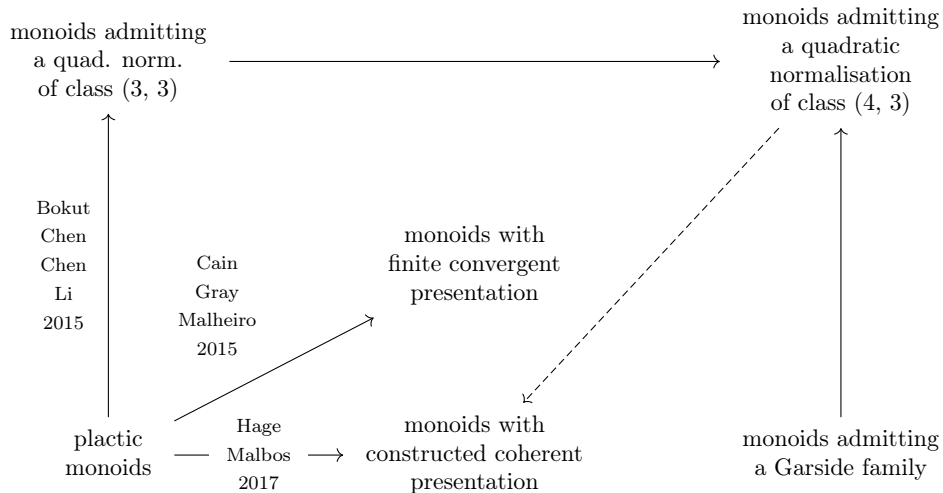
Perspectives. A convergent rewriting system for a monoid gives rise to a collapsing scheme on the bar complex of the monoid (see [6] for demonstration, and [10] for elaboration). It thus provides a smaller complex of the same homotopy type and, hence, computes the homology of the monoid. Somewhat surprising is the result [35, Proposition 9.5] which constructs a collapsing scheme on the bar complex of a factorable monoid. Although this construction is similar to the one given in [6] and [10] for the bar complex of a monoid admitting a convergent rewriting system, it technically circumvents a requirement for the rewriting system to be terminating. As a result, there is a quite small complex computing the homology of factorable monoids.

A tentative goal in the current direction is to develop effective methods for computing homology for certain classes of monoids. We would start by using the established correspondence between factorability structure and quadratic normalisation to adapt the results concerning resolutions arising from the former to the framework of the latter. Then we would try to generalise [35, Proposition 9.5] to monoids admitting a quadratic normalisation of class $(5, 4)$.

Let us sketch another tentative goal. In [16, Example 5.8], it was demonstrated that the Chinese monoid C_X , defined in [9], admits a quadratic normalisation of class $(4, 4)$ if the generating set X has three elements. Guiraud also proved, in an unpublished note, that C_X admits a quadratic normalisation of class $(5, 4)$ if X has four or five elements. Encouraged by Lemma 3.4.1.7, we would use the criterion provided by Theorem 3.4.1.2 to investigate if these Chinese monoids are factorable.

Coherent presentations arising from class $(4, 3)$

In this final chapter, methods considered in Chapter 2 are further applied to a wider class of monoids studied in Chapter 3. Namely, the homotopical completion-reduction procedure is used to construct coherent presentations of monoids admitting a quadratic normalisation of class $(4, 3)$, which generalise the column coherent presentations of plactic monoids. Here is an overview of relevant classes of monoids, with arrows denoting inclusions, the dashed one being the present contribution:



Section 4.1 recalls plactic monoids, focusing on their column coherent presentation. Section 4.2 proposes a construction of coherent presentations for a class of monoids admitting a quadratic normalisation of class $(4, 3)$. Finally, Section 4.3 shows that this construction specialises to that for plactic monoids, given in [33].

4.1. The column coherent presentation of plactic monoids

Subsection 4.1.1 briefly recollects the notion of plactic monoid (for elaboration, the reader is referred to [42]). In Subsection 4.1.2, we recall the notion of column coherent presentation of a plactic monoid, introduced in [33] as an extension of the column presentation introduced in [7].

For simplicity, juxtaposition is used to denote the product in a free monoid in the present section.

4.1.1. Plactic monoids and Young tableaux. Throughout this section, we assume that (A, \leq) is a totally ordered finite set, also called alphabet, unless stated

otherwise. The **plactic monoid** (of type A) over the alphabet A , here denoted by P_A , is the monoid generated by A and subject to the relations

$$(4.1.1) \quad \{acb = cab \mid a \leq b < c\} \cup \{bca = bac \mid a < b \leq c\},$$

called **elementary Knuth transformations**.

An element (a_1, \dots, a_m) of A^* is called a **row** if it is nondecreasing, that is if $a_1 \leq \dots \leq a_m$. Given two rows $u = (a_1, \dots, a_m)$ and $v = (b_1, \dots, b_n)$, we say that u **dominates** v , written $u \triangleright v$, if $m \leq n$ and if, for every k in $\{1, \dots, m\}$, the relation $a_k > b_k$ holds. Relying on the context to provide distinction, here we use the same symbol \leq to denote two different total orders: the former occurrence represents the usual total order on \mathbb{N} , whereas the latter denotes the total order on A .

Every word u in A^* can be decomposed into rows of maximal length, say $u = u_1 u_2 \dots u_q$. A word u is called a **Young tableau**, or simply a tableau¹, over A if every such factor, except for the last one, dominates the next one, i.e. if $u_1 \triangleright u_2 \triangleright \dots \triangleright u_q$. We write YT_A for the set of tableaux on A . A tableau $u_1 u_2 \dots u_q$ can be graphically represented in a planar form, by arranging rows left-aligned and ordered by domination, with u_1 being the top row.²

Let us illustrate all these notions by means of an example.

