Spaces of tensor products on the derived category of a variety

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Espaces de produits tensoriels sur la catégorie dérivée d'une variété

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Résumé

Dans cette thèse on est intéressé à l'étude des catégories dérivées d'une variété lisse et projective sur un champ.
En particulier on étude l'information géométrique et catégorielle d'une variété et sa catégorie dérivée pour mieux comprendre l'ensemble de structures monoïdales on peut munir la catégorie dérivée.
La motivation de ce projet s'inspire en deux théorèmes. L'un c'est le théorème de réconstruction de Bondal-Orlov qu'établit que la catégorie dérivée d'une variété avec diviseur (anti-)canonique ample, c'est assez pour récupérer la variété. D'une autre côté, on a la construction du specturm de Balmer qu'utilise le produit tensoriel dérivé pour récupérer un nombre plus grande de variétés à partir de sa catégorie dérivée de complexes parfaits comme une categorie monoïdale.
L'existence de différentes structures monoïdales est par contre garanti par l'existence des variétés avec des catégories dérivées équivalentes. On a pour but alors comprendre quel est-le rôle de les produits tensoriels dans l'existence (ou non existence) de ces types de variétés. Les résultats principaux qu'on a obtenu sont:

1. Si X est une variété avec diviseur (anti-)canonique ample, et $\boxtimes$ est une structure de catégorie tensoriel triangulée sur $D^b(X)$ tel que le spectrum de Balmer $Spc(D^b(X), \boxtimes)$ est isomorphe à X, alors pour tous $F, G \in D^b(X)$, on a $F \boxtimes G \simeq F \otimes^{\mathbb{L}}_X G$.

2. On utilise le théorème de Morita pour les dg-catégories de Toën pour donner une caractérisation d'une structure tronquée en termes de bimodules sur un produit des dg-algèbres, qu'induisent une structure de catégorie tensoriel triangulée sur la catégorie homotopique.

3. On a étudie la théorie de déformation de cettes structures dans le sens de la cohomologie de Davydov-Yetter. On montre que il existe une correspondence entre un des groupes de cohomologie et l'ensemble de associateurs dont le produit tensoriel peut s'en déformer.

On utilise des techniques à un niveau des catégories triangulées et aussi des perspectives de la théorie des catégories supérieurs comme des dg-catégories et quasi-catégories.
Mots-clés Catégories dérivées, géométrie algébrique dérivée non-commutative, catégorie tensoriel triangulée, dg-catégorie, transformations de Fourier-Mukai.

Abstract

In this thesis we are interested in studying derived categories of smooth projective varieties over a field. Concretely, we study the geometric and categorical information from the variety and from it's derived category in order to understand the set of monoidal structures one can equip the derived category with. The motivation for this project comes from two theorems. The first is Bondal-Orlov reconstruction theorem which says that the derived category of a variety with ample (anti-)canonical bundle is enough to recover the variety. On the other hand, we have Balmer's spectrum construction which uses the derived tensor product to recover a much larger number of varieties from it's derived category of perfect complexes as a monoidal category.
The existence of different monoidal structure is in turn guaranteed by the existence of varieties with equivalent derived categories. We have as a goal then to understand the role of the tensor products in the existence (or not) of these sort of varieties.
The main results we obtained are

1. If X is a variety with ample (anti-)canonical bundle, and $\boxtimes$ is a tensor triangulated category on $D^b(X)$ such that the Balmer spectrum $Spc(D^b(X), \boxtimes)$ is isomorphic to X, then for any $F, G \in D^b(X)$ we have $F \boxtimes G \simeq F \otimes^{\mathbb{L}}_X G$.

2. We have used Toën's Morita theorem for dg-categories to give a characterization of a truncated structure in terms of bimodules over a product of dg-algebras, which induces a tensor triangulated category at the level of homotopy categories.

3. We studied the deformation theory of these structures in the sense of Davydov-Yetter cohomology, concretely showing that there is a relationship between one of these cohomology groups and the set of associators that the tensor product can deform into.

We utilise techniques at the level of triangulated categories and also perspectives from higher category theory like dg-categories and quasi-categories.

Keywords: Derived categories, noncommutative derived algebraic geometry, tensor triangulated category, dg-category, Fourier-Mukai transforms.

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Contents

Introduction

Conventions

Through the rest of this work and unless said otherwise we will be working exclusively over an algebraically closed field k of characteristic 0. Our grading is always cohomological, so that differentials rise degree.

The goal of this work is to provide some results about monoidal category structures on the derived category of a given variety X. The interest in this goal is broad as derived categories arise in connected but still different corners of mathematics, and similarly the interest in studying the extra data of monoidal structures on such a category can be motivated from different points of view.
Our adopted point of view in general is that of derived noncommutative geometry in the specific sense of trying to understand spaces (varieties, schemes, stacks, or higher versions of thereof) through the lense of their derived categories. This point of view can be traced back to the different duality phenomena that pops in different branches of mathematics and we could perhaps trace it back to classical Stone duality, and passing through a number of generalizations and analogue versions in different contexts along many different branches of mathematics.
In the algebraic geometric case we can start the story by the Gabriel-Rosenberg reconstruction theorem ([Ros96, Bra18]), a celebrated result which says that for a large class of schemes it is enough to look at the abelian category of coherent sheaves on it to completely determine the space, so that for such a space X there exists a way to extract information from $Coh(X)$ in such a way that we can get back X. We could then call a certain sort of Grothendieck categories noncommutative schemes keeping in mind then that these categories must be in some way categories of coherent sheaves over some possibly non-realizable space. This is perhaps too good of a result in the sense that while the equivalence between categories and spaces can provide a fruitful difference in point of views, in practice the fact

that this is a lossless correspondence means it is expected that studying these categories is at least as complicated as studying the spaces themselves.
On the other hand, the use of homological methods in algebraic geometry has proven to be an extremely useful tool since the introduction of derived categories and derived functors, just as in algebraic topology one very potent tool in classification problems is to restrict our parameters in terms of homological invariants. As such, one immediate question is whether these derived invariants and the derived category from which they stem contains all the possible information about the space.
Let us briefly recall the different notions of derived category in our context,

Definition. *Let $\mathscr{A}$ be an abelian category, the derived category $D(\mathscr{A})$ is the localization of the category of chain complexes on $\mathscr{A}$ by quasi-isomorphisms.*

In the geometric context we should consider, for a scheme X the abelian category of coherent sheaves on X, $Coh(X)$ and we can construct its derived category, which we denote by $D(Coh(X))$ (which we will often denote simply by $D(X)$), as described above. In full generality this category can be difficult to work with, and so in applications one would often run into the subcategories $D^-(X), D^+(X), D^b(X)$ of bounded below, bounded above, and bounded coherent sheaves on X, respectively, and of special interest for us the derived category $Perf(X)$ of bounded perfect complexes on X.
One of our interests is in understanding equivalences between these derived categories, to do so one has to move to the more general notion of triangulated category introduced by Grothendieck and Verdier to point out the relevant structure that we would like to conserve under a functor. Roughly, a triangulated category consists of the data of a distinguished set of triples of morphisms which one calls triangles and which we further require to satisfy a number of axioms. It is these triangles that we would like to preserve under equivalences, and so when we deal with equivalences of derived categories one is interested in these so-called triangulated functor which are in addition an equivalence of categories.
Now to answer our question from before, we know that $Coh(X)$ is able to recover the space X in a wide class of situations but is the passing to $D^b(X)$ too much loss in information for this reconstruction to be broken? The answer to this has been known for many years, originally it was Mukai who noticed while studying abelian varieties that it is possible for two non isomorphic abelian varieties to have isomorphic derived categories. Concretely in [Muk78], it is shown that if A is an abelian variety and $\hat{A}$ is its dual abelian variety, then $D^b(A)$ is equivalent to $D^b(\hat{A})$, while non-polarized abelian varieties are not isomorphic to their duals.
Mukai named the equivalence between these derived categories a Fourier functor as they resemble a categorification of the classical Fourier transform between spaces of functions. Nowadays two given spaces with equivalent derived categories are said to be Fourier-Mukai partners. Despite these categories not being a complete invariant, it is still reasonable to wonder to which degree does the derived category determine a space and if there are spaces X which are completely determined by $D^b(X)$.
An answer to this latter question comes from the following celebrated result of Bondal and Orlov ([BO01])

Theorem. *Let X be a smooth projective variety and suppose that its (anti-)canonical sheaf ω_X is ample. If Y is another smooth projective variety such that there is an equivalence*

$$F : D^b(X) \to D^b(Y)$$

then there is an isomorphism $X \cong Y$.

Combined with the situation described by Mukai, the hypothesis of this theorem might hint that properties of the canonical bundle ω_X - or equivalently the associated divisor K_X - play an interesting role in how fine of an invariant is the derived category, on one side Calabi-Yau varieties might have many different Fourier-Mukai partners while varieties with ample (anti-)canonical bundles however are uniquely determined by the category.
These phenomena has been recognized in the following conjecture, appearing for example in [Kaw02]

Conjecture. *(K-equivalence implies D-equivalence) Let X and Y be birationally equivalent smooth projective varieties. Then the following are equivalent.*

1. *There exists an equivalence of triangulated categories $D^b(X) \simeq D^b(Y)$.*
2. *There exists a smooth projective variety Z and birational morphisms $f : Z \to X$ and $g : Z \to Y$ such that $f^*K_X \cong g^*K_Y$*

In fact in the same work Kawamata shows the following

Theorem ([Kaw02], Theorem 1.4)**.** *Let X and Y be smooth projective varieties. Assume that $D^b(X) \simeq D^b(Y)$ as triangulated categories. Then the following hold:*

1. *$dimX = dimY = n$.*
2. *If K_X (resp. $-K_X$) is nef, then K_Y (resp. $-K_Y$) is also nef, and the numerical Kodaira dimensions $\nu(X)$, $\nu(Y)$ coincide (resp. $\nu(X, -K_X) = \nu(Y, -K_Y)$).*
3. *If X is of general type, or if the Kodaira dimension $\kappa(X, -K)$ is equal to n, then X and Y are birationally equivalent. Moreover there exists birational morphisms $f : Z \to X$, $g : Z \to Y$ from a smooth projective variety Z such that $f^*K_X \simeq g^*K_Y$*

We recall that a line bundle $\mathscr{L}$ is said to be nef if for every irreducible curve C then $D \cdot C \geqslant 0$ where D is the divisor associated to $\mathscr{L}$. Here $\nu(X, \mathscr{L})$ (or $\nu(X)$ in the case $\mathscr{L} = \omega_X$) denotes the numerical Kodaira dimension of the line bundle $\mathscr{L}$, which is defined as the maximal integer m such that there exists a proper morphism $\phi : W \dashrightarrow X$ W of dimension m with $([\phi^*(\mathscr{L})]^m \cdot W) = 0$.
What the theorem is telling us is that the birational geometry of spaces is captured to a certain degree in the derived category at least in good enough situations, and it then would be interesting to understand

to what extent does the categorical data know about the geometry in this sense.

Using the language of triangulated categories it is possible for example to answer the question of whether it is possible to recover the space from the derived category if we add some extra structure. This is the theme of Balmer's tensor (triangulated) geometry when applied to derived categories of a space. In [Bal02, Bal05] Balmer exploited the classification theorem of Thomason ([Tho97]) to show the following

Theorem. *Let $Perf(X)$ be the derived category of perfect complexes on a noetherian scheme equipped with the monoidal structure given by the derived tensor product, then the set $Spc(Perf(X))$ of thick prime tensor ideals can be equipped with a topology and a locally ringed space structure such that $Spc(Perf(X)) \cong X$.*

Here by tensor thick ideal we mean a thick triangulated subcategory of $Perf(X)$ such that it is closed by taking derived tensor products by any object of $Perf(X)$ and such that if $x \otimes^{\mathbb{L}}_X y$ is in this subcategory, then either x or y were in the subcategory to begin with.
The topology that this Balmer spectrum carries is strikingly similar to the Zariski topology of affine schemes defined for commutative rings, and the structure sheaf comes from choosing an appropriate basis for the topology and defining on each of these opens a ring of endomorphisms of the unit object for the derived tensor product.
In other words, Balmer showed that when the derived category of perfect complexes carries a given monoidal structure then it is always possible to recover the space. The combination of the theorems of Bondal-Orlov and Kawamata with this result leads to the question of the nature of the monoidal structure on such derived category.
Balmer's construction inputs a triangulated category equipped with a monoidal structure which is compatible with the given triangulated structure, i.e. a tensor triangulated category, and outputs a locally ringed space. If we are able to completely classify all the possible tensor triangulated category structures on a given derived category of perfect complexes of a space X, the classification of all Fourier-Mukai partners must follow along as they will all correspond to different monoidal structure son the same underlying triangulated category. It is entirely possible however that the derived category is able to be equipped with tensor structures resulting -after passing through Balmer's construction- in locally ringed spaces which are not derived equivalent to the space X.
This is in fact the main motivation of our work, through this thesis we are mainly interested in understanding results like Bondal-Orlov through the lenses of tensor structures on a triangulated category and what are the properties that the space X reflects on the derived category $D^b(X)$ such that it admits (or not) different tensor structures.
We believe that this point of view can be of importance, for example as Balmer's spectrum yields a locally ringed space of which one can consider its abelian category of sheaves of modules, or under the assumption this space is a scheme we obtain an abelian category of coherent sheaves. We would like to draw the attention that similar things occurs when studying t-structures on triangulated categories,

the heart of such structure is an abelian category one can construct out of the triangulated category and in particular good situations we get a comparison between the derived category of the heart and the derived category of the space, or more generally in a homotopy category of a different nature.

While the context of triangulated categories is enough for much of the work that has been done about derived categories, it has been known from their inception that they are not always well-behaved. For example, one limitation comes from the mapping cone construction.
Just as in the derived category case in any triangulated category $\mathscr{T}$, given a morphism $f : X \to Y$, there is always an object $cone(f)$ such that

$$X \to Y \to cone(f) \to X[1]$$

is a distinguished triangle. This object is determined up to isomorphism but the construction is not canonical in the sense that if we consider the category of arrows $Arr(\mathscr{T})$ with objects given by morphisms $f : X \to Y$ in $\mathscr{T}$ and morphisms between arrows $f : X \to Y$ and $g : X' \to Y'$ are given by commutative squares

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} \\ X' & \longrightarrow & Y' \end{array} .$$

And the mapping cone construction induces a morphism $cone(f) \to cone(g)$, resulting in a diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & cone(f) & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow & & \downarrow \\ X' & \xrightarrow{g} & Y' & \longrightarrow & cone(g) & \longrightarrow & X'[1] \end{array}$$

However this induced map is not in general functorial, if we give another morphism $h : X'' \to Y''$ and morphisms (a', b')

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \longrightarrow & cone(f) & \longrightarrow & X[1] \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \downarrow & & \downarrow \\ X' & \xrightarrow{g} & Y' & \longrightarrow & cone(g) & \longrightarrow & X'[1] \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle b'} & & \downarrow & & \downarrow \\ X'' & \xrightarrow{h} & Y'' & \longrightarrow & cone(h) & \longrightarrow & X''[1] \end{array}$$

Then the morphism $cone(f) \to cone(h)$ induced by the morphisms $f, h, a' \circ a$, and $b' \circ b$ is not in general the morphism $cone(f) \to cone(h)$ given by the composition $cone(f) \to cone(g) \to cone(h)$.
Another such aspect which is of importance to us in this work, is that the category of triangulated functors between triangulated categories does not carry a canonical triangulated structure and so it is complicated to use some of the existing tools to study general triangulated categories.
It was with the development of homotopy theory and keeping this and other problems in mind, that

the notion of enhanced category was found and developed. From a pure homotopy theory perspective, the notion of stable ∞-category serves as a model for an object which contains the structure of a triangulated category in its 1-categorical shadow along with extra data that doesn't show up when working classically. Another possible model for this, and the one we will be using, is that of dg-categories, these are simply categories enriched over chain complexes and so they are easy to describe and work with, and in the same fashion as their ∞-categorical versions, they can be used to enhance triangulated categories under the right conditions.

An important feature of dg-categories is that they can be equipped with model structures which can then help us refine some of the behavior that was not showing up in full when working directly with triangulated categories.

The key insight for us is that a tensor product $\boxtimes : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ in a tensor triangulated structure is among other things a triangulated functor on each variable and as such if we were able to work with a space of such functors as an ambient space we would be able to work with more concrete description of the objects we are looking for.

Indeed in full analogy with the classical Morita theory for rings and algebras, Toën showed the following theorem for dg-categories of perfect complexes over smooth proper schemes:

Theorem (Thm 8.15, [Toë07]). *Let X and Y be two smooth proper schemes over $Speck$. Then, there exists an isomorphism in the homotopy category $Ho(dg-Cat)$*

$$\mathbb{R}\underline{Hom}(Perf_{dg}(X), Perf_{dg}(Y)) \simeq Perf_{dg}(X \times_k Y).$$

Where the homotopy category $Ho(dg-Cat)$ is the homotopy category of one of the model structures we can equip the category of dg-categories with, and where $Perf_{dg}(X)$ denotes an enhancement of the bounded derived category of perfect complexes on X.

What this theorem is essentially telling us is that the relevant functors in the homotopy theory of dg-categories between these enhancements is parametrized by perfect complexes on the product of the spaces. In other words functors come from a sort of generalized bimodules, the category of which has a dg-category structure itself.

This interpretation has a direct relationship with the Fourier-Mukai side of things, in fact one important feature of these sort of equivalences is that they are precisely given by functors of this form, where one picks an object (the kernel of the transform, following the integral transform nomenclature) and then the transform can be written as a tensor product of a pullback followed by a pushforward by the projection morphisms $X \leftarrow X \times_k Y \to Y$.

This context of dg-categories is then where we expect to be able to work with collections of structures, as Toën's Morita theorem allows us to work comfortably with them unlike the triangulated setting in which one would have to keep track of whether each step preserves the structure we are working on.

To make use of this theory we have too to understand the theory of dg-enhancements for our derived categories. This means that there exists dg-categories such that their homotopy categories are triangulated equivalent to these derived categories of perfect complexes and they are in a certain sense unique.

With the motivating background laid out, let us summarize the work developed in this thesis:

Organization of this work

In the first chapter we go over the basic theory of derived categories and review the necessary concepts and results that would allow us to sketch Bondal and Orlov's original proof of their reconstruction for varieties with ample (anti-)canonical bundle, and similarly with Kawamata's result. We introduce the notions of exceptional collections, semi-orthogonal decompositions, Serre functors, and Fourier-Mukai transforms along with some examples of known families of spaces where we have some knowledge on the Fourier-Mukai partners.
In chapter two we go over general tensor triangulated categories and Balmer's reconstruction theorem, we give a brief overview of the developments of the theory and some general structural results about the Balmer spectrum and tools used to understand it.
Chapter three introduces our first results, we approach Bondal-Orlov's reconstruction and Kawamata's result from the point of view of tensor triangulated categories and develop some general ideas on the collection of tensor triangulated structures we can impose on a derived category coming from a variety with ample (anti-)canonical bundle, or in general a variety of general type. Concretely we have:

Corollary. *(Corollary 3.0.11) Let X be a variety of general type and let $\boxtimes$ a tensor triangulated category structure on $D^b(X)$ with unit $\mathcal{O}_X$. Then for any $\boxtimes$-invertible object U such that $U \boxtimes I_{X*} \subseteq I_{X*}$, the equivalence $U\widehat{\boxtimes} : D^b(X)/I_{X*} \to D^b(X)/I_{X*}$ induced by $U \boxtimes \_$ is equivalent to an equivalence given by objects in the group $Pic(D^b(X)/I_{X*}, \widehat{\otimes^{\mathbb{L}}})$ of invertible $\widehat{\otimes^{\mathbb{L}}}$-objects.*

Here the proper ideal I_{X*} is determined by the complement of the augmented base locus of the canonical bundle (Definition 3.0.3). Our corollary above follows from our definition of almost spanning class relative to an ideal (Definition 3.0.7) and its relationship with the hypothesis on the canonical bundle.

Theorem 0.0.1. *(Theorem 3.0.8) Let X be a smooth projective variety of general type. Then the collection of tensor powers $(\omega_X^{\otimes i})_{i\in\mathbb{Z}}$ forms an almost spanning class with respect to the thick tensor ideal I_{X*} in the tensor triangulated category $(D^b(X), \otimes_X^{\mathbb{L}})$.*

Here the ideal I_{X*} might not be a $\boxtimes$-ideal and so the Verdier quotient $D^b(X)/I_{X*}$ might not carry a tensor triangulated category structure, however any autoequivalence of the form $U \boxtimes \_$, for $U \in Pic(D^b(X), \boxtimes)$ such that $U \boxtimes I_{X*} \subseteq I_{X*}$ induces an autoequivalence functor $D^b(X)/I_{X*} \to D^b(X)/I_{X*}$ by the universal property of the Verdier quotient, and what this result is saying is that all these induced equivalences of $D^b(X)/I$ are given by $\otimes^{\mathbb{L}}$-invertible objects.
In case the ideal I_{X*} happens to be a $\boxtimes$-ideal too then what we obtain is that the Picard group

$Pic(D^b(X)/I_{X*}, \widehat{\boxtimes})$ is isomorphic to a subgroup of $Pic(D^b(X)/I_{X*}, \otimes^{\mathbb{L}})$.
The specialization to the case where the canonical bundle of the variety X is ample gives us that any other tensor triangulated structure has a Picard group isomorphic to a subgroup of shifted line bundles on X. This is because in that case the ideal I_{X*} from the corollary above is the 0 ideal and so we obtain $Pic(D^b(X), \boxtimes) < Pic(D^b(X), \otimes^{\mathbb{L}})$.
By the reasoning used for these results, we have a criterion for varieties with ample (anti-)canonical bundle:

Corollary. *(Corollary 3.0.14) Let X be a smooth projective variety over k with ample (anti-)canonical bundle. If $\boxtimes$ is a tensor triangulated category structure on $D^b(X)$ such that ω_X is $\boxtimes$-invertible, then we have that $\boxtimes$ coincides on objects with the derived tensor product $\otimes^{\mathbb{L}}_X$.*

Using this results we can give a proof of Bondal-Orlov reconstruction theorem (Theorem 3.0.20) under some additional slightly stronger conditions.
Using the same reasoning, we see that if one has any tensor structure $\boxtimes$ with Balmer spectrum isomorphic to our variety X then $\boxtimes$ and $\otimes^{\mathbb{L}}_X$ have to coincide on objects. Concretely we have:

Theorem. *(Theorem 3.0.23) Let X be a smooth projective variety with (anti-)canonical bundle. Consider a tensor triangulated category structure $\boxtimes$ on $D^b(X)$ such that $\mathscr{O}_X$ is its unit and $Spc(\boxtimes)$ is isomorphic to X, then $\boxtimes$ and $\otimes^{\mathbb{L}}_X$ coincide on objects.*

In chapter four we present the general theory of dg-categories and review the theory behind dg-enhancements of derived categories, the model structure(s) on the category of dg-categories over a field along with their derived tensor product and Toën's homotopy Morita theory result.
We make use of this theorem in a strong way in chapter five to give a correspondence between tensor triangulated category structures on a given derived category and the data of a certain bimodule and coherent isomorphisms. This follows analogous results by Hovey in [Hov11] where the classical Morita theory is used for the purposes of classifying symmetric closed monoidal structures on a category of R-modules for a ring R. To be more precise, as our derived categories have generating objects and it is known that one can show these derived categories are then homotopy equivalent to a category of dg-modules over the dg-algebra of endomorhisms of a generator, then our result from the previous chapter saying that a tensor triangulated structure corresponds roughly to the data of a 2-fold dg-bimodule over such a dg-algebra together with coherent morphisms corresponding to the unit, associators, and symmetry isomorphisms.
We reproduce some structural results about the bimodule controlling the bifunctor of the tensor triangulated structure and other conditions on the unit or the dg-algebra of coefficients.
Formally we introduce the definitions of pseudo dg-tensor structure (Definition 5.1.5), perfect pseudo dg-tensor structure (Definition 5.1.6), and pseudo dg-tensor functor (Definition 5.1.8). We then see that these structures indeed induce tensor triangulated structures at the homotopy category level.

Lemma (Lemma 5.1.7)**.** *A perfect pseudo dg-tensor structure Γ on a dg-category $\mathscr{T}$ induces a tensor*

triangulated category structure on $H^0(\mathscr{T}_{pe})$.

Using this lemma it is possible to reproduce a series of structural results about the objects involved in this characterization of tensor triangulated structures. We show for example:

Corollary (Corollary 5.1.14)**.** *Let A be a dg-algebra and let Γ be a perfect pseudo dg-tensor structure on A_{pe}. Then Γ is faithful as a $H^0(A)$-module with either multiplication structure.*

Here A_{pe} denotes the dg-enhancement of perfect complexes over A, and we recall that a module is faithful if its annihilator is the zero ideal.
Then, we dedicate some time to discuss symmetric monoidal dg-categories as commutative algebra objects in the $(\infty,1)-$-category of dg-categories induced from the model category structure we discussed in chapter four. We show that our perfect pseudo dg-tensor structures are truncations of these structures in Theorem 6.2.4.
In chapter seven we review the deformation theory of monoidal categories in the sense of Davydov-Yetter. This is a cohomology theory developed for tensor categories which controls in lower cohomology groups the deformation of some of the structural morphisms in the monoidal data. For example in degree 2 the cohomology parametrizes deformations of the coherent associators

$$\alpha : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

We develop an analogous theory for our tensor triangulated structures exploiting the correspondence from the previous chapter by constructing a double complex which contains information too about the deformation of the associators in our derived context.
In concrete terms, we show that we can relate our generalized Davydov-Yetter cohomology HDY^*_{dg} as described in Definition 7.2.4 in lower degrees and deformations of the associator of a perfect pseudo dg-tensor structure.

Theorem (Theorem 7.2.7)**.** *Let $\mathscr{T}$ be a dg-category and let Γ be a perfect pseudo dg-tensor structure on $\mathscr{T}$. Then to any cocycle in $HDY^3_{dg}(\mathscr{T})$ corresponds a first order infinitesimal deformation of the associativity condition of Γ up to equivalence.*

In this chapter we study in more depth the collection $TTS(D^b(X))$ of tensor triangulated structures on $D^b(X)$ and we give some structural results, in particular we take interest in the subcollection of such structures with a fixed tensor unit $TTS(D^b(X), \mathscr{O})$.
After reviewing different moduli spaces and deformation problems relevant to our situation, we show that the space of tensor triangulated structures with a fixed tensor bifunctor and unit has an affine scheme structure

Theorem (Theorem 7.3.1)**.** *Let $\mathscr{T}$ be a dg-category, A a k-algebra, a perfect module U and Γ a 2-fold dg-bimodule over $\mathscr{T} \otimes^{\mathbb{L}} A$. Then the set $TTS_A(\Gamma)$ of perfect pseudo dg-tensor structures over $\mathscr{T} \otimes^{\mathbb{L}} A$*

which has Γ as a 2-fold dg-bimodule and U as a unit, has a structure of a quotient of an affine scheme by an algebraic affine variety.

In our final chapter we take time to pose a number of questions and conjectures that arose during this work and investigation of tensor triangulated geometry from this point of view. We intend to motivate these conjectures and give partial answers and evidence for their validity as legitimate questions. These include the interaction with invariants of the derived category such as t-structures or exceptional collections and decompositions, and we go over some remarks on the nature of the Balmer spectrum of higher homotopical spaces. We mention some results given in the literature we could not include elsewhere.

Chapter 1

Derived categories

In this chapter we give a brief review of the use of derived categories in algebraic geometry. Most of the material here explained is already classical and so we source our exposition from [Huy06]. Through this chapter and unless said otherwise, a space or a variety (often denoted by X or V) will refer to a smooth projective variety over k.

1.1 Generalities

Let us first start by recalling some basic definitions. If X is a variety we can associate to it its abelian category $Coh(X)$ of coherent sheaves on X and its abelian category of quasi-coherent sheaves $QCoh(X)$.

Our main interest is in a localization of the category of chain complexes on $Coh(X)$. We say that the derived category $D(X)$ of X as the localization of the category of chain complexes $C(Coh(X))$ by quasi-isomorphisms. That is, we will invert every chain complex morphism $f : F^* \to G^*$ with the property that $H^n(f)$ is an isomorphism for every $n \in \mathbb{N}$.

This category while still important can be difficult to work with and we then restrict ourselves to the better behaved bounded derived category $D^b(X)$ of X which consists of bounded chain complexes. Similarly one can consider the bounded above and below derived categories, $D^-(X)$ and $D^+(X)$ respectively.

It is a feature of X being a smooth projective variety that in fact this bounded derived category coincides with the even better behaved category of perfect complexes $Perf(X)$, the subcategory of bounded complexes locally free of finite type.

When we refer to **the** derived category of X we will be referring thus to this category of perfect complexes, equivalent to the bounded derived category of X.
In general given an abelian category $\mathcal{A}$ we can consider its derived categories $D(\mathcal{A})$ and $D^b(\mathcal{A})$ but the category $Perf(\mathcal{A})$ doesn't have to be equivalent to the bounded derived category nor does it have to share some of the good properties it has in the case where $\mathcal{A}$ is a category of coherent sheaves on a space X. In the geometric case we also have the following very useful identification:

Theorem 1.1.1. *Let X be a noetherian scheme (in particular quasi-compact and quasi-separated), then the natural functor $D^b(X) \to D^b(QCoh(X))$ identifies $D^b(X)$ with the derived category $D^b_{coh}(QCoh(X))$ of bounded complexes of quasi-coherent sheaves with coherent cohomology.*

Proof. See [Huy06, Proposition 3.5] □

Derived categories on their own are a great framework to study homological phenomena for a given space X, but trying to deduce abstract general properties of derived categories require an axiomatizatiom of those in itself. One such attempt for putting derived categories in an abstract setting was that of triangulated categories by Verdier and Grothendieck. Let us recall briefly that a triangulated category is a k-linear category $\mathscr{T}$ equipped with a shift autoequivalence:

$$[\,] : \mathscr{T} \to \mathscr{T}$$

And a family of morphisms $X \to Y \to Z \to X[1]$ called distinguished triangles, satisfying a number of axioms (cf. [May01]). In particular a derived category carries a triangulated category structure by declaring distinguished triangles to be the induced triangles coming from exact sequences, and the shift autoequivalence being the degree shift in chain complexes.
Among the axioms the structure must satisfy, let us single out that given a morphism $f : X \to Y$ in a triangulated category $\mathscr{T}$ there always exists an object Z such that the triangle

$$X \to Y \to Z \to X[1]$$

is a distinguished triangle. We call Z the cone of $f : X \to Y$ and often we denote it too by $cone(f)$ or $cone(X \to Y)$.
A full subcategory is a triangulated subcategory if it is triangulated and its triangulation coincides at the larger category. Alternatively, if it is closed under cones and shifts.
A triangulated functor consists of the data $(\mathscr{F}, u)$ where $\mathscr{F} : \mathscr{T} \to \mathscr{T}'$ is a functor sending triangles to triangles and u is an equivalence $\mathscr{F} \circ [n] \simeq [n] \circ \mathscr{F}$ for every $n \in \mathbb{N}$.
Similarly a full subcategory $\mathscr{T}'$ of a triangulated category $\mathscr{T}$ is a triangulated subcategory if the inclusion functor $\mathscr{T}' \to \mathscr{T}$ is exact.
Our goal through this section is to provide some exposition about the relationship between the formal properties of a derived category $D^b(X)$ and the geometry of the space. For this we will sometimes work with general triangulated categories and sometimes will provide concrete properties of derived

categories.
Working directly with a triangulated or even with a derived category can be complicated without further information, one important property that helps us manipulate the category is that of it being generated by some object. Let us be more precise with the following series of definitions.

Definition 1.1.2. *Let $\mathscr{T}$ be a triangulated category, and $\mathscr{I} \subseteq \mathscr{T}$ a full triangulated subcategory, we say that it is thick if it is closed under direct summands. So that if $A \oplus B \in \mathscr{I}$ then $A, B \in \mathscr{I}$.*

Definition 1.1.3. *We will say that a thick subcategory $\mathcal{I} \subset \mathscr{T}$ of a triangulated category admitting coproducts, is dense if it is closed under coproducts.*

It is perhaps interesting to mention that this is not the only characterization of a thick subcategory, the previous definition is equivalent to asking for a full triangulated subcategory to be closed under extensions of distinguished triangles. For a proof of this see [Ric89, Proposition 1.3].
If $\mathcal{I} \subseteq \mathscr{T}$ is a class of objects of a triangulated category $\mathscr{T}$, we denote by $\langle \mathcal{I} \rangle$ the thick subcategory generated by $\mathcal{I}$, meaning the smallest thick subcategory of $\mathscr{T}$ containing $\mathcal{I}$.
For two classes $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathscr{T}$ of objects, we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of objects E that sit in a distinguished triangle

$$E_1 \to E \to E_2 \to E_1[1]$$

with $E_i \in \mathcal{I}_i$.
We write $\mathcal{I}_1 \Diamond \mathcal{I}_2$ for the subcategory $\langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$, the smallest thick subcategory containing $\mathscr{I}_1 * \mathscr{I}_2$.
Finally, for an object $E \in \mathscr{T}$, we put $\langle E \rangle_i = \langle E \rangle_{i-1} \Diamond \langle E \rangle_1$ where $\langle E \rangle_1 = \langle E \rangle$. This means for example that given an object $E \in \mathscr{T}$, the subcategory $\langle E \rangle_2$ consists of all the objects F such that F is sitting in a triangle

$$E' \to F \to E'' \to E'$$

Where E' and E'' are direct sums of shifts of direct summands of E.

Definition 1.1.4. *We will say that T is classically generated by an object $E \in \mathscr{T}$ if $\mathscr{T} = \bigcup_k \langle E \rangle_k$. If there exist a $k \in \mathbb{Z}$ such that $\mathscr{T} = \langle E \rangle_k$ then we say in turn that $\mathscr{T}$ is strongly generated.*

In [Orl09] Orlov introduced the following invariants for a triangulated category related to the number of generators that are required to obtain the whole category.

Definition 1.1.5. *For a triangulated category T, its Orlov spectrum is the subset of integers $k \in \mathbb{Z}$ for which there is an object $E \in T$ that generates T in k-many steps. That is $\langle E \rangle_{k+1} = T$.*

Definition 1.1.6. *The (Rickard) dimension of T, $dim(T)$, is the minimum integer appearing in its Orlov spectrum.*

As an example, a well known result for the derived category of $X = \mathbb{P}^1$ says that $D^b(X)$ is strongly generated by $E = \mathcal{O} \oplus \mathcal{O}(-1)$, and does so in 1 step so that $\langle E \rangle_2 = D^b(X)$, which implies that the

dimension of the category is 1. It is possible however that there might be other complexes in $D^b(X)$ such that they generated $D^b(X)$ in more steps, for example if we take $F = \mathcal{O} \oplus \mathcal{O}_p$ for $p \in X$ a closed point, then we have that F is a strong generator but instead generates the category in two steps.
In other words, the Orlov spectrum of the derived category $D^b(X)$ contains $\{1, 2\}$ and what would be left to see is that there are no other integers showing up in this set. But this follows from observing that for any object $\mathscr{F}$ which strongly generates $D^b(X)$ in $\geqslant 2$ steps there is a line bundle and a torsion sheaf as a direct summand, and an extension of two such objects is a different line bundle contained in $\langle F \rangle_2$ and as any two line bundles generate the category, we would have $\langle F \rangle_3 = D^b(X)$.
It is however still an open conjecture whether the dimension of the space coincides with the dimension of the triangulated category in the case of smooth projective varieties, and whether the Orlov spectrum forms a full integer interval for these derived categories.
We have however that the dimension must be finite.

Theorem 1.1.7. *Let X be a quasi-projective scheme and let L be an ample line bundle, then $E = \bigoplus L^i$ is a classical generator for $Perf(X)$.*

As generators in the case of a triangulated category relate directly to the triangulated structure and how one can obtain every object as a sequence of cones of direct sums of shifts of direct summands of the given object, there exists a general (and in the case of triangulated categories, related) notion of spanning class:

Definition 1.1.8. *A collection of objects $\{X_i\} \subseteq \mathscr{T}$ of a triangulated category is called a spanning class if:*

1. *If $Hom(X_i, D[j]) = 0$ for all i, j then $D \simeq 0$*
2. *If $Hom(D[j], X_i) = 0$ for all i, j then $D \simeq 0$*

Before giving out examples, we point that it's possible to cut down the work by half if the derived category has good duality properties.

Definition 1.1.9. *Let $\mathscr{T}$ be a triangulated category an autoequivalence $S : \mathscr{T} \to \mathscr{T}$ satisfying $Hom(A, B) \cong Hom(B, S(A))^*$ for all objects $A, B \in \mathscr{T}$, is called a Serre functor.*

Example 1.1.1. *Specifically if for example the triangulated category is a derived category of a smooth projective scheme of dimension n, we have Grothendieck-Verdier duality which implies that for every pair of objects $M, N \in \mathcal{D}(X)$, $Hom(M, N) = Hom(N, M \otimes \omega_X[n])^*$ where ω_X is the canonical bundle of X.*

We will come back to the general study of Serre functors later, but for now let us observe that if $\mathscr{T}$ has such a functor, then a collection of objects $\{X_i\}$ satisfying any of the two conditions of

spanning classes satisfies the other automatically, indeed if for example it satisfies the first one and $Hom(D[j], X_i) = 0$ for all i, j, then

$$Hom(D[j], X_i) = Hom(X_i, S(D[j]))^* = 0$$

implies $S(D) \simeq 0$ which implies $D \simeq 0$.

Proposition 1.1.10. *[Huy06, 3.17] Let X be a smooth projective variety. The collection $\{\mathcal{O}_x \mid x \in X$ for x a closed point$\}$ forms a spanning class on $D^b(X)$.*

Proof. For $\mathcal{F} \in D^b(X)$ we pick a maximal m such that $H^m(\mathcal{F}) \neq 0$ and $x \in Supp(H^m(\mathcal{F})$. Then there then a nontrivial morphism $H^m(\mathcal{F})) \to \mathcal{O}_x$, by shifting by m and composing with the morphism $\mathcal{F} \mapsto H^m(\mathcal{F})[m]$ we see that $Hom^m(\mathcal{F}, \mathcal{O}_x) \neq 0$ □

There are some relationships between the generators of a triangulated category, specially with the existence of Serre functors. We have,

Definition 1.1.11. *An object $E \in \mathscr{T}$ of a triangulated category is called a weak generator if for any non-zero object D, there is j with $Hom(E, D[j]) \neq 0$.*

In other words, an one-object spanning class is a weak generator. The notion of classical generator and weak generator are related in one direction in general:

Lemma 1.1.12. *If $E \in \mathscr{T}$ is a classical generator then it is a weak generator.*

Proof. If $E' \in \mathscr{T}$ is such that $Hom(E, E'[k]) = 0$ for all $k \in \mathbb{Z}$, implies that for any $E'' \in \langle E \rangle_n$ we have $Hom(E'', E') = 0$, in particular $Hom(E', E') = 0$ which is not possible unless $E' = 0$. □

The converse of this lemma is not in general true, however it is known in some concrete situations for derived categories as we will see by using the following Brown representation theorem for triangulated categories due to Neeman.

Theorem 1.1.13. *[Nee96] Let $\mathscr{T}$ be a triangulated category with small coproducts, if $E \in \mathscr{T}^c$ is a compact object, the following are equivalent:*

1. *E is a classical generator of $\mathscr{T}^c$ and $\mathscr{T}$ is compactly generated*
2. *E is a weak generator*

We should begin first by formally introducing the terms we have yet not defined that appear in the statement of the previous theorem

Definition 1.1.14. *Let $\mathscr{T}$ be a triangulated category admitting coproducts, an object $X \in \mathscr{T}$ is called compact if for every collection of objects Y_i we have*

$$Hom(X, \bigoplus Y_i) \cong \bigoplus Hom(X, Y_i)$$

The full subcategory of compact objects of a triangulated category $\mathscr{T}$ will be denoted by $\mathscr{T}^c$. We will also say that $\mathscr{T}$ is triangulated category compactly generated if there is a collection of compact objects $X_i \in \mathscr{T}^c$ such that $\oplus X_i$ is a weak generator.

Definition 1.1.15. *Given a sequence of objects in a triangulated category,*

$$X_0 \to X_1 \to \dots$$

with transitions $f_n : X_n \to X_{n+1}$, we define the homotopy colimit $HocolimX_n$ of the sequence as the object that fits in a distinguished triangle

$$\oplus X_i \to \oplus X_i \to HocolimX_i \to \oplus X_i[1]$$

Where the map $\oplus X_i \to \oplus X_i$ is the map $1 - f_n$.

We will need a few more lemmas:

Lemma 1.1.16. *[Sta22, Tag 09SN] Let $\mathscr{T}$ be a triangulated category with coproducts, such that $\oplus E_i$ is a weak generator $\mathscr{T}$ with E_i compact objects, then every object $M \in \mathscr{T}$ can be written as $X = HocolimX_n$.*
Where $X_1 \in \langle E_i \rangle_1$ for some $n_i \in \mathbb{Z}$, and there is a distinguished triangle

$$Y_n \to X_n \to X_{n+1} \to Y_n[1]$$

And where $Y_n \in \langle E_i \rangle_1$

Proof. Let X_1 be the direct sum $\oplus_{(i,m,\phi)} E_i[m]$ where $m \in \mathbb{Z}$ and $\phi : E_i[m] \to X$, and let us consider the canonical morphism $X_1 \to X$. Inductively if we are given a morphism $X_n \to X$, construct Y_n as a sum $\oplus E_i[n_i]$ such that the composition of morphisms $E_i \to X_n \to X$ is zero. Thus there exists a triangle $Y_n \to X_n \to X_{n+1} \to Y_n[1]$ and it's possible to pick $X_n \to X_{n+1} \to X$ in such a way that these compositions are the morphisms $X_n \to X$.
This gives us a morphism $HocolimX_n \to X$ that we can fit in a triangle

$$C \to HocolimX_n \to X \to C[1]$$

Compactness of the E_i and the way we constructed the Y_n show that any morphism $E_n[n_i] \to C \to HocolimX_n$ must compose to zero. This in turn means any such morphism $E_n[n_i] \to C$ must be zero, as it factorizes as $E_n[n_i] \to X[-1] \to C \to HocolimX_n$. Since $\oplus E_i$ is a weak generator, $C \simeq 0$ and so $X \simeq HocolimX_n$. □

Lemma 1.1.17. *If $C \to X \in \mathscr{T}$ is a morphism from a compact object, it can be factorized as $C \to E \to X$ with $E \in \langle E_{r_1} \oplus \cdots \oplus E_{r_k} \rangle$ for some indices $r_1, \ldots, r_k$.*

Proof. Proceeding by induction on n. For $n = 1$ nothing needs to be done. If $n > 1$ then given a morphism $C \to X_n$, as we have the triangle $C \to X_n \to Y_{n-1}[1]$, since $Y_{n-1}[1]$ is in $\langle \oplus E_i \rangle_1$ then we can factor the morphism through $C \to E[1] \to Y_{n-1}[1]$ with $E \in \langle E_{r_1} \oplus \cdots \oplus E_{r_k} \rangle$.
This produces a morphism of triangles $(E, C', C) \to (Y_{n-1}, X_{n-1}, X_n)$. Using the induction hypothesis we see that we can factor $C' \to X_{n-1}$ in a similar fashion by $C' \to E' \to X_{n-1}$. This in turn gives us another morphism of triangles $(E, C', C) \to (E, E', \oplus E_i) \to (Y_{n-1}, X_{n-1}, X_n)$. Then all that remains is to show that it is possible to show the existence of E'' such that the vertical morphisms compose correctly, but this can be done again by induction hypothesis and we obtain $E'' \oplus E' \to X_n$. □

With these two lemmas in hand, we can proceed to the proof of Neeman's Brown representability theorem.

Proof of Theorem 1.1.13. One direction was shown before, we then assume that E is a weak generator of $\mathscr{T}$, then we write any compact object as $X = HocolimX_n$, however compactness implies $X \to HocolimX_n$ factors through X_n for some n. By the previous lemma this factors as $X \to E' \to X_n$ and by construction this means $X \in \langle E \rangle$. □

As a consequence of this theorem we can now give a proof of Theorem 1.1.7. Let us first point out the important properties we will use about ample line bundles.

Definition 1.1.18. *We call a collection of objects of an abelian category $\mathcal{A}$, $\{L_i\} \subset \mathcal{A}$ an ample sequence if the following conditions are met: For $i << 0$, and all $A \in \mathcal{A}$*

1. *$Hom(L_i, A) \otimes_k L_i \to A$ is surjective.*
2. *$Hom(A, L_i) = 0$*
3. *$Ext^j(L_i, A) = 0$, $j \neq 0$*

The canonical example of such sequence is of course a family of ample line bundles on a scheme X.
In general ample sequences are very useful and we will make partially use of this concept in the future. To illustrate this let us mention the following useful theorem of Bondal and Orlov which makes use of an ample sequence to fully determine the nature of an autoequivalence. We give the theorem without proof.

Theorem 1.1.19. *[BO01, Prop. A.3] Let $F : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ be a an exact autoequivalence between the derived category of an abelian category $\mathcal{A}$ of finite homological dimension, and let $\{L_i\}$ be an*

ample sequence of $\mathcal{A}$, if there is an isomorphism $id\,|_{L_i} \cong F\,|_{L_i}$ of functors on the full subcategory of $\{L_i\} \subseteq \mathcal{D}(\mathcal{A})$ then it can be extended to an isomorphism $id_{\mathcal{D}(\mathcal{A})} \cong F$

To continue with our proof of theorem 1.1.7, we need the following lemma showing an ample sequence in an abelian category with finite homological dimension forms a spanning class. As anticipated we also have the following result:

Lemma 1.1.20. *Let $\mathcal{A}$ be an abelian category of finite homological dimension and let $\{L_i\}$ be an ample sequence, then the collection $\{L_i\}$ seen as objects of $\mathcal{D}(\mathcal{A})$ form a spanning class*

Proof. We will assume $\mathcal{D}(\mathcal{A})$ has a Serre functor and so will only show that $Hom(L_i, D[j]) = 0$ for all i, j implies $D \simeq 0$. For the proof that doesn't assume the existence of the Serre functor see [Huy06, Proposition 2.73].
Suppose $Hom(L_i, D[j]) = 0$ for all i, j, then D is quasi-isomorphic to a complex

$$D' = \cdots \to 0 \to 0 \to D^n \to D^{n+1} \to \dots$$

with $H^n(D') \neq 0$ and so $Hom(L_i, H^n(D'))$ embeds in $Hom(L_i, D'[n]) = 0$ for every i, however as $\{L_i\}$ is an ample sequence $Hom(L_i, H^n(D')) \otimes_k L_i \to H^n(D')$ is surjective for some i, so it can't be that $D \not\simeq 0$. □

Combining Lemma 1.1.20 and Theorem 1.1.13 by picking as weak generator the direct sum of the spanning class formed from the induced objects from the ample sequence, we obtain the proof of theorem 1.1.7.

1.1.1 Decomposition of triangulated categories

Besides the study of generators as seen previously, one very important tool for the study of derived categories is their decompositions. Roughly speaking the idea is that one is able to study the category in pieces that don't interact much with each other, or if they do we ought to be able to understand the nature of their gluing. Using the language of dg-categories and enhancements we will see in later chapters, this idea of gluing can be put into a more formal language.
In our geometric setting, it is precisely the information of the space that will reflect on the structure of its derived category and will inform us about these decompositions.

Definition 1.1.21. *An object $E \in \mathscr{T}$ of a triangulated category is called exceptional if $Hom(E, E[r]) = 0$ for all $r \neq 0$ and $Hom(E, E) = k$.*

Similarly this definition can be extended to an ordered sequence of objects

Definition 1.1.22. *A collection $E_1, \dots, E_n \in \mathscr{T}$ is called an exceptional collection if $Hom(E_i, E_j[r]) = 0$ if $i > j$ and $r \neq 0$, and $Hom(E_i, E_j[r]) = k$ otherwise.*

As an example there is the well known theorem of Beilinson of the resolution of the diagonal which allows us to conclude that in $\mathbb{P}^n$ we have an exceptional collection given by line bundles. Namely:

Theorem 1.1.23. *The collection $\{\mathcal{O}, \dots, \mathcal{O}(n)\}$ is an exceptional collection in the derived category $D^b(\mathbb{P}^n)$.*

Definition 1.1.24. *An exceptional collection in a triangulated category which generates the category is called a full exceptional collection.*

In particular we see that the collection $\{\mathcal{O}, \dots, \mathcal{O}(n)\}$ generates $D^b(\mathbb{P}^1)$ and so it is also full.

Definition 1.1.25. *An exceptional collection such that $Hom(E_i, E_j[r]) = 0$ for all $r \neq 0$ is called strong.*

We have talked about how an exceptional collection allows us to break down the category in simpler pieces, what this means formally is that the inclusion functor from the subcategory of the triangulated subcategory generated by the exceptional objects to the ambient category admits a right adjoint. In general we have the following definition.

Definition 1.1.26. *Let $\mathscr{T}$ be a triangulated category and $\mathscr{T}'$ a full triangulated subcategory of $\mathscr{T}$, we say that $\mathscr{T}'$ is an admissible subcategory of $\mathscr{T}$ if the inclusion functor $\mathscr{T}' \to \mathscr{T}$ has a right adjoint.*

In particular,

Lemma 1.1.27. *Let $\mathscr{T}$ be a triangulated category such that $dim \bigoplus_n Hom(A, B[n]) < \infty$. If $E \in \mathscr{T}$ is an exceptional object then $\langle E \rangle$ is an admissible subcategory.*

In this case when E is an exceptional object the category $\langle E \rangle$ is equivalent to the derived category of Speck.
In general we see that in the presence of an exceptional collection we have a decomposition in terms of the subcategories generated by the objects on the sequence, we can define

Definition 1.1.28. *A sequence of admissible triangulated subcategories $\mathscr{T}_1, \dots, \mathscr{T}_n \subset \mathscr{T}$ is semi-orthogonal if for $i > j$, $\mathscr{T}_j \subseteq \mathscr{T}_i^{\perp}$.*
In addition this is semi-orthogonal decomposition if the subcategories $\mathscr{T}_i$ classically generate $\mathscr{T}$

From the lemma above we can deduce that in the presence of a full exceptional collection, we can obtain a semi-orthogonal decomposition of our category by considering the triangulated subcategories generated by each object.

