Homogeneous Sobolev and Besov spaces on half-spaces

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INSTITUT de MATHÉMATIQUES de MARSEILLE

Car il n'est pas irrationnel d'être optimiste aussi longtemps que nous vivrons.

— Karl Popper

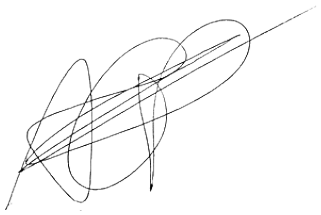
À ma mère Pascale et à mes soeurs Léonie et Bertille.

Affidavit

Je soussigné, Anatole Gaudin, déclare par la présente que le travail présenté dans ce manuscrit est mon propre travail, réalisé sous la direction scientifique de Sylvie Monniaux, dans le respect des principes d'honnêteté, d'intégrité et de responsabilité inhérents à la mission de recherche. Les travaux de recherche et la rédaction de ce manuscrit ont été réalisés dans le respect à la fois de la charte nationale de déontologie des métiers de la recherche et de la charte d'Aix-Marseille Université relative à la lutte contre le plagiat.

Ce travail n'a pas été précédemment soumis en France ou à l'étranger dans une version identique ou similaire à un organisme examinateur.

Fait à Marseille, le 5 Mai 2023



Liste de publications et participation aux conférences

Liste des publications réalisées dans le cadre du projet de thèse :

(i) A. Gaudin. On homogeneous Sobolev and Besov spaces on the whole and the half space. *arXiv e-prints* :arXiv :2211.07707, arXiv :2211.07707, novembre 2022. arXiv : 2211.07707 [math.AP] **Submitted**

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Participation aux conférences et écoles d'été au cours de la période de thèse :

(i) **Rencontre du projet ANR 'Rage'**
Bordeaux (33), France. January 2023.
Meeting on advances in Real Analysis and Geometry, and related areas.

(ii) **MathFlows 7^{th} Edition**
Marseille (13), France. December 2022.
Conference on mathematical aspects of fluid mechanics and related PDEs at the CIRM.

(iii) **Oberwolfach Seminar : Operator-Adapted Spaces in Harm. Anal. and PDEs.**
Oberwolfach (77), Germany. November 2022.
Several lectures by the organizers on Operator-Adapted Spaces.

(iv) **Journées Équations aux Dérivées Partielles.**
Obernai (67), France. June 2021, June 2022.
Conference on general PDEs, which includes several lectures.

(v) **Jean-Morlet Chair : Nonlinear Partial Differential Equations in Fluid Dynamics.**
Marseille (13), France. January - May 2022.
Semester of research, dedicated to non-linear PDEs and fluid dynamics. Three different conferences each with one week duration.

(vi) **Journées du GDR Analyse Fonctionnelle, Harmonique et Probabilités.**
Besançon (25), France. September 2021.
Conference on Probabilities, Harmonic and Functional Analysis.

(vii) **Mécanique des fluides : étude qual. et comportement asympt. des solutions.**
Peyresq (04), France. August 2021.
Conference on asymptotical behavior of solutions of PDEs that come from fluid mechanics.

Résumé

Cette thèse s'attèle principalement au problème de réalisation des espaces de Besov et Sobolev homogènes sur l'espace entier et certains demi-espaces. Ce problème de réalisation des espaces de fonctions apparaît naturellement lors de l'étude du caractère bien posé global en temps et des problèmes de régularité de certaines équations paraboliques dans les domaines non-bornés. Les constructions proposées dans cette thèse étendent celles initiées par Bahouri, Chemin, Danchin, Hieber, Mucha et Tolksdorf au cours de différents articles et monographies. On passera en revue principalement les résultats de densité, d'interpolation réelle et complexe, ainsi que les résultats de trace sur le bord. Une difficulté majeure vient du fait que certains des espaces vectoriels normés considérés ne peuvent pas être complets, ni complétés, au risque de ne plus être constitués d'éléments identifiables à des distributions.

Le manque de complétude pour certains espaces requiert alors une nouvelle construction des outils afin de pouvoir exploiter la théorie des opérateurs et en particulier la régularité maximale parabolique globale en temps dans les espaces de Lebesgue. Une reconstruction de la théorie de l'interpolation et des opérateurs homogènes a été effectuée par Danchin, Hieber, Mucha et Tolksdorf, afin d'obtenir dans ce cadre des estimées globales en temps pour une régularité maximale du type Da Prato-Grisvard pour des équations paraboliques issues d'opérateurs sectoriels injectifs, non-inversibles. On se sert de ce cadre afin d'établir un nouveau type de régularité maximale globale en temps, avec une estimation de trace adaptée, où l'on remplace l'espace de Lebesgue en temps par un espace de Sobolev homogène.

La théorie revisitée de l'interpolation et des opérateurs homogènes en combinaison avec notre construction des espaces homogènes et leurs propriétés sont appliqués à l'étude du Laplacien de Hodge sur le demi-espace plat en dimension arbitraire. On déduit de cette analyse la décomposition de Hodge/Helmholtz, pour tout degré de formes différentielles, des espaces de Sobolev et Besov homogènes qui se trouvent être essentiellement optimale du point de vue de la régularité. En outre, cela nous permet de déduire de nombreux résultats de régularité maximale pour divers systèmes d'évolution de Stokes ou Maxwell assujettis à diverses conditions au bord. Ceux-ci peuvent être d'un intérêt certain en mécanique des fluides et en électromagnétisme.

Enfin, on se concentrera sur la construction et la réalisation des espaces de fonctions homogènes sur les ouverts qui sont des épigraphes de fonctions uniformément lipschitziennes à valeurs réelles. On proposera également une construction des espaces homogènes sur le bord, ainsi qu'un théorème de trace essentiellement optimal du point de vue de la régularité avec des estimées homogènes du point de vue des normes.

Mots clés : Espaces de Sobolev homogènes, Espaces de Besov homogènes, Traces, Régularité maximale, Décomposition de Hodge, Système d'évolution de Stokes

Abstract

This thesis is mainly concerned with the problem of realization of homogeneous Besov and Sobolev spaces on the whole space, and some half-spaces. This problem of realization of function spaces appears naturally when studying the global well-posedness in-time and the regularity problems of some parabolic equations in unbounded domains. The constructions proposed in this thesis extend those initiated by Bahouri, Chemin, Danchin, Hieber, Mucha and Tolksdorf in various articles and monographs. We mainly review the density, real and complex interpolation results, as well as some results for traces on the boundary. A major difficulty comes from the fact that some considered normed vector spaces cannot be complete, nor be completed at the risk of not being constituted of elements identifiable with distributions.

The lack of completeness for some spaces requires then a new construction of the tools in order to be able to exploit the operator theory, in particular for the trace estimates in the case of global-in-time parabolic maximal regularity for Lebesgue spaces. A reconstruction of the theory of interpolation and homogeneous operators has been carried out by Danchin, Hieber, Mucha and Tolksdorf, in order to obtain global-in-time estimates for maximal regularity of the Da Prato-Grisvard type for parabolic equations arising from injective, non-invertible sectorial operators. We use this framework to establish a new type of global-in-time maximal regularity results, with adapted trace estimates, where we replace the Lebesgue space in time by a homogeneous Sobolev space.

The revisited theory of interpolation and homogeneous operators in combination with our construction of homogeneous spaces and their properties are applied to the study of the Hodge Laplacian on the flat upper half-space in arbitrary dimension. From this analysis, we derive the Hodge/Helmholtz decomposition, for any degree of differential forms, of homogeneous Sobolev and Besov spaces which is essentially optimal from the regularity viewpoint. Moreover, it also allows us to deduce many parabolic maximal regularity results for various Stokes or Maxwell evolutionary systems subject to various boundary conditions with global-in-time estimates. Those results may be of interest in fluid mechanics and electromagnetism.

Finally, we also focus on the construction and realization of homogeneous function spaces on the open sets given by epigraphs of uniformly Lipschitz real-valued functions. We also propose a construction of homogeneous function spaces on the boundary, as well as a trace theorem essentially optimal from the point of view of regularity with homogeneous estimates from the point of view of norms.

Keywords: Homogeneous Sobolev spaces, Homogeneous Besov spaces, Traces, Maximal regularity, Hodge decomposition, Evolutionary Stokes system

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1. parfois presque malgré lui et indépendamment de nos volontés respectives ; les placements lors des repas à Oberwolfach étant ce qu'il est...

de mots pour dire à quel point il aura marqué extrêmement positivement ces trois années, me laissant indéfiniment redevable.

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2. avec qui j'ai aussi l'immense honneur, avec **Angelo Zenni**, de partager la 3ème place de la meilleure présentation au séminaire 2247b d'Oberwolfach

3. et bon sang qu'il est important et déterminant d'avoir de telles figures dans son parcours !

4. je n'ai dû finir jusqu'à aujourd'hui que 3 des livres qu'elle m'a recommandés (et aucun depuis que j'ai commencé cette thèse, principalement par manque de temps, dirons-nous).

5. toujours cette histoire du retour d'Oberwolfach, mais pas seulement...

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Table des matières

Introduction

En Français

À l'origine

Un certain nombre d'équations aux dérivées partielles non-linéaires décrivant une évolution temporelle peuvent être écrites sous la forme d'une partie parabolique linéaire et d'un terme non-linéaire. L'intérêt de telles équations provient du fait qu'elles soient issues de disciplines appliquées telles que la physique, la chimie, ou encore la biologie.

L'une des plus importantes d'entre elles servira de support pour nous guider au cours de l'introduction des problèmes soulevés et (partiellement) résolus dans cette thèse. Il s'agit des équations de Navier-Stokes décrivant l'évolution des champs de vitesses pour un fluide homogène incompressible

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla \mathfrak{p} &=& -(u\cdot\nabla)u, & \text{on } (0,T)\times\Omega, \\ \operatorname{div} u &=& 0, & \text{on } (0,T)\times\Omega, \\ u(0) &=& u_0 & . \end{array}\right. \tag{NSE}$$

d'inconnue $(u,\mathfrak{p}) : (0,T)\times\Omega \longrightarrow \mathbb{R}^n\times\mathbb{R}$, où Ω est un ouvert de $\mathbb{R}^n$ (avec éventuellement un jeu de conditions au bord pour fermer le système (NSE)), $T\in(0,+\infty]$, u est la vitesse du fluide considéré, $\mathfrak{p}$ la pression du système, et $u_0 : \Omega\longrightarrow\mathbb{R}^n$ un champ de vitesse initial donné vérifiant $\operatorname{div} u_0 = 0$. La condition dite de divergence nulle $\operatorname{div} u = 0$ traduit ici l'incompressibilité et l'homogénéité du fluide. On incitera le lecteur à consulter [Lem16, Chapters 1, 2 & 3] pour plus de contexte historique et physique autour du système (NSE) dans le cas $\Omega=\mathbb{R}^n$.

Si l'on s'intéresse au caractère bien posé global en temps de (NSE), *i.e.* $T=+\infty$, il est alors naturel d'essayer de déterminer un espace de fonctions adapté pour le choix de la donnée initiale u_0, disons pour que l'on ait un contrôle global en temps sur $(\partial_t u, \Delta u)$ dans $\mathrm{L}^q(\mathbb{R}_+,\mathrm{L}^p(\Omega))$ avec $p,q\in(1,+\infty)$. On peut également traiter cette question dans le cadre d'un problème de Cauchy linéaire abstrait

$$\left\{\begin{array}{rcl} \partial_t u(t) + Au(t) &=& f(t)\,,\ 0<t<+\infty, \\ u(0) &=& u_0 \text{ dans } X. \end{array}\right. , \tag{ACP}$$

où A est un opérateur raisonnable générateur d'un semi-groupe fortement continu et uniformément borné $(e^{-tA})_{t\geqslant 0}$ sur un espace de Banach X. Le problème étant linéaire, on peut pour l'instant supposer $f=0$ et vérifier que l'espace moralement induit par la (semi-)norme

$$\begin{aligned} \|u_0\|_{\mathring{\mathcal{D}}_A(1-\frac{1}{q},q)} := \left(\int_0^{+\infty} (t^{1-(1-\frac{1}{q})}\|Ae^{-tA}u_0\|_X)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} &= \left(\int_0^{+\infty} (\|Ae^{-tA}u_0\|_X)^q \mathrm{d}t\right)^{\frac{1}{q}} \\ &= \left(\int_0^{+\infty} (\|\partial_t(e^{-tA}u_0)\|_X)^q \mathrm{d}t\right)^{\frac{1}{q}} \end{aligned}$$

semble être un candidat naturel, pour lequel on a même $t\mapsto e^{-tA}u_0$ est continue à valeurs dans $\mathring{\mathcal{D}}_A(1-\frac{1}{q},q)$. Plus généralement, cela nous amène à considérer les espaces moralement induits par les (semi-)normes

$$\|x\|_{\mathring{\mathcal{D}}_A(\theta,q)} = \left(\int_0^{+\infty} (t^{1-\theta}\|Ae^{-tA}x\|_X)^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, \qquad \theta\in(0,1),\ q\in[1,+\infty]. \tag{0.1}$$

Le cas particulier de $A=-\Delta$ sur $X=\mathrm{L}^p(\mathbb{R}^n)$ est très bien documenté et connu, voir par exemple

[BCD11, Theorem 2.34], on a, pour $\theta \in (0,1)$, $p,q \in [1,+\infty]$,

$$\|v\|_{\mathring{\mathcal{D}}_{-\Delta}(\theta,q)} = \left(\int_0^{+\infty} (t^{1-\theta}\|\Delta e^{t\Delta} v\|_{\mathrm{L}^p(\mathbb{R}^n)})^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \sim_{p,q,\theta,n} \|v\|_{\dot{\mathrm{B}}^{2\theta}_{p,q}(\mathbb{R}^n)}, \tag{0.2}$$

où $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ est l'espace de Besov homogène défini par exemple dans [BCD11, Chapitre 2], [BL76, Chapter 6, Section 6.3], [Tri83, Chapter 5]. Le lecteur qui n'est pas familier avec les espaces de Besov, en particulier ceux homogènes, peut se raccrocher à l'égalité $\dot{\mathrm{B}}^s_{2,2}(\mathbb{R}^n) = \dot{\mathrm{H}}^s(\mathbb{R}^n)$ avec (semi-)norme équivalente $\|v\|_{\dot{\mathrm{H}}^s(\mathbb{R}^n)} := \|(-\Delta)^{\frac{s}{2}} v\|_{\mathrm{L}^2(\mathbb{R}^n)}$, et pour $s=1$ $\|v\|_{\dot{\mathrm{H}}^1(\mathbb{R}^n)} = \|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^n)}$.

Par l'équivalence de normes (0.2), il paraît naturel de nommer la famille d'espaces vectoriels qui émergent de la famille de (semi-)normes données dans (0.1) **espaces de Besov adaptés à l'opérateur** A. Lorsque A est un opérateur sectoriel, inversible ou injectif, une caractérisation simple a été donnée grâce l'interpolation réelle, plus exactement

$$\|x\|_{\mathring{\mathcal{D}}_A(\theta,q)} \sim_{\theta,q} \|x\|_{(X,\mathrm{D}(\mathring{A}))_{\theta,q}} \tag{0.3}$$

où $\mathrm{D}(\mathring{A})$ correspond à la complétion du domaine $\mathrm{D}(A)$ pour la norme $\|A\cdot\|_X$. Lorsque A est inversible, on mentionne que $\mathrm{D}(\mathring{A}) = \mathrm{D}(A)$ avec équivalence des normes. Ce fait a été grandement revu dans la littérature, par exemple par Haase [Haa06, Chapter 6] ou encore Lunardi [Lun18, Chapter 6]. Cela permet souvent de décrire complètement l'espace lorsque le domaine de A a lui-même une description convenable. On mentionne également le récent travail de Batty et Chen [BC20] qui reprend la construction d'une façon plus générale pour les opérateurs sectoriels en donnant un certain nombre de caractérisations équivalentes, par exemple de type Littlewood-Paley, ainsi qu'une longue liste non-exhaustive de propriétés complémentaires.

Une description effective des espaces de fonctions adaptés lorsque l'opérateur A a une forme explicite a aussi été largement étudié dans la littérature. Le cas des opérateurs elliptiques sous-forme divergence sur les espaces L^p s'est vu porter une attention toute particulière. On citera par exemple, sous l'hypothèse d'estimées Gaussiennes et de Poisson sur le noyau du semi-groupe engendré, les travaux de Bui, Duong et Yan [BDY12] et de Cao et Grygor'yan [CG20] où la coïncidence avec des espaces de Besov classiques est également établie dans certains cas. Sous des hypothèses plus générales, la coïncidence est montrée pour des espaces 'assez proches' de L^2 dans le monographe par Amenta et Auscher [AA18, Chapter 5].

Problèmes émergeant : la réalisation des espaces homogènes

L'apparition d'espaces de Besov homogènes (0.2) pour le choix conditions initiales, pose la question du choix de réalisation de ces espaces, si l'on souhaite revenir à l'étude du problème non-linéaire (NSE) pour $\Omega = \mathbb{R}^n$.

En effet, la construction des espaces de Sobolev et Besov homogènes est faite usuellement, [BL76, Chapter 6, Section 6.3], [Tri83, Chapter 5], à partir de l'espace des distributions tempérées quotienté par les polynômes $\mathcal{S}'(\mathbb{R}^n) \Big/ \mathbb{C}[x]$, afin d'en faire des espaces vectoriels normés. Il n'aura pas échappé au lecteur attentif que les normes homogènes, par exemple $\|\cdot\|_{\dot{\mathrm{H}}^s(\mathbb{R}^n)}$, ne séparent pas les polynômes, et donc ne sont pas réellement des normes sur $\mathcal{S}'(\mathbb{R}^n)$. Cela pose un problème pour définir la non-linéarité de (NSE).

Pour $[u],[v] \in \mathcal{S}'(\mathbb{R}^n) \Big/ \mathbb{C}[x]$, et $u+P, v+Q \in \mathcal{S}'(\mathbb{R}^n)$ deux représentants de $[u]$ et $[v]$,

$$\begin{aligned}((u+P)\cdot\nabla)(v+Q) - ((u+\tilde{P})\cdot\nabla)(v+\tilde{Q}) =& [(P-\tilde{P})\cdot\nabla]v + [u\cdot\nabla](Q-\tilde{Q}) \\ &+ (P\cdot\nabla)Q - (\tilde{P}\cdot\nabla)\tilde{Q}.\end{aligned}$$

De là, sous réserve d'une loi de (para-)produits qui ait du sens, bien que $(P\cdot\nabla)Q - (\tilde{P}\cdot\nabla)\tilde{Q}$ soit un polynôme, ce n'est pas le cas de $[(P-\tilde{P})\cdot\nabla]v + [u\cdot\nabla](Q-\tilde{Q})$ de telle sorte que le produit **dépend du choix des représentants !**

Une autre possibilité serait de compléter naïvement la classe de Schwartz pour la norme homogène qui nous intéresse. Cette construction a le défaut de produire des éléments qui peuvent ne plus être des distributions. On peut vérifier par exemple que

$$\mathrm{C}_c^\infty(\mathbb{R}^n) \not\subset \dot{\mathrm{H}}^{-\frac{n}{2}}(\mathbb{R}^n).$$

Cela empêche *a priori* d'identifier les éléments de $\dot{\mathrm{H}}^{\frac{n}{2}}(\mathbb{R}^n)$ (en tant que complétion) comme étant des distributions. Ce phénomène est celui de la *divergence infrarouge* relatant les problèmes de convergences pour les basses fréquences au sens de Fourier.

Ainsi, pour une réalisation des espaces homogènes, nous ne pouvons choisir que deux parmi trois des propriétés suivantes

(i) des espaces dont les éléments sont des distributions, en un sens raisonnable ;

(ii) des lois produits bien définies ;

(iii) des espaces tous complets.

Pour le traitement d'équations aux dérivées partielles non-linéaires, on choisit d'abandonner la complétude pour tous les espaces de fonctions homogènes. C'est ce choix qui a été retenu par Bahouri, Chemin et Danchin dans leur livre [BCD11]. Leur idée a été de considérer un sous-espace des distributions tempérées $\mathcal{S}'(\mathbb{R}^n)$ qui ne contient pas les polynômes. L'espace en question, parfois appelé **espaces des distributions homogènes de Chemin**, est donné par

$$\mathcal{S}'_h(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \forall \Theta \in \mathrm{C}_c^\infty(\mathbb{R}^n),\ \|\Theta(\lambda\mathfrak{D})u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \xrightarrow[\lambda\to+\infty]{} 0 \right\}.$$

où $\Theta(\lambda\mathfrak{D}) = \mathcal{F}^{-1}\Theta(\lambda\cdot)\mathcal{F}$, avec $\mathcal{F}$ la transformée de Fourier. Dans leur livre [BCD11, Chapter 2], Bahouri, Chemin et Danchin se sont surtout attelés à la construction des espaces de Besov homogènes

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n), \quad s \in \mathbb{R},\ p,q \in [1,+\infty],$$

pour lesquels ils ont montré, [BCD11, Theorem 2.25], qu'ils étaient complets si et seulement si

$$\left[s < \frac{n}{p} \right] \text{ ou } \left[q = 1 \text{ et } s \leqslant \frac{n}{p} \right].$$

La construction a été ensuite étendue aux espaces de Besov sur le demi-espace $\mathbb{R}^n_+$ par Danchin et Mucha [DM09 ; DM15], puis par Danchin, Hieber, Mucha et Tolksdorf [DHMT21] aux espaces de Sobolev homogènes à indices de régularité entiers et positifs $\dot{\mathrm{H}}^{m,p}$ sur $\mathbb{R}^n$ et $\mathbb{R}^n_+$, pour $m \in \mathbb{N}$, $p \in (1,+\infty)$. Ces constructions d'espaces homogènes ont été un atout central pour le caractère bien posés de certaines équations de mécanique des fluides qui généralisent les équations de Navier-Stokes (NSE) dans [DM09 ; DM15 ; DHMT21]. Il aura fallu d'ailleurs attendre [DHMT21] pour les premiers vrais résultats de densité, de traces et d'interpolation pour ces espaces.

Objectif 1 *Étendre et synthétiser la construction et les propriétés des espaces de Sobolev et Besov homogènes initiés par Bahouri, Chemin, Danchin, Hieber, Mucha et Tolksdorf sur* $\mathbb{R}^n$ *et* $\mathbb{R}^n_+$ *: résultats de densité, de traces et d'interpolation pour les espaces*

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\ \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+), \quad s \in \mathbb{R},\ p \in (1,+\infty), q \in [1,+\infty].$$

Le Chapitre 2 y sera dédié.

Réajustement : espaces de fonctions pour les données initiales et la régularité maximale

Le nouveau choix de réalisation des espaces de Sobolev et Besov homogènes a pour conséquence que les outils développés par Haase dans [Haa06] pour obtenir l'équivalence des normes (0.3) avec l'égalité des espaces, ne sont plus adaptés. Par exemple, dans le cas du laplacien sur $\mathrm{L}^p(\mathbb{R}^n)$, nous ne pouvons plus nous servir de la complétion pour obtenir $\mathrm{D}(\mathring{\Delta}) = \dot{\mathrm{H}}^{2,p}(\mathbb{R}^n)$.

Deux remarques, cependant : dans un premier temps, l'interpolation réelle d'espaces vectoriels normés non nécessairement complets a bien un sens, on pourra consulter [BL76, Chapter 3]. Dans un second temps, le problème de complétion pour construire le domaine homogène $\mathrm{D}(\mathring{A})$ n'intervient que lorsque l'opérateur A n'est pas inversible sur X. Dans le cas où A est inversible, $\mathrm{D}(A)$ et $\mathrm{D}(\mathring{A})$ coïncident avec équivalence des normes.

Danchin, Hieber, Mucha et Tolksdorf ont revu la théorie de l'interpolation et des opérateurs homogènes de Haase [Haa06, Chapter 6] dans [DHMT21, Chapter 2]. Leur approche permet de considérer des domaines homogènes $\mathrm{D}(\mathring{A})$ non-complets, sous quelques hypothèses et considérations techniques supplémentaires, souvent vérifiées dans la pratique, de sorte que (0.3) reste valide.

On dit que l'opérateur A admet la propriété de régularité maximale parabolique L^q (globale en temps) sur Y, lorsque pour $u_0 = 0$, $q \in [1, +\infty]$, $f \in \mathrm{L}^q(\mathbb{R}_+, Y)$, (ACP) admet une unique solution $u \in \mathrm{C}^0(\mathbb{R}_+, Y)$, telle que

$$\|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,Y)} \lesssim_{q,Y,A} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,Y)}. \tag{0.4}$$

Danchin, Hieber, Mucha et Tolksdorf ont alors pu montrer, dans ce cadre, [DHMT21, Theorem 2.20], lorsque l'on a $Y = \mathring{\mathcal{D}}_A(\theta, q)$, $q \in [1, +\infty)$, $\theta \in (0, \frac{1}{q})$, et $u_0 \in \mathring{\mathcal{D}}_A(\theta + 1 - \frac{1}{q}, q)$, alors (ACP) admet une unique solution $u \in \mathrm{C}^0_b(\mathbb{R}_+, \mathring{\mathcal{D}}_A(\theta + 1 - \frac{1}{q}, q))$, telle que

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(\theta+1-\frac{1}{q},q))} + \|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,\mathring{\mathcal{D}}_A(\theta,q))} \lesssim_{q,\theta,A} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,\mathring{\mathcal{D}}_A(\theta,q))} + \|u_0\|_{\mathring{\mathcal{D}}_A(\theta+1-\frac{1}{q},q)}. \tag{0.5}$$

L'opérateur A, lorsqu'il est injectif, a en particulier la propriété de régularité maximale L^q sur $\mathring{\mathcal{D}}_A(\theta, q)$, pour tout $q \in [1, +\infty)$, $\theta \in (0, \frac{1}{q})$. Le résultat était déjà connu lorsque A est inversible depuis le travail de Da Prato et Grisvard [DG75, Théoreme 4.15].

La propriété de la régularité maximale L^q (0.4) pour A sur $Y = X$ est cependant une question bien plus compliquée. Une réponse définitive a été apportée par Weis [Wei01] qui caractérise totalement de tels opérateurs, sous la condition que l'espace de Banach X ait la propriété dite **UMD**, qui sont les opérateurs $\mathcal{R}$-sectoriels ; on se réfèrera par exemple à [DHP03 ; KW04 ; PS16] pour plus d'informations. Il est également nécessaire d'avoir dans ce cas $q \in (1, +\infty)$.

Lorsque A a la propriété de régularité maximale L^q sur X, $q \in (1, +\infty)$, pour $u_0 \in \mathring{\mathcal{D}}_A(1 - \frac{1}{q}, q)$, $f \in \mathrm{L}^q(\mathbb{R}_+, X)$, alors (ACP) admet une unique solution u vérifiant $t \mapsto u(t) - e^{-tA}u_0 \in \mathrm{C}^0(\mathbb{R}_+, X)$, et telle que

$$\|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,A} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(1-\frac{1}{q},q)}. \tag{0.6}$$

On notera cependant que (0.6) souffre de sa comparaison avec (0.5), où l'on a un contrôle sur u comme fonction continue et bornée en temps. Le résultat est connu lorsque A est inversible, on pourra par exemple consulter [Ama95, Chapter 4, Theorem 4.10.2]. On aimerait également savoir si l'on a l'inégalité

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1-\frac{1}{q},q))} \lesssim_{q,A} \|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \tag{0.7}$$

lorsque A est seulement injectif.

Certains travaux récents mettent en jeu le transfert de régularité en espaces via des estimées

aux dérivées mixtes du type

$$\|(\partial_t)^\beta A^{1-\beta} u\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,A} \|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,X)}, \qquad \beta \in [0,1],$$

pour traiter des problèmes non-linéaires proches de (NSE), par exemple le travail de Brandolese et Monniaux [BM21] pour le système de Boussinesq.

On pourrait vouloir augmenter ce jeu sur la régularité en espace-temps pour gagner un peu en souplesse dans les estimations, en proposant de remplacer L^q dans (0.4) par un espace de Sobolev homogène $\dot{\mathrm{H}}^{\alpha,q}$, pour $q \in (1,+\infty)$, $\alpha \in (-1+\frac{1}{q}, \frac{1}{q})$. Voir [Pru02 ; PS16] pour plus d'informations concernant les estimations aux dérivées mixtes, et la régularité maximale (inhomogène) Sobolev en temps.

Si l'on peut remplacer L^q par $\dot{\mathrm{H}}^{\alpha,q}$, il resterait alors à savoir par quoi remplacer l'espace $\mathring{\mathcal{D}}_A(1-\frac{1}{q}, q)$ dans (0.6) et (0.7).

Objectif 2 *Prouver qu'une classe pas trop restrictive d'opérateurs injectifs qui admettent la propriété régularité maximale* L^q *sur* X *un Banach UMD, admettent également la propriété régularité maximale* $\dot{\mathrm{H}}^{\alpha,q}$*, pour* $q \in (1,+\infty)$*,* $\alpha \in (-1+\frac{1}{q}, \frac{1}{q})$*. Pour un tel opérateur* A*, on voudrait montrer dans ce cas*

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+, \mathring{\mathcal{D}}_A(\alpha+1-\frac{1}{q},q))} \lesssim_{A,q,\alpha} \|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \lesssim_{A,q,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(\alpha+1-\frac{1}{q},q)}.$$

Le Chapitre 3 y sera dédié, et contiendra une présentation synthétique de la théorie de l'interpolation et des opérateurs homogènes revisitée par Danchin, Hieber, Mucha et Tolksdorf [DHMT21, Chapter 2].

Retour : le système de Stokes et la décomposition de Helmholtz

Nous avons été quelque peu négligents. En effet, nous avons introduit l'étude d'un problème de Cauchy abstrait (ACP), à partir de (NSE), en traitant du Laplacien sur $\mathbb{R}^n$. Cependant, le terme de pression $\mathfrak{p}$ est également une inconnue du problème qui ne semble pas apparaître lors de la mise en forme en un problème de Cauchy abstrait.

Il se trouve qu'en fait, quand $\Omega = \mathbb{R}^n$, le problème (NSE) peut-être réécrit sous la forme

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& -\mathbb{P}[(u\cdot\nabla)u], & \text{on } (0,T)\times\mathbb{R}^n, \\ \operatorname{div} u &=& 0, & \text{on } (0,T)\times\mathbb{R}^n, \\ u(0) &=& u_0 & . \end{array} \right. \tag{NSE'}$$

où $\mathbb{P}$ est le projecteur de Leray sur les fonctions à divergence nulles sur $\mathbb{R}^n$ au sens des distributions. Cela motive en particulier l'étude de l'équation de la chaleur comme exemple principal lorsque $\Omega = \mathbb{R}^n$. Ceci est possible puisque l'opérateur $\mathbb{P}$ commute avec le Laplacien sur $\mathbb{R}^n$, et qu'il est donné par la formule

$$\mathbb{P}f = \mathcal{F}^{-1}\left[\xi \mapsto \mathcal{F}f(\xi) - \frac{\xi\cdot\mathcal{F}f(\xi)}{|\xi|^2}\xi\right], \qquad \xi\in\mathbb{R}^n, \qquad f\in \mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^n),$$

induisant un opérateur linéaire borné,

$$\mathbb{P} : \mathrm{L}^p(\mathbb{R}^n,\mathbb{C}^n) \longrightarrow \mathrm{L}^p_\sigma(\mathbb{R}^n) := \{\, u\in \mathrm{L}^p(\mathbb{R}^n,\mathbb{C}^n) \mid \operatorname{div} u = 0\}, \qquad p\in(1,+\infty)$$

induisant lui-même une décomposition topologique appelée **décomposition de Helmholtz**

$$\mathrm{L}^p(\mathbb{R}^n,\mathbb{C}^n) = \mathrm{L}^p_\sigma(\mathbb{R}^n) \oplus \overline{\nabla \mathrm{H}^{1,p}(\mathbb{R}^n,\mathbb{C})}, \qquad p\in(1,+\infty).$$

Ce n'est en général pas possible de se ramener à l'équation de la chaleur si l'on veut traiter (NSE) avec un domaine à bord $\Omega \subset \mathbb{R}^n$ avec des conditions au bord (nécessaires pour fermer le système).

Avant de continuer, on se doit de faire une clarification sur ce qu'est le projecteur de Leray sur un ouvert Ω de $\mathbb{R}^n$. Dans ce cadre, le projecteur de Leray est l'unique projecteur orthogonal

$$\mathbb{P}_\Omega \,:\, \mathrm{L}^2(\Omega,\mathbb{C}^n) \longrightarrow \mathrm{L}^2_\sigma(\Omega) := \overline{\{\, u \in \mathrm{C}^\infty_c(\Omega,\mathbb{C}^n) \mid \operatorname{div} u = 0 \,\}}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}}.$$

Lorsque $p \in (1,+\infty)$ et $p \neq 2$, la question de la bornitude de $\mathbb{P}_\Omega$ sur L^p à valeurs dans L^p_σ (en gardant la même définition pour L^p_σ) n'a rien d'automatique, comme exhibé dans [Bog86]. Lorsque $1 < p < +\infty$ et que Ω est raisonnable, voir par exemple [Soh01, Lemma 2.5.3] pour $p = 2$, on a la description exacte

$$\mathrm{L}^p_\sigma(\Omega) = \{\, u \in \mathrm{L}^p(\Omega,\mathbb{C}^n) \mid \operatorname{div} u = 0 \;\&\; u \cdot \nu_{|\partial\Omega} = 0 \,\},$$

où $u \cdot \nu_{|\partial\Omega}$ est une trace prise au sens faible et ν est le vecteur unitaire normal au bord de Ω. Le caractère borné de $\mathbb{P}_\Omega$ sur L^p a été étudié dans de nombreux contextes. On pourra consulter par exemple [FMM98 ; SS92] pour les domaines bornés à bord C^1 et Lipschitz.

Ainsi, lors de la présence d'une frontière et d'un jeu de conditions au bord, on doit considérer les conditions au bord et la condition d'incompressibilité $\operatorname{div} u = 0$ dans les espaces de fonctions en jeu pour définir l'opérateur qui remplace le Laplacien. Cela permet de considérer la pression $\mathfrak{p}$ (plus précisément son gradient) comme un multiplicateur de Lagrange. Un tel opérateur est appelé un opérateur de Stokes.

Pour (NSE) avec condition au bord de Dirichlet sur le demi-espace $\mathbb{R}^n_+$, l'opérateur à introduire qui remplace $-\Delta$ dans (NSE') est

$$\begin{aligned} \mathrm{D}_p(A_\mathcal{D}) &= \mathrm{L}^p_\sigma(\mathbb{R}^n_+) \cap \mathrm{H}^{1,p}_0(\mathbb{R}^n_+) \cap \mathrm{H}^{2,p}(\mathbb{R}^n_+) \\ A_\mathcal{D} &= \mathbb{P}_{\mathbb{R}^n_+}(-\Delta)u, \end{aligned}$$

appelé opérateur de Stokes-Dirichlet. Pour une fonction $u \in \mathrm{D}_p(A_\mathcal{D})$, rien n'assure *a priori* que $\mathbb{P}_{\mathbb{R}^n_+} u \in \mathrm{H}^{1,p}_0(\mathbb{R}^n_+)$. Plus généralement, étant donné un jeu de conditions au bord, il n'y *a priori* aucune raison que le projecteur de Leray commute avec le laplacien avec les mêmes conditions au bord. Pour une revue des différents opérateurs Stokes avec conditions au bord, on renvoie à la revue sur le sujet faite par Monniaux et Shen [MS18].

Il est connu, voir [Gal11, Remark III.1.2], que $\mathbb{P}_{\mathbb{R}^n_+}$ est borné sur $\mathrm{L}^p(\mathbb{R}^n_+)$ pour tout $p \in (1,+\infty)$. On soulève les deux points suivants :

(i) Pour des questions de régularité concernant (NSE) et des systèmes similaires, il pourrait être intéressant de savoir si $\mathbb{P}_{\mathbb{R}^n_+}$ est borné sur les espaces

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C}^n),\ \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\mathbb{C}^n), \qquad s \in \mathbb{R},\ p \in (1,+\infty), q \in [1,+\infty].$$

Ce résultat est connu dans le cas des espaces de Besov homogènes pour $s \in (-1+\frac{1}{p}, \frac{1}{p})$, [DM09, Lemma 1].

(ii) Pour $\Omega \subset \mathbb{R}^3$ borné et lipschitzien, en suivant par exemple [MS18, Section 4], il est connu que le laplacien dit de Hodge, donné par

$$\begin{aligned} \mathrm{D}(\Delta_\mathcal{H}) = \{\, u \in \mathrm{L}^2(\Omega,\mathbb{C}^3) \mid &\operatorname{curl} u,\ \operatorname{div} u,\ \operatorname{curl}\operatorname{curl} u - \nabla \operatorname{div} u \in \mathrm{L}^2(\Omega,\mathbb{C}^3), \\ &\& \, u \cdot \nu_{|\partial\Omega} = 0,\ \nu \times \operatorname{curl} u_{|\partial\Omega} = 0 \,\} \\ -\Delta_\mathcal{H} u := \operatorname{curl}\operatorname{curl} u - \nabla \operatorname{div} u, & \end{aligned}$$

est "compatible" avec le projecteur de Leray au sens où il vérifie

$$\begin{aligned} &\mathbb{P}_\Omega \mathrm{D}(\Delta_\mathcal{H}) \subset \mathrm{D}(\Delta_\mathcal{H}), \\ &\mathbb{P}_\Omega(-\Delta_\mathcal{H})u = -\Delta_\mathcal{H}\mathbb{P}_\Omega u, \qquad u \in \mathrm{D}(\Delta_\mathcal{H}). \end{aligned}$$

Pour un champ de vecteurs tridimensionnel régulier $w : \Omega \longrightarrow \mathbb{C}^3$, curl w est défini par

$$\text{curl } w = (\partial_{x_2} w_3 - \partial_{x_3} w_2, \partial_{x_3} w_1 - \partial_{x_1} w_3, \partial_{x_1} w_2 - \partial_{x_2} w_1).$$

En supposant que l'on puisse donner un sens à l'inverse au laplacien de Hodge, on tire moralement l'égalité

$$\mathbb{P}_\Omega v = v + \nabla \operatorname{div} (-\Delta_{\mathcal{H}})^{-1} v$$

et dans ce cas le couple $(\Delta_{\mathcal{H}}, \mathbb{P}_\Omega)$ sur Ω reproduit le comportement de $(\Delta, \mathbb{P}_{\mathbb{R}^n})$ sur $\mathbb{R}^n$. L'opérateur $\mathbb{A}_{\mathcal{H}} = -\Delta_{\mathcal{H}} \mathbb{P}_\Omega$ est l'opérateur de Stokes avec conditions au bord de Hodge, aussi appelé opérateur de Hodge-Stokes.

En dimension 3, les actions conjointes du projecteur de Leray et du laplacien de Hodge semblent préserver certaines caractéristiques géométriques, dont il n'est pas rendu compte en dimensions supérieures. Par exemple, il est particulièrement approprié d'étudier (NSE) dans sa forme utilisant la vorticité puisque

$$\mathbb{A}_{\mathcal{H}} u = \text{curl } (\text{curl } u), \qquad u \in \mathbb{P}_\Omega \mathrm{D}(\Delta_{\mathcal{H}}).$$

Il se trouve qu'il est certainement possible d'extrapoler ces propriétés géométriques en dimension supérieure grâce aux formes différentielles comme mis en avant dans [MM18] sur les ouverts lipschitziens, le formalisme permettant de généraliser naturellement l'opérateur curl . Cela permet également de considérer des systèmes de magnetohydrodynamique en dimension arbitraire [Mon21 ; Den22].

Si sur les espaces

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C}^n) \quad s \in (-1 + 1/p, 1/p),\ p \in (1, +\infty),$$

l'opérateur $-\Delta_{\mathcal{H}}$ génère un semi-groupe analytique, et $\mathbb{P}_{\mathbb{R}^n_+}$ est bien défini borné. En considérant les normes données par (0.1) :

$$\|v\|_{\mathring{\mathcal{D}}^{s,p}_{-\Delta_{\mathcal{H}}}(\theta,q)} = \left(\int_0^{+\infty} (t^{1-\theta} \|\Delta_{\mathcal{H}} e^{t\Delta_{\mathcal{H}}} v\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)})^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, \ \theta \in (0,1),\ p, q \in [1, +\infty],$$

l'action de $\mathbb{P}_{\mathbb{R}^n_+}$ serait aussi bornée pour ces normes, de sorte que l'on obtient un espace de données initiales pour (NSE) avec conditions au bord de Hodge généralisées. On profite dans ce cas de la régularité maximale de Da Prato-Grisvard (0.5). Si le laplacien de Hodge $-\Delta_{\mathcal{H}}$ a la propriété de régularité maximale L^q sur $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, on profiterait d'une régularité maximale $\dot{\mathrm{H}}^{\alpha,q}_t(\dot{\mathrm{H}}^{s,p}_x)$ globale en temps pour l'opérateur de Hodge-Stokes $\mathbb{A}_{\mathcal{H}}$.

Objectif 3 *Montrer qu'en toute dimension $n \geqslant 2$, quitte à considérer des sous espaces fermés, le projecteur de Leray $\mathbb{P}_{\mathbb{R}^n_+}$ est borné sur les espaces*

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C}^n),\ \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+, \mathbb{C}^n), \quad s \in (-1 + 1/p, 2 + 1/p),\ p \in (1, +\infty), q \in [1, +\infty].$$

et que pour ces mêmes espaces le laplacien de Hodge $-\Delta_{\mathcal{H}}$ admet la propriété de régularité maximale L^q (pour le même q). On voudra plutôt introduire les mêmes espaces de formes différentielles et vérifier les propriétés correspondantes.

Il s'agit de l'objectif du Chapitre 4.

Au-delà : vers les frontières lipschitziennes

Il est intéressant de savoir si on peut étendre les résultats sur le laplacien de Hodge et la bornitude du projecteur de Leray, à des espaces de fonctions homogènes sur des ouverts plus irréguliers que le demi-espace $\mathbb{R}^n_+$. Une généralisation naturelle est celle d'un demi-espace donné

par l'épigraphe d'une fonction uniformément lipschitzienne sur $\mathbb{R}^{n-1}$ à valeurs dans $\mathbb{R}$, aussi appelé ouvert spécial Lipschitz qu'on désignera par Ω ici.

Une construction/réalisation des espaces homogènes sur les ouverts spéciaux Lipschitz, contenant même des outils adaptés, comme les opérateurs potentiels ou de correction de Bogovskiĭ, a été faite par Costabel, M$^{\text{c}}$Intosh et Taggart dans [CMT13]. Cependant, cette construction est faite "modulo les polynômes", cela empêche donc de l'exploiter pour le traitement de problèmes non-linéaires, et laisse également peu clair le sens des traces sur la frontière de l'ouvert.

La construction sur $\mathbb{R}^n$ établie dans le Chapitre 2 offre la possibilité d'être stable par composition (possiblement par un difféomorphisme ou une transformation bi-Lipschitzienne). Cela permet de transférer une partie des propriétés juste par changement de carte globale pour peu qu'on puisse préserver les estimées homogènes.

En général, la preuve du théorème de trace avec régularité optimale dans le cas des espaces inhomogènes est fastidieuse sur les ouverts, en particulier lipschitziens. Il existe cependant quelques preuves simples pour les espaces de Sobolev de type L^2

$$[\cdot]_{|\partial\Omega} \,:\, \mathrm{H}^s(\Omega) \longrightarrow \mathrm{H}^{s-\frac{1}{2}}(\partial\Omega), \quad s \in (1/2, 3/2)$$

dans les papiers de Costabel et Ding [Cos88 ; Din96], en utilisant des espaces avec une régularité anisotrope. On va montrer que cette stratégie, moralement simple, est toujours valable pour les espaces de Sobolev et Besov homogènes plus généraux, mais demande plus de technologie.

Objectif 4 *Étendre et synthétiser la construction et les propriétés des espaces de Sobolev et Besov homogènes sur $\mathbb{R}^n_+$ initiés dans le Chapitre 2 aux ouverts Ω spéciaux Lipschitz : résultats de densité et d'interpolation pour les espaces*

$$\dot{\mathrm{H}}^{s,p}(\Omega),\, \dot{\mathrm{B}}^s_{p,q}(\Omega), \quad s \in \mathbb{R},\, p \in (1, +\infty), q \in [1, +\infty].$$

Construire les espaces homogènes correspondants sur la frontière $\partial\Omega$ et montrer dans ce cadre le théorème de trace optimal

$$[\cdot]_{|\partial\Omega} \,:\, \dot{\mathrm{H}}^{s,p}(\Omega),\, \dot{\mathrm{B}}^s_{p,q}(\Omega) \longrightarrow \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\partial\Omega), \quad s \in (1/p, 1+1/p),\, p \in (1, +\infty),\, q \in [1, +\infty].$$

Il s'agit du but du Chapitre 5 qui clôture cette thèse.

Un dernier commentaire : Les constructions sur $\mathbb{R}^n_+$ et les ouverts spéciaux Lipschitz Ω sont traitées séparément à cause de plusieurs obstructions techniques pour les opérateurs d'extensions. De plus, la construction pour les ouverts spéciaux Lipschitz utilise certains résultats de la construction sur le demi-espace, mais aussi des outils développés dans le Chapitre 3.

In English

At the origin

Numerous nonlinear partial differential equations describing temporal evolution can be written in the form of a linear parabolic part and a nonlinear term. The interest in such equations arises from the fact that they originate from applied disciplines such as physics, chemistry, and biology.

One of the most important of these equations will serve as guide through the introduction of the problems raised and (partially) solved in this thesis. These are the Navier-Stokes equations, which describe the evolution of velocity fields for a homogeneous, incompressible fluid

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u + \nabla \mathfrak{p} &=& -(u\cdot\nabla)u, & \text{on } (0,T)\times\Omega, \\ \operatorname{div} u &=& 0, & \text{on } (0,T)\times\Omega, \\ u(0) &=& u_0 & . \end{array}\right. \tag{NSE}$$

with unknown $(u,\mathfrak{p}) : (0,T)\times\Omega \longrightarrow \mathbb{R}^n\times\mathbb{R}$, where Ω is an open set in $\mathbb{R}^n$ (possibly with a set of boundary conditions to close the system (NSE)), $T\in(0,+\infty]$, u is the velocity of the considered fluid, $\mathfrak{p}$ is the pressure of the system, and $u_0 : \Omega\longrightarrow\mathbb{R}^n$ is a given initial velocity field satisfying $\operatorname{div} u_0 = 0$.

The condition known as zero divergence, $\operatorname{div} u = 0$, expresses the incompressibility and homogeneity of the fluid. The reader is encouraged to refer to [Lem16, Chapters 1, 2, & 3] for more historical and physical context surrounding the system (NSE) in the case of $\Omega=\mathbb{R}^n$.

If we are interested in the global-in-time well-posedness of (NSE), i.e. $T=+\infty$, it is natural to try to determine a suitable function space for the choice of the initial data u_0, such that we have global time control on $(\partial_t u, \Delta u)$ in $\mathrm{L}^q(\mathbb{R}_+,\mathrm{L}^p(\Omega))$ with $p,q\in(1,+\infty)$. This question can also be addressed in the framework of an abstract linear Cauchy problem

$$\left\{\begin{array}{rcl} \partial_t u(t) + Au(t) = & f(t)\,, & 0<t<+\infty, \\ u(0) = & u_0 \text{ in } X. & \end{array}\right. , \tag{ACP}$$

where A is a reasonable operator generating a strongly continuous and uniformly bounded semigroup $(e^{-tA})_{t\geqslant 0}$ on a Banach space X. Since the problem is linear, we can assume $f=0$ for now and verify that the space morally induced by the (semi-)norm

$$\begin{aligned}\|u_0\|_{\mathring{\mathcal{D}}_A(1-\frac{1}{q},q)} := \left(\int_0^{+\infty}(t^{1-(1-\frac{1}{q})}\|Ae^{-tA}u_0\|_X)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} &= \left(\int_0^{+\infty}(\|Ae^{-tA}u_0\|_X)^q\mathrm{d}t\right)^{\frac{1}{q}} \\ &= \left(\int_0^{+\infty}(\|\partial_t(e^{-tA}u_0)\|_X)^q\mathrm{d}t\right)^{\frac{1}{q}}\end{aligned}$$

seems to be a natural candidate, for which we even have $t\mapsto e^{-tA}u_0$ continuous with values in $\mathring{\mathcal{D}}_A(1-\frac{1}{q},q)$. More generally, this leads us to consider the spaces morally induced by the (semi-)norms

$$\|x\|_{\mathring{\mathcal{D}}_A(\theta,q)} = \left(\int_0^{+\infty}(t^{1-\theta}\|Ae^{-tA}x\|_X)^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, \qquad \theta\in(0,1),\ q\in[1,+\infty]. \tag{0.8}$$

The case of $A=-\Delta$ on $X=\mathrm{L}^p(\mathbb{R}^n)$, this is very well documented and known, see for instance [BCD11, Theorem 2.34], we have, for $\theta\in(0,1)$, $p,q\in[1,+\infty]$,

$$\|v\|_{\mathring{\mathcal{D}}_{-\Delta}(\theta,q)} = \left(\int_0^{+\infty}(t^{1-\theta}\|\Delta e^{t\Delta}v\|_{\mathrm{L}^p(\mathbb{R}^n)})^q\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \sim_{p,q,\theta,n} \|v\|_{\dot{\mathrm{B}}^{2\theta}_{p,q}(\mathbb{R}^n)}, \tag{0.9}$$

where $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ is the homogeneous Besov space defined for example in [BCD11, Chapter 2],

[BL76, Chapter 6, Section 6.3], [Tri83, Chapter 5]. The reader who is not familiar with Besov spaces, especially homogeneous ones, can rely on the equality $\dot{\mathrm{B}}^s_{2,2}(\mathbb{R}^n) = \dot{\mathrm{H}}^s(\mathbb{R}^n)$ with equivalent (semi-)norm $\|v\|_{\dot{\mathrm{H}}^s(\mathbb{R}^n)} := \|(-\Delta)^{\frac{s}{2}} v\|_{\mathrm{L}^2(\mathbb{R}^n)}$, and for $s = 1$, $\|v\|_{\dot{\mathrm{H}}^1(\mathbb{R}^n)} = \|\nabla v\|_{\mathrm{L}^2(\mathbb{R}^n)}$.

By the equivalence of norms (0.9), it seems natural to name the family of vector spaces that emerge from the family of (semi-)norms given in (0.8) **Besov spaces adapted to the operator** A. When A is a sectorial, invertible or injective operator, a simple characterization has been given thanks to real interpolation, more precisely

$$\|x\|_{\mathring{\mathcal{D}}_A(\theta,q)} \sim_{\theta,q} \|x\|_{(X,\mathrm{D}(\mathring{A}))_{\theta,q}} \tag{0.10}$$

where $\mathrm{D}(\mathring{A})$ corresponds to the completion of the domain $\mathrm{D}(A)$ for the norm $\|A\cdot\|_X$. When A is invertible, it is mentioned that $\mathrm{D}(\mathring{A}) = \mathrm{D}(A)$ with norm equivalence. This fact has been extensively reviewed in the literature, for example by Haase [Haa06, Chapter 6] or Lunardi [Lun18, Chapter 6]. This often allows for a complete description of the space when the domain of A itself has a suitable description. We also mention the recent work of Batty and Chen [BC20] which revisits the construction in a more general way for sectorial operators by giving a number of equivalent characterizations, such as Littlewood-Paley type, as well as a non-exhaustive list of complementary properties.

An effective description of adapted function spaces when the operator A has an explicit form has also been widely investigated in the literature. The case of elliptic operators in divergence form on L^p spaces has received particular attention. For example, under the assumptions of Gaussian or Poisson estimates on the kernel of the generated semigroup, the works of Bui, Duong and Yan [BDY12] and Cao and Grygor'yan [CG20] establish coincidence with classical Besov spaces in some cases. Under more general assumptions, coincidence is shown for spaces that are "close enough" to L^2 in the monograph by Amenta and Auscher [AA18, Chapter 5].

Emerging problems : realization of homogeneous functions spaces

The appearance of homogeneous Besov spaces (0.9) for the choice of initial conditions raises the question of the choice of realization of these spaces, if one wishes to return to the study of the nonlinear problem (NSE) for $\Omega = \mathbb{R}^n$.

Indeed, the construction of homogeneous Sobolev and Besov spaces as it is usually done, [BL76, Chapter 6, Section 6.3], [Tri83, Chapter 5], is based on the space of tempered distributions quotiented by polynomials $\mathcal{S}'(\mathbb{R}^n) \Big/ \mathbb{C}[x]$, in order to make them normed vector spaces. The attentive reader will have noticed that homogeneous (semi-)norms, for instance $\|\cdot\|_{\dot{\mathrm{H}}^s(\mathbb{R}^n)}$, do not separate polynomials, and therefore are not truly norms on $\mathcal{S}'(\mathbb{R}^n)$. This induces a problem for defining nonlinearities such as the one in (NSE).

For $[u], [v] \in \mathcal{S}'(\mathbb{R}^n) \Big/ \mathbb{C}[x]$, and $u + P, v + Q, u + \tilde{P}, v + \tilde{Q} \in \mathcal{S}'(\mathbb{R}^n)$ two representatives of $[u]$ and $[v]$, we have

$$\begin{aligned}((u+P)\cdot\nabla)(v+Q) - ((u+\tilde{P})\cdot\nabla)(v+\tilde{Q}) =& [(P-\tilde{P})\cdot\nabla]v + [u\cdot\nabla](Q-\tilde{Q}) \\ &+ (P\cdot\nabla)Q - (\tilde{P}\cdot\nabla)\tilde{Q}.\end{aligned}$$

Therefore, subject to a (para-)product law that makes sense, although $(P\cdot\nabla)Q - (\tilde{P}\cdot\nabla)\tilde{Q}$ is a polynomial, this is not the case for $[(P-\tilde{P})\cdot\nabla]v + [u\cdot\nabla](Q-\tilde{Q})$ so that the product **depends on the choice of representatives !**

Another possibility would be to naively complete the Schwartz class with respect to the homogeneous norm we want to consider. This construction has the disadvantage of producing elements that may no longer be distributions. For example, one can check that

$$\mathrm{C}^\infty_c(\mathbb{R}^n) \not\subset \dot{\mathrm{H}}^{-\frac{n}{2}}(\mathbb{R}^n).$$

This prevents us, *a priori*, from identifying elements of $\dot{\mathrm{H}}^{\frac{n}{2}}(\mathbb{R}^n)$ (as a completion) as distributions. This phenomenon is known as *infrared divergence* and relates to convergence problems for low frequencies in the sense of Fourier.

Thus, for a realization of homogeneous function spaces, we can only choose two out of the following three properties :

(i) spaces whose elements are distributions, in a reasonable sense ;

(ii) well-defined product laws ;

(iii) all spaces are complete.

For the treatment of nonlinear partial differential equations, we choose to give up completeness for all homogeneous function spaces. This choice was made by Bahouri, Chemin, and Danchin in their book [BCD11]. Their idea instead was to consider a subspace of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ that does not contain polynomials. The space in question, sometimes called **Chemin's space of homogeneous distributions**, is defined by

$$\mathcal{S}'_h(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \forall \Theta \in \mathrm{C}_c^\infty(\mathbb{R}^n),\ \|\Theta(\lambda\mathfrak{D})u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \xrightarrow[\lambda\to+\infty]{} 0 \right\},$$

where $\Theta(\lambda\mathfrak{D}) = \mathcal{F}^{-1}\Theta(\lambda\cdot)\mathcal{F}$, with $\mathcal{F}$ denoting the Fourier transform. In their book [BCD11, Chapter 2], Bahouri, Chemin, and Danchin focused mainly on the construction of homogeneous Besov spaces

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n), \qquad s \in \mathbb{R},\ p, q \in [1, +\infty],$$

for which they proved, [BCD11, Theorem 2.25], that those spaces are complete if and only if

$$\left[s < \frac{n}{p} \right] \text{ or } \left[q = 1 \text{ and } s \leqslant \frac{n}{p} \right].$$

The construction was then extended to homogeneous Besov spaces on the half-space $\mathbb{R}^n_+$ by Danchin and Mucha [DM09 ; DM15], and then by Danchin, Hieber, Mucha, and Tolksdorf [DHMT21] to homogeneous Sobolev spaces with integer and positive regularity indices $\dot{\mathrm{H}}^{m,p}$ on $\mathbb{R}^n$ and $\mathbb{R}^n_+$, for $m \in \mathbb{N}$, $p \in (1, +\infty)$. These constructions of homogeneous function spaces have been a central tool for the global-in-time well-posedness of certain fluid mechanics equations that generalize the Navier-Stokes equations (NSE) in [DM09 ; DM15 ; DHMT21]. In fact, it was necessary to wait until [DHMT21] for the first real results on density, traces, and interpolation for such realizations of homogeneous function spaces.

Goal 1 *To extend and summarize the construction and properties of homogeneous Sobolev and Besov spaces initiated by Bahouri, Chemin, Danchin, Hieber, Mucha, and Tolksdorf on* $\mathbb{R}^n$ *and* $\mathbb{R}^n_+$ *: including density results, and traces and interpolation for the spaces*

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\ \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+), \qquad s \in \mathbb{R},\ p \in (1, +\infty), q \in [1, +\infty].$$

This is the goal of Chapter 2.

Reajustement : function spaces for initial datas and maximal regularity

The new realization of homogeneous Sobolev and Besov spaces has the consequence that the tools developed by Haase in [Haa06] to obtain the equivalence of norms (0.3) with equality of spaces are no longer adapted. For example, in the case of the Laplacian on $\mathrm{L}^p(\mathbb{R}^n)$, we can no longer use completion to obtain $\mathrm{D}(\mathring{\Delta}) = \dot{\mathrm{H}}^{2,p}(\mathbb{R}^n)$.

Two remarks, however : firstly, real interpolation of normed vector spaces that are not necessarily complete does make sense, as can be found in [BL76, Chapter 3]. Secondly, the completion problem for constructing the homogeneous domain $\mathrm{D}(\mathring{A})$ only arises when the operator A is not invertible on X. When A is invertible, $\mathrm{D}(A)$ and $\mathrm{D}(\mathring{A})$ coincide with equivalent norms.

Danchin, Hieber, Mucha, and Tolksdorf reviewed the theory of interpolation and homogeneous operators of Haase [Haa06, Chapter 6] in [DHMT21, Chapter 2]. Their approach allows for the consideration of non-complete homogeneous domains $\mathrm{D}(\mathring{A})$, under certain additional technical assumptions and considerations, often satisfied in practice, so that (0.10) remains valid.

We say that the operator A has the parabolic (global-in-time) L^q-maximal regularity property on Y, when for $u_0 = 0$, $q \in [1, +\infty]$, $f \in \mathrm{L}^q(\mathbb{R}_+, Y)$, (ACP) has a unique solution $u \in \mathrm{C}^0(\mathbb{R}_+, Y)$, such that

$$\|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,Y)} \lesssim_{q,Y,A} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,Y)}. \tag{0.11}$$

Danchin, Hieber, Mucha, and Tolksdorf were then able to show in this framework, [DHMT21, Theorem 2.20], that for $Y = \mathring{\mathcal{D}}_A(\theta, q)$, $q \in [1, +\infty)$, $\theta \in (0, \frac{1}{q})$, and $u_0 \in \mathring{\mathcal{D}}_A(\theta + 1 - \frac{1}{q}, q)$, (ACP) admits a unique solution $u \in \mathrm{C}^0_b(\mathbb{R}_+, \mathring{\mathcal{D}}_A(\theta + 1 - \frac{1}{q}, q))$, such that

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(\theta+1-\frac{1}{q},q))} + \|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,\mathring{\mathcal{D}}_A(\theta,q))} \lesssim_{q,\theta,A} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,\mathring{\mathcal{D}}_A(\theta,q))} + \|u_0\|_{\mathring{\mathcal{D}}_A(\theta+1-\frac{1}{q},q)}. \tag{0.12}$$

The operator A, when injective, has in particular the (global-in-time) L^q-maximal regularity property on $\mathring{\mathcal{D}}_A(\theta, q)$, for all $q \in [1, +\infty)$ and $\theta \in (0, \frac{1}{q})$. This result was already known when A is invertible, from the work of Da Prato and Grisvard [DG75, Theorem 4.15].

The question of the L^q-maximal regularity property (0.11) for A on $Y = X$ is, however, much more involved. A definitive answer was provided by Weis [Wei01], who fully characterizes such operators, under the condition that the Banach space X has the so-called **UMD** property, as being the $\mathcal{R}$-sectorial operators. For more informations and broader review, see for instance [DHP03 ; KW04 ; PS16]. It is also necessary to have $q \in (1, +\infty)$ in this case.

When A has the property of L^q-maximal regularity on X, $q \in (1, +\infty)$, for $u_0 \in \mathring{\mathcal{D}}_A(1 - \frac{1}{q}, q)$ and $f \in \mathrm{L}^q(\mathbb{R}_+, X)$, we have that (ACP) admits a unique solution u satisfying $t \mapsto u(t) - e^{-tA}u_0 \in \mathrm{C}^0(\mathbb{R}_+, X)$ and

$$\|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,A} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(1-\frac{1}{q},q)}. \tag{0.13}$$

However, we should notice that (0.13) suffers from its comparison with (0.12), where we have a control on u as a continuous and bounded function with respect to the time variable. The result is known when A is invertible, and one can consult, for example, [Ama95, Chapter 4, Theorem 4.10.2]. It would also be interesting to know whether one has the inequality

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1-\frac{1}{q},q))} \lesssim_{q,A} \|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \tag{0.14}$$

whenever A is only injective.

Some recent works involve the transfer of regularity space-time via mixed derivative estimates of the form.

$$\|(\partial_t)^\beta A^{1-\beta} u\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,A} \|(\partial_t u, Au)\|_{\mathrm{L}^q(\mathbb{R}_+,X)}, \qquad \beta \in [0, 1],$$

to deal with nonlinear problems similar to (NSE), such as the work of Brandolese and Monniaux [BM21] on the Boussinesq system.

One may want to improve the range of space-time regularity by proposing for example to replace L^q in (0.11) with a homogeneous Sobolev space $\dot{\mathrm{H}}^{\alpha,q}$, for $q \in (1,+\infty)$, $\alpha \in (-1+\frac{1}{q}, \frac{1}{q})$, in order to gain some flexibility in the estimates. See [Pru02 ; PS16] for more information on mixed derivative estimates and (inhomogeneous) Sobolev maximal regularity.

If we can replace L^q by $\dot{\mathrm{H}}^{\alpha,q}$, then the question remains of the space we have to use in order to replace the space $\mathring{\mathcal{D}}_A(1-\frac{1}{q},q)$ with in (0.13) and (0.14).

Goal 2 *Prove that a not-too-restrictive class of injective operators that have the L^q-maximal regularity property on a UMD Banach space X also have the $\dot{\mathrm{H}}^{\alpha,q}$-maximal regularity property, whenever $q \in (1,+\infty)$, $\alpha \in (-1+\frac{1}{q}, \frac{1}{q})$. For such an operator A, we aim to show in this case that*

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(\alpha+1-\frac{1}{q},q))} \lesssim_{A,q,\alpha} \|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \lesssim_{A,q,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(\alpha+1-\frac{1}{q},q)}.$$

Chapter 3 will be dedicated to this, and will contain a brief presentation of the interpolation theory and homogeneous operators revisited by Danchin, Hieber, Mucha, and Tolksdorf [DHMT21, Chapter 2].

Coming back : the Stokes system and the Helmholtz decomposition

We have been somewhat neglectful. Indeed, we introduced the study of an abstract Cauchy problem (ACP), based on (NSE), by dealing with the Laplacian on $\mathbb{R}^n$. However, the pressure term $\mathfrak{p}$ is also an unknown of the problem that does not seem to appear when formatting it into an abstract Cauchy problem.

It turns out that when $\Omega = \mathbb{R}^n$, the problem (NSE) can be rewritten in the form

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u &=& -\mathbb{P}[(u\cdot\nabla)u], & \text{on } (0,T)\times\mathbb{R}^n, \\ \text{div } u &=& 0, & \text{on } (0,T)\times\mathbb{R}^n, \\ u(0) &=& u_0 & . \end{array}\right. \tag{NSE'}$$

where $\mathbb{P}$ is the Leray projector onto the functions with zero divergence on $\mathbb{R}^n$ in the sense of distributions. This motivates in particular the study of the heat equation as a main example when $\Omega = \mathbb{R}^n$. This is possible since $\mathbb{P}$ commutes with the Laplacian on $\mathbb{R}^n$, as it is defined by the formula

$$\mathbb{P}f = \mathcal{F}^{-1}\left[\xi \mapsto \mathcal{F}f(\xi) - \frac{\xi\cdot\mathcal{F}f(\xi)}{|\xi|^2}\xi\right], \qquad \xi\in\mathbb{R}^n, \qquad f\in \mathrm{L}^2(\mathbb{R}^n,\mathbb{C}^n),$$

inducing a bounded linear operator

$$\mathbb{P} : \mathrm{L}^p(\mathbb{R}^n,\mathbb{C}^n) \longrightarrow \mathrm{L}^p_\sigma(\mathbb{R}^n) := \{\, u\in \mathrm{L}^p(\mathbb{R}^n,\mathbb{C}^n) \mid \text{div } u = 0\}, \qquad p\in(1,+\infty)$$

inducing itself a topological decomposition called the **Helmholtz decomposition**

$$\mathrm{L}^p(\mathbb{R}^n,\mathbb{C}^n) = \mathrm{L}^p_\sigma(\mathbb{R}^n) \oplus \overline{\nabla \mathrm{H}^{1,p}(\mathbb{R}^n,\mathbb{C})}, \qquad p\in(1,+\infty).$$

In general, it is not possible to reduce (NSE) to the heat equation when dealing with a domain with boundary $\Omega \subset \mathbb{R}^n$ and boundary conditions (necessary to close the system).

Before continuing, we must clarify what the Leray projector is on an open set Ω of $\mathbb{R}^n$. In this context, the Leray projector is defined as the unique orthogonal projector

$$\mathbb{P}_\Omega : \mathrm{L}^2(\Omega,\mathbb{C}^n) \longrightarrow \mathrm{L}^2_\sigma(\Omega) := \overline{\{\, u\in \mathrm{C}^\infty_c(\Omega,\mathbb{C}^n) \mid \text{div } u = 0\,\}}^{\|\cdot\|_{\mathrm{L}^2(\Omega)}}.$$

When $p\in(1,+\infty)$ and $p\neq 2$, the question of the boundedness of $\mathbb{P}_\Omega$ on L^p with values in L^p_σ (keeping the same definition for L^p_σ) is not automatic, as exhibited in [Bog86]. When $1<p<+\infty$

and Ω is reasonable, see for example [Soh01, Lemma 2.5.3] for $p = 2$, we have the exact description

$$\mathrm{L}^p_\sigma(\Omega) = \{\, u \in \mathrm{L}^p(\Omega, \mathbb{C}^n) \mid \operatorname{div} u = 0 \;\&\; u \cdot \nu_{|\partial\Omega} = 0 \,\},$$

where $u \cdot \nu_{|\partial\Omega}$ is given as the weak partial trace on the boundary, with ν being the unit normal vector on the boundary of Ω. The boundedness of $\mathbb{P}_\Omega$ on L^p has been studied in many contexts. See, for example, [FMM98 ; SS92] for bounded domains with C^1 and Lipschitz boundary.

Thus, in the presence of a boundary and a set of boundary conditions, we must consider the boundary conditions and the incompressibility condition $\operatorname{div} u = 0$ in the involved function spaces that are considered to define the corresponding operator which replaces the Laplacian. This allows us to consider the pressure $\mathfrak{p}$ (more precisely, its gradient) as a Lagrange multiplier. Such an operator is called a Stokes operator.

For example, in the case of the operator for (NSE) with Dirichlet boundary conditions on the half-space $\mathbb{R}^n_+$, the operator to be introduced that replaces $-\Delta$ in (NSE') is

$$\begin{aligned} \mathrm{D}_p(A_{\mathcal{D}}) &= \mathrm{L}^p_\sigma(\mathbb{R}^n_+) \cap \mathrm{H}^{1,p}_0(\mathbb{R}^n_+) \cap \mathrm{H}^{2,p}(\mathbb{R}^n_+) \\ A_{\mathcal{D}} &= \mathbb{P}_{\mathbb{R}^n_+}(-\Delta)u, \end{aligned}$$

called the Stokes-Dirichlet operator. For a function $u \in \mathrm{D}_p(A_{\mathcal{D}})$, there is no guarantee *a priori* that $\mathbb{P}_{\mathbb{R}^n_+} u \in \mathrm{H}^{1,p}_0(\mathbb{R}^n_+)$. More generally, given a set of boundary conditions, there is *a priori* no reason for the Leray projector to commute with the Laplacian with the same boundary conditions. For a survey on various Stokes operators with boundary conditions, we refer to the review on the topic by Monniaux and Shen [MS18].

It is known, see [Gal11, Remark III.1.2], that $\mathbb{P}_{\mathbb{R}^n_+}$ is bounded on $\mathrm{L}^p(\mathbb{R}^n_+)$ for all $p \in (1, +\infty)$. We raise the two following points :

(i) For regularity questions related to (NSE) and similar systems, it could be interesting to know whether $\mathbb{P}_{\mathbb{R}^n_+}$ is bounded on the function spaces

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C}^n),\ \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+, \mathbb{C}^n), \qquad s \in \mathbb{R},\ p \in (1, +\infty), q \in [1, +\infty].$$

The result is known for in the case of Besov spaces $s \in (-1 + \frac{1}{p}, \frac{1}{p})$, [DM09, Lemma 1].

(ii) For $\Omega \subset \mathbb{R}^3$, following [MS18, Section 4], it is known that the Laplacian, named the Hodge Laplacian, given by

$$\begin{aligned} \mathrm{D}(\Delta_{\mathcal{H}}) = \{\, u \in \mathrm{L}^2(\Omega, \mathbb{C}^3) \mid &\operatorname{curl} u,\ \operatorname{div} u,\ \operatorname{curl}\operatorname{curl} u - \nabla \operatorname{div} u \in \mathrm{L}^2(\Omega, \mathbb{C}^3), \\ &\& \, u \cdot \nu_{|\partial\Omega} = 0,\ \nu \times \operatorname{curl} u_{|\partial\Omega} = 0 \,\} \\ -\Delta_{\mathcal{H}} u := \operatorname{curl}\operatorname{curl} u - \nabla \operatorname{div} u, & \end{aligned}$$

is "compatible" with the Leray projector in the sense that it satisfies

$$\begin{aligned} &\mathbb{P}_\Omega \mathrm{D}(\Delta_{\mathcal{H}}) \subset \mathrm{D}(\Delta_{\mathcal{H}}), \\ &\mathbb{P}_\Omega(-\Delta_{\mathcal{H}})u = -\Delta_{\mathcal{H}} \mathbb{P}_\Omega u, \qquad u \in \mathrm{D}(\Delta_{\mathcal{H}}). \end{aligned}$$

For a regular three-dimensional vector field $w : \Omega \longrightarrow \mathbb{C}^3$, curl w is defined by

$$\operatorname{curl} w = (\partial_{x_2} w_3 - \partial_{x_3} w_2, \partial_{x_3} w_1 - \partial_{x_1} w_3, \partial_{x_1} w_2 - \partial_{x_2} w_1).$$

assuming that we can give a meaning to the inverse of the Hodge Laplacian, we can derive the following identity :

$$\mathbb{P}_\Omega v = v + \nabla \operatorname{div} (-\Delta_{\mathcal{H}})^{-1} v$$

and in this case, the pair $(\Delta_{\mathcal{H}}, \mathbb{P}_\Omega)$ on Ω reproduces the behavior of $(\Delta, \mathbb{P}_{\mathbb{R}^n})$ on $\mathbb{R}^n$. The operator

$\mathbb{A}_{\mathcal{H}} = -\Delta_{\mathcal{H}}\mathbb{P}_{\Omega}$ is the Stokes operator with Hodge boundary conditions, also known as the Hodge-Stokes operator.

In dimension 3, the combined actions of the Leray projector and the Hodge Laplacian seem to preserve certain geometric features, which are not accounted for in higher dimensions. For example, it is particularly suitable to investigate (NSE) in vorticity form since

$$\mathbb{A}_{\mathcal{H}}u = \operatorname{curl}(\operatorname{curl} u), \qquad u \in \mathbb{P}_{\Omega}\mathrm{D}(\Delta_{\mathcal{H}}).$$

It turns out that it is certainly possible to extrapolate these geometric properties to higher dimensions using differential forms as highlighted in [MM18] on Lipschitz domains, the formalism allowing for a natural generalization of the curl operator. This also allows for the consideration of magnetohydrodynamic systems in arbitrary dimensions [Mon21 ; Den22].

If on the function spaces

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C}^n) \quad s \in (-1+1/p, 1/p),\, p \in (1,+\infty),$$

the operator $-\Delta_{\mathcal{H}}$ generates an analytic semigroup, and $\mathbb{P}_{\mathbb{R}^n_+}$ is well-defined and bounded, considering the norms given by (0.8) :

$$\|v\|_{\mathring{\mathcal{D}}^{s,p}_{-\Delta_{\mathcal{H}}}(\theta,q)} = \left(\int_0^{+\infty} (t^{1-\theta}\|\Delta_{\mathcal{H}}e^{t\Delta_{\mathcal{H}}}v\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)})^q \frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}}, \ \theta \in (0,1),\, p,q \in [1,+\infty],$$

we obtain that the action of $\mathbb{P}_{\mathbb{R}^n_+}$ would also be bounded for these norms. Therefore, we obtain an initial data space for (NSE) with generalized Hodge boundary conditions (due to the arbitrary dimension). In this case, we could benefit from the maximal regularity of Da Prato-Grisvard (0.12). Moreover, if the Hodge Laplacian $-\Delta_{\mathcal{H}}$ has the L^q-maximal regularity property on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, we would benefit from a global-in-time maximal regularity $\dot{\mathrm{H}}^{\alpha,q}_t(\dot{\mathrm{H}}^{s,p}_x)$ for the Hodge-Stokes operator $\mathbb{A}_{\mathcal{H}}$.

Goal 3 *To show that in any dimension $n \geqslant 2$, possibly up to consider some closed subspaces, the Leray projector $\mathbb{P}_{\mathbb{R}^n_+}$ is bounded on the spaces*

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C}^n),\ \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+,\mathbb{C}^n), \quad s \in (-1+1/p, 2+1/p),\, p \in (1,+\infty), q \in [1,+\infty],$$

and that for these same spaces, the Hodge Laplacian $-\Delta_{\mathcal{H}}$ has the L^q-maximal regularity property (for the same q). We will introduce the corresponding spaces of differential forms instead, and verify the corresponding properties.

This is the goal of the Chapter 4.

Beyond : towards Lipschitz boundaries

It is interesting to know whether the results for the Hodge Laplacian and the boundedness of the Leray projector can be extended to homogeneous functions spaces on domains that are more irregular than the half-space $\mathbb{R}^n_+$. An example of a natural generalization would be a half-space given by the epigraph of a uniformly Lipschitz function on $\mathbb{R}^{n-1}$ with values in $\mathbb{R}$, also called a special Lipschitz domain denoted by Ω here.

A construction/realization of homogeneous spaces on special Lipschitz open sets, containing even adapted tools, such as potential operators or Bogovskiĭ operators, was made by Costabel, M$^{\mathrm{c}}$Intosh and Taggart in [CMT13]. However, this construction is made "modulo polynomials", which prevents it from being exploited for the treatment of nonlinear problems, and the meaning of traces on the boundary of the domain was left unclear.

The construction on $\mathbb{R}^n$ established in Chapter 2 offers the possibility of being stable by composition (possibly by a diffeomorphism, or a bi-Lipschitz transformation). This allows for the transfer of some properties just by a change of global chart as long as homogeneous estimates can be preserved.

In general, proving the trace theorem with optimal regularity in the case of inhomogeneous spaces is tedious on open sets, especially the ones with Lipschitz boundary. However, there are some simple proofs for Sobolev spaces of type L^2

$$[\cdot]_{|\partial\Omega} \,:\, \mathrm{H}^s(\Omega) \longrightarrow \mathrm{H}^{s-\frac{1}{2}}(\partial\Omega), \quad s \in (1/2, 3/2)$$

in the papers of Costabel and Ding [Cos88 ; Din96], using function spaces with anisotropic regularity. We will show that this strategy, which is conceptually simple, is still valid for more general homogeneous Sobolev and Besov spaces but requires more technology.

Goal 4 *Extend and synthesize the construction and properties of homogeneous Sobolev and Besov spaces on $\mathbb{R}^n_+$ initiated in Chapter 2 to special Lipschitz open sets Ω : density and interpolation results for spaces*

$$\dot{\mathrm{H}}^{s,p}(\Omega),\, \dot{\mathrm{B}}^s_{p,q}(\Omega), \quad s \in \mathbb{R},\, p \in (1, +\infty),\, q \in [1, +\infty].$$

To construct the corresponding homogeneous spaces on the boundary $\partial\Omega$ and prove the optimal trace theorem in this context

$$[\cdot]_{|\partial\Omega} \,:\, \dot{\mathrm{H}}^{s,p}(\Omega),\, \dot{\mathrm{B}}^s_{p,q}(\Omega) \longrightarrow \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\partial\Omega), \quad s \in (1/p, 1+1/p),\, p \in (1, +\infty),\, q \in [1, +\infty].$$

This is the goal of Chapter 5, and it concludes this dissertation.

One final comment : The constructions on $\mathbb{R}^n_+$ and special Lipschitz open sets Ω are treated separately due to several technical obstructions for extension operators. Moreover, the construction for special Lipschitz open sets uses some results from the construction on the half-space, as well as tools developed in Chapter 3.

1 Tools and general concepts

Sommaire

Résumé du chapitre.

Ce chapitre est dédié à l'introduction des notations, définitions et propositions concernant les principaux concepts et outils qui seront utilisés tout au long de cette thèse.

On présente dans un premier temps les principaux concepts de la théorie de l'interpolation réelle et complexe d'espaces vectoriels normés. Celle-ci est au cœur de tous les chapitres du manuscrit.

Dans un second temps, on présente les bonnes définitions et les principaux résultats issus de la théorie des opérateurs (bi-)sectoriels sur un espace de Banach. Bien que les outils présentés dans cette partie soient utilisés en filigrane dans tous les chapitres de la thèse, ils n'interviennent réellement qu'au cours des chapitres 3 et 4.

On prend soin de préciser ici que l'introduction de ces outils n'est pas complète, de façon volontaire, afin d'avoir une présentation aussi claire et concise que possible. Par exemple, nous n'introduisons pas la notion de **calculs fonctionnel abstrait** sur un espace de Banach pour introduire le cas particulier du **calcul fonctionnel holormorphe** pour les opérateurs (bi-)sectoriels, on renvoie le lecteur intéressé au livre de Haase [Haa06, Chapter 1], ou la thèse d'Egert [Ege15, Chapter 3, Section 3.1]. Nous n'introduisons pas non plus la notion de **catégories**, ou de **foncteurs** qui permettent une bonne formalisation de certains résultats en théorie de l'interpolation en toute généralité, pas nécessaire ici. Le lecteur intéressé est invité à consulter la synthèse très compréhensible dans la thèse d'Egert [Ege15, Chapter 1, Section 1.3.1] ainsi que les références qui s'y trouvent.

Summary of the chapter.

This chapter is dedicated to the introduction of notations, definitions and propositions concerning the main concepts and tools that will be used throughout this thesis.

First, the main concepts of the theory of real and complex interpolation of normed vector spaces are presented. This theory is at the heart of all the chapters of the manuscript.

In a second part, we present the good definitions and the main results from the theory of (bi-)sectorial operators on a Banach space. Although the tools presented in this part are used in all the chapters of the thesis, they really intervene only during the chapters 3 and 4.

We take care to specify here, that the introduction of these tools is not complete, in a voluntary way, in order to have a presentation as clear and concise as possible. For example, we do not introduce the notion of **abstract functional calculus** on a Banach space to introduce the particular case of the **holormorphic functional calculus** for (bi-)sectoral operators, we refer the interested reader to Haase's book [Haa06, Chapter 1], or Egert's dissertation [Ege15, Chapter 3, Section 3.1].

We do not introduce either the notion of **categories**, or of **functors** which allows a good formalization of some results from interpolation theory at the highest level of generality, not necessary here. The interested reader is invited to consult the very comprehensive synthesis in Egert's thesis [Ege15, Chapter 1, Section 1.3.1] and the references therein.

1 Notations and definitions

Throughout this dissertation the dimension will be $n \geqslant 2$, and $\mathbb{N}$ will be the set of non-negative integers, $\mathbb{Z}$ denotes the set of integers. The notation $\mathbb{R}$ stands for the field of real numbers, $\mathbb{C}$ for the field of complex numbers. We also define $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. The symbols $\Re$ and $\Im$ are respectively the real part and imaginary part operators from $\mathbb{C}$ to $\mathbb{R}$. We denote by $\mathbb{C}_+$ the set of complex numbers with non-negative real parts, and $\mathbb{R}_+ = [0, +\infty)$ the set of non-negative real numbers. The set $\mathbb{C}_+^*$ is the set of complex numbers with positive real part.

The upper half-space of the vector space $\mathbb{R}^n$ is given by $\mathbb{R}^n_+ := \mathbb{R}^{n-1} \times (0, +\infty)$. For $z \in \mathbb{C}^m$, with $m \geqslant 1$ an integer, $|z|$ is the Euclidean norm of z, and if $w \in \mathbb{C}^m$, $z \cdot \overline{w}$ is the standard inner product between z and w, satisfying $z \cdot \overline{z} = |z|^2$ with $\overline{z}$ the component-wise complex conjugate of z. For $a, b \in \mathbb{R}$ with $a \leqslant b$, we write $[\![a, b]\!] := [a, b] \cap \mathbb{Z}$.

For $x \in \mathbb{R}^n$, the (open) ball centered in x of radius $r > 0$, is given by

$$B(x, r) := \{\, y \in \mathbb{R}^n \,|\, |x - y| < r \,\}.$$

For two real numbers $A, B \in \mathbb{R}$, $A \lesssim_{a,b,c} B$ means that there exists a constant $C > 0$ depending on a, b, c such that $A \leqslant CB$. When both $A \lesssim_{a,b,c} B$ and $B \lesssim_{a,b,c} A$ are true, we simply write $A \sim_{a,b,c} B$. When the number of indices is overloaded, we allow ourselves to write $A \lesssim^{d,e,f}_{a,b,c} B$ instead of $A \lesssim_{a,b,c,d,e,f} B$.

Denote by $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ the space of complex valued Schwartz functions, and $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ its topological dual called the space of tempered distributions. The Fourier transform on $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ is written $\mathcal{F}$, and is pointwise defined for any $f \in \mathrm{L}^1(\mathbb{R}^n, \mathbb{C})$ by

$$\mathcal{F}f(\xi) := \int_{\mathbb{R}^n} f(x)\, e^{-ix\cdot\xi}\, \mathrm{d}x,\ \xi \in \mathbb{R}^n.$$

Additionally, for $p \in [1 + \infty]$, we write $p' = \frac{p}{p-1}$ its ***Hölder conjugate***.

For any $m \in \mathbb{N}$, the map $\nabla^m : \mathcal{S}'(\mathbb{R}^n, \mathbb{C}) \longrightarrow \mathcal{S}'(\mathbb{R}^n, \mathbb{C}^{n^m})$ is defined as $\nabla^m u := (\partial^\alpha u)_{|\alpha|=m}$. We denote by ∇' and Δ' respectively the gradient and the Laplacian on $\mathbb{R}^{n-1}$ identified with the $n-1$ first components of $\mathbb{R}^n$, *i.e.* $\nabla' = (\partial_{x_1}, \ldots, \partial_{x_{n-1}})$ and $\Delta' = \partial^2_{x_1} + \ldots + \partial^2_{x_{n-1}}$.

When Ω is an open set of $\mathbb{R}^n$, $\mathrm{C}^\infty_c(\Omega, \mathbb{C})$ is the set of smooth compactly supported function in Ω, and $\mathcal{D}'(\Omega, \mathbb{C})$ is its topological dual. For $p \in [1, +\infty)$, $\mathrm{L}^p(\Omega, \mathbb{C})$ is the normed vector space of complex valued (Lebesgue-) measurable functions whose p-th power is integrable with respect to the Lebesgue measure, $\mathcal{S}(\overline{\Omega}, \mathbb{C})$ (*resp.* $\mathrm{C}^\infty_c(\overline{\Omega}, \mathbb{C})$) stands for functions which are restrictions on Ω of elements of $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$ (*resp.* $\mathrm{C}^\infty_c(\mathbb{R}^n, \mathbb{C})$). Unless the contrary is explicitly stated, we will always identify $\mathrm{L}^p(\Omega, \mathbb{C})$ (resp. $\mathrm{C}^\infty_c(\Omega, \mathbb{C})$) as the subspace of function in $\mathrm{L}^p(\mathbb{R}^n, \mathbb{C})$ (resp. $\mathrm{C}^\infty_c(\mathbb{R}^n, \mathbb{C})$) supported in $\overline{\Omega}$ through the extension by 0 outside Ω. $\mathrm{L}^\infty(\Omega, \mathbb{C})$ stands for the space of essentially bounded (Lebesgue-) measurable functions.

For $s \in \mathbb{R}$, $p \in [1, +\infty)$, $\ell^p_s(\mathbb{Z}, \mathbb{C})$ denotes for the normed vector space of p-summable sequences of complexes numbers with respect to the counting measure $2^{ksp}\mathrm{d}k$; $\ell^\infty_s(\mathbb{Z}, \mathbb{C})$ is the space of complex-valued sequences $(x_k)_{k\in\mathbb{Z}}$ such that $(2^{ks}x_k)_{k\in\mathbb{Z}}$ is bounded. More generally, when X is

a Banach space, for $p \in [1, +\infty]$, one may also consider $\mathrm{L}^p(\Omega, X)$ which stands for the space of (Bochner-)measurable functions $u : \Omega \longrightarrow X$, such that $t \mapsto \|u(t)\|_X \in \mathrm{L}^p(\Omega, \mathbb{R})$, similarly one may consider $\ell^p_s(\mathbb{Z}, X)$. Finally, $\mathrm{C}^0(\Omega, X)$ stands for the space of continuous functions on $\Omega \subset \mathbb{R}^n$ with values in X. The subspace $\mathrm{C}^0_b(\mathbb{R}, X)$ consists of uniformly bounded continuous functions and $\mathrm{C}^0_0(\mathbb{R}, X)$ is the set of continuous functions that vanish at infinity. For $\mathcal{C} \in \{\mathrm{C}^0, \mathrm{C}^0_b, \mathrm{C}^0_0, \mathrm{C}^\infty_c\}$, we set $\mathcal{C}(\overline{\Omega}, X)$ to be the set of continuous functions on $\overline{\Omega}$ which are restrictions of elements that belongs to $\mathcal{C}(\mathbb{R}^n, X)$.

For Ω an open set of $\mathbb{R}^n$, we say that Ω is a special Lipschitz domain, if there exists, up to a rotation, a globally Lipschitz function $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, such that

$$\Omega = \{\, (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, x_n > \phi(x') \,\}.$$

In other words, a special Lipschitz domain of $\mathbb{R}^n$ is the epigraph of a real valued Lipschitz function defined on $\mathbb{R}^{n-1}$.

2 Real and complex interpolation of normed vector spaces

Interpolation theory is the heart of the matter of the present dissertation. We will focus on two different kinds of interpolation, the real and the complex one. For our presentation, we are going to gather definition and main properties and theorems from [BL76] and [Lun18].

Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two complex normed vector spaces. We write $X \hookrightarrow Y$ if X embeds continuously in Y.

We will assume through this whole section, there is a common Hausdorff topological vector space $\mathfrak{Z}$, such that $X, Y \subset \mathfrak{Z}$. In this case, $X \cap Y$ and $X + Y$ are normed vector spaces with their canonical norms

$$\|z\|_{X\cap Y} := \max\left(\|z\|_X, \|z\|_Y\right) \quad \text{and} \quad \|z\|_{X+Y} := \inf_{\substack{(x,y)\in X\times Y,\\ x=x+y}} \left[\|x\|_X + \|y\|_Y\right].$$

Such a couple (X, Y) is called an **interpolation couple**.

The main idea of interpolation theory is deeply related to the following general problem.

Problem 1.1 *Given two interpolation couples (X_0, Y_0) and (X_1, Y_1), and T linear operator from $X_0 + Y_0$ to $X_1 + Y_1$ such that*

(i) T is bounded,

(ii) $T_{|_{Z_0}} : Z_0 \longrightarrow Z_1$ is well-defined and bounded for $Z \in \{X, Y\}$.

Provided normed vector spaces $\tilde{Z}_j$, $j \in \{0, 1\}$, such that

$$X_j \cap Y_j \hookrightarrow \tilde{Z}_j \hookrightarrow X_j + Y_j,\ j \in \{0, 1\},$$

when is the operator $T : \tilde{Z}_0 \longrightarrow \tilde{Z}_1$ still well-defined and bounded ?

While the Problem 1.1 has a very general setting, one of the first results that motivated such investigation goes back to Riesz-Thorin's Theorem, [Tho48], and Marcinkiewicz's Theorem, [Mar39 ; Zyg56], where X_0, Y_0, X_1, Y_1 were given (weak-)Lebesgue spaces.

2.1 Real interpolation

Before we continue, we recall that up to now, **no completeness assumption is made** on either X or Y.

Definition 1.2 One can define the K-functional of $z \in X + Y$, for any $t > 0$ by

$$K(t, z, X, Y) := \inf_{\substack{(x,y)\in X\times Y,\\ z=x+y}} \left(\|x\|_X + t\|y\|_Y\right).$$

This allows us to construct, for any $\theta \in (0,1)$, $q \in [1,+\infty]$, the real interpolation spaces between X and Y with indexes θ, q as

$$(X,Y)_{\theta,q} := \left\{ x \in X+Y \,\middle|\, t \longmapsto t^{-\theta} K(t,x,X,Y) \in \mathrm{L}^q_*(\mathbb{R}_+) \right\},$$

where $\mathrm{L}^q_*(\mathbb{R}_+) := \mathrm{L}^q((0,+\infty), \mathrm{d}t/t)$. We may also define

$$(X,Y)_{\theta} := \left\{ x \in X+Y \,\middle|\, t \longmapsto t^{-\theta} K(t,x,X,Y) \in \mathrm{C}^0_0(\mathbb{R}_+) \right\} .$$

They are all normed vector spaces with respect to their natural norm

$$\|x\|_{(X,Y)_{\theta,q}} := \|t \mapsto t^{-\theta} K(t,x,X,Y)\|_{\mathrm{L}^q_*(\mathbb{R}_+)} \text{ and } \|x\|_{(X,Y)_\theta} := \|x\|_{(X,Y)_{\theta,\infty}}.$$

Proposition 1.3 ([BL76, Theorems 3.4.1 & 3.4.2]) *Let $\theta \in (0,1)$, and $p,q \in [1,+\infty)$. It holds that,*

(i) if $X = Y$, then $(X,Y)_{\theta,p} = X$ with equivalence of norms,

(ii) if X and Y are complete, then so are $(X,Y)_{\theta,p}$ and $(X,Y)_\theta$,

(iii) if $p \leqslant q$, one has continuous embedding

$$X \cap Y \hookrightarrow (X,Y)_{\theta,p} \hookrightarrow (X,Y)_{\theta,q} \hookrightarrow (X,Y)_\theta \hookrightarrow (X,Y)_{\theta,\infty} \hookrightarrow X+Y,$$

(iv) if $p < +\infty$, $X \cap Y$ is dense in $(X,Y)_{\theta,p}$,

(v) the closure of $X \cap Y$ in $(X,Y)_{\theta,\infty}$ is the space $(X,Y)_\theta$.

And in this case one can answer to Problem 1.1, by the following theorem.

Theorem 1.4 ([Lun18, Theorem 1.6]) *Let (X_0,X_1) and (Y_0,Y_1) be two interpolation couple. For $\theta \in (0,1)$, $p \in [1,+\infty]$, we write $Z^p_\theta := (Z_0,Z_1)_{\theta,p}$, $Z_\theta := (Z_0,Z_1)_\theta$, for $Z \in \{X,Y\}$.*

Let $T : X_0 + X_1 \longrightarrow Y_0 + Y_1$ be a linear operator, and we assume that

$$T_{|X_j} : X_j \longrightarrow Y_j,$$

is well-defined and bounded for $j \in \{0,1\}$.

Then for all $\theta \in (0,1)$, $p \in [1,+\infty]$, we have that $T_{|X^p_\theta}$ is bounded from X^p_θ to Y^p_θ with bound

$$\|T\|_{X^p_\theta \to Y^p_\theta} \leqslant \|T\|^{1-\theta}_{X_0 \to Y_0} \|T\|^{\theta}_{X_1 \to Y_1} .$$

The conclusion still holds with (X_θ, Y_θ) instead of $(X^\infty_\theta, Y^\infty_\theta)$.

When the real interpolation spaces are complete, one has stability under the *reiteration* of the real interpolation procedure, in the sense of the next theorem.

Theorem 1.5 (Reiteration theorem [BL76, Theorem 3.5.3]) *Let $\eta, \theta_0, \theta_1 \in (0,1)$, $\theta_0 \neq \theta_1$, $p_0, p_1 \in [1,+\infty]$, and set*

$$\theta := (1-\eta)\theta_0 + \eta\theta_1 \in (0,1).$$

We assume that, for $j = 0,1$, $(X,Y)_{\theta_j,p_j}$ is a Banach space.

Then for all $p \in [1,+\infty]$, so is $(X,Y)_{\theta,p}$, and one has

$$\left((X,Y)_{\theta_0,p_0}, (X,Y)_{\theta_1,p_1}\right)_{\eta,p} = (X,Y)_{\theta,p},$$

with equivalence of norms.

We can also recover dual spaces under additional conditions, that are satisfied in many standard applications.

Theorem 1.6 (Duality theorem [BL76, Theorem 3.7.1]) *Let $\theta \in (0,1)$, $p \in [1,+\infty)$, and assume that X and Y are **Banach spaces** such that $X \cap Y$ is dense in both X and Y.*
Then, we have

$$(X,Y)'_{\theta,p} = (X',Y')_{\theta,p'}, \text{ and } (X,Y)'_{\theta} = (X',Y')_{\theta,1},$$

with equivalence of norms.

There are few other real interpolation methods called either the J-method or the Trace method, but those require the considered normed vector spaces to be complete in order to make sense of their involved quantities. This is really inconvenient for us, since we want to deal with non-complete normed vector spaces. However, when the considered spaces are complete, those methods are known to be equivalent with the K-method presented here. See for instance [BL76, Chapter 3, Section 3.12], [Tri78, Section 1.8], [Lun18, Chapter 1, Section 1.2] for the trace method, and [BL76, Chapter 3, Section 3.2], [Tri78, Section 1.6] for the J-method.

2.2 Fundamental examples : real interpolation

We exhibit here some fundamental examples of explicit computations of real interpolation spaces that are of paramount importance throughout this dissertation.

Theorem 1.7 ([BL76, Theorem 5.6.1]) *Let $p_0, p_1 \in [1,+\infty]$, $s_0, s_1 \in \mathbb{R}$, such that $s_0 \neq s_1$. Let E be a Banach space. If we set for $\theta \in (0,1)$*

$$s := (1-\theta)s_0 + \theta s_1,$$

then for all $p \in [1,+\infty]$, we have

$$(\ell^{p_0}_{s_0}(\mathbb{Z},E), \ell^{p_1}_{s_1}(\mathbb{Z},E))_{\theta,p} = \ell^p_s(\mathbb{Z},E)$$

with equivalence of norms.

Theorem 1.8 ([Tri78, Theorem, Section 1.18.4]) *Let $p_0, p_1 \in [1,+\infty)$, and (Ω,μ) be a sigma-finite measure space. We assume that X and Y are Banach spaces. If we set for $\theta \in (0,1)$*

$$\frac{1}{p} := (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1},$$

then we have

$$(\mathrm{L}^{p_0}(\Omega,\mu,X), \mathrm{L}^{p_1}(\Omega,\mu,Y))_{\theta,p} = \mathrm{L}^p(\Omega,\mu,(X,Y)_{\theta,p})$$

with equivalence of norms.

Corollary 1.9 *Let $p \in [1,+\infty)$, $q_0, q_1 \in [1,+\infty]$, $s_0, s_1 \in \mathbb{R}$, such that $s_0 \neq s_1$, (Ω,μ) be a sigma-finite measure space and E a Banach space. For $\theta \in (0,1)$, if we set*

$$s := (1-\theta)s_0 + \theta s_1,$$

then for all $q \in [1,+\infty]$, we have

$$\big(\mathrm{L}^p(\Omega,\mu,\ell^{q_0}_{s_0}(\mathbb{Z},E)), \mathrm{L}^p(\Omega,\ell^{q_1}_{s_1}(\mathbb{Z},E),\mu)\big)_{\theta,q} = \ell^q_s(\mathbb{Z},\mathrm{L}^p(\Omega,\mu,E))$$

with equivalence of norms.

