



Metric theory of approximation and geometry of discrete self-similar measures

Edouard Daviaud

► To cite this version:

Edouard Daviaud. Metric theory of approximation and geometry of discrete self-similar measures. Mathematics [math]. Université Gustave Eiffel, 2023. English. ⟨NNT : ⟩. ⟨tel-04162415⟩

HAL Id: tel-04162415

<https://hal.science/tel-04162415v1>

Submitted on 21 Jul 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

École Doctorale MSTIC

UMR 8050 - Laboratoire d'Analyse et de Mathématiques
Appliquées

Thèse

Présentée pour l'obtention du grade de DOCTEUR

DE L'UNIVERSITE PARIS-EST

par

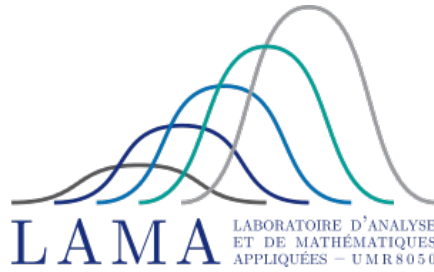
Edouard Daviaud

Metric theory of approximation and geometry of discrete self-similar measures

Spécialité : Mathématiques

Soutenue le 9 Juin 2023 devant un jury composé de :

Directeur de thèse	Prof. Stéphane Seuret	(Université Paris-Est Créteil)
Directeur de thèse	Prof. Julien Barral	(Université Sorbonne Paris-Nord)
Rapporteur	Prof. Yann Bugeaud	(Université de Strasbourg)
Rapporteur	Prof. Sanju Velani	(University of York)
Examineur	Prof. Faustin Adiceam	(Université Paris-Est Créteil)
Examineur	Prof. Céline Esser	(Université de Liège)
Examineur	Prof. Stéphane Jaffard	(Université Paris-Est Créteil)
Examineur	Prof. Maarit Järvenpää	(University Oulu)



**UMR 8050 - Laboratoire d'Analyse et de Mathématiques
Appliquées**

Université de Paris-Est Créteil
61, avenue du Général de Gaulle
94010 Créteil
France

Contents

1	Remerciements	1
2	Introduction en français	3
2.1	Théorèmes d'ubiquité	4
2.1.1	Un historique des théorèmes d'ubiquité	4
2.1.2	Un théorème général d'ubiquité hétérogène	7
2.1.3	Etude de l'optimalité du théorème d'ubiquité	9
2.2	Application en approximation diophantienne	12
2.3	Cibles rétrécissantes faiblement conformes	14
2.4	Application aux mesures discrètes	17
2.5	Conclusion et perspectives	20
3	Introduction	23
3.1	Ubiquity theorems	23
3.1.1	An historic of ubiquity theorems	23
3.1.2	A general ubiquity theorem	27
3.1.3	Study of optimality in ubiquity theorems	29
3.2	An application in Diophantine approximation	32
3.3	Weakly conformal shrinking targets	34
3.4	Application to discrete measure	37
3.5	Conclusion and perspectives	40
4	Covering Lemmas, Hausdorff measures and IFS	45
4.1	Covering lemmas	45
4.2	Hausdorff dimension, dimension of measures	49
4.2.1	Hausdorff measures, dimension and content	49
4.2.2	Packing measure and packing dimension	52

4.2.3	Dimension of measures	53
4.2.4	Multifractal analysis of measures	54
4.3	Iterated function systems	55
4.3.1	IFS and separation condition	57
4.3.2	Generalities on conformal and weakly conformal IFS . . .	60
4.3.3	Weakly conformal systems satisfying AWSC	67
4.3.4	Dimension properties of self-similar measures	73
5	Anisotropic ubiquity for quasi-Bernoulli measures	81
5.1	Anisotropic ubiquity and geometric quasi-Bernoulli measures	81
5.2	Proof of Theorem 5.1.5	84
6	μ-a.c sequences of balls	99
6.1	limsup and μ -a.c property	99
6.1.1	Consequences of the μ -asymptotic covering property . . .	100
6.1.2	Proof of Theorem 6.1.2	103
6.1.3	A version of Borel-Cantelli Lemma	104
6.2	Essential Hausdorff content	106
6.2.1	Computation of essential content for the Lebesgue measure	107
6.2.2	Essential content for weakly conformal measures	108
7	Heterogeneous ubiquity theorem	117
7.1	A general heterogeneous ubiquity theorem	117
7.2	Proof of Theorem 7.1.2 and Corollary 7.1.6	119
7.2.1	Construction of the Cantor set and the measure	123
7.2.2	Proof of Corollary 7.1.6	133
7.3	Ubiquity theorems and self-conformal measures	135
8	Upper-bound and ubiquity	143
8.1	Weakly redundant sequences of balls	144
8.2	Extraction of μ -a.c sub-sequences of balls	146
8.2.1	Extraction of weakly redundant μ -a.c subsequences . . .	148
8.2.2	Extraction of sub-sequences of balls with conditioned mea- sure	149
8.2.3	G_δ sets of full measure and extraction of μ -a.c sub-sequences	155

8.3	Study of the optimality of ubiquity theorems	156
8.3.1	An upper-bound for ubiquity Theorem in the self-similar case	156
8.3.2	Application in the case of balls and rectangles	160
9	Example middle-third Cantor set	163
9.1	Hausdorff dimension of $K_{1/3}^0$	165
9.2	Proof of Theorem 9.0.3	168
10	weakly conformal shrinking targets	171
10.1	Proof of item (1) of Theorem 10.0.1	172
10.2	Proof of item (2) of Theorem 10.0.1	174
10.3	Proof of item (3) of Theorem 10.0.1	175
10.3.1	Variational principle and C^1 weakly conformal IFS . . .	175
10.3.2	Proof of item(3) of Theorem 10.0.1	179
10.4	case of self-similar IFS	181
11	Discrete self-similar measures	183
11.1	Definition and first result	183
11.2	Scaling function of discrete self-similar measures	188
11.3	Proof of Proposition 11.1.6	193
11.4	Multifractal analysis of χ under AWSC	195
11.4.1	Estimates for D_χ for $h \geq \lambda_c + t$	198
11.4.2	Multifractal spectrum of χ for $h \in [0, \lambda_c + t]$	201

Chapter 1

Remerciements

Je souhaite d'abord remercier chaleureusement Stéphane Seuret et Julien Barral (mes deux directeurs de thèse) pour les conseils qu'ils m'ont prodigué et la patience (parfois le courage, il faut bien le dire) dont ils ont fait preuve tout au long de ma thèse, même après avoir constaté que la 20^{ème} version de tel ou tel article changeait du tout au tout pour rendre un résultat un petit peu plus général. En tout cas, il me semblait inévitable de ne pas énoncer l'adage suivant qui m'a été répété de très très (très) nombreuses fois:

“Le mieux est l'ennemi du bien.”

Je souhaite aussi remercier particulièrement les deux rapporteurs de cette thèse, Yann Bugeaud et Sanju Velani pour avoir pris le temps de lire, corriger et commenter mon manuscrit, ainsi que Faustin Adiceam, Céline Esser, Stéphane Jaffard et Maarit Järvenpää pour avoir accepté de faire partie de mon jury.

Je remercie aussi tous les membres du LAMA pour les nombreuses et intéressantes discussions scientifiques et pédagogiques.

Je souhaite également bon courage à Elie, Clément, Nadia (même si elle n'est plus vraiment doctorante), Irène, Sana, Asma, Teo, Thomas, Houssam et Valentin pour finir leur thèse. C'est dur mais ça vaut le coup!

Enfin, je remercie plus particulièrement mes co-bureau fractalsites Wejdene, Quentin, Qian et Guillaume. L'excellent environnement dans lequel nous avons pu travailler (et pas que travailler) a largement contribué à un certain épanouissement scientifique. Je vous souhaite également bon courage pour finir votre thèse pour les trois premiers.

Mes amis, Sakina, Magda, Jimmy, Antoine, Charlie, Paul, Antoine, (oui ça fait deux fois Antoine, mais de nos jours, tout le monde s'appelle Antoine ou Stéphane alors....), Thimotée, Pierre et David, merci pour vos conseils et votre

soutien durant ces difficiles années de thèses (et plus particulièrement pendant les trois dernières).

Enfin, je dédie aussi cette thèse à ma mère, mon père, ma soeur, César, Nico, mon frère, Ani, à Papi, à mes oncles et tantes et mes cousins.

Chapter 2

Introduction en français

La théorie de l'approximation métrique cherche à décrire géométriquement (souvent via le calcul de dimensions de Hausdorff) l'ensemble des éléments approximables par une famille de points $(x_n)_{n \in \mathbb{N}}$ choisie à l'avance dans un espace métrique (X, d) . Historiquement, cette théorie est née des travaux de Dirichlet portant sur l'approximation des nombres réels par des nombres rationnels au XIXème siècle. Ces travaux furent complétés et enrichis plus tard, dans les années 1920, par Jarnik et Besicovitch [47] et récemment, dans les années 2000, par les travaux de Jaffard [46], de Bugeaud [15], de Beresnevich et de Velani [11]. Ce n'est qu'à partir des années 1990, que certains de ces problèmes d'approximation diophantienne ont été reformulés en terme d'approximation par des orbites de systèmes dynamiques [42]. Dans ce cadre, on étudie le cas où la famille $(x_n)_{n \in \mathbb{N}}$ est une orbite, i.e., $x_n = T^n(x)$, $x \in X$, où X est un espace métrique et $T : X \rightarrow X$ est une application mesurable. Pour des exemples où T est le doublement de l'angle sur le cercle, une fonction Markovienne dilatante sur $[0, 1]$ ou bien pour d'autres exemples de fonctions dilatantes, voir [33, 55, 65].

Ces ensembles jouent également un rôle important en analyse multifractale. Par exemple, l'étude des séries de Riemann [46] fait intervenir la vitesse d'approximation d'un réel par des nombres rationnels. Pour d'autres exemples de travaux mêlant analyse multifractale et approximation diophantienne, citons également les travaux de Barral, Seuret, Liao, Fan, Shmel'ing, Troubetzkoy, Jaffard et Persson [10, 33, 45, 55, 64].

2.1 Théorèmes d'ubiquité

2.1.1 Un historique des théorèmes d'ubiquité

Fixons $d \in \mathbb{N}^*$ et munissons \mathbb{R}^d de la norme $\|(x_1, \dots, x_d)\|_\infty = \max_{1 \leq i \leq d} |x_i|$ (le choix de la norme n'importe en fait pas du tout pour les résultats de ce manuscrit).

Une façon de donner du sens à la notion d'approximation d'un point $x \in \mathbb{R}^d$ par une suite de points $(x_n)_{n \in \mathbb{N}}$ de \mathbb{R}^d et d'en caractériser la vitesse, est de se donner des suites d'ensembles $(U_n(\delta))_{n \in \mathbb{N}}$, où $\delta \geq 1$ (parfois on prend un multi-paramètre $\delta \in [1, +\infty)^k$, où $k \in \mathbb{N}$ est fixé) vérifiant:

- pour tout $n \in \mathbb{N}$, pour tout $\delta \geq 1$, $x_n \in U_n(\delta)$,
- pour tout $\delta \geq 1$, $|U_n(\delta)| \rightarrow 0$,
- pour tout $\delta' > \delta$, pour tout $n \in \mathbb{N}$, $U_n(\delta') \subset U_n(\delta)$.

On s'intéresse alors pour chaque δ , à l'ensemble E_δ des points x pour lesquels on peut trouver une infinité d'entiers $(n_k)_{k \in \mathbb{N}}$ telle que $x \in U_{n_k}(\delta)$. L'ensemble de ces points est appelé la *limsup*¹ des ensembles $(U_n(\delta))_{n \in \mathbb{N}}$ et s'écrit

$$E_\delta = \limsup_{n \rightarrow +\infty} U_n(\delta) := \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} U_n(\delta).$$

On dit alors qu'un point x est approximable à vitesse δ si $x \in F_\delta := E_\delta \setminus \bigcup_{\delta' > \delta} E_{\delta'}$ et on cherche en général à calculer les dimensions de Hausdorff (voir Section 4.2, Lemma 4.2.3 pour une définition de la dimension de Hausdorff) $\dim_H(E_\delta)$ et $\dim_H(F_\delta)$ (dans les faits, on arrive souvent à calculer $\dim_H(F_\delta)$ quand on sait que $\mathcal{H}^\zeta(E_\delta) > 0$ où ζ est une jauge appropriée et \mathcal{H}^ζ est la mesure de Hausdorff associée).

Le premier exemple historique de ce genre de calcul est le théorème de Jarnik-Besicovitch.

Théorème 2.1.1 (Jarnik-Besicovitch [47]). *Pour $x \in \mathbb{R}$ et $r > 0$, notons $B(x, r)$ la boule fermée centrée en x et de rayon r . Alors, pour tout $\delta \geq 1$, on a, notant $E_\delta := \limsup_{q \in \mathbb{N}, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta}})$ et $F_\delta := E_\delta \setminus \bigcup_{\delta' > \delta} E_{\delta'}$:*

$$\dim_H(E_\delta) = \dim_H(F_\delta) = \frac{1}{\delta} \quad (2.1)$$

¹Notons ici que la définition de l'ensemble $\limsup_{n \rightarrow +\infty} U_n$ est indépendante de la numérotation choisie pour les ensembles $\{U_n\}_{n \in \mathbb{N}}$, c'est à dire que pour toute bijection $\phi : \mathbb{N} \rightarrow \mathbb{N}$, on a $\limsup_{n \rightarrow +\infty} U_n = \limsup_{n \rightarrow +\infty} U_{\phi(n)}$. En particulier, nous prendrons dans ce manuscrit parfois des limsups d'ensembles sans préciser explicitement la numérotation choisie.

Mentionnons aussi d'autres travaux portant sur l'approximation rationnelle [11, 15, 16, 27] (approximation rationnelle homogène et inhomogène dans \mathbb{R}^d) et [1, 17, 18, 54] (approximation rationnelle d'éléments du Cantor triadique).

Usuellement, pour calculer les dimensions de ces ensembles, on procède séparément pour la minoration et la majoration. Dans la pratique, majorer $\dim_H(E_\delta)$ s'avère souvent ou relativement simple ou très complexe (le lecteur notera que dans ce manuscrit, toutes les preuves des majorations sont relativement plus faciles que celle des minoration de $\dim_H(E_\delta)$).

Les théorèmes d'ubiquité, plus souvent appelés “principes de transfert de masse” ([11]), sont des outils établis pour traiter de la minoration de $\dim_H(E_\delta)$.

Ces théorèmes fonctionnent de la façon suivante: on cherche une mesure $\mu \in \mathcal{M}(\mathbb{R}^d) := \{\text{mesures de probabilité sur } \mathbb{R}^d\}$ (qu'on espère adaptée au problème) de sorte que $\mu(E_1) = 1$. Ensuite, en utilisant les propriétés géométriques de la mesure μ (auto-similarité, analyse multifractale par exemple), on cherche à établir une minoration de $\dim_H E_\delta$ pour tout $\delta > 1$. Voici un exemple important de ce genre de théorème. On dit qu'une mesure $\mu \in \mathcal{M}(\mathbb{R}^d)$ est s -Alfhors régulière, où $0 \leq s \leq d$, s'il existe deux constantes $0 < C_1 \leq C_2$ de sorte que, pour tout boule $B(x, r)$ avec $x \in \text{supp}(\mu) := \text{le support topologique de } \mu$ et $0 < r \leq 1$, on a

$$C_1 r^s \leq \mu(B(x, r)) \leq C_2 r^s.$$

Un exemple de telle mesure est la mesure de Lebesgue \mathcal{L}^d sur $[0, 1]^d$.

Théorème 2.1.2 (Beresnevich-Velani [11]). *Soit $0 \leq s \leq d$ et $\mu \in \mathcal{M}(\mathbb{R}^d)$ une mesure s -Alfhors régulière. Soit $(B_n)_{n \in \mathbb{N}}$ une suite de boules fermées centrées sur $\text{supp}(\mu)$ et satisfaisant:*

- $|B_n| \rightarrow 0$,
- $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$.

Soit $\delta > 1$, et B une boule centrée sur $\text{supp}(\mu)$. Notons pour tout $s_0 \geq 0$, \mathcal{H}^{s_0} la mesure de Hausdorff de dimension s_0 (voir (4.2.2)) et pour $n \in \mathbb{N}$, $B_n^\delta = B(x_n, r_n^\delta)$ si $B_n = B(x_n, r_n)$. On a

$$\mathcal{H}^{\frac{s}{\delta}}(\limsup_{n \rightarrow +\infty} B_n^\delta \cap B) = +\infty.$$

En particulier, on retrouve un résultat établi par Jaffard (1999),

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{s}{\delta}.$$

Dans le cas où $\mu = \mathcal{L}^d$, on peut aussi traiter le cas plus général où au

lieu de déterminer une minoration de $\limsup_{n \rightarrow +\infty} B_n^\delta$ pour tout $\delta > 1$, on décrit une quantité géométrique, le contenu de Hausdorff (voir Section 4.2, equation (4.7)), à estimer pour minorer (efficacement) $\dim_H(\limsup_{n \rightarrow +\infty} U_n)$ pour n'importe quelle suite d'ouverts $(U_n)_{n \in \mathbb{N}}$ satisfaisant $U_n \subset B_n$.

Théorème 2.1.3 (Koivusalo-Rams [51]). *Soit $(B_n)_{n \rightarrow +\infty}$ une suite de boules de $[0, 1]^d$ vérifiant $|B_n| \rightarrow 0$, $(U_n)_{n \in \mathbb{N}}$ une suite d'ouverts vérifiant pour tout $n \in \mathbb{N}$, $U_n \subset B_n$ et $0 \leq s \leq d$.*

Supposons que:

- la suite $(B_n)_{n \in \mathbb{N}}$ vérifie

$$\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1,$$

- pour tout $n \in \mathbb{N}$ assez grand

$$\mathcal{H}_\infty^s(U_n) \geq \mathcal{L}^d(B_n).$$

Alors

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) \geq s.$$

Ajoutons ici que des études de dimensions de Hausdorff d'ensemble \limsup générés par des ensembles aléatoires faisaient déjà apparaître explicitement le rôle du contenu de Hausdorff pour caractériser la dimension de l'ensemble \limsup [28, 38, 48, 49] et que récemment, d'autres résultats d'ubiquité basés sur le contenu de Hausdorff ont également été établis [29]. Un exemple d'application est le cas où chaque U_n est un rectangle ouvert. En effet, fixons $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$. Soit $(B_n := x_n + \prod_{i=1}^d [-r_n, r_n])_{n \in \mathbb{N}}$ une suite de boules de $[0, 1]^d$ vérifiant $|B_n| \rightarrow 0$ et $\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$. Posons

$$R_n = x_n + \prod_{i=1}^d (-r_n^{\tau_i}, r_n^{\tau_i}). \quad (2.2)$$

Alors, pour tout $0 \leq s \leq d$, le contenu de Hausdorff de dimension s des rectangles R_n peut être estimé explicitement en fonction de r_n et des nombres τ_1, \dots, τ_d à une constante multiplicative près C qui ne dépend que de d . En particulier, on obtient, que pour $s_0 = \min_{1 \leq i \leq d} \frac{d + \sum_{1 \leq i \leq d} \tau_k - \tau_i}{\tau_k}$ ([51]),

$$\mathcal{H}_\infty^{s_0}(R_n) \geq C r_n^d \geq C \mathcal{L}^d(B_n).$$

On vérifie facilement que ceci implique que pour tout $s < s_0$ et n assez grand,

$$\mathcal{H}_\infty^s(R_n) \geq r_n^d = \mathcal{L}^d(B_n).$$

Par application du Théorème 2.1.3, on obtient un résultat qui avait été étblit par Wang, Wu et Xu dans [75],

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq s_0 = \min_{1 \leq i \leq d} \frac{d + \sum_{1 \leq i \leq d} \tau_k - \tau_i}{\tau_k}.$$

Lorsque la mesure n'est pas homogène (Alfhors régulière), le problème est plus complexe.

Le théorème de Barral-Seuret traite le cas où la mesure est multinômiale de support $[0, 1]^d$ (mais on en déduit relativement simplement le cas auto-similaire satisfaisant l'Open Set Condition qu'on raccourcira en OSC, voir Section 4.3, Définition 4.3.3 pour plus de détails sur les différentes conditions de séparation). Pour μ une mesure auto-similaire, notons $\dim(\mu)$ dimension de μ (voir Section 4.2, Définition 4.2.12 et Section 4.3, Théorème 4.3.15).

Théorème 2.1.4 (Barral-Seuret [8]). *Soit $\mu \in \mathbb{R}^d$ une mesure multinômiale de support $[0, 1]^d$, $(B_n)_{n \in \mathbb{N}}$ une suite de boules de $[0, 1]^d$ vérifiant $|B_n| \rightarrow 0$.*

Si

$$\mu(\limsup_{n \rightarrow +\infty} B_n) = 1,$$

alors, pour tout $\delta \geq 1$,

$$\dim_H(\limsup_{n \rightarrow +\infty}) = \frac{\dim(\mu)}{\delta}. \quad (2.3)$$

Dans ce manuscrit, on développe un outil qui joue le même rôle que le contenu de Hausdorff pour le Théorème 2.1.3 mais dans le cas où la mesure $\mu \in \mathcal{M}(\mathbb{R}^d)$ est quelconque. Nous montrerons aussi que le théorème d'ubiquité résultant étend à la fois les Théorèmes 2.1.3 et 2.1.4.

2.1.2 Un théorème général d'ubiquité hétérogène

Commençons par introduire la quantité géométrique suivante.

Définition 2.1.5 (D. [25]). *Soit $\mu \in \mathcal{M}(\mathbb{R}^d)$ $0 \leq s \leq d$ et $A \subset \mathbb{R}^d$ un ensemble mesurable. Le contenu de Hausdorff μ -essentiel s -dimensionnel de A est défini par*

$$\mathcal{H}_{\infty}^{\mu, s}(A) = \inf \{ \mathcal{H}_{\infty}^s(E) : E \subset A, \mu(E) = \mu(A) \}. \quad (2.4)$$

Pour $\mu \in \mathcal{M}(\mathbb{R}^d)$, notons $\underline{\dim}_H(\mu)$ la dimension de Hausdorff inférieure de μ (définie en Section 4.2, Définition 4.2.12). Le théorème d'ubiquité associé à cette quantité géométrique est le suivant (il s'agit en fait d'une version simplifiée du Théorème 7.1.2).

Théorème 2.1.6 (D. [25]). Soit $\mu \in \mathcal{M}(\mathbb{R}^d)$, $(B_n)_{n \in \mathbb{N}}$ une suite de boules satisfaisant $|B_n| \rightarrow 0$, $(U_n)_{n \in \mathbb{N}}$ une suite d'ouverts satisfaisant, pour tout $n \in \mathbb{N}$, $U_n \subset B_n$ et $0 \leq s < \underline{\dim}_H(\mu)$.

Supposons que

- $\mu(\limsup_{n \rightarrow +\infty} \frac{1}{2} B_n) = 1$,
- pour n assez grand,

$$\mathcal{H}_\infty^{\mu,s}(U_n) \geq \mu(B_n).$$

Alors on a

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) \geq \min \{s, \underline{\dim}_H(\mu)\}. \quad (2.5)$$

Dans le Chapitre 7, le résultat que nous montrerons est en réalité un peu plus général que le Théorème 2.1.6.

Evidemment, pour pouvoir appliquer ce théorème à une mesure μ , il faut comprendre cette mesure suffisamment finement et être capable d'estimer $\mathcal{H}_\infty^{\mu,s}(U_n)$ pour tout $0 \leq s \leq \underline{\dim}_H(\mu)$ et chacun des ouverts U_n . Il se trouve que lorsque la mesure est faiblement conforme (Section 4.3, Définition 4.3.1), on peut estimer précisément le contenu μ -essentiel de n'importe quel ouvert **sans aucune hypothèse de séparation sur l'IFS**.

Théorème 2.1.7 (D. [23]). Soit $\mu \in \mathcal{M}(\mathbb{R}^d)$ une mesure faiblement conforme de support K .

Pour tout $0 \leq s < \dim(\mu)$, pour tout $\varepsilon > 0$, il existe une constante $c = c(d, \mu, s, \varepsilon) > 0$ qui dépend de d, μ, s et ε , telle que pour toute boule fermée $B = B(x, r)$ centrée sur K et $r \leq 1$, pour tout ouvert Ω , on a

$$\begin{aligned} c(d, \mu, s, \varepsilon) |B|^{s+\varepsilon} &\leq \mathcal{H}_\infty^{\mu,s}(\overset{\circ}{B}) \leq \mathcal{H}_\infty^{\mu,s}(B) \leq |B|^s \\ c(d, \mu, s, \varepsilon) \mathcal{H}_\infty^{s+\varepsilon}(\Omega \cap K) &\leq \mathcal{H}_\infty^{\mu,s}(\Omega) \leq \mathcal{H}_\infty^s(\Omega \cap K). \end{aligned} \quad (2.6)$$

Pour tout $s > \dim(\mu)$, $\mathcal{H}_\infty^{\mu,s}(\Omega) = 0$.

Ainsi la combinaison des Théorèmes 2.1.6 et 2.1.7 permet de retrouver le Théorème 2.1.3 en prenant $\mu = \mathcal{L}^d$ (dans ce cas, $K = [0, 1]^d$) et de montrer les corollaires suivants.

Corollaire 2.1.8. Soit $\mu \in \mathcal{M}(\mathbb{R}^d)$ une mesure faiblement conforme et $(B_n)_{n \in \mathbb{N}}$ une suite de boules centrées sur $\text{supp}(\mu)$ vérifiant $|B_n| \rightarrow 0$.

Si

$$\mu(\limsup_{n \rightarrow +\infty} B_n) = 1,$$

alors, pour tout $\delta \geq 1$,

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{\dim(\mu)}{\delta}.$$

Corollaire 2.1.9. Soit $\mu \in \mathcal{M}(\mathbb{R}^d)$ une mesure faiblement conforme de support $[0, 1]^d$ et $(B_n)_{n \in \mathbb{N}}$ une suite de boules de $[0, 1]^d$ vérifiant $|B_n| \rightarrow 0$. Soit $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$ et R_n , défini par (2.2).

Si

$$\mu(\limsup_{n \rightarrow +\infty} B_n) = 1,$$

alors

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq \min_{1 \leq i \leq d} \frac{\dim(\mu) + \sum_{1 \leq i \leq d} \tau_k - \tau_i}{\tau_k}.$$

Notons que le Corollaire 2.1.8 étend directement le Théorème 2.1.4.

Mentionnons aussi que pour tout $1 \leq \tau_1 \leq \dots \leq \tau_d$, toute suite de matrices de rotations $(O_{k,n})_{k \in \mathbb{N}, 0 \leq n \leq 2^k - 1} \in \mathcal{O}_d(\mathbb{R})^{\mathbb{N}}$, presque pour toute suite de réalisations de variables aléatoires i.i.d. et uniformément distribuées dans $[0, 1]^d$, $(x_{k,n})_{k \in \mathbb{N}, 0 \leq n \leq 2^k - 1}$, il est démontré dans [48] (et la preuve utilise implicitement le Corollaire 2.1.9 dans le cas de la mesure de Lebesgue) que la famille de rectangles

$$\mathcal{R} = \left\{ x_{k,n} + O_{k,n} \prod_{i=1}^d (-2^{-k\tau_i}, 2^{k\tau_i}), 0 \leq n \leq 2^k - 1, k \in \mathbb{N} \right\}$$

vérifie

$$\dim_H(\limsup_{R \in \mathcal{R}} R) = \min_{1 \leq k \leq d} \left\{ \frac{d + \sum_{j=1}^k \tau_k - \tau_j}{\tau_k} \right\}.$$

En particulier la borne donnée par le Corollaire 2.1.9 est optimale. Nous verrons un peu plus tard qu'elle l'est même en un sens plus fort.

Les Corollaires 2.1.8 et 2.1.9 illustrent l'intérêt du Théorème 2.1.6: pour minorer la quantité $\dim_H(\limsup_{n \rightarrow +\infty} U_n)$, on établit pour tout $n \in \mathbb{N}$ une relation qui ne dépend que de B_n , U_n et μ .

Dans le Chapitre 8, on étudie l'optimalité des bornes données par le Théorème 2.1.6.

2.1.3 Etude de l'optimalité du théorème d'ubiquité

Les minoration établies par les Théorèmes 2.1.3, 2.1.4 et 2.1.6 n'ont d'intérêt que si elles sont, en un sens satisfaisant, optimales. Une façon de traiter la

question de l'optimalité de ces bornes est de montrer que si l'on ne dispose pas d'informations supplémentaires (par rapport à celles données par les hypothèses de ces théorèmes) sur la suite de boules $(B_n)_{n \in \mathbb{N}}$, il n'y a pas de raison que $\dim_H(\limsup_{n \rightarrow +\infty} U_n)$ soit supérieur à la borne donnée. Une méthode pour établir que c'est le cas est d'extraire de $(B_n)_{n \in \mathbb{N}}$, quand c'est possible, une sous-suite $(B_{\phi(n)})_{n \in \mathbb{N}}$ qui vérifie toujours les hypothèses des théorèmes d'ubiquité et telle que $\dim_H(\limsup_{n \rightarrow +\infty} U_{\phi(n)})$ est précisément la borne donnée par le théorème. Dans cette situation, nous sommes assurés que l'on ne peut pas obtenir de meilleure borne sous les mêmes hypothèses.

Introduisons la définition suivante, qui est une variante (équivalente) de la notion de suite de boules faiblement redondantes introduite par Barral et Seuret et rappelée également plus tard dans ce manuscrit (Définition 8.1.1).

Définition 2.1.1. Soit $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ une suite de boules vérifiant $|B_n| \rightarrow 0$. Pour $k \in \mathbb{N}$, définissons

$$\begin{aligned} \mathcal{T}_k(\mathcal{B}) &= \{B_n \in \mathcal{B} : 2^{-k-1} \leq |B_n| < 2^{-k}\} \\ N_k(\mathcal{B}) &= \max_{B \in \mathcal{T}_k(\mathcal{B})} \#\{B' \in \mathcal{T}_k(\mathcal{B}) : B \cap B' \neq \emptyset\}. \end{aligned} \quad (2.7)$$

On dit que \mathcal{B} est faiblement redondante si

$$\lim_{k \rightarrow +\infty} \frac{\log N_k(\mathcal{B})}{k} = 0.$$

Heuristiquement, une suite de boules $(B_n)_{n \in \mathbb{N}}$ est faiblement redondante si pour chaque $k \in \mathbb{N}$, chaque boule B_n satisfaisant $|B_n| \approx 2^{-k}$ intersecte moins de $2^{o(k)}$ autres boules $B_{n'}$ satisfaisant aussi $|B_{n'}| \approx 2^{-k}$. Une remarque importante est que la faible redondance est une hypothèse naturelle pour étudier l'optimalité des théorèmes d'ubiquité et qu'elle porte uniquement sur la famille de boules (il n'y pas d'hypothèse impliquant une mesure). La conséquence techniquement importante de cette propriété est que, pour tout mesure $\mu \in \mathcal{M}([0, 1]^d)$ et tout $\varepsilon > 0$, on a

$$\sum_{n \in \mathbb{N}} |B_n|^\varepsilon \mu(B_n) < +\infty. \quad (2.8)$$

Beaucoup de suites de boules sont faiblement redondantes, comme la suite des boules rationnelles $\left(B\left(\frac{p}{q}, \frac{1}{q^2}\right)\right)_{q \in \mathbb{N}, 0 \leq p \leq q, q \wedge p = 1}$, mais il existe aussi des exemples de suites naturelles de boules ne satisfaisant pas cette hypothèse. C'est le cas par exemple de la suite $\left(B\left(\frac{p}{q}, \frac{1}{q^2}\right)\right)_{q \in \mathbb{N}, 0 \leq p \leq q}$. Cependant, on peut montrer pour certains de ces exemples que, bien qu'elles ne satisfassent pas l'hypothèse de faible redondance, ces suites vérifient tout de même l'équation (2.8) pour une

mesure naturelle $\left(B\left(\frac{p}{q}, \frac{1}{q^2}\right)\right)_{q \in \mathbb{N}, 0 \leq p \leq q}$ pour la mesure de Lebesgue). Ajoutons aussi que d'autres auteurs utilisent des hypothèses comparables mais légèrement différentes, telles que l'hypothèse de régularité optimale d'un système de points approximants (voir [15]).

Dans le cas où la mesure est faiblement conforme, l'hypothèse de faible redondance permet d'établir le théorème de majoration suivant.

Théorème 2.1.10 (D.[24]). *Soit $\mu \in \mathcal{M}(\mathbb{R}^d)$ une mesure faiblement conforme, $(B_n)_{n \in \mathbb{N}}$ une suite de boules faiblement redondante, centrées sur $\text{supp}(\mu)$ et satisfaisant $|B_n| \rightarrow 0$, $(U_n)_{n \in \mathbb{N}}$ une suite d'ouverts vérifiant $U_n \subset B_n$ et $0 \leq s < \dim(\mu)$.*

Si pour tout n assez grand,

$$\mathcal{H}_{\infty}^{\mu,s}(U_n) \leq \mu(B_n),$$

alors on a

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) \leq s.$$

Notons qu'étant donnés $\mu \in \mathcal{M}(\mathbb{R}^d)$, $(B_n)_{n \in \mathbb{N}}$ et $(U_n)_{n \in \mathbb{N}}$ satisfaisant pour tout $n \in \mathbb{N}$, $U_n \subset B_n$, il n'est pas vrai en général qu'il existe un exposant critique s_0 de sorte que pour tout $s < s_0$, pour n assez grand, $\mathcal{H}_{\infty}^{\mu,s}(U_n) \geq \mu(B_n)$ et pour tout $s > s_0$, pour n assez grand $\mathcal{H}_{\infty}^{\mu,s}(U_n) \leq \mu(B_n)$.

Néanmoins, dans le cas des mesures faiblement conformes, lorsque le choix des ensembles U_n est fait de "façon cohérente" (par exemple, si l'on prend des rectangles contractés via les mêmes facteurs de contraction $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$ ou bien des boules B_n^{δ} , $\delta \geq 1$ étant fixé à l'avance), le Théorème 2.1.10 permet de conclure à l'optimalité des bornes données par les Corollaires 2.1.8 et 2.1.9.

Corollaire 2.1.11. *Soit $\mu \in \mathcal{M}(\mathbb{R}^d)$ une mesure faiblement conforme et $(B_n)_{n \in \mathbb{N}}$ une suite de boules centrées sur $\text{supp}(\mu)$ vérifiant $\mu(\limsup_{n \rightarrow +\infty} \frac{1}{2} B_n) = 1$ et $|B_n| \rightarrow 0$. Alors:*

1. *Il existe une sous-suite $(B_{\phi(n)})_{n \in \mathbb{N}}$ vérifiant $\mu(\limsup_{n \rightarrow +\infty} B_{\phi(n)}) = 1$ et, pour tout $\delta \geq 1$,*

$$\dim_H(\limsup_{n \rightarrow +\infty} B_{\phi(n)}^{\delta}) = \frac{\dim(\mu)}{\delta}.$$

2. *Si de plus $\text{supp}(\mu) = [0, 1]^d$, alors il existe une sous-suite $(B_{\phi(n)})_{n \in \mathbb{N}}$ vérifiant $\mu(\limsup_{n \rightarrow +\infty} B_{\phi(n)}) = 1$ et, pour tout $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$,*

$$\dim_H(\limsup_{n \rightarrow +\infty} R_{\phi(n)}) = \min_{1 \leq i \leq d} \frac{\dim(\mu) + \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_k},$$

où R_n est donné par (2.2).

En particulier, les bornes données par les Corollaires 2.1.8 et 2.1.9 sont optimales en un sens relativement satisfaisant: sous des hypothèses raisonnables sur la suite $(B_n)_{n \in \mathbb{N}}$, on peut toujours extraire une sous-suite $(B_{\phi(n)})_{n \in \mathbb{N}}$ qui vérifie encore $\mu(\limsup_{n \rightarrow +\infty} B_{\phi(n)}) = 1$ et telle que la lmsup générée par les ensembles contractés associés (boules ou rectangles) ont les dimensions attendues pour tous les ratios de contractions.

La prochaine section présente une application en approximation diophantienne. Ces résultats sont établis dans le Chapitre 9.

2.2 Application en approximation diophantienne

Comme mentionné précédemment, une des problématiques classiques en approximation diophantienne est de déterminer les dimensions de divers ensembles de points approximables par des nombres rationnels. Considérons $f_0 : \mathbb{R} \rightarrow \mathbb{R}$, $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ et $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ définies par $f_0 : x \rightarrow \frac{1}{3}x$, $f_1(x) = \frac{1}{3}x + \frac{1}{3}$ et $f_2(x) = \frac{2}{3} + \frac{1}{3}x$ et notons $K_{1/3}$ l'attracteur (voir Section 4.3, Proposition 4.3.1) de $\{f_0, f_2\}$. Un problème posé par Mahler consiste à déterminer la dimension de l'ensemble $K_{1/3} \cap F_\delta$ (ou F_δ est défini dans le Théorème 2.1.1) des points du Cantor triadique $K_{1/3}$ qui sont approximables par des rationnels à vitesse $\delta \geq 1$ fixée. Ce problème reste non résolu encore aujourd'hui mais les théorèmes d'ubiquité ont tout de même permis d'établir des résultats partiels.

Théorème 2.2.1 (Beresnevich-Velani [11], Barral-Seuret [8]). *Pour tout $\delta \geq 1$,*

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right) \geq \frac{\dim_H(K_{1/3})}{2\delta}.$$

La valeur de $\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right)$, où $\delta \geq 1$, a donnée lieu à plusieurs conjectures

(1) Levesley, Salp, Velani [54]:

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right) = \frac{\dim_H(K_{1/3})}{\delta}.$$

(2) Bugeaud-Durand [18]:

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right) = \max \left\{ \frac{\dim_H(K_{1/3})}{\delta}, \frac{2}{\delta} + \dim_H(K_{1/3}) - 1 \right\}.$$

Grâce au Théorème 2.1.3 (qui est impliqué par Théorème 2.1.6), on peut obtenir une caractérisation complète des points approximables par des rationnels dans un ensemble qui contient $K_{1/3}$ et qui a même dimension.

Notons $\Lambda = \{0, 1, 2\}$ et le shift sur $\Lambda^{\mathbb{N}}$, σ , défini par $\sigma((i_1, i_2, \dots)) = (i_2, \dots)$. La projection canonique de $\Lambda^{\mathbb{N}}$ sur $[0, 1]$ est l'application

$$\pi : x = (x_n)_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow +\infty} f_{x_1} \circ \dots \circ f_{x_n}(0). \quad (2.9)$$

Définition 2.2.2. Soit $\phi : \Lambda^{\mathbb{N}} \rightarrow \{0, 1\}$ définie par

$$\begin{cases} \phi(x) = 1 & \text{si } x_1 = 1 \\ \phi(x) = 0 & \text{si } x_1 = 0 \text{ ou } 2 \end{cases}$$

et

$$K_{1/3}^{(0)} = \pi \left(\left\{ x \in \Lambda^{\mathbb{N}} : \liminf_{k \rightarrow +\infty} \frac{S_k \phi(x)}{k} = 0 \right\} \right),$$

où $(S_k \phi)_{k \in \mathbb{N}}$ est la somme de Birkhoff de ϕ définie pour $x \in \Lambda^{\mathbb{N}}$ par

$$S_k(\phi)(x) = \frac{1}{k} \sum_{i=0}^{k-1} \phi(\sigma^i(x)).$$

Il est démontré dans ce manuscrit (Proposition 9.1.5) que

$$\dim_H K_{1/3}^{(0)} = \frac{\log 2}{\log 3} (= \dim_H K_{1/3}).$$

On a le résultat suivant:

Théorème 2.2.3 (D.[25]). Pour tout $\delta \geq 1$,

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3}^{(0)} \right) = \min \left\{ \frac{\log 2}{\log 3}, \frac{1}{\delta} \right\}. \quad (2.10)$$

On observe un phénomène de saturation: pour $1 \leq \delta \leq \frac{\log 3}{\log 2}$, la dimension $\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3}^{(0)} \right)$ est constante égale à $\frac{\log 2}{\log 3}$, puis décroît en $\frac{1}{\delta}$ lorsque $\delta \geq \frac{\log 3}{\log 2}$.

Dans la prochaine section, nous présentons une application du Corollaire 2.1.8 aux problèmes de cibles rétrécissantes faiblement conformes.

2.3 Cibles rétrécissantes faiblement conformes

Soit $m \geq 2$ et $S = \{f_1, \dots, f_m\}$ un système faiblement conforme et notons $\Lambda = \{1, \dots, m\}$, $\Lambda^* = \bigcup_{k \geq 0} \Lambda^k$ et K son attracteur. Etant donné $x \in K$, on s'intéresse à l'ensemble des éléments approximables par l'orbite de x sous S , c'est-à-dire, aux ensembles de la forme

$$W(x, \psi) = \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), \psi(\underline{i})),$$

où $\psi : \Lambda^* \rightarrow \mathbb{R}$ vérifie $\lim_{|\underline{i}| \rightarrow +\infty} \psi(\underline{i}) = 0$ et pour $\underline{i} = (i_1, \dots, i_k)$, $f_{\underline{i}} = f_{i_1} \circ \dots \circ f_{i_k}$.

Historiquement, l'étude de ces ensembles est reliée à celle des cibles rétrécissantes sur l'attracteur d'un système dynamique répulsif [42].

Plus précisément, supposons pour simplifier que l'IFS S soit auto-similaire et satisfasse pour tout $1 \leq i \neq j \leq m$, $\text{Im}(f_i) \cap \text{Im}(f_j) = \emptyset$. Alors son attracteur K peut être vu comme l'attracteur associé à la fonction mesurable répulsive F définie par $F(x) = f_i^{-1}(x)$ si $x \in \text{Im}(f_i)$. Soit $\delta \geq 1$ et $\psi : \Lambda^* \rightarrow \mathbb{R}$ la fonction définie par $\psi(\underline{i}) = c_{\underline{i}} e^{-n\delta}$. L'ensemble $W(x, \psi)$ se réécrit simplement comme

$$W(x, \psi) = \{z : F^n(z) \in B(x, e^{-n\delta}) \text{ pour une infinité d'entiers } n \in \mathbb{N}\}, \quad (2.11)$$

qui sont justement les ensembles introduits et étudiés par Hill et Velani dans [42].

Il existe cependant a priori plusieurs façons naturelles de choisir les classes de fonctions ψ que l'on considère :

- D'un point de vue dynamique, il est logique de considérer des fonctions ψ de la forme $\psi(\underline{i}) = |f_{\underline{i}}(K)|^\delta$, où $\delta \geq 1$, puisqu'on a naturellement,

$$\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), |f_{\underline{i}}(K)|) = K.$$

Dans ce cas, le Théorème 2.1.6 permet de montrer le résultat suivant :

Théorème 2.3.1 (D. [23]). *Notons $\dim(S)$ la dimension de conformalité de l'IFS S (Définition (4.3.19)). Supposons que $\dim_H(K) = \dim(S)$.*

Alors pour tout $\delta \geq 1$, pour tout $x \in K$, on a

$$\dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), |f_{\underline{i}}(K)|^\delta) \right) = \frac{\dim_H(K)}{\delta}. \quad (2.12)$$

- En approximation diophantienne, on préfère prendre des fonctions ψ de la forme $\psi(\underline{i}) = |f_{\underline{i}}(K)|g(|\underline{i}|)$ où $g : \mathbb{N} \rightarrow \mathbb{R}$ est une fonction décroissante

(pour l'étude des ensemble $W(x, \psi)$ dans ce cadre sous l'hypothèse que le système vérifie l'OSC, citons les travaux de Hill et Velani et de Allen et Bárány [2, 42]).

Cette différence de point de vue est justifiée par (2.11) et provient également du fait que l'on cherche à faire des analogies entre l'étude de ces ensembles et le théorème de Khintchine dans le cadre rationnel que l'on rappelle. Notons $\phi : \mathbb{N} \rightarrow \mathbb{N}$ la fonction d'Euler, définie par

$$\phi(q) = \# \{1 \leq p \leq q : p \wedge q = 1\}.$$

Théorème 2.3.2 (Khintchine [50]). *Soit $\psi : \mathbb{N} : \mathbb{R}^+$ une fonction telle que $q \rightarrow \frac{\psi(q)}{q}$ soit monotone. Alors*

$$\mathcal{L}^d \left(\limsup_{q \in \mathbb{N}, 0 \leq p \leq q, p \wedge q = 1} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \right) = 1 \Leftrightarrow \sum_{q \geq 1} \frac{\phi(q)\psi(q)}{q} = +\infty.$$

Ajoutons qu'il était conjecturé par Duffin et Schaeffer que le Théorème 2.3.2 restait valide sans l'hypothèse de monotonie de $q \rightarrow \frac{\psi(q)}{q}$ et que cette conjecture a été résolue récemment par Maynard et Koukoulopoulos [52].

Pour rendre cette analogie plus concrète, supposons pour simplifier que le système S soit auto-similaire et appelons $0 < c_1, \dots, c_m < 1$ les ratios de contraction des fonctions f_1, \dots, f_m . Rappelons que dans le cas auto-similaire, $\dim(S)$ est le réel solution de

$$\sum_{1 \leq i \leq m} c_i^{\dim(S)} = 1.$$

On dispose toujours sur le codage $\Lambda^{\mathbb{N}}$ d'une mesure naturelle ν définie pour tout $\underline{i} = (i_1, \dots, i_k) \in \Lambda^*$, par

$$\nu(\underline{i}) = c_{\underline{i}}^{\dim(S)}.$$

Soit $\mu = \nu(\pi^{-1})$, où π est la projection canonique de $\Lambda^{\mathbb{N}}$ sur K (définie ci-dessous en (2.9)). L'idée est d'obtenir des résultats de dichotomies de la forme suivante (où $s \geq 0$ est non spécifié ici):

$$\begin{cases} \text{si } \sum_{\underline{i} \in \Lambda^*} \psi(\underline{i})^s = +\infty, \text{ alors } \mu(W(x, \psi)) = 1 \\ \text{si } \sum_{\underline{i} \in \Lambda^*} \psi(\underline{i})^s < +\infty, \text{ alors } \mu(W(x, \psi)) = 0. \end{cases}$$

Ceci en tête, il est clair que la fonction ψ ne peut pas être arbitraire et que $\psi(\underline{i})$ doit dépendre de la génération du mot \underline{i} , si on espère recouvrir

un ensemble de large mesure par des boules $\{B(f_{\underline{i}}(x), \psi(\underline{i}))\}_{\underline{i} \in \Lambda^n}$, pour une infinité d'entiers $n \in \mathbb{N}$.

Pour des exemples de tels théorèmes dans le cadre auto-similaire, référons à [54] et mentionnons aussi le théorème suivant:

Théorème 2.3.3 (Baker [3]). *Supposons que le système S satisfasse l'une des conditions suivantes*

$$\begin{cases} -\sum_{i=1}^m c_i^{\dim(S)} \log(c_i^{\dim(S)}) < -2 \log(\sum_{i \leq i \leq m} c_i^{2 \dim(S)}) \\ c_1 = \dots = c_m. \end{cases}$$

Soit $\psi : \Lambda^* \rightarrow \mathbb{R}$ de la forme $\psi(\underline{i}) = |f_{\underline{i}}(K)|g(|\underline{i}|)$, où $g : \mathbb{N} \rightarrow \mathbb{R}$ est positive et décroissante. Si

$$\sum_{n \geq 0} ng(n)^{\dim(S)} = +\infty,$$

alors, pour tout $x \in K$,

$$\mu(W(x, \psi)) = 1.$$

Le Théorème 2.3.3 combiné au Théorème 2.1.6 nous permet de prouver le résultat suivant:

Théorème 2.3.4 (D. [23]). *Soit $g : \mathbb{N} \rightarrow (0, +\infty)$ une fonction positive décroissante. Posons*

$$s_g = \inf \left\{ s \geq 0 : \sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^k} k(|f_{\underline{i}}(K)|g(k))^s < +\infty \right\}. \quad (2.13)$$

Alors, sous les hypothèses du Théorème 2.3.3, si de plus $\dim(\mu) = \dim(S)$, pour tout $\delta \geq 1$, on a

$$\dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), (|f_{\underline{i}}(K)|g(|\underline{i}|))^{\frac{\delta s_g}{\dim(S)}}) \right) = \frac{\dim(S)}{\delta}. \quad (2.14)$$

Remarquons que (2.14) donne une caractérisation complète de la dimension des ensembles $W(x, \psi)$ lorsque ψ est de la forme $\psi(\underline{i}) = (|f_{\underline{i}}(K)|g(|\underline{i}|))^s$, pour tout $s \geq 0$ et $g : \mathbb{N} \rightarrow \mathbb{N}$ positive décroissante.

La prochaine section présente une application en analyse multifractale de mesures discrètes, développée au Chapitre 11.

2.4 Application à l'analyse multifractale de mesures discrètes

Soit $m \geq 2$ et $S = \{f_1, \dots, f_m\}$ un système de similitudes contractantes de rapports de contraction $0 < c_1, \dots, c_m < 1$ et soit K l'attracteur de S . Etant donné $(q_1, \dots, q_{m+1}) \in (0, 1)^{m+1}$ un vecteur de probabilité, nous étudions les mesures de probabilité solutions de l'équation perturbée

$$\chi(\cdot) = \sum_{1 \leq i \leq m} q_i \chi(f_i^{-1}(\cdot)) + q_{m+1} \delta_{x_0}, \quad (2.15)$$

où δ_x désigne la mesure de Dirac au point x . Ces mesures ont été introduites par Snigireva et Olsen dans [70] sous l'appellation de “mesures auto-similaires in-homogènes de mesure de condensation δ_{x_0} ”.

On peut montrer (cf Proposition 11.1.1) que de telles mesures sont discrètes et sont de la forme

$$\chi(\cdot) = C \sum_{\underline{i} \in \Lambda^*} p_{\underline{i}} c_{\underline{i}}^t \delta_{f_{\underline{i}}(x_0)}, \quad (2.16)$$

où $t > 0$, (p_1, \dots, p_m) est un autre vecteur de probabilité, $C > 0$ est une constante de renormalisation afin que χ soit une mesure de probabilité.

Des mesures très similaires ont été obtenues par Mandelbrot et Riedi [56] en considérant des mesures réelles obtenues en inversant les fonctions de réparations associées à des mesures auto-similaires satisfaisant l'OSC et par Barral et Seuret en considérant le cas d'une mesure de Gibbs sur un Cookie-cutter dans [9]. Les propriétés multifractales de ces mesures ont été étudiées par Falconer dans [31], par Barral et Seuret dans [7, 9] et par Mandelbrot et Riedi dans [56]. Ces derniers, en ne considérant seulement des dimensions locales obtenues via des limites exactes, Mandelbrot et Riedi n'ont pas relevé un phénomène intéressant mis en lumière par Falconer, Barral et Seuret.

L'analyse fine des propriétés multifractales de mesures est un sujet qui a vu le jour dans les années 1970 avec les travaux de Mandelbrot [57, 58] et de Olsen [60–62]. Cette étude consiste, étant donnée $\mu \in \mathcal{M}(\mathbb{R}^d)$, à déterminer d'une part le spectre des dimensions des ensembles suivants (voir Section 4.2, Définition 4.2.12)

$$\begin{aligned} E_\mu^h &= \{x \in \text{supp}(\mu) : \underline{\dim}(\mu, x) = h\}, \\ \widetilde{E}_\mu^h &= \{x \in \text{supp}(\mu) : \underline{\dim}(\mu, x) = \overline{\dim}(\mu, x) = h\}, \end{aligned} \quad (2.17)$$

c'est-à-dire à déterminer pour tout $h \in \mathbb{R}$,

$$\begin{cases} D_\mu(h) = \dim_H(E_\mu^h) \text{ et} \\ \widetilde{D}_\mu(h) = \dim_H(\widetilde{E}_\mu^h). \end{cases}$$

On cherche généralement à comparer les dimensions de ces ensembles à la transformée de Legendre d'une fonction d'échelle, notée τ_μ et définie pour tout $q \in \mathbb{R}$ par

$$\tau_\mu(q) = \liminf_{n \rightarrow +\infty} \frac{-\log \left(\sum_{D \in \mathcal{D}_n : \mu(D) \neq 0} \mu(D)^q \right)}{n}$$

ou \mathcal{D}_n dénote l'ensemble des cubes dyadiques de génération n .

Fixons μ la mesure auto-similaire associée à S et (p_1, \dots, p_m) , c'est-à-dire la mesure solution de l'équation

$$\mu(\cdot) = \sum_{i=1}^m p_i \mu(f_i^{-1}(\cdot)).$$

Nous allons voir que l'analyse multifractale de la mesure χ , donnée par (2.16), dépend à la fois de l'analyse multifractale de la mesure auto-similaire μ et de la vitesse d'approximation d'un point x par les points de l'orbite $(f_{\underline{i}}(x_0))_{\underline{i} \in \Lambda^*}$. Introduisons la définition suivante.

Définition 2.4.1. Soit $x \in K$. Le degré d'approximation de x par rapport à $(f_{\underline{i}}(x_0))_{\underline{i} \in \Lambda^*}$, noté Δ_x , est défini par

$$\Delta_x = \sup \left\{ \Delta : x \in \limsup_{n \rightarrow +\infty} B(f_{\underline{i}}(x_0), (2c_{\underline{i}})^\Delta) \right\}.$$

Dans le cas où le point de base de (2.16) x_0 n'appartient pas à l'attracteur K de S , tous les points de $\text{supp}(\chi)$, excepté $\{f_{\underline{i}}(x_0)\}_{\underline{i} \in \Lambda^*}$, qui est de dimension 0, sont mal approchés par ces points, i.e, ont pour degré 1. Dans ce cas l'analyse multifractale de χ se déduit facilement de celle de μ .

Théorème 2.4.2 (D.[26]). Soit $x_0 \notin K$. Alors, pour $t > 0$, la mesure χ définie par (2.16) existe. De plus, pour tout $h \geq 0$,

$$D_\chi(h) = D_\mu(h - t). \quad (2.18)$$

Remarquons que ce résultat est vrai sans hypothèse de séparation sur la mesure μ . Une conséquence directe de ce résultat est que, lorsque l'attracteur K a mesure de Lebesgue 0, pour Lebesgue presque tout x_0 , la mesure χ a pour spectre multifractal un translaté du spectre de la mesure μ . En d'autres termes, pour simuler une mesure qui a pour spectre multifractal le spectre de

la mesure μ translaté de t , il suffit de tirer aléatoirement un point x_0 suivant la loi uniforme et de simuler la mesure χ associée. Ajoutons aussi que l'on montre dans ce manuscrit que χ ne satisfait pas le formalisme multifractal même si μ le satisfait (car $\tau_\chi(q) = 0$ pour $q \geq 1$).

Lorsque x_0 est choisi sur K , le Corollaire 2.1.8 permet, sous certaines hypothèses de calculer le spectre de la mesure χ associée.

Théorème 2.4.3 (D. [26]). *Soit $m \geq 2$, $S = \{f_1, \dots, f_m\}$ IFS auto-similaire satisfaisant AWSC (Definition 4.3.3) sans overlaps exacts et $K \subset \mathbb{R}^d$ l'attracteur de S . Soit (p_1, \dots, p_m) un vecteur de probabilité, μ la mesure auto-similaire associée avec (p_1, \dots, p_m) , $x_0 \in K$ et $C > 0$ la constante de normalisation telle que la mesure*

$$\chi(\cdot) = C \sum_{\underline{i} \in \Lambda^*} c_{\underline{i}}^t p_{\underline{i}} \delta_{f_{\underline{i}}(x_0)},$$

soit une mesure de probabilité.

Posons $q_c = \min \{q : \tau_\mu(q) + qt = 0\}$ et $\lambda_c = \tau'_\mu(q_c)$. Alors:

1. *La fonction τ_χ satisfait*

$$\begin{cases} \text{pour tout } q \geq q_c, & \tau_\chi(q) = 0 \\ \text{pour tout } 0 \leq q \leq q_c, & \tau_\chi(q) = \tau_\mu(q) + t \\ \text{pour tout } q \leq 0, & \tau_\chi(q) \geq \tau_\mu(q) + t. \end{cases} \quad (2.19)$$

En particulier, pour tout $h \geq 0$, on a

$$\tau_\chi^*(h) \leq \begin{cases} hq_c & \text{si } 0 \leq h \leq \lambda_c + t \\ \tau_\mu^*(h - t) & \text{si } h \geq \lambda_c + t. \end{cases} \quad (2.20)$$

2. *Le sepectre multifractal D_χ de χ satisfait*

$$\begin{cases} \text{pour tout } 0 < h \leq \lambda_c + t, & D_\chi(h) = hq_c \\ \text{pour tout } \lambda_c + t < h \leq \tau'_\mu(0^+) + t, & D_\chi(h) = \tau_\mu^*(h - t). \end{cases} \quad (2.21)$$

3. *Pour tout $h > \tau'_\mu(0^+) + t$, on a*

$$\dim_H(\widetilde{E}_\mu^{h-t} \cap \{x : \Delta_x = 1\}) \leq D_\chi(h) \leq \tau_\mu^*(h - t).$$

Remarque 2.4.4.

- Des exemples de mesures satisfaisant les hypothèses du Théorème 2.4.3 sont donnés par le Corollaire 4.3.27.

- $\tau'_\mu(q_c)$ existe si S vérifie AWSC et n'a pas d'overlaps exacts, voir Théorème 4.3.39.
- Supposons en plus des hypothèses du Théorème 2.4.3 que pour $h \geq \tau'_\mu(0^+)$, il existe $m_h \in \mathcal{M}(\mathbb{R}^d)$ telle que

$$\dim_H(m_h) = \tau_\mu^*(h) \text{ et } m_h(\widetilde{E}_\mu^h) > 0,$$

alors

$$D_\chi(h+t) = \tau_\mu^*(h)$$

et χ satisfait le formalisme multifractal en $h+t$.

Remarquons que l'analyse multifractale est en toute généralité ardue pour h grand. Ce phénomène n'est pas étonnant puisque l'on peut construire des exemples de mesures auto-similaires associées à des systèmes satisfaisant des hypothèses de séparations très raisonnables mais qui ne satisfont pas le formalisme multifractal pour h grand (voir [72]).

2.5 Conclusion et perspectives

Comme expliqué dans cette introduction, les principes de transferts de masse (où les théorèmes d'ubiquité) sont des outils servant en théorie métrique des nombres, en systèmes dynamiques et en analyse multifractale. Ainsi, un grand nombre de questions et d'extensions possibles se dégagent naturellement de ces travaux:

- Que peut-on dire du contenu μ -essentiel d'autres mesures que les mesures associées à des IFS faiblement conformes? Que dire lorsque l'IFS est par exemple auto-affine? Que peut-on dire sous la seule hypothèse que la mesure est exactement dimensionnelle et ergodique?
- Que peut-on dire de la dimension des shrinking targets associées à ces systèmes dynamiques?
- Que dire de la mesure χ satisfaisant (2.16) lorsque l'IFS sous-jacent est auto-affine, satisfait l'hypothèse de séparation exponentielle ou bien est simplement donné par une orbite d'un système dynamique muni d'une mesure ergodique?
- Quels ensembles fractals peut-on distinguer ou caractériser en considérant l'approximation par des rectangles ou des ensembles d'autres formes

au lieux de boules? De telles questions apparaissent naturellement par exemple lorsqu'on regarde l'approximation diophantienne multiplicative. Plus précisément, étant donné $\delta \geq 1$, que peut-on dire de la dimension, de la mesure de Hausdorff (éventuellement pour une jauge adaptée) de l'ensemble

$$A_\delta = \left\{ (x_1, \dots, x_d) : \exists (p_1, \dots, p_d) \in \mathbb{N}^d, \prod_{i=1}^d \left| x_i - \frac{p_i}{q} \right| \leq q^{-\delta}, q \in \mathbb{N} \text{ i.s.} \right\},$$

ou i.s. signifie que l'égalité ci-dessus à lieu pour une infinité d'entiers q . Notons que dans ce cas-ci, les ensembles approximants sont des hyperboles.

Nous commencerons dans le Chapitre 4 par rappeler et préciser les notions requises pour comprendre ce manuscrit. Ensuite, au Chapitre 5, nous donnerons une première construction qui est instructive concernant le résultat d'ubiquité que l'on souhaite montrer.

Au Chapitre 6, nous définirons la quantité géométrique au centre de notre théorème d'ubiquité général. Etant données une suite de boules $(B_n)_{n \in \mathbb{N}}$ et une mesure $\mu \in \mathcal{M}(\mathbb{R}^d)$, nous établirons aussi un lien techniquement utile entre vérifier $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$ et vérifier une certaine propriété de recouvrement pour $(B_n)_{n \in \mathbb{N}}$. Lorsque $(B_n)_{n \in \mathbb{N}}$ vérifie cette propriété, nous dirons que $(B_n)_{n \in \mathbb{N}}$ est μ -asymptotiquement couvrante.

Dans le Chapitre 7, nous démontrons notre résultat principal d'ubiquité, le Théorème 2.1.6 et nous l'appliquons aux mesures faiblement conformes.

Dans le chapitre 8, nous étudions l'optimalité des théorèmes d'ubiquités établis. En particulier, nous montrons que les Corollaires 2.1.8 et 2.1.9 sont, en un certain sens, optimaux.

Les Chapitres 9, 10 et 11 donnent respectivement des applications en théorie métrique des nombres, en systèmes dynamiques ainsi qu'en analyse multifractale. On y montre les Théorèmes 2.3.1, Théorème 2.2.3 et Théorème 2.4.3.

Chapter 3

Introduction

The theory of metric approximation aims at describing geometrically (often by computing Hausdorff dimensions) the set of elements approximable by a family of points $(x_n)_{n \in \mathbb{N}}$ chosen in advance in a metric space (X, d) . Historically, this theory was born from the work of Dirichlet about the approximation of real numbers by rational numbers in the XIXth century. This work was completed and extended around 1920, by Jarnik and Besicovitch [47] and recently, around 2000, by the work of Jaffard [46], Beresnevich and Velani [11]. It is only since 1990 that certain problems in Diophantine approximation were formulated in terms of approximation by orbits of dynamical systems [42]. In those settings, one studies the case where the family of points $(x_n)_{n \in \mathbb{N}}$ is an orbit, i.e, $x_n = T^n(x)$, $x \in X$, where X is a metric space and $T : X \rightarrow X$ is a measurable mapping. For examples where T is the doubling map on the torus, a Markovian expanding map on the torus or other examples of expanding maps, see [33, 55, 65].

Those sets also plays an important role in multifractal analysis. For example, the study of Riemann's series [46] depends on the speed of approximation of a real number by rational numbers. For other work mixing multifractal analysis and Diophantine approximation, let us refer to the work of Barral, Seuret, Jaffard, Liao, Fan, Shmeling Troubetzkoy and Persson [10, 33, 45, 55, 64].

3.1 Ubiquity theorems

3.1.1 An historic of ubiquity theorems

Fix $d \in \mathbb{N}^*$ and endow \mathbb{R}^d with the norm $\|(x_1, \dots, x_d)\|_\infty = \max_{1 \leq i \leq d} |x_i|$ (in this manuscript the choice of the norm actually matters very little).

One way to give meaning to the notion of speed of approximation of a

point x by a sequence of points $(x_n)_{n \in \mathbb{N}}$ of \mathbb{R}^d , and to characterize this speed, is to provide sequences of sets $(U_n(\delta))_{n \in \mathbb{N}}$, where $\delta \geq 1$ (sometimes one can take a multi-parameter $\delta \in [1, +\infty)^k$, where $k \in \mathbb{N}$ is fixed) verifying:

- for every $n \in \mathbb{N}$, for every $\delta \geq 1$, $x_n \in U_n(\delta)$,
- for every $\delta \geq 1$, $|U_n(\delta)| \rightarrow 0$,
- for every $\delta' > \delta$, for every $n \in \mathbb{N}$, $U_n(\delta') \subset U_n(\delta)$.

For each δ , we are interested in the sets E_δ of points x for which there exists an infinity of integers $(n_k)_{k \in \mathbb{N}}$ such that $x \in U_{n_k}(\delta)$. The set of all those points is called the \limsup ¹ of the sequence $(U_n(\delta))_{n \in \mathbb{N}}$ and can be written as

$$E_\delta = \limsup_{n \rightarrow +\infty} U_n(\delta) := \bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} U_n(\delta).$$

We say that a point x is approximable at speed δ if $x \in F_\delta := E_\delta \setminus \bigcup_{\delta' > \delta} E_{\delta'}$ and we aim in general at computing the Hausdorff dimensions (see Section 4.2, Lemma 4.2.3 for a definition of the Hausdorff dimension) $\dim_H(E_\delta)$ and $\dim_H(F_\delta)$ (in practice, one can derive the value of $\dim_H(F_\delta)$ when one knows $\mathcal{H}^\zeta(E_\delta) > 0$ where ζ is an appropriate gauge and \mathcal{H}^ζ is the Hausdorff measure associated with ζ).

The first historical example of such computations is the theorem of Jarnik-Besicovitch.

Theorem 3.1.1 (Jarnik-Besicovitch [47]). *For $x \in \mathbb{R}$ and $r > 0$, denote by $B(x, r)$ the closed ball centered in x and of radius r . Then, for every $\delta \geq 1$, denoting $E_\delta := \limsup_{q \in \mathbb{N}, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta}})$ and $F_\delta := E_\delta \setminus \bigcup_{\delta' > \delta} E_{\delta'}$, one has:*

$$\dim_H(E_\delta) = \dim_H(F_\delta) = \frac{1}{\delta}. \quad (3.1)$$

The reader interested in other works around rational approximation may refer to [11, 15, 16, 27] (homogeneous and inhomogeneous rational approximation) and [1, 17, 18, 54] (rational approximation of elements of the middle-third Cantor set).

Usually, to compute the Hausdorff dimension of those sets, one proceeds separately for the lower-bound and the upper-bound. In practice, getting an

¹Remark here that the definition of $\limsup_{n \rightarrow +\infty} U_n$ does not depend on the manner we enumerate the sets $\{U_n\}_{n \in \mathbb{N}}$ in the sense that for any bijective mapping $\phi : \mathbb{N} \rightarrow \mathbb{N}$, one has $\limsup_{n \rightarrow +\infty} U_n = \limsup_{n \rightarrow +\infty} U_{\phi(n)}$. In particular later on in the manuscript, there might be some \limsup sets taken over families of sets for which the enumeration we choose is not explicitly stated.

upper-bound for $\dim_H(E_\delta)$ is often either straightforward, either very complex (the reader will notice that in this manuscript, the upper-bound for $\dim_H(E_\delta)$ are often much easier to establish than the corresponding lower-bound).

The ubiquity theorems, also called “mass transference principles” ([11]), are tools built to provide lower-bounds for $\dim_H(E_\delta)$.

Those theorems work in the following way: one look for a measure $\mu \in \mathcal{M}(\mathbb{R}^d) := \{\text{probability measures on } \mathbb{R}^d\}$ (hopefully adapted to our problem) such that $\mu(E_1) = 1$. Then, using the geometric property of the measure μ (self-similarity, multifractal analysis for example), one establishes a lower-bound for $\dim_H E_\delta$ for every $\delta > 1$. Here is an important example of such theorems. One says that a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is s -Alfhors regular, where $0 \leq s \leq d$, if there exists two constants $0 < C_1 \leq C_2$ such that, for every ball $B(x, r)$ with $x \in \text{supp}(\mu) := \text{the topological support of } \mu$ and $0 < r \leq 1$, we have

$$C_1 r^s \leq \mu(B(x, r)) \leq C_2 r^s.$$

The Lebesgue measure \mathcal{L}^d on $[0, 1]^d$ is an example of such a measure.

Theorem 3.1.2 (Beresnevich-Velani [11]). *Let $0 \leq s \leq d$ be a real number and $\mu \in \mathcal{M}(\mathbb{R}^d)$ an s -Alfhors-regular measure. Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of closed ball centered on $\text{supp}(\mu)$ and satisfying:*

- $|B_n| \rightarrow 0$,
- $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$.

Let $\delta > 1$ be a real number and B a ball centered on $\text{supp}(\mu)$. For $s_0 \geq 0$, denote by \mathcal{H}^{s_0} the Hausdorff measure of dimension s_0 (see (4.2.2)) and for $n \in \mathbb{N}$, $B_n^\delta = B(x_n, r_n^\delta)$ if $B_n = B(x_n, r_n)$. One has

$$\mathcal{H}^{\frac{s}{\delta}}(\limsup_{n \rightarrow +\infty} B_n^\delta \cap B) = +\infty.$$

In particular, we recover a result previously established by Jaffard ([46] 1999),

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{s}{\delta}.$$

In the case $\mu = \mathcal{L}^d$, one can also deal with the more general situation where instead of giving a lower-bound for the dimension of $\limsup_{n \rightarrow +\infty} B_n^\delta$ for every $\delta > 1$, one describes a geometric quantity, the Hausdorff content (see Section 4.2, equation (4.7)), to estimate in order to get accurate lower-bounds for $\dim_H(\limsup_{n \rightarrow +\infty} U_n)$ where $(U_n)_{n \in \mathbb{N}}$ is any sequence of open sets satisfying $U_n \subset B_n$.

Theorem 3.1.3 (Koivusalo-Rams [51]). *Let $(B_n)_{n \rightarrow +\infty}$ be a sequence of balls of $[0, 1]^d$ verifying $|B_n| \rightarrow 0$, $(U_n)_{n \in \mathbb{N}}$ a sequence of opens sets satisfying for every $n \in \mathbb{N}$, $U_n \subset B_n$ and $0 \leq s \leq d$ a real number.*

Assume that:

- *for every $n \in \mathbb{N}$ large enough, $\mathcal{H}_\infty^s(U_n) \geq \mathcal{L}^d(B_n)$,*
- *$\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$.*

Then , one has

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) \geq s.$$

One should mention here that previous works on limsup generated by random sets already highlighted the role of the Hausdorff content to characterize the dimension of those limsup sets [28, 38, 48, 49] and recently, some interesting techniques based on the Hausdorff where also developed in [29].

One can for instance take U_n an open rectangle for every $n \in \mathbb{N}$. More precisely, fix $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$ and let $(B_n := x_n + \prod_{i=1}^d [-r_n, r_n])_{n \in \mathbb{N}}$ be a sequence of closed balls of $[0, 1]^d$ verifying $|B_n| \rightarrow 0$ and $\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$. Set

$$R_n = x_n + \prod_{i=1}^d (-r_n^{\tau_i}, r_n^{\tau_i}). \quad (3.2)$$

Then, for every $0 \leq s \leq d$, the s -dimensional Hausdorff content of the the rectangles R_n can be explicitly determined and depends on r_n and the numbers τ_1, \dots, τ_d up to some multiplicative constant C which only depends on d . As a consequence of this computation, one obtains that, for $s_0 = \min_{1 \leq i \leq d} \frac{d + \sum_{1 \leq i \leq d} \tau_k - \tau_i}{\tau_k}$,

$$\mathcal{H}_\infty^{s_0}(R_n) \geq C r_n^d \geq C \mathcal{L}^d(B_n).$$

It is also easily verified that this implies that for every $s < s_0$ and every large enough n ,

$$\mathcal{H}_\infty^s(R_n) \geq r_n^d = \mathcal{L}^d(B_n),$$

Applying Theorem 3.1.3, one obtains

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq s_0 = \min_{1 \leq i \leq d} \frac{d + \sum_{1 \leq i \leq d} \tau_k - \tau_i}{\tau_k}.$$

When the measure is not homogeneous, the situation is more complex.

Barral-Seuret Theorem deals with the case where the measure is self-similar satisfying the Open Set Condition (OSC in short, see Section 4.3, Definition 4.3.3 for more details about the different separation conditions). For μ

a self-similar measure, denote by $\dim(\mu)$ the dimension of μ (see Section 4.2, Definition 4.2.12 and Section 4.3, Theorem 4.3.15).

Theorem 3.1.4 (Barral-Seuret [8]). *Let $\mu \in \mathbb{R}^d$ be a self-similar satisfying the OSC and $(B_n)_{n \in \mathbb{N}}$ a sequence of balls centered on $\text{supp}(\mu)$ verifying $|B_n| \rightarrow 0$.*

If

$$\mu(\limsup_{n \rightarrow +\infty} B_n) = 1,$$

then, for every $\delta \geq 1$,

$$\dim_H(\limsup_{n \rightarrow +\infty}) = \frac{\dim(\mu)}{\delta}. \quad (3.3)$$

In this manuscript, we develop a tool which plays the same role as the Hausdorff content in Theorem 3.1.3 but in the case where μ is any probability measure of $\mathcal{M}(\mathbb{R}^d)$. We will show in particular that the resulting ubiquity theorem extends both the Theorems 3.1.3 and 3.1.4.

3.1.2 A general ubiquity theorem

Let us start by introducing the following geometric quantity.

Definition 3.1.5 (D. [25]). *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure, $0 \leq s \leq d$ and $A \subset \mathbb{R}^d$ a measurable set. The s -dimensional μ -essential Hausdorff content of A is defined as*

$$\mathcal{H}_\infty^{\mu,s}(A) = \inf \{ \mathcal{H}_\infty^s(E) : E \subset A, \mu(E) = \mu(A) \}. \quad (3.4)$$

For $\mu \in \mathcal{M}(\mathbb{R}^d)$, denote by $\underline{\dim}_H(\mu)$ the lower Hausdorff dimension of μ (defined in Section 4.2, Definition 4.2.12). The ubiquity theorem associated with the essential Hausdorff content is the following (the version stated here is simplified compared to Theorem 7.1.2).

Theorem 3.1.6 (D. [25]). *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure, $(B_n)_{n \in \mathbb{N}}$ a sequence of balls $|B_n| \rightarrow 0$, $(U_n)_{n \in \mathbb{N}}$ a sequence of open sets satisfying for every $n \in \mathbb{N}$, $U_n \subset B_n$ and $0 \leq s \leq \underline{\dim}_H(\mu)$.*

Assume that

- *one has*

$$\mu(\limsup_{n \rightarrow +\infty} \frac{1}{2} B_n) = 1,$$

- *for every n large enough,*

$$\mathcal{H}_\infty^{\mu,s}(U_n) \geq \mu(B_n).$$

Then, one has

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) \geq \min \{s, \underline{\dim}_H(\mu)\}. \quad (3.5)$$

In Chapter 7, the result we show is actually slightly more general than Theorem 3.1.6.

