# Values for multi-choice cooperative games with applications to sharing the cost of a gas distribution network

David Lowing

N° d'ordre NNT : 2022LYSES022

**THESE de DOCTORAT DE L'UNIVERSITE DE LYON**

opérée au sein de

**L'Université Jean Monnet Saint-Etienne**

**Ecole Doctorale** N° 486
**Ecole Doctorale Sciences Economiques et de Gestion**

**Spécialité de doctorat** : Sciences Economiques
**Discipline** : Théorie des jeux coopératifs

Soutenue publiquement le 27/06/2022, par :
**David Lowing**

---

# Values for multi-choice cooperative games with applications to sharing the cost of a gas distribution network

---

**Devant le jury composé de :**

Josune ALBIZURI, Professeure, Universidad del País Vasco, Rapporteure

Justin LEROUX, Professeur, HEC Montréal, Rapporteur

Inés MACHO-STADLER, Professeure, Universitat Autònoma de Barcelona, Examinatrice

Noemí NAVARRO, Maîtresse de conférences HDR, Université de Rennes 1, Examinatrice

Philippe SOLAL, Professeur, Université de Saint-Étienne, Directeur de thèse

Stéphane GONZALEZ, Maître de conférences, Université de Saint-Étienne, Co-directeur de thèse

Christophe AGOGUE, Administrateur ENGIE GRDF, Encadrant entreprise

## Déclaration



David Lowing

Mars 2022

# Remerciements

Ces trois dernières années ont été particulières, voire difficiles pour beaucoup de monde. Pour ma part, elles se sont excessivement bien passées, et pour cela, je pense qu'il me faut remercier les nombreuses personnes qui m'ont entouré lors de ce long et fastidieux périple qu'est le Doctorat.

Mes premières pensées vont tout naturellement vers mes encadrants, tant gaziers qu'académiques. Sans leur soutien et leur bienveillance, je n'aurais jamais pu produire ce document, ni même passer de si belles années à faire de la recherche. C'est donc de tout cœur que je remercie mes encadrants chez GRDF, à commencer par Vincent Chevallier, mon ancien responsable hiérarchique, pour avoir été un guide et un ami tout au long de ma thèse, notamment lors des moments les plus difficiles. Je remercie Christophe Agogue, administrateur ENGIE chez GRDF, pour sa gentillesse, sa bienveillance et pour avoir accepté de participer à mon jury de thèse. Merci à Laurent Renat, ancien directeur économie régulation chez GRDF, pour ses excellents conseils, sa bonne humeur et pour sa curiosité sans fin concernant mes travaux.

Du côté académique, je dois énormément à mes deux encadrants qui m'ont apporté tellement que je ne saurais énumérer tout ce que j'ai appris à leur côté. Merci donc à mon directeur de thèse, Philippe Solal, pour son encadrement exceptionnel durant ces trois années. Sa patience, sa pédagogie et ses innombrables relectures m'ont formé à ce métier de chercheur, que jamais je n'aurais pensé autant aimer au début de cette thèse. Philippe a toujours été là quand j'étais dans le besoin. Pas un seul instant je me suis retrouvé perdu et désorienté en trois ans, puisque Philippe a su me guider et m'orienter systématiquement dans la bonne direction.

Merci aussi à mon co-directeur de thèse, Stéphane Gonzalez, pour sa bienveillance et pour tous ses bons conseils. C'est lors du Symposium de théorie des jeux organisé par l'Institut Henri Poincaré en 2018 que j'ai rencontré Stéphane et que j'ai pu être témoin, pour la première fois, de sa créativité. Cette même créativité qui m'a motivé et inspiré pendant toute ma thèse. Ce que m'a apporté Stéphane complète parfaitement ce que m'a apporté Philippe, et pour cette raison, je pense que jamais je n'aurais pu rêver meilleur duo en guise de direction de thèse.

Enfin, je tiens à remercier Justin Leroux pour avoir participé au développement de ma thèse. Malgré la distance géographique qui nous sépare, Justin a su faire acte de présence pour me guider tout du long. Je lui renouvelle mes remerciements pour avoir accepté de faire partie de mon jury en qualité de rapporteur.

Je tiens également à adresser mes sincères remerciements aux autres membres de mon jury. Merci à Josune Albizuri pour avoir accepté d'être rapporteure, et merci à Inés Macho-Stadler ainsi qu'à Noemí Navarro d'avoir examiné ce manuscrit.

Mes moments passés au sein de GRDF m'auront permis de rencontrer des gens formidables. Merci donc à Françoise Penaud, Élodie Cador et Alban Thomas pour leur travail en tant que directrice et délégués de la direction économie régulation. Merci à Régis Boa, économiste chez GRDF, pour toutes les fois où il m'a aidé et accompagné. Sa présence m'aura été précieuse, tant pour sa bonne humeur que pour son travail rigoureux. Merci à Nikola Gerasimovic pour m'avoir accompagné dans tous les restaurants de la rue Saint-Anne à Paris et pour avoir écouté mes problèmes de chercheur des heures durant. Enfin, merci à tous ceux avec qui j'ai eu la chance d'échanger au sein de GRDF, à savoir, Adeline Lecomte, Frederic Trongneux, Alexandre Pierru, Olena Pierret, Salifou Fofana, Nicolas Hubert, Morgan Kazmierczak, Sébastien boiteux, Eloise Cabrol, Margaux Jossier et Mohamed Es Sbai.