Proof. — First, if $p = q$, by Theorem 1.8 and Fubini-Tonelli, we obtain that

$$\big(\mathrm{L}^p(\Omega,\mu,\ell^{q_0}_{s_0}(\mathbb{Z},E)), \mathrm{L}^p(\Omega,\mu,\ell^{q_1}_{s_1}(\mathbb{Z},E))\big)_{\theta,p} = \mathrm{L}^p(\Omega,\mu,\ell^p_s(\mathbb{Z},E)) = \ell^p_s(\mathbb{Z},\mathrm{L}^p(\Omega,\mu,E)).$$

Now, for $q \neq p$, we decompose $\theta := (1-\eta)\theta_0 + \eta\theta_1$ for some $\theta_0, \theta_1, \eta \in (0,1)$, so that the reiteration theorem, Theorem 1.5, and above equality yield

$$\big(\mathrm{L}^p(\Omega,\mu,\ell^{q_0}_{s_0}(\mathbb{Z},E)),\mathrm{L}^p(\Omega,\mu,\ell^{q_1}_{s_1}(\mathbb{Z},E))\big)_{\theta,q} = \big(\ell^p_{\tilde{s}_0}(\mathbb{Z},\mathrm{L}^p(\Omega,\mu,E)),\ell^p_{\tilde{s}_1}(\mathbb{Z},\mathrm{L}^p(\Omega,\mu,E))\big)_{\eta,q}$$

where $\tilde{s}_j = (1-\theta_j)s_0 + \theta_j s_1$ for $j = 0, 1$. Thus, Theorem 1.7 gives

$$\big(\mathrm{L}^p(\Omega,\mu,\ell^{q_0}_{s_0}(\mathbb{Z},E)),\mathrm{L}^p(\Omega,\mu,\ell^{q_1}_{s_1}(\mathbb{Z},E))\big)_{\theta,q} = \ell^q_{(1-\eta)\tilde{s}_0+\eta\tilde{s}_1}(\mathbb{Z},\mathrm{L}^p(\Omega,\mu,E))$$

which concludes the proof since $(1-\eta)\tilde{s}_0 + \eta\tilde{s}_1 = (1-\theta)s_0 + \theta s_1 = s$ by construction. ■

2.3 Complex interpolation

From now and until the end of the current section, we assume that X and Y are complex Banach spaces, so that in particular $X \cap Y$ and $X + Y$ are complete too. We consider $\mathrm{F}(X,Y)$ the set of all continuous functions $f : \overline{S} \longmapsto X + Y$, S being the strip of complex numbers whose real part is between 0 and 1, with f holomorphic in S, and such that

$$t \longmapsto f(it) \in \mathrm{C}^0_b(\mathbb{R},X) \quad \text{and} \quad t \longmapsto f(1+it) \in \mathrm{C}^0_b(\mathbb{R},Y).$$

We can endow the space $\mathrm{F}(X,Y)$ with the norm

$$\|f\|_{\mathrm{F}(X,Y)} := \max\left(\sup_{t\in\mathbb{R}}\|f(it)\|_X, \sup_{t\in\mathbb{R}}\|f(1+it)\|_Y\right),$$

which makes $\mathrm{F}(X,Y)$ a Banach space since it is a closed subspace of $\mathrm{C}^0(\overline{S}, X+Y)$.

Definition 1.10 For $\theta \in (0,1)$, the normed vector space given by

$$\begin{aligned}[X,Y]_\theta &:= \big\{\, f(\theta) \,\big|\, f \in \mathrm{F}(X,Y) \,\big\}, \\ \|x\|_{[X,Y]_\theta} &:= \inf_{\substack{f\in\mathrm{F}(X,Y),\\ f(\theta)=x}} \|f\|_{\mathrm{F}(X,Y)},\end{aligned}$$

is a Banach space called the complex interpolation space between X and Y associated with θ.

The definition is motivated by the **Phragmen-Lindelöf inequality**.

Lemma 1.11 *Let Z be a complex Banach space, For all $f \in \mathrm{F}(Z,Z)$, and all $z \in S$*

$$\|f(z)\|_Z \leqslant \max\left(\sup_{t\in\mathbb{R}}\|f(it)\|_Z, \sup_{t\in\mathbb{R}}\|f(1+it)\|_Z\right).$$

In particular, for all $g \in \mathrm{F}(X,Y) \subset \mathrm{F}(X+Y, X+Y)$,

$$\|g(z)\|_{X+Y} \leqslant \max\left(\sup_{t\in\mathbb{R}}\|g(it)\|_X, \sup_{t\in\mathbb{R}}\|g(1+it)\|_Y\right).$$

Proposition 1.12 ([Lun18, Lemma 2.2, Proposition 2.4]) *Let $\theta \in (0,1)$. It holds that,*

(i) if $X = Y$, then $[X,Y]_\theta = X$ with equality of norms,

(ii) one has a continuous embedding

$$X \cap Y \hookrightarrow [X,Y]_\theta \hookrightarrow X + Y,$$

(iii) the set $X \cap Y$ is dense in $[X,Y]_\theta$.

We also have an answer to Problem 1.1 in the complex interpolation theory. The result is even more general, called Stein's interpolation theorem.

Theorem 1.13 ([Lun18, Theorem 2.7]) *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces. For $\theta \in (0,1)$, we write $Z_{[\theta]} := [Z_0, Z_1]_\theta$, for $Z \in \{X, Y\}$.*

Let $(T_z)_{z\in\overline{S}}$ be family of linear operators from $X_0 \cap X_1$ to $Y_0 + Y_1$, such that

(i) for all $x \in X_0 \cap X_1$, $z \mapsto T_z x$ is holomorphic in the interior of S and bounded with values in $Y_0 + Y_1$,

(ii) for $j \in \{0,1\}$, $x \in X_0 \cap X_1$, $t \mapsto T_{j+it}x$ is continuous from $\mathbb{R}$ to Y_j,

(iii) there exist $M_0, M_1 \geqslant 0$, such that, for $j \in \{0,1\}$,

$$\|T_{j+it}x\|_{Y_j} \leqslant M_j \|x\|_{X_j},\ x \in X_0 \cap X_1,\ t \in \mathbb{R}.$$

Then for all $\theta \in (0,1)$, T_θ extends uniquely to a bounded linear operator from $X_{[\theta]}$ to $Y_{[\theta]}$ with bound

$$\|T_\theta\|_{X_{[\theta]}\to Y_{[\theta]}} \leqslant M_0^{1-\theta} M_1^\theta\ .$$

In fact, we can modify the theorem in a way that we get rid of the condition of completeness when $X_0 = X_1 = E$. This is somewhat important since we aim to carry interpolation theory for non-complete spaces.

Theorem 1.14 *Let E be a complex normed vector space and (Y_0, Y_1) be an interpolation couple of Banach spaces. For $\theta \in (0,1)$, we write $Y_{[\theta]} := [Y_0, Y_1]_\theta$. Let D be a dense subspace of E. Let $(T_z)_{z\in\overline{S}}$ be family of linear operators from D to $Y_0 + Y_1$, such that*

(i) for all $x \in D$, $z \mapsto T_z x$ is holomorphic in the interior of S with values in $Y_0 + Y_1$,

(ii) for $j \in \{0,1\}$, $x \in D$, $t \mapsto T_{j+it}x$ is continuous from $\mathbb{R}$ to Y_j,

(iii) there exist $M_0, M_1 \geqslant 0$, such that, for $j \in \{0,1\}$,

$$\|T_{j+it}x\|_{Y_j} \leqslant M_j \|x\|_E,\ x \in D,\ t \in \mathbb{R}.$$

Then for all $\theta \in (0,1)$, T_θ extends uniquely to a bounded linear operator from E to $Y_{[\theta]}$ with bound

$$\|T_\theta\|_{E\to Y_{[\theta]}} \leqslant M_0^{1-\theta} M_1^\theta\ .$$

We also have a reiteration and a duality theorem.

Theorem 1.15 (Reiteration theorem [BL76, Theorem 4.6.1]) *Let $\eta, \theta_0, \theta_1 \in (0,1)$, $\theta_0 \neq \theta_1$, and set*

$$\theta := (1-\eta)\theta_0 + \eta\theta_1 \in (0,1).$$

We assume that $X \cap Y$ is dense in X, Y and $[X,Y]_{\theta_j}$, for $j = 0, 1$.

Then, one has

$$[[X,Y]_{\theta_0}, [X,Y]_{\theta_1}]_\eta = [X,Y]_\theta,$$

with equivalence of norms.

Theorem 1.16 (Duality theorem [BL76, Corollary 4.5.2]) *Let $\theta \in (0,1)$, $p \in [1, +\infty)$, and assume that $X \cap Y$ is dense in both X and Y, and that either X or Y is reflexive.*

Then, we have

$$[X,Y]'_\theta = [X', Y']_\theta$$

with equivalence of norms.

2.4 Fundamental examples : complex interpolation

We exhibit here some fundamental examples of explicit computations of complex interpolation spaces that are of paramount importance throughout this dissertation.

Theorem 1.17 ([BL76, Theorem 5.6.3]) *Let $p_0, p_1 \in [1, +\infty]$, $s_0, s_1 \in \mathbb{R}$. If we set for $\theta \in (0,1)$,*

$$\left(s, \frac{1}{p}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}\right) + \theta\left(s_1, \frac{1}{p_1}\right),$$

then, we have

$$[\ell^{p_0}_{s_0}(\mathbb{Z}, X), \ell^{p_1}_{s_1}(\mathbb{Z}, Y)]_\theta = \ell^p_s(\mathbb{Z}, [X, Y]_\theta)$$

with equivalence of norms.

Theorem 1.18 ([Tri78, Theorem, Section 1.18.4]) *Let $p_0, p_1 \in [1, +\infty)$, and (Ω, μ) be a sigma-finite measure space. If we set for $\theta \in (0,1)$,*

$$\frac{1}{p} := (1-\theta)\frac{1}{p_0} + \theta\frac{1}{p_1},$$

then we have

$$[\mathrm{L}^{p_0}(\Omega, \mu, X), \mathrm{L}^{p_1}(\Omega, \mu, Y)]_\theta = \mathrm{L}^p(\Omega, \mu, [X, Y]_\theta)$$

with equivalence of norms.

Corollary 1.19 *Let $p_0, p_1 \in [1, +\infty)$, $q_0, q_1 \in [1, +\infty]$, $s_0, s_1 \in \mathbb{R}$, (Ω, μ) be a sigma-finite measure space. For $\theta \in (0,1)$, if we set*

$$\left(s, \frac{1}{p}, \frac{1}{q}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right),$$

then, we have

$$\begin{aligned}\left[\mathrm{L}^{p_0}(\Omega, \mu, \ell^{q_0}_{s_0}(\mathbb{Z}, X)), \mathrm{L}^{p_1}(\Omega, \mu, \ell^{q_1}_{s_1}(\mathbb{Z}, Y))\right]_\theta &= \mathrm{L}^p(\Omega, \mu, \ell^q_s(\mathbb{Z}, [X, Y]_\theta)) \\ \left[\ell^{q_0}_{s_0}(\mathbb{Z}, \mathrm{L}^{p_0}(\Omega, \mu, X)), \ell^{q_1}_{s_1}(\mathbb{Z}, \mathrm{L}^p(\Omega, \mu, Y))\right]_\theta &= \ell^q_s(\mathbb{Z}, \mathrm{L}^p(\Omega, \mu, [X, Y]_\theta))\end{aligned}$$

with equivalence of norms.

Thanks to the fact that for all $a, b > 0$, $\theta \in [0,1]$,

$$a + a^{1-\theta}b^\theta \leqslant (a+b)^\theta a^{1-\theta} \leqslant 2^\theta(a + a^{1-\theta}b^\theta)\ ,$$

and since for $q \in [1, +\infty]$, $s_0, s_1 \in \mathbb{R}$, $\theta \in (0,1)$, with $s := (1-\theta)s_0 + \theta s_1$, we have with equivalence of norms

$$\ell^q_{s_0}(\mathbb{Z}) \cap \ell^q_s(\mathbb{Z}) = \ell^q(\mathbb{Z}, (2^{ks_0q} + 2^{ksq})\mathrm{d}k) = \ell^q(\mathbb{Z}, (2^{ks_0q} + 2^{ks_1q})^\theta 2^{ks_0q(1-\theta)}\mathrm{d}k),$$

by the mean of [Tri78, Theorem, Section 1.18.5], dealing in particular with complex interpolation of weighted ℓ^q spaces, we obtain

Proposition 1.20 *Let $q \in [1, +\infty)$, $s_0, s_1 \in \mathbb{R}$, consider a complex Banach space E, and for $\theta \in (0,1)$, we introduce $s := (1-\theta)s_0 + \theta s_1$. The following equality holds with equivalence of norms*

$$[\ell^q_{s_0}(\mathbb{Z}, E), \ell^q_{s_0}(\mathbb{Z}, E) \cap \ell^q_{s_1}(\mathbb{Z}, E)]_\theta = \ell^q_{s_0}(\mathbb{Z}, E) \cap \ell^q_s(\mathbb{Z}, E)\ .$$

The result still holds with $\mathbb{N}$ *instead of* $\mathbb{Z}$.

2.5 Retractions

We present here a useful and powerful argument to compute concrete real and complex interpolation spaces, when one can realize an interpolation couple of normed vector spaces as some subspace of another one. This procedure is called **the retraction-corectraction argument**. Its name comes from the abstract Category theoretic arguments one can use to build interpolation theory, see for instance [BL76, Sections 2.1 & 6.4], [Tri78, Section 1.2] and [Ege15, Section 1.3.1] for a more recent short and self-contained review.

Theorem 1.21 ([Tri78, Theorem, Section 1.2.4]) *Let* (X_0, X_1) *and* (Y_0, Y_1) *be two interpolation couples, and let*

$$\mathfrak{R} : Y_0 + Y_1 \longrightarrow X_0 + X_1 \quad \text{and} \quad \mathfrak{E} : X_0 + X_1 \longrightarrow Y_0 + Y_1$$

be two bounded linear operators such that $\mathfrak{R}\mathfrak{E} = \mathrm{I}_{X_0+X_1}$, *i.e.* $\mathfrak{E}$ *is as right bounded inverse for* $\mathfrak{R}$.
Then for all $\theta \in (0,1)$, $p \in [1, +\infty]$, *one has*

$$(X_0, X_1)_{\theta,p} = \mathfrak{R}(Y_0, Y_1)_{\theta,p}.$$

The result still holds, replacing $(\cdot,\cdot)_{\theta,p}$ *by* $(\cdot,\cdot)_\theta$.
Furthermore, if (X_0, X_1) *and* (Y_0, Y_1) *are two interpolation couples of complex Banach spaces, then the result still holds for* $[\cdot,\cdot]_\theta$, *instead of* $(\cdot,\cdot)_{\theta,p}$.

In actual applications of this dissertation, one may usually not apply these results straight forward for technical reasons, and have to reproduce the retraction-coretraction arguments manually. But, one has to keep Theorem 1.21 in mind through this whole thesis.

3 Sectorial and bi-sectorial operators on Banach spaces

When one comes to study some linear elliptic partial differential equation with its associated linear elliptic operator A, we are generally concerned by the well-posedness, the regularity theory, and the existence of an evolution operator for its parabolic counterpart with the associated well-posedness and regularity theory.

The well-posedness and the regularity theory for the given elliptic and parabolic problems morally reduce to look for a good meaning and a precise description of the operators

$$A^\alpha, \quad e^{-zA}, \qquad \alpha, z \in \mathbb{C},$$

in other words we want to take functions of (unbounded) operators, in a way that respects the formalism of resolvents written as $(\lambda \mathrm{I} - A)^{-1}$, with λ in some subset of $\mathbb{C}$.

It turns out that the class of sectorial operator is precisely a class of operators that allows to consider such objects, and that most of the elliptic operators considered in the literature are indeed sectorial.

Therefore, this section will be dedicated to a short review of the properties of sectorial and bisectorial operators on a Banach space and their properties by the point of view of the holomorphic functional calculus.

Our presentation will closely follow [Haa06, Chapters 2-5] and [Ege15, Chapter 3]. In this dissertation, mainly in Chapters 3 and 4, we have a focus on some sectorial operators. Hence, the statements in the current section will be given with priority to sectorial operators, while corresponding one for bisectorial operators are also available and mentioned. A pure treatment of bisectorial operators is achieved with explicit proofs in [Ege15, Chapter 3].

For an alternative presentation for the case of strongly continuous semigroups and fractional powers of sectorial operators, there are a lot of references available, see for instance Pazy's book

[Paz83] or the monograph written by Arendt, Batty, Hieber and Neubrander [ABHN11, Chapter 3]. We warn the reader that Pazy's book has generally the global assumption that considered sectorial operators are invertible. This is not the case in general : consider $-\Delta$ on $\mathrm{L}^2(\mathbb{R}^n)$ with domain $\mathrm{H}^2(\mathbb{R}^n)$.

3.1 Generalities

We introduce the following subsets of the complex plane

$$\Sigma_\mu := \{\, z \in \mathbb{C}^* \,:\, |\arg(z)| < \mu \,\}, \text{ if } \mu \in (0,\pi),$$
$$S_\mu := (-\Sigma_\mu) \cup \Sigma_\mu, \text{ if } \mu \in (0, \frac{\pi}{2}),$$

we also define $\Sigma_0 := (0,+\infty)$, $S_0 := \mathbb{R}$, and later we also consider $\overline{\Sigma}_\mu$, and $\overline{S}_\mu$ their closure.

The set Σ_μ is the open *sector* with vertex 0 and opening angle 2μ symmetric around the positive real axis. Similarly, S_μ is the corresponding *bisector*.

Definition 1.22 A closed linear operator $(\mathrm{D}(A), A)$ on complex valued Banach space X is said to be ω-***sectorial***, if for a fixed $\omega \in [0,\pi)$, both conditions are satisfied

(i) $\sigma(A) \subset \overline{\Sigma}_\omega$, where $\sigma(A)$ stands for the spectrum of A ;

(ii) For all $\mu \in (\omega,\pi)$, $\sup_{\lambda \in \mathbb{C}\setminus\overline{\Sigma}_\mu} \|\lambda(\lambda \mathrm{I} - A)^{-1}\|_{X\to X} < +\infty$.

Similarly, $(\mathrm{D}(A), A)$ is said to be ω-***bisectorial***, for a fixed $\omega \in (0, \frac{\pi}{2})$, if $\sigma(A) \subset \overline{S}_\omega$, and for all $\mu \in (\omega, \frac{\pi}{2})$, $\sup_{\lambda \in \mathbb{C}\setminus\overline{S}_\mu} \|\lambda(\lambda \mathrm{I} - A)^{-1}\|_{X\to X} < +\infty$.

An important result is the fact that sectorial operators on a reflexive Banach space X are always densely defined. The next proposition is of paramount importance and is useful in many ways such as, for instance, approximation of some elements of X by elements of $\mathrm{D}(A^k) \cap \mathrm{R}(A^k)$.

Proposition 1.23 ([Ege15, Proposition 3.2.2]) *Let $(\mathrm{D}(A), A)$ be a* ***bisectorial*** *operator on a Banach space X. Then the following assertions hold.*

(i) If $k \in \mathbb{N}$, and $x \in \overline{\mathrm{D}(A)}$, then

$$\lim_{t\to+\infty} (it)^k (it\mathrm{I} + A)^{-k} x = x \quad \text{and} \quad \lim_{t\to+\infty} A^k (it\mathrm{I} + A)^{-k} x = 0.$$

(ii) If $k \in \mathbb{N}$, and $x \in \overline{\mathrm{R}(A)}$, then

$$\lim_{t\to 0} (it)^k (it\mathrm{I} + A)^{-k} x = 0 \quad \text{and} \quad \lim_{t\to 0} A^k (it\mathrm{I} + A)^{-k} x = x.$$

In particular, $\mathrm{N}(A) \cap \overline{\mathrm{R}(A)} = \{0\}$, so that $X = \overline{\mathrm{R}(A)}$ implies that A is injective.

(iii) For every $k \in \mathbb{N}$, $\mathrm{D}(A^k) \cap \mathrm{R}(A^k)$ is dense in $\overline{\mathrm{D}(A)} \cap \overline{\mathrm{R}(A)}$.

(iv) If X is reflexive, then A is densely defined and induces a topological decomposition

$$X = \mathrm{N}(A) \oplus \overline{\mathrm{R}(A)}.$$

The result still holds if $(\mathrm{D}(A), A)$ is a ***sectorial*** *operator, replacing i by 1.*

3.2 Holomorphic functional calculus

For $\Omega \subset \mathbb{C}$ an open set, $\mathbf{H}^\infty(\Omega)$ stands for the set of bounded holomorphic functions on Ω.

For $\mu \in [0,\pi)$, we denote by $\mathbf{H}_0^\infty(\Sigma_\mu)$, the set of holomorphic functions $f : \Sigma_\mu \longrightarrow \mathbb{C}$, such that there exist $C, s > 0$ with

$$\forall z \in \Sigma_\mu,\ |f(z)| \leqslant C \frac{|z|^s}{1+|z|^{2s}}.$$

Both vector spaces $\mathbf{H}_0^\infty(\Sigma_\mu)$ and $\mathbf{H}^\infty(\Omega)$ are algebras, called respectively the algebra of *regularly decaying*, resp. *bounded*, holomorphic functions on Σ_μ, resp. Ω.

For $(\mathrm{D}(A), A)$ an ω-sectorial operator on a Banach space X with $\omega \in [0, \pi)$, given $\mu \in (\omega, \frac{\pi}{2})$ then $f \in \mathbf{H}_0^\infty(\Sigma_\mu)$, and $\nu \in (\omega, \mu)$, we define a bounded linear operator $f(A)$ on X by the mean of the convergent Cauchy integral

$$f(A) := \frac{1}{2i\pi} \int_{\partial\Sigma_\nu} f(z)(z\mathrm{I} - A)^{-1} \, \mathrm{d}z. \tag{1.1}$$

Here, $\partial\Sigma_\nu$ the boundary of Σ_ν surrounds $\sigma(A)$ counterclockwise, given by oriented curves

$$\partial\Sigma_\nu = -\mathbb{R}_+ e^{i\nu} \oplus \mathbb{R}_+ e^{-i\nu}.$$

The integral in (1.1) actually makes sense and is absolutely convergent, since $f \in \mathbf{H}_0^\infty(\Sigma_\mu)$ ensures that $f \in \mathrm{L}^1(\Sigma_\mu, \frac{\mathrm{d}|z|}{|z|})$.

Finally, provided $a, b \in \mathbb{C}$, and $g(z) := f(z) + a(1+z)^{-1} + b$ for all $z \in \Sigma_\nu$, we have

$$g \in \boldsymbol{\mathcal{E}}(\Sigma_\mu) := \mathbf{H}_0^\infty(\Sigma_\mu) \oplus \langle z \mapsto (1+z)^{-1} \rangle \oplus \mathbb{C}$$

and we can extend the definition of the functional calculus as follows

$$g(A) := f(A) + a(\mathrm{I} + A)^{-1} + b\mathrm{I}.$$

This is the *Dunford-Riesz* holomorphic functional calculus for sectorial operators, and $\boldsymbol{\mathcal{E}}(\Sigma_\mu)$ is the *Dunford-Riesz class*.

Similarly, one may have similar consideration with $f \in \mathbf{H}_0^\infty(S_\mu)$ for $\mu \in [0, \frac{\pi}{2})$, and S_ν replacing Σ_ν with $0 \leqslant \omega < \nu < \mu < \frac{\pi}{2}$ whenever $(\mathrm{D}(A), A)$ is ω-bisectorial. The Dunford-Riesz class is built the same way with $(i+z)^{-1}$ instead of $(1+z)^{-1}$.

We provide here the fundamental result known as M$^{\mathrm{c}}$Intosh's approximation theorem which was proved during the proof of [McI86, Theorem, Section 7, p.222-223], see [Haa06, Chapter 5, Section 5.2] or [Ege15, Theorem 3.3.9] for more recent and detailed proofs.

Theorem 1.24 (M$^{\mathrm{c}}$Intosh's approximation) *Let $(\mathrm{D}(A), A)$ be an ω-sectorial operator on a Banach space X with $\omega \in [0, \pi)$, and for $\phi \in (\omega, \pi)$, let $f \in \mathbf{H}_0^\infty(\Sigma_\nu)$ such that*

$$\int_0^{+\infty} f(t) \, \frac{\mathrm{d}t}{t} = 1.$$

Then, for all $x \in \overline{\mathrm{D}(A)} \cap \overline{\mathrm{R}(A)}$,

$$\int_a^b f(tA)x \, \frac{\mathrm{d}t}{t} \xrightarrow[\substack{a \to 0 \\ b \to +\infty}]{} x.$$

The result also holds for bisectorial operators.

Definition 1.25 Let $(\mathrm{D}(A), A)$ be an injective ω-(bi)sectorial operator with $\omega \in [0, \pi)$ (resp. $[0, \frac{\pi}{2})$). For $\mu \in (\omega, \pi)$ (resp. $(\omega, \frac{\pi}{2})$) we say that A admits a ***bounded*** (or $\mathbf{H}^\infty(\Sigma_\mu)$- (resp. $\mathbf{H}^\infty(S_\mu)$-)) ***holomorphic functional calculus*** on X (of angle μ), if there exists a constant K_μ, such that for all $f \in \mathbf{H}^\infty(\Sigma_\mu)$ (resp. $\mathbf{H}^\infty(S_\mu)$), we have that

$$\|f(A)\|_{X \to X} \leqslant K_\mu \|f\|_{\mathrm{L}^\infty(\Sigma_\mu)}. \tag{1.2}$$

We replace Σ_μ by S_μ in (1.2) for the bisectorial case.

The above definition is meaningful. Indeed, since $(\mathrm{D}(A), A)$ is assumed to be injective, the operator $(\mathrm{D}(A) \cap \mathrm{R}(A), (\mathrm{I} + A)^2 A^{-1}))$ is well-defined and closed on X. So if one writes for

$f \in \mathbf{H}^\infty(\Sigma_\mu)$,

$$f(z) = f(z)\frac{z}{(1+z)^2}\frac{(1+z)^2}{z} = \frac{(1+z)^2}{z}g(z), \quad z \in \Sigma_\mu,$$

with $g \in \mathbf{H}^\infty_0(\Sigma_\mu)$, one has, *a priori*, an unbounded closed operator $(\mathrm{D}(f(A)), f(A))$ on X, with $\mathrm{D}(A) \cap \mathrm{R}(A) \subset \mathrm{D}(f(A))$.

One can perform a similar argument for bisectorial operators, replacing $(1+z)$ and $(\mathrm{I}+A)$ respectively by $(i+z)$ and $(i\mathrm{I}+A)$.

The question about bounded holomorphic functional calculus can be reduced to the behavior of the $\mathbf{H}^\infty_0$-functional calculus, which is a consequence of the convergence result [McI86, Theorem, Section 5].

Theorem 1.26 ([Ege15, Corollary 3.3.6]) *Let $(\mathrm{D}(A), A)$ be an ω-sectorial operator with $\omega \in [0, \pi)$ on a Banach space X, with dense range and dense domain. Let $\mu \in (\omega, \pi)$.*

If there exists $K_\mu > 0$, such that for all $f \in \mathbf{H}^\infty_0(\Sigma_\mu)$

$$\|f(A)\|_{X\to X} \leqslant K_\mu \|f\|_{\mathrm{L}^\infty(\Sigma_\mu)},$$

then the $\mathbf{H}^\infty$-functional calculus of same angle μ for A is bounded with the same bound.

The result still holds for bisectorial operators up to appropriate modifications.

We end up the presentation of bounded holomorphic functional calculus with standard application in the case of self-adjoint operators on a Hilbert space, which is a consequence of [McI86, Theorem, Section 8] and [Haa06, Propositions C.4.2 & C.4.3], see also [Ege15, Section 3.4].

Proposition 1.27 *Let $(\mathrm{D}(A), A)$ be an injective self-adjoint operator on a Hilbert space H.*

Then $(\mathrm{D}(A), A)$ is 0-bisectorial with the resolvent estimate

$$\sup_{\lambda \in \mathbb{C}\setminus\overline{S}_\theta} \left\|\lambda(\lambda\mathrm{I}-A)^{-1}\right\|_{H\to H} \leqslant \frac{1}{|\sin(\theta)|}, \quad \theta \in (0, \frac{\pi}{2})$$

and has $\mathbf{H}^\infty(S_\mu)$-functional calculus for all $\mu \in (0, \frac{\pi}{2})$.

If moreover $(\mathrm{D}(A), A)$ is non-negative, i.e. $\langle Au, u\rangle_H \geqslant 0$ for all $u \in \mathrm{D}(A)$, then it is also 0-sectorial with the resolvent estimate

$$\sup_{\lambda \in \mathbb{C}\setminus\overline{\Sigma}_\theta} \left\|\lambda(\lambda\mathrm{I}-A)^{-1}\right\|_{H\to H} \leqslant \frac{1}{|\cos(\theta)|}, \quad \theta \in (\frac{\pi}{2}, \pi)$$

and has $\mathbf{H}^\infty(\Sigma_\mu)$-functional calculus for all $\mu \in (0, \pi)$.

Interesting and fundamental applications to illustrate those results will show up in next sections and chapters, mainly the Hodge-Dirac operator $D = \mathrm{d} + \delta$ and the ((negative) Hodge) Laplacian $-\Delta = D^2$.

We notice that in general, proving boundedness of holomorphic functional calculus for a given (concrete) (bi)sectorial operator on a Banach space X may be tedious. Usual techniques require starting from a Hilbert space H, for which one may realize the (bi)sectorial operator on it with a bounded holomorphic functional calculus provided by [McI86, Theorem, Section 8].

From this point, the goal will be to try to extrapolate the estimates up to the desired Banach space X via (extra)interpolation and perturbation argument when X, H are interior points of a same complex interpolation scale. In this case, the results are often carried over by (possibly vector-valued) Harmonic Analysis techniques.

For instance, in the case of elliptic operators on L^p spaces : M$^\mathrm{c}$Intosh and Monniaux performed the argument for the Hodge-Dirac operators and the Hodge Laplacian on L^p for bounded Lipschitz domains [MM18], Blunck and Kuntsmann gave sufficient conditions to extrapolate the bounded holomorphic functional calculus for elliptic operator in divergence form [BK03].

3.3 Fractional powers, bounded imaginary powers

The construction of the positive iterated powers of a (bi)sectorial operator, as Proposition 1.23 told us, does not pose any difficulty. When the operator is sectorial, one can easily extend the construction to complex powers with positive real part thanks to the Dunford-Riesz functional calculus.

For $\alpha \in \mathbb{C}_+^*$, $m \in \mathbb{N}$ such that $\alpha < m$, we can write for $z \in \mathbb{C} \setminus (-\infty, 0]$,

$$z^\alpha = (1+z)^m \frac{z^\alpha}{(1+z)^m} = (1+z)^m f_{\alpha,m}(z),$$

where z^α is defined by the mean of the complex logarithm on its principal branch. Then for $(D(A), A)$ an ω-sectorial operator on a Banach space X, since $f_{\alpha,m} \in \mathbf{H}_0^\infty(\Sigma_\mu)$, $\mu \in (\omega, \pi)$, we can define the closed operator $(\mathrm{D}(A^\alpha), A^\alpha)$ with $\mathrm{D}(A^m) \subset \mathrm{D}(A^\alpha)$, writing

$$A^\alpha := (\mathrm{I} + A)^m f_{\alpha,m}(A).$$

Fractional powers of sectorial operators behave as expected in many ways.

Proposition 1.28 ([Haa06, Propositions 3.1.1, 3.1.2, 3.1.9]) *Let $(\mathrm{D}(A), A)$ be an ω-sectorial operator on a Banach space X, with $\omega \in [0, \pi)$. Let $\alpha, \beta \in \mathbb{C}_+^*$ and $\varepsilon > 0$. Then the following assertion hold.*

(i) The law of exponents $A^{\alpha+\beta} = A^\alpha A^\beta$ is satisfied. In particular, the fractional powers of A coincides with the usual powers of A if $\alpha \in \mathbb{N}$.

(ii) If $\Re\alpha < \Re\beta$, then $\mathrm{D}(A^\beta) \subset \mathrm{D}(A^\alpha)$. If in addition A is densely defined, then $\mathrm{D}(A^\beta)$ is dense in $\mathrm{D}(A^\alpha)$.

(iii) Let $\gamma \in \mathbb{C}_+$, such that either $\Re\gamma > 0$, or $\gamma = 0$. If $\Re\gamma < \Re\beta < \Re\alpha$, for $\theta = \frac{\Re\beta - \Re\gamma}{\Re\alpha - \Re\gamma} \in (0,1)$, one has for all $x \in \mathrm{D}(A^\alpha)$,

$$\|A^\beta x\|_X \lesssim_\theta \|A^\gamma x\|_X^{1-\theta} \|A^\alpha x\|_X^\theta.$$

(iv) It holds $\mathrm{N}(A) = \mathrm{N}(A^\alpha)$ and if A is invertible so is A^α.

(v) One has the equality $\mathrm{D}(A^\alpha) = \mathrm{D}((A+\varepsilon)^\alpha)$ with equivalence of graph norms, and for all $x \in \mathrm{D}(A^\alpha)$

$$(A+\varepsilon)^\alpha \xrightarrow[\varepsilon \to 0_+]{} A^\alpha x.$$

(vi) If $\alpha \in (0, \frac{\pi}{\omega})$, then the operator $(\mathrm{D}(A^\alpha), A^\alpha)$ is $(\alpha \cdot \omega)$-sectorial on X.

It is more difficult to give an appropriate definition of fractional powers in the case of bisectorial operators. The difficulty arise from the absence of a branch of the logarithm on a bisector that allows the compatibility with usual powers of the operator. The raised issue could be partially bypassed, considering for $\alpha \in \mathbb{C}_+^*$, the *pseudo fractional powers* of A,

$$[A]^\alpha := (A^2)^{\frac{\alpha}{2}}.$$

Since A^2 is a sectorial operator, see the discussions at the beginning and the end of [Ege15, Sections 3.2.4 & 3.2.5], the properties are then carried over by Proposition 1.28. We summarize with the following proposition.

Proposition 1.29 *Let $(\mathrm{D}(A), A)$ be an ω-bisectorial operator on a Banach space X, with $\omega \in [0, \frac{\pi}{2})$. For $\alpha \in (0, \frac{\pi}{\omega})$, the operator $(\mathrm{D}([A]^\alpha), [A]^\alpha)$ is $(\alpha \cdot \omega)$-sectorial on X.*

In practice, it is of manifest interest to obtain a description for the domain of the fractional powers of a sectorial operator. Under some circumstances, this is possible for a specific class of sectorial operators, the so called BIP class, where BIP stands for Bounded Imaginary Powers.

For $s \in \mathbb{R}^*$, using the principal branch of the logarithm on $\mathbb{C} \setminus (-\infty, 0]$, we may write for $z \in \Sigma_\mu$, $\mu \in [0, \pi)$,

$$z^{is} = \frac{z^{1+is}}{(1+z)^2} \frac{(1+z)^2}{z} = f_{1+is,2}(z) \frac{(1+z)^2}{z}.$$

Since $f_{1+is,2} \in \mathbf{H}_0^\infty(\Sigma_\mu)$, the next definition is then meaningful.

Definition 1.30 We say that a densely defined, injective, sectorial operator $(\mathrm{D}(A), A)$ on a Banach space X has **bounded imaginary powers** (BIP) of type $\theta_A \in [0, +\infty]$, if A^{is} is a bounded operator linear on X for all $s \in \mathbb{R}$, and we have

$$\theta_A := \inf \left\{ \omega \geqslant 0 \;\middle|\; \sup_{s \in \mathbb{R}} e^{-\omega |s|} \|A^{is}\|_{X \to X} < +\infty \right\} .$$

The description of the domains of fractional powers for an operator that has BIP arise from the computation of the associated complex interpolation space. It follows from the two results [Haa06, Proposition 3.5.5 & Theorem 6.6.9].

Theorem 1.31 *Let $(\mathrm{D}(A), A)$ be a densely defined operator on X such as it has BIP. Let $\alpha, \beta, \gamma \in \mathbb{C}_+$, such that $\Re\alpha < \Re\gamma < \Re\beta$, and set $\theta = \frac{\Re\beta - \Re\gamma}{\Re\beta - \Re\alpha} \in (0,1)$. We have*

$$[\mathrm{D}(A^\alpha), \mathrm{D}(A^\beta)]_\theta = \mathrm{D}(A^\gamma),$$

whenever, either $\alpha = 0$ or $\Re\alpha > 0$.

Another important fact : sectorial operators that have BIP of type $\theta < \frac{\pi}{2}$ play an important role in the theory of evolution equation, mainly well-posedness and regularity for inhomogeneous Cauchy problems, we refer to Chapter 3 for more details.

3.4 Bounded holomorphic semigroups

For $(\mathrm{D}(A), A)$ an unbounded closed operator on X, one may want to solve the abstract ordinary differential equation

$$\begin{cases} \partial_t u(t) + Au(t) = & 0\,,\ t > 0, \\ u(0) = & u_0 \in X. \end{cases}$$

Therefore, we are looking for building the linear flow map $\mathbf{T} = (T(t))_{t \geqslant 0}$ that should behave morally as $(e^{-tA})_{t \geqslant 0}$, *i.e.* such that

(i) one has $\mathbf{T}u_0 \in \mathrm{C}^0(\mathbb{R}_+, X)$, and $T(0)u_0 = u_0$, for all $u_0 \in X$,

(ii) for all $t \in \mathbb{R}_+$, $T(t)$ is a bounded linear operator on X, and for all $s \geqslant 0$,

$$T(t+s) = T(t)T(s),$$

(iii) for all $t > 0$, $u_0 \in X$, one has $T(t)u_0 \in \mathrm{D}(A)$, and $\mathbf{T}u_0 \in \mathrm{C}^1((0, +\infty), X)$, with

$$\partial_t (T(t)u_0) = -AT(t)u_0.$$

When it exists, such family $\mathbf{T}$ of bounded linear operators is called the **strongly continuous semigroup of bounded linear operators on X with (negative) generator** $(\mathrm{D}(A), A)$. For simplicity, we prefer to call it the **C^0-semigroup generated by (negative) A**.

It turns out that such family is well-defined when $(\mathrm{D}(A), A)$ is an ω-sectorial operator on X, with $\omega \in [0, \frac{\pi}{2})$, thanks to the Dunford-Riesz holmorphic functional calculus. Indeed, for all $t > 0$,

the operator $(\mathrm{D}(A), tA)$ is sectorial of the same angle, with the same resolvent bound. Since one has for all $z \in \Sigma_\omega$

$$e^{-z} = \left(e^{-z} - \frac{1}{1+z}\right) + \frac{1}{1+z} = g(z) + \frac{1}{1+z}$$

with $g \in \mathbf{H}_0^\infty(\Sigma_\phi)$, $\varphi \in [0, \frac{\pi}{2})$, we can define for all $t \geqslant 0$

$$T(t) := e^{-tA} := g(tA) + (\mathrm{I} + tA)^{-1}.$$

This construction yields even a uniformly bounded C^0-semigroup.

One can even go a bit further, extending the construction on a sector near the half-line. Indeed, for $\varphi \in [0, \frac{\pi}{2} - \omega)$, one can check that $(\mathrm{D}(A), e^{i\varphi}A)$ is an $(\omega+\varphi)$-sectorial operator with $\omega+\varphi < \frac{\pi}{2}$. The Dunford-Riesz holomorphic functional calculus still apply, and we get the uniformly bounded families of operators

$$(T(z))_{z \in \Sigma_{\frac{\pi}{2}-\omega-\varepsilon} \cup \{0\}}, \qquad \text{for all } \varepsilon \in (0, \frac{\pi}{2} - \omega].$$

The next result summarize the main expected properties of such **bounded holomorphic semigroup** generated by a sectorial operator, and even more. The original statement comes from [Haa06, Proposition 3.4.1], one may also see [ABHN11, Proposition 3.7.2].

Proposition 1.32 *Let $(\mathrm{D}(A), A)$ be an ω-sectorial operator on a Banach space X, with $\omega \in [0, \frac{\pi}{2})$. We set*

$$T(z) := e^{-zA}, \qquad \text{for all } z \in \Sigma_{\frac{\pi}{2}-\omega}.$$

The following assertions hold :

(i) For all $\phi \in (\omega, \frac{\pi}{2})$, $(T(z))_{z \in \Sigma_{\frac{\pi}{2}-\phi} \cup \{0\}}$ is a family of uniformly bounded operators such that

$$T(0) = \mathrm{I} \quad \text{and} \quad T(z+z') = T(z)T(z') \quad \text{for all } z, z' \in \Sigma_{\frac{\pi}{2}-\omega} \cup \{0\}.$$

(ii) If $x \in \overline{\mathrm{D}(A)}$, then for each $\psi \in (0, \frac{\pi}{2} - \omega)$,

$$T(z)x \xrightarrow[\substack{|z|\to 0 \\ |\arg(z)| \leqslant \psi}]{} x \quad \text{and} \quad T(z)x \xrightarrow[\substack{|z|\to +\infty \\ |\arg(z)| \leqslant \psi}]{} 0.$$

(iii) If $x \in X$, then $T(z)x \in \mathrm{D}(A^n)$, for all $z \in \Sigma_{\frac{\pi}{2}-\omega}$, $n \in \mathbb{N}$, and the map $z \longmapsto T(z)x$ is holomorphic on $\Sigma_{\frac{\pi}{2}-\omega}$ with derivatives of order $k \in \mathbb{N}$,

$$T^{(k)}(z)x = (-A)^k T(z)x \quad \text{for all } z \in \Sigma_{\frac{\pi}{2}-\omega}.$$

(iv) The identity

$$(\lambda \mathrm{I} + A)^{-1} = \int_0^{+\infty} e^{-\lambda t} T(t)\, \mathrm{d}t$$

holds true for all $\lambda \in \Sigma_{\frac{\pi}{2}}$.

(v) For $\alpha \in \mathbb{C}_+$, $z \in \Sigma_{\frac{\pi}{2}-\phi}$, with $\phi \in (\omega, \frac{\pi}{2})$, one has

$$\|A^\alpha T(z)\|_{X \to X} \lesssim_{A,\phi,\alpha} \frac{1}{|z|^{\Re\alpha}}.$$

The appearance of point *(v)* is a consequence of the fact that $z \mapsto z^\alpha e^{-z} \in \mathbf{H}_0^\infty(\Sigma_\phi)$, $\phi \in [0, \frac{\pi}{2})$.

When a C^0-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space X satisfies the point *(v)* for $\alpha = 1$, and $t \in (0, +\infty)$, we say that $(T(t))_{t\geqslant 0}$ is a bounded **analytic C^0-semigroup**. When it occurs, then one can find that its generator $(\mathrm{D}(A), A)$ is an ω-sectorial operator on X for some $\omega \in [0, \frac{\pi}{2})$, and Proposition 1.32 applies. See [ABHN11, Theorem 3.7.19] for more details.

One can also use the semigroup generated by a sectorial operator to recover fractional powers of the said generator, thanks to M$^{\mathrm{c}}$Intosh's approximation Theorem 1.24.

Proposition 1.33 *Let $(\mathrm{D}(A), A)$ be an ω-sectorial operator on a Banach space X, with $\omega \in [0, \frac{\pi}{2})$. Let $\alpha \in \mathbb{C}$, such that $0 < \Re\alpha < 1$. The following assertions hold.*

(i) For all $x \in \mathrm{D}(A^\alpha)$,

$$A^\alpha x = \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} t^{1-\alpha} A^{1-\alpha} e^{-tA} A^\alpha x \, \frac{\mathrm{d}t}{t} = \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} t^{-\alpha} A e^{-tA} x \, \mathrm{d}t$$

hold true.

(ii) If $(\mathrm{D}(A), A)$ is injective, for all $x \in \mathrm{R}(A^\alpha)$,

$$A^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^\alpha e^{-tA} x \, \frac{\mathrm{d}t}{t}$$

holds true.

A lot of other representation formulas are available for fractional powers of sectorial operators, called generally Balakrishnan representation formulas. See for instance [Haa06, Proposition 3.1.12], [Lun18, Chapter 4], [ABHN11, Chapter 3, Section 3.8], [BC20, Chapter 2] and the references therein.

2 Homogeneous Sobolev and Besov spaces on the whole and the half-space.

Sommaire

Résumé du chapitre

Dans ce chapitre, nous proposons une construction d'espaces de Sobolev homogènes d'ordre fractionnaire sur $\mathbb{R}^n$ et $\mathbb{R}^n_+$. Cette construction complète la construction d'espaces de Besov homogènes sur $\mathcal{S}'_h(\mathbb{R}^n)$ commencée par Bahouri, Chemin et Danchin sur $\mathbb{R}^n$. Nous étendrons également le traitement effectué par Danchin et Mucha sur $\mathbb{R}^n_+$, et la construction d'espaces de Sobolev homogènes d'ordre entier commencée par Danchin, Hieber, Mucha et Tolksdorf sur $\mathbb{R}^n$ et $\mathbb{R}^n_+$.

Les propriétés d'interpolation réelle et complexe, de dualité et de densité sont discutées. Les résultats de traces sont également revus. Notre approche repose principalement sur la théorie de l'interpolation exposée dans la Section 2 du précédent Chapitre 1, et donne des preuves plus simples de certains résultats précédemment connus dans le cas des espaces de Besov.

Le manque de complétude sur toute l'échelle conduira à considérer des intersection d'espaces des estimations découplées pour contourner ce problème.

Comme applications standards et simples, nous traitons les problèmes des laplaciens de Dirichlet et de Neumann dans ces espaces de fonctions homogènes qui seront utiles dans le Chapitre 4.

Summary of the chapter

In this chapter, we propose a construction of homogeneous Sobolev spaces of fractional order on $\mathbb{R}^n$ and $\mathbb{R}^n_+$. This construction completes the construction of homogeneous Besov spaces on $\mathcal{S}'_h(\mathbb{R}^n)$ started by Bahouri, Chemin and Danchin on $\mathbb{R}^n$. We will also extend the treatment done by Danchin and Mucha on $\mathbb{R}^n_+$, and the construction of homogeneous Sobolev spaces of integer orders started by Danchin, Hieber, Mucha and Tolksdorf on $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Properties of real and complex interpolation, duality, and density are discussed. Trace results are also reviewed. Our approach relies mostly on interpolation theory exposed in Section 2 of previous Chapter 1, and yields simpler proofs of some already known results in the case of Besov spaces.

The lack of completeness on the whole scale will lead to consideration of intersection spaces with decoupled estimates to circumvent this issue.

As standard and simple applications, we treat the problems of Dirichlet and Neumann Laplacians in these homogeneous functions spaces that are going to be useful in the later Chapter 4.

1 Introduction

We want to give an appropriate construction of homogeneous Sobolev spaces as subspaces of tempered distributions instead of a quotient space of distributions by polynomials. This construction is motivated by the fact that one would make sense of (para)products laws, stability under global diffeomorphism, or to look at boundary conditions, and therefore traces, when one restrict those spaces on a domain. This could be somewhat difficult if we work with tempered distributions up to a polynomial. Indeed, it is not clear that one can perform previous operations in a way that does not depend on a choice of a representative $u+P \in \mathcal{S}'(\mathbb{R}^n)$ of $[u] \in \mathcal{S}'(\mathbb{R}^n) \Big/ \mathbb{C}[x]$. This is inconvenient when it comes to study non-linear partial differential equations, or partial differential equations on a domain with boundary conditions. However, the interested reader could consult, for instance [BL76, Chapter 6, Section 6.3], [Tri78, Chapter 5], or [Saw18, Chapter 2, Section 2.4] for such a construction.

To circumvent those issues, the idea of Bahouri, Chemin and Danchin in [BCD11, Chapter 2] was to introduce a subspace of $\mathcal{S}'(\mathbb{R}^n)$ such that we get rid of polynomials, see [BCD11, Examples, p.23]. The aforementioned subspace of $\mathcal{S}'(\mathbb{R}^n)$ is

$$\mathcal{S}'_h(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \forall \Theta \in \mathrm{C}_c^\infty(\mathbb{R}^n),\ \|\Theta(\lambda\mathfrak{D})u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \xrightarrow[\lambda\to+\infty]{} 0 \right\}$$

where $\Theta(\lambda\mathfrak{D}) = \mathcal{F}^{-1}\Theta(\lambda\cdot)\mathcal{F}$, with $\mathcal{F}$ denoting the Fourier transform. The condition of uniform convergence for low frequencies in the definition above ensures that for $u \in \mathcal{S}'_h(\mathbb{R}^n)$, the series

$$\sum_{j\leqslant 0} \dot{\Delta}_j u$$

converges in $\mathrm{L}^\infty(\mathbb{R}^n)$, and then, by [BCD11, Proposition 2.14], the following equality holds in $\mathcal{S}'(\mathbb{R}^n)$

$$u = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u,$$

where $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ is the homogeneous Littlewood-Paley decomposition on $\mathbb{R}^n$. With $\mathcal{S}'_h(\mathbb{R}^n)$ as an ambient space, Bahouri, Chemin and Danchin gave a construction of homogeneous Besov spaces $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ which are complete whenever $(s,p,q) \in \mathbb{R}\times(1,+\infty)\times[1,+\infty]$ satisfies

$$\left[s < \frac{n}{p} \right] \text{ or } \left[q = 1 \text{ and } s \leqslant \frac{n}{p} \right].$$

Later, this has also led Danchin and Mucha to consider homogeneous Besov spaces on $\mathbb{R}^n_+$ and on exterior domains, see [DM09 ; DM15], and Danchin, Hieber, Mucha and Tolksdorf [DHMT21] to consider homogeneous Sobolev spaces $\dot{\mathrm{H}}^{m,p}$ on $\mathbb{R}^n$ and $\mathbb{R}^n_+$, for $m \in \mathbb{N}$, $p \in (1,+\infty)$. Each iteration led to various important applications in fluid dynamics, such as Navier-Stokes equations with variable density in [DM09 ; DM15], or free boundary problems as in [DHMT21]. This highlights

the needs of stability under global diffeomorphism, and (para)product laws that do not rely on a choice of a representative up to a polynomial.

We want to summarize, complete and extend the given construction of Besov spaces in [BCD11, Chapter 2] and the one of homogeneous Sobolev spaces started in [DHMT21, Chapter 3]. We are going to discuss in Section 2 their construction and usual properties, especially their behavior through complex and real interpolation. The whole space case is treated first, then the case of the half-space will follow.

Due to the lack of completeness, for homogeneous Sobolev (and Besov) spaces with high regularity exponents, one will need to consider intersection spaces $\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}$, with either, $\dot{\mathrm{H}}^{s_0,p_0}$ or $\dot{\mathrm{H}}^{s_1,p_1}$ known to be complete (*i.e.* $s_j < n/p_j$). Therefore, one will have to check boundedness of operators with decoupled estimates.

In Section 5, we will review the meaning of traces on the boundary. As an application, in Section 6, we treat the well-posedness of Neumann and Dirichlet Laplacians on the half-space with fine enough behavior of solutions. The "fine enough behavior" have to be understood in the sense that the decay to 0 at infinity is given a very precise sense.

2 Homogeneous function spaces on the whole space

All the function spaces considered here are scalar complex valued. Hence, to alleviate notations during this whole section, we will write $\mathrm{L}^p(\Omega)$ instead of $\mathrm{L}^p(\Omega, \mathbb{C})$, and similarly for any other function spaces : we drop the arrival space $\mathbb{C}$.

2.1 Definition and usual properties

To deal with Besov spaces on the whole space, we need to introduce Littlewood-Paley decomposition given by $\phi \in \mathrm{C}_c^\infty(\mathbb{R}^n)$, radial, real-valued, non-negative, such that

- $\mathrm{supp}\ \phi \subset B(0, 4/3)$;
- $\phi_{|_{B(0,3/4)}} = 1$;

so we define the following functions for any $j \in \mathbb{Z}$ for all $\xi \in \mathbb{R}^n$,

$$\phi_j(\xi) := \phi(2^{-j}\xi), \qquad \psi_j(\xi) := \phi_j(\xi/2) - \phi_j(\xi),$$

and the family $(\psi_j)_{j\in\mathbb{Z}}$ has the following properties

- $\mathrm{supp}(\psi_j) \subset \{\, \xi \in \mathbb{R}^n \,|\, 3 \cdot 2^{j-2} \leqslant |\xi| \leqslant 2^{j+3}/3 \,\}$;
- $\forall \xi \in \mathbb{R}^n \setminus \{0\},\ \sum_{j=-M}^{N} \psi_j(\xi) \xrightarrow[N,M\to+\infty]{} 1.$

Such a family $(\phi, (\psi_j)_{j\in\mathbb{Z}})$ is called a Littlewood-Paley family. Now, we consider the two following families of operators associated with their Fourier multipliers :

- The ***homogeneous*** family of Littlewood-Paley dyadic decomposition operators $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$, where
$$\dot{\Delta}_j := \mathcal{F}^{-1}\psi_j\mathcal{F},$$
- The ***inhomogeneous*** family of Littlewood-Paley dyadic decomposition operators $(\Delta_k)_{k\in\mathbb{Z}}$, where
$$\Delta_{-1} := \mathcal{F}^{-1}\phi\mathcal{F},$$
$\Delta_k := \dot{\Delta}_k$ for any $k \geqslant 0$, and $\Delta_k := 0$ for any $k \leqslant -2$.
- The low frequency cut-off operators $(\dot{S}_j)_{j\in\mathbb{Z}}$, given for all $j \in \mathbb{Z}$ by
$$\dot{S}_j := \mathcal{F}^{-1}\phi_j\mathcal{F}.$$

One may notice, as a direct application of Young's inequality for the convolution, that they are all uniformly bounded families of operators on $\mathrm{L}^p(\mathbb{R}^n)$, $p \in [1,+\infty]$.

Both family of operators lead for $s \in \mathbb{R}$, $p,q \in [1,+\infty]$, $u \in \mathcal{S}'(\mathbb{R}^n)$ to the following quantities,

$$\|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} = \left\| (2^{ks} \|\Delta_k u\|_{\mathrm{L}^p(\mathbb{R}^n)})_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})} \text{ and } \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} = \left\| (2^{js} \left\|\dot{\Delta}_j u\right\|_{\mathrm{L}^p(\mathbb{R}^n)})_{j\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})},$$

respectively named the inhomogeneous and homogeneous Besov norms, but the homogeneous norm is not really a norm since $\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} = 0$ does not imply that $u = 0$. Thus, following [BCD11, Chapter 2] and [DHMT21, Chapter 3], we introduce a subspace of tempered distributions such that $\|\cdot\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}$ is point-separating, say

$$\mathcal{S}'_h(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \forall \Theta \in \mathrm{C}^\infty_c(\mathbb{R}^n),\ \|\Theta(\lambda\mathfrak{D})u\|_{\mathrm{L}^\infty(\mathbb{R}^n)} \xrightarrow[\lambda\to+\infty]{} 0 \right\},$$

where for $\lambda > 0$, $\Theta(\lambda\mathfrak{D})u = \mathcal{F}^{-1}\Theta(\lambda\cdot)\mathcal{F}u$. Notice that $\mathcal{S}'_h(\mathbb{R}^n)$ does not contain any polynomials, and for any $p \in [1,+\infty)$, $\mathrm{L}^p(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$.

One can also define the following quantities called the inhomogeneous and homogeneous Sobolev spaces' potential norms

$$\|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} := \left\|(\mathrm{I}-\Delta)^{\frac{s}{2}} u\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \text{ and } \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} := \left\| \sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u \right\|_{\mathrm{L}^p(\mathbb{R}^n)},$$

where $(-\Delta)^{\frac{s}{2}}$ is understood on $u \in \mathcal{S}'_h(\mathbb{R}^n)$ by the action on its dyadic decomposition, *i.e.*

$$(-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u := \mathcal{F}^{-1}(\xi \mapsto |\xi|^s \mathcal{F}\dot{\Delta}_j u(\xi)),$$

which gives a family of C^∞ functions with at most polynomial growth. Thanks to [DHMT21, Lemma 3.3, Definition 3.4],

$$\sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u \in \mathcal{S}'_h(\mathbb{R}^n)$$

holds for all $u \in \mathcal{S}'_h(\mathbb{R}^n)$, whenever $s \in [0,+\infty)$.

When $u \in \mathcal{S}'_h(\mathbb{R}^n)$ and $\sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u \in \mathcal{S}'_h(\mathbb{R}^n)$, for $s \in \mathbb{R}$, one will simply write without distinction,

$$(-\Delta)^{\frac{s}{2}} u = \sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u \in \mathcal{S}'_h(\mathbb{R}^n),$$

which is somewhat consistent in this case with the fact that $(-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u = \dot{\Delta}_j (-\Delta)^{\frac{s}{2}} u$, $j \in \mathbb{Z}$.

Hence, for any $p,q \in [1,+\infty]$, $s \in \mathbb{R}$, we define

- the inhomogeneous and homogeneous Sobolev (Bessel and Riesz potential) spaces,
  $\mathrm{H}^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} < +\infty \right\}$, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n) \,\middle|\, \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} < +\infty \right\}$;
- and the inhomogeneous and homogeneous Besov spaces,
  $\mathrm{B}^s_{p,q}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \,\middle|\, \|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)} < +\infty \right\}$, $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'_h(\mathbb{R}^n) \,\middle|\, \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} < +\infty \right\}$,

which are all normed vector spaces. We also introduce the following closures

$$\mathcal{B}^s_{p,\infty}(\mathbb{R}^n) = \overline{\mathcal{S}(\mathbb{R}^n)}^{\|\cdot\|_{\mathrm{B}^s_{p,\infty}(\mathbb{R}^n)}} \text{ and } \dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n) = \overline{\mathcal{S}_0(\mathbb{R}^n)}^{\|\cdot\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)}}.$$

Notice that the following equalities holds with equivalence of norms for $s > 0$, $p \in (1,+\infty)$, $q \in [1,+\infty]$,

$$\mathrm{L}^p(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) = \mathrm{H}^{s,p}(\mathbb{R}^n) \text{ and } \mathrm{L}^p(\mathbb{R}^n) \cap \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n) = \mathrm{B}^s_{p,q}(\mathbb{R}^n),$$

see [BL76, Theorem 6.3.2] for more details.

The treatment of homogeneous Besov spaces $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p, q \in [1, +\infty]$, defined on $\mathcal{S}'_h(\mathbb{R}^n)$ has been done in an extensive manner in [BCD11, Chapter 2]. However, the corresponding construction for homogeneous Sobolev spaces $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $p \in (1, +\infty)$ has only been done in the case $(p, s) \in (\{2\}, \mathbb{R}) \cup ((1, +\infty), \mathbb{N})$. See [BCD11, Chapter 1] for the case $p = 2$, [DHMT21, Chapter 3] for the case $s \in \mathbb{N}$.

We first mention the following equivalences of norms.

Proposition 2.1 *For all $s \in \mathbb{R}$, $p \in (1, +\infty)$, $q \in [1, +\infty]$, $m \in \mathbb{N}$, and all $u \in \mathcal{S}'_h(\mathbb{R}^n)$,*

$$\sum_{k=1}^{n} \|\partial^m_{x_k} u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \sim_{s,m,p,n} \|\nabla^m u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \sim_{s,m,p,n} \|u\|_{\dot{\mathrm{H}}^{s+m,p}(\mathbb{R}^n)}, \tag{2.1}$$

$$\sum_{j=1}^{n} \|\partial^m_{x_j} u\|_{\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)} \sim_{s,m,p,n} \|\nabla^m u\|_{\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)} \sim_{s,m,p,n} \|u\|_{\dot{\mathrm{B}}^{s+m}_{p,q}(\mathbb{R}^n)}, \tag{2.2}$$

where (2.1) is a direct consequence of the proof [DHMT21, Proposition 3.7], and (2.2) a consequence of [BCD11, Lemma 2.1].

The following subspace of Schwartz functions, say

$$\mathcal{S}_0(\mathbb{R}^n) := \{\, u \in \mathcal{S}(\mathbb{R}^n) \mid 0 \notin \operatorname{supp}(\mathcal{F}f) \,\},$$

is a nice dense subspace in many cases, to be more precise

Proposition 2.2 *For all $p \in (1, +\infty)$, $q \in [1, +\infty)$, $s \in \mathbb{R}$, $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\mathrm{L}^p(\mathbb{R}^n)$, $\mathrm{H}^{s,p}(\mathbb{R}^n)$, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ and $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$.*

Proof. — The result for $\mathrm{L}^p(\mathbb{R}^n)$ and $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ is known respectively from [DHMT21, Lemma 2.6] and [BCD11, Proposition 2.27]. The case of $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is carried over by the case $\mathrm{L}^p(\mathbb{R}^n)$. Let $s \in \mathbb{R}$, $p \in (1, +\infty)$, and $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. Then let us introduce

$$f := (-\Delta)^{\frac{s}{2}} u \in \mathrm{L}^p(\mathbb{R}^n),$$

so from the L^p case there exists $(f_k)_{k\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}^n)$ such that $f_k \longrightarrow f$ in $\mathrm{L}^p(\mathbb{R}^n)$. Now, for all $k \in \mathbb{N}$ we set $u_k := (-\Delta)^{-\frac{s}{2}} f_k \in \mathcal{S}_0(\mathbb{R}^n)$, it follows

$$\|u - u_k\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} = \big\|(-\Delta)^{\frac{s}{2}} u - (-\Delta)^{\frac{s}{2}} u_k\big\|_{\mathrm{L}^p(\mathbb{R}^n)} = \|f - f_k\|_{\mathrm{L}^p(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0. \qquad \blacksquare$$

The inhomogeneous spaces $\mathrm{L}^p(\mathbb{R}^n)$, $\mathrm{H}^{s,p}(\mathbb{R}^n)$, and $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ are all complete for all $p, q \in [1, +\infty]$, $s \in \mathbb{R}$, but in this setting homogeneous spaces are no longer always complete (see [BCD11, Proposition 1.34, Remark 2.26]). Indeed, it can be shown (see [BCD11, Theorem 2.25]) that homogeneous Besov spaces $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ are complete whenever $(s, p, q) \in \mathbb{R} \times (1, +\infty) \times [1, +\infty]$ satisfies

$$\left[s < \frac{n}{p}\right] \text{ or } \left[q = 1 \text{ and } s \leqslant \frac{n}{p}\right], \tag{$\mathcal{C}_{s,p,q}$}$$

From now, and until the end of this paper, we write $(\mathcal{C}_{s,p})$ for the statement $(\mathcal{C}_{s,p,p})$. Similarly, we show $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete whenever $(\mathcal{C}_{s,p})$ is satisfied, see Proposition 2.4 below.

To prove completeness for our homogeneous Sobolev spaces, we have to check validity of Sobolev embeddings in our setting, manually. (Therefore, completeness of homogeneous Sobolev spaces is not known at this stage.)

Proposition 2.3 *Let $p, q \in (1, +\infty)$, $s \in (0, n)$, such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

We have dense embeddings,

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)},\ \forall u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n),$$
$$\|u\|_{\dot{\mathrm{H}}^{-s,q}(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)},\ \forall u \in \mathrm{L}^p(\mathbb{R}^n).$$

Proof. — Let us first recall, the fact for all $f \in \mathcal{S}(\mathbb{R}^n)$, $s \in (0,n)$, we have that $(-\Delta)^{-\frac{s}{2}} f \in \mathrm{C}^\infty(\mathbb{R}^n)$ with at most polynomial growth, in particular if $f \in \mathcal{S}_0(\mathbb{R}^n)$, we have $(-\Delta)^{-\frac{s}{2}} f \in \mathcal{S}_0(\mathbb{R}^n)$ and the Hardy-Littlewood-Sobolev inequality, see [Gra14b, Section 1.2, Theorem 1.2.3], states that for q such that $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$, we have

$$\left\|(-\Delta)^{-\frac{s}{2}} g\right\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|g\|_{\mathrm{L}^p(\mathbb{R}^n)}, \quad \forall g \in \mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n).$$

Therefore, by density of $\mathcal{S}_0(\mathbb{R}^n)$ in $\mathrm{L}^p(\mathbb{R}^n)$, see above Proposition 2.2, and completeness of $\mathrm{L}^q(\mathbb{R}^n)$, there exists a unique $v \in \mathrm{L}^q(\mathbb{R}^n)$, such that if $(f_\ell)_{\ell\in\mathbb{N}} \subset \mathcal{S}_0(\mathbb{R}^n)$ converge to $f \in \mathrm{L}^p(\mathbb{R}^n)$, we obtain

$$(-\Delta)^{-\frac{s}{2}} f_\ell \xrightarrow[\ell\to+\infty]{\mathrm{L}^q} v,$$

then necessarily for all $k \in \mathbb{Z}$, the following convergence holds in $\mathrm{L}^q(\mathbb{R}^n)$ then in particular in $\mathcal{S}'(\mathbb{R}^n)$

$$(-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f_\ell \xrightarrow[\ell\to+\infty]{} \dot{\Delta}_k v.$$

Hence, for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$\left\langle (-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f_\ell, \phi \right\rangle = \left\langle \dot{\Delta}_k f_\ell, (-\Delta)^{-\frac{s}{2}} [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}]\phi \right\rangle$$

so that

$$\left\langle (-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f_\ell, \phi \right\rangle \xrightarrow[\ell\to+\infty]{} \left\langle \dot{\Delta}_k f, (-\Delta)^{-\frac{s}{2}} [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}]\phi \right\rangle = \left\langle (-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f, \phi \right\rangle.$$

Consequently, we deduce that $(-\Delta)^{-\frac{s}{2}} \dot{\Delta}_k f = \dot{\Delta}_k v$ in $\mathcal{S}'(\mathbb{R}^n)$, and since $v \in \mathrm{L}^q(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$,

$$v = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j v = \sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j f \in \mathrm{L}^q(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n),$$

which give the full meaning of the Hardy-Littlewood-Sobolev inequality in our setting *i.e.*,

$$\left\|(-\Delta)^{-\frac{s}{2}} u\right\|_{\mathrm{L}^q(\mathbb{R}^n)} = \|u\|_{\dot{\mathrm{H}}^{-s,q}(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)},\ \forall u \in \mathrm{L}^p(\mathbb{R}^n).$$

Now if $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, $(-\Delta)^{\frac{s}{2}} u \in \mathrm{L}^p(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$, and $u \in \mathcal{S}'_h(\mathbb{R}^n)$, so it follows, for all $k \in \mathbb{Z}$, that the next chain of equalities must hold pointwise,

$$\begin{aligned}
\dot{\Delta}_k u &= \dot{\Delta}_k [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}] u \\
&= \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F} \dot{\Delta}_k \mathcal{F}^{-1} |\xi|^s \mathcal{F} [\dot{\Delta}_{k-1} + \dot{\Delta}_k + \dot{\Delta}_{k+1}] u \\
&= \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F} \dot{\Delta}_k \left[\sum_{j=k-1}^{k+1} \mathcal{F}^{-1} |\xi|^s \mathcal{F} \dot{\Delta}_j u \right],
\end{aligned}$$

where we notice the property $\dot{\Delta}_k \dot{\Delta}_j = 0$, whenever $|j-k| \geqslant 2$, so that $\mathcal{F}^{-1}|\xi|^{-s}\mathcal{F}\dot{\Delta}_k \mathcal{F}^{-1}|\xi|^s \mathcal{F}\dot{\Delta}_j = 0$. Thus, it comes,

$$\dot{\Delta}_k u = \mathcal{F}^{-1} |\xi|^{-s} \mathcal{F} \dot{\Delta}_k (-\Delta)^{\frac{s}{2}} u,$$

then in $\mathcal{S}'(\mathbb{R}^n)$,

$$(-\Delta)^{-\frac{s}{2}}(-\Delta)^{\frac{s}{2}}u = \sum_{k\in\mathbb{Z}} \mathcal{F}^{-1}|\xi|^{-s}\mathcal{F}\dot{\Delta}_k(-\Delta)^{\frac{s}{2}}u = \sum_{k\in\mathbb{Z}} \dot{\Delta}_k u = u.$$

The function $(-\Delta)^{\frac{s}{2}}u$ is in $\mathrm{L}^p(\mathbb{R}^n)$ so one can apply the freshly adapted Hardy-Littlewood-Sobolev inequality to it and obtain that

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$ ■

Proposition 2.4 *Let $s \in \mathbb{R}$, $p \in (1,+\infty)$, then $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is a Banach space whenever exponents satisfy $(\mathcal{C}_{s,p})$ (i.e. when $s < \frac{n}{p}$).*

Proof. — For $s \in \mathbb{R}$, $p \in (1,+\infty)$ satisfying $s < \frac{n}{p}$, the case $s = 0$ is already done since $\mathrm{L}^p(\mathbb{R}^n)$ is complete. Hence, we have to treat cases $s < 0$, $s \in (0, \frac{n}{p})$.

(i) **The case** $s \in (0, \frac{n}{p})$.
Now, let us consider a Cauchy sequence $(v_k)_{k\in\mathbb{N}} \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, we deduce from Proposition 2.3 both that $(v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^q(\mathbb{R}^n)$, and $((-\Delta)^{\frac{s}{2}}v_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathrm{L}^p(\mathbb{R}^n)$. Thus, by completeness, there exists a unique couple $(v,w) \in \mathrm{L}^q(\mathbb{R}^n) \times \mathrm{L}^p(\mathbb{R}^n)$, such that

$$\|v - v_k\|_{\mathrm{L}^q(\mathbb{R}^n)} + \left\|w - (-\Delta)^{\frac{s}{2}}v_k\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0.$$

In particular, we have that $v, w \in \mathcal{S}'_h(\mathbb{R}^n)$ and by continuity, for all $j \in \mathbb{Z}$

$$(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j v = \dot{\Delta}_j w$$

so that, we have the following equalities in $\mathcal{S}'(\mathbb{R}^n)$

$$(-\Delta)^{\frac{s}{2}}v = \sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}}\dot{\Delta}_j v = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j w = w,$$

hence $(-\Delta)^{\frac{s}{2}}v = w \in \mathrm{L}^p(\mathbb{R}^n)$, which means exactly that $v \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, then $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete.

(ii) **The case** $s < 0$.
Let $(v_k)_{k\in\mathbb{N}} \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ be a Cauchy sequence in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, by completeness of $\mathrm{L}^p(\mathbb{R}^n)$, there exists a unique $w \in \mathrm{L}^p(\mathbb{R}^n)$, such that,

$$\left\|w - (-\Delta)^{\frac{s}{2}}v_k\right\|_{\mathrm{L}^p(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0.$$

In particular, we get that $w \in \mathcal{S}'_h(\mathbb{R}^n)$. Applying [DHMT21, Lemma 3.3], we have that $(-\Delta)^{-\frac{s}{2}}w \in \mathcal{S}'_h(\mathbb{R}^n)$. Then by construction $v := (-\Delta)^{-\frac{s}{2}}w \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, and

$$\|v - v_k\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0,$$

so that $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete. ■

A direct consequence of it is the following corollary

Corollary 2.5 *Let $p \in (1,+\infty)$, $s \in \mathbb{R}$, if $(\mathcal{C}_{s,p})$ is satisfied then*

$$(-\Delta)^{\frac{s}{2}} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) \longrightarrow \mathrm{L}^p(\mathbb{R}^n)$$

is an isometric isomorphism of Banach spaces.

Remark 2.6 In particular, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is a reflexive Banach space, for all $p \in (1,+\infty)$, $s < n/p$.

According to [BL76, Section 6.4], for all $s \in \mathbb{R}$, $p,q \in (1,+\infty)\times[1,+\infty]$, $\mathrm{H}^{s,p}(\mathbb{R}^n)$ and $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ are both complete, and moreover they are reflexive when $q \neq 1,+\infty$, and we have

$$(\mathrm{H}^{s,p}(\mathbb{R}^n))' = \mathrm{H}^{-s,p'}(\mathbb{R}^n),\ (\mathrm{B}^s_{p,q}(\mathbb{R}^n))' = \mathrm{B}^{-s}_{p',q'}(\mathbb{R}^n),$$
$$(\mathcal{B}^s_{p,\infty}(\mathbb{R}^n))' = \mathrm{B}^{-s}_{p',1}(\mathbb{R}^n),\ (\mathrm{B}^s_{p,1}(\mathbb{R}^n))' = \mathrm{B}^{-s}_{p',\infty}(\mathbb{R}^n).$$

We introduce via the next lemma the equivalent homogeneous Triebel-Lizorkin norm, which is somewhat important to carry over effortless usual results like the action of real and complex interpolation on our homogeneous function spaces.

Lemma 2.7 *For all $s \in \mathbb{R}$, $p \in (1,+\infty)$, let us introduce the following quantity for all $u \in \mathcal{S}'_h(\mathbb{R}^n)$,*

$$\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} := \Big\| \big\| (2^{js}\dot{\Delta}_j u)_{j\in\mathbb{Z}} \big\|_{\ell^2(\mathbb{Z})} \Big\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

Then $\|\cdot\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}$ is an equivalent norm on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, i.e. for all $u \in \mathcal{S}'_h(\mathbb{R}^n)$,

$$\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} \sim_{p,n,s} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

This is a very well known result, based on extensive use of Khintchine's inequality ($\mathrm{L}^p(\mathbb{R}^n)$ square estimates) and the Hörmander-Mikhlin Fourier multiplier theorem, but we need a proof for our specific setting, see for instance [Tri92, Remark 3, p.25] and [Gra14a, Proposition 6.1.2] for the case of $\mathcal{S}'(\mathbb{R}^n)$ when $s = 0$.

Proof. — **Step 1 :** $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \sim_{p,n} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$, $u \in \mathcal{S}'_h(\mathbb{R}^n)$.

To show the inequality $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$, $u \in \mathcal{S}'_h(\mathbb{R}^n)$, we can assume that $u \in \mathrm{L}^p(\mathbb{R}^n)$ otherwise $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \leqslant +\infty$ is always true.

So if $u \in \mathrm{L}^p(\mathbb{R}^n)$, we may consider (Ω,μ) to be a probability space, and for $(\varepsilon_k)_{k\in\mathbb{Z}}$ to be a family of independent identically distributed random variables such that for all $k \in \mathbb{Z}$,

$$\mu(\{\varepsilon_k = -1\}) = \mu(\{\varepsilon_k = 1\}) = \frac{1}{2}.$$

One deduce from Khintchine's inequality, see for instance either [MS13, Lemma 5.5], [KW04, Section I, Lemma 2.2] or [Gra14a, Appendix C], that

$$\begin{aligned}\|u\|^p_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} &\sim_p \int_{\mathbb{R}^n}\int_\Omega \Big|\sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\dot{\Delta}_j u(x)\Big|^p \mathrm{d}\mu(\omega)\,\mathrm{d}x \\ &\sim_p \int_\Omega\int_{\mathbb{R}^n} \Big|\sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\dot{\Delta}_j u(x)\Big|^p \mathrm{d}x\,\mathrm{d}\mu(\omega),\end{aligned}$$

where the last estimate comes from an application of Fubini-Tonelli's theorem. Hence, it suffices to investigate the L^p-boundedness of the following random Fourier multiplier operator, defined for almost all $\omega \in \Omega$,

$$T(\omega) := \sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\dot{\Delta}_j,$$

whose Fourier symbol is given by the function $K(\omega)$, such that for all $\xi \in \mathbb{R}^n$,

$$K(\omega)(\xi) = \sum_{j\in\mathbb{Z}} \varepsilon_j(\omega)\psi(2^{-j}\xi).$$

It is not difficult to see that one can make a partition $\mathbb{R}^n$ into annulus of size $|\xi| \sim 2^j$, $j \in \mathbb{Z}$, to check that for all $\ell \in \mathbb{N}$, $\xi \in \mathbb{R}^n \setminus \{0\}$,

$$|\nabla^\ell K(\omega)(\xi)| \lesssim_{n,\ell,\psi} \frac{1}{|\xi|^\ell},$$

where the implicit constant does not depend on ω. Therefore, one may apply Hörmander-Mikhlin's Fourier multiplier Theorem to deduce that $T(\omega)$ is bounded on L^p and admits a uniform bound with respect to ω, and use the fact that $\mu(\Omega) = 1$ to obtain,

$$\int_\Omega \int_{\mathbb{R}^n} \left|T(\omega)u(x)\right|^p \mathrm{d}x \, \mathrm{d}\mu(\omega) \lesssim_{p,n} \int_{\mathbb{R}^n} |u(x)|^p \mathrm{d}x.$$

Thus, we have obtained $\|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)}$, for all $u \in \mathcal{S}'_h(\mathbb{R}^n)$.

Now, to prove $\|u\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)}$, $u \in \mathcal{S}'_h(\mathbb{R}^n)$, we are going to argue by duality. Let $u \in \mathcal{S}'_h(\mathbb{R}^n)$, and $v \in \mathcal{S}(\mathbb{R}^n)$. We can decompose the action of u on v as

$$\left\langle u, \, v \right\rangle_{\mathbb{R}^n} = \sum_{j \in \mathbb{Z}} \left\langle \dot{\Delta}_j u, \, [\dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}] v \right\rangle_{\mathbb{R}^n}$$

so that by $\mathrm{L}^p(\ell^2)$-$\mathrm{L}^{p'}(\ell^2)$ Hölder's inequality, we obtain

$$\begin{aligned} \left|\left\langle u, \, v \right\rangle_{\mathbb{R}^n}\right| &\leqslant \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \left\| \left\| ([\dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}] v)_{j \in \mathbb{Z}} \right\|_{\ell^2(\mathbb{Z})} \right\|_{\mathrm{L}^{p'}(\mathbb{R}^n)} \\ &\leqslant 3 \, \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \, \|v\|_{\dot{\mathrm{F}}^0_{p',2}(\mathbb{R}^n)} \, . \end{aligned}$$

One may apply the previous estimate $\|v\|_{\dot{\mathrm{F}}^0_{p',2}(\mathbb{R}^n)} \lesssim_{p',n} \|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}$, to deduce

$$\left|\left\langle u, \, v \right\rangle_{\mathbb{R}^n}\right| \lesssim_{p',n} \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \, \|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)} \, .$$

Therefore, taking the supremum on $v \in \mathcal{S}(\mathbb{R}^n)$ such that $\|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)} \leqslant 1$, yields

$$\|u\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n} \|u\|_{\dot{\mathrm{F}}^0_{p,2}(\mathbb{R}^n)} \, .$$

Step 2 : $\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} \sim_{p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}$, $u \in \mathcal{S}'_h(\mathbb{R}^n)$, $s \neq 0$.

The proof starts similarly, introducing the following random Fourier multiplier operator

$$T^s(\omega) := \sum_{j \in \mathbb{Z}} 2^{js} (-\Delta)^{-\frac{s}{2}} \varepsilon_j(\omega) \dot{\Delta}_j,$$

from which one fairly obtains, for all $v \in \mathcal{S}'_h(\mathbb{R}^n)$,

$$\left\| \|(2^{js}(-\Delta)^{-\frac{s}{2}} \dot{\Delta}_j v)_{j \in \mathbb{Z}}\|_{\ell^2(\mathbb{Z})} \right\|_{\mathrm{L}^p(\mathbb{R}^n)} \sim_{p,n,s} \|v\|_{\mathrm{L}^p(\mathbb{R}^n)} \, .$$

Now, if $u \in \mathcal{S}'_h(\mathbb{R}^n)$, we can assume that $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ otherwise $\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} \leqslant +\infty$ is always true. One may plug $v = (-\Delta)^{\frac{s}{2}} u = \sum_{j \in \mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u$, to obtain first, from $(-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u = \dot{\Delta}_j (-\Delta)^{\frac{s}{2}} u$,

$$\left\| \|(2^{js} \dot{\Delta}_j u)_{j \in \mathbb{Z}}\|_{\ell^2(\mathbb{Z})} \right\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n,s} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \, .$$

For the reverse estimate, similarly, provided $u \in \mathcal{S}'_h(\mathbb{R}^n)$, we can assume that $\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} < +\infty$

otherwise $\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \leqslant +\infty$ is always true. The Fatou lemma yields

$$\begin{aligned}\Big\| \sum_{j\in\mathbb{Z}} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u \Big\|_{\mathrm{L}^p(\mathbb{R}^n)} &\leqslant \liminf_{N\to+\infty} \Big\| \sum_{j\in[\![-N,N]\!]} (-\Delta)^{\frac{s}{2}} \dot{\Delta}_j u \Big\|_{\mathrm{L}^p(\mathbb{R}^n)} \\ &\lesssim_{p,n,s} \liminf_{N\to+\infty} \Big(\Big\| \Big(\sum_{j\in[\![-N,N]\!]} |2^{js}\dot{\Delta}_j u|^2 \Big)^{\frac{1}{2}} \Big\|_{\mathrm{L}^p(\mathbb{R}^n)} \\ &\qquad + 2^{-(N+1)s} \|\dot{\Delta}_{-(N+1)} u\|_{\mathrm{L}^p(\mathbb{R}^n)} + 2^{(N+1)s} \|\dot{\Delta}_{N+1} u\|_{\mathrm{L}^p(\mathbb{R}^n)} \Big) \\ &\lesssim_{p,n,s} \|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}.\end{aligned}$$

This, shows that $\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}$ is finite, which ends the proof. ∎

One may use it to obtain interpolation inequalities,

Lemma 2.8 *Let $p_0, p_1 \in (1,+\infty)$, $s_0, s_1 \in \mathbb{R}$, we set*

$$\left(\frac{1}{p}, s\right) := (1-\theta)\left(\frac{1}{p_0}, s_0\right) + \theta\left(\frac{1}{p_1}, s_1\right).$$

For all $u \in \mathcal{S}'_h(\mathbb{R}^n)$, we have

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p_0,p_1,s_0,s_1,n} \|u\|^{1-\theta}_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)}.$$

Proof. — For $u \in \mathcal{S}'_h(\mathbb{R}^n)$, as a direct consequence of Hölder's inequality, we have

$$\left(\sum_{j\in\mathbb{Z}} |2^{js}\dot{\Delta}_j u|^2 \right)^{\frac{1}{2}} \leqslant \left(\sum_{j\in\mathbb{Z}} |2^{js_0}\dot{\Delta}_j u|^2 \right)^{\frac{1-\theta}{2}} \left(\sum_{j\in\mathbb{Z}} |2^{js_1}\dot{\Delta}_j u|^2 \right)^{\frac{\theta}{2}}.$$

Thus, one may take the L^p-norm of the above inequality, and use again Hölder's inequality, so that

$$\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)} \leqslant \Big\| \|(2^{js_0}\dot{\Delta}_j u)_{j\in\mathbb{Z}}\|^{1-\theta}_{\ell^2(\mathbb{Z})} \|(2^{js_1}\dot{\Delta}_j u)_{j\in\mathbb{Z}}\|^{\theta}_{\ell^2(\mathbb{Z})} \Big\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant \|u\|^{1-\theta}_{\dot{\mathrm{F}}^{s_0}_{p_0,2}(\mathbb{R}^n)} \|u\|^{\theta}_{\dot{\mathrm{F}}^{s_1}_{p_1,2}(\mathbb{R}^n)}.$$ ∎

Lemma 2.9 *Let $p_j \in (1,+\infty)$, $s_j \in \mathbb{R}$, for $j \in \{0,1\}$. If $(\mathcal{C}_{s_0,p_0})$ is satisfied then $\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$ is a Banach space for which $\mathcal{S}_0(\mathbb{R}^n)$ is dense in it.*

Proof. — The completeness is straightforward. Concerning the claim about density, we follow the proof of [BCD11, Proposition 2.27] with minor modifications, in order to adapt it to our setting.

For $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$, and fixed $\varepsilon > 0$, for $k \in \{0,1\}$ there exists $N \in \mathbb{N}$ such that for all $\tilde{N} \geqslant N$

$$\|u - u_{\tilde{N}}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} < \varepsilon.$$

Here, for any $K \in \mathbb{N}$,

$$u_K := \sum_{|j|\leqslant K} \dot{\Delta}_j u.$$

For $M \in [\![\tilde{N}+1, +\infty[\![$, $R > 0$, provided $\Theta \in \mathrm{C}^\infty_c(\mathbb{R}^n)$, real valued, supported in $B(0,2)$, such that $\Theta_{|_{B(0,1)}} = 1$, and $\Theta_R := \Theta(\cdot/R)$, we introduce

$$u^R_{\tilde{N},M} := (\mathrm{I} - \dot{S}_{-M})[\Theta_R u_{\tilde{N}}].$$

Since $\dot{\Delta}_k u_{\tilde{N}} = 0$, $k \leqslant -M-1$, we have $\dot{S}_{-M} u_{\tilde{N}} = 0$, then

$$u^R_{\tilde{N},M} - u_{\tilde{N}} = (\mathrm{I} - \dot{S}_{-M})[(\Theta_R - 1)u_{\tilde{N}}].$$

If one sets $m_k := \max(0, \lfloor s_k \rfloor + 2)$, since $0 \notin \operatorname{supp} \mathcal{F}(u^R_{\tilde{N},M} - u_{\tilde{N}})$ by construction, we apply [BL76, Theorem 6.3.2] and decreasing embedding of inhomogeneous Sobolev spaces to deduce

$$\begin{aligned}\|u^R_{\tilde{N},M} - u_{\tilde{N}}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} &\lesssim_{M,s_k,p_k} \|u^R_{\tilde{N},M} - u_{\tilde{N}}\|_{\mathrm{H}^{s_k,p_k}(\mathbb{R}^n)} \\ &\lesssim_{M,s_k,p_k} \|(\mathrm{I} - \dot{S}_{-M})[(\Theta_R - 1)u_{\tilde{N}}]\|_{\mathrm{H}^{m_k,p_k}(\mathbb{R}^n)} \\ &\lesssim_{M,s_k,p_k} \|[(\Theta_R - 1)u_{\tilde{N}}]\|_{\mathrm{H}^{m_k,p_k}(\mathbb{R}^n)}.\end{aligned}$$

Since one may check that $u_{\tilde{N}} \in \mathrm{H}^{m_k,p_k}(\mathbb{R}^n)$ for $k \in \{0,1\}$, by dominated convergence theorem it follows that

$$\|u^R_{\tilde{N},M} - u_{\tilde{N}}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} \xrightarrow[R\to+\infty]{} 0.$$

Thus, for $R > 0$ big enough, we have for $k \in \{0,1\}$

$$\|u - u^R_{\tilde{N},M}\|_{\dot{\mathrm{H}}^{s_k,p_k}(\mathbb{R}^n)} < 2\varepsilon.$$

The proof ends here since $u^R_{\tilde{N},M} \in \mathcal{S}_0(\mathbb{R}^n)$. ■

2.2 Interpolation, duality, embeddings

We recall also the usual interpolation properties,

$$\begin{aligned}&[\mathrm{H}^{s_0,p_0}(\mathbb{R}^n), \mathrm{H}^{s_1,p_1}(\mathbb{R}^n)]_\theta = \mathrm{H}^{s,p_\theta}(\mathbb{R}^n), \qquad &&(\mathrm{B}^{s_0}_{p,q_0}(\mathbb{R}^n), \mathrm{B}^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta,q} = \mathrm{B}^s_{p,q}(\mathbb{R}^n), \\ &(\mathrm{H}^{s_0,p}(\mathbb{R}^n), \mathrm{H}^{s_1,p}(\mathbb{R}^n))_{\theta,q} = \mathrm{B}^s_{p,q}(\mathbb{R}^n), \qquad &&[\mathrm{B}^{s_0}_{p_0,q_0}(\mathbb{R}^n), \mathrm{B}^{s_1}_{p_1,q_1}(\mathbb{R}^n)]_\theta = \mathrm{B}^s_{p_\theta,q_\theta}(\mathbb{R}^n),\end{aligned}$$

whenever $(p_0,q_0),(p_1,q_1),(p,q) \in [1,+\infty]^2 (p \neq 1,+\infty$, when dealing with Sobolev (Riesz potential) spaces), $\theta \in (0,1)$, $s_0 \neq s_1$ two real numbers, such that

$$\left(s, \frac{1}{p_\theta}, \frac{1}{q_\theta}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right),$$

see [BL76, Theorem 6.4.5]. A similar statement is available for our $\mathcal{S}'_h$-realization of homogeneous function spaces homogeneous function spaces.

Theorem 2.10 *Let $(p_0,p_1,p,q,q_0,q_1) \in (1,+\infty)^3 \times [1,+\infty]^3$, $s_0,s_1 \in \mathbb{R}$, such that $s_0 \neq s_1$, and for $\theta \in (0,1)$ let*

$$\left(s, \frac{1}{p_\theta}, \frac{1}{q_\theta}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

Assuming $(\mathcal{C}_{s_0,p})$ (resp. $(\mathcal{C}_{s_0,p,q_0})$), we get the following

$$(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))_{\theta,q} = (\dot{\mathrm{B}}^{s_0}_{p,q_0}(\mathbb{R}^n), \dot{\mathrm{B}}^{s_1}_{p,q_1}(\mathbb{R}^n))_{\theta,q} = \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n). \tag{2.3}$$

If moreover $(\mathcal{C}_{s_0,p_0})$ and $(\mathcal{C}_{s_1,p_1})$ are true then also is $(\mathcal{C}_{s,p_\theta})$ and

$$[\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)]_\theta = \dot{\mathrm{H}}^{s,p_\theta}(\mathbb{R}^n), \tag{2.4}$$

and similarly if $(\mathcal{C}_{s_0,p_0,q_0})$ and $(\mathcal{C}_{s_1,p_1,q_1})$ are satisfied then $(\mathcal{C}_{s,p_\theta,q_\theta})$ is also satisfied and

$$[\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n), \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n)]_\theta = \dot{\mathrm{B}}^s_{p_\theta,q_\theta}(\mathbb{R}^n). \tag{2.5}$$

Proof. — **Step 1 :** Let us deal with the real interpolation identity (2.3). Let us consider first the case of Sobolev spaces, with $u \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) + \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. For $(a,b) \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \times \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, such that $u = a + b$, by Lemma 2.7 we have

$$(\dot{\Delta}_j u)_{j\in\mathbb{Z}} = (\dot{\Delta}_j a)_{j\in\mathbb{Z}} + (\dot{\Delta}_j b)_{j\in\mathbb{Z}} \in \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z})) + \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z})).$$

Therefore, by the definition of the K-functional and Lemma 2.7, for $t > 0$,

$$\begin{aligned} K(t, (\dot{\Delta}_j u)_{j\in\mathbb{Z}}, \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z})), \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))) &\leqslant \|a\|_{\dot{\mathrm{F}}^{s_0}_{p,2}(\mathbb{R}^n)} + t\|b\|_{\dot{\mathrm{F}}^{s_1}_{p,2}(\mathbb{R}^n)} \\ &\lesssim_{p,s_0,s_1,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} + t\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)}. \end{aligned}$$

We then take the infimum on an all such pairs (a,b),

$$K(t, (\dot{\Delta}_j u)_{j\in\mathbb{Z}}, \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z})), \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))) \lesssim_{p,s_0,s_1,n} K(t, u, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)). \tag{2.6}$$

Now, we want to prove the reverse estimate. Since $(\dot{\Delta}_j u)_{j\in\mathbb{Z}} \in \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z})) + \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))$, let $(A,B) \in \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z})) \times \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))$ such that

$$(\dot{\Delta}_j u)_{j\in\mathbb{Z}} = A + B. \tag{2.7}$$

For $(w_j)_{j\in\mathbb{Z}} \subset \mathcal{S}'(\mathbb{R}^n)$, say, for simplicity, with finite support in the discrete variable, we define the map

$$\tilde{\Sigma}((w_j)_{j\in\mathbb{Z}}) := \sum_{j=-\infty}^{+\infty} \dot{\Delta}_j[w_{j-1} + w_j + w_{j+1}], \tag{2.8}$$

and satisfies for $v \in \mathcal{S}'_h(\mathbb{R}^n)$

$$\tilde{\Sigma}((\dot{\Delta}_j v)_{j\in\mathbb{Z}}) = v.$$

By Lemma 2.7 and [Gra14a, Proposition 6.1.4], one can check that

$$\tilde{\Sigma} : \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z})) \longrightarrow \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \tag{2.9}$$

is well-defined and bounded since $(\mathcal{C}_{s_0,p})$ is satisfied. Now, we apply $\tilde{\Sigma}$ to (2.7) to deduce from $\tilde{\Sigma}(\dot{\Delta}_j u)_{j\in\mathbb{Z}} = u \in \mathcal{S}'_h(\mathbb{R}^n)$, and $\tilde{\Sigma}A \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$, that

$$\tilde{\Sigma}B = u - \tilde{\Sigma}A \in \mathcal{S}'_h(\mathbb{R}^n).$$

By the mean of [Gra14a, Proposition 6.1.4], we obtain

$$\|\tilde{\Sigma}B\|_{\dot{\mathrm{F}}^{s_1}_{p,2}(\mathbb{R}^n)} = \|(\dot{\Delta}_j \tilde{\Sigma}B)_{j\in\mathbb{Z}}\|_{\mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))} \lesssim_{p,s_1,n} \|B\|_{\mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))}. \tag{2.10}$$

Hence, by Lemma 2.7, $\tilde{\Sigma}B$ is an element of $\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. Therefore by the definition of the K-functional, the boundedness properties of $\tilde{\Sigma}$, Lemma 2.7 and [Gra14a, Proposition 6.1.4], for $t > 0$,

$$\begin{aligned} K(t, u, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) &\leqslant \|\tilde{\Sigma}A\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} + t\|\tilde{\Sigma}B\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)} \\ &\lesssim_{p,s_0,s_1,n} \|A\|_{\mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z}))} + t\|B\|_{\mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))}. \end{aligned}$$

Thus, let us take the infimum on all such pairs (A,B), and invoke (2.6) to obtain for all $t > 0$, and all $u \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) + \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$,

$$K(t, (\dot{\Delta}_j u)_{j\in\mathbb{Z}}, \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_0}(\mathbb{Z})), \mathrm{L}^p(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z}))) \sim_{p,s_0,s_1,n} K(t, u, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)). \tag{2.11}$$

By Corollary 1.9, we recall that we have the real interpolation identity

$$(\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z})))_{\theta,q}=\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n)). \tag{2.12}$$

Thus, up to multiply the estimate (2.11) by $t^{-\theta}$ and taking its L^q_*-norm, it can be turned into

$$\begin{aligned}\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}=\|(\dot\Delta_j u)_{j\in\mathbb{Z}}\|_{\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))}&\sim_{p,s_0,s_1,\theta,n}\|(\dot\Delta_j u)_{j\in\mathbb{Z}}\|_{(\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z})))_{\theta,q}}\\&\sim_{p,s_0,s_1,\theta,n}\|u\|_{(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))_{\theta,q}}.\end{aligned}$$

Therefore (2.3) is proved.

Step 2 : For $p\in(1,+\infty)$, $q\in[1,+\infty]$ and $s\in\mathbb{R}$ such that $(\mathcal{C}_{s,p,q})$ is satisfied, for $\tilde\Sigma$ introduced in (2.8), we want to show the boundedness of

$$\tilde\Sigma\,:\,\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))\longrightarrow\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n).$$

Let $(u_j)_{j\in\mathbb{Z}}\in\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))$ with finite support with respect to the discrete variable, by the real interpolation identity (2.12), it holds that, for some fixed $s_0<s<s_1$,

$$(u_j)_{j\in\mathbb{Z}}\in\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))+\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z})).$$

Let $(a,b)\in\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))\times\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))$, such that $(u_j)_{j\in\mathbb{Z}}=a+b$. Up to restrict a and b to the discrete support of $(u_j)_{j\in\mathbb{Z}}$, denoting those restriction by respectively $\tilde a$ and $\tilde b$, we obtain

$$(u_j)_{j\in\mathbb{Z}}=\tilde a+\tilde b.$$

Therefore, by finite support of $\tilde a$ and $\tilde b$ in the discrete variable and by Lemma 2.7,

$$\tilde\Sigma\tilde a\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)\text{ and }\tilde\Sigma\tilde b\in\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n),$$

so that

$$\tilde\Sigma(u_j)_{j\in\mathbb{Z}}\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)+\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n).$$

Hence, by the definition of the K-functional, Lemma 2.7 and the boundedness properties of $\tilde\Sigma$ (2.9) and (2.10),

$$\begin{aligned}K(t,\tilde\Sigma(u_j)_{j\in\mathbb{Z}},\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))&\lesssim_{p,s_0,s_1,n}\|\tilde\Sigma\tilde a\|_{\dot{\mathrm{F}}^{s_0}_{p,2}(\mathbb{R}^n)}+t\|\tilde\Sigma\tilde b\|_{\dot{\mathrm{F}}^{s_1}_{p,2}(\mathbb{R}^n)}\\&\lesssim_{p,s_0,s_1,n}\|\tilde a\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))}+t\|\tilde b\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}\\&\lesssim_{p,s_0,s_1,n}\|a\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z}))}+t\|b\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}.\end{aligned}$$

Now, one can take the infimum on all such pairs (a,b), to deduce that for all $t>0$,

$$K(t,\tilde\Sigma(u_j)_{j\in\mathbb{Z}},\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))\lesssim_{p,s_0,s_1,n}K(t,(u_j)_{j\in\mathbb{Z}},\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_0}(\mathbb{Z})),\mathrm{L}^p(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))).$$

Multiplying this estimate by $t^{-\theta}$ then taking the L^q_*-norm, thanks to (2.3) and (2.12), yield the estimate

$$\|\tilde\Sigma(u_j)_{j\in\mathbb{Z}}\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}\lesssim_{p,s_0,s_1,\theta,n}\|(u_j)_{j\in\mathbb{Z}}\|_{\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))}.$$

Then, the map $\tilde\Sigma$ extends uniquely to a bounded map whenever $(\mathcal{C}_{s,p,q})$ is satisfied and $q<+\infty$.

For the case $q=+\infty$ when $(\mathcal{C}_{s,p,q})$ is satisfied, *i.e.* when $s<n/p$ is satisfied, the result follows in fact directly from the Step 1.

In fact, the above manual real interpolation procedure was only needed to reach the endpoint couple $(\dot{\mathrm{B}}^{n/p}_{p,1}(\mathbb{R}^n),\ell^1_{n/p}(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n)))$.

Step 3 : For the real interpolation identity (2.3) in the case of Besov spaces, by the previous Step 2, the proof presented in Step 1 is still valid if we replace $(\dot{\mathrm{H}}^{s_0,p},\dot{\mathrm{H}}^{s_1,p})$ and the condition $(\mathcal{C}_{s_0,p})$ by $(\dot{\mathrm{B}}^{s_0}_{p,q_0},\dot{\mathrm{B}}^{s_1}_{p,q_1})$ with the condition $(\mathcal{C}_{s_0,p,q_0})$.

Step 4 : As in the proof of [BL76, Theorem 6.4.5], being aware of Theorem 1.21, we can claim, thanks to previous steps, that

- thanks to its definition, for all $s\in\mathbb{R}$, $p\in(1,+\infty)$, $q\in[1,+\infty]$, when $(\mathcal{C}_{s,p,q})$ is satisfied, $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ is a retraction of $\ell^q_s(\mathbb{Z},\mathrm{L}^p(\mathbb{R}^n))$ on $\mathcal{S}'_h(\mathbb{R}^n)$ through the homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$, and projection map $\tilde{\Sigma}$;
- similarly, due to Lemma 2.7, for all $s\in\mathbb{R}$, $p\in(1,+\infty)$, when $(\mathcal{C}_{s,p})$ is satisfied $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is a retraction of $\mathrm{L}^p(\mathbb{R}^n,\ell^2_s(\mathbb{Z}))$ on $\mathcal{S}'_h(\mathbb{R}^n)$ through the homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$, and projection map $\tilde{\Sigma}$.

Thus, one may apply Theorem 1.21, with Corollary 1.19 for complex interpolation of Besov spaces and Corollary 1.9 for complex interpolation of Sobolev spaces, to obtain respectively (2.5) and (2.4).

