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Omar Anza Hafsa

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Omar Anza Hafsa. Relaxation and homogenization of unbounded integrals in calculus of variations. Mathematics [math]. Université de Nîmes, 2023. tel-04144559

HAL Id: tel-04144559

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UNIVERSITÉ DE NÎMES

HABILITATION À DIRIGER DES RECHERCHES EN MATHÉMATIQUES

**RELAXATION AND HOMOGENIZATION OF UNBOUNDED
INTEGRALS IN CALCULUS OF VARIATIONS**

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REMERCIEMENTS

Je tiens à exprimer ma profonde gratitude envers les rapporteurs qui ont consacré leur temps à lire attentivement mon manuscrit et dont les commentaires constructifs m'ont permis de voir les choses sous un angle différent, élargissant ainsi ma perspective de manière significative.

Je suis également extrêmement reconnaissant envers les membres du jury qui ont accepté de participer à ma soutenance. Leurs questions perspicaces et leurs perspectives nouvelles m'ont ouvert les yeux sur des aspects que je ne connaissais pas auparavant.

Je souhaite exprimer ma sincère gratitude envers l'Université de Nîmes et le laboratoire MIPA pour leur soutien financier et leur assistance technique tout au long de ce processus.

Enfin, je souhaite exprimer ma reconnaissance envers ma famille pour leur soutien inébranlable.

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INTRODUCTION

In this descriptive manuscript, we present our works on the calculus of variations. By this we mean the study of problems involving the minimization of integral functionals. We are interested in relaxation, homogenization and dimension reduction in frameworks going beyond polynomial growth conditions as well as in frameworks going beyond the Euclidean setting.

An integral functional of the calculus of variations can be interpreted in the context of hyperelasticity as an energy of a material occupying the closure of an open set. The relation with nonlinear elasticity comes from the fact that within the framework of hyperelasticity one can formally replace the equilibrium equations by problems of minimization of integral functionals. In hyperelasticity, there are two basic conditions on the stored energy function which are incompatible with polynomial growths: the need for an infinite energy to reduce a finite volume of a matter to a zero volume and the non-interpenetration of the matter, i.e. to prevent the determinants of the strain gradients to be nonpositive.

We have chosen to present three directions of research in which we have made contributions. We have structured this report along these three axes. We first give the essential motivations for each of these directions.

We have been interested in dimension reduction problems since our doctoral studies. One of the founding results is the Le Dret-Raoult [LR93] theorem which, under polynomial growths, shows the Γ -convergence toward a nonlinear membrane model. However the polynomial growths are not compatible with the two basic conditions of hyperelasticity. Our work in this area has focused on taking into account these conditions. To go beyond polynomial growths and allow the integrands to take the infinite value, we have introduced the notion of ample integrands which is adapted to this type of problems. These class of integrands can take the infinite value on not too large subsets of matrices, so that the quasiconvex envelope is finite. We have also studied the relaxation of integral functionals of the calculus of variations with ample integrands and we obtained, for example, a relaxation result where we could take into account the following singular behavior of the stored energy function:

$$\lim_{\det F \rightarrow 0} f(F) = \infty.$$

Motivated by similar reasons, we have studied the relaxation and the homogenization of unbounded integral functionals of the calculus of variations, in particular those having a convex effective domain (bounded and non bounded). In the scalar case a wide literature exists on relaxation and homogenization problems with constraints on the gradient, see the book [CD02] and the reference therein. However, the techniques cannot be easily generalized to tackle problems in the vectorial case. The constraints on the gradient linked to the conditions of hyperelasticity cannot be treated with methods resulting from the scalar case. It is then of interest to develop techniques in the vectorial case for the relaxation and the homogenization of integrals when the integrands can take the infinite value. The difference with the scalar case is that the central notion involved is quasiconvexity and not convexity. This leads to significant technical complications. To deal with the case where the effective domain is no

longer necessarily bounded, we have considered G -growth assumptions on the integrands, i.e. we replaced the polynomial growth conditions by growths governed by a convex function G , and then we studied the case where G is quasiconvex and verifies additional hypothesis. We have introduced the notion of radially uniformly upper semicontinuous function allowing to radially extend the integrands on the boundary of their domains. The reason is that during the relaxation (or the homogenization) process, the weak limits of the sequences of gradients may eventually concentrate on the boundary of the effective domain.

For a long time we have been interested to go beyond the Euclidean framework in the problems of minimization of integral functionals. The reason is to place ourself in an appropriate framework for the study of thin structures in hyperelasticity. Indeed, the only existing theory of elasticity is three-dimensional, so it is important to develop a mathematical framework for low-dimensional elastic structures (see [BBS97]). The setting of metric measure spaces is appropriate, since the support of a Borel measure can be interpreted as a low-dimensional hyperelastic structure with its singularities like for example thin dimensions, corners, junctions, etc. Another motivation is the development of the calculus of variations on ‘singular’ spaces, which are of interest to geometers or physicists, like Carnot groups, Gromov-Hausdorff limit spaces, spaces satisfying generalized Ricci bounds etc.

Next, we give an introduction to each chapter.

Chapter 1: Calculus of variations in Cheeger-Sobolev spaces. In this first chapter, we present our works concerning lower semicontinuity, relaxation and Γ -convergence of integral functionals defined on Cheeger-Sobolev spaces. Similarly to the Euclidean case, it is important to find the necessary conditions for the lower semicontinuity of integrals of the calculus of variations defined on Cheeger-Sobolev spaces. We introduce the $H_\mu^{1,p}$ -quasiconvexity as a necessary condition playing an analogous role to the one of $W^{1,p}$ -quasiconvexity in the Euclidean case. It required a totally different technique of proof from the one of the Euclidean case. We then state an integral representation result obtained recently for local functionals defined on Cheeger-Sobolev spaces. Here again, it was necessary to develop a technique of proof different from that usually used in the Euclidean framework. We explain this method in the section 4 of this chapter, it is based on the Vitali envelope of the local Dirichlet minimization problems associated with the functional which reduces the proof of the upper and lower bounds to cut-off techniques. We present also a relaxation result which is a consequence of our integral representation result and of the application of De Giorgi-Letta criterion [DL77]. In Section 3, we give a brief overview of our work on Γ -convergence of integrals defined on Cheeger-Sobolev spaces.

Chapter 2: Relaxation and homogenization of unbounded integrals. We present our works on the relaxation and homogenization of unbounded integrals. We begin by an exposition of the notion of radially uniformly upper semicontinuous function. Then we present relaxation and homogenization results when the effective domain is bounded convex with nonempty interior. The difficulty here is that in the vectorial case we do not have a natural way to approximate quasiconvex integrand which can take the infinite value by quasiconvex integrand having polynomial growths. Next, we present our works on the homogenization with G -growth, i.e. when the growths are governed by a convex integrand G first, and then a quasiconvex integrand verifying additional hypothesis. Here, the effective domain is not necessarily bounded. While in the case where the growth G is convex, we use the approximation by continuous piecewise affine functions of Sobolev functions to prove the upper bound, when G is quasiconvex we use techniques described in Section 4 of Chapter 1 based on the representation

of the Vitali envelope. Then we end the chapter by a brief exposition of a relaxation result in Cheeger-Sobolev spaces with G -growths.

Chapter 3: Relaxation, homogenization and dimension reduction with ample integrands. In this last chapter, we present our works on relaxation, homogenization and dimension reduction with ample integrands. We first introduce the ample integrands which appear naturally in reduction dimension problems. These class of integrands can take the infinite value on not too large subsets of matrices, so that the quasiconvex envelope is finite. We have obtained relaxation results with ample integrands which can be applied to the conditions determinant not equal to zero, i.e. $|\det F| \neq 0$ with $F \in \mathbb{M}_{33} = \mathbb{M}$, and cross-product not equal to zero, i.e. $|F_1 \wedge F_2| \neq 0$ with $F = (F_1|F_2) \in \mathbb{M}_{32}$. This last condition appearing naturally in the 3D-2D passage by Γ -convergence. We then present two results on homogenization with ample integrands. The proof of the first homogenization result essentially follows the same spirit as the one of the relaxation result. However, we need a continuity assumption on the space variable, which is unsatisfactory for homogenization problems. We present a second homogenization result in which we do not need to assume any continuity assumption with respect to space variable. The last part of this chapter is devoted to the presentation of 3D-2D passage by Γ -convergence with determinant conditions. We show how to obtain in a simple way a Γ -convergence result, under determinant condition of the type $|\det F| \neq 0$, by using the Le Dret-Raoult result. Then we show our work on the 3D-2D passage by Γ -convergence under the two basic conditions of hyperelasticity. In fact, the determinant condition of non-interpenetration of the matter $\det F > 0$ is transformed, during the passage 3D-2D, into the condition $|F_1 \wedge F_2| \neq 0$ for $F = (F_1|F_2|F_3) \in \mathbb{M}$ which is, as we mentioned above, an ‘ample condition’.

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- [AMb] O. Anza Hafsa and J.-P. Mandallena. Integral representation of unbounded variational functionals on Sobolev spaces. To appear in *Ricerche di matematica*.
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CHAPTER 1

CALCULUS OF VARIATIONS IN CHEEGER-SOBOLEV SPACES

This chapter is devoted to the presentation of the work on the extension of the calculus of variations in the setting of metric measure spaces. In Section 1 the Cheeger-Sobolev spaces are introduced and allow to define integral functionals of the calculus of variations in a similar way to the Euclidean framework. Then we present our work on L_μ^p -lower semicontinuity of integral functionals defined on Cheeger-Sobolev spaces. We introduce a necessary condition for the lower semicontinuity which will play the same role as quasiconvexity in the Euclidean setting. In Section 2 we give an exposition of our work on integral representation and relaxation of local functionals defined on Cheeger-Sobolev spaces. In Section 3 we present our works on Γ -convergence. We give, in Section 4, a description reduced to essential features of the methods for establishing lower and upper bound in relaxation and Γ -convergence problems. These methods are useful when we cannot use approximation of Sobolev functions by smoother ones, this is a priori the case in the framework of metric measure spaces.

The works involved in this chapter are [AM22; AM20a; AM20b; AM18; AM17; AM15].

1. LOWER SEMICONTINUITY AND $H_\mu^{1,p}$ -QUASICONVEXITY

Let $p > 1$ be a real number, let (X, d, μ) be a metric measure space, where μ is a nontrivial locally finite Borel regular measure on X and (X, d) is a separable metric space. In what follows, we assume that μ is doubling, i.e. there exists a constant C_d (called doubling constant) such that

$$\forall x \in X \quad \forall \rho > 0 \quad \mu(B_\rho(x)) \leq C_d \mu(B_{\frac{\rho}{2}}(x)). \quad (1.1)$$

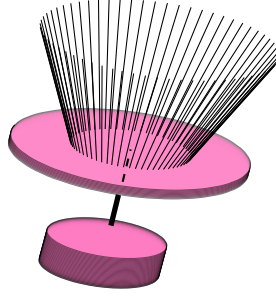
The metric measure space (X, d, μ) enjoys a $(1, p)$ -Poincaré inequality with $p \in]1, \infty[$ if there exist $C_p > 0$ and $\sigma \geq 1$ such that for every $x \in X$ and every $\rho > 0$,

$$\int_{B_\rho(x)} \left| f(y) - \int_{B_\rho(x)} f d\mu \right| d\mu(y) \leq \rho C_p \left(\int_{B_{\sigma\rho}(x)} g^p d\mu \right)^{\frac{1}{p}} \quad (1.2)$$

for every $f \in L_\mu^p(X)$ and every generalized upper gradient (p -weak upper gradient) $g \in L_\mu^p(X)$ for f (see Definition A.2).

The following result is essentially due to Cheeger [Che99, Theorem 4.38] (see also Keith [Kei04, Definition 2.1.1 and Theorem 2.3.1]) who made a major breakthrough in the analysis of metric measure space by establishing the existence of a differential structure on general enough metric measure spaces. We denote by $\text{Lip}(X)$ the algebra of Lipschitz functions from X to \mathbb{R} .

THEOREM 1.1. *If μ is doubling, i.e. (1.1) holds, and X enjoys a $(1, p)$ -Poincaré inequality, i.e. (1.2) holds, then there exist a countable family $\{(X_k, \gamma^k)\}_{k \in \mathbb{N}}$ of μ -measurable disjoint subsets X_k of X with $\mu(X \setminus \bigcup_{k \in \mathbb{N}} X_k) = 0$ and of functions $\gamma^k = (\gamma_1^k, \dots, \gamma_{N(k)}^k) : X \rightarrow \mathbb{R}^{N(k)}$ with $\gamma_i^k \in \text{Lip}(X)$ satisfying the following properties:*


 FIGURE 1.1. Illustration of an example of a metric measure space X .

- (i) there exists an integer $N \geq 1$ such that $N(k) \in \{1, \dots, N\}$ for all $k \in \mathbb{N}$;
 (ii) for every $k \in \mathbb{N}$ and every $f \in \text{Lip}(X)$ there is a unique $D_\mu^k f \in L_\mu^\infty(X_k; \mathbb{R}^{N(k)})$ such that for μ -a.e. $x \in X_k$

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho} \|f - f_x\|_{L_\mu^\infty(B_\rho(x))} = \lim_{\rho \rightarrow 0} \sup_{y \in B_\rho(x)} \frac{|f(y) - f(x) + D_\mu^k f(x) \cdot (\gamma^k(y) - \gamma^k(x))|}{\rho} = 0,$$

where $f_x \in \text{Lip}(X)$ is given by $f_x(y) := f(x) + D_\mu^k f(x) \cdot (\gamma^k(y) - \gamma^k(x))$; in particular

$$D_\mu^k f_x(y) = D_\mu^k f(x) \text{ for } \mu\text{-a.a. } y \in X_k;$$

- (iii) the operator $D_\mu : \text{Lip}(X) \rightarrow L_\mu^\infty(X; \mathbb{R}^N)$ given by

$$D_\mu f := \sum_{k \in \mathbb{N}} \mathbf{1}_{X_k} D_\mu^k f,$$

where $\mathbf{1}_{X_k}$ denotes the characteristic function of X_k , is linear and, for each $f, g \in \text{Lip}(X)$, one has

$$D_\mu(fg) = f D_\mu g + g D_\mu f;$$

We set $\mathbb{M} = \mathbb{R}^{m \times N}$ where N is given by Theorem 1.1 (i). Let $\text{Lip}(X; \mathbb{R}^m) := [\text{Lip}(X)]^m$. Let $\nabla_\mu : \text{Lip}(X; \mathbb{R}^m) \rightarrow L_\mu^\infty(X; \mathbb{M})$ given by

$$\nabla_\mu u := \begin{pmatrix} D_\mu u_1 \\ \vdots \\ D_\mu u_m \end{pmatrix} \text{ with } u = (u_1, \dots, u_m).$$

DEFINITION 1.2. The p -Cheeger–Sobolev space $H_\mu^{1,p}(X; \mathbb{R}^m)$ is defined as the completion of the space of Lipschitz functions $\text{Lip}(X; \mathbb{R}^m)$ with respect to the norm

$$\|u\|_{H_\mu^{1,p}(X; \mathbb{R}^m)} := \|u\|_{L_\mu^p(X; \mathbb{R}^m)} + \|\nabla_\mu u\|_{L_\mu^p(X; \mathbb{M})}. \quad (1.3)$$

Since $\|\nabla_\mu u\|_{L_\mu^p(X; \mathbb{M})} \leq \|u\|_{H_\mu^{1,p}(X; \mathbb{R}^m)}$ for all $u \in \text{Lip}(X; \mathbb{R}^m)$ the linear map ∇_μ from $\text{Lip}(X; \mathbb{R}^m)$ to $L_\mu^p(X; \mathbb{M})$ has a unique extension to $H_\mu^{1,p}(X; \mathbb{R}^m)$ which will still be denoted by ∇_μ .

For more details on the various possible extensions of the classical theory of the Sobolev spaces to the setting of metric measure spaces, we refer to [Hei07, pp. 10–14] and [Hei+15] (see also [Che99; Sha00; GT01; Haj03]).

This allows us to define integral functionals of the calculus of variations on Cheeger-Sobolev spaces $H_\mu^{1,p}(X; \mathbb{R}^m)$ on X by

$$H_\mu^{1,p}(X; \mathbb{R}^m) \ni u \mapsto I(u) = \int_X f(x, \nabla_\mu u(x)) d\mu(x) \quad (1.4)$$

where $f : X \times \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable integrand.

In the setting of Euclidean space $(X, d, \mu) = (\Omega, |\cdot - \cdot|, \mathcal{L}_N|_\Omega)$ where $\Omega \subset \mathbb{R}^N$ is a bounded open subset and $\mathcal{L}_N|_\Omega$ is the Lebesgue measure on Ω , a necessary condition on finite integrands for the sequential weak lower semicontinuity of (1.4) is the quasiconvexity of $f(x, \cdot)$. This condition was introduced by Morrey [Mor52] (see also [Dac08; AF84; Mar85]). Later a generalisation for not necessarily finite integrand, called $W^{1,p}$ -quasiconvexity, was developed by Ball and Murat [BM84]. An interesting problem was to find necessary conditions in the spirit of [Mor52] and [BM84] for the sequential weak lower semicontinuity of (1.4) in $H_\mu^{1,p}(X; \mathbb{R}^m)$. The main difficulty is that we do not have a natural ‘‘periodic’’ structure on X allowing to mimic the proof of [BM84] (or [Mor52]). It means that we need completely new arguments to answer this problem. Cheeger [Che99, Theorem 3.7] in his seminal paper shows that a Lipschitz function $f : X \rightarrow \mathbb{R}$, defined on a metric measure space with doubling measure and satisfying a $(1, p)$ -Poincaré inequality, is asymptotically generalized linear: for μ -a.e. $x \in X$

$$(g_f)^p(x) = \lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x))} \int_{\overline{B}_\rho(x)} (g_{f+\phi})^p(y) d\mu(y) \quad (1.5)$$

where g_f is the minimal generalized upper gradient (see Definition A.2 (ii)). Since the definition of the minimal generalized upper gradient, the functional

$$f \mapsto |g_f|_{L_\mu^p(\overline{B}_\rho(x))}^p := \int_{\overline{B}_\rho(x)} (g_f)^p(y) d\mu(y) \quad (1.6)$$

is $L_\mu^p(X)$ -lower semicontinuous, this is due to Rellich’s theorem see [Che99, Theorem 2.5]. In fact, observing the proof of Cheeger of [Che99, Theorem 3.7], we see that (1.5) is in some sense a necessary condition of the lower semicontinuity of (1.6). Now, taking inspiration from the arguments above, we introduce the following condition playing the role of $W^{1,p}$ -quasiconvexity, by saying that a Borel measurable function $f : X \times \mathbb{M} \rightarrow [0, \infty]$ is $H_\mu^{1,p}$ -quasiconvex at $\xi \in \mathbb{M}$ if for μ -a.e. $x \in X$

$$f(x, \xi) \leq \lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{\overline{B}_\rho(x)} f(y, \xi + \nabla_\mu \phi(y)) d\mu(y). \quad (1.7)$$

The formula on the right hand side of (1.7) already appears in our previous works [AM15; AM18; AM17] in relaxation and homogenization problems. We proved the following result.

THEOREM 1.3 ([AM20a, Theorem 2]). *Let $f : X \times \mathbb{M} \rightarrow [0, \infty]$ be a p -coercive Borel measurable integrand, i.e. satisfying for some $c > 0$, for μ -a.e. $x \in X$ and for every $\xi \in \mathbb{M}$*

$$f(x, \xi) \geq c|\xi|^p. \quad (1.8)$$

Assume that for every $u, \{u_\varepsilon\}_{\varepsilon>0} \subset H_\mu^{1,p}(X; \mathbb{R}^m)$ satisfying $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{L_\mu^p(X; \mathbb{R}^m)} = 0$, it holds

$$\lim_{\varepsilon \rightarrow 0} \int_X f(x, \nabla_\mu u_\varepsilon) d\mu \geq \int_X f(x, \nabla_\mu u) d\mu. \quad (1.9)$$

Then f is $H_\mu^{1,p}$ -quasiconvex at every $\xi \in \mathbb{M}$ satisfying $f(\cdot, \xi) \in L_\mu^1(X)$.

The proof uses a Vitali covering of the set where (1.7) does not hold. Then using the p -coercivity we can construct a sequence of $H_\mu^{1,p}(X; \mathbb{R}^m)$ which strongly converges to 0 in $L_\mu^p(X; \mathbb{R}^m)$. The last step consists in using the lower semicontinuity of I and the finiteness condition $f(\cdot, \xi) \in L_\mu^1(X)$ to conclude that set is necessarily of zero measure.

PERSPECTIVE 1. It is of interest to search the general conditions on X allowing to remove the coercivity condition (1.8) in Theorem 1.3. More precisely, to find the general conditions on X in order to have: if for every $u, \{u_\varepsilon\}_{\varepsilon>0} \subset H_\mu^{1,p}(X; \mathbb{R}^m)$ satisfying $u_\varepsilon \rightharpoonup u$ weakly in $H_\mu^{1,p}(X; \mathbb{R}^m)$, it holds

$$\liminf_{\varepsilon \rightarrow 0} \int_X f(x, \nabla_\mu u_\varepsilon) d\mu \geq \int_X f(x, \nabla_\mu u) d\mu, \quad (1.10)$$

then f is $H_\mu^{1,p}$ -quasiconvex at every $\xi \in \mathbb{M}$.