Je me dois aussi de remercier d'autre personnalités académiques, sans qui je n'aurais certainement pas fait le choix de poursuivre en thèse. Merci à Lisa Anouliès, pour m'avoir donné mon tout premier cours de théorie des jeux en Licence 3 et m'avoir fait aimer cette discipline. Merci à Bruno Nazaret, pour m'avoir redonné le goût des mathématiques lors de son cours d'analyse fonctionnelle en Master 1. Enfin, merci à Michel Grabisch, pour m'avoir introduit aux jeux coopératifs, pour avoir accepté d'encadrer mon mémoire de Master 2 et pour m'avoir transmis cette fameuse offre de thèse.

Toujours dans les personnalités académiques, notamment celles du GATE, je tiens à remercier Julien Salanié pour avoir participé au lancement de cette thèse et pour être un excellent directeur de laboratoire. Merci à Eric Rémila d'avoir participé à la supervision de mes travaux aux côtés de Stéphane et Philippe. Merci à Sylvain Ferrières pour toutes les enrichissante discutions qu'il a eues avec moi, tant personnelle qu'académique. Merci à Sylvie Grenier et Sylvia Coperey Llorca pour leur soutien administratif sans faille. Enfin, merci à tous les autres membres du GATE que je n'ai pas pu citer pour les pauses-déjeuner animées, les anecdotes sur les anciens élèves en finance ou sur la téléréalité française.

En-dehors du GATE, je dois témoigner ma gratitude envers Annick Laruelle pour l'échange enrichissant qu'elle m'a accordé lors de mon premier comité de suivi de thèse. Merci à Sylvain Béal d'avoir été présent pour le second comité et pour m'avoir invité à présenter un travail de recherche au séminaire du CRESE en 2021. J'ai passé un moment fantastique lors de ma visite au CRESE et pendant mes échanges avec Sylvain. Ce sont ces moments qui me font aimer la recherche et le monde académique. Merci à Alexandre Skoda pour m'avoir invité à présenter un travail de recherche au séminaire MDOD en 2022. Je remercie Vincent Merlin pour m'avoir invité à la Summer School en théorie du choix social lors de l'été 2019, Encarnación Algaba pour m'avoir invité à la conférence EURO en 2021, et Michel Grabisch pour m'avoir invité au Workshop OSGAD en 2021. Enfin, merci à Yukihiko Funaki, Frank Huettner et Michel Grabisch pour leurs questions pertinentes lors de mes interventions aux conférences internationales SING et EURO en 2021.

Cette thèse n'aurait pas été aussi enivrante sans la présence de mes camarades doctorants et ex-doctorants. En particulier, je tiens à remercier Adhen Benlahlou pour m'avoir accueilli au sein du GATE et m'avoir fait découvrir les bases de la recherche. Merci à Alain N'Ghauran pour m'avoir accueilli et pour davoir partagé son expérience de doctorant CIFRE au GATE. Merci à Rachid Achbah, Rodrigue Dogble Codjovi, tous deux doctorants à COACTIS, et à Ivan Mitrouchev, dorénavant maître de conférences à l'Université de Strasbourg, pour avoir été de si sympathiques voisins de bureau. Merci à Nesrine Bouzeghoub, Ignacio Hauser, Cyril Chambrefort et à Jordan Nagadzina-Sanchez, tous doctorants au GATE, pour être de si bons compagnons de route. Enfin, je tiens à remercier tout particulièrement mon co-auteur, mon "Kouhai", Kevin Techer. C'est lors du premier confinement en mars 2020 que nos directeurs de thèse nous ont incité à collaborer sur notre premier projet commun. Depuis, Kevin fait partie intégrante de mon processus de réflexion et de recherche. Sans lui, je doute que cette thèse se soit déroulée de la même façon.

Mes dernières pensées vont bien sûr à ma famille et à mes amis. Il me faut remercier mes amis qui m'ont accompagné lors de mon parcours universitaire prédoctoral, à savoir Baptiste, Florianne, Guillaume, Inès, Jad et Nhi. Je les remercie pour avoir été là lors des hauts et des bas qui ont jonché ma thèse. Merci à mes autres amis proches, Camille, Clément, Estelle, Guillaume, Nicolas, Sébastien et bien d'autres, pour m'avoir accompagné et soutenu pendant ces trois années et au-delà. Merci à mon parrain Marc, ma tante Jacqueline, ma nourrice Andrée et Marie-Claude pour avoir été présents pour ma famille en toute circonstance.

Je dédie cette thèse à ma mère qui a toujours été là pour moi, et à mon père qui veille sur moi depuis là où il est.

# Contents

# Chapitre 1: Introduction

## 1.1 La distribution de gaz en France

Le gaz naturel est une énergie fossile (aussi appelée énergie primaire) car il n'est pas produit par la transformation d'une autre énergie (contrairement à l'électricité). Le gaz naturel peut être utilisé à des fins domestiques, mais il est également utile dans d'autres domaines tels que l'industrie, l'agronomie et les transports. Sa consommation n'a cessé de croître ces dernières années, principalement grâce à ses qualités environnementales : il n'émet pratiquement pas de particules fines et est l'hydrocarbure libérant le moins de gaz à effet de serre dans l'atmosphère. Avant d'être livré aux consommateurs, le gaz naturel doit être extrait du sol, traité et enfin transporté jusqu'au lieu de consommation.