The completeness assumption is necessary in the case of complex interpolation, since one can not provide in general an appropriate sense of holomorphic functions (then of the definition of complex interpolation spaces) in non-complete normed vector spaces. ■

Proposition 2.11 *For any $s\in\mathbb{R}$, $p\in(1,+\infty)$,*

$$\left\{\begin{array}{ccc}\dot{\mathrm{H}}^{s,p}\times\dot{\mathrm{H}}^{-s,p'} & \longrightarrow & \mathbb{C}\\ (u,v) & \longmapsto & \displaystyle\sum_{|j-j'|\leq 1}\big\langle\dot{\Delta}_j u,\dot{\Delta}_{j'}v\big\rangle_{\mathbb{R}^n}\end{array}\right.$$

defines a continuous bilinear functional on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\times\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$. Denote by $\mathcal{V}^{-s,p'}$ the set of functions $v\in\mathcal{S}(\mathbb{R}^n)\cap\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$ such that $\|v\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)}\leqslant 1$. If $u\in\mathcal{S}'_h(\mathbb{R}^n)$, then we have

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}=\sup_{v\in\mathcal{V}^{-s,p'}}\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big|.$$

Moreover, if $(\mathcal{C}_{s,p})$ is satisfied, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is reflexive and we have

$$(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'=\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n).$$

Proof. — For simplicity, we will first work with the norm provided by the Lemma 2.7, by equivalence of norms, the result will remain true. Let $(u,v)\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\times\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$, the $\mathrm{L}^p(\ell^2)$-$\mathrm{L}^{p'}(\ell^2)$ Hölder's inequality gives,

$$\begin{aligned}\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big|&\leqslant\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}\Big\|\big\|(2^{-js}[\dot{\Delta}_{j-1}+\dot{\Delta}_j+\dot{\Delta}_{j+1}]v)_{j\in\mathbb{Z}}\big\|_{\ell^2(\mathbb{Z})}\Big\|_{\mathrm{L}^{p'}(\mathbb{R}^n)}\\&\leqslant(2^{|s|+1}+1)\,\|u\|_{\dot{\mathrm{F}}^s_{p,2}(\mathbb{R}^n)}\,\|v\|_{\dot{\mathrm{F}}^{-s}_{p',2}(\mathbb{R}^n)}.\end{aligned}$$

Now, we know that it is a well-defined quantity, we can compute

$$\begin{aligned}\big\langle u,v\big\rangle_{\mathbb{R}^n}&=\sum_{|j-j'|\leq 1}\big\langle\dot{\Delta}_j u,\dot{\Delta}_{j'}v\big\rangle_{\mathbb{R}^n}\\&=\sum_{|j-j'|\leqslant 1}\big\langle(-\Delta)^{\frac{s}{2}}\dot{\Delta}_j u,(-\Delta)^{-\frac{s}{2}}\dot{\Delta}_{j'}v\big\rangle_{\mathbb{R}^n}\\&=\big\langle(-\Delta)^{\frac{s}{2}}u,(-\Delta)^{-\frac{s}{2}}v\big\rangle_{\mathbb{R}^n}.\end{aligned}$$

Hence, Hölder's inequality gives

$$\big|\big\langle u,v\big\rangle_{\mathbb{R}^n}\big|\leqslant\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\|v\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)},$$

which can be turned effortless into

$$\sup_{v\in\mathcal{V}^{-s,p'}} \left|\left\langle u,v\right\rangle_{\mathbb{R}^n}\right| \leqslant \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\,.$$

This also proves the continuous embedding $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\hookrightarrow(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$. For the reverse inequality, but not the reverse embedding, from $\mathrm{L}^p-\mathrm{L}^{p'}$ duality, by density of $\mathcal{S}_0(\mathbb{R}^n)$, we have

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} = \sup_{\substack{v\in\mathcal{S}_0(\mathbb{R}^n),\\ \|v\|_{\mathrm{L}^{p'}}\leqslant 1}} \left|\left\langle(-\Delta)^{\frac{s}{2}}u,v\right\rangle_{\mathbb{R}^n}\right| = \sup_{\substack{w\in\mathcal{S}_0(\mathbb{R}^n),\\ \|w\|_{\dot{\mathrm{H}}^{-s,p'}}\leqslant 1}} \left|\left\langle u,w\right\rangle_{\mathbb{R}^n}\right| \leqslant \sup_{v\in\mathcal{V}^{-s,p'}} \left|\left\langle u,v\right\rangle_{\mathbb{R}^n}\right|.$$

In particular, the embedding $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\hookrightarrow(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$ always holds and is isometric.

Now, assume that $(\mathcal{C}_{s,p})$ holds. We recall that Remark 2.6 yields the reflexivity of $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. Let $\tilde{U}\in(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$, we have

$$\left|\left\langle\tilde{U},(-\Delta)^{\frac{s}{2}}v\right\rangle\right| \leqslant \|\tilde{U}\|_{(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'}\|v\|_{\mathrm{L}^{p'}(\mathbb{R}^n)},\quad v\in\mathcal{S}_0(\mathbb{R}^n).$$

Since the space $\mathcal{S}_0(\mathbb{R}^n)$ is dense in $\mathrm{L}^{p'}(\mathbb{R}^n)$, we deduce there exists a unique function $w\in\mathrm{L}^p(\mathbb{R}^n)$ such that,

$$\left\langle\tilde{U},v\right\rangle = \left\langle w,(-\Delta)^{-\frac{s}{2}}v\right\rangle_{\mathbb{R}^n},\quad v\in\mathcal{S}(\mathbb{R}^n).$$

Thus $u:=(-\Delta)^{-\frac{s}{2}}w\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ by Corollary 2.5, and yields that $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)\hookrightarrow(\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n))'$ is surjective. ■

Proposition 2.12 *For any $s\in\mathbb{R}$, $p\in(1,+\infty)$, $q\in[1,+\infty]$,*

$$\begin{cases}\dot{\mathrm{B}}^s_{p,q}\times\dot{\mathrm{B}}^{-s}_{p',q'} & \longrightarrow\mathbb{C}\\ (u,v) & \longmapsto\displaystyle\sum_{|j-j'|\leq 1}\left\langle\dot{\Delta}_j u,\dot{\Delta}_{j'}v\right\rangle_{\mathbb{R}^n}\end{cases}$$

defines a continuous bilinear functional on $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)\times\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$. Denote by $\mathcal{Q}^{-s}_{p',q'}$ the set of functions $v\in\mathcal{S}(\mathbb{R}^n)\cap\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ such that $\|v\|_{\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)}\leqslant 1$. If $u\in\mathcal{S}'_h(\mathbb{R}^n)$, then we have

$$\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}\lesssim_{p,s,n}\sup_{v\in\mathcal{Q}^{-s}_{p',q'}}\left|\left\langle u,v\right\rangle_{\mathbb{R}^n}\right|.$$

Moreover, if $-n/p'<s<n/p$ is satisfied and $q\in(1,+\infty]$ then

$$(\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n))'=\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)\ \text{and}\ (\dot{\mathcal{B}}^{-s}_{p',\infty}(\mathbb{R}^n))'=\dot{\mathrm{B}}^s_{p,1}(\mathbb{R}^n).$$

The space $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ is reflexive whenever both $(\mathcal{C}_{s,p,q})$ and $q\neq 1,+\infty$ are satisfied.

Proof. — The first part of the claim is just [BCD11, Proposition 2.29]. The claimed part about reflexivity and duality follows directly from the application of Theorem 1.6 and of Theorem 2.10 and Proposition 2.11. ■

We recall that Besov spaces satisfy usual Sobolev-Lebesgue spaces embeddings, say,

Proposition 2.13 ([BCD11, Proposition 2.39]) *Let $p,q\in[1,+\infty]$, $s\in(0,n)$, such that*

$$\frac{1}{q}=\frac{1}{p}-\frac{s}{n}.$$

The following estimates hold

$$\begin{aligned}
\|u\|_{\mathrm{L}^q(\mathbb{R}^n)} &\lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n)},\ \forall u \in \dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n),\ r \in [1,q],\\
\|u\|_{\dot{\mathrm{B}}^{-s}_{q,r}(\mathbb{R}^n)} &\lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)},\ \forall u \in \mathrm{L}^p(\mathbb{R}^n),\ r \in [q,+\infty],\\
\|u\|_{\mathrm{L}^p(\mathbb{R}^n)} &\lesssim_{n,s,p} \|u\|_{\dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n)},\ \forall u \in \dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n),\ r \in [1,\min(2,p)],\\
\|u\|_{\dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n)} &\lesssim_{n,s,p} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)},\ \forall u \in \mathrm{L}^p(\mathbb{R}^n),\ r \in [\max(2,p),+\infty].
\end{aligned}$$

Moreover, if p is finite, we also have $\dot{\mathrm{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_0(\mathbb{R}^n)$ and, each embedding is dense whenever p, q and r are finite.

We also have a Sobolev-Besov multiplier result, which is useful for the construction of homogeneous Sobolev and Besov space on domains. The first presentation of this result in the inhomogeneous setting is due to Strichartz [Str67, Chapter II, Corollary 3.7], one may also check [JK95, Proposition 3.5].

Proposition 2.14 *For all $p \in (1,+\infty)$, for all $s \in [0,\frac{1}{p})$, for all $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, we have $\mathbb{1}_{\mathbb{R}^n_+} u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$ with estimate*

$$\|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}.$$

We are going to use it to prove a straightforward generalization. The next result was known but only stated for homogeneous Besov spaces up to now, see e.g. [DM09, Appendix].

Proposition 2.15 *For all $p \in (1,+\infty)$, $q \in [1,+\infty]$, for all $s \in (-1+\frac{1}{p},\frac{1}{p})$, for all $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ (resp. $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$),*

$$\|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \quad (\text{resp. } \|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}\ ).$$

The same results still hold with (H,B) instead of $(\dot{\mathrm{H}},\dot{\mathrm{B}})$.

Proof. — We start from the result stated in the inhomogeneous case Proposition 2.14, which states the following in the case of the upper half-space, for all $p \in (1,+\infty)$, for all $s \in [0,\frac{1}{p})$, for all $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$

$$\|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)},$$

If $s = 0$, there is nothing to achieve since $\mathrm{H}^{0,p}(\mathbb{R}^n) = \dot{\mathrm{H}}^{0,p}(\mathbb{R}^n) = \mathrm{L}^p(\mathbb{R}^n)$ with equality of norms. Now for $s > 0$, by the equivalence of norms, we obtain

$$\|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\mathrm{L}^p(\mathbb{R}^n)} + \|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)} + \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Plugging $u_\lambda := u(\lambda\cdot)$ in the above inequality, provided λ is a positive real number, since one has $\mathbb{1}_{\mathbb{R}^n_+}(\lambda\cdot)u_\lambda = \mathbb{1}_{\mathbb{R}^n_+} u_\lambda$, we obtain that

$$\lambda^{-\frac{n}{p}} \|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\mathrm{L}^p(\mathbb{R}^n)} + \lambda^{s-\frac{n}{p}} \|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \lambda^{-\frac{n}{p}} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)} + \lambda^{s-\frac{n}{p}} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Thus one may divide by $\lambda^{s-\frac{n}{p}}$, and then as λ tends to infinity, we deduce

$$\|\mathbb{1}_{\mathbb{R}^n_+} u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Therefore, the result follows by density argument.

The result for $s \in (-1+\frac{1}{p},0)$ is a consequence of duality and density using the duality bracket defined on $\mathcal{S}_0(\mathbb{R}^n) \times \mathcal{S}_0(\mathbb{R}^n)$.

The Besov space case follows by real interpolation. ∎

3 Function spaces on the upper half-space

3.1 Function spaces on domains by restriction

Let $s \in \mathbb{R}$, $p \in (1, +\infty)$, $q \in [1, +\infty]$ and Ω an open set of $\mathbb{R}^n$. For any $\mathrm{X} \in \{\mathrm{B}^s_{p,q}, \dot{\mathrm{B}}^s_{p,q}, \mathrm{H}^{s,p}, \dot{\mathrm{H}}^{s,p}\}$, we define

$$\mathrm{X}(\Omega) := \mathrm{X}(\mathbb{R}^n)_{|\Omega},$$

with the quotient norm $\|u\|_{\mathrm{X}(\Omega)} := \inf_{\substack{\tilde{u} \in \mathrm{X}(\mathbb{R}^n), \\ \tilde{u}_{|\Omega} = u}} \|\tilde{u}\|_{\mathrm{X}(\mathbb{R}^n)}$. A direct consequence from the definition of those spaces is the density of $\mathcal{S}_0(\overline{\Omega}) \subset \mathcal{S}(\overline{\Omega})$ in each of them, when $\mathcal{S}_0(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$ is dense in their counterpart on the whole space. The completeness and reflexivity are also carried over when their counterpart on $\mathbb{R}^n$ are respectively complete and reflexive. We can also define

$$\mathrm{X}_0(\Omega) := \left\{ u \in \mathrm{X}(\mathbb{R}^n) \,\middle|\, \operatorname{supp} u \subset \overline{\Omega} \right\},$$

with its natural induced norm $\|u\|_{\mathrm{X}_0(\Omega)} := \|u\|_{\mathrm{X}(\mathbb{R}^n)}$. We always have the canonical continuous injection,

$$\mathrm{X}_0(\Omega) \hookrightarrow \mathrm{X}(\Omega).$$

Since there is a natural embedding $\mathcal{S}'(\mathbb{R}^n) \hookrightarrow \mathcal{D}'(\Omega)$, we also have the inclusion

$$\mathrm{X}(\Omega) \subset \mathcal{D}'(\Omega),$$

where we recall that $\mathcal{D}'(\Omega) = (\mathrm{C}^\infty_c(\Omega))'$ is the topological vector space of distributions on Ω.

If X and Y are different function spaces

- if one has a continuous embedding,

  $$\mathrm{Y}(\mathbb{R}^n) \hookrightarrow \mathrm{X}(\mathbb{R}^n).$$

  A direct consequence of the definition is that

  $$\mathrm{Y}(\Omega) \hookrightarrow \mathrm{X}(\Omega),$$

  and, similarly with X_0 and Y_0.
- We write $[\mathrm{X} \cap \mathrm{Y}](\Omega)$ the restriction of $\mathrm{X}(\mathbb{R}^n) \cap \mathrm{Y}(\mathbb{R}^n)$ to Ω, in general there is nothing to ensure more than

  $$[\mathrm{X} \cap \mathrm{Y}](\Omega) \hookrightarrow \mathrm{X}(\Omega) \cap \mathrm{Y}(\Omega).$$

The results corresponding to those obtained for the whole space $\mathbb{R}^n$ in the previous section are usually carried over by the existence of an appropriate extension operator

$$\mathcal{E} : \mathcal{S}'(\Omega) \longrightarrow \mathcal{S}'(\mathbb{R}^n),$$

bounded from $\mathrm{X}(\Omega)$ to $\mathrm{X}(\mathbb{R}^n)$.

By mean of Theorem 2.10, as for [DHMT21, Proposition 3.22], the definition of function spaces by restriction yields the next result.

Lemma 2.16 *Let $(p, q, q_0, q_1) \in (1, +\infty) \times [1, +\infty]^3$, $s_0, s_1 \in \mathbb{R}$, such that $s_0 < s_1$, and set*

$$s := (1 - \theta)s_0 + \theta s_1.$$

If $(\mathcal{C}_{s_0,p})$ *is satisfied, we have continuous embeddings*

$$\dot{\mathrm{B}}^s_{p,q}(\Omega) \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}(\Omega), \dot{\mathrm{H}}^{s_1,p}(\Omega))_{\theta,q}, \tag{2.13}$$
$$\dot{\mathrm{B}}^s_{p,q,0}(\Omega) \hookleftarrow (\dot{\mathrm{H}}^{s_0,p}_0(\Omega), \dot{\mathrm{H}}^{s_1,p}_0(\Omega))_{\theta,q}, \tag{2.14}$$

Similarly if $(\mathcal{C}_{s_0,p,q_0})$ *is satisfied, we also have*

$$\dot{\mathrm{B}}^s_{p,q}(\Omega) \hookrightarrow (\dot{\mathrm{B}}^{s_0}_{p,q_0}(\Omega), \dot{\mathrm{B}}^{s_1}_{p,q_1}(\Omega))_{\theta,q}, \tag{2.15}$$
$$\dot{\mathrm{B}}^s_{p,q,0}(\Omega) \hookleftarrow (\dot{\mathrm{B}}^{s_0}_{p,q_0,0}(\Omega), \dot{\mathrm{B}}^{s_1}_{p,q_1,0}(\Omega))_{\theta,q}. \tag{2.16}$$

Proof. — For embeddings (2.13) and (2.15), one may follow the first part of the proof of [DHMT21, Proposition 3.22].

The third embedding (2.14), (the fourth one (2.16) can be treated similarly) is straightforward since,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\Omega), \dot{\mathrm{H}}^{s_1,p}_0(\Omega))_{\theta,q} \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))_{\theta,q} = \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n).$$

By definition, $f \in (\dot{\mathrm{H}}^{s_0,p}_0(\Omega), \dot{\mathrm{H}}^{s_1,p}_0(\Omega))_{\theta,q} \subset \dot{\mathrm{H}}^{s_0,p}_0(\Omega) + \dot{\mathrm{H}}^{s_1,p}_0(\Omega)$, hence $\operatorname{supp} f \subset \overline{\Omega}$ and we deduce $f \in \dot{\mathrm{B}}^s_{p,q,0}(\Omega)$. ∎

In this Chapter, we are particularly concerned by the case $\Omega = \mathbb{R}^n_+$.

3.2 Quick overview of inhomogeneous function spaces on (bent) half-spaces

For a suitable extension operator in the case of inhomogeneous function spaces on a (special) Lipschitz domain (in particular the upper half-space $\mathbb{R}^n_+$), a notable approach was achieved by Stein in [Ste70, Chapter VI, Section 3], for Sobolev spaces with non-negative index, and Besov spaces of positive index of regularity (this follows by real interpolation). A full and definitive result for the inhomogeneous case on Lipschitz domains, and even in a more general case (allowing p, q to be less than 1 considering the whole Besov and Triebel-Lizorkin scales), was given by Rychkov in [Ryc99]. In this paper the extension operator Rychkov provided is known to be universal and to cover even negative regularity index.

The extension operator provided by Rychkov can be used to prove, thanks to [BL76, Theorem 6.4.2], if $(\mathfrak{h}, \mathfrak{b}) \in \{(\mathrm{H}, \mathrm{B}), (\mathrm{H}_0, \mathrm{B}_{\cdot,\cdot,0})\}$,

$$[\mathfrak{h}^{s_0,p_0}(\Omega), \mathfrak{h}^{s_1,p_1}(\Omega)]_\theta = \mathfrak{h}^{s,p_\theta}(\Omega), \qquad (\mathfrak{b}^{s_0}_{p,q_0}(\Omega), \mathfrak{b}^{s_1}_{p,q_1}(\Omega))_{\theta,q} = \mathfrak{b}^s_{p,q}(\Omega),$$
$$(\mathfrak{h}^{s_0,p}(\Omega), \mathfrak{h}^{s_1,p}(\Omega))_{\theta,q} = \mathfrak{b}^s_{p,q}(\Omega), \qquad [\mathfrak{b}^{s_0}_{p_0,q_0}(\Omega), \mathfrak{b}^{s_1}_{p_1,q_1}(\Omega)]_\theta = \mathfrak{b}^s_{p_\theta,q_\theta}(\Omega),$$

whenever $(p_0, q_0), (p_1, q_1), (p, q) \in [1, +\infty]^2 (p \neq 1, +\infty$, when dealing with Sobolev (Bessel potential) spaces), $\theta \in (0, 1)$, $s_0 \neq s_1$ two real numbers, such that

$$\left(s, \frac{1}{p_\theta}, \frac{1}{q_\theta}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}, \frac{1}{q_0}\right) + \theta\left(s_1, \frac{1}{p_1}, \frac{1}{q_1}\right).$$

A nice property is that the local description of the boundary yields the following density results, for all $p \in (1, +\infty)$, $q \in [1, +\infty)$, $s \in \mathbb{R}$,

$$\mathrm{H}^{s,p}_0(\Omega) = \overline{\mathrm{C}^\infty_c(\Omega)}^{\|\cdot\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}}, \quad \text{and} \quad \mathrm{B}^s_{p,q,0}(\Omega) = \overline{\mathrm{C}^\infty_c(\Omega)}^{\|\cdot\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)}}.$$

One may check [JK95, Section 2] for the treatment of Sobolev spaces case. The Besov spaces case follows by interpolation argument, see [BL76, Theorem 3.4.2]. As a direct consequence, one has from [JK95, Proposition 2.9] and [BL76, Theorem 3.7.1], that for all $s \in \mathbb{R}$, $p \in (1, +\infty)$,

$q \in [1, +\infty)$,

$$(\mathrm{H}^{s,p}(\Omega))' = \mathrm{H}_0^{-s,p'}(\Omega),\ (\mathrm{B}^s_{p,q}(\Omega))' = \mathrm{B}^{-s}_{p',q',0}(\Omega),$$
$$(\mathrm{B}^s_{p,q,0}(\Omega))' = \mathrm{B}^{-s}_{p',q'}(\Omega).$$

And finally, thanks to a modified version of Proposition 2.15, we also have a particular case of equality of Sobolev spaces, with equivalent norms, for all $p \in (1, +\infty)$, $q \in [1, +\infty]$, $s \in (-1+\frac{1}{p}, \frac{1}{p})$,

$$\mathrm{H}^{s,p}(\Omega) = \mathrm{H}^{s,p}_0(\Omega),\ \mathrm{B}^s_{p,q}(\Omega) = \mathrm{B}^s_{p,q,0}(\Omega).$$

The interested reader may also found an explicit and way more general (and still valid, for the most part of it, in the case of a special Lipschitz domain, and in particular for the upper half-space) treatment for bounded Lipschitz domains in the work of Kalton, Mayboroda and Mitrea [KMM07], where the Triebel-Lizorkin scale, including Hardy spaces, and other endpoint function spaces are also treated.

We also mention that a more recent elementary, geometric and accessible exposition is available in the book of Leoni [Leo23, Chapters 8 & 11]. It deals with inhomogeneous Sobolev-Slobodeckij spaces $\mathrm{W}^{s,p}(\Omega)$, which coincides with usual Sobolev spaces when $s \in \mathbb{Z}$, and with diagonal Besov spaces $\mathrm{B}^s_{p,p}(\Omega)$ when $s \in \mathbb{R} \setminus \mathbb{Z}$. The case of indices $s \in [0, 1]$ is treated in the case of Lipschitz domains, and $s \in [0, m+1]$ in the case where Ω is a $\mathrm{C}^{m,1}$ domain.

All the results presented above will be used without being mentioned and are assumed to be well known to the reader.

3.3 Homogeneous function spaces on $\mathbb{R}^n_+$

One may expect to recover similar results for the scale of homogeneous Sobolev and Besov as the one mentioned in the subsection 3.2. However, due to the setting involving the use of $\mathcal{S}'_h(\mathbb{R}^n)$, we have a lack of completeness so that one can no longer use complex interpolation theory and density argument on the whole scale to provide boundedness of linear operators. A first approach we could have in mind is that one would expect Rychkov's extension operator to preserve $\mathcal{S}'_h$, say $\mathcal{E}(\mathcal{S}'_h(\mathbb{R}^n_+)) \subset \mathcal{S}'_h(\mathbb{R}^n)$ with ***homogeneous*** estimates, which is not known yet.

However, if we consider a more naive extension operator like by reflection around the boundary, as in [DHMT21, Chapter 3], a certain amount of results remains true, up to consider index $s > -1 + \frac{1}{p}$, provided $p \in (1, +\infty)$. This is what we are going to achieve here : this subsection is devoted to proofs of usual results on homogeneous Sobolev and Besov spaces on $\mathbb{R}^n_+$. To be more clear, we are going to show via the previously mentioned extension-restriction operators, few duality arguments, and interpolation theory, that we still have :

- **Expected density results :** Propositions 2.25, 2.26, Lemma 2.32 and Corollaries 2.29 and 2.34.
  For $p \in (1, +\infty)$, $q \in [1, +\infty)$, $s > -1 + \frac{1}{p}$, when $(\mathcal{C}_{s,p,q})$ is satisfied,

  $$\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+) = \overline{\mathrm{C}^\infty_c(\mathbb{R}^n_+)}^{\|\cdot\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}},\quad \text{and}\quad \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+) = \overline{\mathrm{C}^\infty_c(\mathbb{R}^n_+)}^{\|\cdot\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}};$$

  We need to make clear now that this is not a definition but a **consequence** from the definition written at the beginning of Section 3.
- **Expected duality results :** Propositions 2.28 and 2.40.
  For all $p \in (1, +\infty)$, $q \in (1, +\infty]$, $s > -1 + \frac{1}{p}$, when $(\mathcal{C}_{s,p,q})$ is satisfied,

  $$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+),\ (\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))' = \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+),$$
  $$(\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+),\ (\dot{\mathrm{B}}^{-s}_{p',q',0}(\mathbb{R}^n_+))' = \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+).$$
- **Expected interpolation results :** Propositions 2.39 and 2.33.

If $(\dot{\mathfrak{h}},\dot{\mathfrak{b}}) \in \{(\dot{\mathrm{H}},\dot{\mathrm{B}}),(\dot{\mathrm{H}}_0,\dot{\mathrm{B}}_{\cdot,\cdot,0})\}$, with $(p_0,q_0),(p_1,q_1),(p,q) \in (1,+\infty)\times[1,+\infty]$, $\theta\in(0,1)$, $s_j, s > -1+1/p_j$, $j\in\{0,1\}$, with $s>-1+1/p$, where s_0,s_1,s are three real numbers, so that one can set

$$\left(s,\frac{1}{p_\theta},\frac{1}{q_\theta}\right) := (1-\theta)\left(s_0,\frac{1}{p_0},\frac{1}{q_0}\right)+\theta\left(s_1,\frac{1}{p_1},\frac{1}{q_1}\right),$$

such that $(\mathcal{C}_{s,p,q})$ is satisfied. Then, one has

$$[\dot{\mathfrak{h}}^{s_0,p_0}(\mathbb{R}^n_+),\dot{\mathfrak{h}}^{s_1,p_1}(\mathbb{R}^n_+)]_\theta = \dot{\mathfrak{h}}^{s,p_\theta}(\mathbb{R}^n_+), \qquad (\dot{\mathfrak{b}}^{s_0}_{p,q_0}(\mathbb{R}^n_+),\dot{\mathfrak{b}}^{s_1}_{p,q_1}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathfrak{b}}^{s}_{p,q}(\mathbb{R}^n_+),$$
$$(\dot{\mathfrak{h}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathfrak{h}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathfrak{b}}^{s}_{p,q}(\mathbb{R}^n_+), \qquad [\dot{\mathfrak{b}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+),\dot{\mathfrak{b}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)]_\theta = \dot{\mathfrak{b}}^{s}_{p_\theta,q_\theta}(\mathbb{R}^n_+).$$

Note that, due to Proposition 2.15, we have already checked that following equalities of homogeneous Sobolev and Besov spaces remains true, with equivalent norms, for all $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in(-1+\frac{1}{p},\frac{1}{p})$,

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) = \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+),\ \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+) = \dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+).$$

Some already existing density and boundedness results in Besov spaces presented here are already known, but redone here differently giving some minor improvements with regard to [DHMT21, Chapter 3], allowing sometimes to deal sometimes with $s>-1+\frac{1}{p}$ or $q=+\infty$. Some other results, despite being well known in the construction of usual Sobolev and Besov spaces, are quite new due to the ambient framework. This leads to some new proofs in a different spirit than the ones already available in the literature.

This subsection contains 3 subparts : the first one is about extension-restriction and density results for our homogeneous Sobolev spaces, from which for the second, we are going to build corresponding ones for Besov spaces, via some ersatz of real interpolation procedure. Both will be used to build the third subpart, which concerns effective interpolation results for our homogeneous Sobolev and Besov spaces.

We start proving the boundedness of extension operators defined by higher order reflection principle but for homogeneous Sobolev spaces with fractional index of regularity. This is done as in [DHMT21, Lemma 3.15, Proposition 3.19] for homogeneous Besov spaces only.

Proposition 2.17 *For $m\in\mathbb{N}$, there exists a linear extension operator E, depending on m, such that for all $p\in(1,+\infty)$, $-1+\frac{1}{p}<s<m+1+\frac{1}{p}$, so that if either,*

- *$s\geqslant 0$ and $u\in\mathrm{H}^{s,p}(\mathbb{R}^n_+)$;*
- *$s\in(-1+\frac{1}{p},\frac{1}{p})$ and $u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$;*

we have

$$\mathrm{E}u_{|\mathbb{R}^n_+} = u,$$

with the estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,m} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

In particular, $\mathrm{E}:\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)\longrightarrow\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ extends uniquely to a bounded operator whenever $(\mathcal{C}_{s,p})$ is satisfied.

Proof. — As in [DHMT21, Lemma 3.15], let us introduce the higher order reflection operator E, defined for all measurable function $u:\mathbb{R}^n_+\longrightarrow\mathbb{C}$ by

$$\mathrm{E}u(x) := \begin{cases} u(x) & \text{, if } x\in\mathbb{R}^n_+, \\ \sum_{j=0}^m \alpha_j u(x',-\frac{x_n}{j+1}) & \text{, if } x\in\mathbb{R}^n\setminus\mathbb{R}^n_+. \end{cases}$$

where, as in [DHMT21, Lemma 3.15], $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, and $(\alpha_j)_{j \in [\![0,m]\!]}$ is such that E maps C^m-functions on $\mathbb{R}^n_+$ to C^m-functions on $\mathbb{R}^n$. This is indeed true since α_j, $j \in [\![0,m]\!]$, is chosen so that it satisfies for all $\kappa \in [\![0,m]\!]$,

$$\sum_{j=0}^{m} \left(\frac{-1}{j+1}\right)^{\kappa} \alpha_j = 1.$$

By construction, the operator E also maps boundedly $\mathrm{H}^{k,p}(\mathbb{R}^n_+)$ to $\mathrm{H}^{k,p}(\mathbb{R}^n)$ for all $k \in [\![0,m+1]\!]$. The boundedness of the operator E from $\mathrm{H}^{s,p}(\mathbb{R}^n_+)$ to $\mathrm{H}^{s,p}(\mathbb{R}^n)$ for all $s \in [0,m+1]$ follows from complex interpolation.

Notice also that Proposition 2.15 and the formulation, given for $x \in \mathbb{R}^n$,

$$\mathrm{E}u(x) = [\mathbb{1}_{\mathbb{R}^n_+} u](x) + \sum_{j=0}^{m} \alpha_j [\mathbb{1}_{\mathbb{R}^n_+} u](x', -\tfrac{x_n}{j+1})$$

implies that $\mathrm{E} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is bounded for all $s \in (-1+\frac{1}{p}, \frac{1}{p})$.

Now for $p \in (1,+\infty)$, $s \in [0, m+1+\frac{1}{p})$, $s - \frac{1}{p} \notin \mathbb{N}$, $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$, $\mathrm{E} : \mathrm{H}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, we can choose $\ell \in \mathbb{N}$ such that $s - \ell \in (-1+\frac{1}{p}, \frac{1}{p})$ so that

$$\begin{aligned} \partial^{\ell}_{x_k} \mathrm{E}u &= \mathrm{E}[\partial^{\ell}_{x_\ell} u], \text{ provided } k \in [\![1, n-1]\!], \\ \partial^{\ell}_{x_n} \mathrm{E}u &= \mathrm{E}^{(\ell)} \partial^{\ell}_{x_n} u = \sum_{j=0}^{m} \alpha_j \left(\tfrac{-1}{j+1}\right)^{\ell} \partial^{\ell}_{x_n} u(x', -\tfrac{x_n}{j+1}). \end{aligned}$$

For the same reasons as in the beginning of the present proof, $\mathrm{E}^{(\ell)}$ maps $\mathrm{H}^{s,p}(\mathbb{R}^n_+)$ to $\mathrm{H}^{s,p}(\mathbb{R}^n)$ for all $s \in [0, m-\ell+1]$, and $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ to $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ for $s \in (-1+1/p, 1/p)$, thanks to Proposition 2.15.

From the fact that $\partial^{\ell}_{x_j} u \in \dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n_+)$, we deduce

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \sim_{\ell,p,n} \sum_{j=1}^{n-1} \|\partial^{\ell}_{x_j} \mathrm{E}u\|_{\dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)} + \|\mathrm{E}^{(\ell)} \partial^{\ell}_{x_n} u\|_{\dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)} \lesssim_{s,\ell,p,n,m} \sum_{j=1}^{n} \|\partial^{\ell}_{x_j} u\|_{\dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)}. \tag{2.17}$$

To be more synthetic, we have obtained

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,k,n,m} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)},$$

so that $\mathrm{E} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is bounded on the subspace $\mathrm{H}^{s,p}(\mathbb{R}^n_+)$, in particular it extends uniquely to a bounded linear operator on whole $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ when it is complete, i.e. $s < \frac{n}{p}$, this follows from the fact that $\mathcal{S}(\overline{\mathbb{R}^n_+}) \subset \mathrm{H}^{s,p}(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$.

It remains to cover cases when $s - \frac{1}{p} \in [\![0,m]\!]$. To do so, we want to reproduce the above procedure, proving first that E (*resp.* $\mathrm{E}^{(\ell)}$, $\ell \in [\![1,m]\!]$) is bounded from $\dot{\mathrm{H}}^{\frac{1}{p},p}(\mathbb{R}^n_+)$ to $\dot{\mathrm{H}}^{\frac{1}{p},p}(\mathbb{R}^n)$, via some complex interpolation scheme.

Now let $p_0, p_1 \in (1,+\infty)$, $p_1 < n$, $\theta \in (0,1)$. Consider $u \in [\mathrm{L}^{p_0}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta$. Let $f \in F(\mathrm{L}^{p_0}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+))$, such that $f(\theta) = u$, it follows from the above considerations that $\mathrm{E}f \in F(\mathrm{L}^{p_0}(\mathbb{R}^n), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n))$. Thus, from Theorem 2.10, one has

$$\mathrm{E}f(\theta) \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n), \text{ where } \left(\theta, \frac{1}{p}\right) := (1-\theta)\left(0, \frac{1}{p_0}\right) + \theta\left(1, \frac{1}{p_1}\right).$$

So $u = \mathrm{E}f(\theta)_{|\mathbb{R}^n_+} \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)$ with the norm estimate

$$\|u\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)} \lesssim_{m_1,p,n} \|u\|_{[\mathrm{L}^{p_0}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta}$$

which is a direct consequence of the definition of restriction space, the equivalence of the complex interpolation norm (2.4) from Theorem 2.10, the definition of the complex interpolation norm, and then of the boundedness of E from $\mathrm{L}^{p_0}(\mathbb{R}^n)$ to $\mathrm{L}^{p_0}(\mathbb{R}^n_+)$ and from $\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n)$ to $\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)$. Now, if $u \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)$, by definition of restriction spaces there exists $U \in \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)$, such that

$$U_{|\mathbb{R}^n_+} = u, \quad \text{and} \quad \frac{1}{2}\|U\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)} \leqslant \|u\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)} \leqslant \|U\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)}.$$

By Theorem 2.10, there exists $f \in F(\mathrm{L}^{p_0}(\mathbb{R}^n), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n))$ such that $f(\theta) = U$, we deduce $f(\cdot)_{|\mathbb{R}^n_+} \in F(\mathrm{L}^{p_0}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+))$, so $u = f(\theta)_{|\mathbb{R}^n_+} \in [\mathrm{L}^{p_0}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta$ with the following estimate which is a direct consequence of the definition of function spaces by restriction, and complex interpolation spaces,

$$\|u\|_{[\mathrm{L}^{p_0}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{1,p_1}(\mathbb{R}^n_+)]_\theta} \lesssim \|u\|_{\dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+)}.$$

Hence, homogeneous (Riesz potential) Sobolev spaces on the half-space are still a complex interpolation scale provided that $p \in (1,+\infty)$, $s \in [0,1]$, $(\mathcal{C}_{s,p})$ being satisfied, so the boundedness of $\mathrm{E} : \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n_+) \to \dot{\mathrm{H}}^{\theta,p}(\mathbb{R}^n)$ follows by interpolation.

In particular, $\mathrm{E} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is bounded for all $s \in (-1+\frac{1}{p}, \frac{1}{p}]$. Hence, the result has been proved for $s - \frac{1}{p} = 0$. The same result is obtained for $\mathrm{E}^{(\ell)}$, provided $\ell \in [\![1,m]\!]$.

Now, let $p \in (1,+\infty)$, $s - \frac{1}{p} \in [\![1,m]\!]$, for $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$, we have $\mathrm{E}u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, $\nabla^\ell \mathrm{E}u \in \dot{\mathrm{H}}^{s-\ell,p}(\mathbb{R}^n)$, $s - \ell = \frac{1}{p}$, so that, similarly as in (2.17),

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\ell} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

Therefore, we have obtained the desired estimate and can conclude about the boundedness of E via density argument whenever $(\mathcal{C}_{s,p})$ is satisfied. ■

In the proof of Proposition 2.17, we used boundedness of derivatives, *i.e.* for all $p \in (1,+\infty)$, $s \in \mathbb{R}$, $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, $m \in \mathbb{N}$,

$$\|\nabla^m u\|_{\dot{\mathrm{H}}^{s-m,p}(\mathbb{R}^n_+)} \lesssim_{p,s,n,m} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}. \tag{2.18}$$

The above estimate is a direct consequence of definition of function spaces by restriction and can be turned into an equivalence under some additional assumptions.

Proposition 2.18 *Let $p \in (1,+\infty)$, $k \in [\![1,+\infty[\![$, $s > k - 1 + \frac{1}{p}$, for all $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$,*

$$\sum_{j=1}^n \|\partial_{x_j}^k u\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)} \sim_{s,k,p,n} \|\nabla^k u\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)} \sim_{s,k,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

In particular, $\|\nabla^k \cdot\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)}$ and $\sum_{j=1}^n \|\partial_{x_j}^k \cdot\|_{\dot{\mathrm{H}}^{s-k,p}(\mathbb{R}^n_+)}$ provide equivalent norms on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, whenever $(\mathcal{C}_{s-k,p})$ is satisfied.

Proof. — Let us prove it, for $k = 1$, the higher order case can be achieved similarly. Consider $p \in (1,+\infty)$, $s > \frac{1}{p}$, for $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$, we have $\mathrm{E}u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, where E is an extension operator provided by Proposition 2.17 (for some big enough $m \geqslant 1$), $\nabla \mathrm{E}u \in \dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)$, with $s-1 > -1+\frac{1}{p}$.

We can write on $\overline{\mathbb{R}^n_+}^c$

$$\partial_{x_\ell}\mathrm{E}u = \mathrm{E}[\partial_{x_\ell}u], \text{ provided } \ell \in [\![1, n-1]\!], \text{ and } \partial_{x_n}\mathrm{E}u = \sum_{j=0}^{m} \alpha_j \left(\tfrac{-1}{j+1}\right) \partial_{x_n} u(x', -\tfrac{x_n}{j+1}).$$

Hence, we can use definition of restriction space, apply Proposition 2.1, and boundedness of E, since m is large enough, to obtain,

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \leqslant \|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|\nabla \mathrm{E}u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)} \lesssim_{s,p,n,m} \|\nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n_+)}.$$

Therefore by (2.18), the equivalence of norms on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ holds by density when $(\mathcal{C}_{s-k,p})$ is true. ∎

The next proposition is about identifying intersection of homogeneous Sobolev spaces on $\mathbb{R}^n_+$, and give a dense subspace. As we can see later, this will help for real interpolation.

Proposition 2.19 *Let $p_j \in (1, +\infty)$, $s_j > -1 + \frac{1}{p_j}$, $j \in \{0, 1\}$, if $(\mathcal{C}_{s_0,p_0})$ is satisfied then the following equality of vector spaces holds with equivalence of norms*

$$\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+) = [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+).$$

In particular, $\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$ is a Banach space which admits $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ as a dense subspace.

Proof. — Let $p \in (1, +\infty)$, $s_0, s_1 \in \mathbb{R}$, such that $(\mathcal{C}_{s_0,p_0})$. By definition of restriction spaces and Lemma 2.9, $[\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$ is complete and admits $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ as a dense subspace. The following continuous embedding also holds by definition,

$$[\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+).$$

Hence, it suffices to prove the reverse one. To do so, let us choose $\ell \in \mathbb{N}$ such that $(\mathcal{C}_{s_1-\ell,p_1})$ is satisfied, and $s_1 - \ell > -1 + \frac{1}{p_1}$, then choosing E from Proposition 2.17 with $m + 1 + \frac{1}{p_j} > s_j$, $j \in \{0, 1\}$ (m big enough), for all $j \in [\![1, n]\!]$, and all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$, E$u$ makes sense in $\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)$, then in $\mathcal{S}'_h(\mathbb{R}^n)$, and one may use an estimate similar to (2.17), to deduce

$$\sum_{k=1}^{n} \|\partial^\ell_{x_k}\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} = \sum_{k=1}^{n-1} \|\mathrm{E}\partial^\ell_{x_k}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} + \|\mathrm{E}^{(\ell)}\partial^\ell_{x_n}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} \lesssim^{p_1,n}_{s_1,m,\ell} \|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)}.$$

The above operator $\mathrm{E}^{(\ell)}$ is given via the identity $\partial^\ell_{x_n}\mathrm{E} = \mathrm{E}^{(\ell)}\partial^\ell_{x_n}$. Hence, it follows that for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$,

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)} + \sum_{k=1}^{n} \|\partial^\ell_{x_k}\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_1-\ell,p_1}(\mathbb{R}^n)} \lesssim^{p_0,p_1,n}_{s_0,s_1,m,\ell} \|u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)}.$$

In particular, since $\mathrm{E}u \in \mathcal{S}'_h(\mathbb{R}^n)$, and by uniqueness of representation of $\partial^\ell_{x_j}\mathrm{E}u$ in $\mathcal{S}'(\mathbb{R}^n)$, we deduce from Proposition 2.1 that $\mathrm{E}u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$.

Thus $u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$, and by definition of restriction spaces,

$$\|u\|_{[\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)} \leqslant \|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)} + \|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)} \lesssim^{p_0,p_1,n}_{s_0,s_1,m,\ell} \|u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)}.$$

This proves the claim. ∎

So one can deduce the following corollary, which allows separate homogeneous estimates for intersection of homogeneous Sobolev spaces on $\mathbb{R}^n_+$. Since the estimates below are decoupled, it provides an ersatz of extension-restriction operators for homogeneous Sobolev spaces of higher

order, thanks to the taken intersection yielding a complete space. For instance, this will be of use to circumvent the lack of completeness when we will want to (real-)interpolate between a "higher" order homogeneous Sobolev space, and one that is known to be complete.

Corollary 2.20 *Let $p_j \in (1,+\infty)$, $s_j > -1+\frac{1}{p_j}$, $j \in \{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied, consider $m \in \mathbb{N}$ such that $s_j < m+1+\frac{1}{p_j}$, and the extension operator* E *given by Proposition 2.17.*

Then for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$, we have $\mathrm{E}u \in \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)$, $j \in \{0,1\}$, with the estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{s_j,p_j,m,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}.$$

Corollary 2.20 and the proof of Proposition 2.18 lead to

Corollary 2.21 *Let $p_j \in (1,+\infty)$, $m_j \in [\![1,+\infty[\![$, $s_j > m_j - 1 + \frac{1}{p_j}$, $j \in \{0,1\}$, such that $(\mathcal{C}_{s_0-m_0,p_0})$ is satisfied. Then for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n_+)$,*

$$\sum_{k=1}^{n} \|\partial_{x_k}^{m_j} u\|_{\dot{\mathrm{H}}^{s_j-m_j,p}(\mathbb{R}^n_+)} \sim_{s_j,m_j,p_j,n} \|\nabla^{m_j} u\|_{\dot{\mathrm{H}}^{s_j-m_j,p_j}(\mathbb{R}^n_+)} \sim_{s_j,m_j,p_j,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}.$$

Since one may also be interested into Sobolev spaces with 0-boundary condition, we introduce a projection operator that allows to deal with the interpolation property, and to recover, later on, some appropriate density results.

Lemma 2.22 *Let $p \in (1,+\infty)$, $s \in \mathbb{R}$, $m \in \mathbb{N}$, such that $-1+\frac{1}{p} < s < m+1+\frac{1}{p}$, then there exists a bounded projection $\mathcal{P}_0$, depending on m, such that it maps $\mathrm{H}^{s,p}(\mathbb{R}^n)$ to $\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)$.*

If either

- *$s \geqslant 0$ and $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$;*
- *$s \in (-1+\frac{1}{p}, \frac{1}{p})$ and $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$;*

we have the estimate

$$\|\mathcal{P}_0 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,m,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

In particular, $\mathcal{P}_0$ extends uniquely to a bounded projection from $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ to $\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$ whenever $(\mathcal{C}_{s,p})$ is satisfied.

Proof. — Let $p \in (1,+\infty)$, $s > -1+\frac{1}{p}$, $m \in \mathbb{N}$, such that $s < m+1+\frac{1}{p}$. Then we consider the operator E given by Proposition 2.17, but we modify it into an operator E^-, for any measurable function $u : \mathbb{R}^n_- \longrightarrow \mathbb{C}$, we set for almost every $x \in \mathbb{R}^n$

$$\mathrm{E}^- u(x) := \begin{cases} u(x) & , \text{if } x \in \mathbb{R}^n_-, \\ \sum_{j=0}^{m} \alpha_j u(x', -\frac{x_n}{j+1}) & , \text{if } x \in \mathbb{R}^n \setminus \mathbb{R}^n_-. \end{cases}$$

Hence for any measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{C}$, we set for almost every $x \in \mathbb{R}^n$,

$$\mathcal{P}_0 u := u - \mathrm{E}^-[\mathbb{1}_{\mathbb{R}^n_-} u].$$

The fact that $\mathcal{P}_0^2 = \mathcal{P}_0$ is clear by definition, and we have $\mathcal{P}_0 \mathrm{H}^{s,p}(\mathbb{R}^n) \subset \mathrm{H}^{s,p}_0(\mathbb{R}^n_+)$, and that $\mathcal{P}_{0|_{\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)}} = \mathrm{I}$. The boundedness properties, as claimed, follow from Proposition 2.15 and Proposition 2.17. ∎

As well as the extension operator given by higher order reflection principle, the projection operator on "0-boundary condition" homogeneous Sobolev spaces satisfies the homogeneous estimates on intersection spaces. The proof is a direct consequence of Proposition 2.19 and its formula introduced in the proof of Lemma 2.22.

Corollary 2.23 *Let $p_j \in (1,+\infty)$, $s_j > -1+\frac{1}{p_j}$, $j \in \{0,1\}$, $m \in \mathbb{N}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied and $s_j < m+1+\frac{1}{p_j}$, and consider the projection operator $\mathcal{P}_0$ given by Lemma 2.22.*

Then for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$, we have $\mathcal{P}_0 u \in \dot{\mathrm{H}}_0^{s_j,p_j}(\mathbb{R}^n_+)$, $j \in \{0,1\}$, with the estimate

$$\|\mathcal{P}_0 u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{s_j,m,p,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)}.$$

We still have Sobolev embeddings by definition of function spaces by restriction.

Proposition 2.24 *Let $p,q \in (1,+\infty)$, $s \in [0,n)$, such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

We have the estimates

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)},\ \forall u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \tag{2.19}$$

$$\|u\|_{\dot{\mathrm{H}}_0^{-s,q}(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)},\ \forall u \in \mathrm{L}^p(\mathbb{R}^n_+), \tag{2.20}$$

for which each underlying embedding is dense.

Proof. — First, let us recall the Hardy-Littlewood-Sobolev inequality from Proposition 2.3, which says that

$$\|u\|_{\dot{\mathrm{H}}^{-s,q}(\mathbb{R}^n)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\mathbb{R}^n)},\ \forall u \in \mathrm{L}^p(\mathbb{R}^n).$$

Hence, the embedding (2.20) is a direct consequence of plugging $\tilde{u}$, the extension to the whole $\mathbb{R}^n$ of $u \in \mathrm{L}^p(\mathbb{R}^n_+)$.

The embedding (2.19) is a direct consequence of Proposition 2.3 and function spaces defined by restriction. Indeed, for $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, we have for any extension $U \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) \subset \mathrm{L}^q(\mathbb{R}^n)$ such that $u = U_{|_{\mathbb{R}^n_+}} \in \mathrm{L}^q(\mathbb{R}^n_+)$ the estimate

$$\|u\|_{\mathrm{L}^q(\mathbb{R}^n_+)} \leqslant \|U\|_{\mathrm{L}^q(\mathbb{R}^n)} \lesssim_{s,p,q,n} \|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Looking at the infimum on all such U gives the result.

The density for the first embedding follows from the fact that $\mathcal{S}_0(\overline{\mathbb{R}^n_+}) \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ is dense in $\mathrm{L}^q(\mathbb{R}^n_+)$. The density in the second case, follows from the canonical embedding,

$$\mathrm{L}^p(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}_0^{-s,q}(\mathbb{R}^n_+) \hookrightarrow \mathrm{H}_0^{-s,q}(\mathbb{R}^n_+),$$

which turn, by duality into embeddings,

$$\mathrm{H}^{s,q'}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}_0^{-s,q}(\mathbb{R}^n_+))' \hookrightarrow \mathrm{L}^{p'}(\mathbb{R}^n_+).$$

In particular, the following is a dense embedding

$$(\dot{\mathrm{H}}_0^{-s,q}(\mathbb{R}^n_+))' \hookrightarrow \mathrm{L}^{p'}(\mathbb{R}^n_+)$$

hence by reflexivity, the one below also is

$$\mathrm{L}^p(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}_0^{-s,q}(\mathbb{R}^n_+). \qquad \blacksquare$$

Now, all the ingredients are there in order to build the main usual density result for our 0-boundary conditions homogeneous Sobolev spaces.

Proposition 2.25 *For all $p \in (1,+\infty)$, $s \in (-\frac{n}{p'},\frac{n}{p})$, the space $\mathrm{C}_c^\infty(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}_0^{s,p}(\mathbb{R}^n_+)$.*

Proof. — First, let $s \in [0, n)$. Let $p \in (1, +\infty)$, such that $(\mathcal{C}_{s,p})$ is true, and consider $u \in \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$.
In particular, we have $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. Hence, there exists $(u_k)_{k\in\mathbb{N}} \subset \mathrm{H}^{s,p}(\mathbb{R}^n)$ such that

$$u_k \xrightarrow[k\to+\infty]{} u \text{ in } \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n).$$

Thus, it follows from Lemma 2.22, that $(\mathcal{P}_0 u_k)_{k\in\mathbb{N}} \subset \mathrm{H}^{s,p}_0(\mathbb{R}^n_+) \subset \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$ converge to $\mathcal{P}_0 u = u$ in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. For $\varepsilon > 0$, there exists some k_0, such that for all $k \geqslant k_0$, we have

$$\|u - \mathcal{P}_0 u_k\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} < \varepsilon.$$

Now, we use density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ in $\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)$ to assert that there exists $w \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$ so that,

$$\|\mathcal{P}_0 u_k - w\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} \leqslant \|\mathcal{P}_0 u_k - w\|_{\mathrm{H}^{s,p}_0(\mathbb{R}^n_+)} < \varepsilon.$$

This proves the density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ in $\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$, since

$$\|u - w\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} \leqslant \|u - \mathcal{P}_0 u_k\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} + \|\mathcal{P}_0 u_k - w\|_{\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)} < 2\varepsilon.$$

Now let us consider $s \in (0, \frac{n}{p'})$, $u \in \dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)$, applying Proposition 2.24, for $\varepsilon > 0$ there exists a function $v \in \mathrm{L}^q(\mathbb{R}^n_+)$, (with $\frac{1}{p} = \frac{1}{q} - \frac{s}{n}$) such that,

$$\|u - v\|_{\dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)} < \varepsilon.$$

But recalling that $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\mathrm{L}^q(\mathbb{R}^n_+)$, there exists $w \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$ such that

$$\|v - w\|_{\dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \|v - w\|_{\mathrm{L}^q(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \varepsilon,$$

so the triangle inequality gives

$$\|u - w\|_{\dot{\mathrm{H}}^{-s,p}_0(\mathbb{R}^n_+)} \lesssim_{n,s,p,q} \varepsilon,$$

which conclude the proof since $w \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$. ∎

Proposition 2.26 *Let $p_j \in (1, +\infty)$, $s_j \geqslant 0$, $j \in \{0, 1\}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied. The space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}^{s_0,p_0}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p_1}_0(\mathbb{R}^n_+)$.*

Proof. — It suffices to reproduce the first part of the proof of above Proposition 2.25 by the mean of Corollary 2.23. ∎

Corollary 2.27 *For all $p \in (1, +\infty)$, $s \in (-1 + \frac{1}{p}, \frac{1}{p})$,*

$$\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+) = \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+).$$

In particular, $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ for same range of indices.

Proof. — This is a direct consequence of the definition of restriction spaces and Proposition 2.15, the density result follows from Proposition 2.25. ∎

Proposition 2.28 *Let $p \in (1, +\infty)$, $s \in (-\frac{n}{p'}, \frac{n}{p})$, we have*

$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}_0(\mathbb{R}^n_+) \text{ and } (\dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+))' = \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+).$$

Proof. — First, consider $s \in (-\frac{n}{p'}, \frac{n}{p})$, let $\Phi \in \dot{\mathrm{H}}_0^{-s,p'}(\mathbb{R}^n_+) \subset \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$, then using definition of restriction spaces, the following map defines a linear functional on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$,

$$u \longmapsto \big\langle \Phi, \tilde{u} \big\rangle_{\mathbb{R}^n},$$

where $\tilde{u}$ is any extension of u, and notice that the action of Φ does not depend on the choice of such extension of u. Indeed, if $U \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is another extension of u, we obtain that $w := U - \tilde{u} \in \dot{\mathrm{H}}_0^{s,p}(\overline{\mathbb{R}^n_+}^c)$. It follows from Proposition 2.25 that w is a strong limit in $\dot{\mathrm{H}}_0^{s,p}(\overline{\mathbb{R}^n_+}^c)$ of a sequence of functions $(w_k)_{k\in\mathbb{N}} \subset \mathrm{C}_c^\infty(\overline{\mathbb{R}^n_+}^c)$ so that, passing to the limit, in the duality bracket, we obtain

$$\big\langle \Phi, U \big\rangle_{\mathbb{R}^n} - \big\langle \Phi, \tilde{u} \big\rangle_{\mathbb{R}^n} = \big\langle \Phi, w \big\rangle_{\mathbb{R}^n} = 0.$$

This gives a well-defined continuous injective map

$$\left\{ \begin{array}{ccc} \dot{\mathrm{H}}_0^{-s,p'}(\mathbb{R}^n_+) & \longrightarrow & (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' \\ \Phi & \longmapsto & \big\langle \Phi, \tilde{\cdot} \big\rangle_{\mathbb{R}^n} \end{array} \right. . \tag{2.21}$$

Now, let $\Psi \in (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))'$, for all $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, since $\mathbb{1}_{\mathbb{R}^n_+} u = u$, we may write,

$$\langle \Psi, u \rangle = \langle \Psi, \mathbb{1}_{\mathbb{R}^n_+} \widetilde{u} \rangle,$$

for any extension $\tilde{u} \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ of u, hence as a direct consequence of the definition of restriction space $\mathbb{1}_{\mathbb{R}^n_+} \Psi \in (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n))' = \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$, so $\mathbb{1}_{\mathbb{R}^n_+} \Psi \in \dot{\mathrm{H}}_0^{-s,p'}(\mathbb{R}^n_+)$. The following map is well-defined continuous and injective

$$\left\{ \begin{array}{ccc} (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' & \longrightarrow & \dot{\mathrm{H}}_0^{-s,p'}(\mathbb{R}^n_+) \\ \Psi & \longmapsto & \mathbb{1}_{\mathbb{R}^n_+} \Psi \end{array} \right. . \tag{2.22}$$

Both maps (2.21) and (2.22) are even isometric, and we obtain,

$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+))' = \dot{\mathrm{H}}_0^{-s,p'}(\mathbb{R}^n_+),$$

which was the first statement. The second statement follows from duality and reflexivity exchanging roles of involved exponents. ∎

The next result aim to carry over density in intersection spaces to transfer itself as a density result in their real interpolation spaces.

Corollary 2.29 *Let $p \in (1, +\infty)$, $-n/p' < s_0 < s_1 < n/p$, i.e., such that $(\mathcal{C}_{-s_0,p'})$ and $(\mathcal{C}_{s_1,p})$ are both satisfied.*
The space $\mathrm{C}_c^\infty(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+)$.

Proof. — Let $p \in (1, +\infty)$, $-n/p' < s_0 < s_1 < n/p$. There are three subcases, $0 \leqslant s_0 < s_1$, $s_0 < 0 < s_1$, and $s_0 < s_1 \leqslant 0$.

The case $0 \leqslant s_0 < s_1$ follows the lines of Proposition 2.25 thanks to Corollary 2.23.

The case $s_0 < 0 < s_1$, can be done via duality argument as in Proposition 2.25 for the negative index of regularity. Let us consider $\frac{1}{q} = \frac{1}{p} - \frac{s_0}{n}$, the following embeddings are true

$$\mathrm{H}_0^{s_1-s_0,q}(\mathbb{R}^n_+) \hookrightarrow \mathrm{L}^q(\mathbb{R}^n_+) \cap \mathrm{H}_0^{s_1,p}(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+).$$

One may dualize it to deduce

$$\dot{\mathrm{H}}^{-s_1,p'}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+))' \hookrightarrow \mathrm{H}^{s_0-s_1,q'}(\mathbb{R}^n_+).$$

We deduce that the last embedding is dense, since $(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))'$ contains $\dot{\mathrm{H}}^{-s_1,p'}(\mathbb{R}^n_+)$ via canonical embedding, so that by duality and reflexivity of all involved spaces, the following embedding is dense :

$$\mathrm{H}^{s_1-s_0,q}_0(\mathbb{R}^n_+)\hookrightarrow\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+).$$

Since $\mathrm{C}^\infty_c(\mathbb{R}^n_+)\hookrightarrow\mathrm{H}^{s_1-s_0,q}_0(\mathbb{R}^n_+)$ is dense, the result follows.

We end the proof claiming that the third case $s_0<s_1\leqslant 0$ can be done similarly via duality and reflexivity arguments. ∎

We are done with properties of homogeneous Sobolev spaces. We continue with a real interpolation embedding lemma, that allows us to transfer all nice properties, like boundedness of extension and projection operators, from homogeneous Sobolev spaces to homogeneous Besov spaces. The strategy is to use Lemma 2.16 to obtain boundedness of some operators on a sufficiently wide range of indices on Besov spaces via some sort of interpolation method, without the exact description of the interpolation space, see below.

Corollary 2.30 *Let $p\in(1,+\infty), q\in[1,+\infty]$, $s>-1+\frac{1}{p}$, $m\in\mathbb{N}$, such that $s<m+1+\frac{1}{p}$. Let us consider the extension operator* E *(resp. $\mathcal{P}_0$) given by Proposition 2.17 (resp. Lemma 2.22).*

If either

- *$s>0$ and $u\in\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$ (resp. $u\in\mathrm{B}^s_{p,q}(\mathbb{R}^n)$);*
- *$s\in(-1+\frac{1}{p},\frac{1}{p})$ and $u\in\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ (resp. $u\in\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$);*

we have the estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}\lesssim_{s,m,p,n}\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}\ .\ \text{(resp. }\|\mathcal{P}_0u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}\lesssim_{s,m,p,n}\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}\ .\ \text{)}$$

In particular, E *(resp. $\mathcal{P}_0$) is a bounded operator from $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ to $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ (resp. from $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ to $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$) whenever ($\mathcal{C}_{s,p,q}$) is satisfied.*

Proof. — Let $p\in(1,+\infty), q\in[1,+\infty)$, $s>-1+\frac{1}{p}$, $m\in\mathbb{N}$, such that $s<m+1+\frac{1}{p}$. Without loss of generality, it suffices to prove the result for the operator E, since we have the identity $\mathcal{P}_0=\mathrm{I}-\mathrm{E}^-[\mathbb{1}_{\mathbb{R}^n_-}]$, as written in the proof of Lemma 2.22.

The boundedness of E on $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ for $(p,q)\in(1,+\infty)\times[1,+\infty]$, $s\in(-1+\frac{1}{p},\frac{1}{p})$ is again a direct consequence of Proposition 2.15.

It remains to prove boundedness for $s\geqslant\frac{1}{p}$. To do so, we proceed via a manual real interpolation scheme.

Let $u\in\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$, $\theta\in(0,1)$ such that $\theta s_1=s$, where $s_1\in(s,m+1+\frac{1}{p})$. One has

$$u\in(\mathrm{L}^p(\mathbb{R}^n_+),\mathrm{H}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}\hookrightarrow(\mathrm{L}^p(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}\subset\mathrm{L}^p(\mathbb{R}^n_+)+\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+).$$

Hence, for $a\in\mathrm{L}^p(\mathbb{R}^n_+)$, $b\in\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$ such that $f=a+b$, we can deduce that

$$b=u-a\in\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)+\mathrm{L}^p(\mathbb{R}^n_+)\subset\mathrm{L}^p(\mathbb{R}^n_+),$$

so that $b\in\mathrm{L}^p(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)=\mathrm{H}^{s_1,p}(\mathbb{R}^n_+)$ thanks to Proposition 2.19. Hence, $\mathrm{E}u=\mathrm{E}a+\mathrm{E}b$, with $\mathrm{E}a\in\mathrm{L}^p(\mathbb{R}^n_+)$, $\mathrm{E}b\in\mathrm{H}^{s_1,p}(\mathbb{R}^n_+)$, with the homogeneous estimates provided by Proposition 2.17. Then $\mathrm{E}u_{|\mathbb{R}^n_+}=u$, and we have estimates

$$K(t,\mathrm{E}u,\mathrm{L}^p(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))\leqslant\|\mathrm{E}a\|_{\mathrm{L}^p(\mathbb{R}^n)}+t\|\mathrm{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)}\lesssim_{p,m,n}\|a\|_{\mathrm{L}^p(\mathbb{R}^n_+)}+t\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}.$$

Hence, taking infimum on all such functions a and b, and multiplying by $t^{-\theta}$ leads to

$$t^{-\theta}K(t,\mathrm{E}u,\mathrm{L}^p(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n))\lesssim_{p,s,s_1,n}t^{-\theta}K(t,u,\mathrm{L}^p(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)),$$

so one may take the L^q_*-norm of above inequality and use (2.13) from Lemma 2.16 to deduce that

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{p,s,q,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

If $q < +\infty$, then $\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, so that the conclusion holds by density whenever $(\mathcal{C}_{s,p,q})$ is satisfied.

If $q = +\infty$, and $(\mathcal{C}_{s,p,q})$ is satisfied, necessarily $s < \frac{n}{p}$. We introduce $\mathcal{E} := \mathrm{E}[\mathbb{1}_{\mathbb{R}^n_+}\cdot]$ which is bounded, thanks to the above step, seen as an operator

$$\mathcal{E} : \dot{\mathrm{B}}^{s_j}_{p,q_j}(\mathbb{R}^n) \longrightarrow \dot{\mathrm{B}}^{s_j}_{p,q_j}(\mathbb{R}^n),$$

provided $s_0 < s < s_1 < \frac{n}{p}$, and $q_j \in [1,\infty)$, $j \in \{0,1\}$. Thus, by real interpolation argument, thanks to Theorem 2.10, for all $U \in \dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)$, we have

$$\|\mathcal{E}U\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)} \lesssim_{p,s,q,n} \|U\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)}.$$

In particular, for all $u \in \dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n_+)$, and all $U \in \dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)$ such that $U_{|\mathbb{R}^n_+} = u$, we have

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)} \lesssim_{p,s,q,n} \|U\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n)}.$$

Hence, taking the infimum on all such functions U gives the result when $q = +\infty$ and $(\mathcal{C}_{s,p,q})$ is satisfied. ∎

Proposition 2.31 *Let $p, q \in [1,+\infty]$, $s \in (0,n)$, such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

We have the following estimates,

$$\begin{aligned}
\|u\|_{\mathrm{L}^q(\mathbb{R}^n_+)} &\lesssim_{n,s,p,q,r} \|u\|_{\dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n_+)}, \ \forall u \in \dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n_+),\ r \in [1,q]\\
\|u\|_{\dot{\mathrm{B}}^{-s}_{q,r,0}(\mathbb{R}^n_+)} &\lesssim_{n,s,p,q,r} \|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)}, \ \forall u \in \mathrm{L}^p(\mathbb{R}^n_+),\ r \in [q,+\infty],\\
\|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)} &\lesssim_{n,s,p} \|u\|_{\dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n_+)}, \ \forall u \in \dot{\mathrm{B}}^0_{p,r}(\mathbb{R}^n_+),\ r \in [1,\min(2,p)],\\
\|u\|_{\dot{\mathrm{B}}^0_{p,r,0}(\mathbb{R}^n_+)} &\lesssim_{n,s,p,r} \|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)}, \ \forall u \in \mathrm{L}^p(\mathbb{R}^n_+),\ r \in [\max(2,p),+\infty].
\end{aligned}$$

Moreover, we also have $\dot{\mathrm{B}}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_0(\overline{\mathbb{R}^n_+})$, whenever p is finite.

Proof. — Each embedding is a direct consequence of the definition of each space and the corresponding ones on $\mathbb{R}^n$, see Proposition 2.13. ∎

Lemma 2.32 *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$ and $s > 0$. The function space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$ whenever $(\mathcal{C}_{s,p,q})$ is satisfied.*

Proof. — As in the proof of Proposition 2.25, in the case of non negative index : by a successive approximations scheme, we use density of $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ in $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$, to approximate functions in $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$. Then the boundedness of $\mathcal{P}_0$ on $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, and the density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ in $\mathrm{B}^s_{p,q,0}(\mathbb{R}^n_+)$ yields the result. ∎

Proposition 2.33 *Let $(p_0,p_1,p,q) \in (1,+\infty)^3 \times [1,+\infty]$, $s_0, s_1 \in \mathbb{R}$, such that $s_0 < s_1$, let $(\mathfrak{h},\mathfrak{b}) \in \{(\mathrm{H},\mathrm{B}),(\mathrm{H}_0,\mathrm{B}_{\cdot,\cdot,0})\}$, and set*

$$\left(s, \frac{1}{p_\theta}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}\right) + \theta\left(s_1, \frac{1}{p_1}\right).$$

If either one of the following assertions is satisfied,

(i) $q \in [1,+\infty)$, $s_j > -1 + \frac{1}{p_j}$, $j \in \{0,1\}$;

(ii) $q \in [1,+\infty]$, $s_j > -1 + \frac{1}{p_j}$, *and* $(\mathcal{C}_{s_j,p_j})$ *is satisfied,* $j \in \{0,1\}$;

If $p_0 = p_1 = p$ *and* $(\mathcal{C}_{s,p,q})$ *is satisfied, the following equality is true with equivalence of norms*

$$(\dot{\mathfrak{h}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathfrak{h}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathfrak{b}}^s_{p,q}(\mathbb{R}^n_+). \tag{2.23}$$

If $(\mathcal{C}_{s_0,p_0})$ *and* $(\mathcal{C}_{s_1,p_1})$ *are true then also is* $(\mathcal{C}_{s,p_\theta})$ *and*

$$[\dot{\mathfrak{h}}^{s_0,p_0}(\mathbb{R}^n_+), \dot{\mathfrak{h}}^{s_1,p_1}(\mathbb{R}^n_+)]_\theta = \dot{\mathfrak{h}}^{s,p_\theta}(\mathbb{R}^n_+). \tag{2.24}$$

Proof. — We start noticing that (2.24) only makes sense under assertion *(ii)*.

Step 1 : We prove first (2.24) and (2.23) under assertion *(ii)*.

It suffices to assert that $\{\dot{\mathfrak{h}}^{s_0,p_0}(\mathbb{R}^n_+), \dot{\mathfrak{h}}^{s_1,p_1}(\mathbb{R}^n_+)\}$ is a retraction of $\{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)\}$, thanks to Theorem 1.21. Indeed, both retractions are given by

$$\mathrm{E} : \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \text{ and } \mathrm{R}_{\mathbb{R}^n_+} : \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \longrightarrow \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+),$$
$$\iota : \dot{\mathrm{H}}^{s_j,p_j}_0(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \text{ and } \quad \mathcal{P}_0 : \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \longrightarrow \dot{\mathrm{H}}^{s_j,p_j}_0(\mathbb{R}^n_+).$$

Here, $\mathrm{R}_{\mathbb{R}^n_+}$ and ι stand respectively for the restriction and the canonical injection operator. Boundedness and range of E and $\mathcal{P}_0$ provided by Lemma 2.22 and Corollary 2.30 lead to (2.24) and (2.23) under assertion *(ii)*.

Step 2 : We prove (2.23) under assertion *(i)*.

Step 2.1 : $(\mathfrak{h}, \mathfrak{b}) = (\mathrm{H}, \mathrm{B})$.

Thanks to Lemma 2.16, we have continuous embedding,

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}. \tag{2.25}$$

Let us prove the reverse embedding,

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) \hookleftarrow (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}.$$

Without loss of generality, we can assume $s_1 \geqslant \frac{n}{p}$. Let $f \in \mathcal{S}_0(\overline{\mathbb{R}^n_+}) \subset \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, it follows that $f \in (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} \subset \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$. Thus, for all $(a,b) \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) \times \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$ such that $f = a + b$, we have,

$$b = f - a \in (\mathcal{S}_0(\overline{\mathbb{R}^n_+}) + \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+).$$

In particular, we have $a \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)$ and $b \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$. Hence, we can introduce $F := \mathrm{E}a + \mathrm{E}b$, where $F_{|_{\mathbb{R}^n_+}} = f$, $\mathrm{E}a \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)$ and $\mathrm{E}b \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, with the estimates, given by Corollary 2.20,

$$\|\mathrm{E}a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} \lesssim_{s_0,m,p,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)} \text{ and } \|\mathrm{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)} \lesssim_{s_1,m,p,n} \|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}.$$

Then, one may bound the K-functional of F, for $t > 0$,

$$K(t, F, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \leqslant \|\mathrm{E}a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} + t\|\mathrm{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)} \lesssim_{s_j,p,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)} + t\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}$$

Taking the infimum over all such functions a and b, we obtain

$$K(t, F, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \lesssim_{s_j,p,n} K(t, f, \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)),$$

from which we obtain, after multiplying by $t^{-\theta}$, taking the L^q_*-norm with respect to t, and applying

Theorem 2.10,

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \leqslant \|F\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n} \|f\|_{(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}}.$$

Finally, thanks to the first embedding (2.25), we have

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \sim_{p,s,n} \|f\|_{(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}},\ \forall f \in \mathcal{S}_0(\overline{\mathbb{R}^n_+}).$$

Since $q < +\infty$, we can conclude by density of $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ in both $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ and in the interpolation space $(\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q}$. The density argument for the later one is carried over by Lemma 2.9 and Proposition 1.3.

Step 2.2 : $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^s_{p,q,0}$, provided $-1+\frac{1}{p} < s < \frac{1}{p}$, $p \in (1,+\infty)$, $q \in [1,+\infty)$.

Thanks to the above Step 1, one may find $-1+\frac{1}{p} < s_0 < s < s_1 < \frac{1}{p}$, $\theta \in (0,1)$ such that, as a consequence of Proposition 1.3, we have the following dense embedding,

$$\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) = \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).$$

The equality in the above line is a direct consequence of Proposition 2.15. In this case, the density of $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is a straightforward application of Corollary 2.29 by successive approximations.

Step 2.3 : $(\mathfrak{h},\mathfrak{b}) = (\mathrm{H}_0, \mathrm{B}_{\cdot,\cdot,0})$.

Thanks to Lemma 2.16, we have continuous embedding,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q} \hookrightarrow \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).$$

We are going to prove the reverse embedding,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q} \hookleftarrow \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).$$

Again, without loss of generality we can assume $s_1 \geqslant \frac{n}{p}$, otherwise one can go back to the Step 1. Let us consider $u \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$, then, u belongs to $\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) + \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. In particular for $(a,b) \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \times \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, such that $u = a + b$ we have

$$b = u - a \in (\mathrm{C}^\infty_c(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n).$$

in particular we have $a \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)$ and $b \in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. Consequently, we have $u = \mathcal{P}_0 u = \mathcal{P}_0 a + \mathcal{P}_0 b$, with $\mathcal{P}_0 a \in \dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+)$ and $\mathcal{P}_0 b \in \dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+)$, with the estimates

$$\|\mathcal{P}_0 a\|_{\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+)} \lesssim_{s_0,m,p,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} \text{ and } \|\mathcal{P}_0 b\|_{\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+)} \lesssim_{s_1,m,p,n} \|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)},$$

thanks to Corollary 2.23. Thus, one may follow the lines of the step 2.1, to obtain for all $u \in \mathrm{C}^\infty_c(\mathbb{R}^n_+)$,

$$\|u\|_{\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)} \sim_{s,p,n} \|u\|_{(\dot{\mathrm{H}}^{s_0,p}_0(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s_1,p}_0(\mathbb{R}^n_+))_{\theta,q}}.$$

Again, one can conclude via density arguments since $q < +\infty$, and $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$ thanks to the Step 2.2 and Lemma 2.32. ∎

The Step 2.2 in the above proof can be turned more formally into,

Corollary 2.34 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+\frac{1}{p},\frac{1}{p})$. Then the following equality holds with equivalence of norms,*

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) = \dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+).$$

Moreover, the space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense whenever $q < +\infty$.

From general interpolation theory, we are able to deduce the following,

Corollary 2.35 *Let $p \in (1,+\infty)$, $s > -1+1/p$, such that $(\mathcal{C}_{s,p,\infty})$ is satisfied.*

- *The space $\mathrm{C}_c^\infty(\mathbb{R}^n_+)$ is weak* dense in $\dot{\mathrm{B}}^s_{p,\infty,0}(\mathbb{R}^n_+)$.*
- *The space $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ is weak* dense in $\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n_+)$.*

Proof. — The Theorem 1.6 with the remark at the end of its proof in combination with Lemma 2.29, with the use of Proposition 1.3, and Proposition 2.33 imply that, for some $-1+1/p < s_0 < s < s_1$, with $\theta \in (0,1)$, such that $s = (1-\theta)s_0 + \theta s_1$, we have the following strongly dense embedding,

$$\mathrm{C}_c^\infty(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+) \hookrightarrow (\dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+))_\theta,$$

and the following weak* dense embedding

$$(\dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+))_\theta \hookrightarrow (\dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+))''_\theta = (\dot{\mathrm{H}}_0^{s_0,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}_0^{s_1,p}(\mathbb{R}^n_+))_{\theta,\infty} = \dot{\mathrm{B}}^s_{p,\infty,0}(\mathbb{R}^n_+),$$

so that the result follows. We mention that $(\cdot,\cdot)_\theta$ is the real interpolation functor asking the K-functional to decay at infinity and near the origin.

The same argument apply for the weak* density of $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ in $\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n_+)$. ∎

We state below the Besov analogue of Corollary 2.23, Lemma 2.9 and Proposition 2.19, for which the proofs are similar and left to the reader.

Lemma 2.36 *Let $p_j \in (1,+\infty)$, $q_j \in [1,+\infty]$, $s_j > -1+\frac{1}{p_j}$, $j \in \{0,1\}$, $m \in \mathbb{N}$, such that $(\mathcal{C}_{s_0,p_0,q_0})$ is satisfied and $s_j < m+1+\frac{1}{p_j}$, and consider the extension operator E given by Proposition 2.17.*

Then for all $u \in \dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)$, we have $\mathrm{E}u \in \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)$, $j \in \{0,1\}$, with the estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)} \lesssim_{s_j,m,p,n} \|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)}.$$

The same result holds replacing $(\mathrm{E}, \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+), \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n))$ by $(\mathcal{P}_0, \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n), \dot{\mathrm{B}}^{s_j}_{p_j,q_j,0}(\mathbb{R}^n_+))$, where $\mathcal{P}_0$ is the projection operator given in Lemma 2.22.

Proposition 2.37 *Let $p_j \in (1,+\infty)$, $q_j \in [1,+\infty]$, $j \in \{0,1\}$, $-1+\frac{1}{p} < s_0 < s_1$, such that $(\mathcal{C}_{s_0,p_0,q_0})$ is satisfied. Then the following equality of vector spaces holds with equivalence of norms*

$$\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+) = [\dot{\mathrm{B}}^{s_0}_{p_0,q_0} \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\mathbb{R}^n_+).$$

In particular, $\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)$ is a Banach space, and it admits $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ as a dense subspace whenever $q_j < +\infty$, $j \in \{0,1\}$.

Similarly, the following equality with equivalence of norms holds for all $s > 0$, $q \in [1,+\infty]$,

$$\mathrm{L}^p(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) = \mathrm{B}^s_{p,q}(\mathbb{R}^n_+).$$

With direct consequence similar to Corollary 2.21 :

Corollary 2.38 *Let $p_j \in (1,+\infty)$, $q_j \in [1,+\infty]$ $m_j \in [\![1,+\infty[\![$, $s_j > m_j - 1 + \frac{1}{p_j}$, $j \in \{0,1\}$, such that $(\mathcal{C}_{s_0,p_0,q_0})$ is satisfied. For all $u \in [\dot{\mathrm{B}}^{s_0}_{p_0,q_0} \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\mathbb{R}^n_+)$,*

$$\|\nabla^{m_j} u\|_{\dot{\mathrm{B}}^{s_j-m_j}_{p_j,q_j}(\mathbb{R}^n_+)} \sim_{s_j,m_j,p_j,n} \|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)}.$$

Above Proposition 2.37 also implies the expected interpolation result for Besov spaces, for which the proof is similar to the one of Proposition 2.33 and left again to the reader.

Proposition 2.39 *Let $(p_0,p_1,p,q,q_0,q_1)\in(1,+\infty)^3\times[1,+\infty]^3$, $s_0,s_1\in\mathbb{R}$, such that $s_0<s_1$, and let $\mathfrak{b}\in\{\mathrm{B},\mathrm{B}_{\cdot,\cdot,0}\}$, and set*

$$\left(s,\frac{1}{p_\theta},\frac{1}{q_\theta}\right):=(1-\theta)\left(s_0,\frac{1}{p_0},\frac{1}{q_0}\right)+\theta\left(s_1,\frac{1}{p_1},\frac{1}{q_1}\right).$$

such that the following assertion is satisfied,

- *$s_j>-1+\frac{1}{p_j}$, $j\in\{0,1\}$, and $(\mathcal{C}_{s_0,p_0,q_0})$ is true;*

Then if $p_0=p_1=p$, and $(\mathcal{C}_{s,p,q})$ is satisfied, the following equality holds with equivalence of norms

$$(\dot{\mathfrak{b}}^{s_0}_{p,q_0}(\mathbb{R}^n_+),\dot{\mathfrak{b}}^{s_1}_{p,q_1}(\mathbb{R}^n_+))_{\theta,q}=\dot{\mathfrak{b}}^{s}_{p,q}(\mathbb{R}^n_+).$$

If $(\mathcal{C}_{s_0,p_0,q_0})$ and $(\mathcal{C}_{s_1,p_1,q_1})$ are true then also is $(\mathcal{C}_{s,p_\theta,q_\theta})$ and with equivalence of norms,

$$[\dot{\mathfrak{b}}^{s_0}_{p_0,q_0}(\mathbb{R}^n_+),\dot{\mathfrak{b}}^{s_1}_{p_1,q_1}(\mathbb{R}^n_+)]_\theta=\dot{\mathfrak{b}}^{s}_{p_\theta,q_\theta}(\mathbb{R}^n_+).$$

We finish stating a duality result for homogeneous Besov spaces on the half-space.

Proposition 2.40 *Let $p\in(1,+\infty)$, $q\in(1,+\infty]$, $s>-1+\frac{1}{p}$, if $(\mathcal{C}_{s,p,q})$ is satisfied then the following isomorphisms hold*

$$(\dot{\mathrm{B}}^{-s}_{p',q',0}(\mathbb{R}^n_+))'=\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)\ \text{ and }\ (\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))'=\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+).$$

Proof. — We only prove $(\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))'=\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$, the other equality can be shown similarly. First let $q<+\infty$, and choose $u\in\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$, it follows that u induce a linear form on $\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)$,

$$v\longmapsto\langle u,\tilde{v}\rangle_{\mathbb{R}^n}$$

where $\tilde{v}\in\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ is any extension of $v\in\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)$. If one choose v' to be any other extension of v, we have that $\tilde{v}-v'\in\dot{\mathrm{B}}^{-s}_{p',q',0}(\mathbb{R}^n_-)$. Since $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in $\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$, see either Lemma 2.32 or Corollary 2.34, for $(u_k)_{k\in\mathbb{N}}\subset\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ converging to u, we have

$$\langle u,\tilde{v}-v'\rangle_{\mathbb{R}^n}=\lim_{k\to+\infty}\langle u_k,\tilde{v}-v'\rangle_{\mathbb{R}^n}=0$$

due to the fact that $\mathbb{R}^n_+\cap\mathbb{R}^n_-=\emptyset$. Thus, the map does not depend on the choice of the extension but is entirely and uniquely determined by u. We have the continuous canonical embedding

$$\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)\hookrightarrow(\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))'.$$

In fact, the same result holds for $q=+\infty$: the space $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is sequentially weak* dense in $\dot{\mathrm{B}}^{s}_{p,\infty,0}(\mathbb{R}^n_+)$ by Corollary 2.35.

For the reverse embedding, if $U\in(\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+))'$, it induces a continuous linear functional on $\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ by the mean of

$$v\longmapsto\left\langle U,\mathbb{1}_{\mathbb{R}^n_+}\tilde{v}\right\rangle,$$

where again $\tilde{v}\in\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$ is any extension of $v\in\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)$. Thus, $\mathbb{1}_{\mathbb{R}^n_+}U\in(\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n))'$ and by Proposition 2.12 there exists a unique $u\in\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)$ such that, for all $\tilde{v}\in\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n)$,

$$\left\langle U,\mathbb{1}_{\mathbb{R}^n_+}\tilde{v}\right\rangle=\langle u,\tilde{v}\rangle_{\mathbb{R}^n}.$$

Finally, if we test with $\tilde{v}\in\mathrm{C}^\infty_c(\mathbb{R}^n_-)$, it shows that supp $u\subset\overline{\mathbb{R}^n_+}$, then $u\in\dot{\mathrm{B}}^{s}_{p,q,0}(\mathbb{R}^n_+)$ which close the proof. ∎

4 Additional results, notations and some remarks

4.1 Miscellaneous results

We state here few results that will be useful in later parts.

Complex interpolation for intersection of homogeneous Besov spaces

Corollary 2.41 *Let $p \in [1,+\infty]$, $q \in [1,+\infty)$, $s_j \in \mathbb{R}$, $j \in \{0,1\}$ such that $(\mathcal{C}_{s_0,p,q})$ is satisfied. Then for $\theta \in (0,1)$, let's introduce $s := (1-\theta)s_0 + \theta s_1$. Then the following equality holds with equivalence for norms*

$$[\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n), \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n)]_\theta = \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n) \cap \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n) \ .$$

Proof. — Both function spaces $\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n)$, and $\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n)$ are complete normed vector spaces, see [BCD11, Theorem 2.25].

Now, we apply Theorem 1.21 and Proposition 1.20, claiming that, for all $s \in \mathbb{R}$, $\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)$ is a retraction of $\ell^q_s(\mathbb{Z}, \mathrm{L}^p(\mathbb{R}^n))$ through the homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$. ■

Estimates for the Poisson semigroup

Lemma 2.42 *Let $s > 0$, $\alpha \geqslant 0$ and $p,q \in [1,+\infty]^2$. For all $u \in \mathcal{S}'_h(\mathbb{R}^n)$,*

$$\|u\|_{\dot{\mathrm{B}}^{\alpha-s}_{p,q}(\mathbb{R}^n)} \sim_{p,s,\alpha,n,q} \left\| t \mapsto \|t^s(-\Delta)^{\frac{\alpha}{2}} e^{-t(-\Delta)^{\frac{1}{2}}} u\|_{\mathrm{L}^p(\mathbb{R}^n)} \right\|_{\mathrm{L}^q_*(\mathbb{R}_+)} \ .$$

Proof. — It suffices to show the result for $\alpha = 0$. But in this case, the proof is straightforward the same as the one of [BCD11, Theorem 2.34] for the heat semigroup. ■

The following result was already proven in the case of homogeneous Besov spaces only. It is extended here to the case of homogeneous Sobolev spaces, see for instance [DM09, Lemma 2].

Proposition 2.43 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$. The map*

$$T \ : \ f \longmapsto \left[(x', x_n) \mapsto e^{-x_n(-\Delta')^{\frac{1}{2}}} f(x') \right]$$

is such that

(i) Given $s \geqslant 0$, for all $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, we have

$$\|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \ .$$

In particular, T extends uniquely to a bounded linear operator $T \ : \ \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ whenever $(\mathcal{C}_{s,p})$ is satisfied.

(ii) Given $s > 0$, for all $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$, we have

$$\|Tf\|_{\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \ .$$

In particular, T extends uniquely to a bounded linear operator $T \ : \ \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}) \longrightarrow \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+)$ whenever $(\mathcal{C}_{s,p,q})$ is satisfied.

Proof. — **Point *(i)* :** For $p \in (1,+\infty)$, let's consider $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$. We apply Lemma 2.42 to obtain,

$$\begin{aligned}\|Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)} &= \left(\int_0^{+\infty} \|e^{-x_n(-\Delta')^{\frac{1}{2}}} f\|^p_{\mathrm{L}^p(\mathbb{R}^{n-1})}\,\mathrm{d}x_n\right)^{\frac{1}{p}} \\ &= \left(\int_0^{+\infty} \left(t^{\frac{1}{p}}\|e^{-t(-\Delta')^{\frac{1}{2}}} f\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}\right)^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \\ &\lesssim_{p,n} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.\end{aligned}$$

We continue noticing that for all $f \in \mathcal{S}'_h(\mathbb{R}^{n-1})$, $m \in \mathbb{N}$, $\partial^m_{x_n} Tf = (-\Delta')^{\frac{m}{2}} Tf = T(-\Delta')^{\frac{m}{2}} f$ and $Tf \in \mathcal{S}'_h(\mathbb{R}^{n-1})$, thus if $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$ we may apply previous inequality to obtain,

$$\begin{aligned}\|Tf\|_{\dot{\mathrm{H}}^{m,p}(\mathbb{R}^n_+)} &\sim_{p,n,m} \|\partial^m_{x_n} Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)} + \|(-\Delta')^{\frac{m}{2}} Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \\ &\sim_{p,n,m} \|T(-\Delta')^{\frac{m}{2}} f\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \\ &\lesssim_{p,n,m} \|f\|_{\dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.\end{aligned}$$

So that for all $m \in \mathbb{N}$, all $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{H}^{m,p}(\mathbb{R}^n_+)} \lesssim_{p,n,m} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} + \|f\|_{\dot{\mathrm{B}}^{m-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Thus, by complex interpolation and Corollary 2.41, for all $s \geqslant 0$, all $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{H}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} + \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Hence, thanks to Proposition 2.19, $\mathrm{H}^{s,p}(\mathbb{R}^n_+) = \mathrm{L}^p(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$,

$$\|Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)} + \|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} + \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Therefore, if $\lambda \in 2^{\mathbb{N}}$, we can consider f_λ is the dilation by factor λ of f, so that plugging f_λ instead of f in above inequality, and checking the fact that $Tf_\lambda = (Tf)_\lambda$, we obtain

$$\lambda^{-\frac{n}{p}}\|Tf\|_{\mathrm{L}^p(\mathbb{R}^n_+)} + \lambda^{s-\frac{n}{p}}\|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \lambda^{-\frac{n}{p}}\|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} + \lambda^{s-\frac{n}{p}}\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

One may divide above inequality by $\lambda^{s-\frac{n}{p}}$, so as λ tends to infinity, it yields

$$\|Tf\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

So that the result holds by density whenever $(\mathcal{C}_{s,p})$ is satisfied.

Point *(ii)* : Now let $q \in [1,+\infty]$, since for all $s \geqslant 0$, all $f \in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{H}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} + \|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\,.$$

Hence, by real interpolation, using [Haa06, Proposition B.2.7] instead of Corollary 2.41, we obtain

that for all $s>0$ all $f\in \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})\cap \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$,

$$\|Tf\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)}\lesssim_{p,n,s}\|f\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}+\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})}\ .$$

Then the same dilation procedure as before, yields

$$\|Tf\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}\lesssim_{p,n,s}\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})}\ ,$$

which again allows concluding via a density argument if $q<+\infty$ and $(\mathcal{C}_{s,p,q})$ is satisfied. The case $q=+\infty$, when $(\mathcal{C}_{s,p,q})$ is satisfied, follows from real interpolation with the last estimate. ■

Proposition 2.43 can be self-improved as

Corollary 2.44 *Let $p_j\in(1,+\infty)$, $q_j\in[1,+\infty)$, $j\in\{0,1\}$. The map*

$$T\ :\ f\longmapsto\left[(x',x_n)\mapsto e^{-x_n(-\Delta')^{\frac{1}{2}}}f(x')\right]$$

is such that

(i) Let $s_j\geqslant 0$, $j\in\{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied. For all $f\in[\dot{\mathrm{B}}^{s_0-\frac{1}{p_0}}_{p_0,p_0}\cap\dot{\mathrm{B}}^{s_1-\frac{1}{p_1}}_{p_1,p_1}](\mathbb{R}^{n-1})$, we have

$$\|Tf\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}\lesssim_{s,p,n}\|f\|_{\dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,p_j}(\mathbb{R}^{n-1})}\qquad,\ j\in\{0,1\}\ .$$

(ii) Let $s_j>0$, $j\in\{0,1\}$, such that $(\mathcal{C}_{s_0,p_0,q_0})$ is satisfied. For all $f\in[\dot{\mathrm{B}}^{s_0-\frac{1}{p_0}}_{p_0,q_0}\cap\dot{\mathrm{B}}^{s_1-\frac{1}{p_1}}_{p_1,q_1}](\mathbb{R}^{n-1})$, we have

$$\|Tf\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)}\lesssim_{s,p,n}\|f\|_{\dot{\mathrm{B}}^{s-\frac{1}{p_j}}_{p_j,q_j}(\mathbb{R}^{n-1})}\qquad,\ j\in\{0,1\}\ .$$

4.2 Operators on Sobolev and Besov spaces

We introduce domains for an operator A acting on Sobolev or Besov spaces, denoting

- $\mathrm{D}^s_p(A)$ (resp. $\dot{\mathrm{D}}^s_p(A)$) its domain on $\mathrm{H}^{s,p}$ (resp. $\dot{\mathrm{H}}^{s,p}$);
- $\mathrm{D}^s_{p,q}(A)$ (resp. $\dot{\mathrm{D}}^s_{p,q}(A)$) its domain on $\mathrm{B}^s_{p,q}$ (resp. $\dot{\mathrm{B}}^s_{p,q}$);
- $\mathrm{D}_p(A)=\mathrm{D}^0_p(A)=\dot{\mathrm{D}}^0_p(A)$ its domain on L^p.

Similarly, $\mathrm{N}^s_p(A)$, $\mathrm{N}^s_{p,q}(A)$ will stand for its nullspace on $\mathrm{H}^{s,p}$ and $\mathrm{B}^s_{p,q}$, and range spaces will be given respectively by $\mathrm{R}^s_p(A)$ and $\mathrm{R}^s_{p,q}(A)$. We replace N and R by $\dot{\mathrm{N}}$ and $\dot{\mathrm{R}}$ for their corresponding sets on homogeneous function spaces.

If the operator A has different realizations depending on various function spaces and on the considered open set, we may write its domain $\mathrm{D}(A,\Omega)$, and similarly for its nullspace N and range space R. We omit the open set Ω if there is no possible confusion.

4.3 Non-exhaustivity of the construction

The goal of presenting here a definitive construction of homogeneous Sobolev and Besov spaces on the half-space is certainly not reached :

- The way arguments are done Section 3 always requires a ground function space to intersect with so that it ensures we deal with restriction of elements of $\mathcal{S}'_h(\mathbb{R}^n)$, e.g. see the proof of Proposition 2.19. Hence, with this kind of methods, obtaining more general results like an exhaustive description of dual spaces of homogeneous Besov and Sobolev spaces on $\mathbb{R}^n_+$ in the non-complete case seems to be difficult to reach.

- A related problem is that the extension operator we use is not general enough and disallow to cover too much negative index of regularity in case of homogeneous function spaces. It would be of interest to know if one can also recover non-complete positive index independently, without using intersection or density tricks. As mentioned at the beginning of this section, to know if one can construct an operator similar to Rychkov's extension operator, from [Ryc99], $\mathcal{E}$ such that $\mathcal{E}(\mathcal{S}'_h(\overline{\mathbb{R}^n_+})) \subset \mathcal{S}'_h(\mathbb{R}^n)$ with homogeneous estimates would be a sufficiently powerful result to overcome such troubles.
- Other definitions are possible for $\mathcal{S}'_h(\mathbb{R}^n)$. We have chosen here the one with the strongest convergence for the sum of low frequencies to continue the work started in [BCD11, Chapter 2] and [DHMT21, Chapter 3]. The choice of possible definitions and their functional analytic consequences on Besov spaces' construction are reviewed by Cobb in [Cob21, Appendix] and [Cob22].

Not to further burden the actual presentation, we just mention that one could also investigate spaces such as

$$\dot{\mathcal{B}}^s_{p,\infty}(\mathbb{R}^n_+) \text{ and } \dot{\mathcal{B}}^s_{p,\infty,0}(\mathbb{R}^n_+).$$

For those spaces, the space $\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ is dense in the first one by construction, and we can show that $\mathrm{C}^\infty_c(\mathbb{R}^n_+)$ is dense in the second one, and both may be recovered from interpolation of other appropriate homogeneous Sobolev and Besov spaces. We can also prove corresponding duality and traces results. Details are left to the interested reader.

5 On traces of functions

Dealing with function spaces on domains implies that one may need to investigate the meaning of traces on the boundary if those exist, *i.e.* to see in our setting if the trace operator

$$\gamma_0 \,:\, u \longmapsto u_{|_{\partial\mathbb{R}^n_+}}$$

still has the expected behavior on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ and $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$. In fact, in the complete case, it behaves as in the case of inhomogeneous function spaces.

The idea here is to give some appropriate trace theorems for homogeneous Sobolev and Besov spaces. It seems there is no trace theorem for homogeneous function spaces in the literature, except maybe [Jaw78], but in this case corresponding results were obtained in a different framework.

5.1 On inhomogeneous function spaces.

We discuss first about the usual well known trace theorem on $\mathbb{R}^n$ with trace on $\mathbb{R}^{n-1}\times\{0\}$ in the inhomogeneous case, the result is a rewritten weaker version adapted to our context.

Theorem 2.45 ([BL76, Theorem 6.6.1]) *Let $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in(\frac{1}{p},+\infty)$, and consider the following operator*

$$\gamma_0 \,:\, \left\{ \begin{array}{rcl} \mathcal{S}(\mathbb{R}^n) & \longrightarrow & \mathcal{S}(\mathbb{R}^{n-1}) \\ u & \longmapsto & u(\cdot,0) \end{array} \right. ,$$

then following statements are true :

(i) the trace operator $\gamma_0 \,:\, \mathrm{H}^{s,p}(\mathbb{R}^n) \longrightarrow \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$ is a bounded surjection, in particular for all $u\in\mathrm{H}^{s,p}(\mathbb{R}^n)$,

$$\|\gamma_0 u\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s,p,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)};$$

(ii) *the trace operator* $\gamma_0 : \mathrm{B}^s_{p,q}(\mathbb{R}^n) \longrightarrow \mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$ *is a bounded surjection, in particular for all* $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n)$,

$$\|\gamma_0 u\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s,p,n,q} \|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n)};$$

(iii) *the trace operator* $\gamma_0 : \mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n) \longrightarrow \mathrm{L}^p(\mathbb{R}^{n-1})$ *is a bounded surjection, in particular for all* $u \in \mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)$,

$$\|\gamma_0 u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \lesssim_{p,n} \|u\|_{\mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n)};$$

Moreover the trace operator γ_0 *admits a linear right bounded inverse* Ext *in cases (i) and (ii).*

Remark 2.46 One also mentions [Sch10, Theorems 2.2 & 2.10], [JW84, Sections V-VII], which give different proofs of the trace theorem. Notice that in [Sch10, Theorems 2.2 & 2.10] and [Saw18, Theorems 4.47, 4.48] the right bounded inverse they give is not linear but covers case (iii).

Proof. — We only give the proof for the right bounded inverse. The idea here is to complete the approach given in [BL76, Exercices 25, 26, p.166] to recover the full range of exponents since $(\frac{1}{p}, 1]$ were missing. To do so let $p \in (1, +\infty)$, $s > \frac{1}{p}$, and $m \in \mathbb{N}$ such that $s < m + 1 + \frac{1}{p}$. Consider $\chi \in \mathrm{C}^\infty_c(\mathbb{R})$ such that $\operatorname{supp} \chi \subset [-1, 1]$, $0 \leqslant \chi \leqslant 1$ and $\chi(0) = 1$.

We introduce the following operator,

$$\mathcal{L}_+ : f \longmapsto \left[(x', x_n) \mapsto \chi(x_n) e^{-x_n(-\Delta')^{\frac{1}{2}}} f(x')\right].$$

Since $\mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) = \mathrm{L}^p(\mathbb{R}^{n-1}) + \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, see [BL76, Theorem 2.7.1, Theorem 6.3.2], we can apply Lemma 2.42, so that for all $f = a + b \in \mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, where $(a, b) \in \mathrm{L}^p(\mathbb{R}^{n-1}) \times \dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\begin{aligned}
\|\mathcal{L}_+ f\|_{\mathrm{L}^p(\mathbb{R}^n_+)} &= \left(\int_0^{+\infty} \|\chi(x_n) e^{-x_n(-\Delta')^{\frac{1}{2}}} f\|^p_{\mathrm{L}^p(\mathbb{R}^{n-1})} \,\mathrm{d}x_n\right)^{\frac{1}{p}} \\
&= \left(\int_0^{+\infty} \left(t^{\frac{1}{p}} \|\chi(t) e^{-t(-\Delta')^{\frac{1}{2}}} f\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}\right)^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \\
&\leqslant \left(\int_0^1 \|e^{-t(-\Delta')^{\frac{1}{2}}} a\|^p_{\mathrm{L}^p(\mathbb{R}^{n-1})} \mathrm{d}t\right)^{\frac{1}{p}} + \left(\int_0^{+\infty} \left(t^{\frac{1}{p}} \|e^{-t(-\Delta')^{\frac{1}{2}}} b\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}\right)^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}} \\
&\lesssim_{p,n} \|a\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} + \|b\|_{\dot{\mathrm{B}}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})},
\end{aligned}$$

thus, one may take the infimum on all such pair (a, b) to obtain,

$$\|\mathcal{L}_+ f\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \lesssim_{p,n} \|f\|_{\mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

Now, we can use the higher order reflection extension operator E introduced in the proof of Proposition 2.17 to define $\mathcal{L} := \mathrm{E}\mathcal{L}_+$. Thus, due to the above boundedness properties, it follows that

$$\|\mathcal{L} f\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n,m} \|\mathcal{L}_+ f\|_{\mathrm{L}^p(\mathbb{R}^n_+)} \lesssim_{p,n} \|f\|_{\mathrm{B}^{-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

It has been proved, see [BL76, Exercices 25, 26, p.166], that $\mathcal{L}$ also satisfies, for all $f \in$

$\mathrm{B}^{k-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$,

$$\|\mathcal{L}f\|_{\mathrm{H}^{k,p}(\mathbb{R}^n)} \lesssim_{p,n} \|f\|_{\mathrm{B}^{k-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})},$$

for all $1 \leqslant k \leqslant m+1$. Finally, the result follows by complex and real interpolation, and $\mathrm{Ext} = \mathcal{L}$ is the desired right bounded inverse. ∎

Remark 2.47 In the above proof, the extension operator from the boundary to the whole space depends on some fixed regularity degree, which make it non-universal. If one wants an universal extension operator from the boundary to the whole space, one may replace the use of E from the proof of Proposition 2.17 by the use of Stein's extension operator on the half-space, check [Ste70, Section VI, Theorem 5']].

