Symbolic dynamical systems in topological spaces defined via edit distances

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I, undersigned, Firas Ben Ramdhane, hereby declare that the work presented in this manuscript is my own work, carried out under the scientific direction of Mr. Serge Troubetzkoy, Mr. Tarek Sellami and Mr. Pierre Guillon, in accordance with the principles of honesty, integrity and responsibility inherent to the research mission. The research work and the writing of this manuscript have been carried out in compliance with the french national charter for Research Integrity, the Aix-Marseille University charter on the fight against plagiarism and the tunisian national charter for Research Integrity and the Sfax University charter on the fight against plagiarism.

This work has not been submitted previously either in these countries or in other country in the same or in a similar version to any other examination body.

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Liste de publications et participation aux conférences

Liste des publications réalisées dans le cadre du projet de thèse:

1. Generic properties of Besicovitch- and Weyl-like pseudo-metrics, Firas Ben Ramdhane, Pierre Guillon and Serge Troubetzkoy (to be submitted soon).

2. Cellular automata and substitutions in topological spaces defined via edit distances, Firas Ben Ramdhane and Pierre Guillon [RG22] (accepted and to appear in Natural computing journal, Springer).

3. Cellular automata and substitutions in the edit-distance space, Firas Ben Ramdhane [Ram21] (accepted as conference paper in the International Workshop on Cellular Automata 2021: https://hal.science/AUTOMATA2021-EXPLORATORY/hal-03238039v1).

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Résumé

Dans cette thèse, on étudie des systèmes dynamiques symboliques sur des espaces définis à partir de distances d'édition, notamment les espaces de Besicovitch et de Weyl. Ces derniers sont des espaces métriques quotients définis à partir des pseudo-métriques et quotientés par la relation d'équivalence de pseudo-métrique zéro.

À cet effet, nous commençons par étudier ces deux pseudo-métriques qui sont définies à partir de la distance de Hamming. Nous donnons une généralisation de ces deux pseudo-métriques (centrée et glissante) en remplaçant la distance de Hamming par toute distance définie sur l'ensemble de mots finis. Ensuite, nous présentons certaines propriétés de ces deux pseudo-métriques: la mesurabilité, la continuité, l'invariance par le décalage et le comportement sur les configurations périodiques.

D'autre part, ces deux pseudo-métriques sont définies en tant que une limite supérieure. Pour cette raison, on étudie l'existence de la limite pour chaque pseudo-métrique. Nous montrons, que la pseudo-métrique centrée n'est pas toujours une limite, en montrant que dans certain type de sous-shift muni de la topologie de Cantor, l'ensemble où la pseudo-métrique centrée est maximale et la limite inférieure est nulle est résiduel. Cependant, nous montrons que l'ensemble où cette pseudo-métrique est une limite est de mesure pleine pour toute mesure faiblement mélangeante et que cette limite ne dépend pas du choix des configurations. À l'inverse, nous montrons que la pseudo-métrique glissante est en fait toujours une limite; dans une certaine classe de sous-shift munis de la topologie de Cantor, l'ensemble où cette pseudo-métrique est maximale est un G_δ dense. De plus, l'ensemble où cette pseudo-métrique est maximale est de mesure pleine si la mesure est faiblement mélangeante.

Finalement, nous donnons une première étude des dill maps (qui généralisent les automates cellulaires et les substitutions) sur les espaces de Besicovitch, Weyl et Feldman-Katok (ce dernier est obtenu en changeant la distance de Hamming par celle de Levenshtein). Nous prouvons que toutes les dill maps sont bien définies dans ce dernier, contrairement aux espaces de Besicovitch et de Weyl où seulement les dill maps uniformes et constantes sont bien définies. De plus, nous montrons que l'espace de Feldman-Katok est pertinent pour étudier la dynamique des dill maps : nous prouvons que le décalage est égal à l'identité, qu'il n'existe pas d'automates cellulaires expansifs, que chaque substitution admet au moins un point d'équicontinuité.

Mots clés: Systèmes dynamiques symboliques, distances d'édition, automates cellulaires, substitutions, dill maps, distance de Hamming, distance de Levenshtein, pseudo-métrique de Besicovitch, pseudo-métrique de Weyl, espace de Cantor, systèmes dynamiques non-compacts.

Abstract

In this thesis, we study symbolic dynamical systems on spaces defined from edit distances, in particular the spaces of Besicovitch and Weyl. These are metric spaces defined using pseudo-metrics and quotients by the relation of pseudo-metric zero.

For this purpose, we start by studying these two pseudo-metrics which depend on the Hamming distance. We give a generalization of these two pseudo-metrics (centered and sliding) by replacing the Hamming distance by any distance defined on the set of finite words. Then, we present some properties of these two pseudo-metrics: measurability, continuity, shift invariance and behavior on periodic configurations.

On the other hand, these two pseudo-metrics are defined as an upper limit. For this reason, we study the existence of the limit for each pseudo-metric. We show that the centered pseudo-metric is not always a limit. Moreover, we show that in some class of subshifts equipped with the Cantor topology, the set where the centered pseudo-metric reaches the maximum and the lower limit is zero is a dense G_δ. Furthermore, we show that the set where this pseudo-metric is a limit is of full measure for any weakly-mixing measure and that this limit does not depend on the choice of configurations. In contrast, we show that the sliding pseudo-metric is always a limit. Moreover, in some class of subshifts equipped with the Cantor topology, the set where this pseudo-metric reaches the maximum is a dense G_δ. In addition, the set where this pseudo-metric is maximum (within the support of a weakly-mixing measure) is of full measure.

Finally, we give a first study of dill maps (which generalize cellular automata and substitutions) over the Besicovitch, Weyl and the Feldman-Katok spaces (the latter is obtained by changing the Hamming distance by that of Levenshtein). We prove that all dill maps are well-defined over the Feldman-Katok space, in contrast to the Besicovitch and the Weyl spaces where only uniform and constant dill maps are well defined. Furthermore, we show that the Feldman-Katok space is a suitable playground to study the dynamics of dill maps. Indeed, we prove that the shift is equal to the identity, there are no expansive cellular automata, every substitution admits at least one equicontinuous point.

Keywords: Symbolic dynamical systems, edit distances, cellular automata, substitutions, dill maps, Hamming distance, Levenshtein distance, Besicovitch pseudo-metric, Weyl pseudo-metric, Cantor space, non-compact dynamical systems.

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Contents

1 Introduction

Nowadays, dynamical systems play a very important role in the understanding of physical and natural phenomena. Dynamical systems originally arose in the study of systems of differential equations used to model physical phenomena [Bir27]. The motions of the planets [Gla93], or of mechanical systems [JH02], or of molecules in a gas can be modeled by such systems. Besides that, dynamical systems have been used in many other sciences, for example in economics [Gan71], engineering [KSS08], biology and medicine [JR15].

Not all of these models are framed as systems of differential equations. There are also dynamical systems with discrete time, in which the time variable only takes integer values. This is natural when describing population patterns (such as populations of rabbits or bacteria) or seasonal phenomena (numbers of migratory animals that only come for a brief season in a year) or investment models. Here, the concept of symbolic dynamical systems arises.

In mathematics, symbolic dynamics is a rapidly growing area of dynamical systems. Although it originated as a method to study general dynamical systems, it has found significant uses in coding for data storage and transmission as well as in linear algebra. It involves looking at a system by dividing the space into a finite number of regions and by focusing on possible cross-region sequences as the system evolves. By associating a symbol to each region, one can associate a (infinite) sequence of symbols to each path, hence the name of "symbolic dynamics". Symbolic trajectories are of course only an approximation of actual trajectories, but they can reflect some real system properties such as transitivity, expansivity, recurrence or equicontinuity. A general introduction to the field can be found in the monograph by Lind and Marcus [LM21]. Precursor articles include those by Morse and Hedlund [MH38] and Hedlund [Hed69]. Ethan M. Coven and Zbigniew H. Nitecki [CN08] consider that symbolic dynamics, as an autonomous discipline, really begins with Hedlund's article [Hed44].

Most classically, the set of infinite sequences (denoted by $A^{\mathbb{N}}$ where A is a finite set of symbols) is endowed with the product topology of the discrete topology on each copy of A. The topology defined on $A^{\mathbb{N}}$ is metrizable, corresponding to the Cantor distance which depends on the position of the first difference between two infinte sequences. This metric space, is complete, compact, perfect and totally disconnected.

People frequently criticised this topology, but rarely in articles. However, there is a major drawback: The metric is centered on a region close to the origin and anything far from the center is neglected. The amount of neglected information can be extremely important, as " everything else" is an infinite region, while the "near region" is finite. A perturbation of a state is considered small if it is in agreement with the state on a region

(large, but finite) around the origin. The states in the cells of the remaining infinite part of the grid can be arbitrary. Hence, the intuitive idea of "small perturbation" is not satisfied by the Cantor metric concept. Moreover, this topology has other weaknesses. For example, the Cantor distance is not invariant by the shift map. Instead, this topology confers to the shift strong properties of chaoticity, in the sense that it verifies the three conditions of the definition of chaos given by Devaney [Dev22]: transitivity, dense periodic points and sensitivity to initial conditions. However, considered as a shift of the observor point, the shift does not change the configurations.

To overcome this disadvantage, Cattaneao et al. [CFMM97] were inspired by Besicovitch's work on almost periodic functions. The topologies of Besicovitch and Weyl have been introduced in [Bes54] for quasi-periodic function spaces. These topologies for cellular automata take a more global view. The first use of them for one-dimensional cellular automata was only in 1997 [BFK97], and then for cellular automata over finitely generated groups [Cap09] and amenable groups [Cap10]. Note that a cellular automaton consists of an infinite array of cells containing letters of a finite alphabet, which are updated according to a local interaction rule. They have shown considerable interest as a model of symbolic dynamical system. They are a powerful computational model introduced in the early 1950s by John Von Neumann [Neu66], who was then interested in the self-replication of artificial systems. Since their creation they have been studied in various fields such as physics or biology where they allow to model and simulate various phenomena that cannot be analyzed directly.

The dynamics of cellular automata in the spaces of the sequences endowed with the Besicovitch pseudo-metric, which is defined as the asymptotics of the Hamming distance over prefixes of the sequences were studied in [BFK97]. The idea of Besicovitch's pseudo-metric is the point of view of an observer who can only see a fixed and finite part of space, which however becomes larger and larger until each point of the space is reached. In a later work, Blanchard et al. [BFK97] also defined a second pseudo-metric, this time called the Weyl pseudo-metric. The new point of view is that of an observer who not only enlarges the window but also moves it throughout the space.

It was proved in [CFMM97, BFK97, FK09] that the Besicovitch and Weyl spaces are a suitable playground to study the dynamics of cellular automata since they induce (well-defined) Lipschitz maps over these spaces, there are no expansive and no transitive cellular automata over these spaces and the shift map is an isometry over these spaces. In addition, one can find a characterization of cellular automata over these spaces in [MS09] and a characterization of cellular automata that are contracting or isometric over these spaces in [ST12]. Furthermore, in [BCF03] it is proven that every cellular automaton either has a unique fixed point or an uncountable set of periodic points. This result may be considered a further step towards the understanding of cellular automata periodic points behavior. Moreover, the study of symbolic dynamical systems over Besicovitch and Weyl spaces is not focused only on the study of cellular automata, but one can find a study of the geometric properties of Cantor subshifts in the Besicovitch space and a study of canonical projections into subshifts in [ST12].

From another point of view, the Besicovitch pseudo-metric corresponds to the $\bar{d}$-metric defined for ergodic purposes in [Orn74]. [Fel76], and independently [Kat77], proposed to replace the Hamming distance by the Levenshtein distance from [Lev66], and get the $\bar{f}$-metric, which is useful in Kakutani equivalence theory. The Levenshtein distance depends on the minimum number of edit operations (deletion, insertion, substitution) required to change one word into another word. It is extensively used for information theory, linguistics, word algorithmics, statistics.... One can read some properties of the pseudo-metric in [ORW82, Chapter 2], and a nice history of this notion in [KL17]. The recent [GRK20] can be seen as presenting a nice picture of those systems for which the identity map from the Cantor space into the Feldman-Katok space is continuous, after a similar task has been achieved for the Besicovitch space in [GR17].

We adopt in this thesis a complementary point of view, by considering the dynamics within the space itself. Though this task on the Besicovitch space has concerned mainly cellular automata so far, relaxing the pseudo-metric to edit space allows to naturally consider a larger class of systems, that also includes substitutions : the so-called *dill maps* [ST15]. The latter can be seen as a generalisation of cellular automata and substitutions.

This thesis deals with the study of some dynamical properties of dill maps over Besicovitch, Weyl and Feldman-Katok spaces. When dealing with the study of dill maps over quotient spaces two overriding issues arise. The first issue is to determine whether the dill maps are well-defined on these spaces. Indeed, we find that all dill maps are well-defined over the Feldman Katok space, in contrast to the Besicovitch and Weyl spaces, where only uniform and constant dill maps are well-defined. The second one is to determine dynamical properties of the well-defined dill maps over these spaces. Indeed, we treat in this thesis, equicontinuity and expansivity properties of some dill maps since other dynamical properties seem much harder to tackle because of the lack of tools to compute general lower bounds for the Feldman-Katok pseudo-metric (this problem is known to have high algorithmics complexity).

Firstly, in Chapter 2, we give basic definitions and notations used throughout this thesis. We start by giving basic definitions and results appearing in the study of dynamical systems and ergodic theory. After that, we introduce symbolic dynamical systems by giving basic definitions, examples and basic results of particular symbolic dynamical systems like cellular automata, substitutions and dill maps. We finish this chapter by introducing edit distances, in particular the Hamming distance and the Levenshtein distance.

In Chapter 3, after introducing the Besicovitch and Weyl pseudo-metrics (which depend on the Hamming distance), we give a general definition of these pseudo-metrics (sliding and centered pseudo-metrics) by changing the Hamming distance with any distance defined over the set of finite words. Moreover, the two pseudo-metrics are defined as an upper limit. For that, we prove, under some conditions on the distance, that the upper limit in the sliding pseudo-metric is actually a limit (this is the case of the Weyl pseudo-metric) in contrast to the centered pseudo-metric where

the upper limit is not always a limit. Furthermore, we prove some properties of these pseudo-metrics: measurability, continuity, shift invariance and behavior on periodic configurations.

We treat, in Chapter 4, topological and measurable generic behavior of the two pseudo-metrics. We start by showing some properties of distances over specific subshifts. Thereafter, we prove that, in some type of subshift equipped with the Cantor topology, the set where the centered pseudo-metric reaches its maximum and the lower limit equals zero is a dense G_δ. However, the set where this pseudo-metric is a limit is of full measure for any weakly mixing measure, and this limit does not depend on the choice of configurations. On the other hand, we prove that, in some class of subshift equipped with the Cantor topology, the set where the sliding pseudo-metric reaches its maximum is a dense G_δ. Moreover, the set where this pseudo-metric reaches its maximum over the support of a weakly mixing measure, is of full measure. In addition, this limit does not depend on the choice of configurations. We finish this chapter by giving some topological properties which are inherited from the Besicovitch and Weyl spaces.

Finally, Chapter 5 is devoted to the study of the dill maps over the Besicovitch, the Weyl and the Feldman-Katok spaces. Firstly, we prove that all dill maps induce a well-defined map over the Feldman-Katok space, and, only the uniform or constant dill maps are well-defined over the Besicovitch and Weyl spaces. Secondly, we show that any well-defined substitution over Besicovitch and Weyl spaces is equicontinuous, every substitution admits an equicontinuity point over the Feldman-Katok space, and we characterise the substitutions which are equicontinuous over the latter space. We conclude this chapter by proving that there is no expansive cellular automaton on the Feldman-Katok space, like in the case of Besicovitch space.

2 Basic definitions and notations

The aim of this chapter is to introduce basic objects that are encountered in the different parts of this thesis. In the first section, we start with some basic definitions and notations of dynamical systems. The second section, introduces concepts from symbolic dynamical systems. Finally, the third section introduces edit distances.

As usual, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ represent the sets of non-negative integers, integers, rational and real numbers respectively. The number of elements of finite set E is denoted here by $|E|$. We assume the reader to be familiar with the usual basic set operations like union, intersection or set difference: $\bigcup, \bigcap$ or $\setminus$.

2.1 Dynamical systems

2.1.1 Topological dynamical systems

Firstly, we start this section by giving some basic definition from topological dynamical systems. We can refer to [Kůr03, Chapter 1 and chapter 2] for more details. Let us recall the definition of a metric space and uniformly continuous maps, wich are the main objects of a dynamical system. For more details we refer to [Kůr03, Appendix A].

Definition 2.1. *Let X be a non-empty set. A distance d is a map defined over $X \times X$ to $\mathbb{R}_+$ which verifies the following properties:*

1. *Separation: $d(x,y) = 0$ if and only if $x = y$, for all $x, y \in X$.*
2. *Symmetry: $d(x,y) = d(y,x)$, for all $x, y \in X$.*
3. *Triangular inequality: $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$*

A metric space $X_d = (X,d)$ is a pair of a non-empty set X and a distance d over $X \times X$.

Definition 2.2. *A map $F : X_d \mapsto X_d$ is said to be:*

1. *Continuous if: $\forall x \in X, \forall \varepsilon > 0, \exists \delta > 0, \forall y \in X, d(x,y) < \delta \implies d(F(x),F(y)) < \varepsilon$.*
2. *Uniformly continuous if:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X, d(x,y) < \delta \implies d(F(x),F(y)) < \varepsilon.$$

3. *α-*Lipschitz*, for $\alpha > 0$, if: $d(F(x),F(y)) \leq \alpha d(x,y)$ for all $x, y \in X$.*

Note that, if F is uniformly continuous then for all $n \in \mathbb{N}$, F^n is uniformly continuous also, where F^n is the n-th iteration of F defined as $F^0(x) = x$, and $F^n(x) = F^{n-1}(F(x))$ for all $x \in X$. In addition, if F is Lipschitz, then F is uniformly continuous.

Definition 2.3. *A (topological)* dynamical system *is a pair (X_d, F) where F is a uniformly continuous map from a metric space $X_d = (X, d)$ to itself.*

Note that in the literature, X_d is usually assumed compact, but in this thesis we will focus on a more general setting, in which many known results cannot be applied.

The main object for studying dynamical systems is to understand the behavior of states within a system, given a rule for how the state evolves and to predict where the system is heading, where it will ultimately go.

Definition 2.4. *For a topological dynamical system (X_d, F), the orbit of a point $x \in X$ denoted by $\mathcal{O}(x)$ is the set defined as follows: $\mathcal{O}(x) = \left\{ F^t(x) \middle| \, t \in \mathbb{N} \right\}$.*

Definition 2.5. *For a dynamical system (X_d, F), a periodic point of period $p \in \mathbb{N} \setminus \{0\}$ is a point $x \in X$ such that $F^p(x) = x$. The period p is called minimal, if p is the smallest positive period. A point with minimal period 1 is called a fixed point.*

Remark 2.6. *Note that, the orbit $\mathcal{O}(x)$ of a periodic point $x \in X$ is a finite set.*

Before introducing some topological properties of a dynamical system (X_d, F), let us introduce some notations that will be used in the different part of this thesis.

Notations 2.7. *Let X_d be a metric space, $F \subseteq X$ and $x \in F$.*

1. *The open (resp. closed) ball of radius $r > 0$ centered at x denoted by $B(x, r)$ (resp. $\overline{B}(x, r)$) and defined as:*

$$B(x, r) = \left\{ y \in X \middle| \, d(x, y) < r \right\} \text{ and } \overline{B}(x, r) = \left\{ y \in X \middle| \, d(x, y) \leq r \right\}.$$

2. *The closure set of F denoted by $\overline{F}$ is the set defined as follows:*

$$\overline{F} = \left\{ y \in X \middle| \, \exists (x_n)_{n \in \mathbb{N}} \subseteq F, \lim_{n \to \infty} x_n = y \right\}.$$

Some dynamical systems are predictable, whereas others are not. We start by defining the concept of equicontinuity which represents a high degree of stability. By playing on the observation error of the initial point, one may arbitrarily impose a small maximum error for the observation of its whole orbit.

Definition 2.8. *Let (X_d, F) be a dynamical system.*

1. *A point $x \in X$ is an* equicontinuity point *of (X_d, F) if:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall y \in B(x, \delta), \forall t \in \mathbb{N}, F^t(y) \in B(F^t(x), \varepsilon).$$

2. *The dynamical system* (X_d, F) *is* equicontinuous *if:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in X, \forall y \in B(x,\delta), \forall t \in \mathbb{N}, d(F^t(x), F^t(y)) < \varepsilon.$$

Remark 2.9. *Note that if F is α-Lipschitz, then F^t is α^t-Lipschitz. It is then clear that if F is α-Lipschitz for some $\alpha \leq 1$, then F is equicontinuous (and it is actually an equivalence, up to distance equivalence, as seen for instance in [Kůr03, Proposition 2.41]).*

Expansivity and sensitivity represent a version of a very strong instability, since the smallest difference between two points will inevitably lead to a great in their evolution.

Definition 2.10. *A dynamical system* (X_d, F) *is (positively)* expansive *if:*

$$\exists \varepsilon > 0, \forall x \neq y \in X, \exists t \in \mathbb{N}, d(F^t(x), F^t(y)) > \varepsilon.$$

Definition 2.11. *A dynamical system* (X_d, F) *is* sensitive *if:*

$$\exists \varepsilon > 0, \forall x \in X, \forall \delta > 0, \exists y \in B(x,\delta), \exists t \in \mathbb{N}, d(F^t(x), F^t(y)) > \varepsilon.$$

It is folklore that any compact dynamical system has an expansive iterate if and only if it is expansive; for the sake of self-containment, we give a proof in the general (non necessarily compact) case. A proof for cellular automata (see, Definition 2.44) can be read for instance in [ADF09, Proposition 18]. We can refer also to [Gui08] for more properties.

Proposition 2.12. *A dynamical system* (X_d, F) *is expansive if and only if* (X_d, F^n) *is expansive for all* $n \in \mathbb{N}$.

Proof. It is obvious that if F^n is expansive for $n \in \mathbb{N}$ then F is expansive too. Assume now that there exists $k \in \mathbb{N}$ such that F^k is not expansive. Let $\varepsilon > 0$. Since F is uniformly continuous, F^n is uniformly continuous for any $n \in \mathbb{N}$. Hence, there exists $\delta > 0$ such that for every $n \in [\![0, k[\![$ and all x, y such that $d(x,y) \leq \delta$, we have $d(F^n(x), F^n(y)) \leq \varepsilon$. Now, from the non-expansiveness of F^k, there exist $x_0 \neq y_0$ such that $\forall t \in \mathbb{N}, d(F^{tk}(x_0), F^{tk}(y_0)) \leq \delta$. Hence, for all $s \in \mathbb{N}$, writing $s = kt + r$, with $r \in [\![0, k[\![$, we get:

$$d(F^s(x), F^s(y)) = d(F^r(F^{tk}(x)), F^r(F^{tk}(y))) \leq \varepsilon.$$

This means that F is not expansive for constant ε. □

Transitivity is a concept of great instability: all the openings in the space of the phases, small as they are, are crossed by the orbit of the same point. This behaviour forms the basis of the concept of topological chaos. It is difficult to predict the trajectory of objects, as they are often close to any point.

Definition 2.13. *Let (X_d, F) be a dynamical system.*

1. *A point $x \in X$ is said to be a transitive point if its orbit is dense i.e., $\overline{\mathcal{O}(x)} = X$.*
2. *(X_d, F) is said to be transitive if the orbit of any non-empty open set is dense i.e.,*

$$\forall x, y \in X, \forall \varepsilon > 0, \forall \delta > 0, \exists n > 0, \exists z \in B(x, \delta) : F^n(z) \in B(y, \varepsilon).$$

2.1.2 Measure-preserving dynamical system

In mathematics, a measure-preserving dynamical system is a subject of study in the abstract formulation of dynamical systems, and ergodic theory in particular. One can read a nice introduction to this theory in [Wal00].

Let us start by giving some basic definitions and results from measure and probability theory that we will use in this thesis. We shall refer to [KT08] and [Par05].

Definition 2.14. *Let X be a set. A σ-algebra of X is a collection $\mathscr{B}$ of subsets of X satisfying the following conditions: $X \in \mathscr{B}$; if $B \in \mathscr{B}$ then $X \setminus B \in \mathscr{B}$; if $B_n \in \mathscr{B}$ for $n \geq 1$ then $\bigcup_{n=1}^{\infty} B_n \in \mathscr{B}$. We then call the pair $(X, \mathscr{B})$ a measurable space.*

Example 2.15. *The smallest σ-algebra containing all open sets of a topological space is known as the Borel σ-algebra. Therefore, any topological space equipped with its Borel σ-algebra is a measurable space.*

Definition 2.16. *Let $(X, \mathscr{B})$ be a measurable space.*

1. *A finite measure on $(X, \mathscr{B})$ is a function $\mu : \mathscr{B} \mapsto \mathbb{R}^+$ satisfying $\mu(\emptyset) = 0$ and $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ whenever $\{B_n\}_{n\geq 1}$ is a sequence of members of $\mathscr{B}$ which are pairewise disjoint subsets of X.*
2. *A finite measure space is a triple $(X, \mathscr{B}, \mu)$ where $(X, \mathscr{B})$ is a measurable space and μ is a finite measure on $(X, \mathscr{B})$.*
3. *We say that $(X, \mathscr{B}, \mu)$ is a probability space if $\mu(X) = 1$. We then say μ is a probability measure on $(X, \mathscr{B})$.*
4. *The support of a measure μ on a Borel measurable space $(X, \mathscr{B})$ is the smallest (closed) subset Σ of X with measure 1.*

Example 2.17. *Let $A = \{a_1, \cdots, a_n\}$ be a finite set and $p = \{p_1, \cdots, p_n\}$ be a probability vector i.e., non-negative numbers such that $\sum_{i=1}^{n} p_i = 1$. We define the sample space associated to A by $X = A^{\mathbb{N}}$. The σ-algebra $\mathscr{B}$ on X is the product σ-algebra; that is, it is the (countable) direct product of the σ-algebras of the finite set $\{1, \cdots, N\}$. The associated measure is called the Bernoulli measure and denoted by $\mu_p = \{p_1, \cdots, p_n\}^{\mathbb{N}}$. A basis of $\mathscr{A}$ is the cylinder sets. Given a cylinder set $[a_{i_1}, \cdots, a_{i_n}]$, its measure is: $\mu_p([a_{i_1}, \cdots, a_{i_n}]) = \prod_{j=1}^{n} p_{i_j}$. Thus, the triplet $(X, \mathscr{B}, \mu_p)$ is a probability space.*

Definition 2.18. *Suppose that* $(X_1, \mathscr{B}_1, \mu_1)$, $(X_2, \mathscr{B}_2, \mu_2)$ *are probability spaces.*

1. *A transformation* $T : X_1 \mapsto X_2$ *is measurable if* $T^{-1}(\mathscr{B}_2) \subseteq \mathscr{B}_1$.
2. *A transformation* $T : X_1 \mapsto X_2$ *is measure-preserving if* T *is measurable and* $\mu_1(T^1(B_2)) = \mu_2(B_2)$, *for all* $B_2 \in \mathscr{B}_2$. *In particular if* $(X_1, \mathscr{B}_1, \mu_1) = (X_2, \mathscr{B}_2, \mu_2)$, *we say then that* $(X_1, \mathscr{B}_1, \mu_1, T_1)$ *is a measure-preserving dynamical system.*
3. *We say that* $T : X_1 \mapsto X_2$ *is an invertible measure-preserving transformation if* T *is measure-preserving, bijective, and* T^{-1} *is also measure-preserving.*

Example 2.19. *We shall refer to [Wal00, Chapter 1], for more details.*

1. *The identity transformation* id_X *on* $(X, \mathscr{B}, \mu)$ *is obviously measure-preserving.*
2. *Let* $(A^{\mathbb{N}}, \mathscr{B}, \mu_p)$ *be the Bernoulli probability space for some finite set* A *and a probability vector* p. *The one-sided shift map is the transformation* $T : A^{\mathbb{N}} \mapsto A^{\mathbb{N}}$ *such that* $T(x)_i = x_{i+1}$, *for all* $x \in A^{\mathbb{N}}$ *and* $i \in \mathbb{N}$. *This is an example of non-invertible measure-preserving transformation.*
3. *Let* $(A^{\mathbb{Z}}, \mathscr{B}, \mu_p)$ *be the Bernoulli probability space for some finite set* A *and a probability vector* p. *The two-sided shift map is the transformation* $T : A^{\mathbb{Z}} \mapsto A^{\mathbb{Z}}$ *such that* $T(x)_i = x_{i+1}$, *for all* $x \in A^{\mathbb{Z}}$ *and* $i \in \mathbb{Z}$. *This is an example of invertible measure-preserving transformation.*

Definition 2.20. *A measure-preserving transformation* T *of a probability space* $(X, \mathscr{B}, \mu)$ *is called ergodic if every* B *of* $\mathscr{B}$ *with* $T^{-1}(B) = B$ *satisfy* $\mu(B) = 0$ *or* $\mu(B) = 1$.

There are many other ways to state the condition of ergodicity and we give some of them in the following theorem (we refer for instance to [Wal00, Theorem 1.5 and Theorem 1.6]).

Theorem 2.21. *If* $(X, \mathscr{B}, \mu)$ *is a probability space and* $T : X \mapsto X$ *is measure-preserving then the following statements are equivalents:*

1. T *is ergodic.*
2. *For all* $D, B \in \mathscr{B}$, $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mu(T^{-i}(D) \bigcap B) = \mu(D)\mu(B)$.
3. *For every* $B \in \mathscr{B}$ *with* $\mu(B) > 0$ *we have* $\mu(\bigcup_{n=1}^{\infty} T^{-n}(B)) = 1$.
4. *Whenever* f *is measurable and* $(f \circ T)(x) = f(x), \forall x \in X$, *then* f *is constant a.e.*
5. *Whenever* f *is measurable and* $(f \circ T)(x) = f(x)$ *a.e. then* f *is constant a.e.*

One of main results in ergodic theory was proved by G.D Birkhoff in [Bir31]. Here we state it for a measure-preserving transformation of a finite measure space (we can refer for instance to [Wal00, Theorem 1.14]).

Theorem 2.22. *Suppose T is a measure-preserving transformation of a finite measure space $(X,\mathscr{B},\mu)$, and let f be an integrable function. Then with probability one, we have:*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) := f^*(x),$$

where f^* is T-invariant and $\int f^* d\mu = \int f \mathrm{d}\mu$. Moreover, if $(X,\mathscr{B},\mu)$ is a probability space and μ is ergodic then with probability one, we have:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x)) = \int f(x)\,\mathrm{d}\mu.$$

Remark 2.23. *According to the Birkhoff theorem, if T is a measure-preserving transformation of a finite measure space $(X,\mathscr{B},\mu)$ and $B\in\mathscr{B}$ such that $\mu(B)>0$ then:*

$$\lim_{n\to\infty}\frac{\left\{k\in[\![0,n[\![\;\middle|\; T^k(x)\in B\right\}}{n} = \frac{\mu(B)}{\mu(X)}.$$

Another ergodic theorem was proved by Kingman in [Kin68]: the so-called Kingman's (subadditive ergodic) theorem. We shall refer to [Ste89] for the proof of Kingman's theorem. It can be seen as a generalization of Birkhoff's ergodic theorem. Intuitively, the Kingman theorem is a kind of random variable version of Fekete's lemma.

Let us start by stating the Fekete lemma.

Lemma 2.24 ([Fek23])**.** *Let $(x_n)_{n\in\mathbb{N}}$ be a real sequence. If $x_{n+m}\le x_n+x_m$ for all $n,m\in\mathbb{N}$, then:*

$$\lim_{n\to\infty}\frac{x_n}{n} = \inf_{k\in\mathbb{N}\setminus\{0\}}\frac{x_k}{k}.$$

Theorem 2.25 ([Kin68])**.** *Let T be a measure-preserving transformation on a probability space $(X,\mathscr{B},\mu)$, and let $(g_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions such that $g_{n+m}(x)\le g_n(x)+g_m(T^n x)$. Then with probability one, we have:*

$$\lim_{n\to\infty}\frac{g_n(x)}{n} := \inf_{k\in\mathbb{N}\setminus\{0\}}\frac{g_k(x)}{k}.$$

Definition 2.26. *Let T be a measure-preserving transformation of $(X,\mathscr{B},\mu)$. T is said to be weakly mixing if,*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|\mu(T^{-i}D\cap B)-\mu(D)\mu(B)| = 0, \forall B,D\in\mathscr{B}.$$

The following theorem connects weakly mixing of T with the ergodicity of $T\times T$.

Theorem 2.27 ([Wal00, Theorem 1.24])**.** *If T is a measure-preserving transformation on a probability space $(X,\mathscr{B},\mu)$, then:*

$$T \text{ is weakly mixing} \iff T\times T \text{ is ergodic} \iff T\times T \text{ is weakly mixing.}$$

2.2 Symbolic dynamical systems

The aim of this section is to introduce some terminology in word combinatorics, that will be used throughout this thesis, and to introduce some concepts and basic notations in symbolic dynamics. We shall refer to [Kůr03], [BR10] and [LM21].

2.2.1 Definitions and notations

Combinatorics Let us first introduce some terminology in word combinatorics.

Definition 2.28.

1. *An alphabet A is a non-empty finite set. The elements of A are called letters.*
2. *A* finite word *over an alphabet A is a finite sequence of letters in A.*
3. *The length of finite word u, denoted by $|u|$, is the number of letters appearing in u. The unique word of length 0 is the empty word, denoted by λ.*

We fix once and for all an *alphabet* A of finitely many *letters* (it will be precised in each example, but general in our statements).

Example 2.29. *The set $A = \{0, 1, \cdots, n-1\}$, for some $n \in \mathbb{N}$, is an alphabet. In particular we denote $\mathbf{2} = \{0, 1\}$ the binary alphabet.*

Notations 2.30.

1. *The number of occurrences of some subalphabet $B \subseteq A$ within a finite word u is denoted by $|u|_B$. In particular, the number of occurences of a letter $b \in A$ within a finite word u is denoted by $|u|_b$.*
2. *The set of all finite words over A is denoted by A^*, A^n is the set of words of length $n \in \mathbb{N}$ over A, $A^+ = A^* \setminus \{\lambda\}$ and $(A \times A)^* = \left\{(u, v) \in A^* \times A^* \,|\, |u| = |v|\right\}$.*

The concatenation of two words $u = u_0 \cdots u_n$ and $v = v_0 \cdots v_m$ is the word $uv = u_0 \cdots u_n v_0 \cdots v_m$. This operation is associative and has a unit element, the empty word λ, and thus A^* is endowed with the structure of a monoid.