Example 1.1.2. *In $D^b(\mathbb{P}^n)$ the full exceptional collection $\{\mathcal{O}, \dots, \mathcal{O}(n)\}$ induces a semi-orthogonal decomposition of the category.*

It should be said that not every derived category accepts a full exceptional collection, in fact in the presence of an exceptional collection we can always define the following:

Definition 1.1.29. *Let $\{E_1, \dots, E_n\}$ be an exceptional collection in a triangulated category $\mathscr{T}$, then the category $(E_1^{\perp} \cap \dots E_n^{\perp}) := \{X \in \mathscr{T} \mid Hom(E_i, X[p]) = 0 \ \forall i \in \{1, \dots, n\}, p \in \mathbb{Z}\}$ is called the Kuznetsov component of the collection $K(\mathscr{T}) := \{E_1, \dots, E_n\}$.*

Let us remark that in the previous definition the component depends on the collection we pick. By construction one sees that one has a semi-orthogonal decomposition $\langle K(\mathscr{T}), E_1, \dots, E_n \rangle$.
Evidently if the component vanishes then the exceptional collection is full, but as this is rarely the case then finding concrete descriptions for this component becomes an important task in understanding the decomposition and the category itself.
As an illustration we will quickly list some known cases of decompositions for some varieties. First, a general result for Fano varieties:

Theorem 1.1.30. *[Kuz16, Example 2.11] Let X be a Fano variety of index r with $-K_X = rH$. Then the collection of line bundles $\{\mathscr{O}_X(1-r)H, \dots, \mathscr{O}_X(-H), \mathscr{O}_X\}$ is an exceptional collection and we have a semi-orthogonal decomposition*

$$D^b(X) = \langle K(X), \mathscr{O}_X(1-r)H, \dots, \mathscr{O}_X(-H), \mathscr{O}_X \}.$$

remark is that these exceptional collections don't need to be strong in general even in the Fano case.

Example 1.1.3. *Let $Q^n \subset \mathbb{P}^{n+1}$ be a smooth quadric with n odd. Then there is an exceptional collection*

$$D^b(Q^n) = \langle D^b(k), \mathscr{O}_X(1-n), \dots, \mathscr{O}_x(-1), \mathscr{O}_X \rangle.$$

When n is even the first component is in turn equivalent to $D^b(k \times k)$.
The case of Grassmanians was also studied by Kapranov in [Kap88], Kuznetsov has done a deep study of exceptional collections for Fano 3-folds, a survey can be consulted on [Kuz16].
The following theorem will be explored in depth in more general contexts in the future, but for the moment let us express it in these terms.

Theorem 1.1.31. *[Bon89, Theorem 6.2] Let $E_1, \dots, E_n \in \mathscr{T}$ be a full exceptional collection on the derived category $\mathscr{T}$ of a smooth variety, then there exists an equivalence of triangulated categories*

$$\mathscr{T} \to \mathcal{D}(End(\oplus E_i) - mod)$$

Let us mention that the relevant functor here is $Hom(\ , \oplus E_i)$ composed with a an assignment of an injective resolution.
A perhaps important remark is that one way of justifying the notion that the study of these derived categories is noncommutative in nature is precisely this sort of equivalence, as we have that at the level of derived categories we are interested in categories of modules over a noncommutative dg-algebra.
This result can be thought as classical enough by interpreting this endomorphism algebra as corresponding to the path algebra of a quiver which we can construct from the exceptional collection. Concretely if $\{E_1, \dots, E_n\}$ is a full strong exceptional collection, we construct the quiver with n vertices and between vertices i and j we write n edges corresponding to the dimension of the vector space $Hom(E_i, E_j)$. Let us see with a concrete space.

Example 1.1.4. *In the case of $\mathbb{P}^1$ for example the collection $\{\mathcal{O}, \mathcal{O}(1)\}$ will gives us the Kronecker quiver which is known to produce an equivalence between the derived category of $\mathbb{P}^1$ and the category of modules over the path algebra of the quiver, or equivalently the derived category of quiver representations.*
Concretely we have an endomorphism algebra

$$A = \begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}.$$

This behavior is another witness of the strong relationship between the derived and the geometric world. To give another such example of the geometry influencing the properties of the derived category, let us mention the relationship between the birational geometry of a space, and in particular the MMP in birational geometry and properties about the decomposition of the derived category.
First let us define what it means for a category to be indecomposable:

Definition 1.1.32. *We say a triangulated category $\mathscr{T}$ is indecomposable if for any pair of full subcategories $\mathscr{T}_1, \mathscr{T}_2$ every object $X \in \mathscr{T}$ decomposes as $X_1 \oplus X_2$ with $X_i \in \mathscr{T}_i$ and if $Hom^m(\mathscr{T}_i, \mathscr{T}_j) = 0$ for $i \neq j$ and $n \in \mathbb{Z}$, then either $\mathscr{T}_i = 0$ or $\mathscr{T}_j = 0$.*

In ([Bri99]) Bridgeland proves the following result.

Theorem 1.1.33. *Let X be a scheme, then $D^b(X)$ is indecomposable if and only if X is connected.*

Proof. Suppose X is connected and we have subcategories $\mathscr{T}_1, \mathscr{T}_2 \subset D^b(X)$. Let Y be an integral subscheme of X then $\mathcal{O}_Y$ is indecomposable as an object of $D^b(X)$, and so $\mathcal{O}_Y \in \mathscr{T}_1$, for example, for

similar reasons the sheaves $\mathcal{O}_y$ are also in $\mathscr{T}_i$ for any point $y \in Y$.
We then have $X = X_1 \cup X_2$ for X_i the union of integral closed subschemes Y with $\mathcal{O}_Y$ in $\mathscr{T}_i$. We see that the intersection needs to be empty and thus as X is connected, one of X_1, X_2, say, X_2, must be empty and so for all points $x \in X$ we see that $\mathcal{O}_x \in \mathscr{T}_1$.
It follows by 1.1.10 that any object $\mathcal{F} \in \mathscr{T}_2$ must be zero.
For the converse see [Huy06, Proposition 3.10].

□

As a corollary of this theorem we have that Calabi-Yau varieties, which are minimal models under the MMP, cannot be decomposed as a semiorthogonal decomposition. Indeed, for a Calabi-Yau variety X of dimension n, it can be shown (see Example 1.1.1) that the Serre functor is given by shifting

$$\mathcal{F} \mapsto \mathcal{F}[n].$$

This means that if $D^b(X)$ had a semiorthogonal decomposition $< \mathscr{T}_1, \mathscr{T}_2 >$, we would have

$$Hom^n(\mathscr{T}_1, \mathscr{T}_2) = Hom(\mathscr{T}_2[n], \mathscr{T}_1) = Hom(\mathscr{T}_1, \mathscr{T}_2) = 0$$

which would mean $D^b(X)$ would be decomposable, which is not possible as X is connected.
Besides semiorthogonal decompositions induced by exceptional collections, the next two results due to Orlov ([Orl92]) are an important tool to produce such decompositions in a number of general cases. Theorem 1.1.35 in particular is another instance of the geometry having a direct role in the categorical properties of $D^b(X)$

Theorem 1.1.34. *If V is a vector bundle of rank r over a projective variety Y, and if there exists a full exceptional collection $\{E_1, \dots, E_n\}$ in the derived category $D^b(Y)$ then the derived category $D^b(\mathbb{P}(V)))$ also possesses a full exceptional collection given by $\{p^*E_0 \otimes \mathcal{O}_{\mathbb{P}(V)}(-r+1), \dots, p^*E_n \otimes \mathcal{O}_{\mathbb{P}(V)}(-r+1), \dots, p^*E_0, \dots, p^*E_n\}$. Where $p : \mathbb{P}(V) \to Y$ is the structure morphism.*

We can use this theorem to give a full exceptional collection of $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ by considering Beilinson's full exceptional collection for $\mathbb{P}^1$, we obtain:

$$\{\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(1,0), \mathcal{O}(1,1)\}.$$

Similarly it is possible to produce semi-orthogonal decompositions on the derived category of a blowup.

Theorem 1.1.35. *Let $X = Bl_Z(Y)$ be the blowup of a smooth scheme Y along a smooth subscheme Z of codimension m, if we put $j : E \to X$ be the inclusion of the exceptional divisor and $p : X \to Y$ the blowup morphism, $i : Z \to Y$ the inclusion of the subscheme and $\pi : E \to Z$, then there is a semi-orthogonal decomposition*

$$\langle Rj_*(\mathcal{O}_E(1-m) \otimes L\pi^*(D^b(Z))), \dots, Rj_*(\mathcal{O}_E(-1) \otimes L\pi^*(D^b(Z))), Lp^*(D^b(Y))\rangle.$$

The terms of the form $Rp_*(\mathcal{F} \otimes Lq^*(\_))$ turn out to be of central importance in this story of derived equivalences, indeed these have appeared since the foundational findings of Mukai about abelian varieties and thus bear the name of Fourier-Mukai transforms.

Definition 1.1.36. *Let X, Y be two varieties and consider the projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$, an object $E \in D^b(X \times Y)$ is called the kernel the Fourier-Mukai transform $\Phi^E_{X\to Y} := Rq_*(E \otimes Lp^*(\_))$.*

It should be clear that having such a description for a functor is an strict improvement over the abstract data that one can infer individually, hence the importance of the following nontrivial result by Orlov:

Theorem 1.1.37. *Let $\mathcal{F} : D^b(X) \to D^b(Y)$ be an exact full and faithful functor with right adjoint, then there exists an objet $E \in D^b(X \times Y)$ such that $F \simeq \Phi^E_{X\to Y}$.*

In particular autoequivalences are of this form, a fact that we will be using frequently. In fact we are interested very particularly in such cases.
Mukai first found in [Muk78] found that for an abelian variety A there exists an equivalence between the derived category of A and the derived category of its dual abelian variety $\hat{A}$ and this equivalence is given by a Fourier-Mukai transform whose kernel is the Poincaré bundle.
Two varieties with equivalent derived categories (as triangulated categories) are said to be Fourier-Mukai partners and they are crucial to our current work.
A natural question light of Mukai's transform for abelian varieties is whether every variety admits a non-isomorphic Fourier-Mukai partner. Bondal and Orlov proved a theorem in which it is possible to characterize by a geometric condition a class of varieties without such non-isomorphic partners. We explore briefly the steps of their original proof. We remark that their original proof showed that it is just the graded structure and not the triangulated one what is enough to fully characterize the variety in question, we would also like to mention that as we will present more than one proof of the theorem (with some flexibility in the hypothesis) we will only give a rough sketch of their original proof and would recommend the interested reader to refer to the source material [BO01] for a more detailed exposition.

1.1.2 The Bondal-Orlov reconstruction theorem

In this section we discuss the anticipated Bondal-Orlov reconstruction theorem, which establishes that in the presence of an (anti-)canonical bundle, the variety is entirely determined by its derived category. In fact it should be more precise to say that the variety depends only on the graded structure of the derived category as we will see, since the distinguished triangles dont play an essential role. As a consequence of the reconstruction theorem, we obtain a description of the group of autoequivalences of the derived category in terms of autoequivalences of the variety X itself.
This section follows closely [BO01], a very precise treatment can also be found on [Huy06]. The proofs of the results here will be only sketched without much detail but trying to maintain the essential ideas

behind the arguments.
In the previous section we discussed very briefly the concept of Serre functors and how they play an important role in the structure of the triangulated categories. Here we develop the theory further.
The immediate importance of the canonical or anticanonical bundle being ample is that they induce a Serre functor by duality on the derived category $D^b(V)$.

Lemma 1.1.38. *[BVdB03] If $\mathscr{T}$ is a triangulated category and $S : \mathscr{T} \to \mathscr{T}$ is a Serre functor then it is an exact functor.*

The main example of interest for us is the (anti-)canonical bundle

Example 1.1.5. *Let V be a variety of dimension n over a field k, and let ω_V be its canonical bundle, then the functor $S := \cdot \otimes \omega_V[n] : D^b(V) \to D^b(V)$ is a Serre functor. This is true since it commutes with any shift $[m]$ by the nature of the derived tensor product, and by Serre duality we have the desired isomorphisms $Hom(x, y) \cong Hom(y, x \otimes \omega_w[n])^*$.*

The following couple of lemmas show that the Serre functor, when it exists, is part of the triangulated category and is not an extra piece of data one needs to equip it with.

Lemma 1.1.39. *Let $\mathscr{T}$ be a triangulated category with Serre functor S, and let $\psi : \mathscr{T} \to \mathscr{T}$ be any autoequivalence, then $\psi \circ S \cong S \circ \psi$*

Proof. Combining ψ and the isomorphisms $\phi_{x,y}$ we get equivalences

$$\begin{aligned} Hom(\psi(x), \psi(S(y))) &\cong Hom(y, x)^* \\ &\cong Hom(\psi(x), S(\psi(y)) \end{aligned}$$

for any given $x, y \in \mathscr{T}$, hence ψ being essentially surjective gives us isomorphisms of representable functors $Hom(\cdot, \psi(S(y))) \cong Hom(\cdot, S(\psi(y)))$ which implies the required equivalence. □

Theorem 1.1.40. *Let $\mathscr{T}$ be a triangulated category and let S be a Serre functor in $\mathscr{T}$, then it is unique up to graded isomorphism.*

Proof. Suppose there exists another Serre functor S' then there are isomorphisms $Hom(x, x) \cong Hom(x, S(x))^* \cong Hom(S(x), S'(x))$, then the image of $Id_x \in Hom(x, x)$ in $Hom(S(x), S'(x))$ is a graded isomorphism $S \to S'$ that commutes with $\phi_{x,y}$. □

Now we can proceed to define the necessary terminology to sketch the proof of the reconstruction theorem. From now on we assume our triangulated categories all have a Serre functor S.

Definition 1.1.41. *An object $\mathfrak{p} \in \mathscr{T}$ is called a point object of codimension s if*

1. $S(\mathfrak{p}) \cong \mathfrak{p}[s]$
2. $Hom^i(\mathfrak{p}, \mathfrak{p}) = 0$ *for* $i < 0$
3. $Hom^0(\mathfrak{p}, \mathfrak{p}) = k(\mathfrak{p})$ *where* $k(\mathfrak{p})$ *is a field depending on* $\mathfrak{p}$.

Definition 1.1.42. *An object $\mathcal{L} \in \mathscr{T}$ is called an invertible object if for any point $\mathfrak{p} \in \mathscr{K}$ there is an integer s such that $Hom^s(\mathcal{L}, \mathfrak{p}) = k(\mathfrak{p})$, and $Hom^i(\mathcal{L}, \mathfrak{p}) = 0$ for any $i \neq s$.*

The axiomatization of these two kind of objects correspond to what is expected in the case our category $\mathscr{T}$ is the derived category of a variety V, when the (anti-)canonical bundle is ample.

Lemma 1.1.43. *[BO01, 2.2] Let V be a variety with ample (anti-)canonical bundle, then $\mathfrak{p} \in \mathcal{D}^b(V)$ is a point object if and only if it is isomorphic to the shift of a skyscraper sheaf over a closed point $p \in V$*

Proof. If p is a closed point of V, then it clearly satisfies the condition of a point object in $D^b(V)$. On the other hand if $\mathfrak{p}$ is a point object, then there is a i, with $H^i(\mathfrak{p}) \neq 0$. That $s = dimV$ follows from taking i-th cohomology H^i on $S(\mathfrak{p}) = \mathfrak{p} \otimes \omega_V[dimV] = H^{i+dimV}(\mathfrak{p})$. The (anti-)ampleness of ω_V implies that the support of $H(\mathfrak{p})$ must be zero dimensional, this implies there is a sequence

$$\mathfrak{p}[m] \to H^m(\mathfrak{p}) \to H^n(\mathfrak{p}) \to \mathfrak{p}[n]$$

with $m > n$, where $m = min\{i \mid H^i(\mathfrak{p}) \neq 0\}$ and $n = max\{i \mid H^i(\mathfrak{p}) \neq 0\}$, and then by the second condition $m = n$. The third condition then shows $\mathfrak{p} = \mathcal{O}_p[m]$. □

Lemma 1.1.44. *[BO01, 2.4] Let V be a smooth irreducible variety, and suppose all point objects $\mathfrak{p} \in \mathcal{D}^b(V)$ are of the form $\mathcal{O}_{V,p}[s]$ for some point $p \in V$ and some $s \in \mathbb{Z}$. Then $\mathcal{L} \in \mathcal{D}^b(V)$ is invertible if and only if it is isomorphic to a shift of an invertible sheaf.*

Proof. If we start with a translated line bundle $\mathcal{L}[r]$ then a simple computation shows that it is an invertible object under the definition above.
On the other hand if $\mathcal{L}$ is an invertible object, let $m = \max\{i \mid H^i(\mathcal{L}) \neq 0\}$, then from $\mathrm{Hom}(\mathcal{L}, k(x)[r]) = \mathrm{Hom}(H^m(\mathcal{L}), k(x)[r]) \neq 0$ with $x \in V$ a closed point and $r \in \mathbb{Z}$ we deduce $s = -m$.
The spectral sequence $E_2^{p,q} = Ext^p(H^{-q}(\mathcal{L}), k(x))$ converges to $Ext^{p+q}(\mathcal{L}, k(x))$ and the fact that $Ext^{1-m}(\mathcal{L}, k(x)) = 0$ imply that $Ext^1(H^m(\mathcal{L}), k(x)) = 0$, which in turn gives us that $H^m(\mathcal{L}, k(x))$ is locally free.
The fact that it has rank 1 follows from $Hom(\mathcal{L}, k(x)[-m]) \cong Hom(H^m(\mathcal{L}, k(x))$ which is k. Further analysis of the spectral sequence above shows that $\mathcal{L} \cong H^m(\mathcal{L})[-m]$, as for any $n < m$ we have $H^n(\mathcal{L}) = 0$. We refer to the proof of [BO01, 2.4] or [Huy06, Proposition 4.9] for the explicit details. □

Finally the expected reconstruction theorem

Theorem 1.1.45. *[BO01, 2.5] Let V be an irreducible smooth projective variety with ample (anti-)canonical bundle and if $\mathcal{D}(V) \simeq \mathcal{D}(V')$ for some other smooth algebraic variety V', then $V' \cong V$.*

Proof. We will very roughly sketch the steps to follow in the proof
Step 1: First we use Theorem 1.1.43 to establish an equivalence between the point objects in $\mathcal{D}(V)$ and the ones in $\mathcal{D}(V')$.

Step 2: Using Theorem 1.1.44 we see there is an equivalence between invertible objects in $\mathcal{D}(V)$ and $\mathcal{D}(V')$.

Step 3: By the previous step it is possible to pick and fix an invertible sheaf $\mathcal{L} \in \mathcal{D}(V)$ and realize it as one in $\mathcal{D}(V')$, and by the first step identify the points of V and V' by comparing both sets to the point objects $\mathfrak{p} \in \mathcal{D}(V)$ such that $Hom(\mathcal{L}, \mathfrak{p}) = k(\mathfrak{p})$.

Step 4: For a subset L of invertible objects realized at the same time in both categories, a morphism $\alpha : Hom(\mathcal{L}_1, \mathcal{L}_2)$ for $\mathcal{L}_i \in$ L, induces a morphism $\alpha^*_{\mathfrak{p}} : Hom(\mathcal{L}_2, \mathfrak{p}) \to Hom(\mathcal{L}_2, \mathfrak{p})$ for each point object $\mathfrak{p}$. This identifies the basis of the topology as the subsets of points $U_{\alpha,\mathcal{L}_1,\mathcal{L}_2}$ for which $\alpha^*_{\mathfrak{p}} \neq 0$.

Step 5: We notice that the codimension of the point objects imply the dimensions of V and V' coincide. Additionally using the (anti-)ampleness of ω_V we show that $\omega_{V'}$ is (anti-)ample too. Bondal and Orlov show this by appealing to [BGI77, Illusie Exposé II, Proposition 2.2.3] which characterizes ample families in this context using the basis of the previous point.
In [Huy06] however we can find a more geometric argument for this, which goes by showing that $\omega_{V'}$ separates points and tangent lines.
The graded algebra $A^\bullet := Hom(\mathcal{L}, S^\bullet(\mathcal{L}))$ is isomorphic to canonical algebras of both V and V'. We then can deduce the equivalence between V and V'.
Again we will refer to [BO01] and [Huy06] for the precise details of this proof. □

The following theorem describes in detail the autoequivalences of the derived category of a variety V as above. Additionally it will help us forward in identifying some of the 'rigidness' of a derived category.

Theorem 1.1.46. *[BO01, 3.1] Let V be a smooth irreducible projective variety with ample (anti-)canonical bundle, over a field k. Then the autoequivalences of $\mathcal{D}(V)$ are generated by automorphisms of V, twists by invertible sheaves and translations.*

Corollary 1.1.47. *There is an exact sequence*

$$0 \to PicV \oplus \mathbb{Z} \to Aut(\mathcal{D}(V)) \to Aut(V) \to 0 \tag{1.1.1}$$

And a decomposition $Aut(\mathcal{D}(V)) \cong Aut(V) \rtimes (Pic(V) \oplus \mathbb{Z})$ *as* $Pic(V) \oplus \mathbb{Z}$ *is normal in* $Aut(\mathcal{D}(V))$.

Let us just mention that the proof of this follows from a careful analysis of what the autoequivalences of $D^b(V)$ with their isomorphisms. In particular it can be shown that we can suppose any autoequivalence $F : D^b(V) \to D^b(V)$ fixes $\mathcal{O}_V$ and thus ω_V^k for any k, as tensoring by invertible sheaves and shifting is an autoequivalence. It can be shown that it's possible to pick an isomorphic autoequivalence that acts trivially on the canonical algebra, and with this in hand all that remains is to show that we have an isomorphism of functors $id\,|_{\langle\omega^k\rangle} \cong F\,|_{\langle\omega^k\rangle}$ and apply Theorem 1.1.19.
We have already touched a little bit on the interaction of derived and birational equivalences in the Blowup decomposition formula (Theorem 1.1.35) but as it turns out the relationship runs deeper and we are able to say more.
We will touch briefly on this by proving a theorem of Kawamata which generalizes Bondal and Orlov's reconstruction. It is important to mention too a result of Bridgeland and Maciocia as well as touch briefly on flops and some known results about the converse implication, that is, wether birational equivalence, or the stronger condition of K-equivalence, implies equivalences at the derived level.
We recall a standard definition of big bundles and then present a classical and very helpful characterization of such bundles by Kodaira.

Definition 1.1.48. *A line bundle* L *on a variety* X *is big if there exists* r *such that* X *is birationally equivalent to its image in* $\mathbb{P}(H^0(X, L^r))$ *under the map given by the sections of* L^r.

Theorem 1.1.49. *If* X *is a projective irreducible variety and* ω *is a big line bundle on* X*, then there exists an ample bundle* A *and effective bundle* E*, such that* ω *is rationally equivalent to* $A + E$.

We also will need the concept of nef-ness of a line bundle, and we would also like to recall the notion of numerical Kodaira dimension.

Definition 1.1.50. *Let* $\mathscr{L}$ *be a line bundle on* X *with associated divisor* D *we say it is nef if for every irreducible curve* $C \subseteq X$ *we have that* $D \cdot C \geqslant 0$.

And,

Definition 1.1.51. *The numerical Kodaira dimension* $\nu(X, \mathscr{L})$ *of a line bundle* $\mathscr{L}$ *on* X *is defined as the maximal integer m such that there exists a proper morphism* $\phi : W \dashrightarrow X$ *with* W *of dimension* m *such that* $([\phi^*\mathscr{L}]^m \cdot W) = 0$.

Using Theorems 1.1.49 and 1.1.37 Kawamata showed that

Theorem 1.1.52. *Let* X, Y *be smooth projective varieties with* $\mathcal{F} : D^b(X) \cong D^b(Y)$ *as triangulated categories, then*

1. *dim X = dim Y.*
2. *If the canonical divisor K_X is nef, so is K_Y and there is an equality in the numerical Kodaira dimensions $\nu(X)$ and $\nu(Y)$.*
3. *If X is of general type, then X and Y are birational and furthermore, there is a smooth projective variety $p : Z \to X$, $q : Z \to Y$ such that $p^*K_X \simeq q^*K_Y$.*

Proof. From theorem 1.1.37 we know that $\mathcal{F} \simeq \Phi^E$ for some object $E \in D^b(X \times Y)$. It can be shown that the right and left adjoints of this Fourier-Mukai functor are of the form $\Phi^{E^\vee \otimes \pi_1^* \omega_X[dimX]}$, and $\Phi^{E^\vee \otimes \pi_2^* \omega_Y[dimY]}$, respectively, since this is an equivalence this means that the representing kernels are isomorphic and so $dimX = dimY$.
Now consider the support of $E^\vee$, $Z = \bigcup Supp(H^i(E^\vee)$ and consider its irreducible component decomposition $Z = Z_1 \cup \cdots \cup Z_k$.
Let $v_j : \tilde{Z}_j \to Z_j$ be the normalization morphism and assume Z_j is an irreducible component of a single $H^i(E^\vee)$. We can then deduce from the fact that $\tilde{Z}_j$ that $v_j^* \pi_1^* \omega_X^r \cong v_j^* \pi_2^* \omega_Y^r$ for some integer r (See [Huy06, Lemma 6.9] for details).
On the other hand, if on an irreducible component Z_j dominating Y via the projection π_2 K_X is nef then so is $rv_j^* \pi_1^* K_X$ and so is $rv_j^* \pi_2^* K_Y$ which in turn implies K_Y is nef.
Now for 3) we use theorem 1.1.49 and see that $K_X \simeq A + B$ with A ample and B effective, again let us pick an irreducible component Z_1 of Z that dominates X, the claim is that $\pi_2 \mid_{Z_1} : Z_1 \to Y$ is quasifinite outside of $SuppB$. It's possible to see this if we consider a curve C on $\hat{Z}_j \cap \pi_2^{-1}(y)$ of a point $y \in Y$, as π_2 contracts such curve then the degree of intersection with $\pi_2^* \omega_Y$ must be 0. If we suppose that C is not contained entirely on $SuppB$ then that the degree of intersection with $\pi_1^* K_X$ is larger than that of the intersection with the ample divisor A, but by ampleness this is strictly greater than zero and we have arrived to a contradiction, the fiber outside of $SuppB$ is finite.
We deduce that $dimZ_1 = dimX$, from here we can deduce that the fibers of a point over $x \in X$ must be connected. This is because we can write the fiber as a union of non-empty disjoint subsets $Y_1 \cup Y_2$ which must be equal to the support of $\mathcal{F}(k(x))$, and then it's possible to write this as a direct sum $\mathcal{F}_1 \oplus \mathcal{F}_2$, each with respective support on Y_i, however $k = End(\mathcal{F}(k(x))) = End(k(x))$ which can't be $End(\mathcal{F}_1 \oplus \mathcal{F}_2)$. This implies $\mathcal{Z}$ is the graph of a birational morphism. □

The condition on the last part of the theorem is of interest,

Definition 1.1.53. *We say two varieties X, Y are K-equivalent if they are birational and there exists a smooth variety $p : Z \to X$, $q : Z \to Y$ with $p^*K_X \simeq q^*K_Y$*

Let us notice that theorem 1.1.52 generalizes Bondal and Orlov's theorem as K-equivalent varieties for which one of them has an ample canonical bundle implies the varieties are isomorphic.
Let us note too the following result by Bridgeland and Maciocia:

Theorem 1.1.54. *Let X be a minimal surface of general type. Then the only FM partner of X is X itself.*

An important conjecture is whether being derived equivalent is the same as being K-equivalent. We have given some conditions for derived equivalence to imply K-equivalence, on the other direction general results are more scarce, but we have an interesting concrete example (taken from [Kaw02, Example 5.2]).

Example 1.1.6. *Let X be a smooth projective variety of dimension 2m+1, Z a subvariety isomorphic to $\mathbb{P}^m$ and suppose the normal bundle $\mathcal{N}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^m}(1)^{m+1}$. Let $f : Bl_Z X \to X$ be the blowup along Z. Then exceptional divisor is isomorphic to $\mathbb{P}^m \times \mathbb{P}^m$ and we can blow-down towards X by contracting one of the copies of $\mathbb{P}^m$ but we could also do it on the other copy, so we have a morphism $g : Bl_Z X \to Y$. The composition $h : g \circ f^{-1}$ is called the standard flop.*

Using a form of Theorem 1.1.19 described in [BKR01] it is possible to show that there exist a derived equivalence given as a Fourier-Mukai functor with kernel $\mathcal{O}_Z$.
Before finishing let us give a brief summary of some of the known examples of Fourier-Mukai partners. For abelian varieties Orlov showed the following:

Theorem 1.1.55. *[Orl02] Let A, B be abelian varieties over k. Then there is an equivalence of triangulated categories between the derived categories $D^b(A)$ and $D^b(B)$ if and only if there is an isometric isomorphism between $A \times \hat{A}$ and $B \times \hat{B}$.*

For $K3$ surfaces we have

Theorem 1.1.56. *[Orl97] Let S_1 and S_2 be smooth projective $K3$ surfaces over $\mathbb{C}$. Then the derived categories $D^b(S_1)$, $D^b(S_2)$ are equivalent if and only if there exists a Hodge isometry $f : \tilde{H}(S_1, \mathbb{Z}) \to \tilde{H}(S_2, \mathbb{Z})$ between the Mukai lattices of S_1 and S_2.*

The case of polarised $K3$ surfaces is treated in [HP13], extensions of Bondal-Orlov reconstruction to relative, twisted and singular cases have been studied in [Cal18], and elliptic minimal surfaces in [Ueh04] to name a few.
Along the interest for giving conditions for such derived equivalences to exist, one immediate question is that of the cardinality of non-isomorphic Fourier-Mukai partners. Kawamata conjectured that the number must be finite based on the relationship with birational geometry, however in [Les15] it was shown that it is possible to blow-up $\mathbb{P}^3$ in countably infinite many configurations of 8 points producing then a countably infinite number of non-isomorphic spaces with equivalent derived category.
This turns out to be an upper bound as indeed it was shown in [AT08] that there can only be countably many non-isomorphic Fourier-Mukai partners for a given space.
We finish this chapter by reviewing the theory of t-structures on a derived category.

1.1.3 t-structures on derived categories

Our interest in t-structures in this work is to put in precise terms what is the relationship between a given derived category and the abelian category of coherent sheaves of all its non-isomorphic Fourier-Mukai partners.
Let us start with some basic definitions.

Definition 1.1.57. *Let $\mathscr{T}$ be a triangulated category, a t-structure on $\mathscr{T}$ is the data of two full triangulated subcategories $(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant 0})$ satisfying:*

1. $Hom(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant 0}[-1]) = 0$
2. $\mathscr{T}^{\leqslant 0}[1] \subseteq \mathscr{T}^{\leqslant 0}$ *and* $\mathscr{T}^{\geqslant 0}[-1] \subseteq \mathscr{T}^{\geqslant 0}$
3. *Any object $X \in \mathscr{T}$ sits in a distinguished triangle $X' \to X \to X'' \to X'[1]$ where $X' \in \mathscr{T}^{\leqslant 0}$, $X'' \in \mathscr{T}^{\geqslant 0}[1]$*

As we mentioned above, the idea we can keep in mind is the situation in which $\mathscr{T} = D^b(X)$ and then $\mathscr{T}^{\leqslant 0}$ is the full triangulated subcategory $\{X \in \mathscr{T} \mid H^i(X) = 0 \ \forall i > 0\}$ and $\mathscr{T}^{\geqslant 0} = \{F \in \mathscr{T} \mid H^i(F) = 0 \ \forall i < 0\}$. We are thus picking the two subcategories given by complexes with cohomology on negative degrees, and complexes with cohomology on positive degrees respectively.
The main examples to keep in mind for this note are the structures coming from the situation in which we have two smooth projective varieties and a triangulated equivalence $D^b(X) \cong D^b(Y)$, for a fixed variety X each of these derived equivalent varieties Y determine a t-structure on $D^b(X)$ given by the so-called standard structures we mentioned above, that is, we consider the structures $\mathscr{T}_Y^{\leqslant 0} := \{\mathcal{F} \in Coh(Y) \mid H^i(Y) = 0 \ \forall i \leqslant 0\}$.
We will denote $\mathscr{T}^{\geqslant 0}[n]$, respectively $\mathscr{T}^{\leqslant 0}[-n]$, as $\mathscr{T}^{\geqslant n}$, resp $\mathscr{T}^{\leqslant n}$.

Definition 1.1.58. *We say that a t-structure on a triangulated category $\mathscr{T}$, $(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant})$ is bounded if $\mathscr{T} = \bigcup_{i,j} \mathscr{T}^{\leqslant i} \cap \mathscr{T}^{\geqslant j}$, and it is non-degenerate if $\bigcap \mathscr{T}^{\leqslant i} = \mathscr{T}^{\geqslant i} = 0$. One could check easily that the property of being bounded implies nondegeneracy as any object $X \in \cap \mathscr{T}^{\leqslant i}$ must be in some $\mathscr{T}^{\leqslant n} \cap \mathscr{T}^{\geqslant n+1}$ but these are orthogonal and so X should be 0.*

The heart of a t-structure on $\mathscr{T}$ is just the intersection $\mathscr{T}^{\heartsuit} = \mathscr{T}^{\leqslant 0} \cap \mathscr{T}^{\geqslant 0}$. We have the following useful characterization of hearts of t-structures due to Bridgeland [Bri07].

Theorem 1.1.59. *Let $\mathcal{A} \subseteq \mathscr{T}$ be a full additive subcategory of a triangulated category. then $\mathcal{A}$ is the heart of a bounded t-structure if and only if :*

1. *For all $k_1 > k_2$, $Hom(A[k_1], B[k_2]) = 0$ for any $A, B \in \mathcal{A}$*

2. *For any $E \in \mathscr{T}$ different from 0, there is a filtration*

$$0 = E_0 \to E_1 \to E_2 \to \cdots \to E_n = E$$

such that $Cone(E_i \to E_{i+1}) \in \mathcal{A}[k_{i+1}]$ for integers $k_1 > \cdots > k_n$

Proof. Let $\mathscr{T}^{\leqslant 0}$ be the category generated by $\bigcup_{i \geqslant 0} \mathcal{A}[-i]$, so $\mathscr{T}^{\geqslant 0}$ is $(\mathscr{T}^{\leqslant 0})^{\perp}[-1]$. Clearly $\mathcal{A} = \mathscr{T}^{\leqslant 0} \cap \mathscr{T}^{\geqslant 0}$.
On the other hand, by the second condition, for any object $A \in \mathcal{A}$ there is a nonzero map to an object $A_n \in A[k_n]$ for some integer $k_n \geqslant 0$, and then the distinguished triangle $A' \to A \to A_n$ must have $A' \in A[k_m]$ where $k_m \leqslant 0$, by the first condition.

□

An important property of t-structures is that we have truncation functors, concretely we have

Lemma 1.1.60. *Let $\mathscr{T}$ be a triangulated category with a bounded t-structure $(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant 0})$, then for any integer n the inclusion functors $\iota_n : \mathscr{T}^{\leqslant n} \to \mathscr{T}$ and $\iota^n : \mathscr{T}^{\geqslant 0} \to \mathscr{T}$ have right and left adjoints τ_n, τ^n respectively.*

Proof. If we pick $X \in \mathscr{T}$ then by definition of t-structure there is a unique distinguished triangle $X' \to X \to X'' \to X'[1]$ with $X' \in \mathscr{T}^{\leqslant 0}$ and $\mathscr{T}^{\geqslant 1}$, let $Y \in \mathscr{T}^{\leqslant 0}$, then we have a sequence

$$Hom(Y, X''[-1]) \to Hom(Y, X') \to Hom(Y, X) \to Hom(Y, X'')$$

which becomes

$$0 \to Hom(Y, X') \to Hom(Y, X) \to 0$$

as $Y \in \mathscr{T}^{\leqslant 0}$ and X is any object in $\mathscr{T}$, we see that we can pick $\tau_n(X) = X'$. The description of τ^n is analogous. □

Given a t-structure on a triangulated category $\mathscr{T}$ we can talk about cohomology functors on $\mathscr{T}$ which take values on the heart $\mathcal{A}$ of $\mathscr{T}$ with said t-structure. If $X \in \mathscr{T}$ we set $H^0(X) = \tau_0\tau^0 X$ and $H^i(X) = H^0(X[i])$.
The important fact about the heart of a t-structure is that this is always an abelian category, the proof can be consulted on [GM94], what we must show is that this category has kernels and cokernels, we claim without proof that if $X \to Y \in \mathscr{T}$ the kernel is given by $Ker = \tau_{-1}(Cone(X \to Y))$ and the cokernel is given by $Coker = \tau^0(Cone(X \to Y))$, with morphisms given by the compositions $Ker \to Cone(X \to Y) \to X[1]$ and $Y \to Cone(X \to Y) \to Coker$. And it can be showed that any morphism in $\mathcal{A}$ factors as one would want in an abelian category.
As $\mathcal{A}$ is abelian, we can wonder if it is true that $D^b(\mathcal{A}) \cong \mathscr{T}$, however constructing a functor relating the two categories is an obstruction that we will have to deal with later, but even given the existence of such a functor we are not guaranteed to have an equivalence.
There is however a good criterion to see whether this happens

Definition 1.1.61. *A triangulated functor $F : \mathscr{T} \to \mathscr{T}'$ between triangulated categories with t-structures is said to be a left t-exact if $F(\mathscr{T}^{\geqslant 0}) \subset \mathscr{T}'^{\geqslant 0}$, we have a similar notion of right t-exact functor.*

Remark 1.1.62. *There is a graded algebra given by the Ext functors that we can define on the heart of any t-structure, we simply set $Ext^i(X,Y) = Hom(X, Y[i])$ and consider the composition between i-th and j-th graded parts $Ext^i(X,Y) \times Ext^j(Y,Z) \to Ext^{i+j}(X,Z)$ by composition of morphisms in $\mathscr{T}$. We denote this algebra as $Ext^*(X,Y)$.*

Theorem 1.1.63. *Let $\mathscr{T}$ a triangulated category with a bounded t-structure $(\mathscr{T}^{\leqslant 0}, \mathscr{T}^{\geqslant 0})$ with a heart $\mathcal{A} = \mathscr{T}^{\leqslant 0} \cap \mathscr{T}^{\geqslant 0}$. Suppose there is a t-functor $F : D^b(\mathcal{A}) \to \mathscr{T}$, then F is an equivalence if and only if for any two objects $X, Y \in \mathscr{T}$, $Ext^*(X,Y)$ is generated by a sequence of elements α_j in $Ext^j(X_j, X_{j+1})$ for $X_j \in \mathscr{T}$ and $X_1 = X$, $X_{i+1} = Y$.*

Proof. [BBDG18] □

One way to produce such equivalent derived categories by means of constructing derived categories of the heart of a t-structure on $D^b(X)$ is by tilting. The process consists on taking a torsion pair of certain type and then constructing a new t-structure with cohomology objects taking values in the torsion or torsion free class.

Definition 1.1.64. *Let $\mathcal{A}$ be an abelian category, a torsion pair $(\mathcal{T}, \mathcal{F})$ is a pair of full subcategories of $\mathcal{A}$ such that $\mathscr{T} \subseteq^{\perp} \mathcal{F}$ and such that for any $X \in \mathcal{A}$ there is a short exact sequence*

$$0 \to t(X) \to X \to X/t(X) \to 0$$

with $t(X) \in \mathcal{T}$ and $X/t(X) \in \mathcal{F}$.
We say that $(\mathcal{T}, \mathcal{F})$ is cotilting if $\mathcal{F}$ is a generator, meaning that for any $X \in \mathcal{A}$ there is an epimorphism $Y \to X$ with $Y \in \mathcal{F}$.

Given a t-structure and a torsion pair on its heart $\mathcal{A}$, we can get a t-structure on $D^b(\mathcal{A})$ by defining

$$D^b(\mathcal{A})^{\leqslant 0} := \{X \in D^b(\mathcal{A}) \mid H^i(X) = 0 \ \forall i < 0\,, H^0(X) \in \mathcal{T}\}$$

Which has as heart the objects of $D^b(\mathcal{A})$ with cohomology $H^1(X) \in \mathcal{F}$, $H^0(X) \in \mathcal{T}$ and 0 otherwise. The importance of these t-structures comes from a result due to Happel,Reiten and Smalo ([HRS96])

Theorem 1.1.65. *Let $\mathcal{A}$ be an abelian category and let us consider a t-structure on $D^b(\mathcal{A})$ with heart $\mathcal{B}$ obtained by tilting by a cotilting torsion pair $(\mathcal{T}, \mathcal{F})$ as above.*
If $\mathcal{B}$ has enough projective objects, then there exists an equivalence of derived categories $D^b(\mathcal{A}) \to D^b(\mathcal{B})$

Chapter 2

Tensor triangulated categories

After having studied the general properties of derived categories and having gone through some known results about the reconstruction theorems from a derived category, we now dedicate this chapter to the study of tensor triangulated categories.
As a first example, given a space X, the abelian category of coherent sheaves carries a tensor product which can in turn be derived and thus equip the derived category $D^b(X)$ with a derived tensor product $\otimes^{\mathbb{L}}_X$.
As the tensor product is in fact a derived functor it is compatible, in a way we will explain shortly, with the triangulated structure of the derived category.
In light of Bondal and Orlov's reconstruction and the existence of non-isomorphic Fourier-Mukai partners, one natural question to ask is whether the derived category could when equipped with some extra structure, be capable of reconstructing the full space.
This was shown to be true by Balmer in [Bal02, Bal05] where it was shown that in the case of nice enough spaces this derived tensor product is enough to produce back the space.
However this then implies that if for example a variety has more than one non-isomorphic Fourier-Mukai partner then the derived category admits many different derived tensor structures. As the Balmer construction works for general triangulated categories with compatible tensor structures, one of the main motivating questions of this thesis is then the study of the space of all possible tensor triangulated structures on a fixed derived category.
This chapter is then dedicated to the study of tensor triangulated structures and Balmer's reconstruction theorem. We will go over some known examples, limitations and general theory.

We start with some general definitions:

Definition 2.0.1. *A tensor triangulated category (TTC for short) $\mathcal{T}$ is a triangulated category together with the following data:*

1. *A closed symmetric monoidal structure given by a functor $\otimes : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ additive and exact (with respect to the k-linear structure) on both entries.*
2. *The internal hom functor $\underline{hom} : \mathscr{T} \times \mathscr{T} \to \mathscr{T}$ sends triangles to triangles (up to a sign).*
3. *Coherent natural isomorphisms for each n and m, $r : x \otimes (y[n]) \cong (x \otimes y)[n]$ and $l : (x[n]) \otimes y \cong (x \otimes y)[n]$ giving rise to the commutativity of the following diagrams:*

$$\begin{array}{ccc} x[n] & \longleftarrow & x[n] \otimes 1 \\ & \nwarrow & \downarrow{\scriptstyle l} \\ & & (x \otimes 1)[n] \end{array} \qquad \begin{array}{ccc} x[n] & \longleftarrow & 1 \otimes x[n] \\ & \nwarrow & \downarrow{\scriptstyle r} \\ & & (1 \otimes x)[n] \end{array}$$

Where 1 is the unit of the monoidal structure, and the horizontal and diagonal morphisms are given by the unit multiplication morphisms $\lambda : x \otimes 1 \to x$, $\rho : x \otimes 1 \to x$ respectively in each diagram.
We require also the following compatibility diagram between r and l :

$$\begin{array}{ccc} x[n] \otimes y[m] & \xrightarrow{r^{-1}l^{-1}} & (x \otimes y)[n+m] \\ \downarrow & & \downarrow{\scriptstyle (-1)^{n+m}} \\ y[m] \otimes x[n] & \longrightarrow & (x \otimes y)[n+m] \end{array}$$

And finally compatibility with the associative morphism:

$$\begin{array}{ccccc} (x \otimes y)[n] \otimes z & \longrightarrow & ((x \otimes y) \otimes z)[n] & & \\ \downarrow & & & \searrow & \\ & & & & (x \otimes (y \otimes z))[n] \\ & & & \swarrow & \\ (x[n] \otimes y) \otimes z & \longrightarrow & x[n] \otimes (y \otimes z) & & \end{array}$$

We will refer to a TTC by the triple that defines it $(\mathscr{K}, \otimes, 1)$ or simply by $\mathscr{K}$ interchangeably when there is no risk of confusion.

Remark 2.0.2. *It is important to mention that the precise definition of a tensor triangulated category from author to author might slightly change depending on the intended use. Morally what we are interested in is in a triangulated category with a monoidal structure such that the monoidal product $\otimes$ is exact in each variable.*
Some results and examples however will depend on some extra hypothesis and some conventions might change, for example among the coherence conditions showed above.

As usual structure preserving functors are of great importance:

Definition 2.0.3. *An exact monoidal functor $F : \mathscr{T} \to \mathscr{T}'$ between TTC's $(\mathscr{T}, \otimes_{\mathscr{T}}, 1_{\mathscr{T}})$, $(\mathscr{T}', \otimes_{\mathscr{T}'}, 1_{\mathscr{T}'})$ will be called a $\otimes$-triangulated functor*

With the above defined functors its possible to consider then the category $\mathbb{TTC}$ of essentially small tensor triangulated categories.
An idea to keep in mind is that tensor triangulated categories are categorified algebra objects in a category of triangulated categories. With this analogy then one can wonder if one can develop some basic commutative algebra theory for TTC's. For example:

Definition 2.0.4. *Let $(\mathscr{T}, \otimes_{\mathscr{T}}, 1_{\mathscr{T}})$ be a TTC. We will say that a thick subcategory $\mathscr{I} \subset \mathscr{K}$ is $\otimes -$ ideal if for every $x \in \mathscr{T}$ we have $x \otimes \mathscr{I} \subset \mathscr{I}$*

As done in the previous chapter with derived categories one would like to relate the geometry of the space with categorical properties, in this case with properties corresponding to the monoidal structure.
In [Tho97] Thomason, went on to study some relationship between the topology of a variety and its K_0 group.
Before describing Balmer's reconstruction, we will follow [Rou05] and give a sketch of a proof of Bondal-Orlov's result using the derived tensor structure.

Definition 2.0.5. *Let X be a variety, then the 0th K-group of $Perf(X)$ is the group generated by isomorphism classes of objects $M, N.P \in Perf(X)$ modulo the equivalence $M = N + P$ whenever there is a distinguished triangle*

$$N \to M \to P \to N[1]$$

It is denoted by $K_0(Perf(X))$

The immediate goal is to show the following result which already relates dense subcategories with subgroups of $K_0(\mathscr{T})$.

Lemma 2.0.6. *An object $D \in \mathscr{T}$ is in a dense triangulated subcategory $\mathcal{A}$ if and only if it is 0 in the quotient group $K_0(\mathscr{T})/Im(K_0(\mathcal{A}))$*

Proof. The proof is straight-forward, what needs to be shown explicitly is only that if D is 0 in $K_0(\mathscr{T})/Im(K_0(\mathcal{A}))$ then there are objects $B, B' \in \mathcal{A}$ such that $D \oplus B \cong 0 \oplus B'$ and so $D \oplus B$ is in $\mathcal{A}$, as such there is a triangle $B \to B \oplus D \to B \to D[1]$ in $\mathcal{A}$ which is triangulated, and then $D \in \mathscr{T}$. Indeed the lemma follows as the quotient of isomorphism classes of objects of $\mathscr{T}$ modulo the relation $A \sim B$ iff $A \oplus D \cong B \oplus D$ with $D \in \mathcal{D}$ turn out to be isomorphic to the quotient group $K_0(\mathscr{T})/Im(K_0(\mathcal{A}))$. □

Theorem 2.0.7. *There exist a bijection between full dense triangulated subcategories of $\mathscr{T}$ and subgroups of the 0th K-group $K_0(\mathscr{T})$. The correspondence is given by associating to a full dense triangulated subcategory its own K-group and realize it as a subgroup via the inclusion. The inverse correspondence associates a subgroup H to the full subcategory $\mathcal{A}_H$ of objects whose isomorphism class is in H.*

Proof. The proof of this relies on Lemma 2.0.6.
One checks immediately that indeed the image is a subgroup of $K_0(\mathscr{T})$ and that the category $\mathcal{A}_H$ is full triangulated dense subcategory of $\mathscr{T}$.
One uses the Lemma 2.0.6 to show that the assignment is inverse of one another. □

If $\mathscr{I} \subset \mathscr{T}$ is a full subcategory of a triangulated category, let us denote by $\hat{\mathscr{I}}$ the smallest subcategory of $\mathscr{T}$ closed under infinite direct sums containing $\mathscr{I}$.
Let us recall that given an object $\mathscr{F} \in D^b(X)$ for some space X, we can define the cohomological support of $\mathscr{F}$ as $\bigcup supp(H^i(\mathscr{F}))$ which is a subset of X. If $Z \subset X$ we denote by $D_Z(QCoh(X))$ the full subcategory of $D(QCoh(X))$ of complexes with cohomology sheaves supported in Z. Similarly $Perf(X)_Z$ is the subcategory of perfect complexes in $D_Z(QCoh(X))$.
We close this subsection by proving the following result, and then using it to reprove the Bondal-Orlov reconstruction using the language of $\otimes$-ideals. Mentally the original proofs of Bondal and Orlov and the proof of Kawamata are very algebro-geometric in nature and they come from deep understanding of the particularities of the geometry that results in the hypothesis of the theorem. The proof presented here however relies heavily on the role of Thomason's classification and paints the derived category more as an affine object from our point of view.

Theorem 2.0.8. *[Rou05, Theorem 3.6] Let X be a variety, then the assignment*

$$Z \to Perf(X)_Z$$

from closed subsets of X to the set of $\otimes$-ideals generated by a single object, is a bijection.

The fact that an object $\mathscr{F} \in Perf(X)$ supported in Z generates the whole ideal $Perf(X)_Z$ is proved in [Rou05, Lemma 3.8] and follows from the following results by using Neeman's Brown representability Theorem (1.1.13).

Lemma 2.0.9. *If $Z \subset X$ is a closed subset then $\widehat{Perf(X)}_Z = D_Z(QCoh(X))$*

Proof. By Lemma 1.1.20 it is enough to show that $Perf(X)_Z^{\perp} = 0$. □

The following is a key result describing the decomposition of the category by ideals through subideals and quotients.

Theorem 2.0.10. *Let $Z', Z \subset X$ closed subsets. Let $U = X \backslash Z$ and $j : U \to X$ the inclusion. Then the functor j^* induces a full and faithful functor $Perf(X)_Z / Perf(X)_{Z \cap Z'} \to Perf(X)_{U \cap Z'}$. An object in $Perf(X)_{U \cap Z'}$ is in the image if and only if its class in $K_0(Perf(U)_{U \cap Z'})$ is the image of a class in $K_0(Perf(X)_{Z'})$*

Lemma 2.0.11. *If $Z \subset X$ is closed, then there exists a perfect complex M with support on Z.*

Proof. On an open affine $Spec(A)$ this can be seen by considering the perfect complex $0 \to A^n \to A^n \to 0$ where the map corresponds to the defining equations of Z, assuming Z is irreducible.
Globally if we have an open affine U, by the previous point there exists a perfect complex M with support on $U \cap Z$ such that $M \oplus M[1]$ is supported on Z.
If Z is no longer irreducible, the direct sum of the perfect complexes with support in each irreducible component has support on the whole Z. □

Proposition 2.0.12. *[Tho97, Proposition 3.11] Let X be a quasi-compact quasi-separated scheme and let $\{\omega_i\}$ be an ample sequence. Then a thick triangulated subcategory is an $\otimes-$ideal if and only if $\mathscr{I} \otimes \omega_i$ for any $i << 0$.*

Proof. Follows from Lemma 1.1.20. Indeed the sequence $\{\omega_X^i\}$ forms an ample sequence and thus a spanning class in the derived category. □

The original proof of Thomason relies on a deep and non trivial analysis of this derived category. The proof using Neeman's Brown representability seems to save some steps and this is why we have chosen to present it here instead of the original proof.
Rouquier goes to use theorem 2.0.8 to give an alternative proof of Bondal and Orlov reconstruction just from understanding the $\otimes_X^{\mathbb{L}}$-ideals in $Perf(X)$ for some Fano variety. We echo Rouquier's remark that the result here does depend on the triangulated structure in contrast of the original proof which utilizes only the graded structure.