REMARKS 1.4. A Borel measurable function $f : X \times \mathbb{M} \rightarrow [0, \infty]$ is $H_\mu^{1,p}$ -quasiconvex at $\xi \in \mathbb{M}$ satisfying $f(\cdot, \xi) \in L_{\mu, \text{loc}}^1(X)$ if and only if for μ -a.e. $x \in X$

$$f(x, \xi) = \lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{\overline{B}_\rho(x)} f(y, \xi + \nabla_\mu \phi) d\mu,$$

i.e. we can replace ‘ $\lim_{\rho \rightarrow 0}$ ’ by ‘ $\lim_{\rho \rightarrow 0}$ ’.

A stronger necessary condition for the lower semicontinuity is given by:

PROPOSITION 1.5 ([AM20a, Corollary 1]). *Let $f : X \times \mathbb{M} \rightarrow [0, \infty]$ be a p -coercive Borel measurable integrand. Assume that for every $u, \{u_\varepsilon\}_{\varepsilon>0} \subset H_\mu^{1,p}(X; \mathbb{R}^m)$ satisfying $\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - u\|_{L_\mu^p(X; \mathbb{R}^m)} = 0$, it holds*

$$\liminf_{\varepsilon \rightarrow 0} \int_O f(x, \nabla_\mu u_\varepsilon) d\mu \geq \int_O f(x, \nabla_\mu u) d\mu \quad (1.11)$$

for all open set $O \in \mathcal{O}(X)$ satisfying $\mu(O) < \infty$.

Let $\bar{u} \in H_\mu^{1,p}(X; \mathbb{R}^m)$ be such that¹ $f(\cdot, \nabla_\mu \bar{u}(\cdot)) \in L_{\mu, \text{loc}}^1(X)$. Then for μ -a.e. $x \in X$

$$f(x, \nabla_\mu \bar{u}(x)) = \lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{\overline{B}_\rho(x)} f(y, \nabla_\mu \bar{u} + \nabla_\mu \phi) d\mu. \quad (1.12)$$

To prove that $H_\mu^{1,p}$ -quasiconvexity is sufficient when we assume p -growth on the integrand, we need further assumptions on the metric space X : we say that the metric measure space (X, d, μ) satisfies the annular decay property (which was introduced independently by [Buc99, p. 521 and §2 p. 524] and [CM98]) if

(AD $_X$) there exist $\eta > 0$ and $K \geq 1$ such that for every $x \in X$, every $\rho > 0$ and every $\tau \in]0, 1[$,

$$\mu(B_\rho(x) \setminus B_{\tau\rho}(x)) \leq K(1 - \tau)^\eta \mu(B_\rho(x)).$$

We need also the following property of Alexandrov type (or of Portmanteau type):

(AL $_X$) for every open set $O \in \mathcal{O}(X)$ and for every sequence $\{m_n\}_{n \in \mathbb{N}}$ of nonnegative Borel regular measures on O satisfying $\sup_{n \in \mathbb{N}} m_n(O) < \infty$ there exist a subsequence (not

¹By $L_{\mu, \text{loc}}^1(X)$ we denote the vector space of all measurable functions $u : X \rightarrow \overline{\mathbb{R}}$ such that $\int_B |u| d\mu < \infty$ for all open ball $B \subset X$ satisfying $\mu(B) < \infty$.

relabelled) $\{m_n\}_{n \in \mathbb{N}}$ and m a locally finite Borel regular measure on O such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} m_n(V) &\geq m(V) \quad \text{for all open set } V \subset O. \\ \overline{\lim}_{n \rightarrow \infty} m_n(B) &\leq m(B) \quad \text{for all closed ball } B \subset O. \end{aligned}$$

REMARKS 1.6. The annular decay property holds, for instance, when the metric space is a length space, i.e. metric space in which the distance between points is the infimum of lengths of rectifiable paths joining those points, see [Buc99, Corollary 2.2], [CM98, Lemma 3.3], [Che99, Proposition 6.12] and [Hei+15, Proposition 11.5.3, p. 328]). We remark that, when the annular decay property holds, the boundary of balls is of zero measure. If X is compact or locally compact then the property (AL_X) holds.

We proved the following characterization of the lower semicontinuity of (1.4):

THEOREM 1.7 ([AM20a, Theorem 6]). *Assume that (X, d, μ) satisfies (AD_X) , (AL_X) and μ is finite. Let $f : X \times \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable integrand such that for μ -a.e. $x \in X$ the function $f(x, \cdot)$ is lower semicontinuous. Assume that there exist $c, C > 0$ and $A \in L^1_\mu(X)$ such that for μ -a.e. $x \in X$*

$$c|\xi|^p \leq f(x, \xi) \leq A(x) + C(1 + |\xi|^p) \quad \text{for all } \xi \in \mathbb{M}.$$

Then f is $H_\mu^{1,p}$ -quasiconvex if and only if $I(\cdot)$ is lower semicontinuous with respect to the strong convergence of $L^p_\mu(X; \mathbb{R}^m)$.

PERSPECTIVE 2. The property (AL_X) is a nice tool when we have to show a lower bound in lower semicontinuity, relaxation or generally in Γ -convergence problems. In fact, if for instance we assume that X is complete, then μ is a Radon measure (see Remark 1.8 below) and since the measures involved in the proof of Theorem 1.7 are of the type:

$$\nu_n := f(\cdot, \nabla_\mu u_n(\cdot))\mu|_O \quad \text{and} \quad \lambda_n := |\nabla_\mu u_n|^p \mu|_O$$

these are also Radon ([Hei+15, Remark 3.3.43, p. 67]), so we can use (AL_X) with ν_n and λ_n in the proof. In fact, for the proof of Theorem 1.7 we need a weaker version of (AL_X) :

(AL'_X) for every open set $O \in \mathcal{O}(X)$ and for every sequence $\{m_n\}_{n \in \mathbb{N}}$ of nonnegative Borel regular measures on O satisfying $\sup_{n \in \mathbb{N}} m_n(O) < \infty$ and $m_n \ll \mu$ for all $n \in \mathbb{N}$ there exist a subsequence (not relabelled) $\{m_n\}_{n \in \mathbb{N}}$ and m a locally finite Borel regular measure on O such that

$$\begin{aligned} \liminf_{n \rightarrow \infty} m_n(V) &\geq m(V) \quad \text{for all open set } V \subset O. \\ \overline{\lim}_{n \rightarrow \infty} m_n(B) &\leq m(B) \quad \text{for all closed ball } B \subset O. \end{aligned}$$

It is interesting to find a characterization of metric spaces such that (AL'_X) holds.

2. INTEGRAL REPRESENTATION AND RELAXATION

The integral representation of functionals is an important subject in the calculus of variations. We became interested in extending to metric measure spaces the classical integral representation theorems [BD85; Bou+02]. New arguments are needed, since the proof of [BD85] rests upon existence and approximation properties of continuous piecewise affine functions. The integral representation result of [Bou+02, Theorem 2, p. 189] shows that the integrand can be written as limit, when the radius of balls goes to zero, of the average of minimization Dirichlet problems associated with the functional on small balls. The strategy of the proof, known as the ‘global relaxation method’, uses mainly an intermediate representation result of an envelope, similar

to the Carathéodory construction in measure theory, of local minimization Dirichlet problems associated with the functional, see Section 4 for an exposition of the method. One of the advantages of this method is to avoid the use of approximation by continuous piecewise affine functions. It can therefore be adapted more easily to the framework of Cheeger-Sobolev spaces (we have used this technique in [AM15; AM17; AM18]).

We denote by $\Omega \subset X$ an open set with finite measure $\mu(\Omega) < \infty$ and by $\mathcal{O}(\Omega)$ the class of all open subsets of Ω . We assume that μ is doubling, (X, d, μ) enjoys a $(1, p)$ -Poincaré inequality, (X, d) is a complete separable metric space, and (X, d, μ) satisfies the annular decay property (AD_X) .

REMARKS 1.8. Note that μ is then a Radon measure since X is complete, see [Hei+15, Proposition 3.3.44, p. 67]. Remark also that X is locally compact since a complete metric space with μ doubling is proper, i.e. every closed ball is compact.

We proved the following integral representation theorem in Cheeger-Sobolev spaces:

THEOREM 1.9 ([AM22, Theorem 1]). *Let $F : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ satisfy*

- (H₁) *for every $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$ the set function $F(u, \cdot)$ is the restriction to $\mathcal{O}(\Omega)$ of a positive Radon measure;*
- (H₂) *$F(\cdot, O)$ is local, i.e. $F(u, O) = F(v, O)$ whenever $u = v$ μ -a.e. in O for all $(u, v) \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)^2$ and all $O \in \mathcal{O}(\Omega)$;*
- (H₃) *$F(u + z, O) = F(u, O)$ for all $z \in \mathbb{R}^m$, all $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$ and all $O \in \mathcal{O}(\Omega)$;*
- (H₄) *there exist $c > 0$, $b \geq 0$ and $a \in L_\mu^1(\Omega)$ such that for every $(u, O) \in H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$*

$$c \int_O |\nabla_\mu u(x)|^p d\mu(x) \leq F(u, O) \leq \int_O a(x) + b |\nabla_\mu u(x)|^p d\mu(x)$$

where $\nabla_\mu u$ is the μ -gradient of u .

- (H₅) *for every $O \in \mathcal{O}(\Omega)$ the functional $F(\cdot, O)$ is L_μ^p -lower semicontinuous, i.e. for every $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$ and $\{u_n\}_{n \in \mathbb{N}} \subset H_\mu^{1,p}(\Omega; \mathbb{R}^m)$ satisfying $\lim_{n \rightarrow \infty} \|u_n - u\|_{L_\mu^p(\Omega; \mathbb{R}^m)} = 0$ we have*

$$\varliminf_{n \rightarrow \infty} F(u_n, O) \geq F(u, O).$$

Then there exists a Borel measurable function $f : \Omega \times \mathbb{M} \rightarrow [0, \infty]$, with $\mathbb{M} = \mathbb{R}^{m \times N}$ where N is given by Theorem 1.1 (i), such that

- (i) *for every $O \in \mathcal{O}(\Omega)$ and every $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$*

$$F(u, O) = \int_O f(x, \nabla_\mu u(x)) d\mu(x);$$

- (ii) *for every $k \in \mathbb{N}$, for μ -a.e. $x \in \Omega \cap \Omega_k$ and every $\xi \in \mathbb{M}$*

$$f(x, \xi) := \varliminf_{\rho \rightarrow 0} \inf_{\varphi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \frac{F(\xi \cdot \gamma^k(\cdot) + \varphi, B_\rho(x))}{\mu(B_\rho(x))};$$

- (iii) *the function f is $H_\mu^{1,p}$ -quasiconvex, i.e. for every $\xi \in \mathbb{M}$ and for μ -a.e. $x \in \Omega$*

$$f(x, \xi) = \lim_{\rho \rightarrow 0} \inf_{\varphi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{B_\rho(x)} f(y, \xi + \nabla_\mu \varphi(y)) d\mu(y);$$

(iv) for μ -a.e. $x \in \Omega$ and for every $\xi \in \mathbb{M}$ we have

$$c|\xi|^p \leq f(x, \xi) \leq a(x) + b|\xi|^p \quad (1.13)$$

where $c > 0$, $b \geq 0$ and $a \in L^1_\mu(\Omega)$ are given by (H₄);

(v) if there exists a Borel measurable function $\tilde{f} : \Omega \times \mathbb{M} \rightarrow [0, \infty]$ such that for every $O \in \mathcal{O}(\Omega)$ and every $u \in H^{1,p}_\mu(\Omega; \mathbb{R}^m)$

$$F(u, O) = \int_O \tilde{f}(x, \nabla_\mu u(x)) d\mu(x)$$

then for μ -a.e. $x \in \Omega$ and for every $\xi \in \mathbb{M}$

$$\tilde{f}(x, \xi) = f(x, \xi).$$

PERSPECTIVE 3. It would be interesting to search the general conditions on X such that the conclusions of Theorem 1.9 still hold when we replace condition (H₄) by:

(H'₄) there exist $b \geq 0$ and $a \in L^1_\mu(\Omega)$ such that for every $(u, O) \in H^{1,p}_\mu(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega)$

$$0 \leq F(u, O) \leq \int_O a(x) + b|\nabla_\mu u(x)|^p d\mu(x)$$

and condition (H₅) by:

(H'₅) for every $O \in \mathcal{O}(\Omega)$ the functional $F(\cdot, O)$ is sequentially weakly lower semicontinuous on $H^{1,p}_\mu(\Omega; \mathbb{R}^m)$, i.e. for every $u \in H^{1,p}_\mu(\Omega; \mathbb{R}^m)$ and $\{u_n\}_{n \in \mathbb{N}} \subset H^{1,p}_\mu(\Omega; \mathbb{R}^m)$ satisfying $u_n \rightarrow u$ in $H^{1,p}_\mu(\Omega; \mathbb{R}^m)$ we have

$$\varliminf_{n \rightarrow \infty} F(u_n, O) \geq F(u, O).$$

When the functional is not necessarily L^p_μ -lower semicontinuous we need to consider its envelope defined by

$$H^{1,p}_\mu(\Omega; \mathbb{R}^m) \ni u \longmapsto \bar{F}(u, O) := \inf \left\{ \varliminf_{n \rightarrow \infty} F(u_n, O) : u_n \rightarrow u \text{ in } L^p_\mu(\Omega; \mathbb{R}^m) \right\}.$$

We have the following relaxation theorem:

THEOREM 1.10 ([AM22, Theorem 2]). *Let $F : H^{1,p}_\mu(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ satisfying (H₁)-(H₄). Then there exists a Borel measurable function $\bar{f} : \Omega \times \mathbb{M} \rightarrow [0, \infty]$ such that*

(i) for every $O \in \mathcal{O}(\Omega)$ and every $u \in H^{1,p}_\mu(\Omega; \mathbb{R}^m)$

$$\bar{F}(u, O) = \int_O \bar{f}(x, \nabla_\mu u(x)) d\mu(x);$$

(ii) for every $k \in \mathbb{N}$, for μ -a.e. $x \in \Omega \cap \Omega_k$ and for every $\xi \in \mathbb{M}$

$$\bar{f}(x, \xi) := \overline{\lim}_{\rho \rightarrow 0} \inf_{\varphi \in H^{1,p}_{\mu,0}(B_\rho(x); \mathbb{R}^m)} \frac{F(\xi \cdot \gamma^k(\cdot) + \varphi, B_\rho(x))}{\mu(B_\rho(x))};$$

(iii) the function \bar{f} is $H^{1,p}_\mu$ -quasiconvex;

(iv) for μ -a.e. $x \in \Omega$ and for every $\xi \in \mathbb{M}$ we have

$$c|\xi|^p \leq \bar{f}(x, \xi) \leq a(x) + b|\xi|^p$$

where $c > 0$, $b \geq 0$ and $a \in L^1_\mu(\Omega)$ are given by (H₄).

Theorem 1.10 is an extension of the relaxation result obtained in [AM15, Corollary 2.29, p. 79], where we additionally assumed that X is a compact length space. It is also an extension of [AM17, Corollary 2.4, p. 384], where we additionally assumed that X is a locally compact length space. The proof of Theorem 1.10 is more classical and uses the De Giorgi-Letta criterion [DL77] (see also [But89, Lemma 3.3.6 p. 105]) which establishes sufficient conditions for an increasing set function defined on open subsets to be the restriction of a regular Borel measure.

PERSPECTIVE 4. It would be interesting to search general conditions on X such that the conclusions of Theorem 1.10 still hold when we replace condition (H_4) by (H'_4) .

REMARKS 1.11. Let $f : \Omega \times \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function which satisfies the polynomial growths (1.13). Assume that I given by (1.4) is L_μ^p -lower semicontinuous. Then by using Proposition 1.5, we have for every $\bar{u} \in \text{Lip}_0(\Omega; \mathbb{R}^m)$ and for μ -a.e. $x \in X$

$$f(x, \nabla_\mu \bar{u}(x)) = \lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{\overline{B_\rho(x)}} f(y, \nabla_\mu \bar{u} + \nabla_\mu \phi) d\mu.$$

In fact, by using the proof of Theorem 1.9 we can see that μ -a.e. $x \in X$

$$\lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{\overline{B_\rho(x)}} f(y, \nabla_\mu \bar{u} + \nabla_\mu \phi) d\mu = \tilde{f}(x, \nabla_\mu \bar{u}(x))$$

and that the function $\tilde{f}(x, \xi) := \lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{\overline{B_\rho(x)}} f(y, \xi + \nabla_\mu \phi) d\mu$ is Borel measurable. So, we can use [AM22, Corollary 2] which is a version of an Alberti result [Alb91] on the uniqueness of Borel integrands, to have the following condition: for μ -a.e. $x \in X$ and for all $\xi \in \mathbb{M}$

$$f(x, \xi) = \tilde{f}(x, \xi)$$

which is a necessary condition stronger than $H_\mu^{1,p}$ -quasiconvexity in this setting. It follows from Theorem 1.7 that the $H_\mu^{1,p}$ -quasiconvexity implies the lower semicontinuity, so the $H_\mu^{1,p}$ -quasiconvexity is equivalent in this context to the condition: for μ -a.e. $x \in X$ and for all $\xi \in \mathbb{M}$

$$f(x, \xi) = \lim_{\rho \rightarrow 0} \inf_{\phi \in H_{\mu,0}^{1,p}(B_\rho(x); \mathbb{R}^m)} \int_{\overline{B_\rho(x)}} f(y, \xi + \nabla_\mu \phi) d\mu.$$

3. Γ -CONVERGENCE

Let $\Omega \subset X$ be a nonempty open set. For each $\varepsilon > 0$, we consider $E_\varepsilon : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ defined by

$$E_\varepsilon(u, O) := \int_O f_\varepsilon(x, \nabla_\mu u(x)) d\mu(x), \quad (1.14)$$

where $f_\varepsilon : \Omega \times \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable integrand depending on $\varepsilon > 0$, not necessarily convex with respect to $\xi \in \mathbb{M}$, and having p -growth, i.e. there exist $c, C > 0$, which do not depend on ε , such that

$$c|\xi|^p \leq f_\varepsilon(x, \xi) \leq C(1 + |\xi|^p) \quad (1.15)$$

for all $x \in \Omega$ and all $\xi \in \mathbb{M}$. We deal with the problem of computing the Γ -convergence with respect to the strong convergence of $L_\mu^p(\Omega; \mathbb{R}^m)$ of $\{E_\varepsilon\}_{\varepsilon > 0}$ as $\varepsilon \rightarrow 0$ toward $E : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$ of the type

$$E(u, O) = \int_O f(x, \nabla_\mu u(x)) d\mu(x) \quad (1.16)$$

with $f : \Omega \times \mathbb{M} \rightarrow [0, \infty]$ not depending on ε . For each $\varepsilon > 0$ and each $\rho > 0$, let $f_{\varepsilon, \rho} : \Omega \times \mathbb{M} \rightarrow [0, \infty]$ be given by

$$f_{\varepsilon, \rho}(x, \xi) := \inf \left\{ \int_{B_\rho(x)} f_\varepsilon(y, \xi + \nabla_\mu w(y)) d\mu(y) : w \in H_{\mu, 0}^{1,p}(B_\rho(x); \mathbb{R}^m) \right\}.$$

We assume here that μ is doubling, (X, d, μ) enjoys a $(1, p)$ -Poincaré inequality, (X, d) is a complete separable metric space, and (X, d, μ) satisfies the annular decay property (AD $_X$). We proved the following result:

THEOREM 1.12 ([AM17, Theorem 2.2 and Remark 4.2]). *If (1.15) holds then for every $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$, the set function*

$$\Gamma(L_\mu^p)\text{-}\lim_{n \rightarrow \infty} E_\varepsilon(u, \cdot) : \mathcal{O}(\Omega) \rightarrow [0, \infty]$$

can be uniquely extended to a finite positive Radon measure on \mathcal{O} which is absolutely continuous with respect to μ , and

$$\Gamma(L_\mu^p)\text{-}\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u, O) \geq \int_O \overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} f_{\varepsilon, \rho}(x, \nabla_\mu u(x)) d\mu(x)$$

and

$$\Gamma(L_\mu^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} E_\varepsilon(u, O) = \int_O \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} f_{\varepsilon, \rho}(x, \nabla_\mu u(x)) d\mu(x)$$

for all $O \in \mathcal{O}(\Omega)$.