Remark 2.31. *It is convenient to write a word as $u = u_{[\![0,|u|[\![}$ to express u as the concatenation of the letters $u_0, u_1, \ldots, u_{|u|-1}$, with $[\![0, |u|[\![= \{0, ..., |u| - 1\}$.*

Definition 2.32. *Let A be a finite alphabet.*

1. *A configuration $x = x_0 x_1 x_2 \ldots$ over A is the concatenation of infinitely many letters from A. We denote by $A^{\mathbb{N}}$ the set of all configurations over A.*
2. *We say that u is prefix (resp. suffix, factor) of x if $x_{[\![0,|u|[\![} = u$ (resp. $x_{[\![|x|-|u|,|x|[\![} = u$, $x_{[\![k,k+|u|[\![} = u$ for some $k \in \mathbb{N}$). In these cases, we denote $u \sqsubseteq x$.*

3. *The language of a configuration* $x \in A^{\mathbb{N}}$ *is* $\mathscr{L}(x) = \{u \in A^* \mid u \sqsubseteq x\}$.

4. *For* $u \in A^*$ *and* $k \in \mathbb{N}$, *we denote by* $[u]_k$ *the following set* $[u]_k = \left\{x \in A^{\mathbb{N}} \middle| x_{[\![k,k+|u|[\![} = u\right\}$. *In particular, the cylinder set is* $[u]_0$ *and we simply denote it by* $[u]$.

5. *For a non-empty* $U \subseteq A^n$ *for some* $n \in \mathbb{N}$, *we denote* $[U] = \left\{x \in A^{\mathbb{N}} \middle| \exists u \in U, x_{[\![0,n[\![} = u\right\}$.

Symbolic dynamical systems Let us now introduce some basic notions in symbolic dynamics. We start by giving the definition of symbolic dynamical system:

Definition 2.33. *A symbolic dynamical system is a topological dynamical system* (X_d, F) *where* $X = A^{\mathbb{N}}$ *i.e., is a couple* $(A_d^{\mathbb{N}}, F)$ *where* $A_d^{\mathbb{N}} = (A^{\mathbb{N}}, d)$ *is a metric space and* F *is a uniformly continuous map from* $A^{\mathbb{N}}$ *to itself.*

Most classically, the set $A^{\mathbb{N}}$ is endowed with the product topology of the discrete topology on each copy of A. The topology defined on $A^{\mathbb{N}}$ is metrizable, corresponding to the *Cantor distance* denoted by $\mathfrak{d}_C$ and defined as follows:

$$\mathfrak{d}_C(x,y) = 2^{-\min\{n\in\mathbb{N} \mid x_n \neq y_n\}}, \forall x \neq y \in A^{\mathbb{N}}, \text{ and } \mathfrak{d}_C(x,x) = 0, \forall x \in A^{\mathbb{N}}.$$

Thus, two sequences are close to each other if their first terms coincide. Note that the space $(A^{\mathbb{N}}, d_C)$ is complete as a metric space. Moreover, it is a Cantor set, that is, a totally disconnected compact set without isolated points.

Note that, the cylinder sets are clopen (open and closed) sets and form a basis of open sets for the topology of $(A^{\mathbb{N}}, d_C)$. Indeed, for a finite word $u \in A^+$, the cylinder $[u]$ is identified with both $B(u, 2^{-n})$ and $\overline{B}(u, 2^{-n-1})$ for some $n \in \mathbb{N}$.

The prototypical example of symbolic dynamical systems is the full shift:

Definition 2.34. *Let* A *be an alphabet.*

1. *The shift map (Example 2.19 Item 2) denoted here by* σ *and defined over* $A^{\mathbb{N}}$ *to itself as* $\sigma(x)_i = \sigma(x)_{i+1}$, *for all* $x \in A^{\mathbb{N}}$ *and* $i \in \mathbb{N}$. *Note that this map is uniformly continuous over* $(A^{\mathbb{N}}, d_C)$.

2. *The full shift over* A *is the symbolic dynamical system* $(A_{d_C}^{\mathbb{N}}, \sigma)$.

Subshifts Subshifts were introduced in [Sma67]. They are objects of study in symbolic dynamics and ergodic theory. They describe the set of all possible configurations executed by a set of finite words. There are many class of subshifts, we can cite, subshifts of finite type [Par64] and sofic subshifts [Wei73].

Definition 2.35. *Let* A *be an alphabet. A subshift is any subsystem of a full shift i.e., a closed* σ*-invariant subset* $\Sigma \subseteq A^{\mathbb{N}}$.

A subshift can be defined by its set of forbidden words which cannot occur any point of the subshift.

Proposition 2.36. *For $F \subseteq A^*$ the set $\Sigma_F = \left\{ x \in A^{\mathbb{N}} \middle| \forall u \in A^*, u \sqsubseteq x \Rightarrow u \notin F \right\}$ is a subshift.*

Example 2.37. *Let A be the binary alphabet $\mathbf{2} = \{0,1\}$.*

1. *The golden mean subshift is the subshift Σ_F defined by its set of forbidden words $F = \{11\}$, i.e., $\Sigma_F = \left\{ x \in A^{\mathbb{N}} \middle| x_i x_{i+1} \neq 11, \forall i \in \mathbb{N} \right\}$.*
2. *The even subshift is the subshift defined by its set of forbidden words $F = \left\{ 01^{2n+1}0 \middle| n \in \mathbb{N} \right\}$.*

Definition 2.38. *A subshift $\Sigma \subseteq A^{\mathbb{N}}$ is a subshift of finite type (SFT) if there exists a finite set $F \subseteq A^*$ of forbidden words such that : $\Sigma = \Sigma_F = \left\{ x \in A^{\mathbb{N}} \middle| \forall u \sqsubseteq x, u \notin F \right\}$.*

Note that, the golden mean subshift is of finite type in contrast to the even subshift.

Notations 2.39. *For a subshift Σ, the language of Σ denoted by $\mathscr{L}(\Sigma)$ is the set of all finite words appearing in the configurations over Σ, i.e., $\mathscr{L}(\Sigma) = \left\{ u \in A^* \middle| \exists x \in \Sigma, u \sqsubseteq x \right\}$. For $n \in \mathbb{N}$, we denote, $\mathscr{L}_n(\Sigma) = \mathscr{L}(\Sigma) \bigcap A^n$. Finally, we say that the language is regular if it is accepted by some accepting automaton [Kůr03, Definition 3.51].*

Definition 2.40. *A subshift Σ is said to be:*

1. *Transitive if for all $u, v \in \mathscr{L}(\Sigma)$ there exists $w \in \mathscr{L}(\Sigma)$ such that $uwv \in \mathscr{L}(\Sigma)$.*
2. *Mixing subshift if for all $u, v \in \mathscr{L}(\Sigma)$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ there exists $w \in \mathscr{L}_n(\Sigma)$ such that $uwv \in \mathscr{L}(\Sigma)$.*
3. *Sofic subshift if its language is regular (see [Kůr03, Section 3.7] for more details).*

Specific configurations Let us define some specific configurations.

Definition 2.41. *Let x be a configuration over the binary alphabet. The sequence x is Sturmian if for some $\beta \in [0,1[$ and some irrational $\theta \in]0,1[$, we have:*

$$x_n = \lfloor n\theta + \beta \rfloor - \lfloor (n-1)\theta + \beta \rfloor, \forall n \in \mathbb{N}, \text{ or } x_n = \lceil n\theta + \beta \rceil - \lceil (n-1)\theta + \beta \rceil, \forall n \in \mathbb{N}.$$

It appears that Sturmian admit several equivalent definitions. Indeed, Sturmian are configurations over the binary alphabet that have exactly $n+1$ factors of length n for each $n \geq 0$.

Definition 2.42. *Let A be an alphabet and $x \in A^{\mathbb{N}}$.*

1. The period *of $k \in \mathbb{N}$ in x denoted by $r_k(x)$ and defined as follow:*
$$r_k(x) := \begin{cases} \inf\left\{ p > 0 \middle| \forall n \in \mathbb{N}, x_{k+np} = x_k \right\} \\ \infty \text{ if } \left\{ p > 0 \middle| \forall n \in \mathbb{N}, x_{k+np} = x_k \right\} = \emptyset. \end{cases}$$
2. *We said that x is* Tœplitz *if $r_k(x) \neq \infty$, for all $k \in \mathbb{N}$.*

Note that if a configuration x is periodic over $(A^{\mathbb{N}}, \sigma)$, then it is a Tœplitz configuration.

Definitions 2.43. *Let A be an alphabet and $x \in A^{\mathbb{N}}$ a Tœplitz configuration.*

1. A periodic structure *of x is an integer sequence $(p_i)_{i\in\mathbb{N}}$ such that:*
   - *For all $i \in \mathbb{N}$, p_i divides p_{i+1}.*
   - *For all $k \in \mathbb{N}$, there exist $j \in \mathbb{N}$ such that $r_k(x)$ divides p_j.*
2. *We say that x is* regular *if there exists a periodic structure $(p_i)_{i\in\mathbb{N}}$ of x such that: $\lim_{i\to\infty} \frac{q_i(x)}{p_i} = 0$, where $q_i(x) := |\{k \in [\![0, p_i[\![\,|\, r_k(x) \text{ is not a divisor of } p_i\}|$.*

2.2.2 Cellular automata, substitutions and dill maps

As examples of symbolic dynamical systems, we will be interested in this thesis in cellular automata, substitutions, and in general dill maps. For more details in the case of the Cantor space, we can refer to [FBF+02], [BR10], [HM17] and [Kůr03].

Cellular automata. Cellular automata are a powerful computing model introduced in the early 1950s by mathematician John Von Neumann, who took an interest in the self-reproduction of artificial systems [Neu66]. Since their inception, they have been studied in various fields such as physics or biology where they allow to model and simulate various phenomena that cannot be analysed directly. A cellular automaton consists in an infinite array of cells containing letters of a finite alphabet, which are updated according to a local interaction rule. They have shown considerable interest in being a model for complex dynamical systems. We shall refer to [HM17], [CSC10] and [Kůr03, Chapter 5].

Definition 2.44. *A* cellular automaton *(CA) with diameter $\theta \in \mathbb{N}\setminus\{0\}$ is a map $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$, such that there exists a map called* local rule *$f : A^{\theta} \to A$ such that for all $x \in A^{\mathbb{N}}$ and all $i \in \mathbb{N}$: $F(x)_i = f(x_{[\![i,i+\theta[\![})$.*

Example 2.45.

1. *The shift is the CA with diameter $\theta = 2$ and local rule f defined by $f(u_0u_1) = u_1$ for all $u_0, u_1 \in A$.*
2. *Let $A = \{a, b\}$. The* Xor *is the CA with diameter $\theta = 2$ and local rule f defined by: $f(aa) = f(bb) = a$ and $f(ab) = f(ba) = b$.*
3. *Let $A = \{a, b\}$. The* Min *is the CA with diameter $\theta = 2$ and local rule f defined by: $f(aa) = f(ab) = f(ba) = a$ and $f(bb) = b$.*

In the Cantor space, an elegant characterization of cellular automata was given by Curtis, Hedlund and Lyndon in [Hed69] as follows:

Theorem 2.46. *A function $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a cellular automaton if and only if it is continuous with respect to the Cantor metric and shift-invariant (i.e., $F(\sigma(x)) = \sigma(F(x))$, for all $x \in A^{\mathbb{N}}$).*

The shift map is a simple CA from the point of view of the observer since it does not change the configurations. But we find that the shift map verifies strong properties of chaoticity over Cantor space. Let us recall some of these properties:

Proposition 2.47. *Let $X_{d_C} = (A^{\mathbb{N}}, d_C)$ for some alphabet A.*

1. *There is no equicontinuity point for the dynamical system (X_{d_C}, σ).*
2. *(X_{d_C}, σ) is a sensitive cellular automaton.*
3. *(X_{d_C}, σ) is a transitive cellular automaton.*
4. *The set of periodic point of (X_{d_C}, σ) is a dense set.*

Substitutions. Substitutions are maps which preserve the algebraic structure of the monoid A^* with the operation of concatenation. Substitutions occur in many fields such as geometry (fractal, tilings), number theory, logic, ergodic theory, etc., and one can see substitutions as dynamical systems. We shall refer to [FBF+02].

Definition 2.48. *Let A be an alphabet.*

1. *A* substitution *τ is a nonerasing homomorphism of monoid A^*, (i.e., $\tau^{-1}(\lambda) = \{\lambda\}$ and $\tau(uv) = \tau(u)\tau(v)$, for all $u, v \in A^*$).*
2. *The* lower norm *$|\tau|$ and* upper norm *$\|\tau\|$ of τ are defined by:*

$$|\tau| = \min\{|\tau(a)| \mid a \in A\} \text{ and } \|\tau\| = \max\{|\tau(a)| \mid a \in A\}.$$

   We say that τ is uniform *if $|\tau| = \|\tau\|$.*

3. *τ yields a dynamical system, denoted by $\overline{\tau}$, and defined over $A^{\mathbb{N}}$ by:*

$$\overline{\tau}(z) = \tau(z_0)\tau(z_1)\tau(z_2)\tau(z_3)\ldots, \forall z \in A^{\mathbb{N}}.$$

Let us recall, $f(t) = o_{t\to\infty}(g(t))$ if $\lim_{t\to\infty} \frac{f(t)}{g(t)} = 0$, $f(t) = O_{t\to\infty}(g(t))$ if there is $\alpha > 0$ such that for every sufficiently large $t \in \mathbb{N}$, we have $f(t) \leq \alpha g(t)$, and $f(t) = \Theta_{t\to\infty}(g(t))$ if both $f(t) = O_{t\to\infty}(g(t))$ and $g(t) = O_{t\to\infty}(f(t))$.

Remark 2.49. *Thanks to [BR10, Theorem 4.7.15], one knows that there exists a non-empty subalphabet $A_\tau^+ \subset A$ such that for every letter $b \in A$,*

$$\left|\tau^t(b)\right| = \Theta_{t\to\infty}(\left\|\tau^t\right\|) \text{ if } b \in A_\tau^+, \text{ and}$$
$$\left|\tau^t(b)\right| = o_{t\to\infty}(\left\|\tau^t\right\|) \text{ if } b \in A_\tau^- = A \setminus A_\tau^+.$$

Definition 2.50. *Let τ be a substitution over an alphabet A.*

1. *Following [Pan84], we say that τ is* quasi-uniform *if $A_\tau^+ = A$, i.e., for any letter $a \in A$, $\left|\tau^t(a)\right| = \Theta_{t\to\infty}(\left\|\tau^t\right\|)$.*

2. *We say that τ is irreducible if for every letters $a, b \in A$, b appears in $\tau^t(a)$, for some iteration $t \in \mathbb{N}$.*

3. *The matrix of τ is defined as $M(\tau) = (M(\tau)_{ab})_{a,b\in A}$ such that $M(\tau)_{ab}$ is the number $|\tau(a)|_b$ of occurrences of b in $\tau(a)$.*

Remark 2.51. *Quasi-uniform substitutions include uniform substitutions of course, as well as irreducible substitutions. This comes from the observation that, in general, if b appears in $\tau^t(a)$, then $|\tau^s(b)| = O_{s\to\infty}(|\tau^s(a)|)$ (in particular $b \in A_\tau^+ \implies a \in A_\tau^+$).*
When it is reducible, A_τ^+ can be seen from its irreducible components, i.e., the blocks in the triangular form of the matrix $M(\tau)$.
If the spectral radius ρ of the matrix $M(\tau)$ is strictly greater than 1, then Perron-Frobenius theory (see for example [Kůr03, Theorem A.72]) establishes that $\|\tau^t\| = \Theta_{t\to\infty}(\rho^t)$, and for every $a \in A_\tau^-$, $|\tau^t(a)| = O_{t\to\infty}(\rho_-^t)$ for some $\rho_- < \rho$.

Example 2.52. *Let $A = \{a, b\}$.*

1. *The* Thue-Morse *substitution is defined over A by:*

$$\tau: \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & ba \end{array} \qquad M(\tau) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

   This is an irreducible uniform substitution.

2. *The* Fibonacci *substitution is defined over A by:*

$$\tau: \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & a \end{array} \qquad M(\tau) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad M(\tau)^2 = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

   This is an irreducible non-uniform substitution: $|\tau| = 1 < 2 = \|\tau\|$, and $\|\tau^t\|$ is the t^{th} Fibonacci number, which is $\Theta(\rho^t)$, where ρ is the golden ratio.

3. *The* doubling *substitution is defined over A by:*

$$\tau: \begin{array}{ccc} a & \mapsto & aa \\ b & \mapsto & bb \end{array} \qquad M(\tau) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

   This is a uniform reducible substitution: $\{a\}$ and $\{b\}$ are two disjoint invariant subalphabets.

4. *A uniform substitution τ is Toeplitz if there exists $i \in [\![0, \|\tau\|[\![$ such that for all $a, b \in A, \tau(a)_i = \tau(b)_i$. An example is the Cantor substitution, defined by:*

$$\tau: \begin{array}{ccc} a & \mapsto & aba \\ b & \mapsto & bbb \end{array} \qquad M(\tau) = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

   This is a reducible uniform substitution: $\{b\}$ is an invariant subalphabet.

5. *For $k \in \mathbb{N}$, consider the substitution:*

$$\tau : \begin{array}{ccc} a & \mapsto & a^k b \\ b & \mapsto & b \end{array} \qquad M(\tau) = \begin{bmatrix} k & 1 \\ 0 & 1 \end{bmatrix}$$

If $k > 0$, then this substitution is not quasi-uniform: $A_\tau^+ = \{a\}$ and $A_\tau^- = \{b\}$. If $k > 1$, then $\left|\tau^t(a)\right| = \left\|\tau^t\right\| = \Theta(k^t)$, whereas if $k = 1$, then $\left|\tau^t(a)\right| = \left\|\tau^t\right\| = t+1$.

The following lemma states that the fast-growing orbits of letters must involve many fast-growing letters.

Lemma 2.53. *Let τ be any substitution with spectral radius ρ strictly greater than 1, and $a \in A_\tau^+$. Then $\left|\tau^t(a)\right|_{A_\tau^+} = \Theta_{t\to\infty}(\rho^t)$.*

Proof. Since the spectral radius is $\rho > 1$, Remark 2.51 gives $\alpha, \beta, \rho_- > 0$ such that for every $t \in \mathbb{N}$ and every letter $a \in A_\tau^+$, $\beta\rho^t \le \left|\tau^t(a)\right| \le \alpha\rho^t$, and for every letter $a \in A_\tau^-$, $\left|\tau^t(a)\right| \le \alpha\rho_-^t$. Now let $s, t \in \mathbb{N}$, and let us write:

$$\tau^{s+t}(a) = \tau^s(\tau^t(a)_0)\tau^s(\tau^t(a)_1)\cdots\tau^s(\tau^t(a)_{|\tau^t(a)|}).$$

One can now bound the length:

$$\begin{aligned} \left|\tau^{s+t}(a)\right| &= \sum_{\tau^t(a)_i \in A_\tau^+} \left|\tau^s(\tau^t(a)_i)\right| + \sum_{\tau^t(a)_i \in A_\tau^-} \left|\tau^s(\tau^t(a)_i)\right| \\ &\le \left|\tau^t(a)\right|_{A_\tau^+} \alpha\rho^s + (\left|\tau^t(a)\right| - \left|\tau^t(a)\right|_{A_\tau^+})\alpha\rho_-^s \\ &\le \left|\tau^t(a)\right|_{A_\tau^+} \alpha(\rho^s - \rho_-^s) + \alpha^2\rho^t\rho_-^s. \end{aligned}$$

Since $\left|\tau^{s+t}(a)\right| \ge \beta\rho^{s+t}$, we obtain that:

$$\left|\tau^t(a)\right|_{A_\tau^+} \ge \rho^t \frac{\beta - \alpha^2\left(\frac{\rho_-}{\rho}\right)^s}{\alpha(1 - \left(\frac{\rho_-}{\rho}\right)^s)}.$$

When s grows, this fraction converges to $\frac{\beta}{\alpha}\rho^t$. □

Dill maps. Dill maps were defined in [ST15]. As mentioned in [ST15], the word 'dill' comes from the theory of L-systems [Lin74]: a dill map corresponds to a DIL system, that is, a Deterministic Lindenmayer system with Interactions. They add an extra 'l' since they are interested in the action of these maps on Long (infinite) words (configurations).

From another point of view, dill maps generelize both substitutions and CA. Here we give a simple definition, which is equivalent to [ST15, Definition 2].

Definition 2.54.

1. *A* dill map *F with diameter $\theta \in \mathbb{N} \setminus \{0\}$ is a dynamical system over the set of configurations such that there exists a* local rule *$f : A^\theta \to A^+$ satisfying:*

$$F(x) = f(x_{[\![0,\theta[\![}) f(x_{[\![1,\theta+1[\![}) f(x_{[\![2,\theta+2[\![}) \cdots, \forall x \in A^{\mathbb{N}}.$$

2. *We extend the local rule into a self-map $f^* : A^* \to A^*$ by:*

$$f^*(u) = f(u_{[\![0,\theta[\![}) f(u_{[\![1,1+\theta[\![}) \dots f(u_{[\![|u|-\theta,|u|[\![}),$$

   for u such that $|u| \geq \theta$ and $f^*(u) = \lambda$ if $|u| < \theta$.

3. *We also consider the* cocycle *$s_x^n = \sum_{j \leq n} \left| f(x_{[\![j,j+\theta[\![}) \right| = \left| f^*(x_{[\![0,n+\theta[\![}) \right|$ for $n \in \mathbb{N}$ and $x \in A^{\mathbb{N}}$, which represents the position in which one can read the image at offset n:*

$$F(\sigma^n(x)) = \sigma^{s_x^n}(F(x)).$$

4. *The* lower norm *$|f|$ and the* upper norm *$\|f\|$ of a dill map F with diameter θ and local rule f are defined by:*

$$|f| = \min\left\{ |f(u)| \,\middle|\, u \in A^\theta \right\} \text{ and } \|f\| = \max\left\{ |f(u)| \,\middle|\, u \in A^\theta \right\}.$$

5. *If $\|f\| = |f|$, then we say that F is* uniform. *In that case, the cocycle s_x^n does not depend on the configuration x, and we note it s^n.*

When it is clear from the context, we may identify a dill map with its local rule.

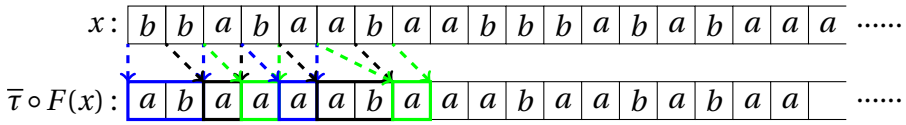
Remark 2.55.

1. *Substitutions are the dill maps with diameter $\theta = 1$.*
2. *Cellular automata are the uniform dill maps with $|f| = \|f\| = 1$.*
3. *The composition of a substitution τ and a cellular automaton local rule f with diameter θ is a dill map local rule $\tau \circ f$ with diameter θ. Actually, every dill map is the composition of a substitution and a shift homomorphism (which is like a cellular automaton, but allowing to change the alphabet).*

Example 2.56. *Let f be the local rule of the Xor CA and τ be the Fibonacci substitution. Then $\tau \circ f$ is a local rule of a dill map with diameter 2 and defined as follows:*

$$\begin{array}{rccc} \tau \circ f : & aa, bb & \mapsto & ab \\ & ba, ab & \mapsto & a \end{array}$$

The following is an example of the action of $\overline{\tau} \circ F$ on a configuration:



Remark 2.57. *For all $x \in A^{\mathbb{N}}$, $n \mapsto s_x^n$ is a one-to-one function.*

Proof. For $n \leq m \in \mathbb{N}$ such that $s_x^n = s_x^m$ we have:

$$\sum_{i \leq m} \left| f(x_{[\![i,i+s[\![}) \right| - \sum_{i \leq n} \left| f(x_{[\![i,i+s[\![}) \right| = \sum_{n < i \leq m} \left| f(x_{[\![i,i+s[\![}) \right| = 0$$

Then for all $n < i \leq m$, $\left| f(x_{[\![i,i+s[\![}) \right| = 0$. Since f is nonerasing (*i.e.*, $|f| > 0$), $n = m$. □

Similarly to the case of cellular automata, we give a characterization of dill maps à la Curtis-Hedlund-Lyndon; it is quite classical but we are not aware of a reference with this exact statement, so we include it for completeness. Recall that $\mathbb{N}$ can be naturally endowed with the discrete topology.

Theorem 2.58. *A function $F : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is a dill map if and only if it is continuous over the Cantor space and there exists a continuous map $s : A^{\mathbb{N}} \to \mathbb{N}$ such that for all $x \in A^{\mathbb{N}}$,*

$$F(\sigma(x)) = \sigma^{s(x)}(F(x)).$$

Proof. Let F be a dill map with diameter θ and local rule f. For $x \in A^{\mathbb{N}}$, $\varepsilon = 2^{-p}$ with $p \in \mathbb{N} \setminus \{0\}$, we take $m = \min\left\{ i \in \mathbb{N} \middle| s_x^i \geq p \right\}$. For $\delta = 2^{-(m+\theta)}$ and $y \in A^{\mathbb{N}}$ such that $\mathfrak{d}_C(x,y) \leq \delta$ we have $x_{[\![0,m+\theta[\![} = y_{[\![0,m+\theta[\![}$. Then $f^*(x_{[\![0,m+\theta[\![}) = f^*(y_{[\![0,m+\theta[\![})$. Hence $F(x)_{[\![0,p[\![} = F(y)_{[\![0,p[\![}$. So $\mathfrak{d}_C(F(x),F(y)) \leq 2^{-p} = \varepsilon$. In conclusion, F is continuous.

Now let us define $s(x) = s_x^1 = \left| f(x_{[\![0,\theta[\![}) \right|$ for all $x \in A^{\mathbb{N}}$. Let $x \in A^{\mathbb{N}}$ and $\varepsilon > 0$. For $y \in A^{\mathbb{N}}$ such that $\mathfrak{d}_C(x,y) < 2^{-\theta}$ we have $x_{[\![0,\theta[\![} = y_{[\![0,\theta[\![}$. Then $f(x_{[\![0,\theta[\![}) = f(y_{[\![0,\theta[\![})$ and hence $\left| f(x_{[\![0,\theta[\![}) \right| = \left| f(y_{[\![0,\theta[\![}) \right|$. So $\left| s(x) - s(y) \right| = 0$.
In conclusion, s is continuous, and it satisfies $F(\sigma(x)) = \sigma^{s(x)}(F(x))$.

Assume now that there exists a continuous map $s : A^{\mathbb{N}} \to \mathbb{N}$ such that $F(\sigma(x)) = \sigma^{s(x)}(F(x))$ for every $x \in A^{\mathbb{N}}$. We can write $A^{\mathbb{N}} = \bigcup_{n \in \mathbb{N}} s^{-1}(\{n\})$. For all $n \in \mathbb{N}$, since $\{n\}$ is clopen and s is continuous, $s^{-1}(\{n\})$ is also open. On the other hand, $(A^{\mathbb{N}}, \mathfrak{d}_C)$ is compact, so that $A^{\mathbb{N}} = \bigcup_{i \in I} s^{-1}(\{i\})$ for some finite set $I \subset \mathbb{N}$. In other words, for every $x \in A^{\mathbb{N}}$, $s(x) \leq \max I$. Since F is continuous, for $\varepsilon = 2^{-\max I}$, there exists $\theta \in \mathbb{N}$ such that for all $y \in A^{\mathbb{N}}$ verifying $\mathfrak{d}_C(x,y) < 2^{-\theta}$, we have $\mathfrak{d}_C(F(x),F(y)) < \varepsilon$. Hence for all $x \in A^{\mathbb{N}}$ and all $y \in A^{\mathbb{N}}$ with $x_{[\![0,\theta[\![} = y_{[\![0,\theta[\![}$, we have $F(x)_{[\![0,s(x)[\![} = F(y)_{[\![0,s(x)[\![}$. So one can define a map $f : A^{\theta} \to A^*$ such that $f(x_{[\![0,\theta[\![}) = F(x)_{[\![0,s(x)[\![}$, for all $x \in A^{\mathbb{N}}$. On the other hand, for $x \in A^{\mathbb{N}}$ and $j \in \mathbb{N}$ we have:

$$f(x_{[\![j,j+\theta[\![}) = f(\sigma^j(x)_{[\![0,\theta[\![}) = F(\sigma^j(x))_{[\![0,s(\sigma^j(x))[\![}.$$

By inductively applying the assumption, we find:

$$F(\sigma^j(x)) = F(\sigma(\sigma^{j-1}(x))) = \sigma^{s(\sigma^{j-1}(x))} F(\sigma^{j-1}(x)) = \sigma^{s_x^j} F(x), \text{ where } s_x^j = \sum_{h=0}^{j-1} s(\sigma^h(x)).$$

Then : $f(x_{[\![j,j+\theta[\![}) = \sigma^{s_x^j} F(x)_{[\![0,s(\sigma^j(x))[\![} = F(x)_{\left[\!\left[s_x^j, s_x^{j+1}\right[\!\right[}$.
Finally, we find that $F(x) = f(x_{[\![0,\theta[\![}) f(x_{[\![1,1+\theta[\![}) f(x_{[\![2,2+\theta[\![}) \cdots$: F is a dill map. □

Corollary 2.59. *Let F be a dill map with local rule f. For all $x \in A^{\mathbb{N}}$, all $n \in \mathbb{N}$ we have:*

$$F \circ \sigma^n(x) = \sigma^{s_x^n} \circ F(x).$$

Proof. Let F be a dill map with diameter θ and local rule f. Then according to the proof of Theorem 2.58 $F(\sigma(x)) = \sigma^{s_x^1}(F(x))$ for all $x \in A^{\mathbb{N}}$. We aim to prove by induction that for all $x \in A^{\mathbb{N}}$, for all $n \in \mathbb{N}$, for $s_x^n = \left| f^*(x_{[\![0,n+\theta[\![}) \right|$ we have:

$$F \circ \sigma^n = \sigma^{s_x^n} \circ F.$$

Let $x \in A^{\mathbb{N}}$. For $k = 1$ we have $F(\sigma(x)) = \sigma^{s_x}(F(x))$. We suppose that our statement is true for $k = n$, so there exists $s_x^n \in \mathbb{N}$ such that $F(\sigma^n(x)) = \sigma^{s_x^n}(F(x))$. Now, for $k = n+1$ we have:

$$F \circ \sigma^{n+1}(x) = F(\sigma(\sigma^n(x))) = \sigma^{s_{\sigma^n(x)}}(F(\sigma^n(x))) = \sigma^{s_{\sigma^n(x)}}(\sigma^{s_x^n}(F(x))) = \sigma^{s_x^{n+1}}(F(x)).$$

Which is the next step of the induction hypothesis.
Hence, for all $x \in A^{\mathbb{N}}$, all $n \in \mathbb{N}$ and for $s_x^n = \sum_{k=0}^{n-1} s_{\sigma^k(x)}$ we have: $F(\sigma^n(x)) = \sigma^{s_x^n}(F(x))$. □

2.3 Edit distances

We can endow the set A^n of words, for $n \in \mathbb{N}$, with some distance, *i.e.,* some application from $A^n \times A^n$ to $\mathbb{R}_+$ satisfying: separation, symmetry, triangle inequality.

The prototypical example is the *Hamming distance* denoted by d_H (see, e.g., [Ham86]). It is usually defined as the number of differences between two finite words of the same length. Let us present a definition in terms of edit distance, that is the minimum number of substitution operations to transform a word into another one.

Definition 2.60. *The* deletion *operation D_j at position $j \in [\![0, |u|[\![$ is defined over word $u \in A^*$ as follows: $D_j(u) = u_0 u_1 \dots u_{j-1} u_{j+1} \dots u_{|u|-1}$.*

Definition 2.61. *Between two finite words with the same length u, v, we define the* Hamming distance*:*

$$d_H(u,v) = \min\left\{ m \in [\![0,|u|[\![\,\middle|\, \exists j_1 < \dots < j_m, D_{j_1} \circ \dots \circ D_{j_m}(u) = D_{j_1} \circ \dots \circ D_{j_m}(v) \right\}.$$

Note that in the literature there exist other edit operations like insertion (insert a letter at a position of a finite word) and substitution (change a letter of a finite word to another letter). Of course, the definition would be completely equivalent by allowing substitution (resp. insertion) operations instead of deletion operations to be performed on both u and v, and is simply the number of differences, letterwise (*i.e.*, $d_H(u,v) = |\{i \in [\![0,|u|[\![\mid u_i \neq v_i\}|$, for all $u, v \in A^*$ with the same length).

Another classical edit distance is the Levenshtein distance [Lev66]. Instead of allowing to edit finite words in the same positions (like for the Hamming distance), we now allow to edit in different positions.

Definition 2.62. *The* Levenshtein distance d_L *is defined over* $u, v \in A^*$ *as follows:*

$$d_L(u,v) = \frac{1}{2}\min\left\{ m+m' \,\middle|\, \exists j_1 < \cdots < j_m, j'_1 < \cdots < j'_{m'}, D_{j_1} \circ \ldots \circ D_{j_m}(u) = D_{j'_1} \circ \ldots \circ D_{j'_{m'}}(v) \right\}.$$

Most frequently, we will consider the distance between two words of equal length, so that the result is an integer, and can be defined equivalently as the minimal length m of two sequences $D_{j_1} \ldots D_{j_m}$ and $D_{j'_1} \ldots D_{j'_m}$ such that $D_{j_1} \circ \ldots \circ D_{j_m}(u) = D_{j'_1} \circ \ldots \circ D_{j'_m}(v)$.

The distance $d_L(u,v)$ can also be defined as $\frac{|u|+|v|}{2} - \ell$, where ℓ is the length of the longest common (possibly non-contiguous) subword between u and v.

Several variants exist in the literature:

Remark 2.63.

- *One may want to remove factor* $\frac{1}{2}$ *in the definition, to make the definition look more natural. Nevertheless, the two points above, as well as the next two remarks, motivate our definition. Anyway, the two distances* d_L *and* $2d_L$ *are equivalent.*
- *If one allows two edition operations, insertion (insert a letter at position* i *in a finite word* u*) and deletion (from Definition 2.60), the purpose could be that it can be defined by performing all operations only on one of the two words. The two distances are here exactly equal because an insertion on one side corresponds to a deletion on the other side. Manipulations are a little more technical because one has to deal with as many insertion operations as there are letters in the alphabet.*
- *If one additionnally allows the substitution operation (change the letter at position* i *of a finite word* u *by another letter from the alphabet), with weight* 1*, then again the two obtained distances are equal, because a substitution corresponds to a sequence of an insertion and a deletion.*
- *If one gives the same weights to the substitution and deletion operations, then one gets an equivalent distance (bounded between* d_L *and* $2d_L$*).*

Example 2.64. *Let $A = \{a, b\}$.*

1. *For $u = ababab$ and $v = bababa$, we have : $d_L(u, v) = 1$.*
   Indeed, $D_0(u) = babab$ (we delete the letter of index 0 in u), then we delete the last letter in the end of the word v and we find $D_0(u) = babab = D_5(v)$. For the sake of comparison, note that $d_H(u, v) = 6$.