Corollary 2.48 *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$, $s \in (\frac{1}{p},+\infty)$, we have continuous embeddings :*

(i) $\mathrm{H}^{s,p}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))$ *;*

(ii) $\mathrm{B}^s_{p,q}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))$ *;*

(iii) $\mathrm{B}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1}))$ *;*

(iv) $\mathrm{B}^s_{p,\infty}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{b,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,\infty}(\mathbb{R}^{n-1}) - \mathit{weak}^*)$.

Proof. — We only check validity of the embedding

$$\mathrm{H}^{s,p}(\mathbb{R}^n) \hookrightarrow \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})).$$

Let $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, for $t > 0$, for almost every $x = (x', x_n) \in \mathbb{R}^n$, we introduce $u_t(x', x_n) := u(x', x_n + t)$, we have $u_t \in \mathrm{H}^{s,p}(\mathbb{R}^n)$, and by Theorem 2.45,

$$\begin{aligned} \|\gamma_0 u_t\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} &\lesssim_{p,s,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}, \\ \|\gamma_0(u_t - u)\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} &\lesssim_{p,s,n} \|u_t - u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}. \end{aligned}$$

Therefore, by strong continuity of translation in Lebesgue spaces, then in Sobolev spaces, we obtain

$$\|\gamma_0(u_t - u)\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u_t - u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)} \xrightarrow[t\to 0]{} 0.$$

Hence, $u \in \mathrm{C}^0_{b,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))$, with the estimate,

$$\left\| t \mapsto \|u(\cdot,t)\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \right\|_{\mathrm{L}^\infty(\mathbb{R})} \lesssim_{p,s,n} \|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n)}.$$

Finally, one can approximate u by Schwartz functions to deduce

$$u \in \mathrm{C}^0_{0,x_n}(\mathbb{R}, \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})).$$

One may perform a similar proof for all other cases, and one may check [Gui91b, Proposition 1.9] for the continuity of translation in Besov spaces, one may also use a density and an interpolation argument. ∎

5.2 On homogeneous function spaces.

Theorem 2.49 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (\frac{1}{p},+\infty)$, then for $(\mathfrak{h},\mathfrak{b}) \in \{(\mathrm{H},\mathrm{B}),(\dot{\mathrm{H}},\dot{\mathrm{B}})\}$, we consider the trace operator*

$$\gamma_0 \,:\, u \longmapsto u(\cdot,0).$$

The following assertions are true.

(i) For all $u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$, we have $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\mathfrak{b}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))$, with the estimate

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathfrak{b}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|u\|_{\mathfrak{h}^{s,p}(\mathbb{R}^n_+)};$$

In particular, the trace operator extends uniquely to a bounded linear operator

$$\gamma_0 \,:\, \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \to \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$$

whenever $(\mathcal{C}_{s,p})$ is satisfied, and the following continuous embedding holds

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})).$$

(ii) For all $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$, we have $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\mathfrak{b}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))$, with the estimate

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathfrak{b}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|u\|_{\mathfrak{b}^{s}_{p,q}(\mathbb{R}^n_+)};$$

In particular, the trace operator extends uniquely to a bounded linear operator

$$\gamma_0 \,:\, \dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+) \to \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$$

whenever $(\mathcal{C}_{s,p,q})$ is satisfied, and the following continuous embedding holds

$$\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})).$$

If $q = +\infty$, the result still holds with uniform boundedness and weak* continuity only.

(iii) For all $u \in \mathrm{B}^{1/p}_{p,1}(\mathbb{R}^n_+)$, we have $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\mathrm{L}^p(\mathbb{R}^{n-1}))$, with the estimate

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathrm{L}^p(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|u\|_{\mathfrak{b}^{1/p}_{p,1}(\mathbb{R}^n_+)};$$

In particular, the trace operator extends uniquely to a bounded linear operator

$$\gamma_0 \,:\, \dot{\mathrm{B}}^{1/p}_{p,1}(\mathbb{R}^n_+) \to \mathrm{L}^p(\mathbb{R}^{n-1})$$

and the following continuous embedding holds

$$\dot{\mathrm{B}}^{1/p}_{p,1}(\mathbb{R}^n_+) \hookrightarrow \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\mathrm{L}^p(\mathbb{R}^{n-1})).$$

Moreover,

(a) If $(\mathfrak{h},\mathfrak{b}) = (\mathrm{H},\mathrm{B})$, the trace operator γ_0 admits a linear right bounded inverse $\mathrm{Ext}_{\mathbb{R}^n_+}$ in cases (i) and (ii).

(b) If $(\mathfrak{h},\mathfrak{b}) = (\dot{\mathrm{H}},\dot{\mathrm{B}})$, the trace operator γ_0 admits a linear right bounded inverse $\underline{\mathrm{Ext}}_{\mathbb{R}^n_+}$ in cases (i) and (ii).

Proof. — We cut the proof in several steps.

Step 1 : The case $(\mathfrak{h},\mathfrak{b})=(\mathrm{H},\mathrm{B})$.

The result is a direct consequence of Corollary 2.45, and the definition of functions space by restriction.

One choose $\mathrm{Ext}_{\mathbb{R}^n_+}=\mathcal{L}_+$ introduced in the proof of Theorem 2.45 which satisfies the desired boundedness properties.

Step 2.1 : The case $(\mathfrak{h},\mathfrak{b})=(\dot{\mathrm{H}},\dot{\mathrm{B}})$. Boundedness of the trace operator.

We only achieve the case *(ii)* other ones can be done similarly. From the Step 1, and for fixed $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s>\frac{1}{p}$, and $u\in\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$, we have

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))}\lesssim_{p,s,n}\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))}\lesssim_{s,p,n}\|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)}.$$

Thus, one may use the fact that $\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)=\mathrm{L}^p(\mathbb{R}^n_+)\cap\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, which comes from Proposition 2.37, to obtain

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))}\lesssim_{s,p,n,q}\|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)}+\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

So that, by a dilation argument, replacing u, by $u_\lambda:=u(\lambda\cdot)$, for $\lambda\in2^{\mathbb{N}}$,

$$\lambda^{s-\frac{n}{p}}\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))}\lesssim_{s,p,n,q}\lambda^{-\frac{n}{p}}\|u\|_{\mathrm{L}^p(\mathbb{R}^n_+)}+\lambda^{s-\frac{n}{p}}\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

Hence, we can divide by $\lambda^{s-\frac{n}{p}}$ on both sides and pass to the limit $\lambda\longrightarrow+\infty$,

$$\|u\|_{\mathrm{L}^\infty_{x_n}(\mathbb{R}_+,\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))}\lesssim_{s,p,n,q}\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

Therefore, if $q<+\infty$, and $(\mathcal{C}_{s,p,q})$ is satisfied, the embedding

$$\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)\hookrightarrow\mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}))$$

holds by density. If $q=+\infty$, and $(\mathcal{C}_{s,p,q})$ is satisfied, the result follows from real interpolation.

Step 2.2 : The case $(\mathfrak{h},\mathfrak{b})=(\dot{\mathrm{H}},\dot{\mathrm{B}})$. Boundedness of the extension operator.

The operator T given by Proposition 2.43 is an appropriate extension operator which satisfies the desired boundedness properties. Thus $\underline{\mathrm{Ext}}_{\mathbb{R}^n_+}:=T$ behaves as expected. ∎

A raised question is about what happens when we want to deal with intersection of homogeneous Sobolev and Besov spaces.

Proposition 2.50 *Let $p\in(1,+\infty)$, $q\in[1,+\infty)$, $-1+\frac{1}{p}<s_0<\frac{1}{p}<s_1$, and $\theta\in(0,1)$ such that*

$$\frac{1}{p}=(1-\theta)s_0+\theta s_1.$$

Then,

(i) For all $u\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)\cap\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)$, we have $\gamma_0u\in\mathrm{B}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$, with the estimate

$$\|\gamma_0u\|_{\mathrm{B}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\lesssim_{s_0,s_1,p,n}\|u\|^{1-\theta}_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n_+)}\|u\|^{\theta}_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}+\|u\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)}.$$

We also have,

$$\|\gamma_0u\|_{\dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\lesssim_{s_0,s_1,p,n}\|u\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n_+)};$$

(ii) For all $u \in \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)$*, we have* $\gamma_0 u \in \mathrm{B}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})$*, with the estimate*

$$\|\gamma_0 u\|_{\mathrm{B}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+)} \|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)} + \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)}.$$

We also have,

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)};$$

(iii) For all $u \in \dot{\mathrm{B}}^{s_0}_{p,\infty}(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^{s_1}_{p,\infty}(\mathbb{R}^n_+)$*, we have* $\gamma_0 u \in \mathrm{L}^p(\mathbb{R}^{n-1})$*, with the estimate*

$$\|\gamma_0 u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,\infty}(\mathbb{R}^n_+)} \|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,\infty}(\mathbb{R}^n_+)}.$$

Proof. — We only start proving the point *(ii)*, and claim that point *(i)* can be achieved in a similar manner. We start noticing, the following continuous embedding,

$$\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+) \overset{\iota}{\hookrightarrow} (\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+), \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+))_{\theta,1} = \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n_+) \overset{\gamma_0}{\hookrightarrow} \mathrm{L}^p(\mathbb{R}^{n-1}).$$

Here, ι is the canonical embedding obtained via standard interpolation theory, and the last embedding via the trace operator is a direct consequence of Theorem 2.49, and everything can be turned into the following inequality,

$$\|\gamma_0 u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \lesssim_{s_0,s_1,p,n} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+)} \|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)}, \ \forall u \in \dot{\mathrm{B}}^{s_0}_{p,q}(\mathbb{R}^n_+) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+).$$

Again, from Theorem 2.49 we obtain for all $u \in \mathcal{S}_0(\overline{\mathbb{R}^n_+})$,

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{s_1,p,n} \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\mathbb{R}^n_+)}.$$

Then one may sum both inequality, notice that $\mathrm{L}^p(\mathbb{R}^{n-1}) \cap \dot{\mathrm{B}}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}) = \mathrm{B}^{s_1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})$ and use the density argument provided by Proposition 2.37 so that each estimate holds. ∎

Remark 2.51 As in Theorem 2.49, above Proposition 2.50 could be turned into a C^0_{0,x_n}-embedding in the appropriate Besov space.

Proposition 2.52 *Let* $p_j \in (1,+\infty)$*,* $q_j \in [1,+\infty)$*,* $s_j > 1/p_j$*,* $j \in \{0,1\}$*, such that* $(\mathcal{C}_{s_0,p_0})$ *(resp.* $(\mathcal{C}_{s_0,p_0,q_0})$*) is satisfied. Then,*

(i) For all $u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$*, we have* $\gamma_0 u \in \dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,p_j}(\mathbb{R}^{n-1})$*,* $j \in \{0,1\}$*, with the estimate*

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,p_j}(\mathbb{R}^{n-1})} \lesssim_{s_j,p_j,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)};$$

(ii) For all $u \in [\dot{\mathrm{B}}^{s_0}_{p_0,q_0} \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\mathbb{R}^n_+)$*, we have* $\gamma_0 u \in \dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,q_j}(\mathbb{R}^{n-1})$*,* $j \in \{0,1\}$*, with the estimate*

$$\|\gamma_0 u\|_{\dot{\mathrm{B}}^{s_j-\frac{1}{p_j}}_{p_j,q_j}(\mathbb{R}^{n-1})} \lesssim_{s_j,p_j,n} \|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n_+)};$$

Remark 2.53 Corollary 2.44 yields the ontoness of the trace operator on intersection spaces given by above Proposition 2.52.

Lemma 2.54 *Let* $p_j \in (1,+\infty)$*,* $s \in (1/p_j, 1+1/p_j)$*,* $j \in \{0,1\}$ *such that* $(\mathcal{C}_{s_0,p_0})$ *is satisfied. For all* $u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+,\mathbb{C})$ *such that* $u_{|_{\partial\mathbb{R}^n_+}} = 0$*, the extension* $\tilde{u}$ *to* $\mathbb{R}^n$ *by* 0*, satisfies*

$$\tilde{u} \in [\dot{\mathrm{H}}^{s_0,p_0}_0 \cap \dot{\mathrm{H}}^{s_1,p_1}_0](\mathbb{R}^n_+,\mathbb{C})$$

with the estimate

$$\|\tilde{u}\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{p_j,s_j,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}\,,\ j\in\{0,1\}.$$

The result still holds replacing $\dot{\mathrm{H}}^{s_j,p_j}$ *by* $\dot{\mathrm{B}}^{s_j}_{p_j,q_j}$, $q_j\in[1,+\infty]$, $j\in\{0,1\}$ *assuming that* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied.*

Proof. — Let $u\in[\dot{\mathrm{H}}^{s_0,p_0}_0\cap\dot{\mathrm{H}}^{s_1,p_1}_0](\mathbb{R}^n_+,\mathbb{C})$ such that $u_{|_{\partial\mathbb{R}^n_+}}=0$, then for all $\phi\in[\dot{\mathrm{H}}^{1-s_j,p_j'}\cap\mathcal{S}](\mathbb{R}^n,\mathbb{C}^n)$, we have

$$\int_{\mathbb{R}^n_+}\nabla u\cdot\phi=-\int_{\mathbb{R}^n_+}\mathrm{div}\,(\phi)\,u.$$

So that introducing the extensions by 0 to $\mathbb{R}^n$, $\tilde{u}$ and $\widetilde{\nabla u}$,

$$\int_{\mathbb{R}^n}\widetilde{\nabla u}\cdot\phi=\int_{\mathbb{R}^n_+}\nabla u\cdot\phi=-\int_{\mathbb{R}^n_+}\mathrm{div}\,(\phi)\,u=-\int_{\mathbb{R}^n}\mathrm{div}\,(\phi)\,\tilde{u}=\big\langle\nabla\tilde{u},\phi\big\rangle_{\mathbb{R}^n}.$$

Therefore, for all $\phi\in[\dot{\mathrm{H}}^{1-s_j,p_j'}\cap\mathcal{S}](\mathbb{R}^n,\mathbb{C}^n)$,

$$\int_{\mathbb{R}^n}\widetilde{\nabla u}\cdot\phi=\big\langle\nabla\tilde{u},\phi\big\rangle_{\mathbb{R}^n}.$$

Hence $\widetilde{\nabla u}=\nabla\tilde{u}$ in $\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^n)$. Thus, by Propositions 2.11 and 2.15, we deduce that

$$\begin{aligned}\big|\big\langle\nabla\tilde{u},\phi\big\rangle_{\mathbb{R}^n}\big|&\leqslant\|\phi\|_{\dot{\mathrm{H}}^{1-s_j,p_j'}(\mathbb{R}^n)}\|\widetilde{\nabla u}\|_{\dot{\mathrm{H}}^{s_j-1,p_j}(\mathbb{R}^n)}\\&\lesssim_{p_j,n,s_j}\|\phi\|_{\dot{\mathrm{H}}^{1-s_j,p_j'}(\mathbb{R}^n)}\|\nabla u\|_{\dot{\mathrm{H}}^{s_j-1,p_j}(\mathbb{R}^n_+)}\\&\lesssim_{p_j,n,s_j}\|\phi\|_{\dot{\mathrm{H}}^{1-s_j,p_j'}(\mathbb{R}^n)}\|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)}.\end{aligned}$$

One may conclude thanks to Proposition 2.11, and Corollary 2.21. The case of Besov spaces follows the same lines. ∎

The following corollary is then immediate

Corollary 2.55 *Let* $p_j\in(1,+\infty)$, $s_j\in(1/p_j,1+1/p_j)$, $j\in\{0,1\}$ *such that* $(\mathcal{C}_{s_0,p_0})$ *is satisfied. We have the following canonical isomorphism of Banach spaces*

$$\{\,u\in[\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+,\mathbb{C})\,|\,u_{|_{\partial\mathbb{R}^n_+}}=0\,\}\simeq[\dot{\mathrm{H}}^{s_0,p_0}_0\cap\dot{\mathrm{H}}^{s_1,p_1}_0](\mathbb{R}^n_+,\mathbb{C}).$$

The result still holds replacing $\dot{\mathrm{H}}^{s_j,p_j}$ *by* $\dot{\mathrm{B}}^{s_j}_{p_j,q_j}$, $q_j\in[1,+\infty]$, $j\in\{0,1\}$ *assuming that* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied.*

6 Applications : the Dirichlet and Neumann Laplacians on the half-space

Before starting the analysis of Dirichlet and Neumann Laplacians on the half space, we introduce two appropriate extension operators. We denote $\mathrm{E}_{\mathcal{J}}$, for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, the extension operator

defined for any measurable function u on $\mathbb{R}^n_+$, for almost every $x=(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}_+$:

$$\mathrm{E}_{\mathcal{D}}u(x',x_n):=\begin{cases}u(x',x_n) & ,\text{ if } (x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}_+,\\ -u(x',-x_n) & ,\text{ if } (x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^*_-\,;\end{cases}$$
$$\mathrm{E}_{\mathcal{N}}u(x',x_n):=\begin{cases}u(x',x_n) & ,\text{ if } (x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}_+,\\ u(x',-x_n) & ,\text{ if } (x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^*_-.\end{cases}$$

Obviously, for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, $s\in(-1+1/p,1/p)$, $p\in(1,+\infty)$, the Proposition 2.15 leads to boundedness of

$$\mathrm{E}_{\mathcal{J}}\,:\,\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)\longrightarrow\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n). \tag{2.26}$$

The same result holds replacing $\dot{\mathrm{H}}^{s,p}$ by either $\mathrm{H}^{s,p}$, $\mathrm{B}^s_{p,q}$, or even by $\dot{\mathrm{B}}^s_{p,q}$, $q\in[1,+\infty]$.

We are going to use the properties of Laplacian acting on the whole space to build resolvent estimates for both the the Dirichlet and the Neumann Laplacian. Usual Dirichlet and Neumann Laplacians are the operators $(\mathrm{D}(\Delta_{\mathcal{J}}),-\Delta_{\mathcal{J}})$, for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, where the subscript $\mathcal{D}$ (resp. $\mathcal{N}$) stands for the Dirichlet (resp. Neumann) Laplacian, with, for $p\in(1,+\infty)$,

$$\mathrm{D}_p(\Delta_{\mathcal{D}}):=\left\{\,u\in\mathrm{H}^{1,p}(\mathbb{R}^n_+,\mathbb{C})\,\middle|\,\Delta u\in\mathrm{L}^p(\mathbb{R}^n_+,\mathbb{C})\text{ and }u_{|_{\partial\mathbb{R}^n_+}}=0\right\},$$
$$\mathrm{D}_p(\Delta_{\mathcal{N}}):=\left\{\,u\in\mathrm{H}^{1,p}(\mathbb{R}^n_+,\mathbb{C})\,\middle|\,\Delta u\in\mathrm{L}^p(\mathbb{R}^n_+,\mathbb{C})\text{ and }\partial_\nu u_{|_{\partial\mathbb{R}^n_+}}=0\right\}.$$

For $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, and all $u\in\mathrm{D}_p(\Delta_{\mathcal{J}})$,

$$-\Delta_{\mathcal{J}}u:=-\Delta u.$$

When $p=2$, one can also realize both Dirichlet and Neumann Laplacians by the mean of densely defined, symmetric, accretive, continuous, closed, sesquilinear forms on $\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$,

$$\mathfrak{a}_{\mathcal{J}}\,:\,\mathrm{D}_2(\mathfrak{a}_{\mathcal{J}})^2\ni(u,v)\longmapsto\int_{\mathbb{R}^n_+}\nabla u\cdot\overline{\nabla v} \tag{2.27}$$

with $\mathrm{D}_2(\mathfrak{a}_{\mathcal{D}})=\mathrm{H}^1_0(\mathbb{R}^n_+,\mathbb{C})$, $\mathrm{D}_2(\mathfrak{a}_{\mathcal{N}})=\mathrm{H}^1(\mathbb{R}^n_+,\mathbb{C})$, so that it is easy to see, and well-known, that both, the Neumann and Dirichlet Laplacians, are closed, densely defined, non-negative self-adjoint operators on $\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, see [Ouh05, Chapter 1, Section 1.2]. We can be even more precise.

Proposition 2.56 *Provided $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, the operator $(\mathrm{D}_2(\Delta_{\mathcal{J}}),-\Delta_{\mathcal{J}})$ is an injective non-negative self-adjoint and 0-sectorial operator on $\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$.*

Moreover, the following hold

(i) $\mathrm{D}_2(\Delta_{\mathcal{J}})$ is a closed subspace of $\mathrm{H}^2(\mathbb{R}^n_+,\mathbb{C})$;

(ii) Provided $\mu\in[0,\pi)$, for $\lambda\in\Sigma_\mu$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$, then $u:=(\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f$ satisfies

$$|\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}+|\lambda|^{\frac12}\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}+\|\nabla^2u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\lesssim_{n,\mu}\|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\,;$$

(iii) The following resolvent identity holds for all $\mu\in[0,\pi)$, $\lambda\in\Sigma_\mu$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\mathbb{C})$,

$$\mathrm{E}_{\mathcal{J}}(\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f=(\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{J}}f.$$

Remark 2.57 For $u\,:\,\mathbb{R}^n_+\longrightarrow\mathbb{C}$, we set

$$\tilde{u}_{\mathcal{J}}:=[\mathrm{E}_{\mathcal{J}}u]_{|_{\mathbb{R}^n_-}}$$

for $\mathcal{J} \in \{\mathcal{D}, \mathcal{N}\}$. We notice that in $\mathcal{D}'(\mathbb{R}^n_-, \mathbb{C})$,

$$\partial_{x_n}[\tilde{u}_{\mathcal{N}}] = \widetilde{[\partial_{x_n} u]}_{\mathcal{D}} \text{ and } \partial_{x_n}[\tilde{u}_{\mathcal{D}}] = \widetilde{[\partial_{x_n} u]}_{\mathcal{N}}.$$

Proof. — One may use self-adjointness and (2.27) which gives, by standard Hilbertian theory, the following resolvent estimate

$$|\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} + \|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} \lesssim_\mu \|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)},$$

where $u := (\lambda \mathrm{I} - \Delta_{\mathcal{J}})^{-1} f$, $f \in \mathrm{L}^2(\mathbb{R}^n_+, \mathbb{C})$, $\lambda \in \Sigma_\mu$, $\mu \in [0, \pi)$.

Now, for fixed $f \in \mathrm{L}^2(\mathbb{R}^n_+, \mathbb{C})$, $\lambda \in \Sigma_\mu$, $\mu \in [0, \pi)$, we consider $u := (\lambda \mathrm{I} - \Delta_{\mathcal{J}})^{-1} f$. Assuming $\mathcal{J} = \mathcal{N}$, we have for $\phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$,

$$\begin{aligned}
\big\langle \mathrm{E}_{\mathcal{N}} u, -\Delta\phi \big\rangle_{\mathbb{R}^n} &= \big\langle u, -\Delta\phi \big\rangle_{\mathbb{R}^n_+} + \big\langle \tilde{u}_{\mathcal{N}}, -\Delta\phi \big\rangle_{\mathbb{R}^n_-} \\
&= \big\langle \nabla u, \nabla\phi \big\rangle_{\mathbb{R}^n_+} + \big\langle u, \nabla\phi \cdot \mathfrak{e}_{\mathfrak{n}} \big\rangle_{\partial\mathbb{R}^n_+} - \big\langle \tilde{u}_{\mathcal{N}}, \nabla\phi \cdot \mathfrak{e}_{\mathfrak{n}} \big\rangle_{\partial\mathbb{R}^n_-} \\
&\quad + \big\langle \widetilde{[\nabla' u]}_{\mathcal{N}}, \nabla'\phi \big\rangle_{\mathbb{R}^n_-} + \big\langle \widetilde{[\partial_{x_n} u]}_{\mathcal{D}}, \partial_{x_n}\phi \big\rangle_{\mathbb{R}^n_-}
\end{aligned}$$

Since $\partial\mathbb{R}^n_+ = \partial\mathbb{R}^n_- = \mathbb{R}^{n-1} \times \{0\}$, with traces $\tilde{u}_{\mathcal{N}|_{\partial\mathbb{R}^n_-}} = u_{|_{\partial\mathbb{R}^n_+}}$, we deduce $\big\langle u_{|_{\partial\mathbb{R}^n_+}}, \nabla\phi \cdot \mathfrak{e}_{\mathfrak{n}} \big\rangle_{\partial\mathbb{R}^n_+} - \big\langle \tilde{u}_{\mathcal{N}|_{\partial\mathbb{R}^n_-}}, \nabla\phi \cdot \mathfrak{e}_{\mathfrak{n}} \big\rangle_{\partial\mathbb{R}^n_-} = 0$. Then, thanks to Remark 2.57 and the boundary condition on u, *i.e.* $\partial_{x_n} u_{|_{\partial\mathbb{R}^n_+}} = 0$, we have

$$\begin{aligned}
\big\langle \mathrm{E}_{\mathcal{N}} u, -\Delta\phi \big\rangle_{\mathbb{R}^n} &= \big\langle \nabla u, \nabla\phi \big\rangle_{\mathbb{R}^n_+} + \big\langle \widetilde{[\nabla' u]}_{\mathcal{N}}, \nabla'\phi \big\rangle_{\mathbb{R}^n_-} + \big\langle \widetilde{[\partial_{x_n} u]}_{\mathcal{D}}, \partial_{x_n}\phi \big\rangle_{\mathbb{R}^n_-} \\
&= \big\langle -\Delta u, \phi \big\rangle_{\mathbb{R}^n_+} + \big\langle \widetilde{[-\Delta' u]}_{\mathcal{N}}, \phi \big\rangle_{\mathbb{R}^n_-} + \big\langle \widetilde{[-\partial^2_{x_n} u]}_{\mathcal{N}}, \phi \big\rangle_{\mathbb{R}^n_-} \\
&\quad - \big\langle \partial_{x_n} u, \phi \big\rangle_{\partial\mathbb{R}^n_+} - \big\langle \widetilde{[\partial_{x_n} u]}_{\mathcal{D}}, \phi \big\rangle_{\partial\mathbb{R}^n_-} \\
&= \big\langle \mathrm{E}_{\mathcal{N}}[-\Delta u], \phi \big\rangle_{\mathbb{R}^n}.
\end{aligned}$$

Thus, $-\Delta \mathrm{E}_{\mathcal{N}} u = \mathrm{E}_{\mathcal{N}}[-\Delta u]$ in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$. One may reproduce above calculations for $\mathcal{J} = \mathcal{D}$. So for $\mathcal{J} \in \{\mathcal{D}, \mathcal{N}\}$, $\mathrm{E}_{\mathcal{J}} u$ is a solution of

$$\lambda U - \Delta U = \mathrm{E}_{\mathcal{J}} f.$$

We have $\mathrm{E}_{\mathcal{J}} f \in \mathrm{L}^2(\mathbb{R}^n, \mathbb{C})$. By uniqueness of the solution provided in $\mathbb{R}^n$, we necessarily have $U = \mathrm{E}_{\mathcal{J}} u$, which can be written as

$$\mathrm{E}_{\mathcal{J}}(\lambda \mathrm{I} - \Delta_{\mathcal{J}})^{-1} f = (\lambda \mathrm{I} - \Delta)^{-1} \mathrm{E}_{\mathcal{J}} f.$$

Thus one deduces point *(iii)*, from the definition of function spaces by restriction, *(ii)* follows, and finally setting $\lambda = 1$ in point *(ii)* yields *(i)*. ∎

We want to show some sharp regularity results on the Dirichlet an Neumann resolvent problems, on the scale of inhomogeneous and homogeneous Sobolev and Besov spaces. To do so, we introduce their corresponding domains on each space. Provided $p \in (1, +\infty)$ $s \in (-1 + 1/p, 1 + 1/p)$, if is satisfied $(\mathcal{C}_{s,p})$:

$$\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{D}}) := \Big\{ u \in [\dot{\mathrm{H}}^{s,p}_0 \cap \dot{\mathrm{H}}^{s+1,p}](\mathbb{R}^n_+, \mathbb{C}) \,\Big|\, \Delta u \in \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+, \mathbb{C}) \text{ and } u_{|_{\partial\mathbb{R}^n_+}} = 0 \Big\} \subset \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+, \mathbb{C}),$$

$$\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{N}}) := \Big\{ u \in [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+1,p}](\mathbb{R}^n_+, \mathbb{C}) \,\Big|\, \Delta u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C}) \text{ and } \partial_\nu u_{|_{\partial\mathbb{R}^n_+}} = 0 \Big\} \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C}).$$

We can also consider their domains on inhomogeneous Sobolev and Besov spaces, as well as

homogeneous spaces, replacing $(\dot{\mathrm{D}}^s_p, \dot{\mathrm{H}}^{s,p})$ by either $(\mathrm{D}^s_p, \mathrm{H}^{s,p})$, $(\mathrm{D}^s_{p,q}, \mathrm{B}^s_{p,q})$ and finally $(\dot{\mathrm{D}}^s_{p,q}, \dot{\mathrm{B}}^s_{p,q})$ provided $q \in [1, +\infty]$, and $(\mathcal{C}_{s,p,q})$ is satisfied.

It is then not difficult to see that the Dirichlet and Neumann Laplacians are well defined unbounded closed linear operators, densely defined, if $q \in [1, +\infty)$ in the case of inhomogeneous and homogeneous Besov spaces. If $q = +\infty$, the domain of the Dirichlet (resp. Neumann) Laplacian is only known to be weak* dense in $\mathrm{B}^s_{p,\infty,0}$ (resp. in $\mathrm{B}^s_{p,\infty}$) and $\dot{\mathrm{B}}^s_{p,\infty,0}$ (resp. $\dot{\mathrm{B}}^s_{p,\infty}$).

Proposition 2.58 *Let $p, \tilde{p} \in (1, +\infty)$, $q, \tilde{q} \in [1, +\infty]$, $s \in (-1 + \frac{1}{p}, 1 + \frac{1}{p})$, $s \neq 1/p$, $\alpha \in (-1 + \frac{1}{\tilde{p}}, 1 + \frac{1}{\tilde{p}})$, $\alpha \neq 1/\tilde{p}$, and $\lambda \in \Sigma_\mu$ provided $\mu \in [0, \pi)$. We assume that $(\mathcal{C}_{s,p})$, and we let $f \in \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+, \mathbb{C})$. Let us consider the **resolvent Dirichlet** problem with homogeneous boundary condition :*

$$\left\{ \begin{array}{rll} \lambda u - \Delta u & = f, & \text{in } \mathbb{R}^n_+, \\ u_{|_{\partial \mathbb{R}^n_+}} & = 0, & \text{on } \partial \mathbb{R}^n_+. \end{array} \right. \tag{$\mathcal{DL}_\lambda$}$$

The problem $(\mathcal{DL}_\lambda)$ admits a unique solution $u \in [\dot{\mathrm{H}}^{s,p}_0 \cap \dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n_+, \mathbb{C})$ with the estimate

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

If moreover $\alpha \neq 1/\tilde{p}$ and $f \in \dot{\mathrm{H}}^{\alpha,\tilde{p}}_0(\mathbb{R}^n_+, \mathbb{C})$, then we also have $u \in [\dot{\mathrm{H}}^{\alpha,\tilde{p}} \cap \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n_+, \mathbb{C})$ with the corresponding estimate

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha,\mu} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)}.$$

The result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}})$ by $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}})$ whenever $(\mathcal{C}_{s,p,q})$ is satisfied.

Remark 2.59 • For this specific Proposition 2.58, we have excluded the cases $s = 1/p$ and $\alpha = 1/\tilde{p}$. Both require to introduce, e.g. in case of Sobolev spaces, the homogeneous counterpart of the Lions-Magenes Sobolev space $\dot{\mathrm{H}}^{1/q,q}_{00}(\mathbb{R}^n_+)$, $q \in \{p, \tilde{p}\}$. See for instance [LM72, Chapter 1, Theorem 11.7] for the inhomogeneous space in the case $q = 2$.

• We bring to the attention of the reader that $(\mathcal{C}_{\alpha,\tilde{p}})$ is **NEVER** assumed, only $(\mathcal{C}_{s,p})$ is. This is in order to echo the principle of decoupled estimates in intersection spaces when one wants to deal with higher regularities involving some non-complete spaces. All the results below are following the same principle.

Proof. — Provided $p \in (1, +\infty)$, and firstly that $s \in (-1 + 1/p, 1/p)$, for $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$, since one is aware of (2.26), which follows from Proposition 2.15, one has for $U := (\lambda \mathrm{I} - \Delta)^{-1} \mathrm{E}_{\mathcal{D}} f$

$$|\lambda| \|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}} \|\nabla U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + \|\nabla^2 U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

Thus, by definition of function space by restriction, we set $u := U_{|_{\mathbb{R}^n_+}}$ which satisfies

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)},$$

then the map $f \mapsto [(\lambda \mathrm{I} - \Delta)^{-1} \mathrm{E}_{\mathcal{D}} f]_{|_{\mathbb{R}^n_+}}$ is a bounded map on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$. Everything goes similarly for $\mathrm{H}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$. One may check, as in the proof of Proposition 2.56, and by a limiting argument, given the density of $[\mathrm{L}^2 \cap \dot{\mathrm{H}}^{s,p}](\mathbb{R}^n_+, \mathbb{C})$ in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$, that $u_{|_{\partial \mathbb{R}^n_+}} = 0$, and

$$\lambda u - \Delta u = f \text{ in } \mathbb{R}^n_+.$$

Again, as in the proof of Proposition 2.56, one may check that any solution u to the above resolvent Dirichlet problem necessarily satisfies $\mathrm{E}_{\mathcal{D}} u = (\lambda \mathrm{I} - \Delta)^{-1} \mathrm{E}_{\mathcal{D}} f$.

Now if $s \in (1/p, 1+1/p)$, $f \in [\dot{\mathrm{H}}^{s-1,p}_0 \cap \dot{\mathrm{H}}^{s,p}_0](\mathbb{R}^n_+, \mathbb{C})$ then we have, thanks to previous considerations, $U := (\lambda \mathrm{I} - \Delta)^{-1} \mathrm{E}_{\mathcal{D}} f \in \dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n, \mathbb{C})$. It suffices to show that $U \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n, \mathbb{C})$, which is true. Indeed, we have

$$\begin{aligned} |\lambda| \|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} &\lesssim_{s,p,n,\mu} |\lambda| \|\nabla U\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)} \\ &\lesssim_{s,p,n,\mu} \|\nabla \mathrm{E}_{\mathcal{D}} f\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)} \\ &\lesssim_{s,p,n,\mu} \sum_{k=1}^{n} \|\partial_{x_k} \mathrm{E}_{\mathcal{D}} f\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)}. \end{aligned}$$

Since equalities $\partial_{x_k} \mathrm{E}_{\mathcal{D}} f = \mathrm{E}_{\mathcal{D}} \partial_{x_k} f$, $k \in [\![1, n-1]\!]$ and $\partial_{x_n} \mathrm{E}_{\mathcal{D}} f = \mathrm{E}_{\mathcal{N}} \partial_{x_n} f$ occur in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$, we deduce

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \leqslant |\lambda| \|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

One may proceed similarly as before to obtain the full estimate

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

Thus the estimates still hold by density for all $f \in \dot{\mathrm{H}}^{s,p}_0(\mathbb{R}^n_+)$, $s \in (-1+1/p, 1+1/p)$, $s \neq 1/p$, whenever $(\mathcal{C}_{s,p})$ is satisfied.

The $\dot{\mathrm{H}}^{\alpha,\tilde{p}}$-estimate for $f \in [\dot{\mathrm{H}}^{s,p}_0 \cap \dot{\mathrm{H}}^{\alpha,\tilde{p}}_0](\mathbb{R}^n_+)$ can be obtained the same way, whenever $(\mathcal{C}_{s,p})$ is satisfied.

The case of Besov spaces $\dot{\mathrm{B}}^s_{p,q,0}$ can be achieved via similar argument for $q < +\infty$, the case $q = +\infty$ is obtained via real interpolation. The case of the $\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q},0}$-estimate for $f \in \dot{\mathrm{B}}^s_{p,q,0} \cap \dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q},0}$ can be done as above. ∎

The proof for the Neumann resolvent problem in the proposition below is fairly similar to the proof of Proposition 2.58, a complex interpolation argument allows values $s = 1/p$ and $\alpha = 1/\tilde{p}$.

Proposition 2.60 *Let $p, \tilde{p} \in (1, +\infty)$, $q, \tilde{q} \in [1, +\infty]$, $s \in (-1+\frac{1}{p}, 1+\frac{1}{p})$, $\alpha \in (-1+\frac{1}{\tilde{p}}, 1+\frac{1}{\tilde{p}})$ and $\lambda \in \Sigma_\mu$ provided $\mu \in [0, \pi)$. We assume that $(\mathcal{C}_{s,p})$, and we let $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$. Let us consider the **resolvent Neumann** problem with homogeneous boundary condition :*

$$\left\{ \begin{array}{rll} \lambda u - \Delta u &= f, & \text{in } \mathbb{R}^n_+, \\ \partial_\nu u_{|_{\partial \mathbb{R}^n_+}} &= 0, & \text{on } \partial \mathbb{R}^n_+. \end{array} \right. \tag{$\mathcal{NL}_\lambda$}$$

The problem $(\mathcal{NL}_\lambda)$ admits a unique solution $u \in [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n_+, \mathbb{C})$ with the estimate

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

If moreover $f \in \dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+, \mathbb{C})$, then we also have $u \in [\dot{\mathrm{H}}^{\alpha,\tilde{p}} \cap \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n_+, \mathbb{C})$ with the corresponding estimate

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha,\mu} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)}.$$

The result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}})$ by $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}})$ whenever $(\mathcal{C}_{s,p,q})$ is satisfied.

Proposition 2.61 *Let $p, \tilde{p} \in (1, +\infty)$, $q, \tilde{q} \in [1, +\infty]$, $s \in (-1+\frac{1}{p}, +\infty)$, $\alpha \in (-1+\frac{1}{\tilde{p}}, +\infty)$ such that $(\mathcal{C}_{s+2,p})$ is satisfied. For $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \mathbb{C})$, $g \in \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \mathbb{C})$, let us consider the*

Dirichlet *problem with inhomogeneous boundary condition :*

$$\left\{\begin{array}{lll} -\Delta u &= f, & \text{in } \mathbb{R}^n_+, \\ u_{|_{\partial\mathbb{R}^n_+}} &= g, & \text{on } \partial\mathbb{R}^n_+. \end{array}\right. \tag{$\mathcal{DL}_0$}$$

The problem ($\mathcal{DL}_0$) *admits a unique solution* u *such that*

$$u \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+,\mathbb{C}) \subset \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}))$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

If moreover $f \in \dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+,\mathbb{C})$ *and* $g \in \dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1},\mathbb{C})$ *then the solution* u *also satisfies* $u \in \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}(\mathbb{R}^n_+,\mathbb{C})$ *with the corresponding estimate*

$$\|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1})}.$$

The result still holds if we replace both $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p})$ *and* $(\dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}, \dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}})$ *by* $(\dot{\mathrm{B}}^{s}_{p,q}, \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,q})$ *and* $(\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{q}})$ *whenever* $(\mathcal{C}_{s+2,p,q})$ *is satisfied,* $q < +\infty$.
If $q = +\infty$, *everything still holds except* $x_n \mapsto u(\cdot, x_n)$ *is no more strongly continuous but only weak* continuous with values in* $\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1},\mathbb{C})$.

Proof. — Let $p \in (1,+\infty)$, $s > -1+1/p$, such that $(\mathcal{C}_{s+2,p})$ is satisfied. Then for $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, $g \in \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C})$ we can write the problem ($\mathcal{DL}_0$) as an evolution problem in the x_n variable,

$$\left\{\begin{array}{lll} -\partial^2_{x_n} u - \Delta' u &= f, & \text{in } \mathbb{R}^{n-1}\times(0,+\infty), \\ u(\cdot,0) &= g, & \text{on } \mathbb{R}^{n-1}. \end{array}\right.$$

Thanks to [ABHN11, Theorem 3.8.3], considering the semigroup $(e^{-x_n(-\Delta')^{1/2}})_{x_n\geqslant 0}$ and its mapping properties given by Proposition 2.43 and Theorem 2.49, if $f = 0$, above problem admits unique solution $u \in \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}))$. Thus, by linearity, we also have uniqueness of the solution u in $\mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}))$ for non-identically zero function f. Therefore, it suffices to construct a solution.

Since $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, by definition, there exists $F \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\mathbb{C})$ such that

$$F_{|_{\mathbb{R}^n_+}} = f, \qquad \text{and} \qquad \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \sim \|F\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Let $v := (-\Delta)^{-1}F \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n,\mathbb{C})$, we also have

$$\|v\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|F\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

So it suffices to prove the result for $w \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+,\mathbb{C})$, such that

$$\left\{\begin{array}{lll} -\Delta w &= 0, & \text{in } \mathbb{R}^{n-1}\times(0,+\infty), \\ w_{|_{\partial\mathbb{R}^n_+}} &= \tilde{g}, & \text{on } \mathbb{R}^{n-1}, \end{array}\right.$$

where $\tilde{g} \in \dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C})$ can be seen as $g - v(\cdot,0)$. But such a w exists and is unique thanks

to Proposition 2.43 and [ABHN11, Theorem 3.8.3], and satisfies

$$\|w\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|\tilde{g}\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

Now, we can set $u := v + w$ which is a solution of $(\mathcal{DL}_0)$, and the triangle inequality leads to

$$\begin{aligned}\|u\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} &\leqslant \|v\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} + \|w\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \\ &\lesssim_{p,n,s} \|v\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} + \|v(\cdot,0)\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \\ &\lesssim_{p,n,s} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\end{aligned}$$

which was the desired bound.

The Besov spaces case for $(f,g) \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\mathbb{C}) \times \dot{\mathrm{B}}^{s+2-1/p}_{p,q}(\mathbb{R}^{n-1},\mathbb{C})$, whenever $(\mathcal{C}_{s+2,p,q})$ is satisfied, follows the same lines as before, except when $q = +\infty$ where the uniqueness argument can only be checked in a weak sense since $(e^{-x_n(-\Delta')^{1/2}})_{x_n\geqslant 0}$ is only weak* continuous in $\dot{\mathrm{B}}^{s+2-1/p}_{p,\infty}(\mathbb{R}^{n-1},\mathbb{C})$.

Now, if we assume that $f \in [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{\alpha,\tilde{p}}](\mathbb{R}^n_+,\mathbb{C})$ and $g \in [\dot{\mathrm{B}}^{s+2-1/p}_{p,p} \cap \dot{\mathrm{B}}^{\alpha+2-1/\tilde{p}}_{\tilde{p},\tilde{p}}](\mathbb{R}^{n-1},\mathbb{C})$, then with the same notations as above, by Proposition 2.52, we have

$$v = (-\Delta)^{-1}F \in [\dot{\mathrm{H}}^{s+2,p} \cap \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n,\mathbb{C}) \text{ and } v(\cdot,0) \in \dot{\mathrm{B}}^{s+2-1/p}_{p,p} \cap \dot{\mathrm{B}}^{\alpha+2-1/\tilde{p}}_{\tilde{p},\tilde{p}}](\mathbb{R}^{n-1},\mathbb{C}).$$

From this, one may reproduce the estimates as above to obtain

$$\|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{\alpha+2-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1})}.$$

The case of intersection of Besov spaces follows the same lines. ■

We state the same result for the corresponding Neumann problem, for which the proof is very close.

Proposition 2.62 *Let $p,\tilde{p} \in (1,+\infty)$, $q,\tilde{q} \in [1,+\infty]$, $s \in (-1+\frac{1}{p},+\infty)$, $\alpha \in (-1+\frac{1}{\tilde{p}},+\infty)$, such that $(\mathcal{C}_{s+2,p})$ is satisfied. For $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, $g \in \dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C})$, let us consider the **Neumann** problem with inhomogeneous boundary condition :*

$$\left\{\begin{array}{rll} -\Delta u &= f, & \text{in } \mathbb{R}^n_+, \\ \partial_\nu u_{|_{\partial\mathbb{R}^n_+}} &= g, & \text{on } \partial\mathbb{R}^n_+. \end{array}\right. \tag{$\mathcal{NL}_0$}$$

The problem $(\mathcal{NL}_0)$ admits a unique solution u such that

$$u \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+,\mathbb{C}) \subset \mathrm{C}^0_{0,x_n}(\overline{\mathbb{R}_+},\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\mathbb{C}))$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{s,p,n} \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}.$$

If moreover $f \in \dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+,\mathbb{C})$ and $g \in \dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1},\mathbb{C})$ then the solution u also satisfies $u \in \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}(\mathbb{R}^n_+,\mathbb{C})$ with the corresponding estimate

$$\|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} \lesssim_{\tilde{p},n,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n_+)} + \|g\|_{\dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}}(\mathbb{R}^{n-1})}.$$

The result still holds, replacing $(\dot{\mathrm{H}}^{s,p},\dot{\mathrm{H}}^{s+2,p},\dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,p},\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,p})$ by $(\dot{\mathrm{B}}^s_{p,q},\dot{\mathrm{B}}^{s+2}_{p,q},\dot{\mathrm{B}}^{s+1-\frac{1}{p}}_{p,q},\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,q})$

and $(\dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}, \dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{p}})$ *by* $(\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{\alpha+1-\frac{1}{\tilde{p}}}_{\tilde{p},\tilde{q}})$ *whenever* $(\mathcal{C}_{s+2,p,q})$ *is satisfied and* $q < +\infty$.

If $q = +\infty$, *everything still holds except* $x_n \mapsto u(\cdot, x_n)$ *is no more strongly continuous but only weak* continuous with values in* $\dot{\mathrm{B}}^{s+2-\frac{1}{p}}_{p,\infty}(\mathbb{R}^{n-1}, \mathbb{C})$.

Similar results but with proofs in a different spirit are available in [DM15, Chapter 3].

3 Global-in-time maximal regularity

Sommaire

Résumé du chapitre

Ce chapitre est principalement consacré à la preuve de la régularité maximale $\dot{H}^{\alpha,q}$ en temps pour une classe d'opérateurs sectoriels injectifs, mais non inversibles, sur un espace de Banach UMD X, à condition que $q \in (1, +\infty)$ et $\alpha \in (-1 + 1/q, 1/q)$. Nous prouvons également l'estimation de trace correspondante, de sorte que la solution du problème de Cauchy abstrait canonique est continue avec des valeurs dans un espace de trace non nécessairement complet. Ceci est fait afin de préserver la possibilité d'utiliser des réalisations d'espaces de fonctions homogènes pour traiter des équations paraboliques non-linéaires (avec une partie linéaire) dans un cadre abstrait approprié.

Ce "nouveau type" (pas tant que ça en fait) de régularité maximale offre de nouvelles perspectives pour améliorer la flexibilité du caratère bien posé global en temps de certaines équations paraboliques et a été principalement inspiré par les travaux de Prüss et Simonett [PS16, Chapitres 3 & 4], Danchin, Hieber, Mucha et Tolksdorf [DHMT21, Chapitre 2] et Brandolese et Monniaux [BM21].

Afin de mettre notre résultat en perspective, nous fournissons également un court état de l'art de la régularité maximale L^q qui inclut quelques avancées récentes telles que la théorie revisitée des opérateurs homogènes et de l'interpolation par Danchin, Hieber, Mucha et Tolksdorf. Cette théorie sera utilisée pour construire l'espace de traces approprié, à partir du quel nous voulons choisir les données initiales, et pour lequel la solution de notre problème de Cauchy abstrait sera continue en temps.

L'estimation de trace établie dans le théorème principal, Théorème 3.21, sera fondamentale dans le dernier Chapitre 5.

Les principaux outils qui seront employés au cours de ce chapitre seront ceux introduits dans la Section 3 du Chapitre 1.

Summary of the chapter

This chapter is mainly concerned by the proof of $\dot{\mathrm{H}}^{\alpha,q}$-maximal regularity for a class of injective, but not invertible, sectorial operators on a UMD Banach space X, provided $q \in (1, +\infty)$, $\alpha \in (-1 + 1/q, 1/q)$. We also prove the corresponding trace estimate, so that the solution to the canonical abstract Cauchy problem is continuous with values in a not necessarily complete trace space. This is done in order to capture the possibility to reach the use of meaningful realizations of homogeneous function spaces to deal with non-linear parabolic equations (with a linear part) with an appropriate abstract framework.

This "new kind" (not that much actually) of maximal regularity offers a new perspective to improve flexibility for global-in-time well-posedness of some parabolic equations and was mainly inspired by the works of Prüss and Simonett [PS16, Chapters 3 & 4], Danchin, Hieber, Mucha and Tolksdorf [DHMT21, Chapter 2] and Brandolese and Monniaux [BM21].

In order to put our result in perspective, we also provide a short review on L^q-maximal regularity which includes some recent advances such as the revisited homogeneous operator and interpolation theory by Danchin, Hieber, Mucha and Tolksdorf. This theory is used to build the appropriate trace space, from which we want to choose the initial data, and the solution of our abstract Cauchy problem to be continuous in time.

The trace estimate inherited in the main theorem, Theorem 3.21, will be central in the last Chapter 5.

The main tools of this chapter are the ones introduced in Section 3 of Chapter 1.

1 Introduction

Before, we start this chapter, we introduce the definition of Banach spaces with the **UMD** property.

We say that a Banach space $(X, \|\cdot\|_X)$ has the **Unconditional Martingale Differences** (UMD) property if the Hilbert transform $\mathbb{H}$, defined by

$$\mathbb{H}f(x) := \frac{1}{\pi}\mathbf{p.v.}\int_{\mathbb{R}} \frac{f(y)}{x-y}\,\mathrm{d}y, \qquad (f \in \mathcal{S}_0(\mathbb{R}, X),\ x \in \mathbb{R})$$

yields a bounded linear operator on $\mathrm{L}^2(\mathbb{R}, X)$.

Where, we (re-)introduce temporarily for the current chapter

$$\mathcal{S}_0(\mathbb{R}, X) := \left\{ u \in \mathcal{S}(\mathbb{R}, X) \,\middle|\, \operatorname{supp}(\mathcal{F}u) \text{ is compact, } 0 \notin \operatorname{supp}(\mathcal{F}u) \right\}.$$

For more details about UMD Banach spaces, see [HvNVW16, Chapters 4 & 5, Theorem 5.1.1], those Banach spaces being also called sometimes Banach spaces of class $\mathcal{HT}$ (for Hilbert Transform), see e.g.[PS16, Chapters 3 & 4].

1.1 Motivations and interests

1.1.1 The example of the Laplacian on $\mathbb{R}^n$

The L^q-maximal regularity is a very powerful and fundamental tool for the study of a wide range parabolic partial differential equations, that comes mainly from physics, geometry, or chemistry.

The usual theory built for sectorial operators on UMD Banach have been widely investigated. However, when it comes to look where the solution of the abstract Cauchy problem lies as a

continuous function of time, we have to restrict ourselves either to control in finite time, or to ask the operator to be invertible.

For instance, let us take a look at the scalar heat equation on $\mathbb{R}^n$, for $T \in (0, +\infty]$, $u_0 \in \mathcal{S}'(\mathbb{R}^n)$, $f \in \mathrm{L}^1_{\mathrm{loc}}([0,T), \mathcal{S}'(\mathbb{R}^n))$,

$$\left\{ \begin{array}{rl} \partial_t u(t) - \Delta u(t) = & f(t)\,,\ 0 < t < T, \\ u(0) = & u_0. \end{array} \right. \quad . \tag{HE}$$

The standard theory, see e.g. [Ama95, Remark 4.10.9], [PS16, Theorem 3.5.5], tells us that provided $f \in \mathrm{L}^q([0,T], \mathrm{L}^p(\mathbb{R}^n))$, $u_0 \in \mathrm{B}^{2-2/q}_{p,q}(\mathbb{R}^n)$, $T < +\infty$, $p, q \in (1, +\infty)$, the Cauchy problem (HE) admits a unique solution $u \in \mathrm{H}^{1,q}([0,T], \mathrm{L}^p(\mathbb{R}^n)) \cap \mathrm{L}^q([0,T], \mathrm{H}^{2,p}(\mathbb{R}^n)) \subset \mathrm{C}^0([0,T], \mathrm{B}^{2-2/q}_{p,q}(\mathbb{R}^n))$ which satisfies the estimates

$$\|u\|_{\mathrm{L}^\infty([0,T],\mathrm{B}^{2-2/q}_{p,q})} \lesssim_{n,p,q,T} \|(u, \partial_t u, \Delta u)\|_{\mathrm{L}^q([0,T],\mathrm{L}^p)} \lesssim_{n,p,q,T} \|f\|_{\mathrm{L}^q([0,T],\mathrm{L}^p)} + \|u_0\|_{\mathrm{B}^{2-2/q}_{p,q}}.$$

In these estimates, implicit constants are dependent of T and blow up as T goes to infinity. It is in fact even worse than that : one cannot expect a global-in-time estimate of this type. Indeed, such a control on the term $\|u\|_{\mathrm{L}^q([0,+\infty),\mathrm{L}^p)}$ would imply that the Laplacian Δ is invertible on $\mathrm{L}^p(\mathbb{R}^n)$, see for instance [CL86, Section 2] or [PS16, Corollary 3.5.3], which is known to be false.

However, when $f \in \mathrm{L}^q(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^n))$, $u_0 \in \dot{\mathrm{B}}^{2-2/q}_{p,q}(\mathbb{R}^n)$, there is still a unique solution u to (HE) such that $\partial_t u, \Delta u \in \mathrm{L}^q(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^n))$ and $u \in \mathrm{C}^0_b(\mathbb{R}_+, \dot{\mathrm{B}}^{2-2/q}_{p,q}(\mathbb{R}^n))$ with the global-in-time estimate

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\dot{\mathrm{B}}^{2-2/q}_{p,q})} \lesssim_{n,p,q} \|(\partial_t u, \Delta u)\|_{\mathrm{L}^q(\mathbb{R}_+,\mathrm{L}^p)} \lesssim_{n,p,q} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,\mathrm{L}^p)} + \|u_0\|_{\dot{\mathrm{B}}^{2-2/q}_{p,q}}. \tag{3.1}$$

This result is well known, but while the right-hand side estimate of (3.1) arises from the usual theory when $u_0 = 0$, see e.g., [Haa06, Proposition 8.3.4, Corollary 9.3.12], this is however not the case for the left hand side **trace estimate** and to obtain the space from which we choose the initial data u_0, see for instance [BCD11, Theorem 2.34]. A reason is that the usual theory for traces in maximal regularity will only produce an inhomogeneous Besov space, which is not suitable : it makes us lose again the uniform control with respect to time on the left-hand side part of (3.1). The same kind of issue would happen for other injective, but non-invertible, sectorial operators. We further hope we have convinced the reader that the general theory cannot be applied for global-in-time estimate for the very well-known Cauchy problem (HE) which is a sufficiently important issue.

1.1.2 On the choice of function spaces.

When it comes to the study of actual partial differential equations, it would be interesting to play with integrability, decay-in-time, or even with some Sobolev regularity in-time of possible solutions for the linear part of the problem. A wide development of the theory of power-weighted fractional Sobolev-in-time maximal regularity is made and applied, and can be found in [PS16, Sections 3.2, 3.4 & 3.5]. Prüss and Simonett gave the complete construction of maximal regularity results for spaces of the type $\mathrm{H}^{\alpha,q}_{\mu,0}(\mathbb{R}_+, X)$, $\alpha \in [0,1]$, $\mu \in (1/q, 1]$, which stands for the space of measurable functions u such that

$$t \mapsto t^{1-\mu} u(t) \in \mathrm{H}^{\alpha,q}_{1,0}(\mathbb{R}_+, X).$$

Here, $\mathrm{H}^{\alpha,q}_{1,0}$ coincides with the standard Sobolev space with zero boundary condition. The applications to general quasilinear parabolic partial differential equations of the L^q_μ-maximal regularity have also been extensively reviewed in [KPW10 ; LPW14].

We also mention [Pru02] which contains a treatment of fractional Sobolev in-time maximal regularity theory as well as a review of mixed derivative estimates. It was recently used, for instance, in [BM21] for the study of the Boussinesq system where mixed derivative estimates were

their main tool in the usual L^q-maximal regularity setting.

However, both of the previous treatments do not allow, again, global-in-time estimates for injective, but not invertible, operators such as the Laplacian on $\mathbb{R}^n$. This is where our idea comes from in order to keep the possibility of playing the Sobolev in-time regularity : we want to show global-in-time homogeneous $\dot{H}^{\alpha,q}$**-maximal regularity** for non-zero initial data, with a trace estimate similar to the one in (3.1), $q \in (1, +\infty)$, $\alpha \in (-1 + 1/q, 1/q)$.

Danchin, Hieber, Mucha and Tolksdorf, in [DHMT21, Chapter 2], provide global-in-time estimates for injective, but not invertible, operators in the framework of the Da Prato-Grisvard L^q-maximal regularity. Let us motivate here their idea of homogeneous interpolation and operator theory for injective sectorial operators from an other point of view. Indeed, in the previous example for the heat equation, if we set $X_p = L^p$ $A = -\Delta$, $D_p(A) = H^{2,p}$, the Besov space used as trace space is given by the real interpolation space

$$B^{2-2/q}_{p,q} = (X_p, D_p(A))_{1-\frac{1}{q},q},$$

and this follows from the general trace theory. See e.g., [Lun18, Section 1.2], [PS16, Section 3.4] or [MV14, Section 4] for even fancier and more general function spaces.

Their idea is to say that the homogeneous Besov space yielding the homogeneous estimate (3.1) would be given by

$$\dot{B}^{2-2/q}_{p,q} = (X_p, D_p(\mathring{A}))_{1-\frac{1}{q},q},$$

where, here, $D_p(\mathring{A}) = \dot{H}^{2,p}$ which is also, at least morally, the closure of $D_p(A)$ under the (semi-) norm $\|A\cdot\|_{X_p} \sim_{p,n} \|\nabla^2\cdot\|_{L^p}$. And this is exactly the kind of construction achieved in [DHMT21, Chapter 2] for abstract sectorial operators, in order to obtain a global-in-time Da Prato-Grisvard L^q-maximal regularity theorem. Moreover, such a construction avoids the need of completeness for $D(\mathring{A})$ which is fundamental in the scope of the treatment of some non-linear partial differential equations with global-in-time estimates. Indeed, realization of homogeneous functions spaces that are usually employed are not necessarily complete on their whole scale, see for instance what we have done in the previous Chapter 2 and the references therein. We notice that the possible lack of completeness of $D(\mathring{A})$ implies that the resulting real interpolation space $(X, D(\mathring{A}))_{\theta,q}$ is not necessarily complete either, but this is somewhat mandatory to deal with actual non-linear or boundary value problems.

Those issues concerning the completion also prevent the use of *standard* homogeneous operator and interpolation theory started in [Haa06, Chapter 6, Sections 6.3 & 6.4], then extended in [HHK06], requiring in the end to work with $D(\mathring{A})$ as a complete space.

We notice that the recent work [ALV23] does not apply in our setting to obtain the desired trace estimate. There are two reasons : $D(\mathring{A})$ is not an actual completion, and their work do not take in consideration homogeneous fractional Sobolev scale for the time variable.

1.2 Road map of the current chapter

In Section 3 : we provide a short construction of the homogeneous Sobolev spaces we need. In order to achieve this, we will need to assume that the Banach space X still has the UMD property. This is to ensure that we have a suitable definition of $\dot{H}^{\alpha,q}(\mathbb{R}_+, X)$, since we will need some complex interpolation theory requiring bounded imaginary powers for the time derivative, see e.g., [LMV18, Theorems 6.7 & 6.8].

Before that, in Section 2, we give a review of the current state of standard L^q-maximal regularity with global-in-time estimates : the treatment will be made first on UMD Banach spaces X. A second part is dedicated to a review of the homogeneous operator and interpolation theory revisited by Danchin, Hieber, Mucha and Tolksdorf, with its application to Da Prato-Girsvard L^q-maximal regularity.

Section 4 is devoted to our main result about $\dot{H}^{\alpha,q}$-maximal regularity for some injective sectorial operators, with trace estimate in the possibly non-complete space $(X, D(\mathring{A}))_{1+\alpha-\frac{1}{q},q}$.

Before proving the main result Theorem 3.21, one has to prove that the quantities involved to solve the Cauchy problem are in fact well-defined, which is the goal of the first subpart.

2 Short state of the art for L^q-maximal regularity

We are going to recall here few facts about L^q-maximal regularity ($q \in (1, +\infty)$) on UMD Banach spaces. We will also deal with the L^q-maximal regularity provided by the Da Prato-Grisvard theory in both versions : inhomogeneous and homogeneous, both allowing under appropriate circumstance $q = 1, +\infty$, allowing also to get rid of the UMD property on X.

2.1 Review for the usual L^q-maximal regularity

First, let us consider $(\mathrm{D}(A), A)$ a densely defined closed operator on a Banach space X. It is known, see [ABHN11, Theorem 3.7.11], that the two following assertions are equivalent :

(i) *A is ω-sectorial on X, with $\omega \in [0, \frac{\pi}{2})$;*

(ii) *$-A$ generates a bounded holomorphic C_0-semigroup on X, denoted by $(e^{-tA})_{t \geqslant 0}$.*

Thus, provided that A is ω-sectorial on X for some $\omega \in [0, \frac{\pi}{2})$, for $T \in (0, +\infty]$, we look at the following abstract Cauchy problem,

$$\left\{ \begin{array}{rl} \partial_t u(t) + Au(t) = & f(t)\,,\ 0 < t < T, \\ u(0) = & u_0. \end{array} \right. , \tag{ACP}$$

where $f \in \mathrm{L}^1_{\mathrm{loc}}((0,T), X)$, $u_0 \in Y$, Y being some normed vector space depending on X and $\mathrm{D}(A)$.

And it turns out, see [ABHN11, Proposition 3.1.16], that in our case for $u_0 \in X$, $f \in \mathrm{L}^1((0,T), X)$, integral solutions $u \in \mathrm{C}^0([0,T), X)$ for (ACP) is unique, also called the **mild solution** of (ACP) and given by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s, \quad 0 \leqslant t < T.$$

The question is : for a given $q \in [1, +\infty]$, can we find an appropriate space Y (depending on X, $\mathrm{D}(A)$ and possibly q), such that if $u_0 \in Y$ and $f \in \mathrm{L}^q((0,T), X)$, then (ACP) admits a unique solution u, satisfying $\partial_t u$, $Au \in \mathrm{L}^q((0,T), X)$, with norm control

$$\|(\partial_t u, Au)\|_{\mathrm{L}^q((0,T),X)} \lesssim_{q,A} \|f\|_{\mathrm{L}^q((0,T),X)} + \|u_0\|_Y \,?$$

The problem (ACP) being linear, we introduce two related subproblems :

- (ACP^0) stands for (ACP) with $f = 0$,
- (ACP_0) stands for (ACP) with $u_0 = 0$,

recalling that according to basic C_0-semigroup theory, $u = 0$ is the unique solution of (ACP^0_0). Hence, if (ACP) admits a solution, such solution is unique due to linearity so that it suffices to treat separately both problem (ACP^0) and (ACP_0).

- **For the** (ACP^0) **problem**, we introduce two quantities for $v \in X + \mathrm{D}(A)$,

$$\|v\|_{\mathring{\mathcal{D}}_A(\theta,q)} := \left(\int_0^{+\infty} (t^{1-\theta} \|Ae^{-tA}v\|_X)^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, \text{ and } \|v\|_{\mathcal{D}_A(\theta,q)} := \|v\|_X + \|v\|_{\mathring{\mathcal{D}}_A(\theta,q)},$$

where $\theta \in (0,1)$, $q \in [1, +\infty]$. This leads to the construction of the vector space

$$\mathcal{D}_A(\theta, q) := \{v \in X \mid \|v\|_{\mathring{\mathcal{D}}_A(\theta,q)} < +\infty\}.$$

The vector space $\mathcal{D}_A(\theta,q)$ is known to be a Banach space under the norm $\|\cdot\|_{\mathcal{D}_A(\theta,q)}$ and moreover it satisfies the following equality with equivalence of norms

$$\mathcal{D}_A(\theta,q) = (X, \mathrm{D}(A))_{\theta,q}, \tag{3.2}$$

see [Haa06, Theorem 6.2.9]. If moreover, $0 \in \rho(A)$ it has been proved, [Haa06, Corollary 6.5.5], that $\|\cdot\|_{\mathring{\mathcal{D}}_A(\theta,q)}$ and $\|\cdot\|_{\mathcal{D}_A(\theta,q)}$ are two equivalent norms on $\mathcal{D}_A(\theta,q)$.

By definition, for all $u_0 \in \mathcal{D}_A(1-1/q,q)$, for $t \mapsto u(t) = e^{-tA}u_0$ the solution of (ACP^0), we have

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1-1/q,q))} \lesssim_{q,A} \|\partial_t u\|_{\mathrm{L}^q(\mathbb{R}_+,X)} = \|Au\|_{\mathrm{L}^q(\mathbb{R}_+,X)} = \|u_0\|_{\mathring{\mathcal{D}}_A(1-1/q,q)},$$

and we also have, for all $T > 0$,

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathcal{D}_A(1-1/q,q))} \lesssim_{q,A} \|u_0\|_{\mathcal{D}_A(1-1/q,q)} \text{ and } \|u\|_{\mathrm{L}^q((0,T),X)} \lesssim_{q,A} T^{\frac{1}{q}}\|u_0\|_X.$$

If moreover, $0 \in \rho(A)$, then

$$\|u\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,A} \|u_0\|_X.$$

- **For the** (ACP_0) **problem**, the question is much more delicate. In fact, the solution u to (ACP_0) is formally given by the Duhamel formula

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s,\ t > 0, \tag{3.3}$$

and since, $\partial_t u = -Au + f$, it suffices to know whether

$$\|Au\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,A} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,X)}. \tag{3.4}$$

This leads to the following definition :

Definition 3.1 The operator A is said to have the **L^q-maximal regularity** property on X if the solution u given by (3.3) satisfies the above estimate (3.4).

Let us remark that the case of finite time $T > 0$ with the corresponding estimate can be easily deduced by (3.4) applied to $\tilde{f}$, the extension of f to $\mathbb{R}_+$ by 0, and the uniqueness of (ACP_0).

It has been proved by Coulhon and Lamberton [CL86], that the property of the L^q-maximal regularity does not depend on $q \in (1,\infty)$. See also [dSim64] for the first version of this result in the Hilbertian-valued case.

Coulhon and Lamberton also showed, see [CL86, Theorem 5.1], that the UMD property is a necessary condition for the Poisson semigroup to have the L^q-maximal regularity property. The canonical example, provided $p \in (1,+\infty)$, is that $X = \mathrm{L}^p(\Omega)$ is a UMD space and so are its closed subspaces, see for instance [HvNVW16, Propositions 4.2.15 & 4.2.17].

The following fact proved by Kalton and Lancien [KL00] : for each non-Hilbertian Banach lattice, there exists a sectorial operator such that (3.4) fails.

However, for UMD Banach spaces, a full and definitive characterization of operators that satisfy L^q-maximal regularity property has been proved by Weis [Wei01, Theorem 4.2]. One may also check [KW04, Theorem 1.11], [DHP03, Theorem 4.4] for other proofs and more details about $\mathcal{R}$-boundedness and its equivalence with L^q-maximal regularity for sectorial operators on a UMD Banach space.

In practice, we rather use other results such has the Dore-Venni Theorem, [DV87, Theorem 2.1], which asserts that the boundedness of imaginary powers of A with type $\theta_A < \frac{\pi}{2}$ is a sufficient condition to recover L^q-maximal regularity for $q \in (1,+\infty)$. We mention [Haa06, Corollary 9.3.12] for the same result that does not require invertibility of A. In particular, the bounded holomorphic functional calculus of A is a sufficient condition to recover L^q-maximal regularity with $q \in (1,+\infty)$.

We may combine all results for (ACP^0) and (ACP_0) to state the following well-known L^q-maximal regularity theorem, where we only state it with the sufficient condition of BIP for convenience.

Theorem 3.2 *Let $\omega \in [0, \frac{\pi}{2})$, $(\mathrm{D}(A), A)$ an ω-sectorial operator on a UMD Banach space X. Assume that A has BIP of type $\theta_A < \frac{\pi}{2}$.*

Let $q \in (1, +\infty)$ and $T \in (0, +\infty]$. For $f \in \mathrm{L}^q((0,T), X)$, $u_0 \in \mathcal{D}_A(1 - 1/q, q)$, the problem (ACP) admits a unique solution u such that $\partial_t u$, $Au \in \mathrm{L}^q((0,T), X)$ with the estimate

$$\|(\partial_t u, Au)\|_{\mathrm{L}^q((0,T),X)} \lesssim_{A,q} \|f\|_{\mathrm{L}^q((0,T),X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(1-1/q,q)}. \tag{3.5}$$

In addition, for all $T_* \leqslant T$, $T_* < +\infty$, we have $u \in \mathrm{C}^0([0, T_*), \mathcal{D}_A(1 - 1/q, q)) \cap \mathrm{L}^q((0, T_*), X)$ with the estimates

$$\|u\|_{\mathrm{L}^\infty([0,T_*],\mathcal{D}_A(1-1/q,q))} \lesssim_{A,q,T_*} \|f\|_{\mathrm{L}^q((0,T_*),X)} + \|u_0\|_{\mathcal{D}_A(1-1/q,q)}, \tag{3.6}$$

$$\|u\|_{\mathrm{L}^q((0,T_*),X)} \lesssim_{A,q} (T_* \|f\|_{\mathrm{L}^q((0,T_*),X)} + T_*^{\frac{1}{q}} \|u_0\|_X). \tag{3.7}$$

If moreover $0 \in \rho(A)$, we also have

$$\|u\|_{\mathrm{L}^q([0,T],X)} \lesssim_{A,q} \|f\|_{\mathrm{L}^q((0,T),X)} + \|u_0\|_X. \tag{3.8}$$

so that (3.6) holds with uniform constant with respect to T_*, hence remains true for $T_* = +\infty$.

We comment the appearance of (3.6) : it is a consequence of the trace theory for initial data in L^q-maximal regularity which is itself a consequence of interpolation theory, see [Ama95, Chapter 4, Theorem 4.10.2], see also [Lun18, Corollary 1.14]. The appearance of (3.8) comes from invertibility of A, so that it suffices to apply (3.4).

However, the approach used to obtain Theorem 3.2 prevents L^1 and L^∞-maximal regularity on X. Moreover, the UMD property requires the space X to be at least reflexive, which is not the case for all spaces that are of use in partial differential equations (one may think about endpoint Besov spaces like $\mathrm{B}^s_{p,1}$ and $\mathrm{B}^s_{p,\infty}$, or even the space of continuous bounded functions C^0_b).

2.2 Revisited homogeneous operator and interpolation theory and global-in-time estimate for the Da Prato-Grisvard L^q-maximal regularity

To overcome such difficulties, we present a theorem due to Da Prato and Grisvard [DG75], where the idea was to replace X by $\mathcal{D}_A(\theta, q)$, and look for L^q-maximal regularity property on it instead of X, allowing $q = 1$.

Theorem 3.3 ([DG75, Theorem 4.15]) *Let $\omega \in [0, \frac{\pi}{2})$, $(\mathrm{D}(A), A)$ an ω-sectorial operator on a Banach space X. Let $q \in [1, +\infty)$, $\theta \in (0, \frac{1}{q})$, $\theta_q := \theta + 1 - 1/q$, and let $T \in (0, +\infty)$.*

For $f \in \mathrm{L}^q((0,T), \mathcal{D}_A(\theta, q))$ and $u_0 \in \mathcal{D}_A(\theta_q, q)$, the problem (ACP) admits a unique mild solution

$$u \in \mathrm{C}^0_b([0,T], \mathcal{D}_A(\theta_q, q)),$$

such that $\partial_t u$, $Au \in \mathrm{L}^q((0,T), \mathcal{D}_A(\theta, q))$ with the estimates

$$\|u\|_{\mathrm{L}^\infty([0,T],\mathcal{D}_A(\theta_q,q))} \lesssim_{A,\theta,q,T} \|(\partial_t u, Au)\|_{\mathrm{L}^q((0,T),\mathcal{D}_A(\theta,q))} \lesssim_{A,\theta,q,T} \|f\|_{\mathrm{L}^q((0,T),\mathcal{D}_A(\theta,q))} + \|u_0\|_{\mathcal{D}_A(\theta_q,q)}. \tag{3.9}$$

If moreover $0 \in \rho(A)$, (3.9) still holds with uniform constant with respect to T, allowing $T = +\infty$.

This Da Prato-Grisvard theorem does not have global in time estimate if $0 \notin \rho(A)$, as was the case for the estimate (3.5) of Theorem 3.2. The estimate (3.5) is uniform in time : this is due to

the fact that the estimate is **homogeneous**. This key point was captured in the work of Danchin, Hieber, Mucha and Tolksdorf [DHMT21, Chapter 2] to build an **homogeneous** version of the Da Prato-Grivard theorem for injective sectorial operators under some additional assumptions on A. We are going to present briefly their construction.

Assumption 3.4 *The operator $(\mathrm{D}(A), A)$ is injective on X, and there exists a normed vector space $(Y, \|\cdot\|_Y)$, such that for all $x \in \mathrm{D}(A)$,*

$$\|Ax\|_X \sim \|x\|_Y .$$

The idea is to construct an homogeneous version of A denoted $\mathring{A}$, defining first its domain

$$\mathrm{D}(\mathring{A}) := \{\, y \in Y \,|\, \exists (x_n)_{n\in\mathbb{N}} \subset \mathrm{D}(A),\ \|y - x_n\|_Y \underset{n\to+\infty}{\longrightarrow} 0 \,\}.$$

Then, for all $y \in \mathrm{D}(\mathring{A})$,

$$\mathring{A}y := \lim_{n\to+\infty} Ax_n.$$

Constructed this way, the operator $\mathring{A}$ is then injective on $\mathrm{D}(\mathring{A})$. We notice that $\mathrm{D}(\mathring{A})$ is a normed vector space, but not necessarily complete. We also need the existence of a Hausdorff topological vector space Z, such that $X, Y \subset Z$, and to consider the following assumption

Assumption 3.5 *The operator $(\mathrm{D}(A), A)$ and the normed vector space Y are such that*

$$X \cap \mathrm{D}(\mathring{A}) = \mathrm{D}(A).$$

As a consequence of all above assumptions, we can extend naturally, see [DHMT21, Remark 2.7], $(e^{-tA})_{t\geqslant 0}$ to a C_0-semigroup,

$$e^{-tA} \,:\, X + \mathrm{D}(\mathring{A}) \longrightarrow X + \mathrm{D}(\mathring{A}) \ ,\ t \geqslant 0,$$

by the mean of the following formula for all $(x_0, a_0) \in X \times \mathrm{D}(\mathring{A})$, $t \geqslant 0$,

$$e^{-tA}(x_0 + a_0) := e^{-tA}x_0 + \left(a_0 - \int_0^t e^{-\tau A}\mathring{A}a_0 \,\mathrm{d}\tau\right). \tag{3.10}$$

and so that for $u_0 \in X + \mathrm{D}(\mathring{A})$, and fixed t, the value above does not depend on the choice of decomposition $u_0 = x_0 + a_0$, see [DHMT21, Proposition 2.6].

Moreover, for all $u_0 = x_0 + a_0 \in X + \mathrm{D}(\mathring{A})$, it is straight forward to see from (3.10) and [DHMT21, Proposition 2.6], that $t \mapsto e^{-tA}u_0$ is strongly differentiable at any order with continuous derivatives on $(0, +\infty)$ taking its values in X. For $k \in [\![1, +\infty[\![$, $t > 0$, by analyticity of the semigroup

$$(-\partial_t)^k(e^{-(\cdot)A}u_0)(t) = A^k e^{-tA}x_0 + A^{k-1}e^{-tA}\mathring{A}a_0 = A^{k-1}\mathring{A}e^{-tA}u_0 \in \mathrm{D}(A) \subset X.$$

From there, one can fully make sense of the following vector space,

$$\mathring{\mathcal{D}}_A(\theta, q) := \left\{ v \in X + \mathrm{D}(\mathring{A}) \,\middle|\, \|v\|_{\mathring{\mathcal{D}}_A(\theta,q)} < +\infty \right\}.$$

Similarly to what happens for $\mathcal{D}_A(\theta, q)$ in (3.2), it has been proved in [DHMT21, Proposition 2.12], that the following equality holds with equivalence of norms,

$$\mathring{\mathcal{D}}_A(\theta, q) = (X, \mathrm{D}(\mathring{A}))_{\theta,q}. \tag{3.11}$$

However, the lack of completeness for $\mathrm{D}(\mathring{A})$ implies that $\mathring{\mathcal{D}}_A(\theta, q)$ is not necessarily complete. This has consequences on how to consider the forcing term f in (ACP), choosing $f \in \mathrm{L}^q((0, T), \mathcal{D}_A(\theta, q))$

instead of $f \in \mathrm{L}^q((0,T),\mathring{\mathcal{D}}_A(\theta,q))$ to avoid definition issues, the latter choice being possible when $\mathring{\mathcal{D}}_A(\theta,q)$ is a Banach space.

Theorem 3.6 ([DHMT21, Theorem 2.20]) *Let $\omega \in [0,\frac{\pi}{2})$, $(\mathrm{D}(A),A)$ an ω-sectorial operator on a Banach space X such that Assumptions (3.4) and (3.5) are satisfied. Let $q \in [1,+\infty)$, $\theta \in (0,\frac{1}{q})$, $\theta_q := \theta + 1 - 1/q$, and let $T \in (0,+\infty]$.*

For $f \in \mathrm{L}^q((0,T),\mathcal{D}_A(\theta,q))$ and $u_0 \in \mathring{\mathcal{D}}_A(\theta_q,q)$, the problem (ACP) admits a unique mild solution

$$u \in \mathrm{C}^0_b([0,T),\mathring{\mathcal{D}}_A(\theta_q,q)),$$

such that $\partial_t u$, $Au \in \mathrm{L}^q((0,T),\mathring{\mathcal{D}}_A(\theta,q))$ with estimates,

$$\|u\|_{\mathrm{L}^\infty([0,T],\mathring{\mathcal{D}}_A(\theta_q,q))} + \|(\partial_t u, Au)\|_{\mathrm{L}^q((0,T),\mathring{\mathcal{D}}_A(\theta,q))} \lesssim_{q,A} \|f\|_{\mathrm{L}^q((0,T),\mathring{\mathcal{D}}_A(\theta,q))} + \|u_0\|_{\mathring{\mathcal{D}}_A(\theta_q,q)}. \tag{3.12}$$

In case $q = +\infty$, we assume in addition that $u_0 \in \mathrm{D}(A^2)$ and then for each $\theta \in (0,1)$,

$$\|(\partial_t u, Au)\|_{\mathrm{L}^\infty([0,T],\mathring{\mathcal{D}}_A(\theta,\infty))} \lesssim_{q,A} \|f\|_{\mathrm{L}^\infty((0,T),\mathcal{D}_A(\theta,\infty))} + \|Au_0\|_{\mathring{\mathcal{D}}_A(\theta,\infty)}.$$

3 Vector-valued Sobolev spaces in a UMD Banach space and the time derivative

3.1 Banach valued Bessel and Riesz potential Sobolev spaces

This subsection is devoted to few reminders on Bessel potential spaces on the whole line with values in a Banach space X which is known to be UMD. This will be based on the constructions provided by [MV12 ; LMV18] and [HvNVW16, Chapter 5, Section 5.6], see also the references therein. From the properties we are going to gather about Bessel potential Sobolev spaces, we will be able give a simple construction of homogeneous (Riesz potential) Sobolev spaces with values in X, for regularity index near 0. Namely, we will focus on the regularity index $\alpha \in (-1+1/q, 1/q)$ when $q \in (1,+\infty)$.

We chose to investigate the Sobolev space on the one the half-line for which we won't have additional compatibility conditions at 0. Notice that the condition on the regularity index is also here in order to avoid troubles of definition. Indeed, the definition of homogeneous function spaces for regularity exponents beyond $1/q$ is not clear and a choice of realization have to be done, even in the scalar case. Such choice implies generally the loss of one, or more, usual and useful properties, like either the loss of distribution theory, the loss of completeness on the whole scale, or the loss of pointwise/meaningful (para-)products (in the scalar case, $X = \mathbb{C}$). See for instance in the previous Chapter 2 and [Tri83 ; BL76 ; BCD11 ; DHMT21 ; Saw18] and the references therein for various constructions and addressed issues in the scalar-valued case.

From there and until the end of this chapter, we assume that X has the UMD property. We recall that such space X is necessarily reflexive.

Definition 3.7 For $q \in (1,+\infty)$, $\alpha \in \mathbb{R}$, we define the vector space

$$\mathrm{H}^{\alpha,q}(\mathbb{R},X) := \left\{ u \in \mathcal{S}'(\mathbb{R},X) \,\middle|\, (\mathrm{I}-\partial_x^2)^{\frac{\alpha}{2}} u \in \mathrm{L}^q(\mathbb{R},X) \right\}$$

with its associated norm

$$\|u\|_{\mathrm{H}^{\alpha,q}(\mathbb{R},X)} := \|(\mathrm{I}-\partial_x^2)^{\frac{\alpha}{2}} u\|_{\mathrm{L}^q(\mathbb{R},X)}.$$

Here, $(\mathrm{I}-\partial_x^2)^{\frac{\alpha}{2}}$ have to be understood as the usual Fourier multiplier operator.

Proposition 3.8 *Let $q \in (1,+\infty)$, $\alpha \in \mathbb{R}$, the following properties are true :*

(i) $\mathrm{H}^{\alpha,q}(\mathbb{R},X)$ *is a reflexive Banach space with*

$$(\mathrm{H}^{\alpha,q}(\mathbb{R},X))^* = \mathrm{H}^{-\alpha,q'}(\mathbb{R},X^*)\,;$$

(ii) $\mathcal{S}_0(\mathbb{R},X)$ *is a dense subspace of* $\mathrm{H}^{\alpha,q}(\mathbb{R},X)$ *;*

(iii) Provided $\alpha \geqslant 0$*, for all* $u \in \mathcal{S}'(\mathbb{R},X)$*,*

$$\|u\|_{\mathrm{H}^{\alpha,q}(\mathbb{R},X)} \sim_{\alpha,q,X} \|u\|_{\mathrm{L}^q(\mathbb{R},X)} + \|(-\partial_x^2)^{\frac{\alpha}{2}} u\|_{\mathrm{L}^q(\mathbb{R},X)}\,;$$

(iv) Provided $\alpha \in [0,1/q)$*,* $\frac{1}{r} = \frac{1}{q} - \alpha$*, for all* $u \in \mathrm{H}^{\alpha,q}(\mathbb{R},X)$*,*

$$\|u\|_{\mathrm{L}^r(\mathbb{R},X)} \lesssim_{\alpha,q,X} \|(-\partial_x^2)^{\frac{\alpha}{2}} u\|_{\mathrm{L}^q(\mathbb{R},X)}\,;$$

(v) Provided $\alpha \in (-1+1/q, 1/q)$*, for all* $u \in \mathrm{H}^{\alpha,q}(\mathbb{R},X)$*,*

$$\|\mathbb{1}_{\mathbb{R}_+} u\|_{\mathrm{H}^{\alpha,q}(\mathbb{R},X)} \lesssim_{\alpha,q,X} \|u\|_{\mathrm{H}^{\alpha,q}(\mathbb{R},X)}.$$

Proof. — Point *(i)* is standard. Point *(ii)* is a direct consequence of the corresponding results for $\alpha = 0$, see [Haa06, Lemma E.5.2]. Point *(iii)* is just [LMV18, Lemma 4.2]. Point *(iv)* follows from [MV12, Corollary 1.4], point *(iii)* and a dilation argument. Point *(v)* is just [LMV18, Theorem 4.1]. ■

For $q \in (1,+\infty)$, $\alpha \in (-\infty, 1/q)$, thanks to the points *(i)*, *(ii)* and *(iv)* from the Proposition 3.8, we introduce the quantity

$$\|u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)} := \|(-\partial_x^2)^{\frac{\alpha}{2}} u\|_{\mathrm{L}^q(\mathbb{R},X)},$$

one can consider the completion

$$\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X) := \overline{\mathcal{S}_0(\mathbb{R},X)}^{\|\cdot\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)}}$$

so that the next definition is meaningful.

Definition 3.9 For $q \in (1,+\infty)$, $\alpha < 1/q$, we define the vector spaces

(i) for $\alpha \geqslant 0$*,* $\frac{1}{r} := \frac{1}{q} - \alpha$*,*

$$\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X) := \left\{ u \in \mathrm{L}^r(\mathbb{R},X) \,\middle|\, (-\partial_x^2)^{\frac{\alpha}{2}} u \in \mathrm{L}^q(\mathbb{R},X) \right\},$$

(ii) for $\alpha \leqslant 0$*,*

$$\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X) := \left\{ u \in \mathrm{H}^{\alpha,q}(\mathbb{R},X) \,\middle|\, (-\partial_x^2)^{\frac{\alpha}{2}} u \in \mathrm{L}^q(\mathbb{R},X) \right\},$$

with their associated norm

$$\|u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)} := \|(-\partial_x^2)^{\frac{\alpha}{2}} u\|_{\mathrm{L}^q(\mathbb{R},X)}.$$

Here, $(-\partial_x^2)^{\frac{\alpha}{2}}$ have to be understood as the usual Fourier multiplier operator.

Similarly, we obtain the following collection of properties.

Proposition 3.10 *Let* $q \in (1,+\infty)$*,* $\alpha < 1/q$*, the following properties are true :*

(i) for $\beta \in \mathbb{R}$*, such that* $\alpha + \beta < 1/q$*,*

$$(-\partial_x^2)^{\frac{\beta}{2}} \,:\, \dot{\mathrm{H}}^{\alpha+\beta,q}(\mathbb{R},X) \longrightarrow \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)$$

is an isomorphism of Banach spaces ;

(ii) $\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)$ *is reflexive, and whenever* $\alpha\in(-1+1/q,1/q)$,

$$(\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X))^*=\dot{\mathrm{H}}^{-\alpha,q'}(\mathbb{R},X^*)\,;$$

(iii) $\mathcal{S}_0(\mathbb{R},X)$ *is a dense subspace of* $\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)$ *;*

(iv) provided $\alpha\in[0,1/q)$, $\frac{1}{r}=\frac{1}{q}-\alpha$, *for all* $u\in\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)$, $v\in\mathrm{L}^{r'}(\mathbb{R},X)$,

$$\|u\|_{\mathrm{L}^r(\mathbb{R},X)}\lesssim_{\alpha,q,X}\|u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)},$$
$$\|v\|_{\dot{\mathrm{H}}^{-\alpha,q'}(\mathbb{R},X)}\lesssim_{\alpha,q,X}\|v\|_{\mathrm{L}^{r'}(\mathbb{R},X)}\,;$$

(v) provided $\alpha\in(-1+1/q,1/q)$, *for all* $u\in\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)$,

$$\|\mathbb{1}_{\mathbb{R}_+}u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)}\lesssim_{\alpha,q,X}\|u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R},X)}.$$

Proof. — Point *(iii)* follows from the definition. Point *(i)* is straightforward by density of $\mathcal{S}_0(\mathbb{R},X)$. Point *(ii)* is a direct consequence of the corresponding results for $\alpha=0$, thanks to the point *(i)*. Point *(iv)* follows from the definition, the corresponding point in Proposition 3.8 and a duality argument provided by the previous point *(ii)*. Point *(v)* follows from points *(iii)* and *(v)* in Proposition 3.8 and a dilation argument when $\alpha\geqslant 0$. The case $\alpha<0$ follows by duality thanks to the current point *(ii)*. ■

Let us start the construction of corresponding function spaces on the half-line.

Definition 3.11 Let $q\in(1,+\infty)$, $\alpha\in\mathbb{R}$, $\mathfrak{h}\in\{\mathrm{H},\dot{\mathrm{H}}\}$. We assume assume moreover that $\alpha<1/q$ when $\mathfrak{h}=\dot{\mathrm{H}}$. We define by restriction, in the sense of distributions, the normed vector space

$$\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X):=\mathfrak{h}^{\alpha,q}(\mathbb{R},X)_{|\mathbb{R}_+}.$$

This is a Banach space with respect to the quotient norm

$$\|u\|_{\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X)}:=\inf_{\substack{U_{|\mathbb{R}_+}=u,\\ U\in\mathfrak{h}^{\alpha,q}(\mathbb{R},X).}}\|U\|_{\mathfrak{h}^{\alpha,q}(\mathbb{R},X)}.$$

Proposition 3.12 *Let* $q\in(1,+\infty)$, $\alpha\in\mathbb{R}$, $\mathfrak{h}\in\{\mathrm{H},\dot{\mathrm{H}}\}$. *We assume moreover that* $\alpha<1/q$ *when* $\mathfrak{h}=\dot{\mathrm{H}}$. *The following properties hold :*

(i) $\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X)$ *is a reflexive Banach space, for which* $\mathcal{S}_0(\mathbb{R},X)_{|\mathbb{R}_+}$ *is a dense subspace ;*

(ii) provided $\alpha\in(-1+1/q,1/q)$, *for all* $u\in\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X)$, *the extension of* u *to the whole line by* 0 *denoted by* $\tilde{u}$ *yields an element of* $\mathfrak{h}^{\alpha,q}(\mathbb{R},X)$

$$\|\tilde{u}\|_{\mathfrak{h}^{\alpha,q}(\mathbb{R},X)}\sim_{\alpha,q,X}\|u\|_{\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X)}\,;$$

(iii) provided $\alpha\in[0,1/q)$, $\frac{1}{r}=\frac{1}{q}-\alpha$, *for all* $u\in\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X)$, $v\in\mathrm{L}^{r'}(\mathbb{R}_+,X)$,

$$\|u\|_{\mathrm{L}^r(\mathbb{R}_+,X)}\lesssim_{\alpha,q,X}\|u\|_{\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X)},$$
$$\|v\|_{\mathfrak{h}^{-\alpha,q'}(\mathbb{R}_+,X)}\lesssim_{\alpha,q,X}\|v\|_{\mathrm{L}^{r'}(\mathbb{R}_+,X)}\,;$$

(iv) for all $\alpha\in(-1+1/q,1/q)$, *the subspace* $\mathrm{C}^\infty_c(\mathbb{R}_+,X)$ *is dense in* $\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X)$ *;*

(v) whenever $\alpha\in(-1+1/q,1/q)$,

$$(\mathfrak{h}^{\alpha,q}(\mathbb{R}_+,X))^*=\mathfrak{h}^{-\alpha,q'}(\mathbb{R}_+,X^*)\,;$$

(vi) provided $\alpha \in [0, 1/q)$*, for all* $u \in \mathrm{H}^{\alpha,q}(\mathbb{R}_+, X)$*,*

$$\|u\|_{\mathrm{L}^q(\mathbb{R}_+,X)} + \|u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \sim_{\alpha,q,X} \|u\|_{\mathrm{H}^{\alpha,q}(\mathbb{R}_+,X)} ;$$

Proof. — Point *(i)* follows from the definition of function spaces by restriction, and the properties for their counterparts on $\mathbb{R}$. The point *(ii)* is a direct consequence of point *(v)* from both Propositions 3.8 and 3.10. Point *(iii)* follows from the definition of function spaces by restriction and the corresponding result in Propositions 3.8 and 3.10.

The point *(iv)* for $\alpha \leqslant 0$ follows from point *(i)* and *(iii)* : indeed, both yields that $\mathrm{L}^r(\mathbb{R}_+, X)$ is dense in $\mathrm{H}^{\alpha,q}(\mathbb{R}_+, X)$, therefore it suffices to approximate functions in $\mathrm{L}^r(\mathbb{R}_+, X)$ by ones in $\mathrm{C}_c^\infty(\mathbb{R}_+, X)$. For $\alpha > 0$, the inhomogeneous case is known to be true, see for instance [LMV18, Proposition 6.4]. The case of homogeneous function space follows since by *(i)*, and by construction, $\mathrm{H}^{\alpha,q}(\mathbb{R}_+, X)$ embeds continuously and densely in $\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+, X)$.

For the point *(v)*, one may use the *(iv)* and the case $\alpha = 0$ in order to reproduce the proof as in the scalar case, e.g., one may reproduce the proof of Proposition 2.28.

Finally, the point *(vi)* can be proved by the mean of current point *(ii)* and the point *(iii)* from Proposition 3.8. ∎

The next lemma is nothing but the Hardy-Sobolev inequality in the vector-valued setting with homogeneous estimate. Its proof is left to the reader and use a complex interpolation argument allowed by [LMV18, Theorem 6.7], then dilation and density arguments by the mean of the points *(iv)* and *(vi)* of Proposition 3.12.

Lemma 3.13 *Let* $q \in (1, +\infty)$*,* $\alpha \in [0, 1/q)$*. For all* $u \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+, X)$ *the following inequality holds*

$$\left\| \tau \mapsto \frac{u(\tau)}{\tau^\alpha} \right\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{\alpha,q} \|u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}.$$

3.2 The derivative on the half-line.

We will not discuss here the construction and properties of inhomogeneous Bessel potential spaces $\mathrm{H}^{s,p}(\mathbb{R}_+, X)$ for $s \in \mathbb{R}$, $p \in (1, +\infty)$ and the meaning of traces at 0. Therefore as in the previous subsection, we refer to [MV12 ; LMV18 ; SSS12] for more details.

We recall that one may define the unbounded operator $\frac{\mathrm{d}}{\mathrm{d}t}$ on $\mathrm{L}^q(\mathbb{R}_+, X)$, also denoted by ∂_t, with domain,

$$\mathrm{D}_q(\partial_t) := \mathrm{H}_0^{1,q}(\mathbb{R}_+, X) := \{f \in \mathrm{L}^q(\mathbb{R}_+, X) \,|\, \partial_t f \in \mathrm{L}^q(\mathbb{R}_+, X),\ f(0) = 0\}.$$

Then, thanks to [DV87, Theorem 3.1] (see also [Haa06, Sections 8.4, 8.5], [LMV18, Theorem 6.8] or [PS16, Theorem 4.3.14]), $\frac{\mathrm{d}}{\mathrm{d}t}$ is an injective sectorial operator on $\mathrm{L}^q(\mathbb{R}_+, X)$ which admits bounded imaginary powers, satisfying, for all $s \in \mathbb{R}$

$$\left\|(\partial_t)^{is}\right\|_{\mathrm{L}^q(\mathbb{R}_+,X)\to\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,X} (1 + s^2)e^{\frac{\pi}{2}|s|},$$

implying that $(\partial_t)^\alpha$, $\alpha \in [0, 1]$, is injective with domain $\mathrm{H}_0^{\alpha,q}(\mathbb{R}_+, X)$, $\alpha \neq \frac{1}{q}$, see [LMV18, Theorems 6.7 & 6.8], and we have an isomorphism, provided $\alpha \in (-1 + 1/q, 1/q)$,

$$(\partial_t)^\alpha \,:\, \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+, X) \longrightarrow \mathrm{L}^q(\mathbb{R}_+, X). \tag{3.13}$$

For $\beta \in (0, 1)$, $\alpha \in [\beta, 1]$, $\alpha \neq 1/q$, $\gamma \in [\beta, \alpha]$, the following representation formula holds for all $f \in (\partial_t)^\beta \mathrm{H}_0^{\alpha,q}(\mathbb{R}_+, X)$,

$$(\partial_t)^{-\beta} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{1}{(t-\tau)^{1-\gamma}} (\partial_t)^{\gamma-\beta} f(\tau)\, \mathrm{d}\tau, t > 0. \tag{3.14}$$

The above formula remains true for $f \in \dot{\mathrm{H}}^{\alpha-\beta,q}(\mathbb{R}_+, X)$, provided $\alpha, \alpha-\beta < 1/q$.

Similarly, the "dual" operator $-\frac{\mathrm{d}}{\mathrm{d}t}$, with domain $\mathrm{D}_q(-\partial_t) := \mathrm{H}^{1,q}(\mathbb{R}_+, X)$, is an injective sectorial operator on $\mathrm{L}^q(\mathbb{R}_+, X)$ which admits bounded imaginary powers. For all $s \in \mathbb{R}$

$$\left\|(-\partial_t)^{is}\right\|_{\mathrm{L}^q(\mathbb{R}_+,X)\to\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{q,X} (1+s^2)e^{\frac{\pi}{2}|s|},$$

which also implies injectivity of $(-\partial_t)^\alpha$, $\alpha \in [0,1]$, with domain $\mathrm{H}^{\alpha,q}(\mathbb{R}_+, X)$.

For $\alpha \in (-1+1/q, 1/q)$, we still have an isomorphism

$$(-\partial_t)^\alpha \,:\, \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+, X) \longrightarrow \mathrm{L}^q(\mathbb{R}_+, X). \tag{3.15}$$

For $\beta \in (0,1)$, $\alpha \in [\beta, 1]$, $\gamma \in [\beta, \alpha]$, the following representation formula holds for all $f \in (-\partial_t)^\beta \mathrm{H}^{\alpha,q}(\mathbb{R}_+, X)$,

$$(-\partial_t)^{-\beta} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^{+\infty} \frac{1}{(\tau-t)^{1-\gamma}} (-\partial_t)^{\gamma-\beta} f(\tau)\,\mathrm{d}\tau, t > 0. \tag{3.16}$$

The above formula remains true for $f \in \dot{\mathrm{H}}^{\alpha-\beta,q}(\mathbb{R}_+, X)$, provided $\alpha, \alpha-\beta < 1/q$.

We specify that both formulas (3.14) and (3.16) are direct consequences of Proposition 1.33.

More details about the functional analytic properties of operators ∂_t and $-\partial_t$ can also be found in [PS16, Section 3.2] and [LMV18, Section 6], where the case of (power-)weighted, but inhomogeneous, Sobolev spaces have been widely treated.