Of course, to apply Theorem 3.1.6 to a measure μ , one needs to understand this measure finely enough so that one can estimate precisely $\mathcal{H}_\infty^{\mu,s}(U_n)$ for every $0 \leq s < \underline{\dim}_H(\mu)$ and each set U_n . It turns out that when the measure is weakly conformal (Section 4.3, Definition 4.3.1), it is possible to estimate precisely the s -dimensional Hausdorff content of every open set and every s **without any separation assumption on the IFS**.

Theorem 3.1.7 (D. [23]). *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure and K its support.*

For every $0 \leq s < \dim(\mu)$, for every $\varepsilon > 0$, there exists a constant $c = c(d, \mu, s, \varepsilon) > 0$ which depends on d, μ, s and ε , such that for every closed balls $B = B(x, r)$ centered on K , for every $0 < r \leq 1$ and for every open set Ω , one has

$$\begin{aligned} c(d, \mu, s, \varepsilon) |B|^{s+\varepsilon} &\leq \mathcal{H}_\infty^{\mu,s}(\overset{\circ}{B}) \leq \mathcal{H}_\infty^{\mu,s}(B) \leq |B|^s \\ c(d, \mu, s, \varepsilon) \mathcal{H}_\infty^{s+\varepsilon}(\Omega \cap K) &\leq \mathcal{H}_\infty^{\mu,s}(\Omega) \leq \mathcal{H}_\infty^s(\Omega \cap K). \end{aligned} \quad (3.6)$$

For every $s > \dim(\mu)$,

$$\mathcal{H}_\infty^{\mu,s}(\Omega) = 0.$$

Combining Theorem 3.1.6 and 3.1.7 we recover Theorem 3.1.3 by taking $\mu = \mathcal{L}^d$ and we obtain the following corollaries.

Corollary 3.1.8. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure and $(B_n)_{n \in \mathbb{N}}$ a sequence of balls centered on $\text{supp}(\mu)$ verifying $|B_n| \rightarrow 0$.*

If

$$\mu(\limsup_{n \rightarrow +\infty} B_n) = 1,$$

then for every $\delta \geq 1$,

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{\dim(\mu)}{\delta}.$$

Corollary 3.1.9. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure with $\text{supp}(\mu) = [0, 1]^d$ and $(B_n)_{n \in \mathbb{N}}$ a sequence of closed ball of $[0, 1]^d$ verifying $|B_n| \rightarrow 0$. Let $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$ and R_n , defined as in (3.2).*

If

$$\mu(\limsup_{n \rightarrow +\infty} B_n) = 1,$$

then

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq \min_{1 \leq i \leq d} \frac{\dim(\mu) + \sum_{1 \leq i \leq d} \tau_k - \tau_i}{\tau_k}.$$

Note that Corollary 3.1.8 extends Theorem 3.1.4.

Let us also mention that for every $1 \leq \tau_1 \leq \dots \leq \tau_d$, every sequence of rotation matrices $(O_{k,n})_{k \in \mathbb{N}, 0 \leq n \leq 2^k - 1} \in \mathcal{O}_d(\mathbb{R})^{\mathbb{N}}$, almost every sequence of realization of i.i.d. uniformly distributed random variables on $[0, 1]^d$, $(x_{k,n})_{k \in \mathbb{N}, 0 \leq n \leq 2^k - 1}$, it is proved in [48] (and the proof uses implicitly the Corollary 3.1.9 in the case $\mu = \mathcal{L}^d$) that the family

$$\mathcal{R} = \left\{ x_{k,n} + O_{k,n} \prod_{i=1}^d (-2^{-k\tau_i}, 2^{k\tau_i}), 0 \leq n \leq 2^k - 1, k \in \mathbb{N} \right\}$$

verifies

$$\dim_H(\limsup_{R \in \mathcal{R}} R) = \min_{1 \leq k \leq d} \left\{ \frac{d + \sum_{j=1}^k \tau_k - \tau_j}{\tau_k} \right\}.$$

In particular, the bound obtained in Corollary 3.1.9 is optimal. We will see later on that it is indeed optimal in a strong sens.

Corollaries 3.1.8 and 3.1.9 shows how Theorem 3.1.6 is used: to get a lower-bound for $\dim_H(\limsup_{n \rightarrow +\infty} U_n)$, one needs to establish for every $n \in \mathbb{N}$ large enough a relation which only depends on B_n , U_n and μ .

In Chapter 8, we study the optimality of the bound provided by Theorem 3.1.6.

3.1.3 Study of optimality in ubiquity theorems

The lower-bounds established in Theorems 3.1.3, 3.1.4 and 3.1.6 are meaningful only if they are, in a sens that is satisfying enough, optimal. One way to deal with this question is to show that, if one does not have more information (compared to the hypotheses of those theorems) about the sequence $(B_n)_{n \in \mathbb{N}}$, there is no reason for $\dim_H(\limsup_{n \rightarrow +\infty} U_n)$ to be larger than the bound provided. We can show this by extracting from the sequence $(B_n)_{n \in \mathbb{N}}$, when possible, a sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ which still verifies the hypotheses of the ubiquity theorems and such that $\dim_H(\limsup_{n \rightarrow +\infty} U_{\phi(n)})$ is precisely the bound given. In this situation, we are ensured that one can not obtain a better bound in general under the same hypotheses.

Let us introduce the following definition, which is equivalent to the notion of weakly redundant sequence of balls introduced by Barral and Seuret (Definition 8.1.1).

Definition 3.1.10. *Let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a sequence of balls verifying $|B_n| \rightarrow 0$. For $k \in \mathbb{N}$, define*

$$\begin{aligned} \mathcal{T}_k(\mathcal{B}) &= \{B_n \in \mathcal{B} : 2^{-k-1} \leq |B_n| < 2^{-k}\} \\ N_k(\mathcal{B}) &= \max_{B \in \mathcal{T}_k(\mathcal{B})} \# \{B' \in \mathcal{T}_k(\mathcal{B}) : B \cap B' \neq \emptyset\}. \end{aligned} \quad (3.7)$$

One says that \mathcal{B} is weakly redundant if

$$\lim_{k \rightarrow +\infty} \frac{\log N_k(\mathcal{B})}{k} = 0.$$

Heuristically, a sequence $(B_n)_{n \in \mathbb{N}}$ is weakly redundant if, for each $k \in \mathbb{N}$, each ball B_n satisfying $|B_n| \approx 2^{-k}$ intersects less than $2^{o(k)}$ other ball $B_{n'}$ satisfying also $|B_{n'}| \approx 2^{-k}$. We emphasize that the weak redundancy property is a natural assumption to study the optimality of the bounds provided by ubiquity theorems but this condition is related only to the sequence of balls itself (no measure is involved) and an important consequence of this property is that, for every measure $\mu \in \mathcal{M}([0, 1]^d)$ and every $\varepsilon > 0$, we have

$$\sum_{n \in \mathbb{N}} |B_n|^\varepsilon \mu(B_n) < +\infty. \quad (3.8)$$

There are many sequences of balls which satisfies the weak redundancy condition, such as the rational balls $\left(B\left(\frac{p}{q}, \frac{1}{q^2}\right)\right)_{q \in \mathbb{N}, 0 \leq p \leq q, q \wedge p = 1}$, but there are also examples of sequences of balls, natural to consider, which does not satisfy this condition. It is for instance the case of $\left(B\left(\frac{p}{q}, \frac{1}{q^2}\right)\right)_{q \in \mathbb{N}, 0 \leq p \leq q}$. However, one can show that, in many situations, those sequences which does not satisfy the weak redundancy property still satisfies (3.8) for a measure characterizing well the system (it is the case for $\left(B\left(\frac{p}{q}, \frac{1}{q^2}\right)\right)_{q \in \mathbb{N}, 0 \leq p \leq q}$ with the Lebesgue measure). Let us also mention that there are other hypotheses used by different authors, that are comparable to the weak redundancy but slightly different. For example, the notion of optimal regularity of sequences of an approximating sequence of point has been used in some settings (see [15]).

In the case of a weakly conformal measure, the weak redundancy condition allows to state an upper-bound theorem.

Theorem 3.1.11 (D.[24]). *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure, $(B_n)_{n \in \mathbb{N}}$ a weakly redundant sequence of balls, centered on $\text{supp}(\mu)$ and satisfying $|B_n| \rightarrow 0$, $(U_n)_{n \in \mathbb{N}}$ a sequence of open set such that $U_n \subset B_n$ and*

$0 \leq s < \dim(\mu)$.

If for every large enough n ,

$$\mathcal{H}_\infty^{\mu,s}(U_n) \leq \mu(B_n),$$

then

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) \leq s.$$

Remark that, given $\mu \in \mathcal{M}(\mathbb{R}^d)$, $(B_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$ satisfying, for every $n \in \mathbb{N}$ $U_n \subset B_n$, it is not true in general that there exists a critical exponent s_0 such that for every $s < s_0$, for every n large enough,

$$\mathcal{H}_\infty^{\mu,s}(U_n) \geq \mu(B_n)$$

and for every $s > s_0$, for every n large enough,

$$\mathcal{H}_\infty^{\mu,s}(U_n) \leq \mu(B_n).$$

Nonetheless, when the measure involved is weakly conformal and the choice of the sets U_n is consistent (for instance, if we consider rectangles all shrunk by the same factors $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$ or balls B_n^δ , where $\delta \geq 1$ is fixed in advance), Theorem 3.1.11 allows to conclude that the bound provided by Corollaries 3.1.8 and 3.1.9 are optimal.

Corollary 3.1.12. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure and $(B_n)_{n \in \mathbb{N}}$ a sequence of balls centered on $\text{supp}(\mu)$ verifying $|B_n| \rightarrow 0$ and*

$$\mu(\limsup_{n \rightarrow +\infty} \frac{1}{2} B_n) = 1.$$

Then

1. *there exists a sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ verifying*

$$\mu(\limsup_{n \rightarrow +\infty} B_{\phi(n)}) = 1$$

and for every $\delta \geq 1$,

$$\dim_H(\limsup_{n \rightarrow +\infty} B_{\phi(n)}^\delta) = \frac{\dim(\mu)}{\delta}.$$

2. *if $\text{supp}(\mu) = [0, 1]^d$, there exists a sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ verifying*

$$\mu(\limsup_{n \rightarrow +\infty} B_{\phi(n)}) = 1$$

and for every $(\tau_1, \dots, \tau_d) \in [1, +\infty)^d$,

$$\dim_H(\limsup_{n \rightarrow +\infty} R_{\phi(n)}) = \min_{1 \leq i \leq d} \frac{\dim(\mu) + \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_k},$$

when R_n is defined as in (3.2).

In particular, the bound given by the Corollaries 3.1.8 and 3.1.9 are optimal in relatively satisfying sens: under reasonable hypotheses on the sequence $(B_n)_{n \in \mathbb{N}}$, one can always extract a sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ which still verifies $\mu(\limsup_{n \rightarrow +\infty} B_{\phi(n)}) = 1$ and such that the limsup generated by the shrunk sets (balls or rectangles) have the expected Hausdorff dimension for every contraction ratios at the same time.

The next section presents some results in Diophantine approximation. Those results are established in Chapter 9.

3.2 An application in Diophantine approximation

As mentioned at the beginning of the introduction, a classical problem in Diophantine approximation consists in determining the Hausdorff dimension of various sets of points approximable by rational numbers.

Consider $f_0 : \mathbb{R} \rightarrow \mathbb{R}$, $f_1 : \mathbb{R} \rightarrow \mathbb{R}$ and $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ defined as $f_0 : x \rightarrow \frac{1}{3}x$, $f_1(x) = \frac{1}{3}x + \frac{1}{3}$ and $f_2(x) = \frac{2}{3} + \frac{1}{3}x$, and denote by $K_{1/3}$ the attractor (see Section 4.3, Proposition 4.3.1) of $\{f_0, f_2\}$. A problem raised by Mahler consists in computing the Hausdorff dimension of the set $K_{1/3} \cap F_\delta$ (where F_δ is defined as in Theorem 3.1.1) of the points of $K_{1/3}$ which are approximable by rational numbers at speed $\delta \geq 1$ fixed in advance. This problem, although intensively studied (see [1, 11, 17, 54] among many other works), remains unsolved to this day but the ubiquity theorems allowed to establish some partial results.

Theorem 3.2.1 (Velani-Beresnevich [11], Barral-Seuret [8]). *For ever $\delta \geq 1$,*

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right) \geq \frac{\dim_H(K_{1/3})}{2\delta}.$$

The value of $\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right)$, where $\delta \geq 1$, has been the subject of various conjectures

- (1) Levesley, Salp, Velani [54]:

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right) = \frac{\dim_H(K_{1/3})}{\delta}.$$

(2) Bugeaud-Durand [18]:

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3} \right) = \max \left\{ \frac{\dim_H(K_{1/3})}{\delta}, \frac{2}{\delta} + \dim_H(K_{1/3}) - 1 \right\}.$$

Thanks to Theorem 3.1.3 (which implied by Theorem 3.1.6), one can obtain a complete characterization of points approximable by rational numbers in a set which contains $K_{1/3}$ and has same Hausdorff dimension.

Denote by $\Lambda = \{0, 1, 2\}$ and by σ the usual shift on $\Lambda^{\mathbb{N}}$, defined as $\sigma((i_1, i_2, \dots)) = (i_2, \dots)$. The canonical projection from $\Lambda^{\mathbb{N}}$ on $[0, 1]$ is the map

$$\pi : x = (x_n)_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow +\infty} f_{x_1} \circ \dots \circ f_{x_n}(0). \quad (3.9)$$

Definition 3.2.2. Let $\phi : \Lambda^{\mathbb{N}} \rightarrow \{0, 1\}$ be the mapping defined as

$$\begin{cases} \phi(x) = 1 & \text{si } x_1 = 1 \\ \phi(x) = 0 & \text{si } x_1 = 0 \text{ ou } 2 \end{cases}$$

and

$$K_{1/3}^{(0)} = \pi \left(\left\{ x \in \Lambda^{\mathbb{N}} : \liminf_{k \rightarrow +\infty} \frac{S_k \phi(x)}{k} = 0 \right\} \right),$$

where $(S_k \phi)_{k \in \mathbb{N}}$ is the Birkhoff average of ϕ defined for every $x \in \Lambda^{\mathbb{N}}$ as

$$S_k(\phi)(x) = \frac{1}{k} \sum_{i=0}^{k-1} \phi(\sigma^i(x)).$$

It is shown in this manuscript (Proposition 9.1.5) that

$$\dim_H K_{1/3}^{(0)} = \frac{\log 2}{\log 3} (= \dim_H K_{1/3}).$$

We have the following result:

Theorem 3.2.3 (D.[25]). For every $\delta \geq 1$,

$$\dim_H \left(\limsup_{0 \leq p \leq q, q \rightarrow +\infty} B\left(\frac{p}{q}, \frac{1}{q^{2\delta}}\right) \cap K_{1/3}^{(0)} \right) = \min \left\{ \frac{\log 2}{\log 3}, \frac{1}{\delta} \right\}. \quad (3.10)$$

Note that there is a saturation phenomena: for every $1 \leq \delta \leq \frac{\log 3}{\log 2}$, the dimension $\dim_H(\limsup_{B \in \mathcal{Q}} B^\delta \cap K_{1/3}^{(0)})$ is constant equal to $\frac{\log 2}{\log 3}$, then decreases as $\frac{1}{\delta}$ when $\delta \geq \frac{\log 3}{\log 2}$.

In the next section, we give an application of Corollary 3.1.8 to self-similar shrinking targets problems.

3.3 Weakly conformal shrinking targets

Let $m \geq 2$ be an integer and $S = \{f_1, \dots, f_m\}$ a weakly conformal IFS. Denote by $\Lambda = \{1, \dots, m\}$, $\Lambda^* = \bigcup_{k \geq 0} \Lambda^k$ and K the attractor of S . Given $x \in K$, we study the set of elements approximable by the orbit of x under S , i.e, the sets

$$W(x, \psi) = \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), \psi(\underline{i})),$$

where $\psi : \Lambda^* \rightarrow \mathbb{R}$ verifies $\lim_{|\underline{i}| \rightarrow +\infty} \psi(\underline{i}) = 0$ and for $\underline{i} = (i_1, \dots, i_k)$, $f_{\underline{i}} = f_{i_1} \circ \dots \circ f_{i_k}$.

Historically, the study of such sets is related to the study of shrinking targets on the attractor of an expanding dynamical system [42].

More precisely, assume for the sake of simplicity that S is self-similar and satisfies for every $1 \leq i \neq j \leq m$, $\text{Im}(f_i) \cap \text{Im}(f_j) = \emptyset$. Then its attractor K can be seen as the attractor associated with the expanding mapping F defined as

$$F(x) = f_i^{-1}(x) \text{ si } x \in \text{Im}(f_i).$$

Let $\delta \geq 1$ be a real number and $\psi : \Lambda^* \rightarrow \mathbb{R}$ the mapping defined as $\psi(\underline{i}) = c_{\underline{i}} e^{-n\delta}$. The set $W(x, \psi)$ can be rewrote as

$$W(x, \psi) = \{z : F^n(z) \in B(x, e^{-n\delta}) \text{ for an infinity of integer } n \in \mathbb{N}\}, \quad (3.11)$$

which are precisely the sets introduced and studied by Hill and Velani in [42].

There are, a priori, several natural classes of mappings one could choose for ϕ :

- From a dynamical standpoint, it is natural to consider ψ of the form

$$\psi(\underline{i}) = |f_{\underline{i}}(K)|^\delta,$$

where $\delta \geq 1$, since one naturally has

$$\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), |f_{\underline{i}}(K)|) = K.$$

In this case, Theorem 3.1.6 can be used to prove the following result:

Theorem 3.3.1 (D. [23]). *Denote by $\dim(S)$ the conformality dimension of the IFS S (Definition (4.3.19)).*

Assume that que

$$\dim_H(K) = \dim(S),$$

then, for every $\delta \geq 1$ and every $x \in K$, one has

$$\dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), |f_{\underline{i}}(K)|^\delta) \right) = \frac{\dim_H(K)}{\delta}. \quad (3.12)$$

- In Diophantine approximation, one often prefers to consider mappings ψ of the form $\psi(\underline{i}) = |f_{\underline{i}}(K)|g(|\underline{i}|)$ where $g : \mathbb{N} \rightarrow \mathbb{R}$ is decreasing. For studies of the sets $W(x, \psi)$ under those settings and the OSC, we refer to the work of Hill and Velani and Allen and Bárány [2, 42].

This difference of point of view is justified by (3.11) and comes as well from the fact that one often aims at proving analogs of Khintchine's Theorem for rational numbers in the settings of IFS. To be more explicit, let us recall Khintchine's Theorem. Denote by $\phi : \mathbb{N} \rightarrow \mathbb{N}$ the Euler mapping, defined as

$$\phi(q) = \# \{1 \leq p \leq q : p \wedge q = 1\}.$$

Theorem 3.3.2 (Khintchine [50]). *Let $\psi : \mathbb{N} : \mathbb{R}^+$ be a mapping such that $q \rightarrow \frac{\psi(q)}{q}$ is monotonic. then*

$$\mathcal{L}^d \left(\limsup_{q \in \mathbb{N}, 0 \leq p \leq q, p \wedge q = 1} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \right) = 1 \Leftrightarrow \sum_{q \geq 1} \frac{\phi(q)\psi(q)}{q} = +\infty.$$

Let us mentioned that it was conjectured by Duffin et Schaeffer that Theorem 3.3.2 remains valid without the monotonicity assumption of $q \rightarrow \frac{\psi(q)}{q}$ and this conjecture was solved recently by Maynard and Koukoulopoulos [52].

To make the analogy between Khintchine's Theorem and comparable result in the settings of IFS, assume to simplify that S is self-similar and call $0 < c_1, \dots, c_m < 1$ the contraction ratios of f_1, \dots, f_m . Remember that in the case of self-similar IFS, $\dim(S)$ is the real number solution to

$$\sum_{1 \leq i \leq m} c_i^{\dim(S)} = 1.$$

One always has a natural measure ν on $\Lambda^{\mathbb{N}}$ ν defined for every $\underline{i} =$

$(i_1, \dots, i_k) \in \Lambda^*$ as

$$\nu(\underline{i}) = c_{\underline{i}}^{\dim(S)}.$$

Consider $\mu = \nu(\pi^{-1})$, where π is the canonical projection of $\Lambda^{\mathbb{N}}$ on K (see (3.9) below). One aims at establishing dichotomies of the following form ($s \geq 0$ is not specified here):

$$\begin{cases} \text{if } \sum_{\underline{i} \in \Lambda^*} \psi(\underline{i})^s = +\infty, \text{ then } \mu(W(x, \psi)) = 1 \\ \text{if } \sum_{\underline{i} \in \Lambda^*} \psi(\underline{i})^s < +\infty, \text{ then } \mu(W(x, \psi)) = 0. \end{cases}$$

That in mind, it is clear that ψ can not be arbitrary and that $\psi(\underline{i})$ must depend on the generation of the word \underline{i} if one hopes to cover a set of large measure by balls $\{B(f_{\underline{i}}(x), \psi(\underline{i}))\}_{\underline{i} \in \Lambda^n}$ for an infinity of integers $n \in \mathbb{N}$.

For examples of such theorems in the self-similar case, one refers to [54] and we mention the following result:

Theorem 3.3.3 (Baker [3]). *Assume that S satisfies one of the following condition*

$$\begin{cases} -\sum_{i=1}^m c_i^{\dim(S)} \log(c_i^{\dim(S)}) < -2 \log(\sum_{i \leq i \leq m} c_i^{2 \dim(S)}) \\ c_1 = \dots = c_m. \end{cases}$$

Let $\psi : \Lambda^ \rightarrow \mathbb{R}$ be a mapping of the form $\psi(\underline{i}) = |f_{\underline{i}}(K)|g(|\underline{i}|)$ where $g : \mathbb{N} \rightarrow \mathbb{R}$ is positive and decreasing. If*

$$\sum_{n \geq 0} ng(n)^{\dim(S)} = +\infty,$$

then for every $x \in K$,

$$\mu(W(x, \psi)) = 1.$$

Theorem 3.3.3 combined with Theorem 3.1.6 allows to prove the following result:

Theorem 3.3.4 (D. [23]). *Let $g : \mathbb{N} \rightarrow (0, +\infty)$ be a positive and decreasing mapping and set*

$$s_g = \inf \left\{ s \geq 0 : \sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^k} k(|f_{\underline{i}}(K)|g(k))^s < +\infty \right\}. \quad (3.13)$$

Then, under the hypotheses of Theorem 3.3.3, if furthermore $\dim(\mu) = \dim(S)$, for every $\delta \geq 1$, one has

$$\dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), (|f_{\underline{i}}(K)|g(|\underline{i}|))^{\frac{\delta s_g}{\dim(S)}}) \right) = \frac{\dim(S)}{\delta}. \quad (3.14)$$

Remark that (3.14) provides a complete characterization of the dimension of the sets $W(x, \psi)$ for $\psi(\underline{i}) = (|f_{\underline{i}}(K)|g(|\underline{i}|))^s$, for every $s \geq 0$ and every $g : \mathbb{N} \rightarrow \mathbb{N}$ positive and decreasing.

The next section presents an application in multifractal analysis of discrete measures, developed in Chapter 11.

3.4 Multifractal analysis of discrete measures

Let $m \geq 2$ be an integer and $S = \{f_1, \dots, f_m\}$ of $m \geq 2$ contracting similarities of contraction ratios $0 < c_1, \dots, c_m < 1$ and K the attractor of S . Given $(q_1, \dots, q_{m+1}) \in (0, 1)^{m+1}$ a probability vector, we study the probability measure solution to the following equation

$$\chi(\cdot) = \sum_{1 \leq i \leq m} q_i \chi(f_i^{-1}(\cdot)) + q_{m+1} \delta_{x_0}, \quad (3.15)$$

where δ_x denotes the dirac measure at x . Such measures were introduced by Olsen and Snigireva in [70] and are called “in-homogeneous self-similar measures with condensation measure δ_{x_0} ”.

We can show (see Proposition 11.1.1) that such measures are discrete can be written as

$$\chi(\cdot) = C \sum_{\underline{i} \in \Lambda^*} p_{\underline{i}} c_{\underline{i}}^t \delta_{f_{\underline{i}}(x_0)}, \quad (3.16)$$

where $t > 0$, (p_1, \dots, p_m) is an other probability vector, $C > 0$ is a renormalization constant so that χ is a probability measure.

Its worth mentioning that comparable measures were obtained and studied by Mandelbrot and Riedi in [56] and by Barral and Seuret in [9] by inverting the distribution function associated with Gibbs measure or self-similar measures satisfying the open set condition. The multifractal properties of those measures have been studied by Falconer [31], Barral and Seuret [7, 9] and Mandelbrot and Riedi [56] but in the last case, by considering only local dimensions that are exact limits (not liminf), the authors missed an interesting phenomenon highlighted by Falconer, Barral and Seuret.

The fine multifractal analysis of measures was born 1970's with the work of Mandelbrot [57, 58] and Olsen [60–62]. Those studies consist, given $\mu \in \mathcal{M}(\mathbb{R}^d)$, in computing the spectrum of the dimension of the following sets (

Section 4.2, Definition 4.2.12)

$$\begin{aligned} E_\mu^h &= \{x \in \text{supp}(\mu) : \underline{\dim}(\mu, x) = h\}, \\ \widetilde{E}_\mu^h &= \{x \in \text{supp}(\mu) : \underline{\dim}(\mu, x) = \overline{\dim}(\mu, x) = h\}, \end{aligned} \quad (3.17)$$

i.e, computing for every $h \in \mathbb{R}$,

$$\begin{cases} D_\mu(h) = \dim_H(E_\mu^h) \text{ and} \\ \widetilde{D}_\mu(h) = \dim_H(\widetilde{E}_\mu^h). \end{cases}$$

One generally aims at comparing the dimensions of those sets to the Legendre transform of a scaling function, denoted by τ_μ ((4.19)) and defined by, for all $q \in \mathbb{R}$,

$$\tau_\mu(q) = \liminf_{n \rightarrow +\infty} \frac{-\log \left(\sum_{D \in \mathcal{D}_n: \mu(D) \neq 0} \mu(D)^q \right)}{n}$$

where, for $n \in \mathbb{N}$, \mathcal{D}_n denotes the set of dyadic cubes of generation n .

Let μ be the self-similar measure associated with S and (p_1, \dots, p_m) , i.e. the measure solution to the equation

$$\mu(\cdot) = \sum_{i=1}^m p_i \mu(f_i^{-1}(\cdot)).$$

We will see that the multifractal analysis of the measure χ , given by (3.16), depends both on the multifractal analysis of the self-similar measure μ and on the speed of approximation of a point x by the orbit $(f_{\underline{i}}(x_0))_{\underline{i} \in \Lambda^*}$. Let us introduce the degree of approximation of a point with respect to $(f_{\underline{i}}(x_0))_{\underline{i} \in \Lambda^*}$.

Definition 3.4.1. *The degree of approximation of a point $x \in K$ with respect to $(f_{\underline{i}}(x_0))_{\underline{i} \in \Lambda^*}$, denoted Δ_x , is defined as*

$$\Delta_x = \sup \left\{ \Delta : x \in \limsup_{n \rightarrow +\infty} B(f_{\underline{i}}(x_0), (2c_{\underline{i}})^\Delta) \right\}.$$

In the case where the base point of (3.16) x_0 does not belong to the attractor K of S , every points of $\text{supp}(\chi)$ outside the set $\{f_{\underline{i}}(x_0)\}_{\underline{i} \in \Lambda^*}$ (which has Hausdorff dimension equal to 0) are badly approximated by those points, i.e, have degree equal to 1. In that case, the multifractal spectrum of χ is a translated of the multifractal spectrum of μ .

Theorem 3.4.2 (D.[26]). *Consider $x_0 \notin K$. Then, for every $t > 0$, the measure*

χ defined as (3.16) exists. Moreover, for every $h \geq 0$,

$$D_\chi(h) = D_\mu(h - t). \quad (3.18)$$

Remark that this result holds without any separation hypotheses on the system S . A direct consequence of this result is that, when the attractor K has Lebesgue measure equal to 0, for Lebesgue almost every x_0 , the spectrum of the measure χ is a translation by t of the spectrum of μ . In other word, to simulate a measure which has the same spectrum as μ but translated by t , it is enough to pick randomly x_0 according to the uniform law and to simulate the measure χ associated (which is discrete). Let us add that is established in this manuscript that in this case, χ does not satisfy the multifractal formalism (one actually has $\tau_\chi(q) = 0$ for $q \geq 1$).

In the case $x_0 \in K$, Corollary 3.1.8 allows, under certain hypotheses, to compute the spectrum of the measure χ .

Theorem 3.4.3 (D. [26]). *Let $m \geq 2$, $S = \{f_1, \dots, f_m\}$ be a self-similar IFS satisfying AWSC (Definition 4.3.3) with no exact overlaps and $K \subset \mathbb{R}^d$ the attractor of S . Let (p_1, \dots, p_m) be a probability vector, μ the self-similar measure associated with (p_1, \dots, p_m) , $x_0 \in K$ and $C > 0$ the normalizing constant such that the measure*

$$\chi(\cdot) = C \sum_{\underline{i} \in \Lambda^*} c_{\underline{i}}^t p_{\underline{i}} \delta_{f_{\underline{i}}(x_0)},$$

is a probability measure.

Define $q_c = \min \{q : \tau_\mu(q) + qt = 0\}$ and $\lambda_c = \tau'_\mu(q_c)$. Then, recalling (4.16):

1. *The mapping τ_χ satisfies*

$$\begin{cases} \text{for every } q \geq q_c, & \tau_\chi(q) = 0 \\ \text{for every } 0 \leq q \leq q_c, & \tau_\chi(q) = \tau_\mu(q) + t \\ \text{for every } q \leq 0, & \tau_\chi(q) \geq \tau_\mu(q) + t. \end{cases} \quad (3.19)$$

In particular for every $h \geq 0$, one has one has

$$\tau_\chi^*(h) \leq \begin{cases} hq_c & \text{if } 0 \leq h \leq \lambda_c + t \\ \tau_\mu^*(h - t) & \text{if } h \geq \lambda_c + t. \end{cases} \quad (3.20)$$

2. The multifractal spectrum D_χ of χ satisfies

$$\begin{cases} \text{for every } 0 < h \leq \lambda_c + t, & D_\chi(h) = hq_c \\ \text{for every } \lambda_c + t < h \leq \tau'_\mu(0^+) + t, & D_\chi(h) = \tau_\mu^*(h - t). \end{cases} \quad (3.21)$$

3. For every $h > \tau'_\mu(0^+) + t$, one has

$$\dim_H(\widetilde{E}_\mu^{h-t} \cap \{x : \Delta_x = 1\}) \leq D_\chi(h) \leq \tau_\mu^*(h - t).$$

Remark 3.4.4.

- Some examples of self-similar measures satisfying the hypotheses of Theorem 3.4.3 are given in Corollary 4.3.27.
- $\tau'_\mu(q_c)$ exists when S satisfies the AWSC with no exact overlaps, see Theorem 4.3.39.
- Assume in addition to the hypotheses of Theorem 3.4.3 that for $h \geq \tau'_\mu(0^+)$, there exists $m_h \in \mathcal{M}(\mathbb{R}^d)$ such that $\dim_H(m_h) = \tau_\mu^*(h)$ and $m_h(\widetilde{E}_\mu^h) > 0$, then

$$D_\chi(h + t) = \tau_\mu^*(h),$$

and χ satisfies the multifractal formalism at $h + t$.

One mentions also that the multifractal analysis of χ for h large is hard in general. This phenomena is not surprising since there exists some examples of self-similar measures satisfying very reasonable separation condition but does not satisfy the multifractal formalism in the decreasing part of the spectrum (see [72]).

3.5 Conclusion and perspectives

As explained in this introduction, the mass transference principles (or ubiquity theorems) are tools used in metric number theory, dynamical systems and multifractal analysis. Hence, there are many natural question and extensions related to these three fields raised by this manuscript:

- Can we estimate the μ -essential contents for a wider class of measures? Can something general be said when the IFS is self-affine? Although it seems likely that there exists exact dimensional ergodic measures for which Corollary 3.1.8 should not hold, is it possible to define a quantity

one can estimate to compute the dimension of shrinking targets under those very natural settings?

- Can we compute the multifractal spectrum of the measure χ satisfying (3.16) when the underlying IFS is self-affine, satisfies the exponential separation condition or is simply given by the orbit of an ergodic dynamical system.
- What fractal sets can we distinguish or characterize by considering the approximation by rectangles or sets of other shapes? Such problems naturally appears in multiplicative Diophantine approximation. More precisely, given $\delta \geq 1$, what is the dimension and the Hausdorff measure (associated with an appropriate gauge function) of the set

$$A_\delta = \left\{ (x_1, \dots, x_d) : \exists (p_1, \dots, p_d) \in \mathbb{N}^d, \prod_{i=1}^d \left| x_i - \frac{p_i}{q} \right| \leq q^{-\delta}, q \in \mathbb{N} \text{ i.o.} \right\},$$

where i.o. means that the inequality holds for infinitely many q . In this case, the approximating sets are hyperboles.

In Chapter 4, some recall about the basic notion frequently used in this manuscript are given. It is also proved in this chapter that C^1 weakly conformal IFS's satisfying AWSC without exact overlaps are dimension regular (Definition 4.3.20) and the scaling function of weakly conformal measures (under these hypothesis) is computed for $q > 0$. A particular case of the general ubiquity Theorem, Theorem 7.1.2 is treated in Chapter 5. More precisely, Chapter 5 deals with the case where the measure involved is quasi-Bernoulli on \mathbb{R}^d .

The Chapter 6 deals with the definition of the basic notion needed to prove Theorem 7.1.2. In particular, given a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ one defines the notion of μ -asymptotically covering sequences of balls and the μ -essential Hausdorff content of a set. The essential content of open sets are also computed when the measure involved is weakly conformal. Finally, as an application, one derive from the Borel-Cantelli Lemma established in [12] for doubling measures a version which is valid for every probability measure on \mathbb{R}^d . Chapter 7 is dedicated to the proof of Theorem 7.1.2 and its consequences when the measure is weakly-conformal. The optimality of the lower-bounds established in Chapter 7 are discussed in Chapter 8. Chapters 9, 10 and 11 gives application of Theorem 7.1.2 respectively, in Diophantine approximation, to weakly conformal shrinking targets and to the multifractal analysis of discrete self-similar measures.

Some notations used in this article are summarized here.

Notations	Definition and references.
\mathbb{N}	set of integers, $\{0, 1, 2, \dots\}$.
\mathbb{Q}	set of rational numbers, $\left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}^*\right\}$.
\mathbb{R}	set of real numbers.
$d \in \mathbb{N}$	dimension of the ambient \mathbb{R}^d .
$\ x\ _\infty, x = (x_1, \dots, x_d) \in \mathbb{R}^d$	norm on \mathbb{R}^d used in this manuscript and defined by $\ x\ _\infty = \max_{1 \leq i \leq d} x_i $.
$\mathcal{D}_p, p \in \mathbb{N}$	set of dyadic cubes of generation p of \mathbb{R}^d . See (4.17).
$\overline{E}, E \subset \mathbb{R}^d$	topological closure of E .
$\mathring{E}, E \subset \mathbb{R}^d$	interior of E .
$\partial E, E \subset \mathbb{R}^d$	boundary of E , defined by $\partial E = \overline{E} \setminus \mathring{E}$.
$B(x, r), x \in \mathbb{R}^d, r > 0$	closed ball with center x and radius r .
$ E , E \subset \mathbb{R}^d$	diameter of the set E .
tB, B a ball of radius $r > 0$ and $t \geq 0$	ball with same center as B and radius tr .
B^δ, B ball of radius $r > 0, \delta \in \mathbb{R}$	ball of same center as B and radius r^δ .
$\mathcal{L}(\mathbb{R}^k, \mathbb{R}^p), k, p \in \mathbb{N}$	space of linear maps from \mathbb{R}^k to \mathbb{R}^p .
$\ \ell\ , \ell \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^p), k, p \in \mathbb{N}$	norm subordinated to $\ \cdot\ _\infty$ on both \mathbb{R}^k and \mathbb{R}^p of ℓ . See (4.21).
$\ \ell\ , \ell \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^p), k, p \in \mathbb{N}$	See (4.21) as well.
$\mathcal{M}_d(\mathbb{R})$	space of $d \times d$ real matrices.
$GL_d(\mathbb{R})$	space of $d \times d$ invertible real matrices.
$\mathcal{O}_d(\mathbb{R})$	space of $d \times d$ orthogonal real matrices.
$\mathcal{S}_d^+(\mathbb{R})$	space of $d \times d$ non neagative symmetric real matrices.
$\mathcal{S}_d^{++}(\mathbb{R})$	space of $d \times d$ positive symmetric real matrices.
$f'(x), f : \mathbb{R}^k \rightarrow \mathbb{R}^p, k, p \in \mathbb{N}$ and $x \in \mathbb{R}^k$	differential of f at x .
$C^n(\mathbb{R}^k, \mathbb{R}^p), n \in \mathbb{N} \cup \infty, k, p \in \mathbb{N}$	space of mappings from $\mathbb{R}^k \rightarrow \mathbb{R}^p$ n times differentiable with continuous n^{th} differential at any $x \in \mathbb{R}^k$.
$C^{n+\varepsilon}(\mathbb{R}^k, \mathbb{R}^p), n \in \mathbb{N} \cup \infty, k, p \in \mathbb{N}, 0 < \varepsilon < 1$	space of mappings from $\mathbb{R}^k \rightarrow \mathbb{R}^p$ n times differentiable with ε -Hölder continuous n^{th} differential at any $x \in \mathbb{R}^k$.
$\mathcal{B}(X)$	Borel σ -algebra of X .
$\mathcal{M}(X)$	space of Borel probability measure on X .
\mathcal{L}^d	Lebesgue measure on \mathbb{R}^d .
μ, ν, η	letters used for measures.
μ -a.c. sequence of balls	μ -asymptotically covering sequence of balls . See Definition 6.1.1.
$\text{supp}(\mu), \mu \in \mathcal{M}(X)$	topological support of μ .
$\text{infess}_\mu, \mu \in \mathcal{M}(X)$	essential infimum with respect to μ .
$\text{supess}_\mu, \mu \in \mathcal{M}(X)$	essential supremum with respect to μ .

$\tau_\mu(q), \mu \in \mathcal{M}(\mathbb{R}^d), q \in \mathbb{R}$	L^q spectrum of μ , see (4.19).
$E_\mu^h, \widetilde{E}_\mu^h, \mu \in \mathcal{M}(\mathbb{R}^d), h \geq 0$	level sets used to compute the multifractal spectrum of μ , see (4.15).
$\mathcal{H}^s(E), s \geq 0, E \subset \mathbb{R}^d$	s -dimensional Hausdorff measure of E , see (4.2.2).
$\mathcal{P}^s(E), s \geq 0, E \subset \mathbb{R}^d$	s -dimensional packing measure of E , see (4.12).
$\dim_H(E), \dim_P(E), E \subset \mathbb{R}^d$	Hausdorff and packing dimension of E , see Lemma 4.2.3 and Proposition 4.2.10.
\mathcal{L}^d	Lebesgue measure on \mathbb{R}^d .
$\mathcal{H}_\infty^s(E), s \geq 0, E \subset \mathbb{R}^d$	s -dimensional Hausdorff content of E , see (4.7).
$\mathcal{H}_\infty^{\mu,s}(E), s \geq 0, \mu \in \mathcal{M}(\mathbb{R}^d), E \subset \mathbb{R}^d$	s -dimensional μ -essential Hausdorff content of E , see Definition 6.2.1.
$\overline{\dim}(\mu, x), \underline{\dim}(\mu, x), \mu \in \mathcal{M}(\mathbb{R}^d), x \in \text{supp}(\mu)$	inferior and superior local dimension of μ at x , see Definition 4.14.
$\underline{\dim}_H(\mu), \overline{\dim}_H(\mu), \underline{\dim}_P(\mu), \overline{\dim}_P(\mu), \dim(\mu), \mu \in \mathcal{M}(\mathbb{R}^d)$	see Definition 4.14.
$f_{\underline{i}}$, where $\underline{i} = (i_1, \dots, i_k) \in \{1, \dots, m\}^k, k \in \mathbb{N}$	$f_{i_1} \circ \dots \circ f_{i_k}$, see Section 4.3.
$c_{\underline{i}}, p_{\underline{i}}$ where $\underline{i} = (i_1, \dots, i_k) \in \{1, \dots, m\}^k, k \in \mathbb{N}$	$c_{i_1} \times \dots \times c_{i_k}$ and $p_{i_1} \times \dots \times p_{i_k}$. See Section 4.3.
$c_{\underline{i}}(x)$ where $\underline{i} = (i_1, \dots, i_k) \in \{1, \dots, m\}^k, k \in \mathbb{N}, x \in \mathbb{R}^d$	$\ f'_{\underline{i}}(x)\ $, S see Section 4.3.2.1.
σ	denote the usual shift operation on $\Lambda^{\mathbb{N}}$, see beginning of Section 4.3.
OSC, SOSC and AWSC	respectively Open Set Condition, Strong Open Set Condition and Asymptotically Weak Separation Condition, See Definition 4.3.3.
$P(s), s \in \mathbb{R}$	pressure associated with a weakly conformal IFS S , See Proposition 4.3.16.

Chapter 4

Covering Lemmas, Hausdorff measures and iterated function systems

4.1 Covering lemmas

This section is dedicated to the statement of various classical and less classical covering lemmas used throughout this manuscript. Let us start the well-known 5r-covering lemma (see [32] Lemma 4.8 p.91).

Lemma 4.1.1 (5r-lemma). *Let (X, d) be a metric space and $(B_n)_{n \in \mathbb{N}}$ a sequence of balls of X satisfying $\sup_{n \in \mathbb{N}} |B_n| < +\infty$. There exists a sub-sequence of balls $(B_{\phi(n)})_{n \in \mathbb{N}}$ such that for any $n < n'$, $B_{\phi(n)} \cap B_{\phi(n')} = \emptyset$ and*

$$\bigcup_{n \in \mathbb{N}} B_n \subset \bigcup_{n \in \mathbb{N}} 5B_{\phi(n)}. \quad (4.1)$$

This lemma is useful when dealing with doubling measures. However in this manuscript, one mainly deals with measures that are not doubling. The rest of the section is dedicated to the establishment of a modified version of the traditional Besicovitch covering theorem more adapted for our purposes.

Proposition 4.1.2 (D. [24]). *For any $0 < v \leq 1$, there exists $Q_{d,v} \in \mathbb{N}^*$, a constant depending only on the dimension d and v , such that for every bounded subset $E \subset \mathbb{R}^d$ and every covering $\mathcal{F} = \{B(x, r(x)) : x \in E, r(x) > 0\}$ of E , there exists $\mathcal{F}_1, \dots, \mathcal{F}_{Q_{d,v}}$ finite or countable sub-families of \mathcal{F} such that:*

- $\forall 1 \leq i \leq Q_{d,v}, \forall L \neq L' \in \mathcal{F}_i$, one has $\frac{1}{v}L \cap \frac{1}{v}L' = \emptyset$.

- E is covered by the families \mathcal{F}_i , i.e.

$$E \subset \bigcup_{1 \leq i \leq Q_{d,v}} \bigcup_{L \in \mathcal{F}_i} L. \quad (4.2)$$

The case $v = 1$ corresponds to the standard Besicovich's covering lemma (see [59], Chapter 2, pp. 28-34 for instance).

A first step towards Proposition 4.1.2 is the next lemma, which allows to split a given family of “weakly” overlapping balls into a finite number of families of disjoint balls.

Lemma 4.1.3. *Let $0 < v < 1$ and $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a countable family of balls such that $\lim_{n \rightarrow +\infty} |B_n| = 0$, and for every $n \neq n' \in \mathbb{N}$, $vB_n \cap vB_{n'} = \emptyset$.*

There exists $\gamma_{d,v} + 1$ ($\gamma_{d,v}$ being the constant appearing in Lemma 4.1.4 below) sub-families of \mathcal{B} , $(\mathcal{F}_i)_{1 \leq i \leq \gamma_{d,v} + 1}$, such that:

- $\mathcal{B} = \bigcup_{1 \leq i \leq \gamma_{d,v} + 1} \mathcal{F}_i$,
- $\forall 1 \leq i \leq \gamma_{d,v} + 1$, $\forall L \cap L' \in \mathcal{F}_i$, one has $L \cap L' = \emptyset$.

Proof. The proof is based on the following lemma, whose proof can be found in [59], Lemma 2.7, pp.30 - there, the result is obtained for $v = 1/2$ but the proof remains valid for any $v < 1$.

Lemma 4.1.4. *For any $0 < v \leq 1$ there exists a constant $\gamma_{v,d} > 0$ depending only on v and the dimension d only, satisfying the following: if a family of balls $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ and a ball B are such that*

- $\forall n \geq 1$, $|B_n| \geq \frac{1}{2}|B|$,
- $\forall n_1 \neq n_2 \geq 1$, $vB_{n_1} \cap vB_{n_2} = \emptyset$,

then B intersects at most $\gamma_{v,d}$ balls of \mathcal{B} .

The families $\mathcal{F}_1, \dots, \mathcal{F}_{\gamma_{d,v} + 1}$ are built recursively.

For $k \in \mathbb{N}$, set

$$\mathcal{G}^{(k)} = \{L \in \mathcal{F} : 2^{-k-1} < |L| \leq 2^{-k}\}.$$

Notice that, since

$$\lim_{n \rightarrow +\infty} |B_n| = 0,$$

each $\mathcal{G}^{(k)}$ is empty or finite.

Observe first that for every $k \in \mathbb{N}$, for every ball $B \in \mathcal{G}^{(k)}$, and for every pair of balls $B_1 \neq B_2 \in \bigcup_{k' \leq k} \mathcal{G}^{(k')} \setminus \{B\}$, one has

$$vB_1 \cap vB_2 = \emptyset$$

as well as

$$|B_i| \geq \frac{|B|}{2}$$

for $i = 1, 2$.

By Lemma 4.1.4, this implies that B intersects at most $\gamma_{d,v}$ balls of $\bigcup_{k' \leq k} \mathcal{G}^{(k')} \setminus \{B\}$.

To get Lemma 4.1.3, we are going to sort the balls of $\bigcup_{k' \leq k} \mathcal{G}^{(k')}$ recursively on k into families $\mathcal{F}_1, \dots, \mathcal{F}_{\gamma_{d,v}+1}$ of pairwise disjoint balls. At each step, a new ball B will be added to one of those families of balls \mathcal{F}_i and the resulting family, $\mathcal{F}_i \cup \{B\}$ will be denoted again by \mathcal{F}_i .

Let k_0 be the smallest integer such that $\mathcal{G}^{(k_0)}$ is non-empty. Consider an arbitrary $L_0 \in \mathcal{G}^{(k_0)}$. By Lemma 4.1.4, L_0 intersects $n_0 \leq \gamma_{d,v}$ other balls of $\mathcal{G}^{(k_0)}$, that are denoted by L_1, \dots, L_{n_0} . The sets \mathcal{F}_i are then set as follows:

- $\forall 1 \leq i \leq n_0, \mathcal{F}_i = \{L_i\},$
- $\forall n_0 + 1 \leq i \leq \gamma_{d,v}, \mathcal{F}_i = \emptyset,$
- $\mathcal{F}_{\gamma_{d,v}+1} = \{L_0\}.$

Further, consider $\tilde{L} \in \mathcal{G}^{(k_0)} \setminus \bigcup_{0 \leq i \leq n_0} \{L_i\}$ (whenever such an \tilde{L} exists). The same argument (Lemma 4.1.4) ensures that \tilde{L} intersects at most $\gamma_{d,v}$ balls of $\mathcal{G}^{(k_0)}$.

In particular there must exist $1 \leq i \leq \gamma_{d,v} + 1$ such that for every $L \in \mathcal{F}_i$, $\tilde{L} \cap L = \emptyset$. Choosing arbitrarily one of those indices i , one adds \tilde{L} to $\mathcal{F}_i := \tilde{L} \cup \mathcal{F}_i$ (we keep the same name for this new family).

The same argument remains valid for any other ball $L'' \notin \bigcup_{1 \leq j \leq \gamma_{d,v}+1} \bigcup_{L \in \mathcal{F}_j} \{L\}$. Hence, proceeding recursively on all balls of $\mathcal{G}^{(k_0)}$ allows to sort the balls of $\mathcal{G}^{(k_0)}$ into $\gamma_{d,v} + 1$ families $(\mathcal{F}_i)_{1 \leq i \leq \gamma_{d,v}+1}$ of pairwise disjoint balls.

Next, let k_1 be the smallest integer such that $k_1 > k_0$ and $\mathcal{G}^{(k_1)}$ is non empty, take an arbitrary $L_0^{(1)} \in \mathcal{G}^{(k_1)}$. It is trivial to check that the family $\mathcal{G}^{(k_0)} \cup \mathcal{G}^{(k_1)}$ and the ball $L_0^{(1)}$ satisfy the conditions of Lemma 4.1.4. Subsequently, $L_0^{(1)}$ intersects at most $\gamma_{d,v}$ balls of $\mathcal{G}^{(k_0)} \cup \mathcal{G}^{(k_1)}$, and there must exist an integer $1 \leq i_0 \leq \gamma_{d,v} + 1$ such that $L_0^{(1)} \cap \bigcup_{L \in \mathcal{F}_{i_0}} L = \emptyset$. As before, we add this ball $L_0^{(1)}$ to the family \mathcal{F}_{i_0} .

Consider $\tilde{L} \in \mathcal{G}^{(k_1)}$ such that $\tilde{L} \notin \bigcup_{1 \leq i \leq \gamma_{d,v}+1} \mathcal{F}_i$ (whenever such a ball exists). The exact same argument shows \tilde{L} intersects at most $\gamma_{d,v}$ balls of $\mathcal{G}^{(k_0)} \cup \mathcal{G}^{(k_1)}$. Hence there exists an integer $1 \leq \tilde{i} \leq \gamma_{d,v} + 1$ such that

$$L \cap \bigcup_{A \in \mathcal{F}_{\tilde{i}}} A = \emptyset.$$

One adds \tilde{L} to the family $\mathcal{F}_{\tilde{i}}$, which remains composed only of pairwise disjoint balls.

One applies this argument to every ball of $\mathcal{G}^{(k_1)}$, hence finally sorting the balls of $\mathcal{G}^{(k_0)} \cup \mathcal{G}^{(k_1)}$ into $\gamma_{d,v} + 1$ families of pairwise disjoint balls, as requested.

It is now easily seen that one can proceed recursively on $k \geq k_0$, ending up with the families $\mathcal{F}_1, \dots, \mathcal{F}_{\gamma_{d,v}+1}$ fulfilling the desired properties. \square

We are now ready to prove Proposition 4.1.2.

Proof. Fix $E \subset [0, 1]^d$ and $\mathcal{F} = \{B(x, r_{(x)}) : x \in E, r_{(x)} > 0\}$.

One applies Besicovich's theorem (i.e. Proposition 4.1.2 with $v = 1$) to $\mathcal{F} = \{B(x, r_{(x)}) : x \in E, r_{(x)} > 0\}$. This provides us with a finite set of families of balls $\mathcal{G}_1, \dots, \mathcal{G}_{\gamma_{d,1}+1}$ composed of pairwise disjoint balls satisfying (4.2), i.e. $E \subset \bigcup_{1 \leq i \leq \gamma_{d,1}+1} \bigcup_{L \in \mathcal{G}_i} L$.

For every $1 \leq i \leq \gamma_{d,1}$, one sets $\mathcal{G}_i^{(v)} = \{\frac{1}{v}L : L \in \mathcal{G}_i\}$, i.e. the sets of balls with same centers as \mathcal{G}_i but with radii multiplied by $v^{-1} > 1$. Notice that by construction, $\forall 1 \leq i \leq \gamma_{d,1}, \forall L \neq L' \in \mathcal{G}_i^{(v)}$, one has $vL \cap vL' = \emptyset$. Hence, Lemma 4.1.3 yields $\gamma_{d,v} + 1$ sub-families $(\mathcal{G}_{i,j}^{(v)})_{1 \leq j \leq \gamma_{d,v}+1}$ of $\mathcal{G}_i^{(v)}$ such that:

- $\forall 1 \leq j \leq \gamma_{d,v} + 1, \forall L \neq L' \in \mathcal{G}_{i,j}^{(v)}$, one has $L \cap L' = \emptyset$,
- $\mathcal{G}_i^{(v)} = \bigcup_{1 \leq j \leq \gamma_{d,v}+1} \mathcal{G}_{i,j}^{(v)}$.

Finally, we set for every $1 \leq i \leq \gamma_{d,1}$ and $1 \leq j \leq \gamma_{d,v} + 1$

$$\mathcal{F}_{i,j} = \{vL : L \in \mathcal{G}_{i,j}^{(v)}\} \quad \text{and} \quad \mathcal{F}_i = \bigcup_{1 \leq j \leq \gamma_{d,v}+1} \mathcal{F}_{i,j}.$$

These sets verify that:

- $\forall 1 \leq i \leq \gamma_{d,1}, \forall 1 \leq j \leq \gamma_{d,v} + 1, \forall L \neq L' \in \mathcal{F}_{i,j}, \frac{1}{v}L \cap \frac{1}{v}L' = \emptyset$ (because the balls of $\mathcal{G}_{i,j}$ are pairwise disjoint),

$$\bullet E \subset \bigcup_{1 \leq i \leq Q_{d,1}} \mathcal{G}_i = \bigcup_{1 \leq i \leq Q_{d,1}} \bigcup_{1 \leq j \leq \gamma_{d,v}+1} \mathcal{F}_{i,j}.$$

This proves the statement and the fact that $Q_{d,v} = Q_{d,1} \cdot (\gamma_{d,v} + 1)$. \square

One also recall the following density-lemma (which holds in metric sapces in which Besicovitch's theorem holds).

Lemma 4.1.5 ([13]). *Let $m \in \mathcal{M}(\mathbb{R}^d)$, $0 < c < 1$ and A be a Borel set with $m(A) > 0$. For every $r > 0$, set*

$$A(r) = \{x \in A : \forall \tilde{r} \leq r, m(B(x, \tilde{r}) \cap A) \geq c m(B(x, \tilde{r}))\} \quad (4.3)$$

Then

$$m\left(\bigcup_{r>0} A(r)\right) = m(A). \quad (4.4)$$

4.2 Hausdorff dimension, dimension of measures

Basic properties of the geometric measure theory are recalled in this section.

4.2.1 Hausdorff measures, dimension and content

Let us start by recalling the definition of an outer measure.

Definition 4.2.1. *Let μ be a non negative function defined on the subsets of \mathbb{R}^d . One says that μ is an outer measure if it verifies*

- $\mu(\emptyset) = 0$,
- for any $A \subset B$,

$$\mu(A) \leq \mu(B),$$

- for any sequence of sets $(E_n)_{n \in \mathbb{N}}$,

$$\mu\left(\bigcup_{n \geq 0} E_n\right) \leq \sum_{n \geq 0} \mu(E_n)$$

(with still " \leq " even when the sets E_n are disjoint measurable sets).

Definition 4.2.2. *Let $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$. Suppose that ζ is increasing in a neighborhood of 0 and $\zeta(0) = 0$. The Hausdorff outer measure at scale $t \in (0, +\infty]$ associated with ζ of a set E is defined by*

$$\mathcal{H}_t^\zeta(E) = \inf \left\{ \sum_{n \in \mathbb{N}} \zeta(|B_n|) : |B_n| \leq t, B_n \text{ closed ball and } E \subset \bigcup_{n \in \mathbb{N}} B_n \right\}. \quad (4.5)$$

The Hausdorff Borel measure associated with ζ of a set E is defined by

$$\mathcal{H}^\zeta(E) = \lim_{t \rightarrow 0^+} \mathcal{H}_t^\zeta(E). \quad (4.6)$$

For $t \in (0, +\infty]$, $s \geq 0$ and $\zeta : x \mapsto x^s$, one simply uses the usual notation $\mathcal{H}_t^\zeta(E) = \mathcal{H}_t^s(E)$ and $\mathcal{H}^\zeta(E) = \mathcal{H}^s(E)$, and these measures are called s -dimensional Hausdorff outer measure at scale $t \in (0, +\infty]$ and s -dimensional Hausdorff measure respectively. Thus,

$$\mathcal{H}_t^s(E) = \inf \left\{ \sum_{n \in \mathbb{N}} |B_n|^s : |B_n| \leq t, B_n \text{ closed ball and } E \subset \bigcup_{n \in \mathbb{N}} B_n \right\}. \quad (4.7)$$

The quantity $\mathcal{H}_\infty^s(E)$ (obtained for $t = +\infty$) is called the s -dimensional Hausdorff content of the set E . Note that it is easily seen that for $0 \leq s < d$, the s -dimensional Hausdorff content is not a measure. For any cube B , one has $\mathcal{H}_\infty^s(B) = |B|^s$ and, if one considers a packing by disjoint smaller cubes B_1, \dots, B_k , one has

$$\begin{aligned} \sum_{1 \leq i \leq k} \mathcal{H}_\infty^s(B_i) &= \sum_{1 \leq i \leq k} |B_i|^s = \sum_{1 \leq i \leq k} (|B_i|^d)^{\frac{s}{d}} \\ &> \left(\sum_{1 \leq i \leq k} |B_i|^d \right)^{\frac{s}{d}} \geq (\mathcal{L}^d(B))^{\frac{s}{d}} = |B|^s = \mathcal{H}_\infty^s(B) = \mathcal{H}_\infty^s\left(\bigcup_{1 \leq i \leq k} B_i\right). \end{aligned}$$

The following lemma defines the Hausdorff dimension of a set.

Lemma 4.2.3. *Let $E \subset \mathbb{R}^d$ be a subset of \mathbb{R}^d . There exists a unique $0 \leq \dim_H(E) \leq d$ such that*

$$\begin{cases} \text{for any } s < \dim_H(E), \mathcal{H}^s(E) = +\infty \\ \text{for any } s > \dim_H(E), \mathcal{H}^s(E) = 0. \end{cases} \quad (4.8)$$

The number $\dim_H(E)$ is called the Hausdorff dimension of the set E .

A very classical, but powerful, tool to estimate Hausdorff dimension is the mass distribution principle, which is stated in the following proposition, together with the Billingsley lemma.

Lemma 4.2.4 ([32]). *Let $E \subset \mathbb{R}^d$ be a Borel set.*

- *Let $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$, be a mapping, increasing in a neighborhood of 0 and such that $\zeta(0) = 0$. Assume that there exists a Borel measure $\mu \in \mathbb{R}^d$ and $r_0 > 0$ such that for any $r \leq r_0$ and any $x \in E$*

$$\mu(B(x, r)) \leq \zeta(r).$$

Then, there exists a constant $C > 0$ depending only on d such that

$$\mathcal{H}^s(E) \geq \frac{\mu(E)}{C}.$$

In particular, if $\mu(E) > 0$, for all r small enough and some $0 \leq s \leq d$, one has $\zeta(r) \leq r^s$ then

$$\dim_H(E) \geq s.$$

- Assume that there exist $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $0 \leq s \leq d$ such that for every $x \in E$, $\underline{\dim}(\mu, x) \leq s$. Then

$$\dim_H(E) \leq s.$$

Let us also give some basic properties of the Hausdorff content.

Proposition 4.2.5. *Let $E \subset \mathbb{R}^d$ be a bounded set subset of \mathbb{R}^d . One has, for any $0 \leq s \leq d$:*

1. $\mathcal{H}_\infty^s(E) \leq |E|^s$,
2. $t \mapsto \mathcal{H}_\infty^t(E)$ is a non increasing function,
3. for any $\delta \geq 1$, $\mathcal{H}_\infty^{\frac{s}{\delta}}(E) \geq (\mathcal{H}_\infty^s(E))^{\frac{1}{\delta}}$,
4. $\mathcal{H}_\infty^s(E) \leq \mathcal{H}^s(E)$ and $\mathcal{H}_\infty^s(E) = 0 \Leftrightarrow \mathcal{H}^s(E) = 0$.

Proof. Item (1) and (2) are easily derived from (4.7). Item (3) is a direct consequence of Hölder's inequalities. The first part of item (4) also readily follows from (4.7) and (4.2.2). Moreover, since $\mathcal{H}_\infty^s(E) \leq \mathcal{H}^s(E)$, one has $\mathcal{H}^s(E) = 0 \Rightarrow \mathcal{H}_\infty^s(E) = 0$.

We prove the second part of item (4). Assume that $\mathcal{H}_\infty^s(E) = 0$. Consider $t > 0$ and $0 < \varepsilon^{\frac{1}{s}} \leq t$. By (4.7), there exists a sequence of balls $(B_n)_{n \in \mathbb{N}}$ such that $E \subset \bigcup_{n \geq 0} B_n$ and

$$\sum_{n \geq 0} |B_n|^s \leq \varepsilon. \quad (4.9)$$

Notice that (4.9) implies that for any $n \in \mathbb{N}$, $|B_n|^s \leq \varepsilon$, so that $|B_n| \leq t$. In particular,

$$\mathcal{H}_t^s(E) \leq \varepsilon.$$

Letting $\varepsilon \rightarrow 0$, one gets $\mathcal{H}_t^s(E) = 0$ and letting $t \rightarrow 0$, $\mathcal{H}^s(E) = 0$. \square

Item (4) indicates that the Hausdorff measure and the Hausdorff content of a set are actually close to give the same geometric information about the distribution of the set E . The major difference between those quantities is that

the Hausdorff measure only gives information at infinitesimal scales compared to the diameter of the set E , while the Hausdorff content gives information about the distribution of the set E taking all scales lower than the diameter of E into account. This fact is characterized efficiently by the following version of the well-known Frostman lemma.

Proposition 4.2.6 ([20]). *Let $s \geq 0$. There is a constant $\kappa_d > 0$ depending only on the dimension d such that for any bounded set $E \subset \mathbb{R}^d$ with $\mathcal{H}_\infty^s(E) > 0$, there exists a probability measure supported by E , that we denote by m_E^s , such that*

$$\text{for every ball } B(x, r), \quad m_E^s(B(x, r)) \leq \kappa_d \frac{r^s}{\mathcal{H}_\infty^s(E)}. \quad (4.10)$$

4.2.2 Packing measure and packing dimension

Let us start by defining the t -packing of a set.

Definition 4.2.7. *Let E be a subset of \mathbb{R}^d and $t > 0$. A collection of balls $\{B_i\}_{i \in I}$ is said to be a t -packing of E if, for every $i \neq j \in I$, B_i is centered on E , $|B_i| \leq t$ and $B_i \cap B_j = \emptyset$.*

In this manuscript, the definition a packing measure associated with a general gauge function will not be used, so that one only defines it in the following traditional way.

Definition 4.2.8. *Let E be a subset of \mathbb{R}^d and $0 \leq s \leq d$. The s -dimensional packing pre-measure of E is defined as*

$$\mathcal{P}_0^s(E) = \lim_{t \rightarrow 0^+} \sup \left\{ \sum_{i \in I} |B_i|^s : \{B_i\}_{i \in I} \text{ is a } t\text{-packing of } E \right\} \quad (4.11)$$

Let us remark that this set function is not an outer-measure for $s < d$. It is easily verified that, for any dense countable set $E = \{x_n\}_{n \in \mathbb{N}}$, $\mathcal{P}_0^s(E) = +\infty$ and for every $n \in \mathbb{N}$, $\mathcal{P}_0^s(\{x_n\}) = 0$.

To overcome this difficulty, one defines the packing measure of a set E as follows.

Definition 4.2.9. *Let E be a subset of \mathbb{R}^d and $0 \leq s \leq d$. The s -dimensional packing measure of E is defined as*

$$\mathcal{P}^s(E) = \inf \left\{ \sum_{i \in \mathbb{N}} \mathcal{P}_0^s(E_i) : E \subset \bigcup_{i \in \mathbb{N}} E_i \right\}. \quad (4.12)$$

Then [32]:

(1) \mathcal{P}^s is an outer measure,

(2) if E_1 and E_2 are two subsets verifying $d(E_1, E_2) > 0$, one has

$$\mathcal{P}^s(E_1 \cup E_2) = \mathcal{P}^s(E_1) + \mathcal{P}^s(E_2).$$

It is standard that any outer measure satisfying item (2) is a Borel measure.

The following proposition defines the packing dimension of a set.

Proposition 4.2.10. *Let E be a subset of \mathbb{R}^d . There exists a unique number $0 \leq \dim_P(E) \leq d$ such that*

$$\begin{cases} \text{for any } s < \dim_P(E), \mathcal{P}^s(E) = +\infty \\ \text{for any } s > \dim_P(E), \mathcal{P}^s(E) = 0. \end{cases} \quad (4.13)$$

The number $\dim_P(E)$ is called packing dimension of E .

Remark 4.2.11. • The definition of packing measures and Hausdorff measures can be adapted to any metric space (X, d) .

- A set E is called a G_δ if E can be written as $E = \bigcap_{n \in \mathbb{N}} \Omega_n$, where the sets Ω_n are non empty open sets. It is proved that any G_δ dense set E has full packing dimension. In particular, when one studies sets of the form $\limsup_{n \rightarrow +\infty} U_n$, with U_n is an open set, denoting $K = \overline{\limsup_{n \rightarrow +\infty} U_n}$ (where \overline{X} denotes the topological closure of X), one always has

$$\dim_P(\limsup_{n \rightarrow +\infty} U_n) = \dim_P(K).$$

This justifies that the packing dimension is not a relevant quantity to study the size of limsup sets and explains why we only consider the Hausdorff dimension in these cases.

4.2.3 Dimension of measures

The Packing and Hausdorff dimension of a Borel measures are defined as follow.

Definition 4.2.12. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. For $x \in \text{supp}(\mu)$, the lower and upper local dimensions of μ at x are defined as

$$\underline{\dim}(\mu, x) = \liminf_{r \rightarrow 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)} \quad \text{and} \quad \overline{\dim}(\mu, x) = \limsup_{r \rightarrow 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)}.$$

Then, the lower and upper Hausdorff dimensions of μ are respectively defined by

$$\underline{\dim}_H(\mu) = \operatorname{ess\,inf}_\mu(\underline{\dim}(\mu, x)) \quad \text{and} \quad \overline{\dim}_P(\mu) = \operatorname{ess\,sup}_\mu(\overline{\dim}(\mu, x)). \quad (4.14)$$

It is known (for more details see [32]) that

$$\begin{aligned} \underline{\dim}_H(\mu) &= \inf\{\dim_H(E) : E \in \mathcal{B}(\mathbb{R}^d), \mu(E) > 0\} \\ \overline{\dim}_P(\mu) &= \inf\{\dim_P(E) : E \in \mathcal{B}(\mathbb{R}^d), \mu(E) = 1\}. \end{aligned}$$

When $\underline{\dim}_H(\mu) = \overline{\dim}_P(\mu)$, this common value is simply denoted by $\dim(\mu)$ and μ is said to be *exact dimensional*. In this case, for μ -almost every $x \in \operatorname{supp}(\mu)$, one has

$$\underline{\dim}(\mu, x) = \overline{\dim}(\mu, x) = \lim_{r \rightarrow 0^+} \frac{\log(\mu(B(x, r)))}{\log(r)} = \dim(\mu).$$

4.2.4 Multifractal analysis of measures

The multifractal analysis of a measure $\mu \in \mathcal{M}(\mathbb{R})$ aims at associating a spectrum which characterizes the pointwise behaviors of μ . The following sets are usually the object of such study.

Definition 4.2.13. For every $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $h \in \mathbb{R}_+$, one sets

$$\begin{aligned} E_\mu(h) &= \{x \in \operatorname{supp}(\mu) : \underline{\dim}(\mu, x) = h\}, \\ \widetilde{E}_\mu(h) &= \{x \in \operatorname{supp}(\mu) : \underline{\dim}(\mu, x) = \overline{\dim}(\mu, x) = h\}. \end{aligned} \quad (4.15)$$

Given $h \geq 0$, a point $x \in \operatorname{supp}(\mu)$ belongs E_μ^h if for any $\varepsilon > 0$, there exists $r_x > 0$ such that for every $0 < r \leq r_x$, $\mu(B(x, r)) \leq r^{h-\varepsilon}$ and there exists a sequence of radii $r_n \rightarrow 0$ for which $\mu(B(x, r_n)) \geq r_n^{h+\varepsilon}$. The multifractal spectra one naturally associates with μ are

$$D_\mu : h \mapsto \dim_H(E_\mu^h) \quad \text{and} \quad \widetilde{D}_\mu : h \mapsto \dim_H(\widetilde{E}_\mu^h). \quad (4.16)$$

In many cases, D_μ can be deduced from the study of the L^q scaling function $\tau_\mu(q)$ defined below.

For $p \in \mathbb{N}$, \mathcal{D}_p stands for the set of closed dyadic cubes of \mathbb{R}^d of generation p , i.e.

$$\mathcal{D}_p = \left\{ \prod_{i=1}^d [k_i 2^{-p}, (k_i + 1) 2^{-p}] : \forall 1 \leq i \leq d, k_i \in \mathbb{N} \right\}. \quad (4.17)$$

When $D \in \mathcal{D}_p$, write also $p = p(D)$ so that, for every $D \in \bigcup_{p \in \mathbb{N}} \mathcal{D}_p$,

$D \in \mathcal{D}_{p(D)}$.

Definition 4.2.14. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $q \in \mathbb{R}$. One defines

$$\Theta_\mu(q, n) = \sum_{D \in \mathcal{D}_n: \mu(D) \neq 0} \mu(D)^q. \quad (4.18)$$

The L^q scaling function τ_μ of μ is defined as

$$\tau_\mu(q) = \liminf_{n \rightarrow +\infty} \frac{-\log_2(\Theta_\mu(q, n))}{n} \in [-\infty, +\infty]. \quad (4.19)$$

The measure μ is said to verify the multifractal formalism at h when $D_\mu(h) = \tau_\mu^*(h)$, where $\tau_\mu^*(h)$ denotes the Legendre transform of τ_μ at h defined as

$$\tau_\mu^*(h) = \inf \{hq - \tau_\mu(q) : q \in \mathbb{R}\}. \quad (4.20)$$

As said above, there are numerous examples of measures satisfying the multifractal formalism, among which a large class of self-similar measures (see Theorem 4.3.39), Gibbs measures,

The basic properties of τ_μ are summarized in the following proposition.

Proposition 4.2.15 ([32, 41]). Let $\mu \in \mathbb{R}^d$.

- The mapping $q \mapsto \tau_\mu(q)$ is concave and decreasing. In particular, the set on which τ_μ does not admit a derivative is at most countable and τ_μ admits right and left derivatives for all q 's for which $-\infty < \tau_\mu(q) < +\infty$.
- One has $\tau_\mu(1) = 0$ and

$$\tau'_\mu(1_-) \leq \dim_H(\mu) \leq \dim_P(\mu) \leq \tau'_\mu(1_+).$$

4.3 Iterated function systems

Let $m \geq 2$ be an integer. An Iterated Function System (IFS in short) is a set $S = \{f_1, \dots, f_m\}$ of mappings $f_i : X \rightarrow X$, where $X \subset \mathbb{R}^d$.

Given an open set $U \subset \mathbb{R}^d$ and $f : U \rightarrow \mathbb{R}^d$ a differentiable map, for any $x \in U$:

- $f'(x)$ is the differential of f at x .
- for $\ell \in \mathcal{L}(\mathbb{R}^k, \mathbb{R}^d)$, one denotes by

$$\|\ell\| = \sup_{x \in \mathbb{R}^k \neq 0} \frac{\|\ell(x)\|}{\|x\|} \text{ and } \|\ell\| = \inf_{x \in \mathbb{R}^k \neq 0} \frac{\|\ell(x)\|}{\|x\|} \quad (4.21)$$

Let us recall the following result.

Proposition 4.3.1 (Hutchinson [44]). *Let $m \geq 2$ be an integer, $U \subset \mathbb{R}^d$ an open set and $S = \{f_1, \dots, f_m\}$ a system of C^1 maps from U to U . Assume that S is uniformly contracting, i.e*

$$\max_{1 \leq i \leq m} \sup_{x \in U} \|f'_i(x)\| < 1.$$

Then there exists a unique non empty compact set K satisfying

$$K = \bigcup_{1 \leq i \leq m} f_i(K).$$

Moreover, for any $(p_1, \dots, p_m) \in (0, 1)^m$, there exists a unique measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ supported on K satisfying

$$\mu = \sum_{1 \leq i \leq m} p_i \mu(f_i^{-1}(\cdot)). \quad (4.22)$$

From now on, an IFS designates a uniformly contracting system of C^1 maps.

The following notations are used throughout the manuscript:

- $\Lambda(S) = \{1, \dots, m\}$ and $\Lambda(S)^* = \bigcup_{k \geq 0} \Lambda(S)^k$. When there is no ambiguity on the system S involved, one simply writes $\Lambda(S) = \Lambda$.
- K_S denotes the attractor of S (or simply K when the context is clear).
- For $\underline{i} = (i_1, \dots, i_k) \in \Lambda^k$, the cylinder $[\underline{i}]$ is defined by

$$[\underline{i}] = \{(i_1, \dots, i_k, x_1, x_2, \dots) : (x_1, x_2, \dots) \in \Lambda^{\mathbb{N}}\}.$$

Moreover, if $(\alpha_n)_{n \in \mathbb{N}}$ is a sequence of real numbers, one sets

$$\alpha_{\underline{i}} = \alpha_{i_1} \times \dots \times \alpha_{i_k}$$

and

$$f_{\underline{i}} = f_{i_1} \circ \dots \circ f_{i_k}.$$

For example, given the probability vector (p_1, \dots, p_m) , $p_{\underline{i}} = p_{i_1} \times \dots \times p_{i_k}$.

- The set $\Lambda^{\mathbb{N}}$ is endowed with the topology generated by the cylinders. The set of probability measures on the Borel sets with respect to this topology is denoted $\mathcal{M}(\Lambda^{\mathbb{N}})$.

- The shift operation $\sigma : \Lambda^{\mathbb{N}} \rightarrow \Lambda^{\mathbb{N}}$ is defined for any $(i_1, i_2, \dots) \in \Lambda^{\mathbb{N}}$ by

$$\sigma((i_1, i_2, \dots)) = (i_2, i_3, \dots). \quad (4.23)$$

- The canonical projection of $\Lambda^{\mathbb{N}}$ on K will be denoted π_{Λ} (or simply π when there is no ambiguity) and, fixing any $x \in K$, is defined, for any $(i_1, i_2, \dots) \in \Lambda^{\mathbb{N}}$, by

$$\pi((i_1, \dots)) = \lim_{k \rightarrow +\infty} f_{i_1} \circ \dots \circ f_{i_k}(x). \quad (4.24)$$

It is easily verified that π is independent of the choice of x .