Theorem 1.12 can be seen as a partial Γ -convergence result. The method of the proof follows the lines described in Section 4 below. What we can see is that if for μ -a.e. $x \in \Omega$ and for all $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$ it holds

$$\overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} f_{\varepsilon, \rho}(x, \nabla_\mu u(x)) = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} f_{\varepsilon, \rho}(x, \nabla_\mu u(x)) \quad (1.17)$$

then we have Γ -convergence and

$$\Gamma(L_\mu^p)\text{-}\lim_{\varepsilon \rightarrow 0} E_\varepsilon(u, O) = \int_O \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} f_{\varepsilon, \rho}(x, \nabla_\mu u(x)) d\mu(x).$$

To go further, we developed in [AM20b] and [AM17] a framework by mimicking what happens in Euclidean spaces for the periodic (and the stochastic) homogenization. First, we assume the existence of a sequence of homeomorphism $\{h_n\}_{n \in \mathbb{N}}$ on X which plays the role of ‘dilations’ $x \mapsto \frac{x}{\varepsilon_n}$ with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then, we set $f_n(x, \xi) = f(h_n(x), \xi)$ for all $(x, \xi) \in X \times \mathbb{M}$. We also assume the existence of a subgroup \mathbb{G} of the homeomorphisms of X which is acting on X and is playing the role of \mathbb{Z}^N acting on \mathbb{R}^N . The measure μ is assumed to be \mathbb{G} -invariant, i.e. $\mu(g(A)) = \mu(A)$ for all $(g, A) \in \mathbb{G} \times \mathcal{B}(X)$ where $\mathcal{B}(X)$ is the Borel σ -algebra of (X, d) . We assume that there exists $\mathbb{U} \in \mathcal{B}(X)$ of positive measure such that $\mu(\partial \mathbb{U}) = 0$ which plays the role of ‘unit cell’ in Euclidean spaces. We also make a meshability assumption with respect to the Borel sets $\{h_k(\mathbb{U})\}_{k \in \mathbb{N}^*}$, roughly we assume that each $h_k(\mathbb{U})$ can be approximated, from below and above, by finite unions of $g \circ h_k(\mathbb{U})$ with $g \in \mathbb{G}$. Now, for each $\xi \in \mathbb{M}$ we consider the subadditive (i.e. for every $A, B \in \mathcal{B}(X)$ it holds $\mathcal{S}^\xi(A \cup B) \leq \mathcal{S}^\xi(A) + \mathcal{S}^\xi(B)$ whenever $A \cap B = \emptyset$) function $\mathcal{S}^\xi : \mathcal{B}(X) \rightarrow [0, \infty]$ defined by

$$\mathcal{S}^\xi(A) := \inf \left\{ \int_A f(y, \xi + \nabla_\mu w(y)) d\mu(y) : w \in H_{\mu, 0}^{1,p}(\overset{\circ}{A}; \mathbb{R}^m) \right\},$$

where $\overset{\circ}{A}$ denotes the interior of A . We proved that for every $\rho > 0$

$$\overline{\lim}_{n \rightarrow \infty} f_{n,\rho}(x, \xi) = \lim_{n \rightarrow \infty} \frac{\mathcal{S}^\xi(h_n(B_\rho(x)))}{\mu(h_n(B_\rho(x)))} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}^\xi(h_k(\mathbb{U}))}{\mu(h_k(\mathbb{U}))} = f_{hom}(\xi).$$

We then see that (1.17) holds, i.e. for μ -a.e. $x \in \Omega$ and for all $u \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$

$$\overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{n \rightarrow \infty} f_{n,\rho}(x, \nabla_\mu u(x)) = \lim_{\rho \rightarrow 0} \overline{\lim}_{n \rightarrow \infty} f_{n,\rho}(x, \nabla_\mu u(x)) = f_{hom}(\nabla_\mu u(x)).$$

PERSPECTIVE 5. Homogenization in Cheeger-Sobolev spaces is at the beginning of its development. The abstract framework for the deterministic and stochastic homogenization developed in [AM20b] and [AM17] is a first attempt in this direction. It will be interesting to find new methods for homogenization adapted to metric measure spaces.

4. ABOUT THE TECHNIQUES TO PROVE Γ -CONVERGENCE RESULTS

In this section, we give a quick explanation of the main techniques we used to prove relaxation and Γ -convergence results.

4.1. GENERAL SCHEME FOR ESTABLISHING UPPER BOUND. This part is devoted to present the general strategy for establishing the upper bound for the problems of relaxation, homogenization, and more generally of Γ -convergence of family of (variational) functionals $\{F_\varepsilon\}_{\varepsilon > 0}$, $F_\varepsilon : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$. The method which was first introduced by [BFM98] in the Euclidean setting and named ‘Global method of relaxation’, allows to avoid any approximation arguments when we want to establish an upper bound under integral form:

$$\Gamma(L_\mu^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u, O) \leq \int_O g(x, \nabla_\mu u(x)) d\mu(x).$$

We consider the set function $G : H_\mu^{1,p}(\Omega; \mathbb{R}^m) \times \mathcal{O}(\Omega) \rightarrow [0, \infty]$

$$G(u, O) := \overline{\lim}_{\varepsilon \rightarrow 0} m_\varepsilon(u, O) \quad \text{where} \quad m_\varepsilon(u, O) := \inf_{\varphi \in H_{\mu,0}^{1,p}(O; \mathbb{R}^m)} F_\varepsilon(u + \varphi, O). \quad (1.18)$$

We cannot expect that $G(u, \cdot)$ is a measure even if $F_\varepsilon(u, \cdot)$ is the restriction of a positive Radon measure, it is the reason that the Vitali envelopes (1.19) and (1.20) of $G(u, \cdot)$ have been introduced. In the following, we present the integral representation of the Vitali envelope of a set function defined on open subsets of Ω , it is partly inspired by [BB00; BFM98; DM86]. For each open set $O \subset \Omega$, we denote by $\mathcal{Bal}(O) \subset \mathcal{O}(O)$ the class of all open balls B of O . Let $G : \mathcal{Bal}(\Omega) \rightarrow \mathbb{R}$ be a set function. We define the lower Vitali envelope of G with respect to μ

$$\mathcal{O}(\Omega) \ni O \mapsto G_-^*(O) := \sup_{\varepsilon > 0} \inf \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\} \quad (1.19)$$

and the upper Vitali envelope with respect to μ

$$\mathcal{O}(\Omega) \ni O \mapsto G_+^*(O) := \inf_{\varepsilon > 0} \sup \left\{ \sum_{i \in I} G(B_i) : \{B_i\}_{i \in I} \in \mathcal{V}^\varepsilon(O) \right\}, \quad (1.20)$$

where for every $\varepsilon > 0$

$$\mathcal{V}^\varepsilon(O) := \left\{ \{B_i\}_{i \in I} \subset \mathcal{Bal}(O) : I \text{ is countable, } \mu \left(O \setminus \bigcup_{i \in I} B_i \right) = 0, \overline{B_i} \subset O, \right. \\ \left. 0 < \text{diam}(B_i) \leq \varepsilon \text{ and } \overline{B_i} \cap \overline{B_j} = \emptyset \text{ for all } i \neq j \right\}.$$

REMARKS 1.13. If G is the trace on $\mathcal{B}al(\Omega)$ of a positive Borel measure λ on Ω which is absolutely continuous with respect to μ then $G_{\pm}^*(O) = \lambda(O)$ for all $O \in \mathcal{O}(\Omega)$.

Let $G : \mathcal{B}al(\Omega) \rightarrow \mathbb{R}$ be a set function. Define the lower and the upper derivatives at $x \in \Omega$ of G with respect to μ as follows

$$D_{\mu}^{-}G(x) := \liminf_{\rho \rightarrow 0} \left\{ \frac{G(B)}{\mu(B)} : x \in B \in \mathcal{B}al(\Omega), 0 < \text{diam}(B) \leq \rho \right\};$$

$$D_{\mu}^{+}G(x) := \limsup_{\rho \rightarrow 0} \left\{ \frac{G(B)}{\mu(B)} : x \in B \in \mathcal{B}al(\Omega), 0 < \text{diam}(B) \leq \rho \right\}.$$

We say that G is μ -differentiable in $O \in \mathcal{O}(\Omega)$ if for μ -a.e. $x \in O$ it holds

$$-\infty < D_{\mu}^{-}G(x) = D_{\mu}^{+}G(x) < \infty.$$

Under domination and subadditivity conditions (see conditions (i) and (ii) below), a set function defined on open sets is μ -differentiable, and the lower and upper Vitali envelopes are equal and admit an integral representation with density its derivative.

THEOREM 1.14 ([AM18, Theorem 3.17, p. 65]). *Let $G : \mathcal{O}(\Omega) \rightarrow \mathbb{R}$ satisfy:*

- (i) *there exists a positive Radon measure α on Ω absolutely continuous with respect to μ , i.e. $\alpha \ll \mu$, satisfying*

$$|G(O)| \leq \alpha(O) \quad \text{for all } O \in \mathcal{O}(\Omega);$$

- (ii) *for every $U, V, O \in \mathcal{O}(\Omega)$ with $U \cap V = \emptyset$, $U \subset O$, $V \subset O$ and $\mu(O \setminus (U \cup V)) = 0$ it holds*

$$G(O) \leq G(U) + G(V).$$

Then G is μ -differentiable with $\Omega \ni x \mapsto \lim_{\rho \rightarrow 0} \frac{G(B_{\rho}(x))}{\mu(B_{\rho}(x))} \in L_{\mu}^1(\Omega)$ and for every $O \in \mathcal{O}(\Omega)$

$$G_{+}^*(O) = G_{-}^*(O) = \int_O \lim_{\rho \rightarrow 0} \frac{G(B_{\rho}(x))}{\mu(B_{\rho}(x))} d\mu(x). \quad (1.21)$$

Now, in practical case (1.18), each functional $F_{\varepsilon}(u, \cdot)$ is the restriction to open sets of a positive Radon measure. Often, it is an integral $F_{\varepsilon}(u, O) = \int_O f_{\varepsilon}(x, \nabla_{\mu} u(x)) d\mu(x)$, and satisfies $F_{\varepsilon}(u, \cdot) \leq \alpha_u(\cdot)$ where $\alpha_u \ll \mu$ is a positive Radon measure, for instance p -growth i.e. $\alpha_u = (a(\cdot) + b|\nabla_{\mu} u(\cdot)|^p)\mu$ (see [AM15; AM17; AM22]) or G -growth (see [AM18]) i.e. $\alpha_u = \beta(1 + G(\cdot, \nabla_{\mu} u(\cdot)))\mu$. Then by applying Theorem 1.14 we show that for every $O \in \mathcal{O}(\Omega)$ and every $u \in H_{\mu}^{1,p}(\Omega; \mathbb{R}^m)$ we have

$$G_{-}^*(u, O) = \int_O \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_{\varepsilon}(u, B_{\rho}(x))}{\mu(B_{\rho}(x))} d\mu(x).$$

The strategy to obtain the upper bound is then to show, by using p -coercivity and Poincaré inequality, that

$$\int_O \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_{\varepsilon}(u, B_{\rho}(x))}{\mu(B_{\rho}(x))} d\mu(x) = G_{-}^*(u, O) \geq \Gamma(L_{\mu}^p) \cdot \overline{\lim}_{\varepsilon \rightarrow 0} F_{\varepsilon}(u, O)$$

then we show, by using cut-off techniques and the growth conditions, that for μ -a.e. $x \in \Omega$

$$\lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_{\varepsilon}(u, B_{\rho}(x))}{\mu(B_{\rho}(x))} \leq \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_{\varepsilon}(u_x, B_{\rho}(x))}{\mu(B_{\rho}(x))} \quad (1.22)$$

where $u_x \in H_\mu^{1,p}(\Omega; \mathbb{R}^m)$ is given by $u_x(y) := u(x) + \nabla_\mu u(x) \cdot (\gamma^k(y) - \gamma^k(x))$ and satisfies

$$\nabla_\mu u_x(y) = \nabla_\mu u(x) \text{ for } \mu\text{-a.a. } y \in X \quad \text{and} \quad \lim_{\rho \rightarrow 0} \frac{1}{\rho} \left(\int_{B_\rho(x)} |u(y) - u_x(y)|^p d\mu(y) \right)^{\frac{1}{p}} = 0. \quad (1.23)$$

In fact, we replace u by (its local approximation around x) u_x on the boundary of the ball $B_\rho(x)$, this part of the proof involves L_μ^p -differentiability arguments as the right hand side of (1.23) and compactness imbeddings theorems when for instance $p > \kappa = \frac{\ln(C_d)}{\ln(2)}$ (or $p > N$ in the Euclidean case). It follows that

$$\Gamma(L_\mu^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u, O) \leq \int_O g(x, \nabla_\mu u(x)) d\mu(x)$$

where $g(x, \nabla_\mu u(x)) := \lim_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, B_\rho(x))}{\mu(B_\rho(x))}$ μ -a.e. $x \in \Omega$.

4.2. GENERAL SCHEME FOR ESTABLISHING LOWER BOUND. Here, we give a description of a usual method (called the localization method or blow-up method which was introduced by [FM92]) to show the lower bound in the Γ -convergence problems. We place ourselves under the assumption (AL_X). To show the lower bound for Γ -limit inf, we first assume that $\Gamma(L_\mu^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u, O) < \infty$, so there exists a sequence $\{u_\varepsilon\}_{\varepsilon > 0}$ such that

$$\lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, O) = \Gamma(L_\mu^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u, O) \quad \text{and} \quad \sup_{\varepsilon > 0} F_\varepsilon(u_\varepsilon, O) < \infty. \quad (1.24)$$

We assume that $F_\varepsilon(v, \cdot)|_O$ is the trace of a Borel measure for all $v \in H_\mu^{1,p}(X; \mathbb{R}^m)$. For each $\varepsilon > 0$ we set $\Theta_\varepsilon := F_\varepsilon(u_\varepsilon, \cdot)$. Note that since (1.24) this sequence is uniformly bounded $\sup_{\varepsilon > 0} \Theta_\varepsilon(O) < \infty$. There exists a Borel measure Θ (in the Euclidean setting, Θ is the weak* limit of a subsequence of $\{\Theta_\varepsilon\}_{\varepsilon > 0}$) such that

$$\Gamma(L_\mu^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u, O) = \lim_{\varepsilon \rightarrow 0} F_\varepsilon(u_\varepsilon, O) = \underline{\lim}_{\varepsilon \rightarrow 0} \Theta_\varepsilon(O) \geq \Theta(O) = \int_O \frac{d\Theta}{d\mu}(x) d\mu(x) + \Theta^s(O)$$

where we used the Lebesgue decomposition theorem and the Radon-Nikodym theorem to write $\Theta = \frac{d\Theta}{d\mu} \mu|_O + \Theta^s$ for some nonnegative Borel measure Θ^s mutually singular with respect to $\mu|_O$ and with for μ -a.e. $x \in O$

$$\frac{d\Theta}{d\mu}(x) = \lim_{\rho \rightarrow 0} \frac{\Theta(B_\rho(x))}{\mu(B_\rho(x))}.$$

Since (AL_X) and $\Theta(O) < \infty$ we can write $(\Theta(\partial B_\rho(x)) = 0$ for all ρ outside a countable set possibly depending on $x \in O$) for μ -a.e. $x \in O$

$$\frac{d\Theta}{d\mu}(x) = \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\Theta_\varepsilon(B_\rho(x))}{\mu(B_\rho(x))} = \lim_{\rho \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon, B_\rho(x))}{\mu(B_\rho(x))}.$$

it follows that

$$\Gamma(L_\mu^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} F_\varepsilon(u, O) \geq \int_O \overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon, B_\rho(x))}{\mu(B_\rho(x))} d\mu(x)$$

Now, to go further, we show, when it is possible, by using cut-off techniques and the growth conditions that for μ -a.e. $x \in O$

$$\overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \frac{F_\varepsilon(u_\varepsilon, B_\rho(x))}{\mu(B_\rho(x))} \geq \overline{\lim}_{\rho \rightarrow 0} \underline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, B_\rho(x))}{\mu(B_\rho(x))}. \quad (1.25)$$

In fact, we try to replace u_ε by a new function which coincides with u_x on the boundary of the ball $B_\rho(x)$, this part is technical and involves L_μ^p -differentiability arguments and compactness

imbeddings when for instance $p > \kappa$ (or $p > N$ in the Euclidean case), and sometimes, under p -growths, we have to use the thin layers technique of De Giorgi [De 75].

4.3. A WAY TO PROVE Γ -CONVERGENCE RESULTS. If we compare the inequalities (1.22) and (1.25) we see that, to get a Γ -convergence result, it is sufficient to show for μ -a.e. $x \in O$

$$\overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, B_\rho(x))}{\mu(B_\rho(x))} = \overline{\lim}_{\rho \rightarrow 0} \overline{\lim}_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(u_x, B_\rho(x))}{\mu(B_\rho(x))} \quad (1.26)$$

this is an important difficulty in the Γ -convergence problems (note that, we already presented this equality in (1.17)).

If we place ourselves in the Euclidean setting, i.e. $(X, d, \mu) = (\mathbb{R}^N, |\cdot - \cdot|, \mathcal{L}_N)$ and we deal with periodic homogenization problems $F_\varepsilon(u_\varepsilon, O) = \int_O f(x/\varepsilon, \nabla u_\varepsilon(x)) dx$ (It also works for stochastic homogenization problems. The only difference is that we need some Lipschitz regularity assumptions (on the integrand) allowing the Γ -limit to hold in a full probability set not depending on Sobolev functions), a way for the equality to hold is to use subadditive theorems. Now, for each $\xi \in \mathbb{M}$ we consider the subadditive function $\mathcal{S}^\xi : \mathcal{B}(\mathbb{R}^N) \rightarrow [0, \infty]$ defined by

$$\mathcal{S}^\xi(A) := \inf \left\{ \int_{\mathring{A}} f(y, \xi + \nabla w(y)) d\mathcal{L}_N(y) : w \in W_0^{1,p}(\mathring{A}; \mathbb{R}^m) \right\},$$

we see that for every $\varepsilon, \rho > 0$ we have

$$\frac{\mathcal{S}^\xi(\frac{1}{\varepsilon}B_\rho(x))}{\mathcal{L}_N(\frac{1}{\varepsilon}B_\rho(x))} = \frac{m_\varepsilon(l_\xi, B_\rho(x))}{\mathcal{L}_N(B_\rho(x))},$$

if $\sup_{A \in \mathcal{B}(\mathbb{R}^N), \mathcal{L}_N(A) > 0} \mathcal{S}^\xi(A)/\mathcal{L}_N(A) < \infty$ then Akcoglu-Krengel subadditive theorem [AK81] (see also [LM02] and [AM20b]) says that for every $\rho > 0$ the following limit exists and then gives a way to have (1.26):

$$\left(\lim_{\varepsilon \rightarrow 0} \frac{m_\varepsilon(l_\xi, B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \right) = \lim_{\varepsilon \rightarrow 0} \frac{\mathcal{S}^\xi(\frac{1}{\varepsilon}B_\rho(x))}{\mathcal{L}_N(\frac{1}{\varepsilon}B_\rho(x))} = \inf_{k \in \mathbb{N}^*} \frac{\mathcal{S}^\xi(kY)}{k^N}.$$

CHAPTER 2

RELAXATION AND HOMOGENIZATION OF UNBOUNDED INTEGRALS

In this chapter, we present our works concerning the extension of relaxation and homogenization results to the case of unbounded integrals, i.e. when the integrand does not have necessarily p -growth. Section 1 gives a presentation of the concept of radially uniformly upper semicontinuous functions. This class of functions is of interest when we want to establish the upper bound in relaxation or Γ -convergence problems. Indeed, one of the difficulties to establish the upper bound is that the weak limits of the sequences of gradients may concentrate on the boundary of the effective domain which is assumed to be convex. It is well known that a convex set with nonempty interior is star-shaped, i.e. every point of the boundary can be brought back radially inside the interior. The strategy is then, first to establish the upper bound when the gradient is uniformly contained in the interior of the effective domain. Then we can prove the upper bound if we can extend radially to the boundary the upper bound obtained for gradient in the interior of the effective domain, this is exactly what radially uniformly upper semicontinuous functions do. In Section 2 we present relaxation and homogenization results of unbounded integral functionals when the effective domain of the integrand is a bounded convex set with nonempty interior. Section 3 presents our works on homogenization when the effective domain is convex but not necessarily bounded. To deal with this case, we have considered G -growth assumptions on the integrands, i.e. we replaced the polynomial growth conditions by growths governed by either a convex function G or a quasiconvex function G satisfying additional hypothesis. Finally Section 4 gives a brief description of a relaxation result in Cheeger-Sobolev spaces with G -growth depending on the space variable $x \in X$.

The works involved in this chapter are [AM18; AMZ15; AM14; AM12a; AM11; Anz10].

1. RADIALLY UNIFORMLY UPPER SEMICONTINUOUS FUNCTIONS

In relaxation or homogenization theory, for convex constraints and when the lower semicontinuous envelope is convex, the radial representation on the boundary holds and allows, under some additional requirements, to extend the integral representation to the whole effective domain of the functional (see for instance [CD02]). However, even for convex constraints, the lower semicontinuous envelope is not necessarily convex in the vectorial case. Therefore, the need for a generalization of the radial representation to a nonconvex setting comes naturally.

The analysis of how the convexity concept plays (see [Roc70, Theorem 7.5, p. 57]) to obtain the radial limit representation involves a uniform upper semicontinuity property, more precisely, when $f : V \rightarrow]-\infty, \infty]$ is convex (V is a vector space) we may write

$$\sup_{u \in \text{dom} f} \frac{f(tu + (1-t)u_0) - f(u)}{1 + |f(u_0)| + |f(u)|} \leq 1 - t \quad (2.1)$$

for any $u_0 \in \text{dom} f$. The left hand side of (2.1) is a kind of uniform semicontinuity modulus which is lower than 0 when $t \rightarrow 1$.

Let V be an Hausdorff topological vector space. For a function $f : V \rightarrow]-\infty, \infty]$ we denote its effective domain by

$$\text{dom}f := \{u \in V : f(u) < \infty\}.$$

For each $a > 0$, $D \subset \text{dom}f$ and $u_0 \in D$, we define $\Delta'_{f,D}$, $\Delta^a_{f,D,u_0} : [0, 1] \rightarrow]-\infty, \infty]$ by

$$\Delta^a_{f,D,u_0}(t) := \sup_{u \in D} \frac{f(tu + (1-t)u_0) - f(u)}{a + |f(u)|} \quad \text{and} \quad \Delta'_{f,D}(t) := \sup_{u \in D} (f(tu + (1-t)u_0) - f(u)).$$

If $D = \text{dom}f$ then we write $\Delta^a_{f,u_0} := \Delta^a_{f,D,u_0}$.

DEFINITION 2.1. (i) [AM14, Definition 2.1] Let $D \subset \text{dom}f$ and $u_0 \in D$. We say that f is radially uniformly upper semicontinuous in D relative to u_0 , if there exists $a > 0$ such that $\overline{\lim}_{t \rightarrow 1} \Delta^a_{f,D,u_0}(t) \leq 0$.

If $D = \text{dom}f$ then we simply say that f is ru-usc relative to $u_0 \in \text{dom}f$.