Theorem 2.0.13. *Let X, Y be smooth projective varieties such that X has ample (anti-)canonical bundle and there is an equivalence $D^b(X) \simeq D^b(Y)$ then $X \cong Y$.*

Proof. We refer to [Rou05, Theorem 3.17] for the full details of the proof.
From Theorem 2.0.8 we can deduce that there is a bijection Φ between the closed subsets of Y and X by noticing that, if $F : D^b(X) \to D^b(Y)$ is the given equivalence, then $F^{-1}(D^b_Z(Y))$ is a subcategory of $D^b(X)$ stable by the Serre functor and thus an ideal and so of the form $D^b_{\Phi(Z)}(X)$.
The assignment $\phi : Y \to X$, defined by $\phi(y) = \Phi(\hat{y})$ sends closed points to closed points, since closed points correspond to minimal thick subcategories $Perf(Y)_y$, property that is preserved under the equivalence, so $Perf(X)_{\phi(y)}$ should be minimal and $\phi(y)$ automatically closed.
Then ϕ must be bijective since the set of all skyscraper sheaves form a spanning class (Lemma

1.1.10). So if there were a point $x \in X$ not in the image of ϕ, $Hom(\mathcal{O}_x, O_{\phi(y)}[i]) = 0$, however $Hom(F(O_x), \mathcal{O}_y[i]) = 0$ for any i. Thus $F(\mathcal{O}_x) = 0$ which is not possible.
We can see, finally, that ϕ is continuous as a point $y \in Y$ is in a closed Z if and only if $Perf(Y)_y \subset Perf(Y)_Z$ if and only if $Perf(X)_{\phi^{-1}(y)} \subset Perf(X)_{\Phi(Z)}$.
This says that there is an homeomorphism between the underlying topological spaces of X and Y, we recover the structure sheaves through the derived equivalence as there are isomorphisms $\Gamma(U) \cong Z(Perf(U))_{red} \cong Z(Perf(\phi(U))) \cong \Gamma(\phi(U))$ for any open $U \subset X$. □

2.0.1 Balmer spectrum and reconstruction of schemes

As promised at the beginning of this chapter we will now dedicate this section to Balmer's reconstruction theorem and general construction of a locally ringed space from a given tensor triangulated category.
A thing to note is that this result goes well beyond algebraic geometry and the general theory of this spectrum has found a place in both homotopy theory and representation theory. Indeed the general theory of classifying tensor ideals on a given tensor triangulated category has turned out to be of great importance as they correspond to certain thick (and localizing subcategories) which turn out to be of great general interest.
Among those applications is for example the category of finite spectra with smash product is of great interest as is the general classification problem for related categories in the so-called telescope conjecture.
This latter result a witness of the deep connections to stable homotopy theory through the work of Thomason and the well known telescope conjecture.
Let us start as usual with a number of definitions.

Definition 2.0.14. *Let $\mathscr{I} \subset \mathscr{T}$ be a subset of objects in a tensor triangulated category. Then $\langle \mathscr{I} \rangle_{\otimes}$ denotes the the $\otimes$-ideal generated by $\mathscr{I}$.*

Mimicking the commutative algebra definition we are interested in localizations and quotients by tensor triangulated ideals. Of great interest are prime $\otimes$-ideals.

Definition 2.0.15. *Let $\mathscr{I}$ be a $\otimes-$ ideal, we will say that it is prime if $x \otimes y \in \mathscr{I}$ implies either $x \in \mathscr{I}$ or $y \in \mathscr{I}$.*

The following implies that this construction yields a nonempty set for a given nonzero TTC $\mathscr{T}$

Definition 2.0.16. *Let $\mathcal{S} \subset \mathscr{T}$ be a family of objects, it is $\otimes$-multiplicative if $1 \in \S$, and $x, y \in \S$ implies $x \otimes y \in S$.*

Lemma 2.0.17. *[Bal05, 2.2] Let $\mathscr{T}$ be a nonzero TTC, $\mathscr{I} \subset \mathscr{T}$ a $\otimes$-ideal and $S \subset \mathscr{T}$ a $\otimes$-multiplicative family of objects with $S \cap \mathscr{I} = \varnothing$, then there is a prime $\otimes$-ideal $\mathfrak{p} \in Spc(\mathscr{T})$ with $\mathscr{I} \subset \mathfrak{p}$ and $\mathfrak{p} \cap S = \varnothing$*

Proof. Let F be the family of thick $\otimes-$ideals $\mathscr{J}$ with $\mathcal{J} \cap S = \varnothing$, $\mathscr{I} \subset \mathscr{J}$ and such that for all $s \in S$ and $x \in \mathscr{T}$ such that $s \otimes x \in \mathscr{J}$ implies $x \in \mathscr{J}$. The set $\mathscr{J}_0 = \{x \in \mathscr{T} \mid \exists s \in S \text{ with } x \otimes s \in \mathscr{I}\}$ is in F, then by Zorn there is a maximal element $\mathfrak{p}$ which can be shown to be prime. □

Corollary 2.0.18. *If $\mathscr{T}$ is a nonzero TTC, then $Spc(\mathscr{T}) \neq \varnothing$*

Proof. If we let for example $S = 1$ and $\mathscr{I} = 0$, applying 2.0.17 guarantees the existence of a thick prime $\otimes -$ ideal $\mathfrak{p}$. □

Definition 2.0.19. *Let $\mathscr{I}$ be a $\otimes$-ideal of a tensor triangulated category $\mathscr{T}$. The radical of $\mathscr{I}$, $\sqrt{\mathscr{I}}$ is the set $\{a \in \mathscr{T} \mid \exists n \geqslant 1, a^{\otimes n} \in \mathscr{I}\}$.*

We say that a $\otimes$-ideal is radical if it is equal to $\sqrt{I}$.
As remarked by Balmer, in practice most of the time all thick tensor ideals on a given tensor triangulated category are in fact radical and so there is no need to make a distinction. We have for example the following characterization of such situations.

Lemma 2.0.20. *The following are equivalent for a TTC $\mathscr{T}$:*

1. *Any thick $\otimes$-ideal is radical*
2. *We have $a \in \langle a \otimes a \rangle_{\otimes}$ for all objects $a \in \mathscr{T}$*

Proof. If $\mathscr{I}$ is radical then so is $\mathscr{I}' = \langle a \otimes a \rangle_{\otimes}$ and so $a \in \mathscr{I}'$. On the other hand we can reason by induction and see that for any $a^{\otimes n} \in \mathscr{I}$ since $a \in \langle a \otimes a \rangle_{\otimes}$. □

In fact as soon as every object in our TTC is dualizable we get that every ideal is radical.
We will be then interested mainly in this case where every tensor ideal is radical, although for the rest of the chapter we will keep the results as general as possible, in practice we will now work under the assumption that in our TTC's every object is dualizable. That is, the TTC is rigid as monoidal category. We shall put another technical condition on our categories, let us recall that an idempotent complete category is a category such that every idempotent $e : A \to A$ splits. From now on we then assume every tensor triangulated category is idempotent and rigid.
As we do in affine algebraic geometry, we are interested in the collection all prime $\otimes$-ideals.

Definition 2.0.21. *Let $\mathscr{T}$ be a tensor triangulated category, the set of all prime $\otimes$-ideals will be denoted by $Spc(\mathscr{T})$.*

We are interested in giving this set a topology akin to the Zariski topology, for that we rely on the concept of support data for tensor triangulated categories.

Definition 2.0.22. *Let $(\mathscr{T}, \otimes, 1)$ be a TTC and let X be a topological space, denote $Cl(X)$ the category of closed subsets of X, a support data on $\mathscr{T}$ is a function $\sigma : \mathscr{T} \to Cl(X)$ such that:*

1. $\sigma(0) = \varnothing$
2. $\sigma(1) = X$
3. $\sigma(x \oplus y) = \sigma(x) \cup \sigma(y)$
4. $\sigma(x[1]) = \sigma(x)$
5. *For any distinguished triangle $a \to b \to c \to a[1]$, $\sigma(a) \subset \sigma(b) \cup \sigma(c)$*
6. $\sigma(x \otimes y) = \sigma(x) \cap \sigma(y)$

Additionally, a morphism of support data over $\mathscr{T}$, $f : (X, \sigma) \to (Y, \tau)$ is given by a continuous map $f : X \to Y$ such that $\sigma(x) = f^{-1}(\tau(x))$ for all $x \in \mathscr{T}$.

A similar notion for commutative rings and distributive lattices were considered by Joyal in [Joy75], to construct the usual Zariski spectrum as a initial object in an obvious way. We have an analogous result for our support data:

Theorem 2.0.23. *Let $(\mathscr{T}, \otimes, 1)$ be a TTC, then there exists an initial support data $(Spc(\mathscr{T}), supp)$.*

To prove this we must first define the support function supp which we will use to generate a topology on $Spc(\mathscr{T})$.

Definition 2.0.24. *Let $(\mathscr{T}, \otimes, 1)$ be a TTC, the support of an object $x \in \mathscr{T}$, denoted $supp(x)$, is the set $\{\mathfrak{p} \in Spc(\mathscr{T}) \mid x \notin \mathfrak{p}\}$.*

Lemma 2.0.25. *The sets of the form $\mathcal{Z}(S) := \bigcap_{x \in S} supph(x)$, for a family of objects $S \subset \mathscr{T}$, form a basis for a topology on $Spc(\mathscr{T})$.*

Proof. We observe first that $\mathcal{Z}(S) = \{\mathfrak{p} \in Spc(\mathscr{T}) \mid S \cap \mathfrak{p} = \varnothing\}$. From this it follows that $\mathcal{Z}(S) \cup \mathcal{Z}(T) = \mathcal{Z}(\{s \oplus t \mid s \in S, t \in T\}$ since prime $\otimes -$ ideals) are thick. Finally we observe $\mathcal{Z}(0) = \varnothing$ and $\mathcal{Z}(1) = Spc(\mathscr{T})$ since the $\otimes -$ ideals are proper subcategories of $\mathscr{T}$. □

Lemma 2.0.26. *For a TTC $(\mathscr{T}, \otimes, 1)$, the pair $(Spc(\mathscr{T}), supp)$ is a support data on $\mathscr{T}$.*

Proof. The remaining conditions to check follow from the fact that the $\otimes$-ideals are all full triangulated subcategories of $\mathscr{T}$, and they are prime. □

The morphism of support data we are interested is the following: If (X, σ) is another support data, then we write $f : X \to Spc(\mathscr{T})$ as the function

$$f(x) = \{a \in \mathscr{T} \mid x \notin a\}.$$

An important result regarding this topology is the following, which restricts the kind of spaces we should be expecting from the construction.

Theorem 2.0.27. *[Bal05, 2.15,2.18] For any TTC $(\mathscr{T}, \otimes, 1)$, the space $Spc(\mathscr{T})$ is a spectral space in the sense of Hochster, meaning it is sober and has a basis of quasi-compact open subsets.*

Before giving the proof of this theorem we need a few basic lemmas:

Lemma 2.0.28. *If $\mathscr{T}$ is not zero, then it has minimal primes*

Proof. If we are given a chain of primes in $Spc(\mathscr{T})$, we just need to pick the intersection $\bigcap p$ which is prime since the primeness of the factors assures the intersection is too. □

Lemma 2.0.29. *Let $a \in \mathscr{T}$ and $S \subseteq \mathscr{T}$, then $U(a) := \{\mathfrak{p} \mid a \in \mathfrak{p}\} \subset U(S)$ if and only if there are $b_1, \dots, b_k \in S$ with $b_1 \otimes \cdots \otimes b_k \in \langle a \rangle_{\otimes}$*

Proof. We may assume S is a multiplicative set the result follows from 2.0.17 □

Lemma 2.0.30. *An open $U(a)$ is quasi-compact and all open quasi-compacts are of the form $U(a)$ for some $a \in \mathscr{T}$*

Proof. By the previous lemma, if we cover $U(a)$ by $U(S_i)$ then there are $b_i, \dots, b_k$ such that $b := b_i \otimes \cdots \otimes b_k \in \langle a \rangle_{\otimes}$, but then b is in some finite union of $U(S_i)$. On the other hand if $U(S)$ is quasi-compact, it is of the form $\bigcup U(a_i)$ for a finite number of a_i, hence of the form $U(\otimes a_i)$. □

Let us recall that a subset $Y \subset X$ of a topological space is called specialization closed if it is a union of closed subsets. Alternatively, if whenever $x \in X$ is such that $x \in \hat{y}$, the closure of some element $y \in Y$, then $x \in Y$.

Definition 2.0.31. *We say that a support data (X, σ) for a tensor triangulated category $\mathscr{T}$ is a classifying support data if the following holds:*

1. *The space X is noetherian and any non-empty irreducible closed subset $Z \subset X$ has a unique generic point.*
2. *There is a bijection between specialization closed subsets of X and the set of radical thick tensor ideals of $\mathscr{T}$ defined by $Y \mapsto \{a \in \mathscr{T} \mid \sigma(a) \subset Y\}$.*

In our terms, Thomason's classification theorem says the following

Theorem 2.0.32. *Let X be a quasi-separated scheme, then $(Spc(Perf(X)), supph)$ is a classifying support data for $(Perf(X), \otimes^{\mathbb{L}}_X)$, where $supph$ is the homological support.*

Let us observe that this topology behaves well and expectedly with respect to $\otimes$-triangulated functors.

Lemma 2.0.33. *If $F : \mathscr{T} \to \mathscr{T}'$ is a $\otimes-$ triangulated functor, the functions $Spc(F) : Spc(\mathscr{T}') \to Spc(\mathscr{T})$ defined by the assignment $\mathfrak{q} \mapsto F^{-1}(\mathfrak{q})$, is continuous.*

Proof. To start off, the function is well defined since $F^{-1}(\mathfrak{q})$ is a thick prime $\otimes-$ ideal because $\mathfrak{q}$ is. IF $x \in \mathscr{T}$ then $Spc(F)^{-1}(supp(x)) = \{\mathfrak{q} \in Spc(\mathscr{T}') \mid F^{-1}(\mathfrak{q}) \in supp(x)\}$, on the other hand $supph(F(x)) = \{\mathfrak{q} \in Spc(\mathscr{T}') \mid F(x) \notin \mathfrak{q}\}$, so $Spc(F)^{-1}(supp(x)) = supp(F(x))$. The function is then continuous as the basic closed sets are intersections of sets of the form $supp(x)$ for some $x \in \mathscr{T}$. For a basic closed subset $Z = \bigcap_i supp(x)$ for objects $x_i \in \mathscr{T}$, $Spc(F)^{-1}$. □

Lemma 2.0.34. *Let $F : \mathscr{T} \to \mathscr{T}'$ is an essentially surjective tensor triangulated functor then the induced map $Spc(F)$ is injective.*

We now recall the notion of localization of a triangulated category. Our interest is of course in localizing along a tensor triangulated ideal.

Definition 2.0.35. *Let $\mathscr{T}$ be a triangulated category and let $\mathscr{T}' \subset \mathscr{T}$ full a triangulated subcategory, the Verdier quotient $\mathscr{T}/\mathscr{T}'$ is the localization of $\mathscr{T}$ with respect to morphisms f such that $Cone(f)$ is isomorphic to an object in $\mathscr{T}'$. This category has a triangulated structure given by the triangles which are images of the triangles in $\mathscr{T}$ under the localization functor. This localization comes equipped with a localization functor $\pi : \mathscr{T} \to \mathscr{T}/\mathscr{T}'$*

We now see Verdier localizations induce tensor triangulated functors.

Lemma 2.0.36. *Let $\mathscr{I} \subset \mathscr{T}$ be thick $\otimes$-ideal, then the localization functor $\pi : \mathscr{T} \to \mathscr{T}/\mathscr{I}$ is a tensor triangulated functor and induces an homeomorphism $Spc(\mathscr{T}/\mathscr{I}) \cong \{\mathfrak{p} \in Spc(\mathscr{T}) \mid \mathscr{I} \subset \mathfrak{p}\}$.*

Lemma 2.0.37. *Let $\mathscr{T}' \subset \mathscr{T}$ be a full tensor triangulated subcategory with the same units and such that for every $x \in \mathscr{T}$ there is $x' \in \mathscr{T}$ such that we have $x \oplus x' \in \mathscr{T}'$. Then $Spc(\mathscr{T}') \to Spc(\mathscr{T})$ is a homeomorphism.*

Now that the topology on $Spc(\mathscr{T})$ has been chosen, the next step is to equip this space with sheaf of rings which will act as the structure sheaf.
To a subset $Y \subset Spc(\mathscr{T})$ we can assign a thick $\otimes$-ideal denoted by $\mathscr{I}_Y$ and defined as the subcategory supported on Y, meaning $\mathscr{I}_Y := \{x \in \mathscr{T} \mid supp(x) \subset Y\}$.

Finally, with Y as above, we denote by 1_{T_Y} the image of the unit 1 of $\mathscr{T}$ under the localization functor $\pi : \mathscr{T} \to \mathscr{T}/\mathscr{I}_Y$.

Definition 2.0.38. *Let $\mathscr{T}$ be a nonzero TTC and we define a structure sheaf $\mathcal{O}_{Spc(\mathscr{T})}$ over $Spc(\mathscr{K})$ as the sheaffification of the assignment $U \mapsto End(1_{T_Z})$ where $Z = Spc(\mathscr{T})\backslash U$.*

In this particular situation where Z is the complement of an open subset let us denote 1_{T_Z} by 1_U so to reduce the notation.
It is not hard to see the assignment $Spc(F)$ respects composition of $\otimes$-triangulated functors, so if $F : \mathscr{T} \to \mathscr{T}'$ is such a functor, we get a morphism of ringed spaces since for a closed $Z = Spc(\mathscr{T})\backslash U$ we have $F(\mathscr{I}_Z) \subset \mathscr{I}_{Z'}$ where $Z' = Spc(\mathscr{T}')\backslash Spc(F)^{-1}(U)$ which implies there is a morphism $\mathcal{O}_{\mathscr{T}} \to Spc(F)_*\mathcal{O}_{\mathcal{L}}$ and so $Spc : \mathbb{TTC} \to RS$ is a functor, and under nice conditions (for example $\mathscr{T}$ being rigid) this can be shown to be a functor $Spc : \mathbb{TTC} \to LRS$.
Indeed this construction recovers the structure sheaf of our schemes:

Theorem 2.0.39 ([Bal02])**.** *Let X be a reduced, topologically noetherian scheme. There is an isomorhism $\mathscr{O}_X \cong \mathscr{O}_{Spc(Perf(X))}$.*

Proof. First let us see that for any tensor triangulated category, the endomorphism ring forms a commutative ring. The key aspect here is in setting ourselves in the case where the two coherence morphisms $1 \otimes 1 \to 1$ coincide. In this situation then we see that for any two endomorphisms $f, g \in End(1)$ we have

$$f \otimes Id_1 \circ Id_1 \otimes g = Id_1 \otimes g \circ f \otimes Id_1 = f \otimes g = g \otimes f.$$

As this endomorphism ring is commutative we see that for any affine open subset $U \subset X$ the morphism $\mathscr{O}_U \to End(1_U)$, which sends $r \in \mathscr{O}_X(U)$ to the multiplication by r endomorphism has an inverse given by taking an endomorphism η and sending it to the image of the unit of 1_U. As X is reduced and affine opens form a subbasis this induces an isomorphism of sheaves $\mathscr{O}_X \to End(1)$. □

We have now Balmer's main theorem in [Bal05].

Theorem 2.0.40 (Balmer's reconstruction theorem)**.** *[Bal05, Theorem 6.3] The functor*

$$X \to (Perf(X), \otimes^{\mathbb{L}}, 1)$$

between reduced topologically noetherian schemes and TTC's is faithful and reflects isomorphisms.

Proof. We know from 2.0.32 that there exists a homeomorphism $Spc(Perf(X)) \cong X$. And from Theorem 2.0.39 we have an isomorphism of structure sheaves. □

Historically the reconstruction first appeared in [Bal02] where Balmer considered presheaves of triangulated categories, in [Bal05] the construction was clarified and changed to the construction of

the spectrum as we present here, while presenting a more systematic treatment of the general theory. We would like to point out the reconstruction recreated for coherent schemes by Kock and Pitsch in [KP17] where the relationship with Hochster duality was fully explored giving a nice conceptual explanation in terms of frames and distributive lattices. In particular we can summarize that the Zariski lattice of radical tensor ideals is the Hochster dual of the Zariski lattice of closed subsets of the original space, and the Balmer spectrum serves as a nice way of encoding this duality in familiar terms.
The case for quasi-compact quasi-separated schemes was originally proved in [BKS07] under different methods.
As we are mainly concerned with smooth projective varieties the generality presented above will suffice, but it is important to mention some of the relevant general theory of Balmer's spectra.
In general Balmer spectrum is tough to calculate, one very useful tool is the following comparison maps studied in [Bal10].

Definition 2.0.41. *An object $u \in \mathscr{T}$ in a tensor triangulated category, is $\otimes$-invertible if there exists $u^{-1} \in \mathscr{T}$ such that $x \otimes x^{-1} \cong 1_{\mathscr{T}}$*

We define the following objects.

Definition 2.0.42. *Let $u \in \mathscr{T}$ be a $\otimes$-invertible object in $\mathscr{T}$. Then for every pair of objects $a, b \in \mathscr{T}$ we denote by $Hom^*_{\mathscr{T}}(a,b)$ the graded abelian group $\oplus Hom_{\mathscr{T}}(a, u^{\otimes i} \otimes b)$ with composition defined, if $f \in Hom^i_T(a,b)$, $g \in Hom^j_T(a,b)$, by*

$$f \circ g := f \circ id^j_u \otimes g : a \to u^{\otimes i+j} \otimes b \in Hom^{i+j}(a,b)$$

Let us point out that in the case where we pick $u = 1_T[1]$, the unit of the tensor structure shifted by 1, then the definition above coincides with the graded Ext algebra from Remark 1.1.62.
As in practice the invertible object $u \in \mathscr{T}$ is understood or fixed permanently, it takes no part in the notation but it is important to keep in mind that this graded group depends strongly on this choice.
The case of the endomorphisms of the unit is of special importance

Definition 2.0.43. *Let $\mathscr{T}$ be a tensor triangulated category, let $u \in \mathscr{T}$ be a $\otimes$-invertible object. We denote by $R^*_{\mathscr{T}}$ the graded ring $Hom^*_{\mathscr{T}}(1_{\mathscr{T}}, 1_{\mathscr{T}})$.*

Recall that for a graded ring R^* we can construct a scheme $Spec^h(R^*)$ of homogeneous ideals with points given by prime homogeneous ideals and a Zariski topology generated by those basic opens defined as $D(s) := \{\mathfrak{p} \in R^* \mid s \in \mathfrak{p}^*\}$ for those $s \in \bigcap R^{2i}$. As usual we have a structure sheaf of graded rings defined on open sets by the rule $\mathscr{O}^*_X(D(s)) = R^*[1/s]$.
We are interested in the following map.
If $\mathfrak{p} \in Spc(\mathscr{T})$ is a prime tensor ideal in the Balmer spectrum of a tensor triangulated category, then we have a continuous map $\rho : Spc(\mathscr{T}) \to Spec^h(R^*)$

$$\mathfrak{p} \mapsto \{f \in R^{hom} \mid cone(f) \notin \mathfrak{p}\}.$$

Here R^{hom} denotes the subset of homogeneous elements.
In fact this map can be seen to extend to a morphism of locally ringed spaces and Balmer shows the following theorem:

Theorem 2.0.44. *Let $\mathscr{T}$ be a tensor triangulated category such that R^* is coherent. Then the map $\rho : Spc(\mathscr{T}) \to Spec^h(R^*_{\mathscr{T}})$ is surjective.*

In general this map is not injective as can be easily seen when taking $Perf(\mathbb{P}^n)$ and $u = \mathscr{O}_X$, but locally for a commutative ring A we do have an homeomorphism when considering the homotopy category $K^b(A - proj)$.
This map is of great utility when wanting to understand some of the structure of $Spc(\mathscr{T})$, for example Balmer gives a counterexample of a Balmer spectrum which is not a scheme by considering the tensor triangulated category SH^{fin} of finite spectra with smash product as the monoidal structure.
In fact in [Bal10, Corollary 9.5] Balmer shows that the Balmer spectrum $Spc(SH^{fin})$ classifies Morava K-theories and uses a topological argument to see that this locally ringed space cannot be a scheme.
Concretely, the category SH^{fin} is local in the sense that $x \otimes y = 0$ implies that $x = 0$ or $y = 0$, which in the spectrum means it has a unique local point.

On the algebraic-geometric side of things we also run into some complications, for example when one wants to recover spaces more complicated than schemes. In fact it is shown in [Kri09] and in more generality in [Hal16] that given a Deligne-Mumford stack, the Balmer spectrum of the derived category of this space only recovers the coarse moduli space associated to the stack. It is observed too that as a consequence of this as non-isomorphic algebraic spaces can have isomorphic moduli spaces, that the Balmer spectrum described as we did cannot recover such higher spaces.
We should note that although computing $Spc(\mathscr{T})$ for a tensor triangulated category can be complicated in general, plenty of work has been done in this direction. For example in [Kel15] the spectrum was constructed for a category of motives, and in [DT12] for noncommutative motives. In modular representation theory we can mention for example that if we pick the stable category of modules over a group algebra kG then the Balmer spectrum recovers $Proj(H^*(G, k))$.
To finish this chapter we will give some definitions and basic results on Picard groups of tensor triangulated categories that we will be using in the future.

Definition 2.0.45. *Let $\mathscr{T}$ be a tensor triangulated category, the group $Pic(\mathscr{T})$ is the set of isomorphism classes $\otimes$-invertible objects in $\mathscr{T}$ equipped with $\otimes$ as group operation.*

In the case of a derived category of a space, we have the following result:

Theorem 2.0.46. *[BF07] Let X be a scheme, then there is a split short exact sequence*

$$0 \to Pic(X) \to Pic(Perf(X)) \to C(X; \mathbb{Z}) \to 0$$

where $C(X;\mathbb{Z})$ denotes the group of locally constant functions from X to $\mathbb{Z}$.

In our case this says that given a smooth projective variety with the tensor triangulated structure given by the derived tensor product, then the Picard group of the category is given by the Picard group of the variety and a copy of $\mathbb{Z}$, which corresponds to the shifting operator. We have seen the description for automorphisms of the derived category of a Fano variety in Chapter 1 as given by $(Pic(X) \times \mathbb{Z}) \rtimes Aut(X)$, in other words, it is just the Picard group of the category itself and the automorphisms of the variety which act by direct pushforward.

Chapter 3

Tensor triangulated structures and Picard groups

So far we have studied the basic theory of derived categories in algebraic geometry and the role that the monoidal structure plays in the reconstruction problem for spaces via the Balmer spectrum. We have seen how the geometry reflects on properties of the derived category but the derived tensor product in Balmer's reconstruction was assumed to be the one coming from the space X we started with.

As we mentioned before, the existence of non-isomorphic Fourier-Mukai partners immediately implies that a given derived category can be equipped by a tensor triangulated structure in more than one way, as any such partner would come with its own derived tensor product which by Balmer's reconstruction cannot produce an equivalence as monoidal categories.

The main object of interest in this thesis is then the set or space of all such tensor triangulated structures one can equip a fixed derived category with. In this chapter we present our first results by giving an alternative proof of Bondal-Orlov's reconstruction theorem by using Balmer's reconstruction. The idea is to study the properties that a possible tensor triangulated structure the derived category of a variety with ample (anti-)canonical can have in contrast with the derived tensor product. By deducing some information about the Picard groups of such tensor triangulated structures we are then able to give proof of the reconstruction under some mild extra hypothesis.

We are however still left with the question of whether there exists tensor triangulated categories which produce extra spaces under Balmer's spectrum. The answer is already known in the case of derived categories equivalent to derived categories of quiver representations, as shown in [LS13] once the derived category comes from representations of a Quiver, the vertex-wise tensor product of quiver representations descends to a derived tensor product which is different than the one from the variety X. The Balmer spectrum of such tensor product produces a discrete space and so it rarely is a Fourier-Mukai partner.

In light of this result we know that a derived category then can hold exotic tensor triangulated structures and we could furthermore wonder if any such structure could potentially produce a locally ringed space not equivalent to a scheme as we saw it possible from the Balmer spectrum of finite spectra in the previous chapter.
Our goal in this chapter is to present these results and expand on the possibilities that the collection of tensor triangulated structures on a given category could entail.
Let us start by looking at the following example extracted from [Sos].

Example 3.0.1. *Let X be a connected smooth projective variety. Consider the derived category of $X \amalg X$. It can be shown that the category $D^b(X \amalg X)$ is not equivalent to the category $D^b(X)$.*
We will equip $D^b(X \amalg X) \simeq D^b(X) \oplus D^b(X)$ with the following tensor product $\boxtimes$

$$(A, B) \boxtimes (C, D) \mapsto (A \otimes^{\mathbb{L}}_X C, A \otimes^{\mathbb{L}}_X D \oplus B \otimes^{\mathbb{L}}_X C).$$

We see that the unit of this product is $(\mathscr{O}, 0)$ under the identification $D^b(X \amalg X) \simeq D^b(X) \oplus D^b(X)$. This product can be directly shown to be symmetric and associative, similarly additivity and invariance under the shift operator is a straightforward calculation.
To check that it is exact, as $\otimes^{\mathbb{L}}_X$ is exact, given a triangle

$$(A, B) \to (C, D) \to (E, F)$$

The first argument after tensoring by a pair (A', B') remains an exact triangle in $D^b(X)$, and on the second one it is simply a direct sum of tensoring the second argument with $\otimes^{\mathbb{L}} B'$ and $\otimes^{\mathbb{L}}_X A'$ and thus remains an exact triangle.
We want to show that $Spc(D^b(X), \otimes^{\mathbb{L}}) \cong Spc(D^b(X \amalg X), \boxtimes)$ as topological spaces and in consequence as schemes too.
Using the decomposition $D^b(X \amalg X) \simeq D^b(X) \oplus D^b(X)$ we see that any prime $\boxtimes$-ideal P of $Spc(D^b(X \amalg X), \boxtimes)$ decomposes as a direct sum $P_1 \oplus P_2$ of thick triangulated subcategories of $D^b(X)$. As we suppose P is $\boxtimes$-prime, we immediately deduce P_1 must be $\otimes^{\mathbb{L}}$-prime in $D^b(X)$.
On the other hand, P_2 can also be deduced to be a $\otimes^{\mathbb{L}}$-ideal, however if P_2 is not a proper ideal, then $(0, \mathscr{O}) \notin P$ but $(0, \mathscr{O}) \boxtimes (0, \mathscr{O}) = (0, 0) \in P$ contradicting the fact that P is $\boxtimes$-prime.
With this in mind the tensor triangulated functor

$$(A, B) \mapsto A$$

induces an homeomorphism $Spc(D^b(X \amalg X) \cong X$. As the endomorphisms of $\mathscr{O}$ coincide with those of $(\mathscr{O}, 0) \in D^b(X \amalg X)$, we see that this extends to an isomorphism as varieties.

Another example we would like to present is the one coming from quiver representations we anticipated before.
Recall that given a Quiver $Q = (V, E)$, a quiver representation P (over k) is nothing but an assignment to each vertex $v \in V$ a vector space $P(v)$ and for each edge $e = v_1 \to v_2 \in E$ a linear transformation

$P(e)$. These form a category $rep(Q)$ which comes equipped with a tensor product given by, for two representations $P, R \in rep(Q)$ the product $P \otimes R$ is the representation where for each vertex $v \in V$, $(P \otimes R)(v) := P(v) \otimes_k R(v)$ and $(P \otimes R)(e) := P(e) \otimes_k R(e)$.
The category of representations is an abelian category and so we can derived it as usual. It can be shown that the tensor product $\otimes$ descends to a derived tensor product $\otimes^{\mathbb{L}}$ in $D^b(rep(Q))$.
In Theorems 2.1.5.1 and 2.2.4.1 of [LS13] it is shown that if Q is a finite quiver then $Spc(D^b(rep(Q))$ is a discrete space with as many points as vertices in Q and such that $\mathscr{O}_Q(W) = k^{\oplus W}$ where $W \subset Spc(D^b(rep(Q))$.
In particular if we have a variety such that its derived category contains a full strong exceptional collection then as we have seen in Theorem 1.1.31 it can be shown that the derived category is equivalent to the derived category of representations of a quiver formed from the Hom spaces between the exceptional objects which act themselves as vertices. For example $D^b(\mathbb{P}^1)$ is equivalent to the derived category of representations of the Kronecker quiver of two vertices and two arrows between them both in the same direction.
But from Balmer's result the derived tensor product of $D^b(\mathbb{P}^1)$ recovers the space $\mathbb{P}^1$ which is not isomorphic to a space of two points which we would recover from the Balmer spectrum of the quiver presentation of the category.
In fact the main theorem of [LS13] allows for relations to be included in the quiver and so a large number of spaces with full exceptional collections fall under this example.
We can conclude from these couple of examples that a derived category can admit tensor triangulated structures producing spaces which are not Fourier-Mukai partners.

In general the behaviour of the dynamics of the Balmer spectrum and taking derived categories can be complex. As we know that the Balmer spectrum is a locally ringed space it has an abelian category of sheaves of modules which admits a tensor product then we can derive this category as usual, however the category of sheaves of modules is in general much more complicated than a category of coherent or even quasi-coherent sheaves. In particular one very important notion we have been exploiting is that the category of perfect complexes which we remind that is equivalent to the bounded derived category in our case, consists of those objects which are compact in the derived category of coherent sheaves. Not only that, but they are also automatically dualizable under the derived tensor product and and as such the theory of classification of thick and ideal subcategories is much better behaved than the theory of so-called big tensor triangulated categories.

Having said that, let us look into the case of smooth projective varieties of general type. Recall a variety is of general type if its canonical bundle is big. In particular varieties with ample canonical bundle are big. One alternative characterization of bigness for a variety is the following,

Theorem 3.0.1. *[Laz17, Example 2.2.9] A smooth projective variety is of general type if and only if, for any sheaf $\mathscr{F} \in Coh(X)$, there exists an integer i_0 depending on $\mathscr{F}$ such that the sheaf $\mathscr{F} \otimes_X \omega_X^i$ is generically globally generated for $i >> i_0$.*

As a consequence of the Kodaira lemma (cf. [Laz17, Prop 2.2.6]) we have the srollary:

Corollary 3.0.2. *Let X be a smooth projective variety of general type, then there exists an open subvariety X^* such that for any $\mathscr{F} \in Coh(X)$, there exists a positive integer i_0 such that for any $i >> i_0$, the sheaf $\mathscr{F}|_{X^*} \otimes_X \omega^i_{X^*}$ on X^* is globally generated.*

Let us explain the pervious corollary and the nature of the open subvariety X^*. We recall some basic definitions.

Definition 3.0.3. *Let X be a projective variety and $\mathscr{L}$ a line bundle on X, the augmented base locus is the Zariski closed set*

$$B_+(\mathscr{L}) := \bigcap_{m\in\mathbb{N}} B(m\mathscr{L} - A)$$

Where A is any ample line bundle, and for any line bundle $\mathscr{L}'$ the set $B(\mathscr{L})$ is defined as the intersection of the base loci of multiples of the line bundle, that is

$$B(\mathscr{L}) := \bigcap_{m\in\mathbb{N}} Bs(m\mathscr{L})$$

In [BCL14] the following theorem characterizing the complement of the augmented base focus is proven:

Theorem 3.0.4. *Let $\mathscr{L}$ be a big line bundle on a normal projective variety X over an algebraically closed field. Then the complement $X\backslash B_+(\mathscr{L})$ of the augmented base locus is the largest Zariski open subset $U \subseteq X\backslash B(\mathscr{L})$ such that for all large and divisible $m(\mathscr{L}) \in \mathbb{Z}$ the restriction of the morphism*

$$\phi_m : X\backslash B(\mathscr{L}) \dashrightarrow \mathbb{P}H^0(X, m\mathscr{L})$$

to U is an isomorphism onto its image.

The following couple important observations follow immediately from the definition, the fact that the augmented base locus is independent of the choice of ample line bundle, and Kodaira's decomposition of big line bundles. See for example [DFEM14, Remark 1.5.14].

Remark 3.0.5.

1. *$B_+(\mathscr{L}) = \varnothing$ if and only if $\mathscr{L}$ is ample.*
2. *$B_+(\mathscr{L}) \neq X$ if and only if $\mathscr{L}$ is big.*

From the remarks above and using Thomason's classification theorem we know that as there exists a correspondence between closed subsets of the Balmer spectrum and radical tensor ideals in the tensor triangulated category then there exists a radical thick tensor ideal corresponding to the augmented base locus $B_+(\mathscr{L})$ for any given line bundle $\mathscr{L}$. In particular the open subvariety X^* from Corollary 3.0.2

is the complement of the augmented base locus, $X\backslash B_+(\omega_X)$ and corresponds to a $\otimes^{\mathbb{L}}_X$-ideal generated by a single object (using Theorem 2.0.8) whose cohomological support gives back the closed subset $B_+(\omega_X)$.

Remark 3.0.6. *Let us denote by I_{X*} the thick tensor ideal corresponding to the open subvariety X^*. By our previous Remark 3.0.5 we have that this ideal must be a proper ideal of $D^b(X)$ and is the ideal 0 precisely when the (anti-)canonical bundle is ample.*

We would like to understand the effect of the positivity of the canonical bundle (in this case the fact that the variety is of general type) on the tensor triangulated structure of the category. We know from Theorem 1.1.40 that the Serre functor in a triangulated category is unique up to degree whenever it exists and so it is a property of the category and not extra data. In our concrete case we know furthermore that the Serre functor is isomorphic to $\_\otimes^{\mathbb{L}}_X \otimes \omega_X[n]$ where $n \in \mathbb{N}$ is the dimension of the variety and ω_X is the dualizing sheaf of X.
Let us start with a definition mimicking that of spanning class:

Definition 3.0.7. *Let $(\mathscr{T}, \otimes)$ be a tensor triangulated category with Serre functor, and let $\mathscr{I} \subseteq \mathscr{T}$ be a thick tensor ideal. We say that a collection of objects $\Omega \subset \mathscr{T}$ is an almost spanning class with respect to $\mathscr{I}$ if for every $X \in \mathscr{T}/\mathscr{I}$ such that $Hom_{\mathscr{T}/\mathscr{I}}(\pi(B), X[j]) = 0$ for all $j \in \mathbb{Z}$ and for all $B \in \Omega$, then $X = 0$. Where $\pi : \mathscr{T} \to \mathscr{T}/\mathscr{I}$ is the localization functor.*

It is immediate to see that the previous definition is equivalent to asking that the collection Ω maps through π to a spanning class on the quotient $\mathscr{T}/\mathscr{I}$. When the ideal in question is the 0 ideal then the definition reduces to that of a spanning class as in Definition 1.1.8.
We would like to generalize Theorem 1.1.20 but for a big canonical bundle instead of an ample one and see that a big bundle induces an almost spanning class in the derived category with respect to an ideal $\mathscr{I}$.

Theorem 3.0.8. *Let X be a smooth projective variety of general type. Then the collection of tensor powers $(\omega_X^{\otimes i})_{i\in\mathbb{Z}}$ forms an almost spanning class with respect to the thick tensor ideal I_{X*} in the tensor triangulated category $(D^b(X), \otimes^{\mathbb{L}}_X)$.*

Proof. We need to show that $\pi(\{\omega_X^{\otimes i}\})$ forms a spanning class in the quotient $D^b(X)/I_{X*}$. As I_{X*} is the ideal corresponding to the open smooth subvariety X^* then we know that there is an isomorphism $Spc(D^b(X)/I_{X*}) \cong X^*$. Since ω_X restricted to X^* is ample by the characterization of Theorem 3.0.4, we get that $\{\omega_X \mid_{X*}\}^{\otimes i}$ forms an spanning class by Lemma 1.1.20 of the derived category of X^* which coincides with the quotient category $D^b(X)/I_{X*}$. □

It is thanks to this result that we are able to have some control over the possible tensor triangulated structures on a given derived category. The key fact to keep in mind is that in this situations any object

admits a resolution in the quotient $D^b(X)/I_{X*}$ by direct sums of iterations of the Serre functor, that is, tensor powers of the dualizing sheaf.
That is, for any $A \in D^b(X)$ the object $\widehat{A} := \pi(A) \in D^b(X)/\mathscr{I}$ has a resolution

$$\cdots \to (\omega_{X*}^{i_0})^{\oplus k_0} \to (\omega_{X*}^{i_1})^{\oplus k_1} \to \cdots \to (\omega_{X*}^{i_m})^{\oplus k_m} \to \hat{A} \to 0$$

for some $m \in \mathbb{N}$.
Let us now consider $\boxtimes$ a tensor triangulated structure on $D^b(X)$ (with unit $\mathscr{O}_X$). Then we have:

Lemma 3.0.9. *Suppose X is a smooth projective variety of general type of dimension n. If $\boxtimes$ is a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$, and U is a $\boxtimes$-invertible object such that $U \boxtimes I_{X*} \subseteq I_{X*}$. Then there is a natural equivalence between the functors induced by $U \boxtimes \_$ and $U \otimes_X^{\mathbb{L}} \_$ in $D^b(X)/I_{X*}$.*

Proof. We know from Theorem 3.0.8 that the $\otimes^{\mathbb{L}}$-powers of the canonical sheaf form an almost spanning class with respect to I_{X*}, and by our previous discussion we know that any object $A \in D^b(X)$ has a resolution in $D^b(X)/I_{X*}$

$$\cdots \to (\omega_{X*}^{i_0})^{\oplus k_0} \to (\omega_{X*}^{i_1})^{\oplus k_1} \to \cdots \to (\omega_{X*}^{i_m})^{\oplus k_m} \to \hat{A} \to 0$$

where $\widehat{A}$ is the image of A in $D^b(X)/I_{X*}$ and we will denote this resolution by $\omega_{\tilde{X}*}^{\bullet} \to \widehat{A} \to 0$.
As the Serre functor in $D^b(X^*)$ is given by $\_ \otimes_X^{\mathbb{L}} \omega_{X*}[n']$, where n' is the dimension of X^* and we know any exact equivalence must commute with it, if we let $U\widehat{\boxtimes}$ and $U\widehat{\otimes^{\mathbb{L}}}$ denote the autoequivalences of $D^b(X)/I_{X*}$ induced by $U\boxtimes$ and $U\otimes^{\mathbb{L}}$ respectively then we have that for any object $A \in D^b(X)$, $(U\widehat{\boxtimes}\hat{A})\widehat{\otimes^{\mathbb{L}}}\omega_{X*}[n'] \cong U\widehat{\boxtimes}(\widehat{A}\widehat{\otimes^{\mathbb{L}}}\omega_{X*}[n'])$.
As $\mathscr{O}_X$ is a unit for both $\otimes_X$ and $\boxtimes$, and after shifting by $[-n']$ we deduce

$$U\widehat{\otimes^{\mathbb{L}}}\omega_{X*} \cong U\widehat{\boxtimes}\omega_{X*}.$$

From this, the exactness of $\otimes^{\mathbb{L}}$ and $\boxtimes$, and the resolutions in terms of ω_{X*}^i, we obtain the isomorphisms

$$U\widehat{\otimes^{\mathbb{L}}}\widehat{A} \cong U\widehat{\boxtimes}\widehat{A}.$$

□

Remark 3.0.10. *Let us point out the slight abuse of notation of the autoequivalence $U\widehat{\otimes^{\mathbb{L}}}$. This functor would formally be denoted by $\widehat{U}\otimes^{\mathbb{L}}_{D^b(X)/I_{X*}}$ as it is induced by the object $\widehat{U}$ in the tensor triangulated categories $(D^b(X)/I_{X*}, \otimes^{\mathbb{L}}_{D^b(X)/I_{X*}})$, but as the only thick tensor ideal we are quotienting by in this section is I_{X*}, we believe our notation is lighter without losing sight of which functors they represent.*

We have the following corollary:

Corollary 3.0.11. *Let X be a variety of general type and let $\boxtimes$ a tensor triangulated category structure on $D^b(X)$ with unit $\mathscr{O}_X$. Then for any $\boxtimes$-invertible object U such that $U\boxtimes I_{X*} \subseteq I_{X*}$, the equivalence $U\widehat{\boxtimes} : D^b(X)/I_{X*} \to D^b(X)/I_{X*}$ induced by $U \boxtimes \_$ is equivalent to an equivalence given by objects in the group $Pic(D^b(X)/I_{X*}, \widehat{\otimes^{\mathbb{L}}})$ of invertible $\widehat{\otimes^{\mathbb{L}}}$-objects.*

Proof. From Lemma 3.0.9 we have that if U^{-1} is such that $U \boxtimes U^{-1} \cong \mathscr{O}_X$ then in the quotient $D^b(X)/I_{X*}$,

$$U\widehat{\otimes^{\mathbb{L}}}\widehat{U^{-1}} \cong U\widehat{\boxtimes}\widehat{U^{-1}} \cong \mathscr{O}_{X*}.$$

As $(D^b(X)/I_{X*}, \widehat{\otimes^{\mathbb{L}}})$ is a tensor triangulated category, we have that $\widehat{U} \in D^b(X)/I_{X*}$ is a $\widehat{\otimes^{\mathbb{L}}}$-invertible objects. □

In Lemma 3.0.9 and Corollary 3.0.11 above, the ideal I_{X*} might not be a $\boxtimes$-tensor ideal and thus the quotient $D^b(X)/I_{X*}$ does not necessarily carry a tensor triangulated category structure induced by $\boxtimes$. However, our result guarantees that after passing to the quotient, the equivalences induced by the functors $U \boxtimes \_$ are equivalent to equivalences given by invertible objects in $(D^b(X)/I_{X*}, \otimes_X^{\mathbb{L}})$ induced by the same object, under the condition that I_{X*} is stable by $U\boxtimes$.
In particular, we have:

Corollary 3.0.12. *Let X be a variety of general type and $\boxtimes$ a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$. If I_{X*} is a $\boxtimes$-ideal then the Picard group $Pic(D^b(X)/I_{X*}, \widehat{\boxtimes})$ is a subgroup of the Picard group $Pic(D^b(X)/I_{X*}, \widehat{\otimes_X^{\mathbb{L}}})$.*

Proof. The proof is as in the previous two, if U is in $Pic(D^b(X)/I_{X*}, \widehat{\boxtimes})$ then it induces an autoequivalence of $D^b(X)/I_{X*}$ and so it commutes with the Serre functor on $D^b(X^*) \simeq D^b(X)/I_{X*}$. As by Theorem 3.0.8 iterations of ω_{X*} form a spanning class in $D^b(X)/I_{X*}$ then if $A \in D^b(X)/I_{X*}$ can be written as a resolution given by ω_{X*}.
Then by the same argument than in the proof of Lemma 3.0.9 we arrive at the isomorphisms

$$U\widehat{\otimes^{\mathbb{L}}}A \cong U\widehat{\boxtimes}A.$$

□

The case when our variety has an ample (anti-)canonical bundle allows us to relate the Picard group of the full derived category to that of any other tensor triangulated category structure on it.
The following result follows from the previous argument.

Corollary 3.0.13. *Let X be a variety with ample (anti-)canonical bundle. Then if $\boxtimes$ is a tensor triangulated category structure on $D^b(X)$ with unit $\mathscr{O}_X$, the Picard group $Pic(D^b(X), \boxtimes)$ is isomorphic to a subgroup of $Pic(D^b(X), \otimes_X)$.*

Proof. We just need to notice that in this case the thick $\otimes_X$-ideal from Corollary 3.0.2 is the 0 ideal and thus we can resolve any object $A \in D^b(X)$ by a sequence of powers of the Serre functor. By the same reasoning as above we see that

$$U \otimes^{\mathbb{L}} A \cong U \boxtimes A.$$

□

One thing to note here is that although Bondal and Orlov had already classified the group of autoequivalences of a derived category of a variety with ample (anti-)canonical bundle, we are working without the condition of an equivalence between the derived category of the Balmer spectrum of $\boxtimes$ and the derived category $D^b(X)$, and as such it is not immediate from their result that the Picard group of $\boxtimes$ must involve invertible sheaves over X.
In other words, as $Spc(\boxtimes)$ is not necessarily isomorphic to X then understanding the autoequivalences of $D^b(X)$ alone does not give us an immediate relationship to the Picard group of $\boxtimes$.
The following two corollaries follow directly from our discussion above.

Corollary 3.0.14. *Let X be as above, then if $\omega_X[n]$ is an invertible object for a tensor triangulated structure $\boxtimes$ on $D^b(X)$ with unit $\mathscr{O}_X$ then $\boxtimes$ and $\otimes^{\mathbb{L}}_X$ coincide on objects.*

Proof. Suppose $\omega_X[n]$ is $\boxtimes$-invertible and let $A, B \in D^b(X)$. Then we know that both A and B have resolutions by powers of ω_X.
In particular we know that $\omega_X \boxtimes A \cong \omega_X \otimes^{\mathbb{L}}_X A$. But as $\boxtimes$ is exact on both variables we have

$$\dots \omega_X^{\otimes i_0} \boxtimes A \to \dots \omega_X^{\otimes i_m} \boxtimes A \to A \boxtimes B \to 0$$

And so $B \boxtimes A \cong B \otimes^{\mathbb{L}}_X A$. □

Corollary 3.0.15. *Let X be a variety with ample (anti-)canonical bundle, suppose $(\mathscr{O}_X, \boxtimes)$ is a tensor triangulated structure on $D^b(X)$ such that $Pic(\boxtimes) \cong Pic(D^b(X))$ via the assignment $U \mapsto U$ then $\boxtimes$ coincides with $\otimes^{\mathbb{L}}_X$ on objects.*

Proof. In this case if this morphism is an isomorphism, then ω_X is $\boxtimes$-invertible and the result follows from the previous corollary. □

In fact if we are under the same hypothesis for X then as soon as we are able to show that the generators of $Pic(D^b(X), \otimes^{\mathbb{L}})$ are $\boxtimes$-invertible then by the previous corollary there must be an equivalence between $\boxtimes$ and $\otimes^{\mathbb{L}}_X$.