3.3 A comment for homogeneous Sobolev spaces and the time derivative on a finite interval

We finish this section with a discussion about Sobolev space on $[0,T]$, $T > 0$, and the related derivative operators. On can define those space similarly.

Definition 3.14 Let $q \in (1, +\infty)$, $-1+1/q < \alpha < 1/q$ when $\mathfrak{h} = \dot{\mathrm{H}}$. We define by restriction, in the sense of distributions, the normed vector space

$$\mathfrak{h}^{\alpha,q}([0,T], X) := \mathfrak{h}^{\alpha,q}(\mathbb{R}, X)_{|[0,T]}.$$

with the induced quotient norm.

But since $\mathbb{1}_{[0,T]} = \mathbb{1}_{\mathbb{R}_+} - \mathbb{1}_{(0,+\infty)}(\cdot - T)$, we obtain

Proposition 3.15 *Let $q \in (1, +\infty)$, $\alpha \in (-1+1/q, 1/q)$. The following properties hold :*

(i) $\dot{\mathrm{H}}^{\alpha,q}([0,T], X)$ *is a reflexive Banach space, for which* $\mathcal{S}_0(\mathbb{R}, X)_{|[0,T]}$ *is a dense subspace ;*

(ii) for all $u \in \dot{\mathrm{H}}^{\alpha,q}([0,T], X)$*, the extension of* u *to the whole line denoted* $\tilde{u}$*, is such that*

$$\|\tilde{u}\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|\tilde{u}\|_{\mathrm{H}^{\alpha,q}(\mathbb{R},X)} \lesssim_{\alpha,q,X,T} \|u\|_{\dot{\mathrm{H}}^{\alpha,q}([0,T],X)}.$$

(iii) $\dot{\mathrm{H}}^{\alpha,q}([0,T], X) = \mathrm{H}^{\alpha,q}([0,T], X)$ *with equivalence of norms (depending on* T*).*

From there, and in particular from point *(ii)* of Proposition 3.15, one may expect that the theory on the half line will carry over the behavior on $[0,T]$ up to extend the elements by 0, or up to the multiplication by $\mathbb{1}_{[0,T]}$. And this is indeed, what actually happens for ∂_t and $-\partial_t$ according to [Haa06, Section 8.5].

4 The global-in-time $\dot{\mathrm{H}}^{\alpha,q}$-maximal regularity with homogeneous trace estimate

Now, we go back to L^q-maximal regularity on a UMD Banach space X. We are going to state few minor improvements of above results, the first one is about global-in-time estimates when the initial data u_0 lies in the homogeneous space $\mathring{\mathcal{D}}_A(\theta,q)$, provided $\theta\in(0,1)$, $q\in(1,+\infty)$.

A second goal is to obtain a $\dot{\mathrm{H}}^{\alpha,q}$-maximal regularity result as a variation of above Theorem 3.2 where we take the advantage of [Pru02, Proposition 2.4], and the isomorphism properties (3.13) and (3.15). Our proof for the corresponding homogeneous trace estimates is mainly inspired by techniques from the proofs of [DHMT21, Lemma 2.19, Theorem 2.20], see also [PS16, Section 3.4] for similar estimates proven similarly.

4.1 About mild solutions in the context of homogeneous operator theory

In the literature, it seems difficult to have a clear and definitive mention of what would be the exact meaning of a mild solution of (ACP) in the context of homogeneous functions spaces with respect to the space variable (here the roles are played by $\mathrm{D}(\mathring{A})$ and $\mathring{\mathcal{D}}_A(\theta,q)$). Here is an attempt.

Definition 3.16 Let $\omega\in[0,\frac{\pi}{2})$, $(\mathrm{D}(A),A)$ an ω-sectorial operator on a Banach space X, such that it satisfies Assumptions (3.4) and (3.5).

Let $T\in(0,+\infty]$, $f\in\mathrm{L}^1_{loc}([0,T),X)$ and $u_0\in X+\mathrm{D}(\mathring{A})$. We say that $u:[0,T)\longrightarrow X+\mathrm{D}(\mathring{A})$ is a **homogeneous**-mild solution of (ACP) if

(i) $u\in\mathrm{C}^0_b([0,T),X+\mathrm{D}(\mathring{A}))$,

(ii) $v(t):=u(t)-e^{-tA}u_0\in X$, for all $t\in[0,T)$,

(iii) $v\in\mathrm{C}^0_b([0,T),X)$ is a mild solution of (ACP_0) in the classical sense, *i.e.* for all $t\in[0,T)$,

$$\int_0^t v(s)\,\mathrm{d}s\in\mathrm{D}(A),$$

and

$$v(t)+A\int_0^t v(s)\,\mathrm{d}s=\int_0^t f(s)\,\mathrm{d}s\text{ in }X.$$

Proposition 3.17 *Let $\omega\in[0,\frac{\pi}{2})$, $(\mathrm{D}(A),A)$ an ω-sectorial operator on a UMD Banach space X, such that it satisfies Assumptions (3.4) and (3.5). Let $T\in(0,+\infty]$, $f\in\mathrm{L}^1_{\mathrm{loc}}([0,T),X)$ and $u_0\in X+\mathrm{D}(\mathring{A})$.*

*The problem (ACP) admits at most one **homogeneous**-mild solution.*

Proof. — Let u_1 and u_2 be two **homogeneous**-mild solutions to (ACP). Then, we set for all $t\geqslant 0$, $V(t):=u_1(t)-u_2(t)=(u_1(t)-e^{-tA}u_0)-(u_2(t)-e^{-tA}u_0)$. It follows that V is a mild solution of (ACP^0_0) in the classical sense. Hence, uniqueness provided by [ABHN11, Proposition 3.1.16] yields $V=0$ in X. ■

4.2 Preliminary lemmas

First, we state a Lemma for the problem (ACP^0), about homogeneous fractional Sobolev in-time estimates for initial data $u_0\in\mathring{\mathcal{D}}_A(\theta,q)$, $q\in(1,+\infty)$, $\theta\in(0,1)$.

Lemma 3.18 *Let $\omega\in[0,\frac{\pi}{2})$, $(\mathrm{D}(A),A)$ an ω-sectorial operator on a UMD Banach space X, such that it satisfies Assumptions (3.4) and (3.5). Let $q\in(1,+\infty)$, $\alpha\in(-1+1/q,1/q)$.*

For all $u_0 \in \mathring{\mathcal{D}}_A(1+\alpha-1/q,q)$, we have

$$t \mapsto \mathring{A}e^{-tA}u_0 \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X) \cap \mathrm{L}^q_{1+\alpha}(\mathbb{R}_+,X)$$

with the estimates

$$\|t \mapsto t^{1-(1+\alpha)}\mathring{A}e^{-tA}u_0\|_{\mathrm{L}^q(\mathbb{R}_+,X)} = \|u_0\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)},$$
$$\|\mathring{A}e^{-(\cdot)A}u_0\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \lesssim_{q,\alpha,A} \|u_0\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)}.$$

Proof. — We just have to prove the estimate in Sobolev space. The equality of norms is straightforward by definition of the $\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)$-norm.

Step 1 : The case $\alpha = 0$ is straightforward.

Step 2 : The case $\alpha \in (0,1/q)$. For $u_0 \in \mathring{\mathcal{D}}_A(1+\alpha-1/q,q)$, one can write $u_0 = x_0 + a_0$, where $(x_0,a_0) \in X \times \mathrm{D}(\mathring{A})$. By [DHMT21, Proposition 2.6], the following equality holds in X, for all $t>0$,

$$\mathring{A}e^{-tA}u_0 = Ae^{-tA}x_0 + e^{-tA}\mathring{A}a_0.$$

Therefore, thanks to the representation formulae (3.16), and integral formulations for fractional powers of A, we have for all $t>0$,

$$\begin{aligned}
A^\alpha \mathring{A}e^{-tA}u_0 &= A^\alpha(Ae^{-tA}x_0 + e^{-tA}\mathring{A}a_0)\\
&= A^{1+\alpha}e^{-tA}x_0 + A^\alpha e^{-tA}\mathring{A}a_0\\
&= \frac{1}{\Gamma(1-\alpha)}\int_t^{+\infty}\frac{1}{(\tau-t)^\alpha}(A^2e^{-\tau A}x_0 + Ae^{-\tau A}\mathring{A}a_0)\,\mathrm{d}\tau\\
&= (-\partial_t)^{\alpha-1}[A^2e^{-(\cdot)A}x_0 + Ae^{-(\cdot)A}\mathring{A}a_0](t)\\
&= (-\partial_t)^{\alpha-1}[A\mathring{A}e^{-(\cdot)A}u_0](t)\\
&= (-\partial_t)^{\alpha}[\mathring{A}e^{-(\cdot)A}u_0](t).
\end{aligned}$$

So that, by the isomorphism property (3.15), we have

$$\|\mathring{A}e^{-(\cdot)A}u_0\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \sim_{\alpha,q} \|A^\alpha\mathring{A}e^{-(\cdot)A}u_0\|_{\mathrm{L}^q(\mathbb{R}_+,X)}\ .$$

From there, we obtain

$$\begin{aligned}
\|A^\alpha\mathring{A}e^{-(\cdot)A}u_0\|_{\mathrm{L}^q(\mathbb{R}_+,X)} &= \left(\int_0^{+\infty}(\tau^{\frac{1}{q}}\|A^\alpha\mathring{A}e^{-\tau A}u_0\|_X)^q\frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}}\\
&\lesssim_{q,\alpha,A} \left(\int_0^{+\infty}(\tau^{\frac{1}{q}-\alpha}\|\mathring{A}e^{-\frac{\tau}{2}A}u_0\|_X)^q\frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}}\\
&\lesssim_{q,\alpha,A} \|u_0\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)}.
\end{aligned}$$

Our last set of inequalities follows from the analyticity of the semigroup $(e^{-tA})_{t>0}$ on X and the fact that one can write for all $\tau>0$,

$$A^\alpha\mathring{A}e^{-\tau A}u_0 = A^\alpha e^{-\frac{\tau}{2}A}\mathring{A}e^{-\frac{\tau}{2}A}u_0\ . \tag{3.17}$$

Step 3 : The case $\alpha \in (-1+1/q,0)$. We play with the integral representations like (3.16) and

fractional powers of A, so that as in the Step 2, we should be able to write for $t>0$,

$$\begin{aligned}(-\partial_t)^\alpha[\mathring{A}e^{-(\cdot)A}u_0](t) &= \int_t^{+\infty}(-\partial_\tau)^{\alpha+1}[\mathring{A}e^{-(\cdot)A}u_0](\tau)\,\mathrm{d}\tau\\ &= \int_t^{+\infty}\tau A^{1+\alpha}\mathring{A}e^{-\tau A}u_0\frac{\mathrm{d}\tau}{\tau}\,.\end{aligned}$$

Notice that the last integral can be understood as an improper Riemann integral, so that it gives a measurable function with values in X.

Therefore, we can bound, thanks to the Fatou Lemma and then to the analyticity of the semigroup (we use the same trick (3.17)),

$$\begin{aligned}\left\|t\mapsto\int_t^{+\infty}\tau A^{1+\alpha}\mathring{A}e^{-\tau A}u_0\frac{\mathrm{d}\tau}{\tau}\right\|^q_{\mathrm{L}^q(\mathbb{R}_+,X)} &\leqslant \liminf_{M\to+\infty}\int_0^{+\infty}\left\|\int_t^M\tau A^{1+\alpha}\mathring{A}e^{-\tau A}u_0\frac{\mathrm{d}\tau}{\tau}\right\|^q_X\mathrm{d}t\\ &\leqslant \int_0^{+\infty}\left(\int_t^{+\infty}\|\tau A^{1+\alpha}\mathring{A}e^{-\tau A}u_0\|_X\frac{\mathrm{d}\tau}{\tau}\right)^q\mathrm{d}t\\ &\lesssim_{q,\alpha,A}\int_0^{+\infty}\left(t^{\frac{1}{q}}\int_t^{+\infty}\tau^{-\alpha}\|\mathring{A}e^{-\frac{\tau}{2}A}u_0\|_X\frac{\mathrm{d}\tau}{\tau}\right)^q\frac{\mathrm{d}t}{t}\,.\end{aligned}$$

Finally, by the mean of Hardy's inequality [Haa06, Lemma 6.2.6], we conclude

$$\begin{aligned}\left\|t\mapsto\int_t^{+\infty}\tau A^{1+\alpha}\mathring{A}e^{-\tau A}u_0\frac{\mathrm{d}\tau}{\tau}\right\|^q_{\mathrm{L}^q(\mathbb{R}_+,X)} &\lesssim_{q,\alpha,A}\int_0^{+\infty}\left(t^{\frac{1}{q}-\alpha}\|\mathring{A}e^{-\frac{t}{2}A}u_0\|_X\right)^q\frac{\mathrm{d}t}{t}\\ &\lesssim_{q,\alpha,A}\|u_0\|^q_{\mathring{\mathrm{D}}_A(1+\alpha-1/q,q)}.\end{aligned}$$

■

Now, the next lemma ensures that the maximal regularity operator applied to a Sobolev in-time function, with negative regularity, still yields an actual measurable function with values in X.

Lemma 3.19 *Let $\omega\in[0,\frac{\pi}{2})$, $(\mathrm{D}(A),A)$ an ω-sectorial operator on a UMD Banach space X, and let $q\in(1,+\infty)$, $\alpha\in[0,1/q)$.*

For $f\in\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)$, the following holds for all $T>0$,

$$t\mapsto\int_0^t e^{-(t-s)A}f(s)\,\mathrm{d}s\in\mathrm{C}^0([0,T],X)$$

with the estimate

$$\left\|t\mapsto\int_0^t e^{-(t-s)A}f(s)\,\mathrm{d}s\right\|_{\mathrm{L}^\infty([0,T],X)}\lesssim_{A,\alpha,q}T^{1+\alpha-1/q}\|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}.$$

Moreover, if A has the L^q-maximal regularity property, the results still holds for $\alpha\in(-1+1/q,0)$.

Proof. — **Step 1 :** For $\alpha=0$, $q\in(1,+\infty)$. Let $f\in\mathrm{L}^q(\mathbb{R}_+,X)$. By uniform boundedness of the semigroup $(e^{-tA})_{t\geqslant0}$ on X and Hölder's inequality yield

$$\begin{aligned}\left\|\int_0^t e^{-(t-s)A}f(s)\,\mathrm{d}s\right\|_X &\lesssim_A\int_0^t\|f(s)\|_X\,\mathrm{d}s\\ &\lesssim_A t^{1-1/q}\|f\|_{\mathrm{L}^q([0,t],\mathbb{R})}\end{aligned}$$

The supremum on $t \in [0,T]$ yields the estimate

$$\left\| t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \right\|_{\mathrm{L}^\infty([0,T],X)} \lesssim_A T^{1-1/q} \|f\|_{\mathrm{L}^q(\mathbb{R}_+,X)}.$$

Continuity in-time follows from the dominated convergence theorem.

Step 2 : For $\alpha \in (0,1/q)$, $f \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)$, for $\frac{1}{r} = \frac{1}{q} - \alpha$, we have $f \in \mathrm{L}^r(\mathbb{R}_+,X)$ by Sobolev embeddings. Therefore, by the Step 1,

$$\begin{aligned}\left\| t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \right\|_{\mathrm{L}^\infty([0,T],X)} &\lesssim_A T^{1-1/r} \|f\|_{\mathrm{L}^r(\mathbb{R}_+,X)} \\ &\lesssim_{A,\alpha,q} T^{1+\alpha-1/q} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}.\end{aligned}$$

Step 3 : Let $\alpha \in (-1+1/q,0)$, $f \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)$ and assume that A has the maximal regularity property. First, by commutation properties for resolvents of ∂_t and A, one can write

$$\begin{aligned}(\partial_t + A)^{-1} f &= (\partial_t)^{-\alpha-1} \partial_t (\partial_t + A)^{-1} (\partial_t)^\alpha f \\ &= (\partial_t)^{-\alpha-1} (\partial_t)^\alpha f - (\partial_t)^{-\alpha-1} A (\partial_t + A)^{-1} (\partial_t)^\alpha f.\end{aligned}$$

So that setting $f^\alpha := (\partial_t)^\alpha f$ and $u^\alpha := (\partial_t + A)^{-1} f^\alpha$, by the representation formula (3.14), we end up with the following expression for $t > 0$

$$(\partial_t + A)^{-1} f(t) = \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s = \frac{1}{\Gamma(1+\alpha)} \int_0^t \frac{1}{(t-s)^{-\alpha}} [f^\alpha(s) - A u^\alpha(s)]\,\mathrm{d}s.$$

Young's inequality for the convolution, then the triangle inequality yield

$$\begin{aligned}\left\| t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \right\|_{\mathrm{L}^\infty([0,T],X)} &\leqslant \frac{1}{\Gamma(1+\alpha)} \left\| t \mapsto t^\alpha \right\|_{\mathrm{L}^{q'}([0,T])} \|f^\alpha - A u^\alpha\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \\ &\leqslant \frac{T^{1+\alpha-1/q}}{\Gamma(1+\alpha)(\alpha q' + 1)^{\frac{1}{q'}}} \left(\|f^\alpha\|_{\mathrm{L}^q(\mathbb{R}_+,X)} + \|A u^\alpha\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \right).\end{aligned}$$

From there, we recall that we have assumed the L^q-maximal regularity property, so that, by the isomorphism property (3.13),

$$\begin{aligned}\left\| t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \right\|_{\mathrm{L}^\infty([0,T],X)} &\lesssim_{A,\alpha,q} T^{1+\alpha-1/q} \|f^\alpha\|_{\mathrm{L}^q(\mathbb{R}_+,X)} \\ &\lesssim_{A,\alpha,q} T^{1+\alpha-1/q} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}.\end{aligned}$$

■

Corollary 3.20 *Let $\omega \in [0, \frac{\pi}{2})$, $(\mathrm{D}(A), A)$ an ω-sectorial operator on a UMD Banach space X, and let $q \in (1,+\infty)$, $\alpha \in [0,1/q)$.*

For $f \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)$, the following holds for all $T > 0$,

$$t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \in \mathrm{C}^0([0,T], \mathcal{D}_A(1+\alpha-1/q,q))$$

with the estimates, for all $T > 0$,

$$\left\| t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \right\|_{\mathrm{L}^\infty([0,T],\mathcal{D}_A(1+\alpha-1/q,q))} \lesssim_{A,\alpha,q} (1+T^{1+\alpha-1/q})\|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \tag{3.18}$$

$$\left\| t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \right\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1+\alpha-1/q,q))} \lesssim_{A,\alpha,q} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}. \tag{3.19}$$

Proof. — Thanks to Lemma 3.19, it suffices to prove the estimate (3.19).

Step 1 : First we assume $\alpha = 0$ and $f \in \mathrm{L}^q(\mathbb{R}_+, X)$, we may extend f to $\mathbb{R}$ by setting $f(t) := 0$ for $t < 0$. Similarly to what has been done in [DHMT21, Lemma 2.19], we can bound

$$\begin{aligned}
\left\| \int_0^t e^{-sA} f(t-s)\,\mathrm{d}s \right\|^q_{\mathring{\mathcal{D}}_A(1-1/q,q)} &= \int_0^{+\infty} \left(\tau^{\frac{1}{q}} \left\| A \int_0^t e^{-(\tau+s)A} f(t-s)\,\mathrm{d}s \right\|_X \right)^q \frac{\mathrm{d}\tau}{\tau} \\
&\lesssim_{A,q} \int_0^{+\infty} \left(\int_0^{+\infty} \frac{1}{\tau+s} \|f(t-s)\|_X \mathrm{d}s \right)^q \mathrm{d}\tau \\
&\lesssim_{A,q} \int_0^{+\infty} \left(\tau^{\frac{1}{q}-1} \int_0^{\tau} \|f(t-s)\|_X \mathrm{d}s \right)^q \frac{\mathrm{d}\tau}{\tau} \\
&\quad + \int_0^{+\infty} \left(\tau^{\frac{1}{q}} \int_{\tau}^{+\infty} \frac{1}{s} \|f(t-s)\|_X \mathrm{d}s \right)^q \frac{\mathrm{d}\tau}{\tau}.
\end{aligned}$$

We can apply Hardy's inequalities, see [Haa06, Lemma 6.2.6], to obtain

$$\left\| \int_0^t e^{-sA} f(t-s)\,\mathrm{d}s \right\|^q_{\mathring{\mathcal{D}}_A(1-1/q,q)} \lesssim_{A,q} \int_0^{+\infty} (\tau^{\frac{1}{q}} \|f(t-\tau)\|_X)^q \frac{\mathrm{d}\tau}{\tau} \lesssim_{A,q} \|f\|^q_{\mathrm{L}^q(\mathbb{R}_+,X)}.$$

Step 2 : For $\alpha \in (0, 1/q)$, $f \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+, X)$, by Sobolev embeddings, we have $f \in \mathrm{L}^r(\mathbb{R}_+, X)$, $r = \frac{q}{1-\alpha q}$, so that by the Step 1, for all $T > 0$:

$$t \mapsto \int_0^t e^{-sA} f(t-s)\,\mathrm{d}s \in \mathrm{C}^0([0,T), \mathcal{D}_A(1-1/r, r)) \ .$$

So that it is well-defined. Let $t > 0$, by [DHMT21, Lemma 2.15], we have

$$\left\| \int_0^t e^{-sA} f(t-s)\,\mathrm{d}s \right\|^q_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)} \sim_{\alpha,q} \underbrace{\int_0^{+\infty} \left(\tau^{1+\frac{1}{q}-\alpha} \left\| A^2 e^{-\tau A} \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \right\|_X \right)^q \frac{\mathrm{d}\tau}{\tau}}_{(I)} \ .$$

Since ∂_t and A have commuting resolvents, we have

$$(\partial_t + A)^{-1} = (\partial_t)^{-\alpha} (\partial_t + A)^{-1} (\partial_t)^{\alpha}.$$

Therefore, setting $f^\alpha := (\partial_t)^\alpha f \in \mathrm{L}^q(\mathbb{R}, X)$ (up to consider, again, the extension of f^α (not f) to the whole line by 0), we can use the representation formula (3.14), to obtain

$$(I) \sim_{\alpha,q} \int_0^{+\infty} \left(\tau^{1+\frac{1}{q}-\alpha} \left\| \int_0^t \frac{1}{(t-u)^{1-\alpha}} \int_0^u A^2 e^{-(\tau+(u-s))A} f^\alpha(s)\,\mathrm{d}s\,\mathrm{d}u \right\|_X \right)^q \frac{\mathrm{d}\tau}{\tau} \ .$$

From there, we can use the triangle inequality, and we can write, provided $0 \leqslant s \leqslant u \leqslant t$,

$$A^2 e^{-(\tau+(u-s))A} = A^{1+\alpha} e^{-\frac{(\tau+(u-s))}{2}A} A^{1-\alpha} e^{-\frac{\tau}{2}A} e^{-\frac{(u-s)}{2}A},$$

so that, by analyticity of the semigroup $(e^{-tA})_{t\geqslant 0}$ on X, and the Fubini-Tonelli theorem, we have

$$(I) \lesssim_{\alpha,q,A} \int_0^{+\infty} \left(\tau^{\frac{1}{q}} \int_0^t \int_0^u \frac{1}{(t-u)^{1-\alpha}} \frac{1}{(\tau+(u-s))^{1+\alpha}} \|f^\alpha(s)\|_X \mathrm{d}s \, \mathrm{d}u \right)^q \frac{\mathrm{d}\tau}{\tau} .$$

Again by Fubini-Tonelli, and since $\int_s^t \frac{1}{(t-u)^{1-\alpha}} \frac{1}{(\tau+(u-s))^{1+\alpha}} \mathrm{d}u = \frac{1}{\alpha} \frac{(t-s)^\alpha}{(\tau+(t-s))\tau^\alpha}$, it follows that

$$(I) \lesssim_{\alpha,q,A} \int_0^{+\infty} \left(\tau^{\frac{1}{q}-\alpha} \int_0^t \frac{s^\alpha}{(\tau+s)} \|f^\alpha(t-s)\|_X \mathrm{d}s \right)^q \frac{\mathrm{d}\tau}{\tau} .$$

We can reproduce the use of Hardy's inequalities [Haa06, Lemma 6.2.6] as in the Step 1, to obtain

$$\int_0^{+\infty} \left(\tau^{\frac{1}{q}-\alpha} \int_0^{+\infty} \frac{s^\alpha}{(\tau+s)} \|f^\alpha(t-s)\|_X \mathrm{d}s \right)^q \frac{\mathrm{d}\tau}{\tau} \lesssim_{\alpha,q} \|f^\alpha\|^q_{\mathrm{L}^q(\mathbb{R}_+,X)} \lesssim_{\alpha,q} \|f\|^q_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} .$$

One may also prove the continuity in-time by a density argument and the estimate (3.18). ■

4.3 The main result

Theorem 3.21 *Let $\omega \in [0, \frac{\pi}{2})$, $(\mathrm{D}(A), A)$ an ω-sectorial operator on a UMD Banach space X, such that it satisfies Assumptions (3.4) and (3.5). Let $q \in (1, +\infty)$, $\alpha \in (-1 + 1/q, 1/q)$ and assume that one of the two following conditions is satisfied*

(i) $\alpha \geqslant 0$ and A has the L^q-maximal regularity property,

(ii) $\alpha < 0$ and A has BIP on X of type $\theta_A < \frac{\pi}{2}$.

Let $T \in (0, +\infty]$. For $f \in \dot{\mathrm{H}}^{\alpha,q}((0,T),X)$, $u_0 \in \mathring{\mathcal{D}}_A(1+\alpha-1/q, q)$, the problem (ACP) admits a unique mild solution $u \in \mathrm{C}^0_b([0,T), \mathring{\mathcal{D}}_A(1+\alpha-1/q,q))$ such that $\partial_t u$, $Au \in \dot{\mathrm{H}}^{\alpha,q}((0,T),X)$ with estimate

$$\|u\|_{\mathrm{L}^\infty([0,T],\mathring{\mathcal{D}}_A(1+\alpha-1/q,q))} \lesssim_{A,q,\alpha} \|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),X)} \lesssim_{A,q,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)}. \tag{3.20}$$

Moreover, if A admits BIP on X of type $\theta_A < \frac{\pi}{2}$, for $f \in \dot{\mathrm{H}}^{\alpha,q}((0,T),X)$, $u_0 \in \mathcal{D}_A(1+\alpha-1/q,q)$ and all $\beta \in [0,1]$,

$$\|(-\partial_t)^{1-\beta} A^\beta u\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),X)} \lesssim_{A,q,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)}. \tag{3.21}$$

Remark 3.22 • In Theorem 3.21, assumptions (3.4) and (3.5) are assumed here in order to ensure that $\mathring{\mathcal{D}}_A(\theta,q)$ is a well-defined, even if not complete, normed vector space.

• If $u_0 = 0$, the estimate (3.21) remains valid if we replace the operator $(-\partial_t)^{1-\beta}$ by $(\partial_t)^{1-\beta}$.

• If one asks instead the initial data u_0 to be in the smaller, but complete, space $\mathcal{D}_A(\theta,q)$ then one can drop assumptions (3.4) and (3.5), and the estimate (3.20) still holds. However, one loose the possibility to compute the corresponding equivalent norm by the mean of real interpolation (3.11).

• The assumption *(ii)* is probably not necessary for the case $\alpha < 0$. However, it is not clear in this case how to prove the left-hand side of the estimate (3.20). Indeed, our approach requires considering the action of $A^{1+\alpha}$, see the Step 3 in the proof.

Proof (of Theorem 3.21). — Let $q \in (1,+\infty)$, $\alpha \in (-1+1/q, 1, q)$. Throughout this proof, and without loss of generality, we assume $T = +\infty$.

Step 1 : The $\dot{\mathrm{H}}^{\alpha,q}$-maximal regularity estimate, for $u_0 = 0$, $f \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+, X)$. Mixed derivatives estimates.

We recall that A has the L^q-maximal regularity property.

Now, we use the fact ∂_t and A have their resolvent that commutes with each other, we have

$$(\partial_t + A)^{-1} = (\partial_t)^{-\alpha}(\partial_t + A)^{-1}(\partial_t)^{\alpha}.$$

This equality and the isomorphism property of $(\partial_t)^{\alpha}$ (3.13) yield

$$\|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \lesssim_{A,q,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} .$$

And for the same reasons, from the L^q-setting, if A has BIP on X of type $\theta_A < \frac{\pi}{2}$, by [Pru02, Proposition 2.4] for all $\beta \in [0,1]$, we have

$$\|(\partial_t)^{1-\beta} A^{\beta} u\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \lesssim_{A,q,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}. \tag{3.22}$$

This estimate will be useful later.

Concerning the estimate (3.21) (with $u_0 \in \mathcal{D}_A(1+\alpha-1/q,q)$), it suffices to assume that the right-hand side of (3.20) holds. Indeed, in this case it suffices to apply Theorem 1.14, to the holomorphic families (of operators)

$$(e^{(z-\beta)^2}(-\partial_t)^{1-z}A^z(\partial_t+A)^{-1})_{0\leqslant \Re(z)\leqslant 1} \quad \text{and} \quad (e^{(z-\beta)^2}(-\partial_t)^{1-z}A^z[e^{-tA}(\cdot)])_{0\leqslant \Re(z)\leqslant 1},$$

provided $\beta \in (0,1)$ is fixed. The proof of the boundedness is then carried over by (3.20) and BIP of A and of $-\partial_t$ respectively. Details are left to the reader.

Step 2 : The trace estimate when $\alpha \in [0, 1/q)$. Let $u_0 \in \mathring{\mathcal{D}}_A(1+\alpha-1/q,q)$, $f \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)$. The solution u must be given for all $t>0$, by

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-sA} f(t-s)\,\mathrm{d}s.$$

Corollary 3.20 tells us that

$$u \in \mathrm{C}^0_b(\mathbb{R}_+, \mathring{\mathcal{D}}_A(1+\alpha-1/q,q)),$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1+\alpha-1/q,q))} \lesssim_{A,q,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)}.$$

Since $f = \partial_t u + Au$ and $e^{-tA}u_0 = u(t) - \int_0^t e^{-sA} f(t-s)\,\mathrm{d}s$, for all $t>0$, the triangle inequality leads to

$$\begin{aligned}\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1+\alpha-1/q,q))} \lesssim_{A,q,\alpha} &\|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \left(\int_0^{+\infty}\left(\tau^{\frac{1}{q}-\alpha}\|Au(\tau)\|_X\right)^q \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}} \\ &+ \left(\int_0^{+\infty}\left(\tau^{\frac{1}{q}-\alpha}\left\|A\int_0^\tau e^{-sA}f(\tau-s)\,\mathrm{d}s\right\|_X\right)^q \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}}.\end{aligned}$$

Thus, by the Hardy-Sobolev inequality, Lemma 3.13, we obtain

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1+\alpha-1/q,q))} \lesssim_{A,q,\alpha} \|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \left\|\tau \mapsto A\int_0^\tau e^{-(\tau-s)A} f(s)\,\mathrm{d}s\right\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}.$$

Now, we may apply the result from the Step 1 to the last term, by the triangle inequality, since $f = \partial_t u + Au$, we deduce

$$\begin{aligned}\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1+\alpha-1/q,q))} &\lesssim_{A,q,\alpha} \|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} \\ &\lesssim_{A,q,\alpha} \|(\partial_t u, Au)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}.\end{aligned}$$

Step 3 : The trace estimate when $\alpha \in (-1+1/q, 0)$. Let $f \in \dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+, X)$.
By Lemma 3.19, we have

$$t \mapsto \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s \in \mathrm{C}^0(\mathbb{R}_+, X)\ .$$

However, for $t, \tau > 0$,

$$e^{-\tau A}\int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s = \int_0^{t+\tau} e^{-(\tau+t-s)A} f(s)\,\mathrm{d}s - \int_0^\tau e^{-(\tau-s)A} f(s+t)\,\mathrm{d}s.$$

So that if we set $v = (\partial_t + A)^{-1} f$, $v_t = (\partial_t + A)^{-1}[f(\cdot + t)]$, we obtain for $t, \tau > 0$

$$e^{-\tau A} v(t) = v(t+\tau) - v_t(\tau).$$

Therefore, by analyticity of the semigroup $(e^{-\tau A})_{\tau>0}$, and the triangle inequality, we obtain for $t > 0$,

$$\|v(t)\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)} \lesssim_{\alpha,q,A} \|A^{1+\alpha} v(\cdot + t)\|_{\mathrm{L}^q(\mathbb{R}_+,X)} + \|A^{1+\alpha} v_t\|_{\mathrm{L}^q(\mathbb{R}_+,X)}.$$

We can now apply (3.22) with $\beta = 1 + \alpha$, and use the translation invariance of Sobolev norms, yielding

$$\|v(t)\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)} \lesssim_{\alpha,q,A} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}.$$

Now, for $u = e^{-(\cdot)A} u_0 + v$, provided $u_0 \in \mathring{\mathcal{D}}_A(1+\alpha-1/q, q)$, we deduce

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}_+,\mathring{\mathcal{D}}_A(1+\alpha-1/q,q))} \lesssim_{\alpha,q,A} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|u_0\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)}.$$

Again, to obtain the left-hand side of (3.20), as in the previous Step 2, it suffices to estimate u_0 in $\mathring{\mathcal{D}}_A(1+\alpha-1/q, q)$-norm. However, such estimate may involve the action of fractional powers of A on $e^{-\tau A} u_0$, for which the meaning is not clear when $u_0 \in \mathring{\mathcal{D}}_A(\theta, q)$. To circumvent this issue, we use the fact that $\mathrm{D}(A)$ is dense in $\mathring{\mathcal{D}}_A(1+\alpha-1/q, q)$ by [DHMT21, Lemma 2.10]. Thus, let $(u_{0,n})_{n\in\mathbb{N}}$ be a sequence in $\mathrm{D}(A)$ which converges to u_0 in $\mathring{\mathcal{D}}_A(1+\alpha-1/q, q)$. We set for all $n \in \mathbb{N}$, $u_n := e^{-(\cdot)A} u_{0,n} + v$.

By analyticity of the semigroup $(e^{-\tau A})_{\tau>0}$, and by the identity

$$(-\partial_t)^\alpha [Au_n - Av](\tau) = (-\partial_t)^\alpha [Ae^{-(\cdot)A} u_{0,n}](\tau) = A^{1+\alpha} e^{-\tau A} u_{0,n},$$

we are able to deduce that

$$\begin{aligned}\|u_{0,n}\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)} &= \left(\int_0^{+\infty} \left(\tau^{\frac{1}{q}-\alpha}\|Ae^{-\tau A} u_{0,n}\|_X\right)^q \frac{\mathrm{d}\tau}{\tau}\right)^{\frac{1}{q}} \\ &\lesssim_{\alpha,q,A} \left(\int_0^{+\infty} \|A^{1+\alpha} e^{-\tau A} u_{0,n}\|_X^q \mathrm{d}\tau\right)^{\frac{1}{q}} \\ &\lesssim_{\alpha,q,A} \left(\int_0^{+\infty} \|(-\partial_t)^\alpha [Au_n - Av](\tau)\|_X^q \mathrm{d}\tau\right)^{\frac{1}{q}}.\end{aligned}$$

Finally, we can use the isomorphism property (3.15) and the triangle inequality to obtain

$$\begin{aligned}\|u_{0,n}\|_{\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)} &\lesssim_{\alpha,q,A} \|Au_n\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|Av\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}\\ &\lesssim_{\alpha,q,A} \|Au_n\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)} + \|f\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}\\ &\lesssim_{\alpha,q,A} \|(\partial_t u_n, Au_n)\|_{\dot{\mathrm{H}}^{\alpha,q}(\mathbb{R}_+,X)}\ .\end{aligned}$$

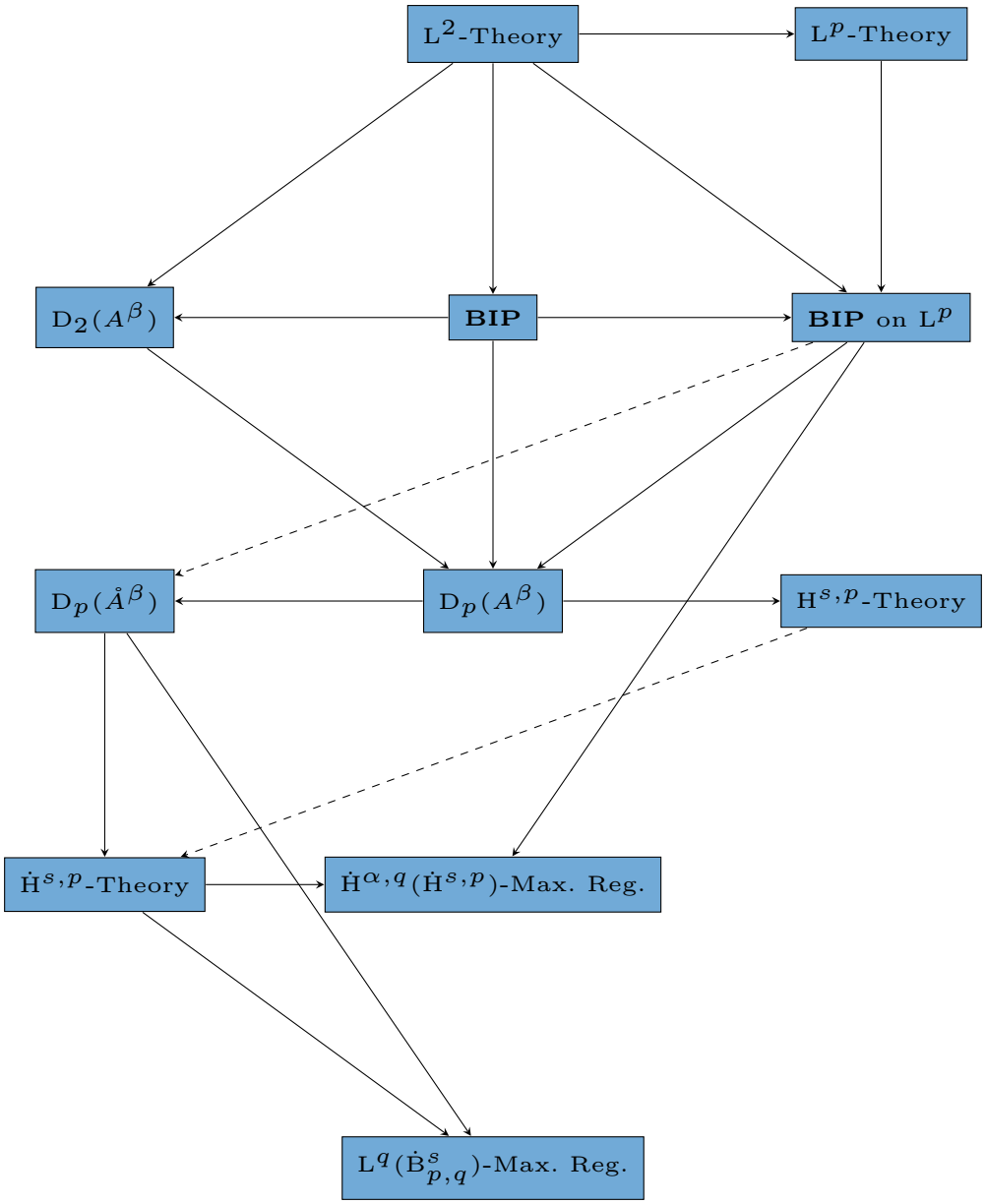
The proof ends here since one can pass to the limit as n goes to infinity.

It remains to prove the continuity in time with values in $\mathring{\mathcal{D}}_A(1+\alpha-1/q,q)$ which follows from a density argument [1]. ■

1. No need of completeness here, since the involved limits are already constructed.

5 A treasure map to reach L^q-maximal regularities in concrete cases

Here, $(\mathrm{D}_2(A), A)$ is a differential operator induced by a continuous, densely defined, closed, accretive, sesquilinear form on (possibly a closed subspace of) $\mathrm{L}^2(\Omega, \mathbb{C}^M)$, $\Omega \subset \mathbb{R}^n$, $M \in \mathbb{N}$. For simplicity, one may have in mind that, for some $m \in \mathbb{N}$, the form domain is a closed subset of $\mathrm{H}^m(\Omega, \mathbb{C}^M)$ with equivalence of norms (usually $m = 1$).



When an arrow indicates towards a (homogeneous) domain of a fractional power, it means it is expected we are able to compute/describe it explicitly. Sometimes additional knowledge maybe necessary, such as the Kato square-root property. For the Kato square-root property in the case of second order elliptic operators in divergence form with rough coefficients, the reader could consult [Aus07] for $\Omega = \mathbb{R}^n$, and [Ege15, Chapter 5] and [Ege18, Theorem 1.2] for the case of mixed boundary conditions when Ω is bounded with minimal regularity on the boundary.

4 Hodge decomposition and maximal regularities for the Hodge Laplacian on the upper half-space

Sommaire

Résumé du chapitre

Ce chapitre s'intéresse à la décomposition de Hodge pour tous les degrés de formes différentielles sur l'espace entier $\mathbb{R}^n$ et le demi-espace $\mathbb{R}^n_+$ sur différentes familles d'espace de fonctions, à savoir les espaces de Besov et de Sobolev homogènes et inhomogènes, $\dot{H}^{s,p}$, $\dot{B}^s_{p,q}$, $H^{s,p}$ et $B^s_{p,q}$, pour tout $p \in (1, +\infty)$, $-1 + 1/p < s < 1/p$. Le calcul fonctionnel holomorphe borné, et d'autres propriétés analytiques fonctionnelles, des laplaciens de Hodge sont également étudiés dans le demi-espace, et donnent des résultats similaires pour les opérateurs de Hodge-Stokes et d'autres opérateurs apparentés via la décomposition de Hodge qui est prouvée.

En conséquence, la théorie des opérateurs homogènes et de l'interpolation revisitée par Danchin, Hieber, Mucha et Tolksdorf appliquée aux espaces de fonctions homogènes soumis à certaines conditions au bord est appliquée et conduit à divers résultats de régularité maximale avec des estimations globales en temps qui pourraient être utiles en dynamique des fluides. De plus, le lien entre le laplacien de Hodge et la décomposition de Hodge nous permettra même d'énoncer la décomposition de Hodge pour des espaces de Sobolev et de Besov d'ordre supérieur avec des conditions de compatibilité supplémentaires, pour un indice de régularité $s \in (-1 + 1/p, 2 + 1/p)$. Afin de donner un sens à toutes ces propriétés dans les espaces de fonctions désirés, nous donnons également un sens approprié aux traces partielles sur le bord.

La stratégie globale est de réaliser les opérateurs de Hodge-Dirac et les laplaciens de Hodge simultanément et indépendamment sur chaque espace de fonctions, puis de prouver et d'extrapoler

les propriétés de régularité sur L^2 aux autres familles d'espaces de fonctions. À partir de là, on peut alors prouver le caractère borné de la projection de Hodge-Leray $\mathbb{P}$ lorsque $s \in (-1+1/p, 1/p)$.

Après l'extrapolation des propriétés de régularité, nous sommes capables de découpler les valeurs au bord en des valeurs au bord homogènes de Dirichlet et de Neumann. A partir de là, nous pouvons utiliser les résultats de la Section 6, du Chapitre 2, pour appliquer la théorie des opérateurs et de l'interpolation homogènes pour chaque opérateur sur tous les espaces considérés et ensuite étendre le caractère borné de $\mathbb{P}$ pour $s \in (-1+1/p, 2+1/p)$.

La principale difficulté provient de l'impossibilité de "voir directement" les valeurs au bord lorsqu'on travaille avec la réalisation homogène de l'opérateur.

Summary of the chapter

This chapter is concerned by the Hodge decomposition for any degree of differential forms is investigated on the whole space $\mathbb{R}^n$ and the half space $\mathbb{R}^n_+$ on different scale of functions space namely homogeneous and inhomogeneous Besov and Sobolev space, $\dot{H}^{s,p}$, $\dot{B}^s_{p,q}$, $H^{s,p}$ and $B^s_{p,q}$, for all $p \in (1, +\infty)$,$-1 + 1/p < s < 1/p$. The bounded holomorphic functional calculus, and other functional analytic properties, of Hodge Laplacians is also investigated in the half space, and yields similar results for Hodge-Stokes and other related operators via the proven Hodge decomposition.

As consequences, the homogeneous operator and interpolation theory revisited by Danchin, Hieber, Mucha and Tolksdorf applied to homogeneous function spaces subject to boundary conditions is applied and leads to various maximal regularity results with global-in-time estimates that could be of use in fluid dynamics. Moreover, the bond between the Hodge Laplacian and the Hodge decomposition even enable us to state the Hodge decomposition for higher order Sobolev and Besov spaces with additional compatibility conditions, for regularity index $s \in (-1+1/p, 2+1/p)$. In order to make sense of all those properties in desired function spaces, we also give appropriate meaning of partial traces on the boundary.

The overall strategy is to realize Hodge-Dirac operators and Hodge Laplacians simultaneously and independently on each space, then prove and extrapolate regularity properties on L^2 to other scales of function spaces. From this point one can then prove the boundedness of the Hodge-Leray projection $\mathbb{P}$ when $s \in (-1+1/p, 1/p)$.

After the extrapolation of regularity, we are able to decouplate boundary values into homogeneous Dirichlet and Neumann boundary values. From this point, we are able to use the results in Section 6, from Chapter 2, to perform the homogeneous operator and interpolation theory for each operator on all the relevant spaces and then extend the boundedness of $\mathbb{P}$ for $s \in (-1+1/p, 2+1/p)$.

The main difficulty arises from the impossibility to "see directly" the boundary values when one deal with the homogeneous realization of the operator.

1 Introduction

1.1 One Laplacian to rule (almost) them all : the differential form formalism and the Hodge decomposition

The study of incompressible fluid dynamics, and in particular the treatment of Navier-Stokes equations, relies mostly on the Helmholtz decomposition of vector fields in appropriate function spaces. The Helmholtz decomposition of vector field $u : \Omega \longrightarrow \mathbb{C}^n$, is given by a vector field $v : \Omega \longrightarrow \mathbb{C}^n$ and and a function $q : \Omega \longrightarrow \mathbb{C}$, such that

$$u = v + \nabla q \text{ and } \operatorname{div} v = 0 \text{ (with possibly } v \cdot \nu_{|\partial\Omega} = 0).$$

This point is central since incompressibility condition for the velocity of a fluid u is carried over by the condition $\operatorname{div} u = 0$.

In the interest of the Navier-Stokes and related equations, one wants the above decomposition to hold topologically in an appropriate normed vector space of functions [1] with uniqueness (up to a constant for q). It is indeed true in $\mathrm{L}^2(\Omega,\mathbb{C}^n)$, since $\mathbb{P}$, the usual Helmholtz-Leray projector on divergence free vector fields with null tangential trace at the boundary, *i.e.* such that

$$\mathbb{P} : \mathrm{L}^2(\Omega,\mathbb{C}^n) \longrightarrow \mathrm{L}^2_\sigma(\Omega) = \{\, u \in \mathrm{L}^2(\Omega,\mathbb{C}^n) \mid \mathrm{div}\ u = 0,\ u\cdot\nu_{|\partial\Omega} = 0 \,\},$$

is well-defined, linear, bounded and unique by construction of the orthogonal projector on a closed subspace of an Hilbert space, here $\mathrm{L}^2_\sigma(\Omega) \subset \mathrm{L}^2(\Omega,\mathbb{C}^n)$. It gives the classical orthogonal and topological Helmholtz decomposition, see [Soh01, Chapter 2, Section 2.5],

$$\mathrm{L}^2(\Omega,\mathbb{C}^n) = \mathrm{L}^2_\sigma(\Omega) \overset{\perp}{\oplus} \overline{\nabla \mathrm{H}^{1,2}(\Omega,\mathbb{C})},$$

for any (bounded) Lipschitz domain Ω, see [Soh01, Lemma 2.5.3]. Here $\mathrm{H}^{1,2}(\Omega,\mathbb{C})$ is the standard L^2-Sobolev space of order 1 on Ω.

The L^2-theory for the Helmholtz decomposition on a domain Ω relies mostly on pure Hilbertian operator theory. However, the question about the L^p-theory, $p \neq 2$, *i.e.,* to know if

$$\mathrm{L}^p(\Omega,\mathbb{C}^n) = \mathrm{L}^p_\sigma(\Omega) \oplus \overline{\nabla \mathrm{H}^{1,p}(\Omega,\mathbb{C})}, \tag{4.1}$$

(or even the Sobolev or Besov counterpart) is actually a harder question, which falls generally in the field of harmonic analysis. The underlying range of Lebesgue and Sobolev exponents for which such decomposition holds will generally depend on the regularity of the boundary and the geometry of the domain Ω.

The L^p setting has been widely studied, we mention the work of Fabes, Mendez and Mitrea, [FMM98, Theorem 12.2], where the result has been proven for bounded Lipschitz domains : (4.1) holds whenever $p \in (3/2-\varepsilon, 3+\varepsilon)$. The work of Sohr and Simader [SS92, Theorem 1.4] yields (4.1) for C^1 bounded and exterior domains, allowing $p \in (1,+\infty)$. For general unbounded domains, when $p \neq 2$, the decomposition (4.1) may fail : see the counterexample by Bogovskiĭ [Bog86, Section 2]. Tolksdorf has shown in his PhD dissertation [Tol17, Theorem 5.1.10] that (4.1) is true for all $p \in (\frac{2n}{(2n+1)} - \varepsilon, \frac{2n}{(2n-1)} + \varepsilon)$, provided Ω is a special Lipschitz domain, $\varepsilon > 0$ depending on Ω. We also mention the works of Farwig, Kozono and Sohr where the decomposition is investigated in a more exotic setting in [FKS05 ; FKS07] for general uniformly C^1 unbounded domains.

Our interest here is the case of the half-space $\mathbb{R}^n_+$, where the Helmholtz is mainly known to be true on $\mathrm{L}^p(\mathbb{R}^n_+,\mathbb{C}^n)$ for all $p \in (1,+\infty)$, see [Gal11, Remark III.1.2] : we aim to generalize this result to the scale of inhomogeneous, and homogeneous Sobolev and Besov spaces on the half-space. To be more precise, we want to investigate decompositions of the type

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C}^n) = \dot{\mathrm{H}}^{s,p}_\sigma(\mathbb{R}^n_+,\mathbb{C}^n) \oplus \overline{\nabla \dot{\mathrm{H}}^{s+1,p}(\mathbb{R}^n_+,\mathbb{C}^n)}, \tag{4.2}$$

and similarly for Besov spaces, and their inhomogeneous counterparts, provided $s \in \mathbb{R}$, $p \in (1,+\infty)$.

In the scale of inhomogeneous and homogeneous Besov and Sobolev spaces on bounded and exterior $\mathrm{C}^{2,1}$ domains the Helmholtz decomposition was shown by Fujiwara and Yamazaki [FY07, Theorem 3.1] : the Helmholtz decomposition holds on $\mathrm{H}^{s,p}(\Omega,\mathbb{C}^n)$ and $\mathrm{B}^s_{p,q}(\Omega,\mathbb{C}^n)$, $p \in (1,+\infty)$, $s \in (-1+1/p, 1/p)$, $q \in [1,+\infty]$, even allowing $p = 1,+\infty$ in case of Besov spaces. We also mention the work of Monniaux and Mitrea [MM08, Proposition 2.16] on bounded Lipschitz domains, where the result is shown to hold true for (inhomogeneous) Sobolev spaces that lie near the family $(\mathrm{H}^{s,2})_{|s|<1/2}$.

It has been notified in several works, e.g., see [GHT13, Introduction], [MS18, Section 4], that

1. From here the divergence will be understood in the distributional sense.

the following Laplace operator acting on vector fields,

$$-\Delta_{\mathcal{H}} := -\Delta u = \operatorname{curl}\operatorname{curl} u - \nabla \operatorname{div} u, \text{ and } [u\cdot\nu_{|\partial\Omega} = 0,\ \nu\times\operatorname{curl} u_{|\partial\Omega} = 0] \tag{4.3}$$

called the [2] Hodge Laplacian, has a strong bond with, and respects, the Helmholtz decomposition in the sense that for all u in the domain of above Laplacian, $\mathbb{P}u$ also lies in, and we have

$$-\mathbb{P}\Delta u = \operatorname{curl}\operatorname{curl} u = -\Delta\mathbb{P}u, \text{ and } [\mathbb{P}u\cdot\nu_{|\partial\Omega} = 0,\ \nu\times\operatorname{curl}\mathbb{P}u_{|\partial\Omega} = 0].$$

Therefore, since the Hodge Laplacian and the Helmholtz-Leray projector seem to copy the corresponding behavior of the whole space, it seems reasonable to infer that

$$\mathbb{P} = \mathrm{I} + \nabla\underline{\operatorname{div}}(-\Delta_{\mathcal{H}})^{-1}, \tag{4.4}$$

where $\underline{\operatorname{div}}$ drives a boundary condition $\nu\cdot u_{|\partial\Omega} = 0$.

But, the above use of curl operators restricts us to the three dimensional case. We can avoid such trouble, by the mean of the differential forms formalism, so that (4.3) becomes

$$-\Delta_{\mathcal{H}}u := -\Delta u = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^* u = (\mathrm{d}+\mathrm{d}^*)^2 u, \text{ and } [\nu\lrcorner u_{|\partial\Omega} = 0,\ \nu\lrcorner \mathrm{d}u_{|\partial\Omega} = 0] \tag{4.5}$$

where $\mathrm{d} : \Lambda^k \longrightarrow \Lambda^{k+1}$ is the exterior derivative, defined on the complexified exterior algebra of $\mathbb{R}^n$, $\Lambda = \Lambda^0\oplus\Lambda^1\oplus\ldots\oplus\Lambda^n$, and satisfies $\mathrm{d}^2 = 0$. The operator $\mathrm{d}^* : \Lambda^k \longrightarrow \Lambda^{k-1}$ is the formal dual operator of d, satisfying also $(\mathrm{d}^*)^2 = 0$ so that on $\mathbb{R}^3$, we can make the identifications

$$\begin{aligned}
\mathrm{d}_{|\Lambda^1} &= \operatorname{curl}\ ,\ \mathrm{d}_{|\Lambda^0} = \nabla,\\
\mathrm{d}^*_{|\Lambda^2} &= \underline{\operatorname{curl}},\ \mathrm{d}^*_{|\Lambda^1} = -\underline{\operatorname{div}},\\
\nu\lrcorner()_{|\Lambda^1} &= \nu\cdot(),\ \nu\lrcorner()_{|\Lambda^2} = \nu\times().
\end{aligned}$$

The $\underline{\operatorname{curl}}$ operator drives a boundary condition $\nu\times u_{|\partial\Omega} = 0$.

Notice this definition still makes sense for differential forms of any degree, in arbitrary dimension. One would check that (4.5) reduce to the Neumann Laplacian in the case of 0-forms identified with scalar-valued functions.

Going back to the case of vector fields, instead of (4.4), the above formalism and the fact that d and d^* are nilpotent, and then commutes (at least formally) with $\Delta_{\mathcal{H}}$, we may infer the next formula, similar to the one mentioned in [ACDH04, Section 5] :

$$\mathbb{P} = \mathrm{I} - \mathrm{d}\mathrm{d}^*(-\Delta_{\mathcal{H}})^{-1} = \mathrm{I} - \mathrm{d}(-\Delta_{\mathcal{H}})^{-1/2}\mathrm{d}^*(-\Delta_{\mathcal{H}})^{-1/2}. \tag{4.6}$$

Under the use of the differential forms formalism, the desired Helmholtz decomposition (4.2) becomes, for $k\in[\![0,n]\!]$ different degrees of differential forms,

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda^k) = \dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+,\Lambda^k)\oplus\overline{\mathrm{d}\dot{\mathrm{H}}^{s+1,p}(\mathbb{R}^n_+,\Lambda^{k-1})} \tag{4.7}$$

which is called the **Hodge decomposition** instead of the Helmholtz decomposition. Here, the space $\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+,\Lambda^k)$ stands for k-differential forms u whose coefficients lies in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\mathbb{C})$, and such that $\mathrm{d}^*u = 0$, and $\nu\lrcorner u_{|\partial\mathbb{R}^n_+} = 0$.

The Hodge decomposition for differential forms is treated by Schwartz [Sch95, Theorem 2.4.2, Theorem 2.4.14] on smooth compact Riemannian manifold M with smooth boundary, where the decomposition is stated on $\mathrm{H}^{k,p}(M)$, $k\in\mathbb{N}$, $p\in(1,+\infty)$. For the case of Ω a bounded Lipschitz domains of $\mathbb{R}^n$, we refer to the work of Monniaux and M$^{\mathrm{c}}$Intosh [MM18, Theorem 4.3, Theorem 7.1] where the Hodge decomposition is proved to be true on $\mathrm{L}^p(\Omega,\Lambda)$ for all $p\in(\frac{2n}{(2n+1)}-\varepsilon,\frac{2n}{(2n-1)}+\varepsilon)$ where $\varepsilon>0$ depends on Ω. The bounded holomorphic functional calculus of the Hodge Laplacian

2. In fact, this is *a* Hodge Laplacian, the one with tangential boundary conditions, we do not make the distinction here for introductory purposes.

is also proved for the same range of indices. One may also consult the work of Mitrea and Monniaux, and Hofmann, Mitrea and Monniaux, [MM09a ; HMM11], for the treatment of the Hodge Laplacian on bounded Lipschitz domains of compact Riemannian manifolds, where functional analytic properties like analyticity of the generated semigroup, or boundedness of associated Riesz transforms are investigated.

One may wonder about the superficiality of proving an identity like (4.7) for general differential forms, instead of vector fields (differential forms of degree $1, n-1$) only. In fact, the differential forms formalism has shown its efficiency, allowing to treat some partial differential equations initially restricted to the three-dimensional setting in arbitrary dimension. See for instance [Mon21 ; Den22], where the magnetohydrodynamical (MHD) system is treated, so that either the triplet $\Lambda^1, \Lambda^2, \Lambda^3$ or the triplet $\Lambda^{n-3}, \Lambda^{n-2}, \Lambda^{n-1}$ are involved. Indeed, the magnetic field is in fact not an effective vector field but a 2-form, identified, when $n = 3$, with a vector field. We also mention that reformulation using differential forms for this kind of system allows looking at vorticity-like formulation of the Navier-Stokes (and related) equations, it is also purely intrinsic so that one can perform a similar treatment on manifolds.

To reach our goal, the idea will be to prove that the formula (4.6) holds on $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$, yielding an operator for which we can also prove its boundedness on Sobolev and Besov spaces, so that we are able to obtain

Theorem 4.1 (see Theorem 4.40 & Corollary 4.51) *Let $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, and let $k \in [\![0, n]\!]$. It holds that*

(i) The (generalized) Helmholtz-Leray projector $\mathbb{P} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+, \Lambda^k)$ is well-defined and bounded. Moreover, the following identity is true

$$\mathbb{P} = \mathrm{I} - \mathrm{d}(-\Delta_{\mathcal{H}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H}})^{-\frac{1}{2}}.$$

(ii) The following Hodge decomposition holds

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k) = \dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+, \Lambda^k) \oplus \dot{\mathrm{H}}^{s,p}_{\gamma}(\mathbb{R}^n_+, \Lambda^k).$$

Moreover, the result remains true if we replace

- $\dot{\mathrm{H}}^{s,p}$ *by* $\dot{\mathrm{B}}^s_{p,q}$, $q \in [1, +\infty]$;
- $(\dot{\mathrm{H}}, \dot{\mathrm{B}})$ *by* (H, B).

The symbol X_γ stands for the range of $\mathrm{I} - \mathbb{P}$ in X. [3]

The way we reach Theorem 4.1 through intermediate results and proofs is so that we recover many properties of the Hodge Laplacian as well as its bounded holomorphic functional calculus on Sobolev and Besov spaces almost for free. This is due to the particular structure of the boundary of $\mathbb{R}^n_+$, and the properties of the Laplacian on the whole space $\mathbb{R}^n$. This, above Theorem 4.1, and the fact that one can define the Hodge-Stokes operator as

$$u \in \dot{\mathrm{D}}^s_p(\mathbb{A}_{\mathcal{H}}) = \mathbb{P}\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H}}) \text{ and } \mathbb{A}_{\mathcal{H}}u := -\Delta_{\mathcal{H}}u = \mathrm{d}^*\mathrm{d}u,$$

will yield automatically

Theorem 4.2 (see Theorem 4.42) *Let $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$. For all $\mu \in (0, \pi)$, the operator $\mathbb{A}_{\mathcal{H}}$ admits a bounded ($\mathbf{H}^\infty(\Sigma_\mu)$-)holomorphic functional calculus on $\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+, \Lambda)$. Moreover, the result remains true if we replace*

- $\dot{\mathrm{H}}^{s,p}$ *by* $\dot{\mathrm{B}}^s_{p,q}$, $q \in [1, +\infty]$;
- $(\dot{\mathrm{H}}, \dot{\mathrm{B}})$ *by* (H, B).

3. The subscript (or exponent in case of Besov spaces) γ is a legacy of the writing of G spaces as spaces of gradients of scalar functions in the case of vector fields.

We mention that our strategy is not morally so different from the one presented in [GHT13, Beginning of Section 4], identifying some Neumann and Dirichlet boundary conditions on various components. However, the treatment of boundary value is done in a more careful way, adapted with the scale of homogeneous function spaces, thanks to a weak-strong correspondence of (partial) traces by the mean of appropriate results in the Section 2.3.1.

1.2 Global-in-time estimates in L^q-maximal regularity : the role of homogeneous function spaces and their interpolation

Another tool which is central in the study of parabolic equations and also for a large class of fluid dynamics problems is the L^q-maximal regularity.

The general problem of global in time L^q-maximal regularity is : for a closed operator $(\mathrm{D}(A), A)$ on a Banach space X, let us consider the evolution equation

$$\left\{ \begin{array}{rl} \partial_t u(t) + Au(t) = & f(t)\,,\, t \in (0,+\infty), \\ u(0) = & 0. \end{array} \right. . \tag{4.8}$$

Provided $q \in [1,+\infty]$ and $f \in \mathrm{L}^q((0,+\infty),X)$, can we solve uniquely (4.8), with *a priori* estimate

$$\|(\partial_t u, Au)\|_{\mathrm{L}^q((0,+\infty),X)} \lesssim \|f\|_{\mathrm{L}^q((0,+\infty),X)} \,?$$

We are particularly concerned in this dissertation for the case where A is a Stokes operator on a suitable function space.

It has been shown in many cases that Stokes operators satisfy the L^q-maximal regularity on $\mathrm{L}^p_\sigma(\Omega)$, for various class of open sets Ω for $p,q \in (1,+\infty)$, with various boundary conditions and this has been widely used to treat various fluid dynamics problems, mainly Navier-Stokes equations. See for instance [GS91 ; Tol18 ; TW20 ; MM09b ; Mon13 ; Mon21 ; HM13 ; HNPS16 ; Hie20].

A major issue is that one cannot reach in general global-in-time L^1 or L^∞-maximal regularity estimates, which is of paramount importance to reach several endpoint spaces. However, for $q = 1$ such maximal regularity estimates have been shown for several non-invertible operators and were of major importance to achieve existence in critical function spaces for some fluid dynamic problems like global well-posedness of Navier-Stokes equations, even for inhomogeneous flows, or free boundary problems, see for instance [Che99 ; DM09 ; DM15 ; OS16 ; OS21 ; OS22].

While the work of Ogawa and Shimizu [OS22] provides a powerful framework for many applications, we are mainly restricted to a specific class of second order elliptic operator with "smooth enough" coefficients. A different and more abstract approach was brought by the recent work of Danchin, Hieber, Mucha and Tolksdorf presented in the previous Chapter 3, where the idea was to give an homogeneous version of the Da Prato-Grisvard theorem [DG75, Theorem 4.15], Theorem 3.6 which holds with implicit constant uniform with respect to the time variable even when $0 \in \sigma(A)$. But further assumptions have to be made, mainly the injectivity of A on X. Their idea is to replace the use of the real interpolation space $Y^\theta_q = (X, \mathrm{D}(A))_{\theta,q}$ by

$$(X, \mathrm{D}(\mathring{A}))_{\theta,q}$$

where $\mathrm{D}(\mathring{A})$ is called the homogeneous domain of A and stands *morally* for the *closure* of $\mathrm{D}(A)$ with respect to the (semi-)norm $\|A\cdot\|_X$. Such kind of investigation was already made in Haase's book [Haa06, Chapter 6] where the *completion* is considered instead of the closure.

If one applies the homogeneous interpolation and then the homogeneous Da Prato-Grisvard theorem as done in [DHMT21, Chapters 2, 3 & 4], choosing $X \subset \mathrm{L}^p(\mathbb{R}^n_+)$ to be a closed subspace, $(\mathrm{D}(A), A)$ an injective elliptic differential operator equal to its principal part, with $\mathrm{D}(A)$ to be a closed subset of $\mathrm{H}^{2m,p}(\mathbb{R}^n_+)$, it would lead to $\mathrm{L}^1_t(\dot{\mathrm{B}}^{2m\theta}_{p,1})$-maximal regularity results with $\theta \in (0,1)$. Proceeding this way disallow to obtain $\mathrm{L}^1_t(\dot{\mathrm{B}}^0_{p,1})$ or $\mathrm{L}^1_t(\dot{\mathrm{B}}^\alpha_{p,1})$-maximal regularity results, for $\alpha < 0$. Our idea is to replace the use of $\mathrm{L}^p(\mathbb{R}^n_+)$ as a ground space by $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, with $p \in (1,+\infty)$,

$s \in (-1+1/p, 1/p)$, so that we may expect to realize A on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ with domain

$$\mathrm{D}(A) \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2m,p}(\mathbb{R}^n_+).$$

Therefore, for $s \in (-1+1/p, 1/p)$ and $\theta \in (0,1)$, it seems reasonable to expect $\mathrm{L}^1_t(\dot{\mathrm{B}}^{s+2m\theta}_{p,1})$-maximal regularity results, and then recover maximal regularity for some non-positive index of regularity.

To reach such realizations of A on homogeneous Sobolev spaces of fractional order on the whole space or on the half-space, we are going to use the construction started [DHMT21, Chapter 3], and continued in the previous Chapter 2. The Section 2.3.1 contains the required informations about the precise meaning of partial traces in such function spaces to ensure that one can realize operators with boundary conditions on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, provided $s \in (-1+1/p, 1/p)$, see also Chapter 2 for usual traces results. We will also provide additional tools that will be useful to compute homogeneous interpolation spaces in presence of boundary conditions, as in Section 3.

In our case, considering first the Hodge Laplacian, then the Hodge-Stokes operator, we are able to apply Danchin, Hieber, Mucha and Tolksdorf's homogeneous Da Prato-Grisvard Theorem 3.6, as well as the usual L^q-maximal regularity for UMD Banach spaces or Theorem 3.21, to reach various maximal regularity results as the next one.

Theorem 4.3 (see Theorems 4.54, 4.55, 4.56 & 4.59) *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$, $s \in (-1+1/p, 1/p+2/q)$, $s, s+2-2/q \notin \mathbb{N}+\frac{1}{p}$, such that $(\mathcal{C}_{s+2-2/q,p,q})$ is satisfied and let $T \in (0,+\infty]$. For any $f \in \mathrm{L}^q((0,T), \dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathcal{H}}(\mathbb{R}^n_+,\Lambda))$, $u_0 \in \dot{\mathrm{B}}^{2+s-\frac{2}{q},\sigma}_{p,q,\mathcal{H}}(\mathbb{R}^n_+,\Lambda)$, there exists a unique mild solution $u \in \mathrm{C}^0_b([0,T], \dot{\mathrm{B}}^{2+s-\frac{2}{q},\sigma}_{p,q,\mathcal{H}}(\mathbb{R}^n_+,\Lambda))$ to*

$$[4] \quad \left\{ \begin{array}{rcll} \partial_t u - \Delta u &=& f, & \text{on } (0,T)\times\mathbb{R}^n_+, \\ \mathrm{d}^* u &=& 0, & \text{on } (0,T)\times\mathbb{R}^n_+, \\ \nu \lrcorner \mathrm{d}u_{|_{\partial\mathbb{R}^n_+}} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+, \\ \nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+, \\ u(0) &=& u_0, & \text{in } \dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}(\mathbb{R}^n_+,\Lambda), \end{array} \right. \tag{HSS}$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty([0,T],\dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}(\mathbb{R}^n_+))} + \big\|(\partial_t u, \nabla^2 u)\big\|_{\mathrm{L}^q((0,T),\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+))} \lesssim_{p,q,s,n} \|f\|_{\mathrm{L}^q((0,T),\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+))} + \|u_0\|_{\dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}(\mathbb{R}^n_+)}.$$

In the case $q = +\infty$, if we assume in addition $u_0 \in \dot{\mathrm{D}}^s_p(\mathbb{A}^2_{\mathcal{H}})$, we have

$$\big\|(\partial_t u, \nabla^2 u)\big\|_{\mathrm{L}^\infty([0,T],\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n_+))} \lesssim_{p,s,n} \|f\|_{\mathrm{L}^\infty((0,T),\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n_+))} + \|\mathbb{A}_{\mathcal{H}} u_0\|_{\dot{\mathrm{B}}^s_{p,\infty}(\mathbb{R}^n_+)}.$$

2 Hodge Laplacians, Hodge decomposition and Hodge-Stokes operators

This section is dedicated to the study of Hodge Laplacians, the Hodge decomposition and Hodge-Stokes operators, on Sobolev and Besov spaces on $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

We first introduce here the formalism of differential forms in the Euclidean setting. Resolvent estimates for the Hodge Laplacian and Hodge-Stokes like operators on the whole space follow from standard Fourier and Harmonic analysis, from which we deduce the related Hodge decomposition on $\mathbb{R}^n$ as well as the boundedness of holomorphic functional calculus for each operator.

4. For introductory purpose, the notations here are either not precise enough or quite redundant. For instance, the condition $\mathrm{d}^* u = 0$ already implies the boundary condition $\nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} = 0$.

Secondly, we are going to give all corresponding similar results for the Hodge Laplacians, the Hodge decomposition and Hodge-Stokes operators on the half-space $\mathbb{R}^n_+$. Those results are going to be built from what happens on the whole space $\mathbb{R}^n$, mimicking the behavior of Dirichlet and Neumann Laplacians on the half-space, see the Section 6 of previous Chapter 2.

2.1 Differential forms on Euclidean space, and corresponding function spaces

Here Ω stands for a domain of $\mathbb{R}^n$ with at least, if not empty, Lipschitz boundary. The open set Ω will be specified later on to be either, the whole space $\mathbb{R}^n$ or the half-space $\mathbb{R}^n_+$. Recall briefly that $\partial\mathbb{R}^n = \emptyset$, and $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$. We also recall that the outer normal unit at $\partial\mathbb{R}^n_+$ is $\nu = -\mathfrak{e}_n$, where $(\mathfrak{e}_k)_{k\in[\![1,n]\!]}$ is the canonical basis of $\mathbb{R}^n$, identified with its dual basis denoted by $(\mathrm{d}x_k)_{k\in[\![1,n]\!]}$, where $\mathrm{d}x_k(\mathfrak{e}_j) = \mathbb{1}_{\{k\}}(j)$, $(k,j) \in [\![1,n]\!]^2$.

Following [MM18 ; Mon21], we introduce the *exterior derivative* $\mathrm{d} := \nabla\wedge = \sum_{k=1}^n \partial_{x_k} \mathfrak{e}_k\wedge$ and the *interior derivative* (or *coderivative*) $\delta := -\nabla\lrcorner = -\sum_{k=1}^n \partial_{x_k} \mathfrak{e}_k\lrcorner$ acting on differential forms on a domain $\Omega \subset \mathbb{R}^n$, *i.e.* acting on functions defined on Ω which take value the complexified exterior algebra $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^n$ of $\mathbb{R}^n$. We allow us a slight abuse of notation : here we do not distinguish vectors of $\mathbb{R}^n$, vector fields, and 1-differential forms.

We also recall that for $k \in [\![0,n]\!]$, $u \in \Lambda^k$ can be uniquely determined by $(u_I)_{I\in\mathcal{I}_n^k} \in \mathbb{C}^{\binom{n}{k}}$ such that

$$u = \sum_{I\in\mathcal{I}_n^k} u_I \,\mathrm{d}x_I,$$

where $\mathcal{I}_n^k = \{(\ell_j)_{j\in[\![1,k]\!]} \in [\![1,n]\!]^k \,|\, \ell_j < \ell_{j+1}\}$, with $|\mathcal{I}_n^k| = \binom{n}{k}$, and u_I and $\mathrm{d}x_I$ stands respectively for $u_{\ell_1\ell_2\ldots\ell_k}$ and $\mathrm{d}x_{\ell_1} \wedge \mathrm{d}x_{\ell_2} \wedge \ldots \wedge \mathrm{d}x_{\ell_k}$ whenever $I = (\ell_j)_{j\in[\![1,k]\!]}$.

One may also notice that such representation of k-differential forms with increasing index is possible due to symmetry properties (*i.e.,* $\mathrm{d}x_\ell \wedge \mathrm{d}x_k = -\mathrm{d}x_k \wedge \mathrm{d}x_\ell$ for all $k,\ell \in [\![1,n]\!]$).

In particular, remark that $\Lambda^0 \simeq \mathbb{C}$, the space of complex scalars, and more generally $\Lambda^k \simeq \mathbb{C}^{\binom{n}{k}}$, so that $\Lambda \simeq \mathbb{C}^{2^n}$. We also set $\Lambda^\ell = \{0\}$ if $\ell < 0$ or $\ell > n$.

On the exterior algebra Λ, the basic operations are

(*i*) the exterior product $\wedge : \Lambda^k \times \Lambda^\ell \to \Lambda^{k+\ell}$,

(*ii*) the interior product $\lrcorner : \Lambda^k \times \Lambda^\ell \to \Lambda^{\ell-k}$,

(*iii*) the Hodge star operator $\star : \Lambda^\ell \to \Lambda^{n-\ell}$,

(*iv*) the inner product $\langle\cdot,\cdot\rangle : \Lambda^\ell \times \Lambda^\ell \to \mathbb{C}$.

If $a \in \Lambda^1, u \in \Lambda^\ell$ and $v \in \Lambda^{\ell+1}$, then

$$\langle a\wedge u, v\rangle = \langle u, a\lrcorner v\rangle.$$

For more details, we refer to, e.g., [AM04, Section 2] and [CM10, Section 3], noting that both these papers contain some historical background (and being careful that δ has the opposite sign in [AM04]). One may also consult [Do 94] for an introduction from the Euclidean setting point of view, and [Jos11, Section 1-3] for basic and usual properties in the more general Riemannian setting [5]. We recall the relation between d and δ via the Hodge star operator :

$$\star\delta u = (-1)^\ell \mathrm{d}(\star u) \quad \text{and} \quad \star\,\mathrm{d}u = (-1)^{\ell-1}\delta(\star u) \quad \text{for an } \ell\text{-form } u.$$

5. Notice that the Riemannian setting presented by Jost deals with compact manifold, but a lot of computations remain true in their full generality, due to local behavior of each operation (Hodge star operator, exterior and interior products etc.)

In dimension $n = 3$, this gives (see [CM10]) for a vector $a \in \mathbb{R}^3$ identified with a 1-form

- u scalar, interpreted as 0 -form : $a \wedge u = ua$, $a \lrcorner u = 0$;
- u scalar, interpreted as 3 -form : $a \wedge u = 0$, $a \lrcorner u = ua$;
- u vector, interpreted as 1 -form : $a \wedge u = a \times u$, $a \lrcorner u = a \cdot u$;
- u vector, interpreted as 2 -form : $a \wedge u = a \cdot u$, $a \lrcorner u = -a \times u$.

From now and until the end of the present paper, if $p \in (1, +\infty)$, $q \in [1, +\infty]$, $s \in \mathbb{R}$, $k \in [\![0, n]\!]$ and $\mathrm{X}^s \in \{\mathrm{H}^{s,p}, \dot{\mathrm{H}}^{s,p}, \mathrm{B}^s_{p,q}, \dot{\mathrm{B}}^s_{p,q}\}$, then $\mathrm{X}^s(\Omega, \Lambda^k)$ stands for k-differential forms whose coefficients lie in $\mathrm{X}^s(\Omega)$, *i.e.* $\mathrm{X}^s(\Omega, \Lambda^k) \simeq \mathrm{X}^s(\Omega, \mathbb{C}^{\binom{n}{k}})$. One may also consider similarly $\mathrm{X}^s_0(\Omega, \Lambda^k)$.

Operators d and δ are differential operators such that $\mathrm{d}^2 = \mathrm{d} \circ \mathrm{d} = 0$ and $\delta^2 = \delta \circ \delta = 0$, and each of them are bounded seen as linear operators $\mathrm{X}^s(\Omega, \Lambda) \longrightarrow \mathrm{X}^{s-1}(\Omega, \Lambda)$.

We recall the following integration by parts formula, for all $u, v \in \mathcal{S}(\overline{\Omega}, \Lambda)$,

$$\int_\Omega \langle \mathrm{d}u, v \rangle = \int_\Omega \langle u, \delta v \rangle + \int_{\partial\Omega} \langle u, \nu \lrcorner v \rangle \, \mathrm{d}\sigma, \tag{4.9}$$

$$\int_\Omega \langle \delta u, v \rangle = \int_\Omega \langle u, \mathrm{d}v \rangle + \int_{\partial\Omega} \langle u, \nu \wedge v \rangle \, \mathrm{d}\sigma, \tag{4.10}$$

which are true since we are in the Euclidean setting and where ν is the outward normal identified as a 1-form. More generally, for all $T \in \mathcal{D}'(\Omega, \Lambda^k) \simeq \mathcal{D}'(\Omega, \mathbb{C}^{\binom{n}{k}})$, $k \in [\![0, n]\!]$, we define

$$\big\langle \mathrm{d}T, \phi \big\rangle_\Omega := \big\langle T, \delta\phi \big\rangle_\Omega \text{ for all } \phi \in \mathrm{C}^\infty_c(\Omega, \Lambda^{k+1}),$$
$$\big\langle \delta T, \psi \big\rangle_\Omega := \big\langle T, \mathrm{d}\psi \big\rangle_\Omega \text{ for all } \psi \in \mathrm{C}^\infty_c(\Omega, \Lambda^{k-1}).$$

In particular, one may see those operators as unbounded ones and introduce their respective domains on $\mathrm{L}^p(\Omega, \Lambda^k)$, $k \in [\![0, n]\!]$, denoted by $\mathrm{D}_p(\mathrm{d}, \Lambda^k)$ and $\mathrm{D}_p(\delta, \Lambda^k)$ defined as

$$\mathrm{D}_p(\mathrm{d}, \Lambda^k) := \left\{ u \in \mathrm{L}^p(\Omega, \Lambda^k) \;\middle|\; \mathrm{d}u \in \mathrm{L}^p(\Omega, \Lambda^{k+1}) \right\} \text{ and } \mathrm{D}_p(\delta, \Lambda^k) := \left\{ u \in \mathrm{L}^p(\Omega, \Lambda^k) \;\middle|\; \delta u \in \mathrm{L}^p(\Omega, \Lambda^{k-1}) \right\}.$$

We can introduce their corresponding counterparts on homogeneous Sobolev spaces scales, $\dot{\mathrm{D}}^s_p(\mathrm{d}, \Lambda^k)$ on $\dot{\mathrm{H}}^{s,p}$, the same goes for inhomogeneous Sobolev spaces $\mathrm{D}^s_p(\mathrm{d}, \Lambda^k)$ on $\mathrm{H}^{s,p}$. The same goes with the interior derivative δ instead of d. One may proceed in a similar fashion, considering their domains on inhomogeneous and homogeneous Besov spaces.

As an exact sequence of (densely defined but not necessarily closed) unbounded operators, we get :

$$\begin{array}{ccccccccccccccc}
\mathrm{d} & : & \mathrm{X}^s(\Omega, \Lambda^0) & \longrightarrow & \mathrm{X}^s(\Omega, \Lambda^1) & \longrightarrow & \mathrm{X}^s(\Omega, \Lambda^2) & \longrightarrow & \dots & \longrightarrow & \mathrm{X}^s(\Omega, \Lambda^{n-1}) & \longrightarrow & \mathrm{X}^s(\Omega, \Lambda^n) & \longrightarrow & 0 \\
0 & \longleftarrow & \mathrm{X}^s(\Omega, \Lambda^0) & \longleftarrow & \mathrm{X}^s(\Omega, \Lambda^1) & \longleftarrow & \mathrm{X}^s(\Omega, \Lambda^2) & \longleftarrow & \dots & \longleftarrow & \mathrm{X}^s(\Omega, \Lambda^{n-1}) & \longleftarrow & \mathrm{X}^s(\Omega, \Lambda^n) & : & \delta.
\end{array}$$

In dimension $n = 3$, one can specialize, by the mean of the identification $\Lambda^k \simeq \mathbb{C}^{\binom{3}{k}}$, as

$$\begin{array}{ccccccccccccc}
\mathrm{d} & : & \mathrm{X}^s(\Omega, \mathbb{C}) & \xrightarrow{\nabla} & \mathrm{X}^s(\Omega, \mathbb{C}^3) & \xrightarrow{\mathrm{curl}} & \mathrm{X}^s(\Omega, \mathbb{C}^3) & \xrightarrow{\mathrm{div}} & \mathrm{X}^s(\Omega, \mathbb{C}) & \longrightarrow & 0 \\
0 & \longleftarrow & \mathrm{X}^s(\Omega, \mathbb{C}) & \xleftarrow{-\mathrm{div}} & \mathrm{X}^s(\Omega, \mathbb{C}^3) & \xleftarrow{\mathrm{curl}} & \mathrm{X}^s(\Omega, \mathbb{C}^3) & \xleftarrow{-\nabla} & \mathrm{X}^s(\Omega, \mathbb{C}) & : & \delta.
\end{array}$$

A very important feature is that, arbitrary dimension n, the operator d restricted to its action on $\mathcal{D}'(\Omega, \Lambda^1)$, with value in $\mathcal{D}'(\Omega, \Lambda^2)$, and δ restricted to its action on $\mathcal{D}'(\Omega, \Lambda^{n-1})$, with value in $\mathcal{D}'(\Omega, \Lambda^{n-2})$, are fair consistent generalizations of the curl operator on $\mathbb{R}^3$. Since in dimension n higher than 4, $n - 1 \neq 2$, we also have to distinguish their dual operators : the operator d restricted to its action on $\mathcal{D}'(\Omega, \Lambda^{n-2})$ and the operator δ restricted to its action on $\mathcal{D}'(\Omega, \Lambda^2)$ which are fair consistent generalizations of the dual operator ${}^t\mathrm{curl}$ (usually fully identified with the curl) on $\mathbb{R}^3$.

We can use (4.9) and (4.10) to consider the adjoints of d and δ in the sense of maximal adjoint operators in the Hilbert space $\mathrm{L}^2(\Omega,\Lambda)$, so that we will see later, e.g. Lemma 4.18, that they have the following exact description of their domains

$$\mathrm{D}_2(\mathrm{d}^*,\Lambda^k)=\{\,u\in\mathrm{D}_2(\delta,\Lambda^k)\;|\,\nu\lrcorner u_{|\partial\Omega}=0\,\}\quad\text{ and }\quad\mathrm{D}_2(\delta^*,\Lambda^k)=\{\,u\in\mathrm{D}_2(\mathrm{d},\Lambda^k)\;|\,\nu\wedge u_{|\partial\Omega}=0\,\}.$$

One can also see those adjoints operators through the following L^2-closures of unbounded operators,

$$(\mathrm{D}_2(\mathrm{d}^*,\Lambda^k),\mathrm{d}^*)=\overline{(\mathrm{C}_c^\infty(\Omega,\Lambda^k),\delta)}\quad\text{ and }\quad(\mathrm{D}_2(\delta^*,\Lambda^k),\delta^*)=\overline{(\mathrm{C}_c^\infty(\Omega,\Lambda^k),\mathrm{d})}.$$

Definition 4.4 (*i*) The **Hodge-Dirac operator** on Ω with **normal boundary conditions** is defined as

$$D_{\mathfrak{n}}:=\delta^*+\delta.$$

Its square denoted by $-\Delta_{\mathcal{H},\mathfrak{n}}:=D_{\mathfrak{n}}^2=\delta^*\delta+\delta\delta^*$, is called the (negative) **Hodge Laplacian** with **relative boundary conditions** (also called **generalized Dirichlet boundary conditions**)

$$\nu\wedge u_{|\partial\Omega}=0,\text{ and }\nu\wedge\delta u_{|\partial\Omega}=0.$$

The restriction to scalar functions $u:\Omega\longrightarrow\Lambda^0$ gives $-\Delta_{\mathcal{H},\mathfrak{n}}u=\delta\delta^*u=-\Delta_{\mathcal{D}}u$, where $-\Delta_{\mathcal{D}}$ is the Dirichlet Laplacian.

(*ii*) The **Hodge-Dirac operator** on Ω with **tangential boundary conditions** is defined as

$$D_{\mathfrak{t}}:=\mathrm{d}+\mathrm{d}^*.$$

Its square denoted by $-\Delta_{\mathcal{H},\mathfrak{t}}:=D_{\mathfrak{t}}^2=\mathrm{d}\mathrm{d}^*+\mathrm{d}^*\mathrm{d}$, is called the (negative) **Hodge Laplacian** with **absolute boundary conditions** (also called **generalized Neumann boundary conditions**)

$$\nu\lrcorner u_{|\partial\Omega}=0,\text{ and }\nu\lrcorner\,\mathrm{d}u_{|\partial\Omega}=0.$$

The restriction to scalar functions $u:\Omega\longrightarrow\Lambda^0$ gives $-\Delta_{\mathcal{H},\mathfrak{t}}u=\mathrm{d}^*\mathrm{d}u=-\Delta_{\mathcal{N}}u$, where $-\Delta_{\mathcal{N}}$ is the Neumann Laplacian.

Notations 4.5 When it does not matter $(\mathfrak{d},D_\cdot,-\Delta_{\mathcal{H}})$ stand either for $(\delta,D_{\mathfrak{n}},-\Delta_{\mathcal{H},\mathfrak{n}})$ or $(\mathrm{d},D_{\mathfrak{t}},-\Delta_{\mathcal{H},\mathfrak{t}})$, just writing

$$-\Delta_{\mathcal{H}}=D_\cdot^2=\mathfrak{d}\mathfrak{d}^*+\mathfrak{d}^*\mathfrak{d}.$$

Remark 4.6 Let's make two independent remarks :

- We recall here that, if $\Omega\subset\mathbb{R}^3$ is an open set with, say at least, Lipschitz boundary, one has formally for u with value in $\Lambda^1\simeq\mathbb{C}^3$ or $\Lambda^2\simeq\mathbb{C}^3$,

$$-\Delta_{\mathcal{H},\mathfrak{n}}u=-\Delta_{\mathcal{H},\mathfrak{t}}u=\text{curl curl }u-\nabla\,\text{div }u$$

with either one of the following couple of boundary conditions

$$[u\cdot\nu_{|\partial\Omega}=0,\,\nu\times\text{curl }u_{|\partial\Omega}=0]\text{ or }[u\times\nu_{|\partial\Omega}=0,\,(\text{div }u)\nu_{|\partial\Omega}=0].$$

- In the case of $\Omega=\mathbb{R}^n$, notice that no boundary value comes in, hence $\mathrm{d}^*=\delta$, $\delta^*=\mathrm{d}$, so that

$$D_\cdot=D_{\mathfrak{n}}=D_{\mathfrak{t}}=(\mathrm{d}+\delta)\qquad\text{and}\qquad-\Delta_{\mathcal{H}}=-\Delta_{\mathcal{H},\mathfrak{n}}=-\Delta_{\mathcal{H},\mathfrak{t}}=(\mathrm{d}+\delta)^2=\mathrm{d}\delta+\delta\mathrm{d}.$$

Definition 4.7 (*i*) The (bounded) orthogonal projector defined on $\mathrm{L}^2(\Omega,\Lambda^k)$ onto $\mathrm{N}_2(\mathrm{d}^*,\Lambda^k)$ is denoted by $\mathbb{P}$ and called the **generalized Helmholtz-Leray** (or **Leray**) **projector**.

(*ii*) The (bounded) orthogonal projector defined on $\mathrm{L}^2(\Omega,\Lambda^k)$ onto $\mathrm{N}_2(\delta,\Lambda^k)$ is denoted by $\mathbb{Q}$.

(*iii*) For $p\in(1,+\infty)$, $s\in\mathbb{R}$ and $k\in[\![0,n]\!]$, we say that $\mathrm{H}^{s,p}(\Omega,\Lambda^k)$ admits a **Hodge decomposition** if $(\mathrm{D}^s_p(\mathrm{d},\Lambda^k),\mathrm{d})$, $(\mathrm{D}^s_p(\delta,\Lambda^k),\delta)$ and their respective adjoints are closable and

$$\begin{aligned}\mathrm{H}^{s,p}(\Omega,\Lambda^k)&=\mathrm{N}^s_p(\mathfrak{d},\Lambda^k)\oplus\overline{\mathrm{R}^s_p(\mathfrak{d}^*,\Lambda^k)},\\&=\overline{\mathrm{R}^s_p(\mathfrak{d},\Lambda^k)}\oplus\mathrm{N}^s_p(\mathfrak{d}^*,\Lambda^k),\end{aligned}\qquad(\mathfrak{H}^s_p)$$

holds in the topological sense. We keep the same definition of the Hodge decomposition on other function spaces replacing $(\mathrm{H}^{s,p},\mathrm{D}^s_p,\mathrm{R}^s_p,\mathrm{N}^s_p)$ by either $(\dot{\mathrm{H}}^{s,p},\dot{\mathrm{D}}^s_p,\dot{\mathrm{R}}^s_p,\dot{\mathrm{N}}^s_p)$, $(\mathrm{B}^s_{p,q},\mathrm{D}^s_{p,q},\mathrm{R}^s_{p,q},\mathrm{N}^s_{p,q})$, or even by $(\dot{\mathrm{B}}^s_{p,q},\dot{\mathrm{D}}^s_{p,q},\dot{\mathrm{R}}^s_{p,q},\dot{\mathrm{N}}^s_{p,q})$, with $q\in[1,+\infty]$.

One can notice that in the case of vector fields (identified with 1-forms, *i.e.,* $\mathrm{L}^2(\Omega,\Lambda^1)\simeq\mathrm{L}^2(\Omega,\mathbb{C}^n)$), we can identify $\mathbb{P}$ as the usual Helmholtz-Leray projector on divergence free vector fields with null tangential trace at the boundary

$$\mathbb{P}\,:\,\mathrm{L}^2(\Omega,\mathbb{C}^n)\longrightarrow\mathrm{L}^2_\sigma(\Omega)=\{\,u\in\mathrm{L}^2(\Omega,\mathbb{C}^n)\,|\,\operatorname{div}u=0,\,u\cdot\nu_{|\partial\Omega}=0\,\}.$$

It gives the following classical orthogonal, topological, Hodge decomposition, see [Soh01, Chapter 2, Section 2.5],

$$\mathrm{L}^2(\Omega,\mathbb{C}^n)=\mathrm{L}^2_\sigma(\Omega)\overset{\perp}{\oplus}\nabla\dot{\mathrm{H}}^1(\Omega,\mathbb{C}),$$

for any sufficiently reasonable domains Ω, say for instance with uniformly Lipschitz boundary, see [Soh01, Lemma 2.5.3].

Before investigating the Hodge decomposition and the functional analytic properties of the Hodge Laplacian on differential forms on function spaces in $\mathbb{R}^n_+$, we want to know a bit more about the whole space case. The next subsection devoted to the whole space, we gather well known facts and results which lack explicit references in the literature to the best of author's knowledge.

2.2 The case of the whole space

On the whole space $\mathbb{R}^n$ the action of the Laplacian and the Hodge decomposition for vector fields is well known in the literature on usual spaces as Lebesgue spaces $\mathrm{L}^p(\mathbb{R}^n,\mathbb{C}^n)$, $p\in(1,+\infty)$, and so is the case of inhomogeneous and homogeneous Sobolev and Besov spaces. Our main goal here is to extend and summarize those results with the formalism of differential forms.

To do so, we introduce an extension of the Fourier transform to differential forms whose coefficients lie in the space of complex valued Schwartz functions $\mathcal{S}(\mathbb{R}^n,\mathbb{C})$, or in the space of tempered distribution $\mathcal{S}'(\mathbb{R}^n,\mathbb{C})$.

- For all $u\in\mathrm{L}^1(\mathbb{R}^n,\Lambda^k)\simeq\mathrm{L}^1(\mathbb{R}^n,\mathbb{C}^{\binom{n}{k}})$, $k\in[\![0,n]\!]$, we define

$$\mathcal{F}u:=\sum_{I\in\mathcal{I}^k_n}\mathcal{F}u_I\,\mathrm{d}\xi_I\in\mathrm{C}^0_0(\mathbb{R}^n,\Lambda^k).$$

  Hence, as in the scalar valued case, the Fourier transform $\mathcal{F}$ induces a topological automorphism of $\mathcal{S}(\mathbb{R}^n,\Lambda^k)\simeq\mathcal{S}(\mathbb{R}^n,\mathbb{C}^{\binom{n}{k}})$.

- For $k\in[\![0,n]\!]$, we write $\mathcal{S}'(\mathbb{R}^n,\Lambda^k):=(\mathcal{S}(\mathbb{R}^n,\Lambda^k))'\simeq\mathcal{S}'(\mathbb{R}^n,\mathbb{C}^{\binom{n}{k}})$. Similarly, the Fourier transform $\mathcal{F}$ is an automorphism of $\mathcal{S}'(\mathbb{R}^n,\Lambda^k)$.

- For all $T \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$, $k \in [\![0, n]\!]$, we define

$$\langle \mathrm{d}T, \phi \rangle_{\mathbb{R}^n} := \langle T, \delta\phi \rangle_{\mathbb{R}^n} \text{ for all } \phi \in \mathcal{S}(\mathbb{R}^n, \Lambda^{k+1}),$$
$$\langle \delta T, \psi \rangle_{\mathbb{R}^n} := \langle T, \mathrm{d}\psi \rangle_{\mathbb{R}^n} \text{ for all } \psi \in \mathcal{S}(\mathbb{R}^n, \Lambda^{k-1}).$$

The following lemma is straightforward and fundamental for our analysis.

Lemma 4.8 *For all $u \in \mathcal{S}(\mathbb{R}^n, \Lambda^k)$, $k \in [\![0, n]\!]$, for all $\xi \in \mathbb{R}^n$,*

$$\mathcal{F}[\mathrm{d}u](\xi) = i\xi \wedge \mathcal{F}u(\xi) \quad \text{and} \quad \mathcal{F}[\delta u](\xi) = -i\xi \lrcorner \mathcal{F}u(\xi).$$

Remark 4.9 This is somewhat consistent, when $n = 3$, with formulas like

$$\mathcal{F}[\text{curl } u](\xi) = i\xi \times \mathcal{F}u(\xi) \text{ and } \mathcal{F}[\text{div } u](\xi) = i\xi \cdot \mathcal{F}u(\xi),\ u \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^3),\ \xi \in \mathbb{R}^3.$$

From there, the identity for all differential forms u of degree k, and all vector $v \in \mathbb{R}^n$,

$$v \wedge (v \lrcorner u) + v \lrcorner (v \wedge u) = |v|^2 u,$$

with the use of Lemma 4.8 yields with Remark 4.6 that, for all $u \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$, $k \in [\![0, n]\!]$,

$$\begin{aligned}\mathcal{F}[-\Delta_{\mathcal{H}} u](\xi) = \mathcal{F}[(\delta\mathrm{d} + \mathrm{d}\delta)u](\xi) = i\xi \wedge (-i\xi \lrcorner \mathcal{F}u(\xi)) + -i\xi \lrcorner (i\xi \wedge \mathcal{F}u(\xi)) &= |\xi|^2 \cdot \mathcal{F}u(\xi) \\ &= \mathcal{F}[-\Delta u](\xi).\end{aligned}$$

Hence, the Hodge Laplacian on the whole space is nothing but the scalar Laplacian applied separately to each component of a differential form so that its properties are carried over by the scalar Laplacian. We state then a very well known result adapted to our setting.

Theorem 4.10 *Let $p, \tilde{p} \in (1, +\infty)$, $q, \tilde{q} \in [1, +\infty]$, $s, \alpha \in \mathbb{R}$, and $k \in [\![0, n]\!]$. The Hodge Laplacian is an injective operator on $\mathcal{S}'_h(\mathbb{R}^n, \Lambda^k)$, and satisfies the following properties*

(i) For $f \in \mathcal{S}'_h(\mathbb{R}^n, \Lambda^k)$, consider the problem

$$-\Delta u = f \quad \text{in } \mathbb{R}^n.$$

a) If $f \in [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{\alpha,\tilde{p}}](\mathbb{R}^n, \Lambda^k)$, and $(\mathcal{C}_{s+2,p})$ is satisfied, then there exists a unique solution $u \in [\dot{\mathrm{H}}^{s+2,p} \cap \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n, \Lambda^k)$ with the estimates,

$$\|\mathrm{d}\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + \|\delta\mathrm{d}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,s} \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,s} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)},$$
$$\|\mathrm{d}\delta u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n)} + \|\delta\mathrm{d}u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n)} \lesssim_{\tilde{p},n,\alpha} \|\nabla^2 u\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n)} \lesssim_{\tilde{p},n,\alpha} \|f\|_{\dot{\mathrm{H}}^{\alpha,\tilde{p}}(\mathbb{R}^n)}.$$

In particular, $-\Delta : [\dot{\mathrm{H}}^{s+2,p} \cap \dot{\mathrm{H}}^{\alpha+2,\tilde{p}}](\mathbb{R}^n, \Lambda^k) \longrightarrow [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{\alpha,\tilde{p}}](\mathbb{R}^n, \Lambda^k)$ is an isomorphism of Banach spaces.

b) The result still holds if we replace $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{\alpha,\tilde{p}}, \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{H}}^{\alpha+2,\tilde{p}})$ by $(\dot{\mathrm{B}}^{s}_{p,q}, \dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}, \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{B}}^{\alpha+2}_{\tilde{p},\tilde{q}})$.

(ii) For $\mu \in [0, \pi)$, $\lambda \in \Sigma_\mu$, $f \in \mathcal{S}'(\mathbb{R}^n, \Lambda^k)$, consider the problem

$$\lambda u - \Delta u = f \quad \text{in } \mathbb{R}^n.$$

a) If $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n, \Lambda^k)$, and $(\mathcal{C}_{s,p})$ is satisfied, then above resolvent problem admits a unique solution $u \in [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n, \Lambda^k)$ with the estimates,

$$|\lambda|\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + |\lambda|^{\frac{1}{2}}\|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)},$$
$$|\lambda|^{\frac{1}{2}}\|(\mathrm{d} + \delta)u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + \|\mathrm{d}\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} + \|\delta\mathrm{d}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

In particular, $\lambda\mathrm{I} - \Delta : [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n, \Lambda^k) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n, \Lambda^k)$ is an isomorphism of Banach spaces.