(ii) [Anz10, Section 1.2] Let $D \subset \text{dom}f$ and $u_0 \in D$. We say that f is strongly radially uniformly upper semicontinuous in D relative to u_0 , if $\overline{\lim}_{t \rightarrow 1} \Delta'_{f,D}(t) \leq 0$.

If $D = \text{dom}f$ then we simply say that f is strongly ru-usc relative to $u_0 \in \text{dom}f$.

(iii) [AM14, Definition 2.1] We say that $D \subset V$ is a strongly star-shaped set relative to $u_0 \in D$, if

$$\forall t \in [0, 1[\quad t\overline{D} + (1-t)u_0 \subset D,$$

where \overline{D} is the closure of D in V .

Radially uniformly upper semicontinuous (resp. strongly radially uniformly upper semicontinuous) will be abbreviated to ru-usc (resp. strongly ru-usc) in what follows.

When $D \subset \text{dom}f$ is star-shaped relative to $u_0 \in D$ and f is ru-usc in D relative to $u_0 \in D$, we say that f is ru-usc in the star-shaped set D relative to $u_0 \in D$.

REMARKS 2.2. (1) Let $u_0 \in V$. One has D strongly star-shaped relative to $u_0 \in D$ if and only if $D - \{u_0\}$ is strongly star-shaped relative to $0 \in D$, and f is ru-usc in D relative to $u_0 \in D$ if and only if $f(\cdot + u_0)$ is ru-usc in $D - \{u_0\}$ relative to $0 \in D - \{u_0\}$.

(2) Note that $\Delta^a_{f(\cdot+u_0), D-\{u_0\}, 0}(\cdot) = \Delta^a_{f,D,u_0}(\cdot)$.

(3) If f is strongly ru-usc in D relative to $u_0 \in D$ then it is ru-usc in D relative to $u_0 \in D$. If f is ru-usc in D relative to $u_0 \in D$ and $\sup_D f < \infty$ then f is strongly ru-usc in D relative to $u_0 \in D$.

(4) If f is ru-usc in D relative to $0 \in D$ then for some $a > 0$ and for every $u \in D \subset \text{dom}f$ and $t \in [0, 1[$ it holds

$$f(tu) = \frac{f(tu) - f(u)}{a + |f(u)|} (a + |f(u)|) + f(u) \leq \Delta^a_{D,f,0}(t) (a + |f(u)|) + f(u)$$

letting $t \rightarrow 1$ we obtain $\overline{\lim}_{t \rightarrow 1} f(tu) \leq f(u)$.

When $D = \text{dom}f$ and $0 \in \text{dom}f$ we simply write $\Delta^a_f(t)$ instead of $\Delta^a_{f,\text{dom}f,0}$ and $\Delta'_f(t)$ for $\Delta'_{f,\text{dom}f}$.

From now, we will always assume that $u_0 = 0$, and we simply write D is strongly star-shaped and f is ru-usc in D , and when $D = \text{dom}f$ then we write f is ru-usc.

We set

$$\forall u \in V \quad \widehat{f}(u) := \varliminf_{t \rightarrow 1^-} f(tu).$$

A preliminary version of this result appeared in [AM11] (see also [AMZ15]):

THEOREM 2.3 ([AM14]). *Let $f : V \rightarrow [0, \infty]$ be a function and let $D \subset \text{dom} f$ be a strongly star-shaped relative to $0 \in D$. Assume that f is ru-usc in D . Then $\widehat{f}(u) = \lim_{t \rightarrow 1^-} f(tu)$ for all $u \in \overline{D}$ and \widehat{f} is ru-usc in $\overline{D} \cap \text{dom} \widehat{f}$. Moreover, $\overline{f}(u) \leq \widehat{f}(u) \leq f(u)$ for all $u \in D$.*

REMARKS 2.4. If $D = \text{dom} f$ then \widehat{f} is ru-usc in its effective domain $\text{dom} \widehat{f}$ since $\overline{f} \leq \widehat{f} \leq f$ implies $\text{dom} f \subset \text{dom} \widehat{f} \subset \text{dom} \overline{f} \subset \overline{\text{dom} f}$.

Now, we can write the parametric version of the ru-usc property. Let (X, \mathcal{X}, ν) be a measure space. For a function $f : X \times V \rightarrow [0, \infty]$ we denote the effective domain of $f(x, \cdot)$ by

$$\mathbb{F}_x := \{u \in V : f(x, u) < \infty\}.$$

For each $a \in L^1_{\text{loc}, \nu}(X;]0, \infty[)$, we set for every $t \in [0, 1]$

$$\Delta_f^a(t) := \sup_{x \in X} \sup_{u \in \mathbb{F}_x} \frac{f(x, tu) - f(x, u)}{a(x) + f(x, u)} \quad \text{and} \quad \Delta'_f(t) := \sup_{x \in X} \sup_{u \in \mathbb{F}_x} (f(x, tu) - f(x, u)).$$

DEFINITION 2.5. (i) [AM11, Definition 3.1] We say that f is radially uniformly upper semicontinuous, if there exists $a \in L^1_{\text{loc}, \nu}(X;]0, \infty[)$ such that $\overline{\lim}_{t \rightarrow 1} \Delta_f^a(t) \leq 0$.

(ii) [AM12a, Definition 4.1] We say that f is strongly radially uniformly upper semicontinuous, if $\overline{\lim}_{t \rightarrow 1} \Delta'_f(t) \leq 0$.

We proved the following representation result of the lower semicontinuous envelope on the boundary of its effective domain:

THEOREM 2.6 ([AM11; AMZ15]). *Let $f : X \times V \rightarrow [0, \infty]$ be a ru-usc (resp. strongly ru-usc) function such that \mathbb{F}_x is star-shaped relative to $0 \in \mathbb{F}_x$ for all $x \in X$. Let $x \in X$. If $f(x, \cdot)$ is lower semicontinuous on \mathbb{F}_x then $\widehat{f}(x, \cdot)$ is ru-usc (resp. strongly ru-usc), is equal to the lower semicontinuous envelope of $f(x, \cdot)$ and for every $u \in V$*

$$\widehat{f}(x, u) = \begin{cases} f(x, u) & \text{if } u \in \mathbb{F}_x \\ \lim_{t \rightarrow 1^-} f(x, tu) & \text{if } u \in \partial \mathbb{F}_x \\ \infty & \text{if } u \in V \setminus \overline{\mathbb{F}_x}. \end{cases}$$

PERSPECTIVE 6. It is of interest to extend Theorem 2.6 to the Γ -convergence setting, i.e. instead of extending the lower semicontinuous envelope, to extend the Γ -limit when it exists of a sequence of functions.

2. RELAXATION AND HOMOGENIZATION OF UNBOUNDED INTEGRALS WITH BOUNDED CONVEX EFFECTIVE DOMAIN

In the following we give a quick summary of key definitions related to quasiconvex functions.

DEFINITION 2.7. (1) We say that a Borel measurable function $g : \mathbb{M} \rightarrow [0, \infty[$ is quasiconvex if for every $\xi \in \mathbb{M}$ we have

$$g(\xi) = \inf \left\{ \int_Y g(\xi + \nabla \phi(y)) dy : \phi \in W_0^{1, \infty}(Y; \mathbb{R}^m) \right\}$$

where $Y =]0, 1[^N$.

(2) The quasiconvex envelope of $f : \mathbb{M} \rightarrow [0, \infty]$ is the function $\mathcal{Q}f : \mathbb{M} \rightarrow [0, \infty]$ defined by

$$\mathcal{Q}f(\xi) := \sup \{g(\xi) : g : \mathbb{M} \rightarrow [0, \infty[\text{ is quasiconvex and } g \leq f\}.$$

(3) For a Borel measurable $f : \mathbb{M} \rightarrow [0, \infty]$ we set $\mathcal{Z}_\infty f : \mathbb{M} \rightarrow [0, \infty]$ defined by

$$\mathcal{Z}_\infty f(\xi) := \inf \left\{ \int_Y f(\xi + \nabla \phi(y)) dy : \phi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}.$$

(For more properties of $\mathcal{Z}_\infty f$ see Section 1 in Chapter 3.)

(4) We say that $h : \mathbb{M} \rightarrow [0, \infty]$ is sup-quasiconvex if there exists a nondecreasing sequence $\{h_n\}_{n \in \mathbb{N}}$, $h_n : \mathbb{M} \rightarrow [0, \infty[$ satisfying

- (i) h_n is quasiconvex for all $n \in \mathbb{N}$;
- (ii) for every $n \in \mathbb{N}$ there exists $\alpha_n > 0$ such that $h_n(\xi) \leq \alpha_n(1 + |\xi|)$ for all $\xi \in \mathbb{M}$;
- (iii) for every $\xi \in \mathbb{M}$ one has $h(\xi) = \sup_{n \in \mathbb{N}} h_n(\xi)$.

REMARKS 2.8. (1) The function $g : \mathbb{M} \rightarrow [0, \infty[$ is continuous when it is quasiconvex.

(2) The quasiconvex envelope $\mathcal{Q}f$ is lower semicontinuous as supremum of continuous functions and satisfies

$$\mathcal{Q}f(\xi) = \inf \left\{ \int_Y \mathcal{Q}f(\xi + \nabla \phi(y)) dy : \phi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}.$$

2.1. RELAXATION THEOREMS. This subsection is devoted to the description of our work on the relaxation of integral functionals of the calculus of variations in Sobolev spaces with bounded constraints on the gradient. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $f : \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function. The effective domain of f is the set $\text{dom} f := \{\xi \in \mathbb{M} : f(\xi) < \infty\}$. We consider a bounded convex subset $C \subset \mathbb{M}$ such that

$$0 \in \text{int} C \quad \text{and} \quad \text{dom} f \subset \overline{C},$$

where $\text{int} C$ is the interior of C . Let $I : L^1(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ be the integral functional defined by

$$I(u) := \begin{cases} \int_\Omega f(\nabla u(x)) dx & \text{if } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \\ \infty & \text{if } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \setminus L^1(\Omega; \mathbb{R}^m). \end{cases}$$

We studied the problem of the integral representation of the relaxation of I

$$\bar{I}(u) := \inf \left\{ \liminf_{n \rightarrow \infty} \int_\Omega f(\nabla u_n(x)) dx : u_n \rightarrow u \text{ in } L^1(\Omega; \mathbb{R}^m) \right\}.$$

We have proved the following two theorems. The first result allows to consider a singular behavior at the boundary of the convex set, i.e. $\lim_{\xi \rightarrow \partial C} f(\xi) = \infty$, an illustration is given by Figure 2.1. For instance, the result applies for integrand of the type

$$\mathbb{M} \ni \xi \mapsto f(\xi) := \begin{cases} g(\xi) + \frac{1}{1 - |\xi|^s} & \text{si } |\xi| < 1 \\ \infty & \text{si } |\xi| \geq 1 \end{cases}$$

where $g : \mathbb{M} \rightarrow [0, \infty[$ is uniformly continuous and $s \in [1, \infty[$.

THEOREM 2.9 ([Anz10, Theorem 1.1. and Theorem 1.2.]). *Assume that*

(H₁) *f is locally bounded in $\text{int} C$, i.e. $\sup_{\xi \in K} f(\xi) < \infty$ for all compact subsets $K \subset \text{int} C$;*

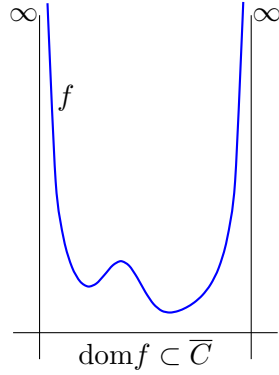


FIGURE 2.1. A possible singular behavior of f in Theorem 2.9.

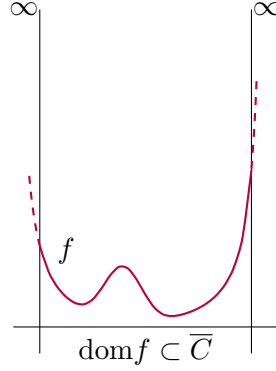


FIGURE 2.2. A possible behavior of f in Theorem 2.10.

(H₂) for every $\alpha > 0$ there exists a compact subset $K_\alpha \subset \text{int}C$ such that $\inf_{\xi \in \overline{C} \setminus K_\alpha} \mathcal{Z}_\infty f(\xi) \geq \alpha$;

(H₃) $\mathcal{Z}_\infty f$ is strongly ru-usc in $\text{dom}f$ (relative to 0).

Then

$$(i) \mathcal{Q}f \text{ is sup-quasiconvex and } \mathcal{Q}f(\xi) = \begin{cases} \mathcal{Z}_\infty f(\xi) & \text{if } \xi \in \text{int}C \\ \infty & \text{otherwise;} \end{cases}$$

$$(ii) \bar{I}(u) = \begin{cases} \int_{\Omega} \mathcal{Q}f(\nabla u(x)) dx & \text{if } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \\ \infty & \text{if } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \setminus L^1(\Omega; \mathbb{R}^m). \end{cases}$$

The second result allows to consider the case of integrands which are bounded on $\text{int}C$, an illustration is given by Figure 2.2.

THEOREM 2.10 ([Anz10, Theorem 1.3. and Theorem 1.4.]). *Assume that*

(H₁) f is bounded in $\text{int}C$, i.e. $\sup_{\xi \in \text{int}C} f(\xi) < \infty$;

(H₂) for every $\xi \in \partial C$ it holds $\lim_{[0,1] \ni t \rightarrow 1} f(t\xi) \leq f(\xi)$;

(H₃) $\mathcal{Z}_\infty f$ is strongly ru-usc in $\text{dom}f$ (relative to 0).

Then

$$(i) \mathcal{Q}f \text{ is sup-quasiconvex and } \mathcal{Q}f = \widehat{\mathcal{Z}_\infty f} = \overline{\mathcal{Z}_\infty f};$$

$$(ii) \bar{I}(u) = \begin{cases} \int_{\Omega} \mathcal{Q}f(\nabla u(x)) dx & \text{if } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \\ \infty & \text{if } u \in W^{1,\infty}(\Omega; \mathbb{R}^m) \setminus L^1(\Omega; \mathbb{R}^m). \end{cases}$$

The proofs of Theorems 2.9 and 2.10 are based on two main ingredients. To establish the lower bound, we use a result of [Mül99a] which states that a sequence of gradients converging in L^1 to \overline{C} can be modified on a small set in a such way that the new sequence converges in L^∞ to \overline{C} . This result allows to show that the quasiconvex envelope of f is the supremum of a nondecreasing sequence of functions with p -growth. Then the lower bound is obtained

by using the convergence monotone theorem and classical lower semicontinuity results with p -growth (see for instance [AF84]). For the upper bound, we first assume that the gradients are compactly contained in the interior of C , then using approximation by continuous piecewise affine functions of Sobolev functions and the continuity property of $\mathcal{Z}_\infty f$ we can establish the upper bound for these particular gradients. Eventually, we use the ru-usc property of $\mathcal{Z}_\infty f$ to extend the upper bound inequality to all gradients, i.e. those contained in \overline{C} .

PERSPECTIVE 7. One interesting further investigation is to obtain relaxation results under star-shaped C but non necessarily convex. For instance, it would be interesting to prove relaxation results when C is the union of two bounded convex sets with the interior of their intersection is nonempty.

2.2. HOMOGENIZATION OF UNBOUNDED INTEGRALS WITH BOUNDED CONVEX EFFECTIVE DOMAIN. Now, we present the problem of homogenization of unbounded integrals $I_\varepsilon : W_0^{1,\infty}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ defined by

$$I_\varepsilon(u) := \int_{\Omega} f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx$$

where $\varepsilon > 0$ and $f : \mathbb{R}^N \times \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable function which is 1-periodic with respect to its first variable, i.e.

$$(H_1) \quad f(x+z, \xi) = f(x, \xi) \text{ for all } (x, z, \xi) \in \mathbb{R}^N \times \mathbb{Z}^N \times \mathbb{M},$$

and infinite outside of a convex bounded set $C \subset \mathbb{M}$ satisfying $0 \in \text{int}C$, i.e.

$$(H_2) \quad \text{dom}f(x, \cdot) = C \text{ for all } x \in \mathbb{R}^N \text{ where } \text{dom}f(x, \cdot) \text{ the effective domain of } f(x, \cdot).$$

We consider the following three assertions:

$$(H_3) \quad f \text{ is locally bounded in } C, \text{ i.e. } \sup_{x \in \mathbb{R}^N} \sup_{\xi \in t\overline{C}} f(x, \xi) < \infty \text{ for all } t \in [0, 1];$$

$$(H_4) \quad f \text{ is singular on the boundary } \partial C \text{ of } C, \text{ i.e.}$$

$$\lim_{t \rightarrow 1} \inf_{x \in \mathbb{R}^N} \inf_{\xi \in \overline{C} \setminus t\overline{C}} f(x, \xi) = \infty;$$

$$(H_5) \quad f \text{ is strongly ru-usc, i.e. } \overline{\lim}_{t \rightarrow 1} \Delta_f(t) := \overline{\lim}_{t \rightarrow 1} \sup_{x \in \mathbb{R}^N} \sup_{\xi \in C} f(x, t\xi) - f(x, \xi) \leq 0.$$

THEOREM 2.11 ([AM12a, Corollary 2.3, p. 189]). *Let $f : \mathbb{R}^N \times \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function satisfying (H_1) - (H_5) . Then $\Gamma(L^1)$ - $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = I_{\text{hom}}$ where $I_{\text{hom}} : W^{1,\infty}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ is defined by*

$$I_{\text{hom}}(u) := \int_{\Omega} f_{\text{hom}}(\nabla u(x)) dx$$

with $f_{\text{hom}} : \mathbb{M} \rightarrow [0, \infty]$ given by $f_{\text{hom}} = \overline{\mathcal{H}_\infty f} = \widehat{\mathcal{H}_\infty f}$ where $\mathcal{H}_\infty f : \mathbb{M} \rightarrow [0, \infty]$ is defined by

$$\mathcal{H}_\infty f(F) := \inf_{k \in \mathbb{N}^*} \inf_{\phi \in W_0^{1,\infty}(kY; \mathbb{R}^m)} \int_{kY} f(y, F + \nabla \phi(y)) dy.$$

and $\overline{\mathcal{H}_\infty f}$ its lower semicontinuous envelope.

The extension into the setting of periodic homogenization of Theorems 2.9 and 2.10 involved some difficulties. The strategy of the proof is similar to the one of the previous relaxation results Theorems 2.9 and 2.10. To adapt the techniques in the context of homogenization, we had to prove a Γ -limit inf lower bound for periodic integrals with 1-growth and for the weak* convergence in $W^{1,\infty}(\Omega; \mathbb{R}^m)$, this played the role of the classical lower semicontinuity

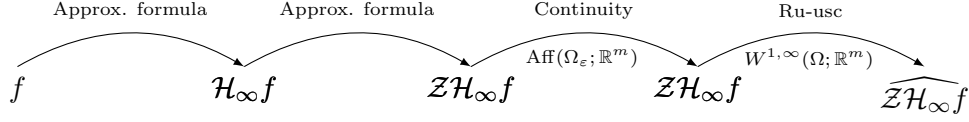


FIGURE 2.3. Schematic representation of the successive steps of the proof of the Γ -limit sup of Theorem 2.11.

results that we used for the relaxation results. For the upper bound, we prove that for every $u \in W^{1,\infty}(\Omega; \mathbb{R}^m)$

$$\Gamma(L^1) \overline{\lim}_{\varepsilon \rightarrow 0} -I_\varepsilon(u) \leq \int_{\Omega} \widehat{\mathcal{Z}\mathcal{H}_\infty f}(\nabla u(x)) dx.$$

where $\mathcal{Z}\mathcal{H}_\infty f(F) := \inf \left\{ \int_Y \mathcal{H}_\infty f(F + \nabla \phi(y)) dy : \phi \in \text{Aff}_0(Y; \mathbb{R}^m) \right\}$. In fact, we can show that $\mathcal{Z}\mathcal{H}_\infty f$ is strongly ru-usc since f is assumed to be strongly ru-usc (see [AM12a, Propositions 3.6 and 3.7]), therefore we can show

$$\overline{\lim}_{t \rightarrow 1} \int_{\Omega} \mathcal{Z}\mathcal{H}_\infty f(t\nabla u(x)) dx \leq \int_{\Omega} \mathcal{Z}\mathcal{H}_\infty f(\nabla u(x)) dx.$$

Using an approximation result of [DM99, Theorem 10.16 and Corollary 10.21] we can find a sequence $\{u_\varepsilon\}_\varepsilon \in W^{1,\infty}(\Omega; \mathbb{R}^m)$ and a sequence of open subsets $\{\Omega_\varepsilon\}_\varepsilon$ of Ω such that $u_\varepsilon|_{\Omega_\varepsilon}$ is a continuous piecewise affine function, $\nabla u_\varepsilon(x) \in (t + \varepsilon)\overline{C} \mathcal{L}_N$ -a.e. $x \in \Omega$, $\lim_{\varepsilon \rightarrow 0} \mathcal{L}_N(\Omega \setminus \Omega_\varepsilon) = 0$ and $u_\varepsilon \rightarrow tu$ strongly in $W^{1,1}(\Omega; \mathbb{R}^m)$. Then we show that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathcal{Z}\mathcal{H}_\infty f(\nabla u_\varepsilon(x)) dx = \int_{\Omega} \mathcal{Z}\mathcal{H}_\infty f(t\nabla u(x)) dx.$$

Now, we can work on each open subsets $\Omega_{\varepsilon,i} \subset \Omega_\varepsilon$ where the gradient $\nabla u_\varepsilon = \xi_i$ is constant, the rest of the proof consists in using approximation formula $\mathcal{Z}\mathcal{H}_\infty f(\xi_i)$ by $\mathcal{H}_\infty f$ (see [AM12a, Proposition 3.19] or Lemma 3.16) and a similar approximation formula for $\mathcal{H}_\infty f$ by f adapted from [Mül87, Lemma 2.1(a)]:

LEMMA 2.12 ([AM12a, Proposition 3.20]). *Let $t \in]0, 1[$, let $\xi \in \mathbb{M}$ and let $O \subset \mathbb{R}^N$ be a bounded open set. If $\xi \in \overline{tC}$ then there exists $\{\phi_\varepsilon\}_\varepsilon \subset W_0^{1,\infty}(O; \mathbb{R}^m)$ such that:*

- (a) $\lim_{\varepsilon \rightarrow 0} \|\phi_\varepsilon\|_{L^1(O; \mathbb{R}^m)} = 0$;
- (b) $\lim_{\varepsilon \rightarrow 0} \int_O f\left(\frac{x}{\varepsilon}, \xi + \nabla \phi_\varepsilon(x)\right) dx = \mathcal{H}_\infty f(\xi)$.