Let us recall the following classical result.

Proposition 4.3.2 ([44]). *Let (p_1, \dots, p_m) be a probability vector and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be the measure solution to (4.22). Define $\nu \in \mathcal{M}(\Lambda^{\mathbb{N}})$ by setting, for any cylinder $[\underline{i}] = [(i_1, \dots, i_k)]$, $\nu([\underline{i}]) = p_{\underline{i}} = p_{i_1} \times \dots \times p_{i_k}$. Then:*

1. $\mu = \nu \circ \pi^{-1}$.
2. ν is ergodic with respect to σ , i.e., for any Borel set $A \subset \Lambda^{\mathbb{N}}$, one has

$$\sigma^{-1}(A) \subset A \Rightarrow \nu(A) = 0 \text{ or } \nu(A) = 1.$$

4.3.1 IFS and separation condition

In the study of the IFS's, understanding the possible overlaps between the images of the maps involved plays a key role. Let $m \geq 2$ and $S = \{f_1, \dots, f_m\}$ be an IFS from an open set $U \subset \mathbb{R}^d$ to U .

Let us first introduce, for all $k \in \mathbb{N}$,

$$\Lambda^{(k)} = \{\underline{i} = (i_1, \dots, i_n) \in \Lambda^* : 2^{-k-1} < |f_{\underline{i}}(K)| \leq 2^{-k}\} \quad (4.25)$$

and

$$\tilde{\Lambda}^{(k)} = \{\underline{i} = (i_1, \dots, i_n) \in \Lambda^* : |f_{\underline{i}}(K)| \leq 2^{-k} < |f_{(i_1, \dots, i_{k-1})}(K)|\}. \quad (4.26)$$

Definition 4.3.3. *The IFS S is said to satisfy:*

1. *the strong separation condition (SSC) when for any $1 \leq i \neq j \leq m$, one has*

$$f_i(K) \cap f_j(K) = \emptyset.$$

2. the open set condition (OSC) when there exists a non empty open set V such that for any $1 \leq i \neq j \leq m$, one has

$$f_i(V) \cap f_j(V) = \emptyset \text{ and } \bigcup_{1 \leq i \leq m} f_i(V) \subset V.$$

3. the strong open set condition (SOSC) when S satisfies the open set condition and the involved open set V verifies $V \cap K \neq \emptyset$.
4. the asymptotically weak separation condition (AWSC) [35] when, writing for $k \in \mathbb{N}$,

$$t_k(S) = \max_{x \in \mathbb{R}^d} \# \left\{ f_{\underline{i}} : \underline{i} \in \tilde{\Lambda}^{(k)} \text{ and } f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset \right\}, \quad (4.27)$$

one has

$$\frac{\log t_k(S)}{k} \rightarrow 0.$$

Let us also add here that when the IFS S has no exact overlaps, (i.e, for any $\underline{i} \neq \underline{j} \in \Lambda^*$, $f_{\underline{i}} \neq f_{\underline{j}}$), one also has

$$t_k(S) = \max_{x \in \mathbb{R}^d} \# \left\{ \underline{i} : \underline{i} \in \Lambda^{(k)} \text{ and } f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset \right\}, \quad (4.28)$$

5. the exponential separation condition (ESC) [43] when S consists in affine invertible transformations and the following separation property is verified.

More precisely, define the metric Δ between affine invertible transformation by using the polar decomposition on the linear part and setting for any $(O, T, a), (O', T', a') \in \mathcal{O}_d(\mathbb{R}) \times \mathcal{S}_d^{++}(\mathbb{R}) \times \mathbb{R}^d$,

$$\Delta(OT + a, O'T' + a') = |\log(\|T\|) - \log(\|T'\|)| + \|O - O'\| + \|a - a'\|.$$

Then S verifies the ESC when there exists $0 < c < 1$ such that, for any $k \in \mathbb{N}$, any $\underline{i} \neq \underline{j} \in \Lambda^k$, one has

$$\Delta(f_{\underline{i}}, f_{\underline{j}}) \geq c^k. \quad (4.29)$$

Remark 4.3.4. It is proved that IFS consisting of similarities behave well in terms of dimension when they satisfy the ESC and additional irreducibility properties (see Theorem 4.3.35). It is worth noticing that this separation condition is not really restrictive. For instance, if one chooses an IFS on \mathbb{R} with

no exact overlaps such that its linear parts have algebraic contracting ratio and the translation parts have algebraic coefficient, then it satisfies the ESC [43].

Let us also mention the following result.

Theorem 4.3.5 ([63]). *Assume that the IFS $S = \{f_1, \dots, f_m\}$ verifies that for any $1 \leq i \leq m$, for any x , $f'_i(x)$ is a similarity and $x \mapsto f'_i(x)$ is Hölder. Then S verifies the OSC $\Leftrightarrow S$ verifies the SOSOC.*

Let (p_1, \dots, p_m) be a probability vector and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be the measure solution to (4.22). When the system satisfies the SSC, one recovers the “intuitive” notion of self-similarity for μ : Let A be a Borel set and $\underline{i} \in \Lambda^k$. Iterating (4.22), one has

$$\mu(f_{\underline{i}}(A)) = \sum_{\underline{j} \in \Lambda^k} p_{\underline{j}} \mu(f_{\underline{j}}^{-1} \circ f_{\underline{i}}(A)).$$

Since, in the case of the SSC, if $\underline{j} \neq \underline{i}$, $f_{\underline{j}}(K) \cap f_{\underline{i}}(K) = \emptyset$, one has $f_{\underline{j}}^{-1} \circ f_{\underline{i}}(A) = \emptyset$ so that

$$\mu(f_{\underline{i}}(A)) = p_{\underline{i}} \mu(A) = \mu(f_{\underline{i}}(K)) \mu(A). \quad (4.30)$$

Let us also mention the following comparable result when S satisfies the SOSOC.

Lemma 4.3.6. *Assume that S satisfies SOSOC and call V the involved open set. Then for any $\mu \in \mathcal{M}(\mathbb{R}^d)$ solution to (4.22), $\mu(V) = 1$ and, for any Borel set $A \subset V$, any $\underline{i} \in \Lambda^*$, one has*

$$\mu(f_{\underline{i}}(A)) = \mu(f_{\underline{i}}(K \cap V)) \mu(A). \quad (4.31)$$

Proof. Recall that $K \cap V \neq \emptyset$ and that $\text{supp}(\mu) = K$. Since V is open, one has $\mu(V) > 0$. Denote by ν the measure on $\Lambda^{\mathbb{N}}$ defined by, for any $[\underline{i}] = [(i_1, \dots, i_k)]$,

$$\nu([\underline{i}]) = p_{i_1} \times \dots \times p_{i_k}.$$

Let π be the canonical projection from $\Lambda^{\mathbb{N}}$ on K , defined by (4.24). One has $\mu = \nu \circ \pi^{-1}$, so that $\nu(\pi^{-1}(V)) > 0$. Moreover,

$$\sigma^{-1} \pi^{-1}(V) = \bigcup_{1 \leq i \leq m} \bigcup_{x \in \pi^{-1}(V)} (i, x) = \pi^{-1} \left(\bigcup_{1 \leq i \leq m} f_i(V) \right) \subset \pi^{-1}(V).$$

Since ν is σ -ergodic, one has $\nu(\pi^{-1}(V)) = 1 = \mu(V)$. The proof of (4.31) is similar to the proof in the case of the SSC when one restricts ourselves to the open set V . \square

Remark 4.3.7. *Theorem 2.1.4 was only stated for self-similar measures satisfying the strong separation condition and for multinomial measures. The main ingredient of the proof is actually (4.30). Since (4.31) holds in an open set of full measure, one easily deduces that its proof actually holds for any self-similar measure satisfying the SOSC, which is proved to be equivalent to satisfy the OSC (Theorem 4.3.5).*

4.3.2 Generalities on conformal and weakly conformal IFS

In this section one restricts ourselves to weakly conformal IFS's. This subclass contains in particular conformal (when the f_i 's are C^1 conformal maps) and self-similar IFS (when the f_i 's are affine similarities). The notation defined at the beginning of Section 4.3 are used.

Let us recall the definition of a weakly conformal map.

Definition 4.3.8. *Let $m \geq 2$ be an integer, $U \subset \mathbb{R}^d$ an open set, $S = \{f_i\}_{i=1}^m$ where each f_i is a C^1 contractions from U to U and K the attractor of S .*

One says that S is weakly conformal when, recalling (4.21),

$$\lim_{k \rightarrow +\infty} \frac{\sup_{(x_i)_{i \in \mathbb{N} \in \{1, \dots, m\}^{\mathbb{N}}}} \left(\log \|f'_{(x_1, \dots, x_k)}(\pi(\sigma^k(x)))\| - \log \|f'_{(x_1, \dots, x_k)}(\pi(\sigma^k(x)))\| \right)}{k} = 0. \quad (4.32)$$

In this case, a measure defined by (4.22) is called a weakly conformal-measure.

Example 4.3.9. • *If the maps f_1, \dots, f_m are affine similarities or conformal maps (i.e verify $\|f'(x)(y)\| = \|f'(x)\| \cdot \|y\|$ for every $x \in U, y \in \mathbb{R}^d$), the system $S = \{f_1, \dots, f_m\}$ is weakly conformal. In this case the IFS is called self-similar or self-conformal and the measures satisfying (4.22) are called respectively, self-similar and self-conformal measures. Note that this class of IFS contains for instance every system of holomorphic contracting mappings.*

- *Assume that for any $1 \leq i \leq m$, $f_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is defined by $f_i(x) = A_i x + b_i$, where for any $1 \leq i \leq m$, $b_i \in \mathbb{R}^d$ and $A_i \in GL_d(\mathbb{R})$ has its eigenvalues equal in modulus to $0 < r_i < 1$ and for any $1 \leq i, j \leq m$, $A_i A_j = A_j A_i$. Then $S = \{f_1, \dots, f_m\}$ is weakly-conformal.*

Let us recall important results about weakly-conformal measures.

4.3.2.1 Geometric properties of C^1 weakly-conformal IFS

Let $m \geq 2$ be an integer. One collects some useful geometric results when dealing with C^1 weakly conformal IFS.

Consider $S = \{f_1, \dots, f_m\}$ a C^1 weakly conformal IFS with attractor K and for every $x \in K$, $k \in \mathbb{N}$ and $\underline{i} = (i_1, \dots, i_k) \in \Lambda^k$, write

$$c_{\underline{i}}(x) = \|f'_{\underline{i}}(x)\|.$$

Let us recall the following result established as [37, Lemma 5.4].

Lemma 4.3.10 ([37]). *For any $c > 1$, there exists a constant $D(c) > 0$ such that, for every $k \in \mathbb{N}$, for every $\underline{i} \in \Lambda^k$ and every $x, y \in K$,*

$$D(c)^{-1}c^{-k}\|f'_{\underline{i}}(x)\| \cdot \|x - y\| \leq \|f_{\underline{i}}(x) - f_{\underline{i}}(y)\| \leq D(c)c^k\|f'_{\underline{i}}(x)\| \cdot \|x - y\| \quad (4.33)$$

$$D(c)^{-1}c^{-k}\|f'_{\underline{i}}(x)\| \leq |f_{\underline{i}}(K)| \leq D(c)c^k\|f'_{\underline{i}}(x)\|. \quad (4.34)$$

Remark 4.3.11. *Let $X \subset U$ be a compact set. It is proven in [37] that equation (4.33) actually holds for any $(x, y) \in X^2$.*

Note that, for every $k \in \mathbb{N}$ and every $x \in K$, one has

$$c^{\pm k}\|f'_{\underline{i}}(x)\| = \|f'_{\underline{i}}(x)\|^{1 + \frac{\pm k \log c}{\log \|f'_{\underline{i}}(x)\|}}.$$

Moreover, since there exists two constants $C_1, C_2 > 0$ such that for every $1 \leq i \leq m$ and every $x \in K$,

$$C_1 \leq \|f'_i(x)\| \leq C_2,$$

there also exists two constants $0 < t_1 \leq t_2$ such that

$$t_1 \leq \frac{k}{\log \|f'_{\underline{i}}(x)\|} \leq t_2.$$

Combining this fact with Lemma 4.3.10, for any $\theta > 0$, there exists $\tilde{C}_\theta > 0$ such that for every $k \in \mathbb{N}$, every $\underline{i} \in \Lambda^k$ and every $x, y \in K$,

$$\tilde{C}_\theta^{-1}c_{\underline{i}}(x)^{1+\theta}\|x - y\| \leq \|f_{\underline{i}}(x) - f_{\underline{i}}(y)\| \leq \tilde{C}_\theta c_{\underline{i}}(x)^{1-\theta}\|x - y\|. \quad (4.35)$$

In particular, there also exists \hat{C}_θ such that for every $\underline{i} \in \Lambda^*$ and every $x \in K$, one has

$$\hat{C}_\theta^{-1}c_{\underline{i}}^{1+\theta}(x)|K| \leq |f_{\underline{i}}(K)| \leq \hat{C}_\theta c_{\underline{i}}^{1-\theta}(x)|K|. \quad (4.36)$$

Let us remark that (4.36) also implies that there exist $0 < \alpha \leq \beta < 1$ as well as $C_\alpha, C_\beta > 0$ such that, for any $k \in \mathbb{N}$,

$$C_\alpha \alpha^k \leq |f_{\underline{i}}(K)| \leq C_\beta \beta^k. \quad (4.37)$$

4.3.2.2 Lyapunov exponent of C^1 weakly conformal IFS's

Let $m \geq 2$ and let us fix a C^1 weakly conformal IFS, $S = \{f_1, \dots, f_m\}$ with attractor K .

Given $x = (x_n)_{n \in \mathbb{N}} \in \Lambda^\mathbb{N}$, the following quantity, called Lyapunov exponent of S at x , defines a logarithmic shrink rate associated with IFS S at x .

Proposition 4.3.12 ([37]). *For any $x = (x_n)_{n \in \mathbb{N}}$, the Lyapunov exponent of S at x , is (well) defined as*

$$\lambda(x) = - \lim_{n \rightarrow +\infty} \frac{\log |f_{x_1} \circ \dots \circ f_{x_n}(K)|}{n}. \quad (4.38)$$

Moreover, for any probability vector $(p_1, \dots, p_m) \in [0, 1]^m$, denoting $\nu \in \mathcal{M}(\Lambda^\mathbb{N})$ the measure defined by $\nu([i]) = p_i$, then there exists $\lambda_\nu \geq 0$ such that for ν -almost any $x = (x_n)_{n \in \mathbb{N}}$,

$$\lambda(x) = \int \lambda(y) d\nu(y) := \lambda_\nu. \quad (4.39)$$

Remark 4.3.13. • By (4.37), the Lyapunov exponent are uniformly (with respect to weakly conformal measures) bounded by above and below by some positive constant.

- When S is a self-similar and $0 < c_1, \dots, c_m < 1$ are the contracting ratio associated with the similarities f_1, \dots, f_m , the lyapunov exponent of ν as in Proposition 4.3.12 is simply

$$\lambda_\nu = - \sum_{1 \leq i \leq m} p_i \log c_i.$$

The following consequence of Proposition 4.3.12 will be useful later on (see Proposition 10.3.2).

Corollary 4.3.14. *Let $((p_1^{(k)}, \dots, p_m^{(k)}))_{k \in \mathbb{N}} \in ([0, 1]^m)^\mathbb{N}$ be a sequence of probability vectors such that $(p_1^{(k)}, \dots, p_m^{(k)}) \rightarrow (p_1, \dots, p_m)$. Denote for $k \in \mathbb{N}$ $\nu, \nu_k \in \mathcal{M}(\Lambda^\mathbb{N})$ the measures defined by, for any cylinder $[(i_1, \dots, i_n)]$,*

$$\nu_k([(i_1, \dots, i_k)]) = p_{i_1}^{(k)} \cdot \dots \cdot p_{i_n}^{(k)} \text{ and } \nu([(i_1, \dots, i_n)]) = p_{i_1} \cdot \dots \cdot p_{i_n}.$$

Then $\nu_k \xrightarrow[k \rightarrow +\infty]{} \nu$ weakly, so that

$$\lim_{k \rightarrow +\infty} \lambda_{\nu_k} = \lambda_\nu.$$

4.3.2.3 Dimension of weakly-conformal IFS's

Let us recall the following fundamental result.

Theorem 4.3.15 (Feng-Hu [37]). *Let $(p_1, \dots, p_m) \in [0, 1]^m$ be a probability vector, $\nu \in \mathcal{M}(\Lambda^\mathbb{N})$ defined by, for any $\underline{i} \in \Lambda^*$, $\nu([\underline{i}]) = p_{\underline{i}}$ and $\mu = \nu \circ \pi^{-1}$.*

There exists $h \geq 0$ such that for μ -almost every $x \in K$, there exists $\mu^{\pi^{-1}(\{x\})} \in \mathcal{M}(\Lambda^\mathbb{N})$ such that:

$$(1) \quad \mu^{\pi^{-1}(\{x\})}(\pi^{-1}(\{x\})) = 1,$$

$$(2) \quad \text{for } \mu^{\pi^{-1}(\{x\})}\text{-almost } y = (y_1, \dots, y_n, \dots),$$

$$\frac{-\log \mu^{\pi^{-1}(\{x\})}([y_1, \dots, y_n])}{n} \rightarrow h. \quad (4.40)$$

$$(3) \quad \text{for every Borel set } A \subset \Lambda^\mathbb{N},$$

$$\nu(A) = \int_K \mu^{\pi^{-1}(\{x\})}(A) d\mu(x). \quad (4.41)$$

- denoting λ the Lyapunov exponent associated with ν ((4.39)), μ is exact-dimensional (Definition 4.14) and

$$\dim(\mu) = \frac{-h - \sum_{1 \leq i \leq m} p_i \log p_i}{\lambda}.$$

In the rest of this sub-section, we define the pressure function associated with a weakly conformal IFS, which is naturally related with the dimension of the attractor. The proof of the good definition of this quantity is very standard and does not diverge much from the proof in the conformal case. Those computations in the weakly conformal case does not seem to be made explicitly in the literature and for the seek of completeness, the proof are made in this manuscript.

Proposition 4.3.16. *Let $m \geq 2$ be an integer, $S = \{f_1, \dots, f_m\}$ be C^1 weakly conformal IFS and K its attractor.*

Let us fix $s \geq 0$ and $z \in K$. The following quantity is well defined and

independent of the choice of z :

$$P_z(s) = \lim_{k \rightarrow +\infty} \frac{1}{k} \log \sum_{\underline{i} \in \Lambda^k} \|f'_{\underline{i}}(z)\|^s. \quad (4.42)$$

Proof. Assume first that the limit exists in $\mathbb{R} \cup \{-\infty\}$ and let us show that it is independent of the choice of z and that the limit is $> -\infty$. Let $c > 1$ be a real number. By (4.3.10), following the notation involved, for any $k \in \mathbb{N}$, one has

$$\log \left(\sum_{\underline{i} \in \Lambda^k} D(c)^{-s} c^{-sk} |f_{\underline{i}}(K)|^s \right) \leq \log \left(\sum_{\underline{i} \in \Lambda^k} \|f'_{\underline{i}}(z)\|^s \right) \leq \log \left(\sum_{\underline{i} \in \Lambda^k} D(c)^s c^{sk} |f_{\underline{i}}(K)|^s \right). \quad (4.43)$$

Since (4.43) holds for any $c > 1$, one gets that

$$\lim_{k \rightarrow +\infty} \frac{1}{k} \left(\log \left(\sum_{\underline{i} \in \Lambda^k} \|f'_{\underline{i}}(z)\|^s \right) - \log \left(\sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^s \right) \right) = 0, \quad (4.44)$$

which proves that this quantity does not depend on z . Moreover, there exists $b > 0$ so that for any $k \in \mathbb{N}$, any $\underline{i} \in \Lambda^k$, any $x \in K$,

$$\|f'_{\underline{i}}(x)\| \geq b^k.$$

This implies that if $P_z(s)$ is well defined, then $P_z(s) > -\infty$.

Let us now prove that the limit exists. For $k \in \mathbb{N}$, write

$$g_k = \log \left(\sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^s \right). \quad (4.45)$$

As in the conformal case, the existence of the pressure relies on a sub-additivity argument.

Lemma 4.3.17. *For any $\varepsilon > 0$, there exists a constant $M_\varepsilon > 0$ such that for any $n, m \in \mathbb{N}$, one has*

$$g_{n+m} \leq M_\varepsilon + m\varepsilon + g_n + g_m. \quad (4.46)$$

Furthermore, any sequence $(g_n)_{n \in \mathbb{N}}$ verifying (4.46) is such that $(\frac{g_n}{n})_{n \in \mathbb{N}}$ converges in $\mathbb{R} \cup \{-\infty\}$.

Proof. Let us start by proving the second statement. Let $(g_n)_{n \in \mathbb{N}}$ be a sequence satisfying (4.46). Fix $\varepsilon > 0$ and M_ε satisfying (4.46). For any $q \in \mathbb{N}$, $b \in \mathbb{N}$,

$0 \leq r < q$, one has

$$\begin{aligned} g_{bq+r} &\leq bg_q + g_r + (bq+r)\varepsilon + (b+1)M_\varepsilon, \\ \Rightarrow \frac{g_{bq+r}}{bq+r} &\leq \frac{bq}{bq+r} \cdot \frac{g_q}{q} + \frac{(b+1)M_\varepsilon + g_r}{bq+r} + \varepsilon. \end{aligned}$$

Fixing q large enough independently of b so that $\frac{(b+1)M_\varepsilon}{bq} \leq \varepsilon$, for any large $b \in \mathbb{N}$, one has

$$\frac{g_{bq+r}}{bq+r} \leq (1+\varepsilon)\frac{g_q}{q} + 2\varepsilon.$$

This implies that

$$\limsup_{n \rightarrow +\infty} \frac{g_n}{n} \leq (1+\varepsilon) \liminf_{n \rightarrow +\infty} \frac{g_n}{n} + 2\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ proves the statement.

One now shows that g_n satisfies (4.46).

Let $k \in \mathbb{N}$ and $\underline{i} \in \Lambda^k$. Let us begin by the following lemma.

Lemma 4.3.18. *Following the notation of Lemma 4.3.10, one has, for any $\underline{j} \in \Lambda^*$,*

$$\frac{1}{2}D(c)^{-2}c^{-2k}|f_{\underline{i}}(K)| \cdot |f_{\underline{j}}(K)| \leq |f_{\underline{ij}}(K)| \leq 2D(c)^2c^{2k}|f_{\underline{i}}(K)| \cdot |f_{\underline{j}}(K)|. \quad (4.47)$$

Proof. Let us start by establishing the lower-bound.

Let $x, y \in K$ such that

$$\|f_{\underline{j}}(x) - f_{\underline{j}}(y)\| \leq |f_{\underline{j}}(K)| \leq 2\|f_{\underline{j}}(x) - f_{\underline{j}}(y)\|. \quad (4.48)$$

By Lemma 4.3.10, one has

$$D(c)^{-1}c^{-k}\|f'_{\underline{i}}(f_{\underline{j}}(x))\| \cdot \|f_{\underline{j}}(x) - f_{\underline{j}}(y)\| \leq \|f_{\underline{ij}}(x) - f_{\underline{ij}}(y)\| \leq |f_{\underline{ij}}(K)|, \quad (4.49)$$

and

$$\|f'_{\underline{i}}(f_{\underline{j}}(x))\| \geq D(c)^{-1}c^{-k}|f_{\underline{i}}(K)|. \quad (4.50)$$

Combining (4.48), (4.49) and (4.50), one obtains

$$\frac{1}{2}D(c)^{-2}c^{-2k}|f_{\underline{i}}(K)| \cdot |f_{\underline{j}}(K)| \leq |f_{\underline{ij}}(K)|.$$

Let us focus now on the upper-bound. Let $x, y \in K$ such that

$$\|f_{\underline{ij}}(x) - f_{\underline{ij}}(y)\| \geq \frac{1}{2}|f_{\underline{ij}}(K)|. \quad (4.51)$$

Using again Lemma 4.3.10, one has

$$\|f_{\underline{i}\underline{j}}(x) - f_{\underline{i}\underline{j}}(y)\| \leq D(c)c^k \|f'_{\underline{i}}(f_{\underline{j}}(x))\| \cdot \|f_{\underline{j}}(x) - f_{\underline{j}}(y)\| \leq D(c)^2 c^{2k} |f_{\underline{i}}(K)| \cdot |f_{\underline{j}}(K)|. \quad (4.52)$$

The upper-bound is obtained by combining (4.51) and (4.52). \square

By Lemma 4.3.18, for any $c > 1$ and any $n, n' \in \mathbb{N}$, one has

$$\begin{aligned} g_{n+n'} &= \log \left(\sum_{\underline{i} \in \Lambda^{n+n'}} |f_{\underline{i}}(K)|^s \right) = \log \left(\sum_{\underline{i} \in \Lambda^n, \underline{j} \in \Lambda^{n'}} |f_{\underline{i}\underline{j}}(K)|^s \right) \\ &\leq \log \left(\sum_{\underline{i} \in \Lambda^n, \underline{j} \in \Lambda^{n'}} 2^s D(c)^{2s} c^{2sn} |f_{\underline{i}}(K)|^s |f_{\underline{j}}(K)|^s \right) \\ &= n \cdot 2s \log(c) + \log(2^s D(c)^{2s}) + \log \left(\left(\sum_{\underline{i} \in \Lambda^n} |f_{\underline{i}}(K)|^s \right) \times \left(\sum_{\underline{j} \in \Lambda^{n'}} |f_{\underline{j}}(K)|^s \right) \right) \\ &\leq 2sn \log(c) + \log(2^s D(c)^{2s}) + g_n + g_{n'}. \end{aligned}$$

Fixing $c = e^{\frac{\varepsilon}{2s}}$, one has $2s \log(c) = \varepsilon$ and setting $M_\varepsilon = \log(2^s D(c)^{2s})$ shows that $(g_n)_{n \in \mathbb{N}}$ satisfies the condition of Lemma 4.3.17. \square

Lemma 4.3.17 together with (4.44) concludes the proof of Proposition 4.3.16. \square

Since $P_z(s)$ does not depend on z , one writes

$$P_z(s) = P(s) = \lim_{k \rightarrow +\infty} \frac{1}{k} \log \left(\sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^s \right).$$

As said above, the pressure function is naturally connected to the dimension of the attractor K associated with the underlying IFS. More precisely, the following quantity is a natural candidate for the dimension K .

Definition 4.3.19. Let $m \geq 2$ be an integer. Let $S = \{f_1, \dots, f_m\}$ be C^1 weakly conformal IFS and K its attractor.

Let us denote by $\dim(S)$ the unique solution to

$$P(s) = 0.$$

One says that $\dim(S)$ is the conformality dimension of S .

The following notion was introduced by Barral and Feng in [4] and will be usefull in this manuscript.

Definition 4.3.20 ([4]). *One says that S is dimension regular if for any weakly conformal measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ associated with the probability vector $(p_1, \dots, p_m) \in [0, 1]^m$ and S , recalling (4.3.12) and denoting $\nu \in \mathcal{M}(\Lambda^{\mathbb{N}})$ verifying $\mu = \nu \circ \pi^{-1}$, one has*

$$\dim(\mu) = \min \left\{ \frac{-\sum_{1 \leq i \leq m} p_i \log(p_i)}{\lambda_\nu}, d \right\}, \quad (4.53)$$

where λ_ν is defined by (4.39).

Remark 4.3.21. • *When S is self-similar, calling $0 < c_1, \dots, c_m < 1$ the contraction ration of the similarities f_1, \dots, f_m , for any probability vector (p_1, \dots, p_m) , μ and ν as in Definition 4.3.20, one has*

$$\dim(\mu) = \min \left\{ \frac{-\sum_{1 \leq i \leq m} p_i \log(p_i)}{\lambda_\nu}, d \right\} = \min \left\{ \frac{\sum_{1 \leq i \leq m} p_i \log(p_i)}{\sum_{i=1}^m p_i \log(c_i)}, d \right\}. \quad (4.54)$$

- *As proved in [43], any self-similar IFS on \mathbb{R} satisfying ESC is dimension regular.*

In the next subsection, we collect some results about weakly-conformal IFS satisfying AWSC. In particuler, we prove that if S is such an IFS and has no exact overlaps, then S is dimension regular.

4.3.3 Weakly conformal systems satisfying AWSC

4.3.3.1 Dimension regularity of C^1 weakly conformal IFS without exact overlaps

We will prove the following result.

Proposition 4.3.22 (D.). *Assume that $S = \{f_1, \dots, f_m\}$ satisfies the AWSC without exact overlaps. Then S is dimension regular and $\dim(S) = \dim_H(K)$.*

Had the IFS been conformal and satisfying some bounded distortion properties, the proof of Proposition 4.3.22 would follow directly from the existence of appropriated Gibbs measures. Unfortunately, such measures does not always exists in the weakly-conformal case but some measures that are close enough from satisfying the desired properties still exist as established by the following lemma.

Lemma 4.3.23. *Let $\varepsilon > 0$ and $s \geq 0$ be a real numbers. There exists $k \in \mathbb{N}$ and a probability vector $(p_{\underline{i}})_{\underline{i} \in \Lambda^k}$ such that the weakly conformal measure ν associated with $S' = \{f_{\underline{i}}\}_{\underline{i} \in \Lambda^k}$ and $(p_{\underline{i}})_{\underline{i} \in \Lambda^k}$ verifies, for any $p \in \mathbb{N}$ and $\underline{i}_1, \dots, \underline{i}_p \in \Lambda^k$,*

$$e^{-kp\varepsilon} \frac{|f_{\underline{i}_1 \dots \underline{i}_p}(K)|^s}{e^{pkP(s)}} \leq \nu([\underline{i}_1 \dots \underline{i}_p]) \leq e^{kp\varepsilon} \frac{|f_{\underline{i}_1 \dots \underline{i}_p}(K)|^s}{e^{pkP(s)}} \quad (4.55)$$

Proof. Fix $\varepsilon > 0$ and $c > 1$ small enough so that $8s \log c \leq \varepsilon$.

By Lemma 4.3.17 there exists $k \in \mathbb{N}$ so large that the constant named $D(c)$ in Lemma 4.3.10 verifies $\frac{\log D(c)}{k} \leq \log c$ and

$$|\frac{1}{k} \log \sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^s - P(s)| \leq \frac{\varepsilon}{2}. \quad (4.56)$$

Writing again $g_k = \log \sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^s$, let us define the probability vector $(p_{\underline{i}})_{\underline{i} \in \Lambda^k}$ by setting

$$p_{\underline{i}} = \frac{|f_{\underline{i}}(K)|^s}{e^{g_k}}.$$

Let ν be the weakly conformal measure associated with $S' = \{f_{\underline{i}}\}_{\underline{i} \in \Lambda^k}$ and $(p_{\underline{i}})_{\underline{i} \in \Lambda^k}$. Applying Lemma 4.47, for any $p \in \mathbb{N}$, $\underline{i}_1, \dots, \underline{i}_p \in \Lambda^k$,

$$D(c)^{-2p} c^{-2kp} \leq \frac{|f_{\underline{i}_1} \circ \dots \circ f_{\underline{i}_p}(K)|}{\prod_{j=1}^p |f_{\underline{i}_j}(K)|} \leq D(c)^{2p} c^{2kp}. \quad (4.57)$$

Also

$$D(c)^{2sp} c^{2skp} = e^{pk2s \cdot (\frac{\log D(c)}{k} + \log c)} \leq e^{\frac{\varepsilon}{2}pk}. \quad (4.58)$$

As a consequence, for any $p \in \mathbb{N}$ and any $\underline{i}_1, \dots, \underline{i}_p \in \Lambda^k$, one has

$$\nu([\underline{i}_1 \dots \underline{i}_p]) = p_{\underline{i}_1} \cdot \dots \cdot p_{\underline{i}_p} = \frac{\prod_{j=1}^p |f_{\underline{i}_j}(K)|^s}{e^{pg_k}} = \frac{\prod_{j=1}^p |f_{\underline{i}_j}(K)|^s}{e^{kp(\frac{g_k}{k} - P(s))} e^{pkP(s)}}.$$

Using (4.56), (4.57) and (4.58) concludes the proof. \square

Remark 4.3.24. *The measure ν can be extended over $\Lambda^{\mathbb{N}}$ by the usual arguments. Moreover, for any $\underline{i} = (i_1, \dots, i_n) \in \Lambda^*$, write $n_1 = k \lfloor \frac{n}{k} \rfloor$ and $n_2 = k(\lfloor \frac{n}{k} \rfloor + 1)$. Consider $\underline{j} \in \Lambda^{n_1}$ such that $[\underline{i}] \subset [\underline{j}]$ and $\underline{\ell} = (\ell_1, \dots, \ell_{n_2-n}) \in \Lambda^{n_2-n}$, one has*

$$e^{-n_2\varepsilon} \frac{|f_{(i_1, \dots, i_n, \ell_1, \dots, \ell_{n_2-n})}(K)|^s}{e^{n_2P(s)}} \leq \nu([\underline{i}\underline{\ell}]) \leq \nu([\underline{i}]) \leq \nu([\underline{j}]) \leq e^{n_1\varepsilon} \frac{|f_{(i_1, \dots, i_{n_1})}(K)|^s}{e^{n_1P(s)}}. \quad (4.59)$$

By (4.47), there exists a constant $C > 0$ such that, uniformly on $\underline{i}, \underline{j}, \underline{i}\underline{\ell}$, one

has

$$C^{-1} \leq \min \left\{ \frac{|f_{\underline{j}}(K)|}{|f_{\underline{i}}(K)|}, \frac{|f_{\underline{i}}(K)|}{|f_{\underline{i}\ell}(K)|} \right\} \leq \max \left\{ \frac{|f_{\underline{j}}(K)|}{|f_{\underline{i}}(K)|}, \frac{|f_{\underline{i}}(K)|}{|f_{\underline{i}\ell}(K)|} \right\} \leq C.$$

Hence there exists a constant $\gamma_{s,\varepsilon}$ such that for any $\underline{i} = (i_1, \dots, i_n) \in \Lambda^*$, one has

$$\gamma_{s,\varepsilon}^{-1} e^{-n\varepsilon} \frac{|f_{\underline{i}}(K)|^s}{e^{nP(s)}} \leq \nu([\underline{i}]) \leq \gamma_{s,\varepsilon} e^{n\varepsilon} \frac{|f_{\underline{i}}(K)|^s}{e^{nP(s)}}. \quad (4.60)$$

Let us now prove Proposition 4.3.22.

Proof. Call K the attractor of S . Let us show first that if any system S satisfying the AWSC also verifies that, for any weakly-conformal measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ associated with a probability vector (p_1, \dots, p_m) and S ,

$$\dim(\mu) = \frac{-\sum_{1 \leq i \leq m} p_i \log p_i}{\lambda_\nu} \quad (4.61)$$

where ν is the measure associated on $\Lambda^\mathbb{N}$, then $\dim(S) = \dim_H(K)$.

Fix $\varepsilon > 0$ consider $k \in \mathbb{N}$, $S' = \{f_{\underline{i}}\}_{\underline{i} \in \Lambda^k}$ and ν as in Lemma 4.3.23 applied with $s = \dim(S)$. Note that, since S satisfies the AWSC, so does S' . Then, considering the measure $\mu = \nu \circ \pi^{-1}$, where π is the canonical projection, one has

$$\dim(S) - \varepsilon \leq \dim(\mu) = \frac{-\sum_{\underline{i} \in \Lambda^k} p_{\underline{i}} \log p_{\underline{i}}}{\lambda_\nu} \leq \dim(S) + \varepsilon.$$

This proves that $\dim_H(K) \geq \dim(S) - \varepsilon$. Since it always holds that $\dim_H(K) \leq \dim(S)$ (see [30]) and ε is arbitrary,

$$\dim_H(K) = \dim(S).$$

Let us show that, for any system satisfying the AWSC, (4.61) holds for every weakly conformal measure μ .

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure associated with S and a probability vector (p_1, \dots, p_m) and $\nu \in \mathcal{M}(\Lambda^\mathbb{N})$ such that $\mu = \nu \circ \pi^{-1}$.

It comes from the proof of Theorem 4.3.15 [37] (applied to μ), that for any $\varepsilon > 0$, for μ -almost any $x \in K$ such that $\mu^{\pi^{-1}(\{x\})}$ exists and satisfies the two first items of Theorem 4.3.15, there exists n_0 large enough so that, for any $n \geq n_0$, there exists $\underline{i}_1, \dots, \underline{i}_{N_n}$ such that:

- for any $1 \leq j \leq N_n$,

$$e^{-n(\lambda+\varepsilon)} \leq |f_{\underline{i}_j}(K)| \leq e^{-n(\lambda-\varepsilon)}, \quad (4.62)$$

- one has

$$\mu^{\pi^{-1}(\{x\})} \left(\bigcup_{1 \leq j \leq N_n} [\underline{i}_j] \right) \geq \frac{1}{2}, \quad (4.63)$$

- for any $1 \leq j \leq N_n$,

$$e^{-n(h+\varepsilon)} \leq \mu^{\pi^{-1}(\{x\})}([\underline{i}_j]) \leq e^{-n(h-\varepsilon)} \quad (4.64)$$

Assume that $h > 0$ and take $0 < \varepsilon < \min \left\{ \frac{h}{2}, \frac{\lambda}{2} \right\}$.

Combining (4.63) and (4.64), one gets

$$N_n \geq \frac{1}{2} e^{n(h-\varepsilon)}. \quad (4.65)$$

Note that $\# \{k : e^{-n(\lambda+\varepsilon)} \leq 2^{-k} \leq e^{-n(\lambda-\varepsilon)}\} \leq \frac{2n\varepsilon}{\log 2}$. As a consequence, there exists $k \in \left[\frac{n(\lambda-\varepsilon)}{\log 2}, \frac{n(\lambda+\varepsilon)}{\log 2} \right]$ such that

$$\# \Lambda^{(k)} \cap \{[\underline{i}_j]\}_{1 \leq j \leq N_n} \geq \frac{N_n}{\frac{2n\varepsilon}{\log 2}} \geq \frac{\frac{1}{2} e^{\frac{nh}{2}}}{\frac{2n\varepsilon}{\log 2}}. \quad (4.66)$$

Since for any $1 \leq j \leq N_n$, $[\underline{i}_j] \cap \pi^{-1}(\{x\}) \neq \emptyset$, one also has $f_{\underline{i}_j}(K) \subset B(x, e^{-n(\lambda-\varepsilon)})$, so that, writing $n' = \lfloor \frac{n(\lambda-\varepsilon)}{\log 2} \rfloor$, one has

$$\# \{ \underline{i} \in \Lambda^{(n')} : f_{\underline{i}}(K) \cap B(x, 2^{-n'}) \} \geq \frac{\frac{1}{2} e^{\frac{nh}{2}}}{\frac{2n\varepsilon}{\log 2}}. \quad (4.67)$$

In particular, recalling (4.28) and Proposition 4.3.28,

$$\frac{\log t_k}{k} \rightarrow 0$$

and S does not satisfy the AWSC. As a consequence, S satisfies the AWSC implies $h = 0$, which, recalling the last item of Theorem 4.3.15, concludes the proof. \square

Remark 4.3.25. *If S is a self-similar system and satisfies the OSC, then it satisfies the AWSC and has no exact overlaps, so that Proposition 4.3.22 holds for S .*

We conclude this sub-section by giving some examples of self-similar measures satisfying AWSC with no exact overlaps. These examples are based on the following result proved by Barral and Feng in [4].

Theorem 4.3.26 ([4]). *Let S be a self-similar IFS on \mathbb{R} satisfying the exponential separation condition (see Definition 4.3.3). Then E satisfies the AWSC if and only if $\dim(S) \leq 1$.*

Corollary 4.3.27. • *Let $m \in \mathbb{N}$ and $c_1, \dots, c_m \in (0, 1)^m$ satisfying*

$$c_1 + \dots + c_m \leq 1.$$

Then by the proof of [43, Theorem 1.8] and Theorem 4.3.26, for \mathcal{L}^m almost all $(a_1, \dots, a_m) \in \mathbb{R}^m$, setting $f_1(x) = c_1x + a_1, \dots, f_m(x) = c_mx + a_m$, the system $S = \{f_1, \dots, f_m\}$ satisfies the AWSC, hence Theorem 11.1.4 applies to any self-similar measures associated with S .

- *One can also provide explicit examples: set $f_1(x) = \frac{1}{4}x$, $f_2(x) = \frac{1}{4}(x+1)$ and $f_3(x) = \frac{1}{4}(x+t)$ where t is irrational. Then, the same proof as in [43, Theorem 1.6] and Theorem 4.3.26 yields that $S = \{f_1, f_2, f_3\}$ satisfies the AWSC and has no exact overlaps.*
- *It is worth mentioning that the proof of Theorem 4.3.26 relies on [68, Theorem 6.6] which is currently established only on \mathbb{R} . An higher dimension version of this result would most likely lead to more examples.*

In the following section, we prove that the AWSC can be equivalently defined using the sets $\Lambda^{(k)}$ and $\tilde{\Lambda}^{(k)}$ (see (4.25)).

4.3.3.2 Equivalent definitions of AWSC

For $k \in \mathbb{N}$ and $x \in \mathbb{R}^d$, recalling (4.26) and (4.25), set

$$\begin{aligned} T_k(x) &= \left\{ f_{\underline{i}} : f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset, \underline{i} \in \tilde{\Lambda}^{(k)} \right\} \\ T'_k(x) &= \left\{ f_{\underline{i}} : f_{\underline{i}}(K) \cap B(x, 2^{-k}) \neq \emptyset, \underline{i} \in \Lambda^{(k)} \right\}. \end{aligned}$$

Note that S satisfies AWSC $\Leftrightarrow \lim_{k \rightarrow +\infty} \max_{x \in \mathbb{R}^d} \frac{\log T_k(x)}{k} = 0$.

Proposition 4.3.28. *One has*

$$\lim_{k \rightarrow +\infty} \max_{x \in \mathbb{R}^d} \frac{\log T_k(x)}{k} = 0 \Leftrightarrow \lim_{k \rightarrow +\infty} \max_{x \in \mathbb{R}^d} \frac{\log T'_k(x)}{k} = 0.$$

Proof. By (4.37), there exists $0 < \alpha < \frac{1}{2} < \beta \leq 1$ such that for every $k \in \mathbb{N}$,

$$\alpha^k \leq |f_{\underline{i}}(K)| \leq \beta^k.$$

Remark 4.3.29. (1): *For every $k \in \mathbb{N}$ and every $\underline{i} = (i_1, \dots, i_n) \in \Lambda^{(k)}$, one has*

$$C(\alpha, \beta)^{-1}k \leq k \frac{-\log 2}{\log \beta} + 1 \leq n \leq 2k \frac{-\log 2}{\log \alpha} \leq C(\alpha, \beta)k.$$

(2): For every $c > 1$, by Lemma 4.47, for every $\underline{i} = (i_1, \dots, i_n) \in \widetilde{\Lambda}^{(k)}$,

$$D(c)^{-2} \min_{1 \leq j \leq m} |f_j(K)| c^{-2C(\alpha, \beta)k} 2^{-k} \leq |f_{\underline{i}}(K)| \leq 2^{-k}.$$

In particular, for any $k \in \mathbb{N}$ large enough, one has

$$c^{-\frac{1}{2}C(\alpha, \beta)k} 2^{-k} \leq |f_{\underline{i}}(K)| \leq 2^{-k}. \quad (4.68)$$

Lemma 4.3.30. For every $\varepsilon_0 > 0$, there exists $k_\varepsilon \in \mathbb{N}$ such that for every $k \geq k_{\varepsilon_0}$, for every $\underline{i} = (i_1, \dots, i_n) \in \Lambda^{(k)}$, there exists $0 \leq p \leq \varepsilon_0 k$ such that $(i_1, \dots, i_{n-p}) \in \widetilde{\Lambda}^{(k)}$.

Proof. Fix $\varepsilon = \frac{\varepsilon_0}{2C(\alpha, \beta)}$ and $c > 1$ such that $c^{1-\varepsilon}\beta^\varepsilon < 1$. By Lemma 4.47, for any $(i_1, \dots, i_n) \in \Lambda^*$ and $0 \leq p \leq n$,

$$|f_{(i_1, \dots, i_n)}(K)| \leq D(c)^2 c^{n-p} |f_{(i_1, \dots, i_{n-p})}(K)| \times |f_{(i_{n-p+1}, \dots, i_n)}(K)|.$$

In particular, for $p \geq n\varepsilon$,

$$|f_{(i_1, \dots, i_{n-p})}(K)| \geq \frac{D(c)^{-2} c^{-(n-p)}}{|f_{(i_{n-p+1}, \dots, i_n)}(K)|} 2^{-k} \geq 2^{-k} \frac{D(c)^{-2}}{(c^{1-\varepsilon}\beta^\varepsilon)^n}.$$

This yields, for k large enough and $p = \lfloor \varepsilon C(\alpha, \beta)k \rfloor + 1 \leq 2C(\alpha, \beta)\varepsilon k$ that

$$|f_{(i_1, \dots, i_{n-p})}(K)| > 2^{-k}.$$

As a consequence, there must exists $p \leq 2C(\alpha, \beta)\varepsilon k$ such that $(i_1, \dots, i_{n-p}) \in \widetilde{\Lambda}^{(k)}$. \square

Lemma 4.3.31. For every $c > 1$, for every $\varepsilon > 0$, for every k large enough (depending on c) and every $x \in \mathbb{R}^d$, one has

$$\#T_k(x) \leq k \lfloor \frac{\frac{C(\alpha, \beta)}{2} \log c}{\log 2} \rfloor C_d c^{kd \frac{C(\alpha, \beta)}{2}} \max_{k \leq k' \leq k(1 + \lfloor \frac{\frac{C(\alpha, \beta)}{2} \log c}{\log 2} \rfloor)} \max_{y \in \mathbb{R}^d} \#T'_{k'}(y) \quad (4.69)$$

$$\text{and } \#T'_k(x) \leq m^{k\varepsilon} \#T_k(x).$$

Proof. Remark that, for each k' such that

$$c^{-\frac{1}{2}C(\alpha, \beta)k} 2^{-k} \leq 2^{-k'} \leq 2^{-k},$$

there exists a dimensional constant C_d so that each ball $B(x, 2^{-k})$ can be covered by less than

$$C_d c^{kd \frac{C(\alpha, \beta)}{2}}$$

balls of radius $2^{-k'}$. This implies that

$$\#\{f_{\underline{i}}, \underline{i} \in T_k(x) \cap \Lambda^{(k')}\} \leq C_d c^{kd \frac{C(\alpha, \beta)}{2}} \max_{y \in \mathbb{R}^d} \#T'_{k'}(y).$$

Since one has

$$\tilde{\Lambda}^{(k)} \subset \bigcup_{k'=k}^{k(1+\lfloor \frac{C(\alpha, \beta)}{2} \frac{\log c}{\log 2} \rfloor)} \Lambda^{(k')},$$

it holds that

$$\#T_k(x) \leq k \lfloor \frac{C(\alpha, \beta)}{2} \frac{\log c}{\log 2} \rfloor C_d c^{kd \frac{C(\alpha, \beta)}{2}} \max_{k \leq k'' \leq k(1+\lfloor \frac{C(\alpha, \beta)}{2} \frac{\log c}{\log 2} \rfloor)} \max_{y \in \mathbb{R}^d} \#T'_{k''}(y). \quad (4.70)$$

Moreover, by Lemma 4.3.30, there exists $\phi_k : T'_k(x) \rightarrow T_k(x)$ defined by

$$\phi_k((i_1, \dots, i_n)) = (i_1, \dots, i_{n-p})$$

with $0 \leq p \leq k\varepsilon$. The mapping ϕ_k verifies that each fiber has cardinality smaller than $m^{k\varepsilon}$. This implies that

$$\#T'_k(x) \leq m^{k\varepsilon} \#T_k(x).$$

□

Taking the log of the estimates of Lemma 4.3.31 and letting k tends to infinity concludes the proof.

□

4.3.4 Dimension properties of self-similar measures

In this section, one recalls what is known about the dimension theory of self-similar measures and sets and the multifractal analysis of these measures. First we start with some preliminaries.

4.3.4.1 Some preliminaries

Let $m \geq 2$ and $S = \{f_1, \dots, f_m\}$ be a self-similar system. It will be particularly convenient here to adopt a slightly different definition of the sets $\tilde{\Lambda}^{(k)}$ and $\Lambda^{(k)}$. Start by setting

$$\beta = \max_{1 \leq i \leq m} c_i \text{ and } \theta = \min_{i=1 \leq m} c_i. \quad (4.71)$$

In this section we set for every $k \in \mathbb{N}$

$$\begin{aligned}\Lambda^{(k)} &= \{\underline{i} = (i_1, \dots, i_n) \in \Lambda^* : \theta^{k+1} < c_{\underline{i}} \leq \theta^k\} \\ \widetilde{\Lambda}^{(k)} &= \{\underline{i} = (i_1, \dots, i_n) \in \Lambda^* : c_{\underline{i}} \leq \theta^k < c_{(i_1, \dots, i_{n-1})}\}.\end{aligned}\tag{4.72}$$

Let us collect few observations.

Remark 4.3.32. (1): *It is easily verified that one can equivalently use the new sets $\Lambda^{(k)}$ or $\widetilde{\Lambda}^{(k)}$ in the definition of AWSC (see Definition 4.3.3 and Proposition 4.3.28).*

(2): *For every $\underline{i} = (i_1, \dots, i_n) \in \widetilde{\Lambda}^{(k)}$, one has*

$$\theta^k \geq c_{\underline{i}} = c_{(i_1, \dots, i_{n-1})} \cdot c_n > \theta^k c_n \geq \theta^{k+1}$$

In particular, one has

$$\widetilde{\Lambda}^{(k)} \subset \Lambda^{(k)}.$$

(3): $\{\Lambda^{(k)}\}_{k \in \mathbb{N}}$ *is a partition of Λ^* .*

(4): *By iterating the self-similarity equation (4.22), for every $k \in \mathbb{N}$, one has*

$$\mu(\cdot) = \sum_{\underline{i} \in \widetilde{\Lambda}^{(k)}} p_{\underline{i}} \mu(f_{\underline{i}}^{-1}(\cdot)).$$

(5): *For every $\underline{i} \in \Lambda^{(k)}$ a similar argument as in Proposition 4.3.28 yields the existence of an integer $0 \leq p \leq 2 \lfloor \frac{\log \theta}{\log \beta} \rfloor$ such that $(i_1, \dots, i_{n-p}) \in \widetilde{\Lambda}^{(k)}$.*

Lemma 4.3.33. *Let $r > 0$ be a real number and $x \in \mathbb{R}^d$. For every $k \geq \lfloor \frac{\log r}{\log \theta} \rfloor + 2$, one has*

$$\mu(B(x, r)) \leq \sum_{\underline{i} \in \widetilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap B(x, r) \neq \emptyset} p_{\underline{i}} \leq \mu(B(x, 2r)).\tag{4.73}$$

Moreover,

$$1 \leq \frac{\sum_{\underline{i} \in \Lambda^{(k)} : f_{\underline{i}}(K) \cap B(x, r) \neq \emptyset} p_{\underline{i}}}{\sum_{\underline{i} \in \widetilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap B(x, r) \neq \emptyset} p_{\underline{i}}} \leq m^{2 \lfloor \frac{\log \theta}{\log \beta} \rfloor}.\tag{4.74}$$

Proof. Fix $k \geq \lfloor \frac{\log r}{\log \theta} \rfloor + 2$. Note that for any $\underline{i} \in \widetilde{\Lambda}^{(k)}$, if $f_{\underline{i}}(K) \cap B(x, r) = \emptyset$, then

$$\begin{aligned}\mu(f_{\underline{i}}^{-1}(B(x, r))) &= \mu(K \cap f_{\underline{i}}^{-1}(B(x, r))) = \mu(f_{\underline{i}}^{-1}(f_{\underline{i}}(K) \cap B(x, r))) \\ &= \mu(\emptyset) = 0.\end{aligned}$$

Also if $f_{\underline{i}}(K) \subset B(x, r)$,

$$1 \geq \mu(f_{\underline{i}}^{-1}(B(x, r))) \geq \mu(f_{\underline{i}}^{-1}(f_{\underline{i}}(K))) = \mu(K) = 1.$$

Using item (4) of Remark 4.3.32 and noticing that for every $\underline{i} \in \Lambda^{(k)}$, $f_{\underline{i}}(K) \subset B(x, 2r)$ yields equation (4.73).

Let us now prove that (4.74) holds. The left-hand side is a straightforward consequence of item (2) of Remark 4.3.32.

By item (5) of Remark 4.3.32, at each $\underline{i} = (i_1, \dots, i_n) \in \Lambda^{(k)}$ corresponds $(i_1, \dots, i_{n-p}) \in \tilde{\Lambda}^{(k)}$ with $p \leq 2 \lfloor \frac{\log \theta}{\log \beta} \rfloor$. Also, one has

$$p_{\underline{i}} = p_{(i_1, \dots, i_{n-p})} \times p_{(i_{n-p+1}, \dots, i_n)} \leq p_{(i_1, \dots, i_{n-p})}.$$

In addition, at each such word $(i_1, \dots, i_{n-p}) \in \tilde{\Lambda}^{(k)}$ can correspond at most $m^{2 \lfloor \frac{\log \theta}{\log \beta} \rfloor}$ words $\underline{i} = (i_1, \dots, i_n) \in \Lambda^{(k)}$. This yields the right-hand side. \square

In particular, for very ball B , and $k \geq \lfloor \frac{\log(\frac{1}{2}|B|)}{\log(\beta)} \rfloor + 1$ one has

$$\mu(B) \leq \sum_{\underline{i} \in \Lambda_{\beta}^{(k)}, f_{\underline{i}}(K) \cap B \neq \emptyset} p_{\underline{i}} \leq \mu(2B). \quad (4.75)$$

4.3.4.2 Dimension of self-similar measures

Let us start with the following remark.

Remark 4.3.34. *If the mappings f_1, \dots, f_m are affine similarities, then the conformality dimension is called the similarity dimension. It is the unique real number s solution to*

$$\sum_{i=1}^m c_i^s = 1. \quad (4.76)$$

The following Theorem summarized some of the known results about the dimension of self-similar measures and the dimension of the attractor associated with those self-similar systems. In the case where the IFS satisfies the OSC, the reader may refer to [32], the case where the IFS satisfies the AWSC is established in this manuscript in Proposition 4.3.22 and in the case where the IFS satisfies the ESC, one refers to [43] for more details.

Theorem 4.3.35. *Let $S = \{f_1, \dots, f_m\}$ be a self-similar IFS. Denote by K its attractor and $0 < c_1, \dots, c_m < 1$ the contracting ratios of f_1, \dots, f_m .*

1. *Assume that S satisfies the AWSC and has no exact overlaps (or the OSC), i.e., for any $\underline{i} \neq \underline{j} \in \Lambda^*$, $f_{\underline{i}} \neq f_{\underline{j}}$. Then*

$$\dim_H(K) = \dim(S)$$

and, for any self-similar measure associated with S and any probability vector (p_1, \dots, p_m) , one has

$$\dim(\mu) = \frac{\sum_{1 \leq i \leq m} p_i \log(p_i)}{\sum_{1 \leq i \leq m} p_i \log(c_i)}.$$

2. Assume that S satisfies the ESC and that no non trivial linear subspace $V \subset \mathbb{R}^d$ is preserved by the linear part of each f_i . Then

$$\dim_H(K) = \min \{\dim(S), d\}$$

and for any self-similar measure μ associated with S and any probability vector (p_1, \dots, p_m) , one has

$$\dim(\mu) = \min \left\{ \frac{\sum_{1 \leq i \leq m} p_i \log(p_i)}{\sum_{1 \leq i \leq m} p_i \log(c_i)}, d \right\}.$$

Item (2) shows that IFS's on \mathbb{R} verifying the ESC actually fall in the more general category of dimension regular self-similar IFS's (Definition 4.3.20).

4.3.4.3 Multifractal analysis of self-similar measures

Let us recall what is known in general about the multifractal formalism of self-similar measures.

Proposition 4.3.36. *Let μ be a self-similar measure. Then:*

1. τ_μ is differentiable at $q = 1^+$ and $\tau'_\mu(1^+) = \dim_H(\mu)$, [69, Theorem 5.1],
2. $\tau_\mu(0) = -\dim_H(\text{supp}(\mu))$,
3. [35] For any $q > 0$,

$$\tau_\mu(q) = \liminf_{n \rightarrow +\infty} \frac{-\log \Theta_\mu(q, n)}{n} = \lim_{n \rightarrow +\infty} \frac{-\log \Theta_\mu(q, n)}{n}.$$

4. [35] At any $q \geq 1$ for which $\tau'_\mu(q)$ exists (which is the case outside an at most countable set of q 's), μ satisfies the multifractal formalism at $h = \tau'_\mu(q)$, so that

$$\tau_\mu^*(h) = \dim_H(E_\mu^h) = qh - \tau_\mu(q). \quad (4.77)$$

Let us also mention that it is known that some self-similar measures do not satisfy the multifractal formalism “for $q < 0$ ” [72] (i.e at $h > \tau'_\mu(0^+)$).

The following quantity plays a particular role in the multifractal analysis of self-similar measures.

Definition 4.3.37. *Let $(p_1, \dots, p_m) \in (0, 1)^m$ be a probability vector and $\mu \in \mathcal{M}(\mathbb{R}^d)$ the self-similar associated with S (i.e., satisfying (4.22)). For $q \in \mathbb{R}$, let us define $T(q)$ as the unique solution to*

$$\sum_{i=1}^m p_i^q r_i^{-T(q)} = 1. \quad (4.78)$$

Proposition 4.3.38 ([32]). *• $q \mapsto T(q)$ is a concave real analytic mapping and in particular, differentiable over \mathbb{R} . Moreover, the equation $T''(q) = 0$ admits only a finite number of solutions so that T is strictly concave.*

- $T(0) = -\dim(S)$.
- $T'(1) = \frac{\sum_{1 \leq i \leq m} p_i \log(p_i)}{\sum_{1 \leq i \leq m} p_i \log(r_i)}$.

One summarizes here what is known about the multifractal analysis of self-similar measures depending on which separation condition they verify. One also compute the L^q scaling function of self-similar measures satisfying AWSC with no exact overlaps. Note that, on \mathbb{R} , this class contains the class of self-similar measures associated with systems satisfying the ESC and with similarity dimension smaller than one.

Theorem 4.3.39. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a self-similar measure associated with S .*

1. [32] *Assume that S satisfies the OSC, then one has*

$$\tau_\mu(q) = T(q). \quad (4.79)$$

Moreover, for any $h \in [\tau'(+\infty), \tau'(-\infty)]$, writing $h = \tau'(q)$,

$$\dim_H(\widetilde{E}_\mu^h) = \dim_H(E_\mu^h) = \tau_\mu^*(h) = qT'(q) - T(q).$$

In particular, μ satisfies the multifractal formalism at any $h \geq 0$ (see Definition 4.2.14).

2. *Assume that S satisfies the AWSC, then for any $q > 0$, μ satisfies the multifractal formalism at any $h \in (\tau'_\mu(+\infty), \tau'_\mu(0+))$ [36]. Moreover, if μ has no exact overlaps, then $\tau_\mu(q) = T(q)$ for any $q > 0$.*
3. [4] *Assume that S is dimension regular, then τ_μ is differentiable on $(0, 1]$ and μ satisfies the multifractal formalism at any $h \in [\tau'_\mu(1), \tau'_\mu(0^+)]$. Moreover, τ_μ is given as follows:*

- If $T'(1) > d$, then, for all $q \in [0, 1]$, $\tau_\mu(q) = d(q - 1)$.
- If $T'(1) < d$ and $T(0) \geq -d$ then $\tau_\mu(q) = T(q)$.
- If $T'(d) > d$ and $T(0) < -d$, then setting $\tilde{q} = \inf \{q : qT'(q) - T(q) \leq d\}$, one has

$$\begin{cases} \text{If } q \in [0, \tilde{q}), \tau_\mu(q) = \frac{q(d+T(\tilde{q}))}{\tilde{q}} - d \\ \text{If } q \in [\tilde{q}, 1], \tau_\mu(q) = T(q). \end{cases} \quad (4.80)$$

4. [68] If $d = 1$ and S satisfies the ESC, then for any $q \geq 1$,

$$\tau_\mu(q) = \min \{q - 1, T(q)\}.$$

In particular, τ_μ is differentiable at any $q \neq q_0$, where q_0 is defined (if possible) by $q_0 - 1 = T(q_0)$. As a consequence, τ_μ satisfies the multifractal formalism at any $h \in [\tau'(+\infty), \tau'(0^+)] \setminus (\tau'_\mu(q_{0-}), \tau'_\mu(q_{0+}))$.

Proof. Let us show the second part of Item (2). Assume that S satisfies the AWSC and has no exact overlaps.

Let $q > 0$, $k \in \mathbb{N}$ and $D \in \mathcal{D}_k$. On one hand, recalling (4.28) and item (4) of Remark 4.3.32,

$$\mu(D)^q \leq \left(\sum_{\underline{i} \in \tilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap D \neq \emptyset} p_{\underline{i}} \right)^q \leq t_k^q \max_{\underline{i} \in \tilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap D \neq \emptyset} p_{\underline{i}}^q \quad (4.81)$$

$$\leq t_k^q \sum_{\underline{i} \in \tilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap D \neq \emptyset} p_{\underline{i}}^q. \quad (4.82)$$

On the other hand,

$$\left(\sum_{D' \in \mathcal{D}_k : \overline{D'} \cap \overline{D} \neq \emptyset} \mu(D') \right)^q \geq \left(\sum_{\underline{i} \in \tilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap D \neq \emptyset} p_{\underline{i}} \right)^q \geq \max_{\tilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap D \neq \emptyset} p_{\underline{i}}^q \quad (4.83)$$

$$\geq t_k^{-1} \sum_{\underline{i} \in \tilde{\Lambda}^{(k)} : f_{\underline{i}}(K) \cap D \neq \emptyset} p_{\underline{i}}^q. \quad (4.84)$$

Note also that

$$\mu(D)^q \leq \left(\sum_{D' \in \mathcal{D}_k : \overline{D'} \cap \overline{D} \neq \emptyset} \mu(D') \right)^q \leq 2^{dq} \max_{D' \in \mathcal{D}_k : \overline{D'} \cap \overline{D} \neq \emptyset} \mu(D')^q \quad (4.85)$$

$$\leq 2^{dq} \sum_{D' \in \mathcal{D}_k : \overline{D'} \cap \overline{D} \neq \emptyset} \mu(D')^q \quad (4.86)$$

and

$$\sum_{D \in \mathcal{D}^k} \mu(D)^q \leq \sum_{D \in \mathcal{D}^k} \sum_{D' \in \mathcal{D}^k: \overline{D'} \cap \overline{D} \neq \emptyset} \mu(D')^q \leq 2^d \sum_{D \in \mathcal{D}^k} \mu(D)^q. \quad (4.87)$$

Let us also remark, that, since, for any $D \neq \widetilde{D} \in \mathcal{D}_k$, $|D| = |\widetilde{D}| = 2^{-k}$ and $D \cap \widetilde{D} = \emptyset$, by Lemma 4.1.4, for any $\underline{i} \in \widetilde{\Lambda}^{(k)}$, $f_{\underline{i}}(K)$ intersects at most γ_d dyadic cubes of generation k .

Combining this remark with (4.81), (4.83), (4.85) and (4.87), one gets

$$2^{-q(d+1)} t_k^{-1} \sum_{\underline{i} \in \widetilde{\Lambda}^{(k)}} p_{\underline{i}}^q \leq \sum_{D \in \mathcal{D}_k} \mu(D)^q \leq t_k^q \gamma_d \sum_{\underline{i} \in \widetilde{\Lambda}^{(k)}} p_{\underline{i}}^q. \quad (4.88)$$

By definition of $T(q)$,

$$\frac{2^{T(q)(-k-1)}}{|K|} \leq \sum_{\underline{i} \in \widetilde{\Lambda}^{(k)}} p_{\underline{i}}^q = \sum_{\underline{i} \in \widetilde{\Lambda}^{(k)}} c_{\underline{i}}^{T(q)} c_{\underline{i}}^{-T(q)} p_{\underline{i}}^q \leq 2^{-T(q)k} |K|. \quad (4.89)$$

Recalling that $\lim_{k \rightarrow +\infty} \frac{\log t_k}{k} = 0$, (4.88) and (4.89), one obtains

$$\lim_{k \rightarrow +\infty} \frac{\log(\sum_{D \in \mathcal{D}_k} \mu(D)^q)}{-k} = T(q). \quad (4.90)$$

□

Now that we recall the standard notion required to our study, we start by giving an instructive construction.

Chapter 5

Anisotropic ubiquity for quasi-Bernoulli measures

In this chapter, one presents a construction which mixes some of the approach mentioned in introduction. Consider a sequence of balls $(B_n)_{n \in \mathbb{N}}$. The main theorem of this chapter, Theorem 5.1.5, gives a lower-bound for the Hausdorff dimension of limsup sets generated by rectangles very much as in [74], in which the measure they consider is the Lebesgue measure, but we assume here that $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$ where μ is a geometrical realization of a quasi-Bernoulli measure.

Although this approach only deals with a particular case (in terms of measure and shape of shrunk sets), we will see that some important geometric quantities already appears naturally.

The first Section, Section 5.1 states the main result, Theorem 5.1.5 and Section 5.2 is dedicated to the proof of Theorem 5.1.5.

5.1 Anisotropic ubiquity and geometric quasi-Bernoulli measures

As mentioned at the beginning of this chapter, a case one can treat to help us understanding the geometric quantities involved when one deals with heterogeneous and anisotropic mass transference principles, is the specific case where the sequence $(B_n)_{n \in \mathbb{N}}$ verifies $\mu\left(\limsup_{n \rightarrow +\infty} B_n\right) = 1$ and μ is a geometric realization on the dyadic cubes of a quasi-Bernoulli measure fully supported on $[0, 1]^d$. In this case, it is possible, given $1 \leq \tau_1 \leq \dots \leq \tau_d$, to establish a lower-bound for $\limsup_{n \rightarrow +\infty} R_n$, where $R_n \subset B_n$ is a rectangle of length-sides $\prod_{i=1}^d |B_n|^{\tau_i}$.

Let us start by recalling what a geometric realization of a quasi-Bernoulli measure is.

Definition 5.1.1. Let $\mu \in \mathcal{P}([0, 1]^d)$. For $D \in \mathcal{B}([0, 1]^d)$ such that $\mu(D) > 0$, define

$$\mu_D = \frac{\mu|_D}{\mu(D)}.$$

When D is a closed dyadic subcube of $[0, 1]^d$, $T_D : D \rightarrow [0, 1]^d$ stands for the canonical affine mapping which sends D onto $[0, 1]^d$. In addition, when $\mu(D) > 0$ one defines

$$\mu^D = T_D \mu_D \in \mathcal{P}([0, 1]^d).$$

The measure μ is said to be quasi-Bernoulli when there exists a constant $C_\mu \geq 1$ such that for every $p \in \mathbb{N}$ and every $D \in \mathcal{D}_p$ with $\mu(D) > 0$, one has

$$\frac{1}{C_\mu} \mu \leq \mu^D \leq C_\mu \mu. \quad (5.1)$$

The measure μ_D is the renormalized restriction of μ to D and μ^D is the rescaled version of μ_D on the unit cube.

Exemple 5.1.2. Define $\Lambda = \{0, 1\}$, $\Sigma = \Lambda^\mathbb{N}$, σ be the shift operator on Σ , and endow Σ with the standard ultra-metric distance. Let π the canonical projection of Σ onto $[0, 1]$. For any Hölder potential ϕ on Σ , denote by ν_ϕ the unique equilibrium state associated with ϕ on Σ (see [14]). Then the measure $\mu_\phi = \nu_\phi \circ \pi^{-1}$ is quasi-Bernoulli, and ν_ϕ is also called a Gibbs measure associated with ϕ . This follows from the fact that there exists a number $P(\phi)$, the topological pressure of ϕ , and $C \geq 1$, such that for all $x \in \Sigma$, for all $n \in \mathbb{N}$:

$$C^{-1} \leq \frac{\nu_\phi(\{y = (y_i)_{i=1}^\infty \in \Sigma : y_i = x_i \text{ for all } 1 \leq i \leq n\})}{e^{-nP(\phi) + \sum_{k=0}^{n-1} \phi(\sigma^k x)}} \leq C.$$

Note that there exist quasi-Bernoulli measures obtained as projections of measures of Gibbs type associated to potentials ϕ with much weaker regularity properties (see [19, 73]).

Remark 5.1.3. It is easily seen that a quasi-Bernoulli measure μ , if not supported on an affine hyperplane, is such that $\mu(\partial[0, 1]^d) = 0$. For otherwise its orthogonal projection onto at least one of the sets $\{0\}^i \times [0, 1] \times \{0\}^{d-i-1}$, which is quasi-Bernoulli as well, would have an atom at $(0, \dots, 0)$ or $(\underbrace{0, \dots, 0}_i, 1, \underbrace{0, \dots, 0}_{d-i-1})$. This should imply that it is a Dirac mass, hence μ is supported on a hyperplane. This property will be used in the proof of our main result.

Let us recall the following result.

Proposition 5.1.4 ([41]). *A quasi-Bernoulli probability measure is exact dimensional.*

One establishes the following ubiquity theorem:

Theorem 5.1.5 (D. [22]). *Let $\mu \in \mathcal{P}([0, 1]^d)$ be a quasi-Bernoulli probability measure fully supported on $[0, 1]^d$. Let $(B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ be a sequence of balls in $[0, 1]^d$ such that $\lim_{n \rightarrow +\infty} r_n = 0$ and $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$.*

Let $1 \leq \tau_1 \leq \dots \leq \tau_d$ be d real numbers, $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ and $(O_n)_{n \in \mathbb{N}} \in \mathcal{O}_d(\mathbb{R})^{\mathbb{N}}$ be a sequence of orthogonal matrices. For $n \in \mathbb{N}$, set

$$R_n = x_n + O_n \tilde{R}_n, \text{ where } \tilde{R}_n = \text{diag}(r_n^{\tau_1}, \dots, r_n^{\tau_d}) \cdot [0, 1]^d \quad (5.2)$$

and

$$s(\mu, \boldsymbol{\tau}) = \min_{1 \leq k \leq d} \left(\frac{\dim(\mu) + \sum_{1 \leq j \leq k} \tau_k - \tau_j}{\tau_k} \right). \quad (5.3)$$

One has

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq s(\mu, \boldsymbol{\tau}). \quad (5.4)$$

Remark 5.1.6. (1) *For convenience, in particular to follow the point of view adopted in [75], the results are stated with \mathbb{R}^d endowed with $\|\cdot\|_{\infty}$ and for balls shrunk into rectangles with one vertex equal to the center of the shrunk ball. However, we emphasize that, up to very slight modifications of the proof (essentially by adding constants at some places), they still hold for another norm and if the balls are shrunk into rectangles containing the center of the initial cube.*

(2) *Given $\tau > 1$, by taking $\tau_i = \tau$ for all $1 \leq i \leq d$ and $O_n = I_d$ for all $n \in \mathbb{N}$, Theorem 5.1.5 reduces to Barral-Seuret's theorem [6] in the special case of quasi-Bernoulli measures.*

(3) *By taking $\mu = \mathcal{L}^d$ and $O_n = I_d$ for all $n \in \mathbb{N}$, we recover the result established in [75], i.e., formula established in [74] in the case of the Lebesgue measure.*

Remark 5.1.7.

- *The proof does not entirely use the exact dimensionality of μ , the key property is the quasi-Bernoulli property 5.1. However, the fact that $\underline{\dim}_H(\mu) = \overline{\dim}_P(\mu)$ can be used to prove $\dim_H(\limsup_{n \rightarrow +\infty} R_n) \leq s(\mu, \boldsymbol{\tau})$ under additional assumptions. The existence of upper bounds for the Hausdorff dimension of lim-sup sets (including of rectangles) included in balls $(B_n)_{n \geq N}$ will be achieved in chapter 8, under more general settings (see [6] for the case $\tau_1 = \dots = \tau_d$).*

- The expression of the lower-bound (5.3) is in fact naturally connected to the Hausdorff content (4.7) of rectangles (which is also, up to some multiplicative constant, the so-called singular value function associated with τ , [51]). More precisely, $s(\mu, \tau)$ is the largest $s \geq 0$ such that, for any $n \in \mathbb{N}$, $\mathcal{H}_{R_n}^s \geq |B_n|^{\dim(\mu)}$. This key remark will be exploited to deal with more general ubiquity theorems.

5.2 Proof of Theorem 5.1.5

Fix once and for all the quasi-Bernoulli measure μ , $1 \leq \tau_1 \leq \dots \leq \tau_d$ and $\tau = (\tau_1, \dots, \tau_d)$. Recall that $\alpha = \dim(\mu)$ is the dimension of μ .

The lower bound of Theorem 5.1.5 will be obtained by constructing a Cantor set included in $\limsup_{n \rightarrow +\infty} R_n$, and of dimension larger than or equal to $s(\mu, \tau)$. Before starting the construction, the following helpful result is recalled.

Lemma 5.2.1 ([11]). *Let $A = B(x, r)$ and $B = B(x', r')$ be two closed balls, and $q \geq 3$ be such that $A \cap B \neq \emptyset$ and $A \setminus (qB) \neq \emptyset$. Then $r' \leq r$ and $qB \subset 5A$.*

Proof. Consider $z \in A \setminus qB$. One has

$$qr' \leq \|z - x'\|_\infty \leq \|z - x\|_\infty + \|x - x'\|_\infty \leq r + r + r'.$$

Hence $\frac{q-1}{2}r' \leq r$, and in particular, one necessarily has $r' \leq r$ and $qr' \leq 2r + r' \leq 3r$.

Furthermore, if $y \in qB$, then

$$\|y - x\|_\infty \leq \|x' - y\|_\infty + \|x' - x\|_\infty \leq qr' + r' + r \leq 5r.$$

This concludes the proof. \square

We construct thereafter a Cantor set K as well as a sequence of strictly positive real numbers $(\varepsilon_p)_{p \in \mathbb{N}}$ and a Borel probability measure η such that:

- $K \subset \limsup_{n \rightarrow +\infty} R_n$ and $\eta(K) = 1$,
- The sequence $(\varepsilon_p)_{p \in \mathbb{N}}$ is decreasing with $\lim_{p \rightarrow +\infty} \varepsilon_p = 0$ and there exists a constant C such that for any $p \in \mathbb{N}$, there exists $r_p > 0$ verifying, for any ball $B \subset \mathbb{R}^d$ of radius r less than r_p ,

$$\eta(B) \leq C.r^{s(\mu, \tau) - 4\varepsilon_p}. \quad (5.5)$$

Then, applying the mass distribution principle (Lemma 4.2.4), since $\eta(K) = 1$ one deduces that, for any $p > 0$,

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq \dim_H(K) \geq s(\mu, \tau) - 4\varepsilon_p,$$

hence, letting $p \rightarrow +\infty$ will conclude.