Example 3.0.2. *Let $X = \mathbb{P}^n$ be the projective space, in this case we know that $Pic(D^b(X)) = \mathbb{Z} \oplus \mathbb{Z}$ corresponding to the line bundles plus their shifts. The result above then says that whenever there is a tensor triangulated structure $\boxtimes$ on $D^b(X)$ with unit $\mathscr{O}_X$ then the Picard group of this tensor structure must necessarily be a subgroup of $\mathbb{Z} \oplus \mathbb{Z}$.*
If $\omega_X = \mathscr{O}_X(-n-1)$ is $\boxtimes$-invertible then we get that $\boxtimes$ coincides with $\otimes^{\mathbb{L}}_X$. Similarly if $\mathscr{O}_X(-1)$ is $\boxtimes$-invertible.

Again under the hypothesis of X having an ample (anti-)canonical bundle, by using Theorem 2.0.46 we see that if $Spc(\boxtimes)$ is a scheme then the Picard group of $Spc(\boxtimes)$ must be a subgroup of the

Picard group of X. So a line bundle in $Spc(\boxtimes)$ has to be a line bundle of X.
From Bondal-Orlov's reconstruction original proof we know that it is actually possible to fully characterize line bundles up to a shift from purely categorical properties, and so one possible question is whether one can obtain the derived tensor product directly from the category without passing through a reconstruction result.

Remark 3.0.16. *In [ha] Antieau sketches a construction in which by considering invertible objects (in the sense of Bondal and Orlov) one can define the derived tensor product $\otimes^{\mathbb{L}}_X$ by exploiting the resolution by powers (compositions) of the Serre functor $S(\_)$.*
The idea is as follows: Let U be an invertible object in the sense of Bondal-Orlov, which we know must be of the form $\mathscr{L}[r]$ for $\mathscr{L}$ a line bundle on X and r an integer. For any object $\mathscr{F} \in D^b(X)$ we have a resolution

$$\cdots \to (\omega_X \otimes^{\mathbb{L}}_X U)^{\otimes i_0} \to \cdots \to (\omega_X \otimes^{\mathbb{L}}_X U)^{\otimes i_m} \to \mathscr{F} \to 0$$

by iterations of the composition of the Serre functor with U, which forms a spanning class since powers of the canonical sheaf is, and the fact that $U = \mathscr{L}[r]$.
We define then for any other object $\mathscr{G} \in D^b(X)$ the product $\otimes_U$, by setting $(\omega_X \otimes^{\mathbb{L}}_X U)^i \otimes_U \mathscr{G} := S^i(\mathscr{G})$, the composition of the Serre functor with $\mathscr{G}$ i-times.
We can extend this then to a full product $\mathscr{F} \otimes_U \mathscr{G}$ which forms a tensor triangulated structure on $D^b(X)$ with unit U.

We have now justified enough the following definition:

Definition 3.0.17. *Let $\mathscr{T}$ be a triangulated category, denote by $TTS(\mathscr{T})$ the collection of equivalence classes of tensor triangulated category structures on $\mathscr{T}$. Where we consider two tensor triangulated category structures to be equivalent if there is a monoidal equivalence between the two of them.*

Let us recall that by tensor triangulated structure we mean a rigid, closed, symmetric monoidal structure on the idempotent-complete triangulated category $\mathscr{T}$ with biexact tensor product functor, although as pointed out before the hypothesis of rigid and idempotent-complete are not a big impediment as we can always idempotent-complete a triangulated category and the rigidity only simplifies the theory in the known cases, but in general the tensor triangulated structures can consist of closed symmetric monoidal category structures on $\mathscr{T}$ with biexact product.
In general for a triangulated category $\mathscr{T}$ we have an action by $Aut(\mathscr{T})$ on the collection $TTS(\mathscr{T})$.
If $\otimes \in TTS(\mathscr{T})$ and $\phi \in Aut(\mathscr{T})$ we have a tensor structure defined by

$$X \otimes_\phi Y := \phi^{-1}(\phi(X) \otimes \phi(Y)).$$

By this reasoning we are interested then in the quotient $TTS(\mathscr{T})/Aut(\mathscr{T})$ which identifies tensor triangulated structures on $\mathscr{T}$ coming from an autoequivalence.

It is not hard to see that two tensor triangulated structures related by an autoequivalence must produce the same space under Balmer's spectrum.
As in general autoequivalences of a derived category are more complicated than in the ample canonical (or anti-canonical) bundle case which as we saw consists of autoequivalences of the space and shifts of line bundles, then a useful thing to do is to consider tensor triangulated structures with a fixed unit as we did for example in the theorems above.
Let us define then

Definition 3.0.18. *Let $\mathscr{T}$ be a triangulated category and $U \in \mathscr{T}$ an object. Then the set $TTS_U(\mathscr{T})$ is the set of equivalence classes of tensor triangulated structures on $\mathscr{T}$ where U is the unit.*

But after taking the quotient space by autoequivalences, there is still a question of whether there might be more tensor structures on a given derived category which might produce an exotic locally ringed space which does not correspond to a Fourier-Mukai partner.
Let us explore briefly the case for $\mathbb{P}^1$.

Example 3.0.3. *Let $X = \mathbb{P}^1$. We will be interested in understanding the set $TTS(D^b(X))/Aut(D^b(X)))$, in particular let us fix the unit to be $\mathscr{O}_X$ and we would like to see if there are any tensor triangulated structures $\boxtimes$ non equivalent to the canonical $\otimes^{\mathbb{L}}_X$.*
We can exploit the fact that the derived category $D^b(\mathbb{P}^1)$ is generated by the exceptional collection $\{\mathscr{O}, \mathscr{O}(1)\}$ and furthermore it has dimension 1 as a triangulated category, which means that any object can be expressed as a cone of a morphism

$$\bigoplus \mathscr{O}[n_i] \bigoplus \mathscr{O}(1)[n_j] \to \bigoplus \mathscr{O}[m_i] \bigoplus \mathscr{O}(1)[m_j].$$

In addition to this we have for example known distinguished triangles

$$\mathscr{O}(i) \to \mathscr{O}(i+1)^{\oplus 2} \to \mathscr{O}(i+2) \to \mathscr{O}(i)[1]. \tag{3.0.1}$$

Using this fact combined with Corollary 3.0.11 we can discard certain assignments from ever forming a tensor triangulated structure. For example, let us suppose $\boxtimes$ is a tensor triangulated structure on $D^b(\mathbb{P}^1)$ such that $\mathscr{O}$ is the unit.
As the assignment $(F, G) \mapsto F \boxtimes G$ is completely determined by the value of $\mathscr{O}(1) \boxtimes \mathscr{O} \cong \mathscr{O}(1)$, $\mathscr{O}(1) \boxtimes \mathscr{O}(1)$ and the value at the morphisms

$$Hom(\mathscr{O}, \mathscr{O}) = k, Hom(\mathscr{O}, \mathscr{O}(1))) = k \oplus k, Hom(\mathscr{O}(1), \mathscr{O}) = k, Hom(\mathscr{O}(1), \mathscr{O}(1)) = k.$$

Then we can quickly discard the assignments $\mathscr{O}(1) \boxtimes \mathscr{O}(1) = \mathscr{O}$ as this would give us a Picard group $\boxtimes$ not isomorphic to a subgroup of $\mathbb{Z} \times \mathbb{Z}$.
To investigate whether there could be products $\boxtimes$ such that $Pic(\boxtimes) \subset \mathbb{Z} \times \mathbb{Z}$ as a proper subgroup, let

us see what happens for example if we suppose $\mathscr{O}(2)$ was invertible but not necessarily $\mathscr{O}(1)$.
Using the distinguished triangles obtained from (3.0.1):

$$\mathscr{O}(-2) \to \mathscr{O}(-1)^{\oplus 2} \to \mathscr{O} \to \mathscr{O}(-2)[1]$$
$$\mathscr{O} \to \mathscr{O}(1)^{\oplus 2} \to \mathscr{O}(2) \to \mathscr{O}[1]$$

Tensoring them with each other produces a diagram

$$\begin{array}{ccccc}
\mathscr{O}\boxtimes\mathscr{O}(-4) & \longrightarrow & \mathscr{O}(1)^{\oplus 2}\boxtimes\mathscr{O}(-4) & \longrightarrow & \mathscr{O}(2)\boxtimes\mathscr{O}(-4) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}\boxtimes\mathscr{O}(-3)^{\oplus 2} & \longrightarrow & \mathscr{O}(1)^{\oplus 2}\boxtimes\mathscr{O}(-3)^{\oplus 2} & \longrightarrow & \mathscr{O}(2)\boxtimes\mathscr{O}(-3)^{\oplus 2} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}\boxtimes\mathscr{O}(-2) & \longrightarrow & \mathscr{O}(1)^{\oplus 2}\boxtimes\mathscr{O}(-2) & \longrightarrow & \mathscr{O}(2)\boxtimes\mathscr{O}(-2)
\end{array} \tag{3.0.2}$$

As we require $\mathscr{O}$ to be the unit and we look for $\mathscr{O}(2)\boxtimes\mathscr{O}(-2)$ to be isomorphic to $\mathscr{O}$, then we get

$$\begin{array}{ccccc}
\mathscr{O}(-4) & \longrightarrow & \mathscr{O}(1)^{\oplus 2}\boxtimes\mathscr{O}(-4) & \longrightarrow & \mathscr{O}(2)\boxtimes\mathscr{O}(-4) \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}(-3)^{\oplus 2} & \longrightarrow & \mathscr{O}(1)^{\oplus 2}\boxtimes\mathscr{O}(-3)^{\oplus 2} & \longrightarrow & \mathscr{O}(2)\boxtimes\mathscr{O}(-3)^{\oplus 2} \\
\downarrow & & \downarrow & & \downarrow \\
\mathscr{O}(-2) & \longrightarrow & \mathscr{O}(1)^{\oplus 2}\boxtimes\mathscr{O}(-2) & \longrightarrow & \mathscr{O}
\end{array} \tag{3.0.3}$$

Using again the triangle (3.0.1) we can calculate the value of $\mathscr{O}(1)^{\oplus 2}\boxtimes\mathscr{O}(-2)$ as to be $\mathscr{O}(-1)^{\oplus 2}$ since $\mathscr{O}(-2)\boxtimes\mathscr{O}(2) \cong \mathscr{O}$. Again from the triangle

$$\mathscr{O}(-2) \to \mathscr{O}(-1)^{\oplus 2} \to \mathscr{O} \to \mathscr{O}(-2)[1].$$

We get

$$\mathscr{O}(-1) \to \mathscr{O}(-1)^{\oplus 2}\boxtimes\mathscr{O}(1) \to \mathscr{O}(1) \to \mathscr{O}(-1)[1]$$

by operating with $\mathscr{O}(1)\boxtimes$.
This tells us that $\mathscr{O}(1)\boxtimes\mathscr{O}(-1) \cong \mathscr{O}$ and from this we deduce that the assignment $\boxtimes$ must match the usual derived tensor product $\otimes^{\mathbb{L}}$ of $\mathbb{P}^1$.
Using a similar argument, suppose $\mathscr{O}(n)$, $n \geqslant 2$ is an invertible object for an exact tensor product bifunctor which coincides with the assignment $\boxtimes$, then it automatically implies that every object $\mathscr{O}(i)$ is invertible and so the Picard groups of this tensor product and $\otimes^{\mathbb{L}}$ must in fact coincide. Let us summarize this in a proposition.

Proposition 3.0.19. *Suppose $\boxtimes$ is a tensor triangulated category structure on the derived category $D^b(\mathbb{P}^1)$ such that $\mathscr{O}$ is a unit for $\boxtimes$. Suppose furthermore that there exists nontrivial invertible objects. Then every object $\mathscr{O}(n)$, $n \in \mathbb{Z}$ is invertible and $\boxtimes$ coincides with the derived tensor product $\otimes^{\mathbb{L}}$ of $\mathbb{P}^1$ on objects.*

What this is telling us that it is enough to check for one object to be invertible under our new tensor product provided we have $\mathscr{O}$ as a unit. On the other hand we still have the possibility of having a Picard group equivalent to 0.

In the arguments presented above we have to be very careful when computing the grid of products resulting from tensoring two distinguished triangles. It is well known that the cone although determined up to isomorphism is not a functorial construction and so we cannot really reason about the vertical maps at the level of the cones.
In our case however we can be only interested in the objects produced by the computation and not on the morphisms. A similar computation for a space such that its derived category has higher dimension is not possible without getting over this impediment, here we exploit the fact that the dimension of the derived category of $\mathbb{P}^1$ is equal to 1 and so everything is simply a cone of the objects generated by the exceptional objects and there is no need to calculate additional cones for the produced triangles.
The previous result for $\mathbb{P}^1$ is not surprising in light of Bondal-Orlov's reconstruction, the derived category seems to reflect strongly the properties of the space in all known cases.
In fact we can give an alternate proof of a weakened form of Bondal-Orlov's reconstruction.

Theorem 3.0.20. *Let X be a variety with ample (anti-)canonical divisor, and let $\boxtimes$ be a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$. Suppose $Spc(\boxtimes)$ is a smooth projective space with ample (anti-)canonical bundle and that there is an equivalence $D^b(X) \simeq D^b(Spc(\boxtimes))$, then $X \cong Spc(\boxtimes)$*

Proof. In fact the only thing to note here is that as $Spc(\boxtimes)$ has ample (anti-)canonical bundle then ω_X has to be $\boxtimes$-invertible. fo, by applying Corollary 3.0.13 to $Spc(\boxtimes)$ we obtain that $Pic(\otimes^{\mathbb{L}}_X)$ has to isomorphic, via the assignment $\mathscr{L} \mapsto \mathscr{L}$, to a subgroup of $Pic(\boxtimes)$. Since ω_X is $\boxtimes$-invertible, by Corollary 3.0.14 we obtain our result. □

Remark 3.0.21. *We need to explain our choice of hypothesis here. On the first hand the assumption that $Spc(\boxtimes)$ is a smooth projective variety is necessary just as in the original Bondal-Orlov theorem formulation. We have added a couple more assumptions, however. We suppose that the (anti-)canonical bundle of $Spc(\boxtimes)$ is also ample to highlight the use of the monoidal structures in the theorem. This hypothesis is however not necessary as we have seen in the original proof of the theorem it is possible and not hard to conclude that this holds true as soon as we have the equivalence between the derived categories. Alternatively, we can formulate the theorem as follows:*

Theorem 3.0.22. *Let X be a variety with ample (anti-)canonical divisor, and let $\boxtimes$ be a tensor triangulated structure on $D^b(X)$ with unit $\mathscr{O}_X$. Suppose $Spc(\boxtimes)$ is a smooth projective space, and that we have an equivalence $D^b(X) \simeq D^b(Spc(\boxtimes))$, then $X \cong Spc(\boxtimes)$.*

The proof of which is exactly as in Theorem 3.0.20 except we invoke the argument in the proof of Theorem 1.1.45 only to show that the (anti-)canonical bundle is ample.
Of more importance is the choice of unit. We know from our discussion in the introduction of this chapter, that there exist (for example for $\mathbb{P}^1$) tensor triangulated structures on the derived category $D^b(X)$ of smooth projective variety X which don't produce spaces isomorphic to X under the Balmer spectrum. We might recover more control on the variety $Spc(\boxtimes)$ by the existence of the equivalence $D^b(X) \simeq D^b(Spc(\boxtimes))$, but from our discussion in Remark 3.0.16 we know that automorphisms of $D^b(X)$ will produce nonequivalent tensor triangulated structures which produce X by the Balmer spectrum. By fixing the choice of the unit to be $\mathscr{O}_X$ we are fixing one of these automorphisms and discarding spaces $Spc(\boxtimes)$ which we already know cannot satisfy the conclusion of the theorem.

We can summarize our discussion with the following theorem

Theorem 3.0.23. *Let X be a smooth projective variety with (anti-)canonical bundle. Consider a tensor triangulated category structure $\boxtimes$ on $D^b(X)$ such that $\mathscr{O}_X$ is its unit and $Spc(\boxtimes)$ is isomorphic to X, then $\boxtimes$ and $\otimes^{\mathbb{L}}_X$ coincide on objects.*

This however does not fully classify $TTS(D^b(X))/Aut(D^b(X))$ as we require Balmer's spectrum to be a Fourier-Mukai partner, but there is no reason to expect in general a relationship between the derived category of the Balmer spectrum and the original triangulated category.
The lack of morphism between a space X and the Balmer spectrum $Spc(\boxtimes)$ for some tensor triangulated structure, and thus of functors between the derived categories of these two spaces is one of the obstacles to being able to understand the possible structures $\boxtimes$.

Chapter 4

DG-categories

4.1 Introduction

In this chapter we will present a brief overview of the theory of differential graded categories, dg-categories for short, from their basic properties to the derived Morita theory of Toën and their relationship with stable ∞-categories. The entirety of the work presented here is expository but is presented with a focus on the applications we have in mind and so the absence of any material is of course only due to us. Most of the material here is taken from [Dri04, Kel06, Tab05b, Toë07, Toë11] and we recommend going to the source material for full details of the whole theory.

While the general theory of dg-categories have possibly many different origins, it is in the context of triangulated categories where it has seen its biggest use. It is common knowledge among practitioners of homological algebra and more generally homotopical algebra that the theory of triangulated categories developed by Verdier and Grothendieck carries certain difficulties when trying to perform geometric or algebraic reasoning in the same way that one usually does for categories of modules. These issues were identified from the genesis of the theory and so it has been conjectured and now well understood that a notion of enhancement of a triangulated category is required for some arguments to take hold.

This is of course not to say that the theory of triangulated categories requires in all of its instances of an enhancement treatment, but it often offers an explanation for the awkwardness that one can witness by working directly with the axioms that define triangulated categories.

Different notions of enhancements have been proposed and it is possible to consider them in some way to be equivalent for appropriate notions of equivalence. We in particular will deal mostly with the theory of dg-categories for its relatively simpler language. The drawback in comparison with for example the theory of stable ∞-categories is that it is harder to put it in the context with other sort of homotopical categories of different nature.

We begin this chapter by presenting the generalities of dg-categories and the (2-)category of dg-

categories along with some of its properties. Next we approach the lifting problem. This is the question of whether a given triangulated category or a triangulated functor between two has a corresponding dg-categorical version which enhances it. The answer is negative in general but we will present conditions under which it will be possible to have an enhancement.
Next we deal with homotopical nature of the theory, we present a model category structure on the collection of dg-categories and study the basic properties of its homotopy category. We recall one of the main results of [Toë07] on the closed monoidal structure on this homotopy category by describing both the derived tensor product and the derived internal hom object. With all of this at hand we are now in a position to go over the derived Morita theory. Our main goal for this section is to give a detailed treatment of Theorem 7.2 of [Toë07] in the case of perfect complexes over dg-algebras relating dg-functors in the homotopy category with bimodules over this same algebra.

4.2 General facts about dg-categories

As we mentioned in the introduction, our main interest in the theory of dg-categories lies in their relationship with triangulated categories. The main idea behind their relationships as dg-categories enhancing triangulated ones is that a dg-category contain strictly more information than the triangulated one but we are always able to keep in sight how the triangulated category lies within the dg-enhancement. Here we adopt a cohomological notation, differentials increase the degree in the category of cochain complexes $C(k)$ over a commutative ring k. If there is no confusion we denote by $\otimes$ the usual tensor product of complexes defined on objects by

$$(M \otimes N)^n := \bigoplus_{n=i+j} M^i \otimes_k N_j$$

A sizable part of the theory presented here follows from the more general theory of enriched categories and many results could be stated in greater generality. We choose to stick with the case at hand so to keep the notation and the flavor of the arguments consistent across the whole chapter and thesis.

Definition 4.2.1. *A dg-category $\mathscr{T}$ is a category enriched over the closed monoidal category $\mathscr{C}(k)$, the category of cochain complexes over k. More precisely a dg-category $\mathscr{T}$ consists of the data:*

1. *A collection of objects $Ob(\mathscr{T})$.*
2. *For any pair of objects $x, y \in \mathscr{T}$, a cochain complex $Hom(x, y)$.*
3. *An identity morphism 1_x, from the cochain complex concentrated in 0, k to any cochain complex Hom of the form $Hom(x, x)$.*

4. *For any triple $x, y, z \in \mathscr{T}$, there is a composition rule $\mu_{y,z,x} : Hom(y,z) \otimes Hom(x,y) \to Hom(x,z)$.*

For this to form an enriched category it must be that the following conditions must be satisfied, The following unit diagram commutes.

$$\begin{array}{ccc} Hom(x,y) & \xrightarrow{Id \otimes 1_x} & Hom(x,y) \otimes Hom(x,x) \\ & \searrow^{Id} & \downarrow{\scriptstyle \mu_{x,y,x}} \\ & & Hom(x,y) \end{array}$$

Similarly, there is a commutative diagram

$$\begin{array}{ccc} Hom(x,y) & \xrightarrow{1_y \otimes Id} & Hom(y,y) \otimes Hom(x,y) \\ & \searrow^{Id} & \downarrow{\scriptstyle \mu_{y,y,x}} \\ & & Hom(x,y) \end{array}$$

Is the identity morphism of $Hom(x,y)$

For every four objects $x, y, z, w \in \mathscr{T}$ the following diagram commutes

$$\begin{array}{ccc} Hom(z,w) \otimes Hom(y,z) \otimes Hom(x,y) & \xrightarrow{id \otimes \mu_{y,z,x}} & Hom(z,w) \otimes Hom(x,z) \\ \downarrow{\scriptstyle \mu_{z,w,y} \otimes id} & & \downarrow{\scriptstyle \mu_{z,w,x}} \\ Hom(y,w) \otimes Hom_(x,y) & \xrightarrow{\mu_{y,w,x}} & Hom(x,w) \end{array}$$

Let us mention a few important and immediate examples

Example 4.2.1. *1. An important source of examples of dg-categories, crucial for our purposes, is that of dg-algebras. If A is a dg-algebra then there exists a dg-category with exactly one object $\{*\}$ and such that the Hom complex $Hom(*,*)$ is the underlying dg-algebra seen as a complex. Indeed, the multiplication of A serves as our composition morphisms $\mu : Hom(*,*) \otimes Hom(*,*) = A \otimes A \to A$. In this way the theory of dg-categories can be considered as a generalized theory of dg-algebras with many objects.*

2. *By a similar argument we can consider any k-linear category as a dg-category. If $\mathscr{C}$ is such a category then for any pair of objects $x, y \in \mathscr{C}$, the Hom set carries by definition a k-vector space structure which we can consider as a complex concentrated in degree 0. In other words the cochain complex defined as $Hom(x,y)^i := 0$ if $i \neq 0$ and $Hom_{\mathscr{C}}(x,y)$ otherwise, can be used to enrich our category $\mathscr{C}$ so that this forms a dg-category.*

3. *The category of cochain complexes itself if we are to use the internal Hom cochain complex. Recall that for two complexes $\mathscr{E}, \mathscr{F} \in \mathscr{C}(k)$ there exists a cochain complex $\underline{Hom}(\mathscr{E}, \mathscr{F})$ given by*

$$\underline{Hom}(\mathscr{E}, \mathscr{F})^n := \Pi_{j-i=n} Hom(\mathscr{E}^i, \mathscr{F}^j)$$

Using these complexes as the Hom in $\mathscr{C}(k)$, we get a dg-category.

We have mentioned already a few times that a dg-category must be thought as an enhancement of a triangulated category. As triangulated categories are by definition regular 1-categories, we would like to be able to always point towards an underlying 1-category that comes within any given dg-category. Let us define:

Definition 4.2.2. *Let $\mathscr{T}$ be a dg-category. The homotopy category $H^0(\mathscr{T})$ of $\mathscr{T}$ is the category with the same class of objects as $\mathscr{T}$ and, if $x, y \in H^0(\mathscr{T})$ we let the Hom sets be given as*

$$Hom_{H^0(\mathscr{T})}(x, y) := H^0(Hom_{\mathscr{T}}(x, y))$$

For this to actually form a category we need at the least a composition rule and an identity morphism induced from that of the one in $\mathscr{T}$. This follows directly from recalling that the composition and unit morphism are morphisms of complexes. As k is concentrated in degree 0, we immediately have that $d(1_x) = 0$ for all $x \in \mathscr{T}$.
Recall that the differential of the tensor product of complexes is given by

$$d(g \otimes f) = d(g) \otimes f + (-1)^{|g|} g \otimes d(f)$$

If $f \in Z^0(Hom(y, z))$ and $g \in Z^0(x, y)$ then $d(g \otimes f)$ and so $\mu_{y,z,x} \circ d = 0$, thus $\mu_{y,z,x}(g \otimes f) \in H^0(Hom(x, z))$.
Just as in regular category theory the next step and one of great importance is to describe ways in which we can compare different dg-categories. Structure preserving functors can be described in our case as the collection of the following data

Definition 4.2.3. *Let $\mathscr{T}, \mathscr{T}'$ be dg-categories, a dg-functor $\mathscr{F} : \mathscr{T} \to \mathscr{T}'$ is a functor enriched over $\mathscr{C}(k)$, in other words:*

1. *An assignment $x \in \mathscr{T} \mapsto \mathscr{F}(x) \in \mathscr{T}'$*
2. *For every pair of objects $x, y \in \mathscr{T}$, a morphism of cochain complexes $\phi_{x,y} : Hom(x, y) \to Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y))$*
3. *For every three objects $x, y, z \in \mathscr{F}$ we require the commutativity of the diagram:*

$$\begin{array}{ccc} Hom(y,z) \otimes Hom(x,y) & \xrightarrow{\phi_{y,z} \otimes \phi_{x,y}} & Hom_{\mathscr{T}'}(\mathscr{F}(y), \mathscr{F}(z)) \otimes Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y)) \\ \downarrow{\scriptstyle \mu_{y,z,x}} & & \downarrow{\scriptstyle \mu_{\mathscr{F}(y),\mathscr{F}(z),\mathscr{F}(x)}} \\ Hom(x,z) & \xrightarrow{\phi_{x,z}} & Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(z)) \end{array}$$

4. *For every element $x \in \mathscr{T}$, an assignment $e_x : 1_x \mapsto 1_{\mathscr{F}(x)}$*

5. *For every pair of objects $x, y \in \mathscr{T}$, we have commutative diagrams:*

$$\begin{array}{ccc} Hom(x,y) \otimes k & \xrightarrow{\phi_{x,y} \otimes 1_{\mathscr{F}(x)}} & Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y)) \otimes Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(x)) \\ & \searrow^{\phi_{x,y}} & \downarrow \mu_{\mathscr{F}(x),\mathscr{F}(y),\mathscr{F}(x)} \\ & & Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y)) \end{array}$$

$$\begin{array}{ccc} k \otimes Hom(x,y) & \xrightarrow{1_{\mathscr{F}(y)} \otimes \phi_{x,y}} & Hom_{\mathscr{T}'}(\mathscr{F}(y), \mathscr{F}(y)) \otimes Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y)) \\ & \searrow^{\phi_{x,y}} & \downarrow \mu_{\mathscr{F}(y),\mathscr{F}(y),\mathscr{F}(x)} \\ & & Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y)) \end{array}$$

Considering the category of dg-functors in itself will be of great importance as it in regular category theory. In fact, that the category of dg-functors carries itself a dg-category structure as we will see ahead is a basic but rather important property that will allow us obtain information that would have remain hidden in the triangulated category world.

As a special case of the category of dg-functors we describe the category of dg-modules over a given dg-category $\mathscr{T}$. This is in the same fashion that we would study module categories over a ring or over a monoid in a given monoidal category that we are used to in algebra. As we will see ahead the role played by dg-categories of modules is of similar nature, the idea is that modules over a dg-category $\mathscr{T}$ should be thought as representations in the dg-category of cochain complexes.

Just as we do in categories, for any given dg-category $\mathscr{T}$ there exists an opposite dg-category given by the same objects as those of $\mathscr{T}$ and whose Hom complexes are, for two objects $x, y \in \mathscr{T}$, the cochain complex $Hom_{\mathscr{T}}(y, x)$. Indeed it is not hard to verify that with the obvious composition and unit, the necessary conditions are satisfied for this to be a dg-category too.

In particular, if A is a dg-algebra seen as a single-object dg-category as we explained in our examples above, the opposite of this dg-category is nothing but the opposite dg-algebra whose multiplication xy is now defined as $yx \in A$.

Let us

Definition 4.2.4. *Let $\mathscr{T}$ be a dg-category, the category of (right) dg-modules over $\mathscr{T}$ is the category of dg functors $\mathscr{T}^{op} \to \mathscr{C}(k)$ and dg-natural transformations between them as functors. We will denote this category by $\mathscr{T}^{op} - Mod$*

Analogously we have a notion of left $\mathscr{T}$-module if we consider functors $\mathscr{T} \to \mathscr{C}(k)$. This category will be in turn denoted simply by $\mathscr{T} - Mod$.

Notice that this concept above generalizes directly the notion of dg-module over a dg-algebra. Indeed if we take $\mathscr{T}$ to be a one-object dg-category, so a dg-algebra, and we consider (right or left) dg-modules

in the sense of definition 4.2.4, a dg-functor will associate a cochain complex to the single object of $\mathscr{T}$ and the dg-algebra structure corresponding to the endomorphism of the object will act on the cochain complex on the right or on the left. This is precisely the usual definition of right and left dg-module over a dg-algebra.

We have yet to describe dg-natural transformations between general dg-functors. As we will be almost exclusively concerned about them only in the context of dg-modules over a dg-category we will only give an explicit description of this case for clarity, although the general case is completely analogous. With the formal analogy relating dg-modules over a dg-algebra and dg-modules over a dg-category, we ought to think about dg-natural transformations as corresponding to morphisms between dg-modules:

Definition 4.2.5. *Let $\mathscr{T}$ be a dg-category and let us consider $\mathscr{T}^{op} - Mod$ and see that it has a natural dg-category structure. For two dg-functors $\mathscr{F}, \mathscr{G} \in \mathscr{T}^{op} - Mod$ the morphisms $Hom_{\mathscr{T}^{op}-Mod}(\mathscr{F}, \mathscr{G})$ are given by families of morphisms of cochain complexes $(\phi_x)^* \in \underline{Hom}_{\mathscr{C}(k)}(\mathscr{F}(x), \mathscr{G}(x))$ for all $x \in \mathscr{T}$, and such that if $f \in Hom_{\mathscr{T}}(x, y)$ we have that the following diagram commutes up to a sign*

$$\begin{array}{ccc} \mathscr{F}(y) & \xrightarrow{\phi_y} & \mathscr{G}(y) \\ \downarrow{\scriptstyle \mathscr{F}(f)} & & \downarrow{\scriptstyle \mathscr{G}(f)} \\ \mathscr{F}(x) & \xrightarrow{\phi_x} & \mathscr{G}(x) \end{array}$$

And for which the differentials are given objectwise, that is, as $\underline{Hom}_{\mathscr{C}(k)}(\mathscr{F}(x), \mathscr{G}(x))$ is a cochain complex itself, it has a differential and so the differential of the natural transformation $(\phi_x)^*_{x \in \mathscr{T}}$ is calculated by,

$$d(\phi^*_x) = (d\phi_x)^*$$

The more general case of dg-natural transformations between two arbitrary dg-functors is exactly the same, as the target dg-category has complexes as Hom's, it has a differential and so the differential of a dg-natural transformation is calculated object-wise.

In this full general situation, we have that there exists a 2-category $dg - cat_k$ with k-dg-categories as objects, morphisms given by dg-functors and 2-morphisms corresponding to dg-natural transformations. We remark that while it is an important fact that dg-functors carry a dg-category structure, we won't be using the 2-category structure itself.

As in the case of regular 1-categories, a Yoneda theorem will be of great utility. Here we present a weak Yoneda lemma for dg-categories as $\mathscr{C}(k)$-enriched categories, but as we will see soon it is a more subtle Yoneda lemma that takes into account the homotopical structure the version that will suit us more often. In any case, let us briefly recount:

Let $\mathscr{T}$ be a dg-category and $x \in \mathscr{T}$ an object, then the module given by

$$h^x := Hom_{\mathscr{T}}(x, \_) : \mathscr{T}^{op} \to \mathscr{C}(k)$$

which associates $y \in \mathscr{T}$ the cochain complex $Hom_{\mathscr{T}}(x,y)$ and we have morphisms in $\mathscr{C}(k)$

$$Hom_{\mathscr{T}}(y,z) = Hom_{\mathscr{T}^{op}}(z,y) \to \underline{Hom}_{\mathscr{C}(k)}(Hom_{\mathscr{T}}(x,z), Hom_{\mathscr{T}}(x,y))$$

If $\phi \in Hom_{\mathscr{T}}(y,z)^n$ this chain map associates ϕ to the family $(\psi) \in Hom(Hom_{\mathscr{T}}(z,x)^i, Hom_{\mathscr{T}}(y,x)^j)$, with $j - i = n$, which maps $\alpha \in Hom_{\mathscr{T}}(z,x)^i$ to the image $\mu(\alpha \otimes \phi)$ under the composition $Hom_{\mathscr{T}}(z,x)^n \otimes Hom_{\mathscr{T}}(y,z)^{j-n} \to Hom_{\mathscr{T}}(y,x)^j$.
It is natural to consider the following definition

Definition 4.2.6. *Let $\mathscr{T}$ be a dg-category, the Yoneda embedding is the dg-functor:*

$$h^{\_} : \mathscr{T} \to \mathscr{T}^{op} - Mod$$

Which maps $x \in \mathscr{T}$ to h^x

We can justify this nomenclature as it is possible to show that this dg-functor is full and faithful in the sense that there is an isomorphism of complexes for any $x, y \in \mathscr{T}$

$$Hom_{\mathscr{T}^{op}-Mod}(h^x, h^y) \cong Hom_{\mathscr{T}}(x,y)$$

. We also have a co Yoneda embedding by the dg-functor

$$h_{\_} : \mathscr{T}^{op} \to \mathscr{T} - Mod$$

Which takes $x \in \mathscr{T}$ and maps it to the left module $Hom_{\mathscr{T}}(\_, x) : \mathscr{T} \to \mathscr{C}(k)$.
As usual we call a right $\mathscr{T}$-module representable if it is isomorphic in $\mathscr{T}^{op} - Mod$ to a right module of the form h^x for some $x \in T$.

As is the case for enriched categories, one can define a tensor product $\otimes$ of dg-categories which makes $dg - cat$ into a closed monoidal category.

Definition 4.2.7. *If $\mathscr{T}, \mathscr{T}'$ are dg-categories, we define their tensor product as the dg-category $\mathscr{T} \otimes \mathscr{T}'$ with objects given by pairs $(x, x') \in \mathscr{T} \times \mathscr{T}'$, and cochain complexes between pairs of objects*

$$Hom_{\mathscr{T}\otimes\mathscr{T}'}((x,x'),(y,y')) := Hom_{\mathscr{T}}(x,y) \otimes_{\mathscr{C}(k)} Hom_{\mathscr{T}'}(x',y')$$

And with composition here given entry-wise in the obvious way.

Now that we are able to take these tensor products, we can consider a particular sort of dg-module which will play an essential role in our theory.

Definition 4.2.8. *Let $\mathscr{T}, \mathscr{T}'$ be dg-categories. A left module $\mathscr{T} \otimes \mathscr{T}'^{op} \to \mathscr{C}(k)$ will be called a dg-bimodule*

The idea of these objects is as in regular algebra, the action of the endomorphism complexes in the domain must act both on the right and the left of the resulting chain complex. In other words, $\mathscr{F}$ is a $(\mathscr{T}, \mathscr{T}')$-bimodule and if $x \in \mathscr{T}$ and $y \in \mathscr{T}$, there is an action of $End_{\mathscr{T}}(x)$ on $\mathscr{F}(x,y) \in \mathscr{C}(k)$ on the left and $End_{\mathscr{T}'}(y)$ acts on the $\mathscr{F}(x,y)$ on the right.
Notice for example that if we are given an object $x \in \mathscr{T}$ and we a $(\mathscr{T}, \mathscr{T}') - bimodule$ then this object x induces a left $\mathscr{T}'$-module $\mathscr{F}(x, \_) : \mathscr{T}' \to \mathscr{C}(k)$, similarly we can construct right $\mathscr{T}$-modules in the an analogous way.
With this general theory we can now start discussing the homotopy theory of dg-categories.

4.2.1 Homotopy theory of dg-categories

Our goal through this section is provide the basic facts about the homotopy theory of dg-categories. The rough idea is that while the category of dg-categories we considered in the previous section has plenty of structure to work on, if we wish to understand the homological nature that a dg-category encodes we need to work with a much more flexible theory.
What we mean by this is that equivalence of dg-categories as $\mathscr{C}(k)$-enriched categories is too coarse of a notion and the resulting notion would give us dg-categories which contain, in a specific way, the same homological information yet they are not equivalent. We will then present a model category structure on $dg - cat_k$, the Dwyer-Kan structure, and will establish the properties that will allow us to sketch a proof of Toën's Morita theory for dg-categories.
We assume the reader is familiar with the general theory of model categories and so we won't be revisiting the theory in full detail from the beginning, although we will do our best to keep the more technical results well referenced, and if necessary we will point to the main arguments required for their proofs. For a general comprehensive introduction the reader can follow for example [Hov07, Hir03].
In particular the result we are looking forward to is a derived version of the famous Eilenberg-Watts theorem which completely describes exact functors between categories of modules over a ring as functors equivalent to tensoring by a bimodule. The same phenomena occurs here if we have the correct notion of equivalence in place.
We proceed first by recalling some properties of model categories and then we present the so called Dwyer-Kan model structure on $dg - cat_k$, then we explain how to derive the tensor product from definition 4.2.7 so that the homotopy category of the model category carries a symmetric closed monoidal structure. We describe the derived internal Hom object and show that the homotopical version of the Yoneda embedding induces an equivalence between these internal Hom's (Theorem 4.2.6 [Toë07][Theorem 7.2]). Afterwards we will be able to discuss the lifting problems for triangulated categories and triangulated functors between them in the case of smooth projective varieties.

4.2.2 The Dwyer-Kan model structure

The so called Dwyer-Kan model structure was first introduced as a Quillen model structure for simplicial sets. In general terms if we consider categories enriched over a category which itself carries a model category structure, we can exploit this extra structure to declare two enriched categories as weak equivalent if they are locally so. In other words, an enriched functor will be a weak equivalence if it induces an weak equivalence between Hom objects and the functor is in some fashion essentially surjective.
Let us recall a couple definitions and fix some notation

Definition 4.2.9. *Let $\mathscr{C}$ be a model category, the homotopy category of $\mathscr{C}$ is the localization $Ho(\mathscr{C}) := \mathscr{C}[W^{-1}]$ at weak equivalences.*

In a model category $\mathscr{C}$ we denote by $Q : \mathscr{C} \to \mathscr{C}$ and $R : \mathscr{C} \to \mathscr{C}$ its cofibrant and fibrant replacement functors, respectively.

Definition 4.2.10. *A model category is called proper if weak equivalences are preserved by pullback and pushout along fibrations and cofibrations, respectively. In other words, if $f : x \to y$ is a weak equivalence, $g : z \to y$ a fibration, then the left vertical morphism in the following pullback diagram is a weak equivalence*

$$\begin{array}{ccc} x \times_y z & \longrightarrow & z \\ \downarrow & & \downarrow \\ x & \longrightarrow & y \end{array}$$

And, if $h : x \to z$ is a cofibration then the right vertical morphism in the following pushout diagram is too a weak equivalence

$$\begin{array}{ccc} x & \longrightarrow & y \\ \downarrow & & \downarrow \\ z & \longrightarrow & y \sqcup_x z \end{array}$$

We will make use of the fact that the model categories we use have good generating properties, to make this precise we should recall the following series of definitions

Definition 4.2.11. *Let $\mathscr{C}$ be a category with all small colimits and S a collection of morphisms of $\mathscr{C}$. An object $x \in \mathscr{C}$ is small relative to S if there exists a cardinal κ such that for all κ-filtered ordinals λ and all λ-sequences of morphisms in S, of the form*

$$x_0 \to x_1 \to \cdots \to x_\beta \to \dots$$

we have an isomorphism

$$colim_{\beta<\lambda} Hom_{\mathscr{C}}(x, x_\beta) \cong Hom_{\mathscr{C}}(x, colim_{\beta<\lambda} x_\beta)$$

Definition 4.2.12. *Let $\mathscr{C}$ be a category with all colimits and I is a set of morphisms of $\mathscr{C}$, then a relative I-cell complex is a morphism $f : x \to y$ in $\mathscr{C}$ such that f is a transfinite composition $X : \lambda \to \mathscr{C}$, so $f : X_0 = x \to \varinjlim X = y$, and for any $\beta + 1 < \lambda$, there is a morphism $c \to d \in I$ and a pushout diagram,*

$$\begin{array}{ccc} c & \longrightarrow & x_\beta \\ \downarrow & & \downarrow \\ d & \longrightarrow & x_{\beta+1} \end{array}$$

We denote by cell(I) the collection of relative I-cell complexes.

Furthermore, we need the following collections of morphisms

Definition 4.2.13. *Let $\mathscr{C}$ be a category with all colimits and I a set of morphisms of $\mathscr{C}$*

1. *rlp(I) is the collection of morphisms of $\mathscr{C}$ with the right lifting property with respect to I.*
2. *llp(I) is the collection of morphisms of $\mathscr{C}$ with the left lifting property with respect to I.*
3. *cof(I) is the collection llp(rlp(I))*

Definition 4.2.14. *A model category is called cofibrantly generated if there exist sets of morphisms I, J of $\mathscr{C}$ such that cof(I) are the cofibrations of $\mathscr{C}$ and cof(J) are the acyclic cofibrations of $\mathscr{C}$ and both I and J admit the small object argument.*
In this case I will be referred to as the generating cofibrations and J as the generating trivial (acyclic) cofibrations.

Let us recall the small object argument refers to the following very important result

Theorem 4.2.15. *Let $\mathscr{C}$ be a category with all colimits and let I be a collection of morphisms on $\mathscr{C}$ such that it is small relative to $cell(I)$. Then any morphism $f : x \to y \in \mathscr{C}$ has a (functorial) factorization*

$$x \xrightarrow{h} x' \xrightarrow{g} y = x \xrightarrow{f} y$$

Where $h \in cell(I)$ and $g \in rlp(I)$

The fact that a model category is cofibrantly generated has very useful consequences, we list some which we might find useful to keep in mind. First we have the transfer theorem which allows us to put a cofibrantly generated model category structure on a category provided there exists an adjoint functor pair involving one cofibrantly generated model category.

Theorem 4.2.16. *(Kan, cf. [Hir03][Theorem 11.3.2]) Let $\mathscr{C}$ be a cofibrantly generated model category with generating cofibrations I and generating trivial cofibrations J. Let $\mathscr{D}$ be a category with all small limits and colimits and consider an adjoint functor pair*

$$F : \mathscr{C} \rightleftarrows \mathscr{D} : G$$

If the sets $F(I), F(J)$ permit the small object argument and G takes relative $F(J)$-cell complexes to weak equivalences, then there exists a cofibrantly generated model category structure on $\mathscr{D}$ with $F(I)$ and $F(J)$ as the sets of generating cofibrations and the set of generating trivial cofibrations, respectiveley. And for which the set of weak equivalences are the morphisms that correspond to weak equivalences in $\mathscr{C}$ by the functor G.

Definition 4.2.17. *A model category $\mathscr{C}$ is stable if it has a zero object and the loop-suspension adjoint $\Omega \dashv \Sigma$ induce an equivalence of categories at the level of homotopy categories.*

A useful property to keep in mind while working with stable model categories is that their homotopy categories admit a triangulated category structure.
The next theorem is too of great importance as it will allow us to characterize our model categories as categories of modules of certain endomorphism algebra. To us this is crucial as we will greatly reduce our arguments by using this particular model instead.

Theorem 4.2.18. *(Schwede-Shipley [SS03][Theorem 3.1.1]) Let C be a proper, stable, cofibrantly generated model category with a compact generator P. Then there exists a chain of Quillen equivalences between $\mathscr{C}$ and the category of modules over the endomorphism algebra of P.*

$$\mathscr{C} \simeq End(P) - Mod$$

The theorem cited above is given under different hypothesis in [SS03], namely their Theorem 3.1.1 concerns simplicial model categories. These are model categories with a compatible simplicial set enrichment structure, however as they note earlier in their paper it is enough to ask for proper stable, cofibrantly generated model categories since by a result from [RSS01] these categories are Quillen equivalent to a simplicial model category.

Remark 4.2.19. *Now let us remark that there exists a model category structure on the category of (unbounded) cochain complexes $\mathscr{C}(k)$ that can be shown to be proper and cofibrantly generated. In this model structure we declare weak equivalences to be quasi-isomorphisms and fibrations to be degree-wise epimorphisms.*
In fact we can say a little bit more and describe concretely the generating cofibrations. Let $M \in k-Mod$ and let $S^n M$ denote the following cochain complex concentrated in degree n

$$\cdots \to 0 \to M \to 0 \to \dots$$

And $D^n M$ similarly concentrated in degrees n-1 and n,

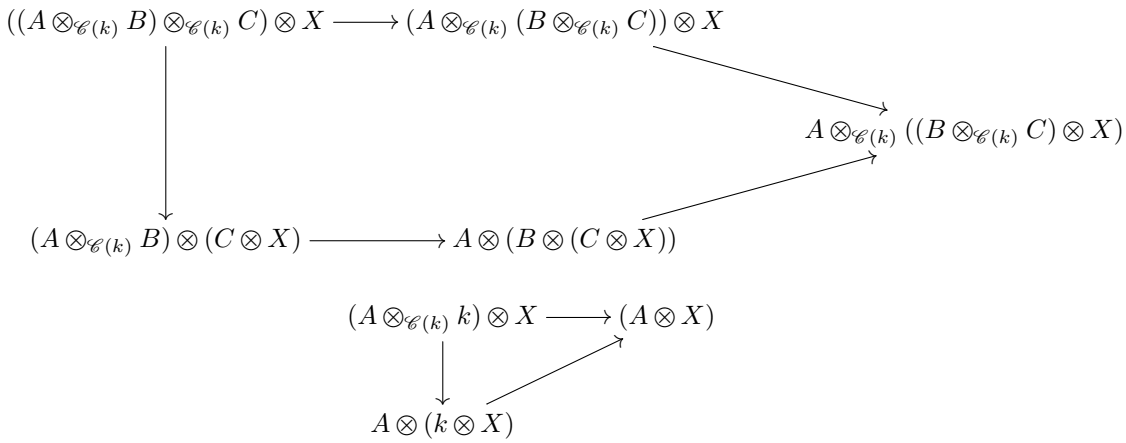
$$\cdots \to 0 \to M \to M \to 0 \to \dots$$

Then we can pick as generating cofibrations of $\mathscr{C}(k)$ the family I of morphisms $S^{n-1}k \to D^n k$, and J the generating trivial cofibrations as the complex morphisms $0 \to D^n k$.

As a remark, when we are working over a field k, all cochain complexes are both fibrant and cofibrant in this model structure.
We will not only need to put a model structure on the whole category of dg-categories, but we will also be interested in putting model category structures on specific sorts of $\mathscr{C}(k)$-enriched categories in a way that makes the enrichment compatible with the model structure. It is in this interaction and compatibility between the global homotopy category of dg-categories and the internal model structure that we can put on a single specific $\mathscr{C}(k)$-enriched category where the core of some arguments will take place so it is important to describe both in some detail.

Definition 4.2.20. *Let $\mathscr{M}$ be a category, a $\mathscr{C}(k)$-model structure on $\mathscr{M}$ consists on a model category structure on $\mathscr{M}$ and a functor $\_\otimes\_ : \mathscr{C}(k) \times \mathscr{M} \to \mathscr{M}$ such that:*

1. *For every $A, B \in \mathscr{C}(k)$ and every $X \in \mathscr{M}$ there are natural isomorphisms $A \otimes (B \otimes X) \cong (A \otimes_{\mathscr{C}(k)} B) \otimes X$, $k \otimes X \cong X$. Satisfying commutativity of the following diagrams:*



2. *Given two cofibrations $i : A \to B$, $j : X \to Y$ in $\mathscr{C}(k)$ and $\mathscr{M}$ respectively, the induced morphism is a cofibration $A \otimes Y \coprod_{A \otimes X} B \otimes X \to B \otimes Y$, and is an equivalence provided i, j are equivalences themselves.*

3. *For every two objects $X, Y \in \mathscr{M}$ there is an internal Hom complex $\underline{Hom}(X, Y) \in \mathscr{C}(k)$ giving natural isomorphisms $Hom_{\mathscr{C}(k)}(A, \underline{Hom}(X, Y)) \cong Hom_{\mathscr{M}}(A \otimes X, Y)$ where $A \in \mathscr{C}(k)$*

In the case where the tensoring operation $\otimes$ of a $\mathscr{C}(k)$-model category might cause some confusion when appearing among other identical symbols we will denote it as $\otimes_M$.