The Figure 2.3 gives a schematic representation of the different steps of the proof of the upper bound.

We present an example (see [AM12a]) in the case $N = m$. Let \mathbb{B} be the unit open ball in \mathbb{M} (note that $\det F > 0$ whenever $F \in I + \mathbb{B}$). Given a continuous function $g : \mathbb{M} \rightarrow [0, \infty[$ and a convex function $h : [0, 1[\rightarrow [0, \infty[$ such that there exist $c > 0$ and $p > 0$

$$h(t) \geq \frac{ct^p}{1-t^p} \text{ for all } t \in [0, 1[, \quad (2.2)$$

we consider $\Phi : \mathbb{M} \rightarrow [0, \infty[$ defined by

$$\Phi(\xi) := \begin{cases} g(I + \xi) + h(|\xi|) & \text{if } \xi \in \mathbb{B} \\ \infty & \text{otherwise,} \end{cases}$$

where I denotes the identity matrix in \mathbb{M} . Given a 1-periodic function $a \in L^\infty(\mathbb{R}^N)$ such that $\alpha \leq a(x) \leq \beta$ for all $x \in \mathbb{R}^N$ and some $\beta > \alpha > 0$, we define $f : \mathbb{R}^N \times \mathbb{M} \rightarrow [0, \infty]$ by

$$f(x, \xi) := a(x)\Phi(\xi).$$

Such a f satisfies the assumptions of Theorem 2.11 (here $C = \mathbb{B}$) as well as some (but not all) conditions of hyperelasticity, i.e. the non-interpenetration of the matter: for every $(x, \xi) \in \mathbb{R}^N \times \mathbb{B}$, $f(x, \xi) < \infty$ if and only if $\det(I + \xi) > 0$, and the necessity of an infinite amount of energy to reduce a finite volume of matter into zero volume: for every $x \in \mathbb{R}^N$, $f(x, \xi) \rightarrow \infty$ as $\det(I + \xi) \rightarrow 0$. However, the material frame-indifference condition is not satisfied, indeed the rotation of angle $\frac{\pi}{2}$ does not belong to $I + \mathbb{B}$ (because $\Phi(R - I) = \infty > \Phi(0)$ with R the rotation of angle $\frac{\pi}{2}$).

3. HOMOGENIZATION OF UNBOUNDED INTEGRALS WITH G -GROWTH

This section is devoted to present our works on the homogenization of periodic unbounded integrals on $W^{1,p}(\Omega; \mathbb{R}^m)$ with $p > N$. Here, the constraints on gradient are now not necessarily bounded. Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded open set with Lipschitz boundary. We consider the periodic homogenization problem of integral functionals by Γ -convergence. More precisely, for each $\varepsilon > 0$, we define $I_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ by

$$I_\varepsilon(u) := \int_{\Omega} f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx,$$

where the integrand $f : \mathbb{R}^N \times \mathbb{M} \rightarrow [0, \infty]$ is Borel measurable and 1-periodic with respect to the first variable. Nonconvex homogenization by Γ -convergence of the family $\{I_\varepsilon\}_{\varepsilon > 0}$ was mainly studied in the framework of p -polynomial growth conditions on f . Unfortunately, this framework is not compatible with two basic conditions of hyperelasticity: the non-interpenetration of the matter, i.e. $f(x, \xi) = \infty$ if and only if $\det(I + \xi) \leq 0$, and the necessity of an infinite amount of energy to reduce a finite volume into zero volume, i.e. for every $x \in \mathbb{R}^N$, $f(x, \xi) \rightarrow \infty$ as $\det(I + \xi) \rightarrow 0$. At present, it seems difficult to take these conditions into account in homogenization problems. Generally, the attempts to go beyond the p -polynomial growth are not easy due to the lack of available techniques. However, in the scalar case, we refer to the book [CD02] where relaxation and homogenization of unbounded functionals were studied (see also [Car+02; Car+04; CCP06; De 07]). In the vectorial case, i.e. when $\min(N, m) > 1$, the homogenization with convex growth see (H₁) (G independent of x) on f was studied in [AM11]. When the convex growth G depends upon the space variable x , the stochastic homogenization problem was studied in [DG16] and their result generalize substantially our work [AM11].

We consider the following assertions:

(H₁) G -growth conditions, i.e. there exist $\alpha, \beta > 0$ such that for every $x \in \mathbb{R}^N$ and every $\xi \in \mathbb{M}$

$$\alpha G(\xi) \leq f(x, \xi) \leq \beta(1 + G(\xi));$$

(H₂) f is p -coercive, i.e. there exists $c > 0$ such that for every $(x, \xi) \in \mathbb{R}^N \times \mathbb{M}$

$$c|\xi|^p \leq f(x, \xi),$$

where $G : \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable function. Denote by \mathbb{G} the effective domain of G , i.e. $\mathbb{G} = \{\xi \in \mathbb{M} : G(\xi) < \infty\}$. We can remark that (H₁) implies that the effective domain of f is independent of x and $\text{dom} f(x, \cdot) = \mathbb{G}$ for all $x \in \mathbb{R}^N$.

We consider the following conditions on G :

(C₁) $0 \in \text{int}(\mathbb{G})$;

(C₂) there exists $C > 0$ such that for every $\xi, \zeta \in \mathbb{M}$ and every $t \in]0, 1[$

$$G(t\xi + (1-t)\zeta) \leq C(1 + G(\xi) + G(\zeta));$$

(C₃) G is $W^{1,p}$ -quasiconvex, i.e. for every $\xi \in \mathbb{M}$

$$G(\xi) = \inf \left\{ \int_Y G(\xi + \nabla\varphi(x))dx : \varphi \in W_0^{1,p}(Y; \mathbb{R}^m) \right\}$$

where $Y =]0, 1[^N$.

Note that (C₂) implies that \mathbb{G} is convex but G is not necessarily convex. The condition (C₂) prevents the possible “strong bumps” of G .

THEOREM 2.13 ([AMZ15, Theorem 1.1]). *Assume that $p > N$. Assume that (C₁)-(C₃), (H₁) and (H₂) hold. If f is periodically ru-usc then $\{I_\varepsilon\}_{\varepsilon > 0}$ Γ -converges with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$ to $I_{\text{hom}} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ defined by*

$$I_{\text{hom}}(u) := \int_\Omega f_{\text{hom}}(\nabla u(x))dx, \quad (2.3)$$

with $f_{\text{hom}} = \widehat{\mathcal{H}f} = \overline{\mathcal{H}f}$ where $\mathcal{H}f$ is the Braides-Müller homogenization formula (3.8) and $\overline{\mathcal{H}f}$ its lower semicontinuous envelope.

Theorem 2.13 is an extension of the homogenization result in [AM11], to the case where f has quasiconvex growth conditions. When the growth G is convex (then (C₂) and (C₃) are fulfilled) we proved the following result.

THEOREM 2.14 ([AM11, Corollary 2.2, p. 170]). *Assume that G is convex. Let $f : \mathbb{R}^N \times \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function. Assume that f is periodically ru-usc and satisfies (H₁) and (H₂). If (C₁) holds, $p > N$ and Ω is strongly star-shaped then $\{I_\varepsilon\}_{\varepsilon > 0}$ Γ -converges with respect to the strong topology of $L^p(\Omega; \mathbb{R}^m)$ to $I_{\text{hom}} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ given by (2.3) with $f_{\text{hom}} = \widehat{\mathcal{H}f} = \overline{\mathcal{H}f}$.*

The assumption that f is periodically ru-usc allows to consider a suitable extension (in a radial way) of the homogenized integrand to the boundary $\partial\mathbb{G}$ of \mathbb{G} . The reason is that the weak limits of the sequences of gradients can be located at $\partial\mathbb{G}$ during the homogenization process by Γ -convergence. The assumption that $p > N$ allows, by using the p -coercivity condition (H₂) and Sobolev compact imbedding, to work with the convergence in $L^\infty(\Omega; \mathbb{R}^m)$ instead of $L^p(\Omega; \mathbb{R}^m)$. Moreover, the functions of $W^{1,p}(\Omega; \mathbb{R}^m)$ are almost everywhere differentiable in Ω since Sobolev imbedding. When it is combined with the fact that G has a local upper bound property, i.e. $\sup_{|\xi| \leq r} G(\xi) < \infty$ for some $r > 0$ (see Remark 2.17), it allows to obtain suitable lower and upper bounds. The main difficulty for proving Theorem 2.13 comes from the proof of the Γ -limit sup. In the setting of convex growth conditions on f we can use mollifier techniques to construct approximations of Sobolev functions by smooth ones (and then use continuous piecewise affine functions). The steps of the proof follows the same lines as the one of Theorem 2.11. Indeed, we prove that for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$\Gamma(L^p) \overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u) \leq \int_\Omega \widehat{\mathcal{Z}\mathcal{H}f}(\nabla u(x))dx.$$

where $\mathcal{Z}\mathcal{H}f(F) := \inf \left\{ \int_Y \mathcal{H}f(F + \nabla\phi(y))dy : \phi \in \text{Aff}_0(Y; \mathbb{R}^m) \right\}$. In fact, we can show that $\mathcal{Z}\mathcal{H}f$ is ru-usc since f is assumed to be ru-usc (see [AM11, Propositions 3.6 and 3.7]), therefore we can show

$$\overline{\lim}_{t \rightarrow 1} \int_\Omega \mathcal{Z}\mathcal{H}f(t\nabla u(x))dx \leq \int_\Omega \mathcal{Z}\mathcal{H}f(\nabla u(x))dx.$$

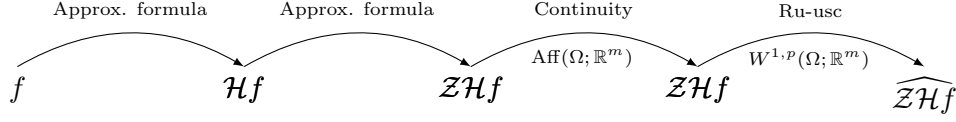


FIGURE 2.4. Schematic representation of the successive steps of the proof of the Γ -limit sup of Theorem 2.14.

Using an approximation result of [Mül87, Lemma 3.6] we can find a sequence $\{u_\varepsilon\}_\varepsilon \in \text{Aff}(\Omega; \mathbb{R}^m)$ such that $\nabla u_\varepsilon(x) \in \text{int}\mathbb{G}$ \mathcal{L}_N -a.e. $x \in \Omega$, $u_\varepsilon \rightarrow tu$ strongly in $W^{1,p}(\Omega; \mathbb{R}^m)$ and $G(\nabla u_\varepsilon) \rightarrow G(\nabla u)$ in $L^1(\Omega)$, then we prove the following general approximation result for nonconvex integrand.

LEMMA 2.15 ([AM11, Propositions 3.15 and 3.16]). *Let $L : \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function with G -convex growth, i.e. there exist a convex function $G : \mathbb{M} \rightarrow [0, \infty]$ and $\alpha, \beta > 0$ such that*

$$\alpha G(\xi) \leq L(\xi) \leq \beta(1 + G(\xi))$$

for all $\xi \in \mathbb{M}$. Let $\Omega \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary which is strongly star-shaped and let $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ be such that

$$\int_{\Omega} L(\nabla u(x)) dx < \infty.$$

(i) *If L is ru-usc and continuous on $\text{int}(\mathbb{G})$ then there exists $\{u_n\}_n \subset \text{Aff}(\Omega; \mathbb{R}^m)$ such that $\lim_{n \rightarrow \infty} \|u_n - u\|_{W^{1,p}(\Omega; \mathbb{R}^m)} = 0$ and*

$$\overline{\lim}_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_n(x)) dx \leq \int_{\Omega} L(\nabla u(x)) dx.$$

(ii) *If L is continuous on $\text{int}(\mathbb{G})$ and $\mathcal{L}_N(\{x \in \Omega : \nabla u(x) \notin \text{int}\mathbb{G}\}) = 0$ then there exists $\{u_n\}_n \subset \text{Aff}(\Omega; \mathbb{R}^m)$ such that $\lim_{n \rightarrow \infty} \|u_n - u\|_{W^{1,p}(\Omega; \mathbb{R}^m)} = 0$ and*

$$\lim_{n \rightarrow \infty} \int_{\Omega} L(\nabla u_n(x)) dx = \int_{\Omega} L(\nabla u(x)) dx.$$

Then¹ by using Lemma 2.15 (ii) we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \mathcal{Z}\mathcal{H}f(\nabla u_\varepsilon(x)) dx = \int_{\Omega} \mathcal{Z}\mathcal{H}f(t\nabla u(x)) dx.$$

Now, we can work on each open subsets $\Omega_{\varepsilon,i} \subset \Omega_\varepsilon$ where the gradient $\nabla u_\varepsilon = \xi_i$ is constant, the rest of the proof consists in using approximation formula $\mathcal{Z}\mathcal{H}f(\xi_i)$ by $\mathcal{H}f$ (see [AM11, Proposition 3.17] or Lemma 3.16) and a similar approximation formula for $\mathcal{H}f$ by f adapted from [Mül87, Lemma 2.1(a)] (see [AM11, Proposition 3.18]).

We had to develop other techniques to deal with quasiconvex growth or more general growth. The strategy we used involves the Vitali envelope of local Dirichlet minimization problems associated to the family $\{I_\varepsilon\}_{\varepsilon>0}$ which reduces the proof of the upper bound to cut-off techniques, avoiding any approximation arguments by continuous piecewise affine functions. We refer to Section 4 where we exposed this method.

As an illustration, we present an example ([AMZ15]) in the case where $N = m = 2$. Consider a the following convex set $\mathbb{G} \subset \mathbb{M}_{22}$ defined by

$$\mathbb{G} := \left\{ \xi \in \mathbb{M}_{22} : \min(1 + \xi_{11}, 1 + \xi_{22}) > \max(|\xi_{12}|, |\xi_{21}|) \right\}$$

¹If $C \subset \mathbb{M}$ is convex with $0 \in \text{int}C$ then $t\bar{C} \subset \text{int}C$ for all $t \in [0, 1]$.

where every $\xi \in \mathbb{M}_{22}$ is denoted by $\xi := \begin{pmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{pmatrix}$, it has the following properties:

- (i) $0 \in \text{int}(\mathbb{G})$;
- (ii) $\det(I + \xi) > 0$ for all $\xi \in \mathbb{G}$;
- (iii) $\text{tr}(\text{cof}(I + \xi)^\top(I + \zeta)) > 0$ for all $\xi, \zeta \in \mathbb{G}$.

The frame-indifference condition is not satisfied by $I + \mathbb{G}$. Indeed, due to the frame-indifference condition, we should have²

$$\text{SO}(2)(I + \mathbb{G}) \subset I + \mathbb{G}, \quad (2.4)$$

but this is not true. Assume that a such $I + \mathbb{G}$ satisfying (2.4) exists then $\text{SO}(2) \subset I + \mathbb{G}$ since (i). Choose any rotation matrix $I + \zeta$ with angle $\theta \in [\frac{\pi}{2}, \frac{3\pi}{2}]$ and $\xi = 0$, then

$$\text{tr}(\text{cof}(I + \xi)^\top(I + \zeta)) = \text{tr}(I + \zeta) \leq 0,$$

so (iii) cannot be satisfied.

Let $G : \mathbb{M}_{22} \rightarrow [0, \infty]$ be defined by

$$G(\xi) := \begin{cases} |\xi|^p + \frac{1}{\det(I + \xi)} & \text{if } \xi \in \mathbb{G} \\ \infty & \text{otherwise.} \end{cases} \quad (2.5)$$

The function G satisfies (C₁), (C₂) and (C₃). Let $f : \mathbb{R}^2 \times \mathbb{M}_{22} \rightarrow [0, \infty]$ be defined by

$$f(x, \xi) := \begin{cases} \Phi(x, \xi) + \frac{1}{\det(I + \xi)} & \text{if } \xi \in \mathbb{G} \\ \infty & \text{otherwise,} \end{cases} \quad (2.6)$$

where $\Phi : \mathbb{R}^2 \times \mathbb{M}_{22} \rightarrow [0, \infty]$ is a quasiconvex function, 1-periodic with respect to its first variable and of p -polynomial growth, i.e. there exist $c, C > 0$ such that

$$c|\xi|^p \leq \Phi(x, \xi) \leq C(1 + |\xi|^p) \quad (2.7)$$

for all $(x, \xi) \in \mathbb{R}^2 \times \mathbb{M}_{22}$. Such a f is consistent with the assumptions of Theorem 2.13 as well as with the two basic conditions of hyperelasticity, i.e. the non-interpenetration of the matter and the necessity of an infinite amount of energy to compress a finite volume of matter into zero volume.

PERSPECTIVE 8. It would be interesting to extend the result of Duerinckx and Gloria [DG16] to the case of quasiconvex growth depending on x satisfying the bump condition (C₂). In our opinion, a three dimensional example, i.e. when $N = m = 3$, similar to the example above would be more convincing.

4. RELAXATION OF UNBOUNDED INTEGRALS WITH G -GROWTH IN CHEEGER-SOBOLEV SPACES

Let (X, d, μ) be a metric measure space, where (X, d) is a length space which is separable and compact, and μ is a positive Radon measure on X . We assume that μ is doubling with $C_d \geq 1$ denoting the doubling constant and that X enjoys a weak $(1, p)$ -Poincaré inequality with $1 < p < \infty$. We suppose $p > \kappa := \frac{\ln(C_d)}{\ln(2)}$, in fact this condition plays the same role as $p > N$ in the Euclidean setting, and allows to have a L_μ^∞ -compactness result similarly to the Euclidean case ([AM18, Corollaries 3.10 and 3.11, p. 62-63]).

²SO(2) is the special orthogonal group of 2×2 matrices.

Let $E : H_\mu^{1,p}(X; \mathbb{R}^m) \times \mathcal{O}(X) \rightarrow [0, \infty]$ be the integral functional defined by

$$E(u, O) := \int_O f(x, \nabla_\mu u(x)) d\mu(x)$$

where $f : X \times \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable function. We assume that f has G -growth, i.e.

(H₁) there exist $\alpha, \beta > 0$ such that for every $(x, \xi) \in X \times \mathbb{M}$

$$\alpha G(x, \xi) \leq f(x, \xi) \leq \beta(1 + G(x, \xi))$$

where $G : X \times \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable function satisfying the following conditions:

(H₂) G is p -coercive, i.e. there exists $c > 0$ such that $G(x, \xi) \geq c|\xi|^p$ for all $(x, \xi) \in X \times \mathbb{M}$;

(H₃) there exists $r > 0$ such that $\sup_{|\xi| \leq r} G(\cdot, \xi) \in L_\mu^1(X)$;

(H₄) there exists $C > 0$ such that for every $x \in X$, every $t \in]0, 1[$ and every $\xi, \zeta \in \mathbb{M}$,

$$G(x, t\xi + (1-t)\zeta) \leq C(1 + G(x, \xi) + G(x, \zeta)).$$

Let $\mathcal{G}, \overline{\mathcal{G}} : H_\mu^{1,p}(X; \mathbb{R}^m) \rightarrow [0, \infty]$ be the functionals defined by:

$$\mathcal{G}(u) := \int_X G(x, \nabla_\mu u(x)) d\mu(x) \quad \text{and} \quad \overline{\mathcal{G}}(u) := \inf \left\{ \varliminf_{n \rightarrow \infty} \mathcal{G}(u_n) : u_n \xrightarrow{L_\mu^p} u \right\}.$$

We consider the following additional assertions:

(H₅) $\text{dom} \mathcal{G} = \overline{\text{dom} \overline{\mathcal{G}}}$;

(H₆) if $u \in \text{dom} \mathcal{G}$ then $\lim_{r \rightarrow 0} \int_{Q_r(x)} |G(y, \nabla_\mu u(x)) - G(x, \nabla_\mu u(x))| d\mu(y) = 0$ μ -a.e. $x \in X$.

In addition, we need to suppose that L is radially uniformly upper semicontinuous, i.e.

(H₇) there exists $a \in L_\mu^1(X;]0, \infty])$ such that $\overline{\lim}_{t \rightarrow 1^-} \Delta_f^a(t) \leq 0$

REMARKS 2.16. If (H₄) holds then $\text{dom} G(x, \cdot)$ is convex. The condition (H₄) prevents the possible ‘strong bumps’ of G . If (H₁) and (H₄) hold then $\text{dom} f(x, \cdot) = \text{dom} G(x, \cdot)$ and so $\text{dom} f(x, \cdot)$ is convex. If (H₁) is satisfied then $\text{dom} \overline{E}(\cdot, X) = \overline{\text{dom} \mathcal{G}}$, and so $\text{dom} \overline{E}(\cdot, X) = \text{dom} \mathcal{G}$ when (H₅) holds.