The construction of (K, η) is decomposed into several steps. Without loss of generality we assume that $s(\mu, \tau) > 0$. Fix a decreasing sequence $(\varepsilon_p)_{p \in \mathbb{N}}$ converging to 0 at ∞ , such that $\varepsilon_0 \leq \max(1, s(\mu, \tau)/4)$.

Step 1: Initialization

Let us start with a definition.

Definition 5.2.2. For $\nu \in \mathcal{P}([0, 1]^d)$, $\beta \geq 0$, and $\varepsilon, \rho > 0$, define

$$E_\nu^{\beta, \varepsilon, \rho} = \{x \in [0, 1]^d : \forall 0 < r \leq \rho, B(x, r) \subset [0, 1]^d \text{ and } \nu(B(x, r)) \leq r^{\beta - \varepsilon}\}.$$

Then set

$$E_\nu^{\beta, \varepsilon} = \bigcup_{n \geq 1} E_\nu^{\beta, \varepsilon, \frac{1}{n}}.$$

With $\beta = \alpha = \underline{\dim}_H(\mu)$, since $\mu(\partial[0, 1]^d) = 0$ (due to Remark 5.1.3 and the assumption that μ is fully supported), for all $\varepsilon > 0$, one has $\mu(E_\mu^{\alpha, \varepsilon}) = 1$. For all $p \in \mathbb{N}$, consider $\rho_p \in (0, 1)$ small enough so that

$$\mu(E_\mu^{\alpha, \varepsilon_p, \rho_p}) \geq \frac{1}{2}. \quad (5.6)$$

For $x \in E_\mu^{\alpha, \varepsilon_1, \rho_1} \cap \limsup_{n \rightarrow +\infty} B_n$, consider $n_x \geq 1$ large enough so that $x \in B_{n_x}$, $4r_{n_x} \leq \rho_1$, and

$$r_{n_x}^{-\varepsilon_1} \geq \max\{4Q_d 4^{\alpha - \varepsilon_1}, \rho_2^{-d/\tau_d}\}. \quad (5.7)$$

Set

$$L_x = B(x, 4r_{n_x}). \quad (5.8)$$

Doing so for every $x \in E_\mu^{\alpha, \varepsilon_1, \rho_1} \cap \limsup_{n \rightarrow +\infty} B_n$ provides us with a Besicovith covering $\mathcal{F}^1 = \{L_x : x \in E_\mu^{\alpha, \varepsilon_1, \rho_1} \cap \limsup_{n \rightarrow +\infty} B_n\}$ such that for every x , the ball L_x is naturally associated with an integer $n_x \geq 1$ such that $x \in B_{n_x}$ and $|L_x| = 8r_{n_x}$. Also, the shrunk rectangle R_{n_x} verifies $R_{n_x} \subset B_{n_x} \subset L_x$. This is illustrated by Figure 5.1.

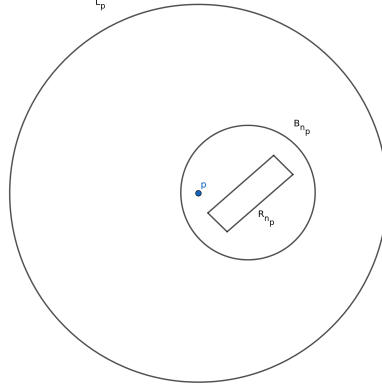


Figure 5.1: Representation of B_{n_p} , $|L_p| = 8r_{n_p}$ and R_{n_p} .

Applying now Proposition 4.1.2 with $v = 1$, from the family \mathcal{F}^1 one can extract $Q_d := Q_{d,1}$ finite or countable families of balls \mathcal{F}_i^1 , $1 \leq i \leq Q_d$, such that:

- $\forall 1 \leq i \leq Q_d$, $\forall L \neq L' \in \mathcal{F}_i^1$, it holds that $L \cap L' = \emptyset$,
- $E_\mu^{\alpha, \varepsilon_1, \rho_1} \cap \limsup_{n \rightarrow +\infty} B_n \subset \bigcup_{1 \leq i \leq Q_d} \bigcup_{L \in \mathcal{F}_i^1} L$.

Since $\mu(E_\mu^{\alpha, \varepsilon_1, \rho_1} \cap (\limsup_{n \rightarrow +\infty} B_n)) \geq \frac{1}{2}$, there exists $1 \leq i_1 \leq Q_d$ such that

$$\mu\left(\bigcup_{L \in \mathcal{F}_{i_1}^1} L\right) \geq \frac{\mu(E_\mu^{\alpha, \varepsilon_1, \rho_1} \cap (\limsup_{n \rightarrow +\infty} B_n))}{Q_d} \geq \frac{1}{2Q_d}. \quad (5.9)$$

Denote by $(L_k^{(1)})_{k \in \mathbb{N}}$ the sequence of balls such that $\mathcal{F}_{i_1}^1 = \{L_k^{(1)}\}_{k \in \mathbb{N}}$, $(x_k^{(1)})_{k \in \mathbb{N}}$ the sequence of points such that for all $k \in \mathbb{N}$, $L_k^{(1)} = L_{x_k^{(1)}}$, and set $r_k^{(1)} = r_{x_k^{(1)}}$. There exists $N_1 \in \mathbb{N}$ so that

$$\mu\left(\bigcup_{1 \leq k \leq N_1} L_k^{(1)}\right) \geq \frac{\mu\left(\bigcup_{L \in \mathcal{F}_{i_1}^1} L\right)}{2}.$$

Set $\mathcal{F}_1 = \{L_k^{(1)}\}_{1 \leq k \leq N_1}$. One has

$$\mu\left(\bigcup_{L \in \mathcal{F}_1} L\right) \geq \frac{1}{4Q_d}. \quad (5.10)$$

Remember that with every ball $L_k^{(1)}$ are naturally associated the ball $B_{n_k^{(1)}}$ and the rectangle $R_{n_k^{(1)}}$, where $n_k^{(1)} = n_{x_k^{(1)}}$; set $R_k^{(1)} = R_{n_k^{(1)}}$. Then define K_1 , the

first generation of the Cantor set by setting

$$\mathcal{K}_1 = \{R_k^{(1)}\}_{1 \leq k \leq N_1} \text{ and } K_1 = \bigcup_{R \in \mathcal{K}_1} R.$$

Finally, measure η_1 on the algebra generated by \mathcal{K}_1 is obtained by concentrating the μ -measure of the balls L_x on the rectangle R_{n_x} . More precisely, for $1 \leq k \leq N_1$ set

$$\eta_1(R_k^{(1)}) = \frac{\mu(L_k^{(1)})}{\sum_{1 \leq k' \leq N_1} \mu(L_{k'}^{(1)})}.$$

Since for all $1 \leq k \leq N_1$, the center $x_k^{(1)}$ of $L_k^{(1)}$ belongs to $E_\mu^{\alpha, \varepsilon_1, \rho_1}$, recalling that $|L_{x_k^{(1)}}|/2 = 4r_{n_k^{(1)}} \leq \rho_1$, the disjointness of the $L_j^{(1)}$, as well as the inequality (5.10), we get that for all $1 \leq k \leq N_1$,

$$\eta_1(R_k^{(1)}) \leq 4Q_d \left(4r_{n_k^{(1)}}\right)^{\alpha - \varepsilon_1} \leq \left(r_{n_k^{(1)}}\right)^{\alpha - 2\varepsilon_1}, \quad (5.11)$$

where (5.7) has been used.

Step 2: Constructing the second generation

This step consists of two sub-steps: First we associate a set of dyadic cubes with each rectangle previously obtained, and then we work inside each of these cubes.

Sub-step 2.1: A set of dyadic cubes inside each R of \mathcal{K}_1

Consider a rectangle R . There exists an orthogonal matrix $O \in \mathcal{O}_d(\mathbb{R})$, a point $x \in [0, 1]^d$ and $0 < \ell_d \leq \ell_2 \leq \dots \leq \ell_1$ such that

$$R = x + O\tilde{R}, \text{ with } \tilde{R} = \prod_{i=1}^d [0, \ell_i].$$

Set $p = -\lfloor \log_2 \left(\frac{\ell_d}{8\sqrt{d}} \right) \rfloor$. Intuitively, $2^{-p} \approx \frac{\ell_d}{8\sqrt{d}}$, so that there are some cubes included in R with side-length 2^{-p} . We associate with R the set of dyadic cubes (see Figure 5.2)

$$\mathcal{C}(R) = \left\{ D \in \mathcal{D}_p : D \subset R, D = \prod_{i=1}^d [k_i 2^{-p}, (k_i + 1) 2^{-p}], 8|k_i, \forall 1 \leq i \leq d \right\}.$$

Observe that $\mathcal{C}(R)$ consists in dyadic cubes of generation p inside R that are quite far from each other. This will ensure that the rectangles used at a

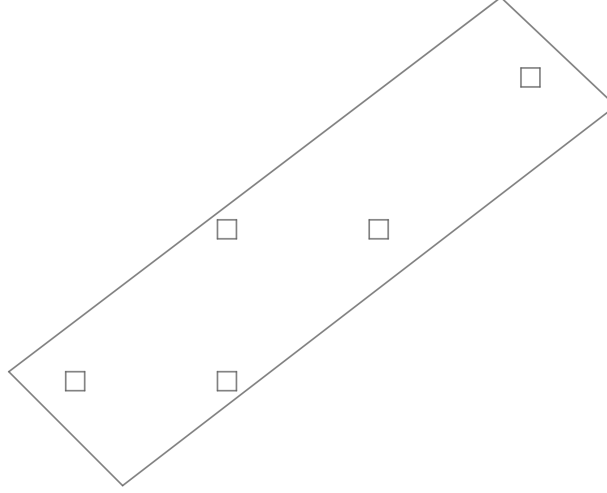


Figure 5.2: Representation of the lacunarity of $\mathcal{C}(R)$ inside R .

given generation of the construction of the Cantor set are well separated. Also, there exist a constant $C_d \geq 1$ depending only on the dimension d , such that the side length 2^{-p} of each $C \in \mathcal{C}(R)$ satisfies $C_d^{-1}\ell_d \leq 2^{-p} \leq C_d\ell_d$, as well as a constant $\kappa_d \geq 1$ such that

$$\kappa_d^{-1} \prod_{i=1}^d \frac{\ell_i}{\ell_d} \leq \#\mathcal{C}(R) \leq \kappa_d \prod_{i=1}^d \frac{\ell_i}{\ell_d}.$$

Recalling (5.2), for every $n \in \mathbb{N}$, one gets

$$\kappa_d^{-1} \cdot r_n^{\sum_{i=1}^d \tau_i - \tau_d} \leq \#\mathcal{C}(R_n) \leq \kappa_d \cdot r_n^{\sum_{i=1}^d \tau_i - \tau_d}. \quad (5.12)$$

Now we construct a measure η_2 , which refines the measure η_1 by distributing the mass uniformly between the cubes of $\mathcal{C}(R)$ for $R \in \mathcal{K}_1$. For every $1 \leq k \leq N_1$ and every $D \in \mathcal{C}(R_k^{(1)})$, set

$$\eta_2(D) = \frac{\eta_1(R_k^{(1)})}{\#\mathcal{C}(R_k^{(1)})}.$$

By construction, $\eta_2(R_k^{(1)}) = \eta_1(R_k^{(1)})$. Recalling (5.11) and (5.12), one gets

$$\eta_2(D) = \frac{\eta_1(R_k^{(1)})}{\#\mathcal{C}(R_k^{(1)})} \leq \frac{\left(r_{n_k^{(1)}}\right)^{\alpha - 2\varepsilon_1}}{\kappa_d^{-1} \cdot \left(r_{n_k^{(1)}}\right)^{\sum_{i=1}^d -\tau_d + \tau_i}} = \kappa_d \cdot \left(r_{n_k^{(1)}}^{\tau_d}\right)^{\frac{\alpha - 2\varepsilon_1 + \sum_{i=1}^d \tau_d - \tau_i}{\tau_d}}. \quad (5.13)$$

Sub-step 2.2: Construction in each cube of $\mathcal{C}(R)$

We start with preliminary observations about the measure μ . Recall Definition 5.1.1. Since μ is a quasi-Bernoulli measure, for every $q \in \mathbb{N}$, every $D \in \mathcal{D}_q$ such that $\mu(D) > 0$, for every $x \in [0, 1]^d$ and $r > 0$ such that $B(x, r) \subset D$, due to (5.1) one has

$$\begin{aligned} \mu(B(x, r)) &= \mu(T_D^{-1}(T_D(B(x, r)))) = \mu(D)\mu^D\left(B\left(T_D(x), \frac{r}{2^{-q}}\right)\right) \\ &\leq C_\mu \mu(D) \mu\left(B\left(T_D(x), \frac{r}{2^{-q}}\right)\right). \end{aligned}$$

Thus, for all $x \in [0, 1]^d$ and $r > 0$ such that $B(x, r) \subset [0, 1]^d$ one has

$$\mu(B(T_D^{-1}(x), r2^{-q})) \leq C_\mu \mu(D) \mu(B(x, r)). \quad (5.14)$$

Also, for every $p \in \mathbb{N}$, (5.1) yields

$$\mu^D(E_\mu^{\alpha, \varepsilon_p, \rho_p}) \geq \frac{\mu(E_\mu^{\alpha, \varepsilon_p, \rho_p})}{C_\mu} \geq \frac{1}{2C_\mu}. \quad (5.15)$$

Moreover,

$$\begin{aligned} T_D^{-1}(E_\mu^{\alpha, \varepsilon_p, \rho_p}) &= \{T_D^{-1}(x) : \forall r \leq \rho_p, B(x, r) \subset [0, 1]^d, \mu(B(x, r)) \leq r^{\alpha - \varepsilon_p}\} \\ &= \left\{T_D^{-1}(x) : \forall r \leq \rho_p, B(x, r) \subset [0, 1]^d, \mu\left(T_D\left(B\left(T_D^{-1}(x), \frac{r}{2^q}\right)\right)\right) \leq r^{\alpha - \varepsilon_p}\right\}, \end{aligned}$$

and using (5.14), one gets

$$\begin{aligned} T_D^{-1}(E_\mu^{\alpha, \varepsilon_q, \rho_q}) &\subset \left\{T_D^{-1}(x) : \forall r \leq \rho_p, B(x, r) \subset [0, 1]^d, \frac{\mu(B(T_D^{-1}(x), r2^{-q}))}{\mu(D)} \leq C_\mu r^{\alpha - \varepsilon_p}\right\} \\ &= \left\{y \in D : \forall r \leq \rho_p 2^{-q}, B(y, r) \subset D, \frac{\mu(B(y, r))}{\mu(D)} \leq C_\mu \left(\frac{r}{2^{-q}}\right)^{\alpha - \varepsilon_p}\right\}. \end{aligned}$$

It follows that if we fix p as above and set

$$E_D^{\varepsilon_p} = \limsup_{n \rightarrow +\infty} B_n \cap \left\{y \in D : \forall r \leq \rho_p 2^{-q}, B(y, r) \subset D, \frac{\mu(B(y, r))}{\mu(D)} \leq C_\mu \left(\frac{r}{2^{-q}}\right)^{\alpha - \varepsilon_p}\right\}, \quad (5.16)$$

then by Definition 5.1.1 and the fact that $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$, we have

$$\mu(E_D^{\varepsilon_p}) = \mu(T_D^{-1}(T_D(E_D^{\varepsilon_p}))) = \mu(D)\mu^D(T_D(E_D^{\varepsilon_p})) \geq \mu(D) \frac{\mu(E_\mu^{\alpha, \varepsilon_p, \rho_p})}{C_\mu} \geq \frac{\mu(D)}{2C_\mu}, \quad (5.17)$$

where we used (5.15).

We now continue the construction. Consider $R \in \mathcal{K}_1$. Fix $D \in \mathcal{C}(R)$. Remember that $p(D)$ is the unique integer such that $D \in \mathcal{D}_{p(D)}$. The set $E_D^{\varepsilon_2}$ is well defined since $\mu(D) > 0$ (the measure μ has been supposed to be fully supported on $[0, 1]^d$). For every $x \in E_D^{\varepsilon_2}$, consider n_x large enough so that:

- $x \in B_{n_x}$,
- $n_x \geq 2$ and

$$4r_{n_x} \leq \rho_2 2^{-p(D)}, \quad \text{and} \quad r_{n_x}^{-\varepsilon_2} \geq \max \left\{ 4C_\mu Q_d \cdot \eta_2(D) (4 \cdot 2^{p(D)})^{\alpha - \varepsilon_2}, \rho_3^{-d/\tau_d} \right\}. \quad (5.18)$$

Set $L_x = B(x, 4r_{n_x})$, as in step 1 (see (5.8)). By repeating the same argument as in step 1, one can extract from $\{L_x : x \in E_D^{\varepsilon_2}\}$ a finite number N_D of balls, $L_1^{(D)} = L_{x_1^{(D)}}, \dots, L_{N_D}^{(D)} = L_{x_{N_D}^{(D)}}$ such that for all $1 \leq k_1 \neq k_2 \leq N_D$ one has $L_{k_1}^{(D)} \cap L_{k_2}^{(D)} = \emptyset$ and by (5.17)

$$\mu \left(\bigcup_{1 \leq k \leq N_D} L_k^{(D)} \right) \geq \frac{\mu(E_D^{\varepsilon_2})}{2} \geq \frac{\mu(D)}{4Q_d C_\mu}. \quad (5.19)$$

and with each ball $L_k^{(D)}$ are associated the ball $B_{n_{x_k^{(D)}}}$ and the rectangle $R_{n_{x_k^{(D)}}}$, that we denote by $B_k^{(D)}$ and $R_k^{(D)}$ respectively; we also set $r_k^{(D)} = r_{n_{x_k^{(D)}}}$. Then define the collection of rectangles of second generation by setting

$$\mathcal{K}(R) = \bigcup_{D \in \mathcal{C}(R)} \{R_k^{(D)}\}_{1 \leq k \leq N_D} \quad \text{and} \quad \mathcal{K}_2 = \bigcup_{R \in \mathcal{K}_1} \mathcal{K}(R),$$

and

$$K_2 = \bigcup_{R \in \mathcal{K}_2} R.$$

One extends further the measure η_1 to the algebra generated by the elements of the set $\mathcal{K}_1 \cup \bigcup_{R \in \mathcal{K}_1} \mathcal{C}(R) \cup \mathcal{K}_2$ by distributing the mass according to μ at that scale. More precisely, for all $R \in \mathcal{K}_1$, $D \in \mathcal{C}(R)$ and $1 \leq k \leq N_D$, one sets

$$\eta_2(R_k^{(D)}) = \eta_2(D) \frac{\mu(L_k^{(D)})}{\sum_{1 \leq k' \leq N_D} \mu(L_{k'}^{(D)})}$$

Note the following facts:

- If $R \in \mathcal{K}_1$, $D, D' \in \mathcal{C}(R)$, $1 \leq k \leq N_D$ and $1 \leq k' \leq N_{D'}$ are such that $R_k^{(D)} \neq R_{k'}^{(D')} \in \mathcal{K}(R)$, then $3B_k^{(D)} \cap 3B_{k'}^{(D')} = \emptyset$.

- If $R \in \mathcal{K}_1$, $D \in \mathcal{C}(R)$ and $1 \leq k \leq N_D$, using the second assertion of (5.18) and the fact that the ball $L_k^{(D)}$ is centered on $E_D^{\varepsilon_2}$, then

$$\frac{\mu(L_k^{(D)})}{\mu(D)} \leq C_\mu \left(\frac{4r_k^{(D)}}{2^{-p(D)}} \right)^{\alpha - \varepsilon_2}$$

so that by (5.19) and the third assertion of (5.18), we get

$$\eta_2(R_k^{(D)}) \leq (\eta_2(D) 4Q_d C_\mu (4 \cdot 2^{p(D)})^{\alpha - \varepsilon_2}) \cdot (r_k^{(D)})^{\alpha - \varepsilon_2} \leq (r_k^{(D)})^{\alpha - 2\varepsilon_2}. \quad (5.20)$$

Further steps: Induction scheme

We proceed as in step 2. Suppose that $p \geq 2$, and for all $1 \leq q \leq p$, a set K_q and a measure η_q , defined on the algebra generated by the elements of $\bigcup_{1 \leq p \leq q} \mathcal{K}_p \bigcup_{R \in \mathcal{K}_p} \mathcal{C}(R)$, have been constructed in such a way that (5.11) holds and:

- (i) For all $1 \leq q \leq p$, \mathcal{K}_q is a finite subset of $\{R_n\}_{n \geq q}$.
- (ii) For all $2 \leq q \leq p$, for all $R \in \mathcal{K}_q$, there exists $R' \in \mathcal{K}_{q-1}$ and $D \in \mathcal{C}(R')$ such that $R \subset D$; one denotes by $\{R_k^{(D)}\}_{1 \leq k \leq N_D}$ the family of rectangles of \mathcal{K}_q included in D .
- (iii) For all $1 \leq q \leq p-1$ and $R \in \mathcal{K}_q$, if r^{τ_d} is the length of the smallest side of R , then

$$(r^{\tau_d})^{-\varepsilon_q} \geq \rho_{q+1}^{-d}. \quad (5.21)$$

- (iv) For all $2 \leq q \leq p$, $R \in \mathcal{K}_{q-1}$, $D \in \mathcal{C}(R)$ and $1 \leq k \leq N_D$, with the rectangle $R_k^{(D)}$ are naturally associated a point $x_k^{(D)} \in E_D^{\varepsilon_q}$, a ball $L_k^{(D)} = B(x_k^{(D)}, 4r_k^{(D)})$, as well as some integer $n_k \in \mathbb{N}$, such that $n_k \geq q$, $x_k^{(D)} \in B_k^{(D)} := B_{n_k} = B(x_{n_k}, r_{n_k})$, $R_k^{(D)} = R_{n_k}$, $r_k^{(D)} = r_{n_k}$ and $4r_k^{(D)} \leq 2^{-p(D)} \rho_q$. In particular, due to (5.16), one has

$$\frac{\mu(L_k^{(D)})}{\mu(D)} \leq C_\mu \left(\frac{4r_{n_k}^{(D)}}{2^{-p(D)}} \right)^{\alpha - \varepsilon_q}. \quad (5.22)$$

- (v) For all $2 \leq q \leq p$, $R \in \mathcal{K}_{q-1}$, $D, D' \in \mathcal{C}(R)$, $1 \leq k \leq N_D$ and $1 \leq k' \leq N_{D'}$ such that $R_k^{(D)} \neq R_{k'}^{(D')}$, one has $3B_k^{(D)} \cap 3B_{k'}^{(D')} = \emptyset$.
- (vi) For all $1 \leq q \leq q' \leq p$ and $R \in \mathcal{K}_q$, $\eta_q(R) = \eta_{q'}(R)$.

(vii) For all $2 \leq q \leq p$, $R \in \mathcal{K}_{q-1}$, $D \in \mathcal{C}(R)$ and $1 \leq k \leq N_D$, one has

$$\eta_q(D) = \frac{\eta_{q-1}(R)}{\#\mathcal{C}(R)} \text{ and } (r_k^{(D)})^{-\varepsilon_q} \geq 4C_\mu Q_d \cdot \eta_q(D) (4 \cdot 2^{p(D)})^{(\alpha-\varepsilon_q)}. \quad (5.23)$$

(viii) For all $2 \leq q \leq p$, $R \in \mathcal{K}_{q-1}$, $D \in \mathcal{C}(R)$ and $1 \leq k \leq N_D$, one has

$$\sum_{1 \leq k' \leq N_D} \mu(L_{k'}^{(D)}) \geq \frac{\mu(D)}{4Q_d} \quad (5.24)$$

and

$$\eta_q(R_k^{(D)}) = \eta_q(D) \cdot \frac{\mu(L_k^{(D)})}{\sum_{1 \leq k' \leq N_D} \mu(L_{k'}^{(D)})}. \quad (5.25)$$

Notice that by (5.22), (5.23) (5.24) and (5.25), for all $2 \leq q \leq p$, $R \in \mathcal{K}_{q-1}$, $D \in \mathcal{C}(R)$ and $1 \leq k \leq N_D$, one has

$$\begin{aligned} \eta_q(R_k^{(D)}) &= \eta_q(D) \cdot \frac{\mu(L_k^{(D)})}{\sum_{1 \leq k' \leq N_D} \mu(L_{k'}^{(D)})} \leq \eta_q(D) \frac{\mu(L_k^{(D)})}{\mu(D)(4Q_d)^{-1}} \\ &\leq \eta_q(D) \left(4r_k^{(D)}\right)^{\alpha-\varepsilon_q} 4Q_d C_\mu \left(2^{p(D)}\right)^{\alpha-\varepsilon_q} \leq \left(r_k^{(D)}\right)^{\alpha-2\varepsilon_q}. \end{aligned} \quad (5.26)$$

Thus, for all $2 \leq q \leq p$, $R \in \mathcal{K}_{q-1}$ and $D \in \mathcal{C}(R)$, denoting by r^{τ_d} the length of the smallest side of R , by (5.11), (5.12), (5.23), (5.26) and (vi), one has

$$\begin{aligned} \eta_q(D) &= \frac{\eta_{q-1}(R)}{\#\mathcal{C}(R)} \leq \frac{r^{\alpha-2\varepsilon_{q-1}}}{\kappa_d^{-1} \cdot r^{\sum_{i=1}^d -\tau_d + \tau_i}} \leq \kappa_d r^{\alpha-2\varepsilon_{q-1} + \sum_{i=1}^d \tau_d - \tau_i} \\ &\leq \kappa_d \left(r^{\tau_d}\right)^{\frac{\alpha-2\varepsilon_{q-1} + \sum_{i=1}^d \tau_d - \tau_i}{\tau_d}}. \end{aligned} \quad (5.27)$$

Let us now explain the induction. Take $R \in \mathcal{K}_p$ and $D \in \mathcal{C}(R)$. For every $x \in E_D^{\varepsilon_{p+1}}$, consider an integer n_x large enough so that:

- $x \in B_{n_x}$,
- $n_x \geq p+1$, $4r_{n_x} \leq \rho_{p+1} 2^{-p(D)}$, and

$$r_{n_x}^{-\varepsilon_{p+1}} \geq \max \left(4^{\alpha-\varepsilon_{p+1}} \frac{\eta_p(R)}{\#\mathcal{C}(R)} 4Q_d C_\mu 2^{p(D)(\alpha-\varepsilon_{p+1})}, \rho_{p+2}^{-d/\tau_d} \right).$$

Using Proposition 4.1.2 with $v = 1$, one can extract from the covering of $E_D^{\varepsilon_{p+1}}$, $\{L_x := B(x, 4r_{n_x}) : x \in E_D^{\varepsilon_{p+1}}\}$, a finite set of balls $\mathcal{F}(D) := \{L_k^{(D)} := L_{x_k}^{(D)}\}_{1 \leq k \leq N_D}$ such that

- $\forall k \neq k' \leq N_D, L_k^{(D)} \cap L_{k'}^{(D)} = \emptyset$. In particular, $3B_{n_{x_k}^{(D)}} \cap 3B_{n_{x_{k'}}^{(D)}} = \emptyset$,
- writing $Q_d = Q_{d,1}$, one has

$$\mu\left(\bigcup_{1 \leq k \leq N_D} L_k^{(D)}\right) \geq \frac{1}{2}\mu(E_D^{\varepsilon_{p+1}}) \geq \frac{\mu(D)}{4Q_d C_\mu}. \quad (5.28)$$

Consider the collection of rectangles naturally associated with the balls $L_k^{(D)}$

$$\mathcal{K}_{p+1}(R) = \bigcup_{D \in \mathcal{C}(R)} \left\{ R_k^{(D)} := R_{n_{x_k}^{(D)}} \right\}_{1 \leq k \leq N_D}.$$

Then define

$$\mathcal{K}_{p+1} = \bigcup_{R \in \mathcal{K}_p} \mathcal{K}(R) \text{ and } K_{p+1} = \bigcup_{R \in \mathcal{K}_{p+1}} R.$$

The probability measure η_p can be extended from the algebra generated by the elements of $\bigcup_{1 \leq p \leq p} \mathcal{K}_q \cup \bigcup_{R \in \mathcal{K}_p} \mathcal{C}(R)$ to the algebra generated by the sets of the union $\bigcup_{1 \leq q \leq p+1} \mathcal{K}_q \cup \bigcup_{R \in \mathcal{K}_p} \mathcal{C}(R)$ as follows: For $R \in \mathcal{K}_p$ and $D \in \mathcal{C}(R)$, we impose that

$$\eta_{p+1}(R) = \eta_p(R) \text{ and } \eta_{p+1}(D) = \frac{\eta_p(R)}{\#\mathcal{C}(R)}. \quad (5.29)$$

And then, for $R \in \mathcal{K}_p$, $D \in \mathcal{C}(R)$ and $1 \leq k \leq N_D$, we set

$$\eta_{p+1}(R_k^{(D)}) = \eta_{p+1}(D) \cdot \frac{\mu(L_k^{(D)})}{\sum_{1 \leq k' \leq N_D} \mu(L_{k'}^{(D)})}. \quad (5.30)$$

It is easily checked that properties (i) to (viii) hold for $p+1$ and this ends the induction.

Last step: the Cantor set and some of its properties.

Set $\mathcal{K}_0 = [0, 1]^d$ and $\eta_0([0, 1]^d) = 1$. Define

$$\mathcal{K} = \bigcup_{p \in \mathbb{N}} \mathcal{K}_p \text{ and } K = \bigcap_{p \in \mathbb{N}} K_p.$$

By construction, item (i) of the recursion implies that $K \subset \limsup_{n \rightarrow \infty} R_n$. Now, for each $p \geq 1$, let $\tilde{\eta}_p$ be the element of $\mathcal{P}([0, 1]^d)$ supported on K_p and such that for every $R \in \mathcal{K}_p$ the restriction of $\tilde{\eta}_p$ to R has $\frac{\eta_p(R)}{\mathcal{L}^d(R)}$ as density with respect to $\mathcal{L}_{|R}^d$. It is easily seen, due to the separation property of the elements

of \mathcal{K}_p , for all $p \in \mathbb{N}$, that $(\tilde{\eta}_p)_{p \in \mathbb{N}^*}$ converges weakly to a Borel probability measure η such that $\eta(R) = \eta_p(R)$ for all $p \in \mathbb{N}$ and $R \in \mathcal{K}_p$.

Note that by construction the following properties hold:

- **Uniform separation property:** For all $p \in \mathbb{N}$ and $n \in \mathbb{N}$ such that $R_n \in \mathcal{K}_p$, if $n_1, n_2 \in \mathbb{N}$ such that $R_{n_1} \neq R_{n_2} \in \mathcal{K}(R_n) = \{R' \in \mathcal{K}_{p+1} : R' \subset R_n\}$, one has $3B_{n_1} \cap 3B_{n_2} = \emptyset$. In the case where R_{n_1} and R_{n_2} are elements of the same $D \in \mathcal{C}(R_n)$, this follows from (v); otherwise, this follows from the fact that two distinct elements D and D' of $\mathcal{C}(R_n)$ are distant from each other by at least $8 \cdot 2^{-p(D)}$, where, as before, $p(D)$ is the unique integer such that $D \in \mathcal{D}_{p(D)}$.
- The estimates (5.26) and (5.27) show (by induction) that for all $p \in \mathbb{N}^*$ and $n \in \mathbb{N}$ such that $R_n \in \mathcal{K}_p$ one has

$$\eta(R_n) \leq r_n^{\alpha-2\varepsilon_p},$$

and for all $D \in \mathcal{C}(R_n)$,

$$\eta(D) \leq \kappa_d r_n^{\alpha-2\varepsilon_p+\sum_{1 \leq i \leq d} \tau_d - \tau_i} = \kappa_d \left(r_n^{\tau_d} \right)^{\frac{\alpha-2\varepsilon_p+\sum_{1 \leq i \leq d} \tau_d - \tau_i}{\tau_d}}. \quad (5.31)$$

Upper bound for the η -measure of a ball.

Let C be a ball (recall that it is an Euclidean cube) of side length r contained in $[0, 1]^d$. Several cases are distinguished.

- When C intersects K_p for at most finitely many $p \in \mathbb{N}$, it is clear that $\eta(C) = 0$, and we set $p_C = +\infty$.

- When C intersects a unique rectangle of \mathcal{K}_p , say $R_{n_C(p)}$, for infinitely many $p \in \mathbb{N}$, then $\eta(C) \leq \eta(R_{n_C(p)}) \leq r_{n_C(p)}^{\alpha-2\varepsilon_p}$ for infinitely many p , so $\eta(C) = 0$. Again, we set $p_C = +\infty$.

- Suppose now that we are not in one of the previous cases. There exists $p_C \in \mathbb{N}$ such that if $p \leq p_C$, C intersects a unique rectangle of \mathcal{K}_p and if $p \geq p_C + 1$, C intersects at least two rectangles of \mathcal{K}_p . Denote by R_{n_C} the unique rectangle in \mathcal{K}_{p_C} intersecting C . Let $v > 0$ be such that $r = r_{n_C}^v$. Again, several cases are distinguished.

(i) Suppose $r \geq r_{n_C}^{\tau_d}$ (i.e. $v \leq \tau_d$): Suppose, moreover, that $r < r_{n_C}$, i.e.

$1 < v \leq \tau_d$. Remember that for $D \in \mathcal{C}(R_{n_C})$ one has (see (5.31))

$$\eta(D) \leq \kappa_d \left(r_{n_C} \right)^{\alpha - 2\varepsilon_{p_C} + \sum_{1 \leq i \leq d} \tau_d - \tau_i}.$$

Also, there exists $\tilde{\kappa}_d > 0$, depending on d so that

$$\begin{aligned} \# \{D \in \mathcal{C}(R_{n_C}) : D \cap C \neq \emptyset\} &\leq \tilde{\kappa}_d \prod_{i: \tau_i < v} \left(\frac{r_{n_C}^v}{r_{n_C}^{\tau_d}} \right) \prod_{i: \tau_i \geq v} \left(\frac{r_{n_C}^{\tau_i}}{r_{n_C}^{\tau_d}} \right) \\ &\leq \tilde{\kappa}_d r_{n_C}^{-d\tau_d + \sum_{i: \tau_i < v} v + \sum_{i: \tau_i \geq v} \tau_i}. \end{aligned}$$

Provided that κ_d was chosen larger than $\tilde{\kappa}_d$ at first, one gets

This gives the following upper bound for $\eta(C)$:

$$\begin{aligned} \eta(C) &\leq \sum_{D \in \mathcal{C}(R_{n_C}) : D \cap C \neq \emptyset} \eta(D) \\ &\leq (\# \{D \in \mathcal{C}(R_{n_C}) : D \cap C \neq \emptyset\}) \cdot \kappa_d \left(r_{n_C} \right)^{\alpha - 2\varepsilon_{p_C} + \sum_{1 \leq i \leq d} \tau_d - \tau_i} \\ &\leq \kappa_d^2 \left(r_{n_C} \right)^{-2\varepsilon_{p_C}} \left(r_{n_C} \right)^{-d\tau_d + \alpha + \sum_{i: \tau_i < v} v + \sum_{i: \tau_i \geq v} \tau_i + \sum_{1 \leq i \leq d} \tau_d - \tau_i} \\ &\leq \kappa_d^2 \left(r_{n_C} \right)^{-2\varepsilon_{p_C}} \left(r_{n_C} \right)^{\alpha + \sum_{i: \tau_i < v} v - \tau_i} \\ &\leq \kappa_d^2 \left(r^{-2\varepsilon_{p_C}} \right) r^{\frac{\alpha + \sum_{i: \tau_i < v} v - \tau_i}{v}}. \end{aligned}$$

The mapping $f : v \mapsto \frac{\alpha + \sum_{i: \tau_i < v} v - \tau_i}{v}$ reaches its minimum at one of the τ_i , say τ_{i_0} with $1 \leq i_0 \leq d$. This can be rephrased as $s(\mu, \boldsymbol{\tau}) = \min_{1 \leq i \leq d} \left(\frac{\alpha + \sum_{1 \leq j \leq i} \tau_i - \tau_j}{\tau_i} \right) = f(\tau_{i_0})$. It follows that

$$\eta(C) \leq \kappa_d^2 r^{s(\mu, \boldsymbol{\tau}) - 2\varepsilon_{p_C}}. \quad (5.32)$$

On the other hand, if $r \geq r_{n_C}$, i.e. $v \leq 1$, then by (5.26), one has

$$\eta(C) \leq \eta(R_{n_C}) \leq r_{n_C}^{\alpha - 2\varepsilon_{p_C}} \leq r^{\alpha - 2\varepsilon_{p_C}},$$

and (5.32) holds as well, since $\alpha = f(\tau_1) \geq s(\mu, \boldsymbol{\tau})$.

(ii) Suppose now that $r < r_{n_C}^{\tau_d}$ (i.e. $v > \tau_d$):

Recall that $r_{n_C}^{\tau_d}$ is the length of the smallest side of the rectangle R_{n_C} . Since C has side length less than $r_{n_C}^{\tau_d}$, and the side length of the cubes of $\mathcal{C}(R_{n_C})$ is larger than or equal to $C_d^{-1} r_{n_C}^{\tau_d}$, one deduces that C intersects at most \tilde{C}_d of those cubes, where \tilde{C}_d depends on d only. For all $D \in \mathcal{C}(R_{n_C})$, such that $C \cap D \neq \emptyset$, denote by $R_{k_1}^{(D)}, \dots, R_{k_{N_C, D}}^{(D)}$ the rectangles included in D that intersect C .

- Suppose first that $20r \leq 2^{-p(D)} \rho_{p_C+1}$ (where $D \in \mathcal{D}_{p(D)}$): Note that

for all $1 \leq i \neq j \leq N_{C,D}$, $3B_{k_i}^{(D)} \cap 3B_{k_j}^{(D)} = \emptyset$. Also, C intersects both $B_{k_i}^{(D)}$ and $B_{k_j}^{(D)}$, and by construction, since $L_{k_i}^{(D)} \cap L_{k_j}^{(D)} = \emptyset$ and $|L_{k_j}^{(D)}| = 4|B_{k_j}^{(D)}|$, we have $r \geq r_{k_j}^{(D)}$. By Lemma 5.2.1 applied to each pair $\{A = C, B = B_{k_j}^{(D)}\}$ and $q = 3$, one gets $\bigcup_{1 \leq i \leq N_{C,D}} 3B_{k_i}^{(D)} \subset 5C$. In particular, $\bigcup_{1 \leq i \leq N_{C,D}} L_{k_i}^{(D)} \subset 10C$ since $L_{k_i}^{(D)} \subset 5B_{k_i}^{(D)}$ for each i . Consequently,

$$\sum_{1 \leq i \leq N_{C,D}} \mu(L_{k_i}^{(D)}) \leq \mu(10C).$$

Further recall that, by item (iv) of the recurrence scheme, for any $1 \leq i \leq N_{C,D}$ the ball $L_{k_i}^{(D)}$ is centered on $E_D^{\varepsilon_{p_C}+1}$. Thus there is $x \in E_D^{\varepsilon_{p_C}+1} \cap 10C$. Since one has $10C \subset B(x, 20r)$ and $\frac{20r}{2^{-p(D)}} \leq \rho_{p_C+1}$, by (5.22) we get

$$\mu(10C) \leq \mu(B(x, 20r)) \leq C_\mu \mu(D) \left(\frac{20r}{2^{-p(D)}} \right)^{\alpha - \varepsilon_{p_C}+1}. \quad (5.33)$$

It follows from (5.28), (5.30) and (5.33) that

$$\begin{aligned} \eta(C \cap D) &\leq \sum_{1 \leq i \leq N_{C,D}} \eta(R_{k_i}^{(D)}) \leq \eta(D) \sum_{1 \leq i \leq N_{C,D}} \frac{\mu(L_{k_i}^{(D)})}{\sum_{1 \leq j \leq N_D} \mu(L_j^{(D)})} \\ &\leq \eta(D) \sum_{1 \leq i \leq N_{C,D}} \frac{\mu(L_{k_i}^{(D)})}{(4Q_d)^{-1} \mu(D)} \\ &\leq 4Q_d \frac{\eta(D)}{\mu(D)} \mu(10C) \\ &\leq C_\mu \eta(D) 4Q_d \left(\frac{20r}{2^{-p(D)}} \right)^{\alpha - \varepsilon_{p_C}+1}. \end{aligned}$$

This yields

$$\begin{aligned} \eta(C) &\leq \sum_{D \in \mathcal{C}(R_{n_C}): C \cap D \neq \emptyset} \eta(C \cap D) \\ &\leq \tilde{C}_d C_\mu \max_{D \in \mathcal{C}(R_{n_C}): C \cap D \neq \emptyset} \eta(D) 4Q_d \left(\frac{20r}{2^{-p(D)}} \right)^{\alpha - \varepsilon_{p_C}+1}. \end{aligned}$$

Moreover by (5.27), for each $D \in \mathcal{C}(R_{n_C})$ such that $C \cap D \neq \emptyset$,

$$\eta(D) \leq \kappa_d 2^{2p(D)\varepsilon_{p_C}} 2^{-s(\mu, \tau)p(D)},$$

hence

$$\eta(C) \leq \tilde{C}_d C_\mu \kappa_d 4Q_d 2^{2p(D)\varepsilon_{p_C}} \left(2^{-p(D)} \right)^{s(\mu, \tau)} \left(\frac{20r}{2^{-p(D)}} \right)^{\alpha - \varepsilon_{p_C}+1}.$$

Since $C_d 2^{-p(D)} \geq r_{n_C}^{\tau_d} \geq r$ and the sequence $(\varepsilon_p)_{p \geq 1}$ is decreasing and bounded, it follows that for some constant γ depending only on the dimension d and μ ,

one has

$$\eta(C) \leq \gamma r^{-3\varepsilon_{p_C}} \frac{r^\alpha}{2^{-p(D)(\alpha-s(\mu,\tau))}} = \gamma r^{-3\varepsilon_{p_C}} \left(\frac{r}{2^{-p(D)}} \right)^{\alpha-s(\mu,\tau)} r^{s(\mu,\tau)}.$$

Thus, as $C_d 2^{-p(D)} \geq r$ and $s(\mu, \tau) \leq \alpha$ (so that $t > 0 \mapsto t^{\alpha-s(\mu,\tau)}$ is non decreasing), we finally obtain

$$\eta(C) \leq \gamma C_d^{\alpha-s(\mu,\tau)} r^{s(\mu,\tau)-3\varepsilon_{p_C}} \leq \gamma C_d^\alpha r^{s(\mu,\tau)-3\varepsilon_{p_C}}.$$

• Suppose now that $20\rho_{p_C+1} 2^{-p(D)} \leq r < r_{n_C}^{\tau_d}$: Again, by definition of $p(D)$, one has $r_{n_C}^{\tau_d} \leq C_d 2^{-p(D)}$. Consequently, C is covered by at most $\lfloor (C_d/20\rho_{p_C+1}) + 1 \rfloor^d$ cubes of side length $20\rho_{p_C+1} 2^{-p(D)}$. Denoting these cubes by D_1, \dots, D_k , and recalling (5.21), the previous estimate yields

$$\begin{aligned} \eta(C) &\leq \sum_{i=1}^k \eta(D_i) \leq \lfloor (C_d/20\rho_{p_C+1}) + 1 \rfloor^d \gamma C_d^\alpha \left(20\rho_{p_C+1} 2^{-p(D)} \right)^{s(\mu,\tau)-3\varepsilon_{p_C}} \\ &\leq \gamma_1 \rho_{p_C+1}^{-d} r^{s(\mu,\tau)-3\varepsilon_{p_C}} \leq \gamma_1 r^{s(\mu,\tau)-4\varepsilon_{p_C}} \end{aligned}$$

for some constant γ_1 depending only on d and μ (we used that $20\rho_{p_C+1} 2^{-p(D)} \leq r$ to get the third inequality, and (iii) as well as the inequality $\varepsilon_{p_C} \geq \varepsilon_{p_C+1}$ to get the fourth one).

To conclude the proof, note that due to the uniform separation property outlined after the last step of the construction of (K, η) ,

$$p(r) = \inf\{p_C : C \text{ is ball of radius } r \text{ included in } [0, 1]^d\}$$

tends to $+\infty$ as r tends to 0.

Combining the previous estimates, setting $\tilde{\gamma}_1 = \max\{\gamma_1, \gamma \cdot C_d^\alpha, \kappa_d^2\}$, we finally get

$$\eta(C) \leq \tilde{\gamma}_1 r^{s(\mu,\tau)-4\varepsilon_{p(r)}}.$$

In particular, for any $p \in \mathbb{N}$, setting $r_p = \frac{1}{2} \sup\{r : p(r) \leq p\}$, it holds that for any $r \leq r_p$, any C of radius r ,

$$\eta(C) \leq \tilde{\gamma}_1 r^{s(\mu,\tau)-4\varepsilon_p}.$$

By Lemma 4.2.4, since $\eta(K) = 1$, it holds that

$$\dim_H(K) \geq s(\mu, \tau) - 4\varepsilon_p.$$

Letting $p \rightarrow +\infty$ proves Theorem [5.1.5](#).

The next section introduces some key notion in order to establish a more general mass transference principle than the one we just established.

Chapter 6

Asymptotically covering sequences of balls and essential content

This chapter comes as a preamble of Chapter 7. First, in Section 6.1 given a sequence of balls $(B_n)_{n \in \mathbb{N}}$, one investigates the link between satisfying the condition $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$ and verifying a certain covering property, we call μ asymptotically covering (Definition 6.1.1). This property is usually used in mass transference principles when the reference measure is doubling on its support ([11]).

In a second time, in Section 6.2, one defines and study the key geometric notion we will need to establish our main ubiquity theorem, Theorem 7.1.2, which is the essential Hausdorff content. Some explicit estimates of this geometric quantity are given and some basic properties are established.

The last section, Section 6.1.3 gives a version of the Borel-Cantelli Lemma suitable for limsup sets of balls.

6.1 Limsup sets of full measure and asymptotically covering sequences of balls

Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of balls of $[0, 1]^d$ satisfying $|B_n| \rightarrow 0$. An important covering property, highlighted in [11] and used to establish ubiquity theorems or mass transference principles in the case of the Lebesgue measure, is that for any open set Ω and any $g \in \mathbb{N}$, assuming that $\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$ it is possible to find disjoint balls $B_{n_1}, \dots, B_{n_k} \subset \Omega$ with $n_i \geq g$ for all i and

$$\mathcal{L}^d\left(\bigcup_{1 \leq i \leq k} B_{n_i}\right) \geq \frac{1}{2} \mathcal{L}^d(\Omega).$$

In this section, one investigates, given any probability measure μ , the relationship between being verifying $\mu\left(\limsup_{n \rightarrow +\infty} B_n\right) = 1$ and satisfying the covering property stated above where \mathcal{L}^d is replaced by μ .

Definition 6.1.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. The sequence $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ of balls of \mathbb{R}^d is said to be μ -asymptotically covering (in short, μ -a.c) when there exists a constant $C > 0$ such that for every open set $\Omega \subset \mathbb{R}^d$ and $g \in \mathbb{N}$, there is an integer $N_\Omega \in \mathbb{N}$ as well as $g \leq n_1 \leq \dots \leq n_{N_\Omega}$ such that:*

- $\forall 1 \leq i \leq N_\Omega, B_{n_i} \subset \Omega,$
- $\forall 1 \leq i \neq j \leq N_\Omega, B_{n_i} \cap B_{n_j} = \emptyset,$
- *one has*

$$\mu\left(\bigcup_{1 \leq i \leq N_\Omega} B_{n_i}\right) \geq C\mu(\Omega). \quad (6.1)$$

In other words, for any open set Ω and any $g > 0$, there exists a finite set of disjoint balls of $\{B_n\}_{n \geq g}$ covering a large part of Ω from the μ -standpoint.

This notion of μ -asymptotically covering is related to the way the balls of \mathcal{B} are distributed according to the measure μ . In particular, given a measure μ , this property turns out to be slightly stronger than being of lim sup of full μ -measure, as illustrated by the following Theorem.

Theorem 6.1.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ be a sequence of balls of \mathbb{R}^d with $\lim_{n \rightarrow +\infty} r_n = 0$.*

1. *If \mathcal{B} is μ -a.c, then $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$.*
2. *If there exists $v < 1$ such that $\mu(\limsup_{n \rightarrow +\infty} (vB_n)) = 1$, then \mathcal{B} is μ -a.c.*

Moreover, it results from the proof of the KGB-Lemma [11] that if the μ is doubling, $\mu\left(\limsup_{n \rightarrow +\infty} B_n\right) = 1 \Leftrightarrow (B_n)_{n \in \mathbb{N}}$ is μ -a.c.

One emphasizes also that, in general, the contracted sets we consider in ubiquity theorems are “much smaller” than vB_n for any $v < 1$. In particular, as it will be highlighted later in this manuscript, this condition is not restrictive at all for the cases we consider.

6.1.1 Consequences of the μ -asymptotic covering property

One first shows that the constant C in Definition 6.1.1 can be replaced by 1 if infinite subsequences of balls are authorized. In fact Definition 6.1.1 ensures

that any open set can be covered (with respect to the μ -measure) by disjoint balls B_n of arbitrary large indices.

Lemma 6.1.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ be a μ -a.c sequence of balls of \mathbb{R}^d with $\lim_{n \rightarrow +\infty} r_n = 0$.*

Then for every open set Ω and every integer $g \in \mathbb{N}$, there exists a subsequence $(B_{(n)}^{(\Omega)}) \subset \{B_n\}_{n \geq g}$ such that:

1. $\forall n \in \mathbb{N}, B_{(n)}^{(\Omega)} \subset \Omega,$
2. $\forall 1 \leq n_1 \neq n_2, B_{(n_1)}^{(\Omega)} \cap B_{(n_2)}^{(\Omega)} = \emptyset,$
3. $\mu\left(\bigcup_{n \geq 1} B_{(n)}^{(\Omega)}\right) = \mu(\Omega).$

In addition, there exists an integer N_Ω such that for the balls $(B_{(n)}^{(\Omega)})_{n=1, \dots, N_\Omega}$, the conditions (1) and (2) are realized, and (3) is replaced by $\mu\left(\bigcup_{n=1}^{N_\Omega} B_{(n)}^{(\Omega)}\right) \geq \frac{3}{4}\mu(\Omega).$

The last part of Lemma 6.1.3 simply follows from item (3) and the σ -additivity of μ .

Proof. The idea consists in covering Ω by pairwise disjoint balls amongst those balls of \mathcal{B} , such that their union has measure at least $C\mu(\Omega)$, then in covering the complementary of the union of those balls in Ω (that is still open) with at least a proportion C of its measure, and so on.

More precisely, this is achieved as follows:

• **Step 1:** By application of Definition 6.1.1 to $\Omega_0 := \Omega$ and $g \in \mathbb{N}$, there exists $C > 0$ and some integers $g \leq n_1 \leq \dots \leq n_{N_0}$ so that the family of balls $\mathcal{F}_0 := \{B_{n_i} := B_i^{(0)}\}_{1 \leq i \leq N_0}$ is pairwise disjoint and $\mu(\bigcup_{1 \leq i \leq N_0} B_{n_i}) \geq C\mu(\Omega).$

• **Step 2:** Setting $\Omega_1 = \Omega \setminus \bigcup_{L \in \mathcal{F}_0} L$, applying Definition 6.1.1 to Ω_1 with the integer g provides us with a family \mathcal{F}_1 of pairwise disjoint balls $B_1^{(1)}, \dots, B_{N_1}^{(1)} \in \{B_n\}_{n \geq g}$ such that $\forall 1 \leq i \leq N_1$ $B_i^{(1)} \subset \Omega_1$ and

$$\mu\left(\bigcup_{1 \leq i \leq N_2} B_i^{(1)}\right) \geq C\mu(\Omega_1).$$

One sets $\mathcal{F}_1 = \mathcal{F}_0 \cup \mathcal{F}_1$. One sees that

$$\begin{aligned} \mu \left(\bigcup_{L \in \mathcal{F}_1} L \right) &= \mu \left(\bigcup_{L \in \mathcal{F}_0} L \right) + \mu \left(\bigcup_{L \in \mathcal{G}_1} L \right) \\ &\geq \mu \left(\bigcup_{L \in \mathcal{F}_0} L \right) + C \left(\mu(\Omega) - \mu \left(\bigcup_{L \in \mathcal{F}_0} L \right) \right) \\ &\geq (1 - C) \mu \left(\bigcup_{L \in \mathcal{F}_0} L \right) + C \mu(\Omega) \\ &\geq (C + C(1 - C)) \mu(\Omega). \end{aligned}$$

Observe that the balls of \mathcal{F}_0 and \mathcal{F}_1 are disjoint by construction.

• **Following steps :** Proceeding recursively, and applying the exact same argument as above, one constructs an increasing sequence of families $(\mathcal{F}_i)_{i \in \mathbb{N}}$ and a decreasing sequence of open sets Ω_i such that:

- $\forall i \in \mathbb{N}, L \in \{B_n\}_{n \geq g}$ and $\forall L \in \mathcal{F}_i, L \subset \Omega_i \subset \Omega$,
- $\forall i \in \mathbb{N}, \forall L \neq L' \in \mathcal{F}_i, L \cap L' = \emptyset$,
- $\forall i \neq j \in \mathbb{N}, \forall L \in \mathcal{F}_i$ and $\forall L' \in \mathcal{F}_j, L \cap L' = \emptyset$,
- $\forall i \in \mathbb{N}, \mu \left(\bigcup_{L \in \mathcal{F}_i} L \right) \geq \mu(\Omega) \sum_{1 \leq k \leq i} C(1 - C)^{k-1}$.

Finally, setting $\mathcal{F} = \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$, one sees that \mathcal{F} is constituted by pairwise disjoint balls chosen amongst $\{B_n\}_{n \geq g}$ satisfying

$$\mu(\Omega) \geq \mu \left(\bigcup_{L \in \mathcal{F}} L \right) \geq \mu(\Omega) \sum_{k \geq 1} C(1 - C)^{k-1} = \mu(\Omega), \quad (6.2)$$

so that \mathcal{F} fulfills the conditions of Lemma 6.1.3. \square

An easy consequence is the following.

Corollary 6.1.4. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $(B_n)_{n \in \mathbb{N}}$ be a μ -a.c sequence of balls. Then for any Borel set E , for any $g \in \mathbb{N}$, there exists a sub-sequence of balls $(B_{(n)}^{(E)}) \subset \{B_n\}_{n \geq g}$ such that:*

1. $\forall 1 \leq n_1 \neq n_2, B_{(n_1)}^{(E)} \cap B_{(n_2)}^{(E)} = \emptyset$,
2. $\mu \left(\bigcup_{n \in \mathbb{N}} B_{(n)}^{(E)} \cap E \right) = \mu(E)$,
3. $\mu \left(\bigcup_{n \in \mathbb{N}} B_{(n)}^{(E)} \right) \leq \mu(E) + \varepsilon$,

Proof. By outer regularity, there exists an open set Ω such that $E \subset \Omega$ and $m(\Omega) \leq \mu(E) + \varepsilon$. Applying Lemma 6.1.3 to Ω , the sequence $(B_n)_{n \in \mathbb{N}}$ fulfills the condition of Corollary 6.1.4. \square

6.1.2 Proof of Theorem 6.1.2

(1) Assume first that $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ is μ -a.c, and let us prove that $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$.

For every $g \in \mathbb{N}$, applying Lemma 6.1.3, there exists a sub-family of balls, $\mathcal{F}_g \subset \{B_n\}_{n \geq g}$ such that $\mu(\bigcup_{L \in \mathcal{F}_g} L) = \mu(\mathbb{R}^d) = 1$. In particular, $\mu(\bigcup_{n \geq g} B_n) = 1$ for every $g \geq 1$, and $\mu(\limsup_{n \rightarrow +\infty} B_n) = \mu(\bigcap_{g \geq 1} \bigcup_{n \geq g} B_n) = 1$.

(2) Suppose next that there exists $v < 1$ such that $\mu(\limsup_{n \rightarrow +\infty} vB_n) = 1$, and let us show that \mathcal{B} is μ -a.c.

Let Ω be an open set in \mathbb{R}^d . Our goal is to find a constant C such that the conditions of Definition 6.1.1 are realized.

Let $E = \Omega \cap \limsup_{n \rightarrow +\infty} vB_n$. For every $y \in E$, consider an integer $n_y \geq g$ large enough so that $y \in vB_{n_y}$ and $B(y, 2r_{n_y}) \subset \Omega$. This is possible since $\lim_{n \rightarrow +\infty} r_n = 0$.

Since $y \in vB_{n_y}$, one has

$$B(y, (1-v)r_{n_y}) \subset B_{n_y} \subset B(y, (1+v)r_{n_y}) \subset B(y, 2r_{n_y}), \quad (6.3)$$

and the family $\mathcal{F} = \{B(y, (1-v)r_{n_y}) : y \in E\}$ covers E by balls centered on E .

Applying Proposition 4.1.2 with constant $v' = \frac{1-v}{2} < 1$ allows to extract from \mathcal{F} finite or countable sub-families $\mathcal{F}_1, \dots, \mathcal{F}_{Q_{d,v'}}$ such that:

- $\forall 1 \leq i \leq Q_{d,v'}, L \neq L' \in \mathcal{F}_i$, one has $\frac{1}{v'}L \cap \frac{1}{v'}L' = \emptyset$.
- E is covered by the families \mathcal{F}_i , i.e. (4.2) holds true.

Now, $\mu(\Omega) = \mu(E) \leq \mu\left(\bigcup_{i=1}^{Q_{d,v'}} \bigcup_{L \in \mathcal{F}_i} L\right)$. There must exist $1 \leq i_0 \leq Q_{d,v'}$ such that

$$\mu\left(\bigcup_{L \in \mathcal{F}_{i_0}} L\right) \geq \frac{1}{Q_{d,v'}} \mu(E) = \frac{1}{Q_{d,v'}} \mu(\Omega).$$

There exist L_1, L_2, \dots, L_N balls of \mathcal{F}_{i_0} such that

$$\mu\left(\bigcup_{1 \leq k \leq N} L_k\right) \geq \frac{1}{2Q_{d,v'}} \mu(\Omega),$$

Notice the following facts:

- $\forall 1 \leq i \leq Q_{d,v'}$, every $L \in \mathcal{F}_i$ is naturally associated with some $y \in E$ and some ball B_{n_y} , with $L \subset B_{n_y} \subset \Omega$,
- $\forall 1 \leq i \leq Q_{d,v'}$, if $L \in \mathcal{F}_i$ is associated with $y \in E$ and B_{n_y} and $L' \in \mathcal{F}_i$ is associated with $y' \in E$ and $B_{n_{y'}}$, then $\frac{1}{v'}L \cap \frac{1}{v'}L' = \emptyset$ implies by (6.3) that $B_{n_y} \cap B_{n_{y'}} = \emptyset$.

The first fact implies that there exist N integers n_1, \dots, n_N such that $B_{n_k} \subset \Omega$ and

$$\mu \left(\bigcup_{1 \leq k \leq N} B_{n_k} \right) \geq \frac{1}{2Q_{d,v'}} \mu(\Omega),$$

The second fact implies that these balls B_{n_k} , $k = 1, \dots, N$ are pairwise disjoint.

This exactly proves that \mathcal{B} is μ -a.c.

6.1.3 A version of Borel-Cantelli Lemma

In this manuscript, one mainly focuses on establishing Hausdorff dimension of limsup sets knowing that a certain limsup set of balls has full measure. In many situations, proving that those limsup sets have full measure is straightforward. Given $\mu \in \mathcal{M}(\mathbb{R}^d)$, when establishing that a sequence of balls $(B_n)_{n \in \mathbb{N}}$ verifies

$$\mu(\limsup_{n \in \mathbb{N}} B_n) = 1$$

is not straightforward, it is convenient to have a tool at our disposal to be able to determine whether or not it is the case. The case where the measure involved is doubling is treated by Beresnevich and Velani.

Theorem 6.1.5 ([12]). *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a doubling measure and $(B_n)_{n \in \mathbb{N}}$ a sequence of balls centered in $\text{supp}(\mu)$ such that $|B_n| \rightarrow 0$. Then the two following assertions are equivalent:*

1. $\mu(\limsup_{n \in \mathbb{N}} B_n) = 1$.
2. *There exists $C > 1$ such that for any open ball B centered on $\text{supp}(\mu)$, there exists a sub-sequence $(L_{B,n})_{n \in \mathbb{N}}$ of $(B_n)_{n \in \mathbb{N}}$ satisfying:*
 - for every $n \in \mathbb{N}$, $L_{B,n} \subset B$,
 - $\sum_{n=0}^{+\infty} \mu(L_{B,n}) = +\infty$,

- for infinitely many $Q \in \mathbb{N}$,

$$\sum_{s,t=1}^Q \mu(L_{B,s} \cap L_{B,t}) \leq \frac{C}{\mu(B)} \left(\sum_{n=1}^Q \mu(L_{B,n}) \right)^2. \quad (6.4)$$

Thanks to Theorem 6.1.2, one can provide a version of this theorem valid for any probability measure on \mathbb{R}^d .

Proposition 6.1.6 (D. [24]). *Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of closed balls satisfying $|B_n| \rightarrow 0$ and $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure.*

(A): *Assume that $(B_n)_{n \in \mathbb{N}}$ is μ -a.c. Then there exists $C > 1$ such that for any open ball B , there exists a sub-sequence of $(B_n)_{n \in \mathbb{N}}$, $(L_{n,B})_{n \in \mathbb{N}}$ satisfying, for any $n \in \mathbb{N}$, $L_{B,n} \subset B$,*

$$\sum_{n \geq 0} \mu(L_{B,n}) = +\infty \quad (6.5)$$

and for infinitely many Q ,

$$\sum_{s,t=1}^Q \mu(L_{B,s} \cap L_{B,t}) \leq \frac{C}{\mu(B)} \left(\sum_{n=1}^Q \mu(L_{B,n}) \right)^2. \quad (6.6)$$

(B): *Assume that there exists $C > 1$ such that for any open ball B , there exists a sub-sequence of $(B_n)_{n \in \mathbb{N}}$ $(L_{n,B})_{n \in \mathbb{N}}$ with, for any $n \in \mathbb{N}$, $L_{n,B} \subset B$, satisfying (6.5) and (6.6). Then $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$, so that for any $\kappa > 1$, $(\kappa B_n)_{n \in \mathbb{N}}$ is μ -a.c.*

Proof. Item A is proved in [12] (this part of the proof does not use the doubling property of the measure). Moreover, it is also proved in [12] that, if there exists $C > 0$ such that for any open ball B , there exists a sub-sequence of $(B_n)_{n \in \mathbb{N}}$ $(L_{n,B})_{n \in \mathbb{N}}$ with, for any $n \in \mathbb{N}$, $L_{n,B} \subset B$, satisfying (6.5) and (6.6), then

$$\mu(\limsup_{n \rightarrow +\infty} B_n \cap B) \geq \frac{1}{C} \mu(B).$$

The following lemma combined with Theorem 6.1.2 finishes the proof of Proposition 6.1.6.

Lemma 6.1.7. *Let $E \subset \mathbb{R}^d$. Assume that there exists $0 < c < 1$ such that for any open ball B , $\mu(E \cap B) \geq c\mu(B)$. Then $\mu(E) = 1$.*

Proof. Assume that $\mu(E) < 1$ and set $A = \mathbb{R}^d \setminus E$. By hypothesis, $\mu(A) > 0$.

By Lemma 4.1.5, there exists an open ball B such that $\mu(B) > 0$ and

$$\mu(B \cap A) \geq (1 - \frac{c}{2})\mu(B).$$

This yields

$$\begin{aligned} \mu(E \cap B \cap A) &= \mu(E \cap B) + \mu(A \cap B) - \mu((E \cap B) \cup (A \cap B)) \\ &\geq (c + 1 - \frac{c}{2} - 1)\mu(B) = \frac{c}{2}\mu(B) > 0, \end{aligned}$$

which implies $\mu(E \cap A) > 0$, which is a contradiction. \square

Taking $c = \frac{1}{C}$ and applying Lemma 6.1.7 finishes the proof of Proposition 6.1.6. \square

Remark 6.1.8. *A version of Proposition 6.1.6 might also be useful in more general metric spaces. The only geometric property we used to prove Proposition 6.1.6 is actually Proposition 4.1.2 (which also implies Lemma 4.1.5), so that Proposition 6.1.6 actually holds in any direction-limited spaces as defined in [34].*

6.2 Essential Hausdorff content

In this section, we introduce the key geometric notion on which our main ubiquity theorem, Theorem 7.1.2, relies.

Definition 6.2.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, and $s \geq 0$. The s -dimensional μ -essential Hausdorff content at scale $t \in (0, +\infty]$ of a set $A \subset \mathcal{B}(\mathbb{R}^d)$ is defined as*

$$\mathcal{H}_t^{\mu,s}(A) = \inf \{ \mathcal{H}_t^s(E) : E \subset A, \mu(E) = \mu(A) \}. \quad (6.7)$$

One will almost exclusively look at these contents at scale $t = +\infty$ and one refers to $\mathcal{H}_\infty^{\mu,s}(A)$ as the s -dimensional μ -essential Hausdorff content of A . Basic properties of those quantities are studied in Sub-section 7.2.2, and precise estimates of $\mathcal{H}_\infty^{\mu,s}(A)$ are achieved for the Lebesgue measure and weakly conformal measures (so that those estimates holds for self-similar measures and self-conformal measures) in this chapter.

Note that in [51, Theorem 3.1] the underlying geometric notion key to handle the variety of shapes of the sets $(U_n)_{n \in \mathbb{N}}$ is the Hausdorff content. It is easily seen from (4.7) that the Hausdorff content also carries some “high scale” geometric information (because there is no restriction concerning the diameter of the balls (B_n) in (4.7)). This will also be the case in this chapter to handle

not only the shape of the sets $(U_n)_{n \in \mathbb{N}}$ but also the geometric behavior related to the measure μ at high scale in the sets $(U_n)_{n \in \mathbb{N}}$.

Also we work in this manuscript mainly with the $\|\cdot\|_\infty$ norm for convenience. Any other norm could have been chosen, the corresponding quantities would have been equivalent.

In (4.7), only closed balls are considered. Choosing open balls does not change the value of (6.7) in Definition 6.2.1.

The following propositions are directly derived from the properties of the standard Hausdorff measures.

Proposition 6.2.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $s \geq 0$ and $A \subset \mathbb{R}^d$ be a Borel set. The s -dimensional $\mathcal{H}_\infty^{\mu,s}(\cdot)$ outer measure satisfies the following properties:*

1. *If $|A| \leq 1$, the mapping $s \geq 0 \mapsto \mathcal{H}_\infty^{\mu,s}(A)$ is decreasing from $\mathcal{H}_\infty^{\mu,0}(A) = 1$ to $\lim_{t \rightarrow +\infty} \mathcal{H}_\infty^{\mu,t}(A) = 0$.*
2. $0 \leq \mathcal{H}_\infty^{\mu,s}(A) \leq \min\{|A|^s, \mathcal{H}_\infty^s(A)\}$.
3. *For every subset $B \subset A$ with $\mu(A) = \mu(B)$, $\mathcal{H}_\infty^{\mu,s}(A) = \mathcal{H}_\infty^{\mu,s}(B)$.*
4. *For every $\delta \geq 1$, $\mathcal{H}_\infty^{\mu, \frac{s}{\delta}}(A) \geq (\mathcal{H}_\infty^{\mu,s}(A))^{\frac{1}{\delta}}$.*
5. *For every $s > \overline{\dim}_H(\mu)$, $\mathcal{H}_\infty^{\mu,s}(A) = 0$.*

Proof. Items (1), (2), (3) directly follow from the definition. Item (4) is obtained by concavity of the mapping $x \mapsto |x|^{1/\delta}$.

(5) By Definition 4.2.12, for any $s > \overline{\dim}_H(\mu)$, there exists a set E with $\dim_H(E) < s$ and $\mu(E) = 1$. Using item (2), one has then $0 \leq \mathcal{H}_\infty^{\mu,s}(A) = \mathcal{H}_\infty^{\mu,s}(A \cap E) \leq \mathcal{H}_\infty^s(A \cap E) \leq \mathcal{H}_\infty^s(E) = 0$.

□

6.2.1 Computation of essential content for the Lebesgue measure

When the measure μ is the Lebesgue measure, the computations are quite easy.

Proposition 6.2.3. *Let $B = B(x, r)$ be a ball in \mathbb{R}^d , and \mathcal{L}^d be the d -dimensional Lebesgue measure. Then for any $0 \leq s \leq d$, $\mathcal{H}_\infty^{\mathcal{L}^d,s}(B) = \mathcal{H}_\infty^{\mathcal{L}^d,s}(\overset{\circ}{B}) = r^s$.*

Proof. One starts first by computing $\mathcal{H}_{\infty}^{\mathcal{L}^d, d}(B)$.

Let $\varepsilon > 0$, and let $E \subset B$ be a Borel set with $\mathcal{L}^d(E) = \mathcal{L}^d(B)$. Notice first that since B covers E , recalling that \mathbb{R}^d is endowed with $\|\cdot\|_{\infty}$ one has

$$\mathcal{H}_{\infty}^{\mathcal{L}^d, d}(E) \leq \mathcal{H}_{\infty}^d(B) \leq |B|^d.$$

Consider a sequence of balls $(L_n)_{n \in \mathbb{N}}$ such that

$$\mathcal{H}_{\infty}^d(E) \leq \sum_{n \geq 0} |L_n|^d \leq (1 + \varepsilon) \mathcal{H}_{\infty}^d(E).$$

This implies

$$\begin{aligned} (1 + \varepsilon)|B|^d &\geq (1 + \varepsilon)\mathcal{H}_{\infty}^d(B) \geq (1 + \varepsilon)\mathcal{H}_{\infty}^d(E) \geq \sum_{n \geq 0} |L_n|^d \\ &\geq \sum_{n \geq 0} \mathcal{L}^d(L_n) \geq \mathcal{L}^d(E) = \mathcal{L}^d(B) = |B|^d. \end{aligned}$$

Taking the infimum on the Borel sets $E \subset B$ such that $\mathcal{L}^d(E) = \mathcal{L}^d(B)$ gives

$$|B|^d \leq (1 + \varepsilon) \mathcal{H}_{\infty}^{\mathcal{L}^d, d}(B).$$

In particular,

$$\frac{1}{1 + \varepsilon} |B|^d \leq \mathcal{H}_{\infty}^{\mathcal{L}^d, d}(B) \leq |B|^d.$$

Letting $\varepsilon \rightarrow 0$ shows that $\mathcal{H}_{\infty}^{\mathcal{L}^d, d}(B) = |B|^d$. This implies, with item (4) of Proposition 6.2.2, that for any $\delta \geq 1$,

$$|B|^{\frac{d}{\delta}} \geq \mathcal{H}_{\infty}^{\mathcal{L}^d, \frac{d}{\delta}}(B) \geq (\mathcal{H}_{\infty}^{\mathcal{L}^d, d}(B))^{\frac{1}{\delta}} = |B|^{\frac{d}{\delta}},$$

hence the result. \square

6.2.2 Essential content for weakly conformal measures

Estimates of essential contents for weakly conformal measures are now established.

Theorem 6.2.4 (D. [23]). *Let S be a C^1 weakly conformal IFS of \mathbb{R}^d .*

Let K be the attractor of S and μ be a weakly conformal measure associated with S . Then,

For any $0 \leq s < \dim(\mu)$, for any $0 < \varepsilon \leq \frac{1}{2}$, there exists a constant $c = c(d, \mu, s, \varepsilon) > 0$ depending on the dimension d , μ , s and ε only, such that

for any ball $B = B(x, r)$ centered on K and $r \leq 1$, for any open set Ω , one has

$$\begin{aligned} c(d, \mu, s, \varepsilon) |B|^{s+\varepsilon} &\leq \mathcal{H}_{\infty}^{\mu, s}(\overset{\circ}{B}) \leq \mathcal{H}_{\infty}^{\mu, s}(B) \leq |B|^s \\ c(d, \mu, s, \varepsilon) \mathcal{H}_{\infty}^{s+\varepsilon}(\Omega \cap K) &\leq \mathcal{H}_{\infty}^{\mu, s}(\Omega) \leq \mathcal{H}_{\infty}^s(\Omega \cap K). \end{aligned} \quad (6.8)$$

For any $s > \dim(\mu)$, $\mathcal{H}_{\infty}^{\mu, s}(\Omega) = 0$.

Remark 6.2.5.

- The system S is not assumed to verify any separation condition.
- When the maps are similarities, one still has, for any $s > \dim(\mu)$, $\mathcal{H}_{\infty}^{\mu, s}(\Omega) = 0$ but for $s < \dim(\mu)$, there exists a constant $c(d, \mu, s)$ such that the following more precise estimates holds true [25]:

$$\begin{aligned} c(d, \mu, s) |B|^s &\leq \mathcal{H}_{\infty}^{\mu, s}(\overset{\circ}{B}) \leq \mathcal{H}_{\infty}^{\mu, s}(B) \leq |B|^s \text{ and} \\ c(d, \mu, s) \mathcal{H}_{\infty}^s(\Omega \cap K) &\leq \mathcal{H}_{\infty}^{\mu, s}(\Omega) \leq \mathcal{H}_{\infty}^s(\Omega \cap K). \end{aligned} \quad (6.9)$$

Proof. Let us first prove the above estimates for balls.

Proposition 6.2.6. *Let μ be a weakly conformal measure as in Definition 4.3.1. For any $0 < \varepsilon \leq \dim(\mu)$, any $0 \leq \varepsilon' \leq \frac{1}{2}$ such that $\dim(\mu) - \varepsilon + \varepsilon' > 0$, there exists a constant $\chi(d, \mu, \varepsilon, \varepsilon') > 0$ such that for any ball $B = B(x, r)$ with $x \in K$ (the attractor of the underlying IFS) and $r \leq 1$, one has*

$$\chi(d, \mu, \varepsilon, \varepsilon') |B|^{\dim(\mu) - \varepsilon + \varepsilon'} \leq \mathcal{H}_{\infty}^{\mu, \dim(\mu) - \varepsilon}(\overset{\circ}{B}) \leq \mathcal{H}_{\infty}^{\mu, \dim(\mu) - \varepsilon}(B) \leq |B|^{\dim(\mu) - \varepsilon}.$$

In addition, for any $s > \dim(\mu)$, $\mathcal{H}_{\infty}^{\mu, s}(B) = 0$.

