Quelques Contributions en Finance Mathématique : Contrôle, Filtrage et Analyse Stochastique

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Année 2017

Habilitation à Diriger les Recherches
Spécialité : Mathématiques Appliquées

présentée par
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Soutenue le DATE, devant le jury composé de :

au vu des rapports de :

Per correr miglior acque alza le vele
omai la navicella del mio ingegno,

Remerciements

Mes premiers remerciements vont naturellement à tous mes co-auteurs, en particulier, Etienne Chevalier, Vathana Ly Vath, Alexandre Roch, Guillaume Bernis, Desirée Teobaldelli, Davide Ticchi, Andrea Vindigni, Giorgia Callegaro, Carlo Sgarra, Ying Jiao, Chao Zhou, Chunhua Ma, Luca Di Persio et Luca Prezioso.

Je suis également reconnaissant envers

Mes remerciements vont également vers mes collègues enseignants-chercheurs et personnels administratifs de l'université Paris Diderot.

J'exprime également ma reconnaissance envers Nathalie, Valérie et Virginie pour toutes leur aides administratives pendant ces années passées à Sophie Germain.

Mes pensées vont enfin aux collègues et amis pour la bonne ambiance et surtout pour leurs amitiés. Je tiens à remercier, en particulier, Vathana, Alexandre, Ying, Chao, Etienne, Marie, Guillaume, Jerome, Nicolas, Sergio, Thomas, Luca, Marco, Alessandro, Francesco, Gianni, Carlo et Giorgia.

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Première partie

CURRICULUM VITAE, ACTIVITES DE RECHERCHE, RESPONSABILITES ADMINISTRATIVES ET ENSEIGNEMENTS

1.1 Curriculum Vitae

Situation :

2010-2017 Maitre de Conférences à l'Université Paris Diderot.

2009-2010 Post-doc chargé de cours à l'Ecole Polytechnique Fédérale de Lausanne.

2006-2009 Chercheur à l'université de Turin (Italie).

2000-2004 Elève fonctionnaire stagiaire à l'Ecole Normale Supérieure de la rue d'Ulm.

1998-2000 Elève à l'Ecole Normale Supérieure de Pisa (Italie).

Cursus universitaire :

2004-2008 Thèse de doctorat en Mathématiques Appliquées, Ecole Nationale des Ponts et Chaussés et Ecole Normale Supérieure de Pisa (Italie), dirigée par Nicolas Bouleau sur l'*Applications de la Théorie des Erreurs par Formes de Dirichlet.*

2002-2003 DEA de Statistiques et Modélisation Aléatoire appliquée à l'Economie et la Finance, Université Paris 7.

2000-2002 Licence, Maitrise et Magistère de Physique à l'ENS.

1998-2000 Etudes de Physique à l'ENS Pise.

1.2 Publications et production scientifique

Articles publiés ou acceptés pour publication:

Alpha-CIR model in sovereign interest rate modelling avec Y. Jiao et C. Ma, 2017, *Finance and Stochastics*, vol 21, issue 3, pp. 789-813.

Optimal investment in markets with over and under-reaction to Information avec G. Callegaro, M. Gaigi et C. Sgarra, 2017, *Mathematics and Financial Economics*, vol 11, issue 3, pp. 299-322.

Alternative to beta coefficients in the context of diffusions, avec G. Bernis, 2017, *Quantitative Finance*, volume 17, issue 2, pp. 275-288.

Optimal execution cost for liquidation through a limit order market, avec E. Chevalier, V. Ly Vath et A. Roch, 2016, *International Journal of Theoretical and Applied Finance*, Vol 19, issue 1.

Exit optimal exit strategies for investment projects, avec E. Chevalier, V. Ly Vath et A. Roch, 2015, *Journal of Mathematical Analysis and Applications*, Vol.425(2), pp.666-694.

Uncertainty and the politics of employment protection, avec C. Tealdi et A. Vindigni, 2015, *Journal of Labor Economics*, vol 33-1, pp 209-267.

Trend detection under erroneous observations - Application to quantitative financial strategies, avec G. Bernis, June 2015, *Risk J. Investment Strategies*, vol 4, n. 3.

Optimal credit allocation under regime uncertainty with sensitivity analysis avec G. Bernis, L. Carassus et G. Docq, Feb 2015, *International Journal of Theoretical and Applied Finance*, volume 18.

An optimal dividend and investment control problem under debt constraints, avec E. Chevalier et V. Ly Vath, 2013, *SIAM J. Finan. Math.*, 4(1), 297 - 326.

Stochastic sensitivity study for optimal credit allocation, avec L. Carassus, 2013 *Arbitrage, Credit and Informational Risks*, Proceedings of the Sino-French Research Program in Financial Mathematics Conference, Beijing June 2013, pp 147-168.

Bid-ask spread modelling, a perturbation approach, avec T. Lim, V. Ly Vath et JM. Sahut, 2013, Seminar on Stochastic Analysis, Random Fields and Applications VII , *Progress in Probability*, 67, pp 411-434.

Travaux soumis:

A branching process approach to power markets avec Y. Jiao, C. Ma et C. Sgarra, 2017.

Stochastic evolution of distributions - applications to CDS indices avec G. Bernis, N. Brunel et A. Kornprobst, 2017.

Sensitivity analysis for marked Hawkes processes - application to CLO pricing, avec G. Bernis et K. Salhi, 2017.

Working papers et travaux en cours:

Wages and Employment in Economies with Multi-Worker Firms, Uncertainty and Labor Turnover Costs, avec B. Bruegemann, V. Ly Vath et A. Vindigni, working paper.

Shadow economy dynamics, avec V. Ly Vath, D. Teobaldelli et D. Ticchi, en cours.

Alpha-Heston model, avec Y. Jiao, C. Ma et C. Zhou, en cours

Optimal harvesting with a reserve policy under uncertain environment, avec V. Ly Vath et M. Gaigi, en cours.

Exponential affine processes and Samuelson effect, avec Y. Jiao, C. Ma et C. Sgarra, en cours.

Optimal dividend and takeover control problem, avec L. Di Persio, V. Ly Vath et L. Prezioso, en cours.

Optimal trading strategies in a credit market with over/under-reaction, avec G. Bernis et V. Ly Vath, en cours.

1.3 Encadrement doctoral et scientifique

Direction thèse: Je suis directeur de thèse de Luca Prezioso, cotutelle entre Paris Diderot et Université de Verona (Italie), co-directeurs: Luca Di Persio (Verona) et Vathana Ly Vath (ENSIIE), 2016-2019. Cette thèse porte sur la théorie du contrôle stochastique et applications à l'économie et la finance.

Encadrement master: J'ai suivi les mémoires M2 de Paul Bichon, Houssein Bouita, Vincent Cochet (ENSIIE) et Yann Monneyron (P1). J'ai aussi suivi plusieurs mémoires M1 à l'université Paris Diderot.

Participation au projet Alma: J'ai été membre du projet Alma research-Paris Diderot pendant les deux premières années à Paris Diderot. Alma Research avait embauché trois post-docs (Thomas Lim, Armand Ngoupeyou et Salvatore Federico) avec comme objectif la creation d'un pont entre le milieu académique et les entreprises financières de la place parisienne.

Porteur du projet α-CIR: J'ai été porteur du projet "on the Cox-Ingersoll-Ross type models driven by stable processes", qui a reçu un financement de 5700 euro de la part de la fondation Hadamard dans le cadre de "Research in Pairs 2014".

Service comme rapporteur: J'ai été referee pour les revues: Finance and Stochastics, Quantitative Finance, International Journal of Theoretical and Applied Finance, Mathematical Finance, Scandinavian Actuarial Journal, Risk Journal of Investment Strategies and Siam Mathematical Finance.

1.4 Responsabilités administratives

Encadrement M2MO: validation et suivi stage, séminaire pro et suivi des anciens: Pendant les quatre premières années à Paris Diderot, j'étais responsable de la validation et du suivi des stages des étudiants du Master 2 en Modélisation Aléatoire (environ soixante-dix étudiants par an avec un pic de plus de cent en 2012). J'étais aussi co-organisateur du séminaire professionnel du meme master. J'ai assuré le suivi des anciens étudiants pour arriver à l'aboutissement avec la creation de l'association Laure Elie Alumni.

Jury M2MO: A compter de ma deuxième année je suis membre du jury du master M2MO.

Participation aux comités de selection: J'ai été membre de trois comités de sélection pour un poste de MCF (2012 et 2014 à Paris Diderot, 2014 à l'ENSIIE).

1.5 Responsabilités pedagogiques

Chargé de cours: Gestion des Risques (M2MO, P7), Calcul Stochastique (M2 Finance des Marchés, P1), Actuariat (M1 ISIFAR, P7), Mathématiques Financières (M2 EPFL, Lausanne), Méthodes Mathématiques pour l'Économie et la Finance (M1 économie, Turin).

Chargé de TD Processus Stochastique et Applications en Finance (M2MO, P7), Mathématiques Financières (M1 ISIFAR, P7), Statistique Inférentielle (L3, P7) Analyse et Algèbre élémentaires I et II (L1, P7), Analyse et Algebre III et IV (L2, P7), Equations Différentielles pour Biologistes (L2, P7) Probabilités (L2, Lausanne) Mathématiques (L1 Turin).

Visiting professor: J'ai donné un cours de "Introduction to Stochastic Control Theory and Applications" dans le cadre des PhD Program in Economics et PhD Program in Control Systems au John von Neumann Institute of Vietnam National University at Ho Chi Minh City en mars 2013.

Visiting professor: J'ai donné un cours de "Stochastic Control with applications in Economics" à l'IMT Lucca en mai 2013.

Visiting professor: J'ai donné un cours de "Real markets, microstructure, clusters and Hawkes with a Branching process point of view" dans le cadre du M2 in Mathematics à l'Université de Verone en mars 2017.

Visiting professor: Le meme cours réadapté aux étudiants du M2 et PhD a été donné au BICMR de l'Université de Pekin en juin 2017.

1.6 Invitations

Juillet 2017: Visiting Researcher à BICMR, invité par Ying Jiao et Chunhua Ma, Beijing, Chine.

Mai 2017: Invited Speaker au "Stochastic Modeling in Finance", Siemreap, Cambodge.

Mai 2017: Invited Speaker au "Second Paris-Asia Conference in Quantitative Finance", Suzhou, Chine.

Novembre 2016: Invited Speaker au "Berlin-Paris Workshop on Stochastic Analysis with applications in Biology and Finance", Berlin, Allemagne.

Juillet 2016: Visiting Researcher à Peking University, invité par Chunhua Ma et Ying Jiao, trois semaines, Chine.

Décembre 2015: Invited Speaker au Canadian Mathematical Society Winter Meeting, Montreal, Canada.

Novembre 2015: Visiting Researcher at NUS, invité par Chao Zhou, Singapour.

Septembre 2015: Invited Speaker au 7th general AMAMEF and Swissquote Conference, Lausanne, Suisse.

Juin 2015: Invited Speaker au "Risk measures and Optimization in Finance and Insurance", Beijing, Chine.

Octobre 2013: Visiting Fellow au IMT Lucca, Laboratoire ICES, Italie.

Juillet 2013: Invited Speaker à la session "Dirichlet Forms in Finance" organisé par Jin Ma dans le cadre de IMS-SWUFE International Conference on Statistics and Probability, Chengdu, Chine.

Printemps 2012: Invité au Centre Interfacultaire Bernoulli dans le cadre du semestre "stochastic analysis and applications", Lausanne, Suisse.

Mai 2011: Invited Speaker au Centro Stefano Franscini dans le cadre du "Seventh Seminar on Stochastic Analysis, Random Fields and Applications", Ascona, Suisse.

Juin 2011: Invited Speaker au "Stochastic Modelling and Applications to Finance" , Beijing, Chine.

Plusieurs séminaires et contribution aux conferences: Beijing, Bern, Boston, Firenze, Grenoble, Institut Henri Poincaré, Lausanne, Lecce, Los Angeles, Lucca, Metabief, Padova, Paris Est, Paris 13, Pisa, Paris Saclay, Singapour, Stresa, Trieste, Torino, Verona.

1.7 Organisations de semestres thématiques, conferences et écoles

— Co-organisateur du semestre thématique "Verona-Paris Stochastic Modelling Semester" qui aura lieu à Verone de mi-décembre 2017 à juin 2018.

— Co-organisateur de la conference "Frontiers in Stochastic Modeling for Finance" à Padoue (2-5 Février 2016).

— Co-organisateur de l'école d'hiver "Frontiers in Stochastic Modeling for Finance" à Padoue (8-12 Février 2016).

— Co-organisateur de la conference "Paris-Southeast Asia Conference in Mathematical Finance" à Siem Reap (7-11 Février 2015).

— Co-organisateur du workshop on "Financial Mathematics and Numerical Probability" à Pekin (19-21 Juin 2013).

Résumé

Dans ce rapport, je présente une partie de mes travaux effectués entre 2010 et 2017. Depuis ma soutenance de thèse, mes travaux peuvent essentiellement se repartir en trois classes sur la base des outils mathématiques utilisés.

- Problèmes de contrôle stochastique appliqué à la finance d'entreprise, au marché du travail et à la modélisation en risque de liquidité ;
- Théorie du filtrage et ses applications en allocation optimale ;
- Application de l'analyse stochastique à la modélisation des risques financiers.

Cette division est certainement arbitraire en vue de la contiguïté des applications traitées. Des chevauchements évidents vont apparaitre et vont en partie justifier l'ordre d'affichage qui s'éloigne de celui de parution des articles liées. Ceci permet tout de même de présenter mes travaux dans des cadres assez homogènes du point de vue mathématique et de pouvoir ensuite souligner les liens, les symétries ou parfois la complémentarité, entre les différents aspects modélisés.

Partie 1 : Contrôle stochastique appliqué à la finance d'entreprise et au marche du travail

Dans cette partie j'ai principalement travaillé sur des problèmes de contrôle stochastique. Dans cet axe de recherche, les principaux défis sont la modélisation et formulation mathématique des problèmes ainsi que leur résolution mathématique, numérique voire parfois la recherche d'une solution explicite dans des cas particulier au fin d'une étude de sensibilité. Les problèmes auxquels je me suis intéressé sont formulés comme des problèmes de contrôle stochastique non standard. Cela nécessite des analyses assez fines au niveau mathématique, aussi bien dans la partie théorique, qui consiste à caractériser les fonctions valeurs à l'aide de l'approche de viscosité pour prouver l'existence et l'unicité, que dans la description des différentes régions caractérisant les politiques optimales des problèmes. Mais un des gros objectifs de la modélisation et de ses applications en économie est de proposer des modèles à la fois capables de capter des phénomènes observés en trouvant une explication simple et parfois surprenante et au même moment d'avoir au moins dans certains cas des solutions quasi explicites pour pouvoir mesurer l'ampleur de ces phénomènes. Cet équilibre entre parcimonie du modèle, richesse de ses prévisions et maniabilité n'est jamais facile et fait ressortir dans notre science dure un certain sens, pas forcement du beau, mais au moins de l'elegance : "Frustra fit per plura quod fieri potest per pauciora".

Cette partie correspond à deux chapitres. Le premier est consacré aux problèmes de finance de marché et il repose sur deux articles, qui se focalisent sur deux aspects différents de la vie d'une entreprise, le premier [A2] écrit avec Etienne Chevalier (Evry) et Vathana

Ly Vath (ENSIIE) traite un problème de dividende et d'investissement sous contrainte de dette. Le deuxième [A6] est une collaboration avec Etienne Chevalier, Vathana Ly Vath et Alexandre Roch (Montréal) qui s'intéresse à l'instant optimal de liquidation pour un investissement qui n'est plus rentable. Du point de vue mathématique, le premier problème est formulé comme un contrôle combiné singulier et de changement multiple de regimes sous contraintes. Le vrai défi est de réussir à caractériser la politique optimale avec un contrôle mixte et en multi-regimes et de pouvoir fournir des interprétations financières de cette politique. Le deuxième problème se focalise sur la stratégie optimale de désinvestissement d'un projet qui n'est plus profitable avec un coût de liquidation stochastique. L'entreprise a le choix entre attendre un éventuel repreneur ou liquider immédiatement en assumant le coût. Le coût de liquidation est gouverné par un processus de retour à la moyenne alors que l'arrivée d'un repreneur est décrite par un processus de Cox dirigé par une chaine de Markov. Nous formulons le problème comme un arrêt optimal en deux dimensions avec horizon aléatoire. Le gros défit du problème vient de l'absence à priori d'unicité de la solution du processus de liquidation qui nous demande une analyse fine pour la caractérisation de la solution. En plus nous réussissons à exhiber les solutions en forme fermée dans le cas d'utilités logarithmique et puissance si le coût de liquidation suit un processus de type CIR.

Le deuxième chapitre s'intéresse à un problème économique central pour l'époque actuelle à savoir la régulation du marché du travail et il est basé sur un article [A4] écrit avec Cristina Tealdi (Lucca) et Andrea Vindigni (Genova). Nous étudions l'impact des coûts de turnover, à savoir les couts d'embauche et de licenciement, sur le chômage. Nous trouvons aussi une explication politique à la forte dichotomie entre pays anglo-saxons et le continent européen et l'augmentation progressive de ce clivage dans les dernieres décennies. Du point de vue mathématique, le problème se configure comme un arrêt optimal avec horizon aléatoire. Le vrai défi de l'article a été de caractériser les distributions ergodiques des entreprises et des travailleurs, leur salaires, etc. L'objectif a toujours été d'obtenir des formules fermées pour pouvoir calculer explicitement les effets d'une variation des paramètres sur les variables économiques comme le taux de chômage et les salaires.

Parmi mes articles publiés et en cours qui ne sont pas resumés dans ce memoire mais ont leur place dans cette partie, je peux citer les travaux sur la liquidation optimale [A8] qui est un sujet assez centrale dans la finance récente, voir [4], [5], [6], et sur le quel je compte revenir avec les méthodes des processus de branchement comme dans le cadre de [7]. Avec Luca Prezioso, qui vient de commencer son doctorat à Verona, je m'intéresse à une extension du problème de la dette en introduisant le cycle économique et un retard dans l'arrivée des opportunités de croissance externe [W20]. Dans la suite du travail sur l'application du contrôle stochastique au marché du travail, je travaille à une extension au cadre d'entreprises avec un continuum de travailleurs [W15]. Je m'intéresse aussi à l'etude de l'économie parallèle et des politiques optimales pour la gestion optimale de la part du gouvernent, [W16].

Partie 2 : Filtrage stochastique et applications à l'allocation optimale

Dans cette partie, je vais détailler mes contributions en allocation optimale en information partielle. Le problème d'allocation optimale a été longuement traité dans la littérature en arrivant au consensus que la stratégie optimale revient à trouver un portefeuille de marché optimal, celui complètement investi en actifs risqués et qui optimise la diversification du portefeuille en donnant le meilleur rapport entre rendement et volatilité, et répartir ensuite la richesse entre ce portefeuille et un actif sans risque pour permettre à l'investisseur de limiter son risque mesuré par la volatilité du portefeuille globale. Ce paradigme, introduit par Markowitz [91], se montre globalement solide si on prend en compte des imperfections de marché comme de la consommation ou des coûts de transaction, par exemple voir [96], [80], [43].

Dans l'article [A10], en collaboration avec Giorgia Callegaro (Padova), M'hamed Gaigi (Tunis) et Carlo Sgarra (Poli-Milano), nous nous concentrons tout d'abord sur une imperfection très différente à savoir la sous ou sur-réaction des actifs à l'arrivée d'actualités les concernants. Ces phénomènes sont très connus en littérature de finance comportementale, voir [47], [48]. Il faut voir que le cadre usuel d'une évolution continue des trajectoires des prix des actifs financiers est lié à l'hypothèse implicite que le flux d'informations se comporte comme un fleuve tranquille. Dans la réalité les nouvelles, surtout les mauvaises, arrivent aux marchés soudainement entrainant des réactions de la part des acteurs de marché qui induisent des brutaux ajustements des prix. Le mot même d'ajustement, usuellement utilisé pour décrire ce phénomène, cache une hypothèse implicite d'efficience qui revient à assumer que les marchés arrivent à trouver immédiatement un nouvel équilibre tenant en compte les informations qui viennent d'être dévoilées. Ceci n'est que rarement le cas. Dans les jours et même les semaines qui suivent, les actifs ont tendance à accentuer l'effet du saut, à la hausse comme à la baisse, ou à retracer en réduisant son effet. On parle alors dans le premier cas de sous-réaction, de sur-réaction dans le deuxième. L'interprétation financière de la sous-réaction est que le marché a sous-estimé l'impact de la nouvelle, le saut n'est donc pas assez important pour que le nouveau prix tienne en compte la nouvelle réalité économique de l'entreprise. A l'opposé dans le cas de une sur-réaction, le marché a surestimé l'impact, le prix va donc se réaligner progressivement en réduisant l'écart. Il n'est évidement pas possible pour un investisseur de savoir a priori si après un fort saut le cours de l'actif va faire état d'une sur- ou sous-réaction et des estimations historiques [63] donnent autant des chances à priori aux deux cas. Le problème auquel nous nous sommes confrontés est donc l'impact qui a cette imperfection de marché sur la politique optimale à la Markowitz.

Du point de vue mathématique, nous avons un problème de filtrage non-standard car la filtration naturelle du processus observé est générée à la fois par un mouvement Brownien et par un processus à sauts. L'approche standard par la méthode des innovations ne peut pas être appliquée directement car il se fonde intrinsèquement sur l'égalité entre la projection

prévisible et celle optionnelle, paradigme usuel dans le cadre d'une filtration brownienne mais clairement faux en présence de sauts observables. Nous avons contourné le problème par une stratégie nouvelle en littérature et à première vue surprenante à savoir par une suite de grossissement initiaux des filtrations. Surprenant est le moins qu'on puisse dire de cette approche : pour réduire l'information on commence par l'augmenter. Pour être plus précis, pour le k-ième grossissement nous rajoutons les marques des k premiers sauts. Des potentiels k-agents initiés seront informés à l'avance de l'amplitude des k premier sauts mais pas des instants de saut. Le point clef de notre démonstration est de remarquer que avec ces filtrations grossies, il est possible d'adapter la méthode des innovations. Enfin un point crucial de notre preuve est que ces k-agents redeviennent des agents classiques après le k-ième saut car ils n'ont plus d'information privilégiée. Le filtrage optimal, et donc ensuite la politique d'investissement, sera donc, entre le saut k et $k+1$, celle qui suivrait le k-agent.

Parmi mes articles publiés et en cours qui ne sont pas resumés dans ce memoire mais ont leur place dans cette partie, je peux citer les travaux sur l'allocation optimale avec incertitude sur le drift sur étude des sensibilités [A3] et [A5]. Il y a aussi un étude sur la detection des trends fondé sur les ondelettes [A7] Je travaille aussi à une extension de l'article sur la sous et sur-reaction à des problèmes de contrôle impulsionel avec application au marché de credit [W21].

Partie 3 : Analyse stochastique et applications à la gestion des risques financières

Dans cette partie, j'ai réuni mes travaux qui font recours à des méthodes de l'analyse stochastique dans le but d'évaluer et gérer les risques financières. Pendant ma thèse sous la direction de Nicolas Bouleau (ENPC) je me suis familiarisé avec les techniques de l'analyse stochastique et en particulier la théorie des capacités et des formes de Dirichlet. Ces méthodes ont fait preuve de leur puissance dans la modélisation et dans la prise en compte des risques extremes mal décrits par le paradigme usuel du vingtième siècle. Ce chemin m'a conduit récemment à m'intéresser aux processus de Hawkes qui sont un des axes de recherche en plein expansion ces derniers années, voir par example [7].

Dans un premiere temps, nous allons nous concentrer sur l'application de la decomposition en chaos pour l'extension du Capital Asset Pricing Model et du coefficient beta dans le but de tenir en compte la presence de risques extremes. Notre méthode repose sur l'expansion en chaos. Dans l'article [A9], écrit en collaboration avec Guillaume Bernis (Natixis Asset Management), nous étendons le cadre Gaussian du CAPM, voir [91], pour prendre en compte les queues épaisses des distributions qui sont observés sur les marchés financiers. Le coefficient beta joue un role central dans la politique d'allocation optimale et dans la couverture des mouvements du portefeuille vis à vis des fluctuations du marché. La simplicité du beta et en particulier le manque de dependance fonctionnelle avec les valeurs du portefeuille et de l'indice sont souvent pointés, voir par example [100] et [102]. Dans ce cadre nous avons proposé une extension du coefficient beta basé sur la decomposition en

chaos du portefeuille et de l'indice en autorisation une ponderation pour chaque chaos. Le beta usuel se retrouve donc on concentrant le poids sur le seul premier chaos. Nous avons aussi montré des résultats financiers intéressant par example en liant le nouveau beta à une insensibilité du portefeuille à un changement de drift de l'indice. Enfin notre cadre se prête bien à fournir des formules fermés ou quasi-fermés pour plusieurs diffusions avec le cas notable des processus de Jacobi. Une autre application interessante est la possibilité de définir des bandes de Bollinger fonctionnelles capables de detecter les changements dans l'evolution des marchés financiers.