EXAMPLE 4.1.1.1. Let $A = \{1, 2, \dots, 100\}$ with the usual order. The word $(2, 3, 1)$ is not a tableau because the first maximal-length row $(2, 3)$ is longer than the second one (1) , so $(2, 3) \not\triangleright (1)$. The word $(1, 3, 1, 2, 3)$ is not a tableau because the first element of the first maximal-length row $(1, 3)$ is not (strictly) greater than the first element of the second maximal row $(1, 2, 3)$, so $(1, 3) \not\triangleright (1, 2, 3)$. The word $(3, 2, 3, 5, 7, 1, 1, 2, 3, 5, 11, 13)$ is a tableau. Its planar representation is

3						
2	3	5	7			
1	1	2	3	5	11	13

An element (a_m, \dots, a_1) of A^* is called a **column** if it is (strictly) decreasing, that is if $a_m > \dots > a_1$. Note that the indices are also written in a decreasing order. A partial order \succeq on the set of columns in A^* is defined as follows. Given two columns $u = (a_m, \dots, a_1)$ and $v = (b_n, \dots, b_1)$, we write $u \succeq v$, if $m \geq n$ and if, for every k in $\{1, \dots, n\}$, the relation $a_k \leq b_k$ holds. Note that, in the planar representation of a tableau, this definition agrees with the usual meaning of a column (e.g. in a matrix or a table).

Tableaux can be read row by row or column by column. The row (resp. column) reading of a tableau t is the product in A^* of all the rows (resp. columns) of t , from top to bottom (resp. from left to right). For example, the column reading of the tableau in Example 4.1.1.1 is $(3, 2, 1, 3, 1, 5, 2, 7, 3, 5, 11, 13)$. The row and column readings provide injective maps from the set of tableaux on A to the free monoid A^* .

There is a product $P : YT_A \times A \rightarrow A^*$, computed as follows. Let $u = u_1 \dots u_q$ be a tableau in A^* , and b a letter in A . Then the tableau that represents the

¹As we do not consider any old tableau in this thesis.

²There is also the opposite convention, with u_1 being the bottom row (see e.g. [24]).

product ub equals

$$P(ub) = \begin{cases} u_1 \cdots u_q b & \text{if } u_q b \text{ is a row} \\ P(u_1 \cdots u_{q-1} b') u'_q & \text{otherwise} \end{cases}$$

where b' denotes the leftmost letter of u_q such that $b' > b$ holds in A , and u'_q denotes the row obtained from u_q by replacing b' with b .

This product is iteratively extended to a product $P : YT_A \times A^* \rightarrow A^*$. Namely, for a tableau u and a word $v = (b_1, \dots, b_n)$, the product $P(uv)$ is calculated as

$$(4.1.2) \quad P(uv) = P(P(ub_1)(b_2, \dots, b_n)).$$

EXAMPLE 4.1.1.2. Let us compute $P(ub)$, with u being the tableau given in Example 4.1.1.1 and b being the A -letter 8. Since the A -word $(1, 1, 2, 3, 5, 11, 13, 8)$ is not a row, we look for the leftmost letter of $(1, 1, 2, 3, 5, 11, 13)$ that is greater than 8. Here, it is 11. So, we replace 11 with 8 in the last row of u to get the last row of $P(ub)$: $(1, 1, 2, 3, 5, 8, 13)$. By the above recipe, we proceed to compute the tableau which represents the product of the tableau

3			
2	3	5	7

and the letter 11. Since $(2, 3, 5, 7, 11)$ is a row, the computation stops, returning the tableau

3						
2	3	5	7	11		
1	1	2	3	5	8	13

In particular, the product (4.1.2) allows one to assign a tableau to a word u in A^* by computing $P(eu)$ with e denoting the empty tableau. The resulting tableau is denoted by $P(u)$.

THEOREM 4.1.1.3 ([36, Theorem 6]). *Let A be a totally ordered finite set. Then, for u and v in A^* , the tableaux $P(u)$ and $P(v)$ are equal if, and only if, u and v represent the same element of the plactic monoid P_A . Furthermore, the product in YT_A that is defined as*

$$(r, s) \mapsto P(r, \text{ any word represented by } s)$$

agrees with the product in P_A .

In particular, the evaluation map $A^* \rightarrow P_A$ has a section whose image consists of all tableaux on A . Put another way, tableaux provide a normal form for P_A . So, one can obtain the normal form of any A -word u by computing $P(u)$. Note that the tableau representing the column reading of a tableau u is u itself ([42, Problem 5.2.4]).

We recall an important combinatorial property of tableaux, to be used in Section 4.3. A **subsequence** of an A -word u is obtained by removing some letters from u , without changing the order of the remaining letters.

THEOREM 4.1.1.4 ([42, Theorem 5.1.1]). *Let A be a totally ordered finite set, a let u be an A -word. Then the number of columns in $P(u)$ equals the maximal length of a nondecreasing subsequence of u ; and the number of rows in $P(u)$ equals the maximal length of a decreasing subsequence of u .*

Observe that a subsequence of a word u is not necessarily a factor of u since a factor of u consists of neighbouring letters of u . Consequently, a nondecreasing subsequence having the maximal length is not to be confused with a row of maximal length. Theorem 4.1.1.4 does say, however, that the lengths in question are equal.

4.1.2. The column coherent presentation. The presentation (4.1.1) of a plactic monoid, equipped with the so-called degree lexicographic order, can be completed to a finite convergent presentation if the cardinality of the generating set equals 3, but not so if the cardinality is greater than 3, as shown in [38]. However, having extended the generating set by adjoining superfluous generators standing for columns, one does obtain a finite convergent presentation, as demonstrated in [4] and [7].