2. *For $u = aaaa$ and $v = aaaab$, we have $d_L(u, v) = \frac{1}{2}$ since it is enough to delete the last letter of v.*

Remark 2.65. *For every $u, v \in A^*$, we have:*

$$\frac{||u| - |v||}{2} \leq d_L(u, v) \leq \frac{|u| + |v|}{2}.$$

Proof. The upper bound comes from the trivial edition sequence producing:

$$D_1 \circ D_2 \circ \cdots \circ D_{|u|}(u) = \lambda = D_1 \circ \ldots \circ D_{|v|}(v).$$

On the other hand, if

$$D_{j_1} \circ D_{j_2} \circ \cdots \circ D_{j_m}(u) = D_{j'_1} \circ D_{j'_2} \circ \cdots \circ D_{j'_{m'}}(v),$$

then

$$\left|D_{j_1} \circ D_{j_2} \circ \cdots \circ D_{j_m}(u)\right| = \left|D_{j'_1} \circ D_{j'_2} \circ \cdots \circ D_{j'_{m'}}(v)\right|.$$

Hence, $|u| - m = |v| - m'$. Then we can conclude that

$$\frac{|u| - |v|}{2} = \frac{\left|m - m'\right|}{2} \leq \frac{m + m'}{2} = d_L(u, v). \quad \square$$

Remark 2.66. *The Hamming distance is an upper bound for the Levenshtein distance, i.e., for all words $u, v \in A^*$ such that $|u| = |v|$, we have $d_L(u, v) \leq d_H(u, v)$.*

Proof. Let $d_H(u, v) = m$. Then there exist $j_1 < \cdots < j_m$ such that for all $h \in [\![1, m]\!]$, $u_{j_h} \neq v_{j_h}$. If we delete u_{j_h} and v_{j_h} for all $h \in [\![1, m]\!]$, then we find $D_{j_1} \circ \ldots \circ D_{j_m}(u) = D_{j_1} \circ \ldots \circ D_{j_m}(v)$. Hence: $d_L(u, v) \leq \frac{2m}{2} = m = d_H(u, v)$. $\square$

Remark 2.67. *Let A, B be two alphabet. Thanks to the characterization of d_L in terms of longest common subword, for $u \in A^*$ and $v \in B^*$, we have*

$$d_L(u, v) = d_L(u', v) + \frac{1}{2}|u|_{A \setminus B},$$

where u' is the subword of all letters in B from u. In particular, if u and v have equal length, then $d_L(u, v) \geq |u|_{A \setminus B}$. This is because every deletion sequence must at least delete these letters and as many letters of v.

If, besides, $|B| = 1$ (i.e., $v = a^{|u|}$ for some a), then there is equality:

$$d_L(u,v) = d_H(u,v) = |u|_{A\setminus\{a\}}.$$

Now let us categorize distances over the set of finite words. Before that, we introduce a notation that will be used throughout this thesis.

Notations 2.68. *For $n \in \mathbb{N}$ and $U \subseteq A^n$, we denote:* $\max d_{|U} = \max_{u,v\in U} d(u,v)$.

Definitions 2.69. *Let d be a distance over the set of finite words A^*.*

1. *We say that d is **additive** if for all u, u', v, v' such that $|u| = |u'|$ and $|v| = |v'|$, we have, $d_H(uv, u'v') = d_H(u,u') + d_H(v,v')$.*
2. *We say that d is **subadditive** if for all $u, v, u', v' \in A^*$ such that $|u| = |v|$ and $|u'| = |v'|$ we have:*
$$d(uu', vv') \leq d(u,v) + d(u',v').$$
3. *d is **infra-superadditive** if for all $u, v \in A^*$ such that $|u| = |v|$ and $|u'| = |v'|$:*
$$\max\big(d(u,v), d(u',v')\big) \leq d(uu', vv').$$
4. *d is **infinitary** if:* $\lim_{\ell\to\infty} \max d_{|A^\ell} = \infty$.

Remark 2.70. *We can remark that additivity implies subadditivity and subadditivity implies that for all $\ell \in \mathbb{N}$, $\max_{|A^\ell} \leq \ell \times \max d_{|A}$ (we get the equality if d is additive).*

Proposition 2.71. *For any subadditive distance d and for any $n \in \mathbb{N}$, we have:*

$$d(u,v) \leq d_H(u,v) \times \max d_{|A}, \forall u, v \in A^n.$$

Proof. Let $u, v \in A^n$. According to subadditivity of d w have:

$$d(u,v) \leq \sum_{i=0}^{n-1} d(u_i, v_i) \leq \sum_{\substack{i\in[\![0,n[\![\\ u_i\neq v_i}} d(u_i,v_i) \leq d_H(u,v) \times \max d_{|A}. \quad \square$$

The Hamming distance satisfies all of these properties. It is actually essentially the only additive distance (up to equivalence).

Proposition 2.72. *If d is an additive distance then it is equivalent to d_H.*

Proof. Let $c_1 = \min_{a\neq b\in A} d(a,b)$ and $c_2 = \max_{a\neq b\in A} d(a,b)$. For $n \in \mathbb{N}$ and $u, v \in A^n$ we have:

$$c_1 \times d_H(u,v) \leq d(u,v) = \sum_{i=0}^{n} d(u_i,v_i) \leq c_2 \times d_H(u,v).$$

Therefore if d is additive then it is equivalent to the Hamming distance. $\square$

Remark 2.73. *The Levenshtein distance is not additive. Indeed, for* $u = 0101$ *and* $v = 1010$*, we find:* $d_{\mathscr{L}}(u,v) = 2$*,* $d_{\mathscr{L}}(u_{[\![0,2[\![}, v_{[\![0,2[\![}) = 2$ *and* $d_{\mathscr{L}}(u_{[\![2,4[\![}, v_{[\![2,4[\![}) = 2$*. Then:* $d_{\mathscr{L}}(u,v) = 2 \neq d_{\mathscr{L}}(u_{[\![0,2[\![}, v_{[\![0,2[\![}) + d_{\mathscr{L}}(u_{[\![2,4[\![}, v_{[\![2,4[\![}) = 2+2 = 4$.

Proposition 2.74. *The Levenshtein distance is subadditive and infra-superadditive.*

Proof. Let us start by subadditivity. Consider $u, u', v, v' \in A^*$, and $m, m', n, n' \in \mathbb{N}$ such that:

$$D_{j_1} \circ \cdots \circ D_{j_m}(u) = D_{j'_1} \circ \cdots \circ D_{j'_{m'}}(v) \text{ and}$$
$$D_{j_{m+1}} \circ \cdots \circ D_{j_{m+n}}(u') = D_{j'_{m'+1}} \circ \cdots \circ D_{j'_{m'+n'}}(v'),$$

for some minimal edition sequences $j_1 < \cdots < j_m < |u|$, $j'_1 < \cdots < j_{m'} < |v|$, $j_{m+1} < \cdots < j_{m+n} < \left|u'\right|$ and $j'_{m'+1} < \cdots < j'_{m'+n'} < \left|v'\right|$, so that $d_L(u,v) = \frac{m+m'}{2}$ and $d_L(u',v') = \frac{n+n'}{2}$. By concatenating the two previous edited words, we obtain:

$$D_{j_1} \circ \cdots \circ D_{j_m} \circ D_{|u|+j_{m+1}} \circ \cdots \circ D_{|u|+j_{m+n}}(uu') = D_{j'_1} \circ \cdots \circ D_{j_{m'}} \circ D_{|v|+j_{m'+1}} \circ \cdots \circ D_{|v|+j'_{m'+n'}}(vv').$$

Therefore $d_L(uu', vv') \leq \frac{m+n+m'+n'}{2} = d_L(u,v) + d_L(u',v')$.

Now, let us prove infra-superadditivity. Consider words u, u', v, v', and m, m' such that:

$$D_{i_1} \circ \cdots \circ D_{i_m}(uu') = D_{i'_1} \circ \cdots \circ D_{i'_{m'}}(vv'),$$

for some minimal editing sequences $i_1 < \cdots < i_m < \left|uu'\right|$, $i'_1 < \cdots < i_{m'} < \left|vv'\right|$, so that $d_L(uu', vv') = \frac{m+m'}{2}$. It is clear that:

$$D_{i_1} \circ \cdots \circ D_{i_p}(u) = D_{i'_1} \circ \cdots \circ D_{i'_{p'}}(v) \text{ and}$$
$$D_{i_{p+1}-|u|} \circ \cdots \circ D_{i_m-|u|}(u') = D_{i'_{p'+1}-|v|} \circ \cdots \circ D_{i'_{m'}-|v|}(v'),$$

where $p = \max\left\{ j \in [\![1, m+1[\![\,\middle|\, i_j < |u| \right\}$ and $p' = \min\left\{ j \in [\![1, m'+1[\![\,\middle|\, i'_j < |v| \right\}$. Hence, $d_L(u,v) \leq \frac{p+p'}{2} \leq d_L(u,v)$ and $d_L(u',v') \leq \frac{m+m'-(p+p')}{2} \leq d_L(uu', vv')$. □

3 Pseudo-metrics defined via distances over finite words

3.1 Besicovitch and Weyl-like pseudo-metrics

3.1.1 The Besicovitch and Weyl pseudo-metrics

The Besicovitch and Weyl pseudo-metrics were introduced and defined in [Bes54] in the aim to study quasi-periodic function. The first use of these pseudo-distances for the study of cellular automata can be seen in [CFMM97] and [BFK97]. In this section, we give basic definitions and first results.

For that, let us start by giving the definition of a pseudo-metric:

Definition 3.1. *Let X be a non-empty set. A pseudo-metric d is a map defined over $X \times X$ to $\mathbb{R}_+$ such that it verifies the following properties:*

1. *For all $x \in X$, $d(x,x) = 0$.*
2. *Symmetry: $d(x,y) = d(y,x)$, for all $x, y \in X$.*
3. *Triangular inequality: $d(x,y) \leq d(x,z) + d(z,y)$, for all $x, y, z \in X$*

The Besicovitch and Weyl pseudo-metrics were defined in [BFK97] as the asymptotical behavior of the Hamming distance.

Definition 3.2.

1. *The Besicovitch pseudo-metric, denoted by $\mathfrak{d}_H$, is defined as follows:*

$$\mathfrak{d}_H(x,y) = \limsup_{\ell \to \infty} \frac{d_H(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell}, \forall x, y \in A^{\mathbb{N}}.$$

2. *The Weyl pseudo-metric, denoted by $\hat{\mathfrak{d}}_H$, is defined as follows:*

$$\hat{\mathfrak{d}}_H(x,y) = \limsup_{\ell \to \infty} \max_{k \in \mathbb{N}} \frac{d_H(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\ell}, \forall x, y \in A^{\mathbb{N}}.$$

Note that the Besicovitch and Weyl pseudo-metrics are sometimes defined differently and the definition above is usually said to be the $\overline{d}$-metric, see for example [KŁO17, Definition 1], [ŁS18, Section 3.2] and [DI88].

However, all these definitions are uniformly equivalent thanks to [KŁO17, Theorem 4 and Corollary 5]. The Besicovitch pseudo-metric appeared also in [Aus59], [FGJ16] and one can read a nice discussion in [KKK23, Section 3] about $\overline{d}$-metric, which is an analogue of Ornstein's metric $\overline{d}_{\mathscr{M}}$ (defined over the space of all Borel invariant probability measures on a shift space X, [Orn74]) on A-valued stationary stochastic processes.

In [BFK97], it was proved that $\mathfrak{d}_H$ and $\hat{\mathfrak{d}}_H$ are two pseudo-metrics and not distances (for more details, see Proposition 3.13 and Example 3.14).

Hence, it is relevant to quotient the space of infinite words by the equivalence of zero pseudo-metrics, in order to get separated topological spaces:

Definition 3.3.

- *The relation* $x \sim_{\mathfrak{d}_H} y \iff \mathfrak{d}_H(x,y) = 0$, *is an equivalence relation.*
- *The quotient space* $A^{\mathbb{N}}/\sim_{\mathfrak{d}_H}$ *(resp.* $A^{\mathbb{N}}/\sim_{\hat{\mathfrak{d}}_H}$*) is a topological space, called the Besicovitch space (resp. the Weyl space), denoted* $X_{\mathfrak{d}_H}$ *(resp.* $X_{\hat{\mathfrak{d}}_H}$*).*

Now we show some basic properties of those spaces and the two pseudo-metrics. Firstly, we show that the Besicovitch pseudo-metric is not always a limit by giving examples deduced from Lemma 3.5 and Lemma 3.6. For that, let us denote,

$$\underline{\mathfrak{d}_H}(x,y) = \liminf_{\ell\to\infty} \frac{d_H(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell}, \ \forall x,y \in A^{\mathbb{N}}.$$

Notations 3.4. *If* $a \neq b \in A$ *and* $(\alpha_i)_{i\in\mathbb{N}}, (\beta_i)_{i\in\mathbb{N}} \in (\mathbb{N}\setminus\{0\})^{\mathbb{N}}$, *then let us denote*

$$(a,b)^{(\alpha_i,\beta_i)_{i\in\mathbb{N}}} = a^{\alpha_0} b^{\beta_0} a^{\alpha_1} b^{\beta_1} a^{\alpha_2} \cdots.$$

Lemma 3.5. *For* $a \neq b \in A$, *if* $x = (a,b)^{(\alpha_i,\beta_i)_{i\in\mathbb{N}}}$, *then:* $\mathfrak{d}_H(x, a^\infty) = \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)}$.

Proof. For $n \in \mathbb{N}$, let $\ell_n = \sum_{i=0}^{n-1}(\alpha_i+\beta_i)$. We have: $\mathfrak{d}_H(x_{[\![0,\ell_n[\![}, a^{\ell_n}) = \sum_{i=0}^{n-1}\beta_i$. Then:

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)} = \limsup_{n\to\infty} \frac{\mathfrak{d}_H(x_{[\![0,\ell_n[\![}, a^{\ell_n})}{\ell_n} \leq \limsup_{\ell\to\infty} \frac{\mathfrak{d}_H(x_{[\![0,\ell[\![}, a^{\ell})}{\ell} = \mathfrak{d}_H(x, a^\infty).$$

On the other hand, for $\ell \in \mathbb{N}$, there exist $n = n(\ell) \in \mathbb{N}$ and $\delta \in [\![0, \alpha_n+\beta_n[\![$ such that $\ell = \sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta$. If $\delta \leq \alpha_n$ then:

$$\begin{aligned}\frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} &= \frac{d_H(x_{[\![0,\sum_{i=0}^{n-1}\alpha_i+\beta_i[\![}, a^{\ell-\delta}) + d_H(x_{[\![\sum_{i=0}^{n-1}\alpha_i+\beta_i, \sum_{i=0}^{n-1}\alpha_i+\beta_i+\delta[\![}, a^\delta)}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta} \\ &= \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta} \leq \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)}.\end{aligned}$$

Otherwise, $\delta > \alpha_n$, then:

$$\begin{aligned}\frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} &= \frac{d_H(x_{[\![0,\sum_{i=0}^{n-1}\alpha_i+\beta_i[\![}, a^{\ell-\delta}) + d_H(x_{[\![\sum_{i=0}^{n-1}\alpha_i+\beta_i, \sum_{i=0}^{n-1}\alpha_i+\beta_i+\delta[\![}, a^\delta)}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta} \\ &= \frac{\sum_{i=0}^{n-1}\beta_i + \delta - \alpha_n}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta} \\ &\leq \frac{\sum_{i=0}^{n-1}\beta_i + \delta - \alpha_n + (\alpha_n+\beta_n-\delta)}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta+(\alpha_n+\beta_n-\delta)} = \frac{\sum_{i=0}^{n}\beta_i}{\sum_{i=0}^{n}(\alpha_i+\beta_i)}.\end{aligned}$$

In both cases, there exists $m(\ell)$ such that :

$$\frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} \leq \frac{\sum_{i=0}^{m(\ell)-1}\beta_i}{\sum_{i=0}^{m(\ell)-1}(\alpha_i+\beta_i)}.$$

Note that $m(\ell) \in \{n, n+1\}$, so that $m(\ell) \to \infty$ when $\ell \to \infty$. It results that:

$$\mathfrak{d}_H(x, a^\infty) = \limsup_{\ell\to\infty} \frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} \leq \limsup_{\ell\to\infty} \frac{\sum_{i=0}^{m(\ell)-1}\beta_i}{\sum_{i=0}^{m(\ell)-1}(\alpha_i+\beta_i)} \leq \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)}. \quad \square$$

Lemma 3.6. *For $a \neq b \in A$, if $x = (a,b)^{(\alpha_i,\beta_i)_{i\in\mathbb{N}}}$, then:*

$$\underline{\mathfrak{d}_H}(x, a^\infty) = \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n}.$$

Proof. For $n \in \mathbb{N}$, let $\ell_n = \sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n$. We have: $d_H(x_{[\![0,\ell_n[\![}, a^{\ell_n}) = \sum_{i=0}^{n-1}\beta_i$. Then:

$$\begin{aligned}\underline{\mathfrak{d}_H}(x, a^\infty) = \liminf_{\ell\to\infty} \frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} &\leq \liminf_{n\to\infty} \frac{d_H(x_{[\![0,\ell_n[\![}, a^{\ell_n})}{\ell_n} \\ &= \liminf_{n\to\infty} \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n}.\end{aligned}$$

On the other hand, for $\ell \in \mathbb{N}$, there exist $n = n(\ell) \in \mathbb{N}$ and $\delta \in [\![0, \beta_n+\alpha_n[\![$ such that $\ell = \sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta$. If $\delta \leq \alpha_n$ then:

$$\begin{aligned}\frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} &= \frac{d_H(x_{[\![0,\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n[\![}, a^{\ell-\delta}) + d_H(x_{[\![\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n, \ell[\![}, a^\delta)}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\delta} \\ &\geq \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n}.\end{aligned}$$

Otherwise, $\delta > \alpha_n$, then:

$$\begin{aligned}\frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} &= \frac{d_H(x_{[\![0,\sum_{i=0}^{n-1}\alpha_i+\beta_i+\alpha_n[\![}, a^{\ell-\delta}) + d_H(x_{[\![\sum_{i=0}^{n-1}\alpha_i+\beta_i+\alpha_n,\ell[\![}, a^\delta)}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n+(\delta-\alpha_n)} \\ &= \frac{\sum_{i=0}^{n-1}\beta_i+\delta-\alpha_n}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n+(\delta-\alpha_n)} \\ &\geq \frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n}.\end{aligned}$$

In both cases, we obtain,

$$\frac{\sum_{i=0}^{n(\ell)-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n} \leq \frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell},$$

where $\sum_{i=0}^{n(\ell)-1}(\alpha_i+\beta_i) \leq \ell \leq \sum_{i=0}^{n(\ell)}(\alpha_i+\beta_i)$, so that $\lim_{\ell\to\infty} n(\ell) = \infty$ (because we assumed that $\alpha_i+\beta_i \in \mathbb{N}\setminus\{0\}$). It results that:

$$\underline{\mathfrak{d}_H}(x, a^\infty) = \liminf_{\ell\to\infty}\frac{d_H(x_{[\![0,\ell[\![}, a^\ell)}{\ell} \geq \liminf_{n\to\infty}\frac{\sum_{i=0}^{n-1}\beta_i}{\sum_{i=0}^{n-1}(\alpha_i+\beta_i)+\alpha_n}. \quad \square$$

Example 3.7. *For $x = (a,b)^{(2^{2i+1},2^{2i})_{i\in\mathbb{N}}}$ we find that: $\mathfrak{d}_H(x, a^\infty) \neq \underline{\mathfrak{d}_H}(x, a^\infty)$. Indeed,*

$$\mathfrak{d}_H(x, a^\infty) = \limsup_{n\to\infty}\frac{\sum_{i=0}^{n-1}2^{2i}}{\sum_{i=0}^{n-1}(2^{2i+1}+2^{2i})} = \limsup_{n\to\infty}\frac{\sum_{i=0}^{n-1}2^{2i}}{3\times\sum_{i=0}^{n-1}2^{2i}} = \frac{1}{3}.$$

And,

$$\begin{aligned}\underline{\mathfrak{d}_H}(x, a^\infty) &= \liminf_{n\to\infty}\frac{\sum_{i=0}^{n-1}2^{2i}}{\sum_{i=0}^{n-1}(2^{2i+1}+2^{2i})+2^{2n+1}} \\ &= \liminf_{n\to\infty}\frac{\sum_{i=0}^{n-1}2^{2i}}{3\times\sum_{i=0}^{n-1}(2^{2i})+2^{2n+1}} \\ &= \liminf_{n\to\infty}\frac{1}{3}\times\frac{2^{2n}-1}{3\times 2^{2n}-1} = \frac{1}{9}.\end{aligned}$$

In several works, the Weyl pseudo-metrics was defined as the upper limit. In constrast to the Besicovitch pseudo-metric, the upper limit appearing in the Weyl pseudo-metric is actually a limit (for more details see Theorem 3.19).

Thanks to the definition of the maximum, one can deduce that the Weyl pseudo-metric is an upper bound of the Besicovitch pseudo-metric. Hence the Weyl topology is finer than the Besicovitch topology. For more details, see Remark 3.15.

Remark 3.8. *For any $x, y \in A^{\mathbb{N}}$, we have, $\mathfrak{d}_H(x,y) \leq \hat{\mathfrak{d}}_H(x,y)$.*

One of the motivations to study those pseudo-metrics is that they are shift-invariant as it was proved in [BFK97] (for more details of the proof, see Theorem 3.24). And thus, the shift map is an isometry over the Besicovitch and Weyl space.

Proposition 3.9 ([BFK97])**.** *For any $x, y \in A^{\mathbb{N}}$, we have,*

$$\mathfrak{d}_H(\sigma(x),\sigma(y)) = \mathfrak{d}_H(x,y) \text{ and } \hat{\mathfrak{d}}_H(\sigma(x),\sigma(y)) = \hat{\mathfrak{d}}_H(x,y).$$

3.1.2 General Definitions and first results

The Besicovitch pseudo-metric is defined as the asymptotics of the Hamming distance over prefixes of the sequences. This pseudo-metric is depending on an edit distance over all finite words of the same length (the Hamming distance). This makes it natural to suggest a general definition, using any distance d over finite words of the same length. Note that any distance over finite words when restricted to words of the same length is still a distance.

Definition 3.10. *Let A be an alphabet and d be a distance over A^*. The **centered pseudo-metric** associated to d is:*

$$\mathfrak{d}_d(x,y) = \limsup_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\max d_{|A^\ell}}, \ \forall x, y \in A^{\mathbb{N}}.$$

Following another point of view, the Weyl pseudometric also based on d_H, measures the density of differences between two given sequences in arbitrary segments of a given length. A general definition, based on any distance d over finite words of the same length, would become the following:

Definition 3.11. *Let A be an alphabet and d be a distance over A^*. The **sliding pseudo-metric** associated to d is:*

$$\hat{\mathfrak{d}}_d(x,y) = \limsup_{\ell\to\infty} \max_{k\in\mathbb{N}} \frac{d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\max d_{|A^\ell}}, \ \forall x, y \in A^{\mathbb{N}}.$$

Example 3.12.

1. *The Besicovitch (resp. Weyl) pseudo-metric is the centered (resp. sliding) pseudo-metric associated to the Hamming distance denoted by $\mathfrak{d}_H = \mathfrak{d}_{d_H}$ (resp. $\hat{\mathfrak{d}}_H = \hat{\mathfrak{d}}_{d_H}$).*
2. *The centered pseudo-metric associated to the Levenshtein distance is called here the Feldman-Katok pseudo-metric.*

The pseudo-metric $\mathfrak{d}_{d_L}$ is often denoted $\bar{f}$ in the literature and Feldman-Katok pseudo-metric is sometimes defined differently (see for example [KM17] and [GRK20]). However, all these definitions are uniformly equivalent thanks to [ORW82]. Moreover, we keep the notation $\mathfrak{d}_L = \mathfrak{d}_{d_L}$ to emphasize the similarity with $\mathfrak{d}_H$.

Proposition 3.13. *For a distance d over A^*, $\mathfrak{d}_d$ and $\hat{\mathfrak{d}}_d$ are pseudo-metrics.*

Proof. The proof of both cases are similar and based on the properties of the distance; so we write only the proof for $\hat{\mathfrak{d}}_d$. Let $x, y, z \in A^{\mathbb{N}}$. If $x = y$ then for all $\ell, k \in \mathbb{N}$ we have $d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![}) = 0$, hence, $\hat{\mathfrak{d}}_d(x,y) = 0$. According to the symmetric property of the distance d we have:

$$\hat{\mathfrak{d}}_d(x,y) = \limsup_{\ell\to\infty} \max_{k\in\mathbb{N}} \frac{d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\max d_{|A^\ell}} = \limsup_{\ell\to\infty} \max_{k\in\mathbb{N}} \frac{d(y_{[\![k,k+\ell[\![}, x_{[\![k,k+\ell[\![})}{\max d_{|A^\ell}} = \hat{\mathfrak{d}}_d(y,x).$$

Finally, since d verifies the triangle inequality, we have:

$$\begin{aligned}
\hat{\mathfrak{d}}_d(x,y) &= \limsup_{\ell\to\infty} \max_{k\in\mathbb{N}} \frac{d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\max d_{|A^\ell}} \\
&\leq \limsup_{\ell\to\infty} \max_{k\in\mathbb{N}} \frac{d(x_{[\![k,k+\ell[\![}, z_{[\![k,k+\ell[\![})}{\max d_{|A^\ell}} + \limsup_{\ell\to\infty} \max_{k\in\mathbb{N}} \frac{d(z_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\max d_{|A^\ell}} \\
&= \hat{\mathfrak{d}}_d(x,z) + \hat{\mathfrak{d}}_d(z,y). \quad \square
\end{aligned}$$

The centered pseudo-metric and the sliding one are not always distances (*i.e.*, one can find two different configurations of pseudo-metric zero). For example,

Remark 3.14. *If d is an infinitary subadditive distance, then $\mathfrak{d}_d$ and $\hat{\mathfrak{d}}_d$ are not distances. Indeed, for $x \in A^{\mathbb{N}}$ and $a \neq b \in A$, we have: $\mathfrak{d}_H(ax, bx) = \mathfrak{d}_d(ax, bx) = 0$ (thanks to Remark 2.71). Similarly, we find that $\hat{\mathfrak{d}}_d(ax, bx) = 0$. However, $ax \neq bx$. Hence $\mathfrak{d}_d$ and $\hat{\mathfrak{d}}_d$ are not distances.*

Remark 3.15. *For a distance d, for all $x, y \in A^{\mathbb{N}}$ we have, $\mathfrak{d}_d(x,y) \leq \hat{\mathfrak{d}}_d(x,y)$. Indeed, for $\ell \in \mathbb{N}$, we have: $d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![}) \leq \max_{k\in\mathbb{N}} d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})$. Dividing by $\max d_{|A^\ell}$ and passing to the upper limit, we find, $\mathfrak{d}_d(x,y) \leq \hat{\mathfrak{d}}_d(x,y)$.*

Remark 3.16. *Thanks to Proposition 2.71, one can deduce that for any subadditive distance d, for all $x, y \in A^{\mathbb{N}}$:*

$$\mathfrak{d}_d(x,y) \leq \max d_{|A} \times \mathfrak{d}_H(x,y) \text{ and } \hat{\mathfrak{d}}_d(x,y) \leq \max d_{|A} \times \hat{\mathfrak{d}}_H(x,y).$$

It is well known that, in the case of the Besicovitch space, the upper limit is not always a limit (see Example 3.7). To see this, let us denote, for all $x, y \in A^{\mathbb{N}}$:

$$\underline{\mathfrak{d}_d}(x,y) = \liminf_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\max d_{|A^\ell}} \text{ and } \underline{\hat{\mathfrak{d}}_d}(x,y) = \liminf_{\ell\to\infty} \max_{k\in\mathbb{N}} \frac{d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\max d_{|A^\ell}}.$$

Proposition 3.17. *For a subadditive distance d and for any subshift $\Sigma \subseteq A^{\mathbb{N}}$, we have:*

$$\delta_{d,\Sigma} := \lim_{\ell\to\infty} \frac{\max d_{|\mathscr{L}_\ell(\Sigma)}}{\ell} = \inf_{n>0} \frac{\max d_{|\mathscr{L}_n(\Sigma)}}{n}.$$

Proof. For all $\ell \in \mathbb{N}\setminus\{0\}$, we denote $D_\ell = \max d_{|\mathscr{L}_\ell(\Sigma)}$. Suppose that $\ell = n+m$ for some $n,m\in\mathbb{N}$. Let $u,v\in\mathscr{L}_\ell(\Sigma)$ such that $d(u,v)=\max d_{|\mathscr{L}_\ell(\Sigma)}$. According to the subadditivity of d, we have:

$$\begin{aligned}\max d_{|\mathscr{L}_\ell(\Sigma)} = d(u,v) &\leq d(u_{[\![0,n[\![}, v_{[\![0,n[\![}) + d(u_{[\![n,n+m[\![}, v_{[\![n,n+m[\![})\\ &\leq \max d_{|\mathscr{L}_n(\Sigma)} + \max d_{|\mathscr{L}_m(\Sigma)}.\end{aligned}$$

Applying Fekete's lemma [Fek23] to D_ℓ, we find that: $\lim_{\ell\to\infty}\frac{D_\ell}{\ell} = \inf_{n>0}\frac{D_n}{n}$.

In conclusion: $\lim_{\ell\to\infty}\frac{\max d_{|\mathscr{L}_\ell(\Sigma)}}{\ell} = \inf_{n>0}\frac{\max d_{|\mathscr{L}_n(\Sigma)}}{n}$. □

Remark 3.18. *For a subadditive distance d, it is clear that $\delta_{d,\Sigma}\leq \max_{a,b\in A} d(a,b)<\infty$.*

In constrast to the Besicovitch pseudo-metric, the sliding pseudo-metric associated to a subadditive distance such that $\delta_d := \delta_{d,A^{\mathbb{N}}}\neq 0$, is always a limit:

Theorem 3.19. *For a subshift Σ and a subadditive distance d, if $\delta_d\neq 0$ then:*

$$\hat{\eth}_d(x,y) = \lim_{\ell\to\infty}\max_{k\in\mathbb{N}}\frac{d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![})}{\max d_{|A^\ell}} = \inf_{n\in\mathbb{N}}\max_{k\in\mathbb{N}}\frac{d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![})}{n\times\delta_d}, \forall x,y\in\Sigma.$$

Proof. Let $x,y\in\Sigma$, and let $D_n(x,y)=\max_{k\in\mathbb{N}} d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![})$ for $n\in\mathbb{N}$. Then for $n,m\in\mathbb{N}$ we find the following:

$$\begin{aligned}D_{n+m}(x,y) &= \max_{k\in\mathbb{N}} d(x_{[\![k,k+n+m[\![}, y_{[\![k,k+n+m[\![})\\ &\leq \max_{k\in\mathbb{N}}(d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![}) + d(x_{[\![k+n,k+n+m[\![}, y_{[\![k+n,k+n+m[\![})) \quad \text{(Subadditivity of } d)\\ &\leq D_n(x,y) + \max_{k\in\mathbb{N}} d(x_{[\![k+n,k+n+m[\![}, y_{[\![k+n,k+n+m[\![})\\ &\leq D_n(x,y) + \max_{k\in\mathbb{N}} d(x_{[\![k,k+m[\![}, y_{[\![k,k+m[\![}) = D_n(x,y) + D_m(x,y).\end{aligned}$$

Hence, according to Fekete's Lemma [Fek23] we deduce that :

$$\limsup_{\ell\to\infty}\max_{k\in\mathbb{N}}\frac{d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\ell} = \inf_{n\in\mathbb{N}}\max_{k\in\mathbb{N}}\frac{d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![})}{n}.$$

Finally according to Proposition 3.17 and since $\delta_d\neq 0$, we have:

$$\begin{aligned}\hat{\eth}_d(x,y) &= \limsup_{\ell\to\infty}\frac{\ell}{\max d_{|A^\ell}}\times\frac{\max_{k\in\mathbb{N}} d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\ell}\\ &= \inf_{n\in\mathbb{N}}\max_{k\in\mathbb{N}}\frac{d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![})}{n\times\delta_d}.\end{aligned}$$ □

Proposition 3.20. *Let Σ be a subshift. For a subadditive distance d, for all $x \in \Sigma$:*

$$\hat{\delta}_{d,\Sigma,x} := \lim_{\ell\to\infty} \max_{u\in\mathscr{L}_\ell(\Sigma),k\in\mathbb{N}} \frac{d(u,x_{[\![k,k+\ell[\![})}{\ell} = \inf_{n>0} \max_{u\in\mathscr{L}_n(\Sigma),k\in\mathbb{N}} \frac{d(u,x_{[\![k,k+n[\![})}{n}.$$

Proof. For all $\ell \in \mathbb{N}\setminus\{0\}$, we denote $D_\ell(x) = \max_{w\in\mathscr{L}_\ell(\Sigma),k\in\mathbb{N}} d(w,x_{[\![k,k+\ell[\![})$. Suppose that $\ell = n+m$ for some $n,m\in\mathbb{N}$. Let $u\in\mathscr{L}_\ell(\Sigma)$ and $k\in\mathbb{N}$, such that:

$$d(u,x_{[\![k,k+\ell[\![}) = \max_{w\in\mathscr{L}_\ell(\Sigma),h\in\mathbb{N}} d(w,x_{[\![h,h+\ell[\![}).$$

According to the subadditivity of d, we have:

$$\begin{aligned} D_\ell(x) &= d(u,x_{[\![k,k+\ell[\![}) \\ &\le d(u_{[\![0,n[\![},x_{[\![k,k+n[\![}) + d(u_{[\![n,n+m[\![},x_{[\![k+n,k+n+m[\![}) \\ &\le \max_{w\in\mathscr{L}_n(\Sigma),h\in\mathbb{N}} d(w,x_{[\![h,h+n[\![}) + \max_{w\in\mathscr{L}_m(\Sigma),h\in\mathbb{N}} d(w,x_{[\![h,h+m[\![}) \\ &= D_n(x) + D_m(x). \end{aligned}$$

Applying Fekete's lemma [Fek23] to D_ℓ, we find that: $\lim_{\ell\to\infty}\frac{D_\ell(x)}{\ell} = \inf_{n>0}\frac{D_n(x)}{n}$.
In conclusion:

$$\lim_{\ell\to\infty} \max_{u\in\mathscr{L}_\ell(\Sigma),k\in\mathbb{N}} \frac{d(u,x_{[\![k,k+\ell[\![})}{\ell} = \inf_{n>0} \max_{u\in\mathscr{L}_n(\Sigma),k\in\mathbb{N}} \frac{d(u,x_{[\![k,k+n[\![})}{n}. \quad \square$$

Notations 3.21. *Note that if $\Sigma = A^{\mathbb{N}}$ we shall write $\hat{\delta}_{d,x} := \hat{\delta}_{d,\Sigma,x}$.*

Remark 3.22. *For a distance d such that $\delta_d \notin \{0,\infty\}$ it is clear that:*

$$\mathfrak{d}_d(x,y) = \frac{1}{\delta_d}\times\limsup_{\ell\to\infty}\frac{d(x_{[\![0,\ell[\![},y_{[\![0,\ell[\![})}{\ell} \text{ and } \hat{\mathfrak{d}}_d(x,y) = \frac{1}{\delta_d}\times\limsup_{\ell\to\infty}\max_{k\in\mathbb{N}}\frac{d(x_{[\![k,k+\ell[\![},y_{[\![k,k+\ell[\![})}{\ell}.$$

In this case we can define a normalized distance $\tilde{d}(u,v) = \dfrac{d(u,v)}{\delta_d}$. Hence $\delta_{\tilde{d}} = 1$, $\mathfrak{d}_{\tilde{d}}(x,y) = \dfrac{\mathfrak{d}_d(x,y)}{\delta_d}$ and $\hat{\mathfrak{d}}_{\tilde{d}}(x,y) = \dfrac{\hat{\mathfrak{d}}_d(x,y)}{\delta_d}$. This motivates to restrict our study to normalized distances d, i.e., such that $\delta_d = 1$.