Ces dernières années, le biogaz est devenu une sérieuse alternative au gaz naturel, car plus écologique. On le produit en cuve par la fermentation de matières organiques en l'absence d'oxygène. Sa production permet de recycler des déchets organiques tout en évitant l'épuisement des ressources naturelles de la planète. Les sites de production de biogaz sont situés relativement proches des lieux de consommation. Ainsi, le biogaz nécessite généralement moins de moyens afin d'être livré aux consommateurs.

Nous étudions l'activité de distribution de gaz, qui est l'étape finale du processus de livraison du gaz. Tout d'abord, expliquons brièvement les différentes étapes constituant ce processus. En premier lieu, les producteurs sont en charge d'extraire le gaz des sols. Celui-ci est alors traité et est ensuite transporté aux points d'entrée frontaliers via des gazoducs internationaux ou des méthaniers. Depuis ces points d'entrée frontaliers, le réseau de transport prend le relais et achemine le gaz à haute pression, via de larges canalisations en acier, jusqu'à l'entrée des villes (ou vers un stockage). Enfin, le réseau de distribution réceptionne le gaz à l'entrée des villes pour ensuite l'acheminer jusqu'au consommateur. Toutes les étapes constituant la livraison de gaz sont résumées dans la Figure 1.1.

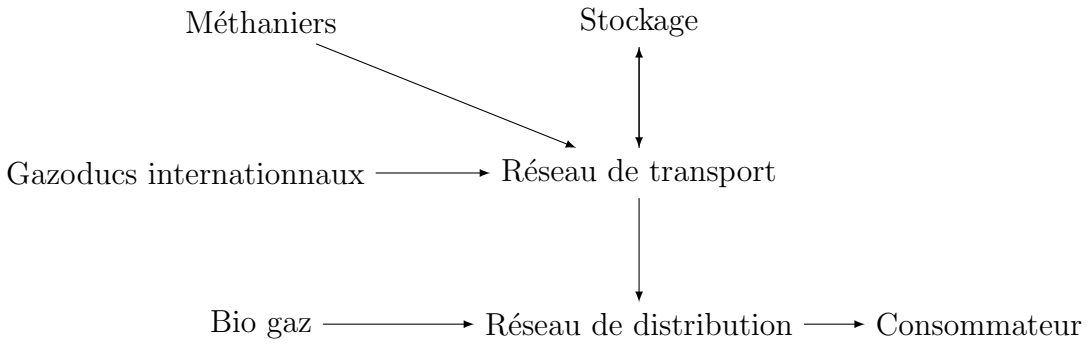



Figure 1.1

Un réseau de distribution de gaz peut être représenté par un graphe arborescent avec une source unique. Les nœuds de l'arbre représentent les consommateurs et les arcs les canalisations. Notons qu'un réseau de distribution de gaz peut éventuellement avoir plusieurs sources. Cependant, dans cette thèse, nous supposons qu'un réseau de distribution de gaz n'a qu'une unique source (voir la figure 1.2 pour un exemple de trois réseaux représentés par des graphes arborescents avec source unique).

En France, le marché de la distribution d'énergie a été ouvert à la concurrence le 1er juillet 2007. Cette ouverture a permis la création d'un marché unique à l'échelle européenne et la possibilité pour un consommateur de choisir librement son gestionnaire de réseau de distribution de gaz parmi la cinquantaine existant en France. La distribution de gaz en France est une mission de service public. Elle s'exerce sous une triple autorité : l'État, les collectivités locales et la Commission de régulation de l'énergie (CRE). L'État fixe le cadre général du marché de l'énergie en garantissant le respect des règles de la concurrence et des missions de service public. Les collectivités locales supervisent la distribution du gaz à plus petite échelle et veillent à son bon fonctionnement. La CRE veille au bon fonctionnement du marché français de l'énergie et à ce que tous les opérateurs de réseaux gaziers bénéficient des mêmes conditions d'accès au réseau de distribution. L'opérateur de distribution de gaz le plus important en France est (de loin) GRDF (Gaz Réseau Distribution France). Cet opérateur gère un réseau de distribution de gaz de plus de 200.000 km de long avec plus de 10.000 salariés pour alimenter plus de 11 millions de clients.