Furthermore, the result still holds, replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p})$ *by* $(\mathrm{H}^{s,p}, \mathrm{H}^{s+2,p})$ *without any restriction on* (s,p).

b) If $(\mathcal{C}_{s,p,q})$ *is satisfied, then the above result still holds replacing* $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p})$ *by* $(\dot{\mathrm{B}}^{s}_{p,q}, \dot{\mathrm{B}}^{s}_{p,q} \cap \dot{\mathrm{B}}^{s+2}_{p,q})$, *or even by* $(\mathrm{B}^{s}_{p,q}, \mathrm{B}^{s+2}_{p,q})$ *without any restriction on* (s,p,q).

(iii) For any $\mu \in (0,\pi)$, *the operator* $-\Delta$ *admits a bounded (or* $\mathbf{H}^{\infty}(\Sigma_{\mu})$*-)holormophic functional calculus on function spaces :* $[\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{\alpha,\tilde{p}}](\mathbb{R}^n, \Lambda^k)$, $(\mathcal{C}_{s,p})$ *being is satisfied,* $\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n, \Lambda^k)$, $(\mathcal{C}_{s,p,q})$ *being satisfied, and on both* $\mathrm{H}^{s,p}(\mathbb{R}^n, \Lambda^k)$ *and* $\mathrm{B}^{s}_{p,q}(\mathbb{R}^n, \Lambda^k)$ *without any restriction on* (s,p,q).

Proof. — **Step 1 :** the scalar Laplacian is injective on $\mathcal{S}'_h(\mathbb{R}^n, \mathbb{C})$. For $f \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$, let u, $v \in \mathcal{S}'_h(\mathbb{R}^n, \mathbb{C})$, such that

$$\forall \phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}),\ \big\langle u, -\Delta\phi \big\rangle_{\mathbb{R}^n} = \big\langle f, \phi \big\rangle_{\mathbb{R}^n} = \big\langle v, -\Delta\phi \big\rangle_{\mathbb{R}^n}.$$

Therefore, it follows that $w := u - v \in \mathcal{S}'_h(\mathbb{R}^n, \mathbb{C})$ satisfies

$$\forall \phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}),\ \big\langle w, -\Delta\phi \big\rangle_{\mathbb{R}^n} = 0.$$

Hence, one may apply the Fourier transform, to obtain

$$\forall \phi \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}),\ \big\langle \mathcal{F}w, |\cdot|^2 \mathcal{F}^{-1}\phi \big\rangle_{\mathbb{R}^n} = 0,$$

so in particular, for test function in the form $\phi := \mathcal{F}\left[\frac{\psi}{|\cdot|^2}\right]$, with $\psi \in \mathrm{C}^{\infty}_c(\mathbb{R}^n \setminus \{0\}, \mathbb{C})$ (notice that one can see that $\mathcal{F}\left[\frac{\psi}{|\cdot|^2}\right] \in \mathcal{S}(\mathbb{R}^n, \mathbb{C})$), we deduce

$$\forall \psi \in \mathrm{C}^{\infty}_c(\mathbb{R}^n \setminus \{0\}, \mathbb{C}),\ \big\langle \mathcal{F}w, \psi \big\rangle_{\mathbb{R}^n} = 0.$$

Following the proof of [DHMT21, Lemma 3.6], we conclude that $w = 0$ (since $w \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$ and $\mathcal{S}'_h(\mathbb{R}^n, \mathbb{C})$ does not contain any polynomial), so $u = v$ in $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$.

Step 2 : For the point *(i)*, it suffices to follow [DM15, Lemma 3.1.1].

For the point *(ii)*, see [ABHN11, Example 3.7.6, Theorem 3.7.11].

For the point *(iii)*, the result on $\mathrm{L}^p(\mathbb{R}^n, \mathbb{C})$ is a consequence of a more general one which is [Haa06, Proposition 8.3.4]. ■

Similarly, thanks again to standard Fourier analysis, we can introduce appropriate differential form-valued version of Riesz transforms for the Hodge Laplacian. Their boundedness on appropriate function spaces are again carried over by their scalar analogue, and a direct consequence is an explicit formula for our generalized Leray projector $\mathbb{P}$ on $\mathbb{R}^n$.

To do so, we notice that one can write associated Fourier symbols, thanks to Lemma 4.8, to obtain

$$\mathrm{d}(-\Delta)^{-\frac{1}{2}} = \sum_{k=1}^{n} R_k \mathfrak{e}_k \wedge \qquad \text{and} \qquad \delta(-\Delta)^{-\frac{1}{2}} = -\sum_{k=1}^{n} R_k \mathfrak{e}_k \lrcorner$$

where for $k \in [\![1,n]\!]$, R_k is the k-th Riesz transform on $\mathbb{R}^n$ given by the Fourier symbol $\xi \mapsto \frac{i\xi_k}{|\xi|}$, which is well known to be bounded on $\mathrm{L}^p(\mathbb{R}^n, \mathbb{C})$, $1 < p < +\infty$, see [Ste70, Chapter 2, Theorem 1 & Chapter 3, Section 1]. Therefore, the next proposition follows naturally.

Proposition 4.11 *Let* $p, \tilde{p} \in (1, +\infty)$, $q, \tilde{q} \in [1, +\infty]$, $s, \alpha \in \mathbb{R}$, *and let* $k \in [\![0,n]\!]$. *The operators*

$$\mathrm{d}(-\Delta)^{-\frac{1}{2}},\ \delta(-\Delta)^{-\frac{1}{2}},\ \mathrm{d}\delta(-\Delta)^{-1} \text{ and } \delta\mathrm{d}(-\Delta)^{-1}$$

are all well-defined bounded linear operators,

- *On* $[\dot{\mathrm{H}}^{s,p}\cap\dot{\mathrm{H}}^{\alpha,\tilde{p}}](\mathbb{R}^n,\Lambda)$, *and on* $[\dot{\mathrm{B}}^{s}_{p,q}\cap\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}}](\mathbb{R}^n,\Lambda)$, ($\mathcal{C}_{s,p,q}$) *being satisfied, with decoupled estimates,*

$$\|Tu\|_{\mathrm{X}^s(\mathbb{R}^n)}\lesssim_{p,n,s}\|u\|_{\mathrm{X}^s(\mathbb{R}^n)}\ \text{ and }\ \|Tu\|_{\mathrm{X}^\alpha(\mathbb{R}^n)}\lesssim_{p,n,\alpha}\|u\|_{\mathrm{X}^\alpha(\mathbb{R}^n)},\ u\in[\mathrm{X}^s\cap\mathrm{X}^\alpha](\mathbb{R}^n,\Lambda^k),$$

where the operator T *is any of the above operators, and* $(\mathrm{X}^s,\mathrm{X}^\alpha)\in\{(\dot{\mathrm{H}}^{s,p},\dot{\mathrm{H}}^{\alpha,\tilde{p}}),(\dot{\mathrm{B}}^{s}_{p,q},\dot{\mathrm{B}}^{\alpha}_{\tilde{p},\tilde{q}})\}$.

- *On* $\mathrm{H}^{s,p}(\mathbb{R}^n,\Lambda)$, *and on* $\mathrm{B}^{s}_{p,q}(\mathbb{R}^n,\Lambda)$, *without any restriction on* (s,p,q).

Moreover, the following identity holds

$$(\mathrm{d}(-\Delta)^{-\frac{1}{2}}+\delta(-\Delta)^{-\frac{1}{2}})^2=\mathrm{I}.$$

Theorem 4.12 *Let* $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in\mathbb{R}$, *and let* $k\in[\![0,n]\!]$. *The following hold*

(i) The following equality is true, whenever ($\mathcal{C}_{s,p}$) *is satisfied,*

$$\dot{\mathrm{N}}^s_p(\mathrm{d},\mathbb{R}^n,\Lambda^k)=\overline{\dot{\mathrm{R}}^s_p(\mathrm{d},\mathbb{R}^n,\Lambda^k)}^{\|\cdot\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}},$$

and still holds replacing d *by* δ.

(ii) The (generalized) Helmholtz-Leray projector $\mathbb{P}:\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k)\longrightarrow\dot{\mathrm{N}}^s_p(\delta,\mathbb{R}^n,\Lambda^k)$ *is well-defined and bounded when* ($\mathcal{C}_{s,p}$) *is satisfied. Moreover, the following identity is true*

$$\mathbb{P}=\mathrm{I}-\mathrm{d}(-\Delta)^{-1}\delta.$$

(iii) The following Hodge decomposition holds whenever ($\mathcal{C}_{s,p}$) *is satisfied,*

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k)=\dot{\mathrm{N}}^s_p(\delta,\mathbb{R}^n,\Lambda^k)\oplus\dot{\mathrm{N}}^s_p(\mathrm{d},\mathbb{R}^n,\Lambda^k).$$

Everything above still holds replacing $(\dot{\mathrm{H}}^{s,p},\dot{\mathrm{N}}^s_p,\dot{\mathrm{R}}^s_p)$ *by either* $(\dot{\mathrm{B}}^{s}_{p,q},\dot{\mathrm{N}}^s_{p,q},\dot{\mathrm{R}}^s_{p,q})$, ($\mathcal{C}_{s,p,q}$) *being satisfied,* $(\mathrm{H}^{s,p},\mathrm{N}^s_p,\mathrm{R}^s_p)$ *or* $(\mathrm{B}^{s}_{p,q},\mathrm{N}^s_{p,q},\mathrm{R}^s_{p,q})$ *without any restriction on* (s,p,q).
In case of Besov spaces with $q=+\infty$, *the density result of point (i) only holds in the weak* sense.*

Remark 4.13 On Λ^1-valued functions identified with vector fields one recover the usual well known formula, *i.e.*

$$\mathbb{P}=\mathrm{I}+\nabla(-\Delta)^{-1}\,\mathrm{div}\ .$$

Proof. — **Step 1 :** The orthogonal projector $\mathbb{P}$ is originally defined only as an operator

$$\mathbb{P}:\mathrm{L}^2(\mathbb{R}^n,\Lambda^k)\longrightarrow\mathrm{N}_2(\delta,\mathbb{R}^n,\Lambda^k).$$

We claim that $\mathbb{P}$ is equal to the operator formally given by

$$\tilde{\mathbb{P}}:=\mathrm{I}-\mathrm{d}(-\Delta)^{-1}\delta.$$

The proof is standard, and works as in the case of vector fields, and then is left to the reader.
Step 2 : Previous step and Proposition 4.11 give that $\mathbb{P}$ is bounded $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k)$.
For $u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k)$, $v\in\mathcal{S}(\mathbb{R}^n)$, we regularize with the resolvent, to compute

$$\begin{aligned}\big\langle\mathbb{P}u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}&=\lim_{\lambda\to0_+}\big\langle u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}-\big\langle\mathrm{d}(\lambda\mathrm{I}-\Delta)^{-1}\delta u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}\\&=\lim_{\lambda\to0_+}\big\langle u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}-\big\langle u,(\lambda\mathrm{I}-\Delta)^{-1}\mathrm{d}\delta\mathrm{d}v\big\rangle_{\mathbb{R}^n}\\&=\lim_{\lambda\to0_+}\big\langle u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}-\big\langle u,(\lambda\mathrm{I}-\Delta)^{-1}(-\Delta)\mathrm{d}v\big\rangle_{\mathbb{R}^n}\\&=\big\langle u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}-\big\langle u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}=0.\end{aligned}$$

Hence $\mathbb{P}\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k)\subset\dot{\mathrm{N}}^s_p(\delta,\mathbb{R}^n,\Lambda^k)$, and we even have $\mathbb{P}_{|_{\dot{\mathrm{N}}^s_p(\delta,\mathbb{R}^n,\Lambda^k)}}=\mathrm{I}$, so that

$$\mathbb{P}\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k)=\dot{\mathrm{N}}^s_p(\delta,\mathbb{R}^n,\Lambda^k).$$

Similarly, $[\mathrm{I}-\mathbb{P}]\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k)\subset\dot{\mathrm{N}}^s_p(\mathrm{d},\mathbb{R}^n,\Lambda^k)$, and $[\mathrm{I}-\mathbb{P}]_{|_{\dot{\mathrm{N}}^s_p(\mathrm{d},\mathbb{R}^n,\Lambda^k)}}=\mathrm{I}$, which comes from

$$[\mathrm{I}-\mathbb{P}]u=\lim_{\lambda\to 0_+}\mathrm{d}(\lambda\mathrm{I}-\Delta)^{-1}\delta u,\ u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda^k).$$

This also gives $\dot{\mathrm{N}}^s_p(\mathrm{d},\mathbb{R}^n,\Lambda^k)=\overline{\dot{\mathrm{R}}^s_p(\mathrm{d},\mathbb{R}^n,\Lambda^k)}^{\|\cdot\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}}$.

The proof is straightforward the same for other function spaces. ∎

2.3 The case of the half-space $\mathbb{R}^n_+$

2.3.1 Interlude : about partial traces on the boundary

We state here a trace theorem for generalized normal and tangential traces of differential forms. The general case for vector fields in inhomogeneous function spaces is well-known, also is the differential form in the inhomogeneous case for bounded Lipschitz subdomains of a Riemannian manifold, see [MMS08, Section 4] and references therein.

We recall that $\nu=-\mathfrak{e}_{\mathfrak{n}}$ is the outer normal unit at the boundary $\partial\mathbb{R}^n_+$.

Theorem 4.14 *Let $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in(-1+\frac{1}{p},\frac{1}{p})$ and let $k\in[\![0,n]\!]$.*

*(i) For all $u\in\mathrm{D}^s_p(\delta,\mathbb{R}^n_+,\Lambda^k)$. Then there exists a unique function $\nu\lrcorner u_{|_{\partial\mathbb{R}^n_+}}\in\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\Lambda^{k-1})$ called the generalized **tangential trace**, such that*

$$\int_{\mathbb{R}^{n-1}}\langle\nu\lrcorner u_{|_{\partial\mathbb{R}^n_+}},\Psi_{|_{\partial\mathbb{R}^n_+}}\rangle=\int_{\mathbb{R}^n_+}\langle u,\mathrm{d}\Psi\rangle-\int_{\mathbb{R}^n_+}\langle\delta u,\Psi\rangle \tag{4.11}$$

for all $\Psi\in\mathrm{H}^{1-s,p'}(\mathbb{R}^n_+,\Lambda)$, with the estimates

$$\|\nu\lrcorner u_{|_{\partial\mathbb{R}^n_+}}\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})}\lesssim_{p,s,n}\|u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n_+)}+\|\delta u\|_{\mathrm{H}^{s,p}(\mathbb{R}^n_+)}.$$

*The same result holds with the corresponding estimate, for $u\in\mathrm{D}^s_p(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)$ we have a partial trace $\nu\wedge u_{|_{\partial\mathbb{R}^n_+}}\in\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1},\Lambda^{k+1})$ called the generalized **normal trace**, satisfying the identity*

$$\int_{\mathbb{R}^{n-1}}\langle\nu\wedge u_{|_{\partial\mathbb{R}^n_+}},\Psi_{|_{\partial\mathbb{R}^n_+}}\rangle=\int_{\mathbb{R}^n_+}\langle\mathrm{d}u,\Psi\rangle-\int_{\mathbb{R}^n_+}\langle u,\delta\Psi\rangle. \tag{4.12}$$

(ii) For all $u\in\mathrm{D}^s_{p,q}(\delta,\mathbb{R}^n_+,\Lambda^k)$ we have $\nu\lrcorner u_{|_{\partial\mathbb{R}^n_+}}\in\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1},\Lambda^{k-1})$, such that formula (4.11) holds for all $\Psi\in\mathrm{B}^{1-s}_{p',q'}(\mathbb{R}^n_+,\Lambda)$. Moreover, we have the estimates

$$\|\nu\lrcorner u_{|_{\partial\mathbb{R}^n_+}}\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})}\lesssim_{p,s,n}\|u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)}+\|\delta u\|_{\mathrm{B}^s_{p,q}(\mathbb{R}^n_+)}.$$

The same results holds with the corresponding estimate, for $u\in\mathrm{D}^s_{p,q}(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)$ we have a partial trace $\nu\wedge u_{|_{\partial\mathbb{R}^n_+}}\in\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1},\Lambda^{k+1})$ such that (4.12) is satisfied.

(iii) *For all* $u \in \mathrm{B}^{s+1}_{p,q}(\mathbb{R}^n_+, \Lambda^k)$, *we have*

$$(\nu \lrcorner u \oplus \nu \wedge u)_{|\partial\mathbb{R}^n_+} \in \mathrm{B}^{s+1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}, \Lambda^{k-1} \oplus \Lambda^{k+1})$$

with the estimate

$$\|(\nu \lrcorner u \oplus \nu \wedge u)_{|\partial\mathbb{R}^n_+}\|_{\mathrm{B}^{s+1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u\|_{\mathrm{B}^{s+1}_{p,q}(\mathbb{R}^n_+)},$$

and everything still hold with $\mathrm{H}^{s+1,p}$ *instead of* $\mathrm{B}^{s+1}_{p,q}$, *when* $q = p$.

Theorem 4.15 *Let* $p \in (1, +\infty)$, $q \in [1, +\infty]$, $s \in (-1 + \frac{1}{p}, \frac{1}{p})$ *and let* $k \in [\![0, n]\!]$.

(i) *For all* $u \in \dot{\mathrm{D}}^s_p(\delta, \mathbb{R}^n_+, \Lambda^k)$,

- *If* $s \leqslant 0$, *then there exists a unique function* $\nu \lrcorner u_{|\partial\mathbb{R}^n_+} \in \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \Lambda^{k-1})$ *such that the formula* (4.11) *holds for all* $\Psi \in \mathrm{H}^{1-s,p'}(\mathbb{R}^n_+, \Lambda)$, *with the estimate*

$$\|\nu \lrcorner u_{|\partial\mathbb{R}^n_+}\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

- *If* $s > 0$, *for* $\frac{1}{r} = \frac{1}{p} - \frac{s}{n} \in (\frac{n-1}{pn}, \frac{1}{p})$, *there exists a unique function* $\nu \lrcorner u_{|\partial\mathbb{R}^n_+} \in \mathrm{B}^{-\frac{1}{r}}_{r,r}(\mathbb{R}^{n-1}, \Lambda^{k-1})$, *such that the formula* (4.11) *holds for all* $\Psi \in \mathrm{H}^{1,r'}(\mathbb{R}^n_+, \Lambda^{k-1})$ *with the estimate*

$$\|\nu \lrcorner u_{|\partial\mathbb{R}^n_+}\|_{\mathrm{B}^{-\frac{1}{r}}_{r,r}(\mathbb{R}^{n-1})} \lesssim_{r,p,s,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

The same result, up to appropriate changes, still holds for $u \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$ *with partial trace* $\nu \wedge u_{|\partial\mathbb{R}^n_+}$ *satisfying the formula* (4.12).

(ii) *For all* $u \in \dot{\mathrm{D}}^s_{p,q}(\delta, \mathbb{R}^n_+, \Lambda^k)$,

- *If* $s < 0$, *there exists a unique* $\nu \lrcorner u_{|\partial\mathbb{R}^n_+} \in \mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}, \Lambda^{k-1})$ *such that the formula* (4.11) *holds for all* $\Psi \in [\mathcal{S} \cap \mathrm{B}^{1-s}_{p',q'}](\mathbb{R}^n_+, \Lambda^{k-1} \oplus \Lambda^{k+1})$, *with the estimates*

$$\|\nu \lrcorner u_{|\partial\mathbb{R}^n_+}\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

- *If* $s > 0$, *for* $\frac{1}{r} = \frac{1}{p} - \frac{s}{n} \in (\frac{n-1}{pn}, \frac{1}{p})$, *there exists a unique* $\nu \lrcorner u_{|\partial\mathbb{R}^n_+} \in \mathrm{B}^{-\frac{1}{\tilde{r}}-\varepsilon}_{\tilde{r},q}(\mathbb{R}^{n-1}, \Lambda^{k-1})$, *for any sufficiently small* $\varepsilon > 0$, *with* $\frac{1}{r} - \frac{\varepsilon}{n} = \frac{1}{\tilde{r}}$, *such that the formula* (4.11) *holds for all* $\Psi \in [\mathcal{S} \cap \mathrm{B}^{1+\varepsilon}_{\tilde{r}',q'}](\mathbb{R}^n_+, \Lambda)$ *with the estimate*

$$\|\nu \lrcorner u_{|\partial\mathbb{R}^n_+}\|_{\mathrm{B}^{-\frac{1}{\tilde{r}}-\varepsilon}_{\tilde{r},q}(\mathbb{R}^{n-1})} \lesssim_{p,s,n,\varepsilon} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}.$$

- *If* $s = 0$, *there exists a unique* $\nu \lrcorner u_{|\partial\mathbb{R}^n_+} \in \mathrm{B}^{-\frac{1}{r}-\varepsilon}_{r,q}(\mathbb{R}^{n-1}, \Lambda^{k-1})$, *where* $\frac{1}{r} = \frac{1}{p} - \frac{\varepsilon}{n}$, *for any sufficiently small* $\varepsilon > 0$, *such that the formula* (4.11) *holds for all* $\Psi \in [\mathcal{S} \cap \mathrm{B}^{1+\varepsilon}_{r',q'}](\mathbb{R}^n_+, \Lambda)$, *with the estimates*

$$\|\nu \lrcorner u_{|\partial\mathbb{R}^n_+}\|_{\mathrm{B}^{-\frac{1}{r}-\varepsilon}_{r,q}(\mathbb{R}^{n-1})} \lesssim_{p,s,n,\varepsilon} \|u\|_{\dot{\mathrm{B}}^0_{p,q}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{B}}^0_{p,q}(\mathbb{R}^n_+)}.$$

The same result, up to appropriate changes, still holds for $u \in \dot{\mathrm{D}}^s_{p,q}(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$ *with partial trace* $\nu \wedge u_{|\partial\mathbb{R}^n_+}$ *satisfying the formula* (4.12).

(iii) For all $u \in [\dot{\mathrm{B}}^s_{p,q} \cap \dot{\mathrm{B}}^{s+1}_{p,q}](\mathbb{R}^n_+, \Lambda^k)$*, if* $q \neq +\infty$*, we have*

$$(\nu \lrcorner u \oplus \nu \wedge u)_{|_{\partial\mathbb{R}^n_+}} \in \mathrm{B}^{s+1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1}, \Lambda^{k-1} \oplus \Lambda^{k+1})$$

with the estimate

$$\|(\nu \lrcorner u \oplus \nu \wedge u)_{|_{\partial\mathbb{R}^n_+}}\|_{\mathrm{B}^{s+1-\frac{1}{p}}_{p,q}(\mathbb{R}^{n-1})} \lesssim_{p,s,n} \|u\|_{[\dot{\mathrm{B}}^s_{p,q} \cap \dot{\mathrm{B}}^{s+1}_{p,q}](\mathbb{R}^n_+)},$$

and everything still hold with $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s+1,p})$ *instead of* $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{B}}^{s+1}_{p,q})$*, when* $q = p$*. If* $q = +\infty$*, we have*

$$(\nu \lrcorner u \oplus \nu \wedge u)_{|_{\partial\mathbb{R}^n_+}} \in \mathrm{L}^p(\mathbb{R}^{n-1}, \Lambda^{k-1} \oplus \Lambda^{k+1})$$

with the corresponding estimate.

Remark 4.16 The proof of Theorem 4.14 in case of inhomogeneous function spaces follows straightforward the same proof provided for corresponding results in [MMS08, Section 4] and is somewhat sharp.

Notice that Theorem 4.15 is certainly not sharp, and investigation of sharp range for partial traces could be of great interest in the treatment of inhomogeneous boundary value problems in homogeneous function spaces.

Proof. — Without loss of generality, we only investigate the case of tangential traces $\nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}}$.

Step 1.1 : Proof of *(i)*, for $s \leqslant 0$. For $u \in \dot{\mathrm{D}}^s_p(\delta, \mathbb{R}^n_+, \Lambda^k)$, for all $\psi \in \mathrm{B}^{\frac{1}{p}-s}_{p',p'}(\mathbb{R}^{n-1}, \Lambda^{k-1})$, and $\Psi \in \mathrm{H}^{1-s,p'}(\mathbb{R}^n_+, \Lambda^{k-1})$ such that $\Psi_{|_{\partial\mathbb{R}^n_+}} = \psi$, we can define the following functional,

$$\kappa_u(\Psi) := \int_{\mathbb{R}^n_+} \langle u, \mathrm{d}\Psi \rangle - \int_{\mathbb{R}^n_+} \langle \delta u, \Psi \rangle.$$

First, the map $(u, \Psi) \mapsto \kappa_u(\Psi)$ is well-defined and bilinear on $\dot{\mathrm{D}}^s_p(\delta, \mathbb{R}^n_+, \Lambda^k) \times \mathrm{H}^{1-s,p'}(\mathbb{R}^n_+, \Lambda^{k-1})$, *i.e.,* in particular only depends on the boundary value ψ of Ψ. It is straightforward from duality that,

$$\begin{aligned}
|\kappa_u(\Psi)| &\lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\mathrm{d}\Psi\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\Psi\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+)} \\
&\lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\Psi\|_{\dot{\mathrm{H}}^{1-s,p'}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\Psi\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+)} \\
&\lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\Psi\|_{\mathrm{H}^{1-s,p'}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\Psi\|_{\mathrm{H}^{-s,p'}(\mathbb{R}^n_+)} \\
&\lesssim_{s,p,n} (\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}) \|\Psi\|_{\mathrm{H}^{1-s,p'}(\mathbb{R}^n_+)},
\end{aligned}$$

where above inequalities follows from $\mathrm{H}^{-s,p'}(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+)$, $\mathrm{H}^{1-s,p'}(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{H}}^{1-s,p'}(\mathbb{R}^n_+)$, since $-1 + \frac{1}{p} < s \leqslant 0$, then from $\mathrm{H}^{1-s,p'}(\mathbb{R}^n_+) \hookrightarrow \mathrm{H}^{-s,p'}(\mathbb{R}^n_+)$.

Now, if we have $\Psi_1, \Psi_2 \in \mathrm{H}^{1-s,p'}(\mathbb{R}^n_+, \Lambda^{k-1})$ such that $\Psi_{1|_{\partial\mathbb{R}^n_+}} = \Psi_{2|_{\partial\mathbb{R}^n_+}} = \psi$, we introduce $\Psi_0 = \Psi_1 - \Psi_2 \in \mathrm{H}^{1-s,p'}_0(\mathbb{R}^n_+, \Lambda^{k-1})$. Therefore, let's consider $(\Phi_k)_{k\in\mathbb{N}} \subset \mathrm{C}^\infty_c(\mathbb{R}^n_+, \Lambda^{k-1})$ such that,

$$\Phi_k \xrightarrow[k\to+\infty]{} \Psi_0 \text{ in } \mathrm{H}^{1-s,p'}_0(\mathbb{R}^n_+, \Lambda^{k-1}).$$

We can deduce,

$$\kappa_u(\Psi_1) - \kappa_u(\Psi_2) = \kappa_u(\Psi_0) = \lim_{k\to+\infty} \left[\int_{\mathbb{R}^n_+} \langle u, \mathrm{d}\Phi_k \rangle - \int_{\mathbb{R}^n_+} \langle \delta u, \Phi_k \rangle \right] = 0.$$

Thus, the following equality occurs, where we also consider the extension operator $\mathrm{Ext}_{\mathbb{R}^n_+}$ from Theorem 2.45,

$$\tilde{\kappa}_u(\psi) := \kappa_u(\mathrm{Ext}_{\mathbb{R}^n_+} \otimes \psi) = \kappa_u(\Psi),$$

and with the estimate, also obtained from Theorem 2.45,

$$|\tilde{\kappa}_u(\psi)| \lesssim_{s,p,n} (\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\mathrm{d}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}) \|\psi\|_{\mathrm{B}^{\frac{1}{p}-s}_{p',p'}(\mathbb{R}^{n-1})}.$$

By duality, there exists a unique function depending linearly on u, $\nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} \in \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1}, \Lambda^{k-1})$, such that (4.11) holds.

To guarantee that the representation formula makes sense, one may use the usual integration by parts formula with $u, \Psi \in \mathcal{S}(\overline{\mathbb{R}^n_+}, \Lambda)$.

Step 1.2 : Proof of *(i)*, for $s > 0$. For the same assumption on u, and Ψ as before, everything works similarly except the way we bounded bilinearly $(u, \Psi) \mapsto \kappa_u(\Psi)$ on $\dot{\mathrm{D}}^s_p(\delta, \mathbb{R}^n_+, \Lambda^k) \times \mathrm{H}^{1-s,p'}(\mathbb{R}^n_+, \Lambda^{k-1})$. For $r \in (1, +\infty)$ such that $\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$, we deduce from Sobolev embeddings and duality that

$$\begin{aligned}
|\kappa_u(\Psi)| &\lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\mathrm{d}\Psi\|_{\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+)} + \|\delta u\|_{\mathrm{L}^r(\mathbb{R}^n_+)} \|\Psi\|_{\mathrm{L}^{r'}(\mathbb{R}^n_+)} \\
&\lesssim_{r,s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\Psi\|_{\mathrm{H}^{1,r'}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \|\Psi\|_{\mathrm{H}^{1,r'}(\mathbb{R}^n_+)} \\
&\lesssim_{r,s,p,n} (\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}) \|\Psi\|_{\mathrm{H}^{1,r'}(\mathbb{R}^n_+)}.
\end{aligned}$$

Thus everything goes similarly.

Step 2.1 : Proof of *(ii)*, for $s < 0$, is very similar to the one of the above Step 1.1.

Step 2.2 : Proof of *(ii)*, for $s > 0$, is somewhat similar to the one of Step 1.2 but needs further explanations. We use Sobolev embeddings, and generalized Hölder inequalities using Lorentz spaces, $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) \hookrightarrow \mathrm{L}^{r,q}(\mathbb{R}^n_+)$, $\mathrm{B}^{\varepsilon}_{\tilde{r}',q'}(\mathbb{R}^n_+) \hookrightarrow \mathrm{L}^{r',q'}(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)$, for $\varepsilon > 0$,

$$\begin{aligned}
|\kappa_u(\Psi)| &\lesssim_{s,p,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \|\mathrm{d}\Psi\|_{\dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+)} + \|\delta u\|_{\mathrm{L}^{r,q}(\mathbb{R}^n_+)} \|\Psi\|_{\mathrm{L}^{r',q'}(\mathbb{R}^n_+)} \\
&\lesssim_{r,s,p,n} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \|\mathrm{d}\Psi\|_{\mathrm{L}^{r',q'}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \|\Psi\|_{\mathrm{L}^{r',q'}(\mathbb{R}^n_+)} \\
&\lesssim_{r,s,p,n} (\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} + \|\delta u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}) \|\Psi\|_{\mathrm{B}^{1+\varepsilon}_{\tilde{r}',q'}(\mathbb{R}^n_+)}.
\end{aligned}$$

Step 2.3 : Proof of *(ii)*, for $s = 0$ is shown via similar Sobolev embeddings arguments and is left to the reader.

Step 3 : Proof of *(iii)*, follows from Proposition 2.52, with explicit representation formula for any suitable k-differential forms u :

$$\begin{aligned}
\nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} = -\mathfrak{e}_n \lrcorner u_{|_{\partial\mathbb{R}^n_+}} &= (-1)^k \sum_{1 \leqslant \ell_1 < \ldots < \ell_{k-1} < n} u_{\ell_1 \ell_2 \ldots \ell_{k-1} n}(\cdot, 0)\, \mathrm{d}x_{\ell_1} \wedge \ldots \wedge \mathrm{d}x_{\ell_{k-1}} \\
&= (-1)^k \sum_{I' \in \mathcal{I}^{k-1}_{n-1}} u_{I',n}(\cdot, 0)\, \mathrm{d}x_{I'}.
\end{aligned}$$

A similar treatment yields the same conclusion for the boundary term $\nu \wedge u_{|_{\partial\mathbb{R}^n_+}}$, so that one ends the proof here. ∎

Remark 4.17 Let's make further comments about the estimates used in above proof of Theorem 4.15, in particular the ones used in Step 2.2.

We recall that from Sobolev embeddings, see Proposition 2.31, for $0 < s_0 < s < s_1 < 1/p$,

$r_0, r_1, p \in (1, +\infty)$, $1/r_j = 1/p - s_j/n$, and appropriate $q_j \in [1, +\infty]$ we have

$$\dot{\mathrm{B}}^{s_j}_{p,q_j}(\mathbb{R}^n_+) \hookrightarrow \mathrm{L}^{r_j}(\mathbb{R}^n_+)\,,\ j \in \{0,1\}.$$

If $(s, 1/r) = (1-\theta)(s_0, 1/r_0) + \theta(s_1, 1/r_1)$, by real interpolation, for $q \in [1, +\infty]$ we obtain,

$$\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n_+) = (\dot{\mathrm{B}}^{s_0}_{p,q_0}(\mathbb{R}^n_+), \dot{\mathrm{B}}^{s_1}_{p,q_1}(\mathbb{R}^n_+))_{\theta,q} \hookrightarrow (\mathrm{L}^{r_0}(\mathbb{R}^n_+), \mathrm{L}^{r_1}(\mathbb{R}^n_+))_{\theta,q} = \mathrm{L}^{r,q}(\mathbb{R}^n_+).$$

And one may proceed similarly for the reverse embedding,

$$\mathrm{L}^{r',q'}(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{B}}^{-s}_{p',q'}(\mathbb{R}^n_+).$$

For more details about Lorentz spaces and their interpolation, one could consult [Lun18, Section 1, Examples 1.10, 1.11 & 1.27] and [BL76, Chapter 5, Section 5.3].

2.3.2 L^2-theory for Hodge Laplacians and the Hodge decomposition

The following lemma is fundamental for the analysis of the L^2 theory of the Hodge Laplacian when one has an explicit access to the boundary, and moreover several proofs presented here do not depend on the open set Ω, here $\Omega = \mathbb{R}^n_+$, and remain valid as long as integration by parts formulas (4.9) and (4.10) and partial traces results for vector fields are available.

Lemma 4.18 *Let $k \in [\![0, n]\!]$. We set*

$$\mathrm{D}_2(\underline{\delta}, \mathbb{R}^n_+, \Lambda^k) := \{\, u \in \mathrm{D}_2(\delta, \mathbb{R}^n_+, \Lambda^k) \,|\, \nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} = 0 \,\},$$
$$\mathrm{D}_2(\underline{\mathrm{d}}, \mathbb{R}^n_+, \Lambda^k) := \{\, u \in \mathrm{D}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k) \,|\, \nu \wedge u_{|_{\partial\mathbb{R}^n_+}} = 0 \,\}.$$

The operator $(\mathrm{D}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k), \mathrm{d})$ is an unbounded densely defined closed operator, with adjoint

$$(\mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^{k+1}), \mathrm{d}^*) = (\mathrm{D}_2(\underline{\delta}, \mathbb{R}^n_+, \Lambda^{k+1}), \delta). \tag{4.13}$$

Similarly, $(\mathrm{D}_2(\delta, \mathbb{R}^n_+, \Lambda^k), \delta)$ is an unbounded densely defined closed operator, with adjoint

$$(\mathrm{D}_2(\delta^*, \mathbb{R}^n_+, \Lambda^{k-1}), \delta^*) = (\mathrm{D}_2(\underline{\mathrm{d}}, \mathbb{R}^n_+, \Lambda^{k-1}), \mathrm{d}). \tag{4.14}$$

Proof. — Closedness and the fact that both are densely defined is straightforward. We just prove the duality identity (4.13), the proof of (4.14) is similar. Let $u \in \mathrm{D}_2(\underline{\delta}, \mathbb{R}^n_+, \Lambda^k)$, then for all $v \in \mathcal{S}_0(\overline{\mathbb{R}^n_+}, \Lambda^k)$, we can use Theorem 4.14, to obtain that

$$\big\langle v, \delta u \big\rangle_{\mathbb{R}^n_+} = \big\langle \mathrm{d}v, u \big\rangle_{\mathbb{R}^n_+}.$$

Thus, by Cauchy-Schwarz inequality

$$\big|\big\langle \mathrm{d}v, u \big\rangle_{\mathbb{R}^n_+}\big| \leqslant \|\delta u\|_{\mathrm{L}^2(\mathbb{R}^n_+)} \|v\|_{\mathrm{L}^2(\mathbb{R}^n_+)}.$$

Hence, $v \mapsto \big\langle \mathrm{d}v, u \big\rangle_{\mathbb{R}^n_+}$ extends uniquely to a bounded linear functional on $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$, so that, necessarily $(\mathrm{D}_2(\underline{\delta}, \mathbb{R}^n_+, \Lambda^k), \delta) \subset (\mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^{k+1}), \mathrm{d}^*)$.

For the reverse inclusion, let $u \in \mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^{k+1})$, for all $v \in \mathrm{D}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$, we have

$$\big\langle v, \mathrm{d}^* u \big\rangle_{\mathbb{R}^n_+} = \big\langle \mathrm{d}v, u \big\rangle_{\mathbb{R}^n_+}.$$

In particular, for $v \in \mathrm{C}^\infty_c(\mathbb{R}^n_+, \Lambda^k)$, it yields that

$$\big\langle v, \mathrm{d}^* u \big\rangle_{\mathbb{R}^n_+} = \big\langle \mathrm{d}v, u \big\rangle_{\mathbb{R}^n_+} = \big\langle v, \delta u \big\rangle_{\mathbb{R}^n_+}.$$

Hence, $\mathrm{d}^* u = \delta u$ in $\mathcal{D}'(\mathbb{R}^n_+, \Lambda^k)$, then in $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$, so that for all $v \in \mathrm{D}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$,

$$\big\langle v, \delta u \big\rangle_{\mathbb{R}^n_+} = \big\langle \mathrm{d} v, u \big\rangle_{\mathbb{R}^n_+}.$$

From above equality, we apply Theorem 4.14 to check $\nu \lrcorner u_{|_{\partial \mathbb{R}^n_+}} = 0$ and deduce

$$(\mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^{k+1}), \mathrm{d}^*) \subset (\mathrm{D}_2(\underline{\delta}, \mathbb{R}^n_+, \Lambda^{k+1}), \delta),$$

the proof being therefore complete. ∎

In particular, since $(\mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k), \mathrm{d}^*)$ and $(\mathrm{D}_2(\delta, \mathbb{R}^n_+, \Lambda^k), \delta)$ are closed operators, both

$$\mathrm{N}_2(\delta, \mathbb{R}^n_+, \Lambda^k) \text{ and } \mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k)$$

are closed subspaces of $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$. Thus, the following orthogonal projection are well-defined and bounded

$$\begin{array}{ll} \mathbb{P} : \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k), & [\mathrm{I} - \mathbb{P}] : \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \overline{\mathrm{R}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)}, \\ \mathbb{Q} : \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \mathrm{N}_2(\delta, \mathbb{R}^n_+, \Lambda^k), & [\mathrm{I} - \mathbb{Q}] : \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \overline{\mathrm{R}_2(\delta^*, \mathbb{R}^n_+, \Lambda^k)}, \end{array}$$

which induce topological Hodge decomposition

$$\begin{aligned} \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) &= \overline{\mathrm{R}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)} \overset{\perp}{\oplus} \mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k), \\ &= \overline{\mathrm{R}_2(\delta^*, \mathbb{R}^n_+, \Lambda^k)} \overset{\perp}{\oplus} \mathrm{N}_2(\delta, \mathbb{R}^n_+, \Lambda^k). \end{aligned} \tag{$\mathfrak{H}_2$}$$

Lemma 4.19 *For $k \in [\![0, n]\!]$, the following Hodge-Dirac operators*

$$\begin{aligned} (\mathrm{D}_2(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k), D_{\mathfrak{t}}) &= (\mathrm{D}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k) \cap \mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k),\ \mathrm{d} + \mathrm{d}^*), \\ (\mathrm{D}_2(D_{\mathfrak{n}}, \mathbb{R}^n_+, \Lambda^k), D_{\mathfrak{n}}) &= (\mathrm{D}_2(\delta^*, \mathbb{R}^n_+, \Lambda^k) \cap \mathrm{D}_2(\delta, \mathbb{R}^n_+, \Lambda^k),\ \delta^* + \delta), \end{aligned}$$

are both densely defined closed operators on $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$.

Proof. — Let $(u_j)_{j \in \mathbb{N}} \subset \mathrm{D}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k) \cap \mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k)$, and $(u, v) \in \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) \times \mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$ such that it satisfies

$$u_j \xrightarrow[j \to +\infty]{} u \quad \text{and} \quad D_{\mathfrak{t}} u_j \xrightarrow[j \to +\infty]{} v \quad \text{in } \mathrm{L}^2(\mathbb{R}^n_+).$$

By the Hodge decomposition, there exists a unique couple $(v_0, v_1) \in \overline{\mathrm{R}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda)} \times \mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda)$ such that $v = v_0 + v_1$. Since u_j goes to u in $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$, by continuity of involved projectors, and uniqueness of decomposition, it follows that

$$\mathrm{d} u_j \xrightarrow[j \to +\infty]{} v_0 \quad \text{and} \quad \mathrm{d}^* u_j \xrightarrow[j \to +\infty]{} v_1 \quad \text{in } \mathrm{L}^2(\mathbb{R}^n_+).$$

But $(u_j)_{j \in \mathbb{N}}$ converge to u in $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$, so in particular in distributional sense, thus necessarily $(v_0, v_1) = (\mathrm{d} u, \mathrm{d}^* u)$ and $v = D_{\mathfrak{t}} u$, *i.e.*, $(\mathrm{D}_2(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k), D_{\mathfrak{t}})$ is closed on $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$. The proof ends here since one can reproduce all above arguments for $(\mathrm{D}_2(D_{\mathfrak{n}}, \mathbb{R}^n_+, \Lambda^k), D_{\mathfrak{n}})$. ∎

Proposition 4.20 *The Hodge-Dirac operator $(\mathrm{D}_2(D_\cdot, \mathbb{R}^n_+, \Lambda), D_\cdot)$ is an injective self-adjoint 0-bisectorial operator on $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$ so that it satisfies the following bound, for all $\theta \in (0, \frac{\pi}{2})$,*

$$\forall \mu \in \mathbb{C} \setminus \overline{S}_\theta, \ \big\| \mu (\mu \mathrm{I} + D_\cdot)^{-1} \big\|_{\mathrm{L}^2(\mathbb{R}^n_+) \to \mathrm{L}^2(\mathbb{R}^n_+)} \leqslant \frac{1}{\sin(\theta)}. \tag{4.15}$$

Moreover, it admits a bounded ($\mathbf{H}^\infty(S_\theta)$-)functional calculus on $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda)$ *with bound* 1, *i.e., for all* $f\in\mathbf{H}^\infty(S_\theta)$, $u\in\mathrm{L}^2(\mathbb{R}^n_+,\Lambda)$,

$$\|f(D.)u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\leqslant\|f\|_{\mathrm{L}^\infty(S_\theta)}\,\|u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\,.$$

Remark 4.21 Proposition 4.20 does not depend on the fact $\Omega=\mathbb{R}^n_+$. See [MM18, Section 2] where the same result is stated for bounded (even weak-)Lipschitz domains.

Proof. — The resolvent bound (4.15) is usual since $(\mathrm{D}_2(D.),D.)$ is self-adjoint by construction, see [Haa06, Proposition C.4.2]. The fact that it admits a bounded holomorphic functional calculus follows from [McI86, Section 10]. ∎

For $k\in[\![0,n]\!]$, the Hodge Laplacian $(\mathrm{D}(\Delta_{\mathcal{H}},\mathbb{R}^n_+,\Lambda^k),-\Delta_{\mathcal{H}})$ can be realized on $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$ by the mean of densely defined, symmetric, accretive, continuous, closed, sesquilinear forms on $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$, for

$$\mathfrak{a}_{\mathcal{H}}\,:\,\mathrm{D}_2(\mathfrak{a}_{\mathcal{H}},\Lambda^k)^2\ni(u,v)\longmapsto\int_{\mathbb{R}^n_+}\langle\mathrm{d}u,\overline{\mathrm{d}v}\rangle+\int_{\mathbb{R}^n_+}\langle\delta u,\overline{\delta v}\rangle \tag{4.16}$$

with $\mathrm{D}_2(\mathfrak{a}_{\mathcal{H},\mathfrak{t}},\Lambda^k)=\mathrm{D}_2(\mathrm{d},\Lambda^k)\cap\mathrm{D}_2(\mathrm{d}^*,\Lambda^k)$, $\mathrm{D}_2(\mathfrak{a}_{\mathcal{H},\mathfrak{n}})=\mathrm{D}_2(\delta^*,\Lambda^k)\cap\mathrm{D}_2(\delta,\Lambda^k)$, so that it is easy to see that both are closed, densely defined, non-negative self-adjoint operators on $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$. See [Ouh05, Chapter 1] for more details about realization of operators via sesquilinear forms on a Hilbert space. The next theorem is a standard consequence.

Theorem 4.22 *Let* $k\in[\![0,n]\!]$. *The operator* $(\mathrm{D}_2(\Delta_{\mathcal{H}},\mathbb{R}^n_+,\Lambda^k),-\Delta_{\mathcal{H}})$ *is an injective non-negative self-adjoint and* 0*-sectorial operator on* $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$, *which admits bounded (or* $\mathbf{H}^\infty(\Sigma_\theta)$*-) holomorphic functional calculus for all* $\theta\in(0,\pi)$.

Moreover, the following hold

(i) $\mathrm{D}_2(\Delta_{\mathcal{H}},\mathbb{R}^n_+,\Lambda^k)$ *is a closed subspace of* $\mathrm{H}^2(\mathbb{R}^n_+,\Lambda)$;

(ii) Provided $\mu\in[0,\pi)$, *for* $\lambda\in\Sigma_\mu$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$, *then* $u:=(\lambda\mathrm{I}-\Delta_{\mathcal{H}})^{-1}f$ *satisfies*

$$|\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}+|\lambda|^{\frac12}\|D.u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}+\|\Delta u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\lesssim_\mu\|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\,; \tag{4.17}$$

$$|\lambda|\|u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}+|\lambda|^{\frac12}\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}+\|\nabla^2 u\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\lesssim_{n,\mu}\|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)}\,; \tag{4.18}$$

(iii) The following resolvent identity holds for all $\mu\in[0,\pi)$, $\lambda\in\Sigma_\mu$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$,

$$\mathrm{E}_{\mathcal{H}}(\lambda\mathrm{I}-\Delta_{\mathcal{H}})^{-1}f=(\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{H}}f.$$

Remark 4.23 In above Theorem 4.22, points *(i)* and *(iii)*, as well as the estimate (4.18) of point *(ii)* are the only points that relies on the fact that the considered openset is $\Omega=\mathbb{R}^n_+$, but mainly the point *(iii)* is used to deduce previous ones. The beginning of the statement, as well as (4.17), does not rely on any particular structure, and remains true on any open set Ω.

Many other results below, in the present subsection about L^2-theory of Hodge Laplacians and the Hodge decomposition, remain true on general domains Ω as long as one can show that Hodge Laplacians are injective.

Before proving Theorem 4.22, following 6 of Chapter 2, for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, we reintroduce the following extension operator defined for any measurable function u on $\mathbb{R}^n_+$, for almost every $x=(x',x_n)\in\mathbb{R}^n$:

$$\mathrm{E}_{\mathcal{D}}u(x',x_n):=\begin{cases}u(x',x_n) & ,\text{ if }(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}_+,\\ -u(x',-x_n) & ,\text{ if }(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^*_-\,;\end{cases}$$

$$\mathrm{E}_{\mathcal{N}}u(x',x_n):=\begin{cases}u(x',x_n) & ,\text{ if }(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}_+,\\ u(x',-x_n) & ,\text{ if }(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}^*_-.\end{cases}$$

Now, we precise the definition of the extension operators $\mathrm{E}_{\mathcal{H},\mathfrak{j}}$, $\mathfrak{j}\in\{\mathfrak{t},\mathfrak{n}\}$, on measurable functions $u:\mathbb{R}^n_+\longrightarrow\Lambda^k$, provided $k\in[\![0,n]\!]$, $I\in\mathcal{I}^k_n$,

$$(\mathrm{E}_{\mathcal{H},\mathfrak{t}}u)_I:=\begin{cases}\mathrm{E}_{\mathcal{D}}u_I & ,\text{ if } n\in I,\\ \mathrm{E}_{\mathcal{N}}u_I & ,\text{ if } n\notin I\,;\end{cases}\quad\text{and}\quad(\mathrm{E}_{\mathcal{H},\mathfrak{n}}u)_I:=\begin{cases}\mathrm{E}_{\mathcal{N}}u_I & ,\text{ if } n\in I,\\ \mathrm{E}_{\mathcal{D}}u_I & ,\text{ if } n\notin I\,;\end{cases}. \tag{4.19}$$

For $u:\mathbb{R}^n_+\longrightarrow\Lambda^k$, we also set

$$\tilde{u}_{\mathfrak{j}}:=[\mathrm{E}_{\mathcal{H},\mathfrak{j}}u]_{|\mathbb{R}^n_-}.$$

By construction, for $\mathfrak{j}\in\{\mathfrak{t},\mathfrak{n}\}$, $s\in(-1+1/p,1/p)$, $p\in(1,+\infty)$, the Proposition 2.15 leads to boundedness of

$$\mathrm{E}_{\mathcal{H},\mathfrak{j}}:\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda)\longrightarrow\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n,\Lambda). \tag{4.20}$$

The same result holds replacing $\dot{\mathrm{H}}^{s,p}$ by either $\mathrm{H}^{s,p}$, $\mathrm{B}^s_{p,q}$, or even by $\dot{\mathrm{B}}^s_{p,q}$, $q\in[1,+\infty]$.

Lemma 4.24 *For all $u\in\mathrm{D}_2(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)$ (resp. $\mathrm{D}_2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda^k)$) we have*

$$\mathrm{E}_{\mathcal{H},\mathfrak{t}}u\in\mathrm{D}_2(\mathrm{d},\mathbb{R}^n,\Lambda^k)\quad(\textit{resp. }\mathrm{D}_2(\delta,\mathbb{R}^n,\Lambda^k)\ )$$

with formulas

$$\mathrm{d}\mathrm{E}_{\mathcal{H},\mathfrak{t}}u=\mathrm{E}_{\mathcal{H},\mathfrak{t}}\mathrm{d}u\qquad(\textit{resp. }\delta\mathrm{E}_{\mathcal{H},\mathfrak{t}}u=\mathrm{E}_{\mathcal{H},\mathfrak{t}}\mathrm{d}^*u).$$

Proof. — Let $u\in\mathrm{D}_2(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)$, for $v\in\mathcal{S}(\mathbb{R}^n,\Lambda^{k+1})$,

$$\begin{aligned}\big\langle\mathrm{E}_{\mathcal{H},\mathfrak{t}}u,\delta v\big\rangle_{\mathbb{R}^n}&=\big\langle u,\delta v\big\rangle_{\mathbb{R}^n_+}+\big\langle\tilde{u}_{\mathfrak{t}},\delta v\big\rangle_{\mathbb{R}^n_-}\\&=\big\langle\mathrm{d}u,v\big\rangle_{\mathbb{R}^n_+}+\big\langle(-\mathfrak{e}_n)\wedge u,v\big\rangle_{\partial\mathbb{R}^n_+}+\big\langle\widetilde{\mathrm{d}u_{\mathfrak{t}}},v\big\rangle_{\mathbb{R}^n_-}+\big\langle(\mathfrak{e}_n)\wedge\tilde{u}_{\mathfrak{t}},v\big\rangle_{\partial\mathbb{R}^n_-}\\&=\big\langle\mathrm{d}u,v\big\rangle_{\mathbb{R}^n_+}+\big\langle\widetilde{\mathrm{d}u_{\mathfrak{t}}},v\big\rangle_{\mathbb{R}^n_-}\\&=\big\langle\mathrm{E}_{\mathcal{H},\mathfrak{t}}\mathrm{d}u,v\big\rangle_{\mathbb{R}^n}.\end{aligned}$$

Which holds, since $(\mathfrak{e}_n)\wedge\tilde{u}_{\mathfrak{t}}(\cdot,0)=(\mathfrak{e}_n)\wedge u(\cdot,0)$.

Now, if $u\in\mathrm{D}_2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda^k)$, for $v\in\mathcal{S}(\mathbb{R}^n,\Lambda^{k-1})$,

$$\begin{aligned}\big\langle\mathrm{E}_{\mathcal{H},\mathfrak{t}}u,\mathrm{d}v\big\rangle_{\mathbb{R}^n}&=\big\langle u,\mathrm{d}v\big\rangle_{\mathbb{R}^n_+}+\big\langle\tilde{u}_{\mathfrak{t}},\mathrm{d}v\big\rangle_{\mathbb{R}^n_-}\\&=\big\langle\delta u,v\big\rangle_{\mathbb{R}^n_+}+\big\langle(-\mathfrak{e}_n)\lrcorner u,v\big\rangle_{\partial\mathbb{R}^n_+}+\big\langle\widetilde{\delta u_{\mathfrak{t}}},v\big\rangle_{\mathbb{R}^n_-}+\big\langle(\mathfrak{e}_n)\lrcorner\tilde{u}_{\mathfrak{t}},v\big\rangle_{\partial\mathbb{R}^n_-}\\&=\big\langle\delta u,v\big\rangle_{\mathbb{R}^n_+}+\big\langle\widetilde{\delta u_{\mathfrak{t}}},v\big\rangle_{\mathbb{R}^n_-}\\&=\big\langle\mathrm{E}_{\mathcal{H},\mathfrak{t}}\mathrm{d}^*u,v\big\rangle_{\mathbb{R}^n}.\end{aligned}$$

Which holds, since $(\mathfrak{e}_n)\lrcorner\tilde{u}_{\mathfrak{t}}(\cdot,0)=-(\mathfrak{e}_n)\lrcorner u(\cdot,0)=0$. ∎

Proof (of Theorem 4.22). — By the realization of Hodge Laplacian by the mean of the sesquilinear form (4.16), we have $(\mathrm{D}_2(\Delta_{\mathcal{H}},\mathbb{R}^n_+,\Lambda^k),-\Delta_{\mathcal{H}})=(\mathrm{D}_2(D^2_{\cdot},\mathbb{R}^n_+,\Lambda^k),D^2_{\cdot})$. Thus, as the square of a self-adjoint 0-bisectorial operator, the Hodge Laplacian is a non-negative self-adjoint 0-sectorial operator on $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$, and it also admits bounded holomorphic functional calculus, see [Ege15, Theorem 3.2.20]. In particular, (4.17) in point *(ii)* holds.

For now, we only consider the case $(\mathrm{D}_2(\Delta_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k),-\Delta_{\mathcal{H},\mathfrak{t}})$, the proof could be achieved in similar fashion for $(\mathrm{D}_2(\Delta_{\mathcal{H},\mathfrak{n}},\mathbb{R}^n_+,\Lambda^k),-\Delta_{\mathcal{H},\mathfrak{n}})$.

For $\lambda\in\Sigma_\mu$, $\mu\in(0,\pi)$, $f\in\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$, we set $U:=(\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{H}}f\in\mathrm{H}^2(\mathbb{R}^n,\Lambda^k)$. By

construction, as in the proof of Theorem 4.22 for $I \in \mathcal{I}_n^k$ we have

$$\begin{aligned} U_{I|_{\partial\mathbb{R}^n_+}} &= 0, \text{ provided } n \in I, \\ \partial_{x_n} U_{I|_{\partial\mathbb{R}^n_+}} &= 0, \text{ provided } n \notin I. \end{aligned}$$

Therefore, we obtain first,

$$\begin{aligned} \nu \lrcorner u_{|\partial\mathbb{R}^n_+} = -\mathfrak{e}_n \lrcorner u_{|\partial\mathbb{R}^n_+} &= (-1)^k \sum_{1\leqslant \ell_1<\ldots<\ell_{k-1}<n} u_{\ell_1\ell_2\ldots\ell_{k-1}n}(\cdot,0)\,\mathrm{d}x_{\ell_1}\wedge\ldots\wedge\mathrm{d}x_{\ell_{k-1}} \\ &= (-1)^k \sum_{I'\in\mathcal{I}_{n-1}^{k-1}} u_{I',n}(\cdot,0)\,\mathrm{d}x_{I'} = 0. \end{aligned}$$

Similarly, we get that

$$\begin{aligned} \nu \lrcorner \mathrm{d}u_{|\partial\mathbb{R}^n_+} = -\mathfrak{e}_n \lrcorner \mathrm{d}u_{|\partial\mathbb{R}^n_+} &= -\sum_{j=1}^{n-1} \sum_{1\leqslant \ell_1<\ldots<\ell_{k-1}<n} \partial_{x_j} u_{\ell_1\ell_2\ldots\ell_{k-1}n}(\cdot,0)\,\mathrm{d}x_j\wedge\mathrm{d}x_{\ell_1}\wedge\ldots\wedge\mathrm{d}x_{\ell_{k-1}} \\ &\quad - \sum_{1\leqslant \ell_1<\ldots<\ell_{k-1}<\ell_k<n} \partial_{x_n} u_{\ell_1\ell_2\ldots\ell_k}(\cdot,0)\,\mathrm{d}x_{\ell_1}\wedge\ldots\wedge\mathrm{d}x_{\ell_k} \\ &= 0. \end{aligned}$$

From above calculations, we deduce that $u := U_{|\mathbb{R}^n_+}$ is such that $u \in \mathrm{H}^2(\mathbb{R}^n_+,\Lambda^k)\cap \mathrm{D}_2(\Delta_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k)$, and

$$\lambda u - \Delta u = f \text{ in } \mathbb{R}^n_+.$$

Hence, by uniqueness $(\lambda\mathrm{I}-\Delta_{\mathcal{H}})^{-1}f = [(\lambda\mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{H}}f]_{\mathbb{R}^n_+}$. One may conclude following the lines of the proof of Proposition 2.56, using Lemma 4.24. ■

Lemma 4.25 *Provided $k \in [\![0,n]\!]$, $\mu\in(0,\pi)$, $\lambda\in\Sigma_\mu$, the following commutation identities hold,*

(i) $\mathbb{P}(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}f = (\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathbb{P}f$, *for all* $f\in\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$ *;*

(ii) $\mathrm{d}(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}f = (\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathrm{d}f$, *for all* $f\in\mathrm{D}_2(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)$ *;*

(iii) $\mathrm{d}^*(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}f = (\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathrm{d}^*f$, *for all* $f\in\mathrm{D}_2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda^k)$.

Every above identities still hold, replacing $(\mathfrak{t},\mathbb{P},\mathrm{d},\mathrm{d}^*)$ *by* $(\mathfrak{n},\mathbb{Q},\delta^*,\delta)$.

Remark 4.26 Lemma 4.25 does not depend on the domain, $\Omega=\mathbb{R}^n_+$ since its proof only relies on the use of the sesquilinear form associated with the Hodge Laplacian.

Proof. — **Step 0 :** For all $u,v\in\mathrm{D}_2(D_{\mathfrak{t}},\mathbb{R}^n_+,\Lambda)$,

$$\begin{aligned} \mathfrak{a}_{\mathcal{H},\mathfrak{t}}(\mathbb{P}u,v) &= \big\langle \mathrm{d}\mathbb{P}u,\mathrm{d}v\big\rangle_{\mathbb{R}^n_+} + \big\langle \mathrm{d}^*\mathbb{P}u,\mathrm{d}^*v\big\rangle_{\mathbb{R}^n_+} \\ &= \big\langle \mathrm{d}u,\mathrm{d}v\big\rangle_{\mathbb{R}^n_+} \\ &= \big\langle \mathrm{d}u,\mathrm{d}\mathbb{P}v\big\rangle_{\mathbb{R}^n_+} \\ &= \big\langle \mathrm{d}u,\mathrm{d}\mathbb{P}v\big\rangle_{\mathbb{R}^n_+} + \big\langle \mathrm{d}^*u,\mathrm{d}^*\mathbb{P}v\big\rangle_{\mathbb{R}^n_+} \\ &= \mathfrak{a}_{\mathcal{H},\mathfrak{t}}(u,\mathbb{P}v). \end{aligned}$$

Step 1 : Let $\mu\in(0,\pi)$, $\lambda\in\Sigma_\mu$, and $f\in\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)$, we set $u := (\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}f$, then for all

$v \in \mathrm{D}_2(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda)$,

$$\begin{aligned}\lambda\big\langle \mathbb{P}u, v\big\rangle_{\mathbb{R}^n_+} + \mathfrak{a}_{\mathcal{H},\mathfrak{t}}(\mathbb{P}u, v) &= \lambda\big\langle u, \mathbb{P}v\big\rangle_{\mathbb{R}^n_+} + \mathfrak{a}_{\mathcal{H},\mathfrak{t}}(u, \mathbb{P}v)\\ &= \big\langle f, \mathbb{P}v\big\rangle_{\mathbb{R}^n_+}\\ &= \big\langle \mathbb{P}f, v\big\rangle_{\mathbb{R}^n_+}.\end{aligned}$$

Hence, by uniqueness of the solution to the resolvent problem in $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$, we deduce $\mathbb{P}u = (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathbb{P}f$.

Step 2 : We use the same notations as the ones introduced in the Step 1, but we assume that $f \in \mathrm{D}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$. For all $v \in \mathrm{D}_2(\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{R}^n_+, \Lambda)$,

$$\begin{aligned}\big\langle \mathrm{d}f, v\big\rangle_{\mathbb{R}^n_+} &= \big\langle f, \mathrm{d}^*v\big\rangle_{\mathbb{R}^n_+}\\ &= \lambda\big\langle u, \mathrm{d}^*v\big\rangle_{\mathbb{R}^n_+} + \mathfrak{a}_{\mathcal{H},\mathfrak{t}}(u, \mathrm{d}^*v)\\ &= \lambda\big\langle u, \mathrm{d}^*v\big\rangle_{\mathbb{R}^n_+} + \big\langle \mathrm{d}u, \mathrm{d}\mathrm{d}^*v\big\rangle_{\mathbb{R}^n_+} + \big\langle \mathrm{d}^*u, (\mathrm{d}^*)^2v\big\rangle_{\mathbb{R}^n_+}\\ &= \lambda\big\langle \mathrm{d}u, v\big\rangle_{\mathbb{R}^n_+} + \big\langle \mathrm{d}^*\mathrm{d}u, \mathrm{d}^*v\big\rangle_{\mathbb{R}^n_+}\\ &= \lambda\big\langle \mathrm{d}u, v\big\rangle_{\mathbb{R}^n_+} + \big\langle \mathrm{d}\mathrm{d}u, \mathrm{d}v\big\rangle_{\mathbb{R}^n_+} + \big\langle \mathrm{d}^*\mathrm{d}u, \mathrm{d}^*v\big\rangle_{\mathbb{R}^n_+}\\ &= \lambda\big\langle \mathrm{d}u, v\big\rangle_{\mathbb{R}^n_+} + \mathfrak{a}_{\mathcal{H},\mathfrak{t}}(\mathrm{d}u, v).\end{aligned}$$

Since the continuous embedding $\mathrm{D}_2(\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{R}^n_+, \Lambda) \hookrightarrow \mathrm{D}_2(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda)$ is dense, the above equality still holds for all $v \in \mathrm{D}_2(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda)$. Hence, by uniqueness of the solution to the resolvent problem in $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$, we obtain $\mathrm{d}u = (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathrm{d}f$. The proof ends here, since all remaining results can be proven similarly. ■

Lemma 4.27 *Let $k \in [\![0, n]\!]$, the following operators*

$$\begin{aligned}\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} &: \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \mathrm{N}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^{k+1});\\ \mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} &: \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^{k-1});\end{aligned}$$

are well-defined bounded linear operators on $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$, and are each-other's adjoint.

Everything still holds, replacing $(\mathfrak{t}, \mathrm{d}, \mathrm{d}^*)$ by $(\mathfrak{n}, \delta^*, \delta)$.

Proof. — We prove the $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$-boundedness of $\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}$ and compute its adjoint.

We use bounded holomorphic functional calculus of $D_{\mathfrak{t}}$ on $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$ provided by Proposition 4.20. By the mean of $z \mapsto \frac{z}{\sqrt{\lambda + z^2}}$, and the boundedness of $\mathbb{P}$, we have, for all $\lambda \geqslant 0$, and all $f \in \mathrm{L}^2(\mathbb{R}^n_+, \Lambda^k)$

$$\begin{aligned}\|\mathrm{d}(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f\|_{\mathrm{L}^2(\mathbb{R}^n_+)} &\leqslant \|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)};\\ \|\mathrm{d}^*(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f\|_{\mathrm{L}^2(\mathbb{R}^n_+)} &\leqslant \|f\|_{\mathrm{L}^2(\mathbb{R}^n_+)}.\end{aligned}$$

The adjoint of $\mathrm{d}(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}$ provided $\lambda > 0$, is $(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^* = \mathrm{d}^*(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}$ up to a dense subset of $\mathrm{L}^2(\mathbb{R}^n_+, \Lambda)$ (here $\mathrm{D}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda)$). Thanks to Lemma 4.25, we can pass to the limit as λ goes to 0 in the L^2 inner product yielding

$$(\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}})^* = \mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}.$$ ■

We can summarize

Theorem 4.28 *Let $k \in [\![0, n]\!]$, the following assertions are true*

(i) The following equality holds

$$\mathrm{N}_2(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)=\overline{\mathrm{R}_2(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)}^{\|\cdot\|_{\mathrm{L}^2(\mathbb{R}^n_+)}},$$

and still holds, replacing d *by* d^*.

(ii) The (generalized) Helmholtz-Leray projector $\mathbb{P}:\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)\longrightarrow\mathrm{N}^2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda^k)$ *satisfies the identity*

$$\mathbb{P}=\mathrm{I}-\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}.$$

(iii) The following Hodge decomposition holds,

$$\mathrm{L}^2(\mathbb{R}^n_+,\Lambda^k)=\mathrm{N}^2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda^k)\overset{\perp}{\oplus}\mathrm{N}^2(\mathrm{d},\mathbb{R}^n_+,\Lambda^k).$$

Moreover, the result remains true if we replace $(\mathfrak{t},\mathrm{d},\mathrm{d}^*,\mathbb{P})$ *by* $(\mathfrak{n},\delta^*,\delta,\mathbb{Q})$.

Remark 4.29 The Theorem 4.28 and the whole construction of this section mainly depends on the injectivity of the Laplacian : the construction is done via resolvent approximation, the abstract functional calculus provided by the Hilbertian structure of $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda)$ and the self-adjointness of the Laplacian. Therefore, such construction and the proof do not depend on the open set $\Omega=\mathbb{R}^n_+$.

To be more precise, the above Theorem 4.28 should remain true for all openset Ω, say at least Lipschitz, which have no harmonic forms. In the case of a bounded domain : the theorem remains true whenever all its Betti numbers vanish.

Proof. — **Step 1 :** Identity for $\mathbb{P}$.

From boundedness of above operators, we deduce that the new operator $\overline{\mathbb{P}}$ defined for all $f\in\mathrm{L}^2(\mathbb{R}^n_+,\Lambda)$ by

$$\overline{\mathbb{P}}f:=f-\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}f,$$

is well-defined and bounded on $\mathrm{L}^2(\mathbb{R}^n_+,\Lambda)$. We are going to check that $\overline{\mathbb{P}}$ is an orthogonal projector, hence, firstly a projector.

$$\begin{aligned}\overline{\mathbb{P}}^2f&=\overline{\mathbb{P}}f-\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\overline{\mathbb{P}}f\\&=\overline{\mathbb{P}}f-\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}f+[\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}]^2f\\&=\lim_{\lambda\to0}\left(\overline{\mathbb{P}}f-\mathrm{d}^*\mathrm{d}(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}f+[\mathrm{d}^*\mathrm{d}(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}]^2f\right)\\&=\lim_{\lambda\to0}\left(\overline{\mathbb{P}}f-\lambda\mathrm{d}^*\mathrm{d}(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-2}f\right)\\&=\overline{\mathbb{P}}f.\end{aligned}$$

By construction and by Lemma 4.27, $\overline{\mathbb{P}}$ is self-adjoint, hence orthogonal.

For all $f\in\mathrm{D}_2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda)$,

$$\begin{aligned}\mathrm{d}^*\overline{\mathbb{P}}f&=\lim_{\lambda\to0}\left(\mathrm{d}^*f-\mathrm{d}^*\mathrm{d}(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^*(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}f\right)\\&=\lim_{\lambda\to0}\left(\mathrm{d}^*f+\Delta_{\mathcal{H},\mathfrak{t}}(\lambda\mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathrm{d}^*f\right)\\&=\lim_{\lambda\to0}\left(\mathrm{d}^*f-\mathrm{d}^*f\right)=0.\end{aligned}$$

Thus, since the embedding $\mathrm{D}_2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda)\hookrightarrow\mathrm{L}^2(\mathbb{R}^n_+,\Lambda)$ is dense, it follows that

$$\mathrm{R}_2(\overline{\mathbb{P}},\mathbb{R}^n_+,\Lambda)\subset\mathrm{N}_2(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda).$$

For all $f \in \mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda)$

$$\begin{aligned}\overline{\mathbb{P}} f &= \lim_{\lambda \to 0} \left(f - \mathrm{d}(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} \mathrm{d}^* (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f \right) \\ &= \lim_{\lambda \to 0} \left(f - \mathrm{d}(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1} \mathrm{d}^* f \right) \\ &= \lim_{\lambda \to 0} (f + 0) = f.\end{aligned}$$

Hence, $\mathbb{P}_{|_{\mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda)}} = \mathrm{I}$.

By construction, we also have $\mathrm{R}_2(\mathrm{I} - \overline{\mathbb{P}}) = \mathrm{R}_2(\mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} \mathrm{d}^* (-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}) \subset \overline{\mathrm{R}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda)}$ and obviously $\overline{\mathbb{P}}^* = \overline{\mathbb{P}}$, so that by uniqueness of the orthogonal projection on $\mathrm{N}_2(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda)$, $\overline{\mathbb{P}} = \mathbb{P}$.

Step 2 : We notice first that the inclusion $\overline{\mathrm{R}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)}^{\|\cdot\|_{\mathrm{L}^2(\mathbb{R}^n_+)}} \subset \mathrm{N}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$ is true.
Now, for the reverse inclusion let $f \in \mathrm{N}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$, we have

$$\begin{aligned}[\mathrm{I} - \mathbb{P}] f &= \lim_{\lambda \to 0} \mathrm{d}\mathrm{d}^* (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1} f \\ &= \lim_{\lambda \to 0} -\Delta_{\mathcal{H},\mathfrak{t}} (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1} f \\ &= f.\end{aligned}$$

By construction, for all $\lambda > 0$, we have $\mathrm{d}\mathrm{d}^* (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1} f \in \mathrm{R}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^{k-1})$, so that the reverse inclusion $\mathrm{N}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k) \subset \overline{\mathrm{R}_2(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)}^{\|\cdot\|_{\mathrm{L}^2(\mathbb{R}^n_+)}}$ holds. ■

2.3.3 $\dot{\mathrm{H}}^{s,p}$ and $\dot{\mathrm{B}}^s_{p,q}$-theory for Hodge Laplacians and the Hodge decomposition

We start this new subsection claiming about closedness of the exterior and interior derivatives with and without 0 boundary conditions. The two following lemmas are straightforward.

Lemma 4.30 *Let $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, $k \in [\![0, n]\!]$. With the same notations as in Lemma 4.18, the operators*

$$(\dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k), \mathrm{d}) \text{ and } (\dot{\mathrm{D}}^s_p(\underline{\delta}, \mathbb{R}^n_+, \Lambda^k), \delta)$$

are densely defined closed operators on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda)$.
Moreover,

- *the result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{D}}^s_p)$ by either $(\mathrm{H}^{s,p}, \mathrm{D}^s_p)$, $(\mathrm{B}^s_{p,q}, \mathrm{D}^s_{p,q})$ or $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{D}}^s_{p,q})$, with $q \in [1, +\infty)$;*
- *in case of $(\mathrm{B}^s_{p,\infty}, \mathrm{D}^s_{p,\infty})$ and $(\dot{\mathrm{B}}^s_{p,\infty}, \dot{\mathrm{D}}^s_{p,\infty})$ above, operators are only weak* densely defined, strongly closed operators ;*
- *all the above results remain true, exchanging the roles of d and δ.*

Lemma 4.31 *Let $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, $k \in [\![0, n]\!]$. With the same notations as in Lemma 4.18, the dual operator of $(\dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda), \mathrm{d})$ on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda)$, is*

$$(\dot{\mathrm{D}}^{-s}_{p'}(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda), \mathrm{d}^*) = (\dot{\mathrm{D}}^{-s}_{p'}(\underline{\delta}, \mathbb{R}^n_+, \Lambda), \delta)$$

as an operator on $\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n_+, \Lambda)$.
Moreover,

- *the result still holds, replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{D}}^s_p, \dot{\mathrm{H}}^{-s,p'}, \dot{\mathrm{D}}^{-s}_{p'})$ by $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{D}}^s_{p,q}, \dot{\mathrm{B}}^{-s}_{p',q'}, \dot{\mathrm{D}}^{-s}_{p',q'})$ with $q \in [1, +\infty)$;*
- *we may replace $(\dot{\mathrm{D}}, \dot{\mathrm{H}}, \dot{\mathrm{B}})$ by $(\mathrm{D}, \mathrm{H}, \mathrm{B})$.*
- *all the above results remain true, exchanging the roles of d and δ.*

Remark 4.32 Notice that talking about $\mathrm{D}^s_p(\underline{\mathrm{d}}, \mathbb{R}^n_+, \Lambda^k)$ in above lemmas, with respect to notation introduced in Lemma 4.18, in particular the involved 0-boundary condition, actually makes sense thanks to Theorem 4.15.

Before we start our investigation of Hodge Laplacians and the Hodge decomposition, we need to show the closedness of Hodge-Dirac operators. In order to verify such a property, the next result is of paramount importance to reproduce the behavior obtained in the L^2 setting on other scales of function spaces. We mention that many results presented here strongly depend on the fact that the considered openset is $\mathbb{R}^n_+$ (mainly Lemma 4.33, and point (ii) of Theorem 4.36 which are widely used to construct other results of the present section).

The proof of the next lemma is identical to the one of Lemma 4.24.

Lemma 4.33 *Let $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, $k \in [\![0, n]\!]$. For all $u \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$ (resp. $\dot{\mathrm{D}}^s_p(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k)$) we have*

$$\mathrm{E}_{\mathcal{H},\mathfrak{t}} u \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n, \Lambda^k) \quad \textit{(resp. } \dot{\mathrm{D}}^s_p(\delta, \mathbb{R}^n, \Lambda^k) \textit{)}$$

with formulas

$$\mathrm{d}\mathrm{E}_{\mathcal{H},\mathfrak{t}} u = \mathrm{E}_{\mathcal{H},\mathfrak{t}} \mathrm{d} u \qquad \textit{(resp. } \delta \mathrm{E}_{\mathcal{H},\mathfrak{t}} u = \mathrm{E}_{\mathcal{H},\mathfrak{t}} \mathrm{d}^* u \textit{)}.$$

Moreover,

- *the result still holds, replacing $\dot{\mathrm{D}}^s_p$ by $\dot{\mathrm{D}}^s_{p,q}$ with $q \in [1, +\infty]$;*
- *we may replace $\dot{\mathrm{D}}$ by D.*
- *all the above results remain true, exchanging the roles of d and δ, and replacing $\mathfrak{t}$ by $\mathfrak{n}$.*

Proposition 4.34 *Let $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, $k \in [\![0, n]\!]$. The Hodge-Dirac operator*

$$(\dot{\mathrm{D}}^s_p(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k), D_{\mathfrak{t}}) = (\dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k) \cap \dot{\mathrm{D}}^s_p(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k), \mathrm{d} + \mathrm{d}^*)$$

is a densely defined closed operator $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda)$.

Moreover,

- *the result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{D}}^s_p)$ by either $(\mathrm{H}^{s,p}, \mathrm{D}^s_p)$, $(\mathrm{B}^s_{p,q}, \mathrm{D}^s_{p,q})$ or $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{D}}^s_{p,q})$, with $q \in [1, +\infty)$;*
- *in case of $(\mathrm{B}^s_{p,\infty}, \mathrm{D}^s_{p,\infty})$ and $(\dot{\mathrm{B}}^s_{p,\infty}, \dot{\mathrm{D}}^s_{p,\infty})$ above Hodge-Dirac operator is only weak* densely defined, and strongly closed ;*
- *all above results remain true, replacing $(\mathfrak{t}, \mathrm{d})$ by $(\mathfrak{n}, \delta)$.*

Proof. — Let $(u_j)_{j \in \mathbb{N}} \subset \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k) \cap \dot{\mathrm{D}}^s_p(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k)$, and $(u, v) \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k) \times \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda)$ satisfying

$$u_j \xrightarrow[j \to +\infty]{} u \quad \text{and} \quad D_{\mathfrak{t}} u_j \xrightarrow[j \to +\infty]{} v \text{ in } \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+).$$

We set for all $j \in \mathbb{N}$, $U_j := \mathrm{E}_{\mathcal{H},\mathfrak{t}} u_j$, $U := \mathrm{E}_{\mathcal{H},\mathfrak{t}} u$. By Lemma 4.33, we have for all $j \in \mathbb{Z}$ $U_j \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n, \Lambda^k) \cap \dot{\mathrm{D}}^s_p(\delta, \mathbb{R}^n, \Lambda^k)$

$$DU_j = \mathrm{E}_{\mathcal{H},\mathfrak{t}} D_{\mathfrak{t}} u_j.$$

We also have,

$$DU_j \xrightarrow[j \to +\infty]{} V := \mathrm{E}_{\mathcal{H},\mathfrak{t}} v \text{ in } \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n).$$

By the Hodge decomposition on $\mathbb{R}^n$, check Theorem 4.12, there exists a unique couple $(V_0, V_1) \in \overline{\dot{\mathrm{R}}^s_p}(\mathrm{d}, \mathbb{R}^n, \Lambda) \times \dot{\mathrm{N}}^s_p(\delta, \mathbb{R}^n, \Lambda)$ such that $V = V_0 + V_1$. Since U_j goes to U in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n, \Lambda^k)$, by continuity of involved projectors, and uniqueness of decomposition, it follows that

$$\mathrm{d}U_j \xrightarrow[j\to+\infty]{} V_0 \quad \text{and} \quad \delta U_j \xrightarrow[j\to+\infty]{} V_1 \text{ in } \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n).$$

In particular, if we set $v_\ell := V_{\ell|_{\mathbb{R}^n_+}}$ for $\ell \in \{0,1\}$, we necessarily have by restriction

$$\mathrm{d}u_j \xrightarrow[j\to+\infty]{} v_0 \quad \text{and} \quad \delta u_j \xrightarrow[j\to+\infty]{} v_1 \text{ in } \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+).$$

But $(u_j)_{j\in\mathbb{N}}$ converge to u in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k)$, so in particular in distributional sense. Thus, necessarily $(v_0, v_1) = (\mathrm{d}u, \delta u)$ and $v = Du$. By continuity of trace provided by Theorem 4.14, we also have $\nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} = 0$, *i.e.*, $(\dot{\mathrm{D}}^s_p(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k), D_{\mathfrak{t}})$ is a closed operator on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k)$. The proof ends here, since one can reproduce all above arguments for $(\dot{\mathrm{D}}^s_p(D_{\mathfrak{n}}, \mathbb{R}^n_+, \Lambda^k), D_{\mathfrak{n}})$, and also for all other kind of function spaces. ■

The next result about closedness of Hodge Laplacian admits a similar proof

Proposition 4.35 *Let $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, $k \in [\![0, n]\!]$. The Hodge Laplacian*

$$(\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k), -\Delta_{\mathcal{H},\mathfrak{t}}) = (\dot{\mathrm{D}}^s_p(D^2_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k), D^2_{\mathfrak{t}})$$

is a densely defined closed injective operator on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda)$. For all $u \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k)$, the following formula holds

$$-\Delta \mathrm{E}_{\mathcal{H},\mathfrak{t}} u = \mathrm{E}_{\mathcal{H},\mathfrak{t}}[-\Delta_{\mathcal{H},\mathfrak{t}} u].$$

Moreover,

- *the result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{D}}^s_p)$ by either $(\mathrm{H}^{s,p}, \mathrm{D}^s_p)$, $(\mathrm{B}^s_{p,q}, \mathrm{D}^s_{p,q})$ or $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{D}}^s_{p,q})$, with $q \in [1, +\infty)$;*
- *in case of $(\mathrm{B}^s_{p,\infty}, \mathrm{D}^s_{p,\infty})$ and $(\dot{\mathrm{B}}^s_{p,\infty}, \dot{\mathrm{D}}^s_{p,\infty})$ the Hodge Laplacian is only weak* densely defined, and strongly closed ;*
- *all above results remain true, replacing $\mathfrak{t}$ by $\mathfrak{n}$.*

From there, the whole context has been established in order to be able to claim the next theorem.

Theorem 4.36 *Let $p \in (1, +\infty)$, $q \in [1, +\infty]$, $s \in (-1 + 1/p, 1/p)$, and $k \in [\![0, n]\!]$.*

(i) For $\mu \in [0, \pi)$, $\lambda \in \Sigma_\mu$, if $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k)$ then the following resolvent problem

$$\lambda u - \Delta_{\mathcal{H}} u = f \quad \text{in } \mathbb{R}^n_+,$$

admits a unique solution $u \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H}}, \mathbb{R}^n_+, \Lambda^k) \subset [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n_+, \Lambda^k)$ with the estimates,

$$|\lambda| \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + |\lambda|^{\frac{1}{2}} \|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)},$$
$$|\lambda|^{\frac{1}{2}} \|(\mathrm{d} + \delta) u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\mathrm{d}\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\delta \mathrm{d} u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\mu} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

In particular, $\lambda \mathrm{I} - \Delta_{\mathcal{H}} : \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H}}, \mathbb{R}^n_+, \Lambda^k) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k)$ is an isomorphism of Banach spaces.
Furthermore, the result still holds replacing $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p}, \dot{\mathrm{D}}^s_p)$ by $(\mathrm{H}^{s,p}, \mathrm{H}^{s+2,p}, \mathrm{D}^s_p)$, $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{B}}^s_{p,q} \cap \dot{\mathrm{B}}^{s+2}_{p,q}, \dot{\mathrm{D}}^s_{p,q})$, or even by $(\mathrm{B}^s_{p,q}, \mathrm{B}^{s+2}_{p,q}, \mathrm{D}^s_{p,q})$.

(ii) *For any $\mu \in (0,\pi)$, the operator $-\Delta_{\mathcal{H}}$ admits a bounded (or $\mathbf{H}^\infty(\Sigma_\mu)$-) holomorphic functional calculus on function spaces : $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda^k)$, $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\Lambda^k)$, $\mathrm{H}^{s,p}(\mathbb{R}^n_+,\Lambda^k)$ and $\mathrm{B}^s_{p,q}(\mathbb{R}^n_+,\Lambda^k)$.*
Moreover, the following resolvent identity holds on any previously mentioned function spaces,

$$\mathrm{E}_{\mathcal{H}}(\lambda \mathrm{I}-\Delta_{\mathcal{H}})^{-1}=(\lambda \mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{H}}.$$

Proof. — For $\mu\in[0,\pi)$, $\lambda\in\Sigma_\mu$, if $f\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda^k)$, thanks to (4.20), we have

$$(\lambda \mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{H}}f\in[\dot{\mathrm{H}}^{s,p}\cap\dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n,\Lambda^k).$$

Thus $u:=[(\lambda \mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{H}}f]_{|_{\mathbb{R}^n_+}}\in[\dot{\mathrm{H}}^{s,p}\cap\dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n_+,\Lambda^k)$, satisfies

$$\lambda u-\Delta u=f \text{ in } \mathbb{R}^n_+,$$

with the estimates

$$|\lambda|\|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}+|\lambda|^{\frac12}\|\nabla u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}+\|\nabla^2 u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\lesssim_{p,n,s,\mu}\|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)},$$
$$|\lambda|^{\frac12}\|(\mathrm{d}+\delta)u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}+\|\mathrm{d}\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}+\|\delta\mathrm{d}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\lesssim_{p,n,s,\mu}\|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

By density of $[\mathrm{L}^2\cap\dot{\mathrm{H}}^{s,p}](\mathbb{R}^n_+,\Lambda^k)$ in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda^k)$ one may use Theorem 4.22 and continuity of traces provided by Theorem 4.15 to show that necessarily $\nu\lrcorner u_{|_{\partial\mathbb{R}^n_+}}=0$ and $\nu\lrcorner \mathrm{d}u_{|_{\partial\mathbb{R}^n_+}}=0$ (or resp. $\nu\wedge u_{|_{\partial\mathbb{R}^n_+}}=0$ and $\nu\wedge\delta u_{|_{\partial\mathbb{R}^n_+}}=0$). Hence $u\in\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H}},\mathbb{R}^n_+,\Lambda^k)$.

Now assume $v\in\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H}},\mathbb{R}^n_+,\Lambda^k)$ satisfies

$$\lambda v-\Delta_{\mathcal{H}}v=f \text{ in } \mathbb{R}^n_+.$$

We apply Proposition 4.35 to claim that $V:=\mathrm{E}_{\mathcal{H}}v$ must satisfy

$$\lambda V-\Delta V=\mathrm{E}_{\mathcal{H}}f \text{ in } \mathbb{R}^n.$$

Thus, necessarily $\mathrm{E}_{\mathcal{H}}(\lambda \mathrm{I}-\Delta_{\mathcal{H}})^{-1}f=(\lambda \mathrm{I}-\Delta)^{-1}\mathrm{E}_{\mathcal{H}}f$.

This resolvent identity leads to the construction of bounded ($\mathbf{H}^\infty(\Sigma_\mu)$-)holomorphic functional calculus, given by the following identity for all $\Psi\in\mathbf{H}^\infty(\Sigma_\mu)$, $\mu\in(0,\pi)$:

$$\mathrm{E}_{\mathcal{H}}\Psi(-\Delta_{\mathcal{H}})=\Psi(-\Delta)\mathrm{E}_{\mathcal{H}}.$$

The result for homogeneous Besov spaces $\dot{\mathrm{B}}^s_{p,q}$, $q<+\infty$, and other similar inhomogeneous function spaces may be achieved similarly. The case of inhomogeneous and homogeneous Besov spaces with $q=+\infty$ follows from real interpolation. ∎

The goal for now is to prove the Hodge decomposition. The idea is to prove that the representation formula of $\mathbb{P}$ (resp. $\mathbb{Q}$) proved in Lemma 4.27 still makes sense on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda)$, $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\Lambda)$, and their inhomogeneous counterparts. To do so, we adapt Lemma 4.25 in the present setting.

Lemma 4.37 *Let $p\in(1,+\infty)$, $s\in(-1+1/p,1/p)$, $\mu\in[0,\pi)$, $\lambda\in\Sigma_\mu$, $t\geqslant 0$, $k\in[\![0,n]\!]$. The following commutation identities hold,*

(i) $\mathrm{d}(\lambda \mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}f=(\lambda \mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathrm{d}f$, *for all* $f\in\dot{\mathrm{D}}^s_p(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)$;

(ii) $\mathrm{d}^*(\lambda \mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}f=(\lambda \mathrm{I}-\Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathrm{d}^*f$, *for all* $f\in\dot{\mathrm{D}}^s_p(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda^k)$.

(iii) $\mathrm{d}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}f=e^{t\Delta_{\mathcal{H},\mathfrak{t}}}\mathrm{d}f$, *for all* $f\in\dot{\mathrm{D}}^s_p(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)$;

(iv) $\mathrm{d}^*e^{t\Delta_{\mathcal{H},\mathfrak{t}}}f=e^{t\Delta_{\mathcal{H},\mathfrak{t}}}\mathrm{d}^*f$, *for all* $f\in\dot{\mathrm{D}}^s_p(\mathrm{d}^*,\mathbb{R}^n_+,\Lambda^k)$.

Every above identities still hold, replacing $(\mathfrak{t}, \mathrm{d}, \mathrm{d}^*)$ *by* $(\mathfrak{n}, \delta^*, \delta)$*, and* $\dot{\mathrm{D}}^s_p$ *by either* D^s_p*,* $\mathrm{D}^s_{p,q}$ *or even by* $\dot{\mathrm{D}}^s_{p,q}$*, with* $q \in [1, +\infty]$*.*

Proof. — Let $f \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k) \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k)$, then by Theorem 4.36 there exists a unique $u \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k) \subset \dot{\mathrm{D}}^s_p(D_{\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k) \subset \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$ such that

$$\lambda u - \Delta_{\mathcal{H},\mathfrak{t}} u = f.$$

Since $u, f \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$, we deduce that $\Delta_{\mathcal{H},\mathfrak{t}} u \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$ and we use $\mathrm{d}^2 = 0$ to deduce

$$\lambda \mathrm{d}u - \Delta_{\mathcal{H},\mathfrak{t}} \mathrm{d}u = \mathrm{d}f.$$

But, we have $\mathrm{d}u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^{k+1})$ the solution of

$$\lambda v - \Delta_{\mathcal{H},\mathfrak{t}} v = \mathrm{d}f.$$

Thus, uniqueness of the solution yields $\mathrm{d}u = (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1} \mathrm{d}f$. If it holds for resolvents, then it holds for semigroups. ∎

Proposition 4.38 *Let* $p \in (1, +\infty)$*,* $s \in (-1+1/p, 1/p)$*,* $k \in [\![0, n]\!]$*. For any* $\lambda \geqslant 0$*, the following operators are well-defined and uniformly bounded with respect to* λ

$$\begin{aligned} \mathrm{d}(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} &: \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \dot{\mathrm{N}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^{k+1}); \\ \mathrm{d}^*(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} &: \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k) \longrightarrow \dot{\mathrm{N}}^s_p(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^{k-1}). \end{aligned}$$

Moreover, the following identities also hold for all $\lambda > 0$

- $\mathrm{d}(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f = (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} \mathrm{d}f$ *for all* $f \in \dot{\mathrm{D}}^s_p(\mathrm{d}, \mathbb{R}^n_+, \Lambda^k)$*;*
- $\mathrm{d}^*(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f = (\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} \mathrm{d}^* f$ *for all* $f \in \dot{\mathrm{D}}^s_p(\mathrm{d}^*, \mathbb{R}^n_+, \Lambda^k)$.

Everything still holds, replacing $(\mathfrak{t}, \mathrm{d}, \mathrm{d}^*)$ *by* $(\mathfrak{n}, \delta^*, \delta)$*, and replacing* $(\dot{\mathrm{H}}^{s,p}, \dot{\mathrm{N}}^s_p)$ *by either* $(\dot{\mathrm{B}}^s_{p,q}, \dot{\mathrm{N}}^s_{p,q})$*,* $(\mathrm{H}^{s,p}, \mathrm{N}^s_p)$ *or even by* $(\mathrm{B}^s_{p,q}, \mathrm{N}^s_{p,q})$ *with* $q \in [1, +\infty]$*.*

Proof. — For $\lambda \geqslant 0$, we introduce the representation formula,

$$(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-\tau\lambda} e^{\tau \Delta_{\mathcal{H},\mathfrak{t}}} f \frac{\mathrm{d}\tau}{\sqrt{\tau}}. \tag{4.21}$$

This representation formula makes sense thanks to holomorphic functional calculus, and the integral is absolutely convergent for every for $f \in \dot{\mathrm{R}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k)$.

We use the definition of function spaces by restriction and the bounded holomorphic functional calculus, with the identity provided by point *(ii)* of Theorem 4.36, *i.e.* $\mathrm{E}_{\mathcal{H},\mathfrak{t}} e^{\tau \Delta_{\mathcal{H},\mathfrak{t}}} = e^{\tau \Delta} \mathrm{E}_{\mathcal{H},\mathfrak{t}}$, to obtain

$$\begin{aligned} \|\mathrm{d}(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|\mathrm{d}^*(\lambda \mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}} f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{k,n} \|\nabla(\lambda \mathrm{I} - \Delta)^{-\frac{1}{2}} \mathrm{E}_{\mathcal{H},\mathfrak{t}} f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \\ &\lesssim_{n,k,s,p} \|f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}. \end{aligned}$$

Therefore, the boundedness follows by density of $\dot{\mathrm{R}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{R}^n_+, \Lambda^k)$ in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+, \Lambda^k)$. Commutations relations when $\lambda > 0$ follow from Lemma 4.37 and the representation formula (4.21). The boundedness on the Besov scale follows from real interpolation. ∎

According to more convenient and usual notations with respect to the field of partial differential equations, we set new symbols.

Notations 4.39 We introduce the following notations

$$\mathrm{H}^{s,p}_{\mathfrak{t},\sigma} := \mathrm{N}^s_p(\mathrm{d}^*),\ \mathrm{H}^{s,p}_{\gamma} := \mathrm{N}^s_p(\mathrm{d}) \text{ and } \mathrm{H}^{s,p}_{\sigma} := \mathrm{N}^s_p(\delta),\ \mathrm{H}^{s,p}_{\mathfrak{n},\gamma} := \mathrm{N}^s_p(\delta^*)\,;$$
$$\mathrm{B}^{s,\sigma}_{p,q,\mathfrak{t}} := \mathrm{N}^s_{p,q}(\mathrm{d}^*),\ \mathrm{B}^{s,\gamma}_{p,q} := \mathrm{N}^s_{p,q}(\mathrm{d}) \text{ and } \mathrm{B}^{s,\sigma}_{p,q} := \mathrm{N}^s_p(\delta),\ \mathrm{B}^{s,\gamma}_{p,q,\mathfrak{n}} := \mathrm{N}^s_{p,q}(\delta^*).$$

Then we are able to obtain the following result.

Theorem 4.40 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+1/p, 1/p)$, and let $k \in [\![0,n]\!]$. It holds that*

(i) The following equality holds with equivalence of norms,

$$\dot{\mathrm{N}}^s_p(\mathrm{d},\mathbb{R}^n_+,\Lambda^k) = \overline{\dot{\mathrm{R}}^s_p(\mathrm{d},\mathbb{R}^n_+,\Lambda^k)}^{\|\cdot\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}},$$

and still holds replacing d *by* d^*.

(ii) The (generalized) Helmholtz-Leray projector $\mathbb{P} : \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda^k) \longrightarrow \dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+,\Lambda^k)$ is well-defined and bounded. Moreover, the following identity is true

$$\mathbb{P} = \mathrm{I} - \mathrm{d}(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H},\mathfrak{t}})^{-\frac{1}{2}}.$$

(iii) The following Hodge decomposition holds

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda^k) = \dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+,\Lambda^k) \oplus \dot{\mathrm{H}}^{s,p}_{\gamma}(\mathbb{R}^n_+,\Lambda^k).$$

Moreover, the result remains true if we replace

- $\dot{\mathrm{H}}^{s,p}$ *by* $\dot{\mathrm{B}}^s_{p,q}$ *;*
- $(\dot{\mathrm{H}},\dot{\mathrm{B}})$ *by* (H,B) *;*
- $(\mathfrak{t},\mathrm{d},\mathrm{d}^*,\mathbb{P},\sigma,\gamma)$ *by* $(\mathfrak{n},\delta^*,\delta,\mathbb{Q},\gamma,\sigma)$.

In case of Besov spaces with $q = +\infty$, the density result of point (i) only holds in the weak sense.*

Proof. — One may reproduce the proofs of Theorem 4.28, thanks to above Proposition 4.38. ■

The following corollary is a direct consequence of the given expression for the Helmholtz-Leray projection in Theorem 4.40.

Corollary 4.41 *Let $p \in (1,+\infty)$, $s \in (-1+1/p,1/p)$, $\mu \in [0,\pi)$, $\lambda \in \Sigma_\mu$, $k \in [\![0,n]\!]$. The following commutation identities hold for all $f \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+,\Lambda^k)$, for all $t \geqslant 0$,*

$$(\lambda\mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1}\mathbb{P}f = \mathbb{P}(\lambda\mathrm{I} - \Delta_{\mathcal{H},\mathfrak{t}})^{-1}f,$$
$$e^{t\Delta_{\mathcal{H},\mathfrak{t}}}\mathbb{P}f = \mathbb{P}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}f.$$

Above identity still holds replacing $(\mathfrak{t},\mathbb{P})$ by $(\mathfrak{n},\mathbb{Q})$, and $\dot{\mathrm{H}}^{s,p}$ by either $\mathrm{H}^{s,p}$, $\mathrm{B}^s_{p,q}$ or even by $\dot{\mathrm{B}}^s_{p,q}$, with $q \in [1,+\infty]$.

2.3.4 Hodge-Stokes and Hodge-Maxwell operators

The present subsection is about discussing properties of Hodge-Stokes and Hodge-Maxwell operators. First, one can define ***Hodge-Stokes operator***'s domain, for all $p \in (1,+\infty)$, $s \in (-1+1/p,1/p)$, $k \in [\![0,n]\!]$, by

$$\dot{\mathrm{D}}^s_p(\mathbb{A}_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k) := \dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+,\Lambda^k) \cap \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k), \tag{4.22}$$

and for all $u \in \dot{\mathrm{D}}^s_p(\mathbb{A}_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k)$

$$\mathbb{A}_{\mathcal{H},\mathfrak{t}}u := \mathrm{d}^*\mathrm{d}u = -\mathbb{P}\Delta u = -\Delta\mathbb{P}u = -\Delta u. \tag{4.23}$$

Above operator $\mathbb{A}_{\mathcal{H},\mathfrak{t}}$ is called the ***Hodge-Stokes operator*** with ***absolute boundary conditions*** which is a closed densely defined operator on $\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+,\Lambda^k)$.

Similarly, one can treat the case of the ***Hodge-Maxwell operator***,

$$\dot{\mathrm{D}}^s_p(\mathbb{M}_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k) := \dot{\mathrm{H}}^{s,p}_{\gamma}(\mathbb{R}^n_+,\Lambda^k)\cap \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k), \tag{4.24}$$

and for all $u\in \dot{\mathrm{D}}^s_p(\mathbb{M}_{\mathcal{H},\mathfrak{t}},\mathbb{R}^n_+,\Lambda^k)$

$$\mathbb{M}_{\mathcal{H},\mathfrak{t}}u := \mathrm{d}\mathrm{d}^*u = -[\mathrm{I}-\mathbb{P}]\Delta u = -\Delta[\mathrm{I}-\mathbb{P}]u = -\Delta u. \tag{4.25}$$

The operator $\mathbb{M}_{\mathcal{H},\mathfrak{t}}$ defined as above is called the ***Hodge-Maxwell operator*** with ***perfectly conductive wall boundary conditions*** which is a closed densely defined operator on $\dot{\mathrm{H}}^{s,p}_{\gamma}(\mathbb{R}^n_+,\Lambda^k)$.