Let us recollect the column presentation introduced in [7]. A key lemma for constructing the column presentation is the following.

LEMMA 4.1.2.1 ([7, Lemma 3.1]). *Let A be a totally ordered finite set, and let u and v be columns in A^* such that $u \not\leq v$, i.e. uv is not a tableau. Then the tableau $P(uv)$ has at most two columns. If, additionally, $P(uv)$ has exactly two columns, then the left column of $P(uv)$ is longer than u .*

Introducing a formal symbol \widehat{u} for a column u in A^* , denote

$$\text{Col}_1(A) = \{\widehat{u} \mid u \text{ is a column in } A^*\}$$

and consider the 2-polygraph $\text{Col}_2(A)$ consisting of a single generating 0-cell, the set $\text{Col}_1(A)$ for the set of generating 1-cells, and the generating 2-cells constructed as follows. For every pair of generating 1-cells \widehat{u} and \widehat{v} such that $\widehat{u}\widehat{v}$ is not a tableau, there is one generating 2-cell

$$\widehat{u}\widehat{v} \Rightarrow P(uv)$$

whose form depends on the number of columns in $P(uv)$:

$$\alpha_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{x}$$

if $P(uv)$ contains only one column x ;

$$\nu_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{x}\widehat{y}$$

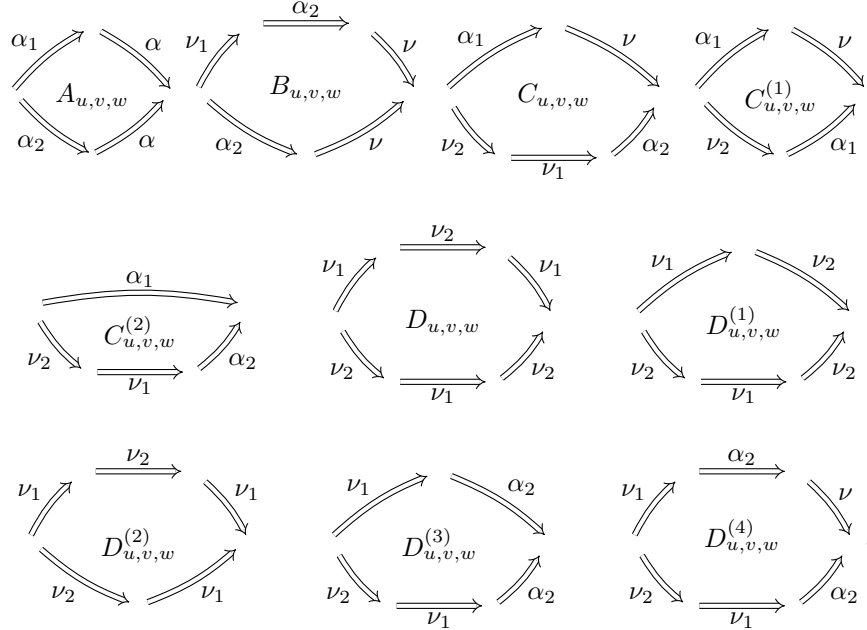
if $P(uv)$ contains two columns, with x and y being the left and right column, respectively.³ All possible cases are, thus, taken into account since, by Lemma 4.1.2.1, the tableau $P(uv)$ can have at most two columns.

It is shown in [7, Section 3] that $\text{Col}_2(A)$ is a finite convergent presentation of P_A , called the **column presentation**.

NOTATION 4.1.2.2. Following Notation 3.2.1.3, we often denote by α_i (resp. ν_i) a 2-cell consisting of the generating 2-cell α (resp. ν) applied to letters at positions i and $i + 1$, leaving other letters unchanged. This is not to be confused with the notation $\alpha_{s,t}$ (resp. $\nu_{s,t}$) where indices denote the letters to which the corresponding generating 2-cell is applied.

³Neither [7] nor [33] uses distinct letters to denote these two forms of arrows.

In [33], the homotopical completion-reduction procedure was performed on the 2-polygraph⁴ $\text{Col}_2(A)$. The resulting **column coherent presentation** of the plactic monoid P_A is the (3, 1)-polygraph $\text{Col}_3(A)$ that extends the 2-polygraph $\text{Col}_2(A)$ with the following families of generating 3-cells:⁵



For simplicity, 1-cells are left out of the diagrams. The sources of all the critical branchings involved are length-three $\text{Col}_1(A)$ -words having no normal length-two factor (i.e. of which no length-two factor forms a tableau, by Proposition 4.1.1.3). Detailed conditions can be found in [33, 3.2.1].

4.2. Coherent presentations of monoids admitting a quadratic normalisation of class (4, 3)

This section presents the final main result of this thesis: a construction of coherent presentations arising from quadratic normalisation of class (4, 3). Section 4.2.1 considers the case of graded monoids. Section 4.2.2 considers the general case, i.e. when there is a quadratic normalisation of class (4, 3) mod e for M , with e being an N -neutral element of S .

4.2.1. The quadratic coherent presentation: the graded case. Assume that (S, N) is a quadratic normalisation of class (4, 3) for a monoid M . Then Propositions 3.3.4.1(1) and 3.3.4.3 on page 77 provide a quadratic reduced convergent presentation of M . We denote this 2-polygraph by $\text{Qua}'_2(S)$ and the generating 2-cells by

$$\nu_{s,t} : s|t \Rightarrow N(s|t)$$

⁴Denoted by $\text{Col}_2(n)$ in [33], where the understood alphabet is $A = \{1, 2, \dots, n\}$.