Proof. Note first that item (5) of Proposition 6.2.2 implies that for any $s > \dim(\mu)$, $\mathcal{H}_{\infty}^{\mu, s}(B) = 0$.

Let us consider $0 \leq s < \dim_H(\mu)$ and start by few remarks.

Set $\alpha = \dim(\mu)$ and let $\varepsilon > 0$ and $\rho > 0$ be two real numbers. One defines

$$E_{\mu}^{\alpha, \rho, \varepsilon} = \{x \in \mathbb{R}^d : \forall r \leq \rho, \mu(B(x, r)) \leq r^{\alpha - \varepsilon}\}.$$

Since μ is α -exact dimensional, for μ -almost every x , $\lim_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log r} = \alpha$. This implies that, for very $\varepsilon > 0$, $\mu\left(\bigcup_{\rho > 0} E_{\mu}^{\alpha, \rho, \varepsilon}\right) = 1$.

Let $\varepsilon > 0$ and $0 < \rho_{\varepsilon} \leq 1$ be two real numbers such that $\mu(E_{\mu}^{\alpha, \rho_{\varepsilon}, \varepsilon}) \geq \frac{1}{2}$ and write $E = E_{\mu}^{\alpha, \rho_{\varepsilon}, \varepsilon}$.

Write $c_{\underline{i}} = |f_{\underline{i}}(K)|$. Let us fix $\underline{i} = (i_1, \dots, i_k) \in \Lambda^*$. For any $x \in K$ and

$r > 0$, by (4.35) and (4.36) applied with $\theta = \varepsilon'$, one has

$$f_{\underline{i}}(B(x, r)) \supset B(f_{\underline{i}}(x_0), \widehat{C}_{\varepsilon'} c_{\underline{i}}(x_0)^{1-\varepsilon'} r) \supset B\left(f_{\underline{i}}(x_0), \frac{\widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}}}{|K|^{-\frac{1+\varepsilon'}{1-\varepsilon'}}} c_{\underline{i}}^{\frac{1+\varepsilon'}{1-\varepsilon'}} r\right).$$

Remember that $\varepsilon' \leq \frac{1}{2}$. Since $\frac{1+\varepsilon'}{1-\varepsilon'} \leq 1 + 4\varepsilon'$,

$$f_{\underline{i}}(B(x, r)) \supset B\left(f_{\underline{i}}(x_0), \widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}} \cdot |K|^{\frac{1+\varepsilon'}{1-\varepsilon'}} c_{\underline{i}}^{1+4\varepsilon'} r\right). \quad (6.10)$$

Writing $\mu_{\underline{i}} = \mu(f_{\underline{i}}^{-1})$, (6.10) yields

$$\begin{aligned} E_{\underline{i}} &:= f_{\underline{i}}(E) \\ &= \left\{ f_{\underline{i}}(x) \in K : \forall r \leq \rho_{\varepsilon}, \mu(B(x, r)) \leq r^{\alpha-\varepsilon} \right\} \\ &\subset \{f_{\underline{i}}(x), x \in K : \forall r \leq \rho_{\varepsilon}, \\ &\mu(f_{\underline{i}}^{-1}(B(f_{\underline{i}}(x_0), \widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}} \cdot |K|^{\frac{1+\varepsilon'}{1-\varepsilon'}} c_{\underline{i}}^{1+4\varepsilon'} r))) \leq \left(\frac{\widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}} \cdot |K|^{\frac{1+\varepsilon'}{1-\varepsilon'}} c_{\underline{i}}^{1+4\varepsilon'} r}{\widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}} \cdot |K|^{\frac{1+\varepsilon'}{1-\varepsilon'}} c_{\underline{i}}^{1+4\varepsilon'}} \right)^{\alpha-\varepsilon} \} \\ &= \left\{ y \in f_{\underline{i}}(K) : \forall r' \leq \widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}} \cdot |K|^{\frac{1+\varepsilon'}{1-\varepsilon'}} c_{\underline{i}}^{1+4\varepsilon'} \rho_{\varepsilon}, \right. \\ &\left. \mu_{\underline{i}}(B(y, r')) \leq \left(\frac{r'}{\widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}} \cdot |K|^{\frac{1+\varepsilon'}{1-\varepsilon'}} c_{\underline{i}}^{1+4\varepsilon'}} \right)^{\alpha-\varepsilon} \right\}. \end{aligned} \quad (6.11)$$

Notice also that

$$\mu_{\underline{i}}(E_{\underline{i}}) = \mu(E) \geq \frac{1}{2}.$$

Let us emphasize that iterating equation (4.22) gives

$$\mu = \sum_{\underline{i}' \in \Lambda^k} p_{\underline{i}'} \mu_{\underline{i}'},$$

which implies that $\mu_{\underline{i}}$ is absolutely continuous with respect to μ (since all $p_{\underline{i}}$'s are strictly positive).

We are now ready to estimate the μ -essential content of a ball B centered in K .

Let us write

$$\gamma(S, \varepsilon') = \widehat{C}_{\varepsilon'}^{-\frac{2}{1-\varepsilon'}} \cdot |K|^{\frac{1+\varepsilon'}{1-\varepsilon'}}. \quad (6.12)$$

Let $B = B(x, r)$ with $x \in K$ and $r \leq c_0 := \min_{z \in K} \min_{1 \leq i \leq m} \|f'_i(z)\|$.

Since $x \in K$, there exists $\underline{i} = (i_1, \dots, i_k) \in \Lambda^*$ such that

- $x \in f_{\underline{i}}(K)$,
- $|f_{\underline{i}}(K)| \leq \frac{1}{3}|B|$,
- $|f_{(i_1, \dots, i_{k-1})}(K)| \geq \frac{1}{3}|B|$.

By (4.36), for any $y \in K$ one has

$$|f_{\underline{i}}(K)| \geq \widehat{C}_{\varepsilon'}^{-1} \|f'_{\underline{i}}(y)\|^{1+\varepsilon'} |K| \quad (6.13)$$

and

$$\begin{aligned} \|f_{\underline{i}}(y)\| &= \|f'_{(i_1, \dots, i_{n-1})}(f_n(x)) \circ f'_{i_n}(x)\| \geq \|f'_{(i_1, \dots, i_{n-1})}(f_n(x))\|_{C_0} \\ &\geq |f_{(i_1, \dots, i_{n-1})}(K)|^{\frac{1}{1-\varepsilon'}} \widehat{C}_{\varepsilon'}^{-\frac{1}{1-\varepsilon'}} \cdot |K|^{\frac{1}{1-\varepsilon'}} C_0. \end{aligned} \quad (6.14)$$

Combining (6.13) and (6.14), one obtains

$$\begin{aligned} c_{\underline{i}} = |f_{\underline{i}}(K)| &\geq \widehat{C}_{\varepsilon'}^{-1-\frac{1+\varepsilon'}{1-\varepsilon'}} |K|^{\frac{-2\varepsilon'}{1-\varepsilon'}} c_0^{1+\varepsilon'} |f_{(i_1, \dots, i_{n-1})}(K)|^{\frac{1+\varepsilon'}{1-\varepsilon'}} \\ &\geq \widehat{C}_{\varepsilon'}^{-1-\frac{1+\varepsilon'}{1-\varepsilon'}} |K|^{\frac{-2\varepsilon'}{1-\varepsilon'}} c_0^{1+\varepsilon'} r^{1+4\varepsilon'}. \end{aligned} \quad (6.15)$$

Note that $E_{\underline{i}} \subset \mathring{B}$.

Consider a set $A \subset B$ verifying $\mu(A) = \mu(B)$. One aims at giving a lower-bound for the Hausdorff content of A which depends only on B , d , ε , ε' and the measure μ .

Consider a sequence of balls $(L_n = B(x_n, \ell_n))_{n \geq 1}$ covering $A \cap E_{\underline{i}}$, such that $\ell_n < \gamma(S, \varepsilon') \rho_{\varepsilon} c_{\underline{i}}^{1+4\varepsilon'}$ and $x_n \in A \cap E_{\underline{i}}$.

Since $\mu_{\underline{i}}$ is absolutely continuous with respect to μ , it holds that $\mu_{\underline{i}}(A) = 1$.

By (6.11) applied to each ball L_n , $n \in \mathbb{N}$, one has $\left(\frac{|L_n|}{\gamma(S, \varepsilon') c_{\underline{i}}^{1+4\varepsilon'}} \right)^{\alpha-\varepsilon} \geq \mu_{\underline{i}}(L_n)$, so that, recalling (6.15),

$$\begin{aligned} \sum_{n \in \mathbb{N}} |L_n|^{\alpha-\varepsilon} &\geq \sum_{n \in \mathbb{N}} \left(\gamma(S, \varepsilon') c_{\underline{i}}^{1+4\varepsilon'} \right)^{\alpha-\varepsilon} \mu_{\underline{i}}(L_n) \geq \left(\gamma(S, \varepsilon') c_{\underline{i}}^{1+4\varepsilon'} \right)^{\alpha-\varepsilon} \mu_{\underline{i}} \left(\bigcup_{n \in \mathbb{N}} L_n \right) \\ &\geq \left(\gamma(S, \varepsilon') c_{\underline{i}}^{1+4\varepsilon'} \right)^{\alpha-\varepsilon} \mu_{\underline{i}}(E_{\underline{i}}) \geq \frac{1}{2} \left(\gamma(S, \varepsilon') c_{\underline{i}}^{1+4\varepsilon'} \right)^{\alpha-\varepsilon} \\ &\geq \kappa(\mu, \varepsilon', \varepsilon) r^{(1+4\varepsilon')^2(\alpha-\varepsilon)} \geq \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon)}, \end{aligned} \quad (6.16)$$

where $\kappa(\mu, \varepsilon', \varepsilon) = \frac{1}{2} \gamma(S, \varepsilon')^{\alpha-\varepsilon} \cdot \left(\widehat{C}_{\varepsilon'}^{-1-\frac{1+\varepsilon'}{1-\varepsilon'}} |K|^{\frac{-2\varepsilon'}{1-\varepsilon'}} c_0^{1+\varepsilon'} \right)^{(1+4\varepsilon')(\alpha-\varepsilon)}$.

This series of inequalities holds for any sequence of balls $(L_n)_{n \in \mathbb{N}}$ with radius less than $\gamma(S, \varepsilon') \rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$ centered in $A \cap E_{\underline{i}}$. One now proves that one can freely remove those constraints on the center and the radius of the balls used to cover $A \cap E_{\underline{i}}$, up to a multiplicative constant.

Consider balls $(L_n = B(x_n, \ell_n))_{n \geq 1}$ covering $A \cap E_{\underline{i}}$ such that $\ell_n < \gamma(S, \varepsilon') \rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$ but x_n does not necessarily belongs to $A \cap E_{\underline{i}}$.

Let $n \in \mathbb{N}$. One constructs recursively a sequence of balls $(L_{n,j})_{1 \leq j \leq J_n}$ such that the following properties hold for any $1 \leq j \leq J_n$:

- $L_{n,j}$ is centered on $A \cap E_{\underline{i}} \cap L_n$;
- $A \cap E_{\underline{i}} \cap L_n \subset \bigcup_{1 \leq j \leq J_n} L_{n,j}$;
- for all $1 \leq j \leq J_n$, $|L_{n,j}| = |L_n|$;
- the center of $L_{n,j}$ does not belong to any $L_{n,j'}$ for $1 \leq j' \neq j \leq J_n$.

To achieve this, simply consider $y_1 \in A \cap E_{\underline{i}} \cap L_n$ and set $L_{1,n} = B(y_1, \ell_n)$. If $A \cap E_{\underline{i}} \cap L_n \not\subset L_{1,n}$, consider $y_2 \in A \cap E_{\underline{i}} \cap L_n \setminus L_{1,n}$ and set $L_{2,n} = B(y_2, \ell_n)$. If $A \cap E_{\underline{i}} \cap L_n \not\subset L_{1,n} \cup L_{2,n}$, consider $y_3 \in A \cap E_{\underline{i}} \cap L_n \setminus L_{1,n} \cup L_{2,n}$ and set $L_{3,n} = B(y_3, \ell_n)$, and so on...

Note that, for any $1 \leq j \leq J_n$, any ball $L_{j,n}$ has radius ℓ_n , intersects L_n (which also has radius ℓ_n) and, because $y_j \notin \bigcup_{1 \leq j' \neq j \leq J_n} L_{j',n}$, it holds that, for any $j \neq j'$, $\frac{1}{3}L_{n,j} \cap \frac{1}{3}L_{n,j'} = \emptyset$. By Lemma 4.1.4, this implies that $J_n \leq Q_{d, \frac{1}{3}}$.

Hence, denoting by $(\tilde{L}_n)_{n \in \mathbb{N}}$ the collection of the corresponding balls centered on $A \cap E_{\underline{i}}$ associated with all the balls L_n , one has by (6.16) applied to $(\tilde{L}_n)_{n \in \mathbb{N}}$:

$$\sum_{n \in \mathbb{N}} |L_n|^{\alpha-\varepsilon} \geq \frac{1}{Q_{d, \frac{1}{3}}} \sum_{n \in \mathbb{N}} |\tilde{L}_n|^{\alpha-\varepsilon} \geq \frac{\kappa(\mu, \varepsilon', \varepsilon)}{Q_{d, \frac{1}{3}}} r^{(1+4\varepsilon')(\alpha-\varepsilon)}.$$

Remark also that any ball of radius smaller than $c_{\underline{i}}$ can be covered by at most $\left(\frac{2c_{\underline{i}}^{-4\varepsilon'}}{\gamma(S, \varepsilon') \rho_\varepsilon} \right)^d$ balls of radius $\gamma(S, \varepsilon') \rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$. Moreover, by (6.15),

$$c_{\underline{i}}^{-4\varepsilon'} \leq \left(\widehat{C}_{\varepsilon'}^{-1-\frac{1+\varepsilon'}{1-\varepsilon'}} |K|^{\frac{-2\varepsilon'}{1-\varepsilon'}} c_0^{1+\varepsilon'} \right)^{-4\varepsilon'} r^{-4\varepsilon' \cdot (1+4\varepsilon')}.$$

Setting

$$\widehat{\kappa}(\mu, \varepsilon, \varepsilon', d) = \left(\frac{2 \left(\widehat{C}_{\varepsilon'}^{-1-\frac{1+\varepsilon'}{1-\varepsilon'}} |K|^{\frac{-2\varepsilon'}{1-\varepsilon'}} c_0^{1+\varepsilon'} \right)^{-4\varepsilon'}}{\gamma(S, \varepsilon') \rho_\varepsilon} \right)^d,$$

any ball of radius less than $c_{\underline{i}}$ can be covered by less than $\widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)r^{-4d\varepsilon' \cdot (1+4\varepsilon')}$ balls of radius less than $\gamma(S, \varepsilon')\rho_\varepsilon c_{\underline{i}}^{1+4\varepsilon'}$.

This proves that, for any sequence of balls \widehat{L}_n with $|\widehat{L}_n| \leq c_{\underline{i}}$ covering $A \cap E_{\underline{i}}$, recalling (6.16), it holds that

$$\sum_{n \in \mathbb{N}} |\widehat{L}_n|^{\alpha-\varepsilon} \geq Q_{d, \frac{1}{3}}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} r^{4d\varepsilon' \cdot (1+4\varepsilon')} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon)} \quad (6.17)$$

$$\geq Q_{d, \frac{1}{3}}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon) + 4d\varepsilon' \cdot (1+4\varepsilon')}. \quad (6.18)$$

Recalling that $|E_{\underline{i}}| \leq c_{\underline{i}}$ and Definition 4.7, since (6.17) is valid for any covering $(\widehat{L}_n)_{n \in \mathbb{N}}$ of $A \cap E_{\underline{i}}$ with $|\widehat{L}_n| \leq c_{\underline{i}}$, one has

$$\begin{aligned} |B|^{\alpha-\varepsilon} &\geq \mathcal{H}_{\infty}^{\alpha-\varepsilon}(A) \geq \mathcal{H}_{\infty}^{\alpha-\varepsilon}(A \cap E_{\underline{i}}) \\ &\geq Q_{d, \frac{1}{3}}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon) + 4d\varepsilon' \cdot (1+4\varepsilon')}. \end{aligned} \quad (6.19)$$

Taking the inf over all the set $A \subset B$ satisfying $\mu(A) = \mu(B)$, one obtains

$$|B|^{\alpha-\varepsilon} \geq \mathcal{H}_{\infty}^{\mu, s}(B) \geq Q_{d, \frac{1}{3}}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon', d)^{-1} \kappa(\mu, \varepsilon', \varepsilon) r^{(1+16\varepsilon')(\alpha-\varepsilon) + 4d\varepsilon' \cdot (1+4\varepsilon')}.$$

The results stands for balls of diameter less than c_0 .

Set

$$\varepsilon'_0 = 16\varepsilon'(\alpha - \varepsilon) + 4d\varepsilon' \cdot (1 + 4\varepsilon')$$

and write

$$\gamma(d, \mu, \varepsilon, \varepsilon'_0) = c_0^{\alpha-\varepsilon+\varepsilon'_0} Q_{d, \frac{1}{3}}^{-1} \widehat{\kappa}(\mu, \varepsilon, \varepsilon'_0, d)^{-1} \kappa(\mu, \varepsilon'_0, \varepsilon).$$

For any ball of radius less than 1 centered on K , one has

$$|B|^{\alpha-\varepsilon} \geq \mathcal{H}_{\infty}^{\mu, \alpha-\varepsilon}(B) \geq \gamma(d, \mu, \varepsilon, \varepsilon'_0) r^{\alpha-\varepsilon+\varepsilon'_0}.$$

□

The estimates of Theorem 6.2.4 are now established in the case of general open sets.

Recall that by item (5) of Proposition 6.2.2, for any $s > \dim(\mu)$ and any set E , $\mathcal{H}_{\infty}^{\mu, s}(E) = 0$.

Let us fix $s < \dim(\mu)$, $\varepsilon' > 0$ and set $\varepsilon' = \min \left\{ \frac{\dim(\mu)-s}{2}, \frac{1}{2} \right\} > 0$.

Since $K \cap \Omega \subset \Omega$ and $\mu(K \cap \Omega) = \mu(\Omega)$, it holds that

$$\mathcal{H}_{\infty}^{\mu,s}(\Omega) \leq \mathcal{H}_{\infty}^s(\Omega \cap K).$$

It remains to show that there exists a constant $c(d, \mu, s, \varepsilon')$ such that for any open set Ω , the converse inequality

$$c(d, \mu, s, \varepsilon') \mathcal{H}_{\infty}^{s+\varepsilon'}(\Omega \cap K) \leq \mathcal{H}_{\infty}^{\mu,s}(\Omega)$$

holds.

Let $E \subset \Omega$ be a Borel set such that $\mu(E) = \mu(\Omega)$ and

$$\mathcal{H}_{\infty}^s(E) \leq 2\mathcal{H}_{\infty}^{\mu,s}(\Omega). \quad (6.20)$$

Let $\{L_n\}_{n \in \mathbb{N}}$ be a covering of E by balls verifying

$$\mathcal{H}_{\infty}^s(L) \leq \sum_{n \geq 0} |L_n|^s \leq 2\mathcal{H}_{\infty}^s(E). \quad (6.21)$$

The covering $(L_n)_{n \in \mathbb{N}}$ will be modified into a covering $(\tilde{L}_n)_{n \in \mathbb{N}}$ verifying the following properties:

- $K \cap \Omega \subset \bigcup_{n \in \mathbb{N}} \tilde{L}_n$,
- $\bigcup_{n \in \mathbb{N}} L_n \subset \bigcup_{n \in \mathbb{N}} \tilde{L}_n$
- $$\sum_{n \geq 0} |\tilde{L}_n|^{s+\varepsilon'} \leq 8 \cdot 2^{s+\varepsilon'} \frac{Q_{d,1}^2}{\gamma(d, \mu, \varepsilon, \varepsilon')} \sum_{n \geq 0} |L_n|^s,$$

where $Q_{d,1}$ and $\gamma(d, \mu, \varepsilon, \varepsilon')$ are the constants arising from Proposition 4.1.2 applied with $v = 1$ and Proposition 6.2.6.

Last item together with (6.20) and (6.21) then immediately imply that

$$\frac{\gamma(d, \mu, \varepsilon, \varepsilon')}{8 \cdot 2^{s+\varepsilon'} Q_{d,1}^2} \mathcal{H}_{\infty}^{s+\varepsilon'}(K \cap \Omega) \leq \mathcal{H}_{\infty}^{\mu,s}(\Omega).$$

Setting $c(d, \mu, \varepsilon, \varepsilon') = \frac{\gamma(d, \mu, \varepsilon, \varepsilon')}{8 \cdot 2^{s+\varepsilon'} Q_{d,1}^2}$ will then conclude the proof.

Let us start the construction of the sequence $(\tilde{L}_n)_{n \in \mathbb{N}}$.

Let $\Delta = (K \setminus \bigcup_{n \in \mathbb{N}} B_n) \cap \Omega$. For every $x \in \Delta$, fix $0 < r_x \leq 1$ such that $B(x, r_x) \subset \Omega$. One of the following alternatives must occur:

1. for any ball L_n such that $L_n \cap B(x, r_x) \neq \emptyset$, $|L_n| \leq r_x$, or
2. there exists $n_x \in \mathbb{N}$ such that $L_{n_x} \cap B(x, r_x) \neq \emptyset$ and $|L_{n_x}| \geq r_x$.

Consider the set S_1 of points of X for which the first alternative holds.

By Proposition 4.1.2 applied with $v = 1$, it is possible to extract from the covering of S_1 , $\{B(x, r_x), x \in S_1\}$, $Q_{d,1}$ families of pairwise disjoint balls, $\mathcal{F}_1, \dots, \mathcal{F}_{Q_{d,1}}$ such that

$$S_1 \subset \bigcup_{1 \leq i \leq Q_{d,1}} \bigcup_{L \in \mathcal{F}_i} L.$$

Now, any ball L_n intersecting a ball $L \in \bigcup_{1 \leq i \leq Q_{d,1}} \mathcal{F}_i$ must satisfy $|L_n| \leq L$. In particular, since for any $1 \leq i \leq Q_{d,1}$, the balls of \mathcal{F}_i are pairwise disjoint, applying Lemma 4.1.4 to the ball of \mathcal{F}_i intersecting L , we get that the ball L_n intersects at most $Q_{d,1}$ balls of \mathcal{F}_i , hence at most $Q_{d,1}^2$ balls of $\bigcup_{1 \leq i \leq Q_{d,1}} \mathcal{F}_i$.

Let $L \in \bigcup_{1 \leq i \leq Q_{d,1}} \mathcal{F}_i$. One aims at replacing all the balls L_n intersecting L by the ball $2L$.

For any $1 \leq i \leq Q_{d,1}$ and any ball $L \in \mathcal{F}_i$, denote by \mathcal{G}_L the set of balls L_n intersecting L . Since $E \subset \bigcup_{n \in \mathbb{N}} L_n$ and $\mu(E) = \mu(\Omega)$, one has $E \cap L \subset \bigcup_{B \in \mathcal{G}_L} B$ and $\mu(E \cap L) = \mu(L)$. By Definition 6.2.1 and Proposition 6.2.6, this implies that

$$\gamma(d, \mu, \varepsilon, \varepsilon') |L|^{s+\varepsilon'} \leq \mathcal{H}_\infty^{\mu, s}(L) \leq \sum_{B \in \mathcal{G}_L} \mathcal{H}_\infty^{\mu, s}(B) \leq \sum_{B \in \mathcal{G}_L} |B|^s. \quad (6.22)$$

Replace the balls of \mathcal{G}_L by the ball $\widehat{L} = 2L$ (recall that $\bigcup_{B \in \mathcal{G}_L} B \subset 2L$). The new sequence of balls so obtained by the previous construction applied to all the balls $L \in \bigcup_{1 \leq i \leq Q_{d,1}} \mathcal{F}_i$ is denoted by $(\widehat{L}_k)_{1 \leq k \leq K}$, where $0 \leq K \leq +\infty$.

It follows from the construction and (6.22) that $S_1 \subset \bigcup_{1 \leq k \leq K} \widehat{L}_k$ and

$$\sum_{1 \leq k \leq K} \left(\frac{|\widehat{L}_k|}{2} \right)^{s+\varepsilon'} \leq \frac{Q_{d,1}^2}{\gamma(d, \mu, \varepsilon, \varepsilon')} \sum_{n \geq 0} |L_n|^s. \quad (6.23)$$

On the other hand, since for any $x \in S_2 = \Delta \setminus S_1$, there exists $n_x \in \mathbb{N}$ such that $L_{n_x} \cap B(x, r_x) \neq \emptyset$ and $r_x \leq |L_{n_x}|$, one has $S_2 \subset \bigcup_{n \in \mathbb{N}} 2L_n$, so that

$$\left(\bigcup_{n \in \mathbb{N}} L_n \right) \cup \left(K \cap \Omega \setminus \bigcup_{n \in \mathbb{N}} L_n \right) \subset \left(\bigcup_{1 \leq k \leq K} \widehat{L}_k \right) \cup \left(\bigcup_{n \in \mathbb{N}} 2L_n \right).$$

Putting the elements of $(\widehat{L}_k)_{1 \leq k \leq K}$ and $(2L_n)_{n \geq 0}$ in a single sequence $(\widehat{L}_n)_{n \geq 0}$, writing $(\widetilde{L}_n := 2\widehat{L}_n)_{n \in \mathbb{N}}$, by construction, $K \cap \Omega \subset \bigcup_{n \in \mathbb{N}} \widetilde{L}_n$ and due to (6.23):

$$\begin{aligned}
\mathcal{H}_\infty^{s+\varepsilon'}(K \cap \Omega) &\leq \sum_{n \geq 0} |\tilde{L}_n|^{s+\varepsilon'} \leq 2^{s+\varepsilon'} \left(\frac{Q_{d,1}^2}{\gamma(d, \mu, \varepsilon, \varepsilon')} + 1 \right) \sum_{n \geq 0} |L_n|^s \\
&\leq 8 \cdot 2^{s+\varepsilon'} \frac{Q_{d,1}^2}{\gamma(d, \mu, \varepsilon, \varepsilon')} \mathcal{H}_\infty^{\mu, s}(\Omega).
\end{aligned}$$

The proof is concluded now by setting

$$c(d, \mu, s, \varepsilon') = \frac{\gamma(d, \mu, \dim(\mu) - s, \varepsilon')}{Q_{d,1}^2 8 \cdot 2^{s+\varepsilon'}}.$$

□

Remark 6.2.7. 1. The proof of Proposition 6.2.6 only relies on the absolute continuity, for any $\underline{i} \in \Lambda^*$, of $\mu(f_{\underline{i}}^{-1})$ with respect to the weakly conformal measure μ .

2. The part of the proof of Theorem 6.8 which handles the case of open sets only relies on the fact that there exists $\gamma(d, \mu, \varepsilon, \varepsilon')$ such that for any $x \in K$, for any $\rho > 0$, there exists $0 < r_x \leq \rho$ so that, writing $B = B(x, r_x)$,

$$\gamma(d, \mu, \varepsilon, \varepsilon') |B|^{\dim(\mu) - \varepsilon + \varepsilon'} \leq \mathcal{H}_\infty^{\mu, \dim(\mu) - \varepsilon}(\overset{\circ}{B}) \leq \mathcal{H}_\infty^{\mu, \dim(\mu) - \varepsilon}(B) \leq |B|^{\dim(\mu) - \varepsilon}. \quad (6.24)$$

In particular Theorem 6.2.4 actually holds for any measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ for which $\text{supp}(\mu) \subset K$ and for any $\underline{i} \in \Lambda^*$, $\mu(f_{\underline{i}}^{-1})$ is absolutely continuous with respect to μ (so that it holds for quasi-Bernoulli measures for instance).

- It is easily verified that the estimates of Proposition 6.2.6 holds in particular if, for $s \geq 0$, there exists a constant $C > 0$ such that for any $x \in \text{supp}(\mu)$, any $0 < r < R$, $\frac{\mu(B(x, r))}{\mu(B(x, R))} \leq C \cdot \left(\frac{r}{R}\right)^s$. This condition is naturally linked to the lower Assouad dimension $\dim_L(\mu)$ of μ defined as [40]

$$\dim_L(\mu) = \inf \left\{ s \geq 0 : \forall x \in \text{supp}(\mu), \forall 0 < r < R, \frac{\mu(B(x, r))}{\mu(B(x, R))} \leq C \left(\frac{r}{R}\right)^s \right\}. \quad (6.25)$$

More precisely, the estimates of Proposition 6.2.6 and Theorem 6.2.4 holds for any $s < \dim_L(\mu)$.

Chapter 7

Heterogeneous ubiquity theorem

In this Chapter, the main ubiquity theorem of this manuscript is proved. Section 7.1 states Theorem 7.1.2. Section 7.2 is dedicated to the proof of Theorem 7.1.2. Finally in Section 7.3, Theorem 7.1.2 is applied to obtain mass transference principles for weakly conformal measures (with no separation conditions).

7.1 A general heterogeneous ubiquity theorem

The s -dimensional μ -essential Hausdorff content is now used to associate a critical exponent to any sequence of open sets $(U_n)_{n \in \mathbb{N}}$ such that $U_n \subset B_n$ for all $n \in \mathbb{N}$. This exponent is involved in our lower bound estimate of $\dim_H(\limsup_{n \rightarrow +\infty} U_n)$.

Definition 7.1.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. If B and U are Borel subsets of \mathbb{R}^d , the μ -critical exponent of (B, U) is defined as*

$$s_\mu(B, U) = \sup \{s \geq 0 : \mathcal{H}_\infty^{\mu, s}(U) \geq \mu(B)\}. \quad (7.1)$$

Let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a sequence of closed balls, $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ a sequence of Borel subsets of \mathbb{R}^d , and $s \geq 0$.

Let

$$\mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s) = \{n \in \mathbb{N} : s_\mu(B_n, U_n) \geq s\}. \quad (7.2)$$

Then, define the μ -critical exponent of $(\mathcal{B}, \mathcal{U})$ as

$$s(\mu, \mathcal{B}, \mathcal{U}) = \sup \{s \geq 0 : (B_n)_{n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s)} \text{ is } \mu\text{-a. c.}\}. \quad (7.3)$$

It is worth noting that, for $s' \leq s$, one has $\mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s) \subset \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s')$.

The main result of this paper is the following.

Theorem 7.1.2. *Let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a sequence of closed balls of \mathbb{R}^d such that $|B_n| \rightarrow 0$ and $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ a sequence of open sets such that $U_n \subset B_n$ for all $n \in \mathbb{N}$.*

Then, for every $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that $\min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\} > 0$ there exists a gauge function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\lim_{r \rightarrow 0^+} \frac{\log \zeta(r)}{\log r} = \min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\}$ and

$$\mathcal{H}^\zeta(\limsup_{n \rightarrow +\infty} U_n) > 0.$$

In particular, for every $\mu \in \mathcal{M}(\mathbb{R}^d)$, one has

$$\dim_H \left(\limsup_{n \rightarrow +\infty} U_n \right) \geq \min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\}. \quad (7.4)$$

Remark 7.1.3. (1) *It is easily verified that the lower-bound in Theorem 7.1.2 equals $-\infty$ if the sequence $(B_n)_{n \in \mathbb{N}}$ is not assumed to be μ -a.c. Consequently, for the previous result to give non trivial information one has to assume that $(B_n)_{n \in \mathbb{N}}$ is μ -a.c. The question is then to give more explicit estimates of $s(\mu, \mathcal{B}, \mathcal{U})$ depending on the specificities of $(\mu, \mathcal{B}, \mathcal{U})$.*

(2) *It is proved in Section 7.2.2 that $s(\mu, \mathcal{B}, \mathcal{U}) \leq \overline{\dim}_H(\mu)$. This implies that for exact dimensional measures, $\min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\} = s(\mu, \mathcal{B}, \mathcal{U})$.*

(3) *The case where μ satisfies $\min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\} = 0$ could also be treated, but although (7.4) is still obviously true, some distinction should further be made when investigating the existence of the gauge function. If $\mathcal{H}_\infty^{\mu, s}(U_n) = 0$ for any $n \in \mathbb{N}$, the set $\limsup_{n \rightarrow +\infty} U_n$ could, for instance, be empty. On the other hand, if $(B_n)_{n \in \mathbb{N}}$ is μ -a.c and $s_\mu(B_n, U_n) > 0$ for any $n \in \mathbb{N}$, a gauge function can be constructed in a similar way than in the proof of Theorem 7.1.2. However the existence of a gauge function in the case $\min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\} = 0$ is of little interest for practical applications and is not treated in this manuscript.*

A quite direct, but useful, corollary of Theorem 7.1.2 is the following:

Corollary 7.1.4. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a μ -a.c. sequence of closed balls of \mathbb{R}^d . Let $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ be a sequence of open sets such that $U_n \subset B_n$ for all $n \in \mathbb{N}$, and $0 \leq s \leq \underline{\dim}_H(\mu)$. If $\limsup_{n \rightarrow +\infty} \frac{\log \mathcal{H}_\infty^{\mu, s}(U_n)}{\log \mu(B_n)} \leq 1$, then $s(\mu, \mathcal{B}, \mathcal{U}) \geq s$, so that*

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) \geq s.$$

In the classical case where the sets U_n are shrunk balls of the form B_n^δ

(with $\delta \geq 1$), it is convenient to consider the following quantity:

Definition 7.1.5. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\varepsilon > 0$ and $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a sequence of balls of \mathbb{R}^d . For every $\delta \geq 1$, set

$$t(\mu, \delta, \varepsilon, \mathcal{B}) = \limsup_{n \rightarrow +\infty} \frac{\log(\mathcal{H}_{\infty}^{\mu, \dim_H(\mu) - \varepsilon}(\mathring{B}_n^{\delta}))}{\log(|B_n^{\delta}|)}. \quad (7.5)$$

Then the (μ, δ) -exponent of the sequence \mathcal{B} is defined as

$$t(\mu, \delta, \mathcal{B}) = \lim_{\varepsilon \rightarrow 0} t(\mu, \delta, \varepsilon, \mathcal{B}). \quad (7.6)$$

It follows from the definitions that $t(\mu, \delta, \mathcal{B})$ exists as a limit, since $\varepsilon \mapsto t(\mu, \delta, \varepsilon, \mathcal{B})$ is monotonic. Moreover, one has $\dim_H(\mu) \leq t(\mu, \delta, \mathcal{B})$ (see the proof of Corollary 7.1.6).

Next result provides a more explicit lower bound estimate of the Hausdorff dimension of the limsup of δ -contracted balls; it is a consequence of Corollary 7.1.4.

Corollary 7.1.6. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ a μ -a.c sequence of closed balls of \mathbb{R}^d . Suppose that $\dim_H(\mu) > 0$. For every $\delta \geq 1$, setting

$$s_{\delta} = \frac{\dim_H(\mu)}{\delta} \cdot \frac{\dim_H(\mu)}{t(\mu, \delta, \mathcal{B})},$$

one has

$$s(\mu, (B_n)_{n \in \mathbb{N}}, (\mathring{B}_n^{\delta})_{n \in \mathbb{N}}) \geq s_{\delta},$$

hence

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^{\delta}) \geq \dim_H(\limsup_{n \rightarrow +\infty} \mathring{B}_n^{\delta}) \geq s_{\delta}.$$

7.2 Proof of Theorem 7.1.2 and Corollary 7.1.6

7.2.0.1 Preliminary facts

We gather in this subsection a series of results on which we will base the proof of Theorem 7.1.2.

The following lemma, which is a version of Besicovitch covering Lemma, as well as the subsequent one, both established in [24], will be used several times.

The following lemma will also be useful later.

Lemma 7.2.1 ([24]). Let \mathcal{L} be a family of pairwise disjoint balls satisfying $\sup_{L \in \mathcal{L}} |L| < +\infty$. Then, for any $v \geq 1$, there exists sub-families $\mathcal{L}_1, \dots, \mathcal{L}_{Q_{d,v}}$

(where $Q_{d,v}$ is the constant of the same name in Proposition 4.1.2) of \mathcal{L} such that $\mathcal{L} = \bigcup_{1 \leq i \leq Q_{d,v}} \mathcal{L}_i$ and for any $L \neq L' \in \mathcal{L}_i$, $vL \cap vL' = \emptyset$.

For $s \geq 0$ and $E \subset \mathbb{R}^d$, a bounded subset such that $\mathcal{H}_\infty^s(E) > 0$, m_E^s will always denote a measure given by Proposition 4.2.6, associated with a (fixed) constant κ_d .

In the next two lemmas, the choice of the interval $[5, 6]$ is convenient to take enough space between the shrunk balls involved in the construction elaborated in Section 7.2.1.

Lemma 7.2.2. *Let $t \in (5, 6)$, $m \in \mathcal{M}(\mathbb{R}^d)$, and $\varepsilon > 0$. Let $x \in \mathbb{R}^d$ be such that $\overline{\dim}(m, x) \leq \beta$. Let $C_{\beta, \varepsilon} = \frac{1}{2}6^{-\frac{\beta}{2\varepsilon}}$. There exists an integer n_x such that for every $n \geq n_x$,*

$$\frac{\#\{0 \leq k \leq n-1 : m(B(x, t^{-k-1})) \geq C_{\beta, \varepsilon} m(B(x, t^{-k}))\}}{n} \geq 1 - \varepsilon. \quad (7.7)$$

Previous lemma is a slight extension of result by Käenmäki [21, Lemma 2.2], which shows such a property at m -almost every point (where one has necessarily $\overline{\dim}(m, x) \leq d$), and uses t integer (a choice that we could make).

Thus, points with a given local dimension with respect to a measure m are for most scales “locally doubling”.

Proof. Observe first that if for a constant $0 < C \leq 1$ and some integer $n \in \mathbb{N}$ one has

$$\frac{\#\{1 \leq k \leq n : m(B(x, t^{-k-1})) \geq C m(B(x, t^{-k}))\}}{n} \leq 1 - \varepsilon,$$

then there necessarily exists $N = \lfloor (n-1)\varepsilon \rfloor$ integers $0 < k_1 < \dots < k_N < n$ such that for every $1 \leq i \leq N$, $m(B(x, t^{-k_i-1})) \leq C m(B(x, t^{-k_i}))$.

In particular, writing $k_{N+1} = n$ and $k_0 = 0$, this implies that

$$\begin{aligned} m(B(x, t^{-n})) &= \prod_{i=0}^N \frac{m(B(x, t^{-k_{i+1}}))}{m(B(x, t^{-k_i}))} \leq \prod_{i=0}^N \frac{m(B(x, t^{-k_{i+1}}))}{m(B(x, t^{-k_i-1}))} \leq C^N \\ &\leq C^{(n-1)\varepsilon} \leq C^{n\varepsilon/2} = (t^{-n})^{\varepsilon \frac{-\log(C)}{2\log(t)}}. \end{aligned}$$

The inequality $C^{(n-1)\varepsilon} \leq C^{n\varepsilon/2}$ occurs when n is large enough. Recalling that $\overline{\dim}(m, x) \leq \beta$, if this happens for infinitely many n , one should have

$$\beta \geq \limsup_{r \rightarrow 0^+} \frac{\log m(B(x, r))}{\log r} \geq \varepsilon \frac{-\log(C)}{2\log(t)},$$

which is equivalent to $C \geq t^{-\frac{\beta}{2\varepsilon}}$.

Setting $C_{\varepsilon, \beta} = \frac{1}{2}6^{-\frac{\beta}{2\varepsilon}}$, one concludes that there exists n_x such that for every $n \geq n_x$, one necessarily has

$$\frac{\#\{0 \leq k \leq n-1 : m(B(x, t^{-k-1})) \geq C_{\varepsilon, \beta} m(B(x, t^{-k}))\}}{n} \geq 1 - \varepsilon,$$

hence the result. \square

Lemma 7.2.3. *Let m and μ be two elements of $\mathcal{M}(\mathbb{R}^d)$, $\beta \geq 0$ and $\varepsilon > 0$. For every $x \in \mathbb{R}^d$ verifying $\overline{\dim}(m, x) \leq \beta$, there exists $\rho_x > 0$ and $t_x \in (5, 6)$ so that for all $0 < r \leq \rho_x$ there exists $r \leq r' \leq r^{1-\varepsilon}$ such that*

$$m(B(x, r'/t_x)) \geq C_{\beta, \frac{\varepsilon}{2}} m(B(x, r')) \text{ and } \mu(\partial B(x, r'/t_x)) = 0. \quad (7.8)$$

Proof. Consider $x \in \mathbb{R}^d$ such that $\overline{\dim}(m, x) \leq \beta$.

We apply Lemma 7.2.2 to x and the measure m , and for an arbitrary $t \in [5, 6]$ and $\varepsilon' = \frac{\varepsilon}{2}$: for $n \geq n_x$, there must be an integer n' such that $n(1 - \varepsilon) \leq n' \leq n$ and $m(B(x, t^{-n'-1})) \geq C_{\beta, \frac{\varepsilon}{2}} m(B(x, t^{-n'}))$.

Let $\rho_x = \min \{t^{-n_x-1}, t^{-\frac{1}{\varepsilon}}\}$. For $r \in (0, \rho_x]$, let n be the integer such that $t^{-n-1} < r \leq t^{-n}$. The previous claim yields an integer $n' \in [n(1 - \frac{\varepsilon}{2}), n]$ such that $m(B(x, t^{-n'})) \geq C_{\beta, \frac{\varepsilon}{2}} m(B(x, t^{-n'+1}))$. Also,

$$r \leq r' = t^{-n'+1} \leq t^{1-(1-\frac{\varepsilon}{2})n} = t^2 \cdot t^{-n-1} \cdot t^{\frac{\varepsilon}{2}n} \leq t^2 \cdot r \cdot r^{-\frac{\varepsilon}{2}} \leq r^{1-\varepsilon}.$$

Consequently,

$$m(B(x, r'/t)) \geq C_{\beta, \frac{\varepsilon}{2}} m(B(x, r')).$$

The desired conclusion holds if we choose $t_x \in (5, t)$ such that $\mu(\partial B(x, r'/t_x)) = 0$. \square

The previous lemma will be used in the case $\beta = d$ in our proof the main theorem (see step 2 of the construction in Section 7.2.1).

Next, we introduce some sets associated to a given element of $\mathcal{M}(\mathbb{R}^d)$, which will play a natural role in our construction.

Definition 7.2.4. *Let $\beta \geq \alpha \geq 0$ be real numbers, $m \in \mathcal{M}(\mathbb{R}^d)$, and $\varepsilon, \rho > 0$ two positive real numbers. Then define*

$$\widetilde{E}_m^{[\alpha, \beta], \rho, \varepsilon} = \left\{ x \in \mathbb{R}^d : \underline{\dim}(m, x) \in [\alpha, \beta] \text{ and } \forall r \leq \rho, m(B(x, r)) \leq r^{\underline{\dim}(m, x) - \varepsilon} \right\} \quad (7.9)$$

and

$$E_m^{[\alpha, \beta], \rho, \varepsilon} = \left\{ x \in \widetilde{E}_m^{[\alpha, \beta], \rho, \varepsilon} : \forall r \leq \rho, \frac{3}{4} m(B(x, r)) \leq m(B(x, r) \cap \widetilde{E}_m^{[\alpha, \beta], \rho, \varepsilon}) \right\}. \quad (7.10)$$

Notice that, for every $0 < \rho < \rho'$, one has $E_m^{[\alpha, \beta], \rho', \varepsilon} \subset E_m^{[\alpha, \beta], \rho, \varepsilon}$.

Definition 7.2.5. Let $\beta \geq \alpha \geq 0$ be real numbers, $m \in \mathcal{M}(\mathbb{R}^d)$, and $\varepsilon > 0$. Define

$$E_m^{[\alpha, \beta], \varepsilon} = \bigcup_{n \geq 1} E_m^{[\alpha, \beta], \frac{1}{n}, \varepsilon}. \quad (7.11)$$

Proposition 7.2.6. For every $m \in \mathcal{M}(\mathbb{R}^d)$, every $\beta \geq \alpha \geq 0$ and $\varepsilon > 0$,

$$m(E_m^{[\alpha, \beta], \varepsilon}) = m(\{x : \underline{\dim}(m, x) \in [\alpha, \beta]\}). \quad (7.12)$$

Notice that, for every $0 < \rho' < \rho$, one has $E_m^{[\alpha, \beta], \rho, \varepsilon} \subset E_m^{[\alpha, \beta], \rho', \varepsilon}$.

These sets play a key role in the proofs of Theorem 7.1.2.

Proof. Note that it is clear from Definition 4.2.12 that

$$\{x : \underline{\dim}(m, x) \in [\alpha, \beta]\} = \bigcup_{\rho > 0} \widetilde{E}_m^{[\alpha, \beta], \rho, \varepsilon}.$$

Let $\varepsilon' > 0$. By Definition 4.2.12, there exists $\rho_{\varepsilon'}$ small enough so that

$$m(\widetilde{E}_m^{[\alpha, \beta], \rho_{\varepsilon'}, \varepsilon}) \geq (1 - \varepsilon')m(\{x : \underline{\dim}(m, x) \in [\alpha, \beta]\}). \quad (7.13)$$

By Lemma 4.1.5 (and the notations therein) applied to $\widetilde{E}_m^{[\alpha, \beta], \rho_{\varepsilon'}, \varepsilon}$, there exists $\tilde{\rho}_{\varepsilon'}$ such that

$$m(\widetilde{E}_m^{[\alpha, \beta], \rho_{\varepsilon'}, \varepsilon}(\tilde{\rho}_{\varepsilon'})) \geq (1 - \varepsilon')m(\widetilde{E}_m^{[\alpha, \beta], \rho_{\varepsilon'}, \varepsilon}). \quad (7.14)$$

Finally for $\rho = \min\{\rho_{\varepsilon'}, \tilde{\rho}_{\varepsilon'}\}$, by Definition 7.2.4 and (4.3), one has $(\widetilde{E}_m^{[\alpha, \beta], \rho_{\varepsilon'}, \varepsilon})_{\tilde{\rho}_{\varepsilon'}} \subset E_m^{[\alpha, \beta], \rho, \varepsilon}$, so that, by (7.13) and (7.14)

$$\begin{aligned} m(E_m^{[\alpha, \beta], \rho, \varepsilon}) &\geq m((\widetilde{E}_m^{[\alpha, \beta], \rho_{\varepsilon'}, \varepsilon}(\tilde{\rho}_{\varepsilon'})) \geq (1 - \varepsilon')m(E_m^{[\alpha, \beta], \varepsilon}) \\ &\geq (1 - \varepsilon')^2 m(\{x : \underline{\dim}(m, x) \in [\alpha, \beta]\}). \end{aligned}$$

In particular

$$m(\{x : \underline{\dim}(m, x) \in [\alpha, \beta]\}) \geq m(E_m^{[\alpha, \beta], \varepsilon}) \geq (1 - \varepsilon')^2 m(\{x : \underline{\dim}(m, x) \in [\alpha, \beta]\}).$$

Letting $\varepsilon' \rightarrow 0$ proves the result. \square

Corollary 7.2.7. For every $m \in \mathcal{M}(\mathbb{R}^d)$, for $\alpha = \underline{\dim}_H(m)$ and $\beta = \overline{\dim}_H(m)$, for any $\varepsilon > 0$, one has

$$m(E_m^{[\alpha, \beta], \varepsilon}) = 1. \quad (7.15)$$

7.2.1 Construction of the Cantor set and the measure

Recall that μ is a probability measure on \mathbb{R}^d , and that $\mathcal{B} = (B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ is a μ -a.c sequence of balls of \mathbb{R}^d with $\lim_{n \rightarrow +\infty} r_n = 0$. Fix $\mathcal{U} = (U_n)_{n \in \mathbb{N}}$ a sequence of open sets satisfying $U_n \subset B_n$ for every $n \in \mathbb{N}$.

Set $\alpha = \underline{\dim}_H(\mu)$, and assume that $\min \{s(\mu, \mathcal{B}, \mathcal{U}), \alpha\} > 0$.

Our goal is to construct a gauge function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\lim_{r \rightarrow 0^+} \frac{\log \zeta(r)}{\log r} = \min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\}$$

as well as $\eta \in \mathcal{M}(\mathbb{R}^d)$ supported on $\limsup_{n \rightarrow \infty} U_n$ such that for all $r \in (0, 1]$ and $x \in \mathbb{R}^d$ one has

$$\eta(B(x, r)) \leq \zeta(2r).$$

Let $(\varepsilon_k)_{k \in \mathbb{N}}$ be a sequence decreasing to 0 and such that $\varepsilon_1 < s(\mu, \mathcal{B}, \mathcal{U})$. For $k \geq 0$, set

$$s_k = \min \{s(\mu, \mathcal{B}, \mathcal{U}), \alpha\} - \varepsilon_k. \quad (7.16)$$

Along the construction of ζ , we only use that $s_k < s(\mu, \mathcal{B}, \mathcal{U})$ and the fact that $s_k < \alpha$ is used at the end of our analysis (see equation (7.52)).

Step 1

We need the following lemma.

Using Lemma 6.1.3 with, $(B_n)_{n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_1)}$ (which is μ -a.c since $s_1 < s(\mu, \mathcal{B}, \mathcal{U})$), $g = 0$ and $\Omega = \mathbb{R}^d$, one finds integers N_1 and $n_1 < \dots < n_{N_1} \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_1)$ such that :

- (i) : $\forall 1 \leq i \leq N_1, B_{n_i} \cap B_{n_j} = \emptyset$,
- (ii) : $\mu(\bigcup_{1 \leq i \leq N_1} B_{n_i}) \geq \frac{1}{2}$.

By Lemma 7.2.1 applied to $\{B_{n_i}\}_{1 \leq i \leq N_1}$ and $v = 4$, the balls $\{B_{n_i}\}_{1 \leq i \leq N_1}$ can be sorted in $Q_{d,4}$ families of balls $\mathcal{L}_1, \dots, \mathcal{L}_{Q_{d,4}}$ such that

- for any $1 \leq i \leq Q_{d,4}$, any $L \neq L' \in \mathcal{L}_i$, $4L \cap 4L' = \emptyset$,
- $\bigcup_{1 \leq i \leq Q_{d,4}} \mathcal{L}_i = \{B_{n_i}\}_{1 \leq i \leq N_1}$.

At least one of these families, \mathcal{L}_{i_0} , must satisfy

$$\mu\left(\bigcup_{L \in \mathcal{L}_{i_0}} L\right) \geq \frac{1}{2Q_{d,4}}.$$

In particular, if one must rename the balls of the family \mathcal{L}_{i_0} , we can assume that the family $\{B_{n_i}\}_{1 \leq i \leq N_1}$ satisfies

(i') : for any $1 \leq i < j \leq N_1$, $4B_{n_i} \cap 4B_{n_j} = \emptyset$

(ii') : and

$$\mu\left(\bigcup_{1 \leq i \leq N_1} B_{n_i}\right) \geq \frac{1}{2Q_{d,4}}. \quad (7.17)$$

Set

$$\mathcal{W}_1 = \{U_{n_i}\}_{1 \leq i \leq N_1} \quad \text{and} \quad W_1 = \bigcup_{1 \leq i \leq N_1} U_{n_i}.$$

Along the construction of the Cantor set, for every $U \in \mathcal{U}$, the ball of \mathcal{B} naturally associated with U will be denoted $B^{[U]}$ (that is $B^{[U_n]} = B_n$).

The pre-measure η on the σ -algebra generated by the sets of \mathcal{W}_1 is defined by

$$\text{for every } U \in \mathcal{W}_1, \quad \eta(U) = \frac{\mu(B^{[U]})}{\sum_{\tilde{U} \in \mathcal{W}_1} \mu(B^{[\tilde{U}]})}. \quad (7.18)$$

It is obvious that $\eta(\mathbb{R}^d) = \eta(W_1) = 1$.

Recalling (7.2) and (7.3), since $s_1 < s(\mu, \mathcal{B}, \mathcal{U})$, the sub-sequence $(B_n)_{n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_1)}$ is μ -a.-c. Remember also that $\lim_{n \rightarrow +\infty} r_n = 0$ and for every $n \in \mathbb{N}$, $|U_n| \leq r_n$.

So, for every $n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_1)$,

$$\mathcal{H}_\infty^{\mu, s_1}(U_n) \geq \mu(B_n) \quad \text{and} \quad |U_n| \leq r_n. \quad (7.19)$$

In particular, by Definition 6.2.1, for every $n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_1)$ for any set $E_n \subset U_n$ with $\mu(E_n) = \mu(U_n)$,

$$\mu(B_n) \leq \mathcal{H}_\infty^{\mu, s_1}(U_n) \leq \mathcal{H}_\infty^{s_1}(E_n).$$

By Proposition 4.2.6, and the notations therein, one has

$$m_{E_n}^{s_1}(U_n) = 1 \leq \frac{\kappa_d |U_n|^{s_1}}{\mathcal{H}_\infty^{s_1}(E_n)} \leq \frac{\kappa_d |U_n|^{s_1}}{\mu(B_n)}.$$

This implies that

$$\mu(B_n) \leq \kappa_d |U_n|^{s_1}. \quad (7.20)$$

By equation (7.20), recalling the fact that the sets $\mathcal{W}_1 \subset \{U_n\}_{n \in \mathbb{N}}$, one has for every $U \in \mathcal{W}_1$,

$$\eta(U) \leq \frac{\mu(B^{[U]})}{\frac{1}{2Q_{d,4}}} \leq 2Q_{d,4} \kappa_d |U|^{s_1}. \quad (7.21)$$

Step 2

This step (and all the following steps) is split into two sub-steps. First, into each open set U of \mathcal{W}_1 , smaller intermediary balls are selected according to the μ -essential content of U . Then in a second time, each intermediary ball will be covered by balls of the sequence $(B_n)_{n \in \mathbb{N}}$ according to the measure μ and, as in step 1, the sets U_n associated with this covering will form the generation \mathcal{W}_2 .

Let $g \in \mathbb{N}$ be such that for every $n \geq g$, $r_n \leq \frac{1}{3} \min(|U| : U \in \mathcal{W}_1)$.

As above, since $s_2 < s(\mu, \mathcal{B}, \mathcal{U})$, the sub-sequence $(B_n)_{n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_2), n \geq g}$ is μ -a.c. The same arguments as above yield for every $n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_2)$,

$$\mathcal{H}_\infty^{\mu, s_2}(U_n) \geq \mu(B_n) \quad \text{and} \quad |U_n| \leq r_n \quad (7.22)$$

and

$$\mu(B_n) \leq \kappa_d |U_n|^{s_2}. \quad (7.23)$$

Covering with respect to the μ -essential content

Consider $U \in \mathcal{W}_1$. Set $\beta = \overline{\dim}_H(\mu)$. For $0 \leq k \leq \lfloor \frac{\beta - \alpha}{\varepsilon_2} \rfloor + 1$, define $\theta_k = \alpha + k\varepsilon_2$. Write

$$E_U = U \cap E_\mu^{[\alpha, \beta], \varepsilon_2} \cap \limsup_{n \rightarrow +\infty} B_n. \quad (7.24)$$

Notice that by Proposition 7.2.6 and by item (1) of Theorem 6.1.2, one has $\mu(E_U) = \mu(U)$.

In addition, using the definition (6.7) of $\mathcal{H}_\infty^{\mu, s_2}$, the fact that $E_U \subset U$ and $\mu(E_U) = \mu(U)$, and finally (7.1) applied with $B_n = B^{[U]}$, one gets

$$\mathcal{H}_\infty^{s_2}(E_U) \geq \mathcal{H}_\infty^{\mu, s_2}(U) \geq \mu(B^{[U]}) > 0. \quad (7.25)$$

This allows us to apply Proposition 4.2.6: there exists a Borel probability measure $m_{E_U}^{s_2}$ supported on E_U such that for every ball $B := B(x, r)$, one has

$$m_{E_U}^{s_2}(B) \leq \kappa_d \frac{r^{s_2}}{\mathcal{H}_\infty^{s_2}(E_U)}.$$

Also, since $m_{E_U}^{s_2}(E_U) = 1$ and $E_U \subset E_\mu^{[\alpha, \beta], \varepsilon_2}$, and recalling (7.11), for any $0 \leq k \leq \lfloor \frac{\beta - \alpha}{\varepsilon_2} \rfloor + 1$, there exists ρ_{k, ε_2} such that

$$m_{E_U}^{s_2}(E_\mu^{[\theta_k, \theta_{k+1}], \rho_{k, \varepsilon_2}, \varepsilon_2}) \geq \frac{1}{2} m_{E_U}^{s_2}(E_\mu^{[\theta_k, \theta_{k+1}], \varepsilon}).$$

Setting $\rho_U = \min_{0 \leq k \leq \lfloor \frac{\beta-\alpha}{\varepsilon_2} \rfloor + 1} \rho_{k, \varepsilon_2}$ one has, for any $0 \leq k \leq \lfloor \frac{\beta-\alpha}{\varepsilon_2} \rfloor + 1$,

$$m_{E_U}^{s_2}(E_\mu^{[\theta_k, \theta_{k+1}], \rho_U, \varepsilon_2}) \geq \frac{1}{2} m_{E_U}^{s_2}(E_\mu^{[\theta_k, \theta_{k+1}], \varepsilon}). \quad (7.26)$$

In particular,

$$m_{E_U}^{s_2}(E_\mu^{[\alpha, \beta], \rho_U, \varepsilon_2}) \geq \frac{1}{2} \quad (7.27)$$

Let

$$S_U := \bigcup_{0 \leq k \leq \lfloor \frac{\beta-\alpha}{\varepsilon_2} \rfloor + 1} E_\mu^{[\theta_k, \theta_{k+1}], \rho_U, \varepsilon_2} \cap E_U \cap \{x \in \mathbb{R}^d : \overline{\dim}(m_{E_U}^{s_2}, x) \leq d\}. \quad (7.28)$$

Recalling that for every probability measure m , $m(\{x = \overline{\dim}(m, x) \leq d\}) = 1$, one necessarily has $m_{E_U}^{s_2}(S_U) \geq 1/2$.

Let $x \in S_U$; consider $0 \leq k_x \leq \lfloor \frac{\beta-\alpha}{\varepsilon_2} \rfloor + 1$ such that $x \in E_\mu^{[\theta_{k_x}, \theta_{k_x+1}], \rho_U, \varepsilon_2}$. Applying Lemma 7.2.3, there exists $0 < r_x < \min(\rho_x, \frac{1}{3} \min\{|V| : V \in \mathcal{W}_1\})$ and $t_x \in (5, 6)$ such that:

$$10r_x < \rho_U; \quad (7.29)$$

$$B(x, r_x) \subset U \quad \text{and} \quad \mu(\partial B(x, r_x/t_x)) = 0; \quad (7.30)$$

$$r_x^{-\varepsilon_2} \geq 5^d \frac{4Q_{d,1}}{C_{\varepsilon_3,d}} \frac{\eta(U)}{\mu(B^{[U]})} \geq 5^{s_2} \frac{4Q_{d,1}}{C_{\varepsilon_2,d}} \frac{\eta(U)}{\mu(B^{[U]})}; \quad (7.31)$$

$$r_x^{\theta_{k_x} + 2\varepsilon_2} \leq \mu(B(x, r_x)) \leq r_x^{\theta_{k_x} - 2\varepsilon_2}; \quad (7.32)$$

$$m_{E_U}^{s_2}(B(x, r_x/t_x)) \geq C_{\varepsilon_2,d} \cdot m_{E_U}^{s_2}(B(x, r_x)). \quad (7.33)$$

Note that in (7.31) the second inequality follows automatically from the first one since $s_2 \leq \alpha \leq d$ and the constant $C_{\varepsilon,d}$ is an increasing function of ε .

The family $\{B(x, r_x) : x \in S_U\}$ forms a covering of S_U . We apply Proposition 4.1.2 with $v = 1$ (i.e., the standard Besicovich covering Theorem) to this family to extract $Q_{d,1}$ subfamilies of balls, $\mathcal{G}_1^U, \dots, \mathcal{G}_{Q_{d,1}}^U$ such that:

- $\forall 1 \leq i \leq Q_{d,1}, \forall B \neq B' \in \mathcal{G}_i^U$, one has $B \cap B' = \emptyset$,
- $S_U \subset \bigcup_{i=1}^{Q_{d,1}} \bigcup_{B \in \mathcal{G}_i^U} B$.

In particular, $m_{E_U}^{s_2}(\bigcup_{i=1}^{Q_{d,1}} \bigcup_{B \in \mathcal{G}_i^U} B) \geq m_{E_U}^{s_2}(S_U) \geq 1/2$.

At least one of these families, say $\mathcal{G}_{i_0}^U$, verifies that

$$m_{E_U}^{s_2}\left(\bigcup_{B \in \mathcal{G}_{i_0}^U} B\right) \geq \frac{m_{E_U}^{s_2}(S_U)}{Q_{d,1}} \geq \frac{1}{2Q_{d,1}}.$$

Writing $\mathcal{G}_{i_0}^U = \{B_{i_0,k}^U\}_{k \in \mathbb{N}}$, one can find an integer N_U so large that

$$m_{E_U}^{s_2} \left(\bigcup_{1 \leq k \leq N_U} B_{i_0,k}^U \right) \geq \frac{1}{4Q_{d,1}}.$$

Remind that each $B_{i_0,k}^U$ is a ball $B(x, r_x)$ satisfying (7.30), (7.31) and (7.33).

Finally, setting $\mathcal{G}^U = \{B(x, r_x/t_x) : B(x, r_x) \in \mathcal{F}_{i_0}^U\}$, one has by construction

$$m_{E_U}^{s_2} \left(\bigcup_{B \in \mathcal{G}^U} B \right) = \sum_{B \in \mathcal{G}^U} m_{E_U}^{s_2}(B) \geq \frac{C_{\varepsilon_2,d}}{4Q_{d,1}}. \quad (7.34)$$

One then extends the pre-measure η to the Borel σ -algebra generated by the balls of \mathcal{G}^U , by the formula

$$\text{for every } B \in \mathcal{G}^U, \quad \eta(B) = \eta(U) \times \frac{m_{E_U}^{s_2}(B)}{\sum_{B' \in \mathcal{G}^U} m_{E_U}^{s_2}(B')}. \quad (7.35)$$

By construction, this formula is consistent since $\eta(U) = \sum_{B \in \mathcal{G}^U} \eta(B)$.

Observe that by (4.10), (7.34) and (7.25), one has for every $B \in \mathcal{G}^U$,

$$\eta(B) \leq \eta(U) \kappa_d \frac{|B|^{s_2}}{\mathcal{H}_{\infty}^{s_2}(E_U)} \frac{4Q_{d,1}}{C_{\varepsilon_2,d}} \leq \frac{4Q_{d,1}\kappa_d}{C_{\varepsilon_2,d}} \frac{\eta(U)}{\mu(B^{[U]})} |B|^{s_2} \leq |B|^{s_2 - \varepsilon_2}, \quad (7.36)$$

where the second inequality of (7.31) was used.

This is achieved simultaneously for all $U \in \mathcal{W}_1$.

Covering with respect to μ

Now, in order to build the second generation of the Cantor set K , we select balls of \mathcal{B} that lie in the interior of these intermediate balls $B \in \mathcal{G}^U$.

Let $U \in \mathcal{W}_1$ and $B \in \mathcal{G}^U$ be one of these intermediary balls. Since \mathcal{B} is μ -a.c., the last part of Lemma 6.1.3 proves the existence of a finite family $\mathcal{F}^B = \{U_{n_i}\}_{1 \leq i \leq N_B}$ such that

(i₁) for every $1 \leq i \leq N_B$, one has $B_{n_i} \subset \overset{\circ}{B}$ and

$$\max \left\{ 2Q_{d,4} \frac{\eta(B)}{\mu(B)}, \frac{5^d 4Q_{d,1}\kappa_d}{C_{\varepsilon_3,d}} \right\} \leq r_{n_i}^{-\varepsilon_2}, \quad (7.37)$$

(i₂) for every $1 \leq i \neq j \leq N_B$, one has $B_{n_i} \cap B_{n_j} = \emptyset$.

In addition, recalling that $\mu(\partial B) = 0$ by (7.30), one has

$$\mu(B_{n_i}) > 0 \quad \text{and} \quad \mu\left(\bigcup_{1 \leq i \leq N_B} B_{n_i}\right) \geq \frac{3\mu(\overset{\circ}{B})}{4} = \frac{3\mu(B)}{4}.$$

Recall the definitions (7.10) and (7.28) of the sets $E_\mu^{[a,b],\rho_U,\varepsilon_2}$ and S_U . By equations (7.29)-(7.33), there exists $\alpha \leq a \leq \beta$ such that the center of B belongs to $S_U \subset E_\mu^{[a,a+\varepsilon_2],\rho_U\varepsilon_2}$, and $|B| \leq \rho_U$, hence one has

$$\mu(B \cap \widetilde{E}_\mu^{[a,a+\varepsilon_2],\rho_U,\varepsilon_2}) \geq \frac{3}{4}\mu(B).$$

By (i₂), and recalling (7.10), one has

$$\begin{aligned} \mu\left(\bigcup_{B_{n_i}: B_{n_i} \cap \widetilde{E}_\mu^{[a,a+\varepsilon_2],\rho_U,\varepsilon_2} \neq \emptyset} B_{n_i}\right) &\geq \mu\left(\bigcup_{1 \leq i \leq N_B} B_{n_i} \cap \widetilde{E}_\mu^{[a,a+\varepsilon_2],\rho_U,\varepsilon_2}\right) \\ &= \mu\left(\bigcup_{1 \leq i \leq N_B} B_{n_i}\right) + \mu\left(\widetilde{E}_\mu^{[a,a+\varepsilon_2],\rho_U,\varepsilon_2}\right) - \mu\left(\widetilde{E}_\mu^{[a,a+\varepsilon_2],\rho_U,\varepsilon_2} \bigcup \bigcup_{1 \leq i \leq N_B} B_{n_i}\right) \\ &\geq \frac{3}{4}\mu(B) + \frac{3}{4}\mu(B) - \mu(B) = \frac{1}{2}\mu(B). \end{aligned}$$

By a slight abuse of notations, up to an extraction, we still denote by $\{B_{n_i}\}_{1 \leq i \leq N_B}$ the balls B_{n_i} such that $B_{n_i} \cap \widetilde{E}_\mu^{[a,a+\varepsilon_2],\varepsilon_2,\rho_U} \neq \emptyset$. The last inequality implies that the family of balls $\{B_{n_i}\}_{1 \leq i \leq N_B}$ can be chosen so that it verifies conditions (i₁) and (i₂), as well as the two following additional conditions:

$$(i_3) \quad \mu(B_{n_i}) > 0 \quad \text{and} \quad \mu\left(\bigcup_{1 \leq i \leq N_B} B_{n_i}\right) \geq \frac{\mu(\overset{\circ}{B})}{2} = \frac{\mu(B)}{2},$$

$$(i_4) \quad \text{for every } 1 \leq i \leq N_B, \quad B_{n_i} \cap \widetilde{E}_\mu^{[a,a+\varepsilon_2],\rho_U,\varepsilon_2} \neq \emptyset.$$

The obtained family is still denoted by \mathcal{F}^B .

Applying again Lemma 7.2.1 to \mathcal{F}^B with $v = 4$, as in step one (see (7.17), (i') and (i'')), if one must consider a subfamily, one can assume that the family \mathcal{F}^B satisfies (i₁) and (i₄) as well as the following condition (i'₂) and (i'₃):

(i'₂) : for every $1 \leq i \neq j \leq N_B$, one has $4B_{n_i} \cap 4B_{n_j} = \emptyset$.

(i'₃) : $\mu(B_{n_i}) > 0$ and $\mu\left(\bigcup_{1 \leq i \leq N_B} B_{n_i}\right) \geq \frac{\mu(\overset{\circ}{B})}{2Q_{d,4}} = \frac{\mu(B)}{2Q_{d,4}},$

Finally one defines

$$\mathcal{W}_2 = \bigcup_{U \in \mathcal{W}_1} \bigcup_{B \in \mathcal{F}^{1,U}} \mathcal{F}^B \quad \text{and} \quad \mathcal{W}_2 = \bigcup_{L \in \mathcal{W}_2} L.$$

The pre-measure η is then extended to the σ -algebra generated by the elements of \mathcal{W}_2 by setting for every $U \in \mathcal{W}_1$, every $B \in \mathcal{G}^U$ and $V \in \mathcal{F}^B$,

$$\eta(V) = \eta(B) \times \frac{\mu(B^{[V]})}{\sum_{V' \in \mathcal{F}^B} \mu(B^{[V']})}. \quad (7.38)$$

By construction, one has $\sum_{V \in \mathcal{F}^B} \eta(V) = \eta(B)$. Also, (7.37), (7.38), and (i'_3) imply

$$\frac{\eta(V)}{\mu(B^{[V]})} \leq 2Q_{d,4} \frac{\eta(B)}{\mu(B)} \leq |V|^{-\varepsilon_2}, \quad (7.39)$$

so that by (7.23) and (7.39) one has

$$\eta(V) \leq 2Q_{d,4} \frac{\eta(B)}{\mu(B)} \times \mu(B^{[V]}) \leq |B^{[V]}|^{-\varepsilon_2} |V|^{s_2} \leq |V|^{s_2 - \varepsilon_2}. \quad (7.40)$$

7.2.1.1 Recurrence scheme and end of the construction

Let $p \in \mathbb{N}^*$ be an integer, and set $\mathcal{W}_0 = \mathbb{R}^d$. Suppose that sets of balls $\mathcal{W}_1, \dots, \mathcal{W}_p$ as well as the measure η are constructed such that :

1. for every $1 \leq q \leq p$, $\mathcal{W}_q \subset \{U_n\}_{n \geq q}$, $\mathcal{W}_q \subset \mathcal{W}_{q-1}$, and η is defined on the σ -algebra generated by the elements of $\bigcup_{q=1}^p \mathcal{W}_q$.
2. For every $1 \leq q \leq p-1$, for every $U \in \mathcal{W}_q$, setting, as in step 2, $E_U = \limsup_{n \in \mathcal{N}_\mu(\mathcal{B}, \mathcal{U}, s_q)} B_n \cap U \cap E_\mu^{[\alpha, \beta], \varepsilon_q}$, then $\mathcal{H}_\infty^{s_q}(E_U) > 0$. If $m_{E_U}^{s_q}$ stands for the measure associated with E_U provided by Proposition 4.2.6, there exists $\rho_U > 0$ such that for every $0 \leq k \leq \lfloor \frac{\beta - \alpha}{\varepsilon_q} \rfloor + 1$, setting $\theta_k = \theta_k^{(q)} = \alpha + k\varepsilon_q$, one has

$$m_{E_U}^{s_q}(E_U \cap E_\mu^{[\theta_k, \theta_{k+1}], \rho_U, \varepsilon_q}) \geq \frac{1}{2} m_{E_U}^{s_q}(E_U \cap E_\mu^{[\theta_k, \theta_{k+1}], \varepsilon_q}).$$

In particular,

$$m_{E_U}^{s_q}(E_U \cap \bigcup_{0 \leq k \leq \lfloor \frac{\beta - \alpha}{\varepsilon_q} \rfloor + 1} E_\mu^{[\theta_k, \theta_{k+1}], \rho_U, \varepsilon_q}) \geq \frac{1}{2}.$$

3. For every $1 \leq q \leq p-1$, for every $U \in \mathcal{W}_q$, there exists a finite family \mathcal{G}^U of balls $B(x, r_x/t_x)$, where $x, r_x < \frac{1}{3} \min \{|\tilde{U}| : \tilde{U} \in \mathcal{W}_q\}$ and t_x satisfy (7.29), (7.30), (7.31), (7.33) and (7.34). Also, if $B \neq B' \in \mathcal{G}^U$, $3B \cap 3B' = \emptyset$.

Also, for every $B \in \mathcal{G}^U$, (7.35) and (7.36) hold true. Moreover $\mathcal{W}_{q+1} \subset \bigcup_{U \in \mathcal{W}_q} \mathcal{G}^U$.

4. For every $1 \leq q \leq p-1$, for every $U \in \mathcal{W}_q$, for every $B \in \mathcal{G}^U$ there exists a family $\mathcal{F}^B \subset \{U_n\}_{n \geq q}$ of pairwise disjoint open sets such that :

- for every $\tilde{U} \neq \hat{U} \in \mathcal{F}^B$, one has

$$4B^{[\tilde{U}]} \cap 4B^{[\hat{U}]} = \emptyset; \quad (7.41)$$

- for every $\tilde{U} \in \mathcal{F}^B$, $\tilde{U} \subset \mathring{B}$, (7.38) and (7.40) hold true, as well as

$$2Q_{d,4} \frac{\eta(B)}{\mu(B)} \leq |B^{[\tilde{U}]}|^{-\varepsilon_{q+1}} \quad (7.42)$$

and

$$B^{[\tilde{U}]} \cap \tilde{E}_\mu^{[\theta_{k_B}, \theta_{k_B+1}], \rho_U, \varepsilon_{q+1}} \neq \emptyset; \quad (7.43)$$

- the following inequality also holds true:

$$\mu \left(\bigcup_{\tilde{U} \in \mathcal{F}^B} B^{[\tilde{U}]} \right) \geq \frac{\mu(B)}{2Q_{d,4}}. \quad (7.44)$$

In item (3), the fact that $3B \cap 3B' = \emptyset$ just follows from the choice of $B(x, r_x/t_x)$ instead of simply $B(x, r_x)$.

The proof follows then exactly and rigorously the same lines as those of Step 2. We do not reproduce it here, the only differences are that \mathcal{W}_1 , \mathcal{W}_2 and s_2 are replaced by \mathcal{W}_p , \mathcal{W}_{p+1} and s_{p+1} .

Finally, define the Cantor set

$$K = \bigcap_{p \geq 1} W_p = \bigcap_{p \geq 1} \bigcup_{V \in \mathcal{W}_p} B^{[V]}.$$

Applying Caratheodory's extension Theorem to the pre-measure η yields a probability outer-measure on \mathbb{R}^d that we still denote by η , which is metric, so that Borel sets are η -measurable and its restriction to Borel sets belongs to $\mathcal{M}(\mathbb{R}^d)$. The so obtained measure η is fully supported on K . Also, for every $p \geq 2$, for any $U \in \mathcal{W}_p$, $B \in \mathcal{G}^U$, and $\tilde{U} \in \mathcal{F}^B$, the inequalities (7.35), (7.36), (7.38) and (7.40) holds with s_p and ε_p instead of s_2 and ε_2 .