Similarly, one may replace $(\mathfrak{t},\mathrm{d},\mathbb{P})$ by $(\mathfrak{n},\delta,\mathbb{Q})$, respectively in, (4.22) and (4.23), and in (4.24) and (4.25). This leads to the construction of

$$(\dot{\mathrm{D}}^s_p(\mathbb{A}_{\mathcal{H},\mathfrak{n}},\mathbb{R}^n_+,\Lambda^k),\mathbb{A}_{\mathcal{H},\mathfrak{n}}) \quad\text{and}\quad (\dot{\mathrm{D}}^s_p(\mathbb{M}_{\mathcal{H},\mathfrak{n}},\mathbb{R}^n_+,\Lambda^k),\mathbb{M}_{\mathcal{H},\mathfrak{n}})$$

called respectively the ***Hodge-Stokes operator*** with ***relative boundary conditions*** and the ***Hodge-Maxwell operator*** with ***relative boundary conditions,*** which are both closed densely defined operator on $\dot{\mathrm{H}}^{s,p}_{\sigma}(\mathbb{R}^n_+,\Lambda^k)$ and $\dot{\mathrm{H}}^{s,p}_{\mathfrak{n},\gamma}(\mathbb{R}^n_+,\Lambda^k)$, respectively.

Those Hodge-Stokes and Hodge-Maxwell operators still have sense on other scale of function spaces replacing $(\dot{\mathrm{H}}^{s,p},\dot{\mathrm{D}}^s_p)$ by $(\dot{\mathrm{B}}^s_{p,q},\dot{\mathrm{D}}^s_{p,q})$, then $(\dot{\mathrm{H}},\dot{\mathrm{B}},\dot{\mathrm{D}})$ by $(\mathrm{H},\mathrm{B},\mathrm{D})$.

Notice again the exception of Besov spaces, homogeneous and inhomogeneous, where the domains of any Hodge-Stokes and Hodge-Maxwell operators are only weak* dense in the case $q=+\infty$.

With the above definitions, Theorem 4.36, Corollary 4.41 and Theorem 4.40, we obtain for free the next theorem.

Theorem 4.42 *Let $p\in(1,+\infty)$, $s\in(-1+1/p,1/p)$. For all $\mu\in(0,\pi)$, the operator $\mathbb{A}_{\mathcal{H},\mathfrak{t}}$ (resp. $\mathbb{M}_{\mathcal{H},\mathfrak{t}}$) admits a bounded (or $\mathbf{H}^\infty(\Sigma_\mu)$-)holomorphic functional calculus on $\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+,\Lambda)$ (resp. $\dot{\mathrm{H}}^{s,p}_{\gamma}(\mathbb{R}^n_+,\Lambda)$).*
Moreover, the result remains true if we replace

- $\dot{\mathrm{H}}^{s,p}$ *by* $\dot{\mathrm{B}}^s_{p,q}$, $q\in[1,+\infty]$;
- $(\dot{\mathrm{H}},\dot{\mathrm{B}})$ *by* (H,B) ;
- $(\mathfrak{t},\sigma,\gamma,\mathbb{A},\mathbb{M})$ *by* $(\mathfrak{n},\gamma,\sigma,\mathbb{M},\mathbb{A})$.

3 L^q-maximal regularities with global-in-time estimates

Before going further, we want to simplify notations. From now on, we only consider function spaces on $\mathbb{R}^n_+$ and no longer on $\mathbb{R}^n$, so that we drop the mention of the open set in domains of operators, we all also drop the mention of degree of differential forms, except when it is necessary. Any discussion involving Dirichlet and Neumann Laplacians will always contain the implicit information that their domains are made of scalar valued functions, whereas talking about Hodge Laplacians and their derived operators will always contain the implicit information we are talking about general differential forms valued functions of any degree, unless it is explicitly stated.

A first aim of this section is about to give an explicit description of homogeneous interpolation spaces, provided $\theta\in(0,1)$, $q\in[1,+\infty]$, see [DHMT21, Proposition 2.12],

$$(\mathrm{X},\mathrm{D}(\mathring{A}))_{\theta,q} = \mathring{\mathcal{D}}_A(\theta,q), \tag{4.26}$$

where $X=\dot{\mathrm{H}}^{s,p},\dot{\mathrm{B}}^s_{p,r}$ and $A\in\{-\Delta_{\mathcal{H}},\mathbb{A}_{\mathcal{H}},\mathbb{M}_{\mathcal{H}}\}$, with $p\in(1,+\infty)$, $-1+1/p<s<1/p$, $r\in[1,+\infty)$. The main task here is about to compute (4.26), in order to compute above space for $A=-\Delta_{\mathcal{H}}$. Indeed, for the related Hodge-Stokes operator, due to the commutations

relations between the Hodge Laplacian and its Helmholtz-Leray projection, see (4.23), (4.25) and Corollary 4.41, we should have (at least formally or up to a dense subset)

$$\mathring{\mathcal{D}}^{s,p}_{\mathbb{A}_{\mathcal{H},\mathfrak{t}}}(\theta,q)=(\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q}=\mathbb{P}(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathfrak{t}}))_{\theta,q}=\mathbb{P}\mathring{\mathcal{D}}^{s,p}_{-\Delta_{\mathcal{H},\mathfrak{t}}}(\theta,q).$$

Obviously similar identities can be obtained with $(\mathfrak{n},\mathbb{Q})$ instead of $(\mathfrak{t},\mathbb{P})$, but also for the Hodge-Maxwell operators up to appropriate changes.

Secondly, we aim to recover global-in-time L^q-maximal regularity estimates for the abstract Cauchy problem (ACP), provided $T\in(0,+\infty]$, for $A\in\{-\Delta_{\mathcal{H}},\mathbb{A}_{\mathcal{H}},\mathbb{M}_{\mathcal{H}}\}$, so that we are able to apply Theorems 3.6 and 3.21.

3.1 Interpolation of homogeneous $\dot{\mathrm{H}}^{s,p}$-domains of operators

We start this section claiming that one can reduce the problem to the computation of interpolation spaces

$$\mathring{\mathcal{D}}^{s,p}_{-\Delta_{\mathcal{D}}}(\theta,q)\ \text{ and }\ \mathring{\mathcal{D}}^{s,p}_{-\Delta_{\mathcal{N}}}(\theta,q).$$

We recall here for convenience that $-\Delta_{\mathcal{D}}$ and $-\Delta_{\mathcal{N}}$ stands respectively for the (negative) Dirichlet and the Neumann Laplacian on the half-space for which a review of their properties in homogeneous function spaces was achieved in the Section 6 of the previous Chapter 2.

To see it, one may stare at Theorem 4.22, Propositions 2.58 and 2.60, formula (4.19), Lemma 4.33 and Theorem 4.36 until the next lemma becomes quite clear

Lemma 4.43 *Let $p\in(1,+\infty)$, $s\in(-1+1/p,1/p)$, $k\in[\![0,n]\!]$. For all $u\in\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}},\Lambda^k)$, we have*

$$\Delta_{\mathcal{H},\mathfrak{t}}u=\sum_{I\in\mathcal{I}^k_{n-1}}\Delta_{\mathcal{N}}u_I\,\mathrm{d}x_I+\sum_{I'\in\mathcal{I}^{k-1}_{n-1}}\Delta_{\mathcal{D}}u_{I',n}\,\mathrm{d}x_{I'}\wedge\mathrm{d}x_n.$$

We also have estimates,

$$\begin{aligned}\|\delta\mathrm{d}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}+\|\mathrm{d}\delta u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}&\sim_{p,s,n}\sum_{I\in\mathcal{I}^k_{n-1}}\|\Delta_{\mathcal{N}}u_I\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}+\sum_{I'\in\mathcal{I}^{k-1}_{n-1}}\|\Delta_{\mathcal{D}}u_{I',n}\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\\&\sim_{p,s,n}\|\nabla^2u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\\&\sim_{p,s,n}\|u\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)}.\end{aligned}$$

The result still holds replacing $(\mathfrak{t},\mathcal{N},\mathcal{D})$ by $(\mathfrak{n},\mathcal{D},\mathcal{N})$.

And for the same reasons, one has more generally,

Lemma 4.44 *Let $p\in(1,+\infty)$, $s\in(-1+1/p,1/p)$, $\alpha\in[0,2]$, such that $s+\alpha\neq1/p,1+1/p$, $k\in[\![0,n]\!]$. For all $u\in\dot{\mathrm{D}}^s_p((-\Delta_{\mathcal{H}})^{\frac{\alpha}{2}},\Lambda)$, we have*

$$\|(-\Delta_{\mathcal{H}})^{\frac{\alpha}{2}}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\sim_{p,s,\alpha,n}\|u\|_{\dot{\mathrm{H}}^{s+\alpha,p}(\mathbb{R}^n_+)}\sim_{p,s,\alpha,n}\|(-\Delta_{\mathcal{H}})^{\frac{s+\alpha}{2}}u\|_{\mathrm{L}^p(\mathbb{R}^n_+)}.$$

Recalling that in particular, $\Delta_{\mathcal{H},\mathfrak{t}|_{\Lambda^0}}=\Delta_{\mathcal{N}}$ and $\Delta_{\mathcal{H},\mathfrak{n}|_{\Lambda^0}}=\Delta_{\mathcal{D}}$.

In general, explicit description for interpolation spaces with boundary condition may be quite tedious. We mention the work of Guidetti [Gui91a ; Gui91b], where such investigation is done. Guidetti's results were used to make a extensive treatment of elliptic boundary value problem with general Lopatinskii-Shapiro boundary conditions in inhomogeneous Besov spaces on the half-space and on bounded domains with smooth boundary.

Thanks to Lemmas 4.43 and 4.44, the current work is reduced to Dirichlet and Neumann boundary conditions in the homogeneous case, which is unknown to the author's knowledge yet.

For $p\in(1,+\infty)$, $q\in[1,+\infty]$, $s\in(-1+1/p,2+1/p)$, such that $(\mathcal{C}_{s,p,q})$ is satisfied, we set :

$$\dot{\mathrm{B}}^s_{p,q,\mathcal{D}}(\mathbb{R}^n_+):=\begin{cases}\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) & ,\text{ if } (s<\frac{1}{p}),\\ \left\{u\in\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)\,\middle|\,u_{|_{\partial\mathbb{R}^n_+}}=0\right\} & ,\text{ if } (s>\frac{1}{p}) \text{ or } (s=\frac{1}{p} \text{ and } q=1).\end{cases}$$

$$\dot{\mathrm{B}}^s_{p,q,\mathcal{N}}(\mathbb{R}^n_+):=\begin{cases}\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+) & ,\text{ if } (s<1+\frac{1}{p}),\\ \left\{u\in\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)\,\middle|\,\partial_\nu u_{|_{\partial\mathbb{R}^n_+}}=0\right\} & ,\text{ if } (s>1+\frac{1}{p}) \text{ or } (s=1+\frac{1}{p} \text{ and } q=1).\end{cases}$$

and similarly $(\mathcal{C}_{s,p})$ is satisfied, we also set :

$$\dot{\mathrm{H}}^{s,p}_{\mathcal{D}}(\mathbb{R}^n_+):=\begin{cases}\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) & ,\text{ if } s<\frac{1}{p},\\ \left\{u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)\,\middle|\,u_{|_{\partial\mathbb{R}^n_+}}=0\right\} & ,\text{ if } s>\frac{1}{p}.\end{cases}$$

$$\dot{\mathrm{H}}^{s,p}_{\mathcal{N}}(\mathbb{R}^n_+):=\begin{cases}\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) & ,\text{ if } s<1+\frac{1}{p},\\ \left\{u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)\,\middle|\,\partial_\nu u_{|_{\partial\mathbb{R}^n_+}}=0\right\} & ,\text{ if } s>1+\frac{1}{p}.\end{cases}$$

And then, for $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$, we introduce the following subspace

$$\mathrm{Y}_{\mathcal{J}}:=\bigcap_{\substack{s\in(-1+1/p,1/p)\\ p\in(1,+\infty)\\ q\in[1,+\infty]}}[\mathrm{D}^s_p\cap\dot{\mathrm{D}}^s_p\cap\mathrm{D}^s_{p,q}\cap\dot{\mathrm{D}}^s_{p,q}](\Delta_{\mathcal{J}}).$$

Proposition 4.45 *Let $p\in(1,+\infty)$, $q\in[1,+\infty)$, $s\in(-1+1/p,2+1/p)$, such that $(\mathcal{C}_{s,p,q})$ is satisfied, we have that*

$$\dot{\mathrm{B}}^s_{p,q,\mathcal{J}}(\mathbb{R}^n_+)=\overline{\mathrm{Y}_{\mathcal{J}}}^{\|\cdot\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)}},$$

whenever

- $\mathcal{J}=\mathcal{D}$, $s\neq 1/p,1+1/p$;
- $\mathcal{J}=\mathcal{N}$, $s\neq 1+1/p$.

When $q=+\infty$, we still have weak density.*

Proof. — We recall that for all $\tilde{p}\in(1,+\infty)$, $\tilde{q}\in[1,+\infty)$, $\tilde{s}\in\mathbb{R}$, such that $(\mathcal{C}_{s,p,q})$ is satisfied, we have that

$$\dot{\mathrm{B}}^{\tilde{s}}_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+)=\overline{\mathcal{S}_0(\overline{\mathbb{R}^n_+})}^{\|\cdot\|_{\dot{\mathrm{B}}^{\tilde{s}}_{\tilde{p},\tilde{q}}(\mathbb{R}^n_+)}}.$$

- First, assume that $s\in(1+1/p,2+1/p)$, for $u\in\dot{\mathrm{B}}^s_{p,q,\mathcal{J}}(\mathbb{R}^n_+)$, we set $f:=-\Delta_{\mathcal{J}}u\in\dot{\mathrm{B}}^{s-2}_{p,q}(\mathbb{R}^n_+)$. For $(f_j)_{j\in\mathbb{N}}\subset\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ such that

$$f_j\xrightarrow[j\to+\infty]{}f,\qquad\text{in }\dot{\mathrm{B}}^{s-2}_{p,q}(\mathbb{R}^n_+).$$

  We set for all $\lambda>0$,

$$u_{\lambda,j}:=(\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f_j\in\mathrm{Y}_{\mathcal{J}}$$

  where belonging to space $\mathrm{Y}_{\mathcal{J}}$ is a consequence of Propositions 2.58 and 2.60. For $\mu,\lambda>0$,

by Lemma 4.43, we have

$$\begin{aligned}\|u_{j,\lambda}-u_{j,\mu}\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} &\lesssim_{s,p,q,n} \left\|-\Delta_{\mathcal{J}}(\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f_j+\Delta_{\mathcal{J}}(\mu\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f_j\right\|_{\dot{\mathrm{B}}^{s-2}_{p,q}(\mathbb{R}^n_+)}\\ &\lesssim_{s,p,q,n} \left\|-\Delta_{\mathcal{J}}(\lambda\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f_j-f_j\right\|_{\dot{\mathrm{B}}^{s-2}_{p,q}(\mathbb{R}^n_+)}\\ &\quad+\left\|f_j+\Delta_{\mathcal{J}}(\mu\mathrm{I}-\Delta_{\mathcal{J}})^{-1}f_j\right\|_{\dot{\mathrm{B}}^{s-2}_{p,q}(\mathbb{R}^n_+)} \xrightarrow[\lambda,\mu\to 0]{} 0.\end{aligned}$$

By uniqueness of the solution for the Neumann (resp. Dirichlet) problem provided by Proposition 2.62 (resp. Proposition 2.61), $(u_{\mu,j})_{\mu>0}$ is a Cauchy net that admits a limit that must be the unique solution u_j. The strong continuity of $(-\Delta_{\mathcal{J}})^{-1}$ concludes the case of $s\in(1+1/p,2+1/p)$.

- For $s\in(1/p,1+1/p)$, we consider first the case $\mathcal{J}=\mathcal{N}$. Again, for $u\in\dot{\mathrm{B}}^s_{p,q,\mathcal{N}}(\mathbb{R}^n_+)=\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, we can introduce $(u_j)_{j\in\mathbb{N}}\subset\mathcal{S}_0(\overline{\mathbb{R}^n_+})$ such that

$$u_j\xrightarrow[j\to+\infty]{\dot{\mathrm{B}}^s_{p,q}}u.$$

We set for all $\tau>0$,

$$u_{\tau,j}:=(\mathrm{I}-\tau\Delta_{\mathcal{J}})^{-1}u_j\in\mathrm{Y}_{\mathcal{J}}$$

where belonging to space $\mathrm{Y}_{\mathcal{J}}$ is a consequence of Propositions 2.58 and 2.60. It is direct to see that

$$u_{\tau,j}\xrightarrow[\tau\to 0]{}u_j\xrightarrow[j\to+\infty]{}u\quad\text{in }\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+).$$

For the case $\mathcal{J}=\mathcal{D}$, since $\dot{\mathrm{B}}^s_{p,q,\mathcal{D}}(\mathbb{R}^n_+)=\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)$ and $\mathrm{C}^\infty_c(\mathbb{R}^n_+)\subset\mathrm{Y}_{\mathcal{D}}$, the result follows from Lemma 2.32.

This argument still works for $s\in(-1+1/p,1+1/p)$, when $\mathcal{J}=\mathcal{N}$.

- Finally $s\in(-1+1/p,1/p)$, we notice that $\dot{\mathrm{B}}^s_{p,q,\mathcal{J}}(\mathbb{R}^n_+)=\dot{\mathrm{B}}^s_{p,q,0}(\mathbb{R}^n_+)=\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$. Since $\mathrm{C}^\infty_c(\mathbb{R}^n_+)\subset\mathrm{Y}_{\mathcal{J}}$, the result follows from Corollary 2.34. ■

The next result has a similar proof, left to the reader.

Proposition 4.46 *Let $p\in(1,+\infty)$, $s_0,s_1\in(1/p,2+1/p)$, $\mathcal{J}\in\{\mathcal{D},\mathcal{N}\}$ such that $(\mathcal{C}_{s_0,p})$ is satisfied, we have*

$$[\dot{\mathrm{H}}^{s_0,p}_{\mathcal{J}}\cap\dot{\mathrm{H}}^{s_1,p}](\mathbb{R}^n_+)=\overline{\mathrm{Y}_{\mathcal{J}}}^{\|\cdot\|_{[\dot{\mathrm{H}}^{s_0,p}\cap\dot{\mathrm{H}}^{s_1,p}](\mathbb{R}^n_+)}},$$

whenever $1/p<s_0,s_1<2+1/p$, except for $s=1+1/p$ when $\mathcal{J}=\mathcal{N}$.

The next lemma is inspired from [Gui91b, Lemma 2.4].

Lemma 4.47 *Let $p_j\in(1,+\infty)$, $q_j\in[1,+\infty)$, and $s_j>1/p_j$, for $j\in\{0,1\}$. Let T be the map*

$$T:f\longmapsto\left[(x',x_n)\mapsto e^{-x_n(-\Delta')^{\frac{1}{2}}}f(x')\right].$$

(i) Assume $s_j\in(1/p_j,1+2/p_j)$, for $j\in\{0,1\}$. Then the operator defined formally by

$$\mathcal{P}_{\mathcal{D}}u:=u-T[u_{|_{\partial\mathbb{R}^n_+}}],$$

is such that

a) *If* $(\mathcal{C}_{s_0,p_0})$ *is satisfied, then* $\mathcal{P}_{\mathcal{D}} : [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+) \longrightarrow [\dot{\mathrm{H}}^{s_0,p_0}_{\mathcal{D}} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$ *is a well defined linear and bounded projection. For all* $u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\mathbb{R}^n_+)$ *the following estimate is true*

$$\|\mathcal{P}_{\mathcal{D}} u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)} \lesssim_{p_j,s_j,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n_+)} \quad , j \in \{0,1\}.$$

b) *If instead* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied, then the above statement still holds with* $(\dot{\mathrm{B}}^{s_0}_{p_0,q_0}, \dot{\mathrm{B}}^{s_1}_{p_1,q_1})$ *replacing* $(\dot{\mathrm{H}}^{s_0,p_0}, \dot{\mathrm{H}}^{s_1,p_1})$.
We also have that $\mathcal{P}_{\mathcal{D}} : \dot{\mathrm{B}}^{s_0}_{p_0,\infty}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{B}}^{s_0}_{p_0,\infty,\mathcal{D}}(\mathbb{R}^n_+)$ *is also well-defined linear and bounded.*

(ii) *Assume* $s_j \in (1 + 1/p_j, 1 + 2/p_j)$, *for* $j \in \{0,1\}$. *Then the operator defined formally by*

$$\mathcal{P}_{\mathcal{N}} u := u + (-\Delta')^{-\frac{1}{2}} T[\partial_{x_n} u_{|_{\partial\mathbb{R}^n_+}}],$$

satisfies points (i)(a) and (i)(b) with $\mathcal{N}$ *instead of* $\mathcal{D}$.

Proof. — This is a direct consequence of Proposition 2.43 and Corollary 2.44. ∎

Proposition 4.48 *Let* $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, $\mathcal{J} \in \{\mathcal{D}, \mathcal{N}, \mathcal{H}\}$, *then* $(\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}}), -\Delta_{\mathcal{J}})$ *satisfies Assumptions* (3.4) *and* (3.5). *In other words,* $-\Delta_{\mathcal{J}}$ *is injective on* $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, *and we can define*

$$\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}) := \{u \in \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+) \,|\, \exists (u_j)_{j\in\mathbb{N}} \subset \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}}),\ \|u - u_j\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \xrightarrow[j\to+\infty]{} 0\}$$

such that it also satisfies

$$\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}) = \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}}). \tag{4.27}$$

The result still holds with either $\mathbb{A}_{\mathcal{H},\mathfrak{t}}$ *(resp.* $\mathbb{M}_{\mathcal{H},\mathfrak{t}}$*) on* $\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+)$ *(resp.* $\dot{\mathrm{H}}^{s,p}_{\gamma}(\mathbb{R}^n_+)$*), and similarly replacing* $(\mathfrak{t}, \sigma, \gamma, \mathbb{A}, \mathbb{M})$ *by* $(\mathfrak{n}, \gamma, \sigma, \mathbb{M}, \mathbb{A})$.

Proof. — We only show (4.27). The following inclusion is clear

$$\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}}) \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}).$$

Now, let $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}})$, then obviously

$$u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+).$$

It suffices to show that u has appropriate boundary conditions. We assume here that $\mathcal{J} = \mathcal{D}$, other cases would be achieved similarly. Let $(u_j)_{j\in\mathbb{N}} \subset \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{D}})$, such that

$$\|u - u_j\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \xrightarrow[j\to+\infty]{} 0.$$

Since $u - u_j \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)$, one may apply Proposition 2.52, and use $u_{j_{|\partial\mathbb{R}^n_+}} = 0$ to obtain

$$\|u_{|\partial\mathbb{R}^n_+}\|_{\dot{\mathrm{B}}^{s+2-1/p}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s,p,n} \|u - u_j\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \xrightarrow[j\to+\infty]{} 0.$$

Therefore $u_{|\partial\mathbb{R}^n_+} = 0$ so that $u \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{D}})$. ∎

The Proposition 4.48 tells us that, for all $p \in (1, +\infty)$, $s \in (-1 + 1/p, 1/p)$, it makes sense to

consider the semigroup,

$$e^{t\Delta_{\mathcal{H}}} \ : \ \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) + \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H}}) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) + \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H}}),$$

thanks to [DHMT21, Chapter 2, Section 1].

For convenience of notations, and for later use, one may think about Lemma 4.43, we also set for all $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+1/p, 1+2/p)$, such that $(\mathcal{C}_{s,p,q})$ is satisfied, and for $k \in [\![0,n]\!]$,

$$\dot{\mathrm{B}}^s_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda^k) := \begin{cases} \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\Lambda^k) & , \ s < \frac{1}{p}, \\ \left\{ u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\Lambda^k) \,\middle|\, \mathfrak{e}_n \lrcorner u_{|_{\partial\mathbb{R}^n_+}} = 0 \right\} & , \ \frac{1}{p} < s < 1+\frac{1}{p}, \\ \left\{ u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\Lambda^k) \,\middle|\, \mathfrak{e}_n \lrcorner u_{|_{\partial\mathbb{R}^n_+}}, \ \mathfrak{e}_n \lrcorner \mathrm{d}u_{|_{\partial\mathbb{R}^n_+}} = 0 \right\} & , \ 1+\frac{1}{p} < s < 2+\frac{1}{p}. \end{cases} \tag{4.28}$$

It is not difficult to see from point *(iii)* of Theorem 4.15, that,

$$\dot{\mathrm{B}}^s_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda^k) \simeq \dot{\mathrm{B}}^s_{p,q,\mathcal{D}}(\mathbb{R}^n_+)^{\binom{n-1}{k-1}} \times \dot{\mathrm{B}}^s_{p,q,\mathcal{N}}(\mathbb{R}^n_+)^{\binom{n-1}{k}},$$

for which one may check for instance the Step 3 of Theorem 4.15's proof.

One can also build in the same fashion $\dot{\mathrm{B}}^s_{p,q,\mathcal{H}_{\mathfrak{n}}}(\mathbb{R}^n_+,\Lambda^k)$, with boundary conditions $\nu \wedge u_{|_{\partial\mathbb{R}^n_+}} = 0$ and $\nu \wedge \delta u_{|_{\partial\mathbb{R}^n_+}} = 0$, so that

$$\dot{\mathrm{B}}^s_{p,q,\mathcal{H}_{\mathfrak{n}}}(\mathbb{R}^n_+,\Lambda^k) \simeq \dot{\mathrm{B}}^s_{p,q,\mathcal{N}}(\mathbb{R}^n_+)^{\binom{n-1}{k-1}} \times \dot{\mathrm{B}}^s_{p,q,\mathcal{D}}(\mathbb{R}^n_+)^{\binom{n-1}{k}}.$$

We denote by $\dot{\mathrm{B}}^s_{p,q,\mathcal{H}}(\mathbb{R}^n_+)$, either $\dot{\mathrm{B}}^s_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda)$ or $\dot{\mathrm{B}}^s_{p,q,\mathcal{H}_{\mathfrak{n}}}(\mathbb{R}^n_+,\Lambda)$.

Proposition 4.49 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, and $s \in (-1+1/p, 1/p)$. For all $\theta \in (0,1)$ such that $(\mathcal{C}_{s+2\theta,p,q})$ is satisfied, provided $\mathcal{J} \in \{\mathcal{D},\mathcal{N},\mathcal{H}\}$, one has*

$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^n_+),$$

with equivalence of norms, whenever

- $s+2\theta \neq 1/p$ *if* $\mathcal{J} = \mathcal{D}$,
- $s+2\theta \neq 1+1/p$ *if* $\mathcal{J} = \mathcal{N}$,
- $s+2\theta \neq 1/p, 1+1/p$ *if* $\mathcal{J} = \mathcal{H}$.

The proof is heavily inspired from the one of [DHMT21, Proposition 4.12].

Proof. — **Step 1 :** We start applying Proposition 2.33 which yields the embedding

$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q} \hookrightarrow (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+).$$

Now, if $q < +\infty$, we recall that $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}) = \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}})$ is dense in $(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q}$ by Proposition 1.3, so that by continuity of traces,

$$(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q} \hookrightarrow \dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^n_+).$$

The case $q = +\infty$ will be done in later steps.

Step 2 : The reverse embedding when $s+2\theta \in (-1+1/p, 1/p)$. Let $f \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}})$, then for all $t > 0$

$$f = e^{t\Delta_{\mathcal{J}}} f + \mathring{\Delta}_{\mathcal{J}} \int_0^t \tau e^{\tau\Delta_{\mathcal{J}}} f \frac{\mathrm{d}\tau}{\tau} =: b + a$$

with obviously $f \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}}) \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) + \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}})$ and by definition of the K-functional, we obtain

$$K(t, f, \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}})) \leqslant \|a\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + t\|\Delta_{\mathcal{J}} b\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

So as in the proof of [DHMT21, Proposition 4.12], we may apply [DHMT21, Lemma 2.11] so that

$$\|f\|_{(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q}} \leqslant \frac{1+\theta}{\theta} \left(\int_0^{+\infty} \|t^{1-\theta} \Delta_{\mathcal{J}} e^{t\Delta_{\mathcal{J}}} f\|^q_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}. \tag{4.29}$$

Now, for $0 < \theta < \varepsilon$, such that $s + 2\theta < s + 2\varepsilon < 1/p$ we want to bound the L^q_*-norm of $\|t^{1-\theta} \Delta_{\mathcal{J}} e^{t\Delta_{\mathcal{J}}} f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}$ by the L^q_*-norm of the K-functional associated with the real interpolation space $(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+))_{\frac{\theta}{\varepsilon},q}$.

Let $(\tilde{a}, \tilde{b}) \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \times \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)$, such that $f = \tilde{a} + \tilde{b}$, the fact that $f \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}})$ implies $\tilde{a}, \tilde{b} \in [\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2\varepsilon,p}](\mathbb{R}^n_+)$. By Lemma 4.44, since $s + 2\varepsilon < 1/p$ we have

$$\tilde{b} \in \dot{\mathrm{D}}^s_p((-\Delta_{\mathcal{J}})^{\varepsilon})$$

Therefore, since the semigroup $(e^{t\Delta_{\mathcal{J}}})_{t>0}$ is analytic, by the use of Lemma 4.44, we have

$$\begin{aligned} \|t^{1-\theta} \Delta_{\mathcal{J}} e^{t\Delta_{\mathcal{J}}} f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\leqslant \|t^{1-\theta} \Delta_{\mathcal{J}} e^{t\Delta_{\mathcal{J}}} \tilde{a}\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|t^{1-\theta} \Delta_{\mathcal{J}} e^{t\Delta_{\mathcal{J}}} \tilde{b}\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \\ &\lesssim_{n,p,s} t^{-\theta} \left(\|\tilde{a}\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + t^{\varepsilon} \|\tilde{b}\|_{\dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)} \right). \end{aligned}$$

Taking the infimum of all such $\tilde{a}, \tilde{b}$, yields

$$\|t^{1-\theta} \Delta_{\mathcal{J}} e^{t\Delta_{\mathcal{J}}} f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} t^{-\theta} K(t^{\varepsilon}, f, \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)).$$

Therefore one may take the L^q_*-norm on both sides, and use Proposition 2.33, to obtain for all $f \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}})$,

$$\|f\|_{(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q}} \sim_{p,s,n,\theta,\varepsilon} \|f\|_{(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2\varepsilon}(\mathbb{R}^n_+))_{\frac{\theta}{\varepsilon},q}} \sim_{p,s,n,\theta,\varepsilon} \|f\|_{\dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+)}.$$

If $q \in [1, +\infty)$, the result follows from Corollary 2.34, since $\mathrm{C}^\infty_c(\mathbb{R}^n_+) \subset \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{J}})$. The case $q = +\infty$ is obtained via the reiteration theorem, Theorem 1.5.

Step 3 : The reverse embedding when $s + 2\theta \in (1/p, 2 + 1/p)$, $\mathcal{J} = \mathcal{D}$. Provided $f \in \mathrm{Y}_{\mathcal{D}}$, as introduced in Proposition 4.45, we may reproduce the above Step 2 up to (4.29). From there, for $0 < \eta < \theta$ such that $1/p < s + 2\eta < s + 2\theta$, we want to prove that one can bound (4.29) by the L^q_*-norm of the K-functional associated with the real interpolation space $(\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+))_{\frac{\theta-\eta}{1-\eta},q}$.

Since $f \in \mathrm{Y}_{\mathcal{D}} \subset \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)$, for $a, b \in \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \times \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)$ such that $f = a + b$, we get

$$a = f - b \in \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \cap (\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+) + \mathrm{Y}_{\mathcal{D}}) \subset \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+),$$

and the same arguments leads to $b \in \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)$. From Lemma 4.47, we have

$$f = \mathcal{P}_{\mathcal{D}} f = \mathcal{P}_{\mathcal{D}} a + \mathcal{P}_{\mathcal{D}} b$$

where $\mathcal{P}_{\mathcal{D}} a, \mathcal{P}_{\mathcal{D}} b \in \dot{\mathrm{H}}^{s+2\eta,p}_{\mathcal{D}}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)$ with the estimates

$$\|\mathcal{P}_{\mathcal{D}} a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} \lesssim_{p,s,\eta,n} \|a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} \text{ and } \|\mathcal{P}_{\mathcal{D}} b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \lesssim_{p,s,n} \|b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)}.$$

Therefore, by the estimate above, the analyticity of the semigroup $(e^{t\Delta_{\mathcal{D}}})_{t>0}$, and Lemma 4.43, we are able to obtain

$$\begin{aligned}\|t^{1-\theta}\Delta_{\mathcal{D}}e^{t\Delta_{\mathcal{D}}}f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\leqslant \|t^{1-\theta}\Delta_{\mathcal{D}}e^{t\Delta_{\mathcal{D}}}\mathcal{P}_{\mathcal{D}}a\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|t^{1-\theta}\Delta_{\mathcal{D}}e^{t\Delta_{\mathcal{D}}}\mathcal{P}_{\mathcal{D}}b\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\\ &\lesssim_{n,p,s} t^{-(\theta-\eta)}\|\mathcal{P}_{\mathcal{D}}a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} + t^{1-\theta}\|\mathcal{P}_{\mathcal{D}}b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)}\\ &\lesssim_{n,p,s} t^{-(\theta-\eta)}\left(\|a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} + t^{1-\eta}\|b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)}\right).\end{aligned}$$

Taking the infimum of all such couple (a,b), yields

$$\|t^{1-\theta}\Delta_{\mathcal{D}}e^{t\Delta_{\mathcal{D}}}f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} t^{-(\theta-\eta)}K(t^{1-\eta}, f, \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)).$$

As in the Step 2, one may take the L^q_*-norm on both sides, and use Proposition 2.33, to obtain for all $f \in \mathrm{Y}_{\mathcal{D}}$,

$$\|f\|_{(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{D}}))_{\theta,q}} \sim_{p,s,n,\theta,\eta} \|f\|_{(\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)))_{\frac{\theta-\eta}{1-\eta},q}} \sim_{p,s,n,\theta,\eta} \|f\|_{\dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+)}.$$

If $q \in [1,+\infty)$, the result follows from Proposition 4.45. The case $q = +\infty$ is obtained via the application of the reiteration theorem, Theorem 1.5, by the mean of Lemma 4.47.

Step 4 : The reverse embedding $s+2\theta \in [1/p, 1+1/p)$, $\mathcal{J} = \mathcal{N}$. One may pick $f \in \mathrm{Y}_{\mathcal{N}}$ so that, as before, we can reproduce above Step 2 up to (4.29). From there, for $0 < \eta < \theta < \varepsilon$ such that $1/p < s+2\eta < s+2\theta < s+2\varepsilon < 1+1/p$, we want to prove that one can bound (4.29) by the L^q_*-norm of the K-functional associated with the interpolation space $(\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+))_{\frac{\theta-\eta}{\varepsilon-\eta},q}$.

Since $f \in \mathrm{Y}_{\mathcal{D}} \subset \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)$, for $a, b \in \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \times \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)$ such that $f = a + b$, we get

$$b = f - a \in \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \cap (\dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+) + \mathrm{Y}_{\mathcal{N}}) \subset \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+).$$

By Proposition 4.46, there exists sequences $(a_j)_{j\in\mathbb{N}}, (b_j)_{j\in\mathbb{N}} \subset Y_{\mathcal{N}}$ such that

$$\|a_j - a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} + \|b_j - b\|_{[\dot{\mathrm{H}}^{s+2\eta,p}\cap\dot{\mathrm{H}}^{s+2\varepsilon,p}](\mathbb{R}^n_+)} \xrightarrow[j\to+\infty]{} 0.$$

Therefore, the analyticity of the semigroup $(e^{t\Delta_{\mathcal{N}}})_{t>0}$ and Lemma 4.43 works together to deliver

$$\begin{aligned}\|t\Delta_{\mathcal{N}}e^{t\Delta_{\mathcal{N}}}a_j\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,\eta} t^{\eta}\|a_j\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)},\\ \|t\Delta_{\mathcal{N}}e^{t\Delta_{\mathcal{N}}}b_j\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,\varepsilon} t^{\varepsilon}\|b_j\|_{\dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)},\end{aligned}$$

so that taking limits, it yields

$$\begin{aligned}\|t\Delta_{\mathcal{N}}e^{t\Delta_{\mathcal{N}}}a\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,\eta} t^{\eta}\|a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)},\\ \|t\Delta_{\mathcal{N}}e^{t\Delta_{\mathcal{N}}}b\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,\varepsilon} t^{\varepsilon}\|b\|_{\dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)}.\end{aligned} \tag{4.30}$$

Therefore, by the estimates (4.30), the following holds

$$\|t^{1-\theta}\Delta_{\mathcal{N}}e^{t\Delta_{\mathcal{N}}}f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{n,p,s,\eta,\varepsilon} t^{-(\theta-\eta)}\left(\|a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} + t^{\varepsilon-\eta}\|b\|_{\dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)}\right).$$

From there, we can take the infimum of all such couple (a,b), and we see that

$$\|t^{1-\theta}\Delta_{\mathcal{N}}e^{t\Delta_{\mathcal{N}}}f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\eta,\varepsilon} t^{-(\theta-\eta)}K(t^{\varepsilon-\eta}, f, \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+), \dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)).$$

As in the Step 2, one may take the L^q_*-norm on both sides, and use Proposition 2.33, to obtain

for all $f \in \mathrm{Y}_{\mathcal{N}}$,

$$\|f\|_{(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{N}}))_{\theta,q}} \sim_{p,s,n,\theta,\eta,\varepsilon} \|f\|_{(\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s+2\varepsilon,p}(\mathbb{R}^n_+)))_{\frac{\theta-\eta}{\varepsilon-\eta},q}} \sim_{p,s,n,\theta,\eta,\varepsilon} \|f\|_{\dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+)}.$$

If $q \in [1,+\infty)$, the result follows from Proposition 4.45. The case $q = +\infty$ is obtained via the application of the reiteration theorem, Theorem 1.5. The case $s = 1/p$ follows from reiteration theorem, Theorem 1.5, between the Step 2 and this one.

Step 5 : The reverse embedding $s+2\theta \in (1+1/p, 2+1/p)$, $\mathcal{J} = \mathcal{N}$. For $f \in \mathrm{Y}_{\mathcal{N}}$, we reproduce again the Step 2 up to (4.29). Now let $0 < \eta < \theta$ such that $1 + 1/p < s + 2\eta < s + 2\theta < 2 + 1/p$, we want to achieve the same estimate obtained at the end of Step 3.

Since $f \in \mathrm{Y}_{\mathcal{N}} \subset \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)$, for $a, b \in \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \times \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)$ such that $f = a + b$, we get

$$b = f - a \in \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \cap (\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+) + \mathrm{Y}_{\mathcal{N}}) \subset \dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+) \cap \dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+).$$

We want to fall in the expected homogeneous domains, *i.e.* to get back the Neumann boundary condition, to do so, we use Lemma 4.47, and we get

$$f = \mathcal{P}_{\mathcal{N}} f = \mathcal{P}_{\mathcal{N}} a + \mathcal{P}_{\mathcal{N}} b,$$

with the estimates

$$\|\mathcal{P}_{\mathcal{N}} a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\eta} \|a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} \text{ and } \|\mathcal{P}_{\mathcal{N}} b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s,\eta} \|b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)}.$$

By Proposition 4.46, there exists sequences $(\mathfrak{a}_j)_{j\in\mathbb{N}}, (\mathfrak{b}_j)_{j\in\mathbb{N}} \subset Y_{\mathcal{N}}$ such that

$$\|\mathfrak{a}_j - \mathcal{P}_{\mathcal{N}} a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} + \|\mathfrak{b}_j - \mathcal{P}_{\mathcal{N}} b\|_{[\dot{\mathrm{H}}^{s+2\eta,p} \cap \dot{\mathrm{H}}^{s+2,p}](\mathbb{R}^n_+)} \xrightarrow[j\to+\infty]{} 0.$$

as in the Step 4, we obtain

$$\begin{aligned} \|t\Delta_{\mathcal{N}} e^{t\Delta_{\mathcal{N}}} \mathcal{P}_{\mathcal{N}} a\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,\eta} t^{\eta} \|a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)}, \\ \|t\Delta_{\mathcal{N}} e^{t\Delta_{\mathcal{N}}} \mathcal{P}_{\mathcal{N}} b\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,} t \|b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)}. \end{aligned}$$

Therefore, by the estimates above, the following estimate holds

$$\|t^{1-\theta}\Delta_{\mathcal{N}} e^{t\Delta_{\mathcal{N}}} f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{n,p,s} t^{-(\theta-\eta)} \left(\|a\|_{\dot{\mathrm{H}}^{s+2\eta,p}(\mathbb{R}^n_+)} + t^{1-\eta} \|b\|_{\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+)} \right).$$

Finally one may finish the present Step 5 with the same arguments present in the Step 3.

Step 6 : The case $\mathcal{J} = \mathcal{H}$. Let $k \in [\![0, n]\!]$, from Lemma 4.43, we deduce that the following holds with equivalence of norms

$$\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathrm{t}}, \Lambda^k) \simeq \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{D}})^{\binom{n-1}{k-1}} \times \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{N}})^{\binom{n-1}{k}}$$

The result is then immediate, by all above steps. The case of the Hodge Laplacian with generalized normal boundary conditions admits a similar proof. ■

Finally, we want to compute interpolation spaces for the Hodge-Stokes and the Hodge-Maxwell operators. To do so, we set for all $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+1/p, 2+1/p)$, such that

($\mathcal{C}_{s,p,q}$) is satisfied, provided, $k \in [\![0,n]\!]$,

$$\dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda^k) := \begin{cases} \dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathfrak{t}}(\mathbb{R}^n_+,\Lambda^k) & , s<\frac{1}{p}, \\ \{\, u\in \dot{\mathrm{B}}^{s}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda^k) \,|\, \delta u = 0\,\} & , \frac{1}{p}<s<2+\frac{1}{p},\ s\neq 1+1/p. \end{cases} \tag{4.31}$$

$$\dot{\mathrm{B}}^{s,\gamma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda^k) := \begin{cases} \dot{\mathrm{B}}^{s,\gamma}_{p,q}(\mathbb{R}^n_+,\Lambda^k) & , s<\frac{1}{p}, \\ \{\, u\in \dot{\mathrm{B}}^{s}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda^k) \,|\, \mathrm{d} u = 0\,\} & , \frac{1}{p}<s<2+\frac{1}{p},\ s\neq 1+1/p. \end{cases} \tag{4.32}$$

One may build similarly $\dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathcal{H}_{\mathfrak{n}}}(\mathbb{R}^n_+,\Lambda^k)$ and $\dot{\mathrm{B}}^{s,\gamma}_{p,q,\mathcal{H}_{\mathfrak{n}}}(\mathbb{R}^n_+,\Lambda^k)$ replacing $(\mathfrak{t},\sigma,\gamma,\mathrm{d},\delta)$ by $(\mathfrak{n},\gamma,\sigma,\delta,\mathrm{d})$.

Proposition 4.50 *Let $p\in(1,+\infty)$, $q\in[1,+\infty]$, and $s\in(-1+1/p,1/p)$. For all $\theta\in(0,1)$ such that ($\mathcal{C}_{s+2\theta,p,q}$) is satisfied, one has*

$$(\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta,\sigma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+), \tag{4.33}$$

$$(\dot{\mathrm{H}}^{s,p}_{\gamma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{M}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta,\gamma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+), \tag{4.34}$$

with equivalence of norms, whenever $s+2\theta\neq 1/p, 1+1/p$.

The same result holds replacing $(\mathfrak{t},\sigma,\gamma,\mathbb{A},\mathbb{M})$ by $(\mathfrak{n},\gamma,\sigma,\mathbb{M},\mathbb{A})$.

Proof. — We only prove (4.33), other equalities have the same proof.

Step 1 : We start with Proposition 2.33 which yields the embedding

$$(\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q} \hookrightarrow (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\dot{\mathrm{H}}^{s+2,p}(\mathbb{R}^n_+))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+).$$

Now, if $q<+\infty$, we recall that $\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+)\cap\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}})$ is dense in $(\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q}$ by Proposition 1.3, so that by continuity of traces,

$$(\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q} \hookrightarrow \dot{\mathrm{B}}^{s+2\theta,\sigma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+) \hookrightarrow \dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+).$$

Again density of $\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+)\cap\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}})$ yields $\delta f=0$ for all $f\in(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q}$. The case $q=+\infty$ is left to the end of Step 3.

Step 2 : We want to extend the range of exponents for the boundedness of $\mathbb{P}$, and get a density result.

Let $f\in\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}})\subset\dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+)$, we have $\mathbb{P}f\in\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}})$ and by Proposition 4.49, [DHMT21, Proposition 2.12], Corollary 4.41 and Theorem 4.40, we obtain successively

$$\begin{aligned}
\|\mathbb{P}f\|_{\dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,\theta} \left(\int_0^{+\infty}\|t^{1-\theta}\Delta_{\mathcal{H},\mathfrak{t}}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}\mathbb{P}f\|^q_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\lesssim_{p,s,n,\theta} \left(\int_0^{+\infty}\|t^{1-\theta}\mathbb{P}\Delta_{\mathcal{H},\mathfrak{t}}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}f\|^q_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\lesssim_{p,s,n,\theta} \left(\int_0^{+\infty}\|t^{1-\theta}\Delta_{\mathcal{H},\mathfrak{t}}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}f\|^q_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{q}} \\
&\lesssim_{p,s,n,\theta} \|f\|_{\dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+)}.
\end{aligned}$$

From the estimates above, if $q<+\infty$, by density of $\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}})$ in $\dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+)$, we have that

$$\mathbb{P}\,:\,\dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+)\longrightarrow\dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+)$$

extends uniquely to a bounded linear projection on $\dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+)$ with range $\dot{\mathrm{B}}^{s+2\theta,\sigma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+)$. The result still holds for $q=+\infty$, by above Step 1, the reiteration theorem, Theorem 1.5, and Proposition 4.49.

In particular, $\dot{\mathrm{D}}^s_p(\mathbb{A}_{\mathcal{H},\mathfrak{t}})=\mathbb{P}\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}})$ is dense in $\dot{\mathrm{B}}^{s+2\theta,\sigma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+)$, when $q<+\infty$.

Step 3 : For the reverse embedding. For $f \in \dot{\mathrm{D}}^s_p(\mathbb{A}_{\mathcal{H},\mathfrak{t}}) \subset \dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+)$ where we recall that $\dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+) = (\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathfrak{t}}))_{\theta,q} \subset \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) + \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathfrak{t}})$

If we let $(a,b) \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+) \times \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathfrak{t}})$ such that $f = a+b$, by Proposition 4.48, it is given that

$$b = f - a \in (\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}}) + \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)) \cap \dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathfrak{t}}) \subset \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}})$$

and for the same reason $a \in \dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H},\mathfrak{t}})$. Therefore,

$$f = \mathbb{P}f = \mathbb{P}a + \mathbb{P}b \in \dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+) + \dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}).$$

By (4.23) and Corollary 4.41, we have

$$\begin{aligned}\|t^{1-\theta}\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}e^{-t\mathbb{A}_{\mathcal{H},\mathfrak{t}}}f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} &\leqslant \|t^{1-\theta}\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}e^{-t\mathbb{A}_{\mathcal{H},\mathfrak{t}}}\mathbb{P}a\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|t^{1-\theta}\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}e^{-t\mathbb{A}_{\mathcal{H},\mathfrak{t}}}\mathbb{P}b\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\\ &\lesssim_{p,n,s} \|t^{1-\theta}\mathbb{P}\Delta_{\mathcal{H},\mathfrak{t}}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}a\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + \|t^{1-\theta}\mathbb{P}\Delta_{\mathcal{H},\mathfrak{t}}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}b\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.\end{aligned}$$

From there, we use analyticity of the semigroup, boundedness of $\mathbb{P}$ given by Theorem 4.40, to obtain

$$\|t^{1-\theta}\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}e^{-t\mathbb{A}_{\mathcal{H},\mathfrak{t}}}f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} t^{-\theta}\|a\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} + t^{1-\theta}\|\Delta_{\mathcal{H},\mathfrak{t}}b\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}.$$

Taking the infimum on all such pairs (a,b) yields

$$\|t^{1-\theta}\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}e^{-t\mathbb{A}_{\mathcal{H},\mathfrak{t}}}f\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,n,s} t^{-\theta}K(t,f,\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathfrak{t}})).$$

One may take the L^q_*-norm of above inequality and apply [DHMT21, Lemma 2.12] and above Proposition 4.49, to deduce that

$$\begin{aligned}\|f\|_{(\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q}} &\lesssim_\theta \left(\int_0^{+\infty} \|t^{1-\theta}\Delta_{\mathcal{H},\mathfrak{t}}e^{t\Delta_{\mathcal{H},\mathfrak{t}}}\mathbb{P}f\|^q_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\frac{\mathrm{d}t}{t}\right)\\ &\lesssim_{p,n,s,\theta} \|f\|_{(\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\Delta}_{\mathcal{H},\mathfrak{t}}))_{\theta,q}}\\ &\lesssim_{p,n,s,\theta} \|f\|_{\dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+)}.\end{aligned}$$

With the Step 1, one has for all $f \in \dot{\mathrm{D}}^s_p(\mathbb{A}_{\mathcal{H},\mathfrak{t}})$,

$$\|f\|_{(\dot{\mathrm{H}}^{s,p}_{\mathfrak{t},\sigma}(\mathbb{R}^n_+),\dot{\mathrm{D}}^s_p(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q}} \sim_{p,n,s,\theta} \|f\|_{\dot{\mathrm{B}}^{s+2\theta}_{p,q}(\mathbb{R}^n_+)}.$$

If $q < +\infty$, then the end of the above Step 2, and [DHMT21, Lemma 2.10] allows concluding by density. If $q = +\infty$, the result follows from the reiteration theorem, Theorem 1.5, and the boundedness and the range of $\mathbb{P}$ in the Step 2 (use a retraction argument, see Theorem 1.21). ■

The Step 2 from the proof above leads to the immediate following corollary.

Corollary 4.51 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+1/p, 2+1/p)$, such that $s \notin \mathbb{N} + \frac{1}{p}$, and $(\mathcal{C}_{s,p,q})$ is satisfied. Then,*

$$\begin{aligned}\mathbb{P} &: \dot{\mathrm{B}}^s_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+),\\ [\mathrm{I}-\mathbb{P}] &: \dot{\mathrm{B}}^s_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+) \longrightarrow \dot{\mathrm{B}}^{s,\gamma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+),\end{aligned}$$

are both well-defined bounded linear projection, so that the following Hodge decomposition holds

$$\dot{\mathrm{B}}^s_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+) = \dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+) \oplus \dot{\mathrm{B}}^{s,\gamma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+).$$

The result still holds if we replace $(\mathfrak{t},\mathbb{P})$ by $(\mathfrak{n},\mathbb{Q})$.

Finally, we mention without its proofs, that follows exactly the same lines, the result for interpolation spaces for the homogeneous operator with a Besov space as an ambient (or ground) function space, say, for $\theta \in (0,1)$, $r,q \in [1,+\infty]$, $p \in (1,+\infty)$, $-1+1/p < s < 1/p$,

$$\mathring{\mathcal{D}}^{s,p,r}_{-\Delta_{\mathcal{H}}}(\theta,q) = (\dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_{p,r}(\mathring{\Delta}_{\mathcal{H}}))_{\theta,q}.$$

We are able to obtain,

Proposition 4.52 *Let $p \in (1,+\infty)$, $r \in [1,+\infty)$ $q \in [1,+\infty]$, and $s \in (-1+1/p, 1/p)$. For all $\theta \in (0,1)$ such that $(\mathcal{C}_{s+2\theta,p,q})$ is satisfied, provided $\mathcal{J} \in \{\mathcal{D},\mathcal{N},\mathcal{H}\}$, one has*

$$(\dot{\mathrm{B}}^s_{p,r}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_{p,r}(\mathring{\Delta}_{\mathcal{J}}))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta}_{p,q,\mathcal{J}}(\mathbb{R}^n_+),$$

with equivalence of norms, whenever

- $s+2\theta \neq 1/p$ *if* $\mathcal{J} = \mathcal{D}$,
- $s+2\theta \neq 1+1/p$ *if* $\mathcal{J} = \mathcal{N}$,
- $s+2\theta \neq 1/p, 1+1/p$ *if* $\mathcal{J} = \mathcal{H}$.

Then by mean of Corollary 4.51,

Proposition 4.53 *Let $p \in (1,+\infty)$,$r \in [1,+\infty)$, $q \in [1,+\infty]$, and $s \in (-1+1/p, 1/p)$. For all $\theta \in (0,1)$ such that $(\mathcal{C}_{s+2\theta,p,q})$ is satisfied, one has*

$$(\dot{\mathrm{B}}^{s,\sigma}_{p,r,\mathfrak{t}}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_{p,r}(\mathring{\mathbb{A}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta,\sigma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+),$$
$$(\dot{\mathrm{B}}^{s,\gamma}_{p,r}(\mathbb{R}^n_+), \dot{\mathrm{D}}^s_{p,r}(\mathring{\mathbb{M}}_{\mathcal{H},\mathfrak{t}}))_{\theta,q} = \dot{\mathrm{B}}^{s+2\theta,\gamma}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+),$$

with equivalence of norms, whenever $s+2\theta \neq 1/p, 1+1/p$.

The same result holds replacing $(\mathfrak{t},\sigma,\gamma,\mathbb{A},\mathbb{M})$ by $(\mathfrak{n},\gamma,\sigma,\mathbb{M},\mathbb{A})$.

3.2 Maximal regularities for Hodge Laplacians and related operators

We present here direct application of Theorem 3.21, and [DHMT21, Theorem 2.20] with appropriate identification of real interpolation spaces, provided $p,r \in (1,+\infty)$, $s \in (-1+1/p, 1/p)$,

$$\mathring{\mathcal{D}}^{s,p}_{A}(\theta,q),\ \mathring{\mathcal{D}}^{s,p,r}_{A}(\theta,q),\ \theta \in (0,1),\ q \in [1,+\infty] \text{ and } A \in \{-\Delta_{\mathcal{H}}, \mathbb{A}_{\mathcal{H}}, \mathbb{M}_{\mathcal{H}}\}$$

subject to either tangential or normal boundary conditions, see Propositions 4.49, 4.50, 4.52 and 4.53.

We recall that the definition of involved spaces are given in Notations 4.39, see also (4.28) and (4.31). To alleviate notations in inequalities, we drop the references to the open set $\mathbb{R}^n_+$.

We give the first two Theorems in the case where the ambient space is an UMD Banach space which is the case of $\dot{\mathrm{H}}^{s,p}$ and $\dot{\mathrm{B}}^s_{p,r}$, provided $p,r \in (1,+\infty)$, $s \in (-1+1/p, 2+1/p)$.

Theorem 4.54 *Let $p,q,r \in (1,+\infty)$, and for $\alpha \in (-1+1/q, 1/q)$ fixed, we set $\alpha_q := 1+\alpha-1/q$.*

Let $s \in (-1+1/p, 2+1/p)$ such that $s, s+2\alpha_q \notin \mathbb{N}+\frac{1}{p}$, $(\mathcal{C}_{s+2\alpha_q,p,q})$ is satisfied, and let $T \in (0,+\infty]$.

For any $f \in \dot{\mathrm{H}}^{\alpha,q}((0,T), \dot{\mathrm{B}}^s_{p,r,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda))$, $u_0 \in \dot{\mathrm{B}}^{s+2\alpha_q}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda)$, there exists a unique mild solution $u \in \mathrm{C}^0_b([0,T], \dot{\mathrm{B}}^{s+2\alpha_q}_{p,q,\mathcal{H}_{\mathfrak{t}}}(\mathbb{R}^n_+,\Lambda))$ of

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u &=& f, & \text{on } (0,T)\times\mathbb{R}^n_+,\\ \nu \lrcorner \mathrm{d}u_{|_{\partial\mathbb{R}^n_+}} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+,\\ \nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+,\\ u(0) &=& u_0, & \text{in } \dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}(\mathbb{R}^n_+,\Lambda), \end{array}\right. \tag{HHS$_{\mathfrak{t}}$}$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty((0,T),\dot{\mathrm{B}}^{s+2\alpha_q}_{p,q})} \lesssim^{\alpha,s}_{p,q,n} \left\|(\partial_t u, \nabla^2 u)\right\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{B}}^{s}_{p,r})} \lesssim^{\alpha,s}_{p,q,n} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{B}}^{s}_{p,r})} + \|u_0\|_{\dot{\mathrm{B}}^{s+2\alpha_q}_{p,q}} .$$

For all $\beta \in [0,1]$, we also have

$$\left\|(-\partial_t)^\beta (-\Delta_{\mathcal{H},\mathfrak{t}})^{1-\beta} u\right\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{B}}^{s}_{p,r})} \lesssim^{s,\alpha}_{p,q,n} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{B}}^{s}_{p,r})} + \|u_0\|_{\dot{\mathrm{B}}^{s+2\alpha_q}_{p,q}} . \tag{4.35}$$

Proof. — From Theorem 4.36 we have the bounded holomorphic calculus of $-\Delta_{\mathcal{H},\mathfrak{t}}$ on $\dot{\mathrm{B}}^{s}_{p,r,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+)$, so that we may apply Theorem 3.21 to obtain maximal regularity estimates, whereas Proposition 4.49 gives an exact description of interpolation spaces. ∎

Theorem 4.55 *Let $p,q \in (1,+\infty)$, $s \in (-1+1/p, 1/p)$, and $\alpha \in (-1+1/q, 1/q)$ fixed, we set $\alpha_q := 1+\alpha-1/q$. We assume that $s+2\alpha_q \notin \mathbb{N}+\frac{1}{p}$, $(\mathcal{C}_{s+2\alpha_q,p,q})$ is satisfied, and let $T \in (0,+\infty]$.*

For any $f \in \dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{H}}^{s,p}_{\mathfrak{n},\gamma}(\mathbb{R}^n_+,\Lambda))$, $u_0 \in \dot{\mathrm{B}}^{s+2\alpha_q,\gamma}_{p,q,\mathcal{H}_\mathfrak{n}}(\mathbb{R}^n_+,\Lambda)$, there exists a unique mild solution $u \in \mathrm{C}^0_b([0,T],\dot{\mathrm{B}}^{s+2\alpha_q,\gamma}_{p,q,\mathcal{H}_\mathfrak{n}}(\mathbb{R}^n_+,\Lambda))$ of

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u &=& f, & \text{on } (0,T)\times\mathbb{R}^n_+,\\ \mathrm{d}u &=& 0, & \text{on } (0,T)\times\mathbb{R}^n_+,\\ \nu \wedge \delta u_{|\partial\mathbb{R}^n_+} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+,\\ \nu \wedge u_{|\partial\mathbb{R}^n_+} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+,\\ u(0) &=& u_0, & \text{in } \dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}(\mathbb{R}^n_+,\Lambda), \end{array}\right. \tag{$\mathrm{HMS}_\mathfrak{n}$}$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty((0,T),\dot{\mathrm{B}}^{s+2\alpha_q}_{p,q})} \lesssim^{s,\alpha}_{p,q,n} \left\|(\partial_t u, \nabla^2 u)\right\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{H}}^{s,p})} \lesssim^{s,\alpha}_{p,q,n} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{H}}^{s,p})} + \|u_0\|_{\dot{\mathrm{B}}^{s+2\alpha_q}_{p,q}} .$$

For all $\beta \in [0,1]$, we also have

$$\left\|(-\partial_t)^\beta (\mathbb{M}_{\mathcal{H},\mathfrak{n}})^{1-\beta} u\right\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{H}}^{s,p})} \lesssim^{s,\alpha}_{p,q,n} \|f\|_{\dot{\mathrm{H}}^{\alpha,q}((0,T),\dot{\mathrm{H}}^{s,p})} + \|u_0\|_{\dot{\mathrm{B}}^{s+2\alpha_q}_{p,q}} . \tag{4.36}$$

Proof. — From Theorem 4.42 we have the bounded holomorphic calculus of $\mathbb{M}_{\mathcal{H},\mathfrak{n}}$ on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, so that we may apply Theorem 3.21 to obtain maximal regularity estimates, whereas Proposition 4.52 gives an exact description of interpolation spaces. ∎

Finally, it remains to apply the homogeneous Da Prato-Grisvard Theorem [DHMT21, Theorem 2.20] to state our last L^q-maximal regularity theorem.

Theorem 4.56 *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$, $s \in (-1+1/p, 1/p+2/q)$, such that $s, s+2-2/q \notin \mathbb{N}+\frac{1}{p}$ and $(\mathcal{C}_{s+2-2/q,p,q})$ is satisfied and let $T \in (0,+\infty]$.*

For any $f \in \mathrm{L}^q((0,T),\dot{\mathrm{B}}^{s,\sigma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+,\Lambda))$, $u_0 \in \dot{\mathrm{B}}^{2+s-\frac{2}{q},\sigma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+,\Lambda)$, there exists a unique mild solution $u \in \mathrm{C}^0_b([0,T],\dot{\mathrm{B}}^{2+s-\frac{2}{q},\sigma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+,\Lambda))$ of

$$\left\{\begin{array}{rcll} \partial_t u - \Delta u &=& f, & \text{on } (0,T)\times\mathbb{R}^n_+,\\ \delta u &=& 0, & \text{on } (0,T)\times\mathbb{R}^n_+,\\ \nu \lrcorner \mathrm{d}u_{|\partial\mathbb{R}^n_+} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+,\\ \nu \lrcorner u_{|\partial\mathbb{R}^n_+} &=& 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+,\\ u(0) &=& u_0, & \text{in } \dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}(\mathbb{R}^n_+,\Lambda), \end{array}\right. \tag{$\mathrm{HSS}_\mathfrak{t}$}$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty([0,T],\dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q})} + \big\|(\partial_t u, \nabla^2 u)\big\|_{\mathrm{L}^q((0,T),\dot{\mathrm{B}}^s_{p,q})} \lesssim^s_{p,q,n} \|f\|_{\mathrm{L}^q((0,T),\dot{\mathrm{B}}^s_{p,q})} + \|u_0\|_{\dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}}.$$

In the case $q = +\infty$, if we assume in addition $u_0 \in \dot{\mathrm{D}}^s_p(\mathbb{A}^2_{\mathcal{H},\mathfrak{t}})$, we have

$$\big\|(\partial_t u, \nabla^2 u)\big\|_{\mathrm{L}^\infty([0,T],\dot{\mathrm{B}}^s_{p,\infty})} \lesssim_{p,s,n} \|f\|_{\mathrm{L}^\infty((0,T),\dot{\mathrm{B}}^s_{p,\infty})} + \|\mathbb{A}_{\mathcal{H},\mathfrak{t}} u_0\|_{\dot{\mathrm{B}}^s_{p,\infty}}.$$

Remark 4.57 Notice that above Theorem 4.56 is the only one presented here that allows L^1 and L^∞ in time maximal regularity estimates.
In particular, one should notice that in the case $q = 1$, that above solution u satisfies for almost every $t \in \mathbb{R}_+$,

$$u(t),\ \partial_t u(t),\ \nabla^2 u(t) \in \dot{\mathrm{B}}^s_{p,1}(\mathbb{R}^n_+).$$

Proof. — We may apply Theorem 3.6 to obtain maximal regularity estimates, since Proposition 4.50 gives an exact description of interpolation spaces. ∎

Remark 4.58 One may perform a cyclic permutation of systems ($\mathrm{HHS}_\mathfrak{t}$), ($\mathrm{HMS}_\mathfrak{n}$) and ($\mathrm{HSS}_\mathfrak{t}$), but also exchange $\mathfrak{t}$ and $\mathfrak{n}$, up to appropriate modification on boundary conditions and considered function spaces, to obtain each type of results for each operator

$$\{-\Delta_{\mathcal{H},\mathfrak{t}}, \mathbb{A}_{\mathcal{H},\mathfrak{t}}, \mathbb{M}_{\mathcal{H},\mathfrak{t}}, -\Delta_{\mathcal{H},\mathfrak{n}}, \mathbb{A}_{\mathcal{H},\mathfrak{n}}, \mathbb{M}_{\mathcal{H},\mathfrak{n}}\}.$$

3.3 Maximal regularity for the Stokes system with Navier-slip boundary conditions

The flatness of $\partial\mathbb{R}^n_+$ has the interesting consequence that, for $u : \mathbb{R}^n_+ \longrightarrow \mathbb{C}^n \simeq \Lambda^1$ regular enough, the tangential Hodge boundary conditions

$$\left\{\begin{array}{rcl} \nu \lrcorner u_{|_{\partial\mathbb{R}^n_+}} &=& 0, \\ \nu \lrcorner \mathrm{d}u_{|_{\partial\mathbb{R}^n_+}} &=& 0. \end{array}\right.$$

are equivalent to the Navier-slip boundary conditions

$$\left\{\begin{array}{rcl} \nu \cdot u_{|_{\partial\mathbb{R}^n_+}} &=& 0, \\ [({}^t\nabla u + \nabla u)\nu]_{\tan|_{\partial\mathbb{R}^n_+}} &=& 0. \end{array}\right. \tag{4.37}$$

Indeed, recalling that $\nu = -\mathfrak{e}_n$, since one has

$$\begin{aligned}[({}^t\nabla u + \nabla u)\nu]_{\tan} &= ({}^t\nabla u + \nabla u)(-\mathfrak{e}_n) - [({}^t\nabla u + \nabla u)(-\mathfrak{e}_n)\cdot(-\mathfrak{e}_n)](-\mathfrak{e}_n) \\ &= -\sum_{k=1}^{n-1}(\partial_{x_k} u_n + \partial_{x_n} u_k)\mathfrak{e}_k.\end{aligned}$$

We may use $-\mathfrak{e}_n \cdot u_{|_{\partial\mathbb{R}^n_+}} = u_n(\cdot,0) = 0$, yielding for all $k \in [\![1, n-1]\!]$

$$0 = \sum_{k=1}^{n-1}(\partial_{x_k} u_n(\cdot,0) + \partial_{x_n} u_k(\cdot,0))\mathfrak{e}_k = \sum_{k=1}^{n-1}\partial_{x_n} u_k(\cdot,0)\mathfrak{e}_k.$$

This implies that u satisfies the exact same $n-1$ Neumann boundary conditions, and a single Dirichlet boundary condition on u_n. This stands exactly as in Lemma 4.43. The converse also holds.

As long as one has enough regularity on u at least in the Sobolev / Besov sense on $\mathbb{R}^n_+$, one is still able to perform the same decoupling for the boundary conditions.

This occurs when $s \in (-1+1/p, 1+1/p)$, $p \in (1,+\infty)$ for the spaces $\mathrm{H}^{s+2,p}$, $\dot{\mathrm{H}}^{s,p} \cap \dot{\mathrm{H}}^{s+2,p}$, $\dot{\mathrm{H}}^{s+2,p}$ when those are complete. It still occurs when we replace $\mathrm{H}^{\cdot,p}$ by $\mathrm{B}^{\cdot}_{p,q}$, $q \in [1,+\infty]$.

Therefore, in each of the previous definitions restricted to $\Lambda^1 \simeq \mathbb{C}^n$, such as e.g. (4.28), one may replace the boundary condition $\nu \lrcorner \mathrm{d}u_{|\partial\mathbb{R}^n_+} = 0$ by $[({}^t\nabla u + \nabla u)\nu]_{\tan|_{\partial\mathbb{R}^n_+}} = 0$.

However, we mention the fact that, as exhibited in [MM09b, Section 2], such identification is no longer true for (even smooth) domains Ω with non-flat boundary. In this case, the equivalence holds up to a correction term $\mathcal{W}u$, *i.e.,* (4.37) is equivalent to

$$\left\{ \begin{array}{rcl} \nu \cdot u_{|\partial\Omega} & = & 0, \\ \nu \lrcorner \mathrm{d}u + \mathcal{W}u_{|\partial\Omega} & = & 0. \end{array} \right.$$

Here, $\mathcal{W}$ is the **Weingarten map**. It is linear in u and its coefficients depends linearly on the first derivatives of the outward unit normal ν, requiring then some smoothness on the boundary $\partial\Omega$. With the flat boundary $\partial\mathbb{R}^n_+$, the outward normal $\nu = -\mathfrak{e}_n$ is constant, which explains why the map $\mathcal{W}$ vanishes.

We can then exhibit the following maximal regularity result, where we identify Λ^1 with $\mathbb{C}^n$. Similar results such as Theorems 4.54 and 4.55 are also available.

Theorem 4.59 *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$, $s \in (-1+1/p, 1/p+2/q)$, such that $s, s+2-2/q \notin \mathbb{N} + \frac{1}{p}$ and $(\mathcal{C}_{s+2-2/q,p,q})$ is satisfied and let $T \in (0,+\infty]$.*

For any $f \in \mathrm{L}^q((0,T), \dot{\mathrm{B}}^s_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+,\mathbb{C}^n))$, $u_0 \in \dot{\mathrm{B}}^{2+s-\frac{2}{q},\sigma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+,\mathbb{C}^n)$, there exists a unique mild solution $(u,\nabla\mathfrak{p}) \in \mathrm{C}^0_b([0,T], \dot{\mathrm{B}}^{2+s-\frac{2}{q},\sigma}_{p,q,\mathcal{H}_\mathfrak{t}}(\mathbb{R}^n_+,\mathbb{C}^n)) \times \mathrm{L}^q((0,T), \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+,\mathbb{C}^n))$ of

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u + \nabla\mathfrak{p} & = & f, & \text{on } (0,T)\times\mathbb{R}^n_+, \\ \mathrm{div}\ u & = & 0, & \text{on } (0,T)\times\mathbb{R}^n_+, \\ [({}^t\nabla u + \nabla u)\nu]_{\tan|_{\partial\mathbb{R}^n_+}} & = & 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+, \\ \nu \cdot u_{|\partial\mathbb{R}^n_+} & = & 0, & \text{on } (0,T)\times\partial\mathbb{R}^n_+, \\ u(0) & = & u_0, & \text{in } \dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}(\mathbb{R}^n_+,\mathbb{C}^n), \end{array} \right. \tag{NSS}$$

with the estimate

$$\|u\|_{\mathrm{L}^\infty([0,T],\dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q})} + \big\|(\partial_t u, \nabla^2 u, \nabla\mathfrak{p})\big\|_{\mathrm{L}^q((0,T),\dot{\mathrm{B}}^s_{p,q})} \lesssim^s_{p,q,n} \|f\|_{\mathrm{L}^q((0,T),\dot{\mathrm{B}}^s_{p,q})} + \|u_0\|_{\dot{\mathrm{B}}^{2+s-\frac{2}{q}}_{p,q}}.$$

In the case $q = +\infty$, if we assume in addition $u_0 \in \dot{\mathrm{D}}^s_p(\mathbb{A}^2_{\mathcal{H},\mathfrak{t}})$, we have

$$\big\|(\partial_t u, \nabla^2 u, \nabla\mathfrak{p})\big\|_{\mathrm{L}^\infty([0,T],\dot{\mathrm{B}}^s_{p,\infty})} \lesssim_{p,s,n} \|f\|_{\mathrm{L}^\infty((0,T),\dot{\mathrm{B}}^s_{p,\infty})} + \|\mathbb{A}_{\mathcal{H},\mathfrak{t}} u_0\|_{\dot{\mathrm{B}}^s_{p,\infty}}.$$

5 Homogeneous function spaces on special Lipschitz domains and the trace theorem

Sommaire

Résumé du chapitre

On clôture cette thèse avec ce chapitre, dans lequel on étend la construction initiée dans le Chapitre 2 au cas des ouverts spéciaux Lipschitz. Pour cela, on utilise les résultats de la Section 3.3, et on améliore l'idée de certaines preuves. Cependant, pour couvrir l'ensemble des indices pertinents, les résultats sont formulés d'une manière plus lourde et moins naturelle.

On montre également un théorème de trace essentiellement optimal du type

$$[\cdot]_{|\partial\Omega} : \dot{\mathrm{H}}^{s,p}(\Omega) \longrightarrow \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\partial\Omega), \qquad (p \in (1,+\infty),\, s \in (1/p, 1+1/p))$$

dont la preuve que l'on propose étend l'idée simple exposée par Costabel et Ding [Cos88 ; Din96] sur l'utilisation de traces pour des espaces de Sobolev anisotropes. Afin d'adapter l'argument à notre cas, on utilise ici le Théorème 3.21 du Chapitre 3, qui donne effectivement des estimées de trace anisotropes lorsque les opérateurs sont bien choisis.

Avant tout cela, on considère des espaces de fonctions homogènes sur le bord d'un ouvert spécial Lipschitz, ce qui semble être une première dans la littérature.

Summary of the chapter

We close this dissertation with this chapter, in which we extend the construction initiated in Chapter 2 to the case of special Lipschitz domains. For this, we use the results of Section 3.3, and we improve the idea of some proofs. However, in order to cover the relevant indices, the results are formulated in a heavier and less natural way.

We also show an essentially optimal trace theorem of the type

$$[\cdot]_{|\partial\Omega} : \dot{\mathrm{H}}^{s,p}(\Omega) \longrightarrow \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\partial\Omega), \qquad (p \in (1,+\infty),\, s \in (1/p, 1+1/p)),$$

whose proof we propose extends the simple idea exposed by Costabel and Ding [Cos88 ; Din96] on the use of traces for anisotropic Sobolev spaces. In order to adapt the argument to our case, we use here the Theorem 3.21, from Chapter 3, which indeed gives anisotropic trace estimates when the operators are well chosen.

First of all, we consider homogeneous function spaces on the boundary of a special Lipschitz domain, which seems to be a first in the literature.

1 Introduction

1.1 Issues for realization of homogeneous function spaces on bent half spaces

It is not clear what would be the actual meaning of homogeneous function spaces on general domains (even smooth), since one cannot perform a composition with the change of coordinates, even in the case of a smooth bent half-space.

Indeed, assuming that $u + P \in \mathcal{S}'(\mathbb{R}^n)$ is a representative of $[u] \in \mathcal{S}'(\mathbb{R}^n) \Big/ \mathbb{C}[x]$, and Ψ is a smooth diffeomorphism of $\mathbb{R}^n$, the meaning of

$$u \circ \Psi + P \circ \Psi$$

is not clear. Even if it has one, it should not depend on the choice of P which is again unclear. It is then a major issue to find a way to transfer properties of homogeneous function spaces from the whole and the half space to a bent one by change of coordinates. In particular, investigation of traces at the boundary seems to be tricky.

Therefore, the realization of homogeneous function spaces provided by Costabel, M$^{\text{c}}$Intosh and Taggart [CMT13], built on $\mathcal{S}'(\mathbb{R}^n) \Big/ \mathbb{C}[x]$, seems to be far from being exploitable for linear problems with boundary values, in addition of being not suitable for non-linear problems. Here again, the $\mathcal{S}'_h(\mathbb{R}^n)$-realization of homogeneous function spaces saves us, and allows us to perform such composition under some circumstances.

In order to be able to give a suitable meaning of traces, we will give a construction of homogeneous Sobolev and Besov spaces on special Lipschitz domains and their boundaries, by improvements of some arguments presented in Section 3 of Chapter 2. The structure of extension and projection operators used here induce the consideration of two families of regularity indices : $(-1 + \frac{1}{p}, 1]$ and $[0, +\infty)$ with common overlap $[0, 1]$. We also note that the naive argument presented for Proposition 2.17 is no longer available for the non-negative family of regularity indices.

This is the main reason why the construction is done separately from the case $\Omega = \mathbb{R}^n_+$ in Chapter 2 : the necessity of two different extension operators further burden the statements for the case of Lipschitz boundary and the strategy proving their boundedness is somewhat different. However, once one has proved nice boundedness properties for the extensions operators, all the remaining proofs follows **extremely closely** the corresponding ones given in Section 3 of Chapter 2, up to appropriate modifications.

1.2 Trace theorems

Trace theorems with sharp regularity are fundamental to study boundary value problems in the field of partial differential equations. The usual theorem for traces of Besov or Sobolev functions on $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, +\infty)$ and on bounded and sufficiently regular domains can be found in [BL76, Subection 6.6], [AF03, Theorem 7.43, Remark 7.45], [Sch10, Theorems 3.16 & 3.19]. A very general result for traces on subsets of $\mathbb{R}^n$ with minimal geometric assumptions can be found in [JW84, Chapters VI & VII]. The usual trace theorem for Lipschitz domains can be stated as follows.

Theorem 5.1 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (\frac{1}{p}, 1+\frac{1}{p})$, and Ω be either a special or a bounded Lipschitz domain,*

(i) the trace operator $[\cdot]_{|\partial\Omega} : \mathrm{H}^{s,p}(\Omega) \longrightarrow \mathrm{B}^{s-\frac{1}{p}}_{p,p}(\partial\Omega)$ is a bounded surjection, in particular for all $u \in \mathrm{H}^{s,p}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,p}(\partial\Omega)} \lesssim_{s,p,n} \|u\|_{\mathrm{H}^{s,p}(\Omega)};$$

(ii) the trace operator $[\cdot]_{|\partial\Omega} : \mathrm{B}^{s}_{p,q}(\Omega) \longrightarrow \mathrm{B}^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ is a bounded surjection, in particular for all $u \in \mathrm{B}^{s}_{p,q}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\mathrm{B}^{s-\frac{1}{p}}_{p,q}(\partial\Omega)} \lesssim_{s,p,n} \|u\|_{\mathrm{B}^{s}_{p,q}(\Omega)};$$

(iii) the trace operator $[\cdot]_{|\partial\Omega} : \mathrm{B}^{\frac{1}{p}}_{p,1}(\partial\Omega) \longrightarrow \mathrm{L}^{p}(\partial\Omega)$ is a bounded surjection, in particular for all $u \in \mathrm{B}^{\frac{1}{p}}_{p,1}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\mathrm{L}^{p}(\partial\Omega)} \lesssim_{p,n} \|u\|_{\mathrm{B}^{\frac{1}{p}}_{p,1}(\Omega)};$$

Moreover, the trace operator $[\cdot]_{|\partial\Omega}$ admits a right bounded inverse for each of the above cases.

Roughly speaking, the goal here is, up to technical modifications, to add dots on every H and B symbols in Theorem 5.1 in the case of special Lipschitz domains. We take a focus on special Lipschitz domains for two main reasons. First, on a bounded Lipschitz domain the localization aspects induce that there are not that much differences between inhomogeneous and homogeneous function spaces, one may think about Poincaré-Wirtinger-Sobolev type inequalities. In a second time, the class of special Lipschitz domains seems to be the only suitable class of domains that admits good extension operators with homogeneous estimates at the moment. For more general unbounded Lipschitz, or smoother, domains one cannot reach homogeneous function spaces by localization with smooth cut-off since this procedure completely destroys the homogeneity.

The reader must know that in the case of inhomogeneous function spaces, one may find simpler proofs for the existence of traces seeing the trace operator as a (compact) operator with value in $\mathrm{L}^p(\partial\Omega)$ (when Ω has compact boundary). Similar results are also available for partial traces of vector fields, even with compactness property in the case of compact boundary, e.g. see [Mon15 ; Den21] and the references therein. A quite general result for partial traces on bounded Lipschitz domains is achieved by Mitrea, Mitrea and Shaw in [MMS08, Section 4] for differential forms, containing the result for vector fields.

Theorem 5.1 and fine properties of simple and double layer potentials were extensively used to study regularity and well-posedness of elliptic boundary value problems and deduce some functional analytic properties of involved elliptic operators (see e.g. [JK95 ; FMM98 ; MM01 ; MMT01]).

The goal of this chapter is to give a proof of the usual trace theorem for scalar-valued homogeneous Sobolev and Besov spaces on special Lipschitz domains. In order to prove the homogeneous version of Theorem 5.1, we aim to follow the strategy exhibited in [Din96], and initially described in [Cos88]. However, the use of the Fourier transform, and the overall strategy restrict everything to the case of inhomogeneous L^2-based Sobolev spaces. The idea we present here is to use the global-in-time $\dot{\mathrm{H}}^{s,p}(\mathrm{L}^p)$-maximal regularity for the Poisson semigroup on $\mathbb{R}^{n-1}$ and interpolation theory to replace the use of L^2 techniques.

1.3 Expected behavior through a Hardy-Littlewood-Sobolev-Kato diagram.

We can represent of Sobolev and Besov spaces $\dot{\mathrm{H}}^{s,p}(\Omega)$ and $\dot{\mathrm{B}}^{s}_{p,q}(\Omega)$ by the choice of a corresponding point $(\frac{1}{p}, s) \in (0,1) \times \mathbb{R}$.

This representation is usually very useful to carryover visually most of the information, especially in the context of interpolation theory. This is also useful to exhibit families of function spaces sharing common properties.

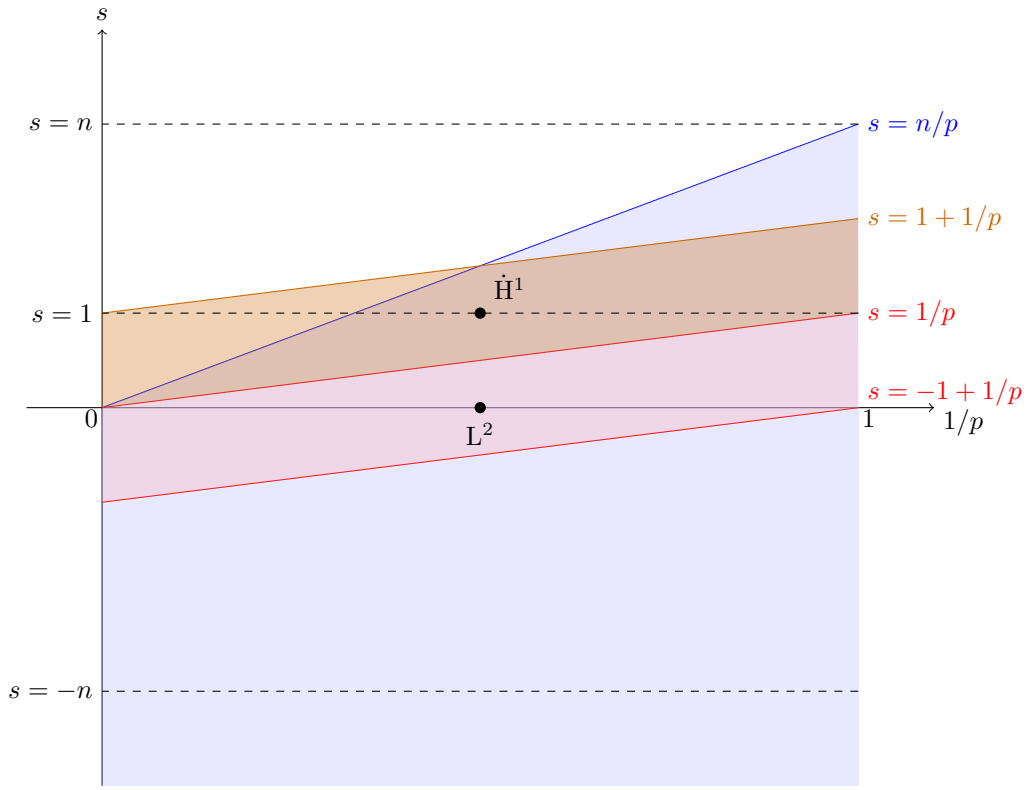


FIGURE 5.1 – Representation of Sobolev and Besov spaces in the plane : a Hardy-Littlewood-Sobolev-Kato diagram. (with $n = 3$)

- The blue part corresponds to the condition $(\mathcal{C}_{s,p,q})$ for Besov spaces, $s < n/p$ for Sobolev spaces. Those are the function spaces that are complete.
- The red part $s \in (-1+1/p, 1/p)$ stands for the spaces for which we can extend the elements of $\dot{\mathrm{H}}^{s,p}(\Omega)$ (resp. $\dot{\mathrm{B}}^{s}_{p,q}(\Omega)$) by 0 to obtain an element of $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ (resp. $\dot{\mathrm{B}}^{s}_{p,q}(\mathbb{R}^n)$). We also have that $\mathrm{C}^{\infty}_{c}(\Omega)$ is dense when $q < +\infty$. See Corollaries 5.14 and 5.19.
- The orange part $s \in (1/p, 1+1/p)$ stands for the spaces for which we have homogeneous trace estimates on a dense subspace. More precisely, for all $u \in \mathrm{H}^{s,p}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\partial\Omega)} \lesssim_{p,s,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}$$

and similarly for Besov spaces, up to appropriate modifications. See Theorem 5.43.

2 Homogeneous Sobolev and Besov spaces on a special Lipschitz domain.

From now on, and until the end of the paper, Ω will be a fixed special Lipschitz domain given by a fixed uniformly Lipschitz function $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$, *i.e.*,

$$\Omega := \{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, x_n > \phi(x') \}.$$

We also set the following global bi-Lipschitz map of $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$,

$$\Psi \,:\, (x', x_n) \longmapsto (x', x_n + \phi(x')).$$

For which, we have

$$\Psi(\mathbb{R}^n_+) = \Omega,\ \Psi^{-1}(\Omega) = \mathbb{R}^n_+ \text{ and } \det(\nabla\Psi) = \det(\nabla(\Psi^{-1})) = 1. \tag{5.1}$$

One may expect to recover similar results for the scale of homogeneous Sobolev and Besov as the one mentioned in Section 3.2. However, it still suffers the same issues as the one raised at the beginning of Section 3.3 in Chapter 2 : the lack of completeness on the whole scale and it is not known whether Rychkov's extension operator satisfies homogeneous estimates.

The extension method employed in [DHMT21, Chapter 3] and Section 3.3 in Chapter 2, by mean of the global change of coordinates and extension operators of higher order reflection around the boundary, will fail for high regularities. Indeed, the global bi-Lipschitz map Ψ suffers from its own lack of regularity, making it impossible to take its derivatives more than once. Moreover, even if it were a smooth global diffeomorphism, higher order derivatives would produce inhomogeneous parts with lower order terms. However, this method still makes sense for regularity indices $s \in (-1 + 1/p, 1]$.

The first idea here is to use Stein's extension operator introduced in [Ste70, Chapter VI], for which we have **homogeneous** estimates for non-negative integers indices of regularity.

The second idea is to fall in a setting so that one just has to use the proofs in Section 3.3 of Chapter 2, or at least to reproduce it with minor modifications. Indeed, those proofs mainly depend on the existence of good extension operator with appropriate homogeneous estimates, and the reflexivity of considered Sobolev spaces, so that everything goes similarly once the boundedness is proved.

2.1 Homogeneous Sobolev spaces

The Sobolev embeddings are a straightforward consequence of the definition of function spaces by restriction, see the proof of Proposition 2.24.

Proposition 5.2 *Let $p, q \in (1, +\infty)$, $s \in [0, n)$, such that*

$$\frac{1}{q} = \frac{1}{p} - \frac{s}{n}.$$

Then the following inequalities hold,

$$\|u\|_{\mathrm{L}^q(\Omega)} \lesssim_{n,s,p,q} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)},\ \forall u \in \dot{\mathrm{H}}^{s,p}(\Omega)\ ,$$
$$\|u\|_{\dot{\mathrm{H}}^{-s,q}_0(\Omega)} \lesssim_{n,s,p,q} \|u\|_{\mathrm{L}^p(\Omega)}\ ,\ \forall u \in \mathrm{L}^p(\Omega)\ .$$

Moreover, each underlying embedding is dense.

In particular, Sobolev embeddings imply that any appropriate extension operator, such as the next one, are already uniquely well-defined on any function in homogeneous Sobolev spaces such that $\dot{\mathrm{H}}^{s,p}(\Omega) \subset \mathrm{L}^{\frac{pn}{n-sp}}(\Omega)$, *i.e.,* for those who are complete. It remains to show the continuity with respect to homogeneous Sobolev norms.

Proposition 5.3 *There exists a universal extension operator $\mathcal{E}$, such that for all $p \in (1,+\infty)$, $s \in \mathbb{R}_+$, all $u \in \mathrm{H}^{s,p}(\Omega)$,*

$$\mathcal{E}u_{|_\Omega} = u,$$

with the estimate

$$\|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} \ .$$

In particular, $\mathcal{E} : \dot{\mathrm{H}}^{s,p}(\Omega) \longrightarrow \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ extends uniquely to a bounded operator whenever $(\mathcal{C}_{s,p})$ is satisfied.

Proof. — Let $\mathcal{E}$ be Stein's extension operator given in [Ste70, Chapter 6, Section 3, Theorem 5']. For all $m \in \mathbb{N}$, $p \in (1,+\infty)$, $u \in \mathrm{H}^{m,p}(\Omega)$, we have

$$\|\nabla^m u\|_{\mathrm{L}^p(\Omega)} \leqslant \|u\|_{\dot{\mathrm{H}}^{m,p}(\Omega)} = \inf_{\substack{\tilde{u}\in\dot{\mathrm{H}}^{m,p}(\mathbb{R}^n),\\ \tilde{u}_{|_\Omega}=u\,.}} \|\nabla^m \tilde{u}\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant \|\nabla^m(\mathcal{E}u)\|_{\mathrm{L}^p(\mathbb{R}^n)} \lesssim_{p,n,m,\partial\Omega} \|\nabla^m u\|_{\mathrm{L}^p(\Omega)} \,.$$

Hence, it satisfies

$$\|\mathcal{E}u\|_{\dot{\mathrm{H}}^{m,p}(\mathbb{R}^n)} \lesssim_{p,n,m,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{m,p}(\Omega)} \ . \tag{5.2}$$

So that $\mathcal{E} : \dot{\mathrm{H}}^{m,p}(\Omega) \longrightarrow \dot{\mathrm{H}}^{m,p}(\mathbb{R}^n)$ is bounded on subspace $\mathrm{H}^{m,p}(\Omega)$, in particular on whole $\dot{\mathrm{H}}^{m,p}(\Omega)$ when it is complete, *i.e.* $m < \frac{n}{p}$, since $\mathcal{S}(\overline{\Omega}) \subset \mathrm{H}^{m,p}(\Omega)$ is dense in $\dot{\mathrm{H}}^{m,p}(\Omega)$.

The estimate (5.2) implies, that for all $U \in \mathrm{H}^{m,p}(\mathbb{R}^n)$, by the definition of function spaces by restriction

$$\|\mathcal{E}[\mathbb{1}_\Omega U]\|_{\dot{\mathrm{H}}^{m,p}(\mathbb{R}^n)} \lesssim_{p,n,m,\partial\Omega} \|U\|_{\dot{\mathrm{H}}^{m,p}(\mathbb{R}^n)} \ .$$

Therefore, if one uses Lemma 2.7,

$$\|(\dot{\Delta}_j\mathcal{E}[\mathbb{1}_\Omega U])_{j\in\mathbb{Z}}\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_m(\mathbb{Z}))} \lesssim_{p,n,m,\partial\Omega} \|(\dot{\Delta}_j U)_{j\in\mathbb{Z}}\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_m(\mathbb{Z}))} \ . \tag{5.3}$$

For $v = (v_j)_{j\in\mathbb{Z}} \in \mathrm{L}^p(\mathbb{R}^n,\ell^2_m(\mathbb{Z}))$ with finite support with respect to the discrete variable, we set

$$\Xi_\Omega v := \Big(\dot{\Delta}_j\mathcal{E}\big[\mathbb{1}_\Omega\big(\sum_{k\in\mathbb{Z}} \dot{\Delta}_k\big[v_{k-1}+v_k+v_{k+1}\big]\big)\big]\Big)_{j\in\mathbb{Z}}$$

and since v has finite support with respect to the discrete variable we may define the auxiliary function $V := \sum_{k\in\mathbb{Z}} \dot{\Delta}_k\big[v_{k-1}+v_k+v_{k+1}\big] \in \mathrm{H}^{m,p}(\mathbb{R}^n)$, and we obtain, by [Gra14a, Proposition 6.1.4],

$$\|\Xi_\Omega v\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_m(\mathbb{Z}))} \lesssim_{p,n,m,\partial\Omega} \|(\dot{\Delta}_j V)_{j\in\mathbb{Z}}\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_m(\mathbb{Z}))} \lesssim_{p,n,m,\partial\Omega} \|v\|_{\mathrm{L}^p(\mathbb{R}^n,\ell^2_m(\mathbb{Z}))} \ .$$

It follows that Ξ_Ω extends uniquely to a bounded linear operator on $\mathrm{L}^p(\mathbb{R}^n,\ell^2_m(\mathbb{Z}))$ for all $p \in (1,+\infty)$, $m \in \mathbb{N}$, which is consistent on elements whose support in the discrete variable is finite. It is still consistent on all element of the form $(\dot{\Delta}_j U)_{j\in\mathbb{Z}}$, provided $U \in \mathrm{H}^{\alpha,p}(\mathbb{R}^n)$, $\alpha \geqslant m$ and we have by construction and uniqueness of the extension

$$\Xi_\Omega[(\dot{\Delta}_j U)_{j\in\mathbb{Z}}] = \big(\dot{\Delta}_j\mathcal{E}\big[\mathbb{1}_\Omega U\big]\big)_{j\in\mathbb{Z}}.$$

The complex interpolation of mixed weighted Lebesgue spaces, see Corollary 1.19, yields that

$$\Xi_\Omega \ : \ \mathrm{L}^p(\mathbb{R}^n,\ell^2_s(\mathbb{Z})) \longrightarrow \mathrm{L}^p(\mathbb{R}^n,\ell^2_s(\mathbb{Z}))$$

is a well-defined bounded linear operator for all $s \geqslant 0$, $p \in (1, +\infty)$. Then, the map

$$\Xi_\Omega[(\dot{\Delta}_j[\cdot])_{j\in\mathbb{Z}}] \,:\, \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n) \longrightarrow \mathrm{L}^p(\mathbb{R}^n, \ell^2_s(\mathbb{Z})) \tag{5.4}$$

is also well-defined and bounded by Lemma 2.7. Provided $s \geqslant 0$, $\mathcal{E}u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$ is already entirely determined for $u \in \mathrm{H}^{s,p}(\Omega)$ by the boundedness on the inhomogeneous Sobolev scale. Hence, for $U \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ such that $U_{|\Omega} = u$, by uniqueness of the extension of Ξ_Ω, we have

$$\Xi_\Omega[(\dot{\Delta}_j U)_{j\in\mathbb{Z}}] = \big(\dot{\Delta}_k \mathcal{E}[\mathbb{1}_\Omega U]\big)_{k\in\mathbb{Z}} = \big(\dot{\Delta}_k \mathcal{E}u\big)_{k\in\mathbb{Z}}.$$

Thus, one may use the estimate (5.4) and Lemma 2.7 to deduce

$$\|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|U\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \ .$$

However, U is an arbitrary $\dot{\mathrm{H}}^{s,p}$-extension of u, so that by the definition of function spaces by restriction, it holds that

$$\|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} \ .$$

When $(\mathcal{C}_{s,p})$ is satisfied, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete, and since $\mathcal{S}_0(\overline{\Omega}) \subset \mathrm{H}^{s,p}(\Omega)$ is dense in $\dot{\mathrm{H}}^{s,p}(\Omega)$, the operator extends uniquely on the whole $\dot{\mathrm{H}}^{s,p}(\Omega)$. ■

Remark 5.4 The method employed here is quite general, and could be adapted to the interpolation of many other kinds of linear operators.

The general idea is to lift the operator to a level for which we can take completion without losing any ambient structure information, here at the level of anisotropic Lebesgue spaces $\mathrm{L}^p(\ell^2_s)$, instead of taking abstract completion of our Sobolev spaces $\dot{\mathrm{H}}^{s,p}$. From this point, one perform the complex interpolation, then one may hope to get back to a (appropriate dense) subset of those spaces for which we can compute explicitly the operator, which was exactly what we have done. This is a key point for complex (and even real) interpolation of operators in the case of non-complete spaces when one wants to preserve homogeneous estimates.

Notice that similar ideas appear in the work of Auscher and Amenta [AA18, Chapter 4, Sections 4.2 & 4.3], where interpolation for realizations of operator-adapted Hardy spaces is concerned.

Proposition 5.3 is already a powerful enough tool to carry many results. However, this Stein's extension operator has its use restricted to non-negative indices of regularity for the Sobolev scale and positive indices of regularity for the Besov scale. It would be of a great interest to be able to look at similar properties for regularity indices $s \in (-1 + 1/p, 1/p)$.

We need to carry over the behavior of the global change of coordinates on the homogeneous scale.