⁵In [33], the fourth and the fifth generating 3-cell are respectively labeled by C and C' . We have relabeled them here, for convenience.

for all s and t in S such that $s|t$ is not N -normal. Let us apply the homotopical completion procedure to $\text{Qua}'_2(S)$.

Critical branchings are of the form

$$\begin{array}{ccc}
 & & N(r|s)|t \\
 & \nearrow^{\nu_{r,s}|t} & \\
 r|s|t & & \\
 & \searrow_{r|\nu_{s,t}} & \\
 & & r|N(s|t)
 \end{array}$$

where $r|s|t$ is a length-three word, of which no length-two factor is N -normal. This branching is confluent by the very definition of the class (4, 3). Relying on Notation 4.1.2.2, we adjoin a generating 3-cell

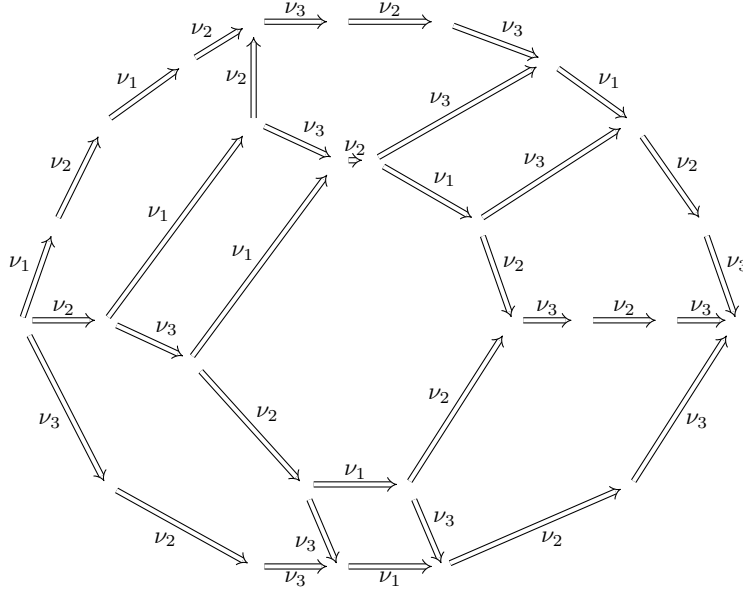
$$\begin{array}{ccccc}
 & & \xrightarrow{\nu_2} & \xrightarrow{\nu_1} & \\
 \nu_1 \nearrow & & & & \searrow \nu_2 \\
 r|s|t & & G_{r,s,t} & & N(r|s|t) \\
 \nu_2 \searrow & & & & \nearrow \nu_2 \\
 & & \xrightarrow{\nu_1} & &
 \end{array}$$

for every such length-three word $r|s|t$ (intermediate 1-cells are left out of the diagram for pragmatic reasons). Let us denote the resulting (3, 1)-polygraph by $\text{Qua}'_3(S)$. Theorem 2.2.2.1 implies that $\text{Qua}'_3(S)$ is a convergent coherent presentation of M .

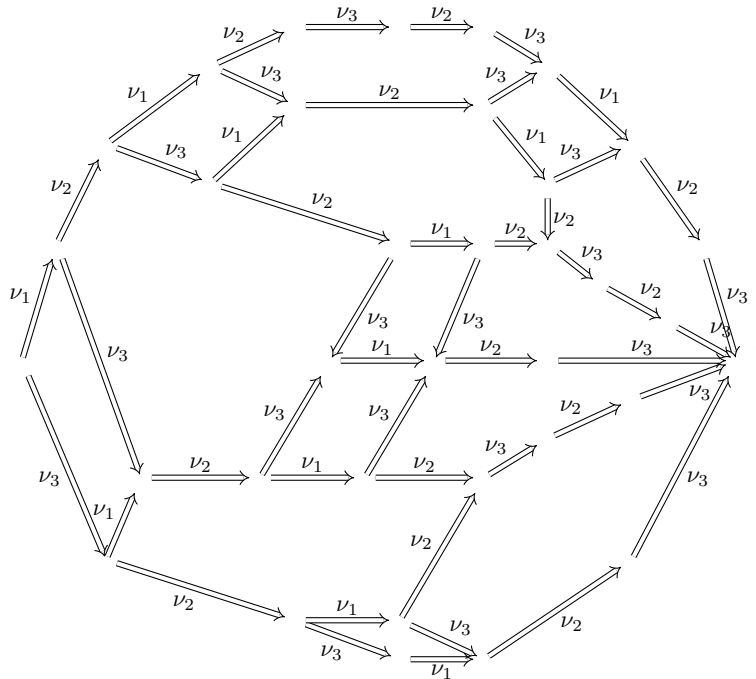
It would be convenient to perform the homotopical reduction procedure on the (3, 1)-polygraph $\text{Qua}'_3(S)$. Let us try to find redundant generating 3-cells by using the particular technique, described in Subsection 2.2.5, which is successfully applied to Garside's coherent presentations in Subsection 2.5.4. Note that there is no critical triple branching starting with a word of length less than 4. Consider a critical triple branching of the form

$$\begin{array}{ccc}
 & & N(q|r)|s|t \\
 & \nearrow^{\nu_1} & \\
 q|r|s|t & \xrightarrow{\nu_2} & q|N(r|s)|t \\
 & \searrow_{\nu_3} & \\
 & & q|r|N(s|t)
 \end{array}$$

where $q|r|s|t$ is a length-four word having no N -normal length-two factor. It produces a 3-sphere Φ whose boundary has the following two parts:⁶



and



Every heptagonal 3-cell is the aforementioned generating 3-cell G , whereas rectangular 3-cells are identities. For simplicity, we display neither 1-cells nor generating 3-cells. Observe that every generating 3-cell in the 3-sphere Φ occurs in a