The first thing to notice after this definition is that the existence of the internal Homs automatically makes a $\mathscr{C}(k)$-model category into a dg-category. Let us note that $\mathscr{C}(k)$ itself with the model category structure described above forms a $\mathscr{C}(k)$-model category trivially when considered as a monoidal category. We have nonetheless a less trivial example:

Example 4.2.2. *Let $\mathscr{T}$ be a dg-category, then the category of $\mathscr{T}$-modules $\mathscr{F} : \mathscr{T} \to \mathscr{C}(k)$, carries a $\mathscr{C}(k)$-model category structure.*
The model structure on $\mathscr{T}$-Mod can be explicitly descried by declaring a morphism $f : \mathscr{F} \to \mathscr{G}$ of $\mathscr{T}$-modules as a weak equivalence if for any $x \in \mathscr{T}$, the induced morphism $f_x : \mathscr{F}(x) \to \mathscr{G}(x)$ is a quasi-isomorphism.
We say that $f : \mathscr{F} \to \mathscr{G}$ is a fibration if in turn it is a fibration in $\mathscr{C}(k)$.
With this structure and using the fact that $\mathscr{C}(k)$ is a cofibrantly generated with generating cofibrations I and generating trivial cofibrations J, then $\mathscr{T}$-Mod is also cofibrantly generated with generating cofibrations given by those morphisms of $\mathscr{T}$-modules of the form

$$h^x \otimes_{\mathscr{C}(k)} C \xrightarrow{id \otimes f} h^x \otimes_{\mathscr{C}(k)} D \tag{4.2.1}$$

Where $x, y \in \mathscr{T}$ and $f : C \to D \in I$.
As $\mathscr{T}$-Mod is a dg-category itself, the internal Hom object will simply be the Hom cochain complex objects of T-Mod and the tensoring by $\mathscr{C}(k)$ is defined degree-wise.
All that rest to check is condition 2 in definition 4.2.20, this can be checked precisely by considering the generating cofibrations we described above.
Let $S^{n-1}k \to D^n k$ be a generating cofibration in $\mathscr{C}(k)$ as we described above in remark 4.2.20 , we can then pick as generating cofibrations a family of morphisms of the form $h^x \otimes S^{n-1}k \to h^x \otimes D^n k$ for some $x \in \mathscr{T}$. If $S^{m-1}k \to D^m k$ is another generating cofibration, we need to calculate the pushout

$$S^{m-1}k \otimes h^x \otimes D^n \coprod_{S^{m-1}k \otimes h^x \otimes S^{n-1}} D^m \otimes h^x \otimes S^{n-1}k$$

As the product $S^m k \otimes S^n k$ is again of the same form $S^{n+m}k$ and the product $D^m k \otimes S^n k$ is of the form $D^{n+m}k$, and the pushouts are calculated degree-wise we see that the pushout is

$$D^{n+m-1}k \otimes h^x \coprod_{S^{m+n-2}k \otimes h^x} D^{m+n-1}k \otimes h^x \cong S^{n+m-2}k \otimes h^x$$

And we can see that the morphism to $D^n k \otimes D^m k \otimes h^x \cong D^{n+m}k \oplus D^{n+m-1}k \otimes h^x$ is a cofibration as it is a right lift of a generating cofibration $S^{n+m-2}k \to D^{n+m-1}k$

In fact the case we describe above is rather similar to a more general situation. If $\mathscr{M}$ is a cofibrantly generated $\mathscr{C}(k)$-model category and $\mathscr{T}$ is a dg-category, then the category of $\mathscr{C}(k)$-enriched functors $\mathscr{M}^{\mathscr{T}}$ carries an object-wise $\mathscr{C}(k)$-model category structure induced by that of $\mathscr{M}$ and it can be shown that it is also cofibrantly generated, and one is able to pick as generating cofibrations morphisms of the form

$$h^x \otimes A \to h^x \otimes B$$

For some object $x \in \mathscr{T}$ and $A \to B$ a generating cofibration of $\mathscr{M}$. This is in exactly the same fashion as we had for $\mathscr{T}$-modules in 4.2.1 above.
In this level of generality we also have that the internal Hom given by the $\mathscr{C}(k)$-model category structure

can be derived into derived internal Hom bifunctor

$$\mathbb{R}\underline{Hom} : Ho(\mathscr{M}^{\mathscr{C}})^{op} \times Ho(\mathscr{M}^{\mathscr{C}}) \to Ho(\mathscr{C}(k))$$

As we have mentioned above, given a $\mathscr{C}(k)$-model category $\mathscr{M}$ we can see this as a dg-category with the internal Hom object provided by the $\mathscr{C}(k)$-model structure. We can do slightly better and consider instead the category $Int(\mathscr{M})$ of fibrant and cofibrant objects of $\mathscr{M}$ as a dg-category with the same Hom cochain complex object. We will use this construction very often in the future, and in particular we rely on the following proposition to link the categories underlying category after taking H^0 and the homotopy category of $\mathscr{M}$

Proposition 4.2.21. *Let $\mathscr{M}$ be a $\mathscr{C}(k)$-model category, then there exists a natural equivalence of categories*

$$H^0(Int(\mathscr{M})) \simeq Ho(\mathscr{M})$$

Proof. This follows from the fact that the derived Hom $\mathbb{R}\underline{Hom}$ in $Ho(\mathscr{M})$ corresponds to the internal $\underline{Hom}$ in $\mathscr{M}$ after taking cofibrant and fibrant replacements. Combining this with the adjunction formula that comes from the fact $\mathscr{M}$ is a $\mathscr{C}(k)$-model category, we obtain the required result

$$H^0(\mathbb{R}\underline{Hom}(x,y)) \simeq Hom_{Ho(\mathscr{C}(k))}(k, \mathbb{R}\underline{Hom}(x,y)) \simeq Hom_{Ho(M)}(x,y)$$

For all $x, y \in \mathscr{M}$. □

Back to the case of $\mathscr{T}$-modules over a dg-category $\mathscr{T}$, the dg-modules $h^x \in \mathscr{T}$-Mod are fibrant as all cochain complexes $Hom_{\mathscr{T}}(y,x)$ are fibrant in the model category of $\mathscr{C}(k)$, to see that all modules of the form h^x for some $x \in \mathscr{T}$ are cofibrant, it is enough to recall that a generating cofibration in $\mathscr{T}$-Mod is of the form

$$h^x \otimes S^{n-1}k \to h^x \otimes D^n k$$

This implies that the Yoneda embedding of Definition 4.2.6 factorizes through $Int(\mathscr{T} - \text{Mod})$.

Definition 4.2.22. *Let $\mathscr{T}$ be a dg-category. A dg-module $\mathscr{F} \in \mathscr{T}^{op}$-Mod is called quasi-representable if it is equivalent in $Ho(\mathscr{T}^{op} - Mod)$ to a module of the form h_x*

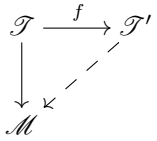
We denote the collection of such modules by $(\mathscr{T} - Mod)^{rqr}$.
Unwrapping this definition, for a quasi-representable module $\mathscr{F} \in \mathscr{T}^{op}$-Mod, for all $i \in \mathbb{Z}$ we have k-Module natural isomorphisms in $y \in \mathscr{T}$

$$H^i\mathscr{F}(y) \cong H^i Hom_{\mathscr{T}}(y,x).$$

Before moving on, let us recall the following general construction. If $f : \mathscr{T} \to \mathscr{T}'$ is a morphism of dg-categories, and $\mathscr{M}$ is a cofibrantly generated $\mathscr{C}(k)$-model structure, recall that there exists a Quillen adjunction

$$f^* : \mathscr{M}^{\mathscr{T}'} \longleftrightarrow \mathscr{M}^{\mathscr{T}} : f_! \tag{4.2.2}$$

Where f^* is simply the composition along f and $f_!$ is the left Kan extension along f:



In general if we have $\mathscr{F} : \mathscr{T} \to \mathscr{M}$ we can calculate $f_!(\mathscr{F})$ at $x \in \mathscr{T}'$ as

$$\int^{y \in \mathscr{T}} h^{f(y)}(x) \otimes \mathscr{F}(y).$$

To better illustrate this situation, we can put $\mathscr{M} = \mathscr{C}(k)$ and $\mathscr{T} = A$ a dg-algebra seen as a one-object dg-category. Here $\mathscr{M}^{\mathscr{T}}$ corresponds to the category of dg-modules over A as we have explained before. Let $\mathscr{F}$ be a dg-module over A and $x \in \mathscr{T}'$ any object. Then

$$f_!(\mathscr{F})(x) \cong Hom_{\mathscr{T}'}(x, f(A)) \otimes \mathscr{F}(A). \tag{4.2.3}$$

The fact that this is a Quillen adjunction follows from the fact that both functor categories inherit a cofibrantly generated $\mathscr{C}(k)$-model category structure, indeed we only need to check that f^* sends generating cofibrations to cofibrations in $\mathscr{M}^{\mathscr{T}'}$ and generating trivial cofibrations to trivial cofibrations in $\mathscr{M}^{\mathscr{T}'}$.
We will be coming back to this particular description in the future, but first we need to describe the homotopy theory on $dg - cat_k$.

Definition 4.2.23. *Let $\mathscr{T}, \mathscr{T}'$ be dg-categories over k. We say that a dg-functor $\mathscr{F} : \mathscr{T} \to \mathscr{T}'$ is a weak equivalence if*

1. *It is quasi-fully faithful. This means that the induced cochain complex morphism $Hom_{\mathscr{T}}(x, y) \to Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y))$ is a quasi-isomorphism for all $x, y \in \mathscr{T}$.*
2. *It is quasi-essentially surjective if $H^0(\mathscr{F}) : H^0(\mathscr{T}) \to H^0(\mathscr{T}')$ is essentially surjective. So for any $x' \in H^0(\mathscr{T}')$, there exists $x \in \mathscr{T}$ such that $\mathscr{F}(x) \simeq x'$ in $H^0(\mathscr{T}')$.*

A dg-functor $\mathscr{F} : \mathscr{T} \to \mathscr{T}'$ is on the other hand a fibration if

1. *The induced morphism of complexes $Hom_{\mathscr{T}}(x, y) \to Hom_{\mathscr{T}'}(\mathscr{F}(x), \mathscr{F}(y))$ is a fibration in the model structure of unbounded complexes.*
2. *For any isomorphism $u' : x' \to y' \in H^0(\mathscr{T}')$ and any $y \in H^0(\mathscr{T})$ such that $\mathscr{F}(y) = y'$, there is an isomorphism $u : x \to y$ in $H^0(\mathscr{T})$ such that $H^0(\mathscr{F})(u) = u'$.*

These classes of morphisms in the category of dg-categories form a model category structure called the Dwyer-Kan model structure on dg-cat. The homotopy category $dg - cat_k[w^{-1}]$, where w is the class of weak equivalences above, of this structure will be denoted by H_{qe}.
We say that a dg-functor $f \in H_{qe}(\mathscr{T}, \mathscr{T}')$ between two dg-categories $\mathscr{T}, \mathscr{T}'$ is a quasi-functor.
Before we introduced the concept of tensor product between dg-categories just as one generally does in the context of enriched categories over a monoidal category. It would be really good if we could induce a monoidal category structure on the homotopy category H_{qe} from this monoidal category, this often is done by using what is known as a monoidal model category by establishing and checking compatibility conditions between the model and the monoidal structures. Let us recall:

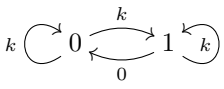
Definition 4.2.24. *A monoidal model category on a model category $\mathscr{M}$ is the data of a monoidal category $\otimes$ such that*

1. *For every pair of cofibrations $f : X \to Y, g : X' \to Y'$ in $\mathscr{M}$ then the induced morphism $X \otimes Y' \amalg Y \times X' \to Y \times Y'$ is a cofibration.*

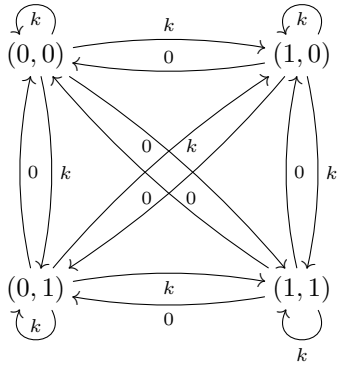
2. *If I is the unit of the monoidal structure, and $QI \to I$ is a cofibrant resolution, then the morphism $QI \times X \to I \otimes X \cong X$ is a weak equivalence.*

If our model category is a monoidal model category then it can be shown that the homotopy category turns into a monoidal category with the induced derived tensor product.
Unfortunately in our case the Dwyer-Kan model structure is not a monoidal model category with the tensor product we have discussed, and the reason of this is that tensoring with a cofibrant object is not cofibrant in general, Toën gives the following example:

Example 4.2.3. *Let Δ^1_k be the following dg-category:*



then the dg-category $\Delta^1_k \otimes \Delta^1_k$:



is not a cofibrant dg-category while the category Δ^1_k is.

It is for this reason then that we cannot directly induce a monoidal structure on the homotopy category directly by localizing at quasi-isomorphisms. However it is entirely possible to derived the tensor product by using a cofibrant replacement, we define

Definition 4.2.25. *The derived tensor product of dg-categories is the bifunctor $\_\otimes^{\mathbb{L}}\_ : dg-Cat \times dg-Cat \to dg-Cat$ defined by*

$$\mathscr{T} \otimes^{\mathbb{L}} \mathscr{T}' := Q(\mathscr{T}) \otimes \mathscr{T}'$$

where $Q(\_)$ is a cofibrant replacement and $\otimes$ is the tensor product of dg-categories.

Remark 4.2.26. *One nice property of the cofibrant replacement in this category is that it can be chosen to be the identity on objects.*

One of the main features of this derived tensor product and one of the main results of [Toë07] is that the monoidal structure turns out to be closed and it is possible to exhibit a precise formula for the internal hom bifunctor.

Theorem 4.2.27. *The monoidal category $(H_{qe}, \otimes^{\mathbb{L}})$ is closed and for any two dg-categories $\mathscr{T}, \mathscr{T}'$ there is a natural isomorphism in H_{qe}*

$$\mathbb{R}\underline{Hom}(\mathscr{T}, \mathscr{T}') \simeq Int((\mathscr{T} \otimes \mathscr{T}')^{op} - Mod^{rqr})$$

This previous result yields a number of corollaries, for example

Corollary 4.2.28. *For any three dg-categories $\mathscr{T}, \mathscr{T}', \mathscr{T}''$, there is a functorial isomorphism of simplicial sets up to equivalence*

$$Map(\mathscr{T} \otimes^{\mathbb{L}} \mathscr{T}', \mathscr{T}'') \simeq Map(\mathscr{T}, \mathbb{R}\underline{Hom}(\mathscr{T}', \mathscr{T}''))$$

Corollary 4.2.29. *Let $\mathscr{T}$ be a dg-category, then the functor*

$$\_\otimes^{\mathbb{L}} \mathscr{T} : dg - cats_k \to dg - cats_k$$

Commutes with homotopy colimits.

And finally,

Corollary 4.2.30. *Given a quasi fully faithful morphism $\mathscr{T} \to \mathscr{T}'$, and a dg-category $\mathscr{T}'$, the induced morphism*

$$\mathbb{R}\underline{Hom}(\mathscr{T}'', \mathscr{T}) \to \mathbb{R}\underline{Hom}(\mathscr{T}'', \mathscr{T}')$$

is quasi-fully faithful.

These results are simply remarking some of the expected properties of a good closed monoidal structure compatible with the rest homotopical information of the category.
With this in mind, let us now proceed to the Toën's Morita theorem for dg-categories.

4.2.3 Morita theory of dg-categories

Coming to this section let us first recall the classical notions of Morita theory for rings. In some way, this theory gives us a weaker form of equivalence from a representation theory point of view in which we identify two rings or two algebras with each other in case their respective categories of modules are equivalent, concretely R, S two non necessarely commutative rings are said to be Morita equivalent if and only if there exists an equivalence of exact categories

$$Mod - R \simeq Mod - S$$

One known nontrivial example of this sort of equivalence comes from a ring and its ring of matrices of any size, in particular this tells us that commutativity is not an invariant for this sort of equivalence.
Nevertheless some important module-theoretic properties do remain invariant under these equivalences and so we are able to deduce certain properties by passing to a better behaved category that we might have more knowledge of.
The nature of the equivalence functor is also very well understood, we have the following celebrated theorem of Eilenberg-Watts:

Theorem 4.2.31. *Given rings R and S and an R-S-bimodule N, the functor*

$$\_ \otimes_R N : Mod - R \to Mod - S$$

between the categories of modules is right exact and preserves small coproducts.
Conversely, if a functor $F : Mod - R \to Mod - S$ is right exact and preserves small coproducts, then it is naturally isomorphic to a functor of the form $\_ \otimes_R N$ with N an R-S-bimodule.

Although versions of this result can be found in many different context for formal reasons, in this particular context we ought to interpret our dg-categories as a sort of many-object dg-algebras and so as a vast generalization of rings. In this sense what remains to do is find a correct analogue concept for the category of modules over such an object and establish exactly what sort of functor is the one we will be using to compare our categories.
Here we would like to mention that this is also a good illustration of the power of dg-categories over triangulated categories as it is hard to give a good description of categories of functors between two derived categories in general, in the dg-category setting however Toën gave a result that reproduces the core idea of the Eilenberg-Watts theorem above but in an adequate context of dg-categories.

Let us start giving some general remarks. By the results discussed in the previous section we have that

$$\hat{\mathscr{T}} := Int(\mathscr{T}^{op} - Mod) \simeq \mathbb{R}\underline{Hom}(\mathscr{T}^{op}, Int(\mathscr{C}(k)))$$

In the homotopy category.
We will make use of the following key lemma about $\mathscr{C}(k)$-enriched categories.

Lemma 4.2.32. *Let M be a $\mathscr{C}(k)$-enriched cofibrantly generated model category. Assume that the domain and codomain of a set of generating cofibrations are cofibrant in M. Let M_0 be a full subcategory of M closed under equivalences, and let $Int(M_0)$ be the full sub dg-category of $Int(M)$ consisting of all objects belonging to M_0. If A is a cofibrant dg-category, letting $Ho(M_0^A)$ denote the full sub-category of $Ho(M^A)$ consisting of objects $\mathscr{F} \in Ho(M^A)$ such that $F(a) \in M_0$ for any $a \in A$. then one has a natural isomorphism*

$$Ho_M(A, Int(M_0) \simeq Iso(Ho(M_0)^A)$$

We refer the reader to the proof of this proof in [Toë07, Lemma 6.2], but let us remark that that idea behind the lemma is that in the case of the Dwyer-Kan model structure for dg-categories, we have that it implies that our dg-categories $\hat{\mathscr{T}}$ of fibrant-cofibrant $\mathscr{T}$-modules correspond to homotopy classes $\mathbb{R}\underline{Hom}(\mathscr{T}^{op}, Int(\mathscr{C}(k)))$ in H_{qe}. Which in turn implies the following useful characterization

$$H_{qe}(\mathscr{T}, \hat{\mathscr{T}'}) \simeq Iso(Ho((\mathscr{T} \otimes^{\mathbb{L}} \mathscr{T}') - Mod)) \simeq H_{qe}(\mathscr{T} \otimes^{\mathbb{L}} \mathscr{T}'^{op}, \hat{k})$$

.

Definition 4.2.33. *Let $\mathscr{T}$ be a dg-category. We denote by $\hat{\mathscr{T}}_{pe}$ the full dg-subcategory of those modules which are compact objects. We call this category the category of perfect $\mathscr{T}$-modules.*

Alternatively, we might write $\mathscr{T}_{pe}$, or even $Perf_{dg}(\mathscr{T})$ instead of $\hat{\mathscr{T}}_{pe}$ specially in the case when working over a field, where there is no need to consider the $(\hat{-})$ operator, or when there is no chance of confusion. For example if A is a dg-algebra, then $Perf(A)$ denotes the category of compact fibrant-cofibrant complexes over A.
Let us denote too by $\mathbb{R}\underline{Hom}_c(\hat{\mathscr{T}}, \mathscr{T}')$ the category of continuous dg-functors between the two dg-categories $\mathscr{T}, \mathscr{T}'$. By this we mean those functors $\hat{\mathscr{T}} \to \hat{\mathscr{T}}'$ such that they commute with direct sums when passing to the homotopy category H_{qe}.

Definition 4.2.34. *We call $\mathbb{R}\underline{Hom}_c(\hat{T}, \hat{T}')$ the dg-category of Morita morphisms from $\mathscr{T} \to \mathscr{T}'$, and perfect Morita morphisms the dg-category $\mathbb{R}\underline{Hom}(\hat{T}_{pe}, \hat{T}'_{pe})$*

We can now formulate the main theorem of this section

Theorem 4.2.35. *[Toë07, Theorem 7.1] Let $\mathscr{T}$ be a dg-category and let $y : \mathscr{T} \to \mathscr{T}^{op} - Mod$ the Yoneda embedding. For any other dg-category $\mathscr{S}$, we have*

1. *The pullback functor* $y^* : \mathbb{R}\underline{Hom}_c(\hat{\mathscr{T}}, \hat{\mathscr{S}}) \to \mathbb{R}\underline{Hom}(\mathscr{T}, \hat{\mathscr{T}})$ *is an isomorphism in* H_{qe}.

2. *The pullback functor* $y^* : \mathbb{R}\underline{Hom}(\hat{\mathscr{T}}_{pe}, \hat{\mathscr{S}}_{pe}) \to \mathbb{R}\underline{Hom}(\mathscr{T}, \hat{\mathscr{S}}_{pe})$ *is an equivalence in* H_{qe}

Proof. We give a rough outline of the proof:
Let $y^* : H_{qe}(\hat{\mathscr{T}} \otimes^{\mathbb{L}} A, \hat{\mathscr{S}})_c \to H_{qe}(\mathscr{T} \otimes^{\mathbb{L}} A, \hat{\mathscr{S}})$ be the pullback of the Yoneda embedding, where $H_{qe}(\hat{\mathscr{T}} \otimes^{\mathbb{L}} A, \hat{\mathscr{S}})_c$ denotes the subset of morphisms such that $\hat{\mathscr{T}} \otimes^{\mathbb{L}} A \to \hat{\mathscr{T}}'$ such that for any $a \in A$ we have that $f(\_, a)$ is a continuous morphism.
Using that $\widehat{A^{op} \otimes^{\mathbb{L}}} \mathscr{S} \simeq Int((A \otimes^{\mathbb{L}} \mathscr{S}) - Mod)$, and Lemma 4.2.32 we see that the equivalence we need can be deduced from showing that the Quillen adjunction

$$y_! : A \otimes^{\mathbb{L}} \mathscr{S} - Mod^{\mathscr{T}} \rightleftarrows A \otimes^{\mathbb{L}} \mathscr{T}^{op} - Mod^{\hat{\mathscr{T}}} : y^*$$

As we have described above. In this case however we can say that the image of $\mathbb{L}y_!$ consists of those $\hat{T}$-modules which corresponds to continuous morphisms.
We have then that there is an equivalence of homotopy categories $\mathbb{L}y_! : Ho(A \otimes^{\mathbb{L}} \mathscr{S} - Mod^{\mathscr{T}}) \simeq A \otimes^{\mathbb{L}} \mathscr{T}^{op} - Mod^{\hat{\mathscr{T}}}$. To show that in fact the image of this morphism are the modules inducing a continuous morphism it is observed that they are precisely the $F \in A \otimes^{\mathbb{L}} \mathscr{S} - Mod^{\mathscr{T}}$ such that for any family of objects $x_i \in \mathscr{T}$, there is an isomorphism in $Ho(A \otimes^{\mathbb{L}} \mathscr{S} - Mod^{\mathscr{T}})$

$$\bigoplus F(x_i) \to F(\oplus x_i)$$

And as $\mathbb{L}y_!$ is fully faithful it is shown that y^* reflects equivalences and then one uses that any $\mathscr{T}$-module can be expressed as a homotopy colimit of objects $Z \otimes^{\mathbb{L}} y_x$ for cochain complexes $Z \in \mathscr{C}(k)$ and $x \in \mathscr{T}$.
The second part of the theorem follows the same strategy, but instead reduces to showing that there is a Quillen equivalence

$$y_! : A \otimes^{\mathbb{L}} \mathscr{S} - Mod^{\mathscr{T}} \rightleftarrows A \otimes^{\mathbb{L}} \mathscr{T}^{op} - Mod^{\hat{\mathscr{T}}_{pe}} : y^*$$

□

Using this result we arrive at the more well known form of the result

Corollary 4.2.36. *Let* $\mathscr{T}$ *and* $\mathscr{S}$ *be two dg-categories, then there exists a natural isomorphism in* H_{qe}

$$\mathbb{R}\underline{Hom}_c(\hat{\mathscr{T}}, \hat{\mathscr{S}}) \simeq \widehat{\mathscr{T}^{op} \otimes^{\mathbb{L}}} \mathscr{S}$$

Proof. Indeed using the theorem, we have

$$\mathbb{R}\underline{Hom}_c(\hat{T}, \hat{\mathscr{S}}) \simeq \mathbb{R}\underline{Hom}(\mathscr{T}, \mathbb{R}\underline{Hom}(\mathscr{S}, \hat{k})) \simeq \mathbb{R}\underline{Hom}(\mathscr{T} \otimes^{\mathbb{L}} \mathscr{S}^{op}, \hat{k}) \simeq \widehat{\mathscr{T}^{op} \otimes^{\mathbb{L}}} \mathscr{S}$$

□

Before going to give the applications of this result we need first to show that to a given derived category we can assign a dg-category, more importantly that this works well with derived categories of perfect complexes and that the choice of enhancement is unimportant for these purposes. That is, that bounded derived categories of perfect complexes over a smooth projective variety have unique dg-enhancements.

4.3 Enhancements of triangulated categories and triangulated functors

We will follow mostly [CS17], [LO10], and [Kel94] on this section.
As we have anticipated through this chapter, the use of dg-categories serves the purpose of enhancing triangulated categories in the sense that to a given triangulated category we would like to have a dg-category which somehow contains the information of the triangulated category. The conceptual simpleness of dg-categories in contrast to triangulated categories will allow for some more systematic use of the dg-categories to perform the required operations.
It is in fact how historically dg-categories were first introduced, in [BK90] dg-categories were introduced precisely with this purpose. Since then different uses have been conceived for similar purposes. Here we present a review of them.
We make our goal a bit more precise

Definition 4.3.1. *Let $\mathscr{K}$ be a triangulated category. We say that a dg-category $\mathscr{T}$ is a dg-enhancement if there exists a triangulated equivalence*

$$\epsilon : H^0(\mathscr{T}) \to \mathscr{K}$$

Definition 4.3.2. *Let $\mathscr{K}$ be a triangulated category and let $\mathscr{T}$ and $\mathscr{S}$ two enhancements $\epsilon : H^0(\mathscr{T}) \to \mathscr{K}, \epsilon' : H^0(\mathscr{S}) \to \mathscr{K}$. We say that $\mathscr{K}$ has a unique enhancement if there is a quasi-functor $f : \mathscr{T} \to \mathscr{S}$ such that $H^0(f)$ is an equivalence of triangulated categories.*

Recall that a Frobenius category is an exact category with enough projectives and injectives, and such that both classes of objects coincide. The stable category of a Frobenius category E, $St(E)$ is the category localized at morphisms which factor through a projective/injective. We say that a sequence of dg-modules over a dg-category $\mathscr{T}$,

$$0 \to L \to M \to N \to 0$$

admits a conflation if there is a section $s \in Hom(M, L)^0$ such that composing with $L \to M$ is the identity.
If $\mathscr{T}$ is a dg-category, then the category of modules over $\mathscr{T}$ is a Frobenius category. We will define a

class of distinguished triangles in $St(\mathscr{T}-Mod)$.
First, let us consider $M \to N$ a morphism of $\mathscr{T}$-modules, and let $M \to I \to M'$ a conflation. Then consider the pushout R given by the roof $I \leftarrow M \to N$.
We declare the triangle

$$M \to N \to R \to M[1]$$

to be a distinguished triangle. It can be shown that these triangles give rise to the structure of a triangulated category on $St(\mathscr{T}-Mod)$ and as it can too be shown that it is equivalent to $H^0(\mathscr{T}-Mod)$, we have that this latter category is canonically triangulated.

Definition 4.3.3. *Let $\mathscr{T}$ be a dg-category. We say that it is pretriangulated if its image under the functor $H^0(y^{\mathscr{T}}) : H^0(\mathscr{T}) \to H^0(\mathscr{T}-Mod)$ is a triangulated subcategory.*

Although not every dg-category is pretriangulated, in light of the fact that $\mathscr{T}-Mod$ is, there exists always a pretriangulated hull.

Definition 4.3.4. *If $\mathscr{T}$ is a dg-category, we let $\mathscr{T}^{pre-tr}$ be the smallest pretriangulated full dg-subcategory of $\mathscr{T}-Mod$. In this way we are adding cones, direct sums and all that might be missing from the original dg-category $\mathscr{T}$*

. Let us denote the triangulated category $H^0(\mathscr{T}^{pre-tr})$ by $tri(\mathscr{T})$, and by $perf(\mathscr{T})$ the full subcategory of compact objects in $tri(\mathscr{T})$.

Remark 4.3.5. *Given a dg-category $\mathscr{T}$, the dg-category $\hat{\mathscr{T}}$ of fibrant-cofibrant objects is a pretriangulated category, as shifts, cones and direct sums are all fibrant and fibrant-cofibrant in the category of dg-modules $\mathscr{T}^{op}-Mod$. We see from Proposition 4.2.21 that in fact $D(\mathscr{T}) \simeq H^0(\hat{\mathscr{T}})$ and we have that $perf(\mathscr{T})$ is enhanced by $\hat{\mathscr{T}}_{pe} = Perf_{dg}(\mathscr{T})$.*

In [BK90] Bondal and Kapranov provided a particular model for the dg-category $\mathscr{T}^{pre-tr}$ by means of twisted complexes. For a dg-category $\mathscr{T}$ let us denote this particular dg-category by $Tw(\mathscr{T})$.
The objects of $Tw(\mathscr{T})$ consist of collections $\{(E_i)_{i\in\mathbb{Z}}, q_ij : E_i \to E_j\}$ where E_i are objects in $\mathscr{T}$ almost all 0, and q_{ij} are morphisms in $\mathscr{T}$ of degree $i-j+1$ such that

$$dq_{ij} + \Sigma_k q_{kj}q_{ik} = 0.$$

Chain complex morphism objects between twisted chains $C = \{E_i, q_{ij}\}$ and $C' = \{E'_i, q'_{ij}\}$ are given by

$$Hom^k(C,C') := \amalg_{l+j-i=k} Hom^l_{\mathscr{T}}(E_i, E'_j).$$

And differential

$$df = d_{\mathscr{T}}f + \Sigma_m(q_{jm}f + (-1)^{l(i-m+1)}fq_{mi}).$$

It can be shown that $Tw(\mathscr{T}^{\oplus})$ is quasi-equivalent to $\mathscr{T}^{pre-tr}$, where $\mathscr{T}^{\oplus}$ is the dg-category $\mathscr{T}$ closed by finite direct sums.

We would like to point out that in the literature both approaches seem to be taken by different authors, since the work of Keller in [Kel94] it has been clear that it is possible to treat dg-categories as we treat categories of complexes and derive them as one does in homological algebra without making explicit the model structure that allows us to do this. In [Toë07] on the other hand one uses the homotopy theory language to relate the different model structures at play.

It is for this reason that in our concrete context of dg-categories of derived categories of varieties, Theorem 4.2.18 is presented in this fashion in the language of model categories, while for example in [Orl16] the result is attributed back to Keller. Both of these claims are evidently correct, but come from different flavors of the same theory. As evidence of this, some of the discussion of dg-enhancements is presented from the latter point of view.

To give a dg-enhancement of the derived category of an abelian category, we recall that by definition this category is simply the localization at quasi-isomorphisms of the category of cochain complexes. Equivalent to this definition is that a derived category is the quotient of the category of cochain complexes at acyclic complexes, those complexes which are cohomologically zero. We denote the collection of all acyclic complexes $Ac(\mathscr{A})$

It should be clear that given an abelian category, its category of cochain complexes $C(\mathscr{A})$ is a pretriangulated dg-category as it is an enhancement of the homotopy category of complexes.

Definition 4.3.6. *Let $\mathscr{T}$ be a dg-category and let $\mathscr{S} \subset \mathscr{T}$ be a full sub dg-category. The quotient $\mathscr{T}/\mathscr{S}$ is the dg-category with the same collection of objects as $\mathscr{T}$ and such that for every $s \in \mathscr{S}$ we add a morphism $s \to s$ in degree -1 so that $d(s \to s) = Id_s$*

With this definition of the quotient of dg-categories it is possible to show that $H^0(\mathscr{T})/H^0(\mathscr{S}) \simeq H^0(\mathscr{T}/\mathscr{S})$, and we can now see that the derived category of an abelian category $\mathscr{A}$ can be enhanced by the quotient $C(\mathscr{A})/Ac(\mathscr{A})$.

The same can be done about bounded (below, above, and both) derived categories by simply taking the corresponding subcategories of $C(\mathscr{A})$ and by consequence of $Ac(\mathscr{A})$.

We can consider the derived category of a general dg-category $\mathscr{T}$ ([Kel94]). Let $Ac(\mathscr{T})$ be the dg-subcategory of dg-modules $\mathscr{T} - Mod$ of those modules acyclic on every object. We have then:

Definition 4.3.7. *Let $\mathscr{T}$ be a dg-category, the derived category $D(\mathscr{T})$ is the quotient $H^0(\mathscr{T} - Mod/Ac(\mathscr{T}))$.*

This derived category is always triangulated as it is a Verdier quotient of the triangulated category $H^0(\mathscr{T} - Mod)$.

For a scheme we have that there always exists a dg-enhancement, but there are two main issues

yet to solve.
The first one is, are these enhancements unique? If they were not unique then we would then need to understand in which ways they are not so to understand when to pick which enhancement. And the second problem is, even if they were unique the model we have describe here is rather complicated to work with in practice, a category of complexes without many properties can be hard to manipulate without much information about the abelian category.
As for the uniqueness, we can immediately present the relevant result in our case

Theorem 4.3.8. *[CS18] Let G be a Grothendieck category with small set of generators S such that*

1. *S is closed under finite coproducts*
2. *Every object of S is a noetherian object*
3. *If $f; A' \to A$ is an epimorphism of G with $A, A' \in S$, then $Ker(f) \in S$.*
4. *For every $A \in S$, there is $N(A) > 0$ such that $D(G)(A, A'[N(A)]) = 0$ for every $A' \in S$*

Then $D(G)^c$ has a unique enhancement

Under this general theorem Canonaco and Stellari are able to show the following very general corollary in the geometric context

Corollary 4.3.9. *If X is a noetherian concentrated algebraic stack with quasi-finite affine diagonal and with enough perfect coherent sheaves, then $Perf(X)$ has a unique enhancement.*

Certainly the spaces we deal with in this thesis fall under the hypothesis above so we can use this to assure ourselves that our derived categories of perfect complexes have unique enhancements.
As for different models, let us present a couple that are usually helpful in applications.

Definition 4.3.10. *Let $\mathscr{T}$ be a dg-category, we call a $\mathscr{T}$-module M homotopically projective (h-projective) if for any acyclic T-module N, $H^0(Hom(N, M)) = 0$.*

The category of h-projective modules $h - proj(\mathscr{T})$ over $\mathscr{T}$ serve as a way to take resolution of T-modules and in this way we can show that $H^0(h - proj(\mathscr{T}))$ is equivalent to $D(\mathscr{T})$ the derived category of $\mathscr{T}$.
This we can use then to give a model for whenever we have an equivalence with a derived category of a dg-category, we can take in turn h-projective modules.
In concrete situations like in geometric settings we have

Theorem 4.3.11. *For a smooth projective space the dg-category of bounded injective complexes of quasi-coherent sheaves with bounded cohomology is an enhancement of the bounded derived category $D^b(X)$*

There is similarly the $Cech$ resolutions in the same situation as above. For this, we consider the sub dg-category of bounded complexes of locally free sheaves of finite rank.
Let us consider an open affine cover $\{U_i, \dots, U_k\}$, then any bounded complex of locally free sheaves of finite rank can be written as

$$\cdots \to 0 \to P^m \to \cdots \to P^{m+p} \to 0 \to \dots$$

Let us consider the resolution of P consisting of coproducts of the form $i_* i^* P^j$ where $i : U \to X$ is the inclusion from an open U of the form $\cap U_i$. The smallest pretriangulated category which contains these resolutions for every bounded complex of locally free sheaves of finite rank can be shown to be an enhancement of $D^b(X)$.
Now that we have seen that our spaces and spaces of geometric nature can be enhanced in great generality, we are still left with the question of whether a triangulated functor can be lifted to a functor between dg-enhancements.
To be more explicit,

Definition 4.3.12. *Let $\mathscr{K}, \mathscr{K}'$ be triangulated categories with dg-enhancements $\mathscr{K}_{dg}, \mathscr{K}'_{dg}$. We say that an exact functor $\mathscr{F} : \mathscr{K} \to \mathscr{K}'$ has a dg-lift if there exists a morphism $f \in H_{qe}(\mathscr{K}_{dg}, \mathscr{K}'_{dg})$ such that $H^0(f) = F$.*

From now on let us fix and denote by $Perf_{dg}(X)$ a dg-enhancement of a derived category of perfect complexes over a space X, and similarly we put $QCoh_{dg}(X)$ for a dg-enhancement of the derived category of quasi coherent sheaves on X.
A consequence of Theorem 4.2.35 is the following

Theorem 4.3.13. *Let X, Y be two quasi-compact and separated schemes over k, assume that one is flat over k. Then there exists an isomorphism in H_{qe}*

$$\mathbb{R}\underline{Hom}_c(QCoh_{dg}(X), QCoh_{dg}(Y)) \simeq QCoh_{dg}(X \times_k Y)$$

Proof. Again we give a rough sketch of the proof.
The first step is to realize that both $QCoh_{dg}(X)$ and $QCoh_{dg}(Y)$ have a compact generator by [BVdB03] and thus are equivalent to a category the category of modules over the endomorphism algebra. To be precise we have an isomorphism in H_{qe}

$$QCoh_{dg}(X) \simeq \hat{A_X}, QCoh_{dg}(Y) \simeq \hat{A}_Y$$

Where A_X and A_Y are respectively the dg endomorphism algebras of the compact generator of X and Y.
It is then shown that the categories of left and right modules over these rings are equivalent, and so there is an isomorphism in H_{qe}, $\hat{A}_X \simeq \hat{A}_X^{op}$ by using the dual perfect complex of the compact generator of $QCoh_{dg}(X)$.

Using that one of the two spaces if flat over k, it can be then shown that $A_X \widehat{\otimes^{\mathbb{L}}} A_Y \simeq QCoh_{dg}(X \times_k Y)$ in H_{qe}.
To show this it is used that the external tensor product of the compact generators is a compact generator of the derived category of the product $X \times_k Y$.
Finally, we obtain

$$\mathbb{R}\underline{Hom}_c(QCoh_{dg}(X), QCoh_{dg}(Y)) \simeq \mathbb{R}\underline{Hom}_c(\hat{A}_X, \hat{A}_Y) \simeq \mathbb{R}\underline{Hom}(A_X, \hat{A}_Y)$$

By Theorem 4.2.35. It follows that

$$\mathbb{R}\underline{Hom}(A_X, \hat{A}_Y) \simeq A_X \widehat{\otimes^{\mathbb{L}}_k} A_Y \simeq QCoh_{dg}(X \times_k Y)$$

□

As a corollary of this, in the smooth case we have

Theorem 4.3.14. *Let X and Y be two smooth and proper schemes over k. Then there exists an isomorphism in H_{qe}*

$$\mathbb{R}\underline{Hom}(Perf_{dg}(X), Perf_{dg}(Y)) \simeq Perf_{dg}(X \times_k Y)$$

Essentially what this theorem is telling us is that there is a correspondence in the smooth case between dg-lifts of exact functors between derived categories of perfect complexes and Fourier-Mukai transforms given by kernels in $Perf_{dg}(X \times_k Y)$.
As it would be good to expand on how to use theorem to see any $\mathscr{F} \in \mathbb{R}\underline{Hom}(Perf_{dg}(X), Perf_{dg}(Y))$ as the corresponding Fourier-Mukai transform with kernel in $Perf_{dg}(X \times_k Y)$, we explain in more detail how to make use of the equivalence.
Let us first start by replacing $Perf_{dg}(X)$ and $Perf_{dg}(Y)$ by the categories of modules $\hat{A}_X$ and $\hat{A}_Y$.
Recall that $\mathbb{R}\underline{Hom}(A, B) \simeq Int(A \otimes^{\mathbb{L}} B^{op} - Mod^{rqr})$ and by the Morita theorem we have that

$$\mathbb{R}\underline{Hom}(Perf(A_X), Perf(A_Y)) \simeq \mathbb{R}\underline{Hom}(A_X, Perf(A_Y)$$

By pulling back the Yoneda embedding along $A_X \to \hat{A}_X$.
Some $\mathscr{F} \in \mathbb{R}\underline{Hom}(Perf(A_X), Perf(A_Y))$ corresponds in $\mathbb{R}\underline{Hom}(A_X, Perf(A_Y)$ to $y^*(\mathscr{F})$ to a right quasi-representable module. This means a $A_X \otimes^{\mathbb{L}} Perf(A_Y)^{op}$-module equivalent in the model structure of dg-modules, to $A_X \otimes^{\mathbb{L}} Perf(A_Y)(\_, M)$ for some $A_X \otimes^{\mathbb{L}} Perf(A_Y)^{op}$-module.
In other words we have that seeing A_X as dg-category with one object $*$, $y^*(\mathscr{F})(*) := \mathscr{F}(Perf(A_X)(*, \_)) \in Perf(A_Y)$ which we see is equivalent to $A_X \otimes^{\mathbb{L}} Perf(A_Y)^{op}(\_, M)$.
This in particular extends Theorem 1.1.37 from smooth projective varieties to smooth proper schemes. Although we for the most part limit ourselves to the former case, we find the arguments presented in this chapter more illuminating for our particular work.

In general, however, it is not true that exact functors between triangulated categories can be lifted not even in nice geometric settings. In [RVdBN19] an explicit example is given of such exact functor that is not of Fourier-Mukai type.
To close this section let us mention the following criteria for uniqueness of dg-enhancements of exact functors between derived categories of quasi-projective spaces.

Theorem 4.3.15. *([Gen16]) Let X, Y be two quasi-projective varieties and let $E, E' \in D^b(X \times Y)$ such that their associated dg-Fourier-Mukai transforms $\Phi^E_{X\to Y}$ and $\Phi^{E'}_{X\to Y}$ are equivalent, and*

$$Hom(\Phi^E_{X\to Y}(\mathscr{O}_X(n)), \Phi^E_{X\to Y}(\mathscr{O}_X(m))[j]) = Hom(\Phi^{E'}_{X\to Y}(\mathscr{O}_X(n)), \Phi^{E'}_{X\to Y}(\mathscr{O}_X(m))[j]) = 0.$$

For all $j < 0$, and for all $n, m \in \mathbb{Z}$ Then $E \cong E'$.

4.4 Morita model structure

In addition to the Dwyer-Kan model structure we have introduced in the previous sections, we can in light of Toën's Morita theorem, we can describe a further refinement of this model structure first introduced by Tabuada in [Tab05a]. In this section we will summarize the results by Tabuada concerning this model category structure.

Definition 4.4.1. *A dg-functor $\mathscr{F} : \mathscr{T} \to \mathscr{T}'$ is a derived Morita equivalence if the extension of scalars functor induces an equivalence between derived categories*

$$D(\mathscr{T}) \simeq D(\mathscr{T}')$$

This is equivalent to asking for the same restriction of scalars functor to induce an exact equivalence between the triangulated categories $perf(\mathscr{T})$ and $perf(\mathscr{T}')$.
Notice too that any quasi-equivalence is a Morita equivalence, as if $\mathscr{T}$ and $\mathscr{T}'$ are quasi-equivalent dg-categories then one can identify quasi-representable modules up to homotopy.
We have in fact that Morita equivalences are the weak-equivalences of a combinatorial model structure

Theorem 4.4.2. *The category $dg - cat_k$ has a combinatorial model category structure with weak-equivalences given by Morita equivalences and cofibrant morphisms given by the cofibrant morphisms of the Dwyer-Kan model structure.*

Let us refer to this model structure as the Morita model structure and denote its homotopy category as H_{mo}. Indeed we can say more,

Theorem 4.4.3. *The Morita model structure is a Bousfield localization of the Dwyer-Kan model structure*

One can show that the fibrant objects of this model structure are precisely those dg-categories $\mathscr{T}$ enhancing $perf(\mathscr{T})$.
Just like the Dwyer-Kan model structure, we have that the homotopy category H_{mo} forms a symmetric monoidal, as the derived tensor product $\otimes^{\mathbb{L}}$ of H_{qe} descends to H_{mo}. This symmetric monoidal category structure is closed and thus we have an internal hom object $\mathbb{R}\underline{Hom}_{mor}$.

Remark 4.4.4. *There exists another model category structure we can impose on the category of dg-categories which also refines quasi-equivalences as weak-equivalences. Instead of asking for the morphisms to induce equivalences between the derived categories of the dg-categories, the quasi-equiconic morphisms are those which induce equivalences between the triangulated hulls $tri(\mathscr{T})$, $tri(\mathscr{T})$.*
These new class of weak-equivalences also forms a cofibrantly generated model category structure and has as cofibrations those of the Dwyer-Kan model structure. Clearly, quasi-equiconic morphisms are also Morita and so this model category structure sits in between the Dwyer-Kan model structure and the Morita model structure.
As in the Morita case, the derived tensor product descends to the homotopy category and gives this a closed symmetric category structure.

Our main interest in this model structure is in the comparison with the homotopy theory of k-linear idempotent complete stable ∞-categories developed in [Coh13] which we will review in Chapter 7.

4.5 Derived noncommutative schemes

We dedicate this section to quickly summarize basic definitions of noncommutative schemes (in the sense of Kontsevich) . This language plays no substantial role in this work but it is in a certain way at the core of the investigation that we have carried out. We follow the notation used in [Orl16, Orl18]
As we have see in Theorem 4.2.18 for a large class of schemes it is possible to give an equivalence in H_{qe} between the dg-enhancement of its derived category with a category of modules over a dg-algebra. In particular this algebra can be shown to be of bounded cohomology and so we obtain a first definition

Definition 4.5.1. *A (derived) noncommutative scheme X over k is a k-linear dg-category equivalent to $Perf_{dg}(R)$ where R is a cohomologically bounded dg-algebra over k.*

A morphism of noncommutative schemes X, Y is simply a quasi-functor $F : X \to Y$. Together with these morphisms we have a category of noncommutative schemes over k which we denote by $NCSch_k$.
We have mentioned that this definition encompasses the geometric case of interest, and so any sufficiently nice scheme (quasi-compact and quasi-separated is enough) is in particular a noncommutative

one.
On the other hand not all noncommutative schemes are schemes, we borrow the following example from [BLS16]

Example 4.5.1. *Let $\Lambda = k[* \to *]$ be the path algebra of the quiver $* \to *$. then the derived category of Λ has a dg-enhancement by [Orl16][Theorem 5.4], however this category has a Serre functor S and it is such that $S^{\otimes 3} = [1]$ and so $D^b(\Lambda)$ cannot be the derived category of a smooth projective variety.*

Now that we know there are noncommutative schemes which are not schemes, it makes sense to describe geometric properties in terms of the properties of the category, for example we have:

Definition 4.5.2. *A noncommutative scheme $X = Perf_{dg}(R)$ is proper if $\bigoplus_{p\in\mathbb{Z}} H^p(Hom_X(M,N))$ are finite dimensional k-vector spaces for any two perfect modules $M, N \in X$.*

It can be shown that this is equivalent to the cohomology algebra $\bigoplus_{p\in\mathbb{Z}} H^p(R)$ being finite dimensional.
Furthermore, if X is a separated scheme, then it is proper if and only if its dg-category $Perf_{dg}(X)$ is proper.

Definition 4.5.3. *A noncommutative scheme X is regular if its triangulated category has a strong generator.*

Similarly, when X is a quasi-compact and separated scheme then it is regular as a noncommutative scheme if and only if it can be covered by open affine's $Spec(R_i)$ with each R_i being of finite global dimension.

Definition 4.5.4. *A noncommutative scheme X is smooth over k if the bimodule $(x, y) \mapsto X(x, y)$ is a compact object of $X \otimes X^{op}$-Mod.*

As with the rest of the properties, in the case where X is of finite type, a commutative scheme is smooth if and only if its category of perfect modules is smooth.
Finally we have:

Definition 4.5.5. *We say a dg-category is saturated if it is proper, smooth and pretriangulated and its triangulated category is idempotent complete.*

A version of the story we have told so far about derived categories of smooth projective varieties regarding Serre functors, semi-orthogonal decompositions and even of Toën's Morita theorem can be reproduced for noncommutative schemes. We refer to [Orl18] for a more through summary of the

theory developed so far.

It is too in this setting where Orlov has developed a more systematic theory of gluing of dg-categories by means of understanding orthogonal decompositions and enhancing the inclusions of such decompositions into bimodules using the theory we reviewed above. By using this one is then able to glue different dg-categories by mean of bimodules and obtain new such categories.

To close, while noncommutative schemes might not be exactly equivalent to a derived category of perfect complexes of a good behaved space, one can still consider the notion of geometric realizations of a noncommutative scheme X. A geometric realization is nothing but a smooth projective scheme Z such that its derived category of quasi-coherent sheaves admits an admissible subcategory such that its enhancement is equivalent to X.

As we saw in the example above not all geometric noncommutative schemes are necessarily of the form of a derived category of perfect complexes over a variety and so they are an interesting class to study in a formal setting.

We denote the full category of saturated noncommutative schemes over k by $NCSch_{sat}(k)$ or $NCSch_{sat}$ if the base is clear from context.

Remark 4.5.6. *One important remark that we will be using in the future is that being smooth and proper is a property closed by taking derived tensor products, but since being triangulated is not preserved in general, we need to pass to perfect modules, so that for $\mathscr{T}, \mathscr{T}' \in NCSch_{sat}(k)$ we have an assignment*

$$NCSch_{sat}(k) \to NCSch_{sat}(\mathscr{T}')$$
$$\mathscr{T} \mapsto \widehat{\mathscr{T} \otimes^{\mathbb{L}}_{k} \mathscr{T}'}_{pe}$$

Chapter 5

Lifting tensor triangulated structures

As we discussed in the previous chapter, triangulated categories often present some technical difficulties. To deal with that we have now seen it is possible to enhance both triangulated categories and functors to the level of dg-categories and there we are then able to obtain some better properties to work around the limitations of triangulated categories. In particular we are interested in lifting the whole notion of tensor triangulated structures to a differential graded setting. Combined with the good homotopical properties of derived categories in their dg-categorical presentations, we are able to deduce some structural properties to help us out in understanding tensor triangulated structures.
The be more concrete, the key ideas rely on exploiting Toën's Morita theorem together with the fact that derived categories of smooth projective varieties are compactly generated and so they are homotopy equivalent to a category of dg-modules over the dg-algebra of endomorphisms of the compact generator. We use the abelian case as a guide following the work of Hovey in [Hov11] where he set astudy closed symmetric monoidal category structures on a category of modules over an algebra. By using classical Morita theory he showed that these correspond to some data composed by a bimodule with multiple compatible right multiplications and a number of morphisms corresponding to the structure morphisms of the monoidal structure. Concretely Hovey uses the Eilenberg-Watts theorem to identify a correspondence between exact bifunctors $\boxtimes$ between module categories, like the tensor product, and $A \otimes (A^{\otimes 2})^{op}$-module Λ

Theorem 5.0.1. *[Hov11, Theorem 2.3] Let Λ be a $A \otimes (A^{\otimes 2})^{op}$-module used to define a bifunctor $\boxtimes$ on the category of R-modules. There is a one-to-one correspondence between additive closed symmetric monoidal structures on R-modules with $\boxtimes$ as the monoidal product and the following data:*

1. *An associativity morphism*
$$\Lambda \otimes \Lambda \to \Lambda \otimes \Lambda$$
2. *A left R-module K and a unit isomorphism $\Lambda \otimes K \to R$ of bimodules*

3. *A commutativity isomorphism* $\Lambda \to \Lambda$

All of which satisfy the pentagon diagram condition, compatibility with left and right unit and the compatibility between associativity, the commutativity morphisms, and the commutativity involution condition.

The advantage of this point of view is that it reduces the study of a structure which can involve many different objects to the structural properties of a single module. In his article, Hovey shows that the classification of such structures in the abelian case can be very disparate, from algebras which have a single such monoidal structure to algebras which admit as many as a proper class (parametrized by isomorphism classes of modules of the algebra).
In our case the task consists first in lifting the tensor product to the dg-setting and then reproducing the correspondence between tensor triangulated structures and the data at the level of dg-modules over a dg-algebra. As we are only interested at the moment in the behavior of the tensor structure at the triangulated level since Balmer's reconstruction depends only on this data, we only need a brute truncation of the homotopical data.