Remark 3.23. *Note that, for a normalized subadditive distance d for all $x,y\in A^{\mathbb{N}}$, $\mathfrak{d}_d(x,y)\le\delta_d$ (resp. $\hat{\mathfrak{d}}_d(x,y)\le\delta_d$) and $\mathfrak{d}_{d,x}(y)\le\delta_{d,x}$ (resp. $\hat{\mathfrak{d}}_{d,x}(x)\le\hat{\delta}_{d,x}$). The proof is a direct conclusion from the definition of the maximum.*

3.2 Shift-invariants pseudo-metrics and periodic configurations

In this section we give some conditions for that pseudo-metrics to be shift-invariants and we give some results about pseudo-metrics on periodic configurations.

3.2.1 Shift-invariants pseudo-metrics

One of the motivations for studying the Besicovitch and Weyl pseudo-metrics is that these are two shift-invariant pseudo-metrics. In this section we give sufficient conditions over a distance so that its associated sliding and centered pseudo-metrics are shift-invariant.

Theorem 3.24. *If d is an infinitary subadditive infra-superadditive distance, then $\mathfrak{d}_d$ and $\hat{\mathfrak{d}}_d$ are shift-invariant.*

Proof. Let us prove that $\hat{\mathfrak{d}}_d$ is shift-invariant. Let $x, y \in A^{\mathbb{N}}$ and $\ell \in \mathbb{N}$. For the first inequality, let h_ℓ be such that:

$$\max_{k\in\mathbb{N}} d(x_{[\![k+1,k+\ell+1[\![}, y_{[\![k+1,k+\ell+1[\![}) = d(x_{[\![h_\ell+1,h_\ell+\ell+1[\![}, y_{[\![h_\ell+1,h_\ell+\ell+1[\![}).$$

According to infra-superadditivity and subadditivity we have:

$$\begin{aligned} d(\sigma(x)_{[\![h_\ell,h_\ell+\ell[\![}, \sigma(y)_{[\![h_\ell,h_\ell+\ell[\![}) &= d(x_{[\![h_\ell+1,h_\ell+\ell+1[\![}, y_{[\![h_\ell+1,h_\ell+\ell+1[\![}) \\ &\leq d(x_{[\![h_\ell,h_\ell+\ell+1[\![}, y_{[\![h_\ell,h_\ell+\ell+1[\![}) \\ &\leq d(x_{[\![h_\ell,h_\ell+\ell[\![}, y_{[\![h_\ell,h_\ell+\ell[\![}) + d(x_{h_\ell+\ell}, y_{h_\ell+\ell}) \\ &\leq \max_{k\in\mathbb{N}} d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![}) + \max_{a,b\in A} d(a,b). \end{aligned}$$

Dividing by $\max_{u,v\in A^\ell} d(u,v)$ we find when ℓ tends to the infinity:

$$\hat{\mathfrak{d}}_d(\sigma(x), \sigma(y)) \leq \hat{\mathfrak{d}}_d(x,y).$$

For the other inequality, for $\ell \in \mathbb{N}$, let h_ℓ be such that:

$$\max_{k\in\mathbb{N}} d(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![}) = d(x_{[\![h_\ell,h_\ell+\ell[\![}, y_{[\![h_\ell,h_\ell+\ell[\![}).$$

According to infra-superadditivity and subadditivity we have:

$$\begin{aligned} d(x_{[\![h_\ell,h_\ell+\ell[\![}, y_{[\![h_\ell,h_\ell+\ell[\![}) &\leq d(x_{[\![h_\ell,h_\ell+\ell+1[\![}, y_{[\![h_\ell,h_\ell+\ell+1[\![}) \\ &\leq d(x_{h_\ell}, y_{h_\ell}) + d(x_{[\![h_\ell+1,h_\ell+\ell+1[\![}, y_{[\![h_\ell+1,h_\ell+\ell+1[\![}) \\ &\leq \max_{a,b\in A} d(a,b) + \max_{k\in\mathbb{N}} d(x_{[\![k+1,k+\ell+1[\![}, y_{[\![k+1,k+\ell+1[\![}) \\ &= \max_{a,b\in A} d(a,b) + \max_{k\in\mathbb{N}} d(\sigma(x)_{[\![k,k+\ell[\![}, \sigma(y)_{[\![k,k+\ell[\![}). \end{aligned}$$

Dividing by $\max_{u,v\in A^\ell} d(u,v)$ we find when ℓ tends to the infinity: $\hat{\mathfrak{d}}_d(x,y) \leq \hat{\mathfrak{d}}_d(\sigma(x),\sigma(y))$. In conclusion, $\hat{\mathfrak{d}}_d(x,y) = \hat{\mathfrak{d}}_d(\sigma(x),\sigma(y))$.

By similar inequalities with $h_\ell = 0$ one can prove that $\mathfrak{d}_d$ is shift-invariant. □

In particular, the Feldman-Katok, Besicovitch and Weyl pseudo-metrics are shift-invariant.

Example 3.25.

1. *Let d be the distance defined as $d(u,v) = |u|$ if $u \neq v$, 0 otherwise for all $(u,v) \in (A\times A)^*$. We can remark that d is infra-superadditive but it is not subadditive (since for $u,w,v \in A^+$ we have $d(uv,wv) = |uv| > d(u,w) + d(v,v) = |u|$). For $A = \mathbf{2}$, for $x = 10^\infty$ and $y = 0^\infty$ we have: $\hat{\mathfrak{d}}_d(x,y) = 1$ and $\hat{\mathfrak{d}}_d(\sigma(x),\sigma(y)) = 0$. Hence $\hat{\mathfrak{d}}_d$ is not shift-invariant.*

2. *Let d be the distance defined as $d(u,v) = 1$ if $u \neq v$, 0 otherwise for all $(u,v) \in (A\times A)^*$. We can remark that d is subadditive and infra-superadditive but not infinitary. For $A = \mathbf{2}$, let $x = 10^\infty$ and $y = 0^\infty$. It is clear that, $\hat{\mathfrak{d}}_d(x,y) = 1$ and $\hat{\mathfrak{d}}_d(\sigma(x),\sigma(y)) = 0$. Hence $\hat{\mathfrak{d}}_d$ is not shift-invariant.*

Proposition 3.26. *If d is a infinitary distance such that there exists $\alpha \geq 0$ such that, for all $a,b \in A$, for all $n \in \mathbb{N}$, all $u,v \in A^n$, $d(au,vb) \leq d(u,v) + \alpha$, then for all $x \in A^{\mathbb{N}}$, $\mathfrak{d}_d(x,\sigma(x)) = \hat{\mathfrak{d}}_d(x,\sigma(x)) = 0$.*

Proof. For $x \in A^{\mathbb{N}}$ we have,

$$\begin{aligned}\hat{\mathfrak{d}}_d(x,\sigma(x)) &= \limsup_{\ell\in\mathbb{N}} \max_{k\in\mathbb{N}} \frac{d(x_{[\![k,k+\ell[\![}, x_{[\![k+1,k+1+\ell[\![})}{\max d_{|A^\ell}} \\ &\leq \limsup_{\ell\in\mathbb{N}} \max_{k\in\mathbb{N}} \frac{d(x_{[\![k+1,k+\ell[\![}, x_{[\![k+1,k+\ell[\![})}{\max d_{|A^\ell}} + \frac{\alpha}{\max d_{|A^\ell}} = 0.\end{aligned}$$

Hence $\hat{\mathfrak{d}}_d(x,\sigma(x)) = \mathfrak{d}_d(x,\sigma(x)) = 0$, for all $x \in A^{\mathbb{N}}$, since $\mathfrak{d}_d(x,\sigma(x)) \leq \hat{\mathfrak{d}}_d(x,\sigma(x))$. □

Notations 3.27. *For a distance d, for all $x \in A^{\mathbb{N}}$, let us denote by $\mathfrak{d}_{d,x}$ and $\hat{\mathfrak{d}}_{d,x}$ the maps defined over $A^{\mathbb{N}}$ as follows: $\mathfrak{d}_{d,x}(y) = \mathfrak{d}_d(x,y)$ and $\hat{\mathfrak{d}}_{d,x}(y) = \hat{\mathfrak{d}}_d(x,y)$, for all $y \in A^{\mathbb{N}}$.*

Corollary 3.28. *If d is an infinitary distance such that there exists α, for all $a,b \in A$, for all $n \in \mathbb{N}$, all $u,v \in A^n$, $d(au,vb) \leq d(u,v) + \alpha$, then for all $x \in A^{\mathbb{N}}$, the maps $\mathfrak{d}_{d,x}$ and $\hat{\mathfrak{d}}_{d,x}$ are shift-invariant.*

Proof. Thanks to the triangular inequality and Proposition 3.26, we have for all $y \in A^{\mathbb{N}}$:

$$\mathfrak{d}_{d,x}(y) = \mathfrak{d}_d(x,y) \leq \mathfrak{d}_d(x,\sigma(y)) + \mathfrak{d}_d(\sigma(y),y) = \mathfrak{d}_d(x,\sigma(y)) = \mathfrak{d}_{d,x}(\sigma(y)),$$

and,

$$\mathfrak{d}_{d,x}(\sigma(y)) = \mathfrak{d}_d(x,\sigma(y)) \leq \mathfrak{d}_d(x,y) + \mathfrak{d}_d(y,\sigma(y)) = \mathfrak{d}_d(x,y) = \mathfrak{d}_{d,x}(y),$$

Hence, for all $y \in A^{\mathbb{N}}$, $\mathfrak{d}_{d,x}(y) = \mathfrak{d}_{d,x}(\sigma(y))$. By the same way we find $\hat{\mathfrak{d}}_{d,x}(y) = \hat{\mathfrak{d}}_{d,x}(\sigma(y))$. □

3.2.2 Pseudo-metrics over periodic configurations

For the Besicovitch and Weyl pseudo-metrics it is well known that each equivalence class contains at most one periodic configuration.

Proposition 3.29 ([CFMM97, Proposition 3]). *If $x \neq y \in A^{\mathbb{N}}$ are periodic configurations, then $0 < \mathfrak{d}_H(x,y) \leq \hat{\mathfrak{d}}_H(x,y)$.*

This remains true if we replace the Hamming distance by any other additive distance thanks to the following theorem:

Proposition 3.30. *If d is an additive distance, then for any p-periodic configurations $x, y \in A^{\mathbb{N}}$ we have:*

$$\mathfrak{d}_d(x,y) = \hat{\mathfrak{d}}_d(x,y) = \frac{d(x_{[\![0,p[\![}, y_{[\![0,p[\![})}{p \times \max d_{|A}}.$$

Proof. Let x, y be two periodic configurations with a period p. Let $\ell > 0$. By the Euclidean division of ℓ by p, we find that $\ell = pq_\ell + r_\ell$ for some $q_\ell \in \mathbb{N}$ and $0 \leq r_\ell < p$. Then:

$$\begin{aligned} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} &= \frac{d(x_{[\![0,pq_\ell+r_\ell[\![}, y_{[\![0,pq_\ell+r_\ell[\![})}{\ell} \\ &= \frac{d(x_{[\![0,pq_\ell[\![}, y_{[\![0,pq_\ell[\![})}{\ell} + \frac{d(x_{[\![pq_\ell,\ell[\![}, y_{[\![pq_\ell,\ell[\![})}{\ell}. \end{aligned}$$

Since: $d(x_{[\![pq_\ell,\ell[\![}, y_{[\![pq_\ell,\ell[\![}) = d(x_{[\![0,r_\ell[\![}, y_{[\![0,r_\ell[\![})$ and $d(x_{[\![0,p[\![}, y_{[\![0,p[\![}) = d(x_{[\![0,r_\ell[\![}, y_{[\![0,r_\ell[\![}) + d(x_{[\![r_\ell,p[\![}, y_{[\![r_\ell,p[\![})$, we have:

$$\frac{d(x_{[\![pq_\ell,\ell[\![}, y_{[\![pq_\ell,\ell[\![})}{\ell} \leq \frac{d(x_{[\![0,p[\![}, y_{[\![0,p[\![})}{\ell} \leq \limsup_{\ell\to\infty} \frac{d(x_{[\![0,p[\![}, y_{[\![0,p[\![})}{\ell} = 0.$$

Therefore: $\mathfrak{d}_d(x,y) = \limsup_{\ell\to\infty} \frac{d(x_{[\![0,pq_\ell[\![}, y_{[\![0,pq_\ell[\![})}{\ell \times \max d_{|A}}$.

On the other hand, we have x and y two p-periodic configurations then : $x_{[\![0,p[\![} = x_{[\![ip,(i+1)p[\![}$ and $y_{[\![0,p[\![} = y_{[\![ip,(i+1)p[\![}$, for all $i \in \mathbb{N}$. Then according to the additivity of d, we find:

$$d(x_{[\![0,pq_\ell[\![}, y_{[\![0,pq_\ell[\![}) = \sum_{i=0}^{q_\ell-1} d(x_{[\![ip,(i+1)p[\![}, y_{[\![ip,(i+1)p[\![}) = q_\ell d(x_{[\![0,p[\![}, y_{[\![0,p[\![}).$$

Hence:

$$\mathfrak{d}_d(x,y) = \limsup_{\ell\to\infty} \frac{q_\ell d(x_{[\![0,p[\![}, y_{[\![0,p[\![})}{\ell \times \max d_{|A}} = \frac{d(x_{[\![0,p[\![}, y_{[\![0,p[\![})}{p \times \max d_{|A}} \times \limsup_{\ell\to\infty} \frac{pq_\ell}{\ell}.$$

Since: $0 \leq r_\ell < p$, we find: $\frac{\ell-p}{\ell} < \frac{\ell-r_\ell}{\ell} = \frac{pq_\ell}{\ell} \leq \frac{\ell}{\ell}$, and then: $\limsup_{\ell\to\infty} \frac{pq_\ell}{\ell} = 1$.

In conclusion, for all p-periodic configurations $x, y \in A^{\mathbb{N}}$ we have:

$$\mathfrak{d}_d(x,y) = \hat{\mathfrak{d}}_d(x,y) = \frac{d(x_{[\![0,p[\![}, y_{[\![0,p[\![})}{p \times \max d_{|A}}.$$

The proof for the case of the sliding pseudo-metric is similar. □

Remark 3.31. *Note that for $k \neq k' \in \mathbb{N}$, if $\sigma^{k'} \sim_{\mathfrak{d}_H} \sigma^k$ (resp. $\sigma^{k'} \sim_{\hat{\mathfrak{d}}_H} \sigma^k$) if and only if $k = k'$. Indeed, suppose that $k \neq k'$. Let $u \in A^*$ such that $|u| \geq \max\{k, k'\}$ and $u_k \neq u_{k'}$. For $x = u^\infty$, it is clear that $\sigma^k(x)_0 = u_k \neq u_{k'} = \sigma^{k'}(x)_0$. Hence, since $\sigma^k(x)$ and $\sigma^{k'}(x)$ are $|u|$-periodic, and thanks to Proposition 3.30 we obtain:*

$$0 < \frac{1}{|u|} \leq \mathfrak{d}_H(\sigma^k(x), \sigma^{k'}(x)) \leq \hat{\mathfrak{d}}_H(\sigma^k(x), \sigma^{k'}(x)).$$

Which contradict the fact that $\sigma^{k'}$ and σ^k are in the same equivalence class.

For a subadditive distance, Proposition 3.30 is not always true. However, in the case of the Feldman-Katok pseudo-metric, for any periodic configuration $x \in A^{\mathbb{N}}$ such that $x \neq \sigma(x)$, $\mathfrak{d}_L(x, \sigma(x)) = 0$ thanks to Proposition 3.26.

Proposition 3.32 ([FK09, Proposition 16])**.** *If $x, y \in A^{\mathbb{N}}$ are Tœplitz configurations, then:*

1. *If $x \neq y$, then $\hat{\mathfrak{d}}_H(x,y) \geq \mathfrak{d}_H(x,y) > 0$. In particular, each equivalence class $x_{\mathfrak{d}_H}$ (resp. $x_{\hat{\mathfrak{d}}_H}$) contains at most one Tœplitz configuration.*
2. *If x, y are regular, then $\hat{\mathfrak{d}}_H(x,y) = \mathfrak{d}_H(x,y)$.*

We have seen that there is at most one periodic configuration in each Besicovitch class. Let us show that there is at most one periodic *orbit* in each Feldman-Katok class.

Remark 3.33. *For every $x, y \in A^{\mathbb{N}}$, if $\mathfrak{d}_L(x,y) = 0$, then for all $p \in \mathbb{N}$ there exist $i, j \in \mathbb{N}$ such that $x_{[\![i,i+p[\![} = y_{[\![j,j+p[\![}$.*

Proof. If $\mathfrak{d}_L(x,y) = 0$ then there exists an increasing sequence $(i_n)_{n\in\mathbb{N}}$ such that:

$$\forall \varepsilon > 0, \exists N > 0, \forall i_n > N, d_L(x_{[\![0,i_n[\![}, y_{[\![0,i_n[\![}) < i_n \cdot \varepsilon.$$

Hence, for $\varepsilon = \dfrac{1}{p+1}$, there exists $N \in \mathbb{N}$ such that for all $i_n > N$,

$$d_L(x_{[\![0,i_n[\![}, y_{[\![0,i_n[\![}) < \frac{i_n}{p+1}.$$

By the characterization of d_L in terms of longest common subword, there exists $u \in A^p$ such that $u \sqsubseteq x_{[\![0,i_n[\![}$ and $u \sqsubseteq y_{[\![0,i_n[\![}$, and thus, there exist $i, j \in [\![0, i_n - p[\![$ such that :

$$u = x_{[\![i,i+p[\![} = y_{[\![j,j+p[\![}. \quad \square$$

Proposition 3.34. *For every p-periodic configurations $x, y \in A^{\mathbb{N}}$, if $\mathfrak{d}_L(x,y) = 0$ (resp. $\hat{\mathfrak{d}}_L(x,y) = 0$) then there exists $k \in [\![0,p[\![$ such that $x = \sigma^k(y)$.*

Proof. Let p be a common period for x and y. According to Remark 3.33, there exist $i, j \in \mathbb{N}$ such that $x_{[\![i,i+p[\![} = y_{[\![j,j+p[\![}$. Since both x and y are p-periodic for the shift, one can write, for every $n \in \mathbb{N}$,

$$x_n = x_{n-i \bmod p+i} = y_{n-i \bmod p+j} = y_{j-i \bmod p+n}.$$

Hence $x = \sigma^{j-i \bmod p}(y)$.

Suppose now that $\hat{\mathfrak{d}}_L(x,y) = 0$, then $\mathfrak{d}_L(x,y) = 0$. Hence $x = \sigma^{j-i \bmod p}(y)$. □

Proposition 3.35. *For all $p \in \mathbb{N} \setminus \{0\}$, all $u, v \in A^p$, we have:*

$$\mathfrak{d}_L(u^\infty, v^\infty) \leq \frac{1}{p} \times \min_{i \in [\![0,p[\![} d_L(\sigma^i(u^\infty)_{[\![0,p[\![}, v).$$

Proof. Let $i \in [\![0,p[\![$ and let $\ell > 0$. By the Euclidean division, we have $\ell - i = pq_\ell + r_\ell$ for some $q_\ell \in \mathbb{N}$ and $r_\ell < p$.

$$\begin{aligned}
\frac{d_L((u^\infty)_{[\![0,\ell[\![}, (v^\infty)_{[\![0,\ell[\![})}{\ell} &\leq \frac{d_L((u^\infty)_{[\![i,\ell[\![}, (v^\infty)_{[\![0,\ell-i[\![}) + i}{\ell} \\
&\leq \frac{d_L(\sigma^i(u^\infty)_{[\![0,\ell-i[\![}, (v^\infty)_{[\![0,\ell-i[\![}) + i}{\ell - i} \\
&\leq \frac{q_\ell \times d_L(\sigma^i(u^\infty)_{[\![0,p[\![}, v) + r_\ell + i}{\ell}
\end{aligned}$$

Hence, by passing to the upper limit and since $\lim_{\ell\to\infty} \frac{q_\ell}{\ell} = \frac{1}{p}$ and $\lim_{\ell\to\infty} \frac{r_\ell+i}{\ell} = 0$, we obtain:

$$\mathfrak{d}_L(u^\infty, v^\infty) \leq \frac{1}{p} \times d_L(\sigma^i(u^\infty)_{[\![0,p[\![}, v).$$

Then we can conclude that,

$$\mathfrak{d}_L(u^\infty, v^\infty) \leq \frac{1}{p} \times \min_{i \in [\![0,p[\![} d_L(\sigma^i(u^\infty)_{[\![0,p[\![}, v).$$

□

We conjecture that the equality is true, but we have not be able to prove it because of the lack of tools to compute general lower bounds for the Feldman-Katok pseudometric.

3.3 Continuity and measurability

This section is devoted to the study of continuity and measurability of the pseudo-metrics and to give some technical results that will be used in Chapter 4.

Remark 3.36. *For an infinitary infra-superadditive subadditive distance d and any $x \in A^{\mathbb{N}}$, the equivalence class $x_{\mathfrak{d}_d}$ (resp. $x_{\hat{\mathfrak{d}}_d}$) intersects any non-empty open set of X_{d_C}.*

More precisely, for $u \in A^*$, we have $y = ux_{[\![|u|,\infty[\![} \in [u]$ and $\mathfrak{d}_d(x,y) = 0$ (resp. $\hat{\mathfrak{d}}_d(x,y) = 0$) and thus $[u] \bigcap x_{\mathfrak{d}_d} \neq \emptyset$ (resp. $[u] \bigcap x_{\hat{\mathfrak{d}}_d} \neq \emptyset$).

Corollary 3.37. *For an infinitary infra-superadditive subadditive distance d, the pseudo-metrics $\mathfrak{d}_d, \hat{\mathfrak{d}}_d : A^{\mathbb{N}} \times A^{\mathbb{N}} \mapsto ([0,1], |.|)$ are not continuous, where $A^{\mathbb{N}} \times A^{\mathbb{N}}$ is equipped with the product topology.*

We can deduce the measurability of pseudo-metrics easily by applying standard results on sequences of measurable maps. The following results also imply measurability, however, they are technical results to be used in Chapter 4.

Notations 3.38. *For a distance d and a nonempty subshift Σ, let us denote $\mathfrak{d}_{d,\Sigma}$ (resp. $\hat{\mathfrak{d}}_{d,\Sigma}$) the restricted pseudo-metric $\mathfrak{d}_d$ (resp. $\hat{\mathfrak{d}}_d$) in $\Sigma \times \Sigma$ and $\mathfrak{d}_{d,\Sigma,x}$ (resp. $\hat{\mathfrak{d}}_{d,\Sigma,x}$) the restricted map $\mathfrak{d}_{d,x}$ (resp. $\hat{\mathfrak{d}}_{d,x}$) in Σ for some $x \in \Sigma$. Let $(A \times A)^*$ be the set of words (u,v) such that $|u| = |v|$.*

Lemma 3.39. *For any map $d : (A \times A)^* \mapsto \mathbb{R}$, for all $\alpha \in \mathbb{R}$ we have:*

$$\left\{ (x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \middle| \limsup_{\ell \to \infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \geq \alpha \right\} = \bigcap_{n \in \mathbb{N}} \bigcup_{j_1 < \ldots < j_n} \bigcap_{h=1}^{n} [V_{h,j_h}],$$

where for all $h \in \mathbb{N} \setminus \{0\}$, $j \in \mathbb{N}$, $V_{h,j} = \left\{ (u,v) \in A^j \times A^j \middle| j(\alpha - \frac{1}{h}) \leq d(u,v) \right\}$.

Proof. If $\alpha \leq \mathfrak{d}_d(x,y)$, then there exists an increasing sequence $(i_p)_{p \in \mathbb{N} \setminus \{0\}}$ such that:

$$\forall \varepsilon > 0, \exists N > 0, \forall p > N, \alpha - \varepsilon \leq \frac{d(x_{[\![0,i_p[\![}, y_{[\![0,i_p[\![})}{i_p}.$$

Hence, for $n \in \mathbb{N} \setminus \{0\}$, $\varepsilon = \dfrac{1}{n}$, there exists $N > 0$ such that for all $p > N$ we have:

$$\alpha - \frac{1}{n} \leq \frac{d(x_{[\![0,i_p[\![}, y_{[\![0,i_p[\![})}{i_p}.$$

In particular, for $j_h = i_{h+N}$, we have:

$$\alpha - \frac{1}{h} \leq \alpha - \frac{1}{n} \leq \frac{d(x_{[\![0,j_h[\![}, y_{[\![0,j_h[\![})}{j_h}, \forall h \in [\![1,n]\!].$$

So that for all $n \in \mathbb{N}$, there exist $j_1 < \cdots < j_n$ such that for all $h \in [\![1,n]\!]$:

$$\alpha - \frac{1}{h} \leq \frac{d(x_{[\![0,j_h[\![}, y_{[\![0,j_h[\![})}{j_h}.$$

Hence, $\mathfrak{d}_d^{-1}([\alpha,\infty[) \subseteq \bigcap_{n \in \mathbb{N}} \bigcup_{j_1 < \ldots < j_n} \bigcap_{h=1}^{n} [V_{h,j_h}]$.

On the other hand, let us suppose that for all $n \in \mathbb{N}$, there exist $j_1 < \cdots < j_n$ such that for all $h \in [\![1,n]\!]$: $\alpha - \frac{1}{h} \leq \dfrac{d(x_{[\![0,j_h[\![}, y_{[\![0,j_h[\![})}{j_h}$. In particular, for all $n \in \mathbb{N}$, there exist $j_1 < \cdots < j_n$ such that: $\alpha - \frac{1}{n} \leq \dfrac{d(x_{[\![0,j_n[\![}, y_{[\![0,j_n[\![})}{j_n}$. Then we can conclude that $\alpha \leq \mathfrak{d}_d(x,y)$ since $j_n \to \infty$ when $n \to \infty$. In conclusion:

$$\mathfrak{d}_d^{-1}([\alpha,\infty[) = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\ldots<j_n} \bigcap_{h=1}^{n} [V_{h,j_h}]. \quad \square$$

Corollary 3.40. *For any map $d : (A\times A)^* \mapsto \mathbb{R}$, for all $\alpha \in \mathbb{R}$, we have:*

$$\left\{(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \middle| \liminf_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \leq \alpha\right\} = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\ldots<j_n} \bigcap_{h=1}^{n} [V_{h,j_h}],$$

where for all $h \in \mathbb{N}\setminus\{0\}$, $j \in \mathbb{N}$, $V_{h,j} = \left\{(u,v) \in A^j \times A^j \middle| d(u,v) \leq j(\frac{1}{h}+\alpha)\right\}$.

Proof. Let $f : (A\times A)^* \mapsto \mathbb{R}$ be such that $f(u,v) = -d(u,v)$. According to Lemma 3.39 for $\beta = -\alpha$, we have:

$$\begin{aligned}
\left\{(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \middle| \mathfrak{d}_f(x,y) \geq \beta\right\} &= \left\{(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \middle| \limsup_{\ell\to\infty} \frac{f(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \geq \beta\right\} \\
&= \left\{(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \middle| \liminf_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \leq \alpha\right\} \\
&= \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\ldots<j_n} \bigcap_{h=1}^{n} [V_{h,j_h}].
\end{aligned}$$

where for all $h \in \mathbb{N}\setminus\{0\}$, $j \in \mathbb{N}$, $V_{h,j} = \left\{(u,v) \in A^j \times A^j \middle| d(u,v) \leq j(\frac{1}{h}+\alpha)\right\}$. $\square$

Lemma 3.41. *Let d be a normalized distance. For all $\alpha \in \mathbb{R}$ we have:*

$$\hat{\mathfrak{d}}_d^{-1}([\alpha,\infty[) = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,i_h}]),$$

where for all $h \in \mathbb{N}\setminus\{0\}$, $j \in \mathbb{N}$, $Y_{h,j} = \left\{(u,v) \in A^j \times A^j \middle| j(\alpha - \dfrac{1}{h}) \leq d(u,v)\right\}$.

Proof. If $\hat{\mathfrak{d}}_d(x,y) \geq \alpha$, then for all $\varepsilon > 0$, there exists $N > 0$ such that for all $n > N$, there exists $k \in \mathbb{N}$ such that: $\alpha - \varepsilon \leq \frac{d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![})}{n}$. Hence, for $n \in \mathbb{N}$ and $\varepsilon = \dfrac{1}{n}$, there exist $j_1 < \cdots < j_n$, such that for all $h \in [\![1,n]\!]$ there exists $k \in \mathbb{N}$ such that:

$$\alpha - \frac{1}{h} \leq \frac{d(x_{[\![k,k+j_h[\![}, y_{[\![k,k+j_h[\![})}{j_h}.$$

Then: $\hat{\mathfrak{d}}_d^{-1}([\alpha,\infty[) \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h}])$.

On the other hand, let us suppose that for all $n \in \mathbb{N}$, there exist $j_1 < \cdots < j_n$ such that for all $h \in [\![1, n]\!]$ there exists $k_h \in \mathbb{N}$ such that: $\alpha - \frac{1}{h} \leq \frac{d(x_{[\![k_h, k_h + j_h[\![}, y_{[\![k_h, k_h + j_h[\![})}{j_h}$. In particular:

$$\alpha - \frac{1}{n} \leq \frac{d(x_{[\![k_n, k_n + j_n[\![}, y_{[\![k_n, k_n + j_n[\![})}{j_n} \leq \max_{k \in \mathbb{N}} \frac{d(x_{[\![k, k + j_n[\![}, y_{[\![k, k + j_n[\![})}{j_n}.$$

Then we can conclude that $\alpha \leq \hat{\mathfrak{d}}_d(x, y)$ since $j_n \to \infty$ when $n \to \infty$. In conclusion:

$$\hat{\mathfrak{d}}_d^{-1}([\alpha, \infty[) = \bigcap_{n \in \mathbb{N}} \bigcup_{j_1 < \cdots < j_n} \bigcap_{h=1}^{n} \bigcup_{k \in \mathbb{N}} \sigma^{-k}([Y_{h, j_h}]). \quad \square$$

Proposition 3.42. *The centered (resp. sliding) pseudo-metric associated to a normalized distance d is a measurable map.*

Proof. The proof is a conclusion from Lemmas 3.39 and 3.41, since $([\alpha, \delta_d])_{\alpha \in \mathbb{R}}$ yield a base for the Borel σ-algebra over the image sets $\mathfrak{d}_d(A^{\mathbb{N}} \times A^{\mathbb{N}}), \hat{\mathfrak{d}}_d(A^{\mathbb{N}} \times A^{\mathbb{N}}) \subseteq [0, \delta_d]$. $\square$

Now we show that, for all $x \in A^{\mathbb{N}}$ and a normalized distance d, $\mathfrak{d}_{d,x}$ and $\hat{\mathfrak{d}}_{d,x}$ are measurable.

Lemma 3.43. *For any map $d : (A \times A)^* \mapsto \mathbb{R}$, for all $x \in A^{\mathbb{N}}$, all $\alpha \in \mathbb{R}$ we have:*

$$\left\{ y \in A^{\mathbb{N}} \middle| \limsup_{\ell \to \infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \geq \alpha \right\} = \bigcap_{k \in \mathbb{N}} \bigcup_{j_1 < \ldots < j_k} \bigcap_{h=1}^{k} [V_{h, j_h, x}],$$

where for all $h \in \mathbb{N} \setminus \{0\}$, $j \in \mathbb{N}$, $V_{h,j,x} = \left\{ u \in A^j \middle| j(\alpha - \frac{1}{h}) \leq d(u, x_{[\![0,j[\![}) \right\}$.

Proof. If $\alpha \leq \mathfrak{d}_{d,x}(y)$ then there exists an increasing sequence $(i_n)_{n \in \mathbb{N} \setminus \{0\}}$ such that for all $\varepsilon > 0$, there exists $N > 0$, for all $i_n > N$: $\alpha - \varepsilon \leq \frac{d(x_{[\![0,i_n[\![}, y_{[\![0,i_n[\![})}{i_n}$. Hence, for $n \in \mathbb{N} \setminus \{0\}$, $\varepsilon = \frac{1}{n}$, there exists $N > 0$ such that for all $i_p > N$ we have:

$$\alpha - \frac{1}{n} \leq \frac{d(x_{[\![0,i_p[\![}, y_{[\![0,i_p[\![})}{i_p}.$$

In particular, for $j_h = i_{h+m}$ where $m = \min\{ p \in \mathbb{N} \middle| i_p > N \}$ and $h \in [\![1, n]\!]$, we have:

$$\alpha - \frac{1}{h} \leq \alpha - \frac{1}{n} \leq \frac{d(x_{[\![0,j_h[\![}, y_{[\![0,j_h[\![})}{j_h}, \forall h \in [\![1, n]\!].$$

So that for all $n \in \mathbb{N}$, there exists $j_1 < \cdots < j_n$ such that:

$$\alpha - \frac{1}{h} \leq \frac{d(x_{[\![0,j_h[\![}, y_{[\![0,j_h[\![})}{j_h}, \forall h \in [\![1, n]\!].$$

Then, $\mathfrak{d}_{d,x}^{-1}([\alpha,\infty[) \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\ldots<j_n} \bigcap_{h=1}^{n} [V_{h,j_h,x}]$.