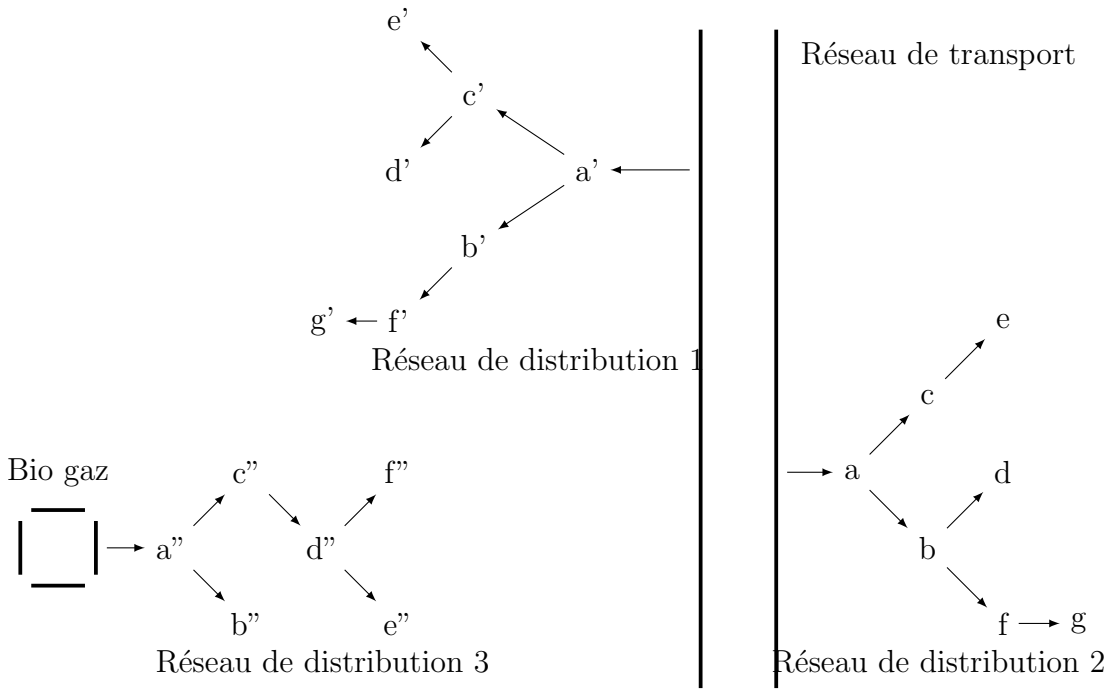



Figure 1.2

## 1.2 Le problème

L'activité de GRDF engendre de nombreux coûts, dont certains ne sont pas directement affectables à un consommateur donné. Par exemple, il existe des coûts liés à l'accès au réseau de transport et des coûts liés à l'accès aux sites de production de biogaz. De plus, l'entretien du réseau, sa surveillance et ses éventuelles extensions génèrent des coûts supplémentaires. GRDF souhaite facturer ces coûts aux consommateurs. Chaque consommateur signe un contrat de distribution avec GRDF (ou un autre gestionnaire de réseau de distribution). Ce contrat contient deux informations majeures. Tout d'abord, il précise la catégorie à laquelle appartient le consommateur. GRDF classe les consommateurs en fonction d'un système de catégorisation bien établis. La demande effective d'un consommateur est l'information la plus déterminante pour sa catégorisation. Cette demande effective correspond au plus haut volume de gaz que le consommateur s'attend à consommer sur une courte période de temps. Typiquement, cela correspond à la quantité qu'un ménage pense consommer pendant les jours les plus froids de l'hiver. Pour une entreprise de transport, cela correspond à la période de l'année où la flotte de l'entreprise est la plus sollicitée. Pour une industrie, cela correspond à la période de l'année où l'activité de production est à son plus haut niveau. Sans perte de généralité, nous supposons que l'ensemble des demandes effectives coïncide avec le système de catégorisation établi par GRDF. Ensuite, le contrat liant le consommateur et GRDF spécifie le montant

que chaque consommateur doit payer en fonction de sa demande effective. Ce montant est déterminé par un tarif de distribution, appelé "Accès des Tiers au Réseau de Distribution" (ATRD). L'ATRD se présente sous forme d'une grille précisant le montant que les consommateurs doivent payer en fonction de leur demande effective. GRDF cherche à recouvrir l'ensemble des coûts via l'ATRD. Pour ce faire, GRDF doit déterminer une méthode afin de partager ces coûts entre les consommateurs. La performance de GRDF dépend fortement de la manière dont ces coûts sont répartis. En particulier, une méthode de partage des coûts est pertinente si elle répond aux principes attendus par le gestionnaire de réseau.

Il existe deux grands principes défendus par GRDF concernant les méthodes de partage des coûts: le principe de péréquation et le principe d'indépendance aux demandes supérieures.[1]. Tout d'abord, le principe de péréquation stipule que deux consommateurs ayant les mêmes demandes effectives doivent se voir facturer le même montant, indépendamment de leur localisation géographique. Par exemple, il ne devrait pas y avoir de différence en termes de tarif appliqué entre les zones rurales et les zones urbaines, bien que les coûts sous-jacents soient différents. Ce principe vise à réduire les inégalités de traitement entre les différents consommateurs. Ce principe est très répandu en France, non seulement dans le domaine de la fourniture de gaz, mais aussi dans celui de la fourniture d'électricité, d'eau et d'autres services publics (voir Fleurbaey & Trannoy (1998)). Ce principe fait même partie du droit français : "Constitution-article 72-2, révision du 28 mars 2003 [. . .] la loi assure des mécanismes de péréquation pour promouvoir l'égalité entre les territoires". Deuxièmement, le principe d'indépendance aux demandes supérieures indique qu'un consommateur ne doit pas avoir à payer pour des demandes supérieures aux siennes. Ce principe permet d'éviter les situations où la présence d'un consommateur ayant une demande élevée implique des coûts supplémentaires pour les consommateurs ayant de plus modestes demandes. Par exemple, il semble injustifié d'augmenter la facture d'un ménage lorsqu'une usine consommant de larges volumes de gaz emménage à proximité.