7.2.1.2 Upper-bound for the mass of a ball

Define the gauge function $\zeta : \mathbb{R}^+ \mapsto \mathbb{R}^+$ as follows:

- if for some $p \geq 1$, $\frac{1}{3} \min \{|U| : U \in \mathcal{W}_{p+1}\} \leq r < \frac{1}{3} \min \{|U| : U \in \mathcal{W}_p\}$, then $\zeta(r) = 2Q_{d,4} 10^d r^{s_p - 5\varepsilon_p}$,

- if $r \geq \frac{1}{3} \min \{|U| : U \in \mathcal{W}_1\}$, $\zeta(r) = 1$,
- $\zeta(0) = 0$.

Since $\varepsilon_p \rightarrow 0$, one checks that $\lim_{r \rightarrow 0^+} \frac{\log(\zeta(r))}{\log(r)} = \min \{s(\mu, \mathcal{B}, \mathcal{U}), \underline{\dim}_H(\mu)\}$.

Let A be a ball of radius r . If there exists $n \in \mathbb{N}$ such that A does not intersect K_n then $\eta(A) = \eta(A \cap K_n) = 0$. Suppose that for every $n \in \mathbb{N}$, A intersects K_n . The goal is to prove that $\eta(A) \leq \zeta(|A|)$ when $|A|$ is small.

Some cases must be distinguished.

First if for every $n \in \mathbb{N}$, A intersects only one contracted set V_n of K_n , then by (7.36)

$$\eta(A) \leq \eta(V_n) \leq |V_n|^{s_n - \varepsilon_n} \xrightarrow{n \rightarrow +\infty} 0.$$

In the other case, there exists $p \in \mathbb{N}$ such that A intersects only one element of \mathcal{W}_p , and at least two elements of \mathcal{W}_{p+1} . Denote by U the unique element of \mathcal{W}_p intersecting A .

1. **Case 1:** If $|A| \geq |U|$, then by (7.40)

$$\eta(A) \leq \eta(U) \leq |U|^{s_p - \varepsilon_p} \leq \zeta(|A|). \quad (7.45)$$

2. **Case 2:** If $|A| < |U|$ and A intersects at least two balls of \mathcal{G}^U : Observe that when A intersects two balls B and B' of \mathcal{G}^U , since by item (3) of the recurrence scheme $3B \cap 3B' = \emptyset$, one necessarily has (by Lemma 5.2.1) $B \cup B' \subset 5A$. Hence, $\bigcup_{B \in \mathcal{G}^U : B \cap A \neq \emptyset} B \subset 5A$ and by (7.35) and (7.34),

$$\eta(A) = \eta(U) \times \frac{\sum_{B \in \mathcal{G}^U : B \cap A \neq \emptyset} m_{E_U}^{s_{p+1}}(B)}{\sum_{B' \in \mathcal{G}^U} m_{E_U}^{s_{p+1}}(B')} \leq \frac{4Q_{d,1}}{C_{\varepsilon_{p+1},d}} \eta(U) m_{E_U}^{s_{p+1}}(5A).$$

Then, by (4.10), (7.25), (7.37) and (7.39)

$$\begin{aligned} \eta(A) &\leq \frac{4Q_{d,1}}{C_{\varepsilon_{p+1},d}} \eta(U) \kappa_d \frac{(5|A|)^{s_{p+1}}}{\mathcal{H}_{\infty}^{\mu, s_{p+1}}(E_U)} \leq 5^{s_{p+1}} \frac{4Q_{d,1} \kappa_d}{C_{\varepsilon_{p+1},d}} \frac{\eta(U)}{\mu(B^{[U]})} |A|^{s_{p+1}} \\ &\leq |A|^{s_{p+1}} |U|^{-2\varepsilon_p} \leq |A|^{s_{p+1} - 2\varepsilon_{p+1}} \leq \zeta(|A|), \end{aligned} \quad (7.46)$$

where we used that $|A| < |U|$, and the mappings $x \mapsto |U|^{-x}$ and $x \mapsto x^{-\varepsilon_{p+1}}$ are decreasing.

3. **Case 3:** If A intersects only one ball of \mathcal{G}^U : calling B this particular ball and r_B its radius (at this stage there should be no confusion with the radii of the terms of the sequence $(B_n)_{n \geq 1}$), two cases must again be distinguished:

(a) **Subcase 3.1:** $|B| \leq |A|$: by (7.36),

$$\eta(A) \leq \eta(B) \leq |B|^{s_{p+1}-\varepsilon_{p+1}} \leq |A|^{s_{p+1}-\varepsilon_{p+1}} \leq \zeta(|A|). \quad (7.47)$$

(b) **Subcase 3.2:** $|A| \leq |B|$: Denote by k_B the integer such that its center belongs to $E_\mu^{[\theta_{k_B}, \theta_{k_B+1}], \rho_U, \varepsilon_{p+1}}$.

The ball A must intersect at least two elements $V \neq V'$ of \mathcal{W}_{p+1} (by definition of p). Note that those sets must belong to \mathcal{F}^B (because A intersects only B). Applying Lemma 5.2.1 to the ball A with any of those ball $V \in \mathcal{F}_{p+1}$, since $A \cap V \neq \emptyset$ and $A \setminus B^{[V]} \neq \emptyset$ (because A intersects an other dilated ball, $B^{[V']}$ by hypothesis and two such balls verifies (7.41)), one has

$$\bigcup_{V \cap A \neq \emptyset} B^{[V]} \subset 5A. \quad (7.48)$$

Then, (7.38) and (7.44) imply that

$$\eta(A) = \eta(B) \cdot \frac{\sum_{V \in \mathcal{W}_{p+1}: V \cap A \neq \emptyset} \mu(B^{[V]})}{\sum_{V' \in \mathcal{F}^B} \mu(B^{[V']})} \leq 2Q_{d,4} \frac{\eta(B)}{\mu(B)} \mu(5A). \quad (7.49)$$

Recalling (7.48), the ball $5A$ contains some of the balls of \mathcal{F}^B : Hence, by (7.43), $\widetilde{E}_\mu^{[\theta_{k_B}, \theta_{k_B+1}], \rho_U, \varepsilon_{p+1}} \cap 5A \neq \emptyset$. Since $|A| \leq |B|$, by (7.29), since $r_B < \frac{1}{10}\rho_U$, for any $x \in \widetilde{E}_\mu^{[\theta_{k_B}, \theta_{k_B+1}], \rho_U, \varepsilon_{p+1}} \cap 5A$ one has

$$\mu(5A) \leq \mu(B(x, 10r)) \leq (10r)^{\theta_{k_B}-2\varepsilon_{p+1}}. \quad (7.50)$$

Recalling (7.32) (applied to the ball B), one has

$$\mu(B) \geq (r_B)^{\theta_{k_B}+2\varepsilon_{p+1}}. \quad (7.51)$$

Using (7.36) (applied to B) (7.49), (7.50) and (7.51), one obtains

$$\begin{aligned} \eta(A) &\leq 2Q_{d,4} r_B^{s_{p+1}-\varepsilon_{p+1}} \frac{(10r)^{\theta_{k_B}-2\varepsilon_{p+1}}}{r_B^{\theta_{k_B}+2\varepsilon_{p+1}}} \\ &= 2Q_{d,4} 10^{\theta_{k_B}-2\varepsilon_{p+1}} \frac{r_B^{s_{p+1}-\theta_{k_B}-\frac{\varepsilon_{p+1}}{\delta}-2\varepsilon_{p+1}}}{r_B^{s_{p+1}-\theta_{k_B}-\varepsilon_{p+1}-2\varepsilon_{p+1}}} r^{s_{p+1}-\theta_{k_B}-\varepsilon_{p+1}-4\varepsilon_{p+1}} \\ &\leq 2Q_{d,4} 10^{\theta_{k_B}-2\varepsilon_{p+1}} r^{s_{p+1}-5\varepsilon_{p+1}}. \end{aligned}$$

Finally, recalling (7.16), $s_{p+1} - 5\varepsilon_{p+1} \leq \alpha \leq \theta_k$, and since $r_B \geq r$

and $s_p \leq s_{p+1}$, one gets

$$\begin{aligned} \eta(A) &\leq 2Q_{d,4}10^{\theta_k - \varepsilon_{p+1}}(r)^{s_{p+1} - 5\varepsilon_{p+1}} \leq 2Q_{d,4}10^{d - \varepsilon_{p+1}}|A|^{s_p - 5\varepsilon_p} \\ &\leq 2Q_{d,4}10^{d - \varepsilon_{p+1}}|A|^{s_p - 5\varepsilon_p}, \end{aligned}$$

hence

$$\eta(A) \leq \zeta(|A|). \quad (7.52)$$

Since for any $p \in \mathbb{N}$ and any ball A satisfying $|A| \leq \frac{1}{3} \min \{|U| : U \in \mathcal{W}_p\}$, if A intersects at most one element of \mathcal{W}_p , the inequalities (7.45), (7.46), (7.47), (7.52) proves that for any such ball, one has $\eta(A) \leq \zeta(|A|)$.

Hence recalling Definition 4.2.2, by the mass distribution principle, one deduces that $\mathcal{H}^\zeta(K) \geq 1$, which concludes the proof of Theorem 7.1.2.

7.2.2 Proof of Corollary 7.1.6

7.2.2.1 Proof of Corollary 7.1.6

One starts with a lemma.

Lemma 7.2.8. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ be a μ -a.c sequence of balls of \mathbb{R}^d with $\lim_{n \rightarrow +\infty} r_n = 0$.*

For any $\varepsilon > 0$, there exists a μ -a.c subsequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ of \mathcal{B} such that for every $n \in \mathbb{N}$, $\mu(B_{\phi(n)}) \leq (r_{\phi(n)})^{\dim_H(\mu) - \varepsilon}$.

Proof. Set $\alpha = \dim_H(\mu)$ and $\gamma = \text{suppess}_\mu(\dim(\mu, x))$.

Let Ω be an open set and $\varepsilon > 0$. By (7.2.6), $\mu(E_\mu^{[\alpha, \gamma], \frac{\varepsilon}{2}}) = 1$ and

$$\mu(\Omega \cap E_\mu^{[\alpha, \gamma], \frac{\varepsilon}{2}}) = \mu(\Omega).$$

For every $x \in \Omega \cap E_\mu^{[\alpha, \gamma], \frac{\varepsilon}{2}}$, there exists $r_x > 0$ such that $B(x, r_x) \subset \Omega$ and $x \in E_\mu^{[\alpha, \gamma], r_x, \frac{\varepsilon}{2}}$.

Recall (7.11) and that the sets $E_\mu^{[\alpha, \gamma], \rho, \varepsilon}$ are non-increasing in ρ . In particular there exists $\rho_\Omega > 0$ such that the set $E_\Omega := \left\{ x \in \Omega \cap E_\mu^{[\alpha, \gamma], \rho_\Omega, \frac{\varepsilon}{2}} : r_x \leq \rho_\Omega \right\}$ verifies

$$\mu(E_\Omega) \geq \frac{3\mu(\Omega)}{4}. \quad (7.53)$$

Let $g \in \mathbb{N}$. Applying Lemma 6.1.3 to Ω , the sequence (B_n) and the measure m , there exists N_Ω as well as $g \leq n_1 \leq \dots \leq n_{N_\Omega}$ verifying:

1. for every $1 \leq i \neq j \leq N_\Omega$, $B_{n_i} \cap B_{n_j} = \emptyset$,

2. for every $1 \leq i \leq N_\Omega$, $2r_{n_i} \leq \rho_\Omega$ and $2^{\alpha - \frac{\varepsilon}{2}} \leq r_{n_i}^{-\frac{\varepsilon}{2}}$,
3. $\mu(\bigcup_{1 \leq i \leq N_\Omega} B_{n_i}) \geq \frac{\mu(\Omega)}{2}$.

We may assume that $\mu(B_{n_i}) > 0$ for every i , otherwise B_{n_i} does not play any role.

Item (3) together with (7.53) implies that

$$\mu\left(\bigcup_{1 \leq i \leq N_\Omega} B_{n_i} \cap E_\Omega\right) \geq \frac{\mu(\Omega)}{4}.$$

Furthermore, for every $1 \leq i \leq N_\Omega$ verifying $B_{n_i} \cap E_\Omega \neq \emptyset$, it holds that $0 < \mu(B_{n_i}) \leq (r_{n_i})^{\alpha - \varepsilon}$. Indeed, let $x \in B_{n_i} \cap E_\Omega$. By item (2), $B_{n_i} \subset B(x, 2r_{n_i})$, and by (7.9), item (2), and (7.2.7), it holds that

$$\mu(B_{n_i}) \leq \mu(B(x, 2r_{n_i})) \leq (2r_{n_i})^{\alpha - \frac{\varepsilon}{2}} \leq (r_{n_i})^{\alpha - \varepsilon}.$$

Writing $\mathcal{B}' = \{B_n : \mu(B_n) \leq r_n^{\alpha - \varepsilon}\}$, the argument above shows that only balls of \mathcal{B}' have been used to cover Ω . This is satisfied for every open set Ω , so that \mathcal{B}' is a sub-sequence of \mathcal{B} satisfying the condition of Definition 6.1.1, which concludes the proof of Lemma 7.2.8. \square

Proof of Corollary 7.1.6. (1) Observe that item (2) of Proposition 6.2.2 implies that $t(\mu, \delta, \varepsilon, \mathcal{B}) \geq \underline{\dim}_H(\mu) - \varepsilon$, and $t(\mu, \delta, \mathcal{B}) \geq \underline{\dim}_H(\mu)$.

Now choose $\varepsilon > 0$ so small that $\frac{(1 - \varepsilon)\underline{\dim}_H(\mu)}{\delta \cdot t(\mu, \delta, \varepsilon, \mathcal{B})} \leq 1$. Recalling Lemma 7.2.8, up to an extraction, one can assume that for any $n \in \mathbb{N}$,

$$\mu(B_n) \leq |B_n|^{(1 - \varepsilon^2) \cdot \underline{\dim}_H(\mu)}.$$

Due to (7.5), there exists $N_\varepsilon \in \mathbb{N}$ such that for any $n \geq N_\varepsilon$,

$$\mathcal{H}_\infty^{\mu, \underline{\dim}_H(\mu) - \varepsilon}(\overset{\circ}{B}_n^\delta) \geq |B_n^\delta|^{(1 + \varepsilon) \cdot t(\mu, \delta, \varepsilon, \mathcal{B})}.$$

Then, Proposition 6.2.2 (4) implies that for every $n \geq N_\varepsilon$,

$$\begin{aligned} \mathcal{H}_\infty^{\mu, \frac{(1 - \varepsilon)\underline{\dim}_H(\mu) \times (\underline{\dim}_H(\mu) - \varepsilon)}{\delta \cdot t(\mu, \delta, \varepsilon, \mathcal{B})}}(\overset{\circ}{B}_n^\delta) &\geq (\mathcal{H}_\infty^{\mu, \underline{\dim}_H(\mu) - \varepsilon}(B_n^\delta))^{\frac{(1 - \varepsilon)\underline{\dim}_H(\mu)}{\delta \cdot t(\mu, \delta, \varepsilon, \mathcal{B})}} \\ &\geq |B_n^\delta|^{\frac{(1 + \varepsilon)t(\mu, \delta, \varepsilon, \mathcal{B})}{\delta \cdot t(\mu, \delta, \varepsilon, \mathcal{B})}(1 - \varepsilon) \cdot \underline{\dim}_H(\mu)} \\ &\geq |B_n|^{(1 + \varepsilon)(1 - \varepsilon)\underline{\dim}_H(\mu)} \geq \mu(B_n). \end{aligned}$$

Thus, setting $s_{\delta,\varepsilon} = \frac{(1-\varepsilon)\dim_H(\mu) \times (\dim_H(\mu) - \varepsilon)}{\delta \cdot t(\mu, \delta, \varepsilon, \mathcal{B})}$, Corollary 7.1.4 yields

$$\dim_H(\limsup_{n \rightarrow +\infty} \mathring{B}_n^\delta) \geq s_{\delta,\varepsilon}.$$

Since the result holds for any $\varepsilon > 0$, one gets the desired conclusion. \square

7.3 Ubiquity theorems and self-conformal measures

Combining Theorem 7.1.2 and Corollary 7.1.6 with Theorem 6.2.4 and Theorem 6.1.2 yield the following results for weakly conformal measures.

7.3.0.1 Mass transference principle from ball to ball

Theorem 7.3.1. *Let S be a weakly conformal IFS of \mathbb{R}^d with attractor K and μ be a weakly conformal measure associated with S . Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of closed balls centered on K , such that $\lim_{n \rightarrow +\infty} |B_n| = 0$.*

1. *Suppose that $(B_n)_{n \in \mathbb{N}}$ is μ -a.c. Then $t(\mu, \delta, (B_n)_{n \in \mathbb{N}}) \leq \dim(\mu)$; consequently $s(\mu, (B_n)_{n \in \mathbb{N}}, (\mathring{B}_n^\delta)_{n \in \mathbb{N}}) \geq \frac{\dim(\mu)}{\delta}$ and there exists a gauge function ζ such that $\lim_{r \rightarrow 0^+} \frac{\log(\zeta(r))}{\log(r)} \geq \frac{\dim(\mu)}{\delta}$ and $\mathcal{H}^\zeta(\limsup_{n \rightarrow \infty} \mathring{B}_n^\delta) > 0$. In particular*

$$\dim_H(\limsup_{n \rightarrow +\infty} \mathring{B}_n^\delta) \geq \frac{\dim(\mu)}{\delta}. \quad (7.54)$$

2. *Suppose that $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$. Then, (7.54) still holds but the existence of the gauge function is not ensured. Furthermore if μ is doubling, then $(B_n)_{n \in \mathbb{N}}$ is μ -a.c, so that the conclusion of item (1) holds.*

Let μ be a weakly similar measure with support K , and set $\alpha = \dim(\mu)$. Let $(B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ be a sequence of balls such that $x_n \in K$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow +\infty} r_n = 0$ and $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$.

Fix $\varepsilon > 0$, $v > 1$ and $\delta \geq 1$ and set $\mathcal{B}_v = \{vB_n\}_{n \in \mathbb{N}}$. Theorem 6.1.2 shows that \mathcal{B}_v is μ -a.c. Then, by Proposition 6.2.6, for n large enough, one has

$$\mathcal{H}_\infty^{\mu, \alpha - \varepsilon}((v\mathring{B}_n)^\delta) \geq \kappa(d, \mu, \varepsilon)(vr_n)^{\delta(\alpha - \varepsilon)} \geq (vr_n)^{\delta(\alpha - \frac{\varepsilon}{2})}.$$

Consequently,

$$t(\mu, \delta, \varepsilon, \mathcal{B}_v) = \limsup_{n \rightarrow +\infty} \frac{\log \mathcal{H}_\infty^{\mu, \alpha - \varepsilon}((v\mathring{B}_n)^\delta)}{\delta \log |vB_n|} \leq \alpha - \frac{\varepsilon}{2}$$

so $t(\mu, \delta, \varepsilon, \mathcal{B}_v) \leq \alpha$. Due to Corollary 7.1.6, one concludes that

$$\dim_H(\limsup_{n \rightarrow +\infty} (vB_n)^\delta) \geq \frac{\alpha}{\delta}.$$

But for any $\varepsilon' > 0$, $\limsup_{n \rightarrow +\infty} (vB_n)^\delta \subset \limsup_{n \rightarrow +\infty} B_n^{\delta-\varepsilon'}$, so that

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^{\delta-\varepsilon'}) \geq \frac{\alpha}{\delta}.$$

It follows that for any $\varepsilon' > 0$ and $\delta \geq 1$ one has

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{\alpha}{\delta + \varepsilon'}.$$

Letting $\varepsilon' \rightarrow 0$ proves that $\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{\dim(\mu)}{\delta}$, hence the result.

Remark 7.3.2.

- If the sequence of balls $(B_n)_{n \in \mathbb{N}}$ is not assumed to be μ -a.c, but only to verify $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$, then the same lower-bound estimate holds for $\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta)$, but the existence of a gauge function as in Theorem 7.1.2 does not hold in general.
- Let us also notice that the computation in the proof of Theorem 7.3.1 actually shows that, under the assumption that $\lim_{n \rightarrow +\infty} \frac{\log \mu(B_n)}{\log |B_n|} = \dim(\mu)$, it holds that, for n large enough, $\mathcal{H}_\infty^{\mu, s}(B_n^\delta) \geq \mu(B_n) \Leftrightarrow s < \frac{\dim(\mu)}{\delta}$.

One also emphasizes that Theorem 7.1.2 allows to deal with “very shrunk balls” as soon as the measure has positive lower Assouad dimension.

Proposition 7.3.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $(B_n)_{n \in \mathbb{N}}$ be a sequence of balls verifying $\lim_{n \rightarrow +\infty} |B_n| = 0$ and $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$. Then, for any $\delta \geq 1$*

1. *if $\delta \geq \frac{\dim_H(\mu)}{\dim_L(\mu)}$, then*

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{\dim_H(\mu)}{\delta}.$$

2. *if $\delta \leq \frac{\dim_H(\mu)}{\dim_L(\mu)}$, then*

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \dim_L(\mu).$$

Proof. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ such that $\beta := \dim_L(\mu) > 0$ and set $\alpha := \dim_H(\mu)$.

(1) :

Fix $\delta > \frac{\alpha}{\beta}$, $\varepsilon > 0$ such that $\frac{\alpha - \varepsilon}{\beta \delta} \geq 1$ and $v > 1$.

Let us set

$$\mathcal{B}_v = \{vB_n\}_{n \in \mathbb{N}}.$$

Theorem 6.1.2 shows that \mathcal{B}_v is μ -a.c.

By Lemma 7.2.8, up to an extraction, one can assume that for any $n \in \mathbb{N}$,

$$\mu(vB_n) \leq (v|B_n|)^{\alpha-\varepsilon}.$$

By applying item (4) of Proposition 6.2.2, Proposition 6.2.6 together with Remark 6.2.7, for every $n \in \mathbb{N}$ large enough,

$$\begin{aligned} \mathcal{H}^{\frac{\alpha-\varepsilon}{\delta}}((v\overset{\circ}{B}_n)^\delta) &\geq (\mathcal{H}^\beta((v\overset{\circ}{B}_n)^\delta))^{\frac{\alpha-\varepsilon}{\beta\delta}} \\ &\geq C^{\frac{\alpha-\varepsilon}{\beta\delta}} (v^{\delta\beta}|B_n|^{\beta\delta})^{\frac{\alpha-\varepsilon}{\beta\delta}} \\ &= C^{\frac{\alpha-\varepsilon}{\beta\delta}} (v|B_n|)^{\alpha-\varepsilon} \geq (v|B_n|)^{\alpha-\varepsilon} \geq \mu(vB_n). \end{aligned}$$

Then, ε being arbitrary, this proves that (as in the proof of 7.3.1) that $s(\mu, \delta, \mathcal{B}_v) \geq \frac{\alpha}{\delta}$. Since for any ε' , $\limsup_{n \rightarrow +\infty} (vB_n)^{\delta+\varepsilon'} \subset \limsup_{n \rightarrow +\infty} B_n^\delta$, one concludes that

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{\alpha}{\delta + \varepsilon'}.$$

Letting $\varepsilon' \rightarrow 0$ proves the result in case (1).

(2) :

Fix $\delta < \frac{\alpha}{\beta}$. Let $\varepsilon > 0$ such that $\delta\beta < \alpha - \frac{\varepsilon}{2}$.

By Lemma 7.2.8, one can assume once again that for any $n \in \mathbb{N}$, $\mu(vB_n) \leq (v|B_n|)^{\alpha-\varepsilon}$.

Using again Proposition 6.2.6 together with remark 6.2.7, one has

$$\mathcal{H}_\infty^{\mu, \beta}(v\overset{\circ}{B}_n)^\delta \geq (v|B_n|)^{\delta\beta} \geq (v|B_n|)^{\alpha-\frac{\varepsilon}{2}} \geq \mu(vB_n).$$

In particular this proves item (2) for $\delta < \frac{\alpha}{\beta}$.

Let $\varepsilon_k \rightarrow 0$. Applying item (2) with $\delta = \frac{\alpha}{\beta} - \varepsilon_k$ proves that item (2) still stands for $\delta = \frac{\alpha}{\beta}$. □

Remark 7.3.4. • *Following the notation of Definition 6.25, one has in particular, for any $x \in \text{supp}(\mu)$, $\underline{\dim}(\mu, x) \geq \dim_L(\mu)$. In particular, this implies $\underline{\dim}_H(\mu) \geq \dim_L(\mu)$.*

- *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ with $\dim_L(\mu) > 0$ and $(B_n)_{n \in \mathbb{N}}$ a μ -a.c sequence of balls of \mathbb{R}^d . Theorem 7.3.3 implies in particular one gets the expected lower*

bound for $\limsup_{n \rightarrow +\infty} B_n^\delta$ for any δ large enough (independently of the sequence $(B_n)_{n \in \mathbb{N}}$).

One emphasizes that there isn't any particular reason for which the Assouad dimension should be a good notion of dimension for ubiquity. In particular, in Theorem 7.3.3, the bound when the lower bound is $\dim_L(\mu)$, in most cases, this lower bound could be not accurate.

7.3.0.2 Mass transference principle from ball to rectangle

One emphasizes again that S is not assumed to verify any separation condition.

Theorem 7.1.2 and Theorem 6.2.4 actually allows to deal with more general sets than shrunk balls. For instance, assume that, under the hypothesis of Theorem 7.3.1, the open sets $U_n \subset B_n$ are shrunk rectangles inside of B_n of length-sides $\prod_{i=1}^d |B_n|^{\tau_i}$, where $1 \leq \tau_1 \leq \dots \leq \tau_d$. Denote $\mathcal{C}(U_n)$ the dyadic cubes $C \subset U_n$ of length-sides $\approx |B_n|^{\tau_d}$ intersecting K . Assume now that, for some $s \geq 0$, for any $n \in \mathbb{N}$ large enough, $\mathcal{H}_\infty^s\left(\bigcup_{C \in \mathcal{C}(U_n)} C\right) \gtrsim |B_n|^\alpha$, then one gets $\dim_H(\limsup_{n \rightarrow +\infty} U_n) \geq s$.

In particular, when the attractor K is the closure of its interior, one obtains the following extension of Theorem 5.1.5.

Definition 7.3.5. Let $1 \leq \tau_1 \leq \dots \leq \tau_d$ and $\tau = (\tau_1, \dots, \tau_d)$. For any $x = (x_i)_{1 \leq i \leq d} \in \mathbb{R}^d$ and $r > 0$, the τ -rectangle centered in x and associated with r is defined by

$$R_\tau(x, r) = \prod_{i=1}^d [x_i - \frac{1}{2}r^{\tau_i}, x_i + \frac{1}{2}r^{\tau_i}]. \quad (7.55)$$

Theorem 7.3.6. Let S be a weakly conformal IFS of \mathbb{R}^d such that the attractor K is equal to the closure of its interior. Let μ be a weakly conformal measure associated with S . Let $1 \leq \tau_1 \leq \dots \leq \tau_d$, $\tau = (\tau_1, \dots, \tau_d)$ and $(B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ be a sequence of balls of \mathbb{R}^d satisfying $r_n \rightarrow 0$ and $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$. Define $R_n = \mathring{R}_\tau(x_n, r_n)$, where

$$R_\tau(x_n, r_n) = x_n + \prod_{i=1}^d [-\frac{1}{2}r_n^{\tau_i}, \frac{1}{2}r_n^{\tau_i}]. \quad (7.56)$$

Then

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq \min_{1 \leq i \leq d} \left\{ \frac{\dim(\mu) + \sum_{1 \leq j \leq i} \tau_j - \tau_i}{\tau_i} \right\}. \quad (7.57)$$

Remark 7.3.7. (1) Since $(\tau_1, \dots, \tau_d) \mapsto \min_{1 \leq i \leq d} \left\{ \frac{\dim(\mu) + \sum_{1 \leq j \leq i} \tau_j - \tau_i}{\tau_i} \right\}$ is continuous, the result stands for the sequence of closed rectangles as well.

(2) One may also apply any rotation to the shrunk rectangles, this wouldn't change the bound (since Hausdorff contents are invariant by rotation).

(3) As said above, when K is the closure of its interior is that it is easy to compute $\mathcal{H}_\infty^s(R_n \cap K)$. Without this assumption, the conclusion of Theorem 7.3.6 fails. Indeed, in general, no formula involving only the dimension of the measure and the contraction ratio can be accurate. For instance, consider a self-similar measure in \mathbb{R}^2 carried by a line D and a sequence $(B_n)_{n \in \mathbb{N}}$ of balls centered on the attractor K and verifying $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$. Then, consider the sequence of rectangles R_n with side-length $|B_n|^{\tau_1} \times |B_n|^{\tau_2}$, $1 \leq \tau_1 \leq \tau_2$ and where the largest side (of side-length $|B_n|^{\tau_1}$) is in the direction of D . In this case, Theorem 7.3.1 yields the lower-bound $\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq \frac{\dim_H(\mu)}{\tau_1}$. Then if \hat{R}_n are the rectangles R_n rotated by $\frac{\pi}{2}$, Theorem 7.3.1 gives that $\dim_H(\limsup_{n \rightarrow +\infty} \hat{R}_n) \geq \frac{\dim_H(\mu)}{\tau_2}$. Moreover, under additional conditions, these lower bounds are equalities.

Proof. Given $\tau_1 = 1 \leq \tau_2 \leq \dots \leq \tau_d$ and $s \geq 0$, set $\tau = (\tau_1, \dots, \tau_d)$ and

$$g_\tau(s) = \max_{1 \leq k \leq d} \left\{ s\tau_k - \sum_{1 \leq i \leq k} \tau_k - \tau_i \right\}.$$

We will need the following lemma (one refers to [51], Proposition 2.1 for the proof, although it is stated in terms of singular values functions).

Lemma 7.3.8. *Let $\tau_1 = 1 \leq \tau_2 \leq \dots \leq \tau_d$.*

The are two positive constants C_1 and C_2 depending on d only such that for all $s \geq 0$, $r > 0$ and $x \in \mathbb{R}^d$ one has

$$C_1 r^{g_\tau(s)} \leq \mathcal{H}_\infty^s(R_\tau(x, r)) = \mathcal{H}_\infty^s(\mathring{R}_\tau(x, r)) \leq C_2 r^{g_\tau(s)}.$$

Recall that K is the closure of its interior, and note that since the weights p_i are taken positive in Definition 4.3.1, one must have $\mu(\mathring{K}) > 0$.

Denote $\tilde{\mu} = \frac{\mu(\cdot)}{\mu(\mathring{K})}$ and $\alpha = \dim(\mu) = \dim(\tilde{\mu})$. It is easily verified that the computation made in the proof of Theorem 6.2.4 implies that, for any, $\varepsilon' > 0$ and any open set $\Omega \subset \mathring{K}$, there exists a constant $c(d, \mu, s, \varepsilon')$ given by Theorem 6.2.4, so that

$$\begin{cases} c(\mu, d, s, \varepsilon') \mathcal{H}_\infty^{s+\varepsilon'}(\Omega) \leq \mathcal{H}_\infty^{\tilde{\mu}, s}(\Omega) \leq \mathcal{H}_\infty^s(\Omega) & \text{if } s < \alpha \\ \mathcal{H}_\infty^{\tilde{\mu}, s}(\Omega) = 0 & \text{if } s > \alpha. \end{cases} \quad (7.58)$$

Also, $\tilde{\mu}$ being absolutely continuous with respect to μ , the sequence $(B_n)_{n \in \mathbb{N}}$ is $\tilde{\mu}$ -a.c. Furthermore, up to a $\tilde{\mu}$ -a.c extraction, we can assume that

each ball $(B_n)_{n \in \mathbb{N}}$ is included in $\overset{\circ}{K}$ (and we will do so).

Let $\varepsilon > 0$. Set $\mathcal{R} = \{R_n\}_{n \geq 0}$. By Lemma 7.2.8, up to a $\tilde{\mu}$ -a.c extraction, one can assume that for every $n \in \mathbb{N}$, the ball B_n satisfies

$$\tilde{\mu}(B_n) \leq r_n^{\alpha - \varepsilon}.$$

Setting $\tau' = (\frac{\tau_i}{\tau_1})_{1 \leq i \leq d}$, for all $0 \leq s < \alpha - \varepsilon$, one has

$$g_{\tau'}(s) = \max_{1 \leq k \leq d} \left\{ \frac{s\tau_k - \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_1} \right\}.$$

From equation (7.58) and Lemma 7.3.8, one deduces that

$$C_1 c(d, \mu, s, \varepsilon') r_n^{\tau_1 g_{\tau'}(s + \varepsilon')} \leq \mathcal{H}_{\infty}^{\tilde{\mu}, s}(R_n). \quad (7.59)$$

In particular, for any s verifying

$$\tau_1 g_{\tau'}(s + \varepsilon') \leq \alpha - \frac{\varepsilon}{2}, \quad (7.60)$$

if $r_n \leq 1$ one has

$$C_1 c(d, \mu, s, \varepsilon') r_n^{\alpha - \frac{1}{2}\varepsilon} \leq C_1 c(d, \mu, s, \varepsilon') r_n^{\tau_1 g_{\tau'}(s + \varepsilon')} \leq \mathcal{H}_{\infty}^{\tilde{\mu}, s}(R_n).$$

Since $r_n \rightarrow 0$, for n large enough, this yields

$$\tilde{\mu}(B_n) \leq r_n^{\alpha - \varepsilon} \leq C_1 c(d, \mu, s, \varepsilon') r_n^{\tau_1 g_{\tau'}(s + \varepsilon')} \leq \mathcal{H}_{\infty}^{\tilde{\mu}, s}(R_n), \quad (7.61)$$

hence $(B_n)_{n \in \mathcal{N}_{\tilde{\mu}}(\mathcal{B}, \mathcal{R}, s)}$ is $\tilde{\mu}$ -a.c., and $s(\tilde{\mu}, \mathcal{R}, \mathcal{B}) \geq s$.

Since ε' is arbitrary, it remains to note that

$$\begin{aligned} (7.60) &\Leftrightarrow \max_{1 \leq k \leq d} \left\{ \frac{s\tau_k - \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_1} \right\} \leq \frac{\alpha - \frac{\varepsilon}{2}}{\tau_1} \\ &\Leftrightarrow \forall 1 \leq k \leq d, \quad \frac{s\tau_k - \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_1} \leq \frac{\alpha - \frac{\varepsilon}{2}}{\tau_1} \\ &\Leftrightarrow \forall 1 \leq k \leq d, \quad s \leq \frac{\alpha - \frac{1}{2}\varepsilon + \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_k} \\ &\Leftrightarrow s \leq \min_{1 \leq k \leq d} \left\{ \frac{\alpha - \frac{1}{2}\varepsilon + \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_k} \right\}. \end{aligned} \quad (7.62)$$

Since $\varepsilon > 0$ was arbitrary, this implies that

$$s(\tilde{\mu}, \mathcal{R}, \mathcal{B}) \geq \min_{1 \leq k \leq d} \left\{ \frac{\alpha + \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_k} \right\},$$

and applying Theorem 7.1.2 gives the desired lower bound estimate.

Remark 7.3.9. *Note that the estimates made in the proof of Theorem 7.3.6, together with Lemma 7.3.8, can be used to show that, under the assumption that $\lim_{n \rightarrow \infty} \frac{\log \mu(B_n)}{\log |B_n|} = \dim(\mu)$, one has the following properties:*

If $s < \min_{1 \leq k \leq d} \left\{ \frac{\dim(\mu) + \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_k} \right\}$ then, for $n \in \mathbb{N}$ large enough,

$$\mathcal{H}_{\infty}^{\mu, s}(R_n) \geq \mu(B_n).$$

If $s > \min_{1 \leq k \leq d} \left\{ \frac{\dim(\mu) + \sum_{1 \leq i \leq k} \tau_k - \tau_i}{\tau_k} \right\}$ then, for n large enough,

$$\mathcal{H}_{\infty}^{\mu, s}(R_n) \leq \mu(B_n).$$

□

Chapter 8

Upper-bounds and optimality in ubiquity theorems

The problem of understanding whether the lower-bound established in Theorem 7.1.2 is sharp is key to ensure that the point of view adopted is relevant. Given $\mu \in \mathcal{M}(\mathbb{R}^d)$, in Section 8.1 we investigate which properties one can ask a μ -a.c sequence of balls $(B_n)_{n \in \mathbb{N}}$ to verify, up to an extraction keeping the sub-sequence μ -a.c (so that, in view of Theorem 7.1.2, the sequence extracted and the original one can not be distinguished). Using those results, in Section 8.3, one proves that, for weakly conformal measures, it is possible to establish a companion theorem to Theorem 7.1.4, which states that $\dim_H \limsup_{n \rightarrow +\infty} U_n \leq s$ as soon as $\mathcal{H}_\infty^{\mu,s}(U_n) \geq \mu(B_n)$ for n large enough and the sequence $(B_n)_{n \in \mathbb{N}}$ is “weakly redundant” (Definition 8.1.1). This property can indeed be obtained by extraction of μ -a.c sub-sequences. As corollaries, one proves that the bounds stated in Theorem 7.3.1 and Theorem 7.3.6 are sharp in a strong way: we may have taken too many balls to begin with, but if one only considers the balls needed to ensure that the sub-sequence is μ -a.c, then the Hausdorff dimension of the limsup set associated with the corresponding U_n ’s is precisely the bound given in those theorems.

In the first section of this chapter, one states an extraction theorem of sub-sequences of balls which verifies the μ -a.c property (or some condition that are related). In the second section, an upper-bound theorem related to Theorem 7.1.2 is stated in the case of weakly conformal measures. The extraction theorem is also applied to show that the bound provided by Theorem 7.3.1 and Theorem 7.3.6 are indeed optimal.

We first start by a small section which study the basic properties of weakly redundant sequences of balls. Then in Section 8.2, we establish the main extraction theorem, Theorem 8.2.1. Section 8.3 is dedicated to the study of the optimality of the bound provided by Theorem 7.3.1 and 7.3.6.

8.1 Weakly redundant sequences of balls

Let us start by defining weakly redundant sequences of balls.

Definition 8.1.1. Let $\mathcal{B} = (B_n =: B(x_n, r_n))_{n \in \mathbb{N}}$ be a family of balls in \mathbb{R}^d . Denote by $\mathcal{T}_k(\mathcal{B}) = \{B_n : 2^{-k-1} < r_n \leq 2^{-k}\}$. The family \mathcal{B} is said to be weakly redundant when for all k , there exists an integer J_k and $\mathcal{T}_{k,1}(\mathcal{B}), \dots, \mathcal{T}_{k,J_k}(\mathcal{B})$ a partition of $\mathcal{T}_k(\mathcal{B})$ such that:

$$(C_1) \quad \mathcal{T}_k(\mathcal{B}) = \bigcup_{1 \leq j \leq J_k} \mathcal{T}_{k,j}(\mathcal{B}),$$

$$(C_2) \quad \text{for every } 1 \leq j \leq J_k \text{ and every pair of balls } B \neq B' \in \mathcal{T}_{k,j}(\mathcal{B}), B \cap B' = \emptyset,$$

$$(C_3) \quad \lim_{k \rightarrow +\infty} \frac{\log_2(J_k)}{k} = 0.$$

So, a sequence of balls $(B_n)_{n \in \mathbb{N}}$ is weakly redundant when at each scale 2^{-k} , the balls of the family $\{B_n\}_{n \in \mathbb{N}}$ that have radii $\approx 2^{-k}$ can be sorted in a relatively small number of families of pairwise disjoint balls. The following proposition illustrates why the weak redundancy property relates to overlaps between balls of the sequence of about the same radii.

Proposition 8.1.2. Let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a sequence of balls of \mathbb{R}^d . Then the following assertions are equivalent:

1. The sequence \mathcal{B} is weakly redundant.
2. The sequence \mathcal{B} satisfies

$$\lim_{n \rightarrow +\infty} \frac{\log \max_{B \in \mathcal{T}_n(\mathcal{B})} \#\{B' \in \mathcal{T}_n(\mathcal{B}) : B' \cap B \neq \emptyset\}}{n} = 0.$$

Proof. We show first (1) \Rightarrow (2).

Fix $\varepsilon > 0$. Since \mathcal{B} is assumed to be weakly redundant, there exists $n_\varepsilon \in \mathbb{N}$ large enough so that for every $n \geq n_\varepsilon$,

$$\frac{\log(J_n)}{n} \leq \varepsilon.$$

Since for every $1 \leq j \leq J_n$ the family $\mathcal{T}_{n,j}$ is composed by pairwise disjoint balls, by Lemma 4.1.4, there exists a constant $C > 0$ depending on the dimension only such that any ball $B \in \mathcal{T}_n(\mathcal{B})$ intersects less than C balls of $\mathcal{T}_{n,j}(\mathcal{B})$. In particular,

$$\frac{\log \#\{B' \in \mathcal{T}_n(\mathcal{B}) : B' \cap B \neq \emptyset\}}{n} \leq \frac{\log(C J_n)}{n} \leq \varepsilon + \frac{\log C}{n}. \quad (8.1)$$

Since C does not depend on B , (8.1) holds for every B provided that n is large enough, letting $\varepsilon \rightarrow 0$, one gets

$$\lim_{n \rightarrow +\infty} \frac{\log \max_{B \in \mathcal{T}_n(\mathcal{B})} \# \{B' \in \mathcal{T}_n(\mathcal{B}) : B' \cap B \neq \emptyset\}}{n} = 0.$$

Let us now prove that (2) \Rightarrow (1).

Assume that

$$\lim_{n \rightarrow +\infty} \frac{\log \max_{B \in \mathcal{T}_n(\mathcal{B})} \# \{B' \in \mathcal{T}_n(\mathcal{B}) : B' \cap B \neq \emptyset\}}{n} = 0.$$

Let $n \in \mathbb{N}$ and set

$$\hat{J}_n = \max_{B \in \mathcal{T}_n(\mathcal{B})} \# \{B' \in \mathcal{T}_n(\mathcal{B}) : B' \cap B \neq \emptyset\} + 1.$$

Note that every $B \in \mathcal{T}_n(\mathcal{B})$ intersects less than $\hat{J}_n - 1$ balls of $\mathcal{T}_n(\mathcal{B})$. Proceeding as in the proof of Lemma 4.1.3, one can sort the balls of $\mathcal{T}_n(\mathcal{B})$ in \hat{J}_n families $\mathcal{F}_{n,1}, \dots, \mathcal{F}_{n,\hat{J}_n}$ such that each family \mathcal{F}_i is composed of pairwise disjoint balls. Since

$$\lim_{n \rightarrow +\infty} \frac{\log \hat{J}_n}{n} = 0,$$

the sequence \mathcal{B} is weakly redundant. □

In practice, one will often use the weak redundancy property in this manuscript through the following lemma.

Lemma 8.1.3. *Let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a weakly redundant sequence of balls of \mathbb{R}^d . Then for every $\mu \in \mathcal{M}(\mathbb{R}^d)$ and any $\varepsilon > 0$, one has*

$$\sum_{n \in \mathbb{N}} |B_n|^\varepsilon \mu(B_n) < +\infty. \quad (8.2)$$

Proof. Let $n \in \mathbb{N}$ and $\mathcal{T}_n(\mathcal{B})$, J_n and $\mathcal{T}_{n,1}(\mathcal{B}), \dots, \mathcal{T}_{n,J_n}(\mathcal{B})$ as in Definition 8.1.1. One has

$$\begin{aligned} \sum_{n \in \mathbb{N}} |B_n|^\varepsilon \mu(B_n) &= \sum_{n \geq 0} \sum_{1 \leq j \leq J_n} \sum_{B \in \mathcal{T}_{n,j}(\mathcal{B})} |B|^\varepsilon \mu(B) \\ &\leq \sum_{n \geq 0} \sum_{1 \leq j \leq J_n} 2^{-n\varepsilon} \sum_{B \in \mathcal{T}_{n,j}(\mathcal{B})} \mu(B). \end{aligned}$$

Since for every $n \in \mathbb{N}$ and every $1 \leq j \leq J_n$, the family $\mathcal{T}_{n,j}(\mathcal{B})$ is composed of

pairwise disjoint balls, one has

$$\sum_{B \in \mathcal{T}_{n,j}(B)} \mu(B) \leq 1.$$

This, recalling that $\frac{\log_2 J_n}{n} \rightarrow 0$, implies that

$$\sum_{n \in \mathbb{N}} |B_n|^\varepsilon \mu(B_n) \leq \sum_{n \geq 0} 2^{-n\varepsilon} J_n < +\infty.$$

□

8.2 Extraction of μ -a.c sub-sequences of balls

The concept of conditioned ubiquity was introduced by Barral and Seuret in [5]. It consists in asking the balls of the sequence (B_n) to verify some specific properties with respect to the measure μ . In numerous practical cases, once the sequence $(B_n)_{n \in \mathbb{N}}$ is assumed to verify certain properties with respect to the measure μ , it is often quite easy to establish that the dimension of the limsup generated by the sets $(U_n)_{n \in \mathbb{N}}$ is indeed provided by a ubiquity theorem. So the point of view is to prove that one can almost always extract from (B_n) a sub-sequence $(B_{\phi(n)})$ which is still μ -a.c. and also verifies those specific properties. By doing so, we prove that one cannot hope in general for a better lower-bound.

Note that in full generality, understanding the optimality of a bound provided by a theorem such as Theorem 7.1.2, means understanding very finely the behavior of the measure μ on the sets U_n (the sequence $(B_n)_{n \in \mathbb{N}}$ being μ -a.c).

In this section, the balls $(B_n)_{n \in \mathbb{N}}$ are supposed to be pairwise distinct and such that $|B_n| \xrightarrow{n \rightarrow +\infty} 0$.

The main theorem of this section is the following:

Theorem 8.2.1 (D. [24]). *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ Let $(B_n)_{n \in \mathbb{N}}$ be a sequence of balls of \mathbb{R}^d .*

1. *If $(B_n)_{n \in \mathbb{N}}$ is μ -a.c, then there exists a μ -a.c sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ which is weakly redundant.*
2. *If there exists $v < 1$ such that $\mu(\limsup_{n \rightarrow +\infty} vB_n) = 1$, then there exists a μ -a.c sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ verifying*

$$\underline{\dim}_H(\mu) \leq \liminf_{n \rightarrow +\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \limsup_{n \rightarrow +\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \overline{\dim}_P(\mu). \quad (8.3)$$

Remark 8.2.2. *Theorem 8.2.1 implies in particular that if the sequence of balls $(B_n)_{n \in \mathbb{N}}$ verifies $\mu(\limsup_{n \rightarrow +\infty} v B_n) = 1$, for some $v < 1$, it is possible to extract a μ -a.c sub-sequence verifying both items (1) and (2).*

One also states here a proposition which is not particularly useful for practical applications, but highlights the fact that, given a measure μ and a μ -a.c sequence of balls, the “relevant balls” for the problem we study are indeed the balls verifying (8.3).

Proposition 8.2.3. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a measure and $(B_n)_{n \in \mathbb{N}}$ a sequence of balls satisfying $|B_n| \rightarrow 0$. Let $\varepsilon > 0$. Let us also define $\mathcal{B}_{>}^\varepsilon = \{B_n : \mu(B_n) \leq |B_n|^{\overline{\dim}_P(\mu) + \varepsilon}\}$ and $\mathcal{B}_{<}^\varepsilon = \{B_n : \mu(B_n) \geq |B_n|^{\underline{\dim}_H(\mu) - \varepsilon}\}$. Then*

$$1. \text{ for any } v < 1, \mu(\limsup_{B \in \mathcal{B}_{>}^\varepsilon} v B) = 0,$$

$$2. \mu(\limsup_{B \in \mathcal{B}_{<}^\varepsilon} B) = 0.$$

Proof. (1) Let us recall that the upper packing dimension of μ satisfies

$$\overline{\dim}_P(\mu) = \sup_{\mu} \{\overline{\dim}(\mu, x)\}.$$

Suppose that there exists $0 < v < 1$ such that $\mu(\limsup_{B \in \mathcal{B}_{>}^\varepsilon} v B) > 0$. Then there exists $x \in \limsup_{B \in \mathcal{B}_{>}^\varepsilon} v B$ such that

$$\limsup_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \leq \overline{\dim}_P(\mu).$$

Consider $r_x > 0$ small enough so that, for any $0 < r \leq r_x$, $\mu(B(x, r)) \geq r^{\overline{\dim}_P(\mu) + \frac{\varepsilon}{2}}$ and $(\frac{1-v}{2})^{\overline{\dim}_P(\mu) + \frac{\varepsilon}{2}} \geq r_x^{\frac{\varepsilon}{4}}$. Let also n be large enough so that $x \in B_n$ and $|B_n| \leq r_x$. Then $B(x, \frac{(1-v)}{2}|B_n|) \subset B_n$, so that

$$\mu(B_n) \geq \mu(B(x, \frac{1-v}{2}|B_n|)) \geq |B_n|^{\overline{\dim}_P(\mu) + \frac{\varepsilon}{2}} (\frac{1-v}{2})^{\overline{\dim}_P(\mu) + \frac{\varepsilon}{2}} \geq |B_n|^{\overline{\dim}_P(\mu) + \frac{3\varepsilon}{4}}. \quad (8.4)$$

This contradicts the definition of $\mathcal{B}_{>}^\varepsilon$.

(2) Assume that $\mu(\limsup_{B \in \mathcal{B}_{<}^\varepsilon} B) > 0$. Then, again, there exists $x \in \limsup_{B \in \mathcal{B}_{<}^\varepsilon} B$ so that

$$\liminf_{r \rightarrow 0} \frac{\log \mu(B(x, r))}{\log r} \geq \underline{\dim}_H(\mu).$$

Consider $r_x > 0$ small enough so such that for any $0 < r \leq r_x$, $\mu(B(x, r)) \leq r^{\underline{\dim}_H(\mu) - \frac{\varepsilon}{2}}$. Consider $n \in \mathbb{N}$ large enough so that $x \in B_n$ and $|B_n| \leq r_x$. One has $B_n \subset B(x, |B_n|)$, hence

$$\mu(B_n) \leq \mu(B(x, |B_n|)) \leq |B_n|^{\underline{\dim}_H(\mu) - \frac{\varepsilon}{2}}.$$

This contradicts the definition of $\mathcal{B}_{<}^\varepsilon$. \square

Remark 8.2.4. For doubling measures, it is straightforward that item (1) can be replaced by simply $\mu(\limsup_{B \in \mathcal{B}_{>}^\varepsilon} B) = 0$. It can be proved that this is also the case for 1-average $d-1$ unrectifiable measures (as a consequence of [53, Theorem 2.11]). Some self-similar measures with open set condition satisfies this property (see [53] again for more details).

Theorem 8.2.1 is obtained by proving first that it is always possible to extract μ -a.c weakly redundant sequences of balls. Then one proves in parallel that it is also possible to extract μ -a.c sequences of balls which verifies (8.3). The two next sub-section are dedicated to those results.

8.2.1 Extraction of weakly redundant μ -a.c subsequences

The main result of this section is stated here.

Proposition 8.2.5. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $(B_n)_{n \in \mathbb{N}}$ be a μ -a.c sequence of balls $\lim_{n \rightarrow +\infty} |B_n| = 0$. There exists a subsequence $(B_{\psi(n)})_{n \in \mathbb{N}}$ of $(B_n)_{n \in \mathbb{N}}$ which is weakly redundant and μ -a.c.

Proof. Let $g_k \in \mathbb{N}$ be large enough so that $\forall n \geq g_k, |B_n| \leq 2^{-k}$. By Lemma 6.1.3, applied with the sequence $(B_n)_{n \in \mathbb{N}}$, $\Omega = \mathbb{R}^d$ for any $k \in \mathbb{N}$, there exists a sub-sequence $(B_{(n,k)})$ of $\{B_n\}_{n \geq g_k}^{\mathbb{N}}$ satisfying

1. $\forall 1 \leq n_1 \neq n_2, B_{(n_1,k)} \cap B_{(n_2,k)} = \emptyset$,
2. $\mu\left(\bigcup_{n \in \mathbb{N}} B_{(n,k)}\right) = 1$.

Define $\mathcal{B}_\psi = (B_{\psi(n)})_{n \in \mathbb{N}}$ as the sub-sequence of balls corresponding to $\bigcup_{k \in \mathbb{N}} \{B_{(n,k)}\}_{n \in \mathbb{N}}$.

Since the following inclusion holds

$$\bigcap_{k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} B_{(n,k)} \subset \limsup_{n \rightarrow +\infty} B_{\psi(n)}, \quad (8.5)$$

by item (2) one has

$$\mu(\limsup_{n \rightarrow +\infty} B_{\psi(n)}) = 1.$$

Note that, for all $k \in \mathbb{N}$, for all $B \in \{B_{(n,k)}\}_{n \in \mathbb{N}}$, $|B| \leq 2^{-k}$. Following the notation of Definition 8.1.1, for any $k \in \mathbb{N}$, $\mathcal{T}_k(\mathcal{B}_\psi)$ can contain only balls of the sequence of the k first families $\{B_{(n,k)}\}_{n \in \mathbb{N}}$, which are composed of pairwise disjoint balls. This proves that $\mathcal{T}_k(\mathcal{B}_\psi)$ can be sorted in at most $k + 1$ families of pairwise disjoint balls. In particular, \mathcal{B}_ψ is weakly redundant.

It remains to show that $(B_{\psi(n)})_{n \in \mathbb{N}}$ is μ -a.c.

Let Ω be an open set and $g \in \mathbb{N}$. One will extract from \mathcal{B}_ψ a finite number of balls satisfying the condition of Definition 6.1.1.

There exists k_0 so large that

$$\begin{cases} \mu(\{x : B(x, 2^{-k_0+1}) \subset \Omega\}) \geq \frac{3\mu(\Omega)}{4} \\ \text{for every } k \geq k_0, \{B_{(n,k)}\}_{n \in \mathbb{N}} \subset \{B_{\psi(n)}\}_{n \geq g} \\ \mu(\limsup_{n \rightarrow +\infty} B_{\psi(n)} \cap \Omega) \geq \frac{3\mu(\Omega)}{4}. \end{cases}$$

Setting

$$\widehat{E} = \left\{ x \in \limsup_{n \rightarrow +\infty} B_{\psi(n)} \cap \Omega : B(x, 2^{-k_0+1}) \subset \Omega \right\},$$

it holds that

$$\mu(\widehat{E}) \geq \frac{1}{2}\mu(\Omega).$$

Recalling (8.5), for every $x \in \widehat{E}$, consider B_x , the ball of $\{B_{(n,k_0)}\}_{n \in \mathbb{N}}$ containing x . Note that, since for $B \in \{B_{(n,k_0)}\}_{n \in \mathbb{N}}$, $|B| \leq 2^{-k_0}$, one has

$$B_x \subset B(x, 2^{-k_0+1}) \subset \Omega.$$

Set

$$\mathcal{F}_1 = \{B_x : x \in \widehat{E}\}.$$

The set \mathcal{F}_1 is composed of pairwise disjoint balls (by item (1) above) of $\{B_{\psi(n)}\}_{n \geq g}$ included in Ω and such that

$$\mu\left(\bigcup_{L \in \mathcal{F}_1} L\right) \geq \mu(\widehat{E}) \geq \frac{1}{2}\mu(\Omega). \quad (8.6)$$

Using the σ -additivity of μ concludes the proof. \square

8.2.2 Extraction of sub-sequences of balls with conditioned measure

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $(B_n)_{n \in \mathbb{N}}$ be an μ -a.c sequence of balls.

This part aims to understand what condition can be assumed about the measure of the ball of the sequence $(B_n)_{n \in \mathbb{N}}$ in general under the μ -a.c condition.

More precisely, item (2) of Theorem 8.2.1 is proved.

Proposition 8.2.6. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$. For any sequence of balls $(B_n)_{n \in \mathbb{N}}$ satisfying $\mu(\limsup_{n \rightarrow +\infty} vB_n) = 1$ for some $0 < v < 1$, there exists a μ -a.c sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ verifying*

$$\underline{\dim}_H(\mu) \leq \liminf_{n \rightarrow +\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \limsup_{n \rightarrow +\infty} \frac{\log \mu(B_{\phi(n)})}{\log |B_{\phi(n)}|} \leq \overline{\dim}_P(\mu).$$

Remark 8.2.7. *For the left part of (8.2.6), the proof actually only uses the fact that $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$.*

Let us introduce some useful sets to prove Lemma 8.2.10, which, together with Lemma 7.2.8 are key in order to prove Proposition 8.2.6.

The following statements are easily deduced from Definition 4.2.12.

Definition 8.2.8. *Let $0 \leq \alpha \leq \gamma$ be real numbers, $\mu \in \mathcal{M}(\mathbb{R}^d)$, and $\varepsilon, \rho > 0$ two positive real numbers. Then define*

$$F_\mu^{[\alpha, \beta], \rho, \varepsilon} = \left\{ x \in \mathbb{R}^d : \overline{\dim}(\mu, x) \in [\alpha, \beta] \text{ and } \forall r < \rho, \mu(B(x, r)) \geq r^{\overline{\dim}(\mu, x) + \varepsilon} \right\}. \quad (8.7)$$

and

$$F_\mu^{[\alpha, \gamma], \varepsilon} = \bigcup_{n \geq 1} F_\mu^{[\alpha, \gamma], \frac{1}{n}, \varepsilon}. \quad (8.8)$$

Proposition 8.2.9. *For every $\mu \in \mathcal{M}(\mathbb{R}^d)$, $\rho > 0$, every $\alpha_2 = \infess_\mu(\overline{\dim}(\mu, x))$ and $\gamma_2 = \overline{\dim}_P(\mu)$, one has*

$$\mu\left(F_\mu^{[\alpha_2, \gamma_2], \varepsilon}\right) = 1. \quad (8.9)$$

Proof. For any $x \in \mathbb{R}^d$, for any $\varepsilon > 0$, there exists $r_x > 0$ such that $\forall r \leq r_x$, $r^{\overline{\dim}(\mu, x) + \varepsilon} \leq \mu(B(x, r)) \leq r^{\underline{\dim}(\mu, x) - \varepsilon}$. This implies

$$F_\mu^{[\alpha, \gamma], \rho, \varepsilon} \subset \left\{ x \in \mathbb{R}^d : \forall r \leq \rho, \mu(B(x, r)) \geq r^{\gamma + \varepsilon} \right\}$$

and

$$F_\mu^{[\alpha, \gamma], \varepsilon} = \left\{ x : \overline{\dim}(\mu, x) \in [\alpha, \gamma] \right\}.$$

Since

$$\mu(\{x : \overline{\dim}(\mu, x) \in [\infess_\mu(\overline{\dim}(\mu, x)), \supess_\mu(\overline{\dim}(\mu, x))]\}) = 1,$$

recalling Definition 4.2.12, it holds that, following the notation of Proposition 8.2.9,

$$\mu\left(F_\mu^{[\alpha_2, \gamma_2], \varepsilon}\right) = 1.$$

□

Before showing Proposition 8.2.6, let us start by the following lemma, companion to Lemma 7.2.8.

Lemma 8.2.10. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$, $v < 1$ and $\mathcal{B} = (B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ a sequence of balls of \mathbb{R}^d verifying $\mu(\limsup_{n \rightarrow +\infty} vB_n) = 1$.*

For all $\varepsilon > 0$, there exists a sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ of \mathcal{B} as well as $0 < v' < 1$ such that

$$\mu(\limsup_{n \rightarrow +\infty} v'B_{\phi(n)}) = 1$$

and for all $n \in \mathbb{N}$, one has

$$\mu(B_{\phi(n)}) \geq (r_{\phi(n)})^{\overline{\dim}_P(\mu) + \varepsilon}.$$

Remark 8.2.11. *The sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ found in Lemma 8.2.10 is in particular μ -a.c by Theorem 6.1.2.*

Proof. Let $\alpha = \inf_{x \in \mathbb{R}^d} \overline{\dim}_\mu(\mu, x)$ and $\gamma = \overline{\dim}_P(\mu)$. Let $\varepsilon > 0$ and $v < v' < 1$.

By (8.9) and Theorem 6.1.2, $\mu(\limsup_{n \rightarrow +\infty} vB_n \cap F_\mu^{[\alpha, \gamma], \frac{3\varepsilon}{2}}) = 1$. For all $x \in \limsup_{n \rightarrow +\infty} vB_n \cap F_\mu^{[\alpha, \gamma], \frac{3\varepsilon}{2}}$, there exists $r_x > 0$ small enough so that

$$r_x^{\frac{\varepsilon}{2}} \leq (v' - v)^{\gamma + \frac{3\varepsilon}{2}} \text{ and } \forall 0 < r \leq r_x, \mu(B(x, r)) \geq r^{\gamma + \frac{3\varepsilon}{2}}. \quad (8.10)$$

Since $x \in \limsup_{n \rightarrow +\infty} vB_n$, for all $n \in \mathbb{N}$, there exists $n_x \geq n$ such that $x \in vB_{n_x}$ and $(v' - v)r_{n_x} \leq r_x$. Note that $B(x, (v' - v)r_{n_x}) \subset v'B_{n_x}$. This implies the following inequalities:

$$\mu(B_{n_x}) \geq \mu(v'B_{n_x}) \geq \mu(B(x, (v' - v)r_{n_x})) \geq ((v' - v)r_{n_x})^{\gamma + \frac{3\varepsilon}{2}} \geq r_{n_x}^{\gamma + 2\varepsilon}.$$

Set $\mathcal{B}_{\gamma, 2\varepsilon} = \{B_n : \mu(B_n) \geq r_{n_x}^{\gamma + 2\varepsilon}\}$. One just showed that

$$\limsup_{n \rightarrow +\infty} vB_n \cap F_\mu^{[\alpha, \gamma], \frac{\varepsilon}{2}} \subset \limsup_{B \in \mathcal{B}_{\gamma, 2\varepsilon}} v'B.$$

This proves that $\mu(\limsup_{B \in \mathcal{B}_{\gamma, 2\varepsilon}} v'B) = 1$.

Since $\varepsilon > 0$ was arbitrary, the results also holds with $\frac{\varepsilon}{2}$, which proves Lemma 8.2.10. □

We are now ready to prove Proposition 8.2.6.

Proof. Set $\alpha = \underline{\dim}_H(\mu)$ and $\beta = \overline{\dim}_P(\mu)$.

Let us fix $(\varepsilon_n)_{n \in \mathbb{N}} \in (\mathbb{R}^{*+})^{\mathbb{N}}$ verifying $\lim_{n \rightarrow +\infty} \varepsilon_n = 0$.

The strategy of the proof consists in constructing recursively coverings of the cube \mathbb{R}^d by using Lemma 7.2.8 and Lemma 8.2.10 and a diagonal argument (on the choice of ε) at each step.

More precisely, at step 1, one will build a sequence of finite families of balls $(\mathcal{F}_{1,i})_{i \in \mathbb{N}}$ verifying:

1. for all $i, j \geq 1$, $\forall L \in \mathcal{F}_{1,i}$, $\forall L' \in \mathcal{F}_{1,j}$ such that $L \neq L'$, one has $L \cap L' = \emptyset$,
2. for all $i \geq 1$, $\mathcal{F}_{1,i}$ is a finite sub-family of $\{B_n\}_{n \geq 1}$,
3. for all $i \geq 1$, for all $L \in \mathcal{F}_{1,i}$, $|L|^{\beta+\varepsilon_i} \leq \mu(L) \leq |L|^{\alpha-\varepsilon_i}$,
4. one has

$$\mu \left(\bigcup_{i \in \mathbb{N}} \bigcup_{L \in \mathcal{F}_{1,i}} L \right) = 1. \quad (8.11)$$

At step 2, a family of balls $(\mathcal{F}_{2,i})_{i \in \mathbb{N}}$ will be constructed such that items 1, 2, 3 and 4 holds with $\varepsilon = \varepsilon_{i+1}$.

Write $\mathcal{F}_2 = \bigcup_{i \geq 1} \mathcal{F}_{2,i}$.

The other steps are achieved following the same scheme.

The construction is detailed below:

Step 1:

Let $\Omega_{1,1} = \mathbb{R}^d$.

Sub-step 1.1:

By Lemma 7.2.8 and Lemma 8.2.10 applied to $\varepsilon = \varepsilon_1$, there exists a μ -a.c sub-sequence $(B_{\psi_{1,1}(n)})_{n \in \mathbb{N}}$, satisfying, for every $n \in \mathbb{N}$,

$$|B_{\psi_{1,1}(n)}|^{\beta+\varepsilon_1} \leq \mu(B_{\psi_{1,1}(n)}) \leq |B_{\psi_{1,1}(n)}|^{\alpha-\varepsilon_1}.$$

By Lemma 6.1.3 applied to $\Omega_{1,1}$, the sequence $(B_{\psi_{1,1}(n)})_{n \in \mathbb{N}}$ and $g = 1$, there exists an integer $N_{1,1}$ as well as some balls $L_{1,1,1}, \dots, L_{1,1,N_{1,1}} \in \{B_n\}_{n \geq 1}$ verifying:

- for all $1 \leq i < j \leq N_{1,1}$, $L_{1,1,i} \cap L_{1,1,j} = \emptyset$,

- for all $1 \leq i \leq N_{1,1}$, $|L_{1,1,i}|^{\beta+\varepsilon_1} \leq \mu(L_{1,1,i}) \leq |L_{1,1,i}|^{\alpha-\varepsilon_1}$,
- $\mu(\bigcup_{1 \leq i \leq N_{1,1}} L_{1,1,i}) \geq \frac{1}{2}$.

Set $\mathcal{F}_{1,1} = \{L_{1,1,i}\}_{1 \leq i \leq N_{1,1}}$.

Sub-step 1.2:

Let $\Omega_{1,2} = \Omega_{1,1} \setminus \bigcup_{L \in \mathcal{F}_{1,1}} L$.

By Lemma 7.2.8 and Lemma 8.2.10 with $\varepsilon = \varepsilon_2$, there exists a μ -a.c sub-sequence $(B_{\psi_{1,2}(n)})_{n \in \mathbb{N}}$ satisfying

$$|B_{\psi_{1,2}(n)}|^{\beta+\varepsilon_2} \leq \mu(B_{\psi_{1,2}(n)}) \leq |B_{\psi_{1,2}(n)}|^{\alpha-\varepsilon_2}.$$

One applies Lemma 6.1.3 to the open set $\Omega_{1,2}$, the sub-sequence of balls $(B_{\psi_{1,2}(n)})_{n \in \mathbb{N}}$ and $g = 1$. There exists $N_{1,2} \in \mathbb{N}$ such that $L_{1,2,1}, \dots, L_{1,2,N_{1,2}}$ verifies:

- for all $1 \leq i < j \leq N_{1,2}$, $L_{1,2,i} \cap L_{1,2,j} = \emptyset$,
- for all $1 \leq i \leq N_{1,2}$, $|L_{1,2,i}|^{\beta+\varepsilon_2} \leq \mu(L_{1,2,i}) \leq |L_{1,2,i}|^{\alpha-\varepsilon_2}$,
- $\mu(\bigcup_{1 \leq i \leq N_{1,2}} L_{1,2,i}) \geq \frac{1}{2}\mu(\Omega_{1,2})$.

The family $\mathcal{F}_{1,2}$ is defined as $\mathcal{F}_{1,2} = \{L_{1,2,i}\}_{1 \leq i \leq N_{1,2}}$.

Proceeding iteratively as in Sub-steps 1.1 and 1.2, for any $i \in \mathbb{N}$, at Sub-step 1. i a finite family of balls $\mathcal{F}_{1,i}$ is constructed so that the items 1, 2, 3 and 4 holds with ε_i (instead of ε_1). Remember that, to justify the last item, this recursive scheme allows to cover \mathbb{R}^d , up to a set of μ -measure 0 (the argument is similar to the one developed at the end of the proof of Lemma 6.1.3 to obtain (6.2)).

Set $\mathcal{F}_1 = \bigcup_{i \geq 1} \mathcal{F}_{1,i}$.

Let us notice that the construction of the family \mathcal{F}_2 does not rely on the existence of the family \mathcal{F}_1 , so that the families \mathcal{F}_k can actually be built independently, following the same scheme, as described below.

Step k :

As in step 1, one constructs a family of balls $(\mathcal{F}_{k,i})_{i \geq 1}$ verifying items 1, 2, 3 and 4 with $\varepsilon = \varepsilon_{k+i-1}$.

Set

$$\mathcal{F}_k = \bigcup_{i \geq 1} \mathcal{F}_{k,i}$$

and

$$\mathcal{F} = \bigcup_{k \geq 1} \mathcal{F}_k.$$

Denote by $(B_{\phi(n)})_{n \in \mathbb{N}}$ the sub-sequence of balls that constitutes the family \mathcal{F} .

By construction, for all $i \in \mathbb{N}$, denoting

$$N_{k,i} = \#\mathcal{F}_{k,i},$$

for every $n \in \mathbb{N}$ there are at most $N \geq \sum_{1 \leq i, k \leq n} N_{k,i}$ balls of $\{B_{\phi(k)}\}_{k \in \mathbb{N}}$ belonging to $\bigcup_{1 \leq i, k \leq n} \mathcal{F}_{i,k}$. As a consequence, for \tilde{N} large enough and every $n' \geq \tilde{N}$, one has

$$|B_{\phi(n')}|^{\beta + \varepsilon_n} \leq \mu(B_{\phi(n')}) \leq |B_{\phi(n')}|^{\alpha - \varepsilon_n}.$$

It follows that

$$\alpha - \varepsilon_n \leq \liminf_{n' \rightarrow +\infty} \frac{\log \mu(B_{\phi(n')})}{\log |B_{\phi(n')}|} \leq \limsup_{n' \rightarrow +\infty} \frac{\log \mu(B_{\phi(n')})}{\log |B_{\phi(n')}|} \leq \beta + \varepsilon_n.$$

Letting $n \rightarrow +\infty$ shows that

$$\underline{\dim}_H(\mu) \leq \liminf_{n' \rightarrow +\infty} \frac{\log \mu(B_{\phi(n')})}{\log |B_{\phi(n')}|} \leq \limsup_{n' \rightarrow +\infty} \frac{\log \mu(B_{\phi(n')})}{\log |B_{\phi(n')}|} \leq \overline{\dim}_P(\mu).$$

It only remains to prove that $(B_{\phi(n)})_{n \in \mathbb{N}}$ is μ -a.c.

Let Ω be an open set and $g \in \mathbb{N}$. We find a finite family of balls $\{L\}_{L \in \mathcal{I}} \subset \{B_{\phi(n)}\}_{n \geq g}$ satisfying the conditions of Definition 6.1.1.

Note that, by (8.11),

$$\text{setting } E = \bigcap_{k \geq 1} \bigcup_{L \in \mathcal{F}_k} L, \quad \text{then } \mu(E) = 1.$$

Let $x \in \Omega \cap E$ and $r_x > 0$ small enough so that $B(x, r_x) \subset \Omega$. Consider $k_x \geq \phi(g) \geq g$ large enough so that, for all $n \geq k_x$, $|B_n| \leq 2r_x$. Recall that $\mathcal{F}_{k_x} \subset \{B_n\}_{n \geq k_x}$. Finally, let us fix k large enough so that $\mu(\widehat{E}) \geq \frac{\mu(\Omega)}{2}$, where $\widehat{E} = \{x \in E : k_x \leq k\}$. For $x \in \widehat{E}$, let $L_x \in \mathcal{F}_k$ be the ball that contains x (the balls of \mathcal{F}_k being pairwise disjoint, L_x is well defined) and $\{L_i\}_{i \geq 1} = \{L_x : x \in \widehat{E}\}$. One has

- for all $1 \leq i < j$, $L_i \cap L_j = \emptyset$,
- for all $i \in \mathbb{N}$, $L_i \in \{B_{\phi(n)}\}_{n \geq g}$ and $L_i \subset \Omega$,

- $\mu(\bigcup_{i \geq 1} L_i) \geq \mu(\widehat{E}) \geq \frac{\mu(\Omega)}{2}$.

By σ -additivity, there exists $N \in \mathbb{N}$ such that $\mu(\bigcup_{1 \leq i \leq N} L_i) \geq \frac{\mu(\Omega)}{4}$, which proves that $(B_{\phi(n)})_{n \in \mathbb{N}}$ satisfies Definition 6.1.1 with $C = \frac{1}{4}$ and is indeed μ -a.c. \square

Proposition 8.2.5 and Proposition 8.2.6 together prove Theorem 8.2.1.

8.2.3 G_δ sets of full measure and extraction of μ -a.c sub-sequences

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure and $(B_n)_{n \in \mathbb{N}}$ be a μ -a.c sequence of balls. In this chapter, we proved that, up to a μ -a.c. extraction, it is always possible to assume that a μ -a.c. sequence of balls satisfies certain properties related to μ and usefull when studying dimensions of limsup sets. In order to understand more finely the specific properties (depending on μ) one can assume $\limsup_{n \rightarrow +\infty} B_n$ to verify, up to a μ -a.c extraction, one investigates in this subsection under which condition a Borel set E satisfying $\mu(E) = 1$, satisfies that there

$$\limsup_{n \rightarrow +\infty} B_{\phi(n)} \subset E$$

for some μ -a.c sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$. We prove that it is the case in general if and only if E contains a G_δ set of full measure.

Proposition 8.2.12. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a probability measure and $(B_n)_{n \rightarrow +\infty}$ be a μ -a.c sequence of balls satisfying $|B_n| \rightarrow 0$.*

If E is a Borel set which containing a G_δ set of full measure, then there exists a μ -a.c sub-sequence $(B_{\phi(n)})_{n \in \mathbb{N}}$ such that $\limsup_{n \rightarrow +\infty} B_{\phi(n)} \subset E$.

Proof. Let $\widetilde{E} \subset E$ be a G_δ set of full measure and let $(\Omega_k)_{k \in \mathbb{N}}$ be a decreasing sequence of open sets such that $\widetilde{E} = \bigcap_{k \in \mathbb{N}} \Omega_k$. By Lemma 6.1.3, there exists some families of balls $(\mathcal{F}_k)_{k \in \mathbb{N}}$ such that:

- for any $k \in \mathbb{N}$, $\mathcal{F}_k \subset \{B_n\}_{n \geq k}$,
- for any $k \in \mathbb{N}$, for any $B \neq B' \in \mathcal{F}_k$, $B \cap B' = \emptyset$,
- for any $k \in \mathbb{N}$, $\bigcup_{B \in \mathcal{F}_k} B \subset \Omega_k$ and $\mu\left(\bigcup_{B \in \mathcal{F}_k} B\right) = \mu(\Omega_k)$.