On the other hand, suppose that for all $n \in \mathbb{N}$, there exist $j_1 < \cdots < j_n$ such that:

$$\alpha - \frac{1}{h} \leq \frac{d(x_{[\![0,j_h[\![}, y_{[\![0,j_h[\![})}{j_h}, \forall h \in [\![1,n]\!].$$

In particular, for all $n \in \mathbb{N}$, there exist $j_1 < \cdots < j_n$ such that:

$$\alpha - \frac{1}{n} \leq \frac{d(x_{[\![0,j_n[\![}, y_{[\![0,j_n[\![})}{j_n}.$$

Hence we can conclude that $\alpha \leq \mathfrak{d}_d(x,y)$ since $j_n \to \infty$ when $n \to \infty$. In conclusion:

$$\mathfrak{d}_{d,x}^{-1}([\alpha,\infty[) = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\ldots<j_n} \bigcap_{h=1}^{n} [V_{h,j_h,x}]. \quad \square$$

Corollary 3.44. *For any map $d : (A\times A)^* \mapsto \mathbb{R}$, for all $x \in A^{\mathbb{N}}$ and all $\alpha \in \mathbb{R}$, we have:*

$$\left\{ y \in A^{\mathbb{N}} \middle| \liminf_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \leq \alpha \right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}],$$

where for all $h \in \mathbb{N}\setminus\{0\}$, $j \in \mathbb{N}$, $V_{h,j,x} = \left\{ u \in A^j \middle| d(u, x_{[\![0,j[\![}) \leq j(\alpha + \frac{1}{h}) \right\}$.

Proof. Let $f_x : (A\times A)^* \mapsto \mathbb{R}$ be such that $f(u,v) = -d(u,v)$. According to Lemma 3.39 for $\beta = -\alpha$, we have:

$$\begin{aligned}
\left\{ y \in A^{\mathbb{N}} \middle| \mathfrak{d}_{f,x}(y) \geq \beta \right\} &= \left\{ y \in A^{\mathbb{N}} \middle| \limsup_{\ell\to\infty} \frac{f(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \geq \beta \right\} \\
&= \left\{ y \in A^{\mathbb{N}} \middle| \liminf_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \leq \alpha \right\} \\
&= \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}],
\end{aligned}$$

where for all $h \in \mathbb{N}\setminus\{0\}$, $j \in \mathbb{N}$, $V_{h,j,x} = \left\{ u \in A^j \middle| d(u, x_{[\![0,j[\![}) \leq j(\alpha + \frac{1}{h}) \right\}$. $\square$

Proposition 3.45. *Let d be a normalized distance. For all $x \in A^{\mathbb{N}}$, we have $\underline{\mathfrak{d}_d}$ and $\underline{\mathfrak{d}_{d,x}}$ are measurable maps, where:*

$$\underline{\mathfrak{d}_d}(x,y) = \liminf_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} \text{ and } \underline{\mathfrak{d}_{d,x}}(y) = \underline{\mathfrak{d}_d}(x,y), \forall x,y \in A^{\mathbb{N}}.$$

Proof. The proof is a direct consequence of Corollary 3.40 and Corollary 3.44, since for all $x \in A^{\mathbb{N}}$, $([0,\beta])_{\beta\in[0,\infty[}$ and $([0,\beta])_{\beta\in[0,\infty[}$ yield a base for the Borel σ-algebra over the image sets $\underline{\mathfrak{d}_d}(A^{\mathbb{N}}) \subseteq [0,\infty[$ and $\underline{\mathfrak{d}_{d,x}}(A^{\mathbb{N}}) \subseteq [0,\infty[$. $\square$

Lemma 3.46. *Let d be a normalized distance. For all $x \in A^{\mathbb{N}}$, all $\alpha \in \mathbb{R}$:*

$$\hat{\mathfrak{d}}_{d,x}^{-1}([\alpha,\infty[) = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}]),$$

where for all $h, j \in \mathbb{N}\setminus\{0\}$, $Y_{h,j,x,k} = \left\{ u \in A^{\mathbb{N}} \middle| j(\alpha - \frac{1}{h}) \leq d(u, x_{[\![k,k+j[\![}) \right\}$.

Proof. If $\alpha \leq \hat{\mathfrak{d}}_{d,x}(y)$, then for all $\varepsilon > 0$, there exists $N > 0$, for all $n > N$, there exists $k \in \mathbb{N}$ such that:

$$\alpha - \varepsilon \leq \frac{d(x_{[\![k,k+n[\![}, y_{[\![k,k+n[\![})}{n} = \max_{m\in\mathbb{N}} \frac{d(x_{[\![m,m+n[\![}, y_{[\![m,m+n[\![})}{n}.$$

Hence, for $n \in \mathbb{N}$ and $\varepsilon = \dfrac{1}{n}$, there exist $j_1 < \cdots < j_n$, for all $h \in [\![1,n]\!]$ there exists $k \in \mathbb{N}$ such that:

$$\alpha - \frac{1}{h} \leq \frac{d(x_{[\![k,k+j_h[\![}, y_{[\![k,k+j_h[\![})}{j_h}.$$

Then $\hat{\mathfrak{d}}_{d,x}^{-1}([\alpha,\infty[) \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}])$.

On the other hand, suppose that for all $n \in \mathbb{N}$, there exists $j_1 < \cdots < j_n$ such that for all $h \in [\![1,n]\!]$ there exists $k \in \mathbb{N}$ such that:

$$\alpha - \frac{1}{h} \leq \frac{d(x_{[\![k,k+j_h[\![}, y_{[\![k,k+j_h[\![})}{j_h} \leq \max_{m\in\mathbb{N}} \frac{d(x_{[\![m,m+j_h[\![}, y_{[\![m,m+j_h[\![})}{j_h}.$$

In particular, for all $n \in \mathbb{N}$ there exists $j_1 < \cdots < j_n$, such that:

$$\alpha - \frac{1}{n} \leq \max_{m\in\mathbb{N}} \frac{d(x_{[\![m,m+j_n[\![}, y_{[\![m,m+j_n[\![})}{j_n}.$$

Hence, $\alpha \leq \hat{\mathfrak{d}}_d(x,y)$, since $j_n \to \infty$ when $n \to \infty$. Then we can conclude that:

$$\bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}]) \subseteq \hat{\mathfrak{d}}_{d,x}^{-1}([\alpha,\infty[).$$

In conclusion:

$$\hat{\mathfrak{d}}_{d,x}^{-1}([\alpha,\infty[) = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}]). \quad \square$$

Proposition 3.47. *Let d be a normalized distance. For all $x \in A^{\mathbb{N}}$, we have $\mathfrak{d}_{d,x}$ and $\hat{\mathfrak{d}}_{d,x}$ are measurable maps.*

Proof. The proof is a direct conclusion from Lemma 3.43 and Lemma 3.46, since for all $x \in A^{\mathbb{N}}$, $([\alpha,\delta_{d,x}])_{\alpha\in\mathbb{R}}$ yield a base for the Borel σ-algebra over the image sets $\mathfrak{d}_{d,x}(A^{\mathbb{N}}), \hat{\mathfrak{d}}_{d,x}(A^{\mathbb{N}}) \subseteq [0,\delta_{d,x}]$. $\square$

4 Generic properties of centered and sliding pseudo-metrics

This chapter is devoted to the study of generic behavior (topological and measurable) of the centered and the sliding pseudo-metrics associated to some distances over the set of finite words. Note that in this chapter we deal with the case of a normalized distance d which verifies subadditivity and infra-superadditivity. This class of distances contains at least the Hamming and the Levenshtein distances.

4.1 Distances over specific subshifts

Before defining the symbolic weak specification property (for more details one can see [KOR16]) and weakly sublinearly mixing subshift, let us mention that definitions and results of this section are mainly given to prove the principal results in this chapter.

Definition 4.1. *A subshift Σ has the (symbolic) weak specification property, if there exists $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f(n) = o_{n\to\infty}(n)$ and for all $u, v \in \mathscr{L}(\Sigma)$ there exists $w \in \mathscr{L}_{f(|v|)}(\Sigma)$ such that $uwv \in \mathscr{L}(\Sigma)$.*

Definition 4.2. *A subshift Σ is weakly sublinearly mixing, if there exists $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $f(n) = o_{n\to\infty}(n)$ and for every $u, u', v, v' \in \mathscr{L}(\Sigma)$ such that $|u| = |u'|$ and $|v| = |v'|$, there exist $\ell \leq f(|v|)$, $w, w' \in \mathscr{L}_\ell(\Sigma)$, such that $uwv, u'w'v' \in \mathscr{L}(\Sigma)$.*

Remark 4.3. *Note that if a subshift has the weak specification property then it is weakly sublinearly mixing subshift. But we have no example of weakly sublinearly mixing subshift without weak specification property. The reason to distinguish them is to prove each result with assumptions as weak as possible.*

Example 4.4.

1. *The golden mean subshift (from Example 2.37) has the weak specification property. Indeed, for all $u, v \in \mathscr{L}(\Sigma)$ we have $u0v \in \mathscr{L}(\Sigma)$. Hence, one can deduce the property by taking $f(n) = 1$ for all $n \in \mathbb{N}$.*

2. *The even subshift (from Example 2.37) has the weak specification property. Indeed, it is enough to take $f(n) = 2$ for all $n \in \mathbb{N}$, and one can deduce that for $u, v \in A^*$, there exist $w \in \mathscr{L}_2(\Sigma)$ such that $uwv \in \mathscr{L}(\Sigma)$.*

3. *It is clear that the subshift of finite type with forbidden set* $F = \{01, 10\}$ *is not weakly sublinearly mixing subshift and it has not the weak specification property, since* $00, 11 \in \mathscr{L}(\Sigma)$ *but for all* $w \in A^*$, $00w11 \notin \mathscr{L}(\Sigma)$.

Before studying the behavior of distances over subshifts defining above, let us give some notations that we will use in this chapter.

Notations 4.5. *Let* Σ *be a subshift,* $x \in \Sigma$ *and* d *be a subadditive distance.*

1. $\delta_{d,\Sigma} = \lim_{\ell\to\infty} \dfrac{\max d_{\mathscr{L}_\ell(\Sigma)}}{\ell}$.
2. $\delta_{d,\Sigma,x} = \limsup_{\ell\to\infty} \max_{u\in\mathscr{L}_\ell(\Sigma)} \frac{d(u,x_{[\![0,\ell[\![})}{\ell}$.
3. $\hat{\delta}_{d,\Sigma,x} = \lim_{\ell\to\infty} \max_{u\in\mathscr{L}_\ell(\Sigma),k\in\mathbb{N}} \frac{d(u,x_{[\![k,k+\ell[\![})}{\ell}$.

Lemma 4.6. *Let* d *be a subadditive infra-superadditive distance and let* Σ *be a subshift with the weak specification property. For any* $x \in \Sigma$, *for all* $\alpha \in [0, \delta_{d,\Sigma,x}[$, *all* $\ell \in \mathbb{N}$ *there exists* $\ell' \in \mathbb{N}$ *such that for all* $u \in \mathscr{L}_\ell(\Sigma)$, *there exists* $w \in \mathscr{L}_{\ell'}(\Sigma)$, *such that* $uw \in \mathscr{L}(\Sigma)$ *and*

$$d(x_{[\![0,\ell+\ell'[\![}, uw) \geq \alpha(\ell + \ell').$$

Proof. Let $\alpha \in [0, \delta_{d,\Sigma,x}[$, $\ell \in \mathbb{N}$ and $u \in \mathscr{L}_\ell(\Sigma)$. By definition of $\delta_{d,\Sigma,x}$, there exists an increasing sequence $(\ell_n)_{n\in\mathbb{N}}$ such that:

$$\forall \varepsilon_0 > 0, \exists N_{\varepsilon_0} > 0, \forall \ell_n > N_{\varepsilon_0}, \frac{\max_{w\in\mathscr{L}_{\ell_n}(\Sigma)} d(x_{[\![0,\ell_n[\![}, w)}{\ell_n} \geq \delta_{d,\Sigma,x} - \varepsilon_0.$$

Since $f(n) = o_{n\to\infty}(n)$, for $\varepsilon \in]0, \delta_{d,\Sigma,x} - \alpha[$, there exists $\ell_m > N_\varepsilon$ such that:

$$\frac{\ell + f(\ell_m)}{\ell + f(\ell_m) + \ell_m} \leq \delta_{d,\Sigma,x} - \alpha - \varepsilon.$$

Let $p = \ell + f(\ell_m) + \ell_m$ and let $z \in \mathscr{L}_p(\Sigma)$ such that:

$$\max_{w\in\mathscr{L}_p(\Sigma)} d(x_{[\![0,p[\![}, w) = d(x_{[\![0,p[\![}, z).$$

By subadditivity of d we have:

$$\begin{aligned}
\delta_{d,\Sigma,x} - \varepsilon &\leq \frac{d(x_{[\![0,p[\![}, z)}{p} \\
&\leq \frac{d(x_{[\![0,\ell+f(\ell_m)[\![}, z_{[\![0,\ell+f(\ell_m)[\![})}{p} + \frac{d(x_{[\![\ell+f(\ell_m),p[\![}, z_{[\![\ell+f(\ell_m),p[\![})}{p} \\
&\leq \frac{\ell + f(\ell_m)}{p} + \frac{d(x_{[\![\ell+f(\ell_m),p[\![}, z_{[\![\ell+f(\ell_m),p[\![})}{p} \\
&\leq \delta_{d,\Sigma,x} - \alpha - \varepsilon + \frac{d(x_{[\![\ell+f(\ell_m),p[\![}, z_{[\![\ell+f(\ell_m),p[\![})}{p}
\end{aligned}$$

Since Σ has the weak specification property and $z' = z_{[\![\ell+f(\ell_m),p[\![}$, $u \in \mathscr{L}(\Sigma)$, there exists $s \in \mathscr{L}_{f(m)}(\Sigma)$ such that, $usz' \in \mathscr{L}(\Sigma)$. On the other hand, thanks to the infra-superadditivity of d we have:

$$\alpha \leq \frac{d(x_{[\![\ell+f(\ell_m),p[\![}, z')}{p} \leq \frac{d(x_{[\![0,p[\![}, usz')}{p}. \quad \square$$

Lemma 4.7. *Let d be a subadditive distance and let Σ be a subshift with the weak specification property. For any $x \in \Sigma$, for all $\alpha > 0$, all $\ell \in \mathbb{N}$ there exists $\ell' \in \mathbb{N}$ such that for all $u \in \mathscr{L}_\ell(\Sigma)$, there exists $w \in \mathscr{L}_{\ell'}(\Sigma)$, such that $uw \in \mathscr{L}(\Sigma)$ and*

$$d(x_{[\![0,\ell+\ell'[\![}, uw) \leq \alpha(\ell+\ell').$$

Proof. Let $\alpha > 0$, $\ell \in \mathbb{N}$ and $u \in \mathscr{L}_\ell(\Sigma)$. According to the weak specification property, there exists $\ell' \in \mathbb{N}$, such that: $\dfrac{f(\ell')}{\ell+\ell'+f(\ell')} \leq \dfrac{\alpha}{2}$ and $\dfrac{d(x_{[\![0,\ell[\![}, u)}{\ell+\ell'+f(\ell')} \leq \dfrac{\alpha}{2}$.

Let $v = x_{[\![\ell+f(\ell'),\ell+\ell'+f(\ell')[\![}$. According to the weak specification property, there exists $s \in \mathscr{L}_{f(\ell')}(\Sigma)$ such that $usv \in \mathscr{L}(\Sigma)$. On the other hand, according to the subadditivity of d we find the following:

$$\begin{aligned}
\frac{d(x_{[\![0,\ell+\ell'+f(\ell')[\![}, usv)}{\ell+\ell'+f(\ell')} &\leq \frac{d(x_{[\![0,\ell[\![}, u)}{\ell+\ell'+f(\ell')} + \frac{d(x_{[\![\ell,\ell+f(\ell')[\![}, s)}{\ell+\ell'+f(\ell')} + \frac{d(v,v)}{\ell+\ell'+f(\ell')} \\
&= \frac{d(x_{[\![0,\ell[\![}, u)}{\ell+\ell'+f(\ell')} + \frac{d(x_{[\![\ell,\ell+f(\ell')[\![}, s)}{\ell+\ell'+f(\ell')} \\
&\leq \frac{\alpha}{2} + \frac{f(\ell')}{\ell+\ell'+f(\ell')} \leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha. \quad \square
\end{aligned}$$

Lemma 4.8. *Let d be a infra-superadditive distance. If Σ is weakly sublinearly mixing then for all $\alpha \in]0, \delta_{d,\Sigma}[$, for all $u, v \in \mathscr{L}_\ell(\Sigma)$, there exist $\ell' \in \mathbb{N} \setminus \{0\}$, $u', v' \in A^{\ell'}$ such that $uu', vv' \in \mathscr{L}_{\ell+\ell'}(\Sigma)$, and:*

$$\alpha(\ell+\ell') < d(uu', vv').$$

Proof. Let $\alpha \in]0, \delta_\Sigma^\infty[$ and $u, v \in \mathscr{L}_\ell(\Sigma)$ for some $\ell \in \mathbb{N} \setminus \{0\}$. Since $f(n) = o_{n\to\infty}(n)$, we have:

$$\lim_{n\to\infty} \frac{\max_{z,z' \in \mathscr{L}_n(\Sigma)} d(z,z')}{\ell+n+f(n)} = \delta_{d,\Sigma}.$$

Then for $\varepsilon = \delta_{d,\Sigma} - \alpha$, there exists $N > 0$ such that for any $n > N$, we have :

$$\frac{\max_{z,z' \in \mathscr{L}_n(\Sigma)} d(z,z')}{\ell+n+f(n)} > \delta_{d,\Sigma} - \varepsilon = \alpha.$$

Hence, for $n > N$ and for $s, s' \in \mathscr{L}_n(\Sigma)$ such that $\max_{z,z' \in \mathscr{L}_n(\Sigma)} d(z,z') = d(s,s')$ we have:

$$\frac{d(s,s')}{n+f(n)+\ell} > \alpha.$$

Then according to the weak sublinear mixing property of Σ and since $u, v, s, s' \in \mathscr{L}(\Sigma)$, there exist w, w' such that $|w| = |w'| \leq f(n)$ and $uws, vw's' \in \mathscr{L}(\Sigma)$. On the other hand, according to the infra-superadditivity of d and since $|w| \leq f(n)$, we deduce that:

$$\alpha < \frac{d(s,s')}{n+f(n)+\ell} \leq \frac{d(uws,vw's')}{n+f(n)+\ell} \leq \frac{d(uws,vw's')}{n+|w|+\ell}.$$

So, we take $u' = ws$ and $v' = w's'$ to find our result. □

Lemma 4.9. *Let d be a subadditive distance. If Σ is weakly sublinearly mixing then for all $\alpha > 0$, for all $u, v \in \mathscr{L}_\ell(\Sigma)$, there exist $\ell' \in \mathbb{N} \setminus \{0\}$, $u', v' \in A^{\ell'}$ which verifies $uu', vv' \in \mathscr{L}_{\ell+\ell'}(\Sigma)$ and: $d(uu', vv') \leq \alpha(\ell + \ell')$.*

Proof. Let $\alpha > 0$ and $u, v \in \mathscr{L}_\ell(\Sigma)$ for some $\ell \in \mathbb{N} \setminus \{0\}$. Let $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) = o_{n\to\infty}(n)$, since Σ is weakly sublinearly mixing. Then, for $\varepsilon = \frac{\alpha}{2}$, there exists $N > 0$, for all $n > N$ we have: $\max_{z,z' \in \mathscr{L}_{(f(n))}(\Sigma)} d(z,z') \leq \frac{n\alpha}{2}$.

Let $m > N$ such that $\frac{d(u,v)}{\ell+m} < \frac{\alpha}{2}$, and let $s \in \mathscr{L}_{(f(m))}(\Sigma)$. Since $u, v, s \in \mathscr{L}(\Sigma)$, then according to the weak sublinear mixing property of Σ, there exist w, w' such that $uws, vw's \in \mathscr{L}(\Sigma)$ and $|w| = |w'| \leq f(m)$. Hence:

$$d(w,w') \leq \max_{z,z' \in \mathscr{L}_{f(m)}(\Sigma)} d(z,z').$$

On the other hand, according to subadditivity property we deduce the following:

$$\begin{aligned} \frac{d(uws,vw's)}{\ell+m+|w|} &\leq \frac{d(u,v)}{\ell+m+|w|} + \frac{d(w,w')}{\ell+m+|w|} + \frac{d(s,s)}{\ell+m+|w|} \\ &\leq \frac{d(u,v)}{\ell+m} + \frac{\max_{z,z' \in \mathscr{L}_{(f(m))}(\Sigma)} d(z,z')}{m} + 0 \\ &\leq \frac{\alpha}{2} + \frac{\alpha}{2} = \alpha. \end{aligned}$$

Finally, we take $u' = ws$ and $v' = w's$ to find our result. □

4.2 Generic behavior of the centered pseudo-metric

In this section, we prove that the set of all couples of configurations (x, y) of a weakly specified subshift Σ such that $\mathfrak{d}_d(x,y) = \delta_{d,\Sigma}$ and $\underline{\mathfrak{d}_d}(x,y) = 0$ is a dense G_δ in $\Sigma \times \Sigma$ equipped with the product topology. In addition, by a similar method we prove that, for any configuration x, the set of all configurations $y \in \Sigma$ such that $\mathfrak{d}_d(x,y) = \delta_{d,\Sigma,x}$ and $\underline{\mathfrak{d}_d}(x,y) = 0$ is also a dense G_δ in (Σ, d_C). We finish this section by proving that the limit exists almost everywhere for any weakly-mixing measure.

4.2.1 Topologically generic behavior

Before showing the main results of this section, let us recall that any closed subset Σ of a compact space equipped with the metric d_C is also a compact space. Moreover, $\Sigma \times \Sigma$ equipped with the product topology is a compact space. And thus, it is a Baire space and we can safely talk about dense G_δ sets. This is the case when Σ is a subshift, since it is closed.

Lemma 4.10. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ is weakly sublinearly mixing, then the following set is a dense G_δ in $\Sigma \times \Sigma$:*

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma}\right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}], \tag{4.1}$$

where $V_{h,j} = \left\{(u,v) \in \mathscr{L}_j(\Sigma) \times \mathscr{L}_j(\Sigma) \middle| d(u,v) \geq j(\delta_{d,\Sigma} - \frac{1}{h})\right\}$ for all $h, j \in \mathbb{N} \setminus \{0\}$.

Proof. Firstly, we can conclude the equality (4.1) from Remark 3.23 and from Lemma 3.39 by setting $\alpha = \delta_{d,\Sigma}$.

Now we aim to prove by induction on $k \in \mathbb{N}$ that $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is dense in $\Sigma \times \Sigma$. For $k = 0$, we have: $\Sigma \times \Sigma \subseteq \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$. And thus, $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is dense in $\Sigma \times \Sigma$. We suppose now that $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is dense. Let W be a non-empty open set. According to the induction hypothesis $W \bigcap (\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}])$ is a non-empty open set, and thus, there exist $j_1 < \ldots < j_k$, $\ell \geq j_k$ and $u, v \in \mathscr{L}_\ell(\Sigma)$ such that:

$$[u] \times [v] \subseteq \bigcap_{h=1}^{k} [V_{h,j_h}] \bigcap W.$$

Hence according to Lemma 4.8, for $\alpha = \delta_{d,\Sigma} - \dfrac{1}{k+1}$, there exists $\ell' \in \mathbb{N}$, $u', v' \in A^{\ell'}$ such that $uu', vv' \in \mathscr{L}_{\ell+\ell'}(\Sigma)$ and $\dfrac{d(uu', vv')}{\ell + \ell'} \geq \delta_{d,\Sigma} - \frac{1}{k+1}$.

Hence for $j_{k+1} = \ell + \ell'$ we have $uu', vv' \in V_{k+1,j_{k+1}}$.

Moreover, since $[uu'] \times [vv'] \subseteq [u] \times [v]$:

$$W \bigcap \left(\bigcup_{j_1<\ldots<j_{k+1}} \bigcap_{h=1}^{k+1} [V_{h,j_h}] \right) \neq \emptyset.$$

In conclusion, for all $k \in \mathbb{N}$ we have $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is a dense open set.

On the other hand, it is clear that $\bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is a G_δ since for all $h, j \in \mathbb{N}$ we have $[V_{h,j}]$ is an open set. Then, according to Baire's theorem we deduce that the following set is a dense G_δ in $\Sigma \times \Sigma$,

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma}\right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]. \quad \square$$

Lemma 4.11. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ is weakly sublinearly mixing, then the following set is a dense G_δ in $\Sigma \times \Sigma$:*

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \underline{\mathfrak{d}_d}(x,y) = 0\right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,i_h}], \tag{4.2}$$

where $V_{h,j} = \left\{(u,v) \in \mathscr{L}_j(\Sigma) \times \mathscr{L}_j(\Sigma) \middle| d(u,v) \leq \frac{j}{h}\right\}$ *for all* $h, j \in \mathbb{N} \setminus \{0\}$.

Proof. First of all, we can conclude (4.2) from Lemma 3.40 by setting $\alpha = 0$. Now we aim to prove by induction on $k \in \mathbb{N}$ that $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is dense. For $k = 0$, it is clear that $\Sigma \times \Sigma \subseteq \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$. We suppose now that $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is dense. Let U be a non-empty open set. According to the induction hypothesis and since $\bigcap_{h=1}^{k} [V_{h,j_h}] \bigcap U$ is an open set, there exist $j_1 < \ldots < j_k$, $\ell \geq j_k$ and $u, v \in \mathscr{L}_\ell(\Sigma)$ such that $[u] \times [v] \subseteq U \bigcap \left(\bigcap_{h=1}^{k} [V_{h,j_h}]\right)$. Hence according to Lemma 4.9, for $\alpha = \dfrac{1}{k+1}$, there exists $\ell' \in \mathbb{N}$, $u', v' \in A^{\ell'}$ such that $uu', vv' \in \mathscr{L}_{\ell+\ell'}(\Sigma)$ and

$$\frac{d(uu', vv')}{\ell + \ell'} \leq \frac{1}{k+1}.$$

Hence for $j_{k+1} = \ell + \ell'$ we have $(uu', vv') \in V_{k+1, j_{k+1}}$.

And thus, since $[uu'] \times [vv'] \subseteq [u] \times [v]$:

$$\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k+1} [V_{h,j_h}] \bigcap U \neq \emptyset.$$

We can conclude now that, for all $k \in \mathbb{N}$ we have $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is dense.

Furthermore, $\bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]$ is a G_δ since for all $h, j \in \mathbb{N}$, $[V_{h,j}]$ is an open set. Finally, according to Baire's theorem we can conclude that the following set is a dense G_δ in $\Sigma \times \Sigma$:

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \underline{\mathfrak{d}_d}(x,y) = 0\right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h}]. \quad \square$$

Theorem 4.12. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ is weakly sublinearly mixing, then the following set is a dense G_δ in $\Sigma \times \Sigma$:*

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma} \text{ and } \underline{\mathfrak{d}_d}(x,y) = 0\right\}.$$

Proof. Since the Cantor space is complete, according to Lemma 4.10, Lemma 4.11 and Baire's theorem we obtain that the following set is a dense G_δ:

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma}\right\} \bigcap \left\{(x,y) \in \Sigma \times \Sigma \middle| \underline{\mathfrak{d}_d}(x,y) = 0\right\},$$

□

Lemma 4.13. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ has the weak specification property, then for all $x \in \Sigma$, the following set is a dense G_δ in Σ:*

$$\left\{ y \in \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma,x} \right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}], \tag{4.3}$$

where for all $h \in \mathbb{N}\setminus\{0\}$, $j \in \mathbb{N}$, $V_{h,j,x} = \left\{ u \in \mathscr{L}_j(\Sigma) \middle| j(\delta_{d,\Sigma,x} - \frac{1}{h}) \leq d(u, x_{[\![0,j[\![}) \right\}$.

Proof. First of all, we can conclude (4.3) from Remark 3.23 and from Lemma 3.43 by setting $\alpha = \delta_{d,\Sigma,x}$. Moreover, it is clear that $\bigcap_{k\in\mathbb{N}}\bigcup_{j_1<\ldots<j_k}\bigcap_{h=1}^{k}[V_{h,j_h,x}]$ is a G_δ since for all $h, j \in \mathbb{N}\setminus\{0\}$, $[V_{h,j,x}]$ is an open.

Now we aim to prove by induction on $k \in \mathbb{N}$ that $\bigcup_{j_1<\ldots<j_k}\bigcap_{h=1}^{k}[V'_{h,j_h}]$ is dense in Σ. For $k = 0$, it is clear that $\Sigma \subseteq \bigcup_{j_1<\ldots<j_k}\bigcap_{h=1}^{k}[V_{h,j_h,x}]$.

We suppose now that for $k > 0$, $\bigcup_{j_1<\ldots<j_k}\bigcap_{h=1}^{k}[V_{h,j_h,x}]$ is dense. Let W be a non-empty open set. According to the induction hypothesis there exist $j_1 < \cdots < j_k$, $\bigcap_{h=1}^{k}[V_{h,j_h,x}]\bigcap W$ is a non-empty open set. Hence, there exists $\ell \geq j_k$ and $u \in \mathscr{L}_\ell(\Sigma)$ such that $[u] \subseteq \bigcap_{h=1}^{k}[V_{h,j_h,x}]\bigcap W$. Moreover, according to Lemma 4.6, for $\alpha = \delta_{d,\Sigma,x} - \frac{1}{k+1}$, there exists $\ell' \in \mathbb{N}\setminus\{0\}$, $u' \in A^{\ell'}$ such that $uu' \in \mathscr{L}_{\ell+\ell'}(\Sigma)$ and

$$\frac{d(x_{[\![0,\ell+\ell'[\![}, uu')}{\ell+\ell'} \geq \delta_{d,\Sigma,x} - \frac{1}{k+1}.$$

Hence for $j_{k+1} = \ell + \ell'$ we have $uu' \in V_{k+1,j_{k+1},x}$. And thus :

$$\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k+1} [V_{h,j_h,x}]\bigcap W \neq \emptyset.$$

Hence, for all $k \in \mathbb{N}$ we have $\bigcup_{j_1<\ldots<j_k}\bigcap_{h=1}^{k}[V_{h,j_h,x}]$ is a dense open set.

Finally, according to Baire's theorem we can conclude that $\left\{ y \in \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma,x} \right\}$ is a dense G_δ in Σ. □

Lemma 4.14. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ has the weak specification property, then for all $x \in \Sigma$, the following set is a dense G_δ in Σ:*

$$\left\{ y \in \Sigma \middle| \underline{\mathfrak{d}_d}(x,y) = 0 \right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}], \tag{4.4}$$

where for all $h \in \mathbb{N}\setminus\{0\}$, $j \in \mathbb{N}$, $V_{h,j,x} = \left\{ u \in \mathscr{L}_j(\Sigma) \middle| d(u, x_{[\![0,j[\![}) \leq \frac{j}{h}) \right\}$.

Proof. First of all, we can conclude (4.4) from Lemma 3.44 by setting $\beta = 0$. Moreover, it is clear that $\bigcap_{k\in\mathbb{N}}\bigcup_{j_1<\ldots<j_k}\bigcap_{h=1}^{k}[V_{h,j_h,x}]$ is a G_δ since for all $k \in \mathbb{N}$, $[V_{k,i_h,x}]$ is an open

set. Now we aim to prove by induction on $k \in \mathbb{N}$ that $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,i_h,x}]$ is dense in Σ. For $k=0$, it is clear that $\Sigma \subseteq \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}]$. We suppose now that for $k>0$, $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}]$ is dense in Σ. Let U be a non-empty open set. According to the induction hypothesis

$$\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}] \bigcap U \neq \emptyset.$$

Then, there exist $j_1 < \ldots < j_k$, $\ell \geq j_k$ and $u \in \mathscr{L}_\ell(\Sigma)$ such that $[u] \subseteq \bigcap_{h=1}^{k} [V_{h,j_h,x}] \bigcap U$.

Hence according to Lemma 4.7, for $\alpha = \frac{1}{k+1}$, there exists $\ell' \in \mathbb{N}$, $u' \in A^{\ell'}$ such that $uu' \in \mathscr{L}_{\ell+\ell'}(\Sigma)$ and $\dfrac{d(x_{[\![0,\ell+\ell'[\![}, uu')}{\ell+\ell'} \leq \dfrac{1}{k+1}$.

Hence for $j_{k+1} = \ell + \ell'$ we have $uu' \in \bigcap_{h=1}^{k+1} [V_{h,j_h,x}]$. And thus:

$$\bigcup_{j_1<\ldots<j_{k+1}} \bigcap_{h=1}^{k+1} [V_{h,j_h,x}] \bigcap U \neq \emptyset.$$

We can conclude now that, for all $k \in \mathbb{N}$ we have $\bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}]$ is a dense open set, and then according to Baire's theorem the following set is a dense G_δ:

$$\left\{ y \in \Sigma \middle| \underline{\mathfrak{d}_d}(x,y) = 0 \right\} = \bigcap_{k\in\mathbb{N}} \bigcup_{j_1<\ldots<j_k} \bigcap_{h=1}^{k} [V_{h,j_h,x}]. \quad \square$$

Theorem 4.15. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ has the weak specification property, then for all $x \in \Sigma$, the following set is a dense G_δ in Σ:*

$$\left\{ y \in \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma,x} \text{ and } \underline{\mathfrak{d}_d}(x,y) = 0 \right\}.$$

Proof. Since Σ equipped with the Cantor topology is a complete space, then according to Lemma 4.13, Lemma 4.14 and Baire's theorem, the following set is a dense G_δ:

$$\left\{ y \in \Sigma \middle| \mathfrak{d}_d(x,y) = \delta_{d,\Sigma,x} \right\} \bigcap \left\{ y \in \Sigma \middle| \underline{\mathfrak{d}_d}(x,y) = 0 \right\}. \quad \square$$

4.2.2 Measurably generic behavior

We have already proved that the centered pseudo-metric is not always a limit (see Exemple 3.7). However, in this section we show that the limit exists almost everywhere for any weakly mixing measure. In addition, we show that this limit does not depend on the choice of the configurations.