REMARKS 2.17. If (H₄) holds and if $0 \in \text{int}(\{\xi \in \mathbb{M} : G(\cdot, \xi) \in L_\mu^1(X)\})$ then (H₃) is verified, see [AM12b, Lemma 4.1]. We can generalize this local upper bound property as follows.

LEMMA 2.18. *Let (M, \mathcal{M}, ν) be a finite complete measure space and $d \in \mathbb{N}^*$. Let $L : M \times \mathbb{R}^d \rightarrow [0, \infty]$ be a $(\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^d), \mathcal{B}([0, \infty]))$ -measurable function. Assume that*

(U₁) *there exists $r > 0$ such that $\int_M L(x, v) d\nu(x) < \infty$ whenever $|v|_\infty \leq r$ for all $v \in \mathbb{R}^d$;*

(U₂) *there exists a nondecreasing concave function $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that for every $(x, v, w) \in M \times \text{dom} L(x, \cdot) \times \text{dom} L(x, \cdot)$ we have*

$$\sup_{t \in]0, 1[} L(x, tv + (1-t)w) \leq \Phi(\max(L(x, v), L(x, w))).$$

Then

$$\int_M \sup_{|v|_\infty \leq r} L(x, v) d\nu(x) < \infty.$$

Let $\mathcal{Q}_\mu f : X \times \mathbb{M} \rightarrow [0, \infty]$ be given by

$$\mathcal{Q}_\mu f(x, \xi) := \overline{\lim}_{\rho \rightarrow 0} \inf \left\{ \int_{Q_\rho(x)} f(y, \xi + \nabla_\mu w(y)) d\mu(y) : w \in H_{\mu,0}^{1,p}(Q_\rho(x); \mathbb{R}^m) \right\}.$$

The following result is a relaxation theorem with G -growth. Such problem with x -dependent G was studied in the Euclidean case in [AM12b], and the techniques of the proof broadly follow the methods exposed in Section 4.

THEOREM 2.19 ([AM18, Theorem 3.3]). *If $p > \kappa$ and (H₁)-(H₇) hold then*

$$\overline{E}(u, X) = \begin{cases} \int_X \widehat{\mathcal{Q}}_\mu f(x, \nabla_\mu u(x)) d\mu(x) & \text{if } u \in \text{dom}\mathcal{G} \\ \infty & \text{if } u \in H_\mu^{1,p}(X; \mathbb{R}^m) \setminus \text{dom}\mathcal{G} \end{cases} \quad (2.8)$$

where $\widehat{\mathcal{Q}}_\mu f : X \times \mathbb{M} \rightarrow [0, \infty]$ is given by $\widehat{\mathcal{Q}}_\mu f(x, \xi) = \underline{\lim}_{t \rightarrow 1^-} \mathcal{Q}_\mu f(x, t\xi)$.

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CHAPTER 3

RELAXATION, HOMOGENIZATION AND DIMENSION REDUCTION WITH AMPLE INTEGRANDS

This chapter presents the works on relaxation, homogenization and dimension reduction with ample integrands. More precisely, Section 1 gives an exposition of our works on ample integrands, such integrands can incorporate singular behavior, for instance in the setting of hyperelasticity, the necessity of an infinite amount of energy to reduce a finite volume to a zero volume. In Section 2 we present our relaxation results with ample integrands. In Section 3 we state our homogenization theorems with ample integrands. Two different homogenization results are presented, the first one involves some assumptions which make the integrand continuous with respect to the space variable, while the second result allows possible discontinuities with respect to the space variable of the integrand. Section 4 provides an exposition of our works on the 3D-2D passage by Γ -convergence which take into account the determinant conditions of hyperelasticity.

The works involved in this chapter are [ACM17; AM12c; ALM11; AM08a; AM08b; AM07; AM06].

1. AMPLE INTEGRANDS

Let $f : \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function which can take the infinite value. We set $\mathcal{Z}_\infty f, \mathcal{Z}f : \mathbb{M} \rightarrow [0, \infty]$ defined by

$$\begin{aligned} \mathcal{Z}_\infty f(F) &:= \inf \left\{ \int_Y f(F + \nabla \phi(y)) dy : \phi \in W_0^{1,\infty}(Y; \mathbb{R}^m) \right\}; \\ \mathcal{Z}f(F) &:= \inf \left\{ \int_Y f(F + \nabla \phi(y)) dy : \phi \in \text{Aff}_0(Y; \mathbb{R}^m) \right\}. \end{aligned}$$

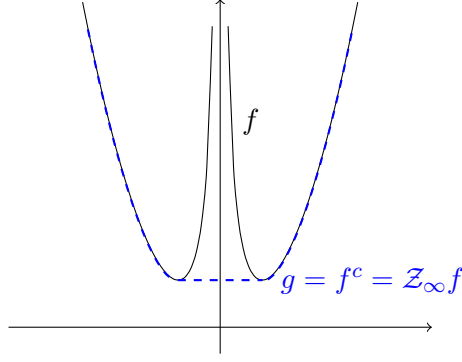
Note that $f \geq \mathcal{Z}f \geq \mathcal{Z}_\infty f \geq \mathcal{Q}f$. Indeed, when $g : \mathbb{M} \rightarrow [0, \infty[$ is quasiconvex and finite with $g \leq f$ we have $g = \mathcal{Z}_\infty g \leq \mathcal{Z}_\infty f$ which, by taking the supremum over all $g \leq f$, gives $\mathcal{Q}f \leq \mathcal{Z}_\infty f$.

In the following example, we can see that the envelope $\mathcal{Z}_\infty f$ is finite although f is not and has a singular behavior at 0 (see Figure 3.1). We assume that $m = N = 1$ and $p > 1$. Let $f : \mathbb{R} \rightarrow [0, \infty]$ be defined by

$$f(x) := \begin{cases} |x|^p + \frac{1}{|x|} & \text{if } x \neq 0 \\ \infty & \text{if } x = 0. \end{cases}$$

It is easy to see that $\mathcal{Z}_\infty f \leq g$ where

$$g(x) := \begin{cases} p^{-\frac{p}{p+1}} + p^{\frac{1}{p+1}} & \text{if } |x| < p^{-\frac{1}{p+1}} \\ f(x) & \text{if } |x| \geq p^{-\frac{1}{p+1}}. \end{cases}$$


 FIGURE 3.1. Illustration of the functions f and $\mathcal{Z}_\infty f$.

The function g is convex and finite with $g \leq f$. It follows that $\mathcal{Z}_\infty f$ is finite, so it is convex (and continuous) since Remark 3.5. The greatest convex function which is lower than f is denoted by f^c with $f^c \leq \mathcal{Z}_\infty f$ since Jensen inequality. We have then $g = f^c = \mathcal{Z}_\infty f$, that means $\mathcal{Z}_\infty f(x) \leq C(1 + |x|^p)$ for all $x \in \mathbb{R}$, with $C := \max(1, p^{\frac{1}{p+1}}) > 0$.

Motivated by this fact, we introduced a novel class of integrands which are called amplex (a word borrowed from the terminology used by Gromov [Gro86]). This class of functions can take the infinite value but not on too large subsets of matrices, so that the \mathcal{Z}_∞ envelope is finite everywhere or more strongly admitting p -growths (therefore, a better and more suggestive terminology should be ‘ \mathcal{Z}_∞ -finite integrands’ or ‘ \mathcal{Z}_∞ - p -growths integrands’ instead of ample integrands).

DEFINITION 3.1. We say that $f : \mathbb{M} \rightarrow [0, \infty]$ is weakly ample if $\mathcal{Z}_\infty f$ is finite, i.e. $\mathcal{Z}_\infty f(\xi) < \infty$ for all $\xi \in \mathbb{M}$.

When f is continuous and finite then $\mathcal{Q}f = \mathcal{Z}f = \mathcal{Z}_\infty f$ (see [Dac82]), we have extended this fact to integrands which are not necessarily finite.

THEOREM 3.2 ([AM07] (see also [AM08a])). *If f is weakly ample then $\mathcal{Q}f = \mathcal{Z}_\infty f$. If $\mathcal{Z}f$ is finite then f is weakly ample and $\mathcal{Q}f = \mathcal{Z}_\infty f = \mathcal{Z}f$.*

Assume for instance that $\mathcal{Z}f$ is finite. Using Proposition 3.3 below, the function $\mathcal{Z}f$ is continuous, so $\mathcal{Q}(\mathcal{Z}f) = \mathcal{Z}(\mathcal{Z}f) = \mathcal{Z}_\infty(\mathcal{Z}f)$. The Lemma 3.4 (2) below gives $\mathcal{Z}f \geq \mathcal{Z}_\infty f \geq \mathcal{Q}f \geq \mathcal{Q}(\mathcal{Z}f) = \mathcal{Z}f = \mathcal{Z}_\infty(\mathcal{Z}f)$.

The proof of Theorem 3.2 is based on the following result of Fonseca [Fon88].

PROPOSITION 3.3. *Let $f : \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function. Then $\mathcal{Z}_\infty f$ (resp. $\mathcal{Z}f$) is continuous in $\text{int}(\text{dom} \mathcal{Z}_\infty f)$ (resp. $\text{int}(\text{dom} \mathcal{Z}f)$).*

The following Lemma 3.4 (1) can be used to prove Proposition 3.3 (see [Fon88] or [Mül99b, Lemma 4.3]).

LEMMA 3.4. *Let $f : \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function.*

- (1) *Let $V \subset \mathbb{M}$ be a nonempty open set. We assume that f is finite in V . Then for every $\xi \in V$*

$$\mathcal{Z}f(\xi) \leq t\mathcal{Z}f(\xi + (1-t)d \otimes \nu) + (1-t)\mathcal{Z}f(\xi - td \otimes \nu)$$

for all $t \in]0, 1[$, all $d \in \mathbb{R}^m$ and all $\nu \in \mathbb{R}^N$.

- (2) *It holds $\mathcal{Z}[\mathcal{Z}_\infty f] = \mathcal{Z}_\infty f$ and $\mathcal{Z}[\mathcal{Z}f] = \mathcal{Z}f$.*

REMARKS 3.5. When $\mathcal{Z}_\infty f$ (resp. $\mathcal{Z}f$) is finite then $\mathcal{Z}_\infty f$ (resp. $\mathcal{Z}f$) is continuous and rank-one convex since Proposition 3.3 and Lemma 3.4 (1).

Assume that $m = N = 3$ and consider the following determinant condition on $f : \mathbb{M} \rightarrow [0, \infty]$ (allowing behavior of the type (3.5)):

$$\forall \delta > 0 \quad \exists c > 0 \quad \forall F \in \mathbb{M} \quad |\det F| \geq \delta \implies f(F) < c. \quad (3.1)$$

THEOREM 3.6 ([AM08a] (see also [AM09])). *Assume that $m = N = 3$. If (3.1) holds then $\mathcal{Z}f$ is finite. In particular, f is weakly ample.*

Now, we need a stronger definition of amplitude in order to prove relaxation theorems. Let $p \in [1, \infty[$.

DEFINITION 3.7. [AM12c, Definition 2.11] We say that f is p -ample if $\mathcal{Z}_\infty f$ has p -growth, i.e. there exists $C > 0$ such that $\mathcal{Z}_\infty f(F) \leq C(1 + |F|^p)$ for all $F \in \mathbb{M}$.

We denote by $\mathcal{R}f$ the rank-one convex envelope of f , i.e. the greatest rank-one convex function which is lower than f . The following result is due to [Ben96] (see also [AM09; AM12c]).

PROPOSITION 3.8. *Assume that $N \leq m$ and*

$$\exists \alpha > 0 \quad \exists \beta > 0 \quad \forall F \in \mathbb{M} \quad v(F) \geq \alpha \implies f(F) \leq \beta(1 + |F|^p) \quad (3.2)$$

where $v(F)$ is the product of the N singular values of F . Then $\mathcal{R}f$ has p -growth.

REMARKS 3.9. If $m = N$ then $v(F) = |\det F|$ for all $F \in \mathbb{M}$. If $N = 2$ and $m = 3$ then $v(F) = |F_1 \wedge F_2|$ for all $F = (F_1 | F_2) \in \mathbb{M}_{32}$ where $\mathbb{M}_{32} = \mathbb{R}^{3 \times 2}$ and (3.2) becomes

$$\exists \alpha > 0 \quad \exists \beta > 0 \quad \forall F \in \mathbb{M} \quad |F_1 \wedge F_2| \geq \alpha \implies f(F) \leq \beta(1 + |F|^p). \quad (3.3)$$

The following lemma is due to [DR04] (see also [CP98]).

LEMMA 3.10. *Assume that $m = N$. Let $F \in \mathbb{M}$ and $\alpha > 0$. Then*

$$|\det F| < \alpha \implies (\exists \varphi \in W_0^{1,\infty}(Y; \mathbb{R}^N) \quad |\det(F + \nabla \varphi(y))| = \alpha \text{ a.e. in } Y).$$

Assume that $m = N$ and consider the following determinant condition on $f : \mathbb{M} \rightarrow [0, \infty]$ (allowing behavior of the type (3.5)):

$$\exists \alpha > 0 \quad \exists \beta > 0 \quad \forall F \in \mathbb{M} \quad |\det F| \geq \alpha \implies f(F) \leq \beta(1 + |F|^p). \quad (3.4)$$

From Lemma 3.10 and Proposition 3.8 we can deduce the following result.

THEOREM 3.11 ([AM12c]). *Assume that $m = N$. If (3.4) holds then f is p -ample (i.e. $\mathcal{Z}_\infty f$ has p -growth).*

Indeed, from (3.4) there exist $\alpha, \beta > 0$ such that $\mathcal{Z}_\infty f(F) \leq f(F) \leq \beta(1 + |F|^p)$ for all $F \in \mathbb{M}$ with $|\det F| \geq \alpha$. Now, consider $F \in \mathbb{M}$ with $|\det F| < \alpha$, then by using Lemma 3.10 there exists $\varphi \in W_0^{1,\infty}(Y; \mathbb{R}^N)$ satisfying $|\det(F + \nabla \varphi(y))| = \alpha$ a.e. in Y . By definition of $\mathcal{Z}_\infty f$ and the growth in (3.4) we have

$$\mathcal{Z}_\infty f(F) \leq \int_Y f(F + \nabla \varphi(y)) dy \leq \beta \left(1 + \int_Y |F + \nabla \varphi(y)|^p dy \right) < \infty.$$

Thus f is weakly ample. By using Remark 3.5 we see that $\mathcal{Z}_\infty f$ is rank-one convex, so $\mathcal{Z}_\infty f \leq \mathcal{R}f$ and $\mathcal{R}f$ has p -growth since Proposition 3.8 and Remarks 3.9.

Now, combining Theorem 3.6, Proposition 3.8 and Remark 3.5, we have proved the following result.

THEOREM 3.12 ([AM12c]). *Assume that $m = N = 3$ and (3.1) holds. Then $\mathcal{Z}f$ has p -growth, in particular f is p -ample.*

In [AM08a] we proved the same result as Theorem 3.12 under additional assumptions.

THEOREM 3.13 ([AM12c]). *Assume that (3.4) holds. If $\mathcal{Z}f$ is finite, then $\mathcal{Z}f$ has p -growth, in particular f is p -ample.*

PERSPECTIVE 9. It would be interesting to know whether it is true that the functions $f : \mathbb{M} \rightarrow [0, \infty]$ such that $\mathcal{Q}f$ has p -growth are ample.

2. RELAXATION THEOREMS WITH AMPLE INTEGRANDS

Let $p > 1$. We consider the integral functional $I : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ defined by

$$I(u) := \int_{\Omega} f(\nabla u(x)) dx$$

where $\Omega \subset \mathbb{R}^N$ is a bounded open set and $f : \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable function. The lower semicontinuous envelope \bar{I} with respect to the L^p distance is defined by

$$\bar{I}(u) = \inf \left\{ \liminf_{\varepsilon \rightarrow 0} I(u_{\varepsilon}) : u_{\varepsilon} \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}.$$

Here, we present our works concerning the relaxation of integrals I when f is p -ample. It is about finding the conditions of the integral representation of \bar{I} . The most interesting singular behavior for us being

$$\lim_{\det F \rightarrow 0} f(F) = \infty, \quad (3.5)$$

which means, in the setting of hyperelasticity, the requirement of an infinite energy to crush the material down to zero volume.

Dacorogna proved in [Dac82] the following relaxation theorem.

THEOREM 3.14. *If f is continuous and has p -growth then*

$$\forall u \in W^{1,p}(\Omega; \mathbb{R}^m) \quad \bar{I}(u) = \int_{\Omega} \mathcal{Q}f(\nabla u(x)) dx. \quad (3.6)$$

We have extended the Dacorogna relaxation theorem to the case of p -ample integrands.

THEOREM 3.15 ([AM07; AM08a]). *If f is p -ample then (3.6) holds. If $\mathcal{Z}f$ has p -growth then (3.6) holds and moreover $\bar{I}(u) = \bar{I}_{\text{Aff}}(u)$ where*

$$\bar{I}_{\text{Aff}}(u) := \inf \left\{ \liminf_{n \rightarrow \infty} I(u_n) : \text{Aff}(\Omega; \mathbb{R}^m) \ni u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}.$$

The proof is based on the following lemma whose proof follows from Vitali covering theorem.

LEMMA 3.16. *Let $f : \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable function and let $O \subset \mathbb{R}^N$ be a nonempty open bounded. For every $F \in \mathbb{M}$ there exists $\{\phi_k\}_{k \in \mathbb{N}} \subset W_0^{1,\infty}(O; \mathbb{R}^m)$ (resp. $\text{Aff}_0(O; \mathbb{R}^m)$) such that*

$$\lim_{k \rightarrow \infty} \int_O f(F + \nabla \phi_k(x)) dx = \mathcal{Z}_{\infty} f(F) \text{ (resp. } \mathcal{Z}f(F)) \text{ and } \phi_k \rightarrow 0 \text{ in } L^p(O; \mathbb{R}^m).$$

If f is p -ample then $\mathcal{Z}_\infty f$ has p -growth and is continuous since Remark 3.5, by Theorem 3.14 we have for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$\begin{aligned} \overline{\mathcal{Z}_\infty I}(u) &= \inf \left\{ \varliminf_{n \rightarrow \infty} \int_{\Omega} \mathcal{Z}_\infty f(\nabla u_n(x)) dx : u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\} = \int_{\Omega} \mathcal{Q}(\mathcal{Z}_\infty f)(\nabla u(x)) dx \\ &= \int_{\Omega} \mathcal{Q}f(\nabla u(x)) dx \end{aligned}$$

where the last equality is due to Theorem 3.2. Now, it is sufficient to show that $\overline{\mathcal{Z}_\infty I} \geq \bar{I}$. This follows from Lemma 3.16, which allows to show that for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$\bar{I}(u) = \overline{\mathcal{Z}_\infty I}_{\text{Aff}}(u) = \inf \left\{ \varliminf_{n \rightarrow \infty} \int_{\Omega} \mathcal{Z}_\infty f(\nabla u_n(x)) dx : \text{Aff}(\Omega; \mathbb{R}^m) \ni u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}.$$

The last step consists in establishing $\overline{\mathcal{Z}_\infty I}_{\text{Aff}} = \overline{\mathcal{Z}_\infty I}$ by using the strong density of $\text{Aff}(\Omega; \mathbb{R}^m)$ in $W^{1,p}(\Omega; \mathbb{R}^m)$.

3. HOMOGENIZATION WITH AMPLE INTEGRANDS

Let $p > 1$. Let $\{I_\varepsilon\}_{\varepsilon > 0}$, $I_\varepsilon : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ be defined by

$$I_\varepsilon(u) := \int_{\Omega} f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx$$

where $\Omega \subset \mathbb{R}^N$ is a nonempty bounded open set and $f : \mathbb{R}^N \times \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable function which is p -coercive, i.e.

$$\exists c > 0 \quad \forall (x, F) \in \mathbb{R}^N \times \mathbb{M} \quad f(x, F) \geq c|\xi|^p,$$

and 1-periodic (with respect to the first variable), i.e.

$$\forall (x, F) \in \mathbb{R}^N \times \mathbb{M} \quad \forall z \in \mathbb{Z}^N \quad f(x + z, F) = f(x, F).$$

Braides [Bra85] and Müller [Mül87] proved the following homogenization theorem.

THEOREM 3.17. *Assume that f is p -coercive, 1-periodic, and has p -growth, i.e.*

$$\exists C > 0 \quad \forall (x, F) \in \mathbb{R}^N \times \mathbb{M} \quad f(x, F) \leq C(1 + |F|^p).$$

Then $\{I_\varepsilon\}_{\varepsilon > 0} \Gamma(L^p)$ -converges toward $I_{\text{hom}} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ given by

$$I_{\text{hom}}(u) = \int_{\Omega} \mathcal{H}f(\nabla u(x)) dx \tag{3.7}$$

where $\mathcal{H}f : \mathbb{M} \rightarrow [0, \infty]$ is the Braides-Müller homogenization formula given by

$$\mathcal{H}f(F) := \inf_{k \in \mathbb{N}^*} \inf_{\phi \in W_0^{1,p}(kY; \mathbb{R}^m)} \int_{kY} f(y, F + \nabla \phi(y)) dy. \tag{3.8}$$

We present here some extensions of the Braides-Müller homogenization theorem for ample integrands.

DEFINITION 3.18. We say that f is p -ample if there exists $C > 0$ such that $\mathcal{Z}_\infty f(x, F) \leq C(1 + |F|^p)$ for all $(x, F) \in \mathbb{R}^N \times \mathbb{M}$.