For any measurable function u on either Ω or $\mathbb{R}^n$, and any measurable function v on either $\mathbb{R}^n_+$ or $\mathbb{R}^n$, we introduce the maps

$$T_\phi u := u \circ \Psi, \quad \text{and} \quad T_\phi^{-1} v = v \circ \Psi^{-1}. \tag{5.5}$$

Proposition 5.5 *Let $p \in (1, +\infty)$, $s \in [-1, 1]$ and $\mathcal{T} \in \{T_\phi, T_\phi^{-1}\}$. We assume that one of the two following conditions is satisfied*

(i) *$(\mathcal{C}_{s,p})$ and $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$,*

(ii) *$s \geqslant n/p$ and $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$.*

Then $\mathcal{T}u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ with the estimate

$$\|\mathcal{T}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}.$$

Proof. — We set $\mathcal{T} = T_\phi$ and $\mathcal{T}^* = T_\phi^{-1}$. First, we let $u \in \mathrm{H}^{1,p}(\mathbb{R}^n)$ and we recall that the following equalities hold almost everywhere

$$\partial_{x_k}(\mathcal{T}u) = \mathcal{T}(\partial_{x_k}u) + \partial_{x_k}\phi\mathcal{T}(\partial_{x_n}u)\ , \qquad \partial_{x_n}(\mathcal{T}u) = \mathcal{T}(\partial_{x_n}u)\ ,\ k \in [\![1, n-1]\!]\ .$$

We recall that $\mathcal{T}$ is bounded on $\mathrm{L}^p(\mathbb{R}^n)$, and that, moreover, the Jacobian determinant of Ψ is 1, see (5.1). Therefore, we obtain

$$\begin{aligned}\|\nabla\mathcal{T}u\|_{\mathrm{L}^p(\mathbb{R}^n)} &\leqslant \|\partial_{x_n}u\|_{\mathrm{L}^p(\mathbb{R}^n)} + \sum_{k=1}^{n-1}\|\partial_{x_k}u\|_{\mathrm{L}^p(\mathbb{R}^n)} + \|\partial_{x_k}\phi\|_{\mathrm{L}^\infty}\|\partial_{x_n}u\|_{\mathrm{L}^p(\mathbb{R}^n)} \\ &\leqslant (1 + (n-1)\|\nabla'\phi\|_{\mathrm{L}^\infty})\|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}.\end{aligned}$$

Similar computations yield,

$$\|\nabla\mathcal{T}^*u\|_{\mathrm{L}^p(\mathbb{R}^n)} \leqslant (1 + (n-1)\|\nabla'\phi\|_{\mathrm{L}^\infty})\|\nabla u\|_{\mathrm{L}^p(\mathbb{R}^n)}.$$

Now, for $v \in \mathrm{L}^2(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{-1,p}(\mathbb{R}^n)$, by Proposition 2.11, and since the Jacobian determinant of Ψ^{-1} is 1,

$$\begin{aligned}\|\mathcal{T}v\|_{\dot{\mathrm{H}}^{-1,p}(\mathbb{R}^n)} &= \sup_{\substack{u\in\mathcal{S}(\mathbb{R}^n)\\ \|u\|_{\dot{\mathrm{H}}^{1,p'}(\mathbb{R}^n)}\leqslant 1}} \left|\int_{\mathbb{R}^n} \mathcal{T}v(x)\,u(x)\,\mathrm{d}x\right| \\ &= \sup_{\substack{u\in\mathcal{S}(\mathbb{R}^n)\\ \|u\|_{\dot{\mathrm{H}}^{1,p'}(\mathbb{R}^n)}\leqslant 1}} \left|\int_{\mathbb{R}^n} v(x)\,\mathcal{T}^*u(x)\,\mathrm{d}x\right| \\ &\leqslant \|v\|_{\dot{\mathrm{H}}^{-1,p}(\mathbb{R}^n)} \left(\sup_{\substack{u\in\mathcal{S}(\mathbb{R}^n)\\ \|u\|_{\dot{\mathrm{H}}^{1,p'}(\mathbb{R}^n)}\leqslant 1}} \|\mathcal{T}^*u\|_{\dot{\mathrm{H}}^{1,p'}(\mathbb{R}^n)}\right) \\ &\lesssim_{p,n,\partial\Omega} \|v\|_{\dot{\mathrm{H}}^{-1,p}(\mathbb{R}^n)}.\end{aligned}$$

The same goes for $\mathcal{T}^*$. Hence, $\mathcal{T}$ (resp. $\mathcal{T}^*$) extends uniquely to a bounded linear operator on $\dot{\mathrm{H}}^{-1,p}(\mathbb{R}^n)$. But since $\mathcal{T}$ (resp. $\mathcal{T}^*$) is known to be bounded on $\mathrm{L}^p(\mathbb{R}^n)$, by complex interpolation given in Theorem 2.10, $\mathcal{T}$ (resp. $\mathcal{T}^*$) is then a bounded linear operator on $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, for all $s \in [-1, 0]$. One may repeat the duality argument, thanks to the boundedness on $\dot{\mathrm{H}}^{-s,p'}(\mathbb{R}^n)$ we just proved, to obtain for $s \in [0, 1]$, $u \in \mathrm{H}^{s,p}(\mathbb{R}^n)$,

$$\|\mathcal{T}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)},$$

and similarly for $\mathcal{T}^*$. Finally, when $(\mathcal{C}_{s,p})$ is satisfied, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ is complete, and since $\mathrm{H}^{s,p}(\mathbb{R}^n)$ is dense in $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$, the operator extends uniquely on the whole $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$. ∎

Remark 5.6 Everything still holds for more general bi-Lipschitz transformation with constant Jacobian determinants. One may probably want to generalize Proposition 5.5 in a way similar to [DM15, Lemma 2.1.1].

We can deduce from Proposition 5.5 many interesting corollaries.

Corollary 5.7 *For all $p \in (1, +\infty)$, for all $s \in (-1 + \frac{1}{p}, \frac{1}{p})$, for all $u \in \dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$,*

$$\|\mathbb{1}_\Omega u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}$$

The same result still holds with H *instead of* $\dot{\mathrm{H}}$.

Proof. — It suffices to write $\mathbb{1}_\Omega u = T_\phi^{-1}\mathbb{1}_{\mathbb{R}^n_+}T_\phi u$ then to apply Propositions 5.5 and 2.15. ■

Corollary 5.8 *Let $p\in(1,+\infty)$, $s\in[0,1]$. If one of the two following conditions is satisfied*

(i) *$(\mathcal{C}_{s,p})$ and $u\in\dot{\mathrm{H}}^{s,p}(\Omega)$,*

(ii) *$s\geqslant n/p$ and $u\in\mathrm{H}^{s,p}(\Omega)$,*

we have $T_\phi u\in\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ with the estimate

$$\|T_\phi u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\lesssim_{p,s,n,\partial\Omega}\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}\ .$$

In particular, $T_\phi:\dot{\mathrm{H}}^{s,p}(\Omega)\longrightarrow\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ is an isomorphism of Banach spaces whenever $(\mathcal{C}_{s,p})$ is satisfied. The result still holds if we replace $(\Omega,\mathbb{R}^n_+,T_\phi)$ by $(\mathbb{R}^n_+,\Omega,T_\phi^{-1})$.

Proof. — Let $u\in\mathrm{H}^{s,p}(\Omega)$, then $T_\phi u=[T_\phi\mathcal{E}u]_{|\mathbb{R}^n_+}\in\mathrm{H}^{s,p}(\mathbb{R}^n_+)\subset\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$, where $\mathcal{E}$ is Stein's universal extension operator. Therefore, by the definition of function spaces by restriction and Propositions 5.5 and 5.3, we have successively

$$\|T_\phi u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)}\leqslant\|T_\phi\mathcal{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\lesssim_{s,p,n,\partial\Omega}\|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\lesssim_{s,p,n,\partial\Omega}\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}.$$

When $(\mathcal{C}_{s,p})$ is satisfied, $\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)$ is complete, and since $\mathcal{S}_0(\overline{\Omega})\subset\mathrm{H}^{s,p}(\Omega)$ is dense in $\dot{\mathrm{H}}^{s,p}(\Omega)$, the operator extends uniquely on the whole $\dot{\mathrm{H}}^{s,p}(\Omega)$. ■

Corollary 5.9 *Let $p\in(1,+\infty)$, $s\in(-1+1/p,1]$. There exists an extension operator E, such that if one of the two following conditions is satisfied*

(i) *$(\mathcal{C}_{s,p})$ and $u\in\dot{\mathrm{H}}^{s,p}(\Omega)$,*

(ii) *$s\geqslant n/p$ and $u\in\mathrm{H}^{s,p}(\Omega)$,*

we have

$$\mathrm{E}u_{|\Omega}=u,$$

with the estimate

$$\|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)}\lesssim_{p,s,n,\partial\Omega}\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}\ .$$

In particular, $\mathrm{E}:\dot{\mathrm{H}}^{s,p}(\Omega)\longrightarrow\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)$ extends uniquely to a bounded linear operator whenever $(\mathcal{C}_{s,p})$ is satisfied.

Proof. — We introduce the extension operator on the half space by even reflection, for any measurable function $u:\mathbb{R}^n_+\longrightarrow\mathbb{C}$, and for almost every $(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$,

$$\tilde{\mathrm{E}}u(x',x_n):=\begin{cases}u(x',x_n) & ,\text{ if }(x',x_n)\in\mathbb{R}^{n-1}\times(0,+\infty),\\ u(x',-x_n) & ,\text{ if }(x',x_n)\in\mathbb{R}^{n-1}\times(-\infty,0).\end{cases}$$

The operator $\tilde{\mathrm{E}}$ is known to have the desired properties when $\Omega=\mathbb{R}^n_+$, see Proposition 2.17. It suffices to set

$$\mathrm{E}:=T_\phi^{-1}\tilde{\mathrm{E}}T_\phi.$$

The boundedness properties follows from Propositions 5.5 and 2.15 when $s\in(-1+1/p,1/p)$. When $s\geqslant 0$, it suffices to apply Corollary 5.8. ■

Proposition 5.10 *Let $p\in(1,+\infty)$, $k\in[\![1,+\infty[\![$, $s>k-1+\frac{1}{p}$, for all $u\in\mathrm{H}^{s,p}(\Omega)$,*

$$\|\nabla^k u\|_{\dot{\mathrm{H}}^{s-k,p}(\Omega)}\sim_{s,k,p,n,\partial\Omega}\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}.$$

In particular, $\|\nabla^k \cdot\|_{\dot{\mathrm{H}}^{s-k,p}(\Omega)}$ *is an equivalent norm on* $\dot{\mathrm{H}}^{s,p}(\Omega)$*, whenever* $(\mathcal{C}_{s-k,p})$ *is satisfied.*

Proof. — For all $k \in [\![1, +\infty[\![$, $p \in (1, +\infty)$, $s \in \mathbb{R}$, $u \in \dot{\mathrm{H}}^{s,p}(\Omega)$, the estimate

$$\|\nabla^k u\|_{\dot{\mathrm{H}}^{s-k,p}(\Omega)} \lesssim_{s,k,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}$$

always holds by the definition of function spaces by restriction. Therefore, it suffices to prove the reverse inequality.

First, we assume $s \in (1/p, 1]$. Let $u \in \mathrm{H}^{s,p}(\Omega)$. In this case, by Corollary 5.9 and the definition of function spaces by restriction,

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} \leqslant \|\mathrm{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|\nabla \mathrm{E}u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\Omega)}.$$

Now, if $s \geqslant 1$, one obtains similarly from Proposition 5.3,

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} \leqslant \|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n)} \lesssim_{s,p,n} \|\nabla \mathcal{E}u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\Omega)}. \tag{5.6}$$

This yields the result when $s > k - 1 + 1/p$, and $k = 1$.

We mention that the estimate (5.6) is indeed legal, because for $k \in [\![1, n]\!]$, there is a linear operator $\mathcal{T}_k$ which has exactly the same boundedness properties as $\mathcal{E}$, and which satisfies the commutation property $\partial_{x_k}\mathcal{E} = \mathcal{E}[\partial_{x_k}] + \mathcal{T}_k[\partial_{x_n}]$, see for instance the argument in the proof of [JK95, Proposition 2.18].

Now, for $s \geqslant 2 - 1 + 1/p$, $k = 2$, by the previous step for $k = 1$, we have

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\Omega)}.$$

Since $s - 1 > 1 - 1 + 1/p$, if $s \geqslant 1$ one may use again Stein's extension operator to obtain

$$\|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} \lesssim_{s,p,n,\partial\Omega} \|\nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\Omega)} \lesssim_{s,p,n,\partial\Omega} \|\nabla \mathcal{E} \nabla u\|_{\dot{\mathrm{H}}^{s-2,p}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|\nabla^2 u\|_{\dot{\mathrm{H}}^{s-2,p}(\Omega)}.$$

Otherwise for $s - 1 \in (1/p, 1]$, one uses the extension operator given by Corollary 5.9.

For $k \geqslant 2$, one obtains the result by induction, reproducing the steps above. ∎

Proposition 5.11 *Let* $p_j \in (1, +\infty)$*,* $s_j > -1 + \frac{1}{p_j}$*,* $j \in \{0, 1\}$*, such that* $(\mathcal{C}_{s_0,p_0})$ *is satisfied. Assume that one of the two following conditions is satisfied*

(i) $s_0, s_1 \leqslant 1$ *and* $\mathbb{E} = \mathrm{E}$*, given by Corollary 5.9,*

(ii) $s_0, s_1 \geqslant 0$ *and* $\mathbb{E} = \mathcal{E}$*, given by Proposition 5.3.*

Then for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p_1}(\Omega)$*, we have* $\mathbb{E}u \in \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)$*,* $j \in \{0, 1\}$*, with the estimate*

$$\|\mathbb{E}u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{s_j,p_j,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\Omega)}. \tag{5.7}$$

Therefore, the following equality of vector spaces holds with equivalence of norms

$$\dot{\mathrm{H}}^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p_1}(\Omega) = [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\Omega).$$

In particular, $\dot{\mathrm{H}}^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p_1}(\Omega)$ *is a Banach space which admits* $\mathcal{S}_0(\overline{\Omega})$ *as a dense subspace.*

Proof. — Let $p_j \in (1, +\infty)$, $s_j > -1 + 1/p_j$, such that $(\mathcal{C}_{s_0,p_0})$. For the beginning, we proceed as in Proposition 2.19 for $\Omega = \mathbb{R}^n_+$, $[\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\Omega)$ is complete and admits $\mathcal{S}_0(\overline{\Omega})$ as a dense subspace and the following continuous embedding also holds by construction,

$$[\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\Omega) \hookrightarrow \dot{\mathrm{H}}^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p_1}(\Omega).$$

For the reverse embedding, one may expect to reproduce the proof of Proposition 2.19, which is only possible when $\mathbb{E} = \mathrm{E}$. Indeed, this is not possible for $\mathbb{E} = \mathcal{E}$, since the aforementioned

proof use the fact that one can take derivatives a finite amount of time to fall in a complete space. When p_1 is large enough $[0, n/p_1)$ is of size less than one, while $(-1+1/p_1, n/p_1) \cap (-1+1/p_1, 1]$ always contains a translation of the interval $[0,1]$.

We let $\mathbb{E} = \mathcal{E}$ and then we get back to the use of the operator Ξ_Ω, as introduced in (5.4). We consider $u \in \dot{\mathrm{H}}^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p_1}(\Omega)$, and let $U \in \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$ such that $U_{|\Omega} = u$. We recall that we have $\mathcal{E}u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \subset \mathcal{S}'_h(\mathbb{R}^n)$ since $(\mathcal{C}_{s_0,p_0})$ is satisfied. One also has

$$\big(\dot\Delta_k \mathcal{E}u\big)_{k\in\mathbb{Z}} = \big(\dot\Delta_k \mathcal{E}[\mathbb{1}_\Omega U]\big)_{k\in\mathbb{Z}} = \Xi_\Omega[(\dot\Delta_k U)_{k\in\mathbb{Z}}] \in \mathrm{L}^{p_1}(\mathbb{R}^n, \ell^2_{s_1}(\mathbb{Z})).$$

Therefore, by Lemma 2.7, since $\mathcal{E}u \in \mathcal{S}'_h(\mathbb{R}^n)$,

$$\begin{aligned}\|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)} &\sim_{p_1,s_1,n} \|\big(\dot\Delta_k \mathcal{E}u\big)_{k\in\mathbb{Z}}\|_{\mathrm{L}^{p_1}(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}\\ &\sim_{p_1,s_1,n} \|\Xi_\Omega[(\dot\Delta_k U)_{k\in\mathbb{Z}}]\|_{\mathrm{L}^{p_1}(\mathbb{R}^n,\ell^2_{s_1}(\mathbb{Z}))}\\ &\lesssim_{p_1,s_1,n,\partial\Omega} \|U\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)}.\end{aligned}$$

As in the proof of Proposition 5.3, since U is an arbitrary extension of u in $\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$, taking the infimum on all such U yields

$$\|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)} \lesssim_{p_1,s_1,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\Omega)}.$$

Thus for $u \in \dot{\mathrm{H}}^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p_1}(\Omega)$, and by the definition of restriction spaces,

$$\|u\|_{[\dot{\mathrm{H}}^{s_0,p_0}\cap\dot{\mathrm{H}}^{s_1,p_1}](\Omega)} \leqslant \|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n)} + \|\mathcal{E}u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)} \lesssim^{p_0,p_1,n}_{s_0,s_1,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s_0,p_0}(\Omega)} + \|u\|_{\dot{\mathrm{H}}^{s_1,p_1}(\Omega)}.$$

This yields the result. ∎

From now on, everything has been set up so that the most part of the remaining proofs in Section 3.3, of Chapter 2, could be reproduced *verbatim* (sometimes up to the appropriate technical modifications).

Now, we want to work with homogeneous Sobolev spaces whose elements are supported in $\overline{\Omega}$.

Proposition 5.12 *Let $p_j \in (1,+\infty)$, $s_j > -1 + \frac{1}{p_j}$, $j \in \{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ is satisfied. Then,*

(i) if $s_0, s_1 \leqslant 1$, there exists a linear operator $\mathcal{P}_0 = \mathrm{P}_0$,

(ii) if $s_0, s_1 \geqslant 0$, there exists a linear operator $\mathcal{P}_0 = \mathcal{P}_0$,

such that for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$, we have $\mathcal{P}_0 u \in \dot{\mathrm{H}}^{s_j,p_j}_0(\Omega)$, $j \in \{0,1\}$, with the estimate

$$\|\mathcal{P}_0 u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)} \lesssim_{s_j,p_j,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)}.$$

Proof. — We notice that $\overline{\Omega}^c$ is also a special Lipschitz domain. If $\mathbb{E} \in \{\mathrm{E}, \mathcal{E}\}$ is an extension operator for Ω provided by Proposition 5.11, we denote by $\mathbb{E}^-$ the extension operator for $\overline{\Omega}^c$, and we set for all $u \in \dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n) \cap \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)$

$$\mathrm{P}_0 u := u - \mathrm{E}^-[\mathbb{1}_{\overline{\Omega}^c} u] \text{ (if } s_0, s_1 \leqslant 1), \text{ and } \mathcal{P}_0 u := u - \mathcal{E}^-[\mathbb{1}_{\overline{\Omega}^c} u] \text{ (if } s_0, s_1 \geqslant 0).$$

In this case, the boundedness properties follow from Proposition 5.11. ∎

The next proposition admits a proof similar to the one of Propositions 2.25 and 2.26. Indeed, all the appropriate tools are available, thanks to Propositions 5.2 and 5.12.

Proposition 5.13 *Let $s, s_0, s_1 \in \mathbb{R}$, $p \in (1,+\infty)$, then the space $\mathrm{C}^\infty_c(\Omega)$ is dense in*

(i) $\dot{\mathrm{H}}^{s,p}_0(\Omega)$, whenever $s \in (-n/p', n/p)$;

(ii) $\dot{\mathrm{H}}_0^{s_0,p}(\Omega) \cap \dot{\mathrm{H}}_0^{s_1,p}(\Omega)$, *if* $s_0, s_1 \geqslant 0$, *and* $(\mathcal{C}_{s_0,p})$ *is satisfied.*

The next corollary is fundamental for a proper theory of Sobolev spaces involving boundary values. This is a direct combination of Proposition 5.13 and Corollary 5.7.

Corollary 5.14 *For all* $p \in (1,+\infty)$, $s \in (-1+\frac{1}{p}, \frac{1}{p})$,

$$\dot{\mathrm{H}}_0^{s,p}(\Omega) = \dot{\mathrm{H}}^{s,p}(\Omega).$$

In particular, $\mathrm{C}_c^\infty(\Omega)$ *is dense in* $\dot{\mathrm{H}}^{s,p}(\Omega)$ *for the same range of indices.*

Proposition 5.15 *Let* $p \in (1,+\infty)$, $s \in (-\frac{n}{p'}, \frac{n}{p})$, *we have*

$$(\dot{\mathrm{H}}^{s,p}(\Omega))' = \dot{\mathrm{H}}_0^{-s,p'}(\Omega) \text{ and } (\dot{\mathrm{H}}_0^{s,p}(\Omega))' = \dot{\mathrm{H}}^{-s,p'}(\Omega).$$

Proof. — It suffices to reproduce the proof of Proposition 2.28, replacing $\mathbb{R}^n_+$ by Ω. ■

Corollary 5.16 *Let* $p \in (1,+\infty)$, $-n/p' < s_0 < s_1 < n/p$. *The space* $\mathrm{C}_c^\infty(\Omega)$ *is dense in* $\dot{\mathrm{H}}_0^{s_0,p}(\Omega) \cap \dot{\mathrm{H}}_0^{s_1,p}(\Omega)$.

Proof. — It suffices to reproduce the proof of Corollary 2.29, replacing $\mathbb{R}^n_+$ by Ω. ■

Proposition 5.17 *Let* $p_j \in (1,+\infty)$, $s_j \in (-n/p_j', n/p_j)$, *for* $j \in \{0,1\}$ *and for* $\theta \in (0,1)$ *we set,*

$$\left(s, \frac{1}{p}\right) := (1-\theta)\left(s_0, \frac{1}{p_0}\right) + \theta\left(s_1, \frac{1}{p_1}\right).$$

We assume that one of the two following condition is satisfied

(i) $s_j \geqslant 0$, *for* $j \in \{0,1\}$;

(ii) $s_j \in (-1+1/p_j, 1]$, $j \in \{0,1\}$.

Then for $\mathfrak{h} \in \{\dot{\mathrm{H}}, \dot{\mathrm{H}}_0\}$,

$$[\mathfrak{h}^{s_0,p_0}(\Omega), \mathfrak{h}^{s_1,p_1}(\Omega)]_\theta = \mathfrak{h}^{s,p}(\Omega) \ ,$$
$$[\mathfrak{h}^{-s_0,p_0'}(\Omega), \mathfrak{h}^{-s_1,p_1'}(\Omega)]_\theta = \mathfrak{h}^{-s,p'}(\Omega).$$

Proof. — We prove the first interpolation equality. It suffices to assert that $\{\dot{\mathfrak{h}}^{s_0,p_0}(\Omega), \dot{\mathfrak{h}}^{s_1,p_1}(\Omega)\}$ is a retraction of $\{\dot{\mathrm{H}}^{s_0,p_0}(\mathbb{R}^n), \dot{\mathrm{H}}^{s_1,p_1}(\mathbb{R}^n)\}$, thanks to Theorem 1.21. Indeed, both retractions are given by

$$\mathbb{E} \ : \ \dot{\mathrm{H}}^{s_j,p_j}(\Omega) \longrightarrow \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \text{ and } \mathrm{R}_\Omega \ : \ \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \longrightarrow \dot{\mathrm{H}}^{s_j,p_j}(\Omega),$$
$$\iota \ : \ \dot{\mathrm{H}}_0^{s_j,p_j}(\Omega) \longrightarrow \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \text{ and } \ \mathcal{P}_0 \ : \ \dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) \longrightarrow \dot{\mathrm{H}}_0^{s_j,p_j}(\Omega).$$

where $\mathbb{E}$ and $\mathcal{P}_0$ are given by Propositions 5.11 and 5.12, respectively. R_Ω stands for the restriction operator, and ι for the canonical embedding.

By construction, $\dot{\mathrm{H}}^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p_1}(\Omega)$ is dense in $\dot{\mathrm{H}}^{s_j,p_j}(\Omega)$, $j \in \{0,1\}$, since it contains $\mathcal{S}_0(\overline{\Omega})$. Corollary 5.16 gives the density of $\dot{\mathrm{H}}_0^{s_0,p_0}(\Omega) \cap \dot{\mathrm{H}}_0^{s_1,p_1}(\Omega)$ in $\dot{\mathrm{H}}_0^{s_j,p_j}(\Omega)$, $j \in \{0,1\}$. Since all involved spaces are reflexive, one deduces the second interpolation identity by duality, see Theorem 1.16. ■

2.2 Homogeneous Besov spaces

Proposition 5.18 *For all* $p \in (1,+\infty)$, $q \in [1,+\infty]$, *for all* $s \in (-1+\frac{1}{p}, \frac{1}{p})$, *for all* $u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$,

$$\|\mathbb{1}_\Omega u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)}$$

The same results still hold with B *instead of* $\dot{\mathrm{B}}$.

Proof. — It suffices to apply real interpolation given by Theorem 2.10 to Corollary 5.7. ∎

Corollary 5.19 *For all $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (-1+\frac{1}{p}, \frac{1}{p})$,*

$$\dot{\mathrm{B}}^s_{p,q,0}(\Omega) = \dot{\mathrm{B}}^s_{p,q}(\Omega).$$

In particular, if $q < +\infty$, the space $\mathrm{C}^\infty_c(\Omega)$ is dense in $\dot{\mathrm{B}}^s_{p,q}(\Omega)$ for the same range of indices.

Proof. — The equality of function spaces is straightforward from Proposition 5.18.

We only prove that $\mathrm{C}^\infty_c(\Omega)$ is dense in $\dot{\mathrm{B}}^s_{p,q}(\Omega)$. We use an interpolation theoretical argument.

By the same equality for homogeneous Sobolev spaces, Corollary 5.14, and the elementary interpolation embeddings (2.13) and (2.14), we obtain the following equality of function spaces with equivalence of norms,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\Omega), \dot{\mathrm{H}}^{s_1,p}_0(\Omega))_{\frac{s-s_0}{s_1-s_0},q} = \dot{\mathrm{B}}^s_{p,q,0}(\Omega) = \dot{\mathrm{B}}^s_{p,q}(\Omega),$$

with $p \in (1,+\infty)$, $-1+1/p < s_0 < s < s_1 < 1/p$, $q \in [1,+\infty]$. By Proposition 1.3, when $q \in [1,+\infty)$, we have the canonical dense embedding

$$\dot{\mathrm{H}}^{s_0,p}_0(\Omega) \cap \dot{\mathrm{H}}^{s_1,p}_0(\Omega) \hookrightarrow \dot{\mathrm{B}}^s_{p,q}(\Omega).$$

But $\mathrm{C}^\infty_c(\Omega)$ is known to be dense in $\dot{\mathrm{H}}^{s_0,p}_0(\Omega) \cap \dot{\mathrm{H}}^{s_1,p}_0(\Omega)$ by Corollary 5.16. ∎

During the proof of the last result, we have shown our first real interpolation identity. In fact, it can be improved. In order to prove general real interpolation statements, we want to improve the boundedness of extension and projection operators up to the scale of Besov spaces.

Proposition 5.20 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s > -1+\frac{1}{p}$. Let us consider the extension operator $\mathbb{E}$ (resp. $\mathcal{P}_0$) as in Proposition 5.11 (resp. Proposition 5.12). We assume moreover that*

- *$s > 0$, if $\mathbb{E} = \mathcal{E}$ (resp. $\mathcal{P}_0 = \mathcal{P}_0$);*
- *$s < 1$, if $\mathbb{E} = \mathrm{E}$ (resp. $\mathcal{P}_0 = \mathrm{P}_0$).*

If one of the two following assertion is satisfied

(i) $s > 0$ and $u \in \mathrm{B}^s_{p,q}(\Omega)$ (resp. $u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n)$);

(ii) $s \in (-1+\frac{1}{p}, 0]$ and $u \in \dot{\mathrm{B}}^s_{p,q}(\Omega)$ (resp. $u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$).

Then $\mathbb{E}u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$, (resp. $\mathcal{P}_0 u \in \dot{\mathrm{B}}^s_{p,q,0}(\Omega)$) and we have the estimate

$$\|\mathbb{E}u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\Omega)}. \text{ (resp. } \|\mathcal{P}_0 u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} .)$$

In particular, E (resp. $\mathcal{P}_0$) is a bounded linear operator from $\dot{\mathrm{B}}^s_{p,q}(\Omega)$ to $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ (resp. from $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$ to $\dot{\mathrm{B}}^s_{p,q,0}(\Omega)$) whenever ($\mathcal{C}_{s,p,q}$) is satisfied.

Proof. — One just has to reproduce the proof of Corollary 2.30. ∎

Proposition 5.21 *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$, $s > -1+1/p$, such that ($\mathcal{C}_{s,p,q}$) is satisfied. The space $\mathrm{C}^\infty_c(\Omega)$ is dense in $\dot{\mathrm{B}}^s_{p,q,0}(\Omega)$.*

Proof. — For $s \in (-1+1/p, 1/p)$, the result is proved in Corollary 5.19. Now assume $s > 0$. Let $u \in \dot{\mathrm{B}}^s_{p,q,0}(\Omega) \subset \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$, then, for $(u_k)_{k\in\mathbb{N}}$ a sequence in $\mathrm{B}^s_{p,q}(\mathbb{R}^n)$ which converges to u in $\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)$. By Proposition 5.20, $\mathcal{P}_0 u_k \in \mathrm{B}^s_{p,q,0}(\Omega)$ and we have

$$\|u - \mathcal{P}_0 u_k\|_{\dot{\mathrm{B}}^s_{p,q,0}(\Omega)} \lesssim_{s,p,q,\partial\Omega} \|u - u_k\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \xrightarrow[k\to+\infty]{} 0.$$

The density of $\mathrm{C}^\infty_c(\Omega)$ in $\mathrm{B}^s_{p,q,0}(\Omega)$ yields the result. ∎

Proposition 5.22 *Let $(p_0,p_1,p,q,q_0,q_1)\in(1,+\infty)^3\times[1,+\infty]^3$, $s_0,s_1\in\mathbb{R}$, such that $s_0<s_1$, let $(\mathfrak{h},\mathfrak{b})\in\{(\mathrm{H},\mathrm{B}),(\mathrm{H}_0,\mathrm{B}_{\cdot,\cdot,0})\}$, and set for $\theta\in(0,1)$,*

$$\left(s,\frac{1}{p_\theta},\frac{1}{q_\theta}\right):=(1-\theta)\left(s_0,\frac{1}{p_0},\frac{1}{q_0}\right)+\theta\left(s_1,\frac{1}{p_1},\frac{1}{q_1}\right).$$

If either one of the following assertions is satisfied,

(i) $s_j\in(-1+\frac{1}{p_j},1]$, $j\in\{0,1\}$;

(ii) $s_j\geqslant 0$, $j\in\{0,1\}$;

If $p_0=p_1=p$ and $(\mathcal{C}_{s,p,q})$ is satisfied, the following equality is true with equivalence of norms

$$(\dot{\mathfrak{h}}^{s_0,p}(\Omega),\dot{\mathfrak{h}}^{s_1,p}(\Omega))_{\theta,q}=\dot{\mathfrak{b}}^{s}_{p,q}(\Omega). \tag{5.8}$$

If $(\mathcal{C}_{s_j,p_j,q_j})$ is true for $j\in\{0,1\}$, with $s_j<1$ in case (i), $s_j>0$ in case (ii), then also is $(\mathcal{C}_{s,p_\theta,q_\theta})$ and

$$[\dot{\mathfrak{b}}^{s_0}_{p_0,q_0}(\Omega),\dot{\mathfrak{b}}^{s_1}_{p_1,q_1}(\Omega)]_\theta=\dot{\mathfrak{b}}^{s}_{p_\theta,q_\theta}(\Omega). \tag{5.9}$$

Proof. — The proof is *verbatim* the one of Proposition 2.33, even if the statement seems quite confusing due to additional conditions, due to the use of two different extension operators. We propose to reassure the reader by giving the proof anyway.

Step 1 : First, we assume $(\mathcal{C}_{s_j,p_j,q_j})$ in the case of Besov space, $(\mathcal{C}_{s_j,p_j})$ in the case of Sobolev spaces $j\in\{0,1\}$. We have retractions given by

$$\begin{array}{rlrl}
\mathbb{E}:\dot{\mathrm{H}}^{s_j,p_j}(\Omega)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) & \text{and } \mathrm{R}_\Omega:\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}(\Omega), \\
\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\Omega)\longrightarrow\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n) & \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)\longrightarrow\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\Omega), \\
\iota:\dot{\mathrm{H}}^{s_j,p_j}_0(\Omega)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n) & \text{and } \mathcal{P}_0:\dot{\mathrm{H}}^{s_j,p_j}(\mathbb{R}^n)\longrightarrow\dot{\mathrm{H}}^{s_j,p_j}_0(\Omega), \\
\dot{\mathrm{B}}^{s_j}_{p_j,q_j,0}(\Omega)\longrightarrow\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n) & \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)\longrightarrow\dot{\mathrm{B}}^{s_j}_{p_j,q_j,0}(\Omega).
\end{array}$$

where $\mathbb{E}$ and $\mathcal{P}_0$ are given by Propositions 5.11, 5.12 and 5.20. R_Ω stands for the restriction operator, and ι for the canonical embedding. In particular, the interpolation identity (5.9) holds.

Step 2 : The interpolation identity (5.8) for $(\mathfrak{h},\mathfrak{b})=(\mathrm{H},\mathrm{B})$ when $q<+\infty$.

Thanks to (2.13) in Lemma 2.16, we have continuous embedding,

$$\dot{\mathrm{B}}^{s}_{p,q}(\Omega)\hookrightarrow(\dot{\mathrm{H}}^{s_0,p}(\Omega),\dot{\mathrm{H}}^{s_1,p}(\Omega))_{\theta,q}.$$

Let us prove the reverse embedding,

$$\dot{\mathrm{B}}^{s}_{p,q}(\Omega)\hookleftarrow(\dot{\mathrm{H}}^{s_0,p}(\Omega),\dot{\mathrm{H}}^{s_1,p}(\Omega))_{\theta,q}.$$

Without loss of generality, we can assume $s_1\geqslant\frac{n}{p}$. Let $f\in\mathcal{S}_0(\overline{\Omega})\subset\dot{\mathrm{B}}^{s}_{p,q}(\Omega)$, if follows that $f\in(\dot{\mathrm{H}}^{s_0,p}(\Omega),\dot{\mathrm{H}}^{s_1,p}(\Omega))_{\theta,q}\subset\dot{\mathrm{H}}^{s_0,p}(\Omega)+\dot{\mathrm{H}}^{s_1,p}(\Omega)$. Thus, for all $(a,b)\in\dot{\mathrm{H}}^{s_0,p}(\Omega)\times\dot{\mathrm{H}}^{s_1,p}(\Omega)$ such that $f=a+b$, we have,

$$b=f-a\in(\mathcal{S}_0(\overline{\Omega})+\dot{\mathrm{H}}^{s_0,p}(\Omega))\cap\dot{\mathrm{H}}^{s_1,p}(\Omega).$$

In particular, we have $a\in\dot{\mathrm{H}}^{s_0,p}(\Omega)$ and $b\in\dot{\mathrm{H}}^{s_0,p}(\Omega)\cap\dot{\mathrm{H}}^{s_1,p}(\Omega)$. Hence, for $F:=\mathbb{E}a+\mathbb{E}b$, we have $F_{|\Omega}=f$, $\mathbb{E}a\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)$ and $\mathbb{E}b\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)\cap\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, with the estimates, given by Proposition 5.11,

$$\|\mathbb{E}a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)}\lesssim_{s_0,\partial\Omega,p,n}\|a\|_{\dot{\mathrm{H}}^{s_0,p}(\Omega)}\text{ and }\|\mathbb{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)}\lesssim_{s_1,\partial\Omega,p,n}\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\Omega)}.$$

Then, one may bound the K-functional of F, for $t>0$,

$$K(t,F,\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \leqslant \|\mathbb{E}a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} + t\|\mathbb{E}b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)} \lesssim_{s_j,p,n} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\Omega)} + t\|b\|_{\dot{\mathrm{H}}^{s_1,p}(\Omega)}$$

Taking the infimum over all such functions a and b, we obtain

$$K(t,F,\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n),\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)) \lesssim_{s_j,p,n} K(t,f,\dot{\mathrm{H}}^{s_0,p}(\Omega),\dot{\mathrm{H}}^{s_1,p}(\Omega)),$$

from which we obtain, after multiplying by $t^{-\theta}$, taking the L^q_*-norm with respect to t, and applying Theorem 2.10,

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\Omega)} \leqslant \|F\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} \lesssim_{s,p,n} \|f\|_{(\dot{\mathrm{H}}^{s_0,p}(\Omega),\dot{\mathrm{H}}^{s_1,p}(\Omega))_{\theta,q}}.$$

Finally, thanks to the first embedding (2.13), we have

$$\|f\|_{\dot{\mathrm{B}}^s_{p,q}(\Omega)} \sim_{p,s,n,\partial\Omega} \|f\|_{(\dot{\mathrm{H}}^{s_0,p}(\Omega),\dot{\mathrm{H}}^{s_1,p}(\Omega))_{\theta,q}},\ \forall f\in \mathcal{S}_0(\overline{\Omega}).$$

Since $q<+\infty$, we can conclude by density of $\mathcal{S}_0(\overline{\Omega})$ in both $\dot{\mathrm{B}}^s_{p,q}(\Omega)$ and in the interpolation space $(\dot{\mathrm{H}}^{s_0,p}(\Omega),\dot{\mathrm{H}}^{s_1,p}(\Omega))_{\theta,q}$. The density argument for the later one is carried over by Proposition 5.11 and Proposition 1.3.

Step 3 : The interpolation identity (5.8) for $(\mathfrak{h},\mathfrak{b})=(\mathrm{H}_0,\mathrm{B}_0)$ when $q<+\infty$.

Thanks to the elementary embedding (2.14) in Lemma 2.16, we have the continuous embedding,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\Omega),\dot{\mathrm{H}}^{s_1,p}_0(\Omega))_{\theta,q} \hookrightarrow \dot{\mathrm{B}}^s_{p,q,0}(\Omega).$$

We are going to prove the reverse embedding,

$$(\dot{\mathrm{H}}^{s_0,p}_0(\Omega),\dot{\mathrm{H}}^{s_1,p}_0(\Omega))_{\theta,q} \hookleftarrow \dot{\mathrm{B}}^s_{p,q,0}(\Omega).$$

Again, without loss of generality we can assume $s_1 \geqslant \frac{n}{p}$, otherwise one can go back to the Step 1. Let us consider $u\in \mathrm{C}^\infty_c(\Omega)$, then, u belongs to $\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)+\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$. In particular for $(a,b)\in \dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)\times\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$, such that $u=a+b$ we have

$$b=u-a\in (\mathrm{C}^\infty_c(\Omega)+\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n))\cap\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n).$$

in particular we have $a\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)$ and $b\in\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)\cap\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)$.

Consequently, we have $u=\mathcal{P}_0u=\mathcal{P}_0a+\mathcal{P}_0b$, with $\mathcal{P}_0a\in\dot{\mathrm{H}}^{s_0,p}_0(\Omega)$ and $\mathcal{P}_0b\in\dot{\mathrm{H}}^{s_0,p}_0(\Omega)\cap\dot{\mathrm{H}}^{s_1,p}_0(\Omega)$, with the estimates

$$\|\mathcal{P}_0a\|_{\dot{\mathrm{H}}^{s_0,p}_0(\Omega)} \lesssim_{s_0,p,n,\partial\Omega} \|a\|_{\dot{\mathrm{H}}^{s_0,p}(\mathbb{R}^n)} \text{ and } \|\mathcal{P}_0b\|_{\dot{\mathrm{H}}^{s_1,p}_0(\Omega)} \lesssim_{s_1,p,n,\partial\Omega} \|b\|_{\dot{\mathrm{H}}^{s_1,p}(\mathbb{R}^n)},$$

thanks to Proposition 5.12. Thus, one may follow the lines of the Step 2, to obtain for all $u\in\mathrm{C}^\infty_c(\Omega)$,

$$\|u\|_{\dot{\mathrm{B}}^s_{p,q,0}(\Omega)} \sim_{s,p,n,\partial\Omega} \|u\|_{(\dot{\mathrm{H}}^{s_0,p}_0(\Omega),\dot{\mathrm{H}}^{s_1,p}_0(\Omega))_{\theta,q}}.$$

Again, one can conclude via density arguments since $q<+\infty$, and $\mathrm{C}^\infty_c(\Omega)$ is dense in $\dot{\mathrm{B}}^s_{p,q,0}(\Omega)$ thanks to Proposition 5.21.

Step 4 : The interpolation identity (5.8) when $q=+\infty$.

In the case of $(\mathfrak{h},\mathfrak{b})=(\mathrm{H},\mathrm{B})$, the reiteration theorem, Theorem 1.5, in combination with the Step 1 and the Step 2 yields the identity.

Similarly, for the case of $(\mathfrak{h},\mathfrak{b})=(\mathrm{H}_0,\mathrm{B}_0)$ with the Step 1 and the Step 3. ■

Lemma 5.23 *Let $p_j\in(1,+\infty)$, $q_j\in[1,+\infty]$, $s_j>-1+\frac{1}{p_j}$, $j\in\{0,1\}$, such that $(\mathcal{C}_{s_0,p_0,q_0})$ is satisfied and consider the extension operator $\mathbb{E}$ given by Proposition 5.20.*

Then for all $u \in \dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\Omega) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\Omega)$, *we have* $\mathbb{E}u \in \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)$, $j \in \{0,1\}$, *with the estimate*

$$\|\mathbb{E}u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n)} \lesssim_{s_j,p_j,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\Omega)}.$$

The same result holds replacing $(\mathbb{E}, \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\Omega), \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n))$ *by* $(\mathcal{P}_0, \dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\mathbb{R}^n), \dot{\mathrm{B}}^{s_j}_{p_j,q_j,0}(\Omega))$, *where* $\mathcal{P}_0$ *is the projection operator given in Proposition 5.20.*

Proof. — It suffices to apply real interpolation to prove the boundedness

$$\Xi_\Omega \,:\, \ell^q_s(\mathbb{Z}, \mathrm{L}^p(\mathbb{R}^n)) \longrightarrow \ell^q_s(\mathbb{Z}, \mathrm{L}^p(\mathbb{R}^n))$$

from the proof of Proposition 5.3, where $s > 0$, $p \in (1,+\infty)$, $q \in [1,+\infty]$.

Finally, the result follows, reproducing the arguments in the proofs of Propositions 5.3 and 5.11. ∎

Proposition 5.24 *Let* $p_j \in (1,+\infty)$, $q_j \in [1,+\infty]$, $j \in \{0,1\}$, $-1+\frac{1}{p} < s_0 < s_1$, *such that* $(\mathcal{C}_{s_0,p_0,q_0})$ *is satisfied. We assume moreover that either* $s_0 > 0$, *or* $s_1 < 1$. *Then the following equality of vector spaces holds with equivalence of norms*

$$\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\Omega) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\Omega) = [\dot{\mathrm{B}}^{s_0}_{p_0,q_0} \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\Omega).$$

In particular, $\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\Omega) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\Omega)$ *is a Banach space, and it admits* $\mathcal{S}_0(\overline{\Omega})$ *as a dense subspace whenever* $q_j < +\infty$, $j \in \{0,1\}$.

Similarly, the following equality with equivalence of norms holds for all $s > 0$, $q \in [1,+\infty]$,

$$\mathrm{L}^p(\Omega) \cap \dot{\mathrm{B}}^s_{p,q}(\Omega) = \mathrm{B}^s_{p,q}(\Omega).$$

Proof. — This is a direct consequence of Lemma 5.23. ∎

Proposition 5.25 *Let* $p_j \in (1,+\infty)$, $q_j \in [1,+\infty]$, $s_j > m_j - 1 + 1/p_j$, $j \in \{0,1\}$, *such that* $(\mathcal{C}_{s_0-m_0,p_0,q_0})$ *is satisfied. For all* $u \in \dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\Omega) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\Omega)$,

$$\|\nabla^{m_j} u\|_{\dot{\mathrm{B}}^{s_j-m_j}_{p_j,q_j}(\Omega)} \sim_{s_j,m_j,p_j,n} \|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\Omega)}.$$

Proof. — The proof follows the lines of the proof of Proposition 5.10. ∎

Proposition 5.26 *Let* $(p,q,q_0,q_1) \in (1,+\infty) \times [1,+\infty]^3$, $s_0, s_1 \in \mathbb{R}$, *such that* $s_0 < s_1$.

Let $\mathfrak{b} \in \{\mathrm{B}, \mathrm{B}_{\cdot,\cdot,0}\}$, *and set for* $\theta \in (0,1)$,

$$s := (1-\theta)s_0 + \theta s_1.$$

Assume $q_j < +\infty$ *if* $(\mathcal{C}_{s_j,p_j,q_j})$ *is* ***not*** *true for* $j \in \{0,1\}$, *and that one of the following assertions is satisfied,*

(i) $s_j \in (-1+\frac{1}{p_j}, 1)$, $j \in \{0,1\}$;

(ii) $s_j > 0$, $j \in \{0,1\}$;

If $(\mathcal{C}_{s,p,q})$ *is satisfied, the following equality is true with equivalence of norms*

$$(\dot{\mathfrak{b}}^{s_0}_{p,q_0}(\Omega), \dot{\mathfrak{b}}^{s_1}_{p,q_1}(\Omega))_{\theta,q} = \dot{\mathfrak{b}}^s_{p,q}(\Omega). \tag{5.10}$$

Proof. — The proof follows the lines of the proof of Proposition 5.22. ∎

Finally, we claim the density result for $\dot{\mathrm{B}}^s_{p,\infty}$ spaces, whose proof is an exact copy of the same statement, Corollary 2.35. The same goes for the duality result that follows, see Proposition 2.40.

Proposition 5.27 *Let* $p \in (1,+\infty)$, $s > -1 + 1/p$, *such that* $(\mathcal{C}_{s,p,\infty})$ *is satisfied.*

- *The space* $\mathrm{C}^\infty_c(\Omega)$ *is weak* dense in* $\dot{\mathrm{B}}^s_{p,\infty,0}(\Omega)$.

- *The space* $\mathcal{S}_0(\overline{\Omega})$ *is weak* dense in* $\dot{\mathrm{B}}^s_{p,\infty}(\Omega)$.

Proposition 5.28 *Let* $p \in (1,+\infty)$, $q \in (1,+\infty]$, $s > -1+\frac{1}{p}$, *if* ($\mathcal{C}_{s,p,q}$) *is satisfied then the following isomorphisms hold*

$$(\dot{\mathrm{B}}^{-s}_{p',q',0}(\Omega))' = \dot{\mathrm{B}}^s_{p,q}(\Omega) \text{ and } (\dot{\mathrm{B}}^{-s}_{p',q'}(\Omega))' = \dot{\mathrm{B}}^s_{p,q,0}(\Omega).$$

3 The trace theorem

In the previous section, an appropriate construction of homogeneous Sobolev and Besov spaces on special Lipschitz domains was given with their interpolation properties. Now, we want to make sense of boundary values in homogeneous function spaces which stand for the main interest of the present chapter.

The first subsection is devoted to the construction of function spaces on the boundary.

The second one concerns the transference of properties on the flat upper half-space to the bent one via the global change of coordinates. However, we want to reach the sharp range of regularity $(1/p, 1+1/p)$ for the trace result. The main issue will occur when $s \in [1, 1+1/p)$, since we do not have more than one full gradient under the action of the global change of coordinate. To circumvent this issue, we introduce an anisotropic trace result inherited from Theorem 3.21. This result is obtained from the $\dot{\mathrm{H}}^{s-1,p}(\mathrm{L}^p)$-maximal regularity for the Poisson semigroup $(e^{-t(-\Delta')^{1/2}})_{t\geqslant 0}$ on $\mathbb{R}^{n-1}$.

The last subsection is devoted to the statement of the main trace theorem, and several straightforward consequences.

3.1 Function spaces on the boundary

To define the trace as in the case of inhomogeneous function spaces, we have to define first (homogeneous) Besov spaces on the boundary $\partial\Omega$. To do so, since we have

$$\Omega = \left\{ (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} \middle| \, x_n > \phi(x') \right\},$$

where $\phi : \mathbb{R}^{n-1} \longrightarrow \mathbb{R}$ is uniformly Lipschitz, we recall that the surface measure on the boundary $\partial\Omega = \left\{(x', \phi(x')),\, x' \in \mathbb{R}^{n-1}\right\} \subset \mathbb{R}^n$ is defined as

$$\sigma(A) := \int_{\mathbb{R}^{n-1}} \mathbb{1}_A(x', \phi(x'))\sqrt{1+|\nabla'\phi(x')|^2}\,\mathrm{d}x',$$

where A is any Lebesgue-measurable set of $\partial\Omega$.

We also recall that σ is the unique Borel measure on $\partial\Omega$ so that we have the integration by parts formula

$$\int_\Omega \partial_{x_k} u(x)v(x)\,\mathrm{d}x = -\int_\Omega u(x)\partial_{x_k}v(x)\,\mathrm{d}x + \int_{\partial\Omega} u(x)v(x)\nu_k(x)\,\mathrm{d}\sigma_x,\ k \in [\![1,n]\!], \tag{5.11}$$

provided $u, v \in \mathrm{C}^{0,1}_c(\mathbb{R}^n)$, the space of complex-valued compactly supported Lipschitz functions. And in (5.11), ν_j stands for the j-th component of the outward unit normal of Ω, defined almost everywhere on $\partial\Omega$ by

$$\nu := \frac{1}{\sqrt{|\nabla'\phi|^2+1}}(\nabla'\phi, -1).$$

We introduce the pushforward map from $\partial\Omega$ to $\mathbb{R}^{n-1}$ for any measurable function $u : \partial\Omega \longrightarrow \mathbb{C}$,

$$S_\phi u(x') := u(x', \phi(x')),\ x' \in \mathbb{R}^{n-1}. \tag{5.12}$$

We also have the pullback map defined for any measurable function $v : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}$,

$$S_\phi^{-1} v(y) := v(y'),\ y \in \partial\Omega. \tag{5.13}$$

To construct the homogeneous function spaces on the boundary, we are going to follow the ideas given in [DM15, Chapter 2, Section 2.2] and [Din96, Section 2].

Definition 5.29 For $p \in [1, +\infty)$, $s \in (0,1)$, for any measurable function f on $\partial\Omega$, we define the following quantities

$$\|f\|^p_{\mathrm{L}^p(\partial\Omega)} := \int_{\partial\Omega} |f(x)|^p \, \mathrm{d}\sigma_x, \qquad \|f\|^p_{\dot{\mathrm{B}}^s_{p,p}(\partial\Omega)} := \int_{\partial\Omega}\int_{\partial\Omega} \frac{|f(x)-f(y)|^p}{|x-y|^{ps+n-1}} \, \mathrm{d}\sigma_x \mathrm{d}\sigma_y,$$

with the usual modifications when $p = +\infty$. We set, assuming $p \neq 1, +\infty$ for the last case,

- $\mathrm{L}^p(\partial\Omega) := \{\, u : \partial\Omega \longrightarrow \mathbb{C} \text{ meas.} \,|\, \|u\|_{\mathrm{L}^p(\partial\Omega)} < +\infty \,\}$,
- $\dot{\mathrm{B}}^s_{p,p}(\partial\Omega) := \{\, u \in \mathrm{L}^1_{\mathrm{loc}}(\partial\Omega) \,|\, S_\phi u \in \mathcal{S}'_h(\mathbb{R}^{n-1}) \,\&\, \|u\|_{\dot{\mathrm{B}}^s_{p,p}(\partial\Omega)} < +\infty \,\}$,
- $\dot{\mathrm{H}}^{1,p}(\partial\Omega) := \{\, u \in \mathrm{L}^1_{\mathrm{loc}}(\partial\Omega) \,|\, S_\phi u \in \mathcal{S}'_h(\mathbb{R}^{n-1}) \,\&\, \|u\|_{\dot{\mathrm{H}}^{1,p}(\partial\Omega)} := \|S_\phi^{-1}[\nabla' S_\phi u]\|_{\mathrm{L}^p(\partial\Omega)} < +\infty \,\}$.

The next lemmas justify that these definitions are meaningful.

Lemma 5.30 *Let $p \in [1, +\infty]$. The map*

$$S_\phi : \mathrm{L}^p(\partial\Omega) \longrightarrow \mathrm{L}^p(\mathbb{R}^{n-1})$$

is well-defined and a is continuous isomorphism of normed vector spaces. The same result holds with $\mathrm{L}^p_{\mathrm{loc}}$, instead of L^p, as complete metric spaces.

Proof. — By direct computations, we obtain for all $p \in [1, +\infty]$, $u \in \mathrm{L}^p(\partial\Omega)$,

$$\|S_\phi u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \leqslant \|u\|_{\mathrm{L}^p(\partial\Omega)} \leqslant (1 + \|\nabla'\phi\|^2_{\mathrm{L}^\infty})^{\frac{1}{2p}} \|S_\phi u\|_{\mathrm{L}^p(\mathbb{R}^{n-1})}.$$

The same goes for S_ϕ^{-1}.

For the $\mathrm{L}^p_{\mathrm{loc}}$ isomorphism property, thanks to the last inequality, it suffices to exhibit two appropriate families of compact subsets.

For $a, b \in \mathbb{R}^{n-1}$, with $a_j < b_j$, $j \in [\![1, n]\!]$, we define

$$Q_{(a,b)} := \prod_{j=1}^{n-1} [a_j, b_j] \text{ and } K_{(a,b)} := \{\, (x', \phi(x')),\ x' \in Q_{(a,b)} \,\}.$$

In this case $(Q_{(a,b)})_{a<b}$ (resp. $(K_{(a,b)})_{a<b}$) is a family of compact subsets of $\mathbb{R}^{n-1}$ (resp. $\partial\Omega$), such that

$$(S_\phi \mathbb{1}_{Q_{(a,b)}})_{a<b} = (\mathbb{1}_{K_{(a,b)}})_{a<b} \text{ and } (S_\phi^{-1} \mathbb{1}_{K_{(a,b)}})_{a<b} = (\mathbb{1}_{Q_{(a,b)}})_{a<b}. \qquad \blacksquare$$

The following corollary is a direct consequence of Lemma 5.30.

Corollary 5.31 *Let $p \in (1, +\infty)$. For $u \in \dot{\mathrm{H}}^{1,p}(\partial\Omega)$, we have $S_\phi u \in \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1}) \cap \mathrm{L}^1_{loc}(\mathbb{R}^{n-1})$, with the estimate*

$$\|u\|_{\dot{\mathrm{H}}^{1,p}(\partial\Omega)} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})}.$$

Conversely, if $v \in \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1}) \cap \mathrm{L}^1_{loc}(\mathbb{R}^{n-1})$, then $S_\phi^{-1} v \in \dot{\mathrm{H}}^{1,p}(\partial\Omega)$ with the corresponding estimate.

The idea behind the definition of Besov spaces on the boundary lies in the fact that for all $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1}) \cap \mathcal{S}'_h(\mathbb{R}^{n-1})$, when $s \in (0,1)$, $p \in [1,+\infty)$,

$$\|f\|^p_{\dot{\mathrm{B}}^s_{p,p}(\mathbb{R}^{n-1})} \sim_{p,s,n} \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}^{n-1}} \frac{|f(x)-f(y)|^p}{|x-y|^{ps+n-1}}\,\mathrm{d}x\mathrm{d}y, \tag{5.14}$$

see [BCD11, Theorem 2.36] for a proof. The case $p = +\infty$ is treated via usual modification with homogeneous Hölder (semi-)norms.

Lemma 5.32 *Let $p \in [1,+\infty]$, $s \in (0,1)$. For all $u \in \dot{\mathrm{B}}^s_{p,p}(\partial\Omega)$, $S_\phi u \in \dot{\mathrm{B}}^s_{p,p}(\mathbb{R}^{n-1}) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ with the estimate*

$$\|u\|_{\dot{\mathrm{B}}^s_{p,p}(\partial\Omega)} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{\mathrm{B}}^s_{p,p}(\mathbb{R}^{n-1})}$$

Conversely, for $v \in \dot{\mathrm{B}}^s_{p,p}(\mathbb{R}^{n-1}) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$, one has $S_\phi^{-1} v \in \dot{\mathrm{B}}^s_{p,p}(\partial\Omega)$ with the corresponding estimate.

Proof. — For $u \in \mathrm{L}^1_{\mathrm{loc}}(\partial\Omega)$, if $p < +\infty$,

$$\begin{aligned}
\|u\|^p_{\dot{\mathrm{B}}^s_{p,p}(\partial\Omega)} &= \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{ps+n-1}}\,\mathrm{d}\sigma_x\mathrm{d}\sigma_y \\
&= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|u(x',\phi(x'))-u(y',\phi(y'))|^p}{|(x'-y',\phi(x')-\phi(y'))|^{ps+n-1}} \sqrt{|\nabla'\phi(x')|^2+1}\sqrt{|\nabla'\phi(y')|^2+1}\,\mathrm{d}x'\mathrm{d}y' \\
&\leqslant [1+\|\nabla'\phi\|^2_{\mathrm{L}^\infty}] \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|S_\phi u(x')-S_\phi u(y')|^p}{|x'-y'|^{ps+n-1}}\,\mathrm{d}x'\mathrm{d}y' \\
&\lesssim_{p,s,n,\partial\Omega} \|S_\phi u\|^p_{\dot{\mathrm{B}}^s_{p,p}(\mathbb{R}^{n-1})}\,.
\end{aligned}$$

The last estimate comes from (5.14).

For the reverse estimate, we start with (5.14) then, we obtain

$$\begin{aligned}
\|S_\phi u\|^p_{\dot{\mathrm{B}}^s_{p,p}(\mathbb{R}^{n-1})} &\lesssim_{p,s,n} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|S_\phi u(x')-S_\phi u(y')|^p}{|x'-y'|^{ps+n-1}}\,\mathrm{d}x'\mathrm{d}y' \\
&\lesssim_{p,s,n} [1+\|\nabla'\phi\|^2_{\mathrm{L}^\infty}]^{\frac{(n-1)+ps}{2}} \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^{n-1}} \frac{|u(x',\phi(x'))-u(y',\phi(y'))|^p}{|(x'-y',\phi(x')-\phi(y'))|^{ps+n-1}}\,\mathrm{d}x'\mathrm{d}y' \\
&\lesssim_{p,s,n,\partial\Omega} \int_{\partial\Omega}\int_{\partial\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{ps+n-1}}\,\mathrm{d}\sigma_x\mathrm{d}\sigma_y = \|u\|^p_{\dot{\mathrm{B}}^s_{p,p}(\partial\Omega)}\,.
\end{aligned}$$

The case $p = +\infty$ is similar and left to the reader with

$$\|w\|_{\dot{\mathrm{B}}^s_{\infty,\infty}(\partial\Omega)} = \sup_{\substack{(x,y)\in\partial\Omega^2,\\ x\neq y.}} \frac{|w(x)-w(y)|}{|x-y|^s}. \qquad \blacksquare$$

Proposition 5.33 *Let $p \in (1,+\infty)$, $s \in (0,1)$. The following equality holds with equivalence of norms*

$$(\mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega))_{s,p} = \dot{\mathrm{B}}^s_{p,p}(\partial\Omega)\,.$$

Proof. — Let $u \in \mathrm{L}^p(\partial\Omega) + \dot{\mathrm{H}}^{1,p}(\partial\Omega) \subset \mathrm{L}^1_{\mathrm{loc}}(\partial\Omega)$, then for $(a,b) \in \mathrm{L}^p(\partial\Omega) \times \dot{\mathrm{H}}^{1,p}(\partial\Omega)$ such that $u = a + b$, by Corollary 5.31, we have

$$S_\phi u = S_\phi a + S_\phi b \in \mathrm{L}^p(\mathbb{R}^{n-1}) + \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1}). \tag{5.15}$$

Therefore, by the definition of the K-functional and Corollary 5.31, we obtain

$$\begin{aligned} K(t, S_\phi u, \mathrm{L}^p(\mathbb{R}^{n-1}), \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})) &\leqslant \|S_\phi a\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} + t\|S_\phi b\|_{\dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})} \\ &\lesssim_{p,n,\partial\Omega} \|a\|_{\mathrm{L}^p(\partial\Omega)} + t\|b\|_{\dot{\mathrm{H}}^{1,p}(\partial\Omega)}. \end{aligned}$$

Looking at the infimum on all such pair (a, b) yields

$$K(t, S_\phi u, \mathrm{L}^p(\mathbb{R}^{n-1}), \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})) \lesssim_{p,n,\partial\Omega} K(t, u, \mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega)).$$

Now, for the reverse estimate, from (5.15), let $(A, B) \in \mathrm{L}^p(\mathbb{R}^{n-1}) \times \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})$, such that on has $S_\phi u = A + B$, since $S_\phi u, A \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$, it follows that

$$B = S_\phi u - A \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1}) \cap \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1}).$$

Hence, by Corollary 5.31,

$$u = S_\phi^{-1} A + S_\phi^{-1} B \in \mathrm{L}^p(\partial\Omega) + \dot{\mathrm{H}}^{1,p}(\partial\Omega).$$

So as before, we obtain,

$$K(t, u, \mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega)) \lesssim_{p,n,\partial\Omega} K(t, S_\phi u, \mathrm{L}^p(\mathbb{R}^{n-1}), \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})).$$

In the end we have obtained for all $u \in \mathrm{L}^p(\partial\Omega) + \dot{\mathrm{H}}^{1,p}(\partial\Omega)$ and all $t > 0$:

$$K(t, u, \mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega)) \sim_{p,n,\partial\Omega} K(t, S_\phi u, \mathrm{L}^p(\mathbb{R}^{n-1}), \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})). \tag{5.16}$$

Finally, if one multiplies by (5.16) by t^{-s}, then take its L^q_*-norm, thanks to (2.3) and Lemma 5.32 we obtain

$$\|u\|_{(\mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega))_{s,p}} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{\mathrm{B}}^s_{p,p}(\mathbb{R}^{n-1})} \sim_{s,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^s_{p,p}(\partial\Omega)}$$

which ends the proof. ∎

Now, we introduce the following definition of homogeneous Besov space on the boundary with third index $q \neq p$, consistent with the case $q = p$.

Definition 5.34 For $p \in (1, +\infty)$, $q \in [1, +\infty]$, $s \in (0, 1)$, we define

$$\dot{\mathrm{B}}^s_{p,q}(\partial\Omega) := (\mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega))_{s,q}.$$

The following results are then a direct consequence of the estimate (5.16) and usual results for homogeneous Sobolev and Besov spaces on $\mathbb{R}^{n-1}$.

Corollary 5.35 *Let $p \in (1, +\infty)$, $s \in (0, 1)$, $q \in [1, +\infty]$. For all $u \in \dot{\mathrm{B}}^s_{p,q}(\partial\Omega)$, $S_\phi u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^{n-1}) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$ with the estimate*

$$\|u\|_{\dot{\mathrm{B}}^s_{p,q}(\partial\Omega)} \sim_{s,p,n,\partial\Omega} \|S_\phi u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^{n-1})}$$

Conversely, for $v \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^{n-1}) \cap \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^{n-1})$, one has $S_\phi^{-1} v \in \dot{\mathrm{B}}^s_{p,q}(\partial\Omega)$ with the corresponding estimate.

Proposition 5.36 *Let $p \in (1, +\infty)$, $q \in [1, +\infty]$, $s \in (0, 1)$. The following assertions are true.*

(i) $\dot{\mathrm{B}}^s_{p,q}(\partial\Omega)$ is a Banach space whenever $(\mathcal{C}_{s+\frac{1}{p},p,q})$ is satisfied.

(ii) If $s \in (0, \frac{n-1}{p})$, for $\frac{1}{r} := \frac{1}{p} - \frac{s}{n-1}$, if $q \in [1, r]$, we have the continuous embedding

$$\dot{\mathrm{B}}^s_{p,q}(\partial\Omega) \hookrightarrow \mathrm{L}^r(\partial\Omega).$$

(iii) When $p > n-1$, we have the continuous embedding

$$\dot{\mathrm{B}}^{\frac{n-1}{p}}_{p,1}(\partial\Omega) \hookrightarrow \mathrm{C}^0_0(\partial\Omega).$$

Remark 5.37 One could also check that the intersection space $\dot{\mathrm{B}}^{s_0}_{p_0,q_0}(\partial\Omega) \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}(\partial\Omega)$ is complete whenever $(\mathcal{C}_{s_0+\frac{1}{p_0},p_0,q_0})$ is satisfied.

From there, it is straightforward to check that one can recover usual and very well known function spaces $\mathrm{H}^{1,p}(\partial\Omega) = \mathrm{L}^p(\partial\Omega) \cap \dot{\mathrm{H}}^{1,p}(\partial\Omega)$, $\mathrm{B}^s_{p,q}(\partial\Omega) = \mathrm{L}^p(\partial\Omega) \cap \dot{\mathrm{B}}^s_{p,q}(\partial\Omega)$, $s \in (0,1)$, $p \in (1,+\infty)$, $q \in [1,+\infty]$.

3.2 Preliminary results for the trace theorem

The strategy of the proof will mainly arise from a flattening procedure of the boundary with anisotropic estimates, as done in [Din96, Lemma 1, Lemma 2]. For the reader's convenience we recall, from (5.1) and (5.5), that for any measurable function $u : \Omega \longrightarrow \mathbb{C}$,

$$T_\phi u(x', x_n) = u(x', x_n + \phi(x')), \quad (x', x_n) \in \mathbb{R}^{n-1} \times [0, +\infty).$$

For $p \in (1,+\infty)$, $s \in [1,2]$, we introduce the function space

$$\mathcal{K}^{s,p}(\mathbb{R}^n_+) := \mathrm{H}^{s,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})) \cap \mathrm{H}^{s-1,p}(\mathbb{R}_+, \mathrm{H}^{1,p}(\mathbb{R}^{n-1}))$$

with its natural norm. We also introduce the homogeneous (semi-)norm

$$\|u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}^n_+)} := \|(\partial_{x_n} u, \nabla' u)\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))}.$$

We notice that $\mathcal{K}^{1,p}(\mathbb{R}^n_+) = \mathrm{H}^{1,p}(\mathbb{R}^n_+)$, and $\|\cdot\|_{\dot{\mathcal{K}}^{1,p}(\mathbb{R}^n_+)} \sim_{p,n} \|\nabla \cdot\|_{\mathrm{L}^p(\mathbb{R}^n_+)}$.

During this section, we will need for a brief moment to use Banach valued (anisotropic) homogeneous Sobolev spaces for non-negative index and with values in a (reflexive) Lebesgue space. Go back to Section 3.1, in Chapter 3, for an elementary construction of homogeneous vector-valued Riesz potential spaces and references therein for a more general review of vector-valued Sobolev (Bessel potential) spaces and their properties.

For $p \in (1,+\infty)$, provided $0 \leqslant \alpha < 1/p$, we define $\frac{1}{r} := \frac{1}{p} - \alpha$,

$$\dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1})) := \left\{ u \in \mathrm{L}^r(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1})) \,\middle|\, (-\partial^2_{x_n})^{\frac{\alpha}{2}} u \in \mathrm{L}^p(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1})) = \mathrm{L}^p(\mathbb{R}^n) \right\}.$$

We also define by restriction, in the sense of distributions, the corresponding space on the half line

$$\dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})) := \dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1}))_{|\mathbb{R}_+}.$$

This is a Banach space with respect to the quotient norm

$$\|u\|_{\dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} := \inf_{\substack{U_{|\mathbb{R}_+} = u, \\ U \in \dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1}))}} \|U\|_{\dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}, \mathrm{L}^p(\mathbb{R}^{n-1}))}.$$

Lemma 5.38 *Let $p \in (1,+\infty)$, $\alpha \in [0, 1/p)$. For all $u \in \dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}^n_+)$, we have the estimate*

$$\|u\|_{\dot{\mathrm{H}}^{\alpha,p}_{x_n}(\mathbb{R}_+, \mathrm{L}^p_{x'}(\mathbb{R}^{n-1}))} \lesssim_{p,\alpha,n} \|u\|_{\dot{\mathrm{H}}^{\alpha,p}(\mathbb{R}^n_+)}.$$

Proof. — On $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$, the result follows from the boundedness of Riesz transforms. The case of the half-space follows from the definition of function spaces by restriction. ∎

Lemma 5.39 *For $p \in (1, +\infty)$, $s \in [1, 1 + 1/p)$. Then the linear operator*

$$T_\phi \,:\, \mathrm{H}^{s,p}(\Omega) \longrightarrow \mathcal{K}^{s,p}(\mathbb{R}^n_+)$$

is well-defined and bounded.

Moreover, for all $u \in \mathrm{H}^{s,p}(\Omega)$ we have the homogeneous estimate

$$\|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}. \tag{5.17}$$

Proof. — For the boundedness of T_ϕ from $\mathrm{H}^{s,p}(\Omega)$ to $\mathcal{K}^{s,p}(\mathbb{R}^n_+)$, it suffices to follow the proof of [Din96, Lemma 2]. One may check the boundedness properties

$$\begin{aligned} T_\phi \,:&\mathrm{H}^{1,p}(\Omega) \longrightarrow \mathrm{H}^{1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})) \\ &\mathrm{H}^{2,p}(\Omega) \longrightarrow \mathrm{H}^{2,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})) \end{aligned}$$

which imply, by complex interpolation, that

$$T_\phi \,:\mathrm{H}^{s,p}(\Omega) \longrightarrow \mathrm{H}^{s,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})) \tag{5.18}$$

is well-defined and bounded for all $s \in [1, 2]$. Similarly, from the boundedness

$$\begin{aligned} T_\phi \,:&\mathrm{H}^{1,p}(\Omega) \longrightarrow \mathrm{L}^p(\mathbb{R}_+, \mathrm{H}^{1,p}(\mathbb{R}^{n-1})) \\ &\mathrm{H}^{2,p}(\Omega) \longrightarrow \mathrm{H}^{1,p}(\mathbb{R}_+, \mathrm{H}^{1,p}(\mathbb{R}^{n-1})), \end{aligned}$$

for $s \in [1, 2]$, we deduce that

$$T_\phi \,:\mathrm{H}^{s,p}(\Omega) \longrightarrow \mathrm{H}^{s-1,p}(\mathbb{R}_+, \mathrm{H}^{1,p}(\mathbb{R}^{n-1})) \tag{5.19}$$

is well-defined and bounded. Thus, (5.18) and (5.19) yield the boundedness of T_ϕ. Now, we prove the estimate (5.17). For $u \in \mathrm{H}^{s,p}(\Omega) \subset \dot{\mathrm{H}}^{s,p}(\Omega)$, we have $T_\phi u \in \mathcal{K}^{s,p}(\mathbb{R}^n_+)$, and since

$$\partial_{x_k}(T_\phi u) = T_\phi(\partial_{x_k} u) + \partial_{x_k}\phi T_\phi(\partial_{x_n} u)\ , \qquad \partial_{x_n}(T_\phi u) = T_\phi(\partial_{x_n} u)\ ,\ k \in [\![1, n-1]\!]\ ,$$

we obtain,

$$\begin{aligned} \|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}^n_+)} &= \|(\partial_{x_n} T_\phi u, \nabla' T_\phi u)\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} \\ &\leqslant \|T_\phi \partial_{x_n} u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} + \|\nabla' T_\phi u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} \\ &\leqslant (1 + (n-1)\|\nabla'\phi\|_{\mathrm{L}^\infty})\|T_\phi \nabla' u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} + \|T_\phi \partial_{x_n} u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} \\ &\lesssim_{p,s,n,\partial\Omega} \|T_\phi \nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))}. \end{aligned}$$

The estimate (5.17) is then a consequence of Lemma 5.38 and Proposition 5.5

$$\begin{aligned} \|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}^n_+)} &\lesssim_{p,s,n,\partial\Omega} \|T_\phi \nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} \\ &\lesssim_{p,s,n,\partial\Omega} \|T_\phi \nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}^n_+)} \\ &\lesssim_{p,s,n,\partial\Omega} \|\nabla u\|_{\dot{\mathrm{H}}^{s-1,p}(\Omega)} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)}. \end{aligned}$$

■

One may notice that, for say all $u \in \mathrm{H}^{1,p}(\Omega)$, we have in $\mathrm{B}^{1-1/p}_{p,p}(\partial\mathbb{R}^n_+)$, the equality

$$[T_\phi u]_{|\partial\mathbb{R}^n_+} = S_\phi[u_{|\partial\Omega}].$$

Lemma 5.40 *Let $p \in (1,+\infty)$, $s \in [1, 1+1/p)$. For all $u \in \mathrm{H}^{s,p}(\Omega)$, we have*

$$T_\phi u \in \mathrm{C}_0^0(\overline{\mathbb{R}_+}, \mathrm{B}^{s-1/p}_{p,p}(\mathbb{R}^{n-1})).$$

Proof. — For $u \in \mathrm{H}^{s,p}(\Omega) \subset \mathrm{H}^{1,p}(\Omega)$, then $T_\phi u \in \mathrm{H}^{1,p}(\mathbb{R}^n_+) \cap \mathcal{K}^{s,p}(\mathbb{R}^n_+)$, and by Corollary 2.48,

$$T_\phi u \in \mathrm{C}_0^0(\overline{\mathbb{R}_+}, \mathrm{B}^{1-1/p}_{p,p}(\mathbb{R}^{n-1})),$$

and $[T_\phi u]_{|\partial\mathbb{R}^n_+} = S_\phi[u_{|\partial\Omega}]$ in $\mathrm{B}^{1-1/p}_{p,p}(\mathbb{R}^{n-1})$.

Therefore, if we set $v(t,x') := T_\phi u(x',t)$, $t \geqslant 0$, $x' \in \mathbb{R}^{n-1}$, we have

$$\begin{aligned} F := \partial_t v + (\mathrm{I}-\Delta')^{\frac12} v \in \mathrm{H}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})) \subset \mathrm{L}^p(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})),\\ \text{and } v(0,\cdot) = [T_\phi u]_{|\partial\mathbb{R}^n_+} = S_\phi[u_{|\partial\Omega}] \in \mathrm{B}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \subset \mathrm{B}^{1-1/p}_{p,p}(\mathbb{R}^{n-1}) \subset \mathrm{L}^p(\mathbb{R}^{n-1}). \end{aligned}$$

By uniqueness of the mild solution, for all $t \geqslant 0$

$$v(t) = e^{-t(\mathrm{I}-\Delta')^{\frac12}} v(0) + \int_0^t e^{-(t-s)(\mathrm{I}-\Delta')^{\frac12}} F(s)\,\mathrm{d}s.$$

Therefore, by Theorem 3.21, since $(\mathrm{I}-\Delta')^{\frac12}$ is invertible on $\mathrm{L}^p(\mathbb{R}^{n-1})$ with its domain $\mathrm{D}_p((\mathrm{I}-\Delta')^{\frac12}) = \mathrm{H}^{1,p}(\mathbb{R}^{n-1})$, we have the following maximal regularity estimate

$$\begin{aligned} \|v\|_{\mathrm{L}^\infty(\mathbb{R}_+, \mathrm{B}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{p,s,n} \|(\partial_t v, (\mathrm{I}-\Delta')^{\frac12} v)\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} \lesssim_{p,s,n} \|F\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))}\\ + \|v(0)\|_{\mathrm{B}^{s-1/p}_{p,p}(\mathbb{R}^{n-1})}, \end{aligned}$$

and $v \in \mathrm{C}_0^0(\overline{\mathbb{R}_+}, \mathrm{B}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}))$. ∎

Corollary 5.41 *Let $p \in (1,+\infty)$, $s \in [1, 1+1/p)$. For all $u \in \mathrm{H}^{s,p}(\Omega)$,*

$$\|T_\phi u\|_{\mathrm{L}^\infty(\mathbb{R}_+, \dot{\mathrm{B}}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{p,s,n,\partial\Omega} \|T_\phi u\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}^n_+)}.$$

Proof. — By Lemma 5.40,

$$T_\phi u \in \mathrm{C}_0^0(\overline{\mathbb{R}_+}, \mathrm{B}^{s-1/p}_{p,p}(\mathbb{R}^{n-1})) \subset \mathrm{C}_0^0(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s-1/p}_{p,p}(\mathbb{R}^{n-1})).$$

As in the proof of Lemma 5.40, for $v(t,x') := T_\phi u(x',t)$, $x' \in \mathbb{R}^{n-1}$ and $t \geqslant 0$, we have

$$\begin{aligned} f := \partial_t v + (-\Delta')^{\frac12} v \in \mathrm{H}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})) \subset \dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1})),\\ \text{and } v(0,\cdot) = [T_\phi u]_{|\partial\mathbb{R}^n_+} = S_\phi[u_{|\partial\Omega}] \in \mathrm{B}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}) \subset \dot{\mathrm{B}}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}). \end{aligned}$$

Therefore, by Theorem 3.21, since the operator $(-\Delta')^{\frac12}$ on $\mathrm{L}^p(\mathbb{R}^{n-1})$ has homogeneous domain $\mathrm{D}_p((-\mathring{\Delta}')^{\frac12}) = \dot{\mathrm{H}}^{1,p}(\mathbb{R}^{n-1})$, we have the following maximal regularity estimate

$$\|v\|_{\mathrm{L}^\infty(\mathbb{R}_+, \dot{\mathrm{B}}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}))} \lesssim_{p,s,n} \|(\partial_t v, (-\Delta')^{\frac12} v)\|_{\dot{\mathrm{H}}^{s-1,p}(\mathbb{R}_+, \mathrm{L}^p(\mathbb{R}^{n-1}))} \sim_{p,s,n} \|v\|_{\dot{\mathcal{K}}^{s,p}(\mathbb{R}^n_+)}.$$ ∎

We may need a Besov counterpart of Corollary 5.8 to carry over the trace result, or more generally to transfer properties from the half-space to Besov spaces on special Lipschitz domains by global change of coordinates. However, our strategy will be a bit different, so that the result will be only used in the very specific case $s = \frac{1}{p}$, $q = 1$, but it still has its own interest.

Lemma 5.42 *Let $p \in (1,+\infty)$, $s \in (-1+1/p, 1)$, $q \in [1,+\infty]$. If one of the two following conditions is satisfied*

(i) $(\mathcal{C}_{s,p,q})$ *and* $u \in \dot{\mathrm{B}}^s_{p,q}(\Omega)$,
(ii) $s \geqslant n/p$ *and* $u \in \mathrm{B}^s_{p,q}(\Omega)$,
we have $T_\phi u \in \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ *with the estimate*

$$\|T_\phi u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\Omega)}.$$

In particular, $T_\phi : \dot{\mathrm{B}}^s_{p,q}(\Omega) \longrightarrow \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$ *is an isomorphism of Banach spaces whenever* $(\mathcal{C}_{s,p,q})$ *is satisfied. The result still holds if we replace* $(\Omega, \mathbb{R}^n_+, T_\phi)$ *by* $(\mathbb{R}^n_+, \Omega, T_\phi^{-1})$.

Proof. — For $s \in (-1+1/p, 1/p)$, the result follows from real interpolation, by Corollary 5.8 and Proposition 5.22. Therefore, without loss of generality, we can assume $s \in [1/p, 1)$. We are going to proceed via a manual real interpolation scheme. Let $u \in \mathrm{B}^s_{p,q}(\Omega)$, then we have

$$u \in \mathrm{L}^p(\Omega) + \mathrm{H}^{1,p}(\Omega) \subset \mathrm{L}^p(\Omega) + \dot{\mathrm{H}}^{1,p}(\Omega).$$

Let $(a, b) \in \mathrm{L}^p(\Omega) \times \dot{\mathrm{H}}^{1,p}(\Omega)$, such that $u = a + b$,

$$b = u - a \in (\mathrm{B}^s_{p,q}(\Omega) + \mathrm{L}^p(\Omega)) \cap \dot{\mathrm{H}}^{1,p}(\Omega) \subset \mathrm{H}^{1,p}(\Omega).$$

We have $T_\phi u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+) \subset \dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)$, and $T_\phi a \in \mathrm{L}^p(\mathbb{R}^n_+)$, $T_\phi b \in \mathrm{H}^{1,p}(\mathbb{R}^n_+)$,

$$T_\phi u = T_\phi a + T_\phi b \in \mathrm{L}^p(\mathbb{R}^n_+) + \mathrm{H}^{1,p}(\mathbb{R}^n_+) \subset \mathrm{L}^p(\mathbb{R}^n_+) + \dot{\mathrm{H}}^{1,p}(\mathbb{R}^n_+).$$

For E, the operator extension from $\mathbb{R}^n_+$ to $\mathbb{R}^n$ by reflection given in the proof of Corollary 5.9, we have $\mathrm{E}T_\phi u \in \mathrm{B}^s_{p,q}(\mathbb{R}^n_+)$, and

$$\mathrm{E}T_\phi u = \mathrm{E}T_\phi a + \mathrm{E}T_\phi b \in \mathrm{L}^p(\mathbb{R}^n) + \mathrm{H}^{1,p}(\mathbb{R}^n) \subset \mathrm{L}^p(\mathbb{R}^n) + \dot{\mathrm{H}}^{1,p}(\mathbb{R}^n).$$

Now, for $t > 0$, by definition of the K-functional and the homogeneous estimates given by Corollaries 5.8 and 5.9, we obtain

$$\begin{aligned} K(t, \mathrm{E}T_\phi u, \mathrm{L}^p(\mathbb{R}^n), \dot{\mathrm{H}}^{1,p}(\mathbb{R}^n)) &\leqslant \|\mathrm{E}T_\phi a\|_{\mathrm{L}^p(\mathbb{R}^n)} + t\|\mathrm{E}T_\phi b\|_{\dot{\mathrm{H}}^{1,p}(\mathbb{R}^n)} \\ &\lesssim_{p,n,\partial\Omega} \|a\|_{\mathrm{L}^p(\Omega)} + t\|b\|_{\dot{\mathrm{H}}^{1,p}(\Omega)}. \end{aligned}$$

Looking at the infimum over such pairs (a, b) yields for $t > 0$,

$$K(t, \mathrm{E}T_\phi u, \mathrm{L}^p(\mathbb{R}^n), \dot{\mathrm{H}}^{1,p}(\mathbb{R}^n)) \lesssim_{p,n,\partial\Omega} K(t, u, \mathrm{L}^p(\Omega), \dot{\mathrm{H}}^{1,p}(\Omega)).$$

One multiply by t^{-s}, and take the L^q_*-norm, so that as a consequence of Lemma 2.16, Theorem 2.10 and the definition of function spaces by restriction

$$\begin{aligned} \|T_\phi u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n_+)} \leqslant \|\mathrm{E}T_\phi u\|_{\dot{\mathrm{B}}^s_{p,q}(\mathbb{R}^n)} &\sim_{p,s,n} \|\mathrm{E}T_\phi u\|_{(\mathrm{L}^p(\mathbb{R}^n), \dot{\mathrm{H}}^{1,p}(\mathbb{R}^n))_{s,q}} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{(\mathrm{L}^p(\Omega), \dot{\mathrm{H}}^{1,p}(\Omega))_{s,q}} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\Omega)}. \end{aligned}$$

If $q < +\infty$ and $(\mathcal{C}_{s,p,q})$ is satisfied, the result follows by density argument. For $q = +\infty$, if $(\mathcal{C}_{s,p,q})$ is satisfied, the result follows from real interpolation by the mean of Proposition 5.26 and the reiteration theorem, Theorem 1.5. One may reproduce a similar proof for T_ϕ^{-1}. ∎

3.3 The trace theorem and related results

Theorem 5.43 *Let $p \in (1,+\infty)$, $q \in [1,+\infty]$, $s \in (1/p, 1+1/p)$. The following statements are true :*

(i) for all $u \in \mathrm{H}^{s,p}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\partial\Omega)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)};$$

In particular, the trace operator $[\cdot]_{|\partial\Omega}$ extends uniquely to a bounded linear operator from $\dot{\mathrm{H}}^{s,p}(\Omega)$ to $\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\partial\Omega)$ whenever $(\mathcal{C}_{s,p})$ is satisfied.

(ii) for all $u \in \mathrm{B}^{s}_{p,q}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\partial\Omega)} \lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{s}_{p,q}(\Omega)};$$

In particular, the trace operator $[\cdot]_{|\partial\Omega}$ extends uniquely to a bounded linear operator from $\dot{\mathrm{B}}^{s}_{p,q}(\Omega)$ to $\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ whenever $(\mathcal{C}_{s,p,q})$ is satisfied.

(iii) for all $u \in \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\mathrm{L}^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega)};$$

(iv) for all $u \in \mathrm{B}^{1+\frac{1}{p}}_{p,1}(\Omega)$,

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{H}}^{1,p}(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)};$$

In particular, the trace operator $[\cdot]_{|\partial\Omega}$ extends uniquely to a bounded linear operator from $\dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)$ to $\dot{\mathrm{H}}^{1,p}(\partial\Omega)$ whenever $p < n-1$ is satisfied.

Proof. — We are going to cut the proof into three parts.

(i) **First part :** homogeneous Sobolev spaces.
Let $u \in \mathrm{H}^{s,p}(\Omega)$. We assume first that $s \in (\frac{1}{p}, 1]$. By Proposition 5.5, we have $T_\phi u \in \mathrm{H}^{s,p}(\mathbb{R}^n_+)$. The standard trace theorem with homogeneous estimates Theorem 2.49 yields that, $x_n \mapsto T_\phi u(\cdot, x_n) \in \mathrm{C}^0_b(\overline{\mathbb{R}_+}, \dot{\mathrm{B}}^{s-1/p}_{p,p}(\mathbb{R}^{n-1}))$, with the estimates

$$\|T_\phi u(\cdot,0)\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s,p,n} \|T_\phi u\|_{\dot{\mathrm{H}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{s,p,n} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} .$$

But for almost every $x' \in \mathbb{R}^{n-1}$, we recall that

$$T_\phi u(x',0) = u(x', 0+\phi(x')) = S_\phi[u_{|\partial\Omega}](x') .$$

Thus, one may apply Lemma 5.32 :

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\partial\Omega)} \lesssim_{p,s,n,\partial\Omega} \|S_\phi[u_{|\partial\Omega}]\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} .$$

Now, we consider $u \in \mathrm{H}^{s,p}(\Omega)$, with $s \in [1, 1+\frac{1}{p})$ it follows from the successive use of Lemma 5.38 and Corollary 5.41, that,

$$\|T_\phi u(\cdot,0)\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,p}(\mathbb{R}^{n-1})} \lesssim_{s,p,n} \|T_\phi u\|_{\dot{\mathcal{C}}^{s,p}(\mathbb{R}^n_+)} \lesssim_{s,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s,p}(\Omega)} .$$

Therefore, one may finish the proof as in the case $s \in (\frac{1}{p}, 1]$.

Hence, if moreover, we assume $(\mathcal{C}_{s,p})$, then every involved space is complete and one may conclude with a density argument.

(ii) **Second part :** homogeneous Besov spaces with traces in Lebesgue/Sobolev spaces.

For $u \in \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega)$, by Lemma 5.42 and Theorem 2.49,

$$T_\phi u \in \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n_+) \subset \mathrm{C}^0_0(\overline{\mathbb{R}_+}, \mathrm{L}^p(\mathbb{R}^{n-1}))$$

with the estimates

$$\|u_{|\partial\Omega}\|_{\mathrm{L}^p(\partial\Omega)} \sim_{p,n,\partial\Omega} \|T_\phi u(\cdot,0)\|_{\mathrm{L}^p(\mathbb{R}^{n-1})} \lesssim_{p,n,\partial\Omega} \|T_\phi u\|_{\dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\mathbb{R}^n_+)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega)}. \tag{5.20}$$

Now, for $u \in \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega) \cap \dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)$, since $\nabla u \in \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega)$, we may use the estimate (5.20),

$$\|[\nabla u]_{|\partial\Omega}\|_{\mathrm{L}^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|\nabla u\|_{\dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)}.$$

But one may check that we have

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{H}}^{1,p}(\partial\Omega)} = \|S_\phi^{-1}\nabla'[S_\phi u_{|\partial\Omega}]\|_{\mathrm{L}^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|[\nabla u]_{|\partial\Omega}\|_{\mathrm{L}^p(\partial\Omega)} \lesssim_{p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)}. \tag{5.21}$$

(iii) **Third part :** homogeneous Besov spaces.

Let $s \in (1/p, 1+1/p)$, $q \in [1,+\infty]$, and let $u \in \mathrm{B}^s_{p,q}(\Omega)$.

$$u \in \mathrm{B}^s_{p,q}(\Omega) \subset \mathrm{B}^{\frac{1}{p}}_{p,1}(\Omega) + \mathrm{B}^{1+\frac{1}{p}}_{p,1}(\Omega) \subset \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega) + \dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega).$$

Let $(a,b) \in \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega) \times \dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)$, such that $u = a + b$,

$$b = a - u \in (\dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega) + \mathrm{B}^s_{p,q}(\Omega)) \cap \dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega) \subset \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega) \cap \dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega).$$

Therefore $u_{|\partial\Omega} \in \mathrm{B}^{s-\frac{1}{p}}_{p,q}(\partial\Omega) \subset \dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\partial\Omega)$ and $(a_{|\partial\Omega}, b_{|\partial\Omega}) \in \mathrm{L}^p(\partial\Omega) \times \mathrm{H}^{1,p}(\partial\Omega)$, so that by definition of the K-functional and homogeneous estimates (5.20) and (5.21), we obtain

$$\begin{aligned} K(t, u_{|\partial\Omega}, \mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega)) &\leqslant \|a_{|\partial\Omega}\|_{\mathrm{L}^p(\partial\Omega)} + t\|b_{|\partial\Omega}\|_{\dot{\mathrm{H}}^{1,p}(\partial\Omega)} \\ &\lesssim_{p,n,\partial\Omega} \|a\|_{\dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega)} + t\|b\|_{\dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)}. \end{aligned}$$

Looking at the infimum over all such pairs (a,b) yields for all $t > 0$,

$$K(t, u_{|\partial\Omega}, \mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega)) \lesssim_{p,n,\partial\Omega} K(t, u, \dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega), \dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega)).$$

Multiplying by $t^{-(s-\frac{1}{p})}$, then taking the L^q_*-norm, by definition of homogeneous Besov spaces on the boundary (see Definition 5.34, and Lemma 2.16), it holds that

$$\begin{aligned} \|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}^{s-\frac{1}{p}}_{p,q}(\partial\Omega)} \sim_{p,s,n} \|u_{|\partial\Omega}\|_{(\mathrm{L}^p(\partial\Omega), \dot{\mathrm{H}}^{1,p}(\partial\Omega))_{s-\frac{1}{p},q}} &\lesssim_{p,s,n,\partial\Omega} \|u\|_{(\dot{\mathrm{B}}^{\frac{1}{p}}_{p,1}(\Omega), \dot{\mathrm{B}}^{1+\frac{1}{p}}_{p,1}(\Omega))_{s-\frac{1}{p},q}} \\ &\lesssim_{p,s,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^s_{p,q}(\Omega)}. \end{aligned}$$

If $q < +\infty$ and $(\mathcal{C}_{s,p,q})$ is satisfied, the result follows by density argument. For $q = +\infty$, if $(\mathcal{C}_{s,p,q})$ is satisfied, the result follows from real interpolation by the mean of Proposition 5.26, Definition 5.34 and the reiteration theorem, Theorem 1.5. ■

We state interesting consequences which concern regularity and integrability of traces in the case of intersection spaces, and identification of function spaces for functions that vanish on the boundary. The proofs are very similar to the proofs one can found at the end of Section 5, in Chapter 2, for Ω being the half-space, therefore we present shortened proofs and the full ones are omitted.

Proposition 5.44 *Let $p \in (1,+\infty)$, $q \in [1,+\infty)$, and $\theta \in (0,1)$, $-1+\frac{1}{p} < s_0 < \frac{1}{p} < s_1 < 1+\frac{1}{p}$ such that*

$$\frac{1}{p} = (1-\theta)s_0 + \theta s_1.$$

Then the following assertions hold.

(i) For all $u \in \dot{\mathrm{H}}^{s_0,p}(\Omega) \cap \dot{\mathrm{H}}^{s_1,p}(\Omega)$, we have $u_{|\partial\Omega} \in \mathrm{B}_{p,p}^{s_1-\frac{1}{p}}(\partial\Omega)$, with the estimate

$$\|u_{|\partial\Omega}\|_{\mathrm{B}_{p,p}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|^{1-\theta}_{\dot{\mathrm{H}}^{s_0,p}(\Omega)} \|u\|^{\theta}_{\dot{\mathrm{H}}^{s_1,p}(\Omega)} + \|u\|_{\dot{\mathrm{H}}^{s_1,p}(\Omega)}.$$

We also have

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}_{p,p}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_1,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{H}}^{s_1,p}(\Omega)};$$

(ii) For all $u \in \dot{\mathrm{B}}^{s_0}_{p,q}(\Omega) \cap \dot{\mathrm{B}}^{s_1}_{p,q}(\Omega)$, we have $u_{|\partial\Omega} \in \mathrm{B}_{p,q}^{s_1-\frac{1}{p}}(\partial\Omega)$, with the estimate

$$\|u_{|\partial\Omega}\|_{\mathrm{B}_{p,q}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,q}(\Omega)} \|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,q}(\Omega)} + \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\Omega)}.$$

We also have

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}_{p,q}^{s_1-\frac{1}{p}}(\partial\Omega)} \lesssim_{s_1,p,n,\partial\Omega} \|u\|_{\dot{\mathrm{B}}^{s_1}_{p,q}(\Omega)};$$

(iii) For all $u \in \dot{\mathrm{B}}^{s_0}_{p,\infty}(\Omega) \cap \dot{\mathrm{B}}^{s_1}_{p,\infty}(\Omega)$, we have $u_{|\partial\Omega} \in \mathrm{L}^p(\partial\Omega)$, with the estimate

$$\|u_{|\partial\Omega}\|_{\mathrm{L}^p(\partial\Omega)} \lesssim_{s_0,s_1,p,n,\partial\Omega} \|u\|^{1-\theta}_{\dot{\mathrm{B}}^{s_0}_{p,\infty}(\Omega)} \|u\|^{\theta}_{\dot{\mathrm{B}}^{s_1}_{p,\infty}(\Omega)}.$$

Proof. — We mention the result Proposition 2.50 for the case $\Omega = \mathbb{R}^n_+$, where the proof only relies on good interpolation inequalities and the appropriate trace estimates. Everything has been made in order to recover the same interpolation inequalities, the result then follows from Theorem 5.43 points *(i)* and *(iii)* for the case of Sobolev spaces, and from points *(ii)* and *(iii)* for the case of Besov spaces. ∎

Proposition 5.45 *Let $p_j \in (1,+\infty)$, $q_j \in [1,+\infty)$, $s_j > 1/p_j$, $j \in \{0,1\}$, such that $(\mathcal{C}_{s_0,p_0})$ (resp. $(\mathcal{C}_{s_0,p_0,q_0})$) is satisfied. Then,*

(i) For all $u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\Omega)$, we have $u_{|\partial\Omega} \in \dot{\mathrm{B}}_{p_j,p_j}^{s_j-\frac{1}{p_j}}(\partial\Omega)$, $j \in \{0,1\}$, with the estimate

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}_{p_j,p_j}^{s_j-\frac{1}{p_j}}(\partial\Omega)} \lesssim_{s_j,p,n} \|u\|_{\dot{\mathrm{H}}^{s_j,p_j}(\Omega)};$$

(ii) For all $u \in [\dot{\mathrm{B}}^{s_0}_{p_0,q_0} \cap \dot{\mathrm{B}}^{s_1}_{p_1,q_1}](\Omega)$, we have $u_{|\partial\Omega} \in \dot{\mathrm{B}}_{p_j,q_j}^{s_j-\frac{1}{p_j}}(\partial\Omega)$, $j \in \{0,1\}$, with the estimate

$$\|u_{|\partial\Omega}\|_{\dot{\mathrm{B}}_{p_j,q_j}^{s_j-\frac{1}{p_j}}(\partial\Omega)} \lesssim_{s_j,p,n} \|u\|_{\dot{\mathrm{B}}^{s_j}_{p_j,q_j}(\Omega)};$$

Proof. — This is a direct consequence of density results provided by Propositions 5.11 and 5.24, and the trace theorem, Theorem 5.43. ■

Lemma 5.46 *Let $p_j \in (1,+\infty)$, $s_j \in (1/p_j, 1+1/p_j)$, $j \in \{0,1\}$ such that $(\mathcal{C}_{s_0,p_0})$ is satisfied. We have the following canonical isomorphism of Banach spaces*

$$\{\, u \in [\dot{\mathrm{H}}^{s_0,p_0} \cap \dot{\mathrm{H}}^{s_1,p_1}](\Omega) \,|\, u_{|\partial\Omega} = 0 \,\} \simeq [\dot{\mathrm{H}}_0^{s_0,p_0} \cap \dot{\mathrm{H}}_0^{s_1,p_1}](\Omega).$$

The result still holds replacing $\dot{\mathrm{H}}^{s_j,p_j}$ by $\dot{\mathrm{B}}^{s_j}_{p_j,q_j}$, $q_j \in [1,+\infty]$, $j \in \{0,1\}$ assuming that $(\mathcal{C}_{s_0,p_0,q_0})$ is satisfied.

Proof. — We mention the result Corollary 2.55 where the proof only rely on integration by parts, the meaning of traces, and the possibility to extends elements of $\dot{\mathrm{H}}^{s,p}$ from Ω to the whole space $\mathbb{R}^n$ by 0 boundedly, whenever $s \in (-1+1/p, 1/p)$. The boundedness of the extension by 0 from Ω to the whole space $\mathbb{R}^n$ is a direct consequence of Corollary 5.7. The same holds for Besov spaces with Proposition 5.18. ■

We conclude with few definitive remarks.

- The naive composition with Poisson's extension only yields right bounded inverse for the trace operator when the regularity index s lies between $1/p$ and 1, for Theorem 5.43.
- One may want a right bounded inverse for the trace operator whenever the regularity index s between $1/p$ and $1+1/p$ in Theorem 5.43. Personal discussions with Patrick Tolksdorf and Moritz Egert persuaded the author that one could adapt Jonsson and Wallin's extension operator [JW84, Chapter VII, Theorem 3] from the boundary to the whole domain in a way so that it preserves homogeneous norms.
- One may use Jonsson and Wallin's extension operator to reprove exactly the same way the proof in Theorem 4.15 but replacing $\mathbb{R}^n_+$ by Ω a special Lipschitz domain.
- Hence, we can make sense of partial traces of differential forms on the boundary. The next step is to show that the potential operators introduced by Costabel, M$^{\mathrm{c}}$Intosh, and Taggart [CMT13] are well-defined for our realization of homogeneous function spaces. Combining this with the arguments in [MM18], we can then extend the results in Chapter 4 to the setting of special Lipschitz domains. Specifically, we aim to establish appropriate generalizations of Theorems 4.42 and 4.40 for Ω a special Lipschitz domain :

For a brighter future. *There exists $\varepsilon > 0$ depending on Ω such that for all $(p,s) \in (1,+\infty) \times (-1+1/p, 1/p)$ that satisfy*

$$\left|\frac{1}{p} - \frac{1}{2}\right| + \frac{|s|}{n} < \frac{1}{2n} + \varepsilon$$

the following assertions hold

(i) The generalized Helmholtz-Leray projector $\mathbb{P} : \dot{\mathrm{H}}^{s,p}(\Omega,\Lambda) \longrightarrow \dot{\mathrm{H}}^{s,p}_{\mathrm{t},\sigma}(\Omega,\Lambda)$ is a well-defined bounded linear operator, and the following identity holds

$$\mathbb{P} = \mathrm{I} - \mathrm{d}(-\Delta_{\mathcal{H}})^{-\frac{1}{2}}\mathrm{d}^*(-\Delta_{\mathcal{H}})^{-\frac{1}{2}}.$$

(ii) The following Hodge decomposition holds

$$\dot{\mathrm{H}}^{s,p}(\Omega,\Lambda) = \dot{\mathrm{H}}^{s,p}_{\mathrm{t},\sigma}(\Omega,\Lambda) \oplus \dot{\mathrm{H}}^{s,p}_{\gamma}(\Omega,\Lambda).$$

(iii) For all $\mu \in (0,\pi)$, the operator $(\dot{\mathrm{D}}^s_p(\Delta_{\mathcal{H}}), -\Delta_{\mathcal{H}})$ admits a bounded $\mathbf{H}^\infty(\Sigma_\mu)$-holomorphic functional calculus on $\dot{\mathrm{H}}^{s,p}(\Omega,\Lambda)$.

Generalizations replacing Ω to be uniformly special Lipschitz, by a uniform special C^1 or by a uniform special $\mathrm{C}^{1,1}$ domain might also be interesting to look at.

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