Proposition 4.16. *Let d be a normalized subadditive distance, then for every shift-invariant measure μ we have for $\mu \otimes \mu$-almost any $(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$:*

$$\mathfrak{d}_d(x,y) = \lim_{\ell\to\infty} \frac{d(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} = \inf_{n>0} \frac{d(x_{[\![0,n[\![}, y_{[\![0,n[\![})}{n}.$$

Proof. Let $D_n(x,y) = d(x_{[\![0,n[\![}, y_{[\![0,n[\![})$, for all $x, y \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$. Since d is subadditive, for all $i \in [\![0,n[\![$ and for all $x, y \in A^{\mathbb{N}}$:

$$D_n(x,y) = d(x_{[\![0,n[\![}, y_{[\![0,n[\![}) \leq d(x_{[\![0,i[\![}, y_{[\![0,i[\![}) + d(x_{[\![i,n[\![}, y_{[\![i,n[\![}) = D_i(x,y) + D_{n-i}(\sigma^i(x), \sigma^i(y)).$$

Then, according to Kingman's theorem [Ste89], for $\mu\otimes\mu$-almost any $(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$ we have:

$$\mathfrak{d}_d(x,y) = \lim_{\ell\to\infty} \frac{D_\ell(x,y)}{\ell} = \inf_{n>0} \frac{D_n(x,y)}{n} = \inf_{n>0} \frac{d(x_{[\![0,n[\![}, y_{[\![0,n[\![})}{n}. \qquad \square$$

Theorem 4.17. *If d is a normalized subadditive infra-superadditive distance, then for any weakly mixing measure μ there exists a constant $c_\mu \geq 0$ such that for $\mu\otimes\mu$-almost any $(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$ we have, $\mathfrak{d}_d(x,y) = \underline{\mathfrak{d}_d}(x,y) = c_\mu$.*

Proof. If μ is weakly mixing, then $\mu\otimes\mu$ is ergodic. By Theorem 3.24, $\mathfrak{d}_d$ is shift-invariant. Hence there exists $\alpha \geq 0$ such that $\mathfrak{d}_d(x,y) = \alpha$ almost everywhere since $\mathfrak{d}_d$ is measurable according to Proposition 3.42. Finally, thanks to Proposition 4.16, we deduce that $\mathfrak{d}_d(x,y) = \underline{\mathfrak{d}_d}(x,y) = \alpha$. $\square$

Proposition 4.18. *For an alphabet A and a weakly mixing measure μ we have for $\mu\otimes\mu$-almost any $(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$,*

$$\mathfrak{d}_H(x,y) = \sum_{a\neq b\in A} \mu([a])\mu([b]).$$

Proof. The system $((A^{\mathbb{N}})^2, \mu^{\otimes 2}, (\sigma,\sigma))$ is ergodic, since μ is weakly mixing. Hence according to the Birkhoff theorem, for $\mu\otimes\mu$-almost every $(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}}$:

$$\begin{aligned}
\lim_{\ell\to\infty} \frac{d_H(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![})}{\ell} &= \lim_{\ell\to\infty} \frac{1}{\ell} \sum_{k=0}^{\ell-1} \mathbb{1}_{\{y_0\neq x_0\}}(\sigma^k(x), \sigma^k(y)) \\
&= \mu\otimes\mu\left(\left\{(x,y) \in A^{\mathbb{N}} \times A^{\mathbb{N}} \,\middle|\, x_0 \neq y_0\right\}\right) \\
&= \mu\otimes\mu\left(\bigsqcup_{a\neq b\in A} [a]\times[b]\right) = \sum_{a\neq b\in A} \mu([a])\mu([b]). \qquad \square
\end{aligned}$$

Theorem 4.19. *If d is a normalized infinitary distance such that there exists $\alpha \geq 0$, for all $a,b \in A$, and all $u,v \in (A\times A)^*$, $d(au,vb) \leq d(u,v) + \alpha$, then for any ergodic measure μ, for all $x \in A^{\mathbb{N}}$ there exists $c_{\mu,x} \geq 0$ such that for μ-almost any $y \in A^{\mathbb{N}}$ we have, $\mathfrak{d}_d(x,y) = c_{\mu,x}$.*

Proof. According to Proposition 3.26, for all $x \in A^{\mathbb{N}}$, $\mathfrak{d}_{d,x}$ is shift-invariant and thus constant μ-almost everywhere since μ is ergodic and $\mathfrak{d}_{d,x}$ is measurable according to Proposition 3.47. $\square$

4.3 Generic behavior of the sliding pseudo-metric

4.3.1 Topologically generic behavior

We have already proved in Theorem 3.17 that the sliding pseudo-metric associated to a subadditive distance is in fact a limit. In this section, in a weakly subblinearly mixing subshift we show that the set of configurations with pseudo-metric equal to $\delta_{d,\Sigma}$ is a dense G_δ in $\Sigma \times \Sigma$.

Theorem 4.20. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ is weakly sublinearly mixing then the following set is a dense G_δ in $\Sigma \times \Sigma$:*

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \hat{\eth}_d(x,y) = \delta_{d,\Sigma}\right\} = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h}]), \tag{4.5}$$

where $Y_{h,j} = \left\{(u,v) \in \mathscr{L}_j(\Sigma) \middle| d(u,v) \geq j(\delta_{d,\Sigma} - \frac{1}{h})\right\}$.

Proof. First of all, we can conclude the equality (4.5) from Remark 3.23 and Lemma 3.41 by setting $\alpha = \delta_{d,\Sigma}$.

It is clear that $\bigcap_{n\in\mathbb{N}} \bigcup_{i_1<\cdots<i_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,i_h}])$ is a G_δ thanks to continuity of σ and openess of $[Y_{h,j}]$ for all h, j. So it only remains for us to prove that it is dense.

On the other hand, according to Remark 3.23 and Remark 3.15 we have:

$$\left\{(x,y) \in \Sigma \times \Sigma \middle| \eth_d(x,y) = \delta_{d,\Sigma}\right\} \subseteq \left\{(x,y) \in \Sigma \times \Sigma \middle| \hat{\eth}_d(x,y) = \delta_{d,\Sigma}\right\}.$$

Finally, thanks to Theorem 4.12 we can conclude that $\left\{(x,y) \in \Sigma \times \Sigma \middle| \hat{\eth}_d(x,y) = \delta_{d,\Sigma}\right\}$ is a dense G_δ in $\Sigma \times \Sigma$. □

Theorem 4.21. *Let d be a normalized subadditive infra-superadditive distance. If a subshift Σ has the weak specification property then for all $x \in \Sigma$, the following set is a dense G_δ in Σ:*

$$\left\{y \in \Sigma \middle| \hat{\eth}_d(x,y) = \hat{\delta}_{d,\Sigma,x}\right\} = \bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}]), \tag{4.6}$$

where for all $h, j \in \mathbb{N}\setminus\{0\}$, $Y_{h,j,x,k} = \left\{u \in \mathscr{L}_j(\Sigma) \middle| j(\alpha - \frac{1}{h}) \leq d(u, x_{[\![k,k+j[\![})\right\}$.

Proof. First of all, we can conclude the equality (4.6) from Remark 3.23 and Lemma 3.46 by setting $\alpha = \hat{\delta}_{d,\Sigma,x}$.

Moreover, $\bigcap_{n\in\mathbb{N}} \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}])$ is a G_δ since $\sigma^{-k}([Y_{h,j_h,x,k}])$ is an open set thanks to continuity of σ and openess of $[Y_{h,j_h,x,k}]$.

Now we aim to prove by induction on $n \in \mathbb{N}$ that $\bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}])$ is dense. For $n = 0$, it is clear that: $\Sigma \subseteq \bigcup_{j_1<\cdots<j_n} \bigcap_{h=1}^{n} \bigcup_{k\in\mathbb{N}} \sigma^{-k}([Y_{h,j_h,x,k}])$.

Suppose now that for $n>0$, $\bigcup_{j_1<\cdots<j_n}\bigcap_{h=1}^{n}\bigcup_{k\in\mathbb{N}}\sigma^{-k}([Y_{h,j_h,x,k}])$ is dense. Let U be a non-empty open set. According to the induction hypothesis:

$$U\bigcap\left(\bigcup_{j_1<\cdots<j_n}\bigcap_{h=1}^{n}\bigcup_{k\in\mathbb{N}}\sigma^{-k}([Y_{h,j_h,x,k}])\right)\neq\emptyset.$$

Then, there exist $j_1<\cdots<j_n$, such that: $U\bigcap\left(\bigcap_{h=1}^{n}\bigcup_{k\in\mathbb{N}}\sigma^{-k}([Y_{h,j_h,x,k}])\right)\neq\emptyset$.

Hence, there exists $y\in U$ such that for all $h\in[\![1,n]\!]$, there exists $k_h\in\mathbb{N}$ such that $y_{[\![k_h,k_h+j_h[\![}\in Y_{h,j_h,x,k_h}$. And since $U\bigcap\left(\bigcap_{h=1}^{n}\bigcup_{k\in\mathbb{N}}\sigma^{-k}([Y_{h,j_h,x,k}])\right)$ is an open set, for $m=\max\left\{k_h\in\mathbb{N}\middle|\, h\in[\![1,n]\!]\right\}$ there exist $\ell\geq j_n+m$ such that for $u=y_{[\![0,\ell[\![}$:

$$[u]\subseteq U\bigcap\left(\bigcap_{h=1}^{n}\bigcup_{k\in\mathbb{N}}\sigma^{-k}([Y_{h,j_h,x,k}])\right).$$

According to Lemma 4.6, for $\alpha=\hat{\delta}_{d,\Sigma,x}-\frac{1}{n+1}$, there exists $\ell'\in\mathbb{N}$, $u'\in A^{\ell'}$ such that $uu'\in\mathscr{L}_{\ell+\ell'}(\Sigma)$ and:

$$\hat{\delta}_{d,\Sigma,x}-\frac{1}{n+1}\leq\frac{d(uu',x_{[\![0,\ell+\ell'[\![})}{\ell+\ell'}.$$

Therefore, for $j_{n+1}=\ell+\ell'$ we have $uu'\in Y_{n+1,j_{n+1},x,0}$. Hence, since $[uu']\subseteq[u]$:

$$[uu']\subseteq U\bigcap\left(\bigcap_{h=1}^{n+1}\bigcup_{k\in\mathbb{N}}\sigma^{-k}([Y_{h,j_h,x,k}])\right).$$

As a consequence:

$$U\bigcap\left(\bigcup_{j_1<\cdots<j_{n+1}}\bigcap_{h=1}^{n+1}\bigcup_{k\in\mathbb{N}}\sigma^{-k}([Y_{h,j_h,x,k}])\right)\neq\emptyset.$$

In conclusion, for all $n\in\mathbb{N}$ we have $\bigcup_{j_1<\ldots<j_n}\bigcap_{h=1}^{n}\bigcup_{k\in\mathbb{N}}[Y_{h,j_h,x,k}]$ is a dense open set, and then according to Baire's theorem, $\left\{y\in\Sigma\middle|\,\hat{\mathfrak{d}}_d(x,y)=\hat{\delta}_{d,\Sigma,x}\right\}$ is a dense G_δ in Σ. □

4.3.2 Measurably generic behavior

We already proved that the sliding pseudo-metric is always a limit for a subadditive distance. In this section, we show that for a weakly mixing measure, the sliding pseudo-metric is a constant μ-almost everywhere. Moreover, we prove that this constant is exactly $\delta_{d,\Sigma}$, where Σ is the support of μ. We finish this section by proving that, for a fixed configuration $x\in A^{\mathbb{N}}$, $\hat{\mathfrak{d}}_{d,x}$ is constant almost everywhere, for any ergodic measure.

Proposition 4.22. *If d is a normalized subadditive infra-superadditive distance, then for every weakly mixing measure μ there exists a constant $c_\mu\geq 0$ such that for $\mu\otimes\mu$-almost any $(x,y)\in A^{\mathbb{N}}\times A^{\mathbb{N}}$ we have, $\hat{\mathfrak{d}}_d(x,y)=c_\mu$.*

Proof. If μ is weakly mixing, then $\mu\otimes\mu$ is ergodic. By Theorem 3.24, $\hat{\mathfrak{d}}_d$ is shift-invariant. Moreover, $\hat{\mathfrak{d}}_d$ is measurable thanks to Proposition 3.42, hence there exists $\alpha\geq 0$ such that $\hat{\mathfrak{d}}_d(x,y)=\alpha$ almost everywhere. □

Remark 4.23. *By a similar argument to Remark 3.23, for any measure μ with support Σ, and any normalized subadditive distance d, we have $\mathfrak{d}_d(x,y)\leq\delta_{d,\Sigma}$ (resp. $\hat{\mathfrak{d}}_d(x,y)\leq\delta_{d,\Sigma}$) for $\mu\otimes\mu$-almost any $(x,y)\in A^{\mathbb{N}}\times A^{\mathbb{N}}$. Indeed, for $\mu\otimes\mu$-almost any $(x,y)\in A^{\mathbb{N}}\times A^{\mathbb{N}}$:*

$$d(x_{[\![k,k+\ell[\![},y_{[\![k,k+\ell[\![})\leq\max d_{|\mathscr{L}_\ell(\Sigma)},\forall\ell,k\in\mathbb{N}.$$

Theorem 4.24. *If d is a normalized subadditive distance, then for every weakly mixing measure μ with support Σ we have $\hat{\mathfrak{d}}_d(x,y)=\delta_{d,\Sigma}$, for $\mu\otimes\mu$-almost any $(x,y)\in A^{\mathbb{N}}\times A^{\mathbb{N}}$.*

Proof. For $\ell\in\mathbb{N}$ let us denote by M_ℓ and W_ℓ the sets:

$$M_\ell=\left\{(u,v)\in\mathscr{L}_\ell(\Sigma)\times\mathscr{L}_\ell(\Sigma)\middle|\, d(u,v)=\max d_{|\mathscr{L}_\ell(\Sigma)}\right\}\text{ and }W_\ell=\bigcup_{k\in\mathbb{N}}\sigma^{-k}\left(\bigcup_{(u,v)\in M_\ell}[(u,v)]\right).$$

According to Remark 4.23, one can deduce that:

$$\bigcap_{\ell\in\mathbb{N}}W_\ell\subseteq\left\{(x,y)\in A^{\mathbb{N}}\times A^{\mathbb{N}}\middle|\,\hat{\mathfrak{d}}_d(x,y)=\delta_{d,\Sigma}\right\}.$$

On the other hand, by definition of the support we have:

$$\mu\otimes\mu\left(\bigcup_{(u,v)\in M_\ell}[(u,v)]\right)=\mu\otimes\mu([M_\ell])>0.$$

And thus, $\mu\otimes\mu(W_\ell)>0$.

Moreover, since μ is weakly mixing, $\mu\otimes\mu$ is ergodic. Then for all $\ell\in\mathbb{N}$, $\mu\otimes\mu(W_\ell)=1$ since W_ℓ is shift-invariant and $\mu\otimes\mu(W_\ell)\neq 0$. Hence

$$\mu\otimes\mu\left(\bigcap_{\ell\in\mathbb{N}}W_\ell\right)=1=\mu\otimes\mu\left(\left\{(x,y)A^{\mathbb{N}}\times A^{\mathbb{N}}\middle|\,\hat{\mathfrak{d}}_d(x,y)=\delta_{d,\Sigma}\right\}\right).$$

In conclusion, for $\mu\otimes\mu$-almost any $(x,y)\in A^{\mathbb{N}}\times A^{\mathbb{N}}$ we have $\hat{\mathfrak{d}}_d(x,y)=\delta_{d,\Sigma}$. □

Theorem 4.25. *If d is a normalized infinitary distance such that there exists $\alpha\geq 0$ such that for all $a,b\in A$, all $(u,v)\in(A\times A)^*$, $d(au,vb)\leq d(u,v)+\alpha$, then for any ergodic measure μ, for all $x\in A^{\mathbb{N}}$ there exists $c_{\mu,x}\geq 0$ such that for μ-almost any $y\in A^{\mathbb{N}}$ we have, $\hat{\mathfrak{d}}_d(x,y)=c_{\mu,x}$.*

Proof. According to Proposition 3.26, for all $x\in A^{\mathbb{N}}$, $\hat{\mathfrak{d}}_{d,x}$ is shift-invariant. $\hat{\mathfrak{d}}_{d,x}$ is measurable thanks to Proposition 3.47. Since μ is ergodic we obtain that μ is constant almost everywhere. □

4.4 Topological properties of the quotient spaces

We finish this chapter by giving some topological properties of spaces defined via the sliding and centered pseudo-metrics. Following the idea behind the Besicovitch and the Weyl space, it is natural to quotient the space of infinite words by the equivalence of zero pseudo-metrics; we obtain metric spaces defined as folllows:

Definitions 4.26.

1. *The **centered space** associated to a distance d over an alphabet A is the metric space $X_{\mathfrak{d}_d} = (A^{\mathbb{N}}_{/\sim_{\mathfrak{d}_d}}, \mathfrak{d}_d)$, where for all $x, y \in A^{\mathbb{N}}$, $x \sim_{\mathfrak{d}_d} y \iff \mathfrak{d}_d(x,y) = 0$.*
2. *The **sliding space** associated to a distance d over an alphabet A is the metric space $X_{\hat{\mathfrak{d}}_d} = (A^{\mathbb{N}}_{/\sim_{\hat{\mathfrak{d}}_d}}, \hat{\mathfrak{d}}_d)$, where for all $x, y \in A^{\mathbb{N}}$, $x \sim_{\hat{\mathfrak{d}}_d} y \iff \hat{\mathfrak{d}}_d(x,y) = 0$.*

Example 4.27.

- *The Besicovitch (resp. the Weyl) space is the centered (resp. sliding) space associated to the Hamming distance.*
- *The centered space associated to the Levenshtein distance is called the Feldman-Katok space.*

Notations 4.28. *Let $\mathfrak{d}$ be any pseudo-metric over $A^{\mathbb{N}}$. We denote by $x_{\mathfrak{d}}$ the equivalence class of $x \in A^{\mathbb{N}}$ in this quotient space $X_{\mathfrak{d}}$.*

Any map $F : A^{\mathbb{N}} \mapsto A^{\mathbb{N}}$ such that $\mathfrak{d}_d(x,y) = 0$ implies $\mathfrak{d}_d(F(x), F(y)) = 0$ for all $x, y \in A^{\mathbb{N}}$, induces a well-defined map $F_{\mathfrak{d}_d} : X_{\mathfrak{d}_d} \to X_{\mathfrak{d}_d}$ over the quotient space.

According to [BFK97], the Besicovitch and Weyl spaces share many topological properties: both spaces are path-connected and infinite-dimensional, but they are neither separable nor locally compact. One of the main differences though is that the Weyl space is not complete, according to [DI88, Proposition 2] contrariwise the Besicovitch space which is complete [BFK97, Proposition 2].

It follows that the topological properties of the centered (resp. sliding) space with respect to an additive distance since this space is equivalent to the Besicovitch (resp. Weyl) space thanks to Proposition 2.72:

Corollary 4.29. *For an additive distance d, the spaces $X_{\mathfrak{d}_d}$ and $X_{\hat{\mathfrak{d}}_d}$ are path-connected, and infinite-dimensional, but they are neither separable nor locally compact. Furthermore, $X_{\mathfrak{d}_d}$ is complete but $X_{\hat{\mathfrak{d}}_d}$ is not complete.*

As we mentioned above (see Remark 3.22), the following results also hold for any distance d such that $\delta_d \notin \{0, \infty\}$. Indeed, almost nothing changes in the proofs of these results since it remains for us just to divide the pseudo-metric by a positive constant.

Proposition 4.30. *If d is a normalized subadditive distance, then $X_{\mathfrak{d}_d}$ and $X_{\hat{\mathfrak{d}}_d}$ are pathwise connected, infinite-dimensional and path-connected.*

Proof. It is clear that for all $x, y \in A^{\mathbb{N}}$, $\mathfrak{d}_d(x,y) \leq \max d_{|A} \times \mathfrak{d}_H(x,y)$ and $\hat{\mathfrak{d}}_d(x,y) \leq \max d_{|A} \times \hat{\mathfrak{d}}_H(x,y)$ since d is subadditive distance. Hence, the identity maps yield continuous surjections $\mathrm{id}_1 : X_{\mathfrak{d}_H} \mapsto X_{\mathfrak{d}_d}$ and $\mathrm{id}_2 : X_{\hat{\mathfrak{d}}_H} \mapsto X_{\hat{\mathfrak{d}}_d}$. Thus these properties are inherited from $X_{\mathfrak{d}_H}$ and $X_{\hat{\mathfrak{d}}_H}$. □

On the other hand, according to [BCF05, Proposition 9], the set of all Tœplitz cofigurations form a dense subset of $X_{\mathfrak{d}_H}$. Hence we can deduce the same results by replacing the Hamming distance by any normalized subadditive distance.

Proposition 4.31. *If d is a normalized subadditive distance, then the classes of Tœplitz configurations form a dense subset of $X_{\mathfrak{d}_d}$.*

Proof. Since d is subadditive, for all $x, y \in A^{\mathbb{N}}$, $\mathfrak{d}_d(x,y) \leq \max d_{|A} \times \mathfrak{d}_H(x,y)$ and $\hat{\mathfrak{d}}_d(x,y) \leq \max d_{|A} \times \hat{\mathfrak{d}}_H(x,y)$. Hence, this property is inherited from $X_{\mathfrak{d}_H}$. □

Proposition 4.32. *If d is a normalized subadditive distance, then $X_{\mathfrak{d}_d}$ is complete.*

Proof. The proof from [BFK97, Proposition 2] generalizes if we replace the Hamming distance by a normalized subadditive distance. Note that the proof of [BFK97, Proposition 2] was already adapted from [Mar39].
Let $(x^{(n)}_{\mathfrak{d}_d})_{n\in\mathbb{N}}$ be a Cauchy sequence of $X_{\mathfrak{d}_d}$. Then:

$$\forall \varepsilon > 0, \exists N > 0, \forall n, m > N, \mathfrak{d}_d(x^{(n)}, x^{(m)}) < \varepsilon.$$

And thus, there exists a subsequence $(x^{(n_j)}_{\mathfrak{d}_d})$ such that:

$$\mathfrak{d}_d(x^{(n_{j+1})}, x^{(n_j)}) < 2^{-j-1}, \forall j \in \mathbb{N}.$$

Let (ℓ_j) be a sequence of positive integers such that for all $j \in \mathbb{N}$ and all $\ell \geq \ell_j$ we have $\ell_{j+1} \geq 2\ell_j$ and $d(x^{(n_{j+1})}_{[\![0,\ell[\![}, x^{(n_j)}_{[\![0,\ell[\![}) < \ell \cdot 2^{-j-1}$. Hence, for all $k > j$ and $\ell \geq \ell_k$ we have:

$$d(x^{(n_k)}_{[\![0,\ell[\![}, x^{(n_j)}_{[\![0,\ell[\![}) < \ell \cdot (2^{-j-1} + \cdots + 2^{-k}) < \ell \cdot 2^{-j}.$$

Let $x \in A^{\mathbb{N}}$ such that $x_{[\![\ell_j,\ell_{j+1}[\![} = x^{(n_j)}_{[\![\ell_j,\ell_{j+1}[\![}$ for all $j \in \mathbb{N}$ and $x_{[\![0,\ell_0[\![}$ is arbitrary. Since d is subadditive, if $k > j$ and $\ell_k \leq \ell < \ell_{k+1}$ we have:

$$\begin{aligned}
d(x_{[\![0,\ell[\![}, x^{(n_j)}_{[\![0,\ell[\![}) &\leq d(x_{[\![0,\ell_j[\![}, x^{(n_j)}_{[\![0,\ell_{\ell_j}[\![}) + \sum_{i=j}^{k-1} d(x_{[\![\ell_i,\ell_{i+1}[\![}, x^{(n_j)}_{[\![\ell_i,\ell_{i+1}[\![}) + d(x_{[\![\ell_k,\ell[\![}, x^{(n_j)}_{[\![\ell_k,\ell[\![}) \\
&= d(x_{[\![0,\ell_j[\![}, x^{(n_j)}_{[\![0,\ell_j[\![}) + \sum_{i=j}^{k-1} d(x^{(n_i)}_{[\![\ell_i,\ell_{i+1}[\![}, x^{(n_j)}_{[\![\ell_i,\ell_{i+1}[\![}) + d(x^{(n_k)}_{[\![\ell_k,\ell[\![}, x^{(n_j)}_{[\![\ell_k,\ell[\![}) \\
&\leq \ell_j + 3\ell \times 2^{-j}.
\end{aligned}$$

Hence, $\mathfrak{d}_d(x, x^{(n_j)}) < 3 \times 2^{-j}$ since $\frac{\ell_j}{\ell}$ tends to 0 when ℓ tends to infinity. Therefore, $(x_{\mathfrak{d}_d}^{(n_j)})$ converges to $x_{\mathfrak{d}_d}$. And thus $(x_{\mathfrak{d}_d}^{(n)})$ converges to $x_{\mathfrak{d}_d}$ also, since $(x_{\mathfrak{d}_d}^{(n)})$ is a Cauchy sequence. In conclusion $X_{\mathfrak{d}_d}$ is complete. □

Other topological properties of $X_{\mathfrak{d}_L}$ have been studied by Alonso Beaumont Llona during his internship with Pierre Guillon and me at I2M. Indeed, he proved Theorem 4.17 and he showed that if there exists a weakly mixing measure μ such that $c_\mu > 0$ then $X_{\mathfrak{d}_L}$ is neither separable nor locally compact. He proved these results with a different method from [BFK97] (but it is adapted from [CS75]). In [BFK97], authors used Sturmian sequences since the Besicovitch pseudo-metric is easy to compute between two Sturmian sequences but this is not the case of $\mathfrak{d}_L$.

5 Dill maps in topological spaces defined via edit distances

This chapter is devoted to the study of dill maps over the sliding and the centered spaces associated to the Hamming and the Levenshtein distances. We start by characterizing the dill maps that are well-defined on these spaces, then we study some dynamical properties of dill maps above these spaces.

5.1 Dill maps over Besicovitch and Weyl space

First of all, let us recall some known results on cellular automata over the Besicovitch and the Weyl spaces which are the main subject of [CFMM97], [BFK97] and [BCF05].

5.1.1 Cellular automata over Besicovitch and Weyl spaces

It was proved in [CFMM97] that any CA induces a well-defined map over Besicovitch and Weyl spaces. We give a proof of general case in Proposition 5.12.

Proposition 5.1 ([CFMM97, Proposition 9])**.** *Any cellular automaton F with diameter θ and local rule f is θ-lipschitz with respect to $\mathfrak{d}_H$ and $\hat{\mathfrak{d}}_H$.*

Remark 5.2. *We shall say that a well-defined map $F_{\mathfrak{d}_d}$ (resp. $F_{\hat{\mathfrak{d}}_d}$) is a dill map (in particular cellular automaton) if there exists a dill map G such that $G \in F_{\mathfrak{d}_d}$, (resp. $G \in F_{\hat{\mathfrak{d}}_d}$), i.e., for all $x \in A^{\mathbb{N}}$, $\mathfrak{d}_d(G(x), F(x)) = 0$ (resp. $\hat{\mathfrak{d}}_d(G(x), F(x)) = 0$).*

[MS09, Theorem 13] goes further, by establishing a characterization à la Curtis-Hedlund-Lyndon of cellular automata in the Besicovitch and Weyl spaces (*i.e.,* necessary and sufficient conditions for a map $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) to be a cellular automaton) by three conditions: shift invariance, a condition in terms of uniform continuity and a condition in terms of periodic configurations.

In [BFK97], they study some dynamical properties of cellular automata over the Besicovitch and Weyl spaces; and the relation between the surjectivity of a cellular automaton and the surjectivity of the induced map over the quotient spaces. Let us recall some of these results.

Proposition 5.3 ([BFK97, Proposition 6])**.** *Let F be a cellular automaton. (X_{d_C}, F) is surjective if and only if $(X_{\mathfrak{d}_H}, F_{\mathfrak{d}_H})$ is surjective if and only if $(X_{\hat{\mathfrak{d}}_H}, F_{\hat{\mathfrak{d}}_H})$ is surjective.*

Let us recall some known results of dynamical properties of cellular automata over Besicovitch and Weyl spaces. We start by equicontinuity. In [BFK97] they proved that equicontinuity over Cantor space implies equicontinuity over Besicovitch and Weyl spaces, and thus, one can deduce that the sensitivity of a cellular automaton $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) implies that the cellular automaton F is sensitive with respect to d_C.

Theorem 5.4. *Let F be a cellular automaton.*

1. *If F has an equicontinuous point with respect to d_C then $F_{\mathfrak{d}_H}$ and $F_{\hat{\mathfrak{d}}_H}$ have an equicontinuous point.*
2. *If (X_{d_C}, F) is equicontinuous, then $(X_{\mathfrak{d}_H}, F_{\mathfrak{d}_H})$ is equicontinuous.*
3. *If $(X_{\mathfrak{d}_H}, F_{\mathfrak{d}_H})$ is sensitive, then (X_{d_C}, F) is sensitive.*

Moreover, since cellular automata are shift-invariant, one can deduce the following:

Proposition 5.5 ([BFK97, Proposition 14])**.** *Let F be a cellular automaton and $m \in \mathbb{N}$. Then we have the following:*

1. *If $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) is sensitive, then $\sigma^m_{\mathfrak{d}_H} \circ F_{\mathfrak{d}_H}$ (resp. $\sigma^m_{\hat{\mathfrak{d}}_H} \circ F_{\hat{\mathfrak{d}}_H}$) is also sensitive.*
2. *If x is an equicontinuity point for $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$), then it is also an equicontinuity point for $\sigma^m_{\mathfrak{d}_H} \circ F_{\mathfrak{d}_H}$ (resp. $\sigma^m_{\hat{\mathfrak{d}}_H} \circ F_{\hat{\mathfrak{d}}_H}$).*
3. *If $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) is equicontinuous, then $\sigma^m_{\mathfrak{d}_H} \circ F_{\mathfrak{d}_H}$ (resp. $\sigma^m_{\hat{\mathfrak{d}}_H} \circ F_{\hat{\mathfrak{d}}_H}$) is equicontinuous.*

One of the main results of [BFK97], is that there is no expansive cellular automata over Besicovitch space [BFK97, Proposition 13]. Contrariwise in the Cantor space, where a simple cellular automata as the shift map is expansive. But the question for the Weyl space is still an open question.

Proposition 5.6 ([BFK97, Proposition 13])**.** *No CA $(X_{\mathfrak{d}_H}, F_{\mathfrak{d}_H})$ is positively expansive.*

Furthermore, it is well known that there is no transitive cellular automata over Besicovitch and Weyl spaces. There are different proofs of this result, one can see [BCF05, Theorem 4], [BS07, Theorem 4] and [ST12, Theorem 9].

Theorem 5.7 ([BCF05, Theorem 14])**.** *No transitive cellular automaton over $X_{\mathfrak{d}_H}$.*

5.1.2 Lipschitz property

We have already shown in Section 5.1.1 that every cellular automata induces a (well-defined) Lipschitz map on the Besicovitch and the Weyl spaces. In contrast, some dill maps are not well-defined. Thus, in this section, we characterise dill maps which induce a well-defined function on these spaces.

Example 5.8. *The Fibonacci substitution from Example 2.52 is not well-defined over the Besicovitch space. Even worse, for every $x \in \{a,b\}^{\mathbb{N}}$ such that $x_{\mathfrak{d}_H} \neq (b^\infty)_{\mathfrak{d}_H}$, altering simply the first letter will induce a shift in the substituted word. Indeed, if $x_0 = b$, then:*

$$\mathfrak{d}_H(\overline{\tau}(S_0^a(x)), \overline{\tau}(x)) = \mathfrak{d}_H(a\overline{\tau}(x), \overline{\tau}(x)).$$

Symmetrically, if $x_0 = a$, then: $\mathfrak{d}_H(\overline{\tau}(S_0^b(x)), \overline{\tau}(x)) = \mathfrak{d}_H(\sigma(\overline{\tau}(x)), \overline{\tau}(x))$.
In both cases, the pseudometric is at least half the frequence $\mathfrak{d}_H(b^\infty, x)$ of a in x. For example, $\mathfrak{d}_H(a^\infty, ba^\infty) = 0$ but $\mathfrak{d}_H(\overline{\tau}(a^\infty), \overline{\tau}(ba^\infty)) = \mathfrak{d}_H((ab)^\infty, (ba)^\infty) = 1$. On the other hand, for all $x \in \{a,b\}^{\mathbb{N}}$, $\mathfrak{d}_H(\overline{\tau}(x), \overline{\tau}(b^\infty)) \leq \mathfrak{d}_H(x, b^\infty)$ (frequence of a in x). So b^∞ is the only continuity point. The same reasoning can be used to prove that the Fibonacci substitution is not well-defined on the Weyl space since $\hat{\mathfrak{d}}_H(a^\infty, ba^\infty) = 0$ and $\hat{\mathfrak{d}}_H(\overline{\tau}(a^\infty), \overline{\tau}(ba^\infty)) = \hat{\mathfrak{d}}_H((ab)^\infty, (ba)^\infty) = 1$.

Before characterising the dill maps which induce a well-defined maps over the Besicovitch and Weyl spaces, let us show that all uniform dill maps are Lipschitz on these spaces, and thus, they induce well-defined maps.

Notations 5.9. *Let us denote, for a uniform dill map F with local rule f and diameter θ: $d_f^+ = \max\left\{ d_H(f(u), f(v)) \middle| u, v \in A^\theta \right\}$ and $d_f^- = \min\left\{ d_H(f(u), f(v)) \middle| u \neq v \in A^\theta \right\}$.*

Lemma 5.10. *Let F be a uniform dill map with diameter θ and local rule f. Then for all $\ell, k \in \mathbb{N}$, for $m = \left\lceil \frac{k}{|f|} \right\rceil$, and for $p = \left\lfloor \frac{\ell+k}{|f|} \right\rfloor - (m+1)$:*

$$d_H(F(x)_{[\![k,k+\ell[\![}, F(y)_{[\![k,k+\ell[\![}) \leq d_H(x_{[\![m,m+p+\theta[\![}, y_{[\![m,m+p+\theta[\![}) \theta d_f^+ + 2\left|f\right|, \forall x, y \in A^{\mathbb{N}}.$$

Proof. Let $x, y \in A^{\mathbb{N}}$ and $\ell, k \in \mathbb{N}$. Since F is uniform, we can write:

$$F(x)_{[\![k,k+\ell[\![} = v f^*(x_{[\![m,m+p+\theta[\![}) w, \text{ and } F(y)_{[\![k,k+\ell[\![} = v' f^*(y_{[\![m,m+p+\theta[\![}) w',$$

where $|v| = |v'| \leq \left|f\right|$ and $|w| = |w'| \leq \left|f\right|$. By additivity, we can write then:

$$\begin{aligned}
d_H(F(x)_{[\![k,k+\ell[\![}, F(y)_{[\![k,k+\ell[\![}) - 2\left|f\right| &\leq \sum_{i=0}^{p} d_H(f(x_{[\![m+i,m+i+\theta[\![}), f(y_{[\![m+i,m+i+\theta[\![})) \\
&\leq \sum_{\substack{i=0 \\ x_{[\![m+i,m+i+\theta[\![} \neq y_{[\![m+i,m+i+\theta[\![}}}^{p} d_H(f(x_{[\![m+i,m+i+\theta[\![}), f(y_{[\![m+i,m+i+\theta[\![})) \\
&\leq \sum_{\substack{i=0 \\ \exists j \in [\![m+i,m+i+\theta[\![, x_j \neq y_j}}^{p} d_f^+ \quad \leq \sum_{\substack{j \in [\![m,m+p+\theta[\![\\ x_j \neq y_j}} \sum_{i \in]\![j-m-\theta, j-m]\!]} d_f^+ \\
&\leq \sum_{\substack{j \in [\![m,m+p+\theta[\![\\ x_j \neq y_j}} \theta d_f^+ \\
&= d_H(x_{[\![m,m+p+\theta[\![}, y_{[\![m,m+p+\theta[\![}) \theta d_f^+. \quad \square
\end{aligned}$$

Lemma 5.11. *Let F be a uniform dill map with diameter θ and local rule f. Then for all $x, y \in A^{\mathbb{N}}$, for all $\ell, k \in \mathbb{N}$ we have:*

$$d_H(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![}) d_f^- - \theta d_f^- \leq d_H(F(x)_{[\![k|f|,(k+\ell)|f|[\![}, F(y)_{[\![k|f|,(k+\ell)|f|[\![}).$$

Proof. Let $D = \left\{ i \in \mathbb{N} \middle| x_i \neq y_i \right\}$ and $D' = \left\{ i \in \mathbb{N} \middle| F(x)_i \neq F(y)_i \right\}$.
If $i \in D$, then $\left|D' \cap \left[\!\left[j\left|f\right|, (j+1)\left|f\right| \right]\!\right]\right| \geq d_f^-$ for each $j \in [\![i-\theta, i[\![$, which implies that $\left|D' \cap \left[\!\left[(i-\theta)\left|f\right|, i\left|f\right| \right]\!\right]\right| \geq \theta d_f^-$, provided that $i \geq \theta$. Hence:

$$\begin{aligned}
d_H(F(x)_{[\![k|f|,(k+\ell)|f|[\![}, F(y)_{[\![k|f|,(k+\ell)|f|[\![}) &= \left|D' \cap \left[\!\left[k\left|f\right|, (k+\ell)\left|f\right| \right]\!\right]\right| \\
&\geq \frac{1}{\theta} \sum_{i=\theta+k}^{k+\ell} \left|D' \cap \left[\!\left[(i-\theta)\left|f\right|, i\left|f\right| \right]\!\right]\right| \\
&\geq \left|D \cap [\![k+\theta, k+\ell[\![\right| d_f^- \\
&\geq \left(d_H(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![}) - \left|D \cap [\![k, k+\theta[\![\right| \right) d_f^- \\
&\geq d_H(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![}) d_f^- - \theta d_f^-. \quad \square
\end{aligned}$$

Proposition 5.12. *Let F be a uniform dill map with diameter θ and local rule f. Then for all $x, y \in A^{\mathbb{N}}$:*

1. $\frac{d_f^-}{|f|} \cdot \mathfrak{d}_H(x,y) \leq \mathfrak{d}_H(F(x),F(y)) \leq \frac{\theta d_f^+}{|f|} \cdot \mathfrak{d}_H(x,y)$.
2. $\frac{d_f^-}{|f|} \cdot \hat{\mathfrak{d}}_H(x,y) \leq \hat{\mathfrak{d}}_H(F(x),F(y)) \leq \frac{\theta d_f^+}{|f|} \cdot \hat{\mathfrak{d}}_H(x,y)$.