La plupart, sinon la totalité, des méthodes de partage des coûts utilisées par GRDF sont inspirés du modèle standard de partage des coûts (version discrete) (voir par exemple Moulin (2003) et Hougaard (2018)). Dans ce qui suit, nous présentons brièvement la méthode d'Aumann-Shapley (version discrète) et la méthode de Shapley-Shubik (version discrète), qui sont deux méthodes de partage des coûts couramment utilisées par GRDF. Fixons un ensemble fini $N$ de $n$ consommateurs de gaz. Chaque consommateur $i \in N$ possède une demande effective $q_i$ déclarée auprès du gestionnaire de réseau. Ce dernier ne dispose d'aucune information concernant les autres demandes qu'un consommateur effectue tout au long de l'année, en dehors de sa demande effective. Par conséquent, chaque consommateur $i \in N$ est doté d'un ensemble discret $\{0, 1, \ldots, q_i\}$ représentant toutes les demandes potentielles que $i$ est en mesure de faire au cours d'une année. Le profil $q = (q_1, \ldots, q_n)$ collecte l'intégralité des demandes effectives des consommateurs. Le problème de partage des coûts de

[1]Le principe de péréquation se traduira par "Uniformity" en anglais.

GRDF peut être représenté par le couple $(q, C)$, où $C$ est une fonction mesurant le coût d'exploitation d'un réseau de distribution en mesure de satisfaire tout profil de demande dans lequel chaque consommateur $i \in N$ demande au plus sa demande effective $q_i$. Formellement, la fonction $C$ est en mesure d'évaluer le coût $C(s)$ de tout réseau de distribution conçu pour satisfaire à un profil de demande $s \leq q$. La fonction $C$ est croissante et vérifie $C(0, \ldots, 0) = 0$, ce qui signifie que $C(t) \leq C(s)$ pour tout $t \leq s \leq q$. Le montant $C(q)$ correspond au coût total du réseau de distribution de gaz. Par exemple, prenons un ensemble de 4 consommateur $N = \{a, b, c, d\}$. Le profil des demandes effectives est $q = (2, 1, 1, 1)$, ce qui signifie que $a$ possède une demande effective de 2 alors que $b, c$ et $d$ ont une demande effective de 1. Une fonction $C$ est capable de mesurer le coût de tout réseau de distribution conçu pour répondre à un profil de demande $s \leq q$. Par exemple, $C$ est en mesure d'évaluer le coût du profil $(1, 1, 0, 0)$ dans lequel $a$ demande 1, $b$ demande 1 et $c$ et $d$ demandent 0. Bien sûr, ce profil est hypothétique et ne correspond pas aux demandes effectives réelles. Mais il est nécessaire de considérer un tel scénario pour pouvoir calculer les méthodes d'Aumann-Shapley et de Shapley-Shubik. GRDF cherche à recouvrir tous les coûts via l'ATRD. Cela signifie que le tarif défini par GRDF doit recouvrir $C(q)$ en l'allouant au sein de l'ensemble des consommateurs. Maintenant, nous introduisons quelques prérequis sur les jeux coopératifs afin de comprendre les méthodes d'Aumann-Shapley et de Shapley-Shubik. Le modèle standard des jeux coopératifs sont les jeux à utilité transférable (jeux TU en abrégé). Les jeux TU modélisent des situations dans lesquelles certains joueurs peuvent former des coalitions pour engendrer une certaine utilité. Formellement, un jeu TU est un couple $(N, v)$, où $N$ est un ensemble fini de joueurs et $v$ est une fonction caractéristique en mesure d'évaluer l'utilité de chaque coalition de joueurs $E \subseteq N$. Un vecteur de gain attribue un certain gain à chacun joueur en fonction des différentes utilitéq engendréeq par la fonction caractéristique. Une solution sur une classe de jeux TU attribue un unique vecteur de gain à chaque jeu de cette classe. La valeur de Shapley (voir Shapley (1953)) est probablement la solution la plus connue pour les jeux TU. Elle est calculée comme suit. Supposons que la grande coalition $N$, dans laquelle tous les joueurs coopèrent, soit formée étape par étape selon un certain ordre linéaire sur l'ensemble des joueurs. À chaque étape, un joueur rejoint la coalition courante et obtient sa contribution marginale (correspondant à la variation de gain engendrée lorsque ce joueur rejoint la coalition). La valeur de Shapley attribue à chaque joueur sa contribution marginale espérée en supposant que chaque ordre linéaire sur l'ensemble des joueurs se survienne avec la même probabilité. Nous avons maintenant les éléments nécessaire pour présenter brièvement la méthode d'Aumann-Shapley et la méthode de Shapley-Shubik.

*La méthode de Shapley-Shubik* : étant donné une coalition $E \subseteq N$ de consommateurs, calculons le coût associé aux demandes effectives des consommateurs dans la coalition. Ce coût est représenté par $v(E) = C(q_E, 0_{N \setminus E})$, où $(q_E, 0_{N \setminus E})$ est le profil dans lequel les consommateurs dans $E$ demandent leur demande effective et les con-

sommateurs en dehors de $E$ ont une demande nulle. La méthode de Shapley-Shubik calcule la valeur de Shapley du jeu TU $(N, v)$. Autrement dit, le coût imputé à un consommateur est déterminé par sa contribution marginale espérée dans ce jeu. Remarquez que la méthode de Shapley-Shubik ne tient pas compte de l'ensemble de demandes $\{0, 1, \ldots, q_i\}$ dont est doté chaque consommateur $i \in N$. Cette méthode suppose que chaque consommateur atteint sa demande effective sans passer par des niveaux de demande intermédiaires.