Dans le meme cadre d'analyse stochastique mais en suivant un tout autre approche, je me suis récemment intéressé aux modèle d'évolution basé sur le champs aléatoires. Dans les articles fondateurs de Dawson et Li [55] et [56], les auteurs introduisent une nouvelle classe d'equation différentielles stochastiques dirigé par des champs aléatoires. La principale innovation est d'accroitre la dimension des champs pour intégrer par rapport à cette nouvelle dimension jusqu'à la valeur du processus lui meme. Dawson et Li arrivent à montrer l'existence et l'unicité de la solution et que cette dernière vérifie la propriété de branchement, à savoir le processus est infiniment divisible par rapport à la donnée initiale. Filipovic avait deja montré un lien très fort entre processus de branchement et structure exponentielle affine, donc les processus ayant une fonction cumulatrice affine. Les modèles à structure exponentielle affine représentent un filon de recherché très prolifique en finance mathématique tout d'abord grace à la forme explicite de la transformé de Laplace ce qui permet de calculer rapidement les prix des produits dérivées et par leur maniabilité. Les applications couvrent quasiment tout les domaines du secteur et des sous-jacents aussi different que les actions, les taux ou les temps de faillite, voir par example [73], [59] et [62]. Le principal intérêt, de mon point de vue, de la forme intégrale proposé par Dawson et Li est de fournir une écriture explicite de l'équation différentielle stochastique satisfaite par le processus. Ceci permet de comprendre plus en profondeur les raisons de certaines phénomènes qui sont anticipés par les processus à l'étude.

Dans ce contexte, je travaille actuellement avec Ying Jiao (ISFA Lyon), Chunhua Ma (Nankai, Chine), Carlo Sgarra (Politecnico di Milano, Italy) et Chao Zhou (National University of Singapore) sur des applications des modèles de branchement en finance. Dans un premier travail [A11] avec Ying et Chunhua, présenté dans le sixième chapitre, nous entendons le modèle CIR [37] pour la modélisation du taux d'interet court en incluant un processus de branchement dirigé par un alpha-stable à spectre positif. Nous donnons deux formes équivalentes pour écrire l'EDS associée, une basé sur l'écriture intégrale de Dawson et Li, l'autre par une équation plus standard mais moins maniable. A l'aide de l'écriture intégrale nous arrivons à mettre en lumière certaines previsions du modèle. Tout d'abord la structure auto-excitante des processus de branchement a au fait un drôle d'effet sur les taux et les obligations, à savoir nous prouvons que le modèle reproduit aisément la persistence des faibles taux d'intérêts ce qui semble en contradiction avec l'idée d'auto-excitation. Ce effet s'explique financièrement par le fait que si l'arrivé d'un saut va augmenter l'intensité

du saut suivant, par symétrie l'absence d'un saut va réduire cette meme intensité. Il y a donc une trappe propre aux branching qui va anéantir le processus qui trop s'approche de zero. Du point de vue mathématique, ce phénomène se traduit par une augmentation brutale de la vitesse de retour à la moyenne entre deux sauts. Cet increment de la vitesse observé est propre à la structure de branchement et nous l'avons mis en evidence grace à la representation intégrale qui permet d'évaluer cette vitesse effective de retour à la moyenne entre deux sauts. De plus, ce terme de correction de la vitesse de retour à la moyenne vient du compensateur du processus à saut qui dirige l'EDS satisfait par le taux d'intérêt. Or ce compensateur augmente clairement si on augmente l'amplitude des sauts, ce que dans un alpha-stable revient à réduire le paramètre α. On a donc un effet très contre-intuitif car la persistence des taux bas est accentué par l'épaisseur de la queue droite de la distribution des sauts, dans une sorte de loi du contra et patior. Ce meme effet est tellement important qu'il est transféré aux taux longs en enduisant une inversion de la dependence des prix des zero-coupons vis à vis des risques de increments brutaux des taux.

Dans le travail [S12] avec Ying, Chunhua et Carlo, nous utilisons un modèle de branchement pour décrire l'evolution des prix spot de l'énergie. La principale raison de ce choix vient de la presence de fortes fluctuations, dites spikes, et du fait que leur distribution n'est pas homogène dans le temps mais des regroupements (cluster) apparaissent. Nous avons essayé de voir les consequences si nous remplaçons le terme martingale du modèle usuel de type Levy Ornstein Uhlenbeck, voir [17], par un terme auto-excitant. Nous avons prouvé que les prix des forwards restent inchangés mais que la structure de la prime de risque change remarquablement montrant naturellement une courbe non-monotone et changeante de signe chose assez difficilement reproduite par les modèles usuels. De plus nous proposons une méthode de calibration astucieuse basé uniquement sur l'observation des temps d'arrivé des sauts. Cette stratégie est basé mathématiquement sur un résultat de convergence du processus de branchement vers l'intensité d'un processus de Hawkes sous condition que la vitesse de retour à la moyenne diverge, ce qui est notre cas. Nous pouvons en suite adapter un résultat connu d'estimation sur les processus ponctuels pour déduire la vraisemblance et l'estimateur qui la maximise.

Je suis en fin en train de travailler sur plusieurs articles en utilisant les processus de branchement avec representation intégrale. Tout d'abord, dans le preprint [S14] avec Guillaume Bernis et Kaouther Salhi, nous travaillons à l'étude de la sensibilité pour un modèle pour les obligations de dette collateralisé (CLO) Ce modèle est naturellement basé sur les processus de Hawkes marqué qui dirige les temps de faillite. Ce travail représente, dans un certain sense, la fermeture d'une boucle commencé pendant ma thèse sur les . Avec Ying Jiao, Chunhua Ma et Chao Zhou, nous travaillons à une extension du modèle de Heston en utilisant des processus de branchement, [W17]. Je travaille aussi à une suite sur le modèle de l'énergie, [W19], dans un cadre plus théorique pour étudier les conditions sous les quelles un modèle multifacteur avec des processus de branchement a un comportement coherent avec l'effet Samuelson.

Deuxième partie

STOCHASTIC CONTROL APPLIED TO CORPORATE FINANCE AND LABOR MARKET

The theory of optimal stochastic control problem, developed in the seventies, has over the recent years once again drawn a significance of interest, especially from the applied mathematics community with the main focus on its applications in a variety of fields including economics and finance. For instance, the use of powerful tools developed in stochastic control theory has provided new approaches and sometime the first mathematical approaches in solving problems arising from corporate finance. It is mainly about finding the best optimal decision strategy for managers whose firms operate under uncertain environment whether it is financial or operational. A number of corporate finance problems have been studied, or at least revisited, with this optimal stochastic control approach. There is a vast literature on firm's investment decisions in stochastic environments, see for instance [26] and [54], [25], [49], [88], [93], [95] and [104].

In this part, we focus on three aspects. First-of-all, we deal with firm managerial decisions on dividend distribution policy and investment decisions. The second chapter focuses on social preferences over employment protection regulation in a general equilibrium model of dynamic labor demand. The third subject is

In Chapter 1 we investigate a number of problems related to optimal dividend distribution policy and investment decisions, which will lead us to a variety of combined singular and switching control problems. One of the first corporate finance problems using singular stochastic control theory was the study of the optimal dividend strategy, see for instance [36], [12] and [74]. The first paper focuses on the study of a singular stochastic control problem arising from the research on optimal dividend policy for a firm whose cash reserve follows a stochastic process. The cash reserve may either grow when the firm makes profits or decrease when it is loss-making. The firm goes into bankruptcy when its cash reserve reaches zero. In these studies, some strong assumptions are made. The firm holds no debt and it is not possible to make any investment for future growth. Furthermore, it is clearly assumed that the firm does have the possibility to dispose of parts of its assets for some cash to avoid bankruptcy when the cash reserve approaches zero. Tackling this new issues is precisely the subject of the study in the first subject.

In [A2], we consider the problem of determining an optimal control on the dividend and investment policy of a firm under debt constraints. We allowed the company to make investment by increasing its outstanding indebtedness, impacting therefore its capital structure and risk profile. The presence of a high-level of debt is a challenging constraint to any firm as it is no other than the threshold below which the firm value should never go to avoid bankruptcy. The formulation of this financial problem has led to a combined singular and multi-switching control problem under constraints. Studying such a combined control problem turns out to be a real challenge to us, especially when our objective is to provide quasi-explicit solutions to our problems.

A second relevant subject is when a firm, engaged in a project that does not produce to its full potential, faces the difficult dilemma of shutting it down or keeping it alive in the hope that it will become profitable once again. In relation to our studies, Dixit and

Pindyck [54] consider various firm's decisions problems with entry, exit, suspension and/or abandonment scenarios in the case of an asset given by a geometric Brownian motion. The firm's strategy can then be described in terms of stopping times given by the time when the value of the assets hit certain threshold levels characterized as free boundaries of a variational problem. Duckworth and Zervos [57], and Lumley and Zervos [90] solve an optimal investment decision problem with switching costs in which the firm controls the production rate and must decide at which time it exits and re-enters production.

In [A6], we consider the problem of a firm having the option to keep the project going while waiting for a buyer, or liquidating the assets at immediate liquidity and termination costs. The liquidity and termination costs are governed by a mean-reverting stochastic process whereas the rate of arrival of buyers is governed by a regime-shifting Markov process. The formulation of this problem has led to a multidimensional optimal stopping time problem with random maturity. The main difficulty arises from the lack of regularity of the SDE inducing the non-uniqueness of the solution. This problem is overcame thanks to a rigorous analysis to show some properties of regularity of the value function thanks to an hypothesis having a nice financial interpretation.

The second problem, in Chapter 2, is formulated as an optimal stopping problem trying to ask to the question about the dichotomy between American and European labor markets, sometimes called the phenomenon of Eurosclerosis. In [A4], we are interested in the employment protection legislation. It varies significantly across OECD countries and an important dichotomy appears between the two sides of Atlantic and the paths diverges during the past forty years. Relatively stringent job security provisions are currently implemented in several Continental European countries whereas other countries such as the U.K. and especially the U.S. are characterised by relatively flexible labor markets, see [19]. We investigate how the interaction of institutional and economic factors affects the emergence and the potential persistence of political support for some form of employment protection regulations. In order to pursue this goal, we develop and fully characterise the solution of a general equilibrium model of dynamic labor demand. Our model is based on three features. First, a firm can hire one worker (one worker per firm model) to be active. Then it becomes active and its productivity evolves stochastically until the exogenous quit decision of the worker. Second, the employed workers appropriate a rent, i.e. an economic benefit in excess of the utility of the unemployed. This benefit is justified by different facts : firms have to cope with an underlying moral hazard problem, etc.. But also by political-institutional factors different across countries. These fundamentals are captured by one parameter linked to the power of extraction of rents of the employed workers. Third, firm can decide to quit operating by firing the worker paying a legislated tax (firing cost). This cost is determined through a political process based on standard majority voting. We show that the Eurosclerosis can be easily explained by the appearance of a "middle working class", i.e. employed workers with intermediate productivity, preferring a rigid labor market with very high firing costs.

Chapitre 2

Optimal dividend, investment and exit strategies under constraints

This Chapter is based on
[A2]. "An optimal dividend and investment control problem under debt constraints", avec V. Ly Vath, et E. Chevalier, 2013, SIAM J. Finan. Math., 4(1), 297 - 326.
[A6]. "Exit Optimal exit strategies for investment projects", avec E. Chevalier, V. Ly Vath et A. Roch, 2015, Journal of Mathematical Analysis and Applications, Vol.425(2), pp.666-694.

Summary. In this chapter, we look at two different problems arising in corporate finance, related in particular to optimal dividend distribution policy and investment or exit decisions. we first consider a combined stochastic control problem which studies the interactions between dividend policy and investment under uncertainty. The formulation of the financial problem under constraints has led to combined singular and multi-switching control problem under constraints. A viscosity solutions approach is employed to carry out an analysis on the associated system of variational inequalities, leading to sharp qualitative characterizations of the switching regions. In the second paper, we study the problem of an optimal exit strategy for an investment project which is unprofitable and for which the liquidation costs evolve stochastically. The firm has the option to keep the project going while waiting for a buyer, or liquidating the assets at immediate liquidity and termination costs. The liquidity and termination costs are governed by a mean-reverting stochastic process whereas the rate of arrival of buyers is governed by a regime-shifting Markov process. We formulate this problem as a multidimensional optimal stopping time problem with random maturity. We characterise the objective function and derive explicit solutions and numerical examples in the case of power and logarithmic utility functions when the liquidity premium factor follows a mean-reverting CIR process. Studying such a combined control problems turns out to be a real challenge to us, especially when our objective is to provide quasi-explicit solutions to our problems.

2.1 Introduction

The first and natural dividend control problem was studied by [74]. They consider a firm whose cash reserve follows a drift brownian motion as follows :

$$dX_t \;=\; \mu dt + \sigma dW_t - dZ_t,\; X_{0-} = x.$$

The objective is to find the best dividend policy which maximizes shareholder's value :

$$\hat{V}_0(x) \;=\; \sup_{Z\in\mathcal{Z}} \mathbb{E}\left[\int_0^{T_0^-} e^{-\rho t} dZ_t\right], \tag{2.1.1}$$

where $T_0 = \inf\{t \geq 0 \;: X_t \leq 0\}$ is the time bankruptcy of the cash reserve in regime 0.

It is known that $\hat{V}_0$, as the value function of a pure singular control problem, is characterized as the unique continuous viscosity solution on $(0,\infty)$, with linear growth condition to the variational inequality :

$$\min\left[\rho\hat{V}_0 - \mathcal{L}_0\hat{V}_0\;,\; \hat{V}_0' - 1\right] \;=\; 0, \quad x > 0, \tag{2.1.2}$$

and boundary data

$$\hat{V}_0(0) \;=\; 0.$$

Actually, $\hat{V}_0$ is C^2 on $(0,\infty)$ and explicit computations of this standard singular control problem are developed in Shreve, Lehoczky and Gaver [101], Jeanblanc and Shiryaev [74], or Radner and Shepp [97] :

$$\hat{V}_0(x) \;=\; \begin{cases} \frac{f_0(x)}{f_0'(\hat{x}_0)} & , 0 \leq x \leq \hat{x}_0 \\ x - \hat{x}_0 + \frac{\mu_0}{\rho}, & x \geq \hat{x}_0, \end{cases}$$

where

$$f_0(x) \;=\; e^{m_0^+ x} - e^{m_0^- x}, \quad \hat{x}_0 \;=\; \frac{1}{m_0^+ - m_0^-} \ln\left(\frac{(m_0^+)^2}{(m_0^-)^2}\right),$$

and $m_0^- < 0 < m_0^+$ are roots of the characteristic equation :

$$\rho - \mu_0 m - \frac{1}{2}\sigma^2 m^2 \;=\; 0.$$

In other words, this means that the optimal cash reserve process is given by the reflected diffusion process at the threshold $\hat{x}_0$ with an optimal dividend process given by the local time at this boundary. When the firm starts with a cash reserve $x \geq \hat{x}_0$, the optimal dividend policy is to distribute immediately the amount $x - \hat{x}_0$ and then follows the dividend policy characterized by the local time.

In this section, the objective is to address related problems when we incorporate the following aspects :

- investment problems : the interaction between dividend policy and investment policy. By investment, we mean the ability of the firm to allow the company to capture growth opportunity which it self-finances on its cash reserve.

- investment under debt constraints : the firm is allowed to make investment and finance it through debt issuance/raising, which in turn would impact its capital structure and risk profile.

- dividend and investment policy under liquidity risk : the firm is allowed to make investment decisions by acquiring or selling productive assets. But we no longer assume that firm assets are either infinitely illiquid or liquid.

The formulation of these financial problems has led to different combined singular and multi-switching control problems under constraints, which turn out to be real challenges to us, especially when our objective is to provide quasi-explicit solutions to our problems.

2.2 An optimal dividend and investment control problem under debt constraints

In this paper, we consider the problem of determining the optimal control on the dividend and investment policy of a firm under debt constraints. As in the Merton model, we consider that firm value follows a geometric Brownian process and more importantly we consider that the firm carries a debt obligation in its balance sheet. However, as in most studies, we still assume that the firm assets is either highly liquid and may be assimilated to cash equivalents or cash reserve, or infinitely illiquid except the cash reserve. We allow the company to make investment and finance it through debt issuance/raising, which would impact its capital structure and risk profile. This debt financing results therefore in higher interest rate on the firm's outstanding debts. More precisely, we model the decisions to raise or redeem some debt obligations as switching decisions controls where each regime corresponds to a specific level debt.

Furthermore, we consider that the manager of the firm works in the interest of the shareholders, but only to a certain extent. Indeed, in the objective function, we introduce a penalty cost P and assume that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy. This penalty cost could represent, for instance, an estimated cost of the negative image upon his/her own reputation due to the bankruptcy under his management leadership. Mathematically, we formulate this problem as a combined singular and multiple-regime switching control problem. Each regime corresponds to a level of debt obligation held by the firm.

The studies that are most relevant to our problem are the one investigating combined singular and switching control problems, see [70]. However, none of the above papers on dividend and investment policies, which provides qualitative solutions, has yet moved away

from the basic Bachelier model or the simplistic assumption that firms hold no debt obligations. In our model switching from one regime, i.e. debt level, to another directly impacts the state process itself. Indeed, the drift of the stochastic differential equation governing the firm value would equally switch as the results of the change in interest rate paid on the outstanding debt. A given level of debt is no other than the threshold below which the firm value should never go to avoid bankruptcy. As such, debt level switching also signifies a change of default constraints on the state process in our optimal control problem. Further original contributions in terms of financial studies of our paper include the feature of the conflicts of interest for firm manager through the presence of the penalty cost in the event of bankruptcy. Studying a mixed singular and multi-switching problem combining with the above financial features including debt constraints and penalty cost turns out to be a major mathematical challenge, especially when our objective is to provide quasi-explicit solutions. In addition, it is always tricky to overcoming the combined difficulties of the singular control with those of the switching control, especially when there are multiple regimes, for instance, building a strict supersolution to our HJB system in order to prove the comparison principle.

2.2.1 The model formulation

We assume that the cash-reserve process of the firm $X^{x,i,\alpha}$, denoted by X when there is no ambiguity and associated to a strategy $\alpha = (Z_t, (\tau_n)_{n\geq 0}, (k_n)_{n\geq 0})$, is governed by the following stochastic differential equation :

$$dX_t = bX_t dt - r_{I_t} D_{I_t} dt + \sigma X_t dW_t - dZ_t + dK_t \tag{2.2.3}$$

where $I_t = \sum_{n\geq 0} k_n 1_{\tau_n \leq t < \tau_{n+1}}$, $I_{0^-} = i$ and $k_n \in \mathbb{I}_N := \{1, ..., N\}$. D_i and r_i represent respectively different levels of debt and their associated interest rate paid on those debts. The process K_t represents the cash-flow due to the change in the firm's indebtedness. More precisely $K_t = \sum_{n\geq 0} \left(D_{\kappa_{n+1}} - D_{\kappa_n} - g\right) 1_{\tau_{n+1}\leq t}$, where g represents the additional cost associated with the change of firm's level of debt.
For a given control strategy $\alpha =$, the bankruptcy time is represented by the stopping time T^α defined as

$$T^\alpha = \inf\{t \geq 0, X_t^{x,i,\alpha} \leq D_{I_t}\}. \tag{2.2.4}$$

We equally introduce a penalty cost or a liquidation cost $P > 0$, in the case of a holding company looking to liquidate one of its own affiliate or activity. In the case of the penalty, it mainly assumes that the manager does not completely try to maximize the shareholders' value since it applies a penalty cost in the case of bankruptcy.
We define the value functions which the manager actually optimizes as follows

$$v_i(x) \;=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}^{(i,x)}\left[\int_0^{T^-} e^{-\rho t} dZ_t - Pe^{-\rho T}\right], \quad x\in\mathbb{R},\; i\in\{1,...,N\}, \tag{2.2.5}$$

where $\mathcal{A}$ represents the set of admissible control strategies, and ρ the discount rate.

2.2.2 PDE characterization

Using the dynamic programming principle, we obtain the associated system of variational inequalities satisfied by the value functions :

$$\begin{aligned} \min\left[-\mathcal{A}_i v_i(x)\;,\; v_i'(x)-1\;,\; v_i(x)-\max_{j\neq i} v_j(x+D_j-D_i-g)\right] &= 0,\; x>D_i,\; i\in\mathbb{I}_N \\ v_i(D_i) &= -P, \end{aligned}$$

where the operator $\mathcal{A}_i$ is defined by $\mathcal{A}_i\phi = \mathcal{L}_i\phi - \rho\phi$, and $\mathcal{L}_i\varphi = [bx - r_iD_i]\varphi'(x) + \frac{1}{2}\sigma^2x^2\varphi''(x)$

We may obtain the following results :

Proposition 2.2.1. *The value functions v_i are continuous on (D_i,∞) and satisfy*

$$v_i(D_i^+) \;:=\; \lim_{x\downarrow D_i} v_i(x) \;=\; -P. \tag{2.2.6}$$

Theorem 2.2.1. *The value functions v_i, $i\in\mathbb{I}_N$, are continuous on (D_i,∞), and are the unique viscosity solutions on (D_i,∞) with linear growth condition and boundary data $v_i(D_i) = -P$, to the system of variational inequalities :*

$$\min\left[-\mathcal{A}_i v_i(x)\;,\; v_i'(x)-1\;,\; v_i(x)-\max_{j\neq i} v_j(x+D_j-D_i-g)\right] = 0,\; x>D_i. \tag{2.2.7}$$

Actually, we obtain some more regularity results on the value functions.

Proposition 2.2.2. *The value functions v_i, $i\in\mathbb{I}_N$, are C^1 on (D_i,∞). Moreover, if we set for $i\in\mathbb{I}_N$:*

$$\mathcal{S}_i \;=\; \left\{x\geq D_i\;, v_i(x) \;=\; \max_{j\neq i} v_j(x+D_j-D_i-g),\right\} \tag{2.2.8}$$

$$\mathcal{D}_i \;=\; \overline{\text{int}\left(\{x\geq D_i\;, v_i'(x)=1\}\right)}, \tag{2.2.9}$$

$$\mathcal{C}_i \;=\; (D_i,\infty)\setminus(\mathcal{S}_i\cup\mathcal{D}_i), \tag{2.2.10}$$

then v_i is C^2 on the open set $\mathcal{C}_i\cup\text{int}(\mathcal{D}_i)\cup\text{int}(\mathcal{S}_i)$ of (D_i,∞), and we have in the classical sense

$$\rho v_i(x) - \mathcal{L}_i v_i(x) \;=\; 0, \quad x\in\mathcal{C}_i.$$

$\mathcal{S}_i$, $\mathcal{D}_i$, and $\mathcal{C}_i$ respectively represent the switching, dividend, and continuation regions when the outstanding debt is at regime i.

2.2.3 Qualitative results on the switching regions

For $i,\ j \in \mathbb{I}_N$ and $x \in [D_i, +\infty)$, we introduce some notations :

$$\delta_{i,j} = D_j - D_i, \quad \Delta_{i,j} = (b - r_j)D_j - (b - r_i)D_i \quad \text{and} \quad x_{i,j} = x + \delta_{i,j} - g.$$

We set $x_i^* = \sup\{x \in [D_i, +\infty) :\ v_i'(x) > 1\}$ for all $i \in \mathbb{I}_N$
We equally define $\mathcal{S}_{i,j}$ as the switching region from debt level i to j.

$$\mathcal{S}_{i,j} = \{x \in (D_i,\ +\infty), v_i(x) = v_j(x_{i,j})\}.$$

and notice that $\mathcal{S}_i\ =\ \cup_{j\neq i}\mathcal{S}_{i,j},\ i \in \mathbb{I}_N$.

We now turn to the first result which that there exists a finite level of cash such that it is optimal to distribute dividends up to this level.

Lemma 2.2.1. *For all $i \in \mathbb{I}_N$, we have $x_i^* := \sup\{x \in [D_i, +\infty) :\ v_i'(x) > 1\} < +\infty$.*

In order to compute the dividend regions, we establish the following lemma.