Through this chapter we let A be a smooth and proper k-dg-algebra. We denote by A_{pe} a dg-enhancement of the derived category of perfect complexes over A. Unless we say otherwise we often think concretely of this enhancement as the h-projective dg-enhancement from the previous chapter.
Let us recall Theorem 4.2.35 :

Theorem 5.0.2. *Let $\mathscr{T}$ be a dg-category and consider the Yoneda embedding $y : \mathscr{T} \to \mathscr{T}^{op} - Mod$. Let $\mathscr{T}'$ be another dg-category.*

1. *The pullback functor y^* : $\mathbb{R}\underline{Hom}_c(\mathscr{T} - Mod, \mathscr{T}' - Mod) \to \mathbb{R}\underline{Hom}(\mathscr{T}, \mathscr{T} - Mod)$ is an isomorphism in Hqe.*

2. *The pullback functor y^* : $\mathbb{R}\underline{Hom}(\mathscr{T}_{pe}, \mathscr{T}'_{pe}) \to \mathbb{R}\underline{Hom}(\mathscr{T}, \mathscr{T}'_{pe})$ is an isomorphism in Hqe.*

If we apply this theorem to a dg-algebra seen as a dg-category with a single object, letting $\mathscr{T} = \mathscr{T}' = A$ then this theorem establishes a relationship between quasi-functors $A_{pe} \to A_{pe}$ and bimodules over A. Recall that we have in general a presentation of $\mathbb{R}\underline{Hom}(\mathscr{T}, \mathscr{T}')$ as the dg-category $hproj(\mathscr{T} \otimes^{\mathbb{L}}_X \mathscr{T}'^{op})^{rqr}$ of right quasi-representable homotopically projective $\mathscr{T} \otimes^{\mathbb{L}}_X \mathscr{T}'$-modules. These are homotopically projective modules $\mathscr{F}$ over $\mathscr{T} \otimes^{\mathbb{L}}_X \mathscr{T}'^{op}$ such that for any $x \in \mathscr{T}$ the module $\mathscr{F}(x, \_) \in \mathscr{T}'^{op} - Mod$ is equivalent in the model category of $\mathscr{T}' - modules$ to a representable module $\mathscr{T}'^{op}(y, \_)$ for some object $y \in \mathscr{T}'^{op}$.
What this means is that any quasi-functor $A_{pe} \to A_{pe}$ corresponds to a right quasi-representable h-projective module over $A \otimes^{\mathbb{L}}_X (A_{pe})^{op}$, thus the object $\mathscr{F}(A)$ is the (A, A)-bimodule we are looking for.

Indeed, the A_{pe}^{op}-module given by $\mathscr{F}(A)$ has a left A-action since there is a chain complex morphism

$$A \to End(\mathscr{F}(A))$$

And as we can identify the representable module $\mathscr{T}'^{op}(y, \_) \simeq \mathscr{F}(A)$ with the object $y \in \mathscr{T}'^{op}$ which in turns has a right action by A, we obtain our right and left actions by A.
Furthermore we know that this bimodule induces a quasi-functor equivalent to $\mathscr{F}$ by the assignment

$$M \mapsto \mathscr{F}(A) \otimes M$$

Where the tensor product is a tensor product of $\mathscr{F}(A)$ seen as a right module and $M \in A - Mod$ seen as left module.
This in turn means that there exists an isomorphism in $H^0(A_{pe})$ between $\mathscr{F}(A) \otimes M$ and $\mathscr{F}(M)$ for all $M \in A_{pe}$.
With this idea in mind we will give a characterization of bimodules which produce tensor triangulated category structures at the homotopy level.

5.1 Pseudo dg-tensor structures

Let us take a moment to recall from [Dri04] the general construction of tensor products between dg-modules. Let $\mathscr{T}, \mathscr{R}, \mathscr{S}$ be cofibrant (in the Dwyer-Kan model structure) dg-categories and let $\mathscr{F} \in \mathscr{T} \otimes \mathscr{S}^{op} - Mod$, $\mathscr{G} \in \mathscr{S} \otimes \mathscr{R} - Mod$.

Definition 5.1.1. *We define the tensor product $\mathscr{F} \otimes_{\mathscr{S}} \mathscr{G} \in \mathscr{T} \otimes \mathscr{R} - Mod$ as, for any $t \in \mathscr{T}$, $r \in \mathscr{R}$, the chain complex calculated as the cokernel of*

$$\bigoplus_{x,y \in \mathscr{S}} \mathscr{F}(t,x) \otimes_k \mathscr{S}(y,x) \otimes_k \mathscr{G}(y,r) \to \bigoplus_{z \in \mathscr{S}} \mathscr{F}(t,z) \otimes_k \mathscr{G}(z,r),$$

where the morphism takes a homogeneous element $v \in \mathscr{F}(t,x)$, an homogeneous element $u \in \mathscr{G}(y,r)$ and a homogeneous morphism $f \in \mathscr{S}(y,x)$ to the homogeneous element

$$\mathscr{F}(t,f)(v) \otimes u - (-1)^{|v||u|} v \otimes \mathscr{G}(f,r)(u)$$

With this construction in mind we define the following notion of n-fold dg-bimodules over a dg-category $\mathscr{T}$

Definition 5.1.2. *Let $\mathscr{T}$ be a dg-category. An n-fold dg-bimodule over $\mathscr{T}$ is a dg-module $\mathscr{F} \in \mathscr{T}^{\otimes n} \otimes \mathscr{T}^{op} - Mod$.*

In particular a 0-fold dg-bimodule is nothing but a $\mathscr{T}^{op} - module$ and a 1-fold bimodule is what we usually call a bimodule over $\mathscr{T}$.
A morphism of n-fold dg-bimodules is simply a morphism of dg-modules and we have then a dg-category

denoted by $Bimod^n_{dg}(\mathscr{T})$ with n-fold dg-bimodules as objects and morphism objects given by morphisms of dg-modules.

Notice that the permutation group Σ_n acts on $Bimod^n_{dg}$ by switching the tensor product $\mathscr{T}^{\otimes n}$ and so if $\mathscr{F} \in Bimod^n_{dg}(\mathscr{T})$, $\sigma \in \Sigma_n$ is a permutation and $x \to y \in \mathscr{T}$ where this $\mathscr{T}$ lies in the k-th slot of the product $\mathscr{T} \otimes \cdots \otimes \mathscr{T}$, $x \to y$ induces a morphism of dg-modules

$$\begin{array}{ccc} \mathscr{F}(\_,\ldots,\underbrace{x}_{k},\ldots,\_,?) & \longrightarrow & \sigma\mathscr{F}(\_,\ldots,\underbrace{x}_{\sigma(k)},\ldots,\_,?) \\ \downarrow & & \downarrow \\ \mathscr{F}(\_,\ldots,\underbrace{y}_{k},\ldots,\_,?) & \longrightarrow & \sigma\mathscr{F}(\_,\ldots,\underbrace{y}_{\sigma(k)},\ldots,\_,?) \end{array}$$

So that x, y now lie in the $\sigma(k)$-th slot of $\sigma\mathscr{F}$.

To avoid confusion and ease the reading we follow Hovey's notation and introduce dummy variables to keep track of which $\mathscr{T}$ factor is being taken into account. For example if $\mathscr{F}$ and $\mathscr{G}$ are 3-fold dg-bimodules and $\sigma = (31)$, a morphism $\eta : \mathscr{F} \to \sigma\mathscr{G}$ is better expressed as

$$\eta : \mathscr{F}_{X,Y,Z} \to \mathscr{G}_{Z,Y,X}$$

to indicate that the action of any morphisms $x \to y \in \mathscr{T}$ at the first slot is now carried to an action to the third one on $\mathscr{G}$.

Using the tensor producf ot bimodules described in 5.1.1 we see that there exists, for any pair of natural numbers n, m a way to tensor n-fold bimodules with m-fold bimodules.

If $\mathscr{F} \in Bimod^n_{dg}(\mathscr{T})$ and $\mathscr{G} \in Bimod^m_{dg}(\mathscr{T})$ then we form the tensor product $\mathscr{F} \otimes_{\mathscr{T}} \mathscr{G}$ by using the leftmost $\mathscr{T}$ factor in $\mathscr{G}$ with the right $\mathscr{T}$ factor of $\mathscr{G}$.

In the case we would want to take this tensor product with any of the other $\mathscr{T}$ right factors of $\mathscr{F}$ we can simply consider a permutation σ which permutes the n-th factor with the factor we want to tensor with.

When using the notation above we are able to drop σ from our expression as it is implied from the order of the subindices which permutation we are operating.

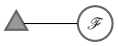
To keep track of which factor is being used to form the tensor product we will extend the notation for morphisms and write for a 2-fold dg-bimodule $\mathscr{F}$, a 3-fold dg-bimodule $\mathscr{G}$

$$\mathscr{F}_{\mathscr{G},X} \otimes \mathscr{G}_{Y,Z,W}$$

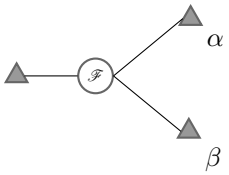
To denote we are forming a 4-fold dg-bimodule by using the first left factor of $\mathscr{G}$ to produce the tensor product with the right factor of $\mathscr{F}$. In our particular case we are for the moment only interested in categories of modules over the a dg-algebra A seen as a single object dg-category and so what we are describing is simply the theory of dg-bimodules over a dg-algebra with multiple compatible left multiplications and the tensor product described here is just the usual tensor product of right and left modules.

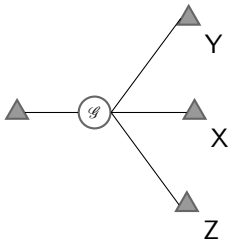
While the notation using subindices is a compact one we find it easier to use a graphical labeled trees notation to keep track of tensor products of the same n-fold dg-bimodule with itself. From now on we specialize to the case of n-fold bimodules over the k-dg-algebra A.

We start first by denoting a right A-module as



The triangle on the left side will denote the right multiplication and the circle labeled by $\mathscr{F}$ is meant to be used to keep track of the module and to its right the right multiplications by A. For example if $\mathscr{F} \in Bimod^2_{dg}(A)$ and $\mathscr{G} \in Bimod^3_{dg}(A)$ then we can express them as:

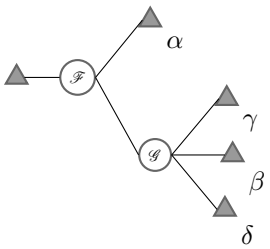


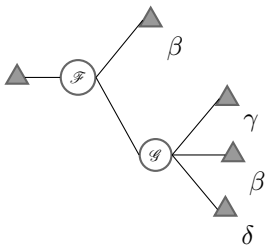


The triangles at each branch will be labeled to keep track of the left A-actions. In the first tree on the left we have two left multiplications the first we label as α while the second by β, meaning that we read these multiplications from top to bottom. Similarly the right tree has has three left A-multiplications although here we have permuted the first and the second actions.
In other words these are respectively the 2-fold dg-bimodule and 3-fold dg-bimodule:

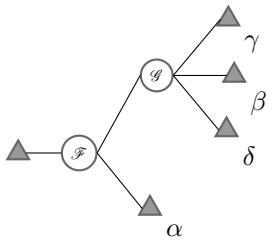
$$\mathscr{F}_{\alpha,\beta}, \mathscr{G}_{Y,X,Z}$$

Finally to express the tensor product of the these two modules we will attach the trees along the right triangle representing the left multiplication we would like to use to form the tensor product and the right A-multiplication of the second dg-bimodule. For example in this case the following tree on the left denotes the product of $\mathscr{F}$ and $\mathscr{G}$ using the second left multiplication of $\mathscr{F}$, while the right tree denotes the tensor product by using the first multiplication of $\mathscr{F}$.





Alternatively we can denote the second tree as



We will make use of Theorem 4.2.35 to show that if have a functor $\mathscr{F} : \mathscr{K}^n \to \mathscr{K}$ from a product of a triangulated category $\mathscr{K}$ such that it is a triangulated functor in each variable, then it is possible to find an appropriate quasi-functor and make it correspond an n-fold dg-bimodule.
Having established our notation we now present a dg-module version of Theorem 2.1 in [Hov11]

Theorem 5.1.3. *Let A be a dg-algebra and let $\boxtimes : H^0(A_{pe}) \times H^0(A_{pe}) \to H^0(A_{pe})$ be an exact functor in each variable. Suppose that for every object $M \in H^0(A_{pe})$, the triangulated functors*

$$M \boxtimes \_ : H^0(A_{pe}) \to H^0(A_{pe})$$
$$\_ \boxtimes M : H^0(A_{pe}) \to H^0(A_{pe})$$

both have unique dg-enhancements R_M and L_M respectively.
Then $L_A(A)$ is a 2-fold dg-bimodule and for any $N \in A_{pe}$ we have

$$H^0(L_A(A) \otimes M \otimes N) \simeq M \boxtimes N$$

Proof. We have to make repeated use of Theorem 4.2.35. First let $R_M \in \mathbb{R}\underline{Hom}(A_{pe}, A_{pe})$ be an enhancement of $M \boxtimes \_$, we know by the derived Morita theorem that this quasi-functor corresponds to a bimodule given by $R_M(A)$. We know by the theorem that for any N, $H^0(R_M(A) \otimes N) \simeq M \boxtimes N$. Now consider the functor $\_ \boxtimes A$ which by hypothesis has a unique enhancement L_A, by using Theorem 4.2.35 again, we know this quasi-functor corresponds to a dg-bimodule $L_A(A)$ and that for any M we have that $H^0(L_A(A) \otimes M) \cong M \boxtimes A$.
However, this latter object is isomorphic to $H^0(R_M(A) \otimes A)$ and as this is a natural isomorphism on M then the functor $R_{\_}(A) \otimes A$ defined by $M \mapsto R_M A \otimes A$ is too an enhancement of $\_ \boxtimes A$. By uniqueness of the enhancement of $\_ \boxtimes A$ we have then $L_A(A) \otimes \_ \simeq R_{\_}(A) \otimes A$ which corresponds to a right quasi-representable A-bimodule.
However, $R_M(A)$ is already a bimodule for every M, in other words $R_A(A)$ has two compatible dg-bimodule structures (in the sense these two same-sided actions by A commute) and so we can consider as an object in $Bimod^2_{dg}(A_{pe})$. Finally we obtain the required isomorphism:

$$H^0(L_A(A) \otimes M \otimes N) \cong H^0(R_M(A) \otimes N) \cong M \boxtimes N$$

□

Even though in general the existence of dg-lifts is not guaranteed as we mentioned in our previous chapter from the example given in [RVdBN19], in our particular context the existence of the lift is only a mild assumption. Indeed as we are coming from a dg-algebra induced from perfect complexes over a smooth projective variety, in that case it is a consequence of the derived Morita theorem that a triangulated functor is of Fourier-Mukai type if and only if it has a dg-lift. Indeed as we saw in 4.3.15 if our functor is the derived functor of an exact functor coming from abelian categories then it is of Fourier-Mukai type. While the condition is a necessary one, as it is unknown whether every triangulated functor that we consider is of Fourier-Mukai type, we find this condition not too strong.
The uniqueness of the lift, however is a stronger condition as we know for sure there are examples of non-uniqueness of the integral kernels that determine these transforms.
Notice, however that we can get away without this, as if the dg-lifts are not unique then in the proof $R_{\_}(A)$ is still an enhancement of $\_ \boxtimes A$ and a quasi-functor corresponding to a dg-bimodule. But as $R_M(A)$ is a dg-bimodule for every M we see that there is a 2-fold dg-bimodule corresponding to $R_{\_}(A) \otimes \_$.
In other words, the uniqueness of the lift guarantees that the bimodule is of the form $L_A(A)$.

With this result in mind, we are finally off to approach tensor triangulated categories as being induced by 2-fold bimodules together with structure maps and their corresponding coherence conditions. Before going further let us remark the following

Remark 5.1.4. *Since any exact functor between derived categories of smooth projective varieties has a right and left adjoints, if $\boxtimes$ is a tensor product of a tensor triangulated category in such triangulated category, the hypothesis of exactness in each variable implies automatically that the symmetric monoidal structure is closed. Similarly any of the triangulated functors that show up in our context are k-linear.*

Let us present the following definition in obvious analogy with the usual (lax) symmetric monoidal category axioms:

Definition 5.1.5. *A pseudo dg-tensor structure in a dg-category $\mathscr{T}$ consists on the data:*

1. *A 2-fold dg-bimodule $\Gamma \in Bimod^2_{dg}(\mathscr{T})$*
2. *An object $U \in \mathscr{T}^{op} - Mod$ called the unit.*
3. *Morphisms of dg-bimodules $\alpha_{X,Y,Z} : \Gamma_{X,\Gamma} \otimes \Gamma_{Y,Z} \to \Gamma_{\Gamma,Z} \otimes \Gamma_{X,Y} \in Bimod^3_{dg}(\mathscr{T})$.*
4. *A morphism of dg-bimodules $\ell_X : \Gamma_{U,X} \otimes U \to \mathscr{T} - Mod \in Bimod^1_{dg}(\mathscr{T})$.*
5. *A morphism of dg-bimodules $r_X : \Gamma_{X,U} \otimes U \to T - Mod \in Bimod^1_{dg}(\mathscr{T})$.*
6. *A morphism $c_{X,Y} : \Gamma_{X,Y} \to \Gamma_{Y,X}$ of dg-bimodules.*

We require that the morphisms $\alpha_{X,Y,Z}$, u_X and $c_{X,Y}$ are all isomorphisms when passing to the homotopy category $H^0(\mathscr{C}(k))$, for all $X, Y, Z \in \mathscr{T}^{op} - Mod$.
Furthermore we require the following homotopy coherence conditions

1. *(Associativity) There exists $\eta \in Hom^{-1}(\Gamma_{X,\Gamma} \otimes \Gamma_{Y,\Gamma} \otimes \Gamma_{Z,W}, \Gamma_{\Gamma,W} \otimes \Gamma_{\Gamma,Z} \otimes \Gamma_{X,Y})$ such that $\alpha^0_{\Gamma_{X,Y},Z,W} \circ \alpha^0_{X,Y,\Gamma_{Z,W}} - \alpha^0_{X,Y,Z} \otimes Id_W \circ \alpha^0_{X,\Gamma_{Y,Z},W} \circ Id_X \otimes \alpha_{Y,Z,W} = d(\eta)$.*

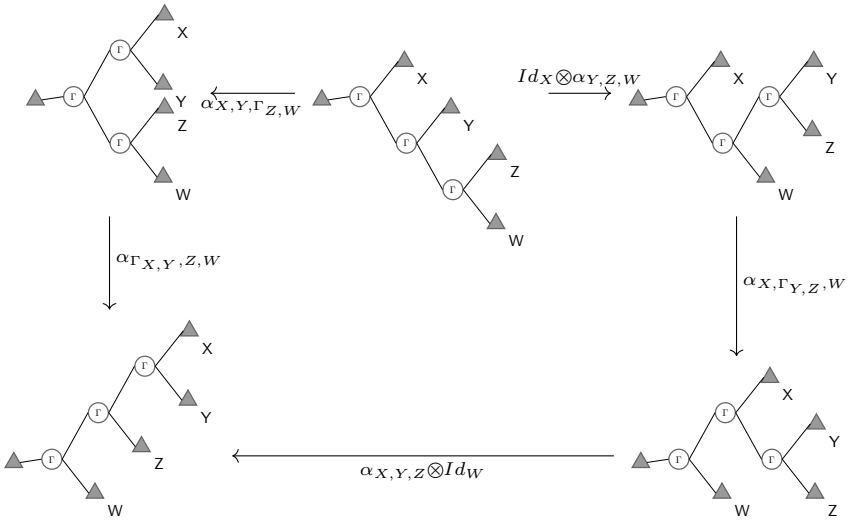
2. *(Unit) There is $\mu \in Hom^{-1}(\Gamma_{X,\Gamma} \otimes \Gamma_{U,Y}, \Gamma_{\Gamma,Y} \otimes \Gamma_{X,U})$ such that $\ell^0_X \otimes Id_Y \circ \alpha^0_{X,U,Y} - Id_X \otimes \ell^0_Y = d(\mu)$*

3. *(Symmetry) The composition $c_{X,Y} \circ c_{Y,X}$ is the identity in $H^0(\mathscr{T} - Mod)$.*

4. *(Unit symmetry) There is $\kappa \in Hom^{-1}(\Gamma_{X,U}, X)$ such that $\ell_X \circ c_{X,U} - r_X = d(\kappa)$*

5. *(Compatibility between associativity and symmetry) There is $\lambda \in Hom^{-1}(\Gamma_{\Gamma,Z} \otimes \Gamma_{X,Y}, \Gamma_{Y,\Gamma} \otimes \Gamma_{Z,X})$ such that $Id_Y \otimes c_{X,Z} \circ \alpha_{X,Y,Z} \circ c_{X,Y} \otimes Id_Z - \alpha_{Y,Z,X} \circ c_{X,\Gamma_{Y,Z}} \circ \alpha_{X,Y,Z} = d(\lambda)$.*

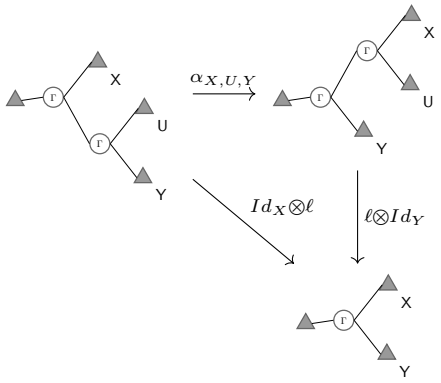
Our requirement that structure morphisms α, u, c are invertible in the homotopy category implicitly implies that the degree 0 part of these morphisms is a cycle.
As the coherence conditions are troublesome to keep track, let us use our graphical notation to draw the diagrams we require



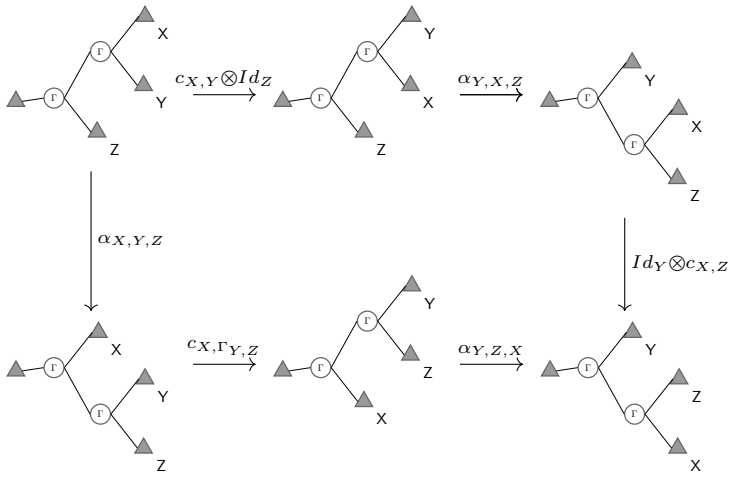
(5.1.1)

This is nothing but the pentagon axiom in a monoidal category except we dont require this composition

to commute but only to commute up to the homotopy η. Similarly,

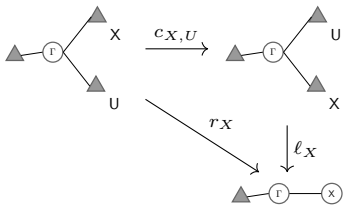


(5.1.2)



(5.1.3)

And finally,



(5.1.4)

We will show that given a pseudo dg-tensor structure on a dg-category $\mathscr{T}$, we can induce a tensor triangulated category structure on A_{pe} under certain conditions.

We could define two such structures to be equivalent if they induce monoidal equivalent tensor triangulated structures on $H^0(\mathscr{T} - Mod)$. This is a weaker notion of equivalence than asking for a functor between dg-enhancements which would respect the pseudo dg-tensor structures through equivalences, it is entirely possible for two non-equivalent 2-fold dg-bimodules Γ, Λ to produce the same tensor triangulated category when passing to the homotopy category, and similarly for the structure morphism and unit object.

An immediate consequence is that if $\mathscr{T}$ and $\mathscr{T}'$ are two Morita equivalent dg-categories, then one

transport any pseudo dg-tensor structure on $\mathscr{T}$ to $\mathscr{T}'$ and vice-versa. It is precisely this principle what we use to study the tensor triangulated structures on a derived category by interesting ourselves in the pseudo dg-tensor structures on A_{pe}.
By abuse of notation and when there is no chance for confusion, we denote by Γ the pseudo dg-tensor structure $(\Gamma, U, \alpha, \ell, r, c)$.
The idea here is that such a structure should correspond to a truncation of a lift of a tensor triangulated structure on $H^0(\mathscr{T} - Mod)$. Let us Let us be more precise in our claims above:

Definition 5.1.6. *A pseudo dg-tensor structure Γ in a dg-category $\mathscr{T}$ is called perfect if the 2-fold dg-bimodule $\Gamma \in Bimod^2_{dg}(\mathscr{T})$ is right quasi-representable and for every $X, Y \in \mathscr{T}^{op} - Mod$, $\Gamma \otimes X \otimes Y$ is quasi-represented by a perfect $\mathscr{T}^{op}$-module.*

Lemma 5.1.7. *A perfect pseudo dg-tensor structure Γ on a dg-category $\mathscr{T}$ induces a tensor triangulated category structure on $H^0(\mathscr{T}_{pe})$.*

Proof. The functor $H^0(\Gamma_{X,Y} \otimes \_ \otimes \_)$ induces a bifunctor on $H^0(\mathscr{T}_{pe})$. Indeed let us denote by $X \boxtimes Y$ the equivalence class of perfect modules quasi-representing $\Gamma_{X,Y} \otimes X \otimes Y$, by hypothesis we know $H^0(X \boxtimes Y)$ is a perfect object and fixing X or Y we get a quasi-representable dg-bimodule which induces triangulated functors

$$\_ \boxtimes Y : H^0(\mathscr{T}_{pe}) \to H^0(\mathscr{T}_{pe})$$

and

$$X \boxtimes \_ : H^0(\mathscr{T}_{pe}) \to H^0(\mathscr{T}_{pe})$$

Using the dg-bimodule morphisms $\alpha_{X,Y,Z}$, ℓ_X, r_X, $c_{X,Y}$ we obtain morphisms $X \boxtimes (Y \boxtimes Z) \to (X \boxtimes Y) \boxtimes Z$, $U \boxtimes X \to X$, $X \boxtimes U \to X$, and $X \boxtimes Y \to Y \boxtimes X$.
The diagrams 5.1.3,5.1.4,5.1.1 and 5.1.2 encode the coherence conditions for this monoidal category with product functor $\boxtimes$. Indeed, the condition that they must commute up homotopy means that when passing to the homotopy category $H^0(\mathscr{T}_{pe})$ they will commute in the usual sense. □

We have to remark a couple things. The first one is that for us, it is necessary to include both right and left unit conditions in the pseudo dg-tensor structure even as if the existence of $c_{X,Y}$ implies that one can obtain one from the other in the 1-categorical setting. In our case however it is necessary to keep track of them as separate entities.
The second thing to mention is that seeing our lemma as a dg-version of Theorem 2.3 of [Hov11] we need to remark that the converse does not hold as-is. Indeed as liftings of objects and morphisms in a triangulated category to a dg-enhancement are far from being unique we cannot expect to have unique -up to isomorphism- dg-bimodules, structure maps and homotopies inducing a certain tensor triangulated category.
And finally, during our review of Balmer's reconstruction and tensor triangulated categories we made

some strong assumptions on the tensor triangulated structures, in particular we supposed that our categories were idempotent complete and rigid in the sense that every object had a tensor dual. Here our perfect pseudo dg-tensor structures will take care of the first assumption as the triangulated category of perfect modules of $\mathscr{T}$ is idempotent complete, however the second assumption needs to be added as a hypothesis.
As in the abelian case we can too encode what a lift of a triangulated tensor endofunctor would be in our dg-setting, and as such a morphism between perfect pseudo dg-tensor structures on a given dg-category $\mathscr{T}$.

Definition 5.1.8. *Let $\mathscr{T}$ be a dg-category and let Γ and Λ be perfect pseudo dg-tensor structures on $\mathscr{T}_{pe}$ with units U and U' respectively. A pseudo dg-tensor functor between Γ and Λ consists of,*

1. *A dg-bimodule Φ*
2. *A morphism of dg-modules $u : U \to \Phi \otimes U'$*
3. *A morphism of dg-bimodules $f : \Gamma_{\Phi,\Phi} \otimes \Phi \otimes \Phi \to \Phi \otimes \Lambda$*

Such that these two morphisms are isomorphisms when passing to H^0.
Furthermore, we need the following coherence conditions

1. *There exists $w \in Hom^{-1}(\Gamma_{\Phi,U'} \otimes \Phi \otimes U, X)$ such that $Id_\Phi \otimes \ell \circ f \circ Id_\Gamma \otimes Id_\Phi \otimes T_{U,\Phi} \circ Id_\Gamma \otimes u \otimes Id_\Phi - \ell \otimes Id_X = d(w)$*
2. *There exists $e \in Hom^{-1}(\Gamma_{\Phi,\Phi} \otimes \Phi \otimes \Phi, \Phi \otimes \Lambda)$ such that $Id_\Phi \otimes c_\Lambda \circ f - f \circ c_\Gamma \otimes T_{\Phi,\Phi} = d(e)$*
3. *There exists $a \in Hom^{-1}(\Gamma_{\Phi,\Gamma} \otimes \Gamma_{\Phi,\Phi} \otimes \Phi \otimes \Phi \otimes \Phi, \Phi \otimes \Lambda \otimes \Lambda)$ such that $f \circ T_{\Phi\otimes\Lambda,\Phi} \circ Id_\Gamma \otimes f \otimes Id_\Phi \circ \alpha_{\Phi,\Phi,\Phi} \otimes T_{\Phi,\Phi} - f \circ T_{\Phi\otimes\Lambda,\Phi} \circ f \otimes Id_\Lambda \otimes Id_\Phi \otimes \alpha = d(a)$*

The structure morphisms and coherence conditions above in the definition are nothing but the structural morphisms and coherence conditions of a monoidal functor with the equivalent underlying category written in terms of bimodules and morphisms between them, with the only difference as in the structure of a pseudo dg-tensor structure being that we have to specify a given homotopy.
The proof of the following is straightforward:

Lemma 5.1.9. *Let $\mathscr{T}$ be a dg-category and let Γ and Λ be perfect pseudo dg-tensor structures on $\mathscr{T}_{pe}$. Then a pseudo dg-tensor functor Φ from Γ to Λ induces a tensor triangulated functor $\mathscr{F}_\Phi : H^0(\mathscr{T}_{pe}, \Gamma) \to H^0(\mathscr{T}_{pe}, \Lambda)$, where $H^0(\mathscr{T}_{pe}, \Gamma)$ and $H^0(\mathscr{T}_{pe}, \Lambda)$ denote the tensor triangulated categories induced by Γ and Λ respectively*

Proof. We saw from Lemma 5.1.7 that Γ and Λ produces tensor triangulated structures on $H^0(\mathscr{T}_{pe})$ and by Toën's Morita theorem, a triangulated functor $\mathscr{T}_{pe} \to \mathscr{T}_{pe}$ corresponds to a $\mathscr{T}$-bimodule Φ.

The structural morphisms u and f of Φ being isomorphisms in $H^0(\mathscr{T}_{pe})$, and the coherence conditions imply that the induced functor between triangulated categories is a symmetric monoidal functor. □

Composition of functors corresponds to tensor product of bimodules. There exists a canonical identity pseudo dg-tensor functor which is given by the dg bimodule $\mathscr{T} - Mod$ (meaning the dg-bimodule which sends $X \in \mathscr{T}^{op}, Y \in \mathscr{T}$ to $\mathscr{T}(X,Y)$) and structural and coherence conditions all given by the canonical isomorphisms $U \to \mathscr{T}-Mod\otimes U'$ and $\Gamma\otimes\mathscr{T}-Mod\otimes\mathscr{T}-Mod \to \mathscr{T}-Mod\otimes\Gamma$. Two pseudo dg-tensor functors Φ and Φ' are said to be equivalent if there exists a morphism of bimodules $\Phi \to \Phi'$ such that $H^0(\Phi) \to H^0(\Phi')$ is an isomorphism and is compatible with the morphisms u and f in the obvious homotopical sense.
We do not describe these natural transformations in detail as we will not be needing the coherence conditions of them but only knowledge that the usual diagrams commute up to a given homotopy.
Previously we said that two perfect pseudo dg-tensor structures were equivalent if they gave rise to equivalent tensor triangulated structures. Although this equivalence is a truncation of the full structure, we have the following lemma.

Lemma 5.1.10. *Let $\mathscr{T}$ be a dg-category, two pseudo dg-tensor structures Γ, Λ are equivalent if there exists a pseudo dg-tensor functor Φ from Γ to Λ given by a dg-bimodule which is invertible under the tensor product of bimodules.*

Proof. Suppose the bimodule Φ is invertible under the tensor product of dg-bimodules, so there exists Φ' such that $\Phi \otimes \Phi'$ is the identity $\mathscr{T} - Mod$ as a bimodule.
If U and U' are the units of Γ and Λ respectively, then we have equivalences $U \to \Phi \otimes \Phi' \otimes U$, and $\Gamma \otimes (\Phi \otimes \Phi') \otimes (\Phi \otimes \Phi') \to (\Phi \otimes \Phi') \otimes \Lambda$. Which are equivalent then to giving an equivalence $U \to U'$ and $\Gamma \to \Lambda$, and so for any $X, Y \in \mathscr{T}_{pe}$, $\Gamma \otimes X \otimes Y \simeq \Lambda \otimes X \otimes Y$, and similarly for the condition about the unit, and then $H^0(\Gamma \otimes X \otimes Y) \cong H^0(\Lambda \otimes X \otimes Y)$.

□

As pointed out before, it is entirely possible that the structures differ at higher degrees and we only need the existence of pseudo dg-tensor functors Φ and Φ' such that $H^0(\Phi\otimes\Phi') \simeq H^0(\mathscr{T}-Mod)$. Let us illustrate what we have so far with an example

Example 5.1.1. *Let X be a smooth projective variety and let us suppose that $D^b(X)$ has a full strong exceptional collection $\{E_1, \ldots, E_m\}$. In this case as we know, the object*

$$E := \bigoplus E_i$$

Is a compact generator and we have thus a homotopy equivalence of dg-categories

$$End(E) - Mod_{dg} \simeq D^b_{dg}(X)$$

between the dg-category of dg-modules over E and a dg-enhancement of $D^b(X)$.
As the exceptional collection is strong there are no higher Ext groups and so this endomorphism algebra is supported in degree 0. Furthermore, we have the following description of this algebra

$$\begin{pmatrix} M_{1,1} & M_{1,2} & \dots & M_{1,m} \\ 0 & M_{2,2} & \dots & M_{2,m} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \dots & k \end{pmatrix}$$

Where $M_{ii} = k$, and M_{ij} is a right module over M_{jl} for every l and a left module over M_{il} for any l.
As we can think of this dg-algebra concentrated in degree 0 as a regular k-algebra, we know that the category of dg-modules over it is simply the category of E-chain complexes.
The usual derived tensor product of $D^b(X)$ can be lifted to a 2-fold dg-bimodule over E and it corresponds to the dg-bimodule given by

$$Hom(E \otimes_X^{\mathbb{L}} E, E) \cong \bigoplus Hom(E_i \otimes_X^{\mathbb{L}} E_j, E) \cong \bigoplus_{i,j,l} Hom(E_i \otimes_X^{\mathbb{L}} E_j, E_l)$$

The best case scenario we can expect is for the variety to have a Picard group isomorphic to $\mathbb{Z}$ and the full strong exceptional collection to be composed of line bundles, in which case one might have a good chance of describing the 2-fold dg-bimodule Γ corresponding to $\otimes_X^{\mathbb{L}}$.

Let us consider an example of this situation and put $X = \mathbb{P}^1$, by using Beilinson's exceptional collection we obtain as we saw before, the endomorphism algebra from Example 1.1.4. The 2-fold bimodule would then correspond to

$$\dots \to \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ k & 0 \end{pmatrix} \to \begin{pmatrix} k & k^2 \\ 0 & k \\ 0 & k \\ 0 & 0 \end{pmatrix} \dots \to$$

Centered in degree 0. Similarly we can calculate the unit object U which corresponds to

$$\begin{pmatrix} k & k^2 \\ 0 & 0 \end{pmatrix}.$$

Following Hovey's paper we can deduce a few things about the classification of tensor triangulated categories on $H^0(A_{pe})$ based on conditions about our dg-algebra A.
Let us mention that unlike tensor triangulated categories, in our setting it is not true that the endomorphism algebra object of the unit is commutative in general. The usual argument for tensor triangulated categories relies in the unit producing isomorphisms $X \otimes U \cong X$ and $U \otimes X \cong X$ and then one applies the Eckmann-Hilton argument to conclude the required commutativity, while in our case all we have is a map from $U \otimes X$ to X and $X \otimes U$ to X.

We know however that when passing to the homotopy category there these maps turn into isomorphisms and so there exists, up to homotopy, maps $X \to U \otimes X$, $X \to X \otimes U$ and any such morphism of modules gives us a product of endomorphisms f, g of U,

$$U \to U \otimes U \overset{f \otimes g}{\to} U \otimes U \to U$$

This product however fails to be associative in any meaningful way as it depends on the choice of the morphisms $U \to U \otimes U$.
As an application of our encoding of tensor triangulated categories through perfect dg-tensor structures at a dg-enhancement we have the following result which is a derived version of a result of Hovey ([Hov11, Prop 4.1]).

Definition 5.1.11. *Let $\mathscr{P}$ be a class of morphisms of chain complexes. We say that it is homotopically replete if for any $f \in \mathscr{P}$ such that there is a square*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{g} & Y \end{array}$$

Where the vertical morphisms induce homotopy equivalences, then the morphism g is in $\mathscr{P}$

Proposition 5.1.12. *Let $\mathscr{T}$ be a dg-category, and Γ a perfect dg-pseudo tensor structure. Let $\mathscr{P}$ is a class of homotopically replete morphisms of chain complexes such that if $f \in \mathscr{P}$ then $X \otimes f$ and $f \otimes Y$, as morphisms of the underlying complexes, are in $\mathscr{P}$ for any pair of dg-modules X, Y.*
If f is a morphism of left $\mathscr{T}$-modules then $f \in \mathscr{P}$ if and only if the morphism $\Gamma \otimes f$ of the underlying complexes is in $\mathscr{P}$. Similarly g a morphism of right $\mathscr{T}$-modules, then $g \in \mathscr{P}$ if and only if $g \otimes \Gamma$ is in $\mathscr{P}$.

Proof. We have that $g \simeq T - Mod \otimes g$ hence as the morphism $\Gamma \otimes U \otimes g \to \mathscr{T} - Mod \otimes g$ induces an isomorphism at the H^0 -level, then $\Gamma \otimes U \otimes g$ is in $\mathscr{P}$. Similarly we have $\Gamma \otimes g \otimes U$. since $\Gamma \otimes g$ is in $\mathscr{P}$ by hypothesis, so is g.
The proof for the other structure on the left is similar. □

With this result in hand, it can be shown that

Corollary 5.1.13. *Let $\mathscr{T}$ be a dg-category and let Γ be a perfect pseudo dg-tensor structure on $\mathscr{T}$. Then let f be a morphism of $\mathscr{T}$-Modules and g a morphism of $\mathscr{T}^{op}$-Modules. Then*

1. *$f \otimes \Gamma \simeq 0$ if and only if $f \simeq 0$. Similarly, $\Gamma \otimes g \simeq 0$ if and only if $g \simeq 0$*
2. *$H^0(f \otimes \Gamma)$ is an isomorphism if and only if $H^0(f)$ is an isomorphism. Similarly for g.*

3. $H^0(f \otimes \Gamma)$ *is a surjection if and only if* $H^0(f)$ *is a surjection.*

Proof. Follows from morphisms $f \simeq 0$, homotopy equivalences and homotopy surjections forming a homotopically replete class. □

As a corollary to this then we have that

Corollary 5.1.14. *Let* A *be a dg-algebra and let* Γ *be a perfect pseudo dg-tensor structure, then* Γ *is faithful as a* $H^0(A)$*-module with either multiplication structure.*

Proof. We let $a \in H^0(A)$ a nonzero class. Then this induces a morphism $A \to A$ of dg-modules given by multiplication, which in turn induces a morphism $\Gamma \otimes H^0(A) \to \Gamma \otimes H^0(A)$, then the the fact that a is not the zero class, implies the induced morphism is also not homotopically zero, thus Γ is faithful as a $H^0(A)$-module. □

In the case where we are working with a dg-category homotopically equivalent to the derived category of a variety which admits a full strong exceptional collection as the ring A is cohomologically concentrated in degree 0, we gain a good amount of control in the sort of 2-fold dg-modules which can show up as a corresponding to a tensor triangulated structure.
Our immediate goal would be then to give a full classification of the tensor triangulated structures on the derived category of $\mathbb{P}^1$ with fixed unit $\mathscr{O}_X$ using the above characterization.
As we have seen, modules over the formal matrix algebra

$$\begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$$

correspond to vector spaces X, Y together with a linear function $\phi : k \otimes X \to Y$, and so the category of dg-modules over this dg-algebra which is supported at 0 is equivalent to the category of chain of complexes of pairs $(X, Y)^*_\phi$.
If we consider a perfect pseudo dg-tensor structure on the category of dg-modules over this algebra, using our corollary above we know that the 2-fold bimodule Γ must be a right dg-module so a chain complex

$$(X, Y)^*_\phi$$

Where both X, Y are different than zero as the module must be faithful in every degree.
We exploit the full strong exceptional decomposition and we see that as the dg-module corresponding to $\mathscr{O}_X$ is fixed as the unit and we know that to calculate Γ we have to calculate the groups of morphisms

$$Hom(\Gamma \otimes \mathscr{O}_X(1) \otimes \mathscr{O}_X(1), \mathscr{O}_X), Hom(\Gamma\otimes, \mathscr{O}_X(1) \otimes \mathscr{O}_X(1), \mathscr{O}_1(1)).$$

Chapter 6

Monoidal dg-categories

In the previous chapters we have discussed the homotopy theory of dg-categories and the symmetric monoidal structure on the homotopy category of dg-categories (both with the Dwyer-Kan and the Morita model structures), and we have also seen how to lift a tensor triangulated structure of derived categories to the setting of dg-categories by use of the homotopical Morita theorem.

However, the way in which we have lifted these structures is not a full lift as we have hinted before. The reason for this is that we are only looking at a truncation of the lift to make sure it behaves as we expect at the H^0-level but we have not looked nor gave any condition on the higher degrees of the pseudo dg-tensor structures.

This is reflected for example when we look at the structural morphisms we have written down as the definition of a pseudo dg-tensor structure, we require for example that our morphisms pass to an isomorphism when the H^0 functor is applied but we have not given any condition on the nature of the inverse at the dg-level nor of the coherence conditions and the homotopies that determine them.

In this chapter we will provide a brief discussion on how to deal with this situation in the abstract from a higher categorical point of view. We point towards [Lur09] and [Lur17] for references to the general theory of ∞-categories as quasicategories in the sense of Boardman-Vogt developed extensively by Joyal and Lurie.

We begin by reviewing the theory of stable ∞-categories by giving a summary of necessary definitions and results which will be of use. Our goal is to give a sketch of the comparison between idempotent stable k-linear $(\infty, 1)-$categories and pretriangulated dg-categories over k described in [Coh13]. Our interest however lies in the comparison as symmetric monoidal $(\infty, 1)-$categories so that the term *monoidal dg-category* often found in the literature can be understood more uniformly.

We would like to understand in which sense does a monoidal dg-category corresponds to a tensor triangulated category. Our choice for a truncated version is however due to the difficulty in performing calculations by hand in the more abstract setting.

6.1 Stable $(\infty, 1)-$categories versus dg-categories

We suppose the reader is familiarized with the basic theory of $(\infty, 1)-$categories and so we present only the definitions that are the most relevant to our work.
Stable $(\infty, 1)-$categories are, along with dg-categories like we saw in Chapter 5, one of the possible higher categorical enhancements for triangulated categories. The enhancement works in essentially the same way, we ought to describe properties of an $(\infty, 1)-$category and then when passing to a homotopy category we must have a natural triangulated category structure on the underlying category.
We follow the presentation in [Lur17]

Definition 6.1.1. *An $(\infty, 1)-$category is pointed if it has a zero object*

Definition 6.1.2. *Let $\mathscr{C}$ be an $(\infty, 1)-$category, a triangle is a diagram*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & Z \end{array}$$

We say that that it is a fiber sequence if it is a pullback diagram and in this case we say that f is a fiber of g. We say that the triangle is a cofiber sequence if it is a pushout diagram, and in this case we say that g is a cofiber of f.
We can denote the triangle simply by

$$X \to Y \to Z$$

Definition 6.1.3. *An $(\infty, 1)-$category is said to be stable if it is pointed, every morphism admits a fiber and a cofiber, and every triangle is a fiber sequence if and only if it is a cofiber sequence.*

Recall that the homotopy category $h\mathscr{C}$ of an $(\infty, 1)-$category $\mathscr{C}$ is a category with objects given by the vertices of $\mathscr{C}$, and morphisms are homotopy classes of edges with the same source and target. This can be shown to form a category which is shown to be equivalent to the homotopy category of $\mathscr{C}$ seen as a simplicial category (cf. [Lur17, Proposition 1.2.3.9]).
To give $h\mathscr{C}$ a triangulated category structure, we describe the suspension as the composition section of the trivial fibration $i : M^{\Sigma} \to \mathscr{C}$ given by evaluation of the initial vertex, where M^{Σ} is the subcategory spanned by diagrams

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0' & \longrightarrow & Y \end{array}$$

And the functor $M^{\Sigma} \to \mathscr{C}$ given by evaluation at the final vertex. We denote this composition of functors by Σ.
Distinguished triangles in the homotopy category will be described as follows

Definition 6.1.4. *Let $\mathscr{C}$ be an $(\infty,1)-$category, a diagram*

$$X \to Y \to Z \to \Sigma X$$

in $h\mathscr{C}$ is a distinguished triangle if and only if there exists a diagram in $\mathscr{C}$ composed of two pushout diagrams

$$\begin{array}{ccccc} X & \xrightarrow{\hat{f}} & Y & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \hat{g}} & & \downarrow \\ 0' & \longrightarrow & Z & \xrightarrow{\hat{h}} & W \end{array}$$

Where $\hat{f},\hat{g}$ represent $X \to Y, Y \to Z$ respectively, and $Z \to \Sigma X$ is given by the equivalence between W and ΣX composed with $\hat{h}$.

We have

Lemma 6.1.5. *Let $\mathscr{C}$ be a pointed $(\infty,1)-$ category which admits cofibers, suppose that Σ is an equivalence. Then $h\mathscr{C}$ is an additive category.*

With this lemma, Lurie obtains

Theorem 6.1.6. *Let $\mathscr{C}$ be a pointed $(\infty,1)-$category which admits cofibers, and suppose that Σ is an equivalence. Then with the suspension functor Σ and the distinguished triangles as above, the category $h\mathscr{C}$ forms a triangulated category.*

We denote by Cat_{∞}^{ex} the ∞-category of stable $(\infty,1)-$categories and ∞-functors which preserve fiber sequences.
We need however to restrict to the ∞-subcategory of idempotent complete stable $(\infty,1)-$categories, meaning those stable $(\infty,1)-$categories such that $h\mathscr{C}$ is idempotent complete as a 1-category ([Lur17] Lemma 1.2.4.6) . We denote this ∞-category by Cat_{∞}^{perf}.
We would now like to put a symmetric monoidal structure on this ∞-category Cat_{∞}^{perf}.

Proposition 6.1.7. *([BZFN10, Proposition 4.4]) The $(\infty,1)-$category $\mathscr{C}_{\infty}^{perf}$ carries a symmetric monoidal structure which for two idempotent complete stable $(\infty,1)-$categories $\mathscr{C}_1,\mathscr{C}_2$ can be characterized by the property that the ∞-category of exact functors $Fun_{ex}(\mathscr{C}_1 \otimes C_2, \mathscr{D})$ is equivalent to the full ∞-category of functors*

$$\mathscr{C}_1 \times \mathscr{C}_2 \to \mathscr{D}$$

preserving colimits in $\mathscr{C}_1$ and $\mathscr{C}_2$ separately.

This tensor product of stable $(\infty,1)-$categories can be seen as described by

$$(Ind(\mathscr{C}_1)\otimes Ind(\mathscr{C}_2))^c$$

Let us unravel this description of the tensor product. Here Ind denotes the Ind-completion of stable $(\infty,1)-$categories and ^{c} the full subcategory of compact objects.
The idea here is that Ind is a functor which sends stable $(\infty,1)-$categories into presentable stable $(\infty,1)-$categories.
Let us recall that a presentable $(\infty,1)-$category $\mathscr{C}$ is a category with all small coproducts, such that $h\mathscr{C}$ is locally small and there is a regular cardinal κ and a κ-compact generator $X\in\mathscr{C}$.
This is equivalent by [Sim07] to say that there exists a combinatorial simplicial model category presenting $\mathscr{C}$ (see [Lur09, A.3.7.6] for the statement in this language).
The product $Ind(\mathscr{C}_1)\otimes Ind(\mathscr{C}_2)$ is thus taken in the symmetric monoidal ∞-category of presentable $(\infty,1)-$categories defined in [Lur07, Lemma 4.1.5].