Let $(B_{\phi(n)})_{n \in \mathbb{N}}$ be the sub-sequence arising from the family of balls $\mathcal{F} = \bigcup_{k \in \mathbb{N}} \mathcal{F}_k$. Since, for each $k \in \mathbb{N}$, the balls of \mathcal{F}_k are pairwise disjoint, any $x \in \limsup_{B_{\phi(n)}}$ must belong to $\bigcup_{B \in \mathcal{F}_k} B \subset \Omega_k$ for an infinite number of k . In

particular, $x \in E$. Moreover, the same argument as the proof of Proposition 8.2.5 shows that $(B_{\phi(n)})_{n \in \mathbb{N}}$ is μ -a.c too. \square

Remark 8.2.13. *One could also choose to work with sequences of open balls $(B_n)_{n \in \mathbb{N}}$. In that case, $\limsup_{n \rightarrow +\infty} B_n = \bigcup_{n \in \mathbb{N}} \bigcup_{k \geq n} B_k$ is a G_δ set. In a particular, if $(B_n)_{n \in \mathbb{N}}$ is μ -a.c and a set E verifies $\limsup_{n \rightarrow +\infty} B_{\phi(n)} \subset E$, for some μ -a.c sub-sequences $(B_{\phi(n)})_{n \in \mathbb{N}}$, then E contains a G_δ set of full measure.*

In the following section, we make good use of Theorem 8.2.1 in order to study the optimality of Theorem 7.1.2.

8.3 Study of the optimality of ubiquity theorems

In the first sub-section, we provide an upper-bound theorem which can be seen as a counter-part of Theorem 7.1.2 in the cas of self-conformal measures. In the second sub-section, we combine this theorem with Theorem 8.2.1 to prove that the bound provided by Theorem 7.3.1 and 7.3.6 are sharp.

8.3.1 An upper-bound for ubiquity Theorem in the self-similar case

The upper-bound we prove in this sub-section is the following:

Theorem 8.3.1. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a C^1 weakly conformal measure, K its support and $(B_n)_{n \rightarrow +\infty}$ be a weakly redundant sequence of balls of \mathbb{R}^d verifying $|B_n| \rightarrow 0$. Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of open sets satisfying $U_n \subset B_n$. For any $0 \leq s < \dim(\mu)$ such that for all large enough $n \in \mathbb{N}$, $\mathcal{H}_\infty^{\mu,s}(U_n) \leq \mu(B_n)$,*

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n \cap K) \leq s. \quad (8.12)$$

Note that if, for any $n \in \mathbb{N}$, the ball B_n intersects K , $\limsup_{n \rightarrow +\infty} U_n \cap K = \limsup_{n \rightarrow +\infty} U_n$.

Theorem 7.1.4 and Theorem 8.3.1 yields the following useful corollary.

Corollary 8.3.2. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure. Let $(B_n)_{n \in \mathbb{N}}$ be a weakly redundant μ -a.c sequence of balls satisfying $|B_n| \rightarrow 0$ and $B_n \cap K \neq \emptyset$ for any $n \in \mathbb{N}$.*

Let $(U_n)_{n \in \mathbb{N}}$ be a sequence of open sets satisfying that, for any $n \in \mathbb{N}$, $U_n \subset B_n$.

Assume that there exists s_0 such that

- for any $s < s_0$, for n large enough, $\mathcal{H}_\infty^{\mu,s}(U_n) \geq \mu(B_n)$,
- for any $s > s_0$, for n large enough, $\mathcal{H}_\infty^{\mu,s}(U_n) \leq \mu(B_n)$.

Then by Theorem 7.1.4 and Theorem 8.3.1,

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) = s_0.$$

Remark 8.3.3. It is easily seen from the proof that the condition $\mathcal{H}_\infty^{\mu,s}(U_n) \leq \mu(B_n)$ in Theorem 8.3.1 can be weakened into $\liminf_{n \rightarrow +\infty} \frac{\log \mathcal{H}_\infty^{\mu,s}(U_n)}{\log \mu(B_n)} \geq 1$.

Let us now prove Theorem 8.3.1 Recall that the sequence $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ is assumed to be weakly redundant. By Lemma 8.1.3, for any $\varepsilon > 0$, one has

$$\sum_{n \geq 0} |B_n|^\varepsilon \mu(B_n) < +\infty. \quad (8.13)$$

Now, for $n \in \mathbb{N}$, consider a sequence of balls $(A_k^n)_{k \in \mathbb{N}}$ satisfying that $|A_k^n| \leq |B_n|$, $U_n \cap K \subset \bigcup_{k \geq 0} A_k^n$ and

$$\sum_{k=0}^{+\infty} |A_{k,n}|^{s+\frac{\varepsilon}{2}} \leq 2\mathcal{H}_\infty^{s+\frac{\varepsilon}{2}}(U_n \cap K).$$

Remember Theorem 6.2.4 and its notations. One has.

$$\begin{aligned} \mathcal{H}_\infty^{s+\frac{\varepsilon}{2}}(U_n \cap K) &\leq \sum_{k \geq 0} |A_k^n|^{s+\frac{\varepsilon}{2}} \leq 2\mathcal{H}_\infty^{s+\frac{\varepsilon}{2}}(U_n \cap K) \\ &\leq \frac{2}{c(d, \mu, s, \frac{\varepsilon}{2})} \mathcal{H}_\infty^{\mu,s}(U_n) \leq \frac{2}{c(d, \mu, s, \frac{\varepsilon}{2})} \mu(B_n). \end{aligned} \quad (8.14)$$

Since for each $n \in \mathbb{N}$, $U_n \cap K \subset \bigcup_{k \geq 0} A_k^n$, it holds that

$$\limsup_{n \rightarrow +\infty} U_n \cap K \subset \limsup_{k, n \rightarrow +\infty} A_k^n.$$

For any $\varepsilon > 0$, one gets

$$\sum_{n \geq 0} \sum_{k \geq 0} |A_k^n|^{s+\varepsilon} \leq \sum_{n \geq 0} |B_n|^{\frac{\varepsilon}{2}} \frac{2}{c(d, \mu, s, \frac{\varepsilon}{2})} \mu(B_n).$$

In particular, by (8.13),

$$\sum_{n \geq 0} \sum_{k \geq 0} |A_k^n|^{s+\varepsilon} < +\infty. \quad (8.15)$$

One concludes that

$$\mathcal{H}^{s+\varepsilon}(\limsup_{n \rightarrow +\infty} U_n \cap K) \leq \mathcal{H}^{s+\varepsilon}(\limsup_{k, n \rightarrow +\infty} A_k^n) < +\infty.$$

This implies that $\dim_H(\limsup_{n \rightarrow +\infty} U_n \cap K) \leq s + \varepsilon$ and ε being arbitrary,

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n \cap K) \leq s.$$

Remark 8.3.4.

- When the self-similar measure verifies $\text{supp}(\mu) = [0, 1]^d$, the existence of s_0 as in Corollary 8.3.2 is ensured as soon as the shapes of the sets U_n are “uniform” in n . For instance, consider the case where $\mu = \mathcal{L}^d$ and $(U_n = R_n)_{n \in \mathbb{N}}$, where R_n is an open rectangle associated with some vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ defined as in Theorem 7.3.6. Recall that by Theorem 6.2.4, the Lebesgue essential Hausdorff content and the classical Hausdorff content are equivalent. It is easily verified that, for any $n \in \mathbb{N}$, $\mathcal{H}_\infty^s(R_n) = |B_n|^{g_\tau(s)}$, for the mapping $g_\tau : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, defined as (see [51])

$$g_\tau(s) = \max_{1 \leq k \leq d} \left\{ s\tau_k - \sum_{1 \leq i \leq k} \tau_k - \tau_i \right\}.$$

Note that $g_\tau(s)$ does not depend on n . Corollary 8.3.2 can therefore be applied with $s_0 = \min_{s: g_\tau(s) \geq d} \{s\}$.

- Unfortunately, when such an s_0 does not exist, the $\dim_H \limsup_{n \rightarrow +\infty} U_n$ has to depend on the structure of the sequence (U_n) itself. For instance given $0 < s_1 < s_2 \leq d$, one could build two scenarios where a sequence of balls $(B_n)_{n \in \mathbb{N}}$ of $[0, 1]^d$ satisfies $\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$ and a sequence of open sets (U_n) with $U_n \subset B_n$ satisfying that for some of the sets U_n , $\mathcal{H}_\infty^{s_1}(U_n) \approx \mathcal{L}^d(B_n)$, for the others $\mathcal{H}_\infty^{s_2}(U_n) \approx \mathcal{L}^d(B_n)$ and, in the case i ,

$$\dim_H \limsup_{n \rightarrow +\infty} U_n = s_i.$$

Here is how.

First scenario Following the scheme of example 3.5 in [51], it is also possible to construct two weakly redundant sequences of balls $(B_{n,1})_{n \in \mathbb{N}}$ and $(B_{n,2})_{n \in \mathbb{N}}$ such that:

- $|B_{n,1}| \rightarrow 0$ and $|B_{n,2}| \rightarrow 0$,
- $0 < \mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_{n,2}) < 1$,

- $\limsup_{n \rightarrow +\infty} R_{n,2} = \emptyset$, where $R_{n,2} \subset B_{n,2}$ is an open rectangle associated with τ_2 ,
- $\limsup_{n \rightarrow +\infty} B_{n,1} \subset [0, 1]^d \setminus \limsup_{n \rightarrow +\infty} B_{n,2}$ and

$$\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_{n,1}) = 1 - \mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_{n,2}).$$

For any $n \in \mathbb{N}$, denote by $R_{n,1} \subset B_{n,1}$ an open rectangle associated with τ_1 .

Those properties implies that that any sequence $(B_n)_{n \rightarrow +\infty}$ corresponding to the family $\{B_{n,i}\}_{n \in \mathbb{N}, i \in \{1,2\}}$ is weakly redundant and satisfies $\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$.

Again, the smallest real number such that the condition of Theorem 8.3.6 holds is s_2 , the largest real number such that the condition of Theorem 7.1.4 holds is s_1 and

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) = s_1.$$

Second scenario: Consider $0 < s_1 < s_2 \leq d$ and two vectors τ_1 and τ_2 such that $s_1 = \min_{s: g_{\tau_1}(s) \geq d} \{s\}$ and $s_2 = \min_{s: g_{\tau_2}(s) \geq d} \{s\}$. Consider a weakly redundant sequence of balls $(B_n)_{n \in \mathbb{N}}$ of $[0, 1]^d$ and a sequence of open sets $(U_n)_{n \in \mathbb{N}}$, $U_n \subset B_n$ satisfying:

- $|B_n| \rightarrow 0$,
- $\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$,
- for any $n \in \mathbb{N}$, $B_n \subset [0, \frac{1}{2}) \times \prod_{i=2}^d [0, 1]$ or $B_n \subset (\frac{1}{2}, 1] \times \prod_{i=2}^d [0, 1]$,
- for any $n \in \mathbb{N}$ such that $B_n \subset [0, \frac{1}{2}) \times \prod_{i=2}^d [0, 1]$, $U_n = (R_n)$ with R_n an open rectangle associated with τ_1 as in Theorem 8.3.6,
- for any $n \in \mathbb{N}$ such that $B_n \subset (\frac{1}{2}, 1] \times \prod_{i=2}^d [0, 1]$, $U_n = R_n$ with R_n an open rectangle associated with τ_2 .

Then smallest real number such that the condition of Theorem 8.3.1 holds is s_2 , the largest real number such that the condition of Theorem 7.1.4 holds is s_1 and

$$\dim_H(\limsup_{n \rightarrow +\infty} U_n) = s_2.$$

8.3.2 Application in the case of balls and rectangles

We can now show in which sense, in view of Theorem 8.2.1, Theorem 7.1.4 is sharp by applying Corollary 8.3.2 to the specific cases where the sets U_n are balls or rectangles.

Corollary 8.3.5. *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be a weakly conformal measure of support K and $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a sequence of balls centered in K satisfying $|B_n| \rightarrow 0$ and $\mu\left(\limsup_{n \rightarrow +\infty} B_n\right) = 1$. Then by Theorem 7.3.1,*

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) \geq \frac{\dim(\mu)}{\delta}.$$

Assume furthermore that \mathcal{B} is weakly redundant and $\limsup_{n \rightarrow +\infty} \frac{\log \mu(B_n)}{\log(|B_n|)} = \dim(\mu)$, then for every $\delta \geq 1$,

$$\dim_H(\limsup_{n \rightarrow +\infty} B_n^\delta) = \frac{\dim(\mu)}{\delta}.$$

Corollary 8.3.6. *Let μ be a weakly conformal measure verifying that its support, K , is the closure of its interior. Let $1 \leq \tau_1 \leq \dots \leq \tau_d$, $\tau = (\tau_1, \dots, \tau_d)$ and $(B_n := B(x_n, r_n))_{n \in \mathbb{N}}$ be a sequence of balls of \mathbb{R}^d satisfying $r_n \rightarrow 0$, $\mu(\limsup_{n \rightarrow +\infty} B_n) = 1$. Define $R_n = \mathring{R}_\tau(x_n, r_n)$, where $R_\tau(x_n, r_n) = x_n + \prod_{i=1}^d [-\frac{1}{2}r_n^{\tau_i}, \frac{1}{2}r_n^{\tau_i}]$. Then, by Theorem 7.3.6,*

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) \geq \min_{1 \leq i \leq d} \left\{ \frac{\dim(\mu) + \sum_{1 \leq j \leq i} \tau_i - \tau_j}{\tau_i} \right\}. \quad (8.16)$$

Assume furthermore that $(B_n)_{n \in \mathbb{N}}$ is weakly redundant and $\lim_{n \rightarrow +\infty} \frac{\log \mu(B_n)}{\log |B_n|} = \dim(\mu)$, then

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) = \min_{1 \leq i \leq d} \left\{ \frac{\dim(\mu) + \sum_{1 \leq j \leq i} \tau_i - \tau_j}{\tau_i} \right\}. \quad (8.17)$$

Remark 8.3.7. • Corollaries 8.3.5 and 8.3.6 are direct consequences of second item of Remark 7.3.2 and Remark 7.3.9, together with Corollary 8.3.2 (applied to, respectively, $s_0 = \frac{\dim(\mu)}{\delta}$ and $s_0 = s(\mu, \tau)$).

- Note that, by Theorem 8.2.1 combined with Corollary 8.3.5 and Corollary 8.3.6, for any sequence of balls $(B_n)_{n \in \mathbb{N}}$ satisfying $\mu\left(\limsup_{n \rightarrow +\infty} \frac{1}{2}B_n\right) = 1$ (μ a self-similar measure satisfying the hypothesis of Corollaries 8.3.5 or 8.3.6 for μ), it is always possible to extract a μ -a.c sub-sequence of balls so that the Hausdorff dimension of the limsup set associated with corresponding U_n 's is the bound provided by Theorems 7.3.1 and 7.3.6 and

recalled in those corollaries. This in particular proves that those bounds are sharp.

- In the case of the Lebesgue measure, it is always verified that

$$\lim_{n \rightarrow +\infty} \frac{\log \mu(B_n)}{\log |B_n|} = \dim(\mu).$$

As a consequence, the lower-bound provided by Theorem 7.1.4 (which is established in [51]) in the case of balls or rectangles is precisely the dimension of $\limsup_{n \rightarrow +\infty} U_n$ as soon as the sequence (B_n) is weakly redundant. More explicitly, given a weakly redundant sequence of balls $(B_n)_{n \in \mathbb{N}}$ of $[0, 1]^d$ satisfying $|B_n| \rightarrow 0$ and $\mathcal{L}^d(\limsup_{n \rightarrow +\infty} B_n) = 1$, for any sequence rectangles associated with a vector τ as in Corollary 8.3.6, one has

$$\dim_H(\limsup_{n \rightarrow +\infty} R_n) = \min_{1 \leq i \leq d} \left\{ \frac{d + \sum_{1 \leq j \leq i} \tau_i - \tau_j}{\tau_i} \right\}.$$

Chapter 9

An application in Diophantine approximation

Let $\mathcal{Q} = \left\{ B\left(\frac{p}{q}, q^{-2}\right) \right\}_{q \in \mathbb{N}^*, 0 \leq p \leq q}$. Remember the following result in Diophantine approximation [47]:

- $\limsup_{B \in \mathcal{Q}} B = [0, 1]$.
- For any $\delta \geq 1$, $\dim_H(\limsup_{B \in \mathcal{Q}} B^\delta) = \frac{1}{\delta}$. (9.1)

Unlike in the case of the points in $[0, 1]$, the approximation by rational numbers of elements of the middle third Cantor set $K_{1/3}$ set is not well understood yet. This question was raised by Mahler, and only some partial results are known (see [11], [8]). Here we consider the set $K_{1/3}^{(0)}$ of points in $[0, 1]$ having an asymptotic lower frequency of appearance of the digit 1 in basis 3 equal to 0. This set contains $K_{1/3}$ and has the same Hausdorff dimension as $K_{1/3}$. We compute the Hausdorff dimension of sets of points in $K_{1/3}^{(0)}$ which are well approximable by rational numbers.

To describe more precisely the problem, let $S = \{f_1, f_2, f_3\}$ where f_1 , f_2 and f_3 are the contracting affine maps of \mathbb{R} defined by $f_0(x) = \frac{1}{3}x$, $f_1(x) = \frac{1}{3}x + \frac{1}{3}$ and $f_2(x) = \frac{1}{3}x + \frac{2}{3}$. The attractor of S is $[0, 1]$. Let $\Lambda = \{0, 1, 2\}$.

The shift operation on the symbolic space $\Lambda^{\mathbb{N}}$ is defined by σ . The canonical projection from $\Lambda^{\mathbb{N}}$ to $[0, 1]$ is the mapping

$$\pi : x = (x_n)_{n \in \mathbb{N}} \mapsto \lim_{n \rightarrow +\infty} f_{(x_1, \dots, x_n)}(0). \quad (9.2)$$

The set $K_{1/3}$ is the attractor of $\{f_0, f_2\}$ and also the image by canonical projection of $\{0, 2\}^{\mathbb{N}}$.

Definition 9.0.1. Let $\phi : \Lambda^{\mathbb{N}} \rightarrow \{0, 1\}$ defined by $\begin{cases} \phi(x) = 1 & \text{if } x_1 = 1 \\ \phi(x) = 0 & \text{if } x_1 = 0 \text{ or } 2. \end{cases}$

and

$$K_{1/3}^{(0)} = \pi \left(\left\{ x \in \Lambda^{\mathbb{N}} : \liminf_{k \rightarrow +\infty} \frac{S_k \phi(x)}{k} = 0 \right\} \right),$$

where $(S_k \phi)_{k \in \mathbb{N}}$ stands for the sequence of Birkhoff sums of ϕ .

It is also proved in Proposition 9.1.5 that

$$\dim_H K_{1/3}^{(0)} = \frac{\log 2}{\log 3} (= \dim_H K_{1/3}). \quad (9.3)$$

As we will study the approximation of numbers of $K_{1/3}^{(0)}$, one introduces the irrationality exponent of a number.

Definition 9.0.2. Let $x \in \mathbb{R}$, we define the irrationality of x $\xi(x)$ by

$$\xi(x) = \sup \left\{ \delta : x \in B\left(\frac{p}{q}, \frac{1}{q^\delta}\right) \text{ for infinitely many } (p, q) \in \mathbb{Z} \times \mathbb{N}^* \right\}.$$

Let us state the main results of this subsection.

Theorem 9.0.3. For every $\delta \geq 1$,

$$\begin{cases} \dim_H \left(\{x : \xi(x) \geq 2\delta\} \cap K_{1/3}^{(0)} \right) = \frac{\log 2}{\log 3} & \text{if } 1 \leq \delta \leq \frac{\log 3}{\log 2}, \\ \dim_H \left(\{x : \xi(x) = 2\delta\} \cap K_{1/3}^{(0)} \right) = \frac{1}{2\delta} & \text{if } \delta \geq \frac{\log 3}{\log 2}. \end{cases} \quad (9.4)$$

Observe that a saturation phenomenon occurs : $\dim_H \left(\{x : \xi(x) \geq 2\delta\} \cap K_{1/3}^{(0)} \right) = \frac{\log 2}{\log 3}$ for $1 \leq \delta \leq \frac{\log 3}{\log 2}$. We also conjecture that for $\delta \in [1, \frac{\log 3}{\log 2}]$, (9.4) should hold for the level set $\{x : \xi(x) = 2\delta\}$.

Remark 9.0.4. Given an integer $b \in \mathbb{N}$ and defining

$$\xi_b(x) = \sup \left\{ \delta \geq 1 : x \in \limsup_{n \in \mathbb{N}, 0 \leq k \leq b^n - 1} B\left(\frac{k}{b^n}, \frac{1}{b^n}\right) \right\},$$

then Theorem 9.0.3 holds with ξ_b instead of ξ .

In Section 9.1, (9.3) is established and Section 9.2 is dedicated to the proof of Theorem 9.0.3.

9.1 Hausdorff dimension of $K_{1/3}^0$

The computation of the Hausdorff dimension of $K_{1/3}^0$ is a straightforward application of the thermodynamical formalism developed for instance in [14].

Let us first introduce the following quantities.

Definition 9.1.1 ([14]). *Let $f : \Sigma \rightarrow \mathbb{R}$ be an Hölder function, $k \in \mathbb{N}$ and $\underline{i} \in \Lambda^k$. Set*

$$Y_{k,f}(\underline{i}) = \sup_{(x_n) \in [\underline{i}]} e^{\sum_{0 \leq i \leq k-1} f(\sigma^i((x_n)_{n \in \mathbb{N}}))}$$

$$Z_{k,f} = \sum_{\underline{i} \in \Lambda^k} Y_{k,f}(\underline{i}).$$

Finally, define the pressure of f as

$$P_f = \lim_{k \rightarrow +\infty} \frac{\log Z_{k,f}}{k}. \quad (9.5)$$

It is classical that, when the potential (i.e the function f) is Hölder, there is an equilibrium measure associated with the Birkhoff averages of f .

Theorem 9.1.2 ([14]). *Let $f : \Sigma \rightarrow \mathbb{R}$ be an Hölder function. Then there exists a unique measure $\mu_f \in \mathcal{M}(\Sigma)$ satisfying, for some constants $C_1, C_2 > 0$, for any $k \in \mathbb{N}$, $\underline{i} \in \Lambda^k$ and $x = (x_n)_{n \in \mathbb{N}} \in [\underline{i}]$ that*

$$C_1 e^{\sum_{0 \leq j \leq k-1} f(\sigma^j(x)) - kP_f} \leq \mu_f([\underline{i}]) \leq C_2 e^{\sum_{0 \leq j \leq k-1} f(\sigma^j(x)) - kP_f}. \quad (9.6)$$

In order to study the equilibrium measure associated with the function ϕ defined in Definition 9.0.1, one estimates the pressure associated with the potentials $q\phi$ for $q \in \mathbb{R}$.

Lemma 9.1.3. *One has $P_\phi = \log(2+e)$ (where ϕ is defined in definition 9.0.1) and for any $q \in \mathbb{R}$, $P_{q(\phi-P_\phi)} = \log(2+e^q) - qP_\phi$.*

Proof. Let $k \in \mathbb{N}$. Then, for any $\underline{i} = (i_1, \dots, i_k) \in \Lambda^k$ and $(x_n)_{n \in \mathbb{N}} \in [\underline{i}]$. Note that $Y_k((x_n)) = e^{\sum_{0 \leq i \leq k-1} \phi(\sigma^i((x_n)_{n \in \mathbb{N}}))}$ does not depends on (x_n) , but only on \underline{i} .

$$Z_{k,\phi} = \sum_{(i_1, \dots, i_k) \in \Lambda^k} e^{\sum_{0 \leq i \leq k-1} \phi(\sigma^i((x_n)_{n \in \mathbb{N}}))} = \sum_{(i_1, \dots, i_k) \in \Lambda^k} e^{\sum_{1 \leq j \leq k} \phi(i_j)}$$

$$= \sum_{0 \leq i \leq k} \binom{k}{i} 2^{k-i} e^i = (2+e)^k.$$

This proves that $P_\phi = \lim_{k \rightarrow +\infty} \frac{1}{k} \log Z_{k,\phi} = \log(2+e)$.

Let $q \in \mathbb{R}$.

$$\begin{aligned} Z_{k,q(\phi-P_\phi)} &= \sum_{(i_1, \dots, i_k) \in \Lambda^k} e^{\sum_{0 \leq i \leq k} q(\phi(\sigma^k((x_n)_{n \in \mathbb{N}}) - P_\phi)} = \sum_{(i_1, \dots, i_k) \in \Lambda^k} e^{\sum_{1 \leq j \leq k} q(\phi(i_j) - P_\phi)} \\ &= e^{-kqP_\phi} \sum_{0 \leq i \leq k} \binom{k}{i} 2^{k-i} e^{qi} = e^{-kqP_\phi} (2 + e^q)^k. \end{aligned}$$

This proves that $P_{q(\phi-P_\phi)} = \log(2 + e^q) - q \log(2 + e)$.

□

The multifractal analysis of the measure μ_q is linked with the set $K_{1/3}^0$ as follows.

Lemma 9.1.4. *Let $x = (x_n)_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$ and $q \in \mathbb{R}$. One has*

$$\begin{aligned} \overline{\dim}(\mu_\phi, x) = \log(2 + e) &\Leftrightarrow \liminf_{k \rightarrow +\infty} \frac{\#\{0 \leq i \leq k : x_i = 1\}}{k+1} = 0 \\ &\Leftrightarrow \overline{\dim}(\mu_{q(\phi-P_\phi)}, x) = \log(2 + e^q) \Leftrightarrow \pi(x) \in K_{1/3}^0. \end{aligned} \quad (9.7)$$

Proof. By Theorem 9.1.2 and Lemma 9.1.3, for any $x = (x_n)_{n \in \mathbb{N}} \in \Lambda^{\mathbb{N}}$, there exists $C_1 > 0$ and $C_2 > 0$ such that for any $k \in \mathbb{N}$,

$$\frac{\log C_1}{k+1} \leq \frac{\log \mu_\phi([x_0, \dots, x_k])}{k+1} - \frac{\#\{0 \leq j \leq k : x_j = 1\}}{k+1} + \log(2 + e) \leq \frac{\log C_2}{k+1}. \quad (9.8)$$

This implies $\pi(x) \in K_{1/3}^0 \Leftrightarrow \liminf_{k \rightarrow +\infty} \frac{\#\{0 \leq i \leq k : x_i = 1\}}{k+1} = 0 \Leftrightarrow \overline{\dim}(\mu_\phi, x) = \log(2 + e)$.

Applying again Theorem 9.1.2 to $q(\phi - P_\phi)$, there exists $C'_1 > 0$ and $C'_2 > 0$ such that for any $k \in \mathbb{N}$,

$$C'_1 \leq \frac{\mu_{q(\phi-P_\phi)}([i])}{e^{\sum_{0 \leq j \leq k} \phi(x_j) - (k+1)qP_\phi - (k+1)P_{q(\phi-P_\phi)}}} \leq C'_2. \quad (9.9)$$

By Lemma 9.1.3, one has

$$e^{\sum_{0 \leq j \leq k} q\phi(x_j) - (k+1)qP_\phi - (k+1)P_{q(\phi-P_\phi)}} = e^{\sum_{0 \leq j \leq k} q\phi(x_j) - (k+1)\log(2+e^q)},$$

So that

$$\frac{\log C'_1}{k+1} \leq \frac{1}{k} \log(\mu_{q(\phi-P_\phi)}([i])) - \frac{\#\{0 \leq i \leq k : x_i = 1\}}{k+1} + \log(2 + e^q) \leq \frac{\log C'_2}{k+1}.$$

In particular it holds that

$$\overline{\dim}(\mu_{q(\phi-P_\phi)}, x) = \log(2 + e^q) \Leftrightarrow \liminf_{k \rightarrow +\infty} \frac{\#\{0 \leq i \leq k : x_i = 1\}}{k+1} = 0,$$

which finishes the proof. \square

Proposition 9.1.5. *One has $\dim_H(K_{1/3}^0) = \frac{\log 2}{\log 3}$.*

Proof. Note that $K_{1/3}^0$ contains the middle-third Cantor set $K_{1/3}$, so that

$$\dim_H(K_{1/3}^0) \geq \frac{\log 2}{\log 3}.$$

We prove that the converse inequality also holds true.

Set $s_0 = \frac{\log 2}{\log 3}$ and let $\varepsilon > 0$. Assume that $\mathcal{H}^{s_0+\varepsilon}(K_{1/3}^0) = +\infty$. Let $q \in \mathbb{R}$ small enough so that $\log(2 + e^q) \leq \log 2 + \frac{\log 3}{2}\varepsilon$.

Let $A \subset K_{1/3}^0$ be a compact set such that $\mathcal{H}^{s_0+\varepsilon}(A) \geq 6$. For every $y \in A$, there exists $x = (x_n)_{n \in \mathbb{N}}$ verifying $\pi(x) = y$ and $\overline{\dim}(\mu_{q(\phi-P_\phi)}, x) = \log(2 + e^q)$. Consider k_y large enough so that, for every $k \geq k_y$,

$$\mu_{q(\phi-P_\phi)}([x_0, \dots, x_{k-1}]) \geq e^{-k(\log(2+e^{-q}) + \frac{\varepsilon \log 3}{2})}.$$

Let k_A be large enough so that

$$\mathcal{H}^{s_0+\varepsilon}(\tilde{A} := \{y \in A : k_y \leq k_A\}) \geq 4.$$

Fix $0 < t < 3^{-k_A}$ and let $k \in \mathbb{N}$ be large enough so that $3^{-k} \leq t$. Set

$$\Lambda(\tilde{A}, k) = \left\{ [x_0, \dots, x_{k-1}] : \pi((x_n)_{n \in \mathbb{N}}) \in \tilde{A} \text{ and } \mu_{q(\phi-P_\phi)}([x_0, \dots, x_{k-1}]) \geq e^{-k(\log(2+e^{-q}) + \frac{\varepsilon \log 3}{2})} \right\}.$$

Note that $\{f_{\underline{i}}([0, 1]) : [\underline{i}] \in \Lambda(\tilde{A}, k)\}$ is a covering of \tilde{A} by balls of diameter smaller than $3^{-k} \leq t$ and

$$\begin{aligned} \sum_{[\underline{i}] \in \Lambda(\tilde{A}, k)} |f_{\underline{i}}([0, 1])|^{s_0+\varepsilon} &= \sum_{[\underline{i}] \in \Lambda(\tilde{A}, k)} 3^{-k(s_0+\varepsilon)} = \sum_{[\underline{i}] \in \Lambda(\tilde{A}, k)} e^{-k \log 3(s_0+\varepsilon)} \\ &= \sum_{[\underline{i}] \in \Lambda(\tilde{A}, k)} e^{-k(\log 2 + \varepsilon \log 3)} \leq \sum_{[\underline{i}] \in \Lambda(\tilde{A}, k)} e^{-k(\log(2+e^q) + \frac{1}{2}\varepsilon \log 3)} \\ &\leq \sum_{[\underline{i}] \in \Lambda(\tilde{A}, k)} \mu_{q(\phi-P_\phi)}([\underline{i}]) \leq 1. \end{aligned}$$

This implies that $4 \leq \mathcal{H}^{s_0+\varepsilon}(\tilde{A}) \leq 1$, which is absurd. One must have then $\mathcal{H}^{s_0+\varepsilon}(K_{1/3}^0) < +\infty$. Since this holds for any $\varepsilon > 0$, $\dim_H(K_{1/3}^0) \leq s_0 = \frac{\log 2}{\log 3}$. \square

9.2 Proof of Theorem 9.0.3

Let us first notice that by Proposition 9.1.5 and (9.1), one has

$$\dim_H \left(\limsup_{B \in \mathcal{Q}} B^\delta \cap K_{1/3}^{(0)} \right) \leq \min \left\{ \frac{1}{\delta}, \frac{\log 2}{\log 3} \right\}.$$

In particular, this proves that the expected upper-bound in Theorem 9.0.3 stands.

Before showing that the lower-bound also holds, let us start with some facts and remarks.

Remark 9.2.1. • One has $\mathcal{H}_\infty^{\frac{\log 2}{\log 3}}(K_{1/3}) > 0$ (this is well known and easily follows from the fact that $K_{1/3}$ carries an Alfhors regular measure of dimension $\frac{\log 2}{\log 3}$).

Moreover, for every $k \in \mathbb{N}$, setting $\mathcal{K}_k = \{f_{\underline{i}}([0, 1])\}_{\underline{i} \in \{0, 2\}^k}$, one has

$$1 = \sum_{I \in \mathcal{K}_k} |I|^{\frac{\log 2}{\log 3}}. \quad (9.10)$$

• For every $k \in \mathbb{N}$, let us define

$$\Omega_k = \bigcup_{I \in \mathcal{K}_k} \overset{\circ}{I}. \quad (9.11)$$

Since $\mathcal{H}_\infty^{\frac{\log 2}{\log 3}}(\bigcup_{I \in \mathcal{K}_k} I \setminus \Omega_k) = 0$ (it is a finite set of points), it follows from (9.10) that

$$C \mathcal{H}_\infty^{\frac{\log 2}{\log 3}}(\Omega_k) \leq \mathcal{H}_\infty^{\frac{\log 2}{\log 3}}(K_{1/3}) \leq \mathcal{H}_\infty^{\frac{\log 2}{\log 3}}\left(\bigcup_{I \in \mathcal{K}_k} I\right) = \mathcal{H}_\infty^{\frac{\log 2}{\log 3}}(\Omega_k) \leq 1, \quad (9.12)$$

with $C = \mathcal{H}_\infty^{\frac{\log 2}{\log 3}}(K_{1/3}) > 0$.

• If $n \in \mathbb{N}$ and $T \in \mathcal{T}_n = \{[\frac{k}{3^n}, \frac{k+1}{3^n}[, 0 \leq k \leq 3^n - 1\}$ is a triadic interval of generation n , denote by F_T the canonical homothetical mapping which sends $[0, 1]$ to \bar{T} . For every $I \in \bigcup_{J \in \mathcal{K}_k} F_T(J)$, for all $n \leq k' \leq n + k$ and all $x = (x_n)_{n \in \mathbb{N}} \in \Sigma$ such that $\pi(x) \in I$, one has

$$\frac{S_{n+k'}\phi(x)}{n+k'} = \frac{S_n\phi(x)}{n} \times \frac{n}{n+k'}. \quad (9.13)$$

We are now ready to finish the proof of Theorem 9.0.3.

Let $(\varepsilon_q)_{q \in \mathbb{N}}$ be a positive sequence such that $\lim_{q \rightarrow \infty} \varepsilon_q = 0$. One constructs a family $\{U_{p,q,\delta}\}_{\delta \geq 1, q \in \mathbb{N}, 0 \leq p \leq q}$ of open sets as follows: Let $\delta \geq 1$, $q \in \mathbb{N}^*$ and $0 \leq p \leq q$. Consider T a triadic interval of generation $n_q = \lfloor \log_3(q^{2\delta}) \rfloor + 1$ included in $B(\frac{p}{q}, q^{-2\delta})$. Let $N_{p,q,\delta}$ be large enough to ensure that for any $x \in \Sigma$ verifying $\pi(x) \in T$, one has

$$\frac{S_{n_q} \phi(x)}{n_q} \times \frac{n_q}{n_q + N_{p,q,\delta}} \leq \varepsilon_q. \quad (9.14)$$

Set

$$U_{p,q,\delta} = F_T(\Omega_{N_{p,q,\delta}}). \quad (9.15)$$

By (9.13) and (9.14), for all $x \in U_{p,q,\delta}$ one has

$$\frac{S_{n_q + N_{p,q}} \phi(x)}{n_q + N_{p,q}} \leq \varepsilon_q.$$

This implies that

$$\bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{0 \leq p \leq q} U_{p,q,\delta} \subset K_{1/3}^{(0)} \cap \bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{0 \leq p \leq q} B\left(\frac{p}{q}, q^{-2\delta}\right).$$

Since $U_{p,q,\delta}$ is an homothetic copy of $\Omega_{N_{p,q,\delta}}$ (see (9.15)), by (9.12), due to the choice of n_q there exists $\tilde{C} > 0$ independent of p , q and δ such that

$$\mathcal{H}_{\infty}^{\frac{\log 2}{\log 3}}(U_{p,q,\delta}) \geq \tilde{C} q^{-2\delta \frac{\log(2)}{\log(3)}}. \quad (9.16)$$

For $1 \leq \delta \leq \frac{\log 3}{\log 2}$, it follows that

$$\mathcal{H}_{\infty}^{\frac{\log 2}{\log 3}}(U_{p,q,\delta}) \geq \tilde{C} q^{-2} = \mathcal{L}\left(B\left(\frac{p}{q}, q^{-2}\right)\right). \quad (9.17)$$

For $\delta \geq \frac{\log 3}{\log 2}$, by concavity of $x \mapsto x^{\frac{\log 3}{\delta \log 2}}$,

$$\mathcal{H}_{\infty}^{\frac{1}{\delta}}(U_{p,q,\delta}) \geq (\mathcal{H}_{\infty}^{\frac{\log 2}{\log 3}}(U_{p,q,\delta}))^{\frac{\log 3}{\delta \log 2}} \geq \tilde{C} (q^{-2\delta \frac{\log 2}{\log 3}})^{\frac{\log 3}{\delta \log 2}} = \tilde{C} \mathcal{L}\left(B\left(\frac{p}{q}, q^{-2}\right)\right). \quad (9.18)$$

By Theorem 7.1.2 applied to $\mathcal{Q} = (B(\frac{p}{q}, \frac{1}{q^2}))_{q \in \mathbb{N}^*, 0 \leq p \leq q}$, $\mathcal{U} = (U_{p,q,\delta})_{q \in \mathbb{N}^*, 0 \leq p \leq q}$ and the Lebesgue measure, there exists a gauge function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $\lim_{r \rightarrow 0^+} \frac{\log \zeta(r)}{\log r} = s_0$ and

$$\mathcal{H}^{\zeta}\left(\limsup_{q \in \mathbb{N}^*, 0 \leq p \leq q} U_{p,q,\delta}\right) > 0.$$

Since $\limsup_{q \in \mathbb{N}^*, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta}}) \subset \{x : \xi(x) \geq 2\delta\}$, for $\delta \in [1, \frac{\log 3}{\log 2}[$ one gets

$$\dim_H \left(\{x : \xi(x) \geq 2\delta\} \cap K_{1/3}^{(0)} \right) = \frac{\log 2}{\log 3}.$$

For $\delta \geq \frac{\log 3}{\log 2}$, observe that

$$\{x : \xi(x) = 2\delta\} = \limsup_{q \in \mathbb{N}^*, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta}}) \setminus \bigcup_{\delta' > \delta} \limsup_{q \in \mathbb{N}^*, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta'}}).$$

Since for every $\delta' > \delta$, $\dim_H \limsup_{q \in \mathbb{N}^*, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta'}}) \cap K_{\frac{1}{3}}^{(0)} \leq \frac{1}{\delta'}$ and $\lim_{r \rightarrow 0^+} \frac{\log \zeta(r)}{\log r} = \frac{1}{\delta}$, one also has

$$\begin{aligned} & \mathcal{H}^\zeta(\{x : \xi(x) = 2\delta\} \cap K_{1/3}^{(0)}) \\ &= \mathcal{H}^\zeta \left(\left(\limsup_{q \in \mathbb{N}^*, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta}}) \setminus \bigcup_{\delta' > \delta} \limsup_{q \in \mathbb{N}^*, 0 \leq p \leq q} B(\frac{p}{q}, \frac{1}{q^{2\delta'}}) \right) \cap K_{1/3}^{(0)} \right) > 0 \end{aligned}$$

so that

$$\dim_H \left(\{x : \xi(x) = 2\delta\} \cap K_{1/3}^{(0)} \right) = \frac{1}{\delta}.$$

Chapter 10

Study of weakly conformal shrinking targets

In this chapter, the Hausdorff dimension of weakly conformal shrinking targets are investigated in a general frame.

Recall the notation of Definition 4.3.19. Our main result is the following.

Theorem 10.0.1. *Let $m \geq 2$ be an integer. Let $S = \{f_1, \dots, f_m\}$ be a C^1 weakly conformal IFS of an open set U with attractor K . Then:*

1. *For any $x_0 \in U$, for any $\delta < 1$,*

$$\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta) = K.$$

2. *For any $x_0 \notin K$, for any $\delta > 1$,*

$$\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta) = \emptyset. \quad (10.1)$$

3. *Assume in addition that $\dim_H(K) = \dim(S)$, then for any $x_0 \in K$, for any $\delta \geq 1$,*

$$\dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta) \right) = \frac{\dim_H(K)}{\delta}. \quad (10.2)$$

Let us provide concrete examples of IFS for which Theorem 10.0.1 applies and examples of IFS for which Theorem 10.0.1 does not:

- Theorem 10.0.1 applies to any self-similar IFS S on \mathbb{R} satisfying the ESC (Definition 4.3.3) with $\dim(S) \leq 1$. For instance, define g_1, g_2 and g_3 three mapping $\mathbb{R} \rightarrow \mathbb{R}$ by setting for every $x \in \mathbb{R}$, $g_1(x) = \frac{1}{4}x$, $g_2(x) = \frac{1}{4}(x+1)$

and $g_3(x) = \frac{1}{4}(x + t)$ where $t \in \mathbb{R} \setminus \mathbb{Q}$. Then Theorem 10.0.1 applies to $S = \{g_1, g_2, g_3\}$.

- Let $m \geq 2$ and $0 < c_1 < \dots < c_m < 1$ m real numbers satisfying

$$c_1 + \dots + c_m \leq 1.$$

Then for Lebesgue-almost every choice of $a_1, \dots, a_m \in \mathbb{R}$, Theorem 10.0.1 applies to the IFS $S = \{f_1, \dots, f_m\}$ where, for every $1 \leq i \leq m$, $f_i(x) = c_i x + a_i$.

- It is relatively easy to see that Theorem 10.0.1 cannot hold when $\dim(S) > d$. One could take for instance so many similarities so that for every $1 \leq \delta \leq 2$,

$$\dim_H \left(\limsup_{i \in \Lambda^*} B(f_i(x_0), |f_i(K)|^\delta) \right) = d.$$

To be more explicit, fix t a badly approximable (by rationals) number and define $\phi_1(x) = \frac{x}{2}$, $\phi_2(x) = \frac{x+1}{2}$, $\phi_3(x) = \frac{x+t}{2}$, $\phi_4(x) = \frac{x+1+t}{2}$ and $\Lambda = \{1, \dots, 4\}$. Then it is proved in [3, Theorem 2.10] that for every $x_0 \in [0, 1+t]$,

$$\dim_H(\limsup_{i \in \Lambda^*} B(\phi_i(x_0), \frac{1}{4^{|i|}})) = 1.$$

The next three sections, Sections 10.1, 10.2 and 10.3 are respectively dedicated to the proof of item (1), (2) and (3) of Theorem 10.0.1. The last section, Section 10.4 gives some complement when the system is self-similar.

10.1 Proof of item (1) of Theorem 10.0.1

Write $s = \dim_H(K)$. The notations of the proof of Theorem 6.2.4 are adopted in this section.

Lemma 10.1.1. *For any $x_0 \in U$ and any $\delta < 1$,*

$$\limsup_{i \in \Lambda^*} B(f_i(x_0), |f_i(K)|^\delta) = K.$$

Proof. Note first that, since K is the (compact) attractor of S ,

$$\limsup_{i \in \Lambda^*} B(f_i(x_0), |f_i(K)|^\delta) \subset K.$$

We now prove the converse inclusion.

Let $c > 1$. By Lemma 4.3.10 and Remark 4.3.11 applied with

$$X = \bigcup_{\underline{i} \in \Lambda^*} f_{\underline{i}}(x_0) \cup K,$$

there exists $D(c) > 0$ such that for any $y \in K$ and any $\underline{i} = (i_1, \dots, i_n) \in \Lambda^*$,

$$\|f_{\underline{i}}(x_0) - f_{\underline{i}}(y)\| \leq D(c)c^n \|f'_{\underline{i}}(y)\| \cdot \|x_0 - y\|. \quad (10.3)$$

By Lemma 4.3.10 and (4.34), one has

$$\|f'_{\underline{i}}(y)\| \leq D(c)c^n |f_{\underline{i}}(K)|. \quad (10.4)$$

Combining (10.3) and (10.4), one gets

$$\|f_{\underline{i}}(x_0) - f_{\underline{i}}(y)\| \leq \max_{z \in K} d(x_0, z) D(c)^2 c^{2n} |f_{\underline{i}}(K)|. \quad (10.5)$$

Recall that there exist $0 < t_1 < t_2$ so that, uniformly on n and $\underline{i} \in \Lambda^n$,

$$t_1 \leq \frac{\log \|f_{\underline{i}}\|}{n} \leq t_2.$$

Let $\varepsilon > 0$, taking $c = e^{\frac{t_1 \varepsilon}{4}}$ and writing $\kappa(S, \varepsilon, x_0) = \max_{z \in K} d(x, z) D(c)^2$, one gets

$$\|f_{\underline{i}}(x_0) - f_{\underline{i}}(y)\| \leq \kappa(S, \varepsilon, x_0) |f_{\underline{i}}(K)|^{1-\frac{\varepsilon}{2}}. \quad (10.6)$$

In particular, since $|f_{\underline{i}}(K)| \rightarrow 0$, for n large enough, for any $\underline{i} \in \Lambda^n$,

$$f_{\underline{i}}(K) \subset B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{1-\varepsilon}).$$

Recalling that

$$K = \bigcup_{\underline{i} \in \Lambda^n} f_{\underline{i}}(K),$$

one concludes that $K \subset \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{1-\varepsilon})$. \square

Remark 10.1.2. In the case where $S = \{f_1, \dots, f_m\}$ is a self-similar system, a more precise statement can be given. Denote $0 < c_1, \dots, c_m < 1$ the contracting ratio of respectively f_1, \dots, f_m . In the self-similar case, for any $z \in K$ and any $\underline{i} \in \Lambda^*$

$$d(f_{\underline{i}}(x_0), f_{\underline{i}}(z)) = c_{\underline{i}} d(x, z) \leq c_{\underline{i}} \max_{y \in K} d(y, x_0).$$

Writing $C(x_0, S) = \max_{y \in K} d(y, x)$, this implies that $f_{\underline{i}}(K) \subset B(f_{\underline{i}}(x_0), C(x_0, S)c_{\underline{i}})$ and

$$K = \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), C(x_0, S)c_{\underline{i}}).$$

10.2 Proof of item (2) of Theorem 10.0.1

Lemma 10.2.1. *For any $\delta > 1$, and any $x_0 \notin K$, one has*

$$\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta) = \emptyset.$$

Proof. We proceed by contradiction. Assume that $\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta) \neq \emptyset$. Consider $x \in \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta)$. Fix $n \geq 1$ and assume that there exists $\underline{i}_n = (i_1, \dots, i_n) \in \Lambda^n$ such that the set

$$S_{[\underline{i}_n]} = \left\{ \underline{j} = (j_1, \dots, j_l) \in [(i_1, \dots, i_n)] \cap \bigcup_{k \geq n} \Lambda^k \text{ such that } x \in B(f_{\underline{j}}(x_0), |f_{\underline{j}}(K)|^\delta) \right\}$$

is infinite. Then there must exist a letter $i_{n+1} \in \{1, \dots, m\}$ such that the set

$$S_{(\underline{i}_n, i_{n+1})} = \left\{ \underline{j} = (j_1, \dots, j_l) \in [(i_1, \dots, i_n, i_{n+1})] \cap \bigcup_{k \geq n} \Lambda^k \text{ such that } x \in B(f_{\underline{j}}(x_0), |f_{\underline{j}}(K)|^\delta) \right\}$$

is also infinite.

Proceeding recursively, one constructs a word $\underline{i} = (i_1, \dots) \in \Lambda^\mathbb{N}$ such that for an infinite number of integers $k \in \mathbb{N}$,

$$x \in B(f_{i_1} \circ \dots \circ f_{i_k}(x_0), |f_{\underline{i}}(K)|^\delta).$$

This implies that $x = \lim_{k \rightarrow +\infty} f_{i_1} \circ \dots \circ f_{i_k}(x_0)$, so that for any $n \in \mathbb{N}$,

$$x = f_{i_1} \circ \dots \circ f_{i_n} \left(\lim_{k \rightarrow +\infty} f_{i_{n+1}} \circ \dots \circ f_{i_{n+k}}(x_0) \right).$$

Writing $z_n = \lim_{k \rightarrow +\infty} f_{i_{n+1}} \circ \dots \circ f_{i_{n+k}}(x_0) \in K$, one has

$$x = f_{i_1} \circ \dots \circ f_{i_n}(z_n).$$

It follows that, for any $n \in \mathbb{N}$,

$$d(f_{i_1} \circ \dots \circ f_{i_n}(x_0), x) = d(f_{i_1} \circ \dots \circ f_{i_n}(x_0), f_{i_1} \circ \dots \circ f_{i_n}(z_n)). \quad (10.7)$$

Write $\underline{i} = (i_1, \dots, i_n)$ and let $\varepsilon > 0$ be small enough so that $1 \leq \frac{1+\varepsilon}{1-\varepsilon} < \delta$. By (4.35) and (4.36) applied with $\theta = \varepsilon$,

$$\begin{aligned} d(f_{\underline{i}}(x_0), f_{\underline{i}}(z_n)) &\geq \widehat{C}_\varepsilon^{-1} \|f'_{\underline{i}}(x)\|^{1+\varepsilon} \|x - z_n\| \\ &\geq d(x, K) \widehat{C}_\varepsilon^{-1} \left(|K| \widehat{C}_\varepsilon \right)^{\frac{1-\varepsilon}{1-\varepsilon}} |f_{\underline{i}}(K)|^{\frac{1+\varepsilon}{1-\varepsilon}} \end{aligned}$$

which implies that, for n large enough,

$$d(f_{\underline{i}}(x_0), x) > |f_{\underline{i}}(K)|^\delta. \quad (10.8)$$

This is a contradiction. \square

The following section is dedicated to the proof of item (3) of Theorem 10.0.1.

10.3 Proof of item (3) of Theorem 10.0.1

10.3.1 Variational principle and C^1 weakly conformal IFS

A modified version of a proposition of Feng and Hu used in the proof of their variational principle [37, Theorem 2.13] is needed to prove item (3) of Theorem 10.0.1. The following sub-section is dedicated to this modification.

The result from Feng-Hu we wish to modify is

Proposition 10.3.1 (Feng-Hu [37]). *Let $m \geq 2$ be an integer and $S = \{f_1, \dots, f_m\}$ a weakly conformal IFS. For any $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ as well as words $\underline{i}_1, \dots, \underline{i}_{n_\varepsilon} \in \Lambda^*$ such*

- *for any $1 \leq j < j' \leq n_\varepsilon$, $f_{\underline{i}_j}(K) \cap f_{\underline{i}_{j'}}(K) = \emptyset$,*
- *writing $S_\varepsilon = \{f_{\underline{i}_1}, \dots, f_{\underline{i}_{n_\varepsilon}}\}$, there exists a probability vector $P_\varepsilon = (p_1, \dots, p_{n_\varepsilon})$ such that the weakly conformal measure μ_ε associated with P_ε and S_ε satisfies $\dim_H(\mu_\varepsilon) \geq \dim_H(K) - \varepsilon$.*

Let us remark that, due to the first item, the IFS $S_\varepsilon = \{T_1, \dots, T_{n_\varepsilon}\}$ satisfies the SSC and might not have K as attractor. We wish to modify this proposition so that the attractor of the IFS S_ε can be taken equal to K .

Note also that in Proposition 10.3.1, because S_ε satisfies the SSC, the dimension of a weakly conformal measure associated with S_ε depends continuously on the choice of the probability vector. Moreover, writing ν_ε the canonical measure on the coding associated with μ_ε , then the Lyapunov exponent $\lambda_{\nu_\varepsilon} > 0$ (see Definition 4.3.12) satisfies for ν_ε -almost $(x_n)_{n \in \mathbb{N}}$ that

$$\lim_{n \rightarrow +\infty} \frac{\log |T_{x_1} \circ \dots \circ T_{x_n}(K)|}{n} = -\lambda_{\nu_\varepsilon}.$$

As announced above, one proves the following modified version of Proposition 10.3.1.

Proposition 10.3.2. *Let $\varepsilon_0 > 0$. There exists an IFS S_{ε_0} and a weakly conformal measure μ_{ε_0} (associated with S_{ε_0}) such that $\text{supp}(\mu_{\varepsilon_0}) = K$ and $\dim_H(\mu_{\varepsilon_0}) \geq s - \varepsilon_0$.*

Remark 10.3.3. *Similarly to the proof of [37, Theorem 2.13], Proposition 10.3.2 yields a measure on $\Lambda^{\mathbb{N}}$ and taking weak limits of ergodic averages of this measure gives an ergodic measure fully supported on K with dimension larger $s - \varepsilon_0$.*

Proof. Fix $\varepsilon = \frac{\varepsilon_0}{2} > 0$. Consider $S_\varepsilon = \{f_{i_1}, \dots, f_{i_{n_\varepsilon}}\}$, P_ε , μ_ε as in Theorem 10.3.1 and $0 < \varepsilon' < \frac{1}{5n_\varepsilon m} \cdot \min_{1 \leq i \leq m} p_i$.

Set

$$\begin{cases} g_j = f_j \text{ for } 1 \leq j \leq m \\ g_j = f_{i_{j-m}} \text{ for } m+1 \leq j \leq n_\varepsilon + m \end{cases}.$$

Also set $\tilde{S}_\varepsilon = \{g_1, \dots, g_{m+n_\varepsilon}\}$ and note that \tilde{S}_ε has attractor K . Denote by $\tilde{P}_{\varepsilon, \varepsilon'} = (\tilde{p}_1, \dots, \tilde{p}_{m+n_\varepsilon})$ the probability vector defined as

$$\begin{cases} \tilde{p}_j = \varepsilon' \text{ for } 1 \leq j \leq m \\ \tilde{p}_j = p_{j-m} - \frac{m}{n_\varepsilon} \varepsilon'. \end{cases}$$

Let $\mu_{\varepsilon, \varepsilon'}$ be the weakly conformal measure associated with \tilde{S}_ε and $\tilde{P}_{\varepsilon, \varepsilon'}$. Applying Theorem 4.3.15 to $\mu_{\varepsilon, \varepsilon'}$, let us prove that the corresponding h (see second item of Theorem 4.3.15) tends to 0 as ε' tends to 0.

Set $\Theta = \{1, \dots, n_\varepsilon + m\}$ and $\Theta^* = \bigcup_{k \geq 0} \Theta^k$. Let us denote by π_Θ the canonical projection of $\Theta^{\mathbb{N}}$ on K . One endows $\Sigma_\Theta = \Theta^{\mathbb{N}}$ with the metric d_Θ defined for any $x = (x_n), y = (y_n) \in \Sigma_\Theta$ by

$$d_\Theta(x, y) = e^{-\min\{i \in \mathbb{N} : x_i \neq y_i\}} \text{ and } d_\Theta(x, x) = 0. \quad (10.9)$$

Let us remark that the metric d allows to define on $\Theta^{\mathbb{N}}$ the Hausdorff dimension and the Packing dimension in a similar way than on \mathbb{R}^d .

Let $\nu_{\varepsilon, \varepsilon'} \in \mathcal{M}(\Theta^{\mathbb{N}})$ be the Bernoulli product verifying $\nu_{\varepsilon, \varepsilon'} \circ \pi_\Theta^{-1} = \mu_{\varepsilon, \varepsilon'}$.

By the strong law of large numbers, for every $x = (x_n)_{n \in \mathbb{N}}$ in a set $\tilde{\Sigma}_\Theta$ of $\nu_{\varepsilon, \varepsilon'}$ -full measure, there exists $N_x \in \mathbb{N}$ such that for any $n \geq N_x$, any $1 \leq i \leq n_\varepsilon + m$,

$$\left| \frac{\#\{1 \leq j \leq n : x_j = i\}}{n} - \tilde{p}_i \right| \leq \varepsilon'. \quad (10.10)$$

For $n \in \mathbb{N}$, write

$$A_n = \{x \in \tilde{\Sigma}_\Theta : N_x \leq n\}.$$

By Theorem 4.3.15, there exists N such that, using the notation involved,

$$\mu_{\varepsilon, \varepsilon'} \left(B_N = \left\{ y : \dim_H(\mu_{\varepsilon, \varepsilon'}^{-1}(\{y\})) = h \text{ and } \mu_{\varepsilon, \varepsilon'}^{-1}(\{y\})(A_N) \geq \frac{1}{2} \right\} \right) \geq \frac{1}{2}.$$

We fix such an N .

The following lemma is useful to estimate the number of cylinders of generation n which intersects A_N .

Lemma 10.3.4. *Consider $N \in \mathbb{N}$, $y \in K$ and $x = (x_n)_{n \in \mathbb{N}}, \tilde{x} = (\tilde{x}_n)_{n \in \mathbb{N}} \in \pi_{\Theta}^{-1}(\{y\})$. Assume that for every $1 \leq k \leq N$,*

$$(x_k \text{ or } \tilde{x}_k \in \{1, \dots, m\}) \Rightarrow (x_k = \tilde{x}_k).$$

Then, for every $0 \leq j \leq N$ such that $x_j \geq m+1$, one also has

$$x_j = \tilde{x}_j.$$

Proof. We proceed by contradiction. Suppose that the claim is not true and let $x_{j_0} \geq m+1$ be such for any $1 \leq i < j_0$, $\tilde{x}_i = x_i$ and $\tilde{x}_{j_0} \neq x_{j_0}$. Write

$$z = \lim_{k \rightarrow +\infty} g_{x_{j_0+1}} \circ g_{x_{j_0+2}} \circ \dots \circ g_{x_{j_0+k}}(0)$$

and

$$\tilde{z} = \lim_{k \rightarrow +\infty} g_{\tilde{x}_{j_0+1}} \circ g_{\tilde{x}_{j_0+2}} \circ \dots \circ g_{\tilde{x}_{j_0+k}}(0).$$

Then, recalling that $x, \tilde{x} \in \pi_{\Theta}^{-1}(\{y\})$,

$$g_{x_1} \circ \dots \circ g_{x_{j_0-1}} \circ g_{x_{j_0}}(z) = g_{x_0} \circ \dots \circ g_{x_{j_0-1}} \circ g_{\tilde{x}_{j_0}}(\tilde{z}) = y,$$

which implies that

$$g_{x_{j_0}}(z) = g_{\tilde{x}_{j_0}}(\tilde{z}).$$

Recalling that the system $\{g_{m+1}, \dots, g_{m+N_\varepsilon}\}$ satisfies the SSC, one also has

$$g_{x_{j_0}}(K) \cap g_{\tilde{x}_{j_0}}(K) = \emptyset$$

which yields a contradiction. □

Continuing the proof of the proposition, we note that, by (10.10), for every $y \in B_N$, $x = (x_n)_{n \in \mathbb{N}} \in \pi_{\Theta}^{-1}(\{y\}) \cap A_N$ and $N' \geq N$,

$$\#\{1 \leq k \leq N' : x_k \in \{1, \dots, m\}\} \leq 2m\varepsilon' N'. \quad (10.11)$$

Lemma 10.3.4 together with (10.11) yields for N large enough,

$$\# \{i \in \Theta^{N'} : [i] \cap A_N \cap \pi_{\Theta}^{-1}(\{y\}) \neq \emptyset\} \leq \sum_{k=0}^{\lfloor 2m\varepsilon' N' \rfloor + 1} \binom{N'}{k} m^k.$$

Since $\varepsilon' < \frac{1}{5m}$ so that $2m\varepsilon' N' < \frac{N'}{2}$,

$$\# \{i \in \Theta^{N'} : [i] \cap A_N \cap \pi_{\Theta}^{-1}(\{y\}) \neq \emptyset\} \leq (\lfloor 2m\varepsilon' N' \rfloor + 2) \binom{N'}{\lfloor 2m\varepsilon' N' \rfloor + 1} m^{\lfloor 2m\varepsilon' N' \rfloor + 1},$$

Using Stierling formula, provided that ε' was chosen small enough at start and N (so N' too) large enough, there exists a constant $C > 0$ such that

$$\begin{aligned} & \# \{i \in \Theta^{N'} : [i] \cap A_N \cap \pi_{\Theta}^{-1}(\{y\}) \neq \emptyset\} \\ & \leq C(\lfloor 2m\varepsilon' N' \rfloor + 2) \frac{(N')^{\lfloor 2m\varepsilon' N' \rfloor + 1} \cdot m^{\lfloor 2m\varepsilon' N' \rfloor + 1}}{\left(\frac{\lfloor 2m\varepsilon' N' \rfloor + 1}{e}\right)^{\lfloor 2m\varepsilon' N' \rfloor + 1} \sqrt{2\pi(\lfloor 2m\varepsilon' N' \rfloor + 1)}} \\ & \leq C(\lfloor 2m\varepsilon' N' \rfloor + 2) \left(\frac{mN'}{\frac{2m\varepsilon' N'}{e}}\right)^{\lfloor 2m\varepsilon' N' \rfloor + 1} \frac{1}{\sqrt{2\pi(\lfloor 2m\varepsilon' N' \rfloor + 1)}} \\ & \leq C(\lfloor 2m\varepsilon' N' \rfloor + 2) \left(\frac{e}{2\varepsilon'}\right)^{3mN'\varepsilon'} \\ & = C(\lfloor 2m\varepsilon' N' \rfloor + 2) e^{3mN'\varepsilon' \log \frac{e}{2\varepsilon'}} \leq e^{\sqrt{\varepsilon'} N'}. \end{aligned} \tag{10.12}$$

Since (10.12) holds for any $N' \geq N$, one obtains that

$$\dim_P(A_N \cap \pi_{\Theta}^{-1}(\{y\})) \leq \sqrt{\varepsilon'}.$$

Recalling that, by definition of B_N , the measure $\mu^{\pi_{\Theta}^{-1}(\{y\})}$ is h -exact dimensional, that $\mu^{\pi_{\Theta}^{-1}(\{y\})}(\pi_{\Theta}^{-1}(\{y\})) = 1$ and that $\mu^{\pi_{\Theta}^{-1}(\{y\})}(A_N) \geq \frac{1}{2}$, one has

$$\begin{aligned} h &= \inf \left\{ \dim_H(A), \text{ A Borel set satisfying } \mu^{\pi_{\Theta}^{-1}(\{y\})}(A) > 0 \right\} \\ &\leq \dim_P(A_N \cap \pi_{\Theta}^{-1}(\{y\})) \leq \sqrt{\varepsilon'}. \end{aligned}$$

By Remark 4.3.13 and the fourth item of Theorem 4.3.15, there exists a constant \tilde{C} depending on the system S such that

$$\dim_H(\mu_{\varepsilon, \varepsilon'}) \geq \frac{\dim_H(\nu_{\varepsilon, \varepsilon'})}{\lambda_{\nu_{\varepsilon, \varepsilon'}}} - \tilde{C}\sqrt{\varepsilon'}, \tag{10.13}$$

where $\lambda_{\nu_{\varepsilon, \varepsilon'}}$ is given by Definition 4.3.12. Also, by Corollary 4.3.14, for any Bernoulli product $\nu \in \mathcal{M}(\Theta)$ associated with a probability vector $\hat{P} \in (0, 1)^{n_{\varepsilon} + m}$, the Lyapunov exponent depends continuously on the vector \hat{P} . Recalling that $\Theta^{\mathbb{N}}$ is endowed with the metric given by (10.9), it is also classical that $\dim_H(\nu)$

depends continuously on the choice of \widehat{P} . Since $\lim_{\varepsilon' \rightarrow 0} P_{\varepsilon, \varepsilon'} = \{0\}^m \times P_\varepsilon$,

$$\lim_{\varepsilon' \rightarrow 0} \frac{\dim_H(\nu_{\varepsilon, \varepsilon'})}{\lambda_{\nu_{\varepsilon, \varepsilon'}}} = \frac{\dim_H(\nu_\varepsilon)}{\lambda_{\nu_\varepsilon}}. \quad (10.14)$$

Equation (10.14) combined with (10.13) proves that for ε' small enough, one has

$$\dim_H(\mu_{\varepsilon, \varepsilon'}) \geq \frac{\dim_H(\nu_\varepsilon)}{\lambda_{\nu_\varepsilon}} - 2\varepsilon \geq s - 2\varepsilon,$$

which concludes the proof of Proposition 10.3.2. \square

We can now finish the proof of item (3) of Theorem 10.0.1.

10.3.2 Proof of item(3) of Theorem 10.0.1

Let us recall that, by Proposition 4.3.16 and Definition 4.3.19, $\dim(S)$ verifies, for any $z \in K$,

$$P(\dim(S)) = \lim_{k \rightarrow +\infty} \frac{1}{k} \log \sum_{\underline{i} \in \Lambda^k} |f_{\underline{i}}(K)|^{\dim(S)} = 0.$$

Fix $x_0 \in K$, $\delta \geq 1$ and write

$$\mathcal{L}(\delta) = \limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^\delta).$$

Let us first show that $\dim_H(\mathcal{L}(\delta)) \leq \frac{\dim(S)}{\delta}$.

Let α and β be as in (4.37). If one must change the constants α and β , one can assume that there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ and every $\underline{i} \in \Lambda^k$

$$\alpha^k \leq |f_{\underline{i}}(K)| \leq \beta^k.$$

Recalling the definition of $\Lambda^{(k)}$ (4.25), for every $\underline{i} = (i_1, \dots, i_n) \in \Lambda^{(k)}$, one has

$$\alpha^n \leq 2^{-k} \leq \beta^{n-1} \Rightarrow n \leq k \frac{-\log(2)}{\log(\beta)} + 1 \leq 2k \frac{-\log(2)}{\log(\beta)}$$

In particular every integer $p \in \mathbb{N}$ and every $(i_{n+1}, \dots, i_{n+p}) \in \Lambda^p$ also verifying that $(i_1, \dots, i_{n+p}) \in \Lambda^{(k)}$ must satisfy

$$\beta^{n+p-1} \geq \alpha^n \Rightarrow p \leq n \times \left(\frac{\log \alpha}{\log \beta} - 1 \right) + 1 \leq 2n \left(\frac{\log \alpha}{\log \beta} - 1 \right) \leq k \times C(\alpha, \beta).$$

This implies that, for any $\nu \in \mathcal{M}(\Lambda^{\mathbb{N}})$,

$$\sum_{\underline{i} \in \Lambda^{(k)}} \nu([\underline{i}]) \leq kC(\alpha, \beta). \quad (10.15)$$

Consider $\varepsilon > 0$. Let us recall that Lemma 4.3.23 applied with

$$\varepsilon' = \frac{\frac{\varepsilon}{2} \log 2}{2 - \frac{\log(2)}{\log(\beta)}}$$

and $s = \dim(S)$ combined with Remark 4.3.24 yield a constant $\gamma_{\varepsilon'} > 0$ and a measure $\nu_{\varepsilon'} \in \mathcal{M}(\Lambda^{\mathbb{N}})$ such that for any $k \in \mathbb{N}$ and every $\underline{i} = (i_1, \dots, i_n) \in \Lambda^{(k)}$,

$$\gamma_{\varepsilon'}^{-1} e^{-k \frac{\varepsilon}{2} \log 2} |f_{\underline{i}}(K)|^{\dim(S)} \leq \nu_{\varepsilon'}([\underline{i}]) \leq \gamma_{\varepsilon'} e^{k \frac{\varepsilon}{2} \log 2} |f_{\underline{i}}(K)|^{\dim(S)}. \quad (10.16)$$

For any $\delta \geq 1$,

$$\begin{aligned} \sum_{\underline{i} \in \bigcup_{k \geq k_0} \Lambda^k} (|f_{\underline{i}}(K)|^{\delta})^{\frac{\dim(S) + \varepsilon}{\delta}} &= \sum_{\underline{i} = (i_1, \dots, i_n) \in \bigcup_{k \geq k_0} \Lambda^{(k)}} |f_{\underline{i}}(K)|^{\dim(S) + \varepsilon} \\ &\leq \sum_{k \geq k_0} \sum_{\underline{i} = (i_1, \dots, i_n) \in \Lambda^{(k)}} 2^{-k\varepsilon} \gamma_{\varepsilon'} e^{k \frac{\varepsilon}{2} \log 2} \nu_{\varepsilon'}([\underline{i}]) \quad (10.17) \end{aligned}$$

$$\leq \gamma_{\varepsilon'} C(\alpha, \beta) \sum_{k \geq k_0} k 2^{-k \frac{\varepsilon}{2}} < +\infty. \quad (10.18)$$

As a consequence,

$$\dim_H(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta})) \leq \frac{\dim(S) + \varepsilon}{\delta},$$

and letting ε tend to 0 establishes the upper-bound.

Now we prove that

$$\dim_H(\mathcal{L}(\delta)) \geq \frac{\dim(S)}{\delta}.$$

Let $\varepsilon > 0$ and μ_{ε} be a weakly conformal measure as in Proposition 10.3.2. For any $k \in \mathbb{N}$, the balls $\{B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|)\}_{\underline{i} \in \Lambda^k}$ are centered on $K = \text{supp}(\mu)$ and their limsup covers K . This implies that $\mu_{\varepsilon}(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|)) = 1$.

Applying Theorem 7.3.1, one gets

$$\frac{s - \varepsilon}{\delta} \leq \dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x_0), |f_{\underline{i}}(K)|^{\delta}) \right).$$

Letting $\varepsilon \rightarrow 0$ finishes the proof.

10.4 Some complements in the case of self-similar IFS

Some cases of self-similar shrinking targets with overlaps have recently been studied in [3], where the following theorem is proved.

Theorem 10.4.1 ([3]). *Let $m \geq 2$ and $S = \{f_1, \dots, f_m\}$ be a system of m similarities of contraction ratio $0 < c_1, \dots, c_m < 1$. Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ be the self-similar measure solution to*

$$\mu(\cdot) = \sum_{i=1}^m c_i^{\dim(S)} \mu(f_i^{-1}(\cdot)). \quad (10.19)$$

Let $g : \mathbb{N} \rightarrow (0, +\infty)$ be a non increasing mapping. Assume that one of the following condition is satisfied:

$$\begin{cases} \sum_{i=1}^m -c_i^{\dim(S)} \log(c_i^{\dim(S)}) < -2 \log \left(\sum_{i=1}^m c_i^{2\dim(S)} \right) \\ \text{or } c_1 = \dots = c_m. \end{cases} \quad (10.20)$$

If

$$\sum_{k \in \mathbb{N}} \sum_{\underline{i} \in \Lambda^k} k \left(|f_{\underline{i}}(K)| g(k) \right)^{\dim(S)} = +\infty,$$

then $\mu \left(\limsup_{\underline{i} \in \Lambda^} B(f_{\underline{i}}(x), |f_{\underline{i}}(K)| g(|\underline{i}|)) \right) = 1$.*

Thanks to Theorem 7.1.2 we are able to complete Theorem 10.4.1.

Theorem 10.4.2 (D. [23]). *Let $g : \mathbb{N} \rightarrow (0, +\infty)$ be a non increasing mapping, define*

$$s_g = \inf \left\{ s \geq 0 : \sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^k} k \left(|f_{\underline{i}}(K)| g(k) \right)^s < +\infty \right\}. \quad (10.21)$$

Assume that (10.20) is satisfied and that $\dim(\mu) = \dim(S)$. Then

$$\begin{cases} \dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B \left(f_{\underline{i}}(x), \left(|f_{\underline{i}}(K)| g(|\underline{i}|) \right)^{\frac{\delta s_g}{\dim(S)}} \right) \right) = \dim(S) \text{ if } 0 < \delta \leq 1 \\ \dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B \left(f_{\underline{i}}(x), \left(|f_{\underline{i}}(K)| g(|\underline{i}|) \right)^{\frac{\delta s_g}{\dim(S)}} \right) \right) = \frac{\dim(S)}{\delta} \text{ if } \delta \geq 1. \end{cases} \quad (10.22)$$

Remark 10.4.3. *As mentioned in Section 4.3, $\dim(\mu) = \dim(S)$ holds in many situations. For instance, any self-similar IFS acting on \mathbb{R}^d with similarity dimension less than d and satisfying Hochman's separation and irreducibility conditions [43] (Definition 4.3.3) satisfies this property.*

Proof. Let us first check that

$$\dim_H \left(\limsup_{\underline{i} \in \Lambda^*} B(f_{\underline{i}}(x), (|f_{\underline{i}}(K))|g(|\underline{i}|))^{\delta \frac{sg}{\dim(S)}} \right) \leq \frac{\dim(S)}{\delta}.$$

Let $\varepsilon > 0$. Recalling (10.21), one has

$$\sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^k} \left((|f_{\underline{i}}(K))|g(k) \right)^{\delta \frac{sg}{\dim(S)}} \frac{\dim(S) + \varepsilon}{\delta} \leq \sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^k} k \left(|f_{\underline{i}}(K))|g(k) \right)^{sg \frac{\dim(S) + \varepsilon}{\dim(S)}} < +\infty.$$

This proves that $\dim_H \limsup_{\underline{i} \in \Lambda^*} B \left(f_{\underline{i}}(x), (|f_{\underline{i}}(K))|g(|\underline{i}|) \right)^{\delta \frac{sg}{\dim(S)}} \leq \frac{\dim(S) + \varepsilon}{\delta}$.
Letting $\varepsilon \rightarrow 0$ concludes this part of the proof.

Now we prove that

$$\dim_H \limsup_{\underline{i} \in \Lambda^*} B \left(f_{\underline{i}}(x), (|f_{\underline{i}}(K))|g(|\underline{i}|) \right)^{\delta \frac{sg}{\dim(S)}} \geq \frac{\dim(S)}{\delta}.$$

Let $\varepsilon > 0$. Note that by (10.21)

$$\sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^k} k \left(|f_{\underline{i}}(K))|g(k) \right)^{\frac{sg}{1+\varepsilon}} = +\infty.$$

By Theorem 10.4.1, for any $\varepsilon > 0$,

$$\mu \left(\limsup_{\underline{i} \in \Lambda^*} B \left(f_{\underline{i}}(x), (|f_{\underline{i}}(K))|g(|\underline{i}|) \right)^{\frac{sg}{(1+\varepsilon)\dim(S)}} \right) = 1.$$

Using Theorem 7.3.1, one gets

$$\dim_H \limsup_{\underline{i} \in \Lambda^*} B \left(f_{\underline{i}}(x), (|f_{\underline{i}}(K))|g(|\underline{i}|) \right)^{\delta \frac{sg}{\dim(S)}} \geq \frac{\dim_H(\mu)}{(1+\varepsilon)\delta}.$$

Letting $\varepsilon \rightarrow 0$, one has $\dim_H \limsup_{\underline{i} \in \Lambda^*} B \left(f_{\underline{i}}(x), (|f_{\underline{i}}(K))|g(|\underline{i}|) \right)^{\delta \frac{sg}{\dim(S)}} \geq \frac{\dim(S)}{\delta}$ and this ends the proof. \square

Chapter 11

Multifractal analysis of discrete self-similar measures

11.1 Definition and first result

Let us fix $m \geq 2$ and $S = \{f_1, \dots, f_m\}$, a system of m similarities of \mathbb{R}^d of contracting ratio $0 < c_1, \dots, c_m < 1$ and of attractor K . In this section, the notation of Section 4.3 are adopted.