Lemma 2.2.2. *Let $i,\ j \in \mathbb{I}_N$ such that $j \neq i$. We assume that there exists $\hat{x}_i$ a left-boundary of $\mathcal{D}_i$.*

i) Assume that $\hat{x}_i \notin \mathcal{S}_i$, then we have $(b - r_i)D_i > -\rho P$ and $\rho v_i(\hat{x}_i) = b\hat{x}_i - r_iD_i$.
As $x \to \rho v_i(x) - bx + r_iD_i$ is increasing, it implies that

$$\rho v_i(x) < bx - r_iD_i \text{ on } (D_i, \hat{x}_i) \quad \text{and} \quad \rho v_i(x) > bx - r_iD_i \text{ on } (\hat{x}_i, +\infty).$$

ii) Assume that $\hat{x}_i \in \mathcal{S}_{i,j}$ then we have
ii.a) $[\hat{x}_i,\ \hat{x}_i + \varepsilon] \subset \mathcal{S}_{i,j}$ and $\hat{x}_i + \delta_{i,j} - g$ is a left-boundary of $\mathcal{D}_j$.
ii.b) $\rho v_i(\hat{x}_i) = b\hat{x}_i - r_iD_i + \Delta_{i,j} - bg$ and $\Delta_{i,j} > 0$.
ii.c) $\forall k \in \mathbb{I}_N - \{i, j\}, \hat{x}_i \notin \mathcal{S}_{i,k}$.
Notice that the last equality implies that $-\rho P + bg < (b - r_j)D_j$.

We now establish an important result in determining the description of the switching regions. The following Theorem states that it is never optimal to expand its operation, i.e. to make investment, through debt financing, should it result in a lower "drift" ($(b - r_i)D_i$) regime. However, when the firm's value is low, i.e. with a relatively high bankruptcy risk, it may be optimal to make some divestment, i.e. sell parts of the company, and use the proceedings to lower its debt outstanding, even if it results in a regime with lower "drift". In other words, to lower the firm's bankruptcy risk, one should try to decrease its volatility, i.e. the diffusion coefficient. In our model, this clearly means making some debt repayment in order to lower the firm's volatility, i.e. σX_t.

Theorem 2.2.2. *Let $i, j \in \mathbb{I}_N$ such that $(b - r_j)D_j > (b - r_i)D_i$. We have the following results :*

1) $x_j^* \notin \mathcal{S}_{j,i}$ *and* $\mathring{\mathcal{D}}_j = (x_j^*,\ +\infty)$.
2) $\mathring{\mathcal{S}}_{j,i} \subset (D_j + g,\ x_j^*)$. *Furthermore, if* $D_j < D_i$, *then* $\mathring{\mathcal{S}}_{j,i} = \emptyset$.

From the above Theorem, we may obtain the following Corollary and Proposition on the determination of different regions. We will in particular see in the next section how from these results, we may obtain the complete results in the two-regime case and above.

Corollary 2.2.1. *Let* $m \in \mathbb{I}_N$ *such that* $(b - r_m)D_m = \max_{i\in\mathbb{I}_N}(b - r_i)D_i$.
1) $x_m^* \notin \mathcal{S}_m$ *and* $\mathring{\mathcal{D}}_m = (x_m^*,\ +\infty)$.
2) For all $i \in \mathbb{I}_N - \{m\}$, *we have :*
i) If $D_m < D_i$, $\mathring{\mathcal{S}}_{m,i} = \emptyset$.
ii) If $D_i < D_m$, $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g,\ x_m^*)$. *Furthermore, if* $b \geq r_i$, *then* $\mathring{\mathcal{S}}_{m,i} \subset (D_m + g,\ (a_i^* + \delta_{i,m} + g) \wedge x_m^*)$, *where* a_i^* *is the unique solution of the equation* $\rho v_i(x) = (bx - r_i D_i)v_i'(x)$. *We further have* $a_i^* \neq x_i^*$.

We now turn to the following results ordering the left-boundaries $(x_i^*)_{i\in\mathbb{I}_N}$ of the dividend regions $(\mathcal{D}_i)_{i\in\mathbb{I}_N}$.

Proposition 2.2.3. *Consider* $i, j \in \mathbb{I}_N$, *such that* $(b - r_i)D_i < (b - r_j)D_j$. *We always have* $x_i^* + \delta_{i,j} - g \leq x_j^*$ *unless there exists a regime* k *such that* $(b - r_j)D_j < (b - r_k)D_k$ *and* $x_i^* \in \mathcal{S}_{i,k}$, *then we have* $x_j^* - \delta_{i,j} + g < x_i^* < x_k^* - \delta_{i,k} + g$.

2.2.4 The two regime-case

Throughout this section, we now assume that $N = 2$, in which case, we will get a complete description of the different regions. We will see that the most important parameter to consider is the so-called "drifts" $((b - r_i)D_i)_{i=1,2}$ and in particular their relative positions. To avoid cases with trivial solution, i.e. immediate consumption, we will assume that $-\rho P < (b - r_i)D_i, i = 1, 2$. Throughout the following Theorems, we provide a complete resolution to our problem in each case.

Theorem 2.2.3. *We assume that* $(b - r_2)D_2 < (b - r_1)D_1$.
We have

$$\mathcal{C}_1 = [D_1, x_1^*),\ \mathcal{D}_1 = [x_1^*,\ +\infty),\ \text{and } \mathring{\mathcal{S}}_1 = \emptyset \text{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

1) If $\mathcal{S}_2 = \emptyset$ *then we have*

$$\mathcal{C}_2 = [D_2, x_2^*),\ \text{and } \mathcal{D}_2 = [x_2^*,\ +\infty) \text{ where } \rho v_2(x_2^*) = bx_2^* - r_2 D_2.$$

2) If $\mathcal{S}_2 \neq \emptyset$ *then there exists* y_2^* *such that* $\mathcal{S}_2 = [y_2^*,\ +\infty)$ *and we distinguish two cases*
a) If $x_2^* + \delta_{2,1} - g < x_1^*$, *then* $y_2^* > x_2^*$, $y_2^* = x_1^* + \delta_{1,2} + g$ *and*

$$\mathcal{C}_2 = [D_2, x_2^*),\ \text{and } \mathcal{D}_2 = [x_2^*,\ +\infty) \text{ where } \rho v_2(x_2^*) = bx_2^* - r_2 D_2.$$

b) If $x_2^* + \delta_{2,1} - g = x_1^*$ *then* $y_2^* \leq x_2^*$, $\rho v_2(x_2^*) = bx_2^* - r_2 D_2 + \Delta_{2,1} - bg$.
We define a_2^* *as the solution of* $\rho v_2(a_2^*) = ba_2^* - r_2 D_2$ *and have two cases*
i) If $a_2^* \notin \mathcal{D}_2$, *we have*

$$\mathcal{D}_2 = [x_2^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_2 = [D_2,\ y_2^*).$$

ii) If $a_2^* \in \mathcal{D}_2$, *there exists* $z_2^* \in (a_2^*, y_2^*)$ *such that*

$$\mathcal{D}_2 = [a_2^*,\ z_2^*] \cup [x_2^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_2 = [D_2,\ a_2^*) \cup (z_2^*,\ y_2^*).$$

Theorem 2.2.4. *We assume that* $(b - r_1)D_1 < (b - r_2)D_2$,
1) we have

$$\begin{aligned} \mathcal{D}_2 &= [x_2^*,\ +\infty) \textit{ where } \rho v_2(x_2^*) = bx_2^* - r_2 D_2 \\ \mathring{\mathcal{S}}_2 &= \emptyset \textit{ or there exist } s_2^*, S_2^* \in (D_2 + g,\ x_2^*) \textit{ such that } \mathring{\mathcal{S}}_2 = (s_2^*, S_2^*). \end{aligned}$$

2) If $\mathring{\mathcal{S}}_1 = \emptyset$ *then we have*

$$\mathcal{C}_1 = [D_1, x_1^*),\ \textit{and } \mathcal{D}_1 = [x_1^*,\ +\infty) \textit{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

3) If $\mathring{\mathcal{S}}_1 \neq \emptyset$ *there exists* y_1^* *such that* $\mathring{\mathcal{S}}_1 = (y_1^*,\ +\infty)$
a) If $x_1^* + \delta_{1,2} - g < x_2^*$, *then* $y_1^* > x_1^*$, $y_1^* = x_2^* + \delta_{2,1} + g$ *and*

$$\mathcal{C}_1 = [D_1, x_1^*),\ \textit{and } \mathcal{D}_1 = [x_1^*,\ +\infty) \textit{ where } \rho v_1(x_1^*) = bx_1^* - r_1 D_1.$$

b) If $x_2^* + \delta_{2,1} - g = x_1^*$, *then* $y_1^* \leq x_1^*$, $\rho v_1(x_1^*) = bx_1^* - r_1 D_1 + \Delta_{1,2} - bg$.
We define a_1^* *as the solution of* $\rho v_1(a_1^*) = ba_1^* - r_1 D_1$ *and have two cases.*
i) If $a_1^* \notin \mathcal{D}_1$, *we have*

$$\mathcal{D}_1 = [x_1^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_1 = [D_1,\ y_1^*).$$

ii) If $a_1^* \in \mathcal{D}_1$, *there exists* $z_1^* \in (a_1^*, y_1^*)$ *such that*

$$\mathcal{D}_1 = [a_1^*,\ z_1^*] \cup [x_1^*,\ +\infty) \quad \textit{and} \quad \mathcal{C}_1 = [D_1,\ a_1^*) \cup (z_1^*,\ y_1^*).$$

2.3 Optimal exit strategies for an investment project

2.3.1 Introduction

There is often a time when a firm is engaged in a project that does not produce to its full potential and faces the difficult dilemma of shutting it down or keeping it alive in the hope that it will become profitable once again. When an investment in not totally irreversible, assets can be sold at their scrap value minus some liquidation and project termination costs, which may include for example termination pay to workers, legal fees and a liquidity premium in the case of fire sale of the assets. Since these closing costs may be substantial, it

may be worthwhile to wait for the project to be profitable again or to wait for an interested buyer that will pay the fair value of the assets and put them to better use. In this study, we give an analytical solution to this problem when the liquidation costs and the value of the assets are diffusion processes and the arrival time of a buyer is modeled by means of an intensity function depending on the current state of a Markov chain.

The firm, we consider, in this paper, must decide between liquidating the assets of an underperforming project and waiting for the project to become once again profitable, in a setting where the liquidation costs and the value of the assets are given by general diffusion processes. We formulate this two-dimensional stochastic control problem as an optimal stopping time problem with random maturity and regime shifting.

Amongst the large literature on optimal stopping problems, we may refer to some related works including Bouchard, El Karoui and Touzi [21], Carr [31], Dayanik and Egami [45], Dayanik and Karatzas [46], Guo and Zhang [71], Lamberton and Zervos [82]. In [46] and [82], the authors study optimal stopping problems with general 1-dimensional processes. Random maturity in optimal stopping problem was introduced in [31] and [21]. It allowed to reduce the dimension of their problems as well as addressing the numerical issues. We may refer to Dayanik and Egami [45] for a recent paper on optimal stopping time and random maturity. For optimal stopping problem with regime shifting, we may refer to Guo and Zhang [71], where an explicit optimal stopping rule and the corresponding value function in a closed form are obtained.

In this paper, our optimal stopping problem combines all the above features, i.e., random maturity and regime shifting, in the bi-dimensional framework. We are able to characterize the value function of our problem and provide explicit solution in some particular cases where we manage to reduce the dimension of our control problem.

In the general bi-dimensional framework, the main difficulty is related to the proof of the continuity property and the PDE characterization of the value function. Since it is not possible to get the smooth-fit property, the PDE characterization may be obtained only via the viscosity approach. To prove the comparison principle, one has to overcome the non-linearity of the lower and upper bounds of the value function when building a strict supersolution to our HJB equation.

In the particular cases where it is possible to reduce our problem to a one-dimensional problem, we are able to provide explicit solution. Our reduced one-dimensional problem is highly related to previous studies in the literature, see for instance Zervos, Johnson and Alezemi [106] and Leung, Li and Wang [83].

2.3.2 The investment project

Let $(\Omega, \mathbb{F}, \mathbb{P})$ be a probability space equipped with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, satisfying the usual conditions. It is assumed that all random variables and stochastic processes are defined on the stochastic basis $(\Omega, \mathbb{F}, \mathbb{P})$. We denote by $\mathcal{T}$ the collection of all $\mathbb{F}-$stopping times.

Let W and B be two correlated $\mathbb{F}$–Brownian motions, with correlation ρ, i.e. $d[W,B]_t = \rho dt$ for all t.

We consider a firm which owns assets that are currently locked up in an investment project which currently produces no output per unit of time. Because the firm is currently not using the assets at its full potential, it considers two possibilities. The first is to liquidate the assets in a fire sale and recover any remaining value. The cash flow obtained in the latter case is the fair value of the assets minus liquidation and project termination costs. We denote by θ the moment at which the firm decides to take this option. The second option is to wait for the project to become profitable once again, or equivalently, to wait for an investor or another firm who will purchase the assets as a whole at their fair value S_τ where τ is the moment when this happens.

The fair value of the assets are given by $S = \exp(X)$, in which

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t, \quad t \geq 0 \\ X_0 &= x. \end{aligned} \tag{2.3.11}$$

Assume that μ and σ are Lipschitz functions on $\mathbb{R}$ satisfying some growth conditions.

Liquidation and Termination Costs. We model the liquidation cost of the assets and terminal costs of the project as a given process $(f(Y_t))_{t\geq 0}$, where f is strictly decreasing C^2 function defined on $\mathbb{R}^+ \to [0,1]$, and satisfies some conditions.

Unlike the value of financial assets, it is natural to model liquidation costs with mean-reverting properties. As such, the costs, given by $f(Y_t)$ at time t, is defined in terms of the mean-reverting non-negative process Y which is governed by the following SDE :

$$\begin{aligned} dY_t &= \alpha(Y_t)dt + \gamma(Y_t)dW_t, \\ Y_0 &= y, \end{aligned} \tag{2.3.12}$$

where α is a Lipschitz function on $\mathbb{R}^+$ and, for any $\varepsilon > 0$, γ is a Lipschitz function on $[\varepsilon, \infty)$. We assume that α and γ satisfy linear growth conditions.

The recovery time. We model the arrival time of a buyer, denoted by τ, or equivalently the time when the project becomes profitable again, by means of an intensity function λ_i depending on the current state i of a continuous-time, time-homogenous, irreducible Markov chain L, independent of W and B, with $m+1$ states. The generator of the chain L under $\mathbb{P}$ is denoted by $A = (\vartheta_{i,j})_{i,j=0,\dots m}$. Without loss of generality we assume $\lambda_0 > \lambda_1 > \dots > \lambda_m > 0$.

Utility function. We let U denote the utility function of the firm. We assume that U satisfies the following assumptions.

Assumption 2.3.1. *$U : \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing, concave and twice continuously differentiable, and satisfies*

$$\lim_{x\to 0} x\, U'(x) < +\infty. \tag{2.3.13}$$

Assumption 2.3.2. *U is supermeanvalued w.r.t. S, i.e.*

$$U(S_t) \geq \mathbb{E}[U(S_\theta)|\mathcal{F}_t] \tag{2.3.14}$$

for any stopping time $\theta \in \mathcal{T}$.

Objective function. The objective of the firm is to maximize the expected profit obtained from the sale of the illiquid asset, either through liquidation or at its fair value at the exogenously given stopping time τ. As such, we consider the following value function :

$$v(i,x,y) := \sup_{\theta \in \mathcal{T}} \mathbb{E}^{i,x,y}\left[h(X_\theta, Y_\theta)\,\mathbf{1}_{\theta \leq \tau} + U(e^{X_\tau})\,\mathbf{1}_{\theta > \tau}\right], \quad x \in \mathbb{R},\, y \in \mathbb{R}^+,\, i \in \{0,\dots,m\} \tag{2.3.15}$$

where $\mathbb{E}^{i,x,y}$ stands for the expectation with initial conditions $X_0 = x$, $Y_0 = y$ and $L_0 = i$, and $h(x,y) = U(e^x f(y))$. Recall that τ is defined through the Markov chain L.

2.3.3 Characterization of the value function

Before characterizing the value function, We first prove the continuity of the value function, which has two main difficulties that need a no-standard treatment. The first one comes from the SDE (2.3.12) satisfied by Y since we do not assume the standard hypothesis of Lipschitz coefficients. We overcome this drawback by showing that the local Lipschitz property is satisfied until the smallest optimal exit time from the investment. The second difficulty is related to the bi-dimensional setting where the classical arguments used to show the regularity of value function are not longer available. We then need to show the continuity in term of limits of sequences and to distinguish different sub-sequences with ad-hoc proofs.

The complexity of the proof of the continuity suggests that a direct proof of differentiability, i.e. smooth-fit property, of the value function is probably out of reach in our setting. We will then turn to the viscosity characterization approach to overcome the impossibility to use a verification approach.

Theorem 2.3.5. *The value functions v_i, $i \in \{0,\dots,m\}$, are continuous on $\mathbb{R} \times \mathbb{R}^+$, and constitute the unique viscosity solution on $\mathbb{R} \times \mathbb{R}^+$ with growth condition*

$$|v_i(x,y)| \leq |U(e^x)| + |U(e^x)f(y)|,$$

and boundary condition

$$\lim_{y \downarrow 0} v_i(x,y) = U(e^x),$$

to the system of variational inequalities :

$$\begin{aligned} \min\Big[-\mathcal{L}v_i(x,y) - \mathcal{G}_i v_\cdot(x,y) - \mathcal{J}_i v_i(x,y)\,,\, v_i(x,y) - U(e^x f(y))\Big] = 0, \\ \forall\,(x,y) \in \times\mathbb{R} \times \mathbb{R}^+_*,\ \textit{and } i \in \{0,\dots,n\}, \end{aligned} \tag{2.3.16}$$

in which $\mathcal{L}$ is the second order differential operator associated to the state processes (X, Y) and $\mathcal{G}_i$ and $\mathcal{J}_i$ are defined as

$$\begin{aligned}\mathcal{G}_i\varphi(.,x,y) &= \sum_{j\neq i}\vartheta_{i,j}\left(\varphi(j,x,\,y)-\varphi(i,x,\,y)\right)\\ \mathcal{J}_i\varphi(i,x,y) &= \lambda_i\left(e^x-\varphi(i,x,\,y)\right).\end{aligned}$$

Remark 2.3.1. *To prove the comparison principle, one has to overcome the non-linearity of the lower and upper bounds of the value function when building a strict supersolution to our HJB equation.*

2.3.4 Liquidation and continuation regions

We now prove useful qualitative properties of the liquidation regions of the optimal stopping problem. We introduce the following liquidation and continuation regions :

$$\begin{aligned}\mathcal{LR} &= \left\{(i,x,y)\in\{0,...,m\}\times\mathbb{R}\times\mathbb{R}^+\,|\,v(i,x,y)=h(x,y)\right\}\\ \mathcal{CR} &= \{0,...,m\}\times\mathbb{R}\times\mathbb{R}^+\setminus\mathcal{LR}.\end{aligned}$$

Clearly, outside the liquidation region $\mathcal{LR}$, it is never optimal to liquidate the assets at the available discounted value. Moreover, the smallest optimal stopping time θ^*_{ixy} verifies

$$\theta^*_{ixy}=\inf\left\{u\geq 0\,|\,\left(L^i_u,X^x_u,Y^y_u\right)\in\mathcal{LR}\right\}.$$

We define the $(i,x)-$sections for every $(i,x)\in\{0,...,m\}\times\mathbb{R}$ by

$$\mathcal{LR}_{(i,x)}=\{y\geq 0\,|\,v(i,x,y)=h(x,y)\}\ \text{ and }\mathcal{CR}_{(i,x)}=\mathbb{R}^+\setminus\mathcal{LR}_{(i,x)}.$$

Proposition 2.3.4 (Properties of liquidation region)**.**

i) $\mathcal{E}$ is closed in $\{0,...,m\}\times\mathbb{R}\times(0,+\infty)$,

ii) Let $(i,x)\in\{0,...,m\}\times\mathbb{R}$.

- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})]=U(e^x)$, then, for all $y\in\mathbb{R}^+$, $v(i,x,y)=U(e^x)$ and $\mathcal{E}_{(i,x)}=\{0\}$.*

- *If $\mathbb{E}^{i,x}[U(e^{X_\tau})]<U(e^x)$, then there exists $x_0\in\mathbb{R}$ such that $\mathcal{E}_{(i,x_0)}\setminus\{0\}\neq\emptyset$ and $\bar{y}^*(i,x):=\sup\mathcal{E}_{(i,x)}<+\infty$.*

2.3.5 Logarithmic utility

Throughout this section, we assume that the diffusion processes X and Y are governed by the following SDE, which are particular cases of (2.3.11) and (2.3.12)

$$dX_t = \mu dt+\sigma(X_t)dB_t;\ X_0=x \tag{2.3.17}$$

$$dY_t = \kappa\left(\beta-Y_t\right)dt+\gamma\sqrt{Y_t}dW_t;\ Y_0=y. \tag{2.3.18}$$

The following theorem shows that in the logarithmic case, we can reduce the dimension of the problem by factoring out the x-variable. For this purpose, we define $\mathcal{T}_{L,W}$ the set of stopping times with respect to the filtration generated by (L, W), and the differential operator $\overline{\mathcal{L}}\phi(y) := \frac{1}{2}\gamma^2 y\frac{\partial^2\phi}{\partial y^2} + \kappa(\beta - y)\frac{\partial\phi}{\partial y} + \mu$, for $\phi \in C^2(\mathbb{R}^+)$.

Theorem 2.3.6. *For $(i, y) \in \{1, ..., m\} \times \mathbb{R}^+$ we define the function :*

$$w(i, y) = \sup_{\theta \in \mathcal{T}_{L,W}} \mathbb{E}^{i,y}[\mu(\theta \wedge \tau) + \ln(f(Y_\theta))\, \mathbb{1}_{\{\theta \leq \tau\}}].$$

Then,

$$v(i, x, y) = x + w(i, y) \text{ on } \{0, ..., m\} \times \mathbb{R} \times \mathbb{R}^+,$$

with w the unique viscosity solution to the system of equations :

$$\min\Big[-\overline{\mathcal{L}}w(i, y) + \lambda_i w(i, y) - \sum_{j \neq i} \vartheta_{i,j}\left(w(j,\, y) - w(i,\, y)\right)\,,\, w(i, y) - g(y)\Big] = 0, \quad (2.3.19)$$

where $g(y) := \ln(f(y))$ Moreover, the functions $w(i, .)$ are of class C^1 on $\mathbb{R}^+$ and C^2 on the open set $\mathcal{C}_{(i,x)} \cup \mathrm{Int}(\mathcal{E}_{(i,x)})$.

2.3.6 Liquidation region

In the logarithmic case, the liquidation region can be characterized in more details.

Proposition 2.3.5. *Let $i \in \{0, ..., m\}$ and set*

$$\hat{y}_i = \inf\{y \geq 0 :\ \mathcal{H}_i g(y) \geq 0\} \text{ with } \mathcal{H}_i g(y) = \overline{\mathcal{L}}g(y) - \lambda_i g(y) + \sum_{j \neq i} \vartheta_{i,j}\left(w(j,\, y) - g(y)\right).$$

There exists $y_i^* \geq 0$ such that $[0, y_i^*] = \mathcal{LR}_{(i,.)} \cap [0, \hat{y}_i]$. Moreover, $w(i, \cdot) - g(\cdot)$ is non-decreasing on $[y_i^*, \hat{y}_i]$.

Proposition 2.3.6. *Assume that the function $y \mapsto \overline{\mathcal{L}}g(y)$ is non-decreasing on $\mathbb{R}^+$, then for all $i \in \{0, ..., m\}$, $w(i, \cdot) - g(\cdot)$ is non-decreasing on $\mathbb{R}^+$ and we have $\mathcal{LR}_{(i,\cdot)} = [0, y_i^*]$, with $y_i^* > 0$.*

2.3.7 Explicit solutions in logarithmic utility in the two regime case

From the above results, we may get completely explicit solution in the two-regime case. In particular, the value function may be written in terms of the confluent hypergeometric functions.

Proposition 2.3.7. *The function w is given by*

$$w(0,y)=\begin{cases} g(y) & y\in[0,y_0^*] \\ \widehat{c}\Phi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa},\frac{2\kappa\beta}{\gamma^2},\frac{2\kappa}{\gamma^2}y\right)+\widehat{d}\Psi\left(\frac{\lambda_0+\vartheta_{0,1}}{\kappa},\frac{2\kappa\beta}{\gamma^2},\frac{2\kappa}{\gamma^2}y\right) & y\in(y_0^*,y_1^*] \\ \quad +\mathcal{I}\left(\frac{2\kappa}{\gamma^2},\beta,-2\frac{\lambda_0+\vartheta_{0,1}}{\gamma^2},2\frac{\vartheta_{0,1}g(\cdot)+\mu}{\gamma^2}\right)(y) & \\ p_0^0\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa},\frac{2\kappa\beta}{\gamma^2},\frac{2\kappa}{\gamma^2}x\right)+\frac{\mu}{\widetilde{\lambda}_0}\right] & y\in(y_1^*,\infty) \\ \quad +p_1^0\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa},\frac{2\kappa\beta}{\gamma^2},\frac{2\kappa}{\gamma^2}x\right)+\frac{\mu}{\widetilde{\lambda}_1}\right] & \end{cases} \tag{2.3.20}$$

$$w(1,y)=\begin{cases} g(y) & y\in[0,y_1^*] \\ p_0^1\left[\widehat{e}\Psi\left(\frac{\widetilde{\lambda}_0}{\kappa},\frac{2\kappa\beta}{\gamma^2},\frac{2\kappa}{\gamma^2}y\right)+\frac{\mu}{\widetilde{\lambda}_0}\right] & y\in(y_1^*,\infty) \\ \quad +p_1^1\left[\widehat{f}\Psi\left(\frac{\widetilde{\lambda}_1}{\kappa},\frac{2\kappa\beta}{\gamma^2},\frac{2\kappa}{\gamma^2}y\right)+\frac{\mu}{\widetilde{\lambda}_1}\right], & \end{cases}$$

where Φ and Ψ denote respectively the confluent hypergeometric function of first and second kind, and $\mathcal{I}$ is a particular solution to the non-homogeneous confluent differential equation. Moreover, $(y_0^*,y_1^*,\widehat{c},\widehat{d},\widehat{e},\widehat{f})$ are such that $w(0,y)$ and $w(1,y)$ belong to $C^1(\mathbb{R}^+)$.