Now just as we have done with dg-categories seen as categories enriched in the symmetric monoidal category of chain complexes, there is a similar notion of spectral categories which are just categories enriched in the symmetric monoidal category of spectra. Just as we have done for dg-categories, we consider a category Cat_S of spectral categories with functors and natural transformations given as enriched functors and natural transformations.
We would like to put model category structures on this category Cat_S, it turns out it is possible to put a Dwyer-Kan model category structure by declaring an spectral functor to be an equivalence if induces an equivalence of morphism spectra and an equivalence in the underlying categories.
In a similar fashion as we have done in the previous chapters, spectral categories have a notion of derived category. If $\mathscr{A}$ is such a category, the category of modules is denoted by $\hat{A}$ and the homotopy category of this category is what we call the derived category and we denote it by $D(\mathscr{A})$.
With this notion we can then talk about a Morita theory like the one for dg-categories. We say that a spectral functor is a Morita equivalence if it induces an equivalence of derived categories.
As remarked in [BGT14], one can obtain a model category of spectral categories where the weak equivalences are given by Morita equivalences.
Then, if we denote by W the class of those Morita equivalences, then one can consider a fibrant replacement of the category Cat_S in the category of simplicial categories and by taking the homotopy coherent nerve of this category and localizing at W, we obtain an $(\infty,1)-$category $N(Cat_S)[W^{-1}]$.
Again, like dg-categories, spectral categories have a symmetric monoidal structure $\wedge$ given by the smash product of spectra. This product while not forming a monoidal model category structure with neither the Dwyer-Kan nor the Morita model category discussed above, can be derived into a derived smash product $\wedge^{\mathbb{L}}$ forming a closed symmetric monoidal category structure in the homotopy category.
Our goal here will be to put a symmetric monoidal category structure in $N(Cat_S)[W^{-1}]$ from this symmetric monoidal structure.
We say that a spectral category $\mathscr{A}$ is pointwise-cofibrant if every morphism spectrum is a cofibrant

spectrum, and we say that it is flat if $\mathscr{A} \wedge \_$ is a flat functor in that it preserves weak equivalences and colimits.
Blumberg, Gepner and Tabuada give the following result which will be of great importance

Proposition 6.1.8. *([BGT14, Propositon 4.1])*

1. *Every spectral category is functorially Morita equivalent to a pointwise-cofibrant spectral category with the same objects.*
2. *The subcategory of point-wise cofibrant spectral categories is closed under the product $\wedge$.*
3. *A point-wise cofibrant spectral category is flat with respect to the smash product of spectral categories.*
4. *If $\mathscr{A}$, $\mathscr{A}'$ are point-wise cofibrant spectral categories, the smash product $\mathscr{A} \wedge \mathscr{A}'$ computes the derived smash product $\mathscr{A} \wedge^{\mathbb{L}} \mathscr{A}'$*

We denote then by Cat_S^{flat} the category of pointwise-cofibrant spectral categories and we remark that it is a symmetric monoidal category by the previous proposition which further more preserves the equivalences in the model category structure.
Using Proposition 4.1.7.4 of [Lur17], we obtain then that the category $N(Cat_S^{flat})[W^{-1}]$ inherits a symmetric monoidal ∞-category structure with underlying symmetric monoidal category Cat_S^{flat}.
We denote then by $N(Cat_S^{flat})[W^1]^{\otimes}$ and $(Cat_\infty^{perf})^{\otimes}$ these symmetric monoidal $(\infty,1)-$categories.
With this in mind, Blumberg, Gepner and Tabuada formulate and prove their Multiplicative Morita theory theorem

Theorem 6.1.9. *([BGT14, Theorem 4.6]) There is an equivalence of symmetric monoidal $(\infty,1)-$categories*

$$N(Cat_S^{flat})[W^1]^{\otimes} \simeq (Cat_\infty^{perf})^{\otimes}$$

Our goal is now to follow the presentation in [Coh13] to give an equivalence of $(\infty,1)-$categories between the category of idempotent-complete k-linear stable $(\infty,1)-$categories and pretriangulated dg-categories over k.
As we need now to work under a base ring, so that morphism objects form modules over such base, we need to introduce a few concepts.
On the side of spectral categories, we do as follows. Let R be an E_∞-ring spectrum and denote by $RMod$ the category of right modules, meaning the category of spectra with an action $R \wedge M \to M$ compatible in the usual way. We denote the category of perfect modules by $Perf(R)$, its full stable subcategory closed under homotopy colimits and retracts.
Furthermore, a module $M \in Perf(R)$ is called a perfect cell module if it is generated by R under finite colimits and tensor with finite spectra, and we denote the symmetric monoidal category of perfect cell modules by $Perf(R)^{cel}$.

We denote by C_{RMod} the category of categories enriched in $RMod$ for some E_∞-ring spectrum, and a we say that a spectral category $\mathscr{A}$ is a module over $Perf(R)^{cell}$ if there is an action $Perf(R)^{cell} \wedge \mathscr{A} \to \mathscr{A}$ with the usual compatibility conditions. We denote by $Mod_{Perf(R)^{cell}}(Cat_S)$ this category of module categories over $Perf(R)^{cell}$.

Both these categories are in a way categories which we might consider to be linear over R, and both categories carry a Morita model category structure, Cat_{RMod} by declaring those spectral functors to be weak equivalences if they Dwyer-Kan equivalences between categories of modules, and a Morita equivalence in $Mod_{Perf(R)^{cell}}(Cat_S)$ to be those functors which induce a Morita equivalence of the underlying spectral categories.
With these two Morita model structures, Cohn shows:

Proposition 6.1.10. *([Proposition 4.3][Coh13]) Let W be the class of Morita equivalences in Cat_{RMod} and W' the class of Morita equivalences in $Mod_{Perf(R)^{cell}}(Cat_S))$. There is an equivalence of underlying $(\infty,1)-$categories*

$$N(Cat_{RMod})[W^1] \simeq N(Mod_{Perf(R)^{cell}}(Cat_S))[W'^{-1}].$$

Given a ring, let us denote by Hk the Eilenberg-MacLane spectrum associated to k. Tabuada showed in [Tab10] that there exists an equivalence as follows:

Theorem 6.1.11. *There exists a Quillen equivalence between $dg-cat_k$ with the Morita model category structure, and Cat_{HkMod} with the Morita model category structure.*

This in turn implies an equivalence of underlying $(\infty,1)-$categories $N(dg-cat_k)[W^{-1}]$ and $N(Cat_{HkMod})[W^{-1}]$.
Now, given an algebra object $\mathscr{M}$ of a monoidal $(\infty,1)-$category $\mathscr{C}$, there exists a notion of $(\infty,1)-$category $RMod_{\mathscr{M}}(\mathscr{C})$ as the category of right $\mathscr{M}$-module objects of $\mathscr{C}$ seen as a $(\infty,1)-$category left tensored over $\mathscr{C}$. We refer to Construction 4.8.3.21 in [Lur17].

Finally, we are able to describe the main theorem in [Coh13]. First, let us define a functor

$$\theta : N(Mod_{Perf(R)^{cell}}(Cat_S)^{flat})[W'^{-1}] \to Mod_{Perf(R)}(N(Cat_S^{flat})[W^{-1}]^{\otimes})$$

which is induced by the localization functor of monoidal $(\infty,1)-$categories

$$N(Cat_S^{flat})^{\otimes} \to N(Cat_S^{flat})[W^{-1}]^{\otimes}.$$

To be more explicit, one can use

Proposition 6.1.12. *([Coh13] Prop 4.44) Let R be an E_∞-ring specturm, then $Perf(R)$ is a commutative algebra object in $(Cat_\infty^{perf})^{\otimes}$.*

Using the equivalence 6.1.9 we have that $Perf(R)$ is then too a commutative algebra object in $N(Cat_S^{flat})^{\otimes}$. The claim is that this is given by an equivalence

$$\begin{aligned} Mod_{Perf(R)}(N(Cat_S^{flat})^{\otimes}) &\simeq N(Mod_{Perf(R)^{cell}}(Cat_S^{flat})) \\ &\simeq N(Mod_{Perf(R)^{cell}}(Cat_S)) \\ &\simeq N(Mod_{Perf(R)^{cell}}(Cat_S)^{flat})[W'^{-1}] \end{aligned}$$

Using that modules over $Perf(R)^{cell}$ when passing to the nerve become modules over $Perf(R)$.

Theorem 6.1.13. *([Coh13, Theorem 5.1]) The functor*

$$\theta : N(Mod_{Perf(R)^{cell}}(Cat_S)^{flat})[W'^{-1}] \to Mod_{Perf(R)}(N(Cat_S^{flat})[W^{-1}]^{\otimes})$$

is an equivalence of presentable $(\infty,1)-$*categories.*

We refer to Cohn's paper for the proof of the theorem but we mention that the theorem follows directly as an application of the Barr-Beck-Lurie theorem by considering functors

$$N(Mod_{Perf(R)^{cell}}(Cat_S)^{flat})[W'^{-1}] \to N(Cat_S^{flat})[W^{-1}]$$

and

$$Mod_{Perf(R)}(N(Cat_S^{flat})[W^{-1}]^{\otimes}) \to N(Cat_S^{flat})[W^{-1}].$$

Then as a corollary, by combining all the previous results and this equivalence, Cohn obtains

Corollary 6.1.14. *([Coh13] Corollary 5.5) There exists an equivalence of* $(\infty,1)-$*categories*

$$N(dg-cat_k)[W^{-1}] \simeq Mod_{Perf(Hk)}((Cat_{\infty}^{perf})^{\otimes})$$

Proof.

$$\begin{aligned} N(dg-cat_k)[W^{-1}] &\overset{(6.1.11)}{\simeq} N(Cat_{HkMod})[W^{-1}] \\ &\overset{(6.1.10)}{\simeq} N(Mod_{Perf(Hk)^{cell}}(Cat_S))[W'^{-1}] \\ &\overset{(6.1.13)}{\simeq} Mod_{Perf(Hk)}(N(Cat_S^{flat})[W^{-1}]^{\otimes}) \\ &\overset{(6.1.9)}{\simeq} Mod_{Perf(Hk)}((Cat_{\infty}^{perf})^{\otimes}). \end{aligned}$$

□

In the references to this equivalence in the literature it is possible to find an implicit (or even explicit) use of it to expand it to an equivalence of monoidal $(\infty,1)-$categories. Concretely, the $(\infty,1)-$category $Mod_{Perf(Hk)}((Cat_{\infty}^{perf})^{\otimes})$ carries itself a monoidal $(\infty,1)-$category structure given

by the tensor product of modules over $Perf(Hk)$.
On the other hand, the $(\infty,1)-$category $N(dg-cat_k)[W^{-1}]$ also carries a symmetric monoidal $(\infty,1)-$category structure given in the same fashion than the one in $N(Cat_S^{flat})[W^{-1}]$ by using [Lur17, Proposition 4.1.7.4] coming from the derived tensor product.
The idea is analogous to what we have seen in the case of spectral categories, we will follow the presentation in [Rob12] to sketch in more detail as this is of particular interest to us.

Definition 6.1.15. *A locally cofibrant dg-category is a dg-category whose Hom chain complexes are cofibrant.*

As the category of chain complexes $\mathscr{C}(k)$ is a monoidal model category, cofibrations are closed under tensor product and then the tensor product of locally cofibrant dg-categories is again a locally cofibrant dg-category.
This means that the subcategory $dg-cat_k^{lc}$ of locally cofibrant dg-categories is closed under tensor products and then inherits a symmetric monoidal category structure. We use then the fact that tensoring weak equivalences by locally cofibrant dg-categories is again a weak equivalence of dg-categories, we see that $(\infty,1)-$category $N(dg-cat_k^{lc})[W_{lc}^{-1}]$, where W_{lc} denotes those Morita equivalences between locally cofibrant dg-categories, inherits a symmetric monoidal $(\infty,1)-$category structure.
Using the fact that we can choose a cofibrant replacement which is the identity on objects (see 4.2.26), then we see that we can in fact pick cofibrant dg-categories which are locally cofibrant, and so we obtain an equivalence:

$$N(dg-cat_k^{lc})[W_{lc}^{-1}] \simeq N(dg-cat_k)[W^{-1}].$$

Combining these two facts, we can extend the symmetric monoidal $(\infty,1)-$category structure from $N(dg-cat_k^{lc})[W_{lc}^{-1}]$ to the $(\infty,1)-$category $N(dg-cat_k)[W^{-1}]$ by taking cofibrant replacements.

In the literature we find that both the symmetric monoidal $(\infty,1)-$category structure on k-linear stable $(\infty,1)-$categories, and the symmetric monoidal $(\infty,1)-$category structure from $N(dg-cat_k^{lc})[W_{lc}^{-1}]$ are used by implied use of the equivalence of the underlying $(\infty,1)-$categories.
We find, for example, in [GR19] that dg-categories are identified with k-linear stable $(\infty,1)-$categories and the monoidal structure is the one from the latter objects, while for example in [TV22] the monoidal structure is the one from the derived tensor product of dg-categories.
A clear comparison between these two monoidal structures would be good to have, and although as we have seen the equivalences between the underlying $(\infty,1)-$categories seem to be extended to the monoidal structure in different works, we have been unable to verify that the Cohn's functor in 6.1.13 is a monoidal equivalence.
As for the rest of the equivalences in the proof, there are a number of intermediate equivalences which are already monoidal. For example, the equivalences from 6.1.9 and 6.1.10 are monoidal.

As an alternative to Cohn's strategy, [Fao17] has defined a Nerve functor from the category of A_∞-

categories to the category of ∞-categories which when restricted to pretriangulated dg-categories lands in the subcategory of stable $(\infty,1)-$categories.
This A_∞-nerve functor is equivalent with the dg-Nerve functor N_{dg} constructed by Lurie in [Lur17] which is given by truncating the Hom chain complex objects of the input dg-category and using the Dold-Kan theorem to produce a simplicial category, a category enriched on the category of simplicial sets, then using the nerve of simplicial categories which in turn produces a simplicial set, which can be shown to be a ∞-category, gives us a functor from dg-categories to quasi-categories.

6.2 Monoidal dg-categories vs Perfect pseudo dg-tensor structures

In the previous chapter we introduced the concept of (perfect) pseudo dg-tensor structure and we have hinted that this structure is a truncation of a higher homotopical object. Here we would like to expand a bit on this phrasing and show that in fact this can be made into a formal statement.
By using the symmetric monoidal structure on the $(\infty,1)-$category of dg-categories (or equivalently the one on the homotopy category) one is able to speak of commutative algebra objects. In the $(\infty,1)-$categorical setting, let us recall that a symmetric monoidal $(\infty,1)-$category is a fibration of operads $\mathscr{C}^\otimes \to N(Fin_*)$ where Fin_* is the category of pointed finite sets, and we have

Definition 6.2.1. *Let $\mathscr{C}^\otimes \to N(Fin_*)$ a symmetric monoidal $(\infty,1)-$category, a commutative algebra object is a section $N(Fin_*) \to \mathscr{C}^\otimes$ which sends inert morphisms to cocartesian morphisms. The category of commutative algebra objects $Fun_{N(Fin_*)}(N(Fin_*),\mathscr{C}^\otimes)$ is denoted $CAlg(\mathscr{C})$.*

We have then

Definition 6.2.2. *A monoidal dg-category is a commutative algebra object in the symmetric monoidal $(\infty,1)-$category $N(dg-cat_k)[W^{-1}]^\otimes$. Let us denote the category of monoidal dg-categories as $MonDG_k$.*

To make precise the notion of truncation, we recall too

Definition 6.2.3. *Let $\mathscr{C}$ be an $(\infty,1)-$category and $k \geqslant -1$. We say that $C \in \mathscr{C}$ is k-truncated if for any $D \in \mathscr{C}$ the homotopy groups of the mapping space $Map_C(D,C)$ vanish for all $m > k$.*

Lurie shows in [Lur09, Proposition 5.5.6.18] that for $k \geqslant -2$, the inclusion of the $(\infty,1)-$subcategory $\tau_{\leqslant k}\mathscr{C}$ of those objects which are k-truncated in $\mathscr{C}$ has a left adjoint denoted by $\tau_{\leqslant k}$.
We arrive to a conclusion in this chapter with the following theorem suggested by Bertrand Toën:

Theorem 6.2.4. *Let $Ho_2(dg-cat_k)$ be the 2-category given by the homotopy category of the 2-truncation $\tau_{\leqslant 2}N(dg-cat_k)[W^{-1}]$ of the $(\infty,1)-$category $N(dg-cat_k)[W^{-1}]$ of dg-categories. Let $\mathscr{M}$ be a dg-category equivalent to a derived noncommutative scheme then any perfect pseudo dg-tensor structure Γ on $\mathscr{M}$ induces an associative monoid in $Ho_2(dg-cat_k)$ and to any associative monoid structure on $\mathscr{M}$ in $Ho_2(dg-cat_k)$ induces a (possibly non unique) perfect pseudo dg-tensor structure on $\mathscr{M}$.*

Chapter 7

Deformation theory

Beyond exploiting the algebraic properties of lifts of tensor triangulated structures via our pseudo dg-tensor structures to understand the underlying structural properties of the structures in themselves, we are also interested in the idea of understanding the deformation theory of them and how they behave in families.

The concept of deforming a plain monoidal category comes with some difficulties, as one has to fist be very precise in what in the structure one is exactly trying to deform and in which sense.

Normally when talking about deformation theory one is interested in studying the tangent space around a certain point in a moduli space of the structures one is interested in deforming. In the classical algebraic deformation theory of algebras of Gerstenhaber, one would take unital associative algebras and calculate the deformation space of associative algebras, and it is this associativity which takes the main focus in the theory as we is mainly interested in extending the product to find classes of cycles which would correspond to associative algebra structure over the new coefficients.

In the case of monoidal categories we are dealing directly with a categorification of this concept, and as such there are a number of moving pieces that did not exist before at the level of algebras but which appear here at the level of monoid objects in some category.

While describing a general theory for symmetric monoidal categories can be devised, it is only for tensor categories that we might expect to have some control via a cohomology theory for the deformations that would show up.

In [Dav97, Yet98] Davydov and Yetter introduced independently the concept of a deformation for tensor structures, one directly by deforming certain structural properties of the tensor category and the other by deforming monoidal functors and deforming the structural conditions it must admit. In this case the deformation of the identity functor seen as a monoidal functor takes the place of the deformation theory of the tensor structure directly.

Davydov-Yetter cohomology mainly parametrizes the associativity structural isomorphisms and pro-

vides an obstruction for these deformations. One important result that is obtained via this theory is Ocneanu's rigidity theorem which establishes that for fusion categories, which geometrically one can think of corresponding to a collection of points, admit no deformations of their tensor structures.
In this chapter we give a brief review of Davydov-Yetter cohomology and we extend it to the case of categories of dg-modules over a dg-algebra A equipped with a pseudo dg-tensor structure. We will see that deformations of the associativity structural morphism can be described in an analogous manner to the abelian situation.
We finish the chapter with an investigation of the general deformation problem of tensor triangulated structures via our dg-enhancements and see how they relate to our extended Davydov-Yetter cohomology.

7.1 Davydov-Yetter cohomology

Let us start by recalling a number of basic definition from the theory of tensor categories. We refer to the canonical reference [EGNO16].

Definition 7.1.1. *A tensor category is a k-linear rigid abelian monoidal category with biexact tensor product.*
We say that it is finite if it is equivalent to a category of finite representations of a finite dimensional algebra.

To a given finite tensor category we will assign a chain complex in the following way:
Through this section $\otimes$ denotes the monoidal product of a tensor category $\mathscr{A}$.
For any collection of objects $X_1, \dots, X_n \in \mathscr{A}$ we denote by $\otimes^n(X_1, \dots, X_n)$ the full right parenthesization

$$X_1 \otimes (X_2 \otimes (\dots (X_{n-1} \otimes X_n) \dots)$$

Similarly we denote by ${}^n\otimes$ the full left parenthesization

$$(\dots (X_1 \otimes X_2) \dots) \otimes X_{n-1}) \otimes X_n$$

For $n = 1$ this assignment is simply the identity $\mathscr{A} \to \mathscr{A}$, and for $n = 0$ then $\otimes^0$ is the constant functor determined by the monoidal unit in $\mathscr{A}$.
Given any parenthesization X of a product of a collection of objects $X_1, \dots, X_n$ there always exists a sequence of isomorphisms from X to $\otimes^n(X_1, \dots, X_n)$ and to ${}^n\otimes(X_1, \dots, X_n)$.
Given any coherent endomorphism f between any parenthesization of objects $X_1, \dots, X_n$, we will write $\underline{f}$ to denote the morphism f composed and precomposed by the coherent associative isomorphisms.
For example, if $f : X_1 \otimes ((X_2 \otimes X_3) \otimes X_4) \to (X_1 \otimes X_2) \otimes (X_3 \otimes X_4)$ then

$$\underline{f} : ((X_1 \otimes X_2) \otimes X_3) \otimes X_4 \to X_1 \otimes ((X_2 \otimes X_3) \otimes X_4) \to (X_1 \otimes X_2) \otimes (X_3 \otimes X_4) \to X_1 \otimes (X_2 \otimes (X_3 \otimes X_4))$$

This turns any such coherent morphism into a natural transformation in $Nat(^n\otimes, \otimes^n)$.
The idea behind this operation is so that we can turn any morphism between different parenthesizations of a collection of objects $X_1, \ldots, X_n$ into the abelian group of morphisms $^n \otimes (X_1, \ldots, X_n) \to \otimes^n(X_1, \ldots, X_n)$. We however still need to keep track of the sign of this padding by associators, and so the morphism $\underline{f}$ comes with the signature $(-1)^{|sgn(f)|}$ where $sgn(f)$ denotes minimal number of associators $\alpha_{X,Y,Z}$ necessary to take a morphism f into a morphism $f :^n \otimes(X_1, \ldots, X_n) \to \otimes^n(X_1, \ldots, X_n)$. For example the morphism 7.1 has sign equal to 2 as we need to compose by the inverse of $Id_{X_1} \otimes \alpha_{X_2,X_3,X_4}$ and then by $\alpha_{X_1\otimes X_2,X_3,X_4}$.

Definition 7.1.2. *Let $\mathscr{A}$ be a finite tensor category, the Davydov-Yetter complex $DY^*(\mathscr{A})$ is the chain complex defined in degree n by $Nat(^n\otimes, \otimes^n)$. If $f \in DY^n$ a homogeneous element, the differential $d^n : DY^n \to DY^{n+1}$ is defined, on a set of objects $X_1, \ldots, X_{n+1}$ by*

$$d^n(f) := Id_{X_1} \otimes f_{X_1,\ldots,X_n} + \Sigma_i(-1)^i f_{X1,\ldots,X_i\otimes X_{i+1},\ldots,X_{n+1}} + (-1)^{n+1} f_{X_1,\ldots,X_n} \otimes Id_{X_{n+1}}$$

It is a routine calculation to see that $d^2 = 0$ and so this forms a chain complex.
In degree 3 for example, we can calculate the component of the Davydov-Yetter complex consists of natural transformations

$$f_{X_1,X_2,X_3} : (X_1 \otimes X_2) \otimes X_3 \to X_1 \otimes (X_2 \otimes X_3)$$

And has differential $d^2(f)_{X_1,X_2,X_3,X_4}$ given by

$$Id_{X_1} \otimes f_{X_1,X_2,X_3} - f_{X_1\otimes X_2,X_3,X_4} + f_{X_1,X_2\otimes X_3,X_4} - f_{X_1,X_2,X_3\otimes X_4} + f_{X_1,X_2,X_3} \otimes Id_{X_4}$$

With this definitions we can now define the Davydov-Yetter cohomology for finite tensor categories

Definition 7.1.3. *Let $\mathscr{A}$ be a finite tensor category, the Davydov-Yetter cohomology $HDY^*(\mathscr{A})$ is the cohomology of the the Davydov-Yetter complex (DY^*, d^*)*

Let us see by a hand calculation what the third cohomology group looks like

Example 7.1.1. *The kernel of d^3 is composed of those natural transformations f_{X_1,X_2,X_3} such that*

$$Id_{X_1} \otimes f_{X_1,X_2,X_3} - f_{X_1\otimes X_2,X_3,X_4} + f_{X_1,X_2\otimes X_3,X_4} - f_{X_1,X_2,X_3\otimes X_4} + f_{X_1,X_2,X_3} \otimes Id_{X_4} = 0$$

While the image of d^2 is

$$Id_{X_1} \otimes f_{X_1,X_2} - f_{X_1\otimes X_2,X_3} + f_{X_1,X_2\otimes X_3} - f_{X_1,X_2} \otimes Id_{X_3}$$

As we said, we are interested in deformations of the associativity constraint of a finite tensor category $\mathscr{A}$, which is a coherent morphism $a_{X_1,X_2,X_3} : (X_1 \otimes X_2) \otimes X_3 \to X_1 \otimes (X_2 \otimes X_3)$.
We have then to say what we mean precisely by a deformation of this structure. In general the idea is

that we should replace the coefficient ground field k by an algebra that we interpret to represent a small neighborhood of it. Classically in the deformation theory of algebras we are interested in deforming by considering infinitesimal deformations of order n, so by replacing k with the local ring $k[x]/x^{n+1}$.
We think of this ring as an augmented k-algebra, and so equipped with a morphism $k[x]/x^{n+1} \to k$ in the usual way. We always have a morphism in the other direction $i : k \to k[x]/x^{n+1}$.

Definition 7.1.4. *An nth order deformation of a finite tensor category $\mathscr{A}$ over a field k is a finite tensor category $\{\{\mathscr{A}\}\}$ over $k[x]/x^{n+1}$ such that there is a monoidal equivalence $\mathscr{F}$ to $\mathscr{A}$, after restriction of scalars $\_ \otimes k$ of Hom objects.*

Definition 7.1.5. *We say that two deformations are equivalent if there is a monoidal equivalence between them, with its underlying functor equal to the identity functor and such that the restriction of scalars of this monoidal equivalence is the identity after restriction of scalars.*

In other words, as Hom objects of the deformation $\{\{\mathscr{A}\}\}$ are modules over $k[x]/x^n$, the restriction of scalars functor given by tensoring by k over $k[x]/x^n$ gives us an object of k-vector spaces.
To classify deformation classes of the associativity (for example) means that we are looking to classify associativity with coefficient in $k[x]/x^n$ which reduce to our original monoidal structure, and then classify the possible monoidal equivalences between them in the sense that we defined above. To say that we are deforming the associativity condition means that we are looking at deformations such that all the structural morphisms are given by extension of scalars except possibly for the associativity conditions. So for example deforming only the associativity condition means that the symmetry condition $X \otimes Y \to Y \otimes X$ in the deformation is exactly the extension by scalars given by $k[x]/x^n \to k$.

Theorem 7.1.6. *[CY98, Theorem 2.2] Let $\mathscr{A}$ be a finite tensor category, then there is a bijection between the 3rd Davydov-Yetter cohomology $HDY^3(\mathscr{A})$ and equivalence classes of first order deformations of the associativity condition of $\mathscr{A}$.*

Proof. As we are working with first order infinitesimal deformations of the associators then we are looking precisely for a family of coherent morphisms

$$\{\{\alpha\}\} = \alpha + \nu x$$

where α is the associator of our finite tensor category $\mathscr{A}$ over k and ν is another natural endomorphism. But as this associator $\{\{\alpha\}\}$ satisfies the pentagon axiom as it is part of a monoidal category structure, then it must be that

$$Id_X \otimes \{\{\alpha\}\}_{X,Y,Z} + \{\{\alpha\}\}_{X,Y\otimes Z,W} + \{\{\alpha\}\}_{X,Y,Z} \otimes Id_W = \{\{\alpha\}\}_{X,Y,Z\otimes W} + \{\{\alpha\}\}_{X\otimes Y,Z,W}$$

Rearranging this and using that α already satisfies the pentagon condition, we see that the equation above holds true if and only if ν satisfies the condition

$$d^3(\nu) = 0$$

Now if we consider two equivalent deformations $\{\{A\}\}$ and $\{\{A\}\}'$ with associators $\{\{\alpha\}\} = \alpha + \nu_1 x$, $\{\{\alpha\}\}' = \alpha + \nu_2 x$ respectively, by definition it means there exists a monoidal functor $F : \{\{A\}\} \to \{\{A\}\}'$ such that when restricting scalars it reduces to the identity functor.
In particular we have that the structure morphism of the monoidal functor must be of the form $Id + \phi x$ for some natural endomorphism ϕ. As the functor F is monoidal and its underlying functor is the identity, we have a commutative diagram

$$\begin{array}{ccc}
(X \otimes Y) \otimes Z & \xrightarrow{(\alpha+\nu_1 x)_{X,Y,Z}} & X \otimes (Y \otimes Z) \\
\downarrow{\scriptstyle \phi_{X,Y} \otimes Id_Z} & & \downarrow{\scriptstyle Id_X \otimes \phi_{Y,Z}} \\
(X \otimes Z) \otimes Z & & X \otimes (Y \otimes Z) \\
\downarrow{\scriptstyle \phi_{X \otimes Y, Z}} & & \downarrow{\scriptstyle \phi_{X, Y \otimes Z}} \\
(X \otimes Y) \otimes Z & \xrightarrow{(\alpha+\nu_2 x)_{X,Y,Z}} & X \otimes (Y \otimes Z)
\end{array}$$

Which after padding appropriately gives us

$$\alpha_{X,Y,Z} + \nu_{X,Y,Z} x + \phi_{X,Y\otimes Z} x + (Id_X \otimes \phi_{Y,Z}) x = \alpha_{X,Y,Z} + \phi_{X\otimes Y,Z} x + (\phi_{X,Y} \otimes Id_Z) x$$

And so as $d(\phi)$ is equal to

$$Id_X \otimes \phi_{Y,Z} - \phi_{X\otimes Y,Z} + \phi_{X,Y\otimes Z} - \phi_{X,Y} \otimes Id_Z$$

We finally get that $\nu_1 - \nu_2 = d(\phi)$, in other words, ϕ is in the image and then $\{\{\alpha\}\}$ and $\{\{\alpha\}\}'$ are equivalent deformations if and only if the natural morphisms ν_1 and ν_2 that defines them are in the same Davydov-Yetter cohomology class. □

Remark 7.1.7. *It is important to recall that there are indeed examples of two non equivalent monoidal categories where the tensor bifunctor coincide but whose associator structural morphisms can be picked to be different. These are classically constructed by noticing the condition is equivalent to the vanishing of some cocycle condition. This can be done for example in the case of graded vector spaces (see [EGNO16, Example 2.3.8]).*

A fact is that the obstruction to these deformations to extend to infinitesimal deformations of 2nd degree seem to be given by the 4th Davydov-Yetter cohomology group (cf. [EGNO16]).
When the category is semisimple we can say something about the deformation theory in this case:

Theorem 7.1.8. *(Ocneanu's rigidity theorem) Let $\mathscr{A}$ be a semisimple finite tensor category, then $HDY^*(\mathscr{A}) = 0$.*

We give no proof of the previous theorem but we do would like to mention that the theorem can be extended and understood in different ways. In [GHS19] Davydov-Yetter cohomology is interpreted as a comonad homology theory and is extended to a theory with coefficients about the deformation

theory of monoidal functors. In this sense one can rephrase Ocneanu's theorem as a theorem about the vanishing of the Davydov-Yetter cohomology of tensor functors between semisimple tensor categories, with coefficients.

This is in contrast with the interpretation given in [EGNO16] where the proof of the theorem consists in comparing the Davydov-Yetter cohomology with the Hochschild cohomology of the so-called canonical algebra of $\mathscr{A}$.

Also important to mention is that the hypothesis of the theorem cannot be omitted, as there exist explicit examples of nonsemisimple finite tensor categories with nontrivial Davydov-Yetter cohomology groups.

Let us mention that in the work of classification of symmetric closed monoidal structures on categories of modules by Hovey, one of the main results is that there exist rings for which every isomorphism class of modules in its category of modules gives rise to a monoidal structure, but it is unclear whether there are non equivalent monoidal structures which correspond to the same isomorphism class of modules.

It is for this reason that we must remark that the deformation classes parametrized by Davydov-Yetter cohomology only involve the associator condition and so we would like to actually look for deformations of the whole structure at once. From the definition of our deformations what we mean by this is that we should be considering deformations where all the structural morphisms are being deformed all at once.

In [DEN18, Proposition 3.1] it is shown that the collection of tensor autoequivalences of a finite tensor category is an affine algebraic group over k.

Let us reproduce this proof

Theorem 7.1.9. *Let $\mathscr{A}$ be a finite tensor category of k. Then $Aut(\mathscr{A})$ has a natural structure of an affine algebraic group over k.*

Proof. Using the fact that finite tensor categories are equivalent to categories of representations of a finite dimensional algebra, and then using this characterization to write a tensor structure on this category as a 2-fold bimodule together with associativity, unit and symmetry structural morphisms as we have done in Theorem 5.0.1. Each of these structures translates to the following:

Suppose then that we have the data of a ring R, and a bimodule Λ together with structural morphisms. In this setting, a tensor autoequivalence $\mathscr{F}$ (a functor $\mathscr{F} : R-Mod \to R-Mod$ such that $F(X \boxtimes Y) = F(X) \boxtimes F(Y)$ satisfying an associativity and unit condition) is the same as giving an isomorphism $m : A \to A$ and an isomorphism of bimodules $J : \Lambda \to \Lambda^m$ where Λ^m is the bimodule with structure given by multiplication by m, with the condition that the morphism J respects structural isomorphisms and coherencies.

To any given unit $x \in R^*$ we can associate one such point given by (Adx, J_x) where Adx is the adjoint matrix associated to the automorphism given by multiplication by x and J_x is the bimodule isomorphism given, on an element $p \in \Lambda$ by $J_x(p) = xp(x^{-1} \otimes x^{-1})$. So that there is a normal algebraic group given by gauge transformations.

The claim is that a tensor autoequivalence $\mathscr{F}$ of $\mathscr{A}$ is isomorphic to the identity functor if and only if there is $x \in R^*$ such that $(Adx, J_x) = \mathscr{F}$. Indeed such a tensor autoequivalence would be given by the scalar multiplication and the condition on the bimodule translates to a trivial scaling of the structural morphisms. □

In a similar way one can construct the functor that associates, to any commutative $k - algebra$, the group of tensor autoequivalences of our finite tensor category $\mathscr{A}$.

$$Aut : B \mapsto Aut_B^{\otimes}(\mathscr{A}).$$

For a commutative k-algebra B.
We should also mention that it is possible to give the deformation complex DY^* a cup product operation

$$\cup : DY^m \times DY^m \to DY^{n+m}$$

Simply by taking, for $f \in DY^n(\mathscr{A})$ and $g \in DY^m(\mathscr{A})$, the product

$$(f \cup g)_{X_1,\dots,X_m,X_{m+1},\dots,X_{m+n}} \mapsto (-1)^{(m-1)(n-1)} f_{X_1,\dots,X_m} \otimes g_{X_1,\dots,X_m}$$

Following the construction in [BD20, Corollary 3.5], this descends into a cup product in cohomology $\cup : HDY^n(\mathscr{A}) \times HDY^m(\mathscr{A}) \to HDY^{n+m+1}(\mathscr{A})$.
As usual this cup product induces bracket operations [,] and [[,]] in cohomology of degree 1 and 2.
By considering the functor of points of the moduli space of tensor structures on a given finite abelian category $\mathscr{A}$,

$$B \mapsto Tens(\mathscr{A} \otimes B).$$

For a commutative k-algebra B. It is possible to show that

Theorem 7.1.10. *Let $\mathscr{A}$ be a finite tensor category, then the 3rd cohomology group $HDY^3(\mathscr{A})$ corresponds to the tangent space of the moduli space $Tens(\mathscr{A})$*

As usual the constructions above generalize to the deformation of tensor functors and our case of deformation of the tensor category corresponds to the identity functor case.
Let us too remark that the functor of points $Tens$ is treated here in the abstract and there is no claim in the representability of the moduli problem.
Our goal for the following is to define a Davydov-Yetter cohomology for tensor triangulated structures and study some of their properties in light of the work done in the abelian case.

7.2 Deforming tensor triangulated structures

In the previous section we saw how to construct a complex such that its cohomology behaves as the tangent space of a finite tensor category, in particular with respect to the associativity structural

morphisms. Here we will construct a similar complex by using our perfect pseudo dg-tensor structures (see 5.1.5).

From now on we will be interested in working with the category smooth proper derived noncommutative schemes (see 4.5.1) $NCSch^{pr}_{sm}$. As we have a homotopical equivalence between the derived categories of interest and a subcategory of such objects we exploit this to work our definitions at the dg-enhancement level. In this sense, whenever we refer to a dg-category $\mathscr{T}$ it is understood to be an object in $NSch^{pr}_{sm}$. We remark too that in general when working with dg-categories and dg-categories of dg-modules the change of basis operation $\_\otimes^{\mathbb{L}} A$ for a dg-algebra A does not in general preserve categories of perfect complexes, in the following however we try to keep notation as simple as possible and so we write $\otimes A$ for the operation $\widehat{\_\otimes^{\mathbb{L}} A_{pe}}$ which takes a dg-category and sends it to the dg-category tensored by the dg-algebra A (in the homotopy category of dg-categories using the derived tensor product) after passing by its perfect closure.

Let us now commence to define our deformation complexes for our lifted tensor triangulated structures.

Definition 7.2.1. *Let $\mathscr{T}$ be a dg-category and Γ a perfect pseudo dg-tensor structure. For any $n \in \mathbb{N}$ we denote by ${}^n\Gamma$ the full left parenthesization of the 2-fold dg-bimodule Γ. This means we use the first multiplication to tensor Γ with itself n-1 times by using this multiplication structure repeatedly.*
This means that if $X_1, \dots, X_n \in \mathscr{T}_{pe}$ then when taking tensor products we have

$$ {}^n\Gamma : X_1, \dots, X_n \mapsto \Gamma_{\Gamma,X_1} \otimes \Gamma_{\Gamma,X_2} \otimes \Gamma_{\Gamma,X_3} \otimes \cdots \otimes \Gamma_{X_{n-1},X_n} \otimes X_n \otimes X_{n-1} $$

Similarly we have Γ^n, which for a collection $X_1, \dots, X_n \in \mathscr{T}_{pe}$

$$ {}^n\Gamma : X_1, \dots, X_n \mapsto \Gamma_{X_1,\Gamma} \otimes X_1 \otimes \Gamma_{X_2,\Gamma} \otimes \cdots \otimes \Gamma_{X_{n-1},X_n} \otimes X_{n-1} \otimes X_n $$

Definition 7.2.2. *Let Γ be a perfect pseudo dg-tensor structure for the dg-algebra A associated to a tensor triangulated structure on $H^0(A_{pe})$. Let $DY^{*,*}_{dg}$ be the double chain complex defined by*

$$ DY^{n,*}_{dg}(A) := Bimod^n_{dg}({}^{n-1}\Gamma,\ \Gamma^{n-1})^{*-1} $$

The complex of morphisms of dg-functors between the n-fold bimodule of totally left parenthesized product of Γ with itself and the n-fold bimodule of totally right parenthesized product of Γ.
The vertical differential $d^{n,m}_v : DY^{n,m}_{dg}(A) \to DY^{n,m+1}_{dg}(A)$ is the differential given by the $Bimod(\ ,\ )$ dg-functor.
The horizontal differential $d^{n,m}_h : DY^{n,m}_{dg}(A) \to DY^{n+1,m}_{dg}(A)$ is given, for $\eta^* \in DY^{n,m}_{dg}(A)$ and a collection $X_1, \dots, X_n \in A_{pe}$

$$ \begin{aligned} d^{n,m}_h(\eta)_{X_1,\dots,X_n} = \\ Id_{X_1} \otimes \eta^m_{\Gamma \otimes_{\mathscr{C}(k)} X_2 \otimes X_3,\dots,X_n} + \\ \Sigma_i (-1)^i \eta^m_{X_1,\dots,\Gamma \otimes_{\mathscr{C}(k)} X_i \otimes X_{i+1},\dots,X_n} + (-1)^{n+1} \eta^m_{X_1,\dots,X_{n-1}} \otimes Id_{X_n} \end{aligned} $$

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow \\
\cdots \longrightarrow & DY_{dg}^{n,m+1}(A) = Bimod_{dg}^{n}({}^{n-1}\Gamma,\ \Gamma^{n-1})^{m} & \xrightarrow{d_h^{n,m+1}} & DY_{dg}^{n+1,m+1}(A) & \longrightarrow \cdots \\
 & & \uparrow{\scriptstyle d_v^{n,m}} & & \uparrow{\scriptstyle d_v^{m+1,n}} \\
\cdots \longrightarrow & DY_{dg}^{n,m}(A) = Bimod_{dg}^{n}({}^{n-1}\Gamma,\ \Gamma^{n-1})^{m-1} & \xrightarrow{d_h^{n,m}} & DY_{dg}^{n+1,m}(A) & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow \\
 & & \vdots & & \vdots
\end{array}$$

A low degree example would be

$$\begin{array}{ccccc}
 & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow \\
\cdots \longrightarrow & Bimod_{dg}^{2}(\Gamma,\Gamma)^{2} & \xrightarrow{d_h^{2,3}} & Bimod^{3}({}^{2}\Gamma,\Gamma^{2})^{2} & \longrightarrow \cdots \\
 & \uparrow{\scriptstyle d_v^{2,1}} & & \uparrow{\scriptstyle d_v^{2,1}} & \\
\cdots \longrightarrow & Bimod^{2}(\Gamma,\Gamma)^{1} & \xrightarrow{d_h^{2,2}} & Bimod^{3}({}^{2}\Gamma,\Gamma^{2})^{1} & \longrightarrow \cdots \\
 & \uparrow & & \uparrow & \\
 & \vdots & & \vdots &
\end{array}$$

To make sense of the expression given by the differential, we proceed as in the abelian case, where the sacrifice that had to be made by working with non strict monoidal categories was that we had to introduce the padding construction to form additive groups of morphisms between different parenthesized products of a collection of objects. In this situation one could instead pass to a strict category using MacLane's coherence result and have that these objects are equal and there is no need for padding.

In our context however our only option is padding our morphisms, so let us explain how do we proceed: Recall that the associativity coherence for a pseudo dg-tensor structure is given by a morphism of dg-bimodules $\alpha_{X,Y,Z} : \Gamma_{\Gamma,Z} \otimes \Gamma_{X,Y} \to \Gamma_{X,\Gamma} \otimes \Gamma_{Y,Z}$ such that it becomes invertible when passing to $H^0(\mathscr{C}(k))$ for every triple of objects X, Y, Z.

Just as before whenever we have a morphism between two parenthesizations we would like to pad it so that it becomes a morphism in $Bimod^*({}^n\Gamma, \Gamma^n)(A)$.

Let us take then $n \in \mathbb{N}$ and a partition $n = n_1 + \cdots + n_k$.

A parenthesization of length $n \in \mathbb{N}$ of Γ is a tensor product of factors of the form ${}^{n_i}\Gamma$ and Γ^{n_j} along any of the two multiplications of Γ, for $0 \leqslant i, j \leqslant k$.

Whenever we have a morphism between two such parenthesizations,

$$f \in Bimod^*({}^{n_1}\Gamma \otimes \cdots \otimes \Gamma^{n_k}, \Gamma^{n'_1} \otimes \cdots \otimes \Gamma^{n'_l})$$

, we would like to turn this into a morphism in $Bimod^*({}^n\Gamma, \Gamma^n)$

The obstruction to do this as we did before is that we only know that the coherent morphism α is

invertible when taking 0th chain cohomology H^0. It is not enough to say that as this is invertible then in each step of the padding to consider a lift of the inverse α^{-1} as there are many and since we are only dealing with the truncation in low degree, the complexes in higher degree can change a lot from one another. So in reality what we must do is choose once and for all the inverse α^{-1} making α invertible. Once the morphism is chosen we proceed as before, and then whenever we have for example a morphism

$$f : \Gamma_{X_1,\Gamma} \otimes \Gamma_{\Gamma,X_4} \otimes \Gamma_{X_2,X_3} \to \Gamma_{\Gamma,\Gamma} \otimes \Gamma_{X_1,X_2} \otimes \Gamma_{X_3,X_4}$$

We can compose by products of the identity between dg-bimodules $\Gamma \to \Gamma$ and α and α^{-1}. In this way we can obtain a morphism

$$\underline{f} : \Gamma_{\Gamma,X_4} \otimes \Gamma_{\Gamma,X_3} \otimes \Gamma_{X_1,X_2} \to \Gamma_{X_1,\Gamma} \otimes \Gamma_{\Gamma,X_4} \otimes \Gamma_{X_2,X_3} \to \Gamma_{\Gamma,\Gamma} \otimes \Gamma X_1, X_2 \otimes \Gamma_{X_3,X_4} \to \Gamma_{X_1,\Gamma} \otimes \Gamma_{X_2,\Gamma} \otimes \Gamma_{X_3,X_4}$$

Now that we have defined our double complex, we can define

Definition 7.2.3. *Let $\mathscr{T}$ be a dg-category and Γ a perfect pseudo dg-tensor structure, the Davydov-Yetter complex is the total complex $Tot(DY_{dg}^{*,*})(\mathscr{T})$ of the double complex $DY_{dg}^{*,*}(\mathscr{T})$ with the usual differential $d_{tot} := d_v + (-1)^{|v||h|} d_h$. Where $\mid v \mid$ and $\mid h \mid$ denote the degree of the differentials d_h and d_v.*

We put ${}^0\Gamma = \Gamma^0$ the bimodule Id corresponding to the identity pseudo functor, and ${}^{-1}\Gamma = \Gamma^{-1}$ is the unit object U, and ${}^n\Gamma = \Gamma^n = 0$ for any $n \leqslant -2$.
As before we calculate the cohomology of the dg-Davydov-Yetter complex

Definition 7.2.4. *Let $\mathscr{T}$ be a dg-category and Γ a perfect pseudo dg-tensor structure and consider its Davydov-Yetter complex $Tot(DY_{dg}^{*,*})(\mathscr{T})$. The total cohomology of this complex is the Davydov-Yetter cohomology of the perfect pseudo dg-tensor structure and we denote it by $HDY_{dg}^{*}(\mathscr{T})$ or by $HDY_{dg}^{*}(\mathscr{T}, \Gamma)$ if there is ambiguity in which perfect pseudo dg-tensor structure is being considered.*

In lower degrees we have components of the total complex given by

$$Tot^3(DY_{dg}^{*,*}) = DY_{dg}^{3,0}(\mathscr{T}) \oplus DY_{dg}^{2,1}(\mathscr{T}) \oplus DY_{dg}^{1,2}(\mathscr{T}) \oplus DY_{dg}^{0,3}(\mathscr{T}) =$$
$$Bimod^3({}^2\Gamma, \Gamma^2)^{-1} \oplus Bimod^2(\Gamma, \Gamma)^0 \oplus Bimod(\mathscr{T}_{pe}, \mathscr{T}_{pe})^1 \oplus \mathscr{T}_{pe}(U,U)^2$$

And

$$Tot^4(DY_{dg}^{*,*}) = DY_{dg}^{4,0}(\mathscr{T}) \oplus DY_{dg}^{3,1}(\mathscr{T}) \oplus DY_{dg}^{1,3}(\mathscr{T}) \oplus DY_{dg}^{2,2}(\mathscr{T}) \oplus DY_{dg}^{3,1}(\mathscr{T}) \oplus DY_{dg}^{4,0} =$$
$$Bimod^4({}^3\Gamma, \Gamma^3)^{-1} \oplus Bimod^3({}^2\Gamma, \Gamma^2)^0 \oplus Bimod^2(\Gamma, \Gamma)^1 \oplus Bimod^1(\mathscr{T}_{pe}, \mathscr{T}_{pe})^2 \oplus \mathscr{T}_{pe}(U,U)^3$$

Let us calculate as before the action of the total differential in these degrees. If we let $\eta^3 =$

$(\eta^{3,1}, \eta^{2,1}, \eta 1, 2, \eta^{0,3})$ then

$$d^3_{tot}(\eta^3) = (d_h(\eta^{3,1}), d_h(\eta^{2,1}) + d_v(\eta^{2,0}), d_v(\eta^{2,1} + d_h(\eta^{2,1}), d_v(\eta^{2,1}) + d_h(\eta^{0,3}), d_v(\eta^{0,3})) =$$
$$(\eta^{3,0}_{X,Y,Z} \otimes Id_W - \eta^{3,0}_{\Gamma\otimes X\otimes Y,Z,W} + \eta^{3,0}_{X,\Gamma\otimes y\otimes Z,W} - \eta^{3,0}_{X,Y,\Gamma\otimes Z\otimes W} + Id_X \otimes \eta^{3,0}_{Y,Z,W},$$
$$\eta^{2,1}_{X,Y} \otimes Id_Z - \eta^{2,1}_{\Gamma\otimes X\otimes Y,Z} + \eta^{2,1}_{X,\Gamma\otimes Y\otimes Z} + Id_X \otimes \eta^{2,1}_{Y,Z} + d_v(\eta^{3,0}),$$
$$\eta^{1,2}_X \otimes Id_Y - \eta^{1,2}_{\Gamma\otimes X\otimes Y} + Id_X \otimes \eta^{2,1}_Y + d_v(\eta^{2,1}),$$
$$d_v(\eta^{0,3}))$$
(7.2.1)

And to calculate the kernel of d^4_{tot}, we see that it corresponds to those $\eta^4 = (\eta^{4,0}, \eta^{3,1}, \eta^{2,2}, \eta^{1,3}, \eta^{0,4})$ such that

$$d^4_{tot}(\eta^4) =$$
$$(d_h(\eta^{4,0}), -d_h(\eta^{3,1}) + d_v(\eta^{4,0}), d_h(\eta^{2,2}) + d_v(\eta^{3,1}), d_h(\eta^{1,3}) + d_v(\eta^{2,2}), d_h(\eta^{0,4}) + d_v(\eta^{1,3}, d_v(\eta^{0,4})) =$$
$$(Id_X \otimes \eta^{4,0}_{Y,Z,W,R} - \eta^{4,0}_{\Gamma\otimes X\otimes Y,Z,W,R} + \eta^{4,0}_{X,\Gamma\otimes Y\otimes Z,W,R} - \eta^{4,0}_{X,Y,\Gamma\otimes Z\otimes W,R} + \eta^{4,0}_{X,Y,Z,\Gamma\otimes W\otimes R} - \eta^{4,0}_{X,Y,Z,W} \otimes Id_R,$$
$$-Id_X \otimes \eta^{3,1}_{Y,Z,W} + \eta^{3,1}_{\Gamma\otimes X\otimes Y,Z,W} - \eta^{3,1}_{X,\Gamma\otimes Y\otimes Z,W} + \eta^{3,1}_{X,Y,\Gamma\otimes Z\otimes W} - \eta^{3,1}_{X,Y,Z} \otimes Id_W + d_v(\eta^{4,0}),$$
$$Id_X \otimes \eta^{2,2}_{Y,Z} - \eta^{2,2}_{\Gamma\otimes X\otimes Y,Z} + \eta^{2,2}_{X,\Gamma\otimes Y\otimes Z} - \eta^{2,2}_{X,Y} \otimes Id_Z + d_v(\eta^{3,1}),$$
$$Id_X \otimes \eta^{1,3}_Y - \eta^{1,3}_{\Gamma\otimes X\otimes Y} + \eta^{1,3}_X \otimes Id_Y + d_v(\eta^{2,2}),$$
$$Id_X \otimes U + d_v(\eta^{1,3}), d_v(\eta^{0,4})) = 0$$
(7.2.2)

As before, let us describe what we mean by a deformation of the structure, Γ. For this we need

Definition 7.2.5. *Let $\mathscr{T}$ be a dg-category, Γ a perfect pseudo dg-tensor structure on $\mathscr{T}$. An n-th order deformation of Γ consists of a perfect pseudo dg-tensor structure $\{\{\Gamma\}\}$ on $\mathscr{T} \otimes_k k[x]/x^{n+1}$ such that*

$$i^*\{\{\Gamma\}\} := \{\{\Gamma\}\} \otimes_{k[x]/x^{n+1}} k$$

is a perfect pseudo dg-tensor structure equivalent to Γ.