Proof. Let us start by proving that F is $\frac{\theta d_f^+}{|f|}$-Lipschitz with respect to $\mathfrak{d}_H$. Let $x, y \in A^{\mathbb{N}}$. According to Lemma 5.10, for large enough ℓ, for $k = 0$, $m = \left\lceil \frac{k}{|f|} \right\rceil = 0$ and $p = \left\lfloor \frac{\ell}{|f|} \right\rfloor - 1$, one can deduce that :

$$d_H(F(x)_{[\![0,\ell[\![}, F(y)_{[\![0,\ell[\![}) \leq d_H(x_{[\![0,p+\theta[\![}, y_{[\![0,p+\theta[\![}) \theta d_f^+ + 2\left|f\right|,$$

Hence:

$$\begin{aligned}
\frac{d_H(F(x)_{[\![0,\ell[\![}, F(y)_{[\![0,\ell[\![})}{\ell} &\leq \frac{d_H(x_{[\![0,p[\![}, y_{[\![0,p[\![}) \theta d_f^+ + \theta^2 d_f^+ + 2\left|f\right|}{\ell} \\
&\leq \frac{\theta d_f^+}{\left|f\right|} \cdot \frac{d_H(x_{[\![0,p[\![}, y_{[\![0,p[\![})}{p} + \frac{\theta^2 d_f^+ + 2\left|f\right|}{p|f|}.
\end{aligned}$$

Finaly, since $p \to \infty$ when $\ell \to \infty$, we find: $\mathfrak{d}_H(F(x),F(y)) \leq \frac{\theta d_f^+}{|f|} \mathfrak{d}_H(x,y)$.

On the other hand, according to Lemma 5.11, for all $x, y \in A^{\mathbb{N}}$, for all $\ell \in \mathbb{N} \setminus \{0\}$,

$$d_H(x_{[\![0,\ell[\![}, y_{[\![0,\ell[\![}) d_f^- - \theta d_f^- \leq d_H(F(x)_{[\![0,\ell|f|[\![}, F(y)_{[\![0,\ell|f|[\![}).$$

Then: $\frac{d_f^-}{|f|}\cdot\frac{d_H(x_{[\![0,\ell[\![},y_{[\![0,\ell[\![})}{\ell}\leq\frac{d_H(F(x)_{[\![0,\ell|f|[\![},F(y)_{[\![0,\ell|f|[\![})}{\ell|f|}+\frac{\theta d_f^-}{\ell|f|}$.
Hence,

$$\begin{aligned}\frac{d_f^-}{|f|}\mathfrak{d}_H(x,y)&\leq\limsup_{\ell\to\infty}\left(\frac{d_H(F(x)_{[\![0,\ell|f|[\![},F(y)_{[\![0,\ell|f|[\![})}{\ell|f|}+\frac{\theta d_f^-}{\ell|f|}\right)\\&=\limsup_{\ell\to\infty}\frac{d_H(F(x)_{[\![0,\ell|f|[\![},F(y)_{[\![0,\ell|f|[\![})}{\ell|f|}\leq\mathfrak{d}_H(F(x),F(y)).\end{aligned}$$

Now, let us prove that F is $\frac{\theta d_f^+}{|f|}$-Lipschitz with respect to $\hat{\mathfrak{d}}_H$. According to Lemma 5.10, for large enough ℓ, for $k\in\mathbb{N}$, $m=\left\lceil\frac{k}{|f|}\right\rceil$ and $p=\left\lfloor\frac{\ell+k}{|f|}\right\rfloor-(m+1)$ we obtain:

$$d_H(F(x)_{[\![k,k+\ell[\![},F(y)_{[\![k,k+\ell[\![})\leq d_H(x_{[\![m,m+p[\![},y_{[\![m,m+p[\![})\theta d_f^+ +\theta^2 d_f^+ +|f|.$$

Hence:

$$\begin{aligned}\frac{d_H(F(x)_{[\![k,k+\ell[\![},F(y)_{[\![k,k+\ell[\![})}{\ell}&\leq\frac{\max_{h\in\mathbb{N}}d_H(x_{[\![h,h+p[\![},y_{[\![h,h+p[\![})\theta d_f^+ +\theta^2 d_f^+ +2|f|}{\ell}\\&\leq\frac{\theta d_f^+}{|f|}\cdot\frac{\max_{h\in\mathbb{N}}d_H(x_{[\![h,h+p[\![},y_{[\![h,h+p[\![})}{p}+\frac{\theta^2 d_f^+ +2|f|}{\ell}.\end{aligned}$$

Since this was true for every k and since $p\to\infty$ when $\ell\to\infty$, we obtain:

$$\hat{\mathfrak{d}}_H(F(x),F(y))\leq\frac{\theta d_f^+}{|f|}\hat{\mathfrak{d}}_H(x,y).$$

For the lower bound, let $x,y\in A^{\mathbb{N}}$, $\ell\in\mathbb{N}\setminus\{0\}$ and $k\in\mathbb{N}$. According to Lemma 5.11,

$$d_H(x_{[\![k,k+\ell[\![},y_{[\![k,k+\ell[\![})d_f^-\leq d_H(F(x)_{[\![k|f|,(k+\ell)|f|[\![},F(y)_{[\![k|f|,(k+\ell)|f|[\![})+\theta d_f^-.$$

Then,

$$\frac{d_f^-}{|f|}\cdot\frac{d_H(x_{[\![k,k+\ell[\![},y_{[\![k,k+\ell[\![})}{\ell}\leq\frac{\max_{h\in\mathbb{N}}d_H(F(x)_{[\![h,h+\ell|f|[\![},F(y)_{[\![h,h+\ell|f|[\![})}{\ell|f|}+\frac{\theta d_f^-}{\ell|f|}.$$

Since this was true for any k, we obtain:

$$\begin{aligned}\frac{d_f^-}{|f|}\hat{\mathfrak{d}}_H(x,y)&\leq\limsup_{\ell\to\infty}\left(\frac{\max_{h\in\mathbb{N}}d_H(F(x)_{[\![h,h+\ell|f|[\![},F(y)_{[\![h,h+\ell|f|[\![})}{\ell|f|}+\frac{\theta d_f^-}{\ell|f|}\right)\\&=\limsup_{\ell\to\infty}\frac{\max_{h\in\mathbb{N}}d_H(F(x)_{[\![h,h+\ell|f|[\![},F(y)_{[\![h,h+\ell|f|[\![})}{\ell|f|}\leq\hat{\mathfrak{d}}_H(F(x),F(y)).\quad\square\end{aligned}$$

In particular, the Cantor and Thue-Morse substitutions are well-defined over this space (we will discuss them later).

Proposition 5.13. *Let F be a dill map with diameter $\theta \in \mathbb{N} \setminus \{0\}$ and local rule f. If $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) is well-defined then F is either constant or uniform.*

Proof. Assume that F is non-uniform, *i.e.,* there are two words u and v of equal length such that $\left|f^*(u)\right| \neq \left|f^*(v)\right|$. One can assume that their longest common suffix has length $\theta - 1$. Indeed, otherwise let $a \in A$, $u' = u_{[\![|u|-\theta+1,|u|[\![}a^{\theta-1}$ and $v' = v_{[\![|u|-\theta+1,|v|[\![}a^{\theta-1}$; one can note that $f^*(ua^{\theta-1}) = f^*(u)f^*(u')$ and $f^*(va^{\theta-1}) = f^*(v)f^*(v')$, so that either $\left|f^*(ua^{\theta-1})\right| \neq \left|f^*(va^{\theta-1})\right|$, or

$$\left|f^*(u')\right| = \left|f^*(ua^{\theta-1})\right| - \left|f^*(u)\right| \neq \left|f^*(va^{\theta-1})\right| - \left|f^*(v)\right| = \left|f^*(v')\right|.$$

Note that both of these pairs of words share a common suffix of length at least $\theta - 1$. Assume also without loss of generality that $k = \left|f^*(u)\right| - \left|f^*(v)\right| > 0$.

- First assume that there exist $w \in A^*$ and $i \in \mathbb{N}$ such that $f^*(w)_i \neq f^*(w)_{i+k}$. By our previous assumption, we know that for $z = w^\infty$ and $w' = u_{[\![|u|-\theta,|u|[\![}z_{[\![0,\theta[\![} = v_{[\![|v|-\theta,|v|[\![}z_{[\![0,\theta[\![}$ we have: $F(uz) = f^*(u)f^*(w')F(z)$, so that for every $j \in \mathbb{N}$,

  $$F(uz)_{\left|f^*(u)\right|+\left|f^*(w')\right|+j\left|f^*(z_{[\![0,|w|+\theta[\![})\right|+i} = f^*(w)_i.$$

  On the other hand:

  $$\begin{aligned} F(vz)_{\left|f^*(u)\right|+\left|f^*(w')\right|+j\left|f^*(z_{[\![0,|w|+\theta[\![})\right|+i} &= F(vz)_{\left|f^*(v)\right|+k+\left|f^*(w')\right|+j\left|f^*(z_{[\![0,|w|+\theta[\![})\right|+i} \\ &= f^*(w)_{i+k} \neq f^*(w)_i. \end{aligned}$$

  We deduce that:

  $$\hat{\mathfrak{d}}_H(F(uz),F(vz)) \geq \mathfrak{d}_H(F(uz),F(vz)) \geq \frac{1}{\left|f^*(z_{[\![0,|w|+\theta[\![})\right|}.$$

  Since $|u| = |v|$, we know $\hat{\mathfrak{d}}_H(uz,vz) = \mathfrak{d}_H(uz,vz) = 0$, so that F is not well-defined over the quotient spaces.

- Otherwise, for all $w \in A^*$, $i \in [\![0,\left|f^*(w)\right|[\![$, we have $f^*(w)_i = f^*(w)_{i+k}$. Let $w' \in A^*$, such that $\left|w'\right| \geq k$ and let $w \in A^*$. Then for all $j \in \mathbb{N}$:

  $$f^*(w)_j = f^*(w'w)_{j+\left|f^*(w')\right|} = f^*(w'w)_{j+\left|f^*(w')\right| \bmod k} = f^*(w')_{j+\left|f^*(w')\right| \bmod k}.$$

  Hence for all $x \in A^{\mathbb{N}}$, $j \in \mathbb{N}$, we have $F(x)_j = f^*(w')_{j+\left|f^*(w')\right| \bmod k}$. So, F is constant. □

We now reach necessary and sufficient conditions for dill maps to induce well-defined dynamical systems over this space.

Theorem 5.14. *Let F be a dill map with diameter θ and local rule f. Then the following statements are equivalent:*

1. *$F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) is well-defined.*
2. *F is $\frac{\theta d_f^+}{|f|}$-Lipschitz with respect to $\mathfrak{d}_H$ (resp. $\hat{\mathfrak{d}}_H$).*
3. *F is either constant or uniform.*

Proof. 2 $\Longrightarrow$ 1 is clear from the definition of Lipschitz maps. Implication 3 $\Longrightarrow$ 2 follows from Proposition 5.12. Implication 1 $\Longrightarrow$ 3 follows from Proposition 5.13. □

In the particular case of uniform substitutions, $\theta d_f^+ = d_f^+ \leq \|f\| = |f|$, which allows the following.

Corollary 5.15. *A substitution τ yields a well-defined dynamical system $\overline{\tau}_{\mathfrak{d}_H}$ (resp. $\overline{\tau}_{\hat{\mathfrak{d}}_H}$) over $X_{\mathfrak{d}_H}$ (resp. $X_{\hat{\mathfrak{d}}_H}$) if and only if it is 1-Lipschitz with respect to $\mathfrak{d}_H$ (resp. $\hat{\mathfrak{d}}_H$).*

5.1.3 Dynamical properties

The following derives directly from Proposition 5.12 (and completeness of $X_{\mathfrak{d}_H}$).

Corollary 5.16. *Let F be a uniform dill map with diameter θ and local rule f.*

1. *If $\theta d_f^+ \leq |f|$, then $F_{\mathfrak{d}_H}$ and $F_{\hat{\mathfrak{d}}_H}$ is equicontinuous.*
   For example, for every uniform substitution τ, $\overline{\tau}_{\mathfrak{d}_H}$ is equicontinuous.
2. *If $\theta d_f^+ < |f|$, then $F_{\mathfrak{d}_H}$ and $F_{\hat{\mathfrak{d}}_H}$ is contracting. So according to the Banach fixed point theorem [Ban22], every orbit converges to a unique fixed point over the Besicovitch space, since it is a complete space. For example, for every uniform substitution τ such that $d_\tau^+ < |\tau|$, $\overline{\tau}_{\mathfrak{d}_H}$ is contracting.*
3. *For every uniform substitution τ such that $d_\tau^- = |\tau|$ (which means that the substitution is everywhere marked: any two images have no letter in common), $\overline{\tau}_{\mathfrak{d}_H}$ and $\overline{\tau}_{\hat{\mathfrak{d}}_H}$ are isometries.*

Example 5.17. *Let τ be a Toeplitz substitution. By definition and by Corollary 5.16, $\overline{\tau}_{\mathfrak{d}_H}$ is contracting, so that all orbits converge towards a unique fixed point: the class for $\sim_{\mathfrak{d}_H}$ of the usual fixed points of the substitution (which is unique if for all $a, b \in A, \tau(a)_0 = \tau(b)_0$, but may not be otherwise, like for the Cantor substitution).*

Example 5.18. *On the contrary, the Thue-Morse substitution is an isometry, thanks to Item 3 of Corollary 5.16. In particular, if Σ_τ is the orbit closure of the two fixed points, then for every $x \notin \Sigma_\tau$, the pseudometric $\mathfrak{d}_H(\tau^t(x), \Sigma_\tau)$ is constantly positive, so that our intuition that orbits converge towards Σ_τ, though justified in the Cantor space, is completely false in the Besicovitch space.*

Remark 5.19. *The behaviors from Example 5.17 and 5.18 give an essentially full picture of what can occur for substitutions. Indeed, if there exists $p \in \mathbb{N}$ such that $d^+_{\tau^p} < |\tau|^p$, then τ^p is contracting; consequently, for every $t \in \mathbb{N}$ the diameter of $\tau^t(A^{\mathbb{N}})$ is bounded by that of $\tau^{p\lfloor \frac{t}{p} \rfloor}(A^{\mathbb{N}})$, which is bounded by $\left(\frac{d^+_{\tau^p}}{|\tau|^p}\right)^{\lfloor \frac{t}{p} \rfloor}$; so all orbits of τ converge towards a unique fixed point.*
If, on the other hand, for every $t \in \mathbb{N}$, $d^+_{\tau^t} = |\tau|$, this means that there exists a subalphabet $A_t \in A$ containing at least two letters, such that $a \neq b \in A_t \implies d_H(\tau^t(a), \tau^t(b)) = |\tau|^t$. Hence for $t \in \mathbb{N}$ and A_t the maximal subalphabet satisfying the previous condition, it is not difficult to see that $A_{t+1} \subset A_t$, and since A_t contains at least two letters for all $t \in \mathbb{N}$, then the subalphabet $A' = \bigcap_{t\in\mathbb{N}} A_t$ contains at least two letters. Finally, since A' is preserved by τ, then the restriction of $\overline{\tau}$ to $A'^{\mathbb{N}}$ is an isometry (because Proposition 5.12 remains true when the minimum and maximum are taken over a subalphabet).

The links between dynamical properties in the Cantor space and in the Besicovitch space appeared for cellular automata in [BFK97], [FK09]: in particular, sensitivity in the Besicovitch space implies sensitivity in the Cantor space, and equicontinuity in the Cantor space implies equicontinuity in the Besicovitch space. Nevertheless, unlike for cellular automata, there exist dill maps which are equicontinuous in the Cantor space but not in the Besicovitch space.

Example 5.20. *Consider the dill map F with diameter 2 defined over the alphabet $A = \{a, b\}$ by the following local rule: $f(bb) = bbb$, and $f(u_0u_1) = u_0u_1$ if $u_0u_1 \neq bb$. This dill map is 1-Lipschitz in the Cantor space (because it either preserves or doubles the common prefix), and hence it is equicontinuous. On the contrary, $F_{\mathfrak{d}_H}$ is not well-defined over the Besicovitch space since it is neither constant nor uniform (thanks to Theorem 5.14).*

Some weak robustness properties of cellular automata from [FK09], though, can be generalized to dill maps, like in the following statement.

Proposition 5.21. *Let F be a uniform dill map and $m \in \mathbb{N}$. Then we have the following:*

1. *If $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) is sensitive, then $\sigma^m_{\mathfrak{d}_H} \circ F_{\mathfrak{d}_H}$ (resp. $\sigma^m_{\hat{\mathfrak{d}}_H} \circ F_{\hat{\mathfrak{d}}_H}$) is also sensitive.*
2. *If x is an equicontinuity point for $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$), then it is also an equicontinuity point for $\sigma^m_{\mathfrak{d}_H} \circ F_{\mathfrak{d}_H}$ (resp. $\sigma^m_{\hat{\mathfrak{d}}_H} \circ F_{\hat{\mathfrak{d}}_H}$).*
3. *If $F_{\mathfrak{d}_H}$ (resp. $F_{\hat{\mathfrak{d}}_H}$) is equicontinuous, then $\sigma^m_{\mathfrak{d}_H} \circ F_{\mathfrak{d}_H}$ (resp. $\sigma^m_{\hat{\mathfrak{d}}_H} \circ F_{\hat{\mathfrak{d}}_H}$) is equicontinuous.*

Proof. The key to prove the three statements is the following:

$$\forall n \in \mathbb{N}, (\sigma^m \circ F)^n = \sigma^{\sum_{k=0}^{n-1} m|f|^k} \circ F^n.$$

Let us prove this by induction on $n \in \mathbb{N}$. The case $n = 0$ is obvious. Suppose that it is true for some n.

$$\begin{aligned}(\sigma^m \circ F)^{n+1} &= (\sigma^m \circ F) \circ (\sigma^m \circ F)^n \\ &= \sigma^m \circ F \circ (\sigma^{\sum_{k=0}^{n-1} m|f|^k} \circ F^n) \\ &= \sigma^m \circ \sigma^{|f|\sum_{k=0}^{n-1} m|f|^k} \circ F \circ F^n,\end{aligned}$$

which is the next step of the induction hypothesis. The last equality comes from the fact that F is uniform, so that $s_x^n = |f| = \|f\|$.

Now we can deduce the proof of the statements: for all $x, y \in A^{\mathbb{N}}$ and for all $n, m \in \mathbb{N}$:

$$\begin{aligned}\mathfrak{d}_H((\sigma^m \circ F)^n(x), (\sigma^m \circ F)^n(y)) &= \mathfrak{d}_H(\sigma^{\sum_{k=0}^{n-1} m|f|^k}(F^n(x)), \sigma^{\sum_{k=0}^{n-1} m|f|^k}(F^n(y))) \\ &= \mathfrak{d}_H(F^n(x), F^n(y)) \quad \text{(since } \mathfrak{d}_H \text{ is shift-invariant).}\end{aligned}$$

Similarly,

$$\begin{aligned}\hat{\mathfrak{d}}_H((\sigma^m \circ F)^n(x), (\sigma^m \circ F)^n(y)) &= \hat{\mathfrak{d}}_H(\sigma^{\sum_{k=0}^{n-1} m|f|^k}(F^n(x)), \sigma^{\sum_{k=0}^{n-1} m|f|^k}(F^n(y))) \\ &= \hat{\mathfrak{d}}_H(F^n(x), F^n(y)) \quad \text{(since } \hat{\mathfrak{d}}_H \text{ is shift-invariant).} \quad \square\end{aligned}$$

It was known that the cellular automata behave well in the Besicovitch and the Weyl spaces, and we have seen in this section that it is also the case for uniform substitutions. But we proved that this is not true for non-uniform substitutions. In the next section, we consider another topological space, in which both cellular automata and all substitutions are well-defined over this space.

5.2 Dill maps in the Feldman-Katok space

5.2.1 Shift and Lipschitz property

Lipschitz property of Dill maps Now, we aim at proving that, unlike in the Besicovitch space, all dill maps are well-defined in the Feldman-Katok space.

The following notion concretizes the notion of a dill map which transforms a deletion into a bounded number of deletions.

Definition 5.22. *A dill map F with local rule f is (M, M')-bounded-deletion-spreading (BDS) for some $M, M' \in \mathbb{N}$ if for every $u \in A^*$ and every $j \in [\![0, |u|[\![$:*

$$d_L(f^*(D_j(u)), f^*(u)) \leq M + \frac{|f^*(u)| - |f^*(D_j(u))|}{2}, \text{ and}$$
$$d_L(f^*(D_j(u)), f^*(u)) \leq M' - \frac{|f^*(u)| - |f^*(D_j(u))|}{2}.$$

Note that this property does not depend on the choice of a local rule, since the definition invoves the application of f over arbitrarily long words.

Proposition 5.23. *Let F be a dill map with local rule f and diameter θ. Then F is $(\theta \|f\|, (\theta-1)\|f\|)$-BDS.*

Proof. Let $u \in A^*$ and $j \in [\![0, |u|[\![$. By definition of f^*, we have the following:

$$f^*(u) = f^*(u_{[\![0,j-\theta[\![}) f^*(u_{[\![j-\theta,j+\theta[\![}) f^*(u_{[\![j+1,|u|[\![})$$

and,

$$f^*(D_j(u)) = f^*(u_{[\![0,j-\theta[\![}) f^*(D_j(u)_{[\![j-\theta,j+\theta-1[\![}) f^*(u_{[\![j+1,|u|[\![}).$$

Let us denote $w = f^*(u_{[\![j-\theta,j+\theta[\![})$ and $w' = f^*(D_j(u)_{[\![j-\theta,j+\theta-1[\![})$. By subadditivity and thanks to Remark 2.65: $d_L(f^*(u), f^*(D_j(u))) \leq d_L(w,w') \leq \dfrac{|w|+|w'|}{2}$.

Obviously,

$$|w| = |f^*(u_{[\![j-\theta,j+\theta[\![})| \leq \theta\|f\| \text{ and } |w'| = |f^*(D_j(u)_{[\![j-\theta,j+\theta-1[\![})| \leq (\theta-1)\|f\|.$$

Moreover: $|f^*(u)| - |f^*(D_j(u))| = |w| - |w'|$. Hence,

$$d_L(f^*(u), f^*(D_j(u))) \leq \frac{|w|+|w'|}{2} = |w| + \frac{|w'|-|w|}{2} \leq \theta\|f\| + \frac{\left|f^*(D_j(u))\right| - \left|f^*(u)\right|}{2}.$$

Also,

$$d_L(f^*(u), f^*(D_j(u))) \leq \frac{|w|+|w'|}{2} = |w'| + \frac{|w|-|w'|}{2} \leq (\theta-1)\|f\| + \frac{\left|f^*(u)\right| - \left|f^*(D_j(u))\right|}{2}. \quad \square$$

Lemma 5.24. *Let F be a (M,M')-BDS dill map for some $M, M' \in \mathbb{N}$ and local rule f. Then for all $\ell \in \mathbb{N}$ and $u, v \in A^\ell$, we have:*

$$d_L(f^*(u), f^*(v)) \leq (M+M') d_L(u,v) - \frac{\left|\left|f^*(u)\right| - \left|f^*(v)\right|\right|}{2}.$$

Proof. Consider words u, v, and $m \in \mathbb{N}$ such that: $D_{j_1} \circ \cdots \circ D_{j_m}(u) = D_{j'_1} \circ \cdots \circ D_{j'_m}(v)$, for some minimal edition sequences $j_1 < \cdots < j_m < |u|$ and $j'_1 < \cdots < j'_m < |v|$, so that $d_L(u,v) = m$. By the triangular inequality, one gets:

$$\begin{aligned} d_L(f^*(u), f^*(v)) \leq & \sum_{k=1}^{m} d_L(f^*(D_{j_{k+1}} \circ \cdots \circ D_{j_m}(u)), f^*(D_{j_k} \circ \cdots \circ D_{j_m}(u))) + \\ & + d_L(f^*(D_{j_1} \circ \cdots \circ D_{j_m}(u)), f^*(D_{j'_1} \circ \cdots \circ D_{j'_m}(v))) + \\ & + \sum_{k=1}^{m} d_L(f^*(D_{j'_k} \circ \cdots \circ D_{j'_m}(v)), f^*(D_{j'_{k+1}} \circ \cdots \circ D_{j'_m}(v))). \end{aligned}$$

Now our two assumptions allow to write:

$$\begin{aligned}
d_L(f^*(u), f^*(v)) &\leq \sum_{k=1}^{m}\left(M + \frac{\left|f^*(D_{j_{k+1}}\circ\cdots\circ D_{j_m}(u))\right| - \left|f^*(D_{j_k}\circ\cdots\circ D_{j_m}(u))\right|}{2}\right) + 0 + \\
&\quad + \sum_{k=1}^{m}\left(M' + \frac{\left|f^*(D_{j'_k}\circ\cdots\circ D_{j'_m}(v))\right| - \left|f^*(D_{j'_{k+1}}\circ\cdots\circ D_{j'_m}(v))\right|}{2}\right) \\
&\leq Mm + \frac{\left|f^*(u)\right| - \left|f^*(D_{j_1}\circ\cdots\circ D_{j_m}(u))\right|}{2} + \\
&\quad + M'm + \frac{\left|f^*(D_{j'_1}\circ\cdots\circ D_{j'_m}(v))\right| - \left|f^*(v)\right|}{2} \\
&\leq (M+M')m + \frac{\left|f^*(u)\right| - \left|f^*(v)\right|}{2}. \quad \square
\end{aligned}$$

Lemma 5.25. *Let F be a (M, M')-BDS dill map with local rule f and diameter θ. Let x be such that for every $j \in \mathbb{N}$, $\left|f(x_{[\![j,j+\theta[\![})\right| \geq L$, for some $L > 0$. Then for every $y \in A^{\mathbb{N}}$,*

$$\mathfrak{d}_L(F(x), F(y)) \leq \frac{M+M'}{L}\mathfrak{d}_L(x,y).$$

Proof. Let $x, y \in A^{\mathbb{N}}$ and $\ell \in \mathbb{N}$. Consider the largest $k \in \mathbb{N}$ such that $\left|f^*(x_{[\![0,k[\![})\right| \leq \ell$. Then $F(x)_{[\![0,\ell[\![}$ can be written $f^*(x_{[\![0,k[\![})w$ for some w of length less than $\|f\|$. Note that $\ell \geq \sum_{i=0}^{k-\theta}\left|f(x_{[\![i,i+\theta[\![})\right| \geq (k-\theta+1)L$. Proposition 2.74 gives the following:

$$d_L(F(x)_{[\![0,\ell[\![}, F(y)_{[\![0,\ell[\![}) \leq d_L(f^*(x_{[\![0,k[\![}), f^*(y_{[\![0,k[\![})) + d_L(w, F(y)_{\left[\!\left[\left|f^*(y_{[\![0,k[\![})\right|,\ell\right[\!\right[}).$$

Note that the previous inequality still holds if $\left|f^*(y_{[\![0,k[\![})\right| \geq \ell$, in which case the second term is $d_L(w,\lambda) = \frac{|w|}{2}$. Otherwise,

$$\begin{aligned}
d_L(w, F(y)_{\left[\!\left[\left|f^*(y_{[\![0,k[\![})\right|,\ell\right[\!\right[}) &\leq \frac{|w| + \left|F(y)_{\left[\!\left[\left|f^*(y_{[\![0,k[\![})\right|,\ell\right[\!\right[}\right|}{2} \\
&\leq \frac{|w| + \left(\ell - \left|f^*(x_{[\![0,k[\![})\right|\right) + \left(\left|f^*(x_{[\![0,k[\![})\right| - \left|f^*(y_{[\![0,k[\![})\right|\right)}{2}. \\
&< \frac{\left(\left|f^*(x_{[\![0,k[\![})\right| - \left|f^*(y_{[\![0,k[\![})\right|\right)}{2} + \|f\| - 1.
\end{aligned}$$

We can use Lemma 5.24 to get:

$$\begin{aligned} d_L(F(x)_{[\![0,\ell[\![}, F(y)_{[\![0,\ell[\![}) &< (M+M')d_L(x_{[\![0,k[\![}, y_{[\![0,k[\![}) - \frac{\left|\left|f^*(x_{[\![0,k[\![})\right| - \left|f^*(y_{[\![0,k[\![})\right|\right|}{2} + \\ &+ \frac{\left(\left|f^*(x_{[\![0,k[\![})\right| - \left|f^*(y_{[\![0,k[\![})\right|\right)}{2} + \|f\| - 1 \\ &\leq (M+M')d_L(x_{[\![0,k[\![}, y_{[\![0,k[\![}) + \|f\| - 1 \end{aligned}$$

Since $\ell \geq (k-\theta+1)L$, one can write:

$$\begin{aligned} \frac{d_L(F(x)_{[\![0,\ell[\![}, F(y)_{[\![0,\ell[\![})}{\ell} &\leq \frac{(M+M')d_L(x_{[\![0,k[\![}, y_{[\![0,k[\![}) + \|f\| - 1}{(k-\theta+1)L} \\ &= (M+M')\frac{d_L(x_{[\![0,k[\![}, y_{[\![0,k[\![})}{k} \cdot \frac{1}{(1-\frac{\theta-1}{k})L} + \frac{\|f\| - 1}{(k-\theta+1)L}. \end{aligned}$$

Finally, since $k \geq \left\lfloor \frac{\ell}{\|f\|} \right\rfloor$ tends to infinity when ℓ tends to infinity, we have:

$$\begin{aligned} \mathfrak{d}_L(F(x), F(y)) &= \limsup_{\ell\to\infty} \frac{d_L(F(x)_{[\![0,\ell[\![}, F(y)_{[\![0,\ell[\![})}{\ell} \\ &\leq \limsup_{k\to\infty} \left((M+M')\frac{d_L(x_{[\![0,k[\![}, y_{[\![0,k[\![})}{k} \cdot \frac{1}{(1-\frac{\theta-1}{k})L} + \frac{\|f\| - 1}{(k-\theta+1)L} \right) \\ &= \limsup_{k\to\infty} \frac{M+M'}{L} \cdot \frac{d_L(x_{[\![0,k[\![}, y_{[\![0,k[\![})}{k} = \frac{M+M'}{L}\mathfrak{d}_L(x,y). \quad \square \end{aligned}$$

Theorem 5.26.

1. *Let F be a (M,M')-BDS dill map for some $M, M' \in \mathbb{N}$, with local rule f. Then F is $\frac{M+M'}{|f|}$-Lipschitz with respect to $\mathfrak{d}_L$.*
2. *Let F be any dill map with local rule f and diameter θ, and x be such that for every $i \in \mathbb{N}$, $\left|f(x_{[\![i,i+\theta[\![})\right| \geq L$, for some $L > 0$. Then, for every $y \in A^{\mathbb{N}}$,*

$$\mathfrak{d}_L(F(x), F(y)) \leq (2\theta-1)\frac{\|f\|}{L}\mathfrak{d}_L(x,y).$$

For example, if τ is a substitution and x is such that $|\tau(x_i)| \geq L$, for every $i \in \mathbb{N}$, then for every $y \in A^{\mathbb{N}}$,

$$\mathfrak{d}_L(\overline{\tau}(x), \overline{\tau}(y)) \leq \frac{\|\tau\|}{L}\mathfrak{d}_L(x,y).$$

3. *In particular, any dill map with local rule f and diameter θ is $(2\theta-1)\frac{\|f\|}{|f|}$-Lipschitz with respect to $\mathfrak{d}_L$.*
For example, any substitution τ yields a $\frac{\|\tau\|}{|\tau|}$-Lipschitz dynamical system.

Proof.