Numerical Simulation

In Figure 2.1, we represent the value functions in the two-regime case, for the cases $\mu=-.05$ and $\mu=-0.3$. Other numerical results, in particular sensitivity analysis for the parameters μ, λ, β, and $\vartheta_{0,1}$ are equally obtained.

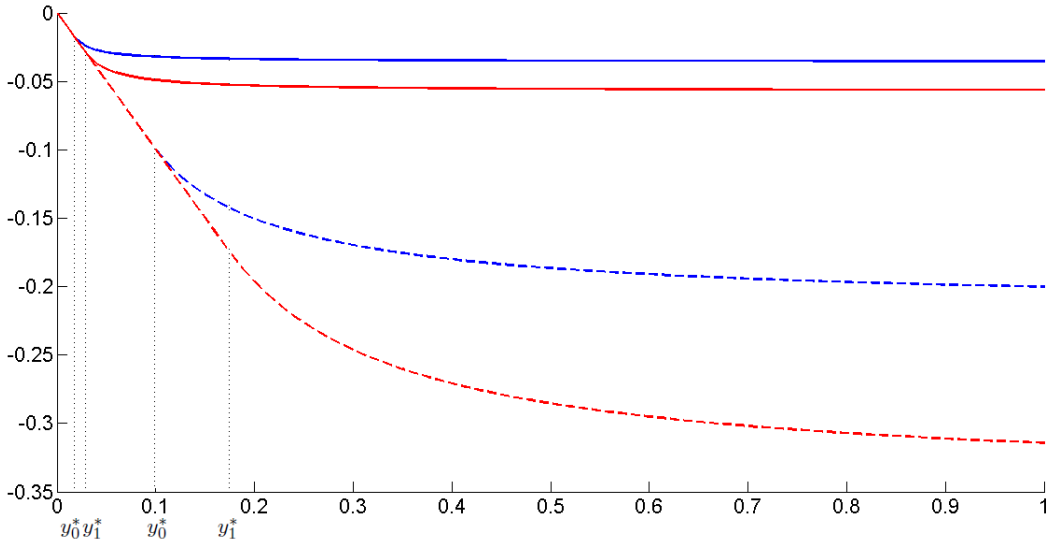


FIGURE 2.1: Value functions in the two-regime case, for the cases $\mu = -0.05$ (solid line) and $\mu = -0.3$ (dashed line). Regime 0 is presented in blue and regime 1 in red. The parameters used are $\lambda_0 = 2, \lambda_1 = 0.5, \vartheta_{0,1} = \vartheta_{1,0} = 1, \kappa = 1, \beta = 0.25, \gamma = 0.5$. The liquidation region are indicated by dashed lines. In the case $\mu = -0.5$, $y_0^* = 0.0172$ and $y_1^* = 0.0288$. In the case $\mu = -0.3$, $y_0^* = 0.0983$ and $y_1^* = 0.1742$.

Chapitre 3

Labor market and firing costs

This Chapter is based on

[A4]. "Uncertainty and the Politics of Employment Protection", with C. Tealdi and A. Vindigni, 2015, Journal of Labor Economics, Vol.33(1), pp.209-267.

Summary.

This chapter investigates social preferences over employment protection regulation, in a general equilibrium model of dynamic labor demand, and sheds some light on the comparative dynamics of Eurosclerosis. When firing costs are low, a transition to a rigid labor market is favoured by all the employed workers with idiosyncratic productivity below some threshold ; when their *status quo* level is high, preserving a rigid labor market is favoured only by the employed with intermediate productivity. A more volatile environment, and a lower rate of productivity growth increase the political support for labor market rigidity only in high rents economies.

Employment protection legislation varies significantly across OECD countries. Relatively stringent job security provisions are currently implemented in several Continental European countries whereas other countries such as the U.S. are characterised by relatively flexible labor markets. There is evidence that in Continental Europe firing costs have gradually become higher since the early 1970's, see for example Caballero and Hammour [29], Blanchard [19], and Blanchard and Wolfers [20], the period traditionally associated with the build-up of "Eurosclerosis,"see Giersch [68], and mildly reduced since the 1990's. During the same period, the structure of the labor markets of the U.S. has instead changed relatively little.

The goal of this work, is to investigate how the interaction of institutional and economic factors affects the *emergence* and the potential *persistence* of political support for some form of employment protection regulations. In order to pursue this goal, we develop and fully characterise the solution of a general equilibrium model of dynamic labor demand, which carries three distinctive features.

3.1 Basic environment

The economy is a small and open one, populated by a continuum of measure one of risk neutral workers who always consume all of their disposable income. Workers can be employed or unemployed, and discount future welfare at rate r equal to the real interest rate. Firms are created by a small set of risk neutral entrepreneurs, by paying a fixed cost C. The Firm produces some amount of output per unit of time by hiring one worker only. The productivity X of each firm varies over time due to the realization of two independent types of random idiosyncratic shocks. Specifically, X follows a geometric Brownian process, whose stochastic differential is represented by

$$dX_t = \mu X_t dt + \sigma X_t dW_t, \tag{3.1.1}$$

where W stands for a Wiener process. In addition, each production unit is also subject to a Poisson shock with arrival rate λ, which reflects a potential exogenous voluntary quit of the worker, driving permanently the productivity of the firm to zero.

Because productivity is variable, a firm may eventually decide to stop producing and to lay-off the worker. When this event happens, the firm pays a mandatory firing cost F for dismissing the worker, which represents a pure deadweight loss, i.e., the corresponding income is entirely wasted.

The value of a firm $\mathcal{J}(\cdot \mid R, \theta)$ active at time $t \in \mathbb{R}_+$ conditionally on the endogenous reservation productivity R and on the endogenous job creation rate θ, can be written as

$$\mathcal{J}(x \mid R, \theta) = \sup_{\bar{T} \in \mathcal{T}_t} \mathbb{E}^x \left\{ \int_t^{\bar{T} \wedge T_\lambda} e^{-r(\tau - t)} \left[X_\tau - w(X_\tau \mid R, \theta) \right] d\tau - F e^{-r\left(\bar{T} - t\right)} \mathbb{I}_{\bar{T} \leq T_\lambda} \mid \mathcal{F}_t \right\}, \tag{3.1.2}$$

Notice that the supremum is taken over the set $\mathcal{T}_t$ of possible stopping times within $[t, +\infty)$. However, the actual random separation time is equal to the minimum between $\bar{T}$, i.e., the random time at which the firm decides to stop producing, and the arrival time T_λ of the exogenous Poisson quitting shock of the worker. By standard arguments, the value function $\mathcal{J}(\cdot \mid R, \theta)$ satisfies the following Bellman equation

$$r\mathcal{J}(x \mid R, \theta) = \max \left\{ x - w(x \mid R, \theta) + \mathcal{L}\mathcal{J} - \lambda \mathcal{J}, -rF \right\}, \tag{3.1.3}$$

which characterizes the optimal stopping problem of the firm.

The value of a firm $\mathcal{J}(\cdot \mid R, \theta)$ also satisfies the initial value condition following from the standard assumption of free entry, which implies that firms earn no pure profits in equilibrium, since the *ex-ante* value of job creation, corresponding to the initial level of productivity $x = 1$ is equalized to the set-up, or hiring cost C. Formally, free entry of vacancies implies that

$$\mathcal{J}(1 \mid R, \theta) = C. \tag{3.1.4}$$

We now describe the wage setting mechanism. By standard arguments, the value $\mathcal{W}(x \mid R, \theta)$ of working in a firm with idiosyncratic productivity $x \in (R, +\infty)$, and the value $\mathcal{U}(R, \theta)$ of unemployment satisfy the following system of functional equations

$$r\mathcal{W}(x \mid R, \theta) = w(x \mid R, \theta) + \mathcal{L}\mathcal{W} + \lambda \left[\mathcal{U}(R, \theta) - \mathcal{W}(x \mid R, \theta)\right], \tag{3.1.5}$$

and,

$$r\mathcal{U}(R, \theta) = b + \theta \left[\mathcal{W}(1 \mid R, \theta) - \mathcal{U}(R, \theta)\right], \tag{3.1.6}$$

where $w(x \mid R, \theta)$ is the wage rate paid by the firm to the worker, b is the exogenous level of unemployment compensation (or value of leisure). We have the following main result.

Proposition 3.1.1 (Firm and Workers value functions and Wages)**.** *We have the following closed relation for firm value function, worker value function and wages.*

$$\mathcal{J}(x \mid R, \theta) = \frac{(1-\beta)\, x}{r + \lambda - \mu} - \frac{b}{r + \lambda} - \frac{\theta\beta}{r + \lambda}\left(\frac{1 - R^{1-\alpha}}{r + \lambda - \mu}\right) - \frac{(1-\beta)R^{1-\alpha}x^{\alpha}}{\alpha\,(r + \lambda - \mu)}, \tag{3.1.7}$$

where α corresponds to the negative root of the characteristic polynomial associated with the differential equation satisfied by $\mathcal{J}(\cdot \mid R, \theta)$, see the appendix of the paper for a closed form.

$$V(x \mid R) = \frac{x}{r + \lambda - \mu} - \frac{R^{1-\alpha}x^{\alpha}}{r + \lambda - \mu}, \tag{3.1.8}$$

$$w(x \mid R, \theta) = b + \theta\beta \frac{1 - R^{1-\alpha}}{r + \lambda - \mu} + \beta x. \tag{3.1.9}$$

3.2 Economic equilibrium

In section 3.1, we have assumed that the job creation rate θ and the job destruction rate δ are given. The objective of the following part is to characterise the equilibrium value of job creation and job destruction.

Because the transition density function of the stochastic process (3.1.1) describing the dynamics of productivity is time-homogenous, the random time $\bar{T}(R) \equiv \bar{T}_1(R) - t$ elapsed since the time t of creation of a firm (with productivity $X_t = 1$), at the moment when absorption takes place, does not depend on the calendar time of creation of the firm. Therefore, we can write the probability distribution of $\bar{T}(R)$ as follows

$$\mathbb{P}\left\{\bar{T}(R) \wedge T_{\lambda} > \tau\right\} = \int_{R}^{+\infty} \tilde{p}_{\lambda}(1, \xi; \tau)\, d\xi,$$

where $\tilde{p}_{\lambda}(1, \cdot; \tau)$ denotes the time-homogenous transition density function of x, conditional on the absence of absorption or exogenous quit at rate λ since the moment of creation of the firm t up to time $t + \tau$.

Proposition 3.2.2 (Steady state equilibrium relation)**.**

In the steady state, we have the labor market flows-balance condition $\delta L = \theta(1-L)$ *where* L *denotes denotes the total mass of employed workers at equilibrium. Moreover we have the following relation for the job destruction rate*

$$\delta = \frac{1}{\int_0^{+\infty} \mathbb{P}\left\{\bar{T}(R) \wedge T_\lambda > t\right\} dt}. \tag{3.2.10}$$

A second result links the job creation and the reservation productivity R and R as fonction of the costs C and F.

Proposition 3.2.3 (Job creation and reservation productivity relations)**.**

We have the two following relations

$$\theta = \frac{r+\lambda}{\pi\beta\left(1-R^{1-\alpha}\right)}\left[(1-\beta)R^{1-\alpha} + \pi(r+\lambda-\mu)\left(\frac{1-\beta}{r+\lambda-\mu} - \frac{b}{r+\lambda} - C\right)\right], \tag{3.2.11}$$

and

$$\frac{1-\beta}{r+\lambda-\mu}\left(1-R+\frac{R-R^{1-\alpha}}{\alpha}\right) = C+F. \tag{3.2.12}$$

Finally, under the condition that the two costs C and F are smaller than some explicit boundaries, we have the following result.

Proposition 3.2.4 (Job destruction and ergodic distribution)**.** *If* $F \in (0, F_{MAX}]$*, the steady state aggregate job destruction rate,* δ*, reads*

$$\delta = \frac{\lambda}{(1-R^{-\phi_2})}, \tag{3.2.13}$$

if $\lambda > 0$*, and*

$$\delta_0 = \frac{\left(\sigma^2/2\right)-\mu}{\hat{R}^+} = \frac{\eta}{\ln(R)}, \tag{3.2.14}$$

if $\lambda = 0$*, with* $\hat{R}^+ \equiv |\ln R|$*, and* $\eta \equiv \left[\mu - \left(\sigma^2/2\right)\right]$*. In addition, the ergodic cross-sectional distribution of productivity across firms,* $\Psi_\lambda(\cdot)$*, has probability density function* $\psi_\lambda(x)$ *represented by*

$$\psi_\lambda(x) = \frac{\phi_2\phi_1}{(\phi_2-\phi_1)}\left\{\left[\left(\mathbb{I}_{x\geq 1} + \mathbb{I}_{R<x<1}x^{\phi_1-\phi_2}\right) - R^{\phi_1-\phi_2}\right]x^{-\phi_1-1}\left(1-R^{-\phi_2}\right)^{-1}\right\}, \tag{3.2.15}$$

where $\mathbb{I}$ *denotes the indicator function defined in the standard way, and* ϕ_1 *and* ϕ_2 *are constant reported in the appendix of the related paper, with* $\phi_1 > 0$*,* $\phi_2 < 0$ *and* $\phi_2 - \phi_1 < 0$*.*

An important corollary concerns how Employment Protection Legislation (EPL) affects the mean level of productivity across establishments, which is equivalent to the mean value of the productivity of labor as one firms hires one worker only.

Corollary 3.2.1. *Higher firing costs reduce the equilibrium average productivity of labor.*

In our economy, this result is not surprising since firing costs correspond to a pure deadweight loss, and have no potential role in improving the competitive allocation of resources. Furthermore, we can compute the expected value of lifetime of a firm conditionally on its productivity.

Proposition 3.2.5. *The expected duration of a job with current productivity x reads*

$$\mathbb{E}\left[\bar{T}_x(R) \wedge T_\lambda\right] = \frac{1}{\lambda}\left[1 - \left(\frac{x}{R}\right)^{\phi_2}\right]. \tag{3.2.16}$$

Corollary 3.2.2. *If $\lambda = 0$ (i.e., $T_\lambda = +\infty$), the expected duration of a firm with productivity x reads*

$$\mathbb{E}\left[\bar{T}_x(R)\right] = \frac{\ln(x) - \ln(R)}{|\eta|}. \tag{3.2.17}$$

The economic equilibrium of the model has a number of comparative statics properties, some of which are non-standard, with the qualification that the firing cost will still be treated as an exogenous parameter at this stage of the analysis of the model (general economic equilibrium). Some numerical and graphic simulations, illustrating the main properties of the model, are reported on the paper.

3.3 The political mechanism

We assume that a given level of the firing cost $F = F_0$ is initially implemented, representing the *status quo* level of employment protection, and that the economy is in the corresponding stationary equilibrium. The *status quo* value of F may be changed as a result of a majority voting process. We assume that voting on the firing cost takes place only once, immediately after an *unexpected* shock to the exogenous variables of the model occurs, potentially affecting the rent extraction power of the workers, and the drift and standard deviation of the Brownian process describing the evolution of productivity, when the economy is in the politico-economic equilibrium corresponding to $F = F_0$. The new legislated firing cost corresponds to any point of the policy space, i.e., the interval $[0, F_{MAX}]$. The main political result of the paper is the following.

Proposition 3.3.6. *If $\beta \leq \hat{\beta}$, where $\hat{\beta}$ is given in the related paper, then the unique unconditional political equilibrium of the model involves setting $R = 1$, and this choice is preferred at unanimity to any alternative.*

Proposition 3.3.6 tells us that as long as the rent extraction power of the employed is relatively low, a fully flexible labor market is politically stable, in the sense that workers prefer it at unanimity to *any* possible alternative, whatever the *status quo* is.

Troisième partie

STOCHASTIC FILTERING AND APPLICATION TO OPTIMAL ALLOCATION

Over the last fifty years stochastic filtering theory in continuous time has drawn more and more attention, due to the variety of possible applications, that cover Biology, Engineering, Physics, Economics and Finance. Optimal filtering can be combined with stochastic control theory, leading, e.g., to the solution of problems of insider trading and of optimal allocation under partial information. So, it is possible to compare optimal investment strategies for informed and uninformed investors, finding estimates for the price of information.

The problem of stochastic filtering spans a very large and different subjects in the area of applied probability and mathematical finance, see for instance [86], [30] and [41]. The objective is to replace the unobservable processes by their best proxy, that is their conditional expectation with respect to filtration generated by the observable processes. From the measure theory point of view, the filtering problem is associated with the construction of the optional projection of the signal process on the observation filtration. We think that the following remark, taken from [52] chapter VII, page 118 could be useful for the reader : *Soit X un processus mesurable : si X n'est pas adapté, la v.a. X_t ne fait pas partie de l'univers connu á l'instant t, et elle peut tout au plus être estimée. Il est d'usage de considérer comme meilleure estimation possible l'espérance conditionnelle $\mathbb{E}[X_t|\mathcal{F}_t]$, et nous ne discuterons pas ce point. Maintenant, supposons que l'observateur suive ainsi à la trace le processus (X_t) tout au long du temps : peut on recoller toutes ces espérances conditionnelles en un seul processus, de manière raisonnable, alors que $\mathbb{E}[X_t|\mathcal{F}_t]$ est en realité une classe de variables aléatoires et qu'il faut donc faire une infinité non dénombrable de choix ? Ce que nous dit le théorème 43, c'est que si l'on impose au processus des estimations d'être optionnel et de realiser aussi la meilleure estimation en certains instants aléatoires, les temps d'arrêt, alors ce processus des estimations peut être défini de manière unique, et c'est évidemment très satisfaisant.*

The role and the properties of the observable filtration, hence forward called partial or investor's filtration, are crucial, whereas the global filtration, that is the one where the unobservable processes are defined, is less relevant. In particular a large part of the literature in stochastic filtering focuses on partial filtration generated by Brownian motion, even if the global filtration contains a discontinuous process as a Markov chain, see [86]. The usual strategy to compute the optimal filter is known as innovation theory, it particular in some case it gives us explicitly the stochastic differential equation satisfied by the optimal filter allowing to write an SDE for the whole system under partial information and to state a control problem for partially informed investors. That is by comparison of two different ways to write the product of the optimal filter and a generic combination of the observable processes. The first way require to apply the Ito formula on the global filtration and then to take the predictable version of the conditional expectation with respect to the investor's filtration. The second one inverts the order : first taking the optional version of the conditional expectation and then writing the SDE satisfied by the product. It is possible then to deduce the explicit SDE satisfied by the optimal filter if its predictable and optional version coincide. The main advance of a partial filtration generated by continuous

stochastic processes, is that the filtration is continuous in the sense of Dellacherie-Meyer [52], that is it is continuous on the set of stopping times, is that the two versions are the same.

Unfortunately, the hypothesis that the investor's filtration is generated by continuous processes is sometimes to restrictive in finance. For instance it is plain that jumps occurs, see [47] and [48]. In this framework, the investor's filtration cannot be longer considered as continuous and a different approach is needed. A recent and increase literature has then focus on the subject in particular for financial applications, see [33] and [34].

In the paper with Giorgia Callegaro, M'hamed Gaigi and Carlo Sgarra [A10], we follow a different strategy, that is we try to reconquer the continuity of the filtration on the stopping times by a countable number of initial enlargements of filtration. Our basic remark is that a stopping time θ is $\mathcal{F}_{\theta^-}$-mesurable. The problem of the lack of continuity does not come from the presence of jumps themselves but by their size and in particular the fact that these size are unpredictable. It can be surprising but a pure counting process, that is a pure jump process with jump size equal to one, generates a continuous filtration in the sense of Dellacherie and Meyer.

Our strategy to reconquer the stochastic differential equation satisfied by the filter is then to look at the jump processes, that are compound Poisson processes, as marked counting processes. That is, our idea is to enlarge the filtration including the size of the jumps. A potential demiurge investor will then known at the beginning the size of the jumps, the only unknown will be the jump times. His/her filtration is then continuous and the usual innovation theory will apply with some technical arrangements. The main drawback is that now, the demiurge has to many information and can of course exploit it. It is then impossible to use this approach without introducing arbitrages. In order to overcome this second problem we remark that a stochastic differential equation is, by definition, a forward equation. Our approach works then up to the first jump with random size. Our solution is then to focus between jump k and $k+1$ and to consider the initial enlargement with the first k size. The interest of this particular enlargement is twice. First-of-all, up to the $(k+1)$-jump time, the related filtration is continuous in the sense of Dellacherie and Meyer allowing to perform the usual filtering strategy above detailed. Moreover, after the k-jump time the k-demiurge, that is the one working under the k-partially informed filtration, loses his/her advantage since the usual investor knows the size of the first k jumps too. To conclude, we have only to recombine the optimal filter over all small intervals between two jumps to obtain the global filter.

Our approach is new in the literature and quite counterintuitive since it is based on enlargement of filtration procedures that are the natural converse to the filtering procedure. This method seems able to be generalised to others cases, I actually work with Guillaume Bernis (Natixis) and Vathana Ly Vath (ENSIIE), [W21], including memory between jumps and self-exciting structure

Chapitre 4

Over and under-reaction

This Chapter is based on
[A10]. Optimal investment in markets with over and under-reaction to Information avec G. Callegaro, M. Gaigi et C. Sgarra, 2017, *Mathematics and Financial Economics*. vol 11, issue 3, pp. 299-322.

Summary.

In this chapter, we introduce a jump-diffusion model of shot-noise type for stock prices, taking into account over and under-reaction of the market to incoming news. We work in a partial information setting, by supposing that standard investors do not have access to the market direction, the drift, after a jump. We focus on the expected utility maximization problem by providing the optimal investment strategy in explicit form, both under full and partial information.

The main mathematical contribution of our paper concerns the determination of the optimal filter dynamics, that is the evolution of the conditional law of the non observable quantities given the available information. On the mathematical side, it is well known that many results obtained when dealing with diffusion models are not robust, i.e., they cannot be immediately generalised in the presence of jumps. Standard proofs in stochastic filtering in continuous time and for the diffusive case crucially use the equivalence between optional and predictable versions of the filter itself. This does not hold true in general when dealing with jump-diffusion models. We introduce, then, a countable increasing family of filtrations, obtained by initially enlarging the observation filtration. We then show that these filtrations satisfy a property, that we can call "continuity" to a certain extent, that does not lead to the equivalence between optional and predictable versions of the process, but that is strong enough to obtain the filter dynamics in the enlarged filtrations. Finally, the desired filter dynamics in the standard investors' filtration is obtained via projection.

We test our results on market data relative to Enron and Ahold stocks. The three main contributions are : the introduction of a new market model dealing with over and under-reaction to news, the explicit computation of the optimal filter dynamics using an original approach combining enlargement of filtrations with Innovation Theory and the application

of the optimal portfolio allocation rule to market data.

4.1 Introduction

In this work, we introduce a new market model in which over-reaction and under-reaction to market news is taken into account. We consider two assets : one risk-free and one risky. The dynamics of the latter exhibits jumps. As usual, jumps are used to model the arrival of important (positive or negative) news about the firm. An extensive literature focuses on empirical studies on cross-section average stock returns and shows the presence of anomalies that are classified as "under-reaction to information" and "over-reaction to information", see for instance De Bondt and Thaler [47], Barberis *et al.* [22] and Fama [63] and references therein.

These phenomena can be described as follows : over-reaction takes place when, after some news related to the firm, the amplitude of the jump in the trajectory of the stock price is extremely large, both in the case of positive jumps related to good news, and in the case of negative jumps related to bad news. Under-reaction describes the opposite situation : the amplitude of the jump after a news is relatively small. The under/over-reaction effects are usually mitigated after the jump : the real relevance of the new information becomes accessible to the whole market as time goes by, so that the jump's effect fades away.

Our first contribution is, then, the introduction, in a market model for stock prices, of over and under-reaction effects under partial information.