*La méthode d'Aumann-Shapley* : étant donné un consommateur $i \in N$, chaque unité demandée par $i$ peut être vue comme une unité à part. Il existe donc un ensemble $N^q = \cup_{i \in N} N_i$ d'unités, où $|N_i| = q_i$. Chaque $N_i$ peut être vu comme un ensemble contenant $q_i$ unités identiques de gaz associées au consomateur $i$. On supposera qu'un consommateur $i$ peut donc effectuer une demande de $d_i \leq q_i$ unités de gaz. Soit $E \subseteq N^q$, on note $d(E) = (|E \cap N_i|)_{i \in N}$ les unités de gaz demandées par chaque consommateurs dans $E$ et on note $v^q(E) = C(d(E))$ le coût associé à ces demandes. La valeur de Shapley du jeu $(N^q, v)$ attribue un certain paiement à chaque élément de $N^q$. Pour chaque $i \in N$, la somme de tous les paiements des éléments de $N_i$ constitue son allocation par la méthode d'Aumann-Shapley. Observons que, contrairement à la méthode de Shapley-Shubik, la méthode d'Aumann-Shapley prend en compte l'ensemble discret des demandes $\{0, 1, \ldots, q_i\}$ de chaque consommateur $i \in N$. Cette méthode suppose donc que chaque consommateur atteint progressivement sa demande effective.

Puisque nous sommes dans le cadre de la distribution de gaz, deux critiques peuvent être formulées à l'égard de ces méthodes. Premièrement, les principes attendus par GRDF ne sont pas satisfais par ces méthodes. Par définition de ces méthodes, l'indépendance aux demandes supérieures n'est clairement pas satisfaite. Si un consommateur augmente sa demande, alors d'autres consommateurs avec des demandes plus faibles peuvent potentiellement être affectés. De plus, le principe de péréquation n'est également pas satisfait par ces deux méthodes. En effet, ces méthodes étant basées sur les contributions marginales des consommateurs dans certain jeux TU, il est peu probable que deux consommateurs ayant la même demande effective se voient facturer le même montant.

Deuxièmement, aucune de ces méthodes ne prend en compte le réseau de distribution comme une information indépendante du problème. Cela est dû au fait que l'information fournie par le réseau est souvent contenue dans la fonction de coûts. En clair, le problème de partage des coûts de GRDF devrait être écrit $(q, C^{Net})$, la fonction $C^{Net}$ mesurant les coûts en se basant sur les informations fournies par le réseau de distribution. Cette fonction peut être vue comme la combinaison de deux informations : le réseau de distribution $Net$ et les caractéristiques économiques de GRDF $C$ (par exemple, les coûts d'accès au réseau de transport et aux producteurs de biogaz, l'activité administrative, la gestion des contrats, etc.) La méthode d'Aumann-Shapley ou de la méthode de Shapley-Shubik ont leurs avantages tech-

niques mais aussi leurs inconvénients puisque $C^{Net}$ comprime une grande quantité d'information. De plus, une telle compression d'information à des conséquences sur la méthode axiomatique, qui est centrale dans cette thèse. La méthode axiomatique est utilisée pour déterminer une solution à partir de certains axiomes qui expriment des principes raisonnables ou désirables qu'une solution pourrait ou devrait satisfaire. Un axiome tire profit de l'information du problème pour formuler une propriété. Ainsi, si le problème comprime les informations fournies par le réseau de distribution et les caractéristiques économiques de GRDF, cette compression aura un effet direct sur la pertinence et le contenu des axiomes. Nous proposons de dissocier $Net$ et $C$ en deux informations distinctes, et de considérer un nouveau problème représenté par le triplet $(q, C, Net)$. De cette façon, le réseau de distribution sera une information indépendante des autres informations du problème. D'un point de vue théorique, cette approche permet une étude axiomatique plus fine du problème de partage des coûts de GRDF. Ce point est détaillé dans la section 1.5. La section suivante présente la méthode axiomatique.

## 1.3 Méthode axiomatique

Cette section est fortement inspirée de Thomson (2001). Une étude axiomatique d'une situation à plusieurs agents commence par la spécification d'une classe de problèmes. Un problème est un ensemble d'alternative disponibles pour des agents ainsi qu'un ensemble d'information sur les agents (s'ils sont joueurs, consommateurs, électeurs, etc.). Des informations supplémentaires peuvent être fournies avec le problème (préférences des agents, une structure sur l'ensemble des agents, etc.) Présentons quelques situations pour illustrer le large éventail de problèmes possibles. Un problème de choix social consiste en un ensemble non structuré d'alternative réalisables, ainsi que les préférences des agents sur cet ensemble. Le problème consiste alors à choisir une ou plusieurs alternative en fonction des préférences des agents. Pour les problèmes de négociation et d'allocation des coûts, l'ensemble des alternative consiste uniquement en un ensemble de vecteurs d'utilités réalisables. Le problème consiste alors à sélectionner un sous-ensemble de ces vecteurs. Dans les problèmes de banqueroute, il existe une quantité donnée d'un bien parfaitement divisible, et chaque agent réclame une certaine quantité de ce bien. Le problème est alors de partager la quantité totale de ce bien entre les agents en supposant que cette quantité n'est pas suffisante pour satisfaire à toutes leurs réclamations.