⁶To the best of our knowledge, it was Matthieu Picantin who first did the calculation.

nonempty context. On the other hand, in order for this generating triple confluence to actually eliminate a generating 3-cell, that generating 3-cell ought to occur (only once and) in an empty context. Consequently, this 3-sphere cannot eliminate any generating 3-cell, using the technique described in Subsection 2.2.5. Furthermore, since all the generating 3-cells affect only length-three factors and since all the generating 2-cells preserve length (in other words, since the monoid is graded), a generating triple confluence starting with a word of length 4 or longer cannot eliminate a generating 3-cell because there is always a nonempty context. Similarly, no generating 2-cell can be eliminated in this way.

4.2.2. The quadratic coherent presentation: the general case. Now, assume that (S, N) is a quadratic normalisation of class (4, 3) mod e for a monoid M . Then Propositions 3.3.4.2(1) and 3.3.4.3 on page 77 provide a quadratic reduced convergent presentation of M . We denote this 2-polygraph by $\text{Qua}_2(S)$. Note that there are two families of generating 2-cells in $\text{Qua}_2(S)$. For s and t in S such that $s|t$ is not N -normal, there is at most one of the following two generating 2-cells: the triangular one

$$a_{s,t} : s|t \Rightarrow st = (\pi_e \circ N)(s, t)$$

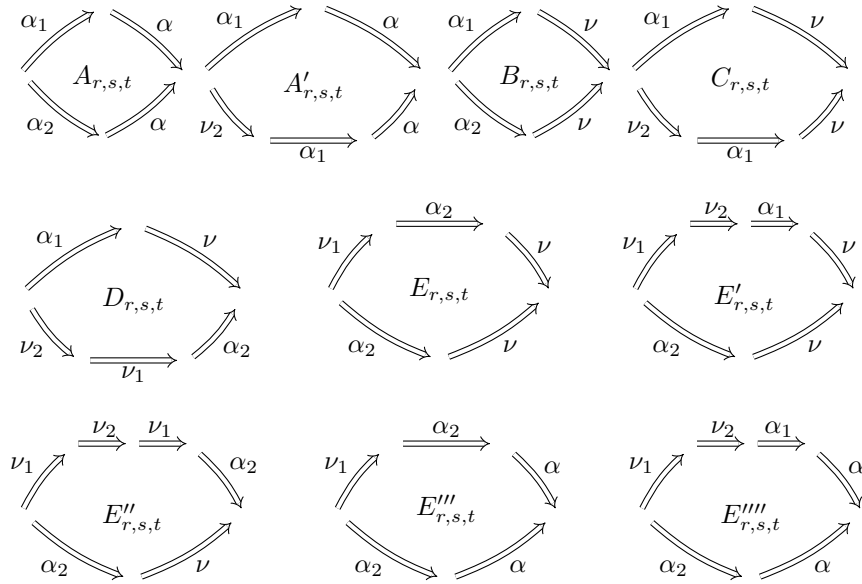
if st is an element of S ; and the quadratic one

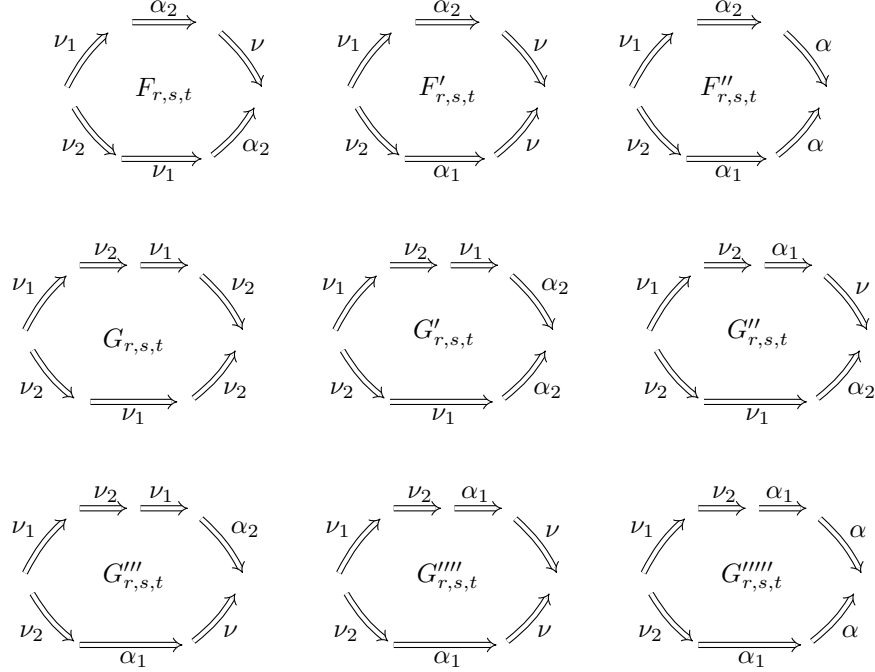
$$\nu_{s,t} : s|t \Rightarrow N(s|t)$$

if st is not an element of S .

Reminding the reader of Notation 4.1.2.2, let us perform the homotopical completion procedure on the 2-polygraph $\text{Qua}_2(S)$.