1. Since, for all $x \in A^{\mathbb{N}}$ and all $i \in \mathbb{N}$, we have $\left|f(x_{[\![i,i+\theta[\![})\right| \geq |f|$, then according to Lemma 5.25, we find that F is $\frac{M+M'}{|f|}$-Lipschitz.
2. According to Proposition 5.23, F is $(\theta \|f\|, (\theta-1)\|f\|)$-BDS. Hence the result follows from Lemma 5.25.
3. Since, for all $x \in A^{\mathbb{N}}$ we have $\left|f(x_{[\![i,i+\theta[\![})\right| \geq |f|$, then we deduce the result from Item 2. □

Likewise the Besicovitch and the Weyl spaces, not all dill maps are defined over the sliding space associated to the Levenshtein distance. See for instance the following example:

Example 5.27. *Let τ be a substitution defined over $A = \{0,1\}^{\mathbb{N}}$ as follows:*

$$\tau : \begin{array}{ccc} 0 & \mapsto & 0 \\ 1 & \mapsto & 11 \end{array}.$$

Let $x = (0,1)^{(n,n)_{n\in\mathbb{N}\setminus\{0\}}}$ and $y = (0,1)^{(n+1,n-1)_{n\in\mathbb{N}\setminus\{0\}}}$. Note that for all $j \in \mathbb{N}$, since $s_j := j(j+1) = \sum_{i=1}^{j} 2i$, we obtain: $d_H(x_{[\![s_j,s_{j+1}[\![}, y_{[\![s_j,s_{j+1}[\![}) = 1$. Let $\ell \in \mathbb{N}\setminus\{0\}$ and $k \in \mathbb{N}$. For $p = \min\{j \in \mathbb{N} \,|\, s_j \geq k\}$, $m = \max\{j \in \mathbb{N} \,|\, s_j \leq k+\ell\}$ and by subadditivity:

$$\begin{array}{rcl} d_H(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![}) & = & d_H(x_{[\![k,s_p[\![}, y_{[\![k,s_p[\![}) + d_H(x_{[\![s_p,s_m[\![}, y_{[\![s_p,s_m[\![}) + d_H(x_{[\![s_m,k+\ell[\![}, y_{[\![s_m,k+\ell[\![}) \\ & \leq & 1 + d_H(x_{[\![s_p,s_m[\![}, y_{[\![s_p,s_m[\![}) + 1 \leq (m-p)+2. \end{array}$$

Moreover, since $\ell + k \geq m^2 + m$ and $k \leq p^2 + p$, we obtain $\frac{m-p}{\ell} \leq \frac{1}{m+p+1}$, and thus:

$$\frac{d_H(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\ell} \leq \frac{m-p+2}{\ell} \leq \frac{2}{m+p+1} + \frac{2}{\ell}.$$

Since m tends to ∞ when ℓ tends to ∞ we obtain:

$$\lim_{\ell\to\infty} \frac{d_H(x_{[\![k,k+\ell[\![}, y_{[\![k,k+\ell[\![})}{\ell} = 0, \forall k \in \mathbb{N}.$$

And thus $\hat{\mathfrak{d}}_L(x,y) = \hat{\mathfrak{d}}_H(x,y) = 0$.

On the other hand, it is clear that $\overline{\tau}(x) = (0,1)^{(n,2n)_{n\in\mathbb{N}\setminus\{0\}}}$ and $\overline{\tau}(y) = (0,1)^{(n+1,2(n-1))_{n\in\mathbb{N}\setminus\{0\}}}$. Let $\ell \in \mathbb{N}\setminus\{0\}$. For $k = \sum_{i=1}^{\ell}(i+1) + \sum_{i=1}^{\ell} 2(i-1)$, one can remark that $\overline{\tau}(y)_{[\![k,k+\ell[\![} = 0^{\ell}$.

In contrast, since $\left(\sum_{i=1}^{\ell} i + \sum_{i=1}^{\ell} 2i\right) - \ell = k$, we obtain $\overline{\tau}(x)_{[\![k,k+\ell[\![} = 1^{\ell}$. And thus:

$$d_L(\overline{\tau}(x)_{[\![k,k+\ell[\![}, \overline{\tau}(y)_{[\![k,k+\ell[\![}) = d_L(0^{\ell}, 1^{\ell}) = \ell.$$

Hence, $\max_{h\in\mathbb{N}} d_L(\overline{\tau}(x)_{[\![h,h+\ell[\![}, \overline{\tau}(y)_{[\![h,h+\ell[\![}) = \ell$. Therefore, $\hat{\mathfrak{d}}_L(\overline{\tau}(x), \overline{\tau}(y)) = 1$. In conclusion, $\overline{\tau}$ is not well defined over the sliding space associated to the Levenshtein space.

Proposition 5.28. *Any uniform dill map F is $(2\theta-1)$-Lipschitz with respect to $\hat{\mathfrak{d}}_L$.*

Proof. Let $x, y \in A^{\mathbb{N}}$. Since F is a uniform dill map, for large enough ℓ and $k \in \mathbb{N}$ we can write:

$$F(x)_{[\![k,k+\ell[\![} = v f^*(x_{[\![m,m+p[\![})\, w \text{ and } F(y)_{[\![k,k+\ell[\![} = v' f^*(y_{[\![m,m+p[\![})\, w'$$

where $m = \left\lceil \frac{k}{\|f\|} \right\rceil$, $p = \left\lfloor \frac{k+\ell}{\|f\|} \right\rfloor - m$ and $v, v', w, w' \in A^*$ such that $|v| = |v'| < \|f\|$ and $|w| = |w'| < \|f\|$. Hence, by subadditivity:

$$\begin{aligned} d_L(F(x)_{[\![k,k+\ell[\![}, F(y)_{[\![k,k+\ell[\![}) &\leq d_L(v,v') + d_L(f^*(x_{[\![m,m+p[\![}), f^*(y_{[\![m,m+p[\![})) + d_L(w,w') \\ &\leq 2\|f\| + d_L(f^*(x_{[\![m,m+p[\![}), f^*(y_{[\![m,m+p[\![})). \end{aligned}$$

According to Lemma 5.24 and Proposition 5.23:

$$d_L(F(x)_{[\![k,k+\ell[\![}, F(y)_{[\![k,k+\ell[\![}) \leq 2\|f\| + (2\theta-1)\|f\|\, d_L(x_{[\![m,m+p[\![}, y_{[\![m,m+p[\![}).$$

Since $\ell = |v| + |w| + (p-\theta)\|f\| \geq (p-\theta)\|f\|$ and by subadditivity:

$$\begin{aligned} \frac{d_L(F(x)_{[\![k,k+\ell[\![}, F(y)_{[\![k,k+\ell[\![})}{\ell} &\leq \frac{\|f\|(2+\theta(2\theta-1))}{\ell} + \frac{(2\theta-1)d_L(x_{[\![m,m+p-\theta[\![}, y_{[\![m,m+p-\theta[\![})}{p-\theta} \\ &\leq \frac{\|f\|(2+\theta(2\theta-1))}{\ell} + \frac{(2\theta-1)d_L(x_{[\![m,m+p-\theta[\![}, y_{[\![m,m+p-\theta[\![})}{p-\theta} \\ &\leq \frac{\|f\|(2+\theta(2\theta-1))}{\ell} + (2\theta-1) \times \max_{h\in\mathbb{N}} \frac{d_L(x_{[\![h,h+p-\theta[\![}, y_{[\![h,h+p-\theta[\![})}{p-\theta}. \end{aligned}$$

Since this was true for every $k \in \mathbb{N}$ and since $p \to \infty$ when $\ell \to \infty$:

$$\hat{\mathfrak{d}}_L(F(x), F(y)) \leq (2\theta-1)\hat{\mathfrak{d}}_L(x,y). \quad \square$$

A natural question is now the following: as in the case of Besicovitch and Weyl spaces, are only uniform and constant dill maps well-defined above the sliding space associated to Levenshtein distance?

Shift One of the motivation to study the Besicovitch space is that the shift is an isometry over this space. In the Feldman-Katok space, this is still true, but even more than this: the shift is exactly the identity.

Proposition 5.29. *The shift over $X_{\mathfrak{d}_L}$ (resp. $X_{\hat{\mathfrak{d}}_L}$) is the identity map.*

Proof. The proof is a direct conclusion from Proposition 3.26. $\square$

Since every equivalence class is invariant by the shift, dynamical systems over this space can be considered as acting on shift orbits.

Let us now see that the shifts are the only dill maps in the class of the identity.

Theorem 5.30. *For every dill map F, $F \in \mathrm{id}_{\mathfrak{d}_L} \iff \exists k \in \mathbb{N}, F = \sigma^k$.*

Proof. Proposition 5.29 proves that if $F = \sigma^k$ for some $k \in \mathbb{N} \setminus \{0\}$ then $F \in \mathrm{id}_{\mathfrak{d}_L}$.

Conversely, let F be a dill map with diameter $\theta \in \mathbb{N} \setminus \{0\}$ and local rule f such that $F \in \mathrm{id}_{\mathfrak{d}_L}$. Let $u \in A^*$ be such that for all $w \in A^\theta$, $w \sqsubseteq u$, $u_{[\![0,\theta[\![} = a^{\theta-1}$, for some $a \in A$, $u_\theta \neq a$ and $u_{|u|} \neq a$. Let $x = (ua^n)^\infty$ for some n strictly larger than both $|u|$ and $\left|f^*(ua^{\theta-1})\right| - |u|$. It is clear that $\mathfrak{d}_L(F(x), x) = 0$ since $F \in \mathrm{id}_{\mathfrak{d}_L}$, then according to Proposition 3.34, there exists $k \in [\![0, |u| + n[\![$ such that:

$$F(x) = (f^*(ua^{n+\theta-1}))^\infty = \sigma^k(x) = \begin{cases} (u_{[\![k,|u|[\![} a^n u_{[\![0,k[\![})^\infty & if \quad k \leq |u|\,; \\ (a^{n+|u|-k} u a^{k-|u|})^\infty & if \quad k \geq |u|\,. \end{cases}$$

Thanks to the assumption that u_θ is the first letter different from a, and that there is no factor a^n in u, we get the aperiodicity property of ua^n: the shortest period of x is $|u| + n$. Since $F(x)$ has the same periods and $\left|f^*(ua^{n+\theta-1})\right|$ is one of them, we deduce:

$$f^*(x_{[\![0,|u|+n+\theta[\![}) = f^*(ua^{\theta-1}) f^*(a^{n+\theta-1}) = \begin{cases} (u_{[\![k,|u|[\![} a^n u_{[\![0,k[\![})^q & if \quad k \leq |u|\,; \\ (a^{n+|u|-k} u a^{k-|u|})^q & if \quad k \geq |u|\,. \end{cases}$$

Firstly, if $q > 1$, then by our assumption that $\left|f^*(ua^{\theta-1})\right| \leq |u| + n$, we can derive that $f^*(a^{n+\theta-1}) = f(a^\theta)^n$ ends by a whole period of $\sigma^k(x)$, and hence is not monochromatic. Hence $q = 1$, *i.e.,*

$$\left|ua^n\right| = \left|f^*(ua^{n+\theta-1})\right| = \sum_{i=0}^{|u|+n-1} \left|f((ua^{n+\theta-1})_{[\![i,i+\theta[\![})\right|.$$

Since f is nonerasing, we get that for every $i \in [\![0, |u| + n[\![$, $\left|f((ua^{n+\theta-1})_{[\![i,i+\theta[\![})\right| = 1$, and since u contains all the words of A^θ, then for all $w \in A^\theta$, $|f(w)| = 1$, hence F is a CA.

Secondly, if $\theta < k \leq |u| < n$, then $f(a^\theta)^n = f^*(a^{n+\theta-1}) = a^{n-k} u_{[\![0,k[\![}$ starts with a but contains $u_\theta \neq a$ at position $n - k + \delta$. Similarly, if $k > |u|$, then $f(a^\theta)^n = f^*(a^{n+\theta-1})$ ends with a but contains $u_{|u|} \neq a$ at position $n - k + |u|$. In both cases, we have contradicted its monochromaticity.

We have proven that $k \leq \theta$. Since u contains all words of A^θ, we can conclude that, $f(w) = w_k$, for all $w \in A^\theta$, which is exactly the local rule of σ^k. □

Corollary 5.31. *For every dill map F, $F \in \mathrm{id}_{\hat{\mathfrak{d}}_L} \iff F \in \mathrm{id}_{\mathfrak{d}_L} \iff \exists k \in \mathbb{N}, F = \sigma^k$.*

Proof. If $F \in \mathrm{id}_{\hat{\mathfrak{d}}_L}$, then $\hat{\mathfrak{d}}_L(x, F(x)) = 0$, for all $x \in A^{\mathbb{N}}$. Hence $\mathfrak{d}_L(x, F(x)) = 0$ for all $x \in A^{\mathbb{N}}$ since $\hat{\mathfrak{d}}_L(x, F(x)) \geq \mathfrak{d}_L(x, F(x))$, and thus, $F \in \mathrm{id}_{\mathfrak{d}_L}$. According to Theorem 5.30, there exists $k \in \mathbb{N}$ such that $F = \sigma^k$. On the other hand, if $F = \sigma^k$ for some $k \in \mathbb{N}$, then for all $x \in A^{\mathbb{N}}$ we have, $\hat{\mathfrak{d}}_L(x, F(x)) = 0$ thanks to Proposition 3.26. Hence $F \in \mathrm{id}_{\hat{\mathfrak{d}}_L}$. □

Corollary 5.32. *If $k \in \mathbb{N}$, then a dill map F is in the class $\sigma^k_{\mathfrak{d}_H}$ if and only if $F = \sigma^k$.*

Proof. If $F \in \sigma^k_{\mathfrak{d}_H}$ then $F \in \mathrm{id}_{\mathfrak{d}_L}$ (thanks to Remark 3.16 and Theorem 5.30). Then, according to Corollary 5.31 there exists $k' \in \mathbb{N}$ such that $F = \sigma^{k'}$. Hence, $\sigma^{k'} \sim_{\mathfrak{d}_H} \sigma^k$. Therefore, thanks to Remark 3.31 we obtain that $k = k'$. And thus, $F = \sigma^k$. □

5.2.2 Dynamical properties

Equicontinuity We can already derive from Corollary 5.26 that all uniform substitutions yield equicontinuous dynamical systems in the Feldman-Katok space. The following theorem generalizes the result by establishing a characterization of equicontinuous substitutions.

Theorem 5.33. *Let τ be any substitution. Consider the dynamical system $\overline{\tau}$ over the Feldman-Katok space. Then:*

1. *The configurations of $(A_\tau^+)^{\mathbb{N}}$ are equicontinuity points.*
2. *If τ has spectral radius $\rho > 1$, then the configurations of $(A_\tau^-)^{\mathbb{N}}$ are not equicontinuity points.*

The following corollary can directly be derived from the theorem, by noting that A_τ^+ is never empty.

Corollary 5.34.

1. *Every substitution yields a dynamical system with at least one equicontinuity configuration.*
2. *A substitution with spectral radius $\rho > 1$ yields an equicontinuous dynamical system if and only if it is quasi-uniform.*

Proof of Theorem 5.33. Let τ be a substitution.

1. Let $x \in (A_\tau^+)^{\mathbb{N}}$, *i.e.*, there exist $\alpha > 0$ such that for every $i \in \mathbb{N}$ and sufficiently large $t \in \mathbb{N}$, $\left|\tau^t(x_i)\right| \geq \alpha \left\|\tau^t\right\|$. From Item 2 of Corollary 5.26, for every $y \in A^{\mathbb{N}}$,

$$\mathfrak{d}_L(\overline{\tau}^t(x), \overline{\tau}^t(y)) \leq \frac{\left\|\tau^t\right\|}{\alpha \left\|\tau^t\right\|} \mathfrak{d}_L(x,y) = \frac{1}{\alpha} \mathfrak{d}_L(x,y).$$

   So all iterates $\overline{\tau}^t_{\mathfrak{d}_L}$ are Lipschitz with a uniform coefficient: $\overline{\tau}_{\mathfrak{d}_L}$ is equicontinuous.

2. Consider any configuration $x \in (A_\tau^-)^{\mathbb{N}}$. Let $k \in \mathbb{N} \setminus \{0\}$, and y defined by $y_i = x_i$ for every $i \notin k\mathbb{N}$, and y_i be any letter from A_τ^+, if $i \in k\mathbb{N}$. Note that $\mathfrak{d}_L(x,y) = \mathfrak{d}_H(x,y) = \frac{1}{k}$. From Remark 2.67, if $u \in (A_\tau^-)^*$, then for every $v \in A^*$ of equal length, $d_L(u,v) \geq \left|\left\{ i \in [\![0, |v|[\![\,\middle|\, v_i \in A_\tau^+ \right\}\right|$. Remark 2.51 gives that A_τ^- is stable by τ. Hence, for every $u \in (A_\tau^-)^*$ and $v \in A^*$ such that $|u| = \left|\tau^t(v)\right|$, $d_L(u, \tau^t(v))$ is at least $\left|\tau^t(v)\right|_{A_\tau^+}$. Also, from Lemma 2.53, there exist $\alpha > 0$ (which does not depend

on k) such that for every $i \in \mathbb{N}$ and sufficently large $t \in \mathbb{N}$, $\left|\tau^t(y_{ik})\right|_{A_\tau^+} \geq \alpha \left|\tau^t(y_{ik})\right|$. So, for every $m \in \mathbb{N}$, and sufficiently large $t \in \mathbb{N}$,

$$d_L\left(\overline{\tau}^t(x)_{\left[\!\left[0,\left|\tau^t(y_{[\![0,mk[\![})\right|\right[\!\right[}, \tau^t(y_{[\![0,mk[\![})\right) \geq \left|\tau^t(y_{[\![0,mk[\![})\right|_{A_\tau^+} = \sum_{i=0}^{m-1} \left|\tau^t(y_{ik})\right|_{A_\tau^+}$$
$$\geq \alpha \sum_{i=0}^{m-1} \left|\tau^t(y_{ik})\right|.$$

On the other hand:

$$\left|\tau^t(y_{[\![0,mk[\![})\right| = \sum_{i=0}^{m-1} \left(\left|\tau^t(y_{ik})\right| + \left|\tau^t(y_{[\![ik,(i+1)k[\![})\right|\right).$$

Overall, we get:

$$\frac{d_L\left(\overline{\tau}^t(x)_{\left[\!\left[0,\left|\tau^t(y_{[\![0,km[\![})\right|\right[\!\right[}, \overline{\tau}^t(y)_{\left[\!\left[0,\left|\tau^t(y_{[\![0,km[\![})\right|\right[\!\right[}\right)}{\left|\tau^t(y_{[\![0,km[\![})\right|} \geq \frac{1}{\frac{1}{\alpha} + k \cdot \frac{o_{t\to\infty}(\|\tau^t\|)}{\Theta_{t\to\infty}(\|\tau^t\|)}}.$$

This converges to $\alpha > 0$, independently of $k \in \mathbb{N}$. Since k was taken arbitrary, y is arbitrarily close to x, so that x is not an equicontinuity point. □

Example 5.35. *The Fibonacci substitution τ is irreducible, so $\overline{\tau}$ is equicontinuous in the Feldman-Katok space, though it admits almost no equicontinuity point in the Besicovitch space (see Example 5.8).*

Example 5.36. *Let τ be a substitution defined as follows :*

$$\tau : \begin{array}{ccc} a & \mapsto & b \\ b & \mapsto & aa \end{array} \qquad M(\tau) = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \qquad M(\tau)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

Hence, $M(\tau)$ is irreducible (though not primitive: no iteration yields a positive matrix, unlike all of our irreducible examples so far). So, according to Corollary 5.34, $\overline{\tau}_{\mathfrak{d}_L}$ is equicontinuous. More precisely, τ^2 is actually the doubling substitution, proven to be 1-Lipschitz by Corollary 5.26. Besides, it can be shown that the latter is even an isometry: the longest common subword of $\tau^2(u)$ and $\tau^2(v)$ is always obtained by doubling a common subword of u and v.

Though quasi-uniform substitutions behave smoothly in our spaces, other substitutions may be more pathological.

Example 5.37. *Let τ be a substitution defined over $A = \{a, b\}^{\mathbb{N}}$ as follows:*

$$\tau : \begin{array}{ccc} a & \mapsto & a \\ b & \mapsto & bb \end{array} \qquad M(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad M(\tau)^n = \begin{bmatrix} 1 & 0 \\ 0 & 2^n \end{bmatrix}.$$

Hence there are two components, $\{a\}$ *and* $\{b\}$*, and* $A_\tau^+ = \{b\}$*. Then, according to Theorem 5.33,* b^∞ *is an equicontinuity point of* $\overline{\tau}_{\mathfrak{d}_L}$ *and* a^∞ *is not, as illustrated in Figures 5.1 and 5.2 (where b is represented in black, and a in red).*

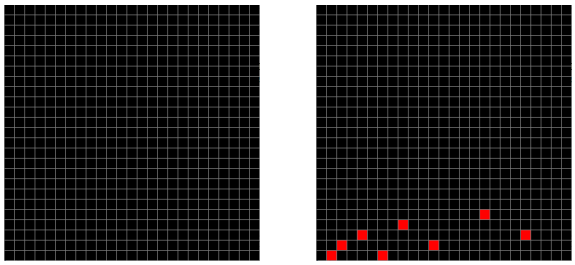


Figure 5.1: b^∞ is an equicontinuity point for $\overline{\tau}$.

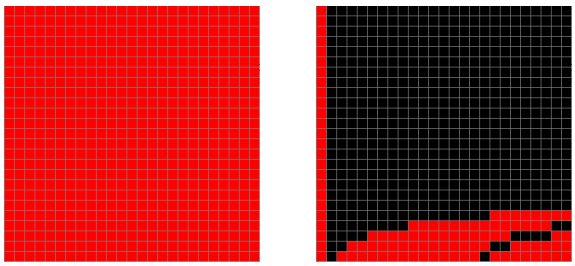


Figure 5.2: a^∞ is a non-equicontinuity point for $\overline{\tau}$.

Moreover, from the proof of Theorem 5.33, remark that if there exists a maximal terminal component, *i.e.,* some letter $a \in A_\tau^+$ such that for every $t \in \mathbb{N}$, $\tau^t(a)$ contains only letters from A_τ^+, then α can be taken equal to 1. This proves some extreme form of non-equicontinuity: whatever the precision $\frac{1}{k}$ with which one initially measures x, there is a neighboring configuration y whose orbit will be nearly maximally distant to that of x.

Example 5.38. *Consider the following substitution* τ*:*

$$\tau : \begin{array}{ccc} a & \mapsto & ab \\ b & \mapsto & b \end{array} \qquad M(\tau) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then $A_\tau^+ = \{a\}$ *and* $A_\tau^- = \{b\}$ *are both nontrivial. Yet, the system is asymptotically nilpotent: every orbit converges to the class of* b^∞ *(even in the Besicovitch space). In particular all configurations are equicontinuous, even* $b^\infty \in A_\tau^{-\mathbb{N}}$*. Here, the spectral radius is* 1 *and the growth is linear.*

We believe that a substitution yields an equicontinuous system in the Feldman-Katok space if and only if it either is quasi-uniform, or admits a unique terminal component, which is a single vertex.

Apart from the subalphabet argument from Remark 2.67, it is usually quite hard to prove lower bounds for the Feldman-Katok pseudometric. In particular, we have no example of a dill map without equicontinuity configuration.

Example 5.39. *Let F be the Xor CA. Then neither $F_{\mathfrak{d}_L}$ nor $F_{\mathfrak{d}_H}$ is equicontinuous. Indeed, let us prove that $x = a^\infty$ is a non-equicontinuity point. Let $k \in \mathbb{N}$, and $y = (a^{2^k-1}b)^\infty$. Then $\mathfrak{d}_L(x,y) \leq \mathfrak{d}_H(x,y) \leq 2^{-k}$. A classical induction on k gives that $F^{2^k-1}(y) = b^\infty$ (for more details see [Kůr03, Example 5.6]). Hence:*

$$\mathfrak{d}_H(F^{2^k-1}(x), F^{2^k-1}(y)) = \mathfrak{d}_L(F^{2^k-1}(x), F^{2^k-1}(y)) = 1.$$

Figure 5.3 illustrates that b^∞ is also a non-equicontinuity point.

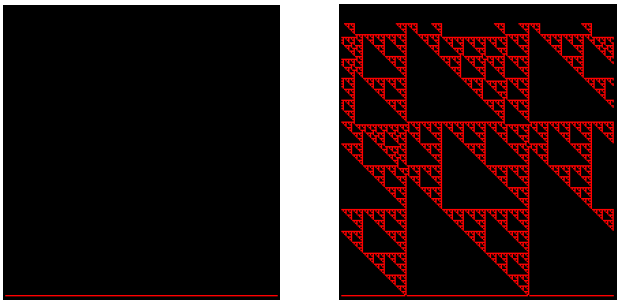


Figure 5.3: A non-equicontinuity point with respect to $\mathfrak{d}_L$ and $\mathfrak{d}_H$.

One natural question is whether, in the Feldman-Katok space, the orbit of a dill map always converges towards its (classical) limit set (the well-studied substitutive subshift in the primitive case). On the one hand, in the Besicovitch space, this is not always the case (see Example 5.18).

On the other hand, the limit set of any primitive substitution (endowed with the shift map) is a particular finite-rank system, hence, as explained in [GRK20, Section 6.1], a particular *topologically loosely Kronecker system*. From [GRK20, Theorem 1.1], this implies that it is a singleton in the Feldman-Katok space. The tools involved may be useful to understand our question.

Expansiveness It was proved in [BFK97] that there is no expansive CA over the Besicovitch space. By a similar method, we prove that there is no expansive CA over the Feldman-Katok space. Note that it does not derive directly from the corresponding result in the Besicovitch space, because some configurations from different Besicovitch classes could be in the same Feldman-Katok class, hence not concerned by the expansiveness property.

Theorem 5.40. *There is no expansive CA in the Feldman-Katok space (over a nontrivial alphabet).*

Proof. Let F be a CA over an alphabet A with diameter θ and local rule f.
Let $\varepsilon > 0$, and $p \in \mathbb{N}$ such that $\frac{1}{1+p} < \varepsilon$. Define $\alpha_i = p^{2i+1}$ and $\beta_i = p^{2i}$, for $i \in \mathbb{N}$. We let $b \in A \setminus \{a\}$ and define $x = a^{(\alpha_i)_{i\in\mathbb{N}}} b^{(\beta_i)_{i\in\mathbb{N}}}$ like in Lemma 3.5, and $y = a^\infty$. According to Remark 2.67 and Lemma 3.5, we have:

$$\mathfrak{d}_L(x,y) = \mathfrak{d}_H(x,y) = \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} p^{2i}}{\sum_{i=0}^{2n-1} p^i} = \frac{1}{1+p} < \varepsilon.$$

On the other hand, for $t > 0$, since α_i goes to infinity, there exists $j \in \mathbb{N}$ such that for all $i \geq j$ we have $\alpha_i \geq t\theta$. Furthermore, at each position $n \in \mathbb{N}$, for all i such that $x_{[\![n,n+\alpha_i[\![} = a^{\alpha_i}$, we have $F^t(x)_{[\![n,n+\alpha_i-t\theta[\![} = a_t^{\alpha_i - t\theta}$, where $a_0 = a$ and $a_{t+1} = f(a_t^\theta)$ for every $t \in \mathbb{N}$. The previous remark, applied in parallel, gives that if b_t is any letter different from a_t, $F^t(x)$ has at least as many occurrences of a_t as $a_t^{(\alpha_i - t\theta)_{i\geq j}} b_t^{(\beta_i + t\theta)_{i\geq j}}$:

$$(a_t^{(\alpha_i - t\theta)_{i\geq j}} b_t^{(\beta_i + t\theta)_{i\geq j}})_m = a_t \implies F^t(a^{(\alpha_i)_{i\geq j}} b^{(\beta_i)_{i\geq j}})_m = a_t.$$

In other words, $\mathfrak{d}_H(F^t(a^{(\alpha_i)_{i\geq j}} b^{(\beta_i)_{i\geq j}}), a_t^\infty) \leq \mathfrak{d}_H(a_t^{(\alpha_i - t\theta)_{i\geq j}} b_t^{(\beta_i + t\theta)_{i\geq j}}, a_t^\infty)$.
As a consequence:

$$\begin{aligned}
\mathfrak{d}_L(F^t(x), F^t(y)) &\leq \mathfrak{d}_H(F^t(a^{(\alpha_i)_{i\in\mathbb{N}}} b^{(\beta_i)_{i\in\mathbb{N}}}), a_t^\infty) \\
&\leq \mathfrak{d}_H(F^t(a^{(\alpha_i)_{i\geq j}} b^{(\beta_i)_{i\geq j}}), a_t^\infty) \text{ (by shift-invariance)} \\
&\leq \mathfrak{d}_H(a_t^{(\alpha_i - t\theta)_{i\geq j}} b_t^{(\beta_i + t\theta)_{i\geq j}}, a_t^\infty) \\
&\leq \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} \beta_i + t\theta}{\sum_{i=0}^{n-1} (\alpha_i + \beta_i)} \quad \text{(according to Lemma 3.5)} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} \beta_i}{\sum_{i=0}^{n-1} (\alpha_i + \beta_i)} + \frac{nt\theta}{\sum_{i=0}^{n-1} (\alpha_i + \beta_i)} \\
&\leq \limsup_{n\to\infty} \frac{\sum_{i=0}^{n-1} \beta_i}{\sum_{i=0}^{n-1} (\alpha_i + \beta_i)} \quad (\text{ Since } n = o_{n\to\infty}(\sum_{i=0}^{n-1} \alpha_i + \beta_i)) \\
&\leq \frac{1}{1+p} < \varepsilon.
\end{aligned}$$

The orbits of these two distinct configurations stay very close forever. Hence, F is not expansive. □

One natural question is whether there is an expansive cellular automata over the sliding space associated to the Levenshtein distance or not.

6 Conclusion

In this thesis, we studied CA, substitutions, and in general dill maps over non-trivial topological spaces. These spaces were constructed using two pseudo-metrics depending on two different edit distances over finite words (the Hamming distance and the Levenshtein distance).

First of all, we studied a generalization of the Besicovitch and Weyl pseudo-metrics by changing the Hamming distance by any other distance defined above the set of finite words. We gave some generic behavior by proving that the upper limit appering in the sliding pseudo-metric is always a limit; the set of configurations with maximal sliding pseudo-metric is a dense G_δ in any weakly sublinearly mixing subshift. In addition, the set where the centered pseudo-metric is maximal and is not a limit is a dense G_δ in any weakly sublinearly mixing subshift.

Then, we gave some measurably generic behavior by proving that the set of configurations such that the sliding pseudo-metric reaches the maximum over the support of a weakly mixing measure has full measure; the centered pseudo-metric is a limit almost everywhere with respect to any shift invariant measure, and it does not depend on the choice of configurations for any weakly mixing measure. In addition, we have given some topological properties which have been deduced from the properties of the Besicovitch and Weyl spaces. The Weyl space shares many properties with the Besicovitch space; one of the main differences though is that it is not complete, according to [DI88]. We may now ask whether the sliding space associated to a subadditive distance is also not complete.

Thereafter, to the best of our knowledge, we gave the first study of dill maps over the Besicovitch, Weyl and Feldman-Katok spaces. We proved that all dill maps are well-defined over the Feldman-Katok space, in contrast to the Besicovitch and the Weyl spaces where only uniform and constant dill maps are well defined.

A relevant question is now the following: which properties of distance d make dill maps well-defined in the corresponding pseudo-metric space?

Moreover, it was known since [CFMM97, BFK97] that the Besicovitch and Weyl spaces are a suitable playground to study the dynamics of cellular automata.

Here we saw that the same can be said for the Feldman-Katok space with respect to the dynamics of dill maps. In this space, the shift is equal to the identity, there are no expansive CA, every substitution admits at least one equicontinuous point.

Other dynamical properties would be interesting to study (like transitivity and sensitivity) but seem much harder to tackle because of the lack of tools to compute general lower bounds for the Feldman-Katok pseudo-metric. In this thesis we have given some results which can be useful to study these properties.

In addition, the problem of the expansiveness of CA on the Weyl (resp. the sliding Feldman-Katok) space has yet to be resolved.

Another interesting work should be to study the dynamical properties of dill maps on the centered and sliding spaces associated with some types of distances such as subadditive or infra-superadditive distances.

Finally, we can mention that generalizations exist of the Besicovitch and Weyl pseudo-metrics over groups (see for instance [LS16, CGN20]). An interesting work would be to generalize more of these pseudo-metrics to this setting. Let us replace $(\mathbb{N},+)$ by any monoid $(\mathbb{M},\cdot)$. A *pattern* with finite support $U \subset \mathbb{M}$ is some coloring $u \in A^U$. If $U = g \cdot V$, then the *translate* by $g \in \mathbb{M}$ of a pattern $u \in A^U$ is the pattern $\sigma^g(u)$ defined over support V such that $\sigma^g(u)_i = u_{i \cdot g}$. Let $\mathscr{G}$ be a (say finite) set of *right-cancelable* elements, that is, $i \cdot g = j \cdot g \implies i = j$ for every $g \in \mathscr{G}$. The *deletion* D_j^g at position $j \in \mathbb{M}$ with respect to $g \in \mathscr{G}$ is the function mapping any pattern u with support U into the pattern v defined over support

$$V = U \setminus \left\{ jg^k \middle| k \in \mathbb{N} \right\} \cup \left\{ jg^k \middle| k \in \mathbb{N}, jg^{k+1} \in U \setminus \{j\} \right\}$$

by $v_i = u_i$ if $i \in U \setminus \left\{ jg^k \middle| k \in \mathbb{N} \right\}$ and $v_{jg^k} = u_{jg^{k+1}}$ otherwise. By the cancelability property, $|V| = \left|U \setminus \{j\}\right|$, so that iterating $|U|$ deletions can exhaust the subset. Now one can consider, as a generalization of the Levenshtein distance, the following: the distance $d_M(u,v)$ between patterns $u \in A^U$ and $v \in A^V$ is the minimal half-number $\frac{m+m'}{2}$ of deletions such that $D_{j_1}^{g_1} \circ D_{j_2}^{g_2} \circ \cdots D_{j_m}^{g_m}(u)$ and $D_{j'_1}^{g'_1} \circ D_{j'_2}^{g'_2} \circ \cdots D_{j'_{m'}}^{g'_{m'}}(v)$ have a common translate, where $j_1,\dots,j_m,j'_1,\dots,j'_{m'} \in \mathbb{N}$ and $g_1,\dots,g_m,g'_1,\dots,g'_{m'} \in \mathscr{G}$. From the cardinality remark above, this distance is at most $\frac{|U|+|V|}{2}$. Note that the classical case can be recovered with $\mathbb{M} = \mathbb{N}$ and $\mathscr{G} = \{1\}$. The Feldman-Katok-like pseudo-metric over configurations from $A^{\mathbb{M}}$ endowed with a spanning sequence $(\mathscr{F}_n)_{n \in \mathbb{N}}$ of finite subsets of $\mathbb{M}$, would then be:

$$\mathfrak{d}_M = \limsup_{n \to \infty} \frac{d_M(x_{\mathscr{F}_n}, y_{\mathscr{F}_n})}{|\mathscr{F}_n|}.$$

From the previous remark, $\mathfrak{d}_M$ is between 0 and 1. To expect the pseudo-metric to enjoy nice properties (like shift-invariance, for example), one should probably assume some Følner-like condition (without any, pathologies are known to appear for the Besicovitch distance [CGN20]).

More generally, once any distance d over finite patterns with some support is fixed, a generalisation of Definition 3.10 is relevant for space $A^{\mathbb{M}}$, where $\mathbb{M}$ is any space endowed with a spanning sequence $(\mathscr{F}_n)$, and a generalisation of Definition 3.11 is also meaningfull, with the additional assumption that $\mathbb{M}$ is a monoid, so as to be able to slide the windows.

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