Étant donné une classe de problèmes $D$, une solution sur $D$ est une fonction qui associe un ensemble d'alternative à chaque problème dans $D$. Une solution est représentée par $f$ et l'ensemble des alternative est représenté par $X$. Par conséquent, une solution peut être écrite sous la forme $f : D \to X$. Les solutions peuvent être un ensemble d'alternative dans certains modèles et peuvent être un singleton dans d'autres. Que l'objectif soit descriptif ou prescriptif, les solutions-singleton (aussi appelées "valeur") sont souvent préférées, car une solution capable de faire

des prédictions ou des recommandations spécifiques a plus de chances d'être selectionnée par un décideur. Cependant, les solutions-singleton peuvent être très difficiles à caractériser et pour de nombreux modèles, les solutions ensemblistes sont donc préférées. Dans la théorie de la négociation et des problèmes de choix social, la plupart des concepts de solution sont des solutions-singleton. Dans les jeux coopératifs et les problèmes de partage des coûts, les deux types de concepts de solution sont populaires.

Une étude axiomatique commence par la spécification d'une classe de problèmes, elle-même suivie par la formulation d'une liste de propriétés souhaitables ou raisonnables (axiomes) pour les solutions aux problèmes de cette classe. L'étude se termine par la description de l'ensemble des solutions satisfaisant à diverses combinaisons d'axiomes. L'étude doit également proposer une analyse de l'indépendance logique entre les axiomes, car c'est un moyen efficace d'évaluer leur puissance relative. En outre, il est tout aussi utile de formuler et d'explorer des variantes des axiomes, car il n'est pas rare que les idées générales qui les inspirent aient pu être obtenues par des formulations mathématiques légèrement différentes mais tout aussi désirable. Une étude axiomatique aboutit souvent à des théorèmes de caractérisation. Ces théorèmes identifient une solution satisfaisant à une liste donnée d'axiomes. Une étude axiomatique peut également produire des théorèmes d'impossibilité, indiquant l'incompatibilité d'une certaine liste d'axiomes sur une certaine classe de problèmes. Pour des raisons pratiques, l'étude peut commencer par les solutions. Bien que les axiomes soient conceptuellement les premiers à devoir être considérés, il est utile d'avoir à notre disposition plusieurs concepts de solution. En effet, cela nous permet d'évaluer la force des axiomes en testant les conjectures concernant la compatibilité des axiomes et l'indépendance logique entre eux. Précisement, étant donné un certain théorème de caractérisation, on dira que les axiomes sont logiquement indépendants si, en supprimant l'un d'entre eux, la caractérisation ne tient plus. L'indépendance logique des axiomes est importante car elle garantit que le théorème de caractérisation soit aussi général que possible. En pratique, pour établir l'indépendance logique d'un axiome par rapport aux autres dans une caractérisation, il suffit d'établir une solution, différente de celle qui est caractérisée, satisfaisant à tous les axiomes sauf celui qui nous intéresse.

## 1.4 Objectif de la thèse

L'objectif de cette thèse est de fournir des méthodes afin de répartir les coûts de GRDF entre ses consommateurs. Contrairement aux méthodes d'Aumann-Shapley et de Shapley-Shubik, nos méthodes doivent réussir à prendre en compte le réseau de distribution comme une information indépendante du reste du problème et être cohérentes avec les principes attendus par GRDF. A l'aide d'un partage des coûts, GRDF souhaite mettre en place une grille tarifaire. L'idée centrale d'un tarif est de fournir une grille spécifiant un montant à payer par les consommateurs en fonction

de leur demande. Une méthode de partage des coûts doit donc être adaptée pour en dériver une grille tarifaire et être capable de spécifier un montant à payer par les consommateurs en fonction de leur demande.

Nous utilisons les jeux coopératifs pour déterminer des méthodes pertinentes. Ces jeux ont été grandement étudiés et appliqués aux problèmes de partage des coûts et autres situations connexes. Cependant, la plupart des concepts de solution des jeux TU allouent un gain par joueur (par exemple, la valeur de Shapley, la valeur de division égalitaire, voir Béal *et al.* (2016), ou les valeurs de Shapley égalitaires, voir Joosten (1996)) sans aucune précision sur la façon dont ce gain varie en fonction de la demande du joueur.[2] Du point de vue de GRDF, les concepts de solution issus des jeux TU ne sont clairement pas suffisants pour établir une grille tarifaire. Pour palier cet inconvénient, la thèse se concentre sur une extension des jeux TU: les jeux multi-choix. Les jeux multi-choix, introduits par Hsiao & Raghavan (1992) et van den Nouweland (1993), sont une extension naturelle des jeux TU dans lesquels chaque joueur peut choisir parmi plusieurs niveaux d'activité auxquels il peut coopérer au sein d'une coalition.

Par le passé, les jeux multi-choix ont été appliqués avec succès en économie. Par exemple, Branzei *et al.* (2009) étudie les jeux multi-choix qui découlent de situations de marché avec deux factions. L'une des factions est constituée d'un groupe de joueurs puissants ayant des choix de type oui ou non. L'autre faction est constituée de joueurs non puissants avec plusieurs choix concernant le degré de coopération au sein de la faction; Grabisch & Rusinowska (2010) généralise un modèle d'influence oui-non à un cadre multi-choix. Les auteurs étudient une situation dans laquelle certains agents font partie d'un réseau social. Chaque agent a un ensemble ordonné d'actions possibles et est influencé par ses voisins dans le réseau lorsqu'il choisit son action; enfin, Techer (2021) aborde le problème du coût social, introduit à l'origine par Coase (1960), en utilisant des jeux multi-choix. L'auteur étudie des situations dans lesquelles un pollueur interagit avec plusieurs victimes potentielles, et vise à négocier un accord stable concernant le niveau de pollution. Le pollueur a plusieurs niveaux auxquels il souhaite produire et polluera proportionnellement, tandis que les victimes peuvent participer ou non aux négociations.