PROPOSITION 4.2.2.1. *Let (S, N) be a quadratic normalisation of class (4, 3) mod e for a monoid M . Then the convergent 2-polygraph $\text{Qua}_2(S)$ admits, as a homotopical completion, the (3, 1)-polygraph $\text{Qua}_3(S)$ that extends $\text{Qua}_2(S)$ with the following families of generating 3-cells, indexed by all possible elements of $S \setminus \{1\}$ (the source of every branching is $r|s|t$, and the rightmost 1-cell is $N(r|s|t)$):*





While some of the branchings may look similar to those in Proposition 2.5.3.1 on page 46, their completions greatly differ because, unlike the generating 2-cell β in Proposition 2.5.3.1, the generating 2-cell ν here necessarily normalises its length-two argument. For instance, here in a more general context, the second main case for the critical branchings in the proof of Proposition 2.5.3.1 does not occur at all (hence, there are no analogues of the generating 3-cells H and I here) because there is at most one generating 2-cell $\nu_{s,t}$ for a each pair s and t . Besides, a care must be taken here not to rely on any specific properties of Garside families.

PROOF. By Propositions 3.3.4.2(1) and 3.3.4.3, the 2-polygraph $\text{Qua}_2(S)$ is convergent, so we can construct a Squier completion.

Let us consider all possible critical branchings. The source of a critical branching has to be a length-three word because a length-two word is the source of at most one generating 2-cell. The elements of $S \setminus \{1\}$ that constitute the source determine the arrangement of the generating 2-cells α and ν occurring in (the completion of) the branching. Note that the sources of generating 2-cells forming such a branching have to overlap on exactly one element of $S \setminus \{1\}$. The following restrictions apply.

- The number of the generating 2-cells α in the upper branch must equal the number of the generating 2-cells α in the lower branch. The reason for this is the fact that the generating 2-cells α do not preserve length of their arguments, while the generating 2-cells ν do.
- Each branch can have at most two generating 2-cells α . The reason for this is that, after two generating 2-cells α , a length-three word becomes a length-one word, i.e. an element of S and thus normal by the property (2) of Definition 3.3.1.1 (of normalisation).

- If two generating 2-cells α occur in a branch (upper or lower), then they have to occur in two consecutive rewriting steps, i.e. there can be no generating 2-cell ν between them. The reason for this is that the generating 2-cell ν after the generating 2-cell α would result in a normal form.

We start by considering critical branchings consisting only of the generating 2-cells α . There is only one such critical branching for each triple (r, s, t) of elements of $S \setminus \{1\}$ such that rs and st lie in S :

$$\begin{array}{ccc}
 & & rs|t \\
 & \nearrow^{\alpha_{r,s}|t} & \\
 r|s|t & & \\
 & \searrow_{r|\alpha_{s,t}} & \\
 & & r|st
 \end{array}$$

If the rst also lies in S , then the homotopical completion procedure adjoins the generating 3-cell $A_{r,s,t}$. Otherwise, the generating 3-cell $B_{r,s,t}$ is adjoined.

Next we consider critical branchings containing the generating 2-cell ν . Since the source of a branching has length 3, one of the 2-cells forming a branching changes the leftmost two generating 1-cells of the source and the other one changes the rightmost two. There are three distinct forms of such branchings. In order to facilitate discussing the cases, we first display a ‘generic’ 2-sphere, that is to say relying only on the defining properties of the class (4, 3) without specifying the forms of the generating 2-cells involved:

$$\begin{array}{ccccccc}
 & & r_1|s_1|t_1 & \xrightarrow{2} & r_3|s_3|t_3 & \xrightarrow{1} & r_5|s_5|t_5 & \xrightarrow{2} & & \\
 & \nearrow^1 & & & & & & & & \\
 r|s|t & & & & & & & & & r_7|s_7|t_7 \\
 & \searrow_2 & & & & & & & = & \\
 & & r_2|s_2|t_2 & \xrightarrow{1} & r_4|s_4|t_4 & & & & & r_6|s_6|t_6 \\
 & & & & & & & & & \\
 & & & & & & & & &
 \end{array}$$

Here, a 2-cell denoted by i in $\{1, 2\}$ consists of a yet unspecified generating 2-cell applied to elements at positions i and $i + 1$, leaving an element on the possible remaining position unchanged (recall Notation 4.1.2.2). Some of the generic letters displayed here are obviously redundant, others may become redundant in some specific cases. For instance, the equalities $t = t_1$, $r = r_2$ always hold; and, if st lies in S , then it is interpreted as $st = r_1$, thus making the letter s_1 (and, hence, all the odd-indexed letters s_{2k+1}) the empty word.

The form

$$\begin{array}{ccc}
 & \alpha_{r,s}|t & \nearrow rs|t = r_1|t_1 \\
 r|s|t & & \\
 & r|\nu_{s,t} & \searrow r|N(s|t) = r_2|s_2|t_2
 \end{array}$$

is defined under the conjunction: rs lies in S , and st does not lie in S . This condition splits into three mutually exclusive cases yielding three families of generating 3-cells as displayed in the following table:

	$r_1t_1 \in S$	$r_1t_1 \notin S$
$r_2s_2 \in S$	$A'_{r,s,t}$	$C_{r,s,t}$
$r_2s_2 \notin S$	\emptyset	$D_{r,s,t}$

The branching

$$\begin{array}{ccc}
 & \nu_{r,s}|t & \nearrow N(r|s)|t = r_1|s_1|t_1 \\
 r|s|t & & \\
 & r|\alpha_{s,t} & \searrow r|st = r_2|s_2
 \end{array}$$

is defined under the conjunction: rs does not lie in S , and st lies in S . This condition splits into four disjoint cases yielding four families of generating 3-cells as displayed in the following table:

	$s_1t_1 \in S$	$s_1t_1 \notin S$
$r_2s_2 \in S$	$E'''_{r,s,t}$	$E''''_{r,s,t}$
$r_2s_2 \notin S$	$E_{r,s,t}$	$E'_{r,s,t}$ if $r_3s_3 \in S$ $E''_{r,s,t}$ if $r_3s_3 \notin S$

The form

$$\begin{array}{ccc}
 & \nu_{r,s}|t & \nearrow N(r|s)|t \\
 r|s|t & & \\
 & r|\nu_{s,t} & \searrow r|N(s|t)
 \end{array}$$

is defined under the condition that neither rs nor st lie in S . This condition splits into two mutually exclusive cases depending on whether r_2s_2 is an element of S . Each of these cases further splits, as shown in the following tables: for r_2s_2 lying in S ,

	$s_1t_1 \in S$	$s_1t_1 \notin S$
$r_4t_4 \in S$	$F''_{r,s,t}$	$G''''_{r,s,t}$
$r_4t_4 \notin S$	$F'_{r,s,t}$	$G'''_{r,s,t}$ if $r_3s_3 \in S$ $G'_{r,s,t}$ if $r_3s_3 \notin S$

and for r_2s_2 not lying in S ,

	$s_1t_1 \in S$	$s_1t_1 \notin S$
$s_4t_4 \notin S$	\emptyset	$G_{r,s,t}$
$s_4t_4 \in S$	$F_{r,s,t}$	$G''_{r,s,t}$ if $r_3s_3 \in S$ $G'_{r,s,t}$ if $r_3s_3 \notin S$

□

4.3. Specialisation to plactic monoids

This final section shows that Proposition 4.2.2.1 generalises [33, Theorem 1]. We begin with the coherent presentation $\underline{\text{Qua}}_3(S)$ and assume additionally that M is the plactic monoid P_A and that S is the set of columns in A^* . These additional assumptions make conditions for certain branchings of $\underline{\text{Qua}}_3(S)$ unattainable, resulting in a simpler presentation.

To facilitate reference, we display a corollary of Theorem 4.1.1.4.

LEMMA 4.3.0.1. *Let A be a totally ordered finite set. Then, a word u in A^* is a column if, and only if, $P(u)$ has exactly one column.*

PROOF. If $P(u)$ has exactly one column, then the maximal length of nondecreasing subsequence of u equals 1, by Theorem 4.1.1.4. Hence, u is a column. Conversely, if u is a column, then the tableau $P(u)$ has exactly one column (equal to u) by the defining computation of $P(u)$. □

Now, we demonstrate that the column coherent presentation of the plactic monoid is a special case of the coherent presentation $\underline{\text{Qua}}_3(S)$.

COROLLARY 4.3.0.2. *Let A be the set $\{1, \dots, n\}$, and let S be the set $\text{Col}_1(A)$. Then the $(3, 1)$ -polygraph $\underline{\text{Qua}}_3(S)$ coincides with the $(3, 1)$ -polygraph $\text{Col}_3(A)$.*

PROOF. The idea is that, in this particular context, for some critical branchings of the $(3, 1)$ -polygraph $\underline{\text{Qua}}_3(S)$, there are no length-three words that meet the defining requirements for the source of a branching. Therefore, such branchings can be discarded. To facilitate the exposition, we often abuse the language and say, for instance, the branching A of $\underline{\text{Qua}}_3(S)$ instead of ‘the branching completion that makes the boundary of the generating 3-cell A of the $(3, 1)$ -polygraph $\underline{\text{Qua}}_3(S)$ ’.

If the sources of two generating 2-cells α overlap, then the normal form of the source of the branching formed by these two generating 2-cells has to consist of a single column. Namely, if a length-three $\text{Col}_1(A)$ -word $\widehat{u}\widehat{v}\widehat{w}$ is such that the tableaux $P(uv)$ and $P(vw)$ consist of a single column each, then uv and vw are columns in A^* , by Lemma 4.3.0.1. Then, by the very definition of column, uvw is a column too. Consequently, no factor of $\widehat{u}\widehat{v}\widehat{w}$ can be source of the generating 2-cell ν since every factor of a column is itself a column. This discards from $\underline{\text{Qua}}_3(S)$ the critical branching that corresponds to the generating 3-cell B .

If any branch of a branching contains two consecutive generating 2-cells α , then the completion of the branching can contain no generating 2-cell ν . Namely, suppose that a length-three $\text{Col}_1(A)$ -word $\widehat{u}\widehat{v}\widehat{w}$ is such that the tableau $P(uv)$ consists of a single column and that $P(P(uv)w)$ consists of a single column. Using Lemma 4.3.0.1 again, we deduce that uvw is a column. The other possibility, that the tableau $P(vw)$ consists of a single column and that $P(uP(vw))$ consists of a single column, is treated in the same fashion. Like in the previous paragraph, we

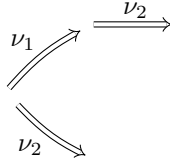
conclude that no factor of $\widehat{u}\widehat{v}\widehat{w}$ can be the source of the generating 2-cell ν . This discards from $\underline{\text{Qua}}_3(S)$ the critical branchings that correspond to the generating 3-cells $A', E''', E''''', F'', G''''''$.

Since the column presentation is of class (3, 3), by [4, Subsections 4.2-4.4], the fourth step in the upper branch is superfluous (if it exists). This immediately discards the critical branchings which correspond to the generating 3-cells E'' and G'' because their fourth step in the upper branch contains the generating 2-cell α . Removing it would make the branching unattainable since α reduces the length of its argument.