Another contribution of this work is the application to a problem of maximisation of expected logarithmic utility from terminal wealth. As a matter of fact, the optimal investment problem in our paper is related to the wide class of trend following trading. Under this point of view, the under-reaction (respectively the over-reaction) of the signal can be interpreted as the dominance of a trend follow (resp. contra-trend) strategy of the active market participants. Trend following trading strategies have attracted increasing interest during the last ten years and a relevant amount of literature deals with portfolio optimisation problems in this context. We just mention the papers by Dai *et al.* [41], [39] and [40]. We shall show how standard (performing our filtering procedure) investors' strategies and naive investment strategies do not exhibit significant differences as far as jumps are not very large. On the contrary, a positive significant difference appears when jumps describe the occurrence of dramatic events, in both over and under-reaction cases.

4.2 The mathematical framework

We consider a complete probability space $(\Omega, \mathcal{G}, \mathbb{P})$, representing all the randomness of our economic context. We denote by $(\mathcal{G}_t)_{t\geq 0}$ the global information filtration (typically the one accessible to a limited set of well-informed investors) and by $(\mathcal{F}_t)_{t\geq 0}$ the investors' filtration, with $\mathcal{F}_t \subset \mathcal{G}_t$ for every $t \geq 0$. The two filtrations will be made precise in Section

4.3. We suppose that they satisfy the usual hypotheses introduced by Dellacherie and Meyer [51, Ch. IV, 48]. The probability space is equipped with a Brownian motion $\{W_t\}_{t\geq 0}$ and a Poisson process $\{C_t\}_{t\geq 0}$, defined as

$$C_t := \sum_{n\geq 1} \mathbb{I}_{\tau_n \leq t}, \tag{4.2.1}$$

independent of W. The arrival times $(\tau_n)_{n\geq 1}$ (notice that, implicitly, we have assumed that $\tau_0 = 0$), are supposed to be exponentially distributed with parameter $\lambda > 0$. We also introduce a triplet $(Z_t, N_t, \Gamma_t)_{t\geq 0}$ as follows :

$$Z_t = \sum_{n=1}^{C_t} \zeta_n \tag{4.2.2}$$

$$N_t = \sum_{n=1}^{C_t} \zeta_n e^{-\rho(t-\tau_n)} \tag{4.2.3}$$

$$\Gamma_t = \sum_{n\geq 1} \gamma_n \mathbb{I}_{t\in[\tau_n,\tau_{n+1})} = \sum_{n=0}^{C_t} (\gamma_n - \gamma_{n-1}), \tag{4.2.4}$$

where $\{\zeta_n, \gamma_n\}_{n\geq 1}$ is a sequence of i.i.d. random variables (henceforth r.v.) independent of C and W, taking values in $\{(-1,\infty)\backslash\{0\}\} \times \{-1, 1\}$. Every r.v. ζ_n is supposed to be $\mathcal{F}_{\tau_n}$-measurable and it will represent the jump size of the stock S at time τ_n. The Bernoulli r.v. $\gamma_n, n \geq 1$, is supposed to be measurable with respect to $\mathcal{G}_{\tau_n}$. We take $\rho > 0$. Even if the over/under-reaction process is not fully accessible to standard investors, since the γ_n are not observable, we suppose that investors know the prior probability of $\gamma_n = 1$, i.e., they have an a priori estimate of the probability $\mathbb{P}(\gamma_n = 1)$ for every $n \geq 1$. We denote this prior distribution, that will be extensively used henceforth, by π_n :

$$\pi_n := \mathbb{P}(\gamma_n = 1) \quad \text{for every} \quad n \geq 1. \tag{4.2.5}$$

With the above elements, we are now ready to introduce the dynamics of S : we have $S_0 > 0$ and

$$dS_t = S_{t-} \left\{ [b_0 + b(N_t)\Gamma_t]\, dt + \sigma(N_t) dW_t + \zeta_{C_t} dC_t \right\}, \tag{4.2.6}$$

where $b_0 \in \mathbb{R}$ is the drift coefficient of S in absence of any shock, $b : \mathbb{R} \to \mathbb{R}, b(0) = 0$, is an increasing function (think, for example, of $b(x) = x$) that measures the impact of the jumps on the drift, and $\sigma : \mathbb{R} \to \mathbb{R}_+$ is a continuous, strictly positive and bounded function.

4.3 Optimal filtering of the over/under-reaction process

Let us recall that in our model the global filtration is denoted by $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, while the investors' filtration is denoted by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. More precisely, the full information filtration

is the one generated by W, N and Γ, while, when the reference filtration is $\mathbb{F}$, the r.v. γ_n in Γ are not observable.

The filtering problem consists, roughly speaking, in evaluating the r.v. Γ_t, measurable w.r.t. $\mathcal{G}_t$, given the σ-algebra $\mathcal{F}_t$, i.e., in computing

$$\widehat{\Gamma}_t := \mathbb{E}[\Gamma_t|\mathcal{F}_t], \quad t \geq 0. \tag{4.3.7}$$

For a given $k \in \mathbb{N}, k \geq 1$, we introduce two filtrations $\mathbb{F}^{(k)}$ and $\mathbb{G}^{(k)}$ obtained by initially enlarging $\mathbb{F}$ and $\mathbb{G}$ with the first k marks $(\zeta_1, \ldots, \zeta_k)$ and $2k$ marks $(\zeta_1, \gamma_1, \ldots, \zeta_k, \gamma_k)$, respectively, i.e., for $t \geq 0$:

$$\begin{array}{rcl} \mathcal{F}_t^{(k)} & := & \mathcal{F}_t \vee \sigma(\zeta_1, \ldots, \zeta_k) \\ \mathcal{G}_t^{(k)} & := & \mathcal{G}_t \vee \sigma(\zeta_1, \gamma_1, \ldots, \zeta_k, \gamma_k)\,. \end{array} \tag{4.3.8}$$

Remark 4.3.1. *It is worth noticing here the embedding properties that hold in our setting and that we are going to use henceforth :* $\forall k \in \mathbb{N}, k \geq 1$

— $\mathbb{F} \hookrightarrow \mathbb{F}^{(k)}$

— $\mathbb{G} \hookrightarrow \mathbb{G}^{(k)}$

— $\mathbb{F}^{(k)} \hookrightarrow \mathbb{G}^{(k)}$

where the symbol $\hookrightarrow$ *denotes immersion according to the definition provided by P. Brémaud and M. Yor [24] (Section 2.4).*

The main result of this section is the following :

Theorem 4.3.1 (SDE verified by the optional version $\widehat{\Gamma}$)**.**

Let $\widehat{\Gamma}$ be the optional version of the optimal filter process : $\widehat{\Gamma}_t = \mathbb{E}\left[\Gamma_t|\mathcal{F}_t\right]$, for all $t \geq 0$. Then, we have

$$\widehat{\Gamma}_t = \sum_{n\geq 1} \mathbb{1}_{\tau_n \leq t < \tau_{n+1}} \widehat{\Gamma}_t^{(n)} = \sum_{n\geq 1} \mathbb{1}_{\tau_n \leq t < \tau_{n+1}} \mathbb{E}\left[\gamma_n|\mathcal{F}_t^{(n)}\right] = \sum_{n\geq 1} \mathbb{1}_{\tau_n \leq t < \tau_{n+1}} \left(2\widehat{\pi}_t^{n,n} - 1\right),$$

where $\widehat{\pi}_t^{n,n} := \mathbb{P}\left(\gamma_n = 1 \middle| \mathcal{F}_t^{(n)}\right)$ and $\mathbb{E}\left[\gamma_n|\mathcal{F}_t^{(n)}\right]$ is a standard conditional expectation with respect to the filtration $\mathbb{F}^{(n)}$ with initial condition $\widehat{\pi}_0^{n,n} := \pi_n$. Moreover, we find the following explicit expression for the dynamics of $\widehat{\pi}^{n,n}$ over the interval $[\tau_n, \tau_{n+1})$:

$$d\widehat{\pi}_t^{n,n} := 2\frac{b(N_t)}{\sigma^2(N_t)} \widehat{\pi}_t^{n,n} \left(1 - \widehat{\pi}_t^{n,n}\right) d\widetilde{Y}_t^{n,2}, \tag{4.3.9}$$

where

$$\widetilde{Y}_t^{n,2} := U_t - \log(s_0) - \int_0^t \left(b_0 + b(N_s)\widehat{\Gamma}_s^{(n)} - \frac{1}{2}\sigma^2(N_s)\right) ds, \ t \geq 0 \tag{4.3.10}$$

is the innovation process.

Remark 4.3.2. Notice that the optimal filter of Γ in Theorem 4.3.1 differs from the optimal filter of a Markov chain usually found in the literature (see for instance Liptser and Shiryaev [86, Th. 9.1]). We deem this is due to the fact that the jump times $\{\tau_n\}_{n\geq 1}$ are observable here.

We shall split the proof of Theorem 4.3.1 into three steps, that we are going to resume in the next lines.

Step 1 : Properties of enlarged filtrations. Studying the initial enlargement in (4.3.8), we will show that for every k, $\mathbb{F}^{(k)}$ is continuous at any stopping times θ strictly smaller than τ_{k+1}, in the sense that $\mathcal{F}^{(k)}_{\theta-} = \mathcal{F}^{(k)}_{\theta}$.

Step 2 : Computation of the filter under $\mathbb{F}^{(k)}$. We obtain the optimal filter of Γ given the observations $\mathbb{F}^{(k)}$, namely the (optional) projections :

$$\widehat{\Gamma}^{(k)}_t := \mathbb{E}[\Gamma_t | \mathcal{F}^{(k)}_t], \quad t \geq 0, \tag{4.3.11}$$

following a direct application of the Innovation Theory in our setting (the continuity of the filtration $\mathbb{F}^{(k)}$ will be crucial). We will see that this reduces to obtaining, for every k, the following (optional) projections on $\mathbb{F}^{(k)}$:

$$\widehat{\gamma}^{j,k}_t := \mathbb{E}\left[\gamma_j | \mathcal{F}^{(k)}_t\right], \quad 0 \leq t < \tau_{k+1}, \tag{4.3.12}$$

where $k \geq j \geq 1$.

Step 3 : Projection on $\mathbb{F}$. We conclude noticing that between two jumps τ_k and τ_{k+1} the investor filtration $\mathbb{F}$ coincides with $\mathbb{F}^{(k)}$, so that we will obtain the optimal filter by pasting together the filters $\widehat{\gamma}^{k,k}_t$ defined in Equation (4.3.12).

4.4 The optimal portfolio problem

In this section we deal with the problem of optimal portfolio selection over a finite time horizon $T > 0$ both under full and under partial information. We consider a financial market model consisting of a risk-free asset (representing the money market account) and a risky asset S, whose dynamics evolves following equation (4.2.6). Without loss of generality, as said before, we assume $r = 0$. We then introduce the fraction α_t of wealth invested in the risky asset at time t, so that the admissibility condition becomes $0 \leq \alpha_t \leq 1$ a.s. for every $t \in [0, T]$. Distinguishing between the complete and the incomplete information settings we introduce the following sets of admissible strategies

$$\mathcal{A}^{\mathbb{H}} := \left\{ \alpha^{\mathbb{H}} \,|\, \alpha^{\mathbb{H}} \text{is } \mathbb{H} - \text{predictable}, \, 0 \leq \alpha^{\mathbb{H}}_t \leq 1 \text{ a.s. } \forall t \in [0, T] \right\}. \tag{4.4.13}$$

where $\mathbb{H}$ can be $\mathbb{F}$ or $\mathbb{G}$.

Following a classical approach, in this paper we consider the problem of a fully informed (resp., partially informed) investor having a logarithmic utility function, who maximizes her

expected utility from terminal wealth, over a finite time horizon T :

$$\sup_{\alpha\in\mathcal{A}^{\mathbb{G}}} \mathbb{E}\left[\log(V_T)\right] \quad \left(\text{resp., } \sup_{\alpha\in\mathcal{A}^{\mathbb{F}}} \mathbb{E}\left[\log(V_T)\right]\right). \tag{4.4.14}$$

In what follows we characterize the optimal investment strategies under full and partial information, following a direct approach.

As we shall see, the solution under partial information will be obtained from the solution in the complete information setting by replacing the unobservable γ by their current conditional expectation via the filter process $\widehat{\Gamma}$. This property is known as *certainty equivalence* (CE) (this concept was originally introduced in the 80's, see e.g. Aoki [10, Ch. II.E] and it has been always used in further papers, such as in e.g. Callegaro *et al.* [30].

The following two propositions provide the optimal investment policies under both full and partial information.

Proposition 4.4.1 (Optimal policy for the fully-informed investor)**.** The optimal fraction of wealth $\alpha_t^{\mathbb{G},\star}$ to be invested in the risky asset at time t is

$$\alpha_t^{\mathbb{G},\star} = \begin{cases} 0 & \text{if } \mathcal{P}_t(0) \le 0 \\ \mathcal{P}_t^{-1}(0) & \text{if } \mathcal{P}_t(0) > 0 \text{ and } \mathcal{P}_t(1) < 0 \\ 1 & \text{if } \mathcal{P}_t(1) \ge 0 \end{cases} \tag{4.4.15}$$

where

$$\mathcal{P}_t(\alpha) := b_0 + b(N_t)\Gamma_t - \alpha\sigma^2(N_t) + \int \frac{z}{1+\alpha z}\nu(dz)\,, \tag{4.4.16}$$

and with $\nu(dz)$ denoting the Lévy measure of the jumps with support $\mathcal{R} \subset \{(-1,\infty)\backslash\{0\}\}$. Moreover, we have :

$$\begin{aligned} V_T \;=\; V_0\, \exp\Bigg\{ & \int_0^T \alpha_t^{\mathbb{G},\star}\sigma(N_t)dW_t + \int_0^T\int_{\mathcal{R}} \log\left(1+\alpha_t^{\mathbb{G},\star}z\right)\widetilde{Q}(dt,dz) \\ & + \int_0^T \Bigg[\alpha_t^{\mathbb{G},\star}[b_0 + b_1 N_t\Gamma_t] - \left(\alpha_t^{\mathbb{G},\star}\right)^2 \frac{\sigma^2(N_t)}{2} \\ & \qquad + \int_{\mathcal{R}} \left[\log\left(1+\alpha_t^{\mathbb{G},\star}z\right)\right] d\nu(z)\Bigg]\, dt\Bigg\} \end{aligned} \tag{4.4.17}$$

where $\widetilde{Q}(dt,dz)$ is the compensated version of the Poisson random measure associated with Z.

Proposition 4.4.2 (Optimal policy for a standard/partially informed investor)**.** The optimal fraction of wealth $\alpha_t^{\mathbb{F},\star}$ to be invested in the risky asset at time t is such that

$$\alpha_t^{\mathbb{F},\star} = \begin{cases} 0 & \text{if } \mathcal{P}_t^{\mathbb{F}}(0) \le 0 \\ (\mathcal{P}_t^{\mathbb{F}})^{-1}(0) & \text{if } \mathcal{P}_t^{\mathbb{F}}(0) > 0 \text{ and } \mathcal{P}_t^{\mathbb{F}}(1) < 0 \\ 1 & \text{if } \mathcal{P}_t^{\mathbb{F}}(1) \ge 0 \end{cases} \tag{4.4.18}$$

where

$$\mathcal{P}_t^{\mathbb{F}}(\alpha) := b_0 + b(N_t)\widehat{\Gamma}_t - \alpha\sigma^2(N_t) + \int \frac{z}{1+\alpha z}\nu(dz)\,. \tag{4.4.19}$$

The two Propositions introduce similar functions $\mathcal{P}_t$ and $\mathcal{P}_t^{\mathbb{F}}$ differing only by one factor, that is independent of α. We can conclude our analysis remarking that the particular case introduced by Kou [81] gives closed formula in our framework and it will be also used in the following. See the related paper for the explicit computation.

4.5 Two relevant examples : empirical facts and statistics

Market analysts usually explain market fluctuations from an "event-based" point of view, i.e., linking every movement to an economic event or to an announcement. It is clear that important and sudden news can induce large market movements. Even if these events rarely occur, their impact is important.

Accounting scandals happened frequently over the last years, the most resounding cases being Enron, Worldcom, Royal Ahold and Parmalat. We focus on Enron's scandal, because of its impact on the US market and Ahold's one, since it was considered the European analogous to Enron, even if the end of the two stories was different.

The Enron scandal revealed in October 2001. As many reports highlight (see for instance Jickling [75]), Enron's problems did not arise in its core business, but in its Internet and high-tech communications investments, in which it entered during the late 1990, i.e., just before the "dot com" bubble burst. Enron falsified its accounts to cover its bad investments, using special-purpose entities to remove investment losses for nearly 18 months. Enron stocks reached the record level of \$90.75 per share in August 2000. When the accounting fictions came to light, over 80% of the profits reported since 2000 vanished and Enron quickly collapsed. Enron's stock price, which quoted $US\$90$ in mid 2000, plummeted to less than \$1 one year later. Left figure 4.1 shows the evolution of Enron's stock price. We highlight using red vertical lines the estimated jump times (see Section 4.5.1 for more details on jumps' detection).

The second example is Ahold, that in the US was even defined "Europe's Enron" (see for instance Knapp and Knapp [79]). On February 24th 2003, Ahold announced that : "... net earnings and earning per share are significantly lower than previously indicated, primarily due to overstatements of income ...". The accounting irregularities were mainly concentrated at two US subsidiaries. As a consequence, Ahold's stock's price collapsed by 63%. After two months, the parallelism with Enron seemed an exaggeration and Ahold stock rose during the next five months.

Right figure 4.1 shows the evolution of Ahold stock's price. We highlight using red vertical lines the estimated jump times.

4.5.1 Statistical analysis

In this subsection we estimate the parameters in the model for S in Equation (4.2.6) and the law of the jumps ζ, by working on market data relative to Enron and Ahold. Our statistical sample for Enron's stock consists of closure prices between January 1997

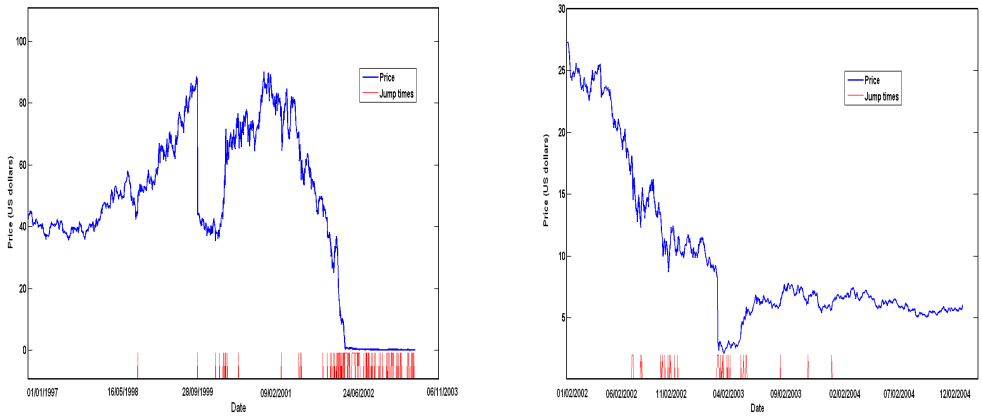


FIGURE 4.1: Closing prices in blu, estimated jump times for Enron and Ahold in red.

and December 2002 (1508 values, see left Figure 4.1), while for Ahold's stock it consists of closure prices between January 2002 and December 2004 (769 values, see right Figure 4.1). For simplicity, we choose $b(x) = x$ and σ constant in Equation (4.2.6). Moreover, we consider the model introduced by Kou in [81], namely we suppose that the law of $\log(\zeta_n + 1)$ is for every $n \geq 1$ equal to

$$\nu_E(dz) = \left[\lambda p \eta_1 e^{-\eta_1 z} \mathbb{I}_{z>0} + \lambda(1-p)\eta_2 e^{-\eta_2 |z|} \mathbb{I}_{z<0}\right] dz,$$

where $0 < p < 1$, $\eta_1 > 1$, $\eta_2 > 0$ and $\lambda > 0$ is the intensity rate of the Poisson process. We obtain the following estimates :

	σ	λ	p	η_1	η_2	ρ
Enron	0.38	21.37	0.40	8.89	8.51	2.85
Ahold	0.43	9.74	0.50	29.07	12.58	1.90

4.5.2 A qualitative analysis of the optimal policies

In this subsection we consider four different market actors :

— An insider trader : an agent who has access to the r.v. γ, i.e., whose filtration is $\mathbb{G}$;

— A partially informed investor : an agent who does not observe the r.v. γ, i.e., whose filtration is $\mathbb{F}$;

— An insider but irrational investor : an agent acting oppositely to the insider trader, i.e., in the worst possible way ;

— A naïve (or blind) investor : an agent supposing that no fade-away effects occur on the market, i.e., $\Gamma \equiv 0$. This investor can observe the jumps but she does not know if the market undergoes over/under-reaction.

Our final aim is comparing their (optimal) portfolios in the two different cases of Enron and Ahold in Figures 4.1. Namely, we consider four investors facing an optimal investment

problem in Enron's (resp., Ahold's) stock in the period January 1997 - December 2002 (resp., January 2002 - December 2004).

Figures 4.2 show, respectively, the information process N (in black) and the optimal filter $\widehat{\Gamma}$ (in blue) for Enron and Ahold.

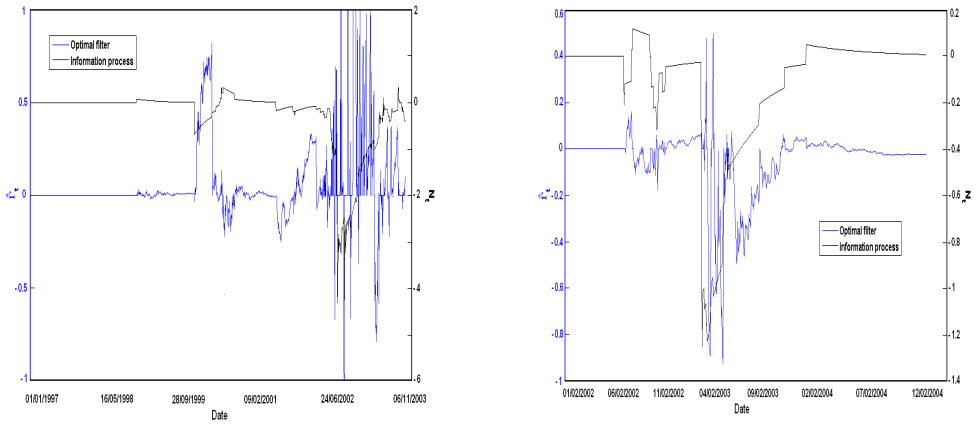


FIGURE 4.2: The information process N and the optimal filter $\widehat{\Gamma}$ for Enron and Ahold

In Figures 4.3 we depict the value of the optimal portfolios for the four agents, when investing in Enron and Ahold, respectively.

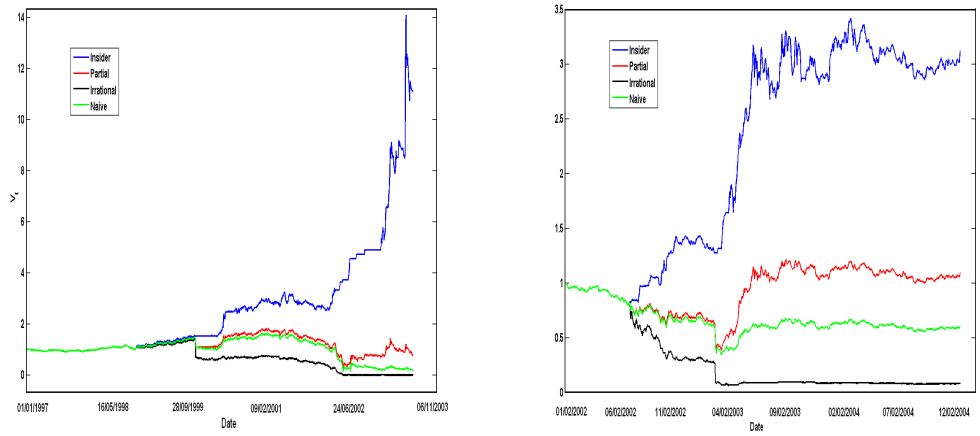


FIGURE 4.3: The four optimal investment strategies for Enron.

It can be noticed that, as long as no jump occurs (recall left Figure 4.1), i.e., $N_t = 0$, all the portfolios are identical. After a few small jumps arrive, i.e., $N_t \neq 0$ and small, only the informed investor takes advantage of her information, while the naïve investor and the partially informed trader perform similarly, but clearly worse than the insider and better than the irrational. The best portfolio performance is the one of the fully informed investor,

followed by the agent who uses the filter, the naïve one and the irrational one.

When the first big (negative) jump occurs, both the naïve and the uninformed investors suffer a similar big loss, but immediately afterwards the partially informed investor can take advantage of the incoming information, since the big jump allows the filtering procedure to distinguish soon the sign of Γ and this introduces a difference in the two portfolios performances. As time increases, this effect tends to disappear because of the exponentially decaying behavior (due to the presence of $\rho > 0$) of the stock price trajectory between two jumps.