We have proved the following homogenization result which is compatible with the singular behavior

$$\forall x \in \mathbb{R}^N \quad \lim_{\det F \rightarrow 0} f(x, F) = \infty. \tag{3.9}$$

THEOREM 3.19 ([ALM11, Theorem 3.2]). *Assume that f is p -ample, p -coercive, 1-periodic, and satisfies for some $\omega : [0, \infty[\rightarrow [0, \infty[$ continuous at 0 with $\omega(0) = 0$*

$$\forall (x_1, x_2, F) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{M} \quad f(x, F) \leq \omega(|x_1 - x_2|)(1 + f(x_2, F)) + f(x_2, F). \quad (3.10)$$

Then $\{I_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$ -converges toward I_{hom} defined by (3.12) and $\mathcal{H}f$ defined by (3.8).

The proof is based on the well known result in the theory of Γ -convergence that we can replace I_ε by the relaxation \bar{I}_ε in the $\Gamma(L^p)$ -limit inf and sup, i.e.

$$\Gamma(L^p)\text{-}\liminf_{\varepsilon \rightarrow 0} I_\varepsilon = \Gamma(L^p)\text{-}\liminf_{\varepsilon \rightarrow 0} \bar{I}_\varepsilon \quad \text{and} \quad \Gamma(L^p)\text{-}\limsup_{\varepsilon \rightarrow 0} I_\varepsilon = \Gamma(L^p)\text{-}\limsup_{\varepsilon \rightarrow 0} \bar{I}_\varepsilon$$

where for each $\varepsilon > 0$

$$\bar{I}_\varepsilon(u) = \inf \left\{ \liminf_{n \rightarrow \infty} I_\varepsilon(u_n) : W^{1,p}(\Omega; \mathbb{R}^m) \ni u_n \rightarrow u \text{ in } L^p(\Omega; \mathbb{R}^m) \right\}. \quad (3.11)$$

With the help of the Serrin type condition (3.10), we proved the following relaxation which is an extension of Theorem 3.15 for x -dependent integrand

$$\bar{I}_\varepsilon(u) = \int_{\Omega} \mathcal{Z}_\infty f \left(\frac{x}{\varepsilon}, \nabla u(x) \right) dx$$

for all $\varepsilon > 0$ and all $u \in W^{1,p}(\Omega; \mathbb{R}^m)$. The integrand $\mathcal{Z}_\infty f$ has p -growth and inherits of all the assumptions of f , we can then apply the Braides-Müller homogenization theorem to obtain that for every $u \in W^{1,p}(\Omega; \mathbb{R}^m)$

$$\Gamma(L^p)\text{-}\lim_{\varepsilon \rightarrow 0} \bar{I}_\varepsilon(u) = \int_{\Omega} \mathcal{H}(\mathcal{Z}_\infty f)(\nabla u(x)) dx.$$

The last step consists in showing that $\mathcal{H}(\mathcal{Z}_\infty f) = \mathcal{H}f$ which is a consequence of a relaxation theorem similar to the one described above with $W^{1,p}(\Omega; \mathbb{R}^m)$ replaced by $W_0^{1,p}(\Omega; \mathbb{R}^m)$ in (3.11).

The Serrin type condition (3.10) implies that $f(\cdot, F)$ is continuous which is not a realistic assumption for homogenization problems. Thus in [ACM17] some attempts to remove the continuity assumption with respect to the first variable were developed. In the following, we present one of these results.

We consider the following assumptions:

(H₁) there exists $C > 0$ such that for every nonempty bounded open set $O \subset \mathbb{R}^N$ and every $F \in \mathbb{M}$

$$\frac{m_{\text{Aff}}(l_F, O)}{\mathcal{L}_N(O)} := \inf_{\phi \in \text{Aff}_0(O; \mathbb{R}^m)} \int_O f(y, F + \nabla \phi(y)) dy \leq C(1 + |F|^p)$$

where $l_F(x) := Fx$ for all $x \in \mathbb{R}^N$;

(H₂) for a.e. $x \in \mathbb{R}^N$ and for every $F \in \mathbb{M}$

$$\tilde{\mathcal{Z}}f(x, F) := \lim_{\rho \rightarrow 0} \frac{m_{\text{Aff}}(l_F, B_\rho(x))}{\mathcal{L}_N(B_\rho(x))} \leq f(x, F);$$

(H₃) for every $F \in \mathbb{M}$ it holds $\mathcal{Z} \left(\mathcal{H}_{\text{Aff}} \left(\tilde{\mathcal{Z}}f \right) \right) (F) \leq \mathcal{H} \left(\tilde{\mathcal{Z}}f \right) (F)$ where

$$\mathcal{H}_{\text{Aff}} \left(\tilde{\mathcal{Z}}f \right) (F) := \inf_{k \in \mathbb{N}^*} \inf_{\phi \in \text{Aff}_0(kY; \mathbb{R}^m)} \int_{kY} \tilde{\mathcal{Z}}f(y, F + \nabla \phi(y)) dy.$$

THEOREM 3.20 ([ACM17, Theorem 2.33]). *Assume that f is p -coercive, 1-periodic, and satisfies (H₁), (H₂) and (H₃). Then $\{I_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$ -converges toward $I_{\text{hom}} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, \infty]$ given by*

$$I_{\text{hom}}(u) = \int_{\Omega} \mathcal{H}(\tilde{\mathcal{Z}}f)(\nabla u(x)) dx. \quad (3.12)$$

Theorem 3.20 applies to integrand of the form $f(x, F) = a(x)H(F)$ where $a \in L^\infty(\mathbb{R}^N; \mathbb{R}^m)$ is 1-periodic, lower semicontinuous and $a(\cdot) \geq \eta$ for some $\eta > 0$, and $H : \mathbb{M} \rightarrow [0, \infty]$ is upper semicontinuous, p -coercive and satisfies the determinant condition (3.4).

The strategy of the proof of Theorem 3.20 is different from the one of Theorem 3.19. In fact, (H₁) and (H₂) implies that $\tilde{\mathcal{Z}}f$ has p -growth and is lower than f , and it is easy to see that $\tilde{\mathcal{Z}}f$ is 1-periodic, so we can apply the Braides-Müller homogenization theorem to obtain a lower bound of the $\Gamma(L^p)$ -limit inf

$$\Gamma(L^p)\text{-}\liminf_{\varepsilon \rightarrow 0} I_\varepsilon(u) \geq \int_{\Omega} \mathcal{H}(\tilde{\mathcal{Z}}f)(\nabla u(x)) dx.$$

For the $\Gamma(L^p)$ -limit sup, since $\mathcal{Z}(\mathcal{H}_{\text{Aff}}(\tilde{\mathcal{Z}}f))$ has p -growth and then is continuous, using the strong density of continuous piecewise affine $\text{Aff}(\Omega; \mathbb{R}^m)$ in $W^{1,p}(\Omega; \mathbb{R}^m)$, it is sufficient to show for every $u \in \text{Aff}(\Omega; \mathbb{R}^m)$

$$\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_\varepsilon(u) \leq \int_{\Omega} \mathcal{Z}(\mathcal{H}_{\text{Aff}}(\tilde{\mathcal{Z}}f))(\nabla u(x)) dx.$$

We prove first that for every $u \in \text{Aff}(\Omega; \mathbb{R}^m)$ there exists a sequence $\{u_\varepsilon\}_{\varepsilon>0} \subset \text{Aff}(\Omega; \mathbb{R}^m)$ of continuous piecewise affine functions such that $u_\varepsilon \rightarrow u$ in $L^p(\Omega; \mathbb{R}^m)$ and

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \tilde{\mathcal{Z}}f\left(\frac{x}{\varepsilon}, \nabla u_\varepsilon(x)\right) dx = \int_{\Omega} \mathcal{Z}(\mathcal{H}_{\text{Aff}}(\tilde{\mathcal{Z}}f))(\nabla u(x)) dx.$$

For the second step we use the strategy involving Vitali envelope exposed in Section 4 to show that for every $\varepsilon > 0$ and every $u \in \text{Aff}(\Omega; \mathbb{R}^m)$

$$\bar{I}_\varepsilon(u) \leq \int_{\Omega} \tilde{\mathcal{Z}}f\left(\frac{x}{\varepsilon}, \nabla u(x)\right) dx.$$

PERSPECTIVE 10. We proved in [ACM17] that the condition (H₃) holds when $\tilde{\mathcal{Z}}f$ has p -growth, i.e. $\sup_{x \in \mathbb{R}^N, F \in \mathbb{M}} \frac{\tilde{\mathcal{Z}}f(x, F)}{1+|F|^p} < \infty$, and $\tilde{\mathcal{Z}}f(x, \cdot)$ is upper semicontinuous for a.e. $x \in \mathbb{R}^N$. The condition (H₃) is related to the method of the proof of the upper bound, i.e. the use of the approximation of Sobolev functions by continuous piecewise affine functions. It would be interesting to try to find a new method to prove the upper bound in Theorem 3.20 without condition (H₃).

4. 3D-2D PASSAGE WITH DETERMINANT TYPE CONSTRAINTS

Let $f : \mathbb{M} \rightarrow [0, \infty]$ be a Borel measurable and p -coercive (with $p > 1$) function. For each $\varepsilon > 0$, let $I_\varepsilon : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \rightarrow [0, \infty]$ be defined by

$$I_\varepsilon(\phi) := \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f(\nabla \phi(x, x_3)) dx dx_3$$

where $\Sigma_\varepsilon := \Sigma \times]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[\subset \mathbb{R}^3$ with $\Sigma \subset \mathbb{R}^2$ Lipschitz, open and bounded, and a point of Σ_ε is denoted by (x, x_3) with $x \in \Sigma$ and $x_3 \in]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}[$. The problem of 3D-2D passage is to pass to

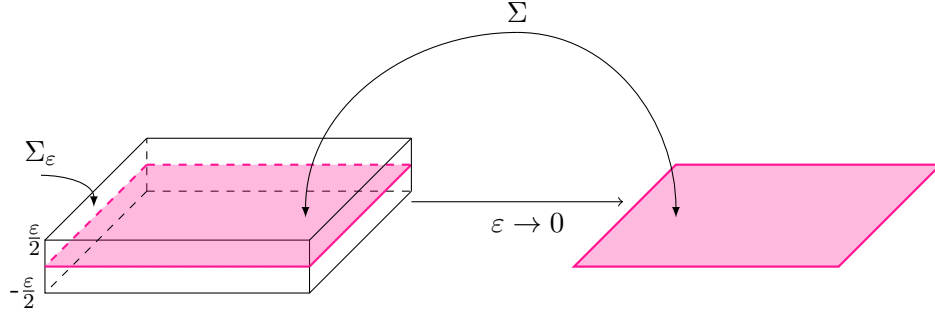


FIGURE 3.2. Illustration of a 3D-2D passage.

the limit as $\varepsilon \rightarrow 0$ by Γ -convergence to obtain $I_{\text{mem}} : W^{1,p}(\Sigma; \mathbb{R}^3) \rightarrow [0, \infty]$ given by

$$I_{\text{mem}}(\psi) := \int_{\Sigma} f_{\text{mem}}(\nabla \psi(x)) dx$$

where $f_{\text{mem}} : \mathbb{M}_{23} \rightarrow [0, \infty]$ with $\mathbb{M}_{23} = \mathbb{R}^{2 \times 3}$ (see Figure 3.2).

In our works [Anz05; AM06; AM08b] we used the $\Gamma(\pi)$ -convergence which is a variation of Γ -convergence, and which seems to us well adapted to dimension reduction problems, it was introduced by Anzellotti, Baldo and Percivale [ABP94] (see also [ABP91]). Let $\pi = \{\pi_{\varepsilon}\}_{\varepsilon}$ be the family of L^p -continuous maps $\pi_{\varepsilon} : W^{1,p}(\Sigma_{\varepsilon}; \mathbb{R}^3) \rightarrow W^{1,p}(\Sigma; \mathbb{R}^3)$ defined by

$$\pi_{\varepsilon}(\phi) := \frac{1}{\varepsilon} \int_{-\frac{\varepsilon}{2}}^{\frac{\varepsilon}{2}} \phi(\cdot, x_3) dx_3.$$

DEFINITION 3.21. We say that $\{I_{\varepsilon}\}_{\varepsilon}$ $\Gamma(\pi)$ -converge to I_{mem} as $\varepsilon \rightarrow 0$, and we write

$$I_{\text{mem}} = \Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_{\varepsilon},$$

if and only if for every $\psi \in W^{1,p}(\Sigma; \mathbb{R}^3)$

$$(\Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_{\varepsilon})(\psi) = (\Gamma(\pi)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_{\varepsilon})(\psi) = I_{\text{mem}}(\psi)$$

with $\Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_{\varepsilon}, \Gamma(\pi)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_{\varepsilon} : W^{1,p}(\Sigma; \mathbb{R}^3) \rightarrow [0, \infty]$ respectively given by

$$\begin{aligned} \Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_{\varepsilon}(\psi) &:= \inf \left\{ \lim_{\varepsilon \rightarrow 0} I_{\varepsilon}(\phi_{\varepsilon}) : \pi_{\varepsilon}(\phi_{\varepsilon}) \rightarrow \psi \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}, \\ \Gamma(\pi)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_{\varepsilon}(\psi) &:= \inf \left\{ \overline{\lim}_{\varepsilon \rightarrow 0} I_{\varepsilon}(\phi_{\varepsilon}) : \pi_{\varepsilon}(\phi_{\varepsilon}) \rightarrow \psi \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}. \end{aligned}$$

For each $\varepsilon > 0$, consider $\mathcal{I}_{\varepsilon} : W^{1,p}(\Sigma; \mathbb{R}^3) \rightarrow [0, \infty]$ defined by $\mathcal{I}_{\varepsilon}(\psi) := \inf \left\{ I_{\varepsilon}(\phi) : \pi_{\varepsilon}(\phi) = \psi \right\}$. It is not difficult to show (see [ABP94]) that $I_{\text{mem}} = \Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_{\varepsilon}$ if and only if $I_{\text{mem}} = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \mathcal{I}_{\varepsilon}$.

Let $f_0 : \mathbb{M}^{3 \times 2} \rightarrow [0, \infty]$ be defined by

$$f_0(\xi) := \inf_{\zeta \in \mathbb{R}^3} f(\xi \mid \zeta).$$

Le Dret-Raoult in their seminal papers [LR93; LR95] have proved the following result:

THEOREM 3.22. *If f is continuous and has p -growth, i.e.*

$$\exists C > 0 \quad \forall F \in \mathbb{M} \quad f(F) \leq C(1 + |F|^p)$$

then

$$\forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon(\psi) = \int_{\Sigma} \mathcal{Q}f_0(\nabla\psi(x))dx.$$

Although Theorem 3.22 is not compatible with the singular behavior (3.5), it established a suitable variational framework to deal with dimensional reduction problems, it has been the starting point for many works on the subject. By using Theorem 3.22 we can prove the following result of which a preliminary version under additional assumptions was proved in [AM06].

THEOREM 3.23 ([AM12c] (see also [AM09])). *Assume that (3.4) holds. Then for every $\psi \in W^{1,p}(\Sigma; \mathbb{R}^3)$*

$$\Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon(\psi) = \int_{\Sigma} \mathcal{Q}f_0(\nabla\psi(x))dx.$$

Let us outline the proof. As the $\Gamma(\pi)$ -limit is stable by substituting I_ε by its relaxed functional \bar{I}_ε , i.e. $\bar{I}_\varepsilon : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \rightarrow [0, \infty]$ given by

$$\bar{I}_\varepsilon(\phi) := \inf \left\{ \liminf_{n \rightarrow \infty} I_\varepsilon(\phi_n) : \phi_n \rightarrow \phi \text{ in } L^p(\Sigma_\varepsilon; \mathbb{R}^3) \right\},$$

it suffices to prove that for every $\psi \in W^{1,p}(\Sigma; \mathbb{R}^3)$

$$\Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} \bar{I}_\varepsilon(\psi) = \int_{\Sigma} \mathcal{Q}f_0(\nabla\psi(x))dx.$$

As f satisfies (3.4) it is p -amplable by Theorem 3.11, and so by Theorem 3.15 and Theorem 3.2 we have for every $\varepsilon > 0$ and every $\phi \in W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3)$

$$\bar{I}_\varepsilon(\phi) = \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} \mathcal{Q}f(\nabla\phi(x, x_3))dx dx_3$$

with $\mathcal{Q}f = \mathcal{Z}_\infty f$. Applying the Le Dret-Raoult theorem we deduce that

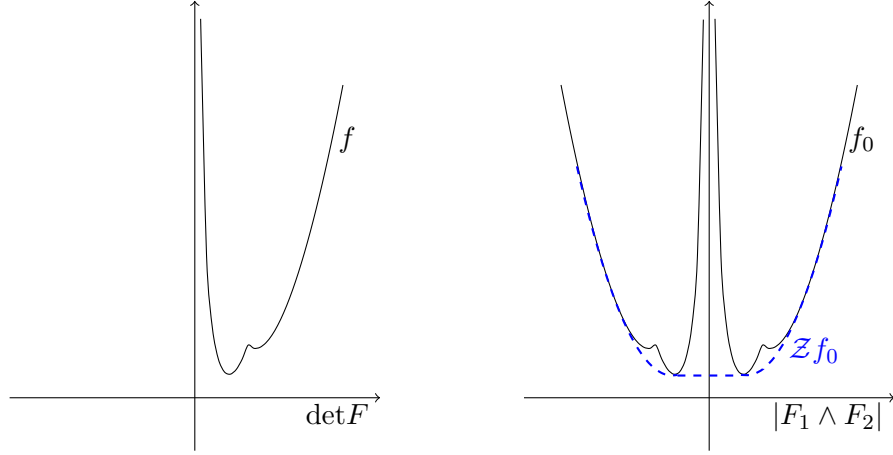
$$\forall \psi \in W^{1,p}(\Sigma; \mathbb{R}^3) \quad \Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} \bar{I}_\varepsilon(\psi) = \int_{\Sigma} \mathcal{Q}[\mathcal{Q}f]_0(\nabla\psi(x))dx$$

with $[\mathcal{Q}f]_0 : \mathbb{M}_{32} \rightarrow [0, \infty]$ given by $[\mathcal{Q}f]_0(\xi) := \inf_{\zeta \in \mathbb{R}^3} \mathcal{Q}f(\xi \mid \zeta)$. Finally, we prove that $\mathcal{Q}[\mathcal{Q}f]_0 = \mathcal{Q}f_0$.

PERSPECTIVE 11. It would be interesting to consider the problem of reduction dimension and homogenization simultaneously under the determinant condition (3.4). It was studied in the p -growth case by [Shu00] (see also [BFF00; BF05]). More precisely, let $\delta, \varepsilon \in]0, 1[$, we consider a periodic distribution of heterogeneities on Σ with size $\delta > 0$. For each $s := (\delta, \varepsilon) \in \mathcal{R}_+ := \mathbb{R}_+^* \times \mathbb{R}_+^*$ we consider the integral functional $E_s : W^{1,p}(\Sigma_\varepsilon; \mathbb{R}^3) \rightarrow [0, \infty]$ defined by:

$$E_s(u) := \frac{1}{\varepsilon} \int_{\Sigma_\varepsilon} f\left(\frac{x}{\delta}, \nabla u(x, x_3)\right) dx dx_3$$

where $f : \mathbb{R}^2 \times \mathbb{M} \rightarrow [0, \infty]$ is a Borel measurable function 1-periodic with respect to the space variable. In [Shu00], a comprehensive study has been carried out by taking into account all the possible limits behavior of the ratio ε/δ when $\mathcal{R}_+ \ni s \rightarrow 0$. It would be interesting to extend the study to ample integrands.


 FIGURE 3.3. The passage from f to f_0 and then to $\mathcal{Z}f_0$.

The following theorem is a 3D-2D passage by Γ -convergence with the assumptions on f compatible with the two basic conditions of hyperelasticity. It is the result of several contributions, the attempt of Percivale in 1991 (see [Per91]), the works in the p -growth case of Le Dret and Raoult (see [LR93; LR95]) and especially the important contributions of Ben Belgacem (see [Ben96; Ben97; Ben00]) who paved the path of the proof.

THEOREM 3.24 ([AM08b] (see also [AM12c; AM09])). *Assume that*

(D₁) f is continuous;

(D₂) $f(F) = \infty \iff \det F \leq 0$ for all $F \in \mathbb{M}_{33}$;

(D₃) $\forall \delta > 0 \quad \exists c_\delta > 0 \quad \forall F \in \mathbb{M}_{33} \quad (\det F \geq \delta \implies f(F) \leq c_\delta(1 + |F|^p))$.

Then for every $\psi \in W^{1,p}(\Sigma; \mathbb{R}^3)$

$$\Gamma(\pi)\text{-}\lim_{\varepsilon \rightarrow 0} I_\varepsilon(\psi) = \int_{\Sigma} \mathcal{Q}f_0(\nabla\psi(x))dx.$$

It is easy to see that if f satisfies (D₁), (D₂) and (D₃) then f_0 is continuous and

$$\forall \alpha > 0 \quad \exists \beta_\alpha > 0 \quad \forall F \in \mathbb{M}_{32} \quad (|F_1 \wedge F_2| \geq \alpha \implies f_0(\xi) \leq \beta_\alpha(1 + |\xi|^p)). \quad (3.13)$$

Somehow the one-sided constraint (D₃) is mirrored in (3.13) by passing to f_0 which turns out to be an ‘ample condition’ (see Figure 3.3).