Let u, v and w be columns in A^* , such that the tableau $P(vw)$ consists of a single column and that $P(uv)$ contains two columns, say $P(uv) = xy$. Then, by [33, Lemma 3], the tableau $P(yw)$ cannot contain two columns. Consequently, there is no $\text{Col}_1(A)$ -word that satisfies the conditions for the branching that corresponds to the generating 3-cell E' in $\underline{\text{Qua}}_3(S)$.

Let u, v and w be columns in A^* , such that the tableaux $P(vw)$ and $P(uv)$ contain two columns each and, say $P(vw) = xy$. Then, by [33, Lemma 5], the tableau $P(ux)$ contains two columns. Consequently, there is no $\text{Col}_1(A)$ -word that satisfies the conditions for the branchings that correspond to the generating 3-cells F', G''' and G'''' in $\underline{\text{Qua}}_3(S)$.

The critical branching corresponding to the generating 3-cell G'' cannot be realised under current assumptions. Namely, a completion of a configuration of the form



cannot be continued with the rewriting step α_1 in the upper branch. We refer the reader to [33, Lemma 5] for discussion.

The remaining critical branchings of $\underline{\text{Qua}}_3(S)$ are A, C, D, E, F and G . To conclude the proof, we need to show that these branchings specialise to all the critical branchings of the (3, 1)-polygraph $\text{Col}_3(A)$. One can readily observe that the generating 3-cells A, D, E, F and G of $\underline{\text{Qua}}_3(S)$ respectively specialise to the generating 3-cells $A, C, B, D^{(4)}$ and D of $\text{Col}_3(A)$.

It is a straightforward verification to establish that the branching that corresponds to the generating 3-cell C of $\underline{\text{Qua}}_3(S)$ amounts to the branching which corresponds to the generating 3-cell $C^{(1)}$ of $\text{Col}_3(A)$. Namely, the third rewriting step in the lower branch of C becomes superfluous as the normal form is reached after only two rewriting steps in the lower branch. For elaboration on the conditions, see Case 1 in the proof of [33, Lemma 4].

Similarly, the branching D of $\underline{\text{Qua}}_3(S)$ also specialises to $C^{(1)}$ of $\text{Col}_3(A)$. This time, the second rewriting step in the upper branch of D becomes superfluous as the normal form is reached after the first step in the upper branch. For details, see Case 2 in the proof of [33, Lemma 4].

In a similar fashion, the branching F of $\underline{\text{Qua}}_3(S)$ also specialises to $D^{(3)}$ of $\text{Col}_3(A)$ by dropping the third rewriting step in the upper branch if the normal

form is reached after two rewriting steps. See Case 4 in the proof of [33, Lemma 5] for conditions.

Similarly, the branching G of $\underline{\text{Qua}}_3(S)$ also specialises to $D^{(1)}$ of $\text{Col}_3(A)$ by dropping the third rewriting step in the upper branch (recall that the fourth rewriting step is already dropped). Details on the conditions can be found in Case 2 in the proof of [33, Lemma 5].

Finally, the branching G of $\underline{\text{Qua}}_3(S)$ also specialises to $D^{(2)}$ of $\text{Col}_3(A)$ by dropping the third rewriting step in the lower branch. See Case 3 in the proof of [33, Lemma 5] for details on conditions. \square

Tentative steps further. The construction of a coherent presentation of a general monoid admitting a quadratic normalisation of class (4, 3) opens two directions for further research: extension and specialisation. Regarding the former, we would like to undertake a fine study of the shapes of higher-dimensional cells arising from quadratic normalisations of class (4, 3). This particular tentative step is similar to the first one stated at the end of Chapter 2, but it is more general (and also ‘more syntactical’) as it considers a larger class of monoids.

As for specialisation, we have already shown that the (3, 1)-polygraph $\underline{\text{Qua}}_3(S)$ specialises to the known column coherent presentation $\text{Col}_3(A)$ of plactic monoids of type A , constructed by Hage and Malbos [33]. Specialising the coherent presentation $\underline{\text{Qua}}_3(S)$ to coherent presentations of the plactic monoids of type C (see e.g. [44]) seems to be a feasible goal, too (see e.g. [8] for definition of types of plactic monoids).

We would also try specialising $\underline{\text{Qua}}_3(S)$ to coherent presentations constructed in Chapter 2. An obvious first attempt would be specialisation of the (3, 1)-polygraph $\underline{\text{Qua}}_3(S)$ to the (3, 1)-polygraph $\underline{\text{Gar}}_3(S)$. However, an obstacle is anticipated as follows. Although the generating 2-cells α of the (3, 1)-polygraph $\underline{\text{Qua}}_3(S)$ do coincide with the generating 2-cells α of the (3, 1)-polygraph $\underline{\text{Gar}}_3(S)$, there are no generating 2-cells of the (3, 1)-polygraph $\underline{\text{Qua}}_3(S)$ that would correspond to the generating 2-cells β of the (3, 1)-polygraph $\underline{\text{Gar}}_3(S)$, as remarked after the statement of Proposition 4.2.2.1. Fortunately, the generating 2-cells β of the (3, 1)-polygraph $\underline{\text{Gar}}_3(S)$ are removed by the homotopical reduction procedure in Subsection 2.5.4, anyway. So, specialising $\underline{\text{Qua}}_3(S)$ to the (3, 1)-polygraph $\underline{\text{Gar}}_3(S)$, rather than to $\underline{\text{Gar}}_3(S)$, seems to be the right way to proceed.

Performing the homotopical reduction procedure on $\underline{\text{Qua}}_3(S)$ would facilitate that task. It is worth a try in its own right, too, as it would greatly simplify further applications.

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