Notice, moreover, that in the Enron's case, both the partially informed and the naïve investors' portfolios are "annihilated" by the big negative jump in 2002. This can be associated to Enron's stock becoming a "penny stock". Nevertheless, even in this case the filtering procedure performs very well : it allows the portfolio of the investor using the filter to rise from its ashes.

Quatrième partie

STOCHASTIC ANALYSIS WITH APPLICATION TO FINANCIAL RISK MODEL

During my PhD thesis, with Nicolas Bouleau as supervisor, I started to work on stochastic analysis and in particular potential theory and Dirichlet forms. They give powerful methods to deal with stochastic sensitivity and they came back to the mathematical finance research mainstream after the subprime crisis that had emphasise the lack of knowledge about the probability measures that we use to describe the financial markets. My PhD thesis was devoted to how to take into account this ambiguity in a parsimonious framework. This research has lead me to focusing during the two last years on self-exciting processes using a random field representation. This field is seething during the last years with many different applications, see for instance [7].

In a first part, we will focus on an extension of the paradigm of Capital Asset Pricing Model that takes into account only the Gaussian framework to different risks. Our approach is based on the chaos decomposition of both the portfolio and the benchmark in order to take into account extreme risks. In a work [A9] with Guillaume Bernis (Natixis AM), we define a new beta index that take into account the fat tails observed on financial markets. The literature, see for instance [100] and [102], has pointed the lack of functional dependency of the coefficient beta. The chaos decomposition is historically related to the Malliavin calculus and then to the concept of stochastic sensitivity, a more direct example on stochastic sensitivity via "weights" à la Malliavin is detailed in [S14]. We have also show interesting financial results as the fact that the new beta is linked to the insensitivity of the portfolio with respect to a change on the index drift. Moreover, our framework is suitable in order to obtain closed formula in many particular cases notably the Jacobi processes. Another financial application is the opportunity to define functional Bollinger bands ables to detect the changing on the financial markets evolution.

This dissertation ends with my most recent subject of research, that is the evolution models based on random fields and the branching processes. In their keystone papers, [55] and [56], Dawson and Li introduces a new class of SDE driven by random fields. Their idea is to increase the dimension of each field in order to integrate along this new direction up to the process itself. The authors show the existence and the uniqueness of the solution and that it satisfies the branching property, i.e. the process is infinitely divisible with respect to the initial condition. Filipovic has shown that branching processes have exponential affine structure and this property is often exploited in finance. The applications span a wide spectrum of underlying, e.g. stocks, interest rates and default times, see [73], [59] and [62]. In my opinion, the most interest of the integral representation proposed by Dawson and Li is to give an explicit form for the SDE satisfied by a branching process allowing to understand some phenomena appearing in financial markets.

In light of this, I actually works with Ying Jiao (ISFA Lyon), Chunhua Ma (Nankai, China), Carlo Sgarra (Politecnico di Milano, Italy) and Chao Zhou (National University of Singapore) on some applications of branching models in finance. In a first paper [A11], we extend the CIR model [37] for interest rates including a spectrally positive alpha-stable branching process. The self-exciting structure, characteristic of branching process, has an

intriguing impact on the interest rates and bonds. We have shown that our model can reproduce the persistency of very low interest rates, intriguing since it seems in contradiction with the idea of self-exciting. This effect can be explained in financial terms by the fact that, if the occurence of a jumps increases the intensity of the next one, by symmetry, in its absence, the intensity decreases. There exists then a trap specific to branching processes able to annihilate the process that approches zero. We link this result to a sudden increment of the mean reverting speed between two jumps. This term is proportional to the compensator of the jump term driving the SDE. It is plain that if we increase the jumps size, i.e. decrease the parameter α, we also increase the compensator. As a consequence, a second counter-intuitive effect is that the persistency of low interest rates is magnified increasing the tail of jumps distribution. This effect is so strong that the same impact is transferred to long interest rates producing an inversion of the dependency of zero-coupon prices toward the risk of large fluctuations in interest rates.

In a second paper, [S12] with Ying, Chunhua and Carlo, we use a branching process in order to describe the evolution of energy prices. The main reason comes from the presence of strong fluctuations, known as spikes, and their inhomogeneous distribution over the time exhibiting clusters. Our idea is to replace the martingale term into the usual Levy Ornstein Uhlenbeck model, see [17], by a self-exciting term. We show that the forward prices are unchanged but the structure of risk premium changes remarkably exhibiting a change of sign. This phenomenon is observed on energy markets but it is difficult to reproduce it using previous models. Moreover, we propose a calibration method cleverly based only on the time arrival of spikes. Our strategy is mathematically based on a convergence result of the branching processes to an Hawkes process assuming that the mean reverting speed diverges, that is coherent with the energy data. We then can use a known result about the estimation of point process in order to deduce the likelihood and the related estimator.

Finally, I work actually on different papers related to branching processes. First of all, in the preprint [S14] with Guillaume Bernis and Kaouther Salhi (Natixis AM) we study the sensitivity in a model for CLO. This model is based on Hawkes process and this work can be seen as a closure of cycle started with my PhD thesis. With Ying, Chunhua and Chao, we work on an extension of Heston model using branching processes, [W17]. I also work on an extension of the model for energy prices, [W19] in a more theoretical framework in order to study the conditions to have that a multi-factor branching model satisfies the Samuelson effect.

Chapitre 5

Alternative to beta coefficients in the context of diffusions

This Chapter is based on
[A9]. Alternative to Beta Coefficients in the Context of Diffusions avec G. Bernis, *Quantitative Finance*, volume 17, issue 2, pp.275-288, 2017.

Summary.

We develop an alternative to the beta coefficient of the CAPM theory. We show the link between this notion and the Wiener chaos expansion of the underlying processes. In the setting of Markov diffusions, we define the drift-neutral beta, which is the quantity of benchmark such that the resulting portfolio is immune to an infinitesimal change of drift on the Brownian motion driving the benchmark. Our approach yields a coefficient which depends on the initial values of both the portfolio and its benchmark, in many practical cases. It can also be used to take into account extreme risks and not only the variance. We study several classical diffusion processes and give a full analysis in the case of Jacobi processes. Examples with credit indices show the efficiency of the method to hedge a portfolio.

5.1 Introduction

The beta coefficient plays a central role in the CAPM theory, developed by Markovitz [91]. It represents the exposition of a portfolio to a benchmark, representing the global market. This coefficient can be used to hedge the movements of the portfolio with respect to the market fluctuations. It also allows the portfolio manager to build an allocation exposed to a specific risk (for instance, the risk of a certain sector), once the global market movements have been neutralised. Some authors point out limitations of the beta coefficient - especially the problems stemming from the underlying regression - and emphasise the lack of a functional relationship between the portfolio and the benchmark, see [100] and [102]. In the first section of this work, we show the link between the beta and the decomposition of the

portfolio and its benchmark into Wiener chaos. The expansion of a Brownian functional into Wiener chaos consists in a decomposition on an orthogonal basis of multiple Itō integrals, see [94]. More precisely, from a static point of view, we define the value of the portfolio and its benchmark as two random variables. The "Wiener chaos beta" coefficient is such that the projection of the portfolio minus beta times the benchmark, on a certain mixture of Wiener chaos, is null. We will show that this new definition coincides with the classic one either if the portfolio and its benchmark are both Gaussian, or if the mixture is reduced to the first chaos, i.e. the Gaussian term. This approach is very general and does not require any restrictive hypothesis on the financial side. In particular, the static approach and the Wiener chaos decomposition do not require the Markov property. However, this general approach has two drawbacks, which unfortunately are intrinsically linked. First, the Wiener chaos decomposition exists under very general assumptions but the proof is not constructive because it is based on the martingale representation theorem. However, Di Persio et al. [53] propose a new and efficient technique to evaluate numerically the chaos decomposition of a random process modelled by a SDE. Moreover, Briand and Labart [27] and Geiss and Labart [66] develop recently algorithms for BSDE based on chaos decomposition confirming that this technique is a powerful tool in numerical schemes.

This is why we propose to adapt our framework to a Markov diffusion setting. We show that our approach can be seen, equivalently, as a notion of resilience of the portfolio with respect to a change of drift on the Brownian motion driving the market risk. Basically, it constructs a portfolio which is immune, at the first order, to a change of drift on the Brownian motion which drives the benchmark. Therefore, we will denote by "drift-neutral beta" the coefficient produced by this method. We will show that the drift-neutral beta coincides with the Wiener chaos beta under technical assumptions assuring, essentially, the existence of the Wiener decomposition. We provide an analysis of this problem in the case of several classical diffusion processes : Ornstein-Uhlenbeck, co-integrated diffusions, geometric Brownian motion and Jacobi processes, see [50].

5.2 Mathematical framework

Let $(B_t, W_t)_{t\geq 0}$ be a 2-dimensional standard Brownian motion, on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the following diffusion process in $\mathbb{R}^2$

$$\begin{aligned} dX_t &= \mu(X_t)dt + \sigma(X_t)dB_t \\ dY_t &= \nu(X_t, Y_t)dt + \gamma(X_t, Y_t)\left[\rho dB_t + \sqrt{1-\rho^2}dW_t\right] \end{aligned} \tag{5.2.1}$$

The process $(Z_t)_{t\geq 0} := (X_t, Y_t)_{t\geq 0}$ is assumed to be a continuous diffusion process on an open rectangle $R :=]m, M[\times]n, N[$ of $\mathbb{R}^2$, with $-\infty \leq m < M \leq +\infty$, and $-\infty \leq n < N \leq +\infty$. In the following, we will work under the natural filtration of the process Z, denoted by $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$. We will also need the natural filtration generated by B, denoted by $\mathbb{F}^B := (\mathcal{F}_t^B)_{t\geq 0}$

Set $T > 0$, the time horizon of our hedging problem. Let $0 \leq t \leq T$ and $z = (x, y)$. Let $F(Z_T) := Y_T - \beta X_T$ be the portfolio, with its hedge. Parameter β has to be chosen in order to offset the risk stemming from the benchmark X in the portfolio Y.

5.2.1 Wiener chaos approach

Analysing the Gaussian framework, we can notice that the standard beta coefficient is given by the projection on the first Wiener chaos of the Brownian X, through the following equation :

$$\mathbb{E}\left[Y_T \int_0^T \overline{\sigma} dB_s\right] = Cov[Y_T, X_T] \equiv \beta_0 Var[X_T] = \beta_0 \mathbb{E}\left[X_T \int_0^T \overline{\sigma} dB_s\right].$$

In this case, a straightforward application of Itō calculus shows that the result does not depend on the drifts, neither on the time.

We now introduce our

Definition 5.2.1. *Let Φ_T be a zero-mean square integrable $\mathcal{F}_T^B$-random variable. The* Wiener chaos beta coefficient *with respect to the zero-mean random variable Φ_T, horizon T, and initial condition $z = (x, y)$, is the real number $\beta_{T,z}^{(\Phi)}$, which solves*

$$\mathbb{E}\left[Y_T \Phi_T\right] = \beta_{T,z}^{(\phi)} \mathbb{E}\left[X_T \Phi_T\right]. \tag{5.2.2}$$

We have the following proposition.

Proposition 5.2.7. *Let X_T, Y_T Φ_T be square integrable respectively $\mathbb{F}^B$-random variable (r.v.), $\mathbb{F}$-r.v. and a zero-mean $\mathbb{F}^B$-r.v.. Then*

$$\beta_{T,z}^{(\Phi)} = \frac{\sum_{k=1}^{\infty} \left\langle f_k^Y, f_k^\Phi \right\rangle_{S_k}}{\sum_{k=1}^{\infty} \left\langle f_k^X, f_k^\Phi \right\rangle_{S_k}} \tag{5.2.3}$$

where f_k^X and f_k^Φ are the k-projection of X_T and Φ_T with respect to the Wiener chaos decomposition of the probability space $(\Omega, \mathcal{F}_T^B, \mathbb{P})$ and whereas f_k^Y is the B-component of the k-projection of Y_T with respect to the Wiener chaos decomposition of the probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$.

The financial consequences of Proposition 5.2.7 are summarised in the following remark.

Remark 5.2.1. *The main financial result of Proposition 5.2.7 is that $\beta_{T,z}^{(\Phi)}$ can be seen as a weighted beta of the chaos of Y and X, where the weights are the chaos of the process Φ, see Equation (5.2.3). As a consequence, it is possible to put more weight on extreme values of X and Y, in order to obtain a hedge $\beta_{T,z}^{(\Phi)}$ adapted to situations of stress on the markets. Thus, our approach can be used in a risk management point of view. A second important point is that the Wiener chaos expansion has the advantage of not depending on a Markovian set-up. Then our Definition 5.2.1 is still valid in a non-Markovian framework. This is another useful property from a risk management perspective.*

5.2.2 A change of drift point of view

The idea that we will follow is to restrict slightly the form of the variable Φ_T. It is given by $\int_t^T h(X_s)dB_s$, where h is a square-integrable and measurable function from $]m, M[$ into $\mathbb{R}$. The reason why we use a mapping of X is twofold. First, we will show, in Proposition 5.2.9, that our method is based on the stability of portfolio F with respect to an infinitesimal change of drift for the Brownian motion B. As the market index X is used to hedge, we restrict our attention to drifts which are mappings of X. Second, this approach leads to a simple form, which is tractable by the means of Itō calculus, in many cases. Of course, this framework is Markovian, then more restrictive than the previous one. We introduce the alternative beta definition that we will call drift-neutral beta coefficient as follows :

Definition 5.2.2. *Let h be a square-integrable and measurable function from from $]m, M[$ into $\mathbb{R}$. The* drift-neutral beta coefficient *at time t, for drift h, horizon $T \geq t$, and initial condition z, is the real number $\beta^{(h)}_{t,T,z}$, which solves, in $\beta \in \mathbb{R}$, the following equation*

$$\mathbb{E}\left\{ Y_T \int_t^T h(X_s)\, dB_s \;\middle|\; \mathcal{F}_t \right\} = \beta \mathbb{E}\left\{ X_T \int_t^T h(X_s)\, dB_s \;\middle|\; \mathcal{F}_t \right\}. \tag{5.2.4}$$

We will remark that, even if h is constant, this approach differs from the traditional beta, when X and Y are not standard Brownian motions. The next proposition links the Definitions 5.2.1 and 5.2.2, the proof is straightforward.

Proposition 5.2.8. *Fix $t = 0$. Let h be as in Definition 5.2.2. Define $\Phi := \int_0^T h(X_t)dB_t$. Under the conditions of Proposition 5.2.7. We have that the Wiener chaos beta coefficient associated to Φ coincides with the drift-neutral beta one for drift h.*

As noticed above, Definition 5.2.2 can be put in relation with a variational approach, in which we consider the resilience of the portfolio with respect to infinitesimal change of drifts on B. The notion of change of drift for a Brownian motion is closely linked to the Girsanov theorem, which provides the form of the density of the probability under which $(B_s - \varepsilon h(X_s))_{s\geq 0}$ is a standard Brownian motion. We will resort to this approach to link the notion of portfolio neutrality with respect to a change of drift with Definition 5.2.2. This is the purpose of the following proposition.

Proposition 5.2.9. *Let us consider a measurable mapping h from $]m, M[$ into $\mathbb{R}$ (endowed with their Borelian sigma-fields) such that there exists $\delta > 0$ such that $\mathbb{E}\left[e^{\delta \int_0^T h^2(X_s)ds}\right] < +\infty$. Then,*

$$\lim_{\varepsilon \to 0^+} \mathbb{E}\left\{ \frac{1}{\varepsilon}\left[\frac{\Lambda_\varepsilon(T)}{\Lambda_\varepsilon(t)} - 1 \right] F(Z_T) \mid \mathcal{F}_t \right\} = \mathbb{E}\left\{ F(Z_T) \int_t^T h(X_s)dB_s \mid \mathcal{F}_t \right\}$$

where $\Lambda_\varepsilon := (\Lambda_\varepsilon(t))_{t\geq 0}$ is the Doléans-Dade exponential of $(\varepsilon h(X_s))_{s\geq 0}$, i.e. the unique strong solution of the following stochastic differential equation :

$$\Lambda_\varepsilon(t') = 1 + \varepsilon \int_0^{t'} \Lambda_\varepsilon(s) h(X_s) dB_s \tag{5.2.5}$$

The following proposition provides a more convenient form for the drift-neutral beta.

Proposition 5.2.10. *Let h be as in Proposition 5.2.9 and set $B[h]_{(s,t)} := \int_s^t h(X_u)dB_u$, for any $t \geq 0$. Assume, for technical reasons, that X, $\sigma(X)$, Y and $\gamma(X,Y)$ have finite moments of order 4 and that*

$$A(t,T) := \mathbb{E}\left\{X_T B[h](t,T) \mid \mathcal{F}_t\right\} \neq 0.$$

Then, the drift-neutral beta coefficient $\beta^{(h)}_{t,T,z}$ is given by

$$\beta^{(h)}_{t,T,z} = \frac{\mathbb{E}^{t,z}\left\{\int_t^T \left[B[h]_{(t,s)}\nu(Z_s) + \rho h(X_s)\gamma(Z_s)\right] ds\right\}}{\mathbb{E}^{t,x}\left\{\int_t^T \left[B[h]_{(t,s)}\mu(X_s) + h(X_s)\sigma(X_s)\right] ds\right\}} \tag{5.2.6}$$

where $\mathbb{E}^{t,z}$ (resp. $\mathbb{E}^{t,x}$) stands for the expectation with initial condition $Z_t = z$ (resp. $X_t = x$).

5.3 Application to credit indices

In this section, we apply the results that we have derived in the previous sections to synthetic credit indices. We consider X as the iTraxx Europe 5 years index and Y as the iTraxx Cross-Over 5 years index. It means that we want to hedge a purely directional credit trend from a high yield index, in order to capture the specific high yield component. We calibrate the parameters of a Jacobi process, see for instance [11], on a set of daily observations from 20/06/2011 to 23/03/2015. We obtain the following set of parameters :

m	M	λ	$\overline{\mu}$	$\overline{\sigma}$	n	N	η	$\overline{\nu}$	$\overline{\gamma}$	ρ
0.408%	2.396%	0.5	1.36%	0.658	1.865%	10.056%	0.5	5.96%	0.652	65%

We consider mapping h localised on a side of the volatility function, whether left or right. For instance, $\sigma_{-25\%}(x)$ is equal to σ if x is below the left quarter of $]m, \overline{\mu}[$ and 0 elsewhere. Figure 5.1 displays $\beta^{(h)}_{0,T,z}$ as a mapping of the initial condition $z = (x, y)$. We observe that it tends to attain high values for small values of x and y. Outside this zone, the surface shows two valleys surrounding the central hump. This general shape also prevails for $h \equiv \sigma$, $h \equiv \sigma_{50\%}$ or h constant.

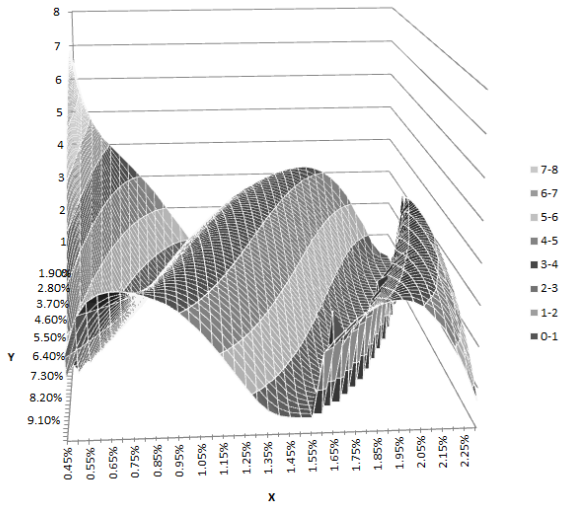


FIGURE 5.1: Beta coefficient with $h = \sigma_{-50\%}$ as a mapping of the initial conditions (x, y) for iTraxx Europe (X) and Cross-Over (Y) (5 year)s indices. The time horizon T is 30 days.

Chapitre 6

Financial models with branching framework and random field

This Chapter is based on

[A11]. "Alpha-CIR model with branching processes", with Y. Jiao and C. Ma, 2017, *Finance and Stochastics*, vol 21, issue 3, pp. 789-813.

[S12]. "A Branching Process Approach to Power Markets", with Y. Jiao, C. Ma and C. Sgarra, 2017, submitted.

Summary.

In this chapter, we look at two different problems arising in mathematical modelisation normally considered as really far, that is the evolution of sovereign interest rate and energy prices. Our main result is to show that a similar model based on Branching processes, see for instance Kawazu and Watanabe [76] or the recent monograph [84], can explain different features appearing in term structure and energy prices. Considering interest rates, they have reached a historically low level in the Euro countries. On the other hand, sovereign bonds can have large variations when uncertainty about unpredictable political or economical events increases such as in the Greek case. Considering energy prices, they exhibits jumps with cluster phenomenon. Moreover, the risk premium sign, according to many empirical investigations performed in different power markets, can change suddenly, and induce a change between a contango to a backwardation regime, see for instance Benth and Sgarra [18], and Benth et al. [17].

Recently, the Hawkes processes introduced in [72] have been widely adopted since they exhibit self-exciting properties which are suitable for such modelling. A large and growing literature is devoted to the financial application of Hawkes processes, see for example Aït-Sahalia, Cacho-Diaz and Laeven [2], Errais, Giesecke and Goldberg [62], Dassios and Zhao [42] and Rambaldi, Pennesi and Lillo [98]. In the above mentioned papers, as naturally in the Hawkes framework, the driving process is at least two-dimensional since both the dynamics of jump process and its intensity are taken into account.

Our idea is to use the self-exciting structure of the "frequency" of Hawkes process. We will introduce a new class of financial models based on CBI (continuous state branching processes with immigration) processes. This class of model can be seen as an extension of Hawkes process adding specific marks for each jumps and including infinite activity. We exploit an integral representation of the model with random fields. From the theoretical point of view, this general representation has been thoroughly studied by Dawson and Li [55], [56] and Li and Ma [85]. In the financial literature, the random field modelling has been adopted to describe financial quantities, see for example Kennedy [78], Albeverio et al. [3] in interest rate framework or Ambit Field-based models in energy modelling introduced by O.E. Barnorff-Nielsen et al. [14]. In our model, we adopt the integral representation to emphasize on the property of branching processes since they arise as the limit of Hawkes processes and exhibit, by their inherent nature, the self-exciting property implying that the jump frequency increases or decreases with the value of the process itself. In the modelling of interest rate, the link between the CBI processes and the nonnegative affine models has been established by the pioneering paper of Filipović [64] where the exponential affine term structure of bond prices for general CBI processes has been highlighted. By consequence the CBI processes prove to be a prolific subject in probability having interesting applications in finance, see for instance Duffie, Filipović and Schachermayer [59]. The most simple and popular CBI process is the continuous CIR model.

6.1 The modeling framework

We now introduce our modeling framework, which is based on stochastic differential equations driven by Lévy random field. We consider a Lévy random field which is a combination of a Gaussian random measure W and a compensated Poisson random measure N independent to W. For background for such general stochastic equations with jumps, we refer the readers to Dawson and Li [56], Li and Ma [85].

Let us briefly present the preliminaries. We fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. A *white noise* W on $\mathbb{R}^2_+$ is a Gaussian random measure such that, for any Borel set $A \in \mathcal{B}(\mathbb{R}^2_+)$ with finite Lebesgue measure $|A|$, $W(A)$ is a normal random variable of mean zero and variance $|A|$; and that if $A_1, \cdots, A_n$ are disjoint Borel sets in $\mathcal{B}(\mathbb{R}^2_+)$, then $W(A_1), \cdots, W(A_n)$ are mutually independent. We denote by N the Poisson random measure on $\mathbb{R}^3_+$ with *intensity* λ which is a Borel measure on $\mathbb{R}^3_+$ in form of the product of the Lebesgue measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with a Borel measure μ on $\mathbb{R}^+$ such that $\int_0^\infty (\zeta \wedge \zeta^2)\mu(d\zeta) < +\infty$. Note that μ is a Lévy measure since $\int_0^\infty (1 \wedge \zeta^2)\mu(d\zeta) < +\infty$. Recall that for each Borel set $A \in \mathcal{B}(\mathbb{R}^3_+)$ with $\lambda(A) < +\infty$, the random variable $N(A)$ has the Poisson distribution with parameter $\lambda(A)$; moreover, if $A_1, \ldots, A_n$ are disjoint Borel sets in $\mathcal{B}(\mathbb{R}^3_+)$, then $N(B_1), \cdots, N(B_n)$ are mutually independent. We let $\widetilde{N} = N - \lambda$ be the compensated Poisson random measure on $\mathbb{R}^3_+$ associated to N.