THEOREM 3.25 ([AM07] (see also [AM08a; AM09; AM12c])). *If f_0 satisfies (3.3) then $\mathcal{Z}f_0$ has p -growth.*

Since (3.13) implies (3.3), $\mathcal{Z}f_0$ has p -growth. Let $\mathcal{I}, \bar{\mathcal{I}} : W^{1,p}(\Sigma; \mathbb{R}^3) \longrightarrow [0, \infty]$ be defined by:

$$\mathcal{I}(\psi) := \int_{\Sigma} f_0(\nabla\psi(x))dx \quad \text{and} \quad \bar{\mathcal{I}}(\psi) := \inf \left\{ \liminf_{n \rightarrow \infty} \mathcal{I}(\psi_n) : \psi_n \rightarrow \psi \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}$$

As $\mathcal{Z}f_0$ has p -growth, by Theorem 3.15 we have for every $\psi \in W^{1,p}(\Sigma; \mathbb{R}^3)$

$$\bar{\mathcal{I}}(\psi) = \int_{\Sigma} \mathcal{Q}f_0(\nabla\psi(x))dx.$$

By using an interchange of infimum and integral result that we developed in [AM03], we can show the following representation of \mathcal{I} :

$$\forall \psi \in C_*^1(\bar{\Sigma}; \mathbb{R}^3) \quad \mathcal{I}(\psi) = \inf_{j \geq 1} \left\{ \inf_{\zeta \in C(\bar{\Sigma}; \mathbb{R}^3), \det(\nabla \psi(\cdot)|\zeta(\cdot)) \geq \frac{1}{j}} \int_{\Sigma} W(\nabla \psi(x)|\zeta(x)) dx \right\}. \quad (3.14)$$

With the help of (3.14) we can show the following inequalities:

$$\Gamma(\pi)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} I_{\varepsilon} \leq \overline{\mathcal{I}}_{\text{diff}_*} \quad \text{and} \quad \overline{\mathcal{I}} \leq \Gamma(\pi)\text{-}\underline{\lim}_{\varepsilon \rightarrow 0} I_{\varepsilon}$$

where $\overline{\mathcal{I}}_{\text{diff}_*} : W^{1,p}(\Sigma; \mathbb{R}^3) \rightarrow [0, \infty]$ is defined by

$$\overline{\mathcal{I}}_{\text{diff}_*}(\psi) := \inf \left\{ \underline{\lim}_{n \rightarrow \infty} \mathcal{I}(\psi_n) : C_*^1(\bar{\Sigma}; \mathbb{R}^3) \ni \psi_n \rightarrow \psi \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}$$

with $C_*^1(\bar{\Sigma}; \mathbb{R}^3)$ is the set of C^1 -immersions from $\bar{\Sigma}$ to \mathbb{R}^3 , i.e.

$$C_*^1(\bar{\Sigma}; \mathbb{R}^3) := \left\{ \psi \in C^1(\bar{\Sigma}; \mathbb{R}^3) : \forall x \in \bar{\Sigma} \quad \partial_1 \psi(x) \wedge \partial_2 \psi(x) \neq 0 \right\}.$$

To prove Theorem 3.24 it is sufficient to show that $\overline{\mathcal{I}}_{\text{diff}_*} \leq \overline{\mathcal{I}}$.

Let $\overline{\mathcal{I}}_{\text{Aff}_{\text{li}}}, \overline{\mathcal{R}\mathcal{I}}, \overline{\mathcal{R}\mathcal{I}}_{\text{Aff}_{\text{li}}} : W^{1,p}(\Sigma; \mathbb{R}^3) \rightarrow [0, \infty]$ be respectively defined by:

- $\overline{\mathcal{I}}_{\text{Aff}_{\text{li}}}(\psi) := \inf \left\{ \underline{\lim}_{n \rightarrow \infty} \mathcal{I}(\psi_n) : \text{Aff}_{\text{li}}(\Sigma; \mathbb{R}^3) \ni \psi_n \rightarrow \psi \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\};$
- $\overline{\mathcal{R}\mathcal{I}}(\psi) := \inf \left\{ \underline{\lim}_{n \rightarrow \infty} \int_{\Sigma} \mathcal{R}f_0(\nabla \psi_n(x)) dx : \psi_n \rightarrow \psi \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\};$
- $\overline{\mathcal{R}\mathcal{I}}_{\text{Aff}_{\text{li}}}(\psi) := \inf \left\{ \underline{\lim}_{n \rightarrow \infty} \int_{\Sigma} \mathcal{R}f_0(\nabla \psi_n(x)) dx : \text{Aff}_{\text{li}}(\Sigma; \mathbb{R}^3) \ni \psi_n \rightarrow \psi \text{ in } L^p(\Sigma; \mathbb{R}^3) \right\}$

with $\text{Aff}_{\text{li}}(\Sigma; \mathbb{R}^3) := \{ \psi \in \text{Aff}(\Sigma; \mathbb{R}^3) : \psi \text{ is locally injective} \}$. Ben Belgacem in his thesis [Ben96] implemented the following strategy: as $\overline{\mathcal{R}\mathcal{I}} \leq \overline{\mathcal{I}}$, a way for proving $\overline{\mathcal{I}}_{\text{diff}_*} \leq \overline{\mathcal{I}}$ is to establish the following three inequalities:

$$\overline{\mathcal{I}}_{\text{diff}_*} \leq \overline{\mathcal{I}}_{\text{Aff}_{\text{li}}} \leq \overline{\mathcal{R}\mathcal{I}}_{\text{Aff}_{\text{li}}} \leq \overline{\mathcal{R}\mathcal{I}}. \quad (3.15)$$

More precisely, the first inequality follows by using the fact that f_0 is continuous and satisfies (3.13) together with the result ([Ben96], see also [AM09]) that we can approximate strongly in $W^{1,p}(\Sigma; \mathbb{R}^3)$ a locally injective continuous piecewise affine function $\psi \in \text{Aff}_{\text{li}}(\Sigma; \mathbb{R}^3)$ by a sequence of C^1 -immersions $\{\psi_n\}_{n \geq 1} \subset C_*^1(\bar{\Sigma}; \mathbb{R}^3)$ satisfying $|\partial_1 \psi_n(x) \wedge \partial_2 \psi_n(x)| \geq \delta$ for all $x \in \bar{\Sigma}$, all $n \geq 1$ and for some $\delta > 0$. The second inequality is obtained by exploiting the Kohn-Strang representation of the rank-one convex envelope $\mathcal{R}f_0$ of f_0 (see [Ben96], see also [AM12c; AM09]). Finally, using Remarks 3.9 and Proposition 3.8 we see that $\mathcal{R}f_0$ has p -growth and so is continuous, we then establish the last inequality by using the strong density of $\text{Aff}_{\text{li}}(\Sigma; \mathbb{R}^3)$ in $W^{1,p}(\Sigma; \mathbb{R}^3)$ which is a consequence of [GÉ71] (see also [AM09]). We can summarize the previous arguments by indicating for each inequality as subscript the (main) result we have been used to prove it:

$$\overline{\mathcal{I}}_{\text{diff}_*} \underset{[\text{Ben96}]}{\leq} \overline{\mathcal{I}}_{\text{Aff}_{\text{li}}} \underset{[\text{KS86}]}{\leq} \overline{\mathcal{R}\mathcal{I}}_{\text{Aff}_{\text{li}}} \underset{[\text{GÉ71}]}{\leq} \overline{\mathcal{R}\mathcal{I}}.$$

PERSPECTIVE 12. It would be interesting to search a different strategy for the proof of $\overline{\mathcal{I}}_{\text{diff}_*} \leq \overline{\mathcal{I}}$ rather than using the sequence of inequalities (3.15).

APPENDIX

A. SOME DEFINITIONS

Heinonen and Koskela introduced in [HK98] the upper gradient.

DEFINITION A.1. A Borel function $g : X \rightarrow [0, \infty]$ is said to be an upper gradient for $f : X \rightarrow \mathbb{R}$ if $|f(c(1)) - f(c(0))| \leq \int_0^1 g(c(s))ds$ for all continuous rectifiable curves $c : [0, 1] \rightarrow X$.

The concept of upper gradient has been generalized by Cheeger as follows (see [Che99, Definition 2.8]).

DEFINITION A.2.

- (i) A function $g \in L^p_\mu(X)$ is said to be a generalized upper gradient (or a p -weak upper gradient) for $f \in L^p_\mu(X)$ if there exist $\{f_n\}_n \subset L^p_\mu(X)$ and $\{g_n\}_n \subset L^p_\mu(X)$ such that for each $n \geq 1$, g_n is an upper gradient for f_n , $f_n \rightarrow f$ in $L^p_\mu(X)$ and $g_n \rightarrow g$ in $L^p_\mu(X)$.
- (ii) A minimal generalized upper gradient for $f \in L^p_\mu(X)$ is a generalized upper gradient g_f , such that $|f|_{1,p} = |f|_{L^p_\mu(X)} + |g_f|_{L^p_\mu(X)}$, where

$$|f|_{1,p} = |f|_{L^p_\mu(X)} + \inf \left\{ \liminf_{n \rightarrow \infty} |g_n|_{L^p_\mu(X)} : \{g_n\}_{n \in \mathbb{N}} \in G(f) \right\}$$

with

$$G(f) := \left\{ \{g_n\}_{n \in \mathbb{N}} : \exists \{f_n\}_{n \in \mathbb{N}} \lim_{n \rightarrow \infty} |f_n - f|_{L^p_\mu(X)} = 0 \text{ and } \forall n \in \mathbb{N} f_n \text{ is an upper gradient for } g_n \right\}.$$

We give the definition of Γ -convergence ([DF75]) on a metric space, see [Dal93; BD98] for more details.

DEFINITION A.3. Let (M, δ) be a metric space. Let $\{E_\varepsilon\}_{\varepsilon > 0}$, $E_\varepsilon : M \rightarrow [0, \infty]$ be a family of functions. We say that $\{E_\varepsilon\}_{\varepsilon > 0}$ $\Gamma(\delta)$ -converge if $\Gamma(\delta)$ - $\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u) = \Gamma(\delta)$ - $\overline{\lim}_{\varepsilon \rightarrow 0} E_\varepsilon(u)$, where

$$\Gamma(\delta)\text{-}\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u) := \inf \left\{ \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) : u_\varepsilon \xrightarrow{\delta} u \right\}$$

and

$$\Gamma(\delta)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} E_\varepsilon(u) := \inf \left\{ \overline{\lim}_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) : u_\varepsilon \xrightarrow{\delta} u \right\}$$

with

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) &:= \inf \left\{ \liminf_{n \rightarrow \infty} E_{\varepsilon_n}(u_{\varepsilon_n}) : \{\varepsilon_n\}_{n \in \mathbb{N}} \subset]0, \infty[, \varepsilon_n \rightarrow 0 \right\} \\ \overline{\lim}_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon) &:= \sup \left\{ \overline{\lim}_{n \rightarrow \infty} E_{\varepsilon_n}(u_{\varepsilon_n}) : \{\varepsilon_n\}_{n \in \mathbb{N}} \subset]0, \infty[, \varepsilon_n \rightarrow 0 \right\} \end{aligned}$$

for all $u \in M$. In this case $\{E_\varepsilon\}_{\varepsilon > 0}$ $\Gamma(\delta)$ -converge towards $E_0 : X \rightarrow [0, \infty]$ with

$$E_0 := \Gamma(\delta)\text{-}\liminf_{\varepsilon \rightarrow 0} E_\varepsilon = \Gamma(\delta)\text{-}\overline{\lim}_{\varepsilon \rightarrow 0} E_\varepsilon.$$

B. SOME PROOFS

PROOF OF THEOREM 2.3. Fix $u \in \overline{D}$. We can assume that $\underline{\lim}_{t \rightarrow 1^-} f(tu) < \infty$. Let $\tau \in]0, 1[$. There exists $t \in]\tau, 1[$ such that

$$\underline{\lim}_{t \rightarrow 1} f(tu) \geq \inf_{t \in]\tau, 1[} f(tu) > \tau - 1 + f(tu).$$

Now, since $\frac{\tau}{t} < 1$ we can write

$$f(\tau u) \leq \Delta_{D,f}^a \left(\frac{\tau}{t} \right) (a + f(tu)) + f(tu) \leq \Delta_{D,f}^a \left(\frac{\tau}{t} \right) (a + \underline{\lim}_{t \rightarrow 1} f(tu) + 1 - \tau) + \underline{\lim}_{t \rightarrow 1} f(tu) + 1 - \tau.$$

Letting $\tau \rightarrow 1$ we have $\frac{\tau}{t} \rightarrow 1$ and then using the ru-usc property, we obtain

$$\overline{\lim}_{\tau \rightarrow 1} f(\tau u) \leq \underline{\lim}_{t \rightarrow 1} f(tu).$$

Assume in addition that $\widehat{f}(u) < \infty$. It is direct to see that for every $s \in]0, 1[$

$$\widehat{f}(su) \leq \overline{\lim}_{t \rightarrow 1} [\Delta_{D,f}^a(s)(a + f(tu)) + f(tu)] = \Delta_{D,f}^a(s) (a + \widehat{f}(u)) + \widehat{f}(u),$$

which yields $\Delta_{D \cap \text{dom} \widehat{f}, \widehat{f}}^a(s) \leq \Delta_{D,f}^a(s)$ and we finish by letting $s \rightarrow 1$.

Finally, the last inequalities hold since (4) and the definitions of \widehat{f} and of the lower semicontinuous envelope of f . ■

PROOF OF THEOREM 2.6. Let $x \in X$ and $u \in \mathbb{F}_x$. Since f is ru-usc, we have $\widehat{f}(x, u) \leq f(x, u)$, see Remarks 2.2 (4). For each $t \in [0, 1[$, $tu \in \mathbb{F}_x$ and $tu \rightarrow u$ as $t \rightarrow 1$, then by lower semicontinuity $\widehat{f}(x, u) = \lim_{t \rightarrow 1^-} \widehat{f}(x, tu) \geq f(x, u)$. By Theorem 2.3 and Remark 2.4 \widehat{f} is ru-usc. We know that $\overline{f} \leq \widehat{f}$ since Remark 2.4. Let $u \in V$ be such that $\overline{f}(x, u) < \infty$. There exists a sequence $u_n \rightarrow u$ such that $\lim_{n \rightarrow \infty} f(u_n) = \overline{f}(u)$, it follows that $u_n \in \mathbb{F}_x$ for all $n \in \mathbb{N}$ and $u \in \overline{\mathbb{F}}_x$. Since $tu_n \in \mathbb{F}_x$ and $tu \in \mathbb{F}_x$ for all $t \in [0, 1[$, we have by the lower semicontinuity of $f(x, \cdot)$ on \mathbb{F}_x and the ru-usc property

$$\overline{f}(x, u) \geq \overline{f}(x, u) + (a + \overline{f}(x, u)) \overline{\lim}_{t \rightarrow 1} \Delta_f^a(t) \geq \underline{\lim}_{t \rightarrow 1} \underline{\lim}_{n \rightarrow \infty} f(x, tu_n) \geq \underline{\lim}_{t \rightarrow 1} f(x, tu) = \widehat{f}(x, u). \blacksquare$$

PROOF OF LEMMA 2.18. Let $\mathcal{S}_r := \{v = (v_1, \dots, v_d) \in \mathbb{R}^d : \forall k \in \{0, \dots, d\} \ v_k \in \{-r, 0, r\}\}$. Since (U₁) we have $L^*(x) := \sup_{v \in \mathcal{S}_r} L(x, v) < \infty$ for all $x \in M$ (Note that L^* is $(\mathcal{M}, \mathcal{B}([0, \infty]))$ -measurable as a supremum of a finite family of $(\mathcal{M}, \mathcal{B}([0, \infty]))$ -measurable functions.). The set $\{(x, v) \in M \times \mathbb{R}^d : L(x, v) = L^*(x)\} \cap (M \times \mathcal{S}_r)$ belongs to $\mathcal{M} \otimes \mathcal{B}(\mathbb{R}^d)$, so there exists a $(\mathcal{M}, \mathcal{B}(\mathbb{R}^d))$ -measurable function $u : M \rightarrow \mathcal{S}_r$ such that $L^*(x) = L(x, u(x))$ (see [CV77, p. 86]). For each $v \in \mathcal{S}_r$ set $M_v := \{x \in M : u(x) = v\} \in \mathcal{M}$. The family $\{M_v\}_{v \in \mathcal{S}_r}$ is a measurable partition of M , i.e. it is pairwise disjoint and satisfies $\cup_{v \in \mathcal{S}_r} M_v = M$. So, by using (U₁) we have

$$\begin{aligned} \int_M L^*(x) d\nu(x) &= \int_M L(x, u(x)) d\nu(x) = \sum_{v \in \mathcal{S}_r} \int_{M_v} L(x, u(x)) d\nu(x) \\ &= \sum_{v \in \mathcal{S}_r} \int_{M_v} L(x, v) d\nu(x) \\ &\leq \sum_{v \in \mathcal{S}_r} \int_M L(x, v) d\nu(x) < \infty. \end{aligned} \quad (\text{B.1})$$

We define d equivalence relations on \mathbb{R}^d . Let $(v, w) \in \mathbb{R}^d \times \mathbb{R}^d$. For each $i \in \{1, \dots, d-1\}$, we say that $v \sim_i w$ if there exists $u = (u_1, \dots, u_k, \dots, u_d) \in \mathcal{S}_r$ such that

$$v - w = \sum_{k=i+1}^d u_k e_k$$

where $\{e_k\}_{k=1}^d$ is the standard basis of \mathbb{R}^d . When $i = d$, we say that $v \sim_d w$ if $v = w$. We denote by $[v]_i$ the equivalence class of v for the relation \sim_i for all $i \in \{0, \dots, d\}$. Let $v \in \mathbb{R}^d$ satisfy $|v|_\infty \leq r$. We claim that is enough to show that for every $x \in M$

$$\forall i \in \{1, \dots, d\} \quad \forall w \in \left[\sum_{k=1}^i v_k e_k \right]_i \quad L(x, w) \leq \Phi^i(L^*(x)). \quad (\text{B.2})$$

where we set for every $t \in \mathbb{R}_+$

$$\Phi^1(t) := \Phi(t) \quad \text{and} \quad \Phi^i(t) := \Phi(\Phi^{i-1}(t)) \quad \text{for all } i \geq 2.$$

Indeed, when $i = d$ we have $\left[\sum_{k=1}^d v_k e_k \right]_d = \{v\}$, so by using (B.2) we get for every $x \in M$

$$L(x, v) \leq \Phi^d(L^*(x)),$$

and then taking the supremum over all $v \in \mathbb{R}^d$ satisfying $|v|_\infty \leq r$ and integrating over M , we obtain

$$\int_M \sup_{|v|_\infty \leq r} L(x, v) d\nu(x) \leq \int_M \Phi^d(L^*(x)) d\nu(x).$$

Now, we can use the Jensen inequality since Φ^d is still concave and the finiteness of the last term is a consequence of (B.1)

$$\int_M \Phi^d(L^*(x)) d\nu(x) \leq \nu(M) \Phi^d\left(\int_M L^*(x) d\nu(x)\right) < \infty.$$

Fix $x \in M$. Let us prove (B.2) by induction. Let $w \in [v_1 e_1]_1$. There exists $u = (u_1, \dots, u_k, \dots, u_d) \in \mathcal{S}_r$ such that $w = v_1 e_1 + \sum_{k=1}^d u_k e_k$, which can be rewritten as

$$w = \frac{|v_1|}{r} \left(\text{sign}(v_1) r e_1 + \sum_{k=2}^d u_k e_k \right) + \left(1 - \frac{|v_1|}{r} \right) \left(0 + \sum_{k=2}^d u_k e_k \right).$$

By (U₂) and using the fact that Φ is nondecreasing, we have

$$L(x, w) \leq \Phi \left(\max \left(L \left(x, \text{sign}(v_1) r e_1 + \sum_{k=2}^d u_k e_k \right), L \left(x, \left(\sum_{k=2}^d u_k e_k \right) \right) \right) \right) \leq \Phi(L^*(x)).$$

So, we proved that $L(x, w) \leq \Phi(L^*(x))$ for all $w \in [v_1 e_1]_1$.

Let $i \in \{1, \dots, d\}$. Assume that

$$\forall j \in \{1, \dots, i\} \quad \forall w \in \left[\sum_{k=1}^j v_k e_k \right]_j \quad L(x, w) \leq \Phi^j(L^*(x)). \quad (\text{B.3})$$

Let $w \in \left[\sum_{k=1}^{i+1} v_k e_k \right]_{i+1}$. We are going to show that $L(x, w) \leq \Phi^{i+1}(L^*(x))$. There exists $u = (u_1, \dots, u_k, \dots, u_d) \in \mathcal{S}_r$ such that $w = \sum_{k=1}^{i+1} v_k e_k + \sum_{k=i+2}^d u_k e_k$, which can be rewritten

as

$$w = \frac{|v_{i+1}|}{r} \left(\sum_{k=1}^i v_k e_k + \text{sign}(v_{i+1})r + \sum_{k=i+2}^d u_k e_k \right) + \left(1 - \frac{|v_{i+1}|}{r} \right) \left(\sum_{k=1}^i v_k e_k + \sum_{k=i+2}^d u_k e_k \right).$$

By (U₂), it follows that

$$L(x, w) \leq \Phi \left(\max \left(L \left(x, \sum_{k=1}^i v_k e_k + \text{sign}(v_{i+1})r + \sum_{k=i+2}^d u_k e_k \right), L \left(x, \sum_{k=1}^i v_k e_k + \sum_{k=i+2}^d u_k e_k \right) \right) \right).$$

We see that

$$\sum_{k=1}^i v_k e_k + \text{sign}(v_{i+1})r + \sum_{k=i+2}^d u_k e_k \in \left[\sum_{k=1}^i v_k e_k \right]_i \quad \text{and} \quad \sum_{k=1}^i v_k e_k + \sum_{k=i+2}^d u_k e_k \in \left[\sum_{k=1}^i v_k e_k \right]_i,$$

so, by (U₂) and using the fact that Φ is nondecreasing, we have by using (B.3)

$$L(x, w) \leq \Phi \left(\Phi^i(L^*(x)) \right) = \Phi^{i+1}(L^*(x)). \blacksquare$$

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