Definition 7.2.6. *We say two n-th order infinitesimal deformations of a perfect pseudo dg-tensor structure Γ are equivalent if there is a pseudo dg-tensor functor Φ in $\mathscr{T} \otimes_k k[x]/x^n$ such that its restriction $\Phi \otimes_{k[x]/x^{n+1}} k$ is equivalent to the pseudo dg-tensor functor given by the identity dg-bimodule $\mathscr{T}_{pe}$.*

Just as before, we will say that a deformation of a perfect pseudo dg-structure Γ is a deformation of the associativity condition if the deformation $\{\{\Gamma\}\}$ has as structure morphisms for the unit and symmetry conditions equivalent equivalent to $\ell \otimes k[x]/x^{n+1}$,$r \otimes k[x]/x^{n+1}$, and $c \otimes k[x]/x^{n+1}$, while we allow for the associativity coherence condition to possibly be different.
The following is an analogue result to Theorem 7.1.6

Theorem 7.2.7. *Let $\mathscr{T}$ be a dg-category and let Γ be a pseudo dg-tensor structure on $\mathscr{T}$. Then to any element of $HDY^4_{dg}(\mathscr{T})$ we can associate an equivalence class of infinitesimal deformations of order 1 of the associativity condition of Γ.*

Proof. Let us recall that

$$DY^4_{dg}(\mathscr{T}) = Bimod^4({}^3\Gamma, \Gamma^3)^{-1} \oplus Bimod^3({}^2\Gamma, \Gamma^2)^0 \oplus Bimod^2(\Gamma, \Gamma)^1 \oplus Bimod^1(\mathscr{T}_{pe}, \mathscr{T}_{pe})^2 \oplus \mathscr{T}_{pe}(U,U)^3$$

As we calculated, the kernel of d^4_{dg} consists of those $\eta \in DY^4_{dg}(\mathscr{T})$ such that the equation 7.2.2 is equal to zero.

We see that the first component is a morphism of 4-fold bimodules, $\eta^{4,0}$ satisfying an hexagon condition. The second component on the other hand, is a 3-fold bimodule morphism $\eta^{3,1} \in Bimod^3({}^2\Gamma, \Gamma^2)^0$ satisfying the pentagon diagram condition up to the homotopy $d_v(\eta^{4,0})$. This is precisely the condition we require as a coherence condition for the associator morphism of a pseudo dg-tensor structure.

To be more precise, we will consider a perfect pseudo dg-tensor structure on $\mathscr{T} \otimes k[x]/x^2$ given by the 2-fold dg-bimodule $\Gamma^{k[e]} := \Gamma \otimes_k k[x]/x^2$ where every one of the structural morphisms of Definition 5.1.5 are given by the extension of scalars $\otimes_k k[x]/x^2$ except the associator.

Our goal is to define a new associator $\{\{\alpha\}\}$ which will restrict back to the associator α.

Let us write then $\{\{\alpha\}\} := \alpha + \eta^{3,1} x \in Bimod({}^2\Gamma^{k[e]}, \Gamma^{k[3]\ (2)})^0$. So, a natural morphism

$$\{\{\alpha\}\}_{X,Y,Z} : \Gamma^{k[e]}_{\Gamma^{k[e]},Z} \otimes \Gamma^{k[e]}_{X,Y} \to \Gamma^{k[e]}_{X,\Gamma^{k[e]}} \otimes \Gamma^{k[e]}_{Y,Z}$$

We need to check that this morphism satisfies the pentagon identity. Recall to add morphisms between different parenthesizations we need first to pad the morphism in the sense that we need to compose and precompose by the associativity morphism $\alpha \otimes k[x]/x^2$ and a fixed choice of an inverse $\alpha^{-1} \otimes k[x]/x^2$. We do this in such a way that addition of morphisms is always between the leftmost parenthesization and the rightmost one.

We proceed as in the abelian case and see that since the associator of the pseudo dg-tensor structure already satisfies the pentagon up to homotopy and x^2, our associator $\{\{\alpha\}\}$ satisfies the pentagon diagram up to homotopy because α already satisfies the condition, and as we are working with coefficients in $k[x]/x^2$ then all that is left is the expression involving α and the morphisms $\eta^{3,1}$ which corresponds to our padding of morphisms and then we obtain the condition $d^3_h(\eta^{3,1}) = d_v(\eta^{4,0})$.

Now we would like to see that two such deformations $\{\{\alpha\}\}, \{\{\alpha\}\}'$ given as above are equivalent then they come from a pseudo dg-tensor functor induced equivalence of perfect pseudo dg-tensor structures. As in the abelian case the underlying functor we are looking for is the identity functor and so our dg-bimodule Φ is nothing but the bimodule $\mathscr{T}_{pe}$.

This means that we are looking for a morphism

$$\Gamma \otimes k[x]/x^2 \simeq \Gamma \otimes \mathscr{T}_{pe} \otimes \mathscr{T}_{pe} \otimes k[x]/x^2 \to \mathscr{T}_{pe} \otimes \Gamma \otimes k[x]/x^2 \simeq \Gamma \otimes k[x]/x^2$$

of the form

$$Id + \beta x$$

where

$$\beta : \Gamma \to \Gamma$$

is a morphism of dg-bimodules.

We write β then as any dg-bimodule morphism such that $\beta^0 = \eta^{2,1}$, and writing the associativity condition for the pseudo dg-tensor we see that $\{\{f\}\} := Id + \beta x$ satisfies this condition if $\eta^{2,1}$ is in the image of d^2_{tot}, as the second component of d^2_{tot} gives us precisely the coherence diagram up to homotopy that $\{\{f\}\}$ has to satisfy.

Indeed as the identity dg-bimodule $\mathscr{T}_{pe}$ is a pseudo dg-tensor functor the identity $\mathscr{T}_{pe} \to \mathscr{T}_{pe}$ satisfies the associativity condition and as in the abelian case we see that the only remaining morphisms are those composed with the associativity condition of the pseudo dg-tensor structure, this in turn corresponds to a padding operation and thus we obtain precisely the expression in the second component of the image of d^2_{tot}.

□

The converse of these theorems don't seem to hold in general and likely require either a more general and coherent setting in which the deformation and space of tensor structures takes place, or under stricter conditions for the pseudo dg-tensor structure itself. At this point we ignore what the higher coherence conditions appearing both in the kernel and the image of the differentials d_{tot} represent in the context of deformations of the lifts of tensor triangulated structures. In all likeness an approach where the lift is meant to produce a tensor structure itself in the dg-enhancement is the correct setting in which one ought to take these deformations. Having said that as our motivation was kept in line with the tensor structure at the triangulated category level and this is the reason for the brute truncation of these tensor structures.

With this in mind one we however need to make two important observations. In our motivation we were interested in tensor triangulated structures in the context of Balmer's reconstruction, however so far we in the theory around deformations we have focused on deformations of one of the structural conditions, the associativity and so one natural question to ask is whether the Balmer spectrum is sensible to tensor triangulated categories equivalent in everything but the associativity condition.

Reviewing the Balmer spectrum construction we see that the precise nature of the associativity is of no importance. Indeed we do need a symmetric monoidal category structure on the triangulated category, and so coherent natural isomorphisms $\alpha_{X,Y,Z}$ but the construction will not distinguish between different equivalence classes of associators, and similarly for the unit and symmetry coherence conditions.

Indeed one can check that at no point in the process of constructing the Balmer spectrum there is a need to involve the equivalence class of the associators, not for the topology not for the structure sheaf. In fact from the work of Hovey in [Hov11, Proposition 3.1] we know that as there is only one (up to monoidal equivalence) tensor product in the category of modules over a commutative ring, then suppose $\boxtimes^{\mathbb{L}}$ is a tensor triangulated structure on $D^b(X)$ which is a derived tensor product of a product $\boxtimes$ of sheaves in the category of coherent modules. By this we mean that locally for any basic open

$U \subset X$ we have that the product $\boxtimes$ is equivalent in objects to the usual tensor product of modules. Then by the result above we know that there is one equivalence class and thus the derived tensor product $\boxtimes^{\mathbb{L}}$ must be the usual tensor product $\otimes^{\mathbb{L}}_X$.

In the abelian case we saw a few more things, namely we had Ocneanu's rigidity theorem, Theorem 7.1.10 exhibiting Davydov-Yetter cohomology as a tangent space of the moduli functor of tensor structures on a given finite tensor category $\mathscr{A}$. Additionally we have Lie algebra structures induced by a cup product defined at the level of Davydov-Yetter complexes.
Our conjecture for Ocneanu's rigidity theorem is that it should hold in the same spirit as in the abelian case, that is we should be looking for dg-categories which decompose in simple parts.
The Lie algebra structure with a Gerstenhaber bracket is possible to construct in the same fashion from the dg-Davydov-Yetter cohomology due to formal reasons. The interpretation of this as a tangent space of the moduli problem remains to be seen and likely reflects once again the truncation we are taking on the deformation problem.
As for the moduli functor itself, as we explained in the introduction of this section, the change of scalars functor for dg-categories can be performed without much issue by using the derived tensor product of dg-categories and then passing to the perfect modules closure, in that sense we can consider too a functor

$$TTS_{dg}(X) : CAlg_k \to NCSch^{pr,\otimes}_{sm}$$
$$A \mapsto TTS(D^b(X))_A$$

Assigning to each commutative k-algebra the groupoid (using pseudo dg-tensor functor equivalences) of perfect pseudo dg-tensor structures on the derived category of perfect complexes with coefficients in A.
In fact there is a number of moduli spaces and deformation problems we can discuss in this direction.

7.3 Some moduli spaces

In the previous section we saw that it is possible to a certain degree to give analogue results to the ones from tensor categories in the derived setting. However as we hinted before the deformation theory that we were occupied with was limited to only one of the structural morphisms which are involved in the whole structure.
We saw before that it is convenient to fix a given unit and wonder only about tensor structures, and in this case their deformations, who share a given fixed unit. There remains then the structural maps of the unit and the structural map of the symmetric condition which one could still deform.
Furthermore, as the 2-fold dg-bimodule Γ is itself a dg-module on a dg-category it can be deformed

together with the whole structure. We investigate these questions and reproduce some of the known relevant deformation and moduli problems that can be involved in this problem.

The theory of ∞-categories developed in the past few decades by a number of authors has seen one important result in the theorem of Lurie and Pridham establishing a long conjectured relationship between deformation problems and dg Lie algebras. Concretely the problem of studying the deformation theory of dg-categories has been done in this language in [BKP18].
The concrete situation is studying the formal deformations of a given k-linear dg-category $\mathscr{C}$, this means that given an artinian algebra A we want to understand the functor

$$A \to DefCat(A) := (\mathscr{C}_A, u)$$

and where $\mathscr{C}_A$ is the category $\mathscr{C} \otimes A$ and u is an equivalence $C_A \otimes k \simeq \mathscr{C}$.
One known condition for this to be a deformation problem that we could hope to understand well is that it satisfies the Schlessinger's conditions for representability. This condition fails in this situation but the authors show that by restricting to the case where the categories are compactly generated then the classifying functor above has a closer behaviour to that of a representable problem.
In particular they show that when considering formal deformations (meaning working with coefficients in $k[[t]]$) this classification is equivalent to a limit over the infinitesimal order deformations.
More so, if $CatDef^c$ denotes this functor classifying compactly generated k-linear dg-categories, they show that if $\mathscr{C}$ is a compactly generated k-linear dg-category admitting a single generator E, and such that $Ext^i(E,E) = 0$ for all $i >> 0$, then

$$CatDef^c_{\mathscr{C}}(k[[t]]) = \{k[u] - \text{linear structures on } \mathscr{C}\}$$

Where u is in cohomological degree 2.
Our interest in this result lies first of all on the fact that our derived categories can be enhanced with k-linear idempotent complete dg-categories like we have already discussed. Furthermore as derived categories of our spaces are compactly generated so are their dg-categorical enhancements.
We could consider then for a compactly generated idempotent-complete k-linear symmetric monoidal stable ∞-category $\mathscr{C}$ and an artinian algebra, the functor

$$CatMonDef^c_{\mathscr{C}} : Alg^{(2)} \to Pr^{\mathbb{L}}$$

From E_2-k-algebras to presentable ∞-categories, which classifies compactly generated idempotent-complete k-linear symmetric monoidal stable ∞-categories $\mathscr{C}_A$ and symmetric monoidal equivalences $u : \mathscr{C}_A \otimes k \simeq \mathscr{C}$.
In other words, what this problem is classifying is the space of every compactly generated symmetric monoidal category inside the larger ∞-category of presentable ∞-categories. Or alternatively we could work in the category of symmetric monoidal ∞-categories.
In this way it is the structure of the whole category together with the monoidal structure which we could deform by studying $CatMonDef^c_{\mathscr{C}}(k[x]/x^n)$ or the formal deformations $CatMonDef^c_{\mathscr{C}}(k[[t]])$,

but in principle it might be hard to work with this moduli functor. We will comment more on this later when we discuss the result announced by Toën in [Toë14].

Another alternative to our moduli problem to consider relies in the deformation of only the 2-fold dg-bimodule Γ of a dg-lift of a tensor triangulated structure. In our case of a perfect pseudo dg-tensor structure.
In this case we consider Γ as a $\mathscr{T}\otimes\mathscr{T}^{\otimes 2,op}$-module. This moduli problem is considered too in [BKP18]. Here they consider for an object $E\in\mathscr{C}$, with $\mathscr{C}$ as above, the functor

$$ObjDef_E : Alg_k^{art}\to S$$

Which takes artinian k-algebras A and produces the space of objects A-module objects E_A together with an equivalence $u : E_A\otimes k\simeq E$. In this case it is the object that is being parametrized as we commented before. In this case under certain conditions it is pointed out that this moduli problem is formal in the sense of Lurie.
A similar problem was studied in [ELO09] where for a given dg-module E over some dg-category, the authors considered the groupoid $Def^h(E)$ for a dg-artin algebra R, of pairs (S,i) such that S is, as a graded object isomorphic to $(E\otimes R)$ and i is a morphism $S\otimes k\to E$ making the composition factored by the restriction of scalars of $(E\otimes R)$, the identity.
In it, under certain conditions they manage to define a Maurer Cartan groupoid which parametrizes the deformations of the dg-module.
There must be some compatibility between these two functors although one has to be careful about what is being compared exactly.

Let us now investigate the space of all tensor structures with a fixed 2-fold bimodule Γ and unit U.

Theorem 7.3.1. *Let $\mathscr{T}$ be a dg-category, A a k-algebra, a perfect module U and Γ a 2-fold dg-bimodule over $\mathscr{T}\otimes^{\mathbb{L}}A$. Then the set $TTS_A(\Gamma)$ of perfect pseudo dg-tensor structures over $\mathscr{T}\otimes^{\mathbb{L}}A$ which has Γ as a 2-fold dg-bimodule and U as a unit, has a structure of a quotient of an affine scheme by an algebraic affine scheme.*

Proof. As both Γ and U are fixed, what we are looking for is simply families of morphisms $\alpha:\Gamma\otimes\Gamma\to\Gamma\otimes\Gamma$, $u:\Gamma\otimes U\to\mathscr{T}_{pe}$, and $c:\Gamma\to\Gamma$ we use that we are under the assumption that our dg-category $\mathscr{T}$ is proper and so locally perfect, which means that every complex of morphisms is bounded and of finite dimension as k-vector spaces.
The polynomial equations that determine morphisms α, c and u together with the differentials in every chain complex determine then an affine scheme and so do they when we restrict them by imposing the coherence conditions.
For two such structures, Γ_1, Γ_2 to be equivalent, we need an invertible pseudo dg-tensor functor Φ

from, say, $\boxtimes_1$ to $\boxtimes_2$ (using Lemma 5.1.10). Meaning that we have Φ' from $\boxtimes_2$ to $\boxtimes_1$ such that $\Phi \otimes \Phi'$ is isomorphic to the dg-bimodule $\mathscr{T}_{pe}$.
As the dg-bimodule Γ is the same in both structures Γ_1 and Γ_2, any such pseudo dg-tensor functor is determined by its structural morphisms

$$\Gamma \to \Gamma$$

and

$$\Gamma \otimes U \to \Gamma$$

Just as before the complexes are bounded and finite dimensional k-vector spaces and then they determine an affine scheme. Using composition as the group operation, taking the identity pseudo dg-tensor functor T_{pe} as the identity element and noting that every pseudo dg-tensor functor has an inverse, we get that the affine scheme of these equivalences forms an algebraic group.
We have then a quotient of the space of pseudo dg-tensor structures with 2-fold dg-bimodule Γ and unit U by the affine group scheme of equivalences between them. □

Notice however that we must use this space with care as it is not invariant under equivalence of perfect pseudo dg-tensor structures. So this definition works for a fixed choice of Γ and more work has to be performed to arrive to a good choice of invariant space.
We would then be heavily interested in understanding in full the tangent space of this space in terms of deformations and be able to relate them to our Davydov-Yetter cohomology.
As it was shown in [TV07], it is possible to prove that there exists a moduli stack of perfect complexes for a given dg-category $\mathscr{T}$. In fact it is shown there that this stack $M_{\mathscr{T}}$ is locally geometric and of finite presentation.
More precisely this moduli space classifies isomorphism classes of perfect complexes on the homotopy category of the dg-category.
As such, a strategy to follow to understand the full space of pseudo dg-tensor structures on a given dg-category $\mathscr{T}' = \mathscr{T} \otimes^{\mathbb{L}} \mathscr{T}^{op} \otimes \mathscr{T}^{op}$, would be to compare $M_{\mathscr{T}'}$ against the space $TTS_A(\Gamma)$ above every point Γ parametrized by Toën-Vaquié's moduli stack.

Chapter 8

Everything I Don't Know About Tensor Triangulated Categories

As a closing chapter I would like to dedicate a few pages to many of the questions that showed up during the work on this thesis but that unfortunately there was not enough time or correct insights to bring them to a concrete conclusion.
There are at least two different directions in which the theory might be expanded upon from the point of view of this work.
On one hand the expansion of Balmer reconstruction results in the algebro-geometric world for spaces of higher homotopical nature has already been successfully explored to a certain degree. By [Kri09, Hal16] as mentioned in a previous chapter it is already known that for a nicely behaved stack the Balmer spectrum is able to recover its associated coarse moduli space, which in itself reflect plenty of the geometric information of such stack.
Concretely Hall presents the following theorem:

Theorem 8.0.1. *Let X be a quasi-compact algebraic stack with quasi-affine and separated diagonal. If X is tame, then there is a natural isomorphism of locally ringed spaces*

$$(\mid X \mid, \mathscr{O}_{XZar}) \to Spc(Perf(X))$$

It is then immediately pointed out that this implies that the Balmer spectrum is not able to fully recover the space X. Indeed already for algebraic spaces it is known that this coarse moduli space does not determine the algebraic space and so the Balmer spectrum of the coarse moduli space and the Balmer spectrum of the algebraic space are isomorphic while the algebraic space and its coarse moduli space are not.
This however does not imply that the information contained in the tensor triangulated structure is

exhausted, we could hope perhaps for a refined spectrum construction just at this level which might throw more information in our way.
Indeed we can recall the following well known result of Lurie in [Lur04].

Theorem 8.0.2. *Suppose that $(S, \mathscr{O}_S)$ is a ringed topos which is local for the étale topology and that X is a geometric stack. Then the functor*

$$f \mapsto f^*$$

induces an equivalence of categories

$$T : Hom(S, X) \to Hom_{\otimes}(QC_X, M_{\mathscr{O}_S}).$$

Here $M_{\mathscr{O}_S}$ denotes the category of sheaves of $\mathscr{O}_S$-modules and $Hom_{\otimes}(QC_X, M_{\mathscr{O}_S})$ denotes monoidal functors between QC_X and $M_{\mathscr{O}_S}$. We have also the following theorem from [FI13, Theorem 5.10].

Theorem 8.0.3. *Let X be an algebraic stack which satisfies either*

1. *X is a noetherian scheme which has a very ample invertible sheaf*
2. *X is a tame separated (Deligne-Mumford) algebraic stack of the form $[X/G]$ where X is a finitely generated noetherian scheme and G is a linear algebraic group acting on X. Suppose further that the coarse moduli space is quasiprojective and X has a G-ample invertible sheaf.*

Let S be a quasi-compact scheme with affine diagonal over k. Then there is a categorical equivalence

$$F : Hom_k(S, X) \to Map_k^{\otimes}(D_{qcoh}^{\otimes}(X), D_{qcoh}^{\otimes}(S))$$

which sends $f : S \to X$ to f^*.

In this case $D_{qcoh}^{\otimes}(X)$ denotes an appropriately defined symmetric monoidal stable ∞-category of quasi-coherent complexes and $Map^{\otimes}$ the space of monoidal functors .
By these results it is expected that along the monoidal structures, these structures provide enough information about the space we would like to recover under certain conditions.
It is nonetheless very interesting to have concrete models of these reconstructions at hand in the same fashion as the Balmer spectrum in which it we can interpret these derived categories and associated data as affine objects of the geometry. On the one hand the functor of points approach that can be inferred from the above results has been extremely useful in algebraic geometry to get away from concrete models when abstracting away and bypassing certain difficulties inherent to the ringed space approach.
We insist however that having a concrete model for a space equipped with a structure sheaf should shed some light towards which specific properties of the derived category are being reflected in the space,

at least in the nice cases. As an illustration Thomason's classification of tensor triangulated ideals as parameterized by certain subsets of the space definitely provide an image that might be on principle hard to devise directly from a purely categorical representation result.

One of the first attempts at the theory of derived spaces comes from dg-schemes, these are spaces which Zariski locally look like commutative dg-algebras. The theory of these objects is however not as well behaved as one would want for a theory, although it is possible to relate a given dg-scheme to a derived stack in the sense of Toën-Vezzosi ([TV04]). Despite these limitations, the presentation of these objects as spaces with a structure sheaf of sorts is appealing for the reconstruction theory we have been considering:
Let us recall the notion of a dg-scheme according to Ciocan-Fonatanine-Kapranov ([CFK01]):

Definition 8.0.4. *A dg-scheme is a pair $(X, \mathscr{O}_X^*)$ where X is a scheme and $\mathscr{O}_X$ is a sheaf of negatively graded dg-algebras on X such that $\mathscr{O}_X^0$ is isomorphic to the structure sheaf on X and such that $\mathscr{O}_X^i$ is a quasi-coherent sheaf over $\mathscr{O}_X^0$*

Morphisms between two dg-schemes $X, Y$are given simply by considering morphisms $f : X \to Y$ between the underlying schemes and a morphism of sheaves of dg-algebras $f^*\mathscr{O}_Y^* \to \mathscr{O}_X^*$.

It is clear that any scheme can be thought of as a dg-scheme by considering an appropriate $\mathscr{O}_X$ and so it extends the original notion. On the other direction we have a truncation to usual schemes:

$$\pi_0(X) := Spec(H^0(\mathscr{O}_X^*))$$

One natural question is whether these spaces can be reconstructed in a similar fashion than schemes using a Balmer spectrum construction.
Let us suppose from now on that the underlying scheme $(X, \mathscr{O}_X^0)$ is quasiprojective and that $\mathscr{O}_X^i$ is furthermore coherent over $\mathscr{O}_X^0$.
As the underlying scheme is quasi-projective we know it can be recovered by Balmer's spectrum just as we have seen before. The question that remains is whether the structure sheaf of X as a dg-scheme can be recovered in a similar fashion. For this we need first to define an appropriate notion of derived categories of coherent sheaves for our dg-schemes.

Definition 8.0.5. *A quasi-coherent sheaf F^* on a dg-scheme X is a sheaf of $\mathscr{O}_X^i$-dg-modules on X such that F^i is quasicoherent over $\mathscr{O}_X^0$. We say that F^* is coherent if the F^i are in addition coherent and if F^i is bounded above.*

Let F^*, G^* be two coherent sheaves on a dg-scheme X, a morphism $\phi^* : F^* \to G^*$ is as usual and two such morphisms ϕ^*, ψ^* are said to be homotopically equivalent if there is a morphism

$h : F^* \to G^{*+1}$ as graded sheaves such that $d_G \circ h = h \circ d_F = \phi - \psi$.
As expected $\phi^* : F^* \to G^*$ is a quasi-isomorphism if it induces an isomorphism in cohomology $H^i(F^*) \to H^i(G^*)$.

Definition 8.0.6. *For a dg-scheme X, the derived category of coherent dg-sheaves $D(Coh(X))$ is the triangulated category obtained from the category of coherent dg-shaves on X by passing to homotopy classes of morphisms and localizing at quasi-isomorphisms.*

We can realize the derived category of the underlying sheaf $(X, \mathscr{O}_X^0)$ as the full triangulated subcategory spanned by those dg-sheaves quasi-isomorphic to a dg-sheaf concentrated in degree 0. Let us denote this category by $DCoh^0(X)$
This triangulated category comes equipped with a tensor triangulated structure given by a derived tensor product:

Proposition 8.0.7 ([CFK01] Prop 2.3.5, 2.4.1)**.** *Let X be a dg-scheme and F^*, G^* be two coherent dg-sheaves on X. The derived tensor product $F^* \otimes_X^{\mathbb{L}} G^*$ is defined as the tensor product $E^* \otimes G^*$ of dg-sheaves where E^{ast} is a resolution of F^* such that it is flat as a graded sheaf and bounded above.*

The structure sheaf $\mathscr{O}_X^*$ acts as usual as the unit of this tensor structure.
If we were to proceed as in Theorem 2.0.39 we would immediately see it is not possible to recover $\mathscr{O}_X^*$ from the endomorphism ring in the derived category as the grading is lost. Let us consider however the graded endomorphism ring of Definition 2.0.42 in this situation,

$$End^i_{DCoh(X)}(\mathscr{O}_X^*, \mathscr{O}_X^*) := Hom(\mathscr{O}_X^*, \mathscr{O}_X^*[i])$$

Let us consider this problem locally as before and let $\{U_i\}$ be an affine cover for X as before. In this situation an endomorphism η^i of degree i, in $End^i_{DCoh(U)}(\mathscr{O}_X(U))$ induces an element of $\mathscr{O}_X(U)^i$ by considering the image $\eta^i(1)$.

One immediate problem with this approach is the failure of these homotopical spaces to be represented by locally ringed spaces. Balmer's reconstruction as presented is inherently a process which outputs a topological space and so the first thing to do to reconstruct, for example, a DM stack would be to extend the construction so that the process would output in this case the étale topos and a way to recover it as a locally ringed topos from the monoidal structure.
This is very clearly a tall order, it would require us to upgrade Thomason's classification theorem to this setting so that we could recover the space.
One possible approach to this would be the point of view of Balmer's reconstruction proposed in [KP17] in which the reconstruction is framed under the pointless philosophy focusing solely on the lattice theoretic aspects of the reconstruction through Hochster duality.
To be more precise the Hochster dual of spectral space consists of the topological space X generated by those open subsets with quasi-compact complement, we denote this space by $X^\vee$. Hochster duality

establishes that $X^\vee$ is again spectral and that $X^{\vee\vee} \cong X$. Taking the coherent frame of open subsets of this space, then the Hochster dual corresponds to the opposite frame of the frame of finite elements.
Let X be a coherent scheme, denote by $D^{\omega}_{qc}(X)$ the derived category of compact objects in the derived category of modules with quasi-coherent cohomology. If X were for example a smooth projective variety this category is equivalent to the category of perfect complexes on X.
In [KP17] the following is the main observation applied to coherent schemes :

Theorem 8.0.8. *Let X be a coherent scheme. If the collection of thick radical $\otimes$-ideals in $D^{\omega}_{qc}(X)$ forms a set, then this set has a coherent frame structure $Zar(Perf(X))$ called the Zariski frame of $(Perf(X), \otimes^{\mathbb{L}}_X)$.*
The Hochster dual of $Zar(Perf(X))$ is equivalent to the coherent frame of Zariski open subsets of X.

Besides the conceptual difference in this approach one of the main advantages of this point of view is that it is possible to avoid having to rely heavily on Thomason's results and also being able to reduce the argument to a local situation.
We hope that the previous discussion supports the raising of the following question:

Question 8.0.1. *Let X be a DM stack over k, can the derived category of X or a dg-enhancement of it recover X as a locally ringed topos?*

We have previously gone through a study of moduli functors associated to our problem. In this regard, in [AT08] the authors remark that the functor of proper smooth geometrically connected varieties over k, $Var^{gc}_{smpr}(k)$ is known to be a stack but it is not algebraic and that one of the ways to turn this space into an algebraic one is by considering instead a functor which classifies varieties with a polarization, so that on each commutative k-algebra A, $Polarized(A)$ is the groupoid of smooth projective geometrically connected varieties over A, together with an invertible bundle L.
Indeed in [Sta22, Tag 0D4X] it is shown that this stack is indeed algebraic.
The purpose of this discussion however is that Anel and Toën argue that while the prestack they consider in their work, which classifies saturated, connected dg-categories over some base k in the homotopy category of dg-categories, is not a stack. It still has some good properties as they show, for example they use heavily that the diagonal of this prestack is representable and this is a key part of the proof of the main result of that article.
In this line of thinking they suggest that considering a prestack of saturated and connected dg-categories with extra structure would allow for better behaviour, like the inclusion of a Bridgeland stability condition, which consists of a heart of a bounded t-structure and a stability function satisfying certain conditions.
The idea of such a structure is that it provides some control in the geometric behaviour of the objects in a given derived category and allows us to treat them like coherent sheaves on a space.
As pointed out before it seems hinted that there is a strong relationship between tensor structures and t-structures, and specifically of their hearts, thus the prestack which assigns to each commutative

k-algebra the groupoid of saturated and connected dg-categories together with a monoidal structure seems like a good alternative to investigate.

Question 8.0.2. *Is the functor $MonDG_k^{stgc} : A \mapsto \{$Saturated connected monoidal dg-categories over $A\}$ a stack?*

8.1 Recovering tensor products from other invariants

As we have seen through this work, derived categories, and triangulated categories in general, can exhibit different behaviors and carry different invariants. In the case of derived categories of spaces one immediate thing one could question is understanding what is the role and interaction of these invariants with the tensor product corresponding to the space.
For example one very important aspect that has helped us in our deductions from studying tensor triangulated category structures on $D^b(\mathbb{P}^1)$ is that we understand well the generating properties of the category. Namely as the category possesses a full strong exceptional collection and we understand not only its dimension but the Orlov spectrum we are able to significantly reduce the space of possible tensor triangulated structures.
Clearly this case is likely the easiest to understand, the further away we go from spaces with ample canonical bundles then the more unlikely are exceptional collections - of different kinds- to show up. It is at the moment unclear to the author if it is possible to give an explicit description of all the tensor triangulated structures which can be detected just from the existence of these collections. Even in a more complicated setting of a derived category with a full strong exceptional collection, and the knowledge of both the dimension and Orlov spectrum it is not possible to distinguish the space, it is clear that the tensor product cannot be uniquely described.
It is possible however to wonder if without further knowledge of the category it is possible to construct a freely generated tensor triangulated category we can construct just from this data.
A very close problem is treated in [Orl16] by Orlov, namely

Theorem 8.1.1. *[Orl16, Theorem 5.8] Let $\mathscr{A}$ be a small dg-category over k such that $H^0(\mathscr{A})$ has a full exceptional collection*

$$(\mathscr{E}_1, \dots, \mathscr{E}_n)$$

Then there are a smooth projective scheme X and an exceptional collection of line bundles $(\mathscr{L}_1, \dots, \mathscr{L}_n)$ such that the dg-subcategory of perfect complexes generated by the line bundles is quasi-equivalent to A. Moreover, X is a sequence of projective bundles and has a full exceptional collection.

As described in the statement itself the way this space is constructed is by taking iterated projective bundles over some gluing of dg-categories determined by the full exceptional collection.
Using this, Orlov considers briefly a construction of what is there called a noncommutative deformation of the plane $\mathbb{P}^2$ by considering a dg-category with a exceptional collection as the one of the plane but

giving consideration to different relations between the compositions of the morphisms. By using the theorem above Orlov describes the smooth projective spaces for which the derived category can be used to embed these categories and their exceptional collections.
Inside the derived category of this smooth projective space there exists a smallest tensor triangulated subcategory containing the full exceptional collection.

Question 8.1.1. *Let $\mathscr{A}$ be a dg-category such that its homotopy category has a full exceptional collection $(\mathscr{E}_1, \ldots, \mathscr{E}_n)$ as in 8.1.1. Let $(\mathscr{T}, \otimes^{\mathbb{L}})$ be the smallest tensor triangulated subcategory in $Perf(X)$ containing the exceptional objects $\mathscr{E}_i$. Is the category $\mathscr{T}$ in any way canonical in the sense that if $(\mathscr{T}', \boxtimes)$ is any other tensor triangulated category with an exceptional collection $(\mathscr{E}_i, \ldots, \mathscr{E}_n)$ then is there a monoidal embedding $\mathscr{T} \to T'$?*

Given the disparate behavior of the Balmer spectrum with respect to exotic tensor structures, it wouldn't be expected that a single tensor structure on a given derived category could inform us about the generating properties of the underlying category.
On the other hand we know from [LS13] the Balmer spectrum of the vertex-wise tensor product on the derived category of representations on a quiver is given by the number of vertices. We interpret each of these vertices corresponding to exceptional objects and the fact that the Balmer spectrum collapses to a finite number of points to the fact that if the tensor product is collapsing to points which then correspond to categories equivalent to $D^b(k)$. Under this interpretation,

Question 8.1.2. *Let $(\mathscr{T}, \boxtimes)$ be a tensor triangulated category such that $Spc(\mathscr{T}, \boxtimes)$ is isomorphic to the Balmer spectrum of a derived category of representations of a finite quiver. Does $\mathscr{T}$ admit a full exceptional collection?*

Suppose $\boxtimes$ on $\mathscr{T}$ is as in the question above. Then $Spc(\mathscr{T}, \boxtimes)$ being a discrete space of $n \in \mathbb{N}$ points means by definition that there are only n prime $\boxtimes$-ideals P_i and the fact that the space is discrete means there exist n objects $\{\mathscr{E}_i\}$ such that $U(\mathscr{E}_i) := \{P \mid \mathscr{E}_i \in P_i\} = P_i$ are the one-point open subsets of $Spc(\boxtimes)$.
It is these latter objects that should then be possible to show form a full exceptional collection.

Besides properties relating to how well generated a derived category is, one classical tool we have already mentioned is that of t-structures. To recall, the heart of a t-structure determines an abelian category which we interpret as sitting inside of the derived category.
In general as mentioned in 1.1.63, in the presence of a functor from the derived category of the heart to the starting derived category, there is a condition that can be checked for this functor to induce an equivalence.
Here we seem to have a situation that parallels our situation with the different tensor structures on a given category, as a heart of a t-structure determines, through methods like the Gabriel-Rosenberg

reconstruction theorem, a space which in the correct setting will recover the original space.
It is natural to wonder what is the relationship between the hearts from this point of view and the tensor products on the derived category. In the same way that coming up with the functor from the derived category of the heart towards the original derived category is challenging, there is no obvious way of relating the derived category of the locally ringed space $Spc(\boxtimes)$ to the original triangulated category.

Chapter 9

Bibliography

[AT08] Mathieu Anel and Bertrand Toën. Dénombrabilité des classes d'équivalences dérivées de variétés algébriques. *Journal of Algebraic Geometry*, 18:257–277, 2008. 37, 147

[Bal02] Paul Balmer. Presheaves of triangulated categories and reconstruction of schemes. *Mathematische Annalen*, 324(3):557–580, 2002. 12, 41, 51

[Bal05] Paul Balmer. The spectrum of prime ideals in tensor triangulated categories. *J. Reine Angew. Math.*, 588:149–168, 2005. 12, 41, 46, 49, 51

[Bal10] Paul Balmer. Spectra, spectra, spectra–tensor triangular spectra versus zariski spectra of endomorphism rings. *Algebraic & Geometric Topology*, 10(3):1521–1563, 2010. 52, 53

[BBDG18] Alexander Beilinson, Joseph Bernstein, Pierre Deligne, and Ofer Gabber. *Faisceaux pervers*. Société mathématique de France, 2018. 40

[BCL14] Sébastien Boucksom, Salvatore Cacciola, and Angelo Felice Lopez. Augmented base loci and restricted volumes on normal varieties. *Mathematische Zeitschrift*, 278(3):979–985, 2014. 58

[BD20] Michael Batanin and Alexei Davydov. Cosimplicial monoids and deformation theory of tensor categories. *arXiv preprint arXiv:2003.13039*, 2020. 131

[BF07] Paul Balmer and Giordano Favi. Gluing techniques in triangular geometry. *Q. J. Math.*, 58(4):415–441, 2007. 53

[BGI77] P Berthelot, A Grothendieck, and L Illusie. Sga 6. *Théorie des intersectiones et Theoreme de Riemann-Roch*, 1977. 34

[BGT14] Andrew J Blumberg, David Gepner, and Gonçalo Tabuada. Uniqueness of the multiplicative cyclotomic trace. *Advances in Mathematics*, 260:191–232, 2014. 118, 119

[BK90] Alexey Igorevich Bondal and Mikhail Mikhailovich Kapranov. Enhanced triangulated categories. *Matematicheskii Sbornik*, 181(5):669–683, 1990. 89, 90

[BKP18] Anthony Blanc, Ludmil Katzarkov, and Pranav Pandit. Generators in formal deformations of categories. *Compositio Mathematica*, 154(10):2055–2089, 2018. 139, 140

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid. The mckay correspondence as an equivalence of derived categories. *Journal of the American Mathematical Society*, 14(3):535–554, 2001. 37

[BKS07] Aslak Bakke Buan, Henning Krause, and Øyvind Solberg. Support varieties: an ideal approach. *Homology, Homotopy and Applications*, 9(1):45–74, 2007. 52

[BLS16] Daniel Bergh, Valery A Lunts, and Olaf M Schnürer. Geometricity for derived categories of algebraic stacks. *Selecta Mathematica*, 22(4):2535–2568, 2016. 97

[BO01] Alexei Bondal and Dmitri Orlov. Reconstruction of a variety from the derived category and groups of autoequivalences. *Compositio Mathematica*, 125(3):327–344, 2001. 10, 25, 31, 33, 34

[Bon89] Alexey Igorevich Bondal. Representation of associative algebras and coherent sheaves. *Izvestiya rossiiskoi akademii nauk. Seriya Matematicheskaya*, 53(1):25–44, 1989. 29

[Bra18] Martin Brandenburg. Rosenberg's reconstruction theorem. *Expo. Math.*, 36(1):98–117, 2018. 9

[Bri99] Tom Bridgeland. Equivalences of triangulated categories and Fourier–Mukai transforms. *Bulletin of the London Mathematical Society*, 31(1):25–34, 1999. 29

[Bri07] Tom Bridgeland. Stability conditions on triangulated categories. *Annals of Mathematics*, pages 317–345, 2007. 38

[BVdB03] Aleksei Igorevich Bondal and Michel Van den Bergh. Generators and representability of functors in commutative and noncommutative geometry. *Moscow Mathematical Journal*, 3(1):1–36, 2003. 32, 93

[BZFN10] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and drinfeld centers in derived algebraic geometry. *Journal of the American Mathematical Society*, 23(4):909–966, 2010. 117

[Cal18] John Calabrese. Relative singular twisted Bondal–Orlov. *Mathematical Research Letters*, 25(2):393–414, 2018. 37

[CFK01] Ionuţ Ciocan-Fontanine and Mikhail Kapranov. Derived quot schemes. In *Annales scientifiques de l'École normale supérieure*, volume 34, pages 403–440. Elsevier, 2001. 145, 146

[Coh13] Lee Cohn. Differential graded categories are k-linear stable infinity categories. *arXiv preprint arXiv:1308.2587*, 2013. 96, 115, 119, 120, 121

[CS17] Alberto Canonaco and Paolo Stellari. A tour about existence and uniqueness of dg enhancements and lifts. *Journal of Geometry and Physics*, 122:28–52, 2017. 89

[CS18] Alberto Canonaco and Paolo Stellari. Uniqueness of dg enhancements for the derived category of a Grothendieck category. *Journal of the European Mathematical Society*, 20(11):2607–2641, 2018. 92

[CY98] Louis Crane and David N Yetter. Deformations of (bi) tensor categories. *Cahiers de topologie et géométrie différentielle catégoriques*, 39(3):163–180, 1998. 128

[Dav97] AA Davydov. Twisting of monoidal structures. *arXiv preprint q-alg/9703001*, 1997. 125

[DEN18] Alexei Davydov, Pavel Etingof, and Dmitri Nikshych. Autoequivalences of tensor categories attached to quantum groups at roots of 1. In *Lie groups, geometry, and representation theory*, volume 326 of *Progr. Math.*, pages 109–136. Birkhäuser/Springer, Cham, 2018. 130

[DFEM14] Tommaso De Fernex, Lawrence Ein, and Mircea Mustata. Vanishing theorems and singularities in birational geometry. *Unpublished monograph*, 2014. 58

[Dri04] Vladimir Drinfeld. Dg quotients of dg categories. *Journal of Algebra*, 272(2):643–691, 2004. 69, 101

[DT12] Ivo Dell'Ambrogio and Gonçalo Tabuada. Tensor triangular geometry of non-commutative motives. *Advances in Mathematics*, 229(2):1329–1357, 2012. 53

[EGNO16] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205. American Mathematical Soc., 2016. 126, 129, 130

[ELO09] Alexander I Efimov, Valery A Lunts, and Dmitri O Orlov. Deformation theory of objects in homotopy and derived categories i: general theory. *Advances in Mathematics*, 222(2):359–401, 2009. 140

[Fao17] Giovanni Faonte. Simplicial nerve of an a_∞-category. *Theory Appl. Categ*, 32(31-52):3607208, 2017. 122

[FI13] Hiroshi Fukuyama and Isamu Iwanari. Monoidal infinity category of complexes from tannakian viewpoint. *Mathematische Annalen*, 356(2):519–553, 2013. 144

[Gen16] Francesco Genovese. The uniqueness problem of dg-lifts and fourier–mukai kernels. *Journal of the London Mathematical Society*, 94(2):617–638, 2016. 95

[GHS19] Azat M Gainutdinov, Jonas Haferkamp, and Christoph Schweigert. Davydov-Yetter cohomology, comonads and ocneanu rigidity. *arXiv preprint arXiv:1910.06094*, 2019. 129

[GM94] Sergej Izrailevivc Gelfand and Yu I Manin. *Homological algebra*, volume 38. Springer Science & Business Media, 1994. 39

[GR19] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*, volume 221. American Mathematical Society, 2019. 122

[ha] Benjamin Antieau (https://mathoverflow.net/users/100/benjamin antieau). Can we define the tensor product in the derived category $D^b_{\mathsf{coh}}(x)$ just from $D^b_{\mathsf{coh}}(x)$ in certain cases? MathOverflow. URL:https://mathoverflow.net/q/143661 (version: 2013-10-01). 63

[Hal16] Jack Hall. The Balmer spectrum of a tame stack. *Annals of K-Theory*, 1(3):259–274, 2016. 53, 143

[Hir03] Philip S Hirschhorn. *Model categories and their localizations*. Number 99. American Mathematical Soc., 2003. 76, 78

[Hov07] Mark Hovey. *Model categories*. Number 63. American Mathematical Soc., 2007. 76

[Hov11] Mark Hovey. Additive closed symmetric monoidal structures on r-modules. *Journal of Pure and Applied Algebra*, 215(5):789–805, 2011. 16, 99, 104, 108, 112, 137

[HP13] Klaus Hulek and David Ploog. Fourier-Mukai partners and polarised K3 surfaces. In *Arithmetic and geometry of K3 surfaces and Calabi–Yau threefolds*, pages 333–365. Springer, 2013. 37

[HRS96] Dieter Happel, Idun Reiten, and Sverre O Smalø. *Tilting in abelian categories and quasitilted algebras*, volume 575. American Mathematical Soc., 1996. 40

[Huy06] Daniel Huybrechts. *Fourier-Mukai transforms in algebraic geometry*. Clarendon Press, 2006. 19, 20, 23, 26, 30, 31, 33, 34, 36

[Joy75] André Joyal. Les théorèmes de chevalley-tarski et remarques sur l'algèbre constructive. In *Deuxieme Colloque sur l'Algebre des Catégories, Centre National de la Recherche Scientifique*, 1975. 48

[Kap88] Mikhail M Kapranov. On the derived categories of coherent sheaves on some homogeneous spaces. *Inventiones mathematicae*, 92(3):479–508, 1988. 28

[Kaw02] Yujiro Kawamata. D-equivalence and K-equivalence. *Journal of Differential Geometry*, 61(1):147–171, 2002. 11, 37

[Kel94] Bernhard Keller. Deriving dg categories. In *Annales scientifiques de l'Ecole normale supérieure*, volume 27, pages 63–102, 1994. 89, 91

[Kel06] Bernhard Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006. 69

[Kel15] Shane Kelly. Some observations about motivic tensor triangulated geometry over a finite field. In *International Conference on Bousfield Classes form a set: in Memory of Testusuke Ohkawa*, pages 221–243. Springer, 2015. 53

[KP17] Joachim Kock and Wolfgang Pitsch. Hochster duality in derived categories and point-free reconstruction of schemes. *Transactions of the American Mathematical Society*, 369(1):223–261, 2017. 52, 146, 147

[Kri09] Amalendu Krishna. Perfect complexes on Deligne-Mumford stacks and applications. *Journal of K-Theory*, 4(3):559–603, 2009. 53, 143

[Kuz16] Alexander Kuznetsov. Derived categories view on rationality problems. In *Rationality problems in algebraic geometry*, pages 67–104. Springer, 2016. 28

[Laz17] Robert K Lazarsfeld. *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, volume 48. Springer, 2017. 57, 58

[Les15] John Lesieutre. Derived-equivalent rational threefolds. *International Mathematics Research Notices*, 2015(15):6011–6020, 2015. 37

[LO10] Valery Lunts and Dmitri Orlov. Uniqueness of enhancement for triangulated categories. *Journal of the American Mathematical Society*, 23(3):853–908, 2010. 89

[LS13] Yu-Han Liu and Susan J Sierra. Recovering quivers from derived quiver representations. *Communications in Algebra*, 41(8):3013–3031, 2013. 55, 57, 149

[Lur04] Jacob Lurie. Tannaka duality for geometric stacks. *arXiv preprint math/0412266*, 2004. 144

[Lur07] Jacob Lurie. Derived algebraic geometry ii: Noncommutative algebra. *arXiv preprint math/0702299*, 2007. 118

[Lur09] Jacob Lurie. *Higher topos theory*. Princeton University Press, 2009. 115, 118, 123

[Lur17] Jacob Lurie. Higher algebra. 2017. 115, 116, 117, 119, 120, 122, 123

[May01] J. P. May. The additivity of traces in triangulated categories. *Adv. Math.*, 163(1):34–73, 2001. 20

[Muk78] Shigeru Mukai. Semi-homogeneous vector bundles on an abelian variety. *Journal of Mathematics of Kyoto University*, 18(2):239–272, 1978. 10, 31

[Nee96] Amnon Neeman. The Grothendieck duality theorem via bousfield's techniques and brown representability. *Journal of the American Mathematical Society*, 9(1):205–236, 1996. 23

[Orl92] Dmitri Olegovich Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya*, 56(4):852–862, 1992. 30

[Orl97] Dmitri O Orlov. Equivalences of derived categories and k3 surfaces. *Journal of Mathematical Sciences*, 84(5):1361–1381, 1997. 37

[Orl02] Dmitri Olegovich Orlov. Derived categories of coherent sheaves on abelian varieties and equivalences between them. *Izvestiya: Mathematics*, 66(3):569, 2002. 37

[Orl09] Dmitri Orlov. Remarks on generators and dimensions of triangulated categories. *Mosc. Math. J.*, 9(1):153–159, back matter, 2009. 21

[Orl16] Dmitri Orlov. Smooth and proper noncommutative schemes and gluing of dg categories. *Advances in Mathematics*, 302:59–105, 2016. 91, 96, 97, 148

[Orl18] Dmitri Olegovich Orlov. Derived noncommutative schemes, geometric realizations, and finite dimensional algebras. *Russian Mathematical Surveys*, 73(5):865, 2018. 96, 97

[Ric89] Jeremy Rickard. Derived categories and stable equivalence. *Journal of pure and applied Algebra*, 61(3):303–317, 1989. 21

[Rob12] Marco Robalo. Noncommutative motives i: A universal characterization of the motivic stable homotopy theory of schemes. *arXiv preprint arXiv:1206.3645*, 2012. 122

[Ros96] Alexander L Rosenberg. Reconstruction of schemes. *MPI preprint*, 1996. 9

[Rou05] Raphaël Rouquier. Catégories dérivées et géométrie birationnelle. *Séminaire Bourbaki*, 946:283–307, 2005. 43, 44, 45

[RSS01] Charles Rezk, Stefan Schwede, and Brooke Shipley. Simplicial structures on model categories and functors. *American Journal of Mathematics*, 123(3):551–575, 2001. 79

[RVdBN19] Alice Rizzardo, Michel Van den Bergh, and Amnon Neeman. An example of a non-Fourier–Mukai functor between derived categories of coherent sheaves. *Inventiones mathematicae*, 216(3):927–1004, 2019. 95, 105

[Sim07] Carlos Simpson. A giraud-type characterization of the simplicial categories associated to closed model categories as ∞-pretopoi. *arXiv preprint math.AT/9903167*, 2007. 118

[Sos] Pawel Sosna. Tensor triangulated categories in algebraic geometry. Master's thesis. 56

[SS03] Stefan Schwede and Brooke Shipley. Stable model categories are categories of modules. *Topology*, 42(1):103–153, 2003. 79

[Sta22] The Stacks project authors. The stacks project. https://stacks.math.columbia.edu, 2022. 24, 147

[Tab05a] Gonçalo Tabuada. Invariants additifs de dg-catégories. *International Mathematics Research Notices*, 2005(53):3309–3339, 2005. 95

[Tab05b] Goncalo Tabuada. Une structure de catégorie de modeles de quillen sur la catégorie des dg-catégories. *Comptes Rendus Mathematique*, 340(1):15–19, 2005. 69

[Tab10] Gonçalo Tabuada. Generalized spectral categories, topological Hochschild homology and trace maps. *Algebr. Geom. Topol.*, 10(1):137–213, 2010. 120

[Tho97] Robert W Thomason. The classification of triangulated subcategories. *Compositio Mathematica*, 105(1):1–27, 1997. 12, 43, 45

[Toë07] Bertrand Toën. The homotopy theory of dg-categories and derived morita theory. *Inventiones mathematicae*, 167(3):615–667, 2007. 14, 69, 70, 76, 85, 87, 91

[Toë11] Bertrand Toën. Lectures on dg-categories. In *Topics in algebraic and topological K-theory*, pages 243–302. Springer, 2011. 69

[Toë14] Bertrand Toën. Derived algebraic geometry and deformation quantization. *arXiv preprint arXiv:1403.6995*, 2014. 140

[TV04] Bertrand Toën and Gabriele Vezzosi. From hag to dag: derived moduli stacks. In *Axiomatic, enriched and motivic homotopy theory*, pages 173–216. Springer, 2004. 145

[TV07] Bertrand Toën and Michel Vaquié. Moduli of objects in dg-categories. In *Annales scientifiques de l'Ecole normale supérieure*, volume 40, pages 387–444, 2007. 141

[TV22] Bertrand Toën and Gabriele Vezzosi. Trace and künneth formulas for singularity categories and applications. *Compositio Mathematica*, 158(3):483–528, 2022. 122

[Ueh04] Hokuto Uehara. An example of Fourier-Mukai partners of minimal elliptic surfaces. *Math. Res. Lett.*, 11(2-3):371–375, 2004. 37

[Yet98] David N Yetter. Braided deformations of monoidal categories and vassiliev. In *Higher Category Theory: Workshop on Higher Category Theory and Physics, March 28-30, 1997, Northwestern University, Evanston, IL*, volume 230, page 117. American Mathematical Soc., 1998. 125