We introduce the filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geqslant 0}$ as the natural filtration generated by the Lévy

random field and satisfying the usual conditions, namely, for any Borel subset $A \in \mathcal{B}(\mathbb{R}_+)$ and $B \in \mathcal{B}(\mathbb{R}_+^2)$ of finite Lebesgue measure, the processes $(W([0,t] \times A), t \geq 0)$ and $(\widetilde{N}([0,t] \times B), t \geq 0)$ are $\mathbb{F}$-martingales. We consider the following stochastic differential equation in the integral form.

Definition 6.1.1. *Let $a, b, \sigma, \gamma \in \mathbb{R}_+$ be constant parameters. Consider the equation*

$$Y(t) = Y(0) + \int_0^t a\,(b - Y(s))\,ds + \sigma \int_0^t \int_0^{Y(s)} W(ds, du) + \gamma \int_0^t \int_0^{Y(s-)} \int_{\mathbb{R}^+} \zeta \widetilde{N}(ds, du, d\zeta) \tag{6.1.1}$$

where $W(ds, du)$ is a white noise on $\mathbb{R}_+^2$ with unit covariance, $\widetilde{N}(ds, du, d\zeta)$ is an independent compensated Poisson random measure on $\mathbb{R}_+^3$ with intensity $\lambda = dsdu\mu(d\zeta)$ with $\mu(d\zeta)$ being a Lévy measure on $\mathbb{R}_+$ and satisfying $\int_0^\infty (\zeta \wedge \zeta^2)\mu(d\zeta) < \infty$.

The integral appearing in the previous formula (and in the following) is in the sense of [105]. It follows from [56, Theorem 3.1] or [85, Theorem 2.1] that the equation (6.1.1) has a unique strong solution. Below are several particular examples.

Example 6.1.1. *(CIR model) In the case where $\gamma = 0$, we have*

$$\begin{aligned} Y(t) &= Y(0) + \int_0^t a\,(b - Y(s))\,ds + \sigma \int_0^t \int_0^{Y(s)} W(ds, du) \\ &= Y^*(t) + \sigma \int_0^t \int_0^{Y(s)} e^{-a(t-s)} W(ds, du), \end{aligned} \tag{6.1.2}$$

where $Y^*(t) = r_0 e^{-at} + ab \int_0^t e^{-a(t-s)} ds$ is a deterministic function. This corresponds to the classic CIR model in the interest rate where an equivalent representation of (6.1.2) *in probability is given as*

$$Y(t) = Y(0) + \int_0^t a(b - Y(s))ds + \sigma \int_0^t \sqrt{Y(s)} dB(s)$$

where $B = (B(t), t \geq 0)$ is a Brownian motion.

Example 6.1.2. *(α-CIR model) As a generalization of the CIR model with jumps, let $\sigma = 0$ and the Lévy measure μ be given by as*

$$\mu(d\zeta) = -\frac{1_{\{\zeta > 0\}} d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1 < \alpha < 2, \tag{6.1.3}$$

then the model (6.1.1) *admits the following representation*

$$Y(t) = Y(0) + \int_0^t a\,(b - Y(s))\,ds + \gamma \int_0^t (Y(s-))^{\frac{1}{\alpha}} dL(s) \tag{6.1.4}$$

where $L = (L(t), t \geq 0)$ is a spectrally positive compensated α-stable Lévy process with parameter $\alpha \in (1, 2]$, which is independent of B and whose Laplace transform is given, for $q \geq 0$, by

$$\mathbb{E}\left[e^{-qL(t)}\right] = \exp\left(-\frac{tq^\alpha}{\cos(\pi\alpha/2)}\right).$$

When $\alpha = 2$, L reduces to a Brownian motion scaled by $\sqrt{2}$ and we recover the standard CIR model.

6.1.1 Connection with CBI and affine processes

The purpose of this section is to highlight the relation between the process class introduced in the previous section and the classes of CBI processes and affine processes. This relation will turn out to be relevant in order to apply transform-based methods to derivatives evaluation. Our model actually belongs to the family of CBI processes. We briefly recall the definition by Kawazu and Watanabe [76]. A Markov process X with state space $\mathbb{R}_+$ is called a CBI process characterized by branching mechanism $\Psi(\cdot)$ and immigration rate $\Phi(\cdot)$, if its characteristic representation is given, for $p \geq 0$, by

$$\mathbb{E}_x\left[e^{-pX(t)}\right] = \exp\left(-xv(t,p) - \int_0^t \Phi\big(v(s,p)\big)ds\right), \tag{6.1.5}$$

where the function $v : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following differential equation

$$\frac{\partial v(t,p)}{\partial t} = -\Psi(v(t,p)), \quad v(0,p) = p \tag{6.1.6}$$

and Ψ and Φ are functions of the variable $q \geq 0$ given by

$$\Psi(q) = \beta q + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-qu} - 1 + qu)\pi(du),$$
$$\Phi(q) = \delta q + \int_0^\infty (1 - e^{-qu})\nu(du),$$

with $\sigma, \gamma \geq 0$, $\beta \in \mathbb{R}$ and π, ν being two Lévy measures such that

$$\int_0^\infty (u \wedge u^2)\pi(du) < \infty, \quad \int_0^\infty (1 \wedge u)\nu(du) < \infty. \tag{6.1.7}$$

It is proved in Dawson and Li [56, Theorem 3.1] that the process (6.1.1) is a CBI process with the branching mechanism Ψ given by

$$\Psi(q) = aq + \frac{1}{2}\sigma^2 q^2 + \int_0^\infty (e^{-q\sigma_Z\zeta} - 1 + q\sigma_Z\zeta)\mu(d\zeta) \tag{6.1.8}$$

and the immigration rate $\Phi(q) = abq$.

The link between CBI processes and the affine interest rate term structure models has been established by Filipović [64]. If the process takes values in $\mathbb{R}_+$ he proves equivalence between the two classes. We recall the joint Laplace transform of a CBI process X and its integrated process, which is given in [64, Theorem 5.3], as follows. For non-negative real numbers ξ and θ, we have

$$\mathbb{E}_x\Big[e^{-\xi X(t) - \theta\int_0^t X(s)ds}\Big] = \exp\Big\{-xv(t,\xi,\theta) - \int_0^t \Phi\big(v(s,\xi,\theta)\big)ds\Big\}, \tag{6.1.9}$$

where $v(t,\xi,\theta)$ is the unique solution of

$$\frac{\partial v(t,\xi,\theta)}{\partial t} = -\Psi(v(t,\xi,\theta)) + \theta, \quad v(0,\xi,\theta) = \xi. \tag{6.1.10}$$

6.2 Interest rate modelling and the persistency of low interest rate

6.2.1 Introduction

On the current European sovereign bond market, there exists a number of well-established and seemingly puzzling facts. On one hand, the interest rate has reached a historically low level in the Euro countries. On the other hand, sovereign bonds can have large variations when uncertainty about unpredictable political or economical events increases such as in the Greek case. In literature, large fluctuations in financial data motivate naturally the introduction of jumps in the interest rate dynamics such as in Eberlein and Raible [61], Filipović et al. [65]. Nevertheless, the presence of jumps conflicts in general with the trend of low interest rates. One way to reconcile large fluctuations with low rates is to use a regime change framework but this may increase the dimension of stochastic processes in order to preserve the Markov property.

Empirical studies underline that the behavior of the bond prices cannot be fully explained by the CIR model which systematically overestimates short interest rates, as pointed out by Brown and Dybvig [28] and Gibbons and Ramaswamy [67]. In our framework the CIR model is the departing model and the inclusion of the α-stable processes allows to better describe the low interest rate behavior. Despite the simplicity and the reduced number of extra parameters compared to the standard CIR, the α-CIR model show several advantages from the financial point of view. First, the α-CIR model exhibits positive jumps and, in particular, by combining heavy-tailed jump distribution with infinite activity, can describe in a unified way both the large fluctuations observed in recent sovereign bond market and the usual small oscillations. Second, the interest rate can be split into different components in a branching process framework which can eventually be interpreted as spreads, each one following the same dynamics. Third, by the link between the α-CIR model and the CBI processes, we deduce the bond prices in an explicit way by using the joint Laplace transform of the affine model in Filipović [64] and we analyze the bond yield behaviors following the paper of Keller-Ressel and Steiner [77].

The main, and perhaps most interesting, forecast of the α-CIR model is that the bond prices decrease with respect to the parameter α, with those given by the standard CIR model being the lowest prices. The parameter α is inversely related to the tail fatness. In general, the standard behavior of bond prices increases with respect to the fatness of tails, such as the case in the extended CIR model with jumps in Duffie and Gârleanu [58] or in the Lévy-Ornstein-Uhlenbeck (LOU) dynamics (e.g. Barndorff-Nielsen and Shephard [13]) in which the jump part is a subordinator. The explanation of this seemingly paradoxical result is based on the features of the α-CIR model. The use of fat-tail distributed positive jumps will imply a large negative compensator, then between two jumps the mean reversion term is magnified whenever α decreases. Moreover, for a given value of α, the branching property

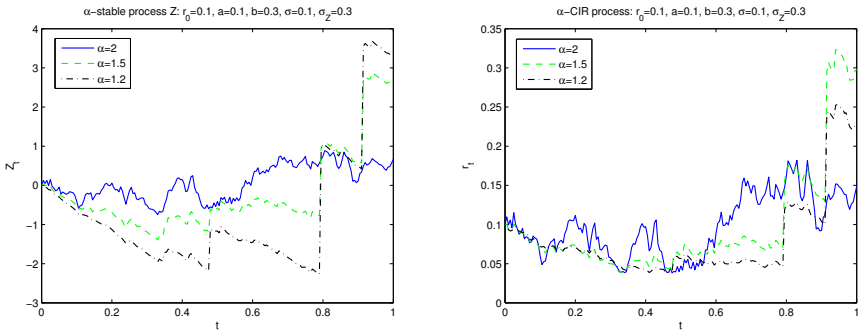


FIGURE 6.1: Lévy process Z and the corresponding short rate r with different values of α.

adds a new phenomenon in the α-CIR model. The frequency of large jumps decreases when interest rates are low thanks to the self-exciting structure which allows some "freezing" effect of low short rates for relatively longer time period compared to the standard CIR model. We are particularly interested in the jump behavior, notably for the large jumps which signify in the interest rate dynamics a sudden increasing sovereign risk. We focus on the number of large jumps occurred during a given time interval and deduce its Laplace transform, with which we obtain the probability law and the expectation for the first large jump time. The impact of the tail index α is emphasized.

6.2.2 Two representations for α-CIR model

We begin by presenting the following root representation for the short interest rate $r = (r_t, t \geq 0)$, which is a direct extension of the standard CIR model as

$$r_t = r_0 + \int_0^t a\,(b - r_s)\,ds + \sigma \int_0^t \sqrt{r_s} dB_s + \sigma_Z \int_0^t r_{s-}^{1/\alpha} dZ_s \tag{6.2.11}$$

where $B = (B_t, t \geq 0)$ is a Browinan motion and $Z = (Z_t, t \geq 0)$ is an spectrally positive α-stable compensated Lévy process with parameter $\alpha \in (1, 2]$ which is independent of B and whose Laplace transform is given, for $q \geq 0$, by

$$\mathbb{E}\big[e^{-qZ_t}\big] = \exp\Big(-\frac{tq^\alpha}{\cos(\pi\alpha/2)}\Big).$$

When $\alpha \in (1, 2)$, the corresponding Lévy measure is given by

$$-\frac{1_{\{z>0\}} dz}{\cos(\pi\alpha/2)\Gamma(-\alpha) z^{1+\alpha}}.$$

Figure 6.1 gives a simulation for the compensated α-stable process Z and the corresponding short interest rate r defined in (6.2.11) with three different values of α : 2, 1.5 and 1.2.

Finally, when the Lévy measure μ is given by

$$\mu_\alpha(d\zeta) = -\frac{1_{\{\zeta>0\}} d\zeta}{\cos(\pi\alpha/2)\Gamma(-\alpha)\zeta^{1+\alpha}}, \quad 1 < \alpha < 2, \tag{6.2.12}$$

the solution of (6.1.1) has the same probability law as that of the equation (6.2.11). Moreover, in an expanded probability space, the solution of the two equations are equal almost surely by using similar arguments as in [84, Theorem 9.32].

6.2.3 Positivity and limit distribution of α-CIR model

We use the CBI characterization to deduce some properties of the α-CIR model. First, we show that the usual condition of inaccessibility of the point 0 is preserved when we extend CIR to α-CIR model .

Proposition 6.2.11. *For the α-CIR $(a, b, \sigma, \sigma_Z, \alpha)$ process with $\alpha \in (1, 2)$, the point 0 is an inaccessible boundary if and only if $2ab \geq \sigma^2$. In particular, a pure jump α-CIR process with $ab > 0$ never reaches 0.*

Finally, we characterize the ergodic distribution of the α-CIR process. Note that the first part of the following result was also shown in Keller-Ressel and Steiner [77, Theorem 3.16].

Proposition 6.2.12. *The α-CIR integral type process defined in* (6.1.1) *has a limit distribution, whose Laplace transform is given by*

$$\mathbb{E}[e^{-pr_\infty}] = \exp\Big(- \int_0^p \frac{\Phi(q)}{\Psi(q)} dq \Big), \quad p \geq 0. \tag{6.2.13}$$

Moreover, the process is exponentially ergodic, namely

$$\|\mathbb{P}[r_t \in \cdot\,] - \mathbb{P}[r_\infty \in \cdot\,]\| \leqslant C\rho^t$$

for some positive constants C and $\rho < 1$, where $\|\cdot\|$ denotes the total variation norm.

6.2.4 Behaviors of bond price and bond yield

We now focus on the properties of bond prices and the corresponding bond yield curves.

Proposition 6.2.13. *Let the short rate r be given by the α-CIR model* (6.2.11) *under the probability measure $\mathbb{Q}$. Then the zero-coupon bond price is given by*

$$B(t, T) = \exp\Big(- r_t v(T - t) - ab \int_0^{T-t} v(s) ds \Big) \tag{6.2.14}$$

where $v(s)$ is the unique solution of the equation

$$\frac{\partial v(t)}{\partial t} = 1 - \Psi_\alpha(v(t)), \quad v(0) = 0, \tag{6.2.15}$$

with $\Psi_\alpha(q) = aq + \frac{\sigma^2}{2} q^2 - \frac{\sigma_Z^\alpha}{\cos(\pi\alpha/2)} q^\alpha$. Moreover, we have

$$v(t) = f^{-1}(t) \ \text{ where } \ f(t) = \int_0^t \frac{dx}{1 - \Psi_\alpha(x)}$$

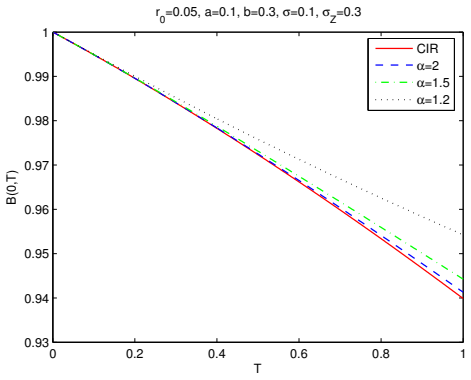


FIGURE 6.2: Bond prices $B(0,T)$ with different values of α, and in comparison with the CIR model.

The next proposition highlight an interesting and surprising result.

Proposition 6.2.14. *The function v is increasing with respect to $\alpha \in (1,2]$. In particular, the bond price $B(0,T)$ is decreasing with respect to α.*

Proposition 6.2.14 seems to be surprising at first sight since the parameter α is an inverse measure of heaviness of distribution tails, more as α is close to 1, more likely that the large jumps appear. In addition, in the α-CIR model, α coincides with the so-called generalized Blumenthal-Getoor index which is defined as $\inf\{\beta > 0 : \sum_{0\leq s\leq T} \Delta r_s^\beta < \infty,\ a.s.\}$ with $\Delta r_s := r_s - r_{s-}$ and T a horizon time (see e.g. Aït-Sahalia and Jacod [1]) and is often used to measure the activity of the small jumps in a semimartingale. When $\mu_\alpha(du)$ is defined by (6.2.12), this index is reduced to $\inf\{\beta > 0 : \int_0^T r_s ds \int_0^1 u^\beta \mu_\alpha(du) < \infty,\ a.s.\}$ and thus is equal to α. The index $\alpha \in (1,2)$ shows that the jumps are of infinite variation. The explanation of Proposition 6.2.14 is based on the self-exciting property. For the compensated α-stable Lévy process Z in the α-CIR model (6.2.11), a smaller α is related to a deeper (negative) compensation and hence a stronger mean-reversion. Then as the interest rate becomes low because of the mean-reversion effect, the self-exciting property will imply a decreasing frequency of jumps and enforce the tendency of low interest rate.

6.2.5 Analysis of jumps

We now focus on the jump part of the short interest rate r. In particular, we are interested in the large jumps which capture significant changes in the interest rate dynamics.

Let us fix a jump threshold $y = \sigma_Z \overline{y} > 0$. In this subsection, we study the following two quantities : the number of large jumps whose jump sizes are larger than y and the first large jump time. For this purpose, we separate the large and small jumps and use the non-compensated version of the Poisson random measure in the integral form (6.1.1).

We introduce the auxiliary process which represents the truncated interest rate r except

the jumps larger than y as

$$\begin{aligned}\widehat{r}_t^{(y)} = r_0 + \int_0^t \widetilde{a}(\alpha, y)\big(\widetilde{b}(\alpha, y) - \widehat{r}_s^{(y)}\big)ds + \sigma \int_0^t \int_0^{\widehat{r}_s^{(y)}} W(ds, du) \\ + \sigma_Z \int_0^t \int_0^{\widehat{r}_{s-}^{(y)}} \int_0^{\overline{y}} \zeta \widetilde{N}(ds, du, d\zeta).\end{aligned} \tag{6.2.16}$$

Where $\widetilde{a}$, $\widetilde{b}$ are defined on the related paper. For any jump threshold $y > 0$, the process $\widehat{r}^{(y)}$ coincides with r up to the first large jump time $\tau_y := \inf\{t > 0 : \Delta r_t > y\}$. The process $\widehat{r}^{(y)}$ is a CBI process with the branching mechanism given by

$$\Psi_\alpha^{(y)}(q) := \Big(a + \sigma_Z^\alpha \int_y^\infty \zeta \mu_\alpha(d\zeta)\Big)q + \frac{1}{2}\sigma^2 q^2 + \sigma_Z^\alpha \int_0^y (e^{-q\zeta} - 1 + q\zeta)\mu_\alpha(d\zeta) \tag{6.2.17}$$

and the immigration rate given by $\Phi(q) = abq$.

Proposition 6.2.15. *Let r be an α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$ process with initial value $r_0 \geq 0$. Then for any $p \geq 0$ and $t \geq 0$,*

$$\mathbb{E}\left[e^{-pJ_t^y}\right] = \exp\left(-l(p, y, t)r_0 - ab\int_0^t l(p, y, s)ds\right) \tag{6.2.18}$$

where $l(p, y, t)$ is the unique solution of the following equation

$$\frac{\partial l(p, y, t)}{\partial t} = \sigma_Z^\alpha \int_y^\infty \big(1 - e^{-p - l(p,y,t)\zeta}\big)\mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}(l(p, y, t)), \tag{6.2.19}$$

with initial condition $l(p, y, 0) = 0$ and $\Psi_\alpha^{(y)}$ given by (6.2.17).

Now we consider the first time when the jump size of the short rate r is larger than $y = \sigma_Z \overline{y}$. We show that this random time also exhibits an exponential affine cumulative distribution function. The following result gives its distribution function as a consequence of the above proposition.

Corollary 6.2.3. *For any $t \geq 0$, we have*

$$\mathbb{P}[\tau_y > t] = \exp\Big(-l(y, t)r_0 - ab\int_0^t l(y, s)ds\Big) \tag{6.2.20}$$

where $l(y, t)$ is the unique solution of the following ODE

$$\frac{dl}{dt}(y, t) = \sigma_Z^\alpha \int_y^\infty \mu_\alpha(d\zeta) - \Psi_\alpha^{(y)}(l(y, t)), \tag{6.2.21}$$

with initial condition $l(y, 0) = 0$ and $\Psi_\alpha^{(y)}$ given by (6.2.17).

Corollary 6.2.3 has an alternative form

$$\mathbb{P}[\tau_y > t] = \mathbb{E}\left[\exp\left(-\sigma_Z^\alpha \left(\int_y^\infty \mu_\alpha(d\zeta)\right)\left(\int_0^t \widehat{r}_s^{(y)} ds\right)\right)\right]. \tag{6.2.22}$$

Proposition 6.2.16. *We have* $\mathbb{P}(\tau_y < \infty) = 1$. *Furthermore,*

$$\mathbb{E}\left[\tau_y\right] = \int_0^{l_y^*} \frac{1}{F(u)} \exp\Big(- ur_0 - \int_0^u \frac{abs}{F(s)} ds\Big) du < \infty, \tag{6.2.23}$$

where l_y^* *is the unique solution of the equation* $F(q) = 0$, *which identifies with* $\lim_{t\to\infty} l(y,t)$ *where the function* $l(y,t)$ *is given by* (6.2.21).

6.3 Modeling the electricity price

We now turn to the application of our modelling framework to the description of electricity price dynamics. We follow the arithmetic model as in [17] in which the power price dynamics is assumed to be the sum of several factors. In our model the factors are assumed to be given by the SDE (6.1.1).

6.3.1 An arithmetic model for the spot price

We assume that the spot price process $S = (S(t), t \geq 0)$ evolves according to the following dynamics

$$S(t) = \alpha(t) + X(t) \tag{6.3.24}$$

where $\alpha(t)$ is a deterministic function representing the seasonality, and the process $X = (X(t), t \geq 0)$ is a superposition of the factors Y_i as

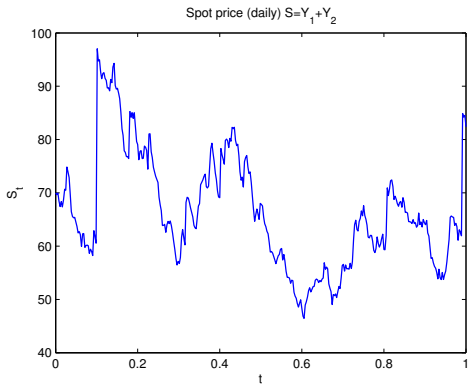
$$X(t) = \sum_{i=1}^{n} Y_i(t). \tag{6.3.25}$$

The factors $Y_i(t)$ evolve according to the equation (6.1.1), more precisely,

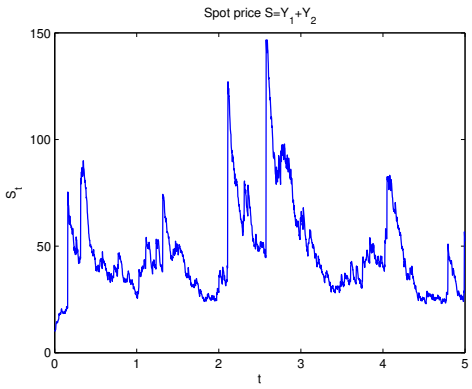
$$\begin{aligned} Y_i(t) = Y_i(0) + \int_0^t a_i \left(b_i - Y_i(s)\right) ds + \sigma_i \int_0^t \int_0^{Y_i(s)} W_i(ds, du) \\ + \gamma_i \int_0^t \int_0^{Y_i(s-)} \int_{\mathbb{R}^+} \zeta \widetilde{N}_i(ds, du, d\zeta) \end{aligned} \tag{6.3.26}$$

Figure 6.3 plots two trajectories of the spot price in a two-factor model where the first factor Y_1 follows a standard CIR model as in Example 6.1.1 and the second factor Y_2 is an α-CIR process as in Example 6.1.2. In order to get a more faithful description of the power prices dynamics it is convenient to increase the values of the parameters, in agreement with the values proposed in the previous literature, see [17], in particular the mean reversion speed and the jumps sizes should be magnified. In Figure 6.3b the trajectory is plotted with same parameters as in the previous case except $Y_1(0) = Y_2(0) = 5$, $a_1 = 10$, $a_2 = 20$, $b_1 = b_2 = 5$, $\gamma_2 = 10$, exhibiting a more realistic behavior.

The following result computes the expected value of the factors Y_i given in (6.3.26) and will be useful for the pricing of energy derivatives.



(a) Small values for the mean reversion speed and jumps size.



(b) Big values for the mean reversion speed and jumps size.

FIGURE 6.3: Simulation of the spot price in a two-factor model

Lemma 6.3.1. *For any $T \geq 0$ and $\tau \leq T$, we have*

$$\mathbb{E}\big[Y_i(T)\big|\mathcal{F}_\tau\big] = b_i + (Y_i(\tau) - b_i)e^{-a_i(T-\tau)}. \tag{6.3.27}$$

In the electricity market, the risk-neutral probability $\mathbb{Q}$ is often chosen by introducing a drift adjustment in the dynamics of an underlying asset. Compared to the original historical probability $\mathbb{P}$, such measure change is done by using Esscher transform such as in [14]. We now define the equivalent probability measures in our model. The following proposition shows that the spot process S defined in (6.3.24) will remain in the same class.