Let $\eta \in \mathcal{M}(\mathbb{R}^d)$ be a measure and $(q_1, \dots, q_{m+1}) \in (0, 1)^{m+1}$ a probability vector. One introduces the following perturbed self-similarity equation (of variable $\chi \in \mathcal{M}(\mathbb{R}^d)$):

$$\chi(\cdot) = \sum_{i=1}^m q_i \chi(f_i^{-1}(\cdot)) + q_{m+1} \eta(\cdot). \quad (11.1)$$

Such measures have been introduced by Snigireva and Olsen in [70] and are called in-homogeneous self-similar measures with condensation measure η .

Proposition 11.1.1 ([26]). *Equation (11.1) admits a unique solution ([70]) which is*

$$\chi(\cdot) = q_{m+1} \sum_{\underline{i} \in \Lambda^*} q_{\underline{i}} \eta(f_{\underline{i}}^{-1}(\cdot)). \quad (11.2)$$

Proof. The existence and the unicity of the solution of (11.1) is established in [70] but for the convenience of the reader, we include the proof here.

Let us recall the definition of the Wasserstein distance on $\mathcal{M}(\mathbb{R}^d)$.

Definition 11.1.2 ([39]). *Let $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ be two compactly supported probability measures and denote by $\text{Lip}_1(\mathbb{R}^d)$ the set of 1-Lipshitz mapping from \mathbb{R}^d*

to \mathbb{R} . The Wasserstein distance between μ and ν is

$$d(\mu, \nu) = \sup \left\{ \left| \int f d\mu - \int f d\nu \right| : f \in \text{Lip}_1(\mathbb{R}^d) \right\}. \quad (11.3)$$

It is known that $(\mathcal{M}(\mathbb{R}^d), d)$ is complete [39].

Let us define $\Phi : \mathcal{M}(\mathbb{R}^d) \rightarrow \mathcal{M}(\mathbb{R}^d)$ by

$$\Phi(\chi) = \sum_{i=1}^m q_i \chi(f_i^{-1}(\cdot)) + q_{m+1} \eta(\cdot). \quad (11.4)$$

For any $\chi \in \mathcal{M}(\mathbb{R}^d)$ and $f \in \text{Lip}_1(\mathbb{R}^d)$. One has

$$\int f d\Phi(\chi) = \sum_{i=1}^m q_i \int f \circ f_i d\chi + q_{m+1} \int f d\eta = \sum_{i=1}^m q_i c_i \int c_i^{-1} f \circ f_i d\chi + q_{m+1} \int f d\eta.$$

Note that, for any $1 \leq i \leq m$, $c_i^{-1} f \circ f_i$ is 1-Lipshitz. In particular, for any $\chi' \in \mathcal{M}(\mathbb{R}^d)$, one has

$$\begin{aligned} \left| \int f d\Phi(\chi) - \int f d\Phi(\chi') \right| &= \left| \sum_{i=1}^m q_i c_i \left(\int c_i^{-1} f \circ f_i d\chi - \int c_i^{-1} f \circ f_i d\chi' \right) \right| \\ &\leq \sum_{i=1}^m q_i c_i \left| \int c_i^{-1} f \circ f_i d\chi - \int c_i^{-1} f \circ f_i d\chi' \right| \\ &\leq \left(\sum_{i=1}^m q_i c_i \right) d(\chi, \chi'). \end{aligned}$$

Taking the supremum over $f \in \text{Lip}_1(\mathbb{R}^d)$,

$$d(\Phi(\chi), \Phi(\chi')) \leq \left(\sum_{i=1}^m q_i c_i \right) d(\chi, \chi'). \quad (11.5)$$

Since $\sum_{i=1}^m q_i c_i < 1$, Φ is contractive and admits a unique fixed point. Let us prove that the measure χ given by (11.2) is indeed this fixed point:

$$\begin{aligned} \Phi(\chi) &= \sum_{i=1}^m q_i \chi(f_i^{-1}(\cdot)) + q_{m+1} \eta(\cdot) \\ &= \sum_{i=1}^m q_i q_{m+1} \sum_{\underline{i} \in \Lambda^*} q_{\underline{i}} f_{\underline{i}}^{-1} \circ \eta(f_i^{-1}(\cdot)) + q_{m+1} \eta(\cdot) \\ &= q_{m+1} \left(\sum_{\underline{i} \in \bigcup_{k \geq 1} \Lambda_k} q_{\underline{i}} \eta(f_{\underline{i}}^{-1}(\cdot)) + \eta \right) = q_{m+1} \sum_{\underline{i} \in \Lambda^*} q_{\underline{i}} \eta(f_{\underline{i}}^{-1}(\cdot)) = \chi. \end{aligned}$$

This finishes the proof of Proposition 11.1.1. \square

Corollary 11.1.3. *Let $t > 0$ be the solution to the equation*

$$\sum_{i=1}^m c_i^{-t} q_i = 1, \quad (11.6)$$

and let us introduce the probability vector $(p_1, \dots, p_m) = (c_1^{-t} q_1, \dots, c_m^{-t} q_m)$.

When $\eta = \delta_{x_0}$, where $x_0 \in \mathbb{R}^d$ and δ_x denotes the Dirac measure at x , the measure χ given by (11.2) is

$$\chi(\cdot) = q_{m+1} \sum_{\underline{i} \in \Lambda^*} q_{\underline{i}} \delta_{f_{\underline{i}}(x_0)} = q_{m+1} \sum_{\underline{i} \in \Lambda^*} c_{\underline{i}}^t p_{\underline{i}} \delta_{f_{\underline{i}}(x_0)}. \quad (11.7)$$

Measures which can be written as the right-hand side of (11.7) are called discrete self-similar measures and their multifractal properties of comparable measures have been studied in depth by Barral and Seuret in [7, 9]. In the following sections, one makes good use of the techniques they developed to study the multifractal spectrum and the scaling function of χ .

In order to cover the largest class of self-similar measure we can, our result will be stated for IFS satisfying the AWSC (Definition 4.3.3) with no exact overlaps. Let us recall here that it is proved in [4] that this class of self-similar IFS contains the class of self-similar IFS on the real line satisfying ESC with $\dim(S) \leq 1$. For explicit examples of IFS satisfying this condition, see Corollary 4.3.27. One also adds that, since the proof of this result relies on Shmerkin's theorem [68] about the L^q spectrum of self-similar measures on the real line, if a similar result held for self-similar measure satisfying ESC and Hochman's irreducibility assumption on \mathbb{R}^d , then our theorem would also cover the case of IFS satisfying those assumption on \mathbb{R}^d with $\dim(S) \leq d$.

Theorem 11.1.4 (D. [26]). *Let $m \geq 2$, $S = \{f_1, \dots, f_m\}$ be a self-similar IFS satisfying AWSC (Definition 4.3.3) with no exact overlaps and $K \subset \mathbb{R}^d$ the attractor of S . Let (p_1, \dots, p_m) be a probability vector, μ the self-similar measure associated with (p_1, \dots, p_m) , $x_0 \in K$ and $C > 0$ the normalizing constant such that the measure*

$$\chi(\cdot) = C \sum_{\underline{i} \in \Lambda^*} c_{\underline{i}}^t p_{\underline{i}} \delta_{f_{\underline{i}}(x_0)},$$

is a probability measure.

Define $q_c = \min \{q : \tau_\mu(q) + qt = 0\}$ and $\lambda_c = \tau'_\mu(q_c)$. Then, recalling (4.16):

1. The mapping τ_χ satisfies

$$\begin{cases} \text{for every } q \geq q_c, & \tau_\chi(q) = 0 \\ \text{for every } 0 \leq q \leq q_c, & \tau_\chi(q) = \tau_\mu(q) + t \\ \text{for every } q \leq 0, & \tau_\chi(q) \geq \tau_\mu(q) + t. \end{cases} \quad (11.8)$$

In particular for every $h \geq 0$, one has

$$\tau_\chi^*(h) \leq \begin{cases} hq_c & \text{if } 0 \leq h \leq \lambda_c + t \\ \tau_\mu^*(h - t) & \text{if } h \geq \lambda_c + t. \end{cases} \quad (11.9)$$

2. The multifractal spectrum D_χ of χ satisfies

$$\begin{cases} \text{for every } 0 < h \leq \lambda_c + t, & D_\chi(h) = hq_c \\ \text{for every } \lambda_c + t < h \leq \tau_\mu'(0^+) + t, & D_\chi(h) = \tau_\mu^*(h - t). \end{cases} \quad (11.10)$$

3. For every $h > \tau_\mu'(0^+) + t$, one has

$$\dim_H(\widetilde{E}_\mu^{h-t} \cap \{x : \Delta_x = 1\}) \leq D_\chi(h) \leq \tau_\mu^*(h - t).$$

Remark 11.1.5. • If S satisfies the AWSC and has no exact overlaps, q_c and $\tau_\mu'(q_c)$ exists by Theorem 4.3.39.

- Assume furthermore that for $h \geq \tau_\mu'(0^+)$, there exists $m_h \in \mathcal{M}(\mathbb{R}^d)$ such that

$$\dim_H(m_h) = \tau_\mu^*(h) \text{ and } m_h(\widetilde{E}_\mu^h) > 0,$$

then

$$D_\chi(h + t) = \tau_\mu^*(h)$$

and χ satisfies the multifractal formalism at $h + t$.

- As a consequence of Theorem 11.1.4, one recovers the estimates of the scaling function and the multifractal spectrum established by Snigireva and Olsen in [70] only for $q \geq 1$ and $q \leq 0$ in the case where the IFS satisfies the strong separation condition and the condensation measure is a Dirac (to see this, note here that the notation of [70] involves the mapping $\beta(q) = -(T(q) + qt)$ and that the sign of $T(q) + qt$ depends on the fact that $q \leq q_c$ or $q \geq q_c$). Regarding the multifractal spectrum, since only limits (not \liminf) are considered in [71], one only recovers their estimates on the multifractal spectrum for $q \geq q_c$.

- The proof of item (1) of Theorem 11.1.4 relies on the existence of self-similar auxiliary measures for μ . It is worth mentioning that in the more general case of systems satisfying AWSC with possible exact overlaps, it is known that μ satisfies the multifractal formalism and auxiliary measures (that are not self-similar) exists [35, 36] for every $h \in (\tau'_\mu(+\infty), \tau'_\mu(0^+))$ even if it is not proved that τ_μ is differentiable on this interval. As a consequence, one might expect that item (1) of Theorem 11.1.4 actually holds assuming only that S satisfies AWSC. We also underline here that, although it is known that IFS on the real line satisfying exponential separation satisfies the multifractal formalism on $\mathbb{R}^+ \setminus I$ for I an interval (see [4, 68]) studying discrete measures under this only assumption should raise problem of a different nature (such as dealing with non weakly redundant sequences of balls).
- Even under assumptions that are stronger than AWSC (with possible exact overlaps), there are self-similar measures which does not satisfy the multifractal formalism for every $h \geq \tau'_\mu(0^+)$ [72]. As a consequence, there is a priori no reason that anything can be said in all generality about χ for h in this interval but in many concrete cases, it is still possible to compute the spectrum of μ for $q < 0$ (see [66, 67, 72]) and one might expect to be able to do it as well for the measure χ .

Note that the case $x_0 \notin K$ can also be dealt with.

Proposition 11.1.6. *Let $m \geq 2$ and $S = \{f_1, \dots, f_m\}$ be a self-similar IFS and K the attractor of S . Let (p_1, \dots, p_m) be a probability vector, $x_0 \notin K$, $t > 0$ and*

$$\chi(\cdot) = \sum_{\underline{i} \in \Lambda^*} p_{\underline{i}} c_{\underline{i}}^t \delta_{f_{\underline{i}}(x_0)}. \quad (11.11)$$

Then for any $h \geq 0$

$$D_\chi(h) = D_\mu(h - t). \quad (11.12)$$

Moreover the estimates of τ_χ given by Theorem 11.1.4 still holds.

Remark 11.1.7. *An interesting consequence is that no separation condition is assumed in Proposition 11.1.6 and D_μ is easily deduced from the spectrum D_χ . For instance if $\mathcal{L}^d(K) = 0$, then, for \mathcal{L}^d -almost every choice of x_0 , D_χ is a translation of D_μ . One must add to this remark that, since it will be proved that $\tau_\chi(q) = 0$ for $q \geq 1$, the measure χ does not satisfy the multifractal formalism in that case.*

To finish this section we recall that examples of IFS's satisfying AWSC with no exact overlaps are given in Corollary 4.3.27.

In the following section we prove Proposition 11.1.6 and in Section 11.2 and 11.4 Theorem 11.1.4 is proved. More precisely, we first study the scaling function τ_χ (Section 11.2) and then we study the multifractal spectrum of χ .

11.2 Scaling function of discrete self-similar measures

Let $(p_1, \dots, p_m) \in (0, 1)^m$, $\chi \in \mathcal{M}(\mathbb{R}^d)$ given by (11.13) and μ the self-similar measure associated with (p_1, \dots, p_m) and S .

Let $(p_1, \dots, p_m) \in (0, 1)^m$ be a probability vector, $x_0 \in K$, $t > 0$ and $\chi \in \mathcal{M}(\mathbb{R}^d)$ defined by

$$\chi(\cdot) = C \sum_{\underline{i} \in \Lambda^*} c_{\underline{i}}^t p_{\underline{i}} \delta_{f_{\underline{i}}(x_0)}, \quad (11.13)$$

where $C > 0$ is chosen so that χ is a probability measure. In this section we study the scaling function τ_χ associated with χ . **For the sake of simplicity (and without loss of generality), in the two following sections, one assumes that $|K| = 1$.** In particular, for any $k \in \mathbb{N}$,

$$\Lambda^{(k)} = \{\underline{i} = (i_1, \dots, i_n) \in \Lambda^* : c_{\underline{i}} \leq 2^{-k} < c_{(i_1, \dots, i_{n-1})}\}.$$

Let us define μ the self-similar measure associated with S and (p_1, \dots, p_m) (i.e satisfying (4.22)). Recalling (4.78), define

$$q_c = \min \left\{ q : \frac{T(q)}{q} = -t \right\}. \quad (11.14)$$

Let us prove that the equation $\frac{T(q)}{q} = -t$ admits a unique solution.

It is known (e.g [32], p 287) that T is C^∞ and strictly concave as soon as there exists $i \neq j \leq m$ such that

$$\frac{\log c_i}{\log p_i} \neq \frac{\log c_j}{\log p_j}.$$

In that case, setting $f(q) = T(q) + tq$, one has

$$f'(q) = T'(q) + t.$$

For $q < 0$ and $q > 1$, one has respectively $T(q) < 0$ and $T(q) > 0$, so that $\frac{T(q)}{q} > 0$ and the equation $\frac{T(q)}{q} = -t$ does not admit a solution in $(-\infty, 0] \cup [1, +\infty)$. For $q \in (0, 1)$, one has $T(q) < 0$ and T is increasing so that $f'(q) > 0$. Since $f(0) = -\dim(S)$ and $f(1) = t$, the equation $f(q) = 0$ admits a unique solution.

In the case where one has for every $1 \leq i \neq j \leq m$,

$$\frac{\log c_i}{\log p_i} = \frac{\log c_j}{\log p_j} = \alpha,$$

it is easily verified that $\alpha = \dim(S)$ and $T(q) = \dim(S)q - \dim(S)$ so that $\frac{T(q)}{q} = -t$ admits a unique solution as well.

We now estimates the mapping τ_χ . The estimates are splitted into 3 lemmas.

Lemma 11.2.1. *For every $q > q_c$, one has*

$$\tau_\chi(q) = 0. \quad (11.15)$$

Proof. Write $\widehat{p}_i = c_i^t p_i$ and for every $q \in \mathbb{R}$,

$$M(q) = \sum_{i=1}^m \widehat{p}_i^q. \quad (11.16)$$

Recalling (4.78), one has

$$M(q) = 1 \Leftrightarrow \sum_{i=1}^m c_i^{tq} p_i^q = 1 \Leftrightarrow tq = -T(q) \Leftrightarrow t = \frac{-T(q)}{q},$$

so that

$$M(q) < 1 \Leftrightarrow q > q_c$$

where $q_c = \inf \left\{ q : \frac{T(q)}{q} = -t \right\}$.

Let us fix $q > q_c$.

It is known that if a measure $\eta \in \mathcal{M}(\mathbb{R}^d)$ satisfies $\eta(\{x\}) > 0$ for some $x \in \mathbb{R}^d$, then $\tau_\eta(q) = 0$ for every $q \geq 1$. For the sake of completeness, the proof is recalled here. Fix $\eta \in \mathcal{M}(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ as above, $q \geq 1$ and $n \in \mathbb{N}$. Write $D_n(x) \in \mathcal{D}_n$ the dyadic cube of generation $n \in \mathbb{N}$ containing x . One has

$$1 \geq \sum_{D \in \mathcal{D}_n} \eta(D)^q \geq \eta(D_n(x))^q \geq (\eta(\{x\}))^q, \quad (11.17)$$

which implies that

$$\lim_{n \rightarrow +\infty} \frac{\log \left(\sum_{D \in \mathcal{D}_n} \eta(D)^q \right)}{n} = 0.$$

We now deal with the case $q_c < q < 1$. For every $n \in \mathbb{N}$, one has

$$\begin{aligned} 1 &\leq \sum_{D \in \mathcal{D}_n} \chi(D)^q = C^q \sum_{D \in \mathcal{D}_n} \left(\sum_{\underline{i}: f_{\underline{i}}(x_0) \in D} c_{\underline{i}}^t p_{\underline{i}} \right)^q \leq C^q \sum_{D \in \mathcal{D}_n} \sum_{\underline{i}: f_{\underline{i}}(x_0) \in D} \widehat{p}_{\underline{i}}^q \quad (11.18) \\ &= C^q \sum_{\underline{i} \in \Lambda^*} \widehat{p}_{\underline{i}}^q = C^q \sum_{k \geq 0} M(q)^k = C^q \frac{1}{1 - M(q)}. \end{aligned}$$

Equations (11.17) and (11.18) together prove that, for any $q > q_c$,

$$\tau_\chi(q) = \lim_{n \rightarrow +\infty} \frac{-\log(\sum_{D \in \mathcal{D}_n} \chi(D)^q)}{n} = 0.$$

□

Lemma 11.2.2. *For every $0 \leq q \leq q_c$, one has*

$$T(q) + qt \leq \tau_\chi(q) \leq \tau_\mu(q) + qt. \quad (11.19)$$

Proof. We first prove that $T(q) + qt \leq \tau_\chi(q)$.

Consider $0 \leq q \leq q_c$, for any $n \in \mathbb{N}$, $D_0 \in \mathcal{D}_n$, one has

$$\begin{aligned} \chi(D_0)^q &\leq \left(\sum_{\underline{i} \in \bigcup_{k \geq n} \Lambda^{(k)}: f_{\underline{i}}(x_0) \in D_0} c_{\underline{i}}^t p_{\underline{i}} \right)^q + \left(\sum_{\underline{i} \in \bigcup_{0 \leq k < n} \Lambda^{(k)}: f_{\underline{i}}(x_0) \in D_0} c_{\underline{i}}^t p_{\underline{i}} \right)^q \\ &\leq \sum_{k \geq n} 2^{-kqt} \left(\sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D_0} \neq \emptyset} \mu(D') \right)^q + \sum_{k=0}^{n-1} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in D_0} c_{\underline{i}}^{tq} p_{\underline{i}}^q \\ &\leq \frac{2^{-qnt}}{1 - 2^{-qt}} \times \left(\sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D_0} \neq \emptyset} \mu(D') \right)^q + \sum_{k=0}^{n-1} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in D_0} c_{\underline{i}}^{tq+T(q)} c_{\underline{i}}^{-T(q)} p_{\underline{i}}^q \\ &\leq \frac{2^{-qnt}}{1 - 2^{-qt}} \times \left(\sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D_0} \neq \emptyset} \mu(D') \right)^q + \sum_{k=0}^{n-1} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in D_0} 2^{-(k-1)(tq+T(q))} c_{\underline{i}}^{-T(q)} p_{\underline{i}}^q. \end{aligned} \quad (11.20)$$

It also holds that

$$\left(\sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D_0} \neq \emptyset} \mu(D') \right)^q \leq (2d)^q \max_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D_0} \neq \emptyset} \mu(D')^q \leq 2^{dq} \sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D_0} \neq \emptyset} \mu(D')^q. \quad (11.21)$$

Combining (11.20) and (11.21), recalling (4.25) and (4.78), one gets

$$\begin{aligned} \sum_{D \in \mathcal{D}_n} \chi(D)^q &\leq 2^{-qnt} \times \frac{2^{dq}}{1 - 2^{-qt}} \times \sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D} \neq \emptyset} \mu(D')^q + \sum_{k=0}^{n-1} \sum_{\underline{i} \in \Lambda^{(k)}} 2^{-(k-1)(tq+T(q))} c_{\underline{i}}^{-T(q)} p_{\underline{i}}^q \\ &\leq 2^{-qnt} \times \frac{2^{dq+d}}{1 - 2^{-qt}} \sum_{D \in \mathcal{D}_n} \mu(D)^q + 2 \cdot 2^{-n(qt+T(q))}, \end{aligned} \quad (11.22)$$

Recalling Remark 4.3.32 items (1) and (5), there exists a constant $C > 0$ such that for ever $k \in \mathbb{N}$

$$\sum_{\underline{i} \in \Lambda^{(k)}} 2^{-(k-1)(tq+T(q))} c_{\underline{i}}^{-T(q)} p_{\underline{i}}^q \leq C \sum_{\underline{i} \in \tilde{\Lambda}^{(k)}} c_{\underline{i}}^{-T(q)} p_{\underline{i}}^q = C 2^{-(k-1)(tq+T(q))}.$$

This implies

$$\sum_{D \in \mathcal{D}_n} \chi(D)^q \leq 2^{-qnt} \times \frac{2^{dq+d}}{1 - 2^{-(k-1)(tq+T(q))}} 2^{-n(qt+T(q))},$$

which proves that $\tau_\chi(q) \geq \inf \{\tau_\mu(q) + qt, T(q) + qt\}$. Since for $0 \leq q \leq 1$, it always holds that $T(q) \leq \tau_\mu(q)$, one gets

$$\tau_\chi(q) \geq T(q) + qt.$$

Let us now prove that $\tau_\chi(q) \leq \tau_\mu(q) + qt$:

First, since $0 \leq q \leq 1$, notice that

$$\sum_{D \in \mathcal{D}_n} \sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D} \neq \emptyset} \chi(D')^q \geq \sum_{D \in \mathcal{D}_n} \left(\sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D} \neq \emptyset} \chi(D') \right)^q.$$

Moreover, for every $D \in \mathcal{D}_n$ and every $\underline{i} \in \Lambda^{(n)}$ such that $f_{\underline{i}}(K) \cap D \neq \emptyset$, one has

$$f_{\underline{i}}(x_0) \in f_{\underline{i}}(K) \subset \bigcup_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D} \neq \emptyset} D'$$

so that, recalling (4.25),

$$\left(\sum_{D' \in \mathcal{D}_n, \overline{D'} \cap \overline{D} \neq \emptyset} \chi(D') \right)^q \geq \left(\sum_{\underline{i} \in \Lambda^{(n)}: f_{\underline{i}}(K) \cap D \neq \emptyset} p_{\underline{i}} c_{\underline{i}}^t \right)^q \geq 2^{-qt(n-1)} \mu(D)^q.$$

It follows that

$$\sum_{D \in \mathcal{D}_n} \chi(D)^q \geq \frac{1}{2d} \sum_{D \in \mathcal{D}_n} \sum_{D' \in \mathcal{D}_n, \bar{D}' \cap \bar{D} \neq \emptyset} \chi(D')^q \geq \frac{1}{2d} \cdot 2^{-qt(n-1)} \sum_{D \in \mathcal{D}_n} \mu(D)^q$$

and

$$\tau_\chi(q) \leq \tau_\mu(q) + qt.$$

□

Lemma 11.2.3. *For every $q < 0$,*

$$\tau_\chi(q) \geq \tau_\mu(q) + qt.$$

Proof. Let us fix $q < 0$. Remark that, for any $n \in \mathbb{N}$, $D_0 \in \mathcal{D}_n$, one has

$$\mu(D_0) \leq \sum_{\underline{i} \in \Lambda^{(n)}, f_{\underline{i}}(K) \cap D_0 \neq \emptyset} p_{\underline{i}} \leq \mu(2D_0). \quad (11.23)$$

In particular, if \widetilde{D}_{D_0} denotes a dyadic cube included in $\frac{1}{2}D_0$ of generation $\tilde{n} = n + 2$, one has

$$\sum_{D \in \mathcal{D}_n} \chi(D)^q \leq \sum_{D \in \mathcal{D}_n} 2^{-\tilde{n}tq} \mu(\widetilde{D}_D)^q \leq 2^{-\tilde{n}tq} \sum_{D' \in \mathcal{D}_{\tilde{n}}} \mu(D')^q. \quad (11.24)$$

Equation (11.24) shows that

$$\tau_\chi(q) \geq \tau_\mu(q) + t. \quad (11.25)$$

□

We summarize in the following proposition our results about the scaling function τ_χ .

Proposition 11.2.4. *The scaling function τ_χ (Definition 4.19) verifies*

$$\begin{cases} \text{for } q > q_c, & \tau_\chi(q) = 0 \\ \text{for } 0 \leq q < q_c, & T(q) + qt \leq \tau_\chi(q) \leq \tau_\mu(q) + qt \\ \text{for } q \leq 0 & \tau_\chi(q) \geq \tau_\mu(q) + qt. \end{cases} \quad (11.26)$$

Remark 11.2.5. • *Note that Proposition 11.2.4 holds without any separation assumption about S .*

- *The case $0 \leq q \leq q_c$ should depend in general on the Diophantine properties of the sequence $(f_{\underline{i}}(x_0))_{\underline{i} \in \Lambda^*}$ (so in particular on the separation condition*

satisfied by S) as the computation of τ_χ involves the high-scale distribution of those points.

Recall that, by Theorem 4.3.39, if S satisfies the AWSC and has no exact overlaps, then for $0 < q < q_c$, recalling (4.78), one has $T(q) = \tau_\mu(q)$. As a consequence, τ_μ is differentiable at q_c and the following corollary holds.

Corollary 11.2.6. *Assume that S satisfies the AWSC and has no exact overlaps. Then, writing $\lambda_c = \tau'_\mu(q_c)$, one has*

$$\begin{cases} \text{for } h \leq \lambda_c + t & \tau_\chi^*(h) = hq_c \\ \text{for } \lambda_c + t \leq h \leq \tau'_\mu(0^+) + t & \tau_\chi^*(h) = \tau_\mu^*(h - t) \\ \text{for } h \geq \tau'_\mu(0^+) + t & \tau_\chi^*(h) \leq \tau_\mu^*(h - t). \end{cases} \quad (11.27)$$

In the following section, we compute the multifractal spectrum of χ when $x_0 \notin K$.

11.3 Proof of Proposition 11.1.6

Let $m \geq 2$ and $S = \{f_1, \dots, f_m\}$ be a self-similar IFS and K the attractor of S .

Let us fix (p_1, \dots, p_m) be a probability vector, $x_0 \notin K$ and $t > 0$. We set

$$\chi(\cdot) = \sum_{\underline{i} \in \Lambda^*} p_{\underline{i}} c_{\underline{i}}^t \delta_{f_{\underline{i}}(x_0)}. \quad (11.28)$$

The proof of Proposition 11.1.6 readily follows from the following lemma.

Lemma 11.3.1. *There exists $C' > 1$ such that for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$ such that for any $0 < r < r_\varepsilon$, recalling Remark 10.1.2 and the notation involved and writing $\kappa(x_0, S) = \max \{C(x_0, S), 1\}$, one has*

$$C'^{-1} r^t \mu\left(B\left(x, \frac{1}{2\kappa(x_0, S)} r\right)\right) \leq \chi(B(x, r)) \quad (11.29)$$

and

$$\chi(B(x, r)) \leq C' r^t \mu(B(x, 2\kappa(x_0, S)r)) + 4 \frac{-\log r}{\log 2} r^{\frac{t}{1+\varepsilon}} \mu(B(x, 2\kappa(x_0, S)r^{\frac{1}{1+\varepsilon}})). \quad (11.30)$$

Proof. By Lemma 10.2.1, for any $\varepsilon > 0$, there exists only a finite set $I \subset \Lambda^*$ for which for any $\underline{i} \in I$,

$$x \in B(f_{\underline{i}}(x_0), c_{\underline{i}}^{1+\varepsilon}).$$

For $r < \min_{\underline{i} \in I} c_{\underline{i}}^{1+\varepsilon}$, no word $\underline{i} \in \Lambda^*$ satisfies that $f_{\underline{i}}(x_0) \in B(x, r)$ and $c_{\underline{i}} > r^{\frac{1}{1+\varepsilon}}$. In particular, if $f_{\underline{i}}(x_0) \in B(x, r)$, then

$$c_{\underline{i}} \leq r^{\frac{1}{1+\varepsilon}}.$$

This implies that, recalling (4.25), for any $0 < r$ small enough, one has

$$\begin{aligned} \chi(B(x, r)) &= \sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &= \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} + \sum_{\frac{1}{1+\varepsilon} \frac{\log(r)}{-\log(2)} \leq k < \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &= T_1(x, r) + T_2(x, r). \end{aligned} \quad (11.31)$$

Let us recall that by Remark 10.1.2, $f_{\underline{i}}(K) \subset B(f_{\underline{i}}(x_0), \kappa(x_0, S)c_{\underline{i}})$ and that $\kappa(x_0, S) \geq 1$.

This yields

$$\begin{aligned} T_1(x, r) &= \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &\leq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-kt} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} p_{\underline{i}} \\ &\leq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-kt} \mu(B(x, 2\kappa(c_0, S)r)) \leq \widehat{C} r^t \mu(B(x, 2\kappa(c_0, S)r)). \end{aligned}$$

Also,

$$\begin{aligned} T_1(x, r) &= \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &\geq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-(k+1)t} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} p_{\underline{i}} \\ &\geq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-(k+1)t} \mu\left(B\left(x, \frac{r}{2\kappa(x_0, S)}\right)\right) \\ &\geq \widehat{C}' r^t \mu\left(B\left(x, \frac{r}{2\kappa(x_0, S)}\right)\right). \end{aligned} \quad (11.32)$$

Moreover, since for each $\frac{1}{1+\varepsilon} \frac{\log(r)}{-\log(2)} \leq k \leq \frac{-\log(r)}{\log(2)} + 2$ and each $\underline{i} \in \Lambda^{(k)}$

satisfying $f_{\underline{i}}(K) \cap B(x, r) \neq \emptyset$, one has $f_{\underline{i}}(K) \subset B(x, 2\kappa(x_0, S)r^{\frac{1}{1+\varepsilon}})$ and we get

$$\sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(K) \cap B(x, r) \neq \emptyset} c_{\underline{i}}^t p_{\underline{i}} \leq r^{\frac{t}{1+\varepsilon}} \mu(B(x, 2\kappa(x_0, S)r^{\frac{1}{1+\varepsilon}}))$$

so that

$$\begin{aligned} T_2(x, r) &= \sum_{1 \leq k \leq \frac{-\log(r)}{\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &\leq 4 \frac{-\log r}{\log 2} r^{\frac{t}{1+\varepsilon}} \mu(B(x, 2\kappa(x_0, S)r^{\frac{1}{1+\varepsilon}})). \end{aligned} \quad (11.33)$$

Equation (11.31) together with (11.32) and (11.33) finishes the proof of Lemma 11.3.1. \square

We derive now Proposition 11.1.6 from Lemma 11.3.1.

By Lemma 11.3.1, for any $x \in K$ and any $\varepsilon > 0$,

$$\frac{1}{1+\varepsilon} \left(t + \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log r} \right) \leq \liminf_{r \rightarrow 0^+} \frac{\log \chi(B(x, r))}{\log(r)} \leq t + \liminf_{r \rightarrow 0^+} \frac{\log \mu(B(x, r))}{\log r}.$$

Since $\varepsilon > 0$ is arbitrary, for any $x \in K$, one has

$$\underline{\dim}(\chi, x) = \underline{\dim}(\mu, x) + t.$$

This proves that

$$E_{\chi}^h = E_{\mu}^{h-t}.$$

11.4 Multifractal analysis of χ under AWSC

In this section, the system is assumed to verify the AWSC (see Definition 4.3.3) with no exact overlaps (i.e, for any $\underline{i} \neq \underline{j} \in \Lambda^*$, $f_{\underline{i}} \neq f_{\underline{j}}$). As a simple consequence of the AWSC (4.28) and Proposition 8.1.2, the sequence $(B(f_{\underline{i}}(x_0), r_{\underline{i}}))_{\underline{i} \in \Lambda^*}$ is weakly redundant (Definition 8.1.1).

The multifractal analysis of χ relies on the following proposition.

Proposition 11.4.1. *Let $x \in K$ and $r > 0$. There exists $C' > 0$ depending on the system S such that $\chi(B(x, r))$ can be written as*

$$\chi(B(x, r)) = T_1(x, r) + T_2(x, r), \quad (11.34)$$

where

$$\begin{aligned} C'^{-1}r^t\mu(B(x, \frac{r}{2})) &\leq T_1(x, r) \leq C'r^t\mu(B(x, 2r)) \\ \max_{\underline{i} \in \Lambda^*: c_{\underline{i}} \geq r, f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} &\leq T_2(x, r) \leq O(\log^2(r)) \max_{\underline{i} \in \Lambda^*: c_{\underline{i}} \geq r, f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}}. \end{aligned} \quad (11.35)$$

Proof. Let $x \in K$ and $r > 0$.

$$\begin{aligned} \chi(B(x, r)) &= \sum_{k \geq 0} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &= \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} + \sum_{k < \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &= T_1(x, r) + T_2(x, r). \end{aligned}$$

On one hand

$$\begin{aligned} T_1(x, r) &= \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\ &\leq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-kt} \sum_{\underline{i} \in \Lambda^{(k)}: f_{\underline{i}}(x_0) \in B(x, r)} p_{\underline{i}}. \end{aligned}$$

Note that for every $k \geq \frac{\log(r)}{-\log(2)} + 2$, $|f_{\underline{i}}(K)| \leq \frac{r}{2}$ so that

$$\bigcup_{\underline{i} \in \Lambda^{(k)}, f_{\underline{i}}(K) \cap B(x, r) \neq \emptyset} f_{\underline{i}}(K) \subset B(x, 2r).$$

By Lemma (4.3.33), this implies that there exists constants $C, \widehat{C} > 0$ such that

$$T_1(x, r) \leq C \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-kt} \mu(B(x, 2r)) \leq \widehat{C} r^t \mu(B(x, 2r)).$$

On the other hand, by Lemma 4.3.33 again, one also has that for, every $k \geq$

$\frac{\log r}{-\log 2} + 2$, for every $\underline{i} \in \Lambda^{(k)}$ such that $f_{\underline{i}}(K) \cap B(x, \frac{r}{2}) \neq \emptyset$, $f_{\underline{i}}(K) \subset B(x, r)$.

$$\begin{aligned}
T_1(x, r) &= \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)} : f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\
&\geq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-(k+1)t} \sum_{\underline{i} \in \Lambda^{(k)} : f_{\underline{i}}(x_0) \in B(x, r)} p_{\underline{i}} \\
&\geq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-(k+1)t} \sum_{\underline{i} \in \Lambda^{(k)} : f_{\underline{i}}(K) \cap B(x, \frac{r}{2}) \neq \emptyset} p_{\underline{i}} \\
&\geq \sum_{k \geq \frac{\log(r)}{-\log(2)} + 2} 2^{-(k+1)t} \mu(B(x, r/2)) \\
&\geq \hat{C}' r^t \mu(B(x, r/2)).
\end{aligned}$$

We now estimates T_2 . Recalling (4.28), one has

$$\begin{aligned}
\max_{\underline{i} \in \Lambda^* : c_{\underline{i}} \geq r, f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} &\leq T_2(x, r) = \sum_{0 \leq k \leq \frac{-\log(r)}{\log(2)} + 2} \sum_{\underline{i} \in \Lambda^{(k)} : f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}} \\
&\leq 4 \frac{-\log r}{\log 2} \max_{0 \leq k \leq \frac{-\log(r)}{\log(2)} + 2} t_k \max_{\underline{i} \in \Lambda^* : c_{\underline{i}} \geq r, f_{\underline{i}}(x_0) \in B(x, r)} c_{\underline{i}}^t p_{\underline{i}}.
\end{aligned}$$

Since $\frac{\log t_k}{k} \rightarrow 0$, one also has $\frac{\log \max_{0 \leq i \leq k} t_i}{k} \rightarrow 0$, so that

$$4 \frac{-\log r}{\log 2} \max_{0 \leq k \leq \frac{-\log(r)}{\log(2)} + 2} t_k = O(\log^2 r),$$

which concludes the proof. □

A consequence of Proposition 11.4.1 is that, for any $x \in K$, $\underline{\dim}(x, \chi)$ is the smallest of the two exponents $h_1(x) = \underline{\dim}(\mu, x) + t = \liminf_{r \rightarrow 0} \frac{\log T_1(x, r)}{\log r}$ and $h_2(x) = \liminf_{r \rightarrow 0} \frac{\log T_2(x, r)}{\log r}$.

Let us remark that whenever $c_{\underline{i}} \geq r$, one has $f_{\underline{i}}(x_0) \in B(x, r) \Leftrightarrow x \in B(f_{\underline{i}}(x_0), c_{\underline{i}}^\delta)$, where $\delta = \frac{\log c_{\underline{i}}}{\log r} \geq 1$. In particular, introducing

$$\begin{aligned}
h(x) &= \\
&\min \left\{ h : \forall \varepsilon > 0, \exists \lambda \geq 0, \delta \geq 1, \frac{\lambda + t}{\delta} \leq h + \varepsilon \text{ and } x \in \limsup_{\substack{\underline{i} \in \Lambda^* : c_{\underline{i}}^{\lambda+\varepsilon} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda-\varepsilon}}} B(f_{\underline{i}}(x_0), c_{\underline{i}}^\delta) \right\}, \\
&\hspace{15em} (11.36)
\end{aligned}$$

one gets the following corollary.

Corollary 11.4.2. *For any $x \in K$,*

$$\underline{\dim}(\chi, x) = \min \{ \underline{\dim}(\mu, x) + t, h(x) \}. \quad (11.37)$$

As a consequence, writing for $h \geq 0$,

$$F_h = \{x : h(x) = h\}, \quad (11.38)$$

One has

$$\begin{aligned} E_\chi^h &= \left((F_h \setminus \bigcup_{h' < h} F_{h'}) \cap \bigcup_{h' \geq h} E_\mu^{h'-t} \right) \cup \left(E_\mu^{h-t} \cap \bigcup_{h' > h} F_{h'} \right) \\ &= \left(F_h \setminus \left(\bigcup_{h' < h} F_{h'} \cup E_\mu^{h'-t} \right) \right) \cup \left(E_\mu^{h-t} \cap \bigcup_{h' > h} F_{h'} \right). \end{aligned} \quad (11.39)$$

Note that this decomposition is exact, so that the multifractal analysis only relies on estimating the dimension of the terms involved.

11.4.1 Estimates for D_χ for $h \geq \lambda_c + t$

In the following sections, in order to avoid any confusion with the Dirac measure, denoted by δ , and the degree of points with respect to an approximating family, one will denote by the letter Δ the degree of approximation. More precisely, the degree of a point $x \in K$ with respect to $\{B(f_{\underline{i}}(x_0), c_{\underline{i}})\}_{\underline{i} \in \Lambda^*}$ is defined as follows.

Definition 11.4.3. *Let $x \in [0, 1]^d$. Define Δ_x , the degree of approximation Δ_x of x with respect to the family $(B(f_{\underline{i}}(x_0), 2c_{\underline{i}}))_{k \in \mathbb{N}, \underline{i} \in \Lambda^k}$ as*

$$\Delta_x = \sup \left\{ \Delta \geq 1 : x \in \limsup_{k \rightarrow +\infty, \underline{i} \in \Lambda^k} \left(B(f_{\underline{i}}(x_0), (2c_{\underline{i}})^\Delta) := B_{\underline{i}}^\Delta \right) \right\}.$$

Let us now deal with the case $h \geq \lambda_c + t$ in Theorem 11.1.4. Note first that the general estimates given in this case in item (2) readily follows from the following remark.

Remark 11.4.4. *As a straightforward consequence of (11.39), one has*

$$\widetilde{E}_\mu^{h-t} \cap \{\Delta_x = 1\} \subset E_\chi^h.$$

Also, since for every $h \geq \lambda_c + t$, $\tau_\chi^(h) \leq \tau_\mu^*(h - t)$, one has*

$$D_\chi(h) \leq \tau_\mu^*(h - t).$$

We now establish item (3) of Theorem 11.1.4.

Proposition 11.4.5. *Assume that there exists a measure $m_{h-t} \in \mathcal{M}(\mathbb{R}^d)$ such that $m_{h-t}(\widetilde{E}_\mu^{h-t}) > 0$ and $\underline{\dim}_H(m_{h-t}) = \tau_\mu^*(h-t)$, then, for any $h \geq \lambda_c$,*

$$\dim_H(E_\chi^h) = \tau_\mu^*(h-t).$$

Proof. By Theorem 4.3.39, it is known that for $q > 0$, $\tau_\mu(q) = T(q)$. Hence, by Corollary 11.2.6, for $h \geq \lambda_c$, one has

$$\tau_\chi^*(h) \leq \tau_\mu^*(h-t),$$

which implies

$$\dim_H(E_\chi^h) \leq \tau_\mu^*(h-t).$$

Let us establish the converse inequality.

We recall the following results.

Lemma 11.4.6 ([5], Lemma 4.2). *Let $\mathcal{B} = (B_n)_{n \in \mathbb{N}}$ be a weakly redundant sequence of balls and, for $k \in \mathbb{N}$ let $T_k(\mathcal{B})$ be defined as in Definition 8.1.1. Then for all $\alpha \in \mathbb{R}$,*

$$\limsup_{k \rightarrow +\infty} \frac{\log \# \{B \in T_k(\mathcal{B}) : |B|^{\alpha+\varepsilon} \leq \mu(B) \leq |B|^{\alpha-\varepsilon}\}}{k} \leq \sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a). \quad (11.40)$$

Let $(B_n)_{n \in \mathbb{N}}$ be a weakly redundant sequence of balls. For any $\varepsilon > 0$ and $\Delta \geq 1$, let us show first that

$$\dim_H \left(\limsup_{n \in \mathbb{N}: |B_n|^{\alpha+\varepsilon} \leq \mu(B_n) \leq |B_n|^{\alpha-\varepsilon}} B_n^\Delta \right) \leq \frac{\sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a)}{\Delta}.$$

Let $\varepsilon' > 0$ and $k_0 \in \mathbb{N}$ be large enough so that for any $k \geq k_0$,

$$\# \{B \in T_k(\mathcal{B}) : |B|^{\alpha+\varepsilon} \leq \mu(B) \leq |B|^{\alpha-\varepsilon}\} \leq 2^{k(\sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a) + \varepsilon')}.$$

Note also that, for any $p \in \mathbb{N}$,

$$\limsup_{n \in \mathbb{N}: |B_n|^{\alpha+\varepsilon} \leq \mu(B_n) \leq |B_n|^{\alpha-\varepsilon}} B_n^\Delta \subset \bigcup_{k \geq p} \bigcup_{L \in \{B \in T_k(\mathcal{B}) : |B|^{\alpha+\varepsilon} \leq \mu(B) \leq |B|^{\alpha-\varepsilon}\}} L.$$

Moreover,

$$\begin{aligned}
& \sum_{k \geq k_0} \sum_{B \in \{B \in \mathcal{T}_k(\mathcal{B}) : |B|^{\alpha+\varepsilon} \leq \mu(B) \leq |B|^{\alpha-\varepsilon}\}} |B|^{\Delta(\frac{\sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a)}{\Delta} + 2\varepsilon')} \\
& \leq \sum_{k \geq k_0} \# \{B \in \mathcal{T}_k(\mathcal{B}) : |B|^{\alpha-\varepsilon} \leq \mu(B) \leq |B|^{\alpha-\varepsilon}\} 2^{-k \sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a) - k2\Delta\varepsilon'} \\
& \leq \sum_{k \geq k_0} 2^{k \sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a) + k\varepsilon'} \cdot 2^{-k \sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a) - k2\Delta\varepsilon'} \leq \sum_{k \geq k_0} 2^{k\varepsilon'(-2\Delta+1)} < +\infty.
\end{aligned}$$

This implies that

$$\dim_H \left(\limsup_{n \in \mathbb{N}: |B_n|^{\alpha+\varepsilon} \leq \mu(B_n) \leq |B_n|^{\alpha-\varepsilon}} B_n^\Delta \right) \leq \frac{\sup_{a \in [\alpha-\varepsilon, \alpha+\varepsilon]} \tau_\mu^*(a)}{\Delta} + 2\varepsilon.$$

Letting $\varepsilon' \rightarrow 0$ proves the claim.

We now prove that for any sequence of balls $(B_n)_{n \in \mathbb{N}}$ with $|B_n| \rightarrow 0$, any $\Delta > 1$ and any $\varepsilon > 0$,

$$\widetilde{E}_\mu^{h-t} \cap \limsup_{n \rightarrow +\infty} B_n^\Delta \subset \limsup_{n \in \mathbb{N}: |B_n|^{h-t+\varepsilon} \leq \mu(B_n) \leq |B_n|^{h-t-\varepsilon}} B_n^\Delta. \quad (11.41)$$

Consider $x \in \widetilde{E}_\mu^{h-t}$ such that $\Delta_x > 1$. Fix ε and $\varepsilon_0 < \varepsilon$. There exists $r_x > 0$, small enough so that $(\frac{1}{5}r_x)^{\varepsilon_0} \geq r_x^\varepsilon$ and for every $0 < r \leq r_x$,

$$r^{h-t+\varepsilon_0} \leq \mu(B(x, r)) \leq r^{h-t-\varepsilon_0}.$$

In particular, for any $n \in \mathbb{N}$ large enough so that $x \in B_n^{\Delta_x}$, $B_n^{\Delta_x} \subset \frac{1}{2}B_n$ and $|B_n| \leq \frac{r_x}{3}$, one has

$$|B_n|^{h-t+\varepsilon} \leq \left| \frac{1}{5}B_n \right|^{h-t+\varepsilon_0} \leq \mu(B(x, \frac{1}{5}|B_n|)) \leq \mu(B_n) \leq \mu(B(x, |B_n|)) \leq |B_n|^{h-t-\varepsilon}$$

which proves that (11.41) holds true.

As a consequence of (11.41), one has for any $\Delta > 1$,

$$\widetilde{E}_\mu^{h-t} \cap \limsup_{n \rightarrow +\infty} B_n^\Delta < \tau_\mu^*(h-t),$$

which implies

$$m_{h-t} \left(\bigcup_{k \geq 2} \limsup_{n \in \mathbb{N}} B_n^{1-\frac{1}{k}} \right) = 0.$$

Since

$$\bigcup_{\Delta > 1} \{x : \Delta_x = \Delta\} \subset \bigcup_{k \geq 2} \limsup_{n \in \mathbb{N}} B_n^{1-\frac{1}{k}},$$

one gets

$$m_{h-t} \left(\widetilde{E}_\mu^{h-t} \setminus \bigcup_{\Delta > 1} \{x : \Delta_x = \Delta\} \right) > 0.$$

Recalling that $\underline{\dim}_H(m_{h-t}) = \tau_\mu^*(h-t)$,

$$D_\chi(h) = \dim_H(E_\chi^h) \geq \dim_H(\widetilde{E}_\mu^{h-t} \cap \{\Delta_x = 1\}) \geq \tau_\mu^*(h-t).$$

This concludes the proof of Proposition 11.4.5. \square

Remark 11.4.7. Remember that for any $q > 0$, $\tau_\mu(q) = T(q)$ so that τ_μ is differentiable on $(\lambda_c, \dim(S))$. It is proved in [35] that in this case, the measure m_h exists for any $h \in [\lambda_c, \dim(S))$, so that $D_\chi(h+t) = \tau_\mu^*(h)$.

11.4.2 Multifractal spectrum of χ for $h \in [0, \lambda_c + t]$

Let us start by proving the following proposition.

Proposition 11.4.8. For any $0 \leq h \leq \lambda_c + t$, one has

$$\dim_H(F_h) = \dim_H(F_h \cup E_\mu^{h-t}) = hq_c.$$

Moreover, there exists a gauge function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\lim_{r \rightarrow 0^+} \frac{\log(\zeta(r))}{\log r} = hq_c \text{ and } \mathcal{H}^\zeta(F_h) > 0.$$

Proof. The proof of Proposition 11.4.8 is divided in three steps:

1. We prove that, recalling the definition of the Legendre transform of a mapping (4.20),

$$q_c = \max_{\lambda \geq 0} \left\{ \frac{T^*(\lambda)}{\lambda + t} \right\}. \quad (11.42)$$

2. Using 1, we show that

$$\dim_H(F_h \cup E_\mu^{h-t}) \leq hq_c.$$

3. Finally we prove that there exists a gauge function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\lim_{r \rightarrow 0^+} \frac{\log(\zeta(r))}{\log r} = hq_c \text{ and } \mathcal{H}^\zeta(F_h) > 0,$$

which also implies that

$$\dim_H(F_h \cup E_\mu^{h-t}) \geq hq_c.$$

Let us start by proving that item 1 holds:

Let $\lambda \geq 0$ be a real number. One has

$$\frac{T^*(\lambda)}{\lambda + t} = \frac{\tau_\mu^*(\lambda)}{\lambda - \frac{T(q_c)}{q_c}} = q_c \cdot \frac{\tau_\mu^*(\lambda)}{q_c \lambda - T(q_c)} \leq q_c.$$

Moreover, recalling that $\lambda_c = T'(q_c)$,

$$\frac{\tau_\mu^*(\lambda_c)}{\lambda_c - \frac{T(q_c)}{q_c}} = q_c \cdot \frac{\tau_\mu^*(\lambda_c)}{q_c \lambda_c - T(q_c)} = q_c.$$

We now show that item 2 holds:

Remark that

$$\tau_\mu^*(h - t) = h \frac{\tau_\mu^*(h - t)}{h} \leq h q_c. \quad (11.43)$$

Also by (11.38),

$$F_h \subset \left\{ x : \forall \varepsilon > 0, \exists \lambda \geq 0, \Delta \geq 1, \frac{\lambda + t}{\Delta} \leq h + \varepsilon \text{ and } x \in \limsup_{\underline{i} \in \Lambda^*: c_{\underline{i}}^{\lambda+\varepsilon} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda+\varepsilon}} B_{\underline{i}}^\Delta \right\}.$$

Fix $\varepsilon_0 > 0$. Since $h \mapsto \tau_\mu^*(h)$ is continuous, there exists $0 \leq \varepsilon \leq \min\{\varepsilon_0, \frac{h}{2}, \frac{t}{2}\}$ such that for any $\lambda, \lambda' \in [0, T'(0)]$ satisfying $|\lambda - \lambda'| \leq 3\varepsilon$, one has $|\tau_\mu^*(\lambda) - \tau_\mu^*(\lambda')| \leq \varepsilon_0$ and for every $x > 0$,

$$\frac{x + t}{x + t - 3\varepsilon} \leq (1 + \varepsilon_0).$$

For any $\lambda \geq 0$, note that if $\Delta' \geq \Delta$, one has

$$\limsup_{\underline{i} \in \Lambda^*: c_{\underline{i}}^{\lambda+\varepsilon} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda+\varepsilon}} B_{\underline{i}}^{\Delta'} \subset \limsup_{\underline{i} \in \Lambda^*: c_{\underline{i}}^{\lambda+\varepsilon} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda+\varepsilon}} B_{\underline{i}}^\Delta.$$

Writing, for $\underline{i} \in \Lambda^*$,

$$\lambda_{\underline{i}} = \frac{\log p_{\underline{i}}}{\log c_{\underline{i}}} \text{ and } \Delta_{\underline{i}} = \frac{\lambda_{\underline{i}} + t}{h + \varepsilon},$$

one has

$$\begin{aligned} F_h &\subset \limsup_{\underline{i} \in \Lambda^*: \lambda_{\underline{i}} \leq h - t + \varepsilon} B_{\underline{i}} \cup \limsup_{\underline{i} \in \Lambda^*: T'(0) - t - 3\varepsilon \geq \lambda_{\underline{i}} \geq h - t + \varepsilon} B_{\underline{i}}^{\Delta_{\underline{i}}} \cup \limsup_{\underline{i} \in \Lambda^*: T'(0) - t - 3\varepsilon \leq \lambda_{\underline{i}}} B_{\underline{i}}^{\Delta_{\underline{i}}} \\ &= G_{h,1} \cup G_{h,2} \cup G_{h,3}. \end{aligned}$$

We now estimate by above the Hausdorff dimension of $G_{h,1}$, $G_{h,2}$ and $G_{h,3}$.

- We start by estimating $\dim_H G_{h,1}$:

Note that $p_i \geq c_i^{h-t+\varepsilon} \Rightarrow \mu(B_i) \geq c_i^{h-t+\varepsilon}$. Since $h-t \leq \lambda_c \leq T'(0)$, by (11.42),

$$\dim_H(G_{h,1}) = \dim_H \left(\limsup_{i \in \Lambda^*: \lambda_i \leq h-t+\varepsilon} B_i \right) \leq \tau_\mu^*(h-t+\varepsilon) = h \frac{\tau_\mu^*(h-t+\varepsilon)}{h} \leq h q_c + \varepsilon_0. \quad (11.44)$$

- Upper-bound for $\dim_H G_{h,2}$:

$$\limsup_{i \in \Lambda^*: \lambda_i \in [h-t-\varepsilon, T'(0)-3\varepsilon]} B_i^{\Delta_i} \subset \bigcup_{k=0}^{\lfloor \frac{T'(0)-h+t}{\varepsilon} \rfloor - 2} \limsup_{i \in \Lambda^*: c_i^{h-t+(k+2)\varepsilon} \leq p_i \leq c_i^{h-t+(k-1)\varepsilon}} B_i^{\frac{(k-1)\varepsilon+h}{h+\varepsilon}}.$$

By Lemma 11.4.6, one has

$$\dim_H \left(\limsup_{i \in \Lambda^*: \lambda_i \in [h-t-\varepsilon, T'(0)-3\varepsilon]} B_i^{\Delta_i} \right) \leq \max_{0 \leq k \leq \lfloor \frac{T'(0)-h+t}{\varepsilon} \rfloor + 1} \sup_{\lambda \in [h-t+(k-1)\varepsilon, h-t+(k+2)\varepsilon]} (h+\varepsilon) \frac{\tau_\mu^*(\lambda)}{(k-1)\varepsilon+h}.$$

By choice of ε and (11.42),

$$\begin{aligned} \dim_H G_{h,2} &= \dim_H \left(\limsup_{i \in \Lambda^*: \lambda_i \in [h-t-\varepsilon, T'(0)-3\varepsilon]} B_i^{\Delta_i} \right) \\ &\leq \max_{0 \leq k \leq \lfloor \frac{T'(0)-h+t}{\varepsilon} \rfloor + 1} (h+\varepsilon) \frac{\tau_\mu^*((k-1)\varepsilon+h-t)+\varepsilon_0}{(k-1)\varepsilon+h} \\ &\leq (1+\varepsilon_0)(h+\varepsilon_0)q_c. \end{aligned} \quad (11.45)$$

- We now estimate the Hausdorff dimension of $G_{h,3}$:

$$\limsup_{i \in \Lambda^*: \lambda_i \geq T'(0)+\varepsilon} B_i^{\Delta_i} \subset \limsup_{i \in \Lambda^*: \lambda_i \geq T'(0)-3\varepsilon} B_i^{\frac{T'(0)+t-3\varepsilon}{h+\varepsilon}}.$$

Defining $\Delta = \frac{T'(0)+t-3\varepsilon}{h+\varepsilon}$ and $\nu \in \mathcal{M}(\Lambda^\mathbb{N})$ defined by $\nu([i]) = c_i^{\dim(S)}$ for every $i \in \Lambda^*$, the same computation as in (10.17) yields, for any $\varepsilon' > 0$,

$$\sum_{i \in \Lambda^*} |B_i^\Delta|^{\frac{\dim(S)+\varepsilon'}{\Delta}} < +\infty.$$

Recalling that $\tau_\mu^*(T'(0)) = \dim(S)$ and (11.42) and letting ε' tends to 0, one

gets

$$\begin{aligned}
 \dim_H G_{h,3} &= \dim_H \left(\limsup_{\substack{\underline{i} \in \Lambda^*: \\ \lambda_{\underline{i}} \geq T'(0) - 3\varepsilon}} B_{\underline{i}}^{\frac{T'(0)+t-3\varepsilon}{h+\varepsilon}} \right) \leq \frac{\dim(S)}{\Delta} = (h+\varepsilon) \frac{\tau_\mu^*(T'(0))}{T'(0)+t-3\varepsilon} \\
 &\leq (h+\varepsilon) \frac{\tau_\mu^*(T'(0))}{T'(0)+t} \times \frac{T'(0)+t}{T'(0)+t-3\varepsilon} \\
 &\leq (1+\varepsilon_0)(h+\varepsilon_0)q_c. \quad (11.46)
 \end{aligned}$$

Since μ satisfies the multifractal formalism at any $h' \in [0, T'(0)]$, combining (11.43), (11.44), (11.46) and (11.45) and letting $\varepsilon_0 \rightarrow 0$, one obtains

$$\dim_H (F_h \cup E_\mu^{h-t}) \leq hq_c,$$

which was the claim of item 2.

Let us now establish item 3 :

Remark first that

$$\bigcap_{\varepsilon > 0} \limsup_{\substack{\underline{i} \in \Lambda^*: \\ c_{\underline{i}}^{\lambda_c + \varepsilon} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda_c - \varepsilon}}} B_{\underline{i}}^{\frac{\lambda_c + t}{h}} \subset F_h. \quad (11.47)$$

Moreover, let $\nu_c \in \mathcal{M}(\Lambda^\mathbb{N})$ be the measure defined by

$$\nu_c([i]) = p_i^{q_c} c_i^{-T(q_c)} \text{ for every } i \in \Lambda^*$$

and μ_c its projection on K . For $x \in \Lambda^\mathbb{N}$, let us denote by $X_k(x)$ the cylinder of generation k containing x . One has, for ν_c -almost every $x \in \Lambda^\mathbb{N}$:

$$\begin{cases} \frac{\log \nu_c(X_k(x))}{\log |\pi(X_k(x))|} \rightarrow \tau_\mu^*(\lambda_c) \\ c_i^{\lambda_c + \varepsilon} \leq p_i \leq c_i^{\lambda_c - \varepsilon} \Leftrightarrow c_i^{\tau_\mu^*(\lambda_c) + q_c \varepsilon} \leq \nu_c([i]) \leq c_i^{\tau_\mu^*(\lambda_c) - q_c \varepsilon}. \end{cases} \quad (11.48)$$

Since $\mu_c(\limsup_{\underline{i} \in \Lambda^*} \frac{1}{2} B_{\underline{i}}) = 1$, the construction made in the proof of Proposition 8.2.6 can easily be adapted to show that there exists a sequence $(\varepsilon_{\underline{i}})_{\underline{i} \in \Lambda^*} \in \mathbb{R}^{+\mathbb{N}}$ such that $\varepsilon_{\underline{i}} \rightarrow 0$ and $\{B_{\underline{i}} : c_{\underline{i}}^{\lambda_c + \varepsilon_{\underline{i}}} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda_c - \varepsilon_{\underline{i}}}\}_{\underline{i} \in \Lambda^*}$ is μ_c -a.c. Let us point out that, recalling (11.38),

$$\limsup_{\substack{\underline{i} \in \Lambda^*: \\ c_{\underline{i}}^{\lambda_c + \varepsilon_{\underline{i}}} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda_c - \varepsilon_{\underline{i}}}}} B_{\underline{i}}^{\frac{\lambda_c + t}{h}} \subset F_h. \quad (11.49)$$

Also, since by Proposition 4.3.22 S is dimension regular (Definition 4.3.20) so that $\dim(\mu_c) = \tau_\mu^*(\lambda_c)$, applying Theorem 7.3.1, there exists a gauge function $\zeta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that

$$\lim_{r \rightarrow 0+} \frac{\log(\zeta(r))}{\log r} = h \frac{\tau_\mu^*(\lambda_c)}{\lambda_c + t} = hq_c,$$

and

$$\mathcal{H}^\zeta\left(\limsup_{\substack{\underline{i} \in \Lambda^*: c_{\underline{i}}^{\lambda_c + \varepsilon_{\underline{i}}} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda_c - \varepsilon_{\underline{i}}}} B_{\underline{i}}^{\frac{\lambda_c + t}{h}}\right) > 0.$$

By (11.49), this implies that

$$\mathcal{H}^\zeta(F_h) \geq \mathcal{H}^\zeta\left(\limsup_{\substack{\underline{i} \in \Lambda^*: c_{\underline{i}}^{\lambda_c + \varepsilon_{\underline{i}}} \leq p_{\underline{i}} \leq c_{\underline{i}}^{\lambda_c - \varepsilon_{\underline{i}}}} B_{\underline{i}}^{\frac{\lambda_c + t}{h}}\right) > 0,$$

which concludes the proof of Proposition 11.4.8. \square

Corollary 11.4.9. *For any $h \leq \lambda_c + t$,*

$$\dim_H\left(F_h \setminus \left(\bigcup_{h' < h} F_{h'} \cup E_\mu^{h'-t}\right)\right) = hq_c \geq \tau_\mu^*(h - t),$$

so that

$$\dim_H(E_\chi^h) = \dim_H\left(\left(F_h \setminus \left(\bigcup_{h' < h} F_{h'} \cup E_\mu^{h'-t}\right)\right) \cup \left(E_\mu^{h-t} \cap \bigcup_{h' > h} F_{h'}\right)\right) = hq_c. \quad (11.50)$$

In particular, Corollary 11.4.9 together with Corollary 11.2.6 implies that χ satisfies the multifractal formalism for $0 \leq h \leq \lambda_c + t$. Moreover Remark 11.4.7 and Corollary 11.4.9 implies Theorem 11.1.4.

Bibliography

- [1] D. Allen, S. Baker, S. Schow, and Y. Han. Dyadic Approximation in the Middle-Third Cantor Set. to appear in Selecta Math, 2022.
- [2] D. Allen and S. Bárány. On the Hausdorff measure of shrinking target sets on self-conformal sets. accepted Mathematika, 2021.
- [3] S. Baker. Intrinsic Diophantine approximation for overlapping iterated function systems. submitted, 2022.
- [4] J. Barral and D. Feng. On multifractal formalism for self-similar measures with overlaps. Math. Z., (298):359–383, 2021.
- [5] J. Barral and S. Seuret. Sums of dirac masses and conditioned ubiquity. C. R. Acad. Sci. Paris, Sér. I 339:787–792, 2004.
- [6] J. Barral and S. Seuret. Heterogeneous ubiquitous systems in \mathbb{R}^d and Hausdorff dimensions. Bull. Brazilian Math. Soc, 38(3):467–515, 2007.
- [7] J. Barral and S. Seuret. The multifractal nature of heterogeneous sums of dirac masses. Math. Proc. Camb. Phil. Soc., 144(3):707–727, 2008.
- [8] J. Barral and S. Seuret. Ubiquity and large intersections properties under digit frequencies constraints. Math. Proc. Camb. Phil. Soc., 145(3):527–548, 2008.
- [9] J. Barral and S. Seuret. The singularity spectrum of the inverse of cookie-cutters. Ergod. Th. Dynam. Sys, 29:1075–1095, 2009.
- [10] J. Barral and S. Seuret. Besov spaces in multifractal environment, and the Frisch-Parisi conjecture. arXiv:2007.00971, 2021.
- [11] V. Beresnevitch and S. Velani. A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures. Ann. Math., 164(3):22 pages, 2006.
- [12] V. Beresnevitch and S. Velani. The divergence Borel-Cantelli lemma revisited. arXiv:2103.12200, 2021.

- [13] A.S. Besicovitch. A general form of the covering principle and relative differentiation of additive functions. Math. Proc. Camb. Phil. Soc., 41:103–110, 1945.
- [14] R. Bowen. Equilibrium states and the ergodic theory of Anosov diffeomorphisms. Lecture Notes in Mathematics, no. 470. Springer-Verlag, Berlin-New York, (236):121–153, 1975.
- [15] Y. Bugeaud. An inhomogeneous Jarnik theorem. Journal d’analyse mathématique, (1)(92):327–349, 2004.
- [16] Y. Bugeaud. Intersective sets and Diophantine approximation. Mich. Math., 2004.
- [17] Y. Bugeaud. Diophantine approximation and Cantor sets. Math. Ann., (341):677–684, 2008.
- [18] Y. Bugeaud. Metric Diophantine approximation on the middle-third Cantor set. J. Europ. Math. Soc., 2016.
- [19] S. Bárány, A. Käenmäki, and I. Morris. Domination, almost additivity, and thermodynamic formalism for planar matrix cocycles. Israël J. Math., (239):173–214, 2020.
- [20] L. Carleson. Selected Problems on Exceptional Sets. Van Nostrand, 1967.
- [21] M. Csörnyei, A. Käenmäki, T. Rajala, and V. Suomala. Upper conical density densities for general measure on \mathbb{R}^n . Proc. Edinburgh Math. Soc., 53:971–992, 2010.
- [22] E. Daviaud. An anisotropic inhomogeneous ubiquity theorem. Submitted, [arXiv:2112.10413](#), 2021.
- [23] E. Daviaud. Dimension of weakly conformal iterated function system shrinking targets with overlaps. Submitted, [arXiv:2207.08458](#), 2022.
- [24] E. Daviaud. Extraction of optimal sub-sequences of balls and application to optimality estimates of mass transference principles. Submitted, [arXiv:2204.01304](#), 2022.
- [25] E. Daviaud. An heterogeneous ubiquity theorem, application to self-similar measures with overlaps. Submitted, [arXiv:2204.01302](#), 2022.
- [26] E. Daviaud. Multifractal analysis of discrete self-similar measure satisfying AWSC. final phase of redaction, 2022.

-
- [27] A. Durand. Sets with large intersection and ubiquity. Math. Proc. Camb. Phil. Soc., 144(1):119–144, 2008.
 - [28] F. Ekström and T. Persson. Hausdorff dimension of random limsup sets. Journal of the London Mathematical Society, 98:661–686, 2018.
 - [29] S. Eriksson-Bique. A new Hausdorff content bound for limsup sets. preprint, 2022.
 - [30] K. Falconer. Fractal Geometry. John Wiley & Sons, 1990.
 - [31] K. Falconer. Representation of families of sets by measures, dimension spectra and Diophantine approximation. Math. Proc. Camb. Phil. Soc., (128, 111), 2000.
 - [32] K. Falconer. Fractal geometry. John Wiley & Sons, Inc., Hoboken, NJ, second edition, 2003. Mathematical foundations and applications.
 - [33] A.-H. Fan, J. Schmeling, and S. Troubetzkoy. A multifractal mass transference principle for Gibbs measures with applications to dynamical diophantine approximation. Proc. London Math. Soc., 107 (5):1173–1219, 2013.
 - [34] H. Federer. Geometric measure theory, volume Band 153 of Die Grundlehren der mathematischen Wissenschaften). Springer-Verlag New York Inc., New York, 1969.
 - [35] D. Feng. Gibbs properties of self-conformal measures and the multifractal formalism. Ergodic Theory Dynam. Systems, 27(3):787–812, 2007.
 - [36] D. Feng. Multifractal analysis of Bernoulli convolutions associated with Salem numbers. Adv. Math., (229(5)):3052–3077, 2012.
 - [37] D. Feng and H. Hu. Dimension theory of iterated function systems. Comm. Pure Appl. Math., 62:1435–1500, 2009.
 - [38] D. Feng, E. Jarvenpaa, M. Jarvenpaa, and M. Suomala. Dimensions of random covering sets in Riemann manifolds. Ann. Probab., 46:1542–1596, 2018.
 - [39] A. Figalli and F. Glaudo. An Invitation to Optimal Transport, Wasserstein Distances, and Gradient Flows. 2021.
 - [40] J. Fraser and L. Stuart. A new perspective on the Sullivan dictionary via Assouad type dimensions and spectra, submitted. Comm. Pure Appl. Math., 2021.

- [41] Y. Heurteaux. Estimations de la dimension inférieure et de la dimension supérieure des mesures. Ann. Inst. H. Poincaré Probab. Statist., 34:309–338, 1998.
- [42] R. Hill and S. Velani. The ergodic theory of shrinking targets. Inv. Math., 119:175–198, 1995.
- [43] M. Hochman. On self-similar sets with overlaps and inverse theorems for entropy in \mathbb{R}^d . To appear in Memoires of the AMS, 2022.
- [44] J.E. Hutchinson. Fractals and self similarity. Indiana Univ. Math. J., 30:713–747, 1981.
- [45] S. Jaffard. Functions with prescribed Hölder exponent. Appl. Comput. Harmon. Anal., 2:400–401, 1995.
- [46] S. Jaffard. The spectrum of singularities of Riemann’s function. Rev. Mat. Iberoamericana., 12:441–460, 1996.
- [47] V. Jarnik. Diophantischen approximationen und Hausdorffsches mass. Mat. Sbornik, 36:371–381, 1929.
- [48] E. Jarvenpää, M. Jarvenpää, H. Koivusalo, B. Li, and V. Suomala. Hausdorff dimension of affine random covering sets in torus. Ann. Inst. Henri Poincaré Probab. Stat., 50 (4):1371–1384, 2014.
- [49] E. Jarvenpää, M. Jarvenpää, H. Koivusalo, B. Li, V. Suomala, and Y. Xiao. Hitting probabilities of random covering sets in torus and metric spaces.
- [50] A Khintchine. Zur Theorie der unbeschränkt teilbaren Verteilungsgesetze. Rec. Math., 2(44):79–119, 1937.
- [51] H. Koivusalo and M. Rams. Mass transference principle: From balls to arbitrary shapes. To appear in I.R.M.N., 2020.
- [52] D. Koukoulopoulos and J. Maynard. On the Duffin-Shaeffer conjecture. to appear in Annals of Mathematics, 2020.
- [53] A. Käenmäki, T. Sahlsten, and P. Shmerkin. Dynamics of the scenery flow and geometry of measures. Proc. Lond. Math. Soc., 110(3):1248–1280, 2015.
- [54] J. Levesly, C. Salp, and S. Velani. On a problem of K. Mahler: Diophantine approximation and Cantor sets. Math. Ann., 42(338):97–118, 2007.

-
- [55] L. Liao and S. Seuret. Diophantine approximation by orbits of expanding Markov maps. Ergod. Th. Dyn. Syst., 33:585–608, 2013.
- [56] B. Mandelbrot and R. Riedi. Exceptions to the multifractal formalism for discontinuous measures. Math. Proc. Cambridge Philos. Soc., 1998.
- [57] B. B. Mandelbrot. Intermittent turbulence in self-similar cascades, divergence of high moments and dimension of the carrier. J. Fluid. Mech., 62:331–358, 1974.
- [58] B. B. Mandelbrot. Multiplications aléatoires itérées et distributions invariantes par moyennes pondérées. C. R. Acad. Sci. Paris, 278:289–292 et 355–358, 1974.
- [59] P. Mattila. Cambridge Studies in Advanced Mathematics, Timestamp = 2008.01.04, Title = Geometry of Sets and Measures in Euclidean Spaces: Fractals and Rectifiability, Year = 1999.
- [60] L. Olsen. A multifractal formalism. Adv. Math., 116:92–195, 1995.
- [61] L. Olsen. Fractal and multifractal dimensions of prevalent measures. Indiana Univ. Math. J., 59 (2):661–690, 2010.
- [62] L. Olsen. Prevalent L^q -dimensions of measures. Math. Proc. Camb. Phil. Soc., 149(3):553–571, 2010.
- [63] P. Peres, M. Rams, K. Simon, and B. Solomyak. Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets. Proc. Amer. Math. Soc., (129(9)):2689–2699, 2001.
- [64] T. Persson. A note on random coverings of tori. Bull. London Math. Soc., 47 (1):7–12, 2015.
- [65] T. Persson and M. Rams. On shrinking targets for piecewise expanding interval maps. Ergod. Th. Dyn. Syst., 37:646–663, 2017.
- [66] A. Rutar. A multifractal decomposition for self-similar measures with exact overlaps. submitted, 2022.
- [67] P. Shmerkin. A modified multifractal formalism for a class of self-similar measures with overlap. Asian J. Math., (3):323–348, 2005.
- [68] P. Shmerkin. On Furstenberg’s intersection conjecture, self-similar measures, and the L^q norms of convolutions. Ann. Math., 189(2):319–391, 2019.

- [69] P. Shmerkin and B. Solomyak. Absolute continuity of self-similar measures, their projections and convolutions. Trans. Amer. Math. Soc., no.7(368), 2016.
- [70] N. Snigireva and L. Olsen. L^q spectra and Rényi dimensions of inhomogeneous self-similar measures. Nonlinearity, 2006.
- [71] N. Snigireva and L. Olsen. Multifractal spectra of in-homogenous self-similar measures. Indiana University Mathematics Journal, 2008.
- [72] B. Testud. Phase transitions for the multifractal analysis of self-similar measures. Nonlinearity, 42:1–35, 2006.
- [73] P. Walters. Invariant measures and equilibrium states for some mappings which expand distances. Trans. Amer. Math. Soc., (236):121–153, 1978.
- [74] B. Wang and J. Wu. Mass transference principle from rectangles to rectangles in Diophantine approximation. Mathematishe Annalen, 110(381):1–75, 2021.
- [75] B. Wang, J. Wu, and J. Xu. Mass transference principle for limsup sets generated by rectangles. Math. Proc. Camb. Phil. Soc., 158:419–443, 2015.