Proposition 6.3.1. *Let $Y_1, Y_2, \cdots, Y_n$ be independent CBI processes where for each $i \in \{1, \cdots, n\}$, Y_i is a CBI process under the probability measure $\mathbb{P}$. Assume that the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated by the random fields $W_1, W_2, \cdots, W_n$ and $\widetilde{N}_1, \widetilde{N}_2, \cdots, \widetilde{N}_n$. For each i, fix $\eta_i \in \mathbb{R}$ and $\theta_i \in \mathbb{R}_+$ and define*

$$U_t := \sum_{i=1}^n \eta_i \int_0^t \int_0^{Y_i(s)} W_i(ds, du) + \sum_{i=1}^n \int_0^t \int_0^{Y_i(s-)} \int_0^\infty (e^{-\theta_i \zeta} - 1)\widetilde{N}_i(ds, du, d\zeta). \tag{6.3.28}$$

Then the Doléans-Dade exponential $\mathcal{E}(U)$ is a martingale and the probability measure $\mathbb{Q}$ defined by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \mathcal{E}(U)_t, \tag{6.3.29}$$

is equivalent to $\mathbb{P}$. Moreover, under $\mathbb{Q}$, the processes $Y_1, Y_2, \cdots Y_n$ are independent of each other and, for each i, Y_i is a CBI process with the parameters $(a_i', b_i', \sigma_i', \gamma_i', \mu_i')$, where

$$a_i' = a_i - \sigma_i\eta_i - \gamma_i \int_0^\infty \zeta(e^{-\theta_i\zeta} - 1)\mu_i(d\zeta), \quad b_i' = a_i b_i / a_i', \quad \sigma_i' = \sigma_i, \quad \gamma_i' = \gamma_i \tag{6.3.30}$$

$$\mu_i'(d\zeta) = e^{-\theta_i\zeta}\mu_i(d\zeta). \tag{6.3.31}$$

6.3.2 Pricing of forward and flow forward

By applying the previous results, we can provide an explicit and simple expression for the (instantaneous) forward contract price in the present modeling framework. Recalling the definition of a forward contract, we have for any $\tau \leq T$ that

$$F(\tau, T) := \mathbb{E}^{\mathbb{Q}}\big[S(T)\big|\mathcal{F}_\tau\big]. \tag{6.3.32}$$

Proposition 6.3.2. *We have the following expression for the forward contract*

$$F(\tau, T) = S(\tau) + \big(\alpha(T) - \alpha(\tau)\big) + \sum_{i=1}^{n} \big(e^{-a_i'(T-\tau)} - 1\big)\Big(Y_i(\tau) - b_i'\Big) \tag{6.3.33}$$

where a_i' and b_i' are as in (6.3.30).

Forward contracts in electricity markets deliver the underlying commodity over a period rather than at a fixed time, and for this reason they are sometimes called *flow forwards*. The future price with delivery period $[T_1, T_2]$ is defined by

$$F(\tau, T_1, T_2) = \frac{1}{T_2 - T_1}\mathbb{E}^{\mathbb{Q}}\Big[\int_{T_1}^{T_2} S(u)du \,|\, \mathcal{F}_t\Big] = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} F(\tau, u)du,$$

which leads directly to the following result.

Corollary 6.3.1. *The value of the flow forward contract $F(\tau; T_1, T_2)$ is given, for $\tau < T_1 < T_2$, by*

$$\begin{aligned} F(\tau, T_1, T_2) \;\; = \;\; & S(\tau) + \Big(\frac{1}{T_2 - T_1}\int_{T_1}^{T_2} \alpha(u)du - \alpha(\tau)\Big) \\ & - \sum_{i=1}^{n} \Big(1 - \frac{e^{-a_i'(T_1-\tau)} - e^{-a_i'(T_2-\tau)}}{a_i'(T_2 - T_1)}\Big)\big(Y_i(\tau) - b_i'\big) \end{aligned} \tag{6.3.34}$$

6.3.3 The risk premium

The risk premium is an important quantity in power markets which provides a link between the forward and expected spot prices. We present an explicit representation formula for this quantity in our modeling framework. The risk premium is defined as the difference between conditional expectations of the underlying price computed with respect to the risk-neutral measure $\mathbb{Q}$ and the historical measure $\mathbb{P}$:

$$R(\tau, T) = \mathbb{E}^{\mathbb{Q}}\left[S(T)|\mathcal{F}_\tau\right] - \mathbb{E}^{\mathbb{P}}\left[S(T)|\mathcal{F}_\tau\right] \tag{6.3.35}$$

Since the factors Y_i remain to be CBI processes under the equivalent measure change, according to Lemma 6.3.1 and Proposition 6.3.2, we have the following result.

Corollary 6.3.2. *Let the risk-neutral probability $\mathbb{Q}$ be given as in Proposition 6.3.1. Then we have*

$$\begin{aligned} R(\tau,T) &= \sum_{i=1}^{n} Y_i(\tau)\left(e^{-a_i'(T-\tau)} - e^{-a_i(T-\tau)}\right) \\ &\quad - \sum_{i=1}^{n}\left[b_i'\left(e^{-a_i'(T-\tau)}-1\right) - b_i\left(e^{-a_i(T-\tau)}-1\right)\right] \end{aligned} \tag{6.3.36}$$

where the parameters a_i', b_i' are related to the parameters a_i and b_i by the relations (6.3.30).

The most relevant feature exhibited by the present modeling framework is that the very simple measure change introduced in the previous section implies a change in the mean speed reversion parameter. This feature is not feasible for Lévy-driven Ornstein-Uhlenbeck models as long as simple measure changes are considered. The Esscher transform for the Lévy driver implies a change in the long-term mean, but not in the mean-reversion speed; this in turn implies a risk premium term structure exhibiting sign change only for very special choice of the exponential tilting parameter θ (see Benth and Sgarra [18]).

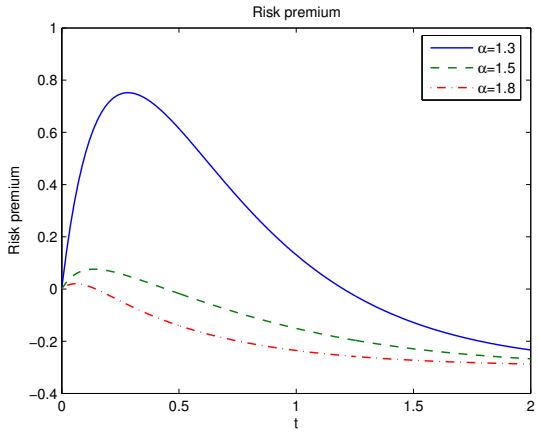


FIGURE 6.4: Risk premium term structure in a two factor model for different values of jump tail parameter α.

6.3.4 Jumps analysis in a two-factor model

In this section, we consider a model with two factors. The first factor is continuous and corresponds to a standard CIR model and the second one is with jumps. Our objective is to make a thorough analysis of the jump behavior, in particular, for large jumps and spikes. The results will also be applied to the statistical analysis. Let the first factor Y_1 be driven by a Gaussian random measure as

$$Y_1(t) = Y_1(0) + \int_0^t a_1\left(b_1 - Y_1(s)\right) ds + \sigma_1 \int_0^t \int_0^{Y_1(s)} W_1(ds,du), \quad t \geq 0 \tag{6.3.37}$$

and the second factor Y_2 be driven by a pure jump Poisson random measure as

$$Y_2(t) = Y_2(0) + \int_0^t a_2 \left(b_2 - Y_2(s)\right) ds + \gamma_2 \int_0^t \int_0^{Y_2(s-)} \int_{\mathbb{R}^+} \zeta \widetilde{N}_2(ds, du, d\zeta), \quad t \geq 0. \tag{6.3.38}$$

The presence of large jumps and spikes is an important feature discussed in literature for electricity prices. A "spike" here means a sudden large rise in the price followed by a rapid drop to its regular level.

6.3.5 Locally equivalent jump process

The purpose of the present section is threefold : the first target is to introduce the spike frequency by identifying a suitable auxiliary process, the second is to prove formally that the last process is, up to a constant factor, the *intensity* of the next spike ; the third is to show how Y_2 can be approximated by the intensity of a Hawkes-type process and this result will allow to adopt the maximum likelihood estimator for Hawkes processes as an approximate estimator for the model described. Consider a family of jump times $\{\tau_k\}_{k\in\mathbb{N}}$ of the process Y_2 which corresponds to significant movements of the market. For example, it can denote jump times larger than a threshold. More precisely, for a given constant z_0, we denote $\{\tau_k^{(z_0)}\}_{k\in\mathbb{N}}$ as the sequence of jump times of Y_2 whose jump sizes are equal or larger than $\gamma_2 z_0$, i.e.,

$$\tau_k^{(z_0)} = \inf\{t > \tau_{k-1}^{(z_0)} : \Delta Y_2(t) > \gamma_2 z_0\}, \quad \tau_0^{(z_0)} = 0. \tag{6.3.39}$$

In the following, we shall suppress the over-quote z_0 for sake of readability. We are interested in the evolution of process Y_2 between two jump times, that is for any $t \in [\tau_k, \tau_{k+1})$,

$$Y_2(t) = Y_2(\tau_k) + \int_{\tau_k}^t a_2 \left(b_2 - Y_2(s)\right) ds + \gamma_2 \int_{\tau_k}^t \int_0^{Y_2(s-)} \int_{\mathbb{R}^+} \zeta \widetilde{N}_2(ds, du, d\zeta). \tag{6.3.40}$$

With these notations, the time τ_{k+1} is the arrival time of the first jump after τ_k larger than z_0 for the measure N_2 or equivalently, larger than $\gamma_2 z_0$ for Y_2. By the following result, we can separate small and large jumps and move the compensation inside the speed and mean coefficients. We remark that the second line of (6.3.41) describes a finite activity pure jump process, then it can be seen as a compounded Poisson process jumping only at an increasing sequence of times, such that only an almost surely finite number of jumps arrive in a compact set of time.

Proposition 6.3.3. *The process Y_2 can be written, for all $t \in [\tau_k, \tau_{k+1})$, as*

$$\begin{aligned} Y_2(t) &= Y_2(\tau_k) + \int_{\tau_k}^t A_2 \left(B_2 - Y_2^{(z_0)}(s)\right) ds + \gamma_2 \int_{\tau_k}^t \int_0^{Y_2^{(z_0)}(s-)} \int_0^{z_0} \zeta \widetilde{N}_2(ds, du, d\zeta) \\ &\quad + \gamma_2 \int_{\tau_k}^t \int_0^{Y_2^{(z_0)}(s-)} \int_{z_0}^{\infty} \zeta N_2(ds, du, d\zeta), \end{aligned} \tag{6.3.41}$$

where $z_0 > 0$ is a fixed constant,

$$A_2 := a_2 + \gamma_2 \int_{z_0}^{\infty} \zeta \mu(dz), \quad B_2 := \frac{a_2 b_2}{A_2} \tag{6.3.42}$$

and $Y_2^{(z_0)}$ is the truncated process, for all $t \in [\tau_k, \tau_{k+1})$, defined by

$$Y_2^{(z_0)}(t) = Y_2(\tau_k) + \int_{\tau_k}^{t} A_2 \left(B_2 - Y_2^{(z_0)}(s) \right) ds + \gamma_2 \int_{\tau_k}^{t} \int_{0}^{Y_2^{(z_0)}(s-)} \int_{0}^{z_0} \zeta \widetilde{N}_2(ds, du, d\zeta). \tag{6.3.43}$$

The main idea of Proposition 6.3.3 is to distinguish the original process Y_2 by the intensity of its jumps, that is, the truncated process $Y_2^{(z_0)}$ given by formula (6.3.41). These two processes coincide up to the next large jump occurred.

The process $Y_2^{(z_0)}$ is related to the intensity of large jumps since it appears inside the last integral in (6.3.41) associated to large jumps. The instantaneous intensity of the jumps of Y_2 at time t is then $Y_2^{(z_0)}(t-)$ times the normalization term associated to the jump measure, i.e. $1_{\{\zeta > z_0\}} \mu(d\zeta)$. We stated that the truncated process $Y_2^{(z_0)}$ is linked to the intensity of the large jumps of Y_2. The following proposition explains in detail this link. Up to a constant, the process $Y_2^{(z_0)}$ is the stochastic intensity or *hazard rate* of the random time of the next big jump.

Proposition 6.3.4. *Let $\{\tau_k\}_{k \in \mathbb{N}}$ be defined by (6.3.39). Then we have*

$$\mathbb{P}\left(\tau_{k+1} - \tau_k > t\right) = \mathbb{E}\left[exp\left\{ -K_Y^{(z_0)} \int_{\tau_k}^{\tau_k + t} Y_2^{(z_0)}(s) ds \right\} \right], \tag{6.3.44}$$

where the renormalisation term $K_Y^{(z_0)} = \gamma_2 \int_{z_0}^{\infty} \zeta \mu_2(d\zeta)$ which is the proper truncated mass of the jumps distribution, and the frequency process $Y_2^{(z_0)}$ is given by (6.3.43).

Finally, we deal with the asymptotic behaviour of $Y_2^{(z_0)}$ when the mean reverting speed a_2 diverges. Let us introduce the process $\widehat{Y}_2^{(z_0)}$ defined as

$$\widehat{Y}_2^{(z_0)}(t) = b_2 + e^{-A_2 t} \left[Y_2(0) - b_2 \right] + \gamma_2 \int_0^t \int_0^{\widehat{Y}_2^{(z_0)}(s-)} \int_{z_0}^{\infty} e^{-A_2(t-s)} \zeta N_2(ds, du, d\zeta). \tag{6.3.45}$$

The next proposition shows that the two processes Y_2 and $Y_2^{(z_0)}$ have the same behaviour when a_2 goes to infinity. As a consequence, we can approximate the frequency of large jumps by the one of the Hawkes process as soon as a_2 is large.

Proposition 6.3.5. *Consider Y_2 with $\mathbb{E}[Y_2(0)] < \infty$. As $a_2 \to +\infty$, we have that for each $t > 0$, $Y_2(t) - \widehat{Y}_2(t)$ goes to zero in probability.*

6.3.6 Statistical analysis

Fitting the model to real data is a relevant target, especially in the present case where a new feature, i.e. the clustering effect, must be taken into account. In this section, we are going to highlight the main issues and to propose a theoretical statistical approach. In particular, we are going to derive the maximum likelihood estimator for the parameters of the intensity process $Y_2^{(z_0)}$. These parameters are in a one-to-one relation with the parameters of the process Y_2. Following the ideas presented in [17], the first step to perform is to de-seasonalise the data. The second step, definitely less trivial, is to split the components Y_1 and Y_2 emerging from the data. We then focus on the process Y_1. Following [17], we look for the ergodic distribution of Y_1 fitting the data. By recalling that the ergodic distribution of a CIR diffusion is of Gamma type [37], our model is in agreement with the previous literature (see subsection 5.4.2 in[17]) and we obtain in a similar way the estimated parameters for Y_1.

The estimation of the parameters of the spike process Y_2 is then our following main issue. We remark that the process Y_2 is not directly observable since the data are given by the sum of three components, i.e. the seasonality function, the base signal and the spike process itself. However, we easily remark that the spikes signal is not invariant by translation, see equation (6.1.1). In our model, the translation is impossible since the long-run value b is related to the frequency of jumps far from the cluster periods as pointed out in Propositions 6.3.4 and 6.3.5. As a consequence, we propose a different strategy to estimate the parameters of the spike signal. Our proposal it based on the following remark. The spike signal is observable only when a spike occurs. That is the observation is reduced to the sequence $(\tau_k, \Delta S(\tau_k))_{k\in\mathbb{N}}$, $\Delta S(\tau_k) = S(\tau_k) - S(\tau_k^-)$, where τ_k is the time of the k^{th} spike and ΔS_{τ_k} is its jump size. Due to the continuity of the seasonality function and the base signal, we have $\Delta S(\tau_k) = \Delta Y_2(\tau_k)$. As a consequence, we can assume that we observe the jump times and the jump sizes of Y_2.

Our idea is then to estimate the intensity process $Y_2^{(z_0)}$ rather than Y_2 itself. As a consequence, we have to deal with a marked point process estimation. Looking at Proposition 6.3.4, we have that the reverting speeds of $Y_2^{(z_0)}$ and Y_2 coincide. This speed is very high with respect to the mean reversion speed of the base signal as it has been pointed out in literature, see for instance [17]. We may then consider that the limit distribution expressed by Proposition 6.3.5 as the approximate distribution of the jump frequency and we can then neglect small jumps. The approximated intensity is then given by (6.3.45).

In looking then at the sequence $(\tau_k, \Delta S(\tau_k))_{k\in\mathbb{N}}$, it can be considered as the realization of a marked Hawkes process N_2 with intensity $Y_2^{(z_0)}$. We remark that the parameters B_2, γ_2 and A_2 can then be estimated by the maximum likelihood estimator.

Proposition 6.3.6 (Marked Hawkes Likelihood function)**.** *Given the observations* $(\tau_k, \Delta S(\tau_k))_{k=1\ldots N}$,

we have the following Likelihood function

$$\begin{aligned}\log L\left(\tau_1, \Delta S(\tau_1), \ldots \tau_N, \Delta S(\tau_N) | B_2, \gamma_2, A_2\right) := \\ -B_2\tau_N + \sum_{i=1}^{N} \frac{\gamma_2 \Delta S(\tau_i)}{A_2}\left[e^{-A_2(\tau_N - \tau_i)} - 1\right] \\ + \sum_{i=1}^{N} \log\left\{B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}\right\}\end{aligned} \tag{6.3.46}$$

Moreover, the MLE estimators are :

$$\frac{\partial \log L}{\partial B_2} = -\tau_N + \sum_{i=1}^{N}\left\{B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}\right\}^{-1} \tag{6.3.47}$$

$$\frac{\partial \log L}{\partial \gamma_2} = \sum_{i=1}^{N} \frac{\Delta S(\tau_i)}{A_2}\left[e^{-A_2(\tau_n - \tau_i)} - 1\right] + \sum_{i=1}^{N} \frac{\sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}}{B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}} \tag{6.3.48}$$

$$\begin{aligned}\frac{\partial \log L}{\partial A_2} = & \sum_{i=1}^{N} \frac{\gamma_2 \Delta S(\tau_i)}{A_2^2}\left\{1 - \left[A_2(\tau_n - \tau_i) + 1\right] e^{-A_2(\tau_n - \tau_i)}\right\} \\ & - \sum_{i=1}^{N} \frac{\gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j)(\tau_i - \tau_j) e^{-A_2(\tau_i - \tau_j)}}{B_2 + \gamma_2 \sum_{j=1}^{i-1} \Delta S(\tau_j) e^{-A_2(\tau_i - \tau_j)}}\end{aligned} \tag{6.3.49}$$

By applying the procedure just outlined, the estimated value of the parameters for the two factor model illustrated from the historical series of power prices available on the public site of the National Energy Authority, are the following : $a_1 = 2.9$, $b_1 = 31.24$, $\sigma_1 = 2.86$, for the first factor, while for the second factor (including just the jump contribution) we obtained $a_2 = 5.1$, $b_2 = 34.7$, $\gamma_2 = 4.7$ and the tail parameter $\alpha = 1.54$. The values are very close to those assumed for the simulations illustrated in Section 6.3.

Cinquième partie

ONGOING RESEARCH AND PROJETS

Chapitre 7

Ongoing research and projects

In this Chapter, I present some on-going and future research projects. At the moment, my future research projects concerns in particular interest/credit rates modelling with branching mechanism, stochastic control applied to economic problems and applications of Hawkes processes and their sensitivity.

7.1 Alpha-Heston model

Joint project with Y. Jiao, C. Ma and C. Zhou.

We want ot extend Heston model [73] including self-exciting jumps into the equation satisfied by the volatility. Our objective is to propose a parsimonious model explaining in a unified way some features of derivatives assets, in particular the long memory of the volatility and the roughness for short maturities.

7.2 Sensitivity analysis for marked Hawkes processes - application to CLO pricing

Working paper, with G. Bernis and K. Salhi

In this work, we introduce a model for pricing Collateralized Loan Obligations, where the underlying credit risk is driven by a marked Hawkes process, involving both clustering effects on defaults and random recovery rates. We provide a sensitivity analysis of the CLO price with respect to the parameters of the Hawkes process using a change of probability and a variational approach. We also provide a simplified version of the model where the intensity of the Hawkes process is taken as the instantaneous default rate. In this setting, we give a moment-based formula for the expected survival probability.

7.3 Optimal dividend and takeover control problem

Joint project with L. Di Persio, V. Ly Vath and L. Prezioso.

We consider an optimal mixed singular/switching control under constraints and with jumps and regime changes.

In this work, we extend our paper [A2] including regime changes modelling economic cycles and taking into account that the external growth or the reduction of debt are related to the arrival of proposal to external acquisition or takeover (resp. disinvestments or handover). Our objective is to characterise the value function using viscosity approach and to describe rigorously the optimal policy.

7.4 Wages and employment in economies with multi-worker firms, uncertainty and labor turnover costs

Working paper, with B. Bruegemann, V. Ly Vath and A. Vidigni

We present a dynamic general equilibrium model of the labor market where multi-worker firms, producing with decreasing returns to scale technology subject to a number of different productivity shocks, bargain à la Stole-Zwiebel (a generalization of Nash bargaining) over wages, in presence of hiring and dismissal costs. We show that the optimal employment policy of firms lets the marginal value of labor fluctuate persistently in an interval, defined as the inaction range, and hirings or dismissals take place only when the two reflecting barriers characterizing it are hit. We prove that the uncertainty generated by random shocks which directly affect the size of the firm, increases the size of the inaction range by making firms more cautious in both hiring and dismissal, and decreases job creation and employment. Higher uncertainty generated by shocks to the productivity of firms, also reduces unambiguously long run aggregate employment, consistently with recently provided empirical evidence, but has no effect on the employment policy of each particular firm**.** Additionally, we provide formal proofs for a number of well-established empirical regularities, such as the existence of wage dispersion across observationally equivalent workers, and the fact that larger firms tend to pay higher wages. We also account for the fact that the differential growth rate of employment in large vs. small firms appears to be strongly procyclical, along many dimensions. Furthermore, we demonstrate that Gibrat's law holds on and off the stationary equilibrium, if idiosyncratic productivity follows a particular diffusion process. The causal mechanism at work in our theory does not rely on search frictions and convex vacancy creation costs (which are intentionally ignored), but only on the interaction between labor turnover costs, the existence of firms of (endogenously) variable size due to stochastic shocks, and to the relatively standard production technology and wage setting rule assumed. Methodologically, our problem is formulated as a bi-dimensional singular control problem and we use the viscosity theory to characterize and solve explicitly or quasi-explicitly our problem.

7.5 Shadow economy dynamics

joint project with V. Ly Vath, D. Teobaldelli and D. Ticchi

We consider a two dimensional switching control problem under constraints. We focus on an economy with firms that can produce in the formal and in the informal sector with two production functions. We assume that the technology of the competitive firm is described by a constant returns to scale Cobb-Douglas on capital and labor. Firms can move capital from informal to formal sector and viceversa paying some proportional cost. formal sector is taxed but receive some long term support by the state thanks for instance to the infrastructure funded through taxes. We are interested on the labor and firm equilibria and in particular on the ratio between shadow and formal sector.

7.6 Optimal trading strategies in a credit market with over/under-reaction

joint project with G. Bernis and V. Ly Vath.

We consider an impulse control problem under constraints and partial information. We consider a market model in which over-reaction and under-reaction to market news is taken into account as in [A10]. As usual, jumps are used to model the arrival of important (positive or negative) news about the firm. The objective is to extend the previous model taking into account a self-exciting structure for large jumps and memory between jumps. Moreover, we have to take into account the peculiarities of credit markets described in [A5]. Market data will be used for calibration purposes.

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[S13] Stochastic Evolution of Distributions - Applications to CDS indices with G. Bernis, N. Brunel and A. Kornprobst, 2017.

[S14] Sensitivity Analysis for Marked Hawkes Processes - Application to CLO Pricing, with G. Bernis and K. Salhi , working paper

Working papers and works in progress

[W15] Wages and Employment in Economies with Multi-Worker Firms, Uncertainty and Labor Turnover Costs, with B. Bruegemann, V. Ly Vath and A. Vindigni, working paper.

[W16] Shadow economy dynamics, with V. Ly Vath, D. Teobaldelli and D. Ticchi, work in progress.

[W17] Alpha-Heston model, with Y. Jiao, C. Ma and C. Zhou, work in progress.

[W18] Optimal harvesting with a reserve policy under uncertain environment, with V. Ly Vath and M. Gaigi, work in progress.

[W19] Exponential Affine Processes and Samuelson effect, with Y. Jiao, C. Ma and C. Sgarra, work in progress.

[W20] Optimal Dividend and Takeover Control Problem, with L. Di Persio, V. Ly Vath and L. Prezioso, work in progress.

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