Soit $N$ un ensemble de joueurs. Dans un jeu multi-choix, chaque joueur $i \in N$ est en mesure de choisir parmi plusieurs niveaux d'activité auxquels il peut coopérer. Le niveau 0 signifie qu'il ne coopère pas et le niveau $m_i$ représente son niveau d'activité maximal. Le profil $m = (m_1, \ldots, m_n)$ rassemble tous les niveaux d'activité maximaux des joueurs. Un profil $s \leq m$ est une coalition (multi-choix). Un jeu multi-choix est un couple $(m, v)$ où $v$ est la fonction caractéristique qui associe à chaque coalition $s \leq m$ une valeur réelle et qui vérifie $v(0, \ldots, 0) = 0$, où $(0, \ldots, 0)$ est la coalition vide. La sous-classe des jeux multi-choix où $m = (1, \ldots, 1)$ peut être vue comme la classe complète des jeux TU sur $N$. De plus, la classe des problèmes de partage des coûts (discrets) peut être vue comme la sous-classe des jeux multi-choix non

[2]Les concepts de solution-singletons des problèmes de partage des coûts discrets allouent également un gain unique par joueur.

décroissants. Lorsque l'on se place dans le cadre du problème de la conception d'un tarif, les niveaux d'activité représentent les demandes des consommateurs. Un vecteur de paiement (multi-choix) décrit dans quelle mesure le paiement de chaque joueur varie en fonction de son niveau d'activité, ce qui permet d'établir une grille tarifaire. Une solution-singleton (que l'on appellera "valeur" par la suite) sur une classe de jeux multi-choix attribue un vecteur de paiement unique à chaque jeu dans cette classe. Une solution ensembliste définie sur une classe de jeux multi-choix attribue un ensemble (éventuellement vide) de vecteurs de paiements à chaque jeu dans cette classe.

Dans cette thèse, nous introduisons de nouveaux concepts de solution pour les jeux multi-choix. En particulier, nous étudions des situations dans lesquelles le jeu est doté d'une structure qui tire partie de l'information fournie par le réseau de distribution afin d'établir des relations entre les consommateurs. De cette façon, les concepts de solution pourront prendre en compte l'information fournie par le réseau de distribution. De plus, certains des concepts de solution réussiront à satisfaire aux principes attendus par GRDF. Nous soulignons que tous les concepts de solution proposés dans cette thèse ne parviennent pas forcément à satisfaire les principes poursuivis par GRDF et à prendre en compte le réseau en même temps.

Le reste de cette introduction est organisé comme suit. Nous discutons de l'interaction entre les jeux coopératifs et les structures dans la section 1.5. Les concepts de solution des jeux multi-choix et le principe d'indépendance aux demandes supérieures sont abordés dans la section 1.6. La section 1.7 discute du principe de péréquation. Enfin, la section 1.8 présente un résumé de cette thèse et détaille les contributions des différents chapitres.

## 1.5 Jeux et structures

Dans un jeu TU, on suppose généralement que la coopération entre un groupe de joueurs n'est pas entravée par un quelconque facteur exogène. Cependant, en pratique, plusieurs structures exogène peuvent avoir un impact sur la formation des coalitions, l'évaluation des coalitions ou le processus d'allocation. Des jeux TU dotés d'une structure sur l'ensemble des joueurs ont été introduits pour modéliser des situations où la formation ou l'évaluation des coalitions peuvent dépendre de contraintes de communication, de coalition ou hiérarchiques.

Les contraintes de communication, modélisées par un graphe non orienté (par exemple, la figure 1.3), sont abordées dans Myerson (1977). Les joueurs sont représentés par les nœuds d'un graphe non orienté dans lequel les arêtes représentent les canaux de communication entre les joueurs. Une coalition de joueurs ne peut coopérer que si elle peut communiquer par un chemin qui se trouve à l'intérieur de cette coalition. Le modèle des jeux TU avec contraintes de communication peut être représenté par un triplet $(N, v, L)$, où $(N, v)$ est un jeu TU et $(N, L)$ est un graphe non orienté représentant les contraintes de communication. Ce modèle distingue la fonction car-

actéristique de la structure. Comme mentionné dans la Section 1.2, cette approche permet de mener une étude axiomatique sur un problème $(N, v, L)$ où $(N, v)$ et $(N, L)$ sont deux informations distinctes. Myerson (1977) caractérise axiomatiquement un concept de solution calculé comme la valeur de Shapley du jeu $(N, v^L)$, où $v^L$ est une fonction caractéristique qui combine $v$ et $L$. La valeur $v^L(E)$ d'une coalition $E \subseteq N$ dans le jeu $(N, v^L)$ est calculée comme la somme de la valeur de chaque coalition connectée maximale (par rapport au graphe de communication) contenue dans $E$. Ainsi, les contraintes de communication affectent l'évaluation des coalitions.

La façon dont $v$ et $L$ sont combinés en $v^L$ est entièrement déterminée par les axiomes, ce qui signifie qu'un autre ensemble d'axiomes pourrait conduire à un résultat différent. Cela illustre bien l'un des avantages théoriques de la distinction entre la fonction caractéristique et la structure : il existe différents ensembles d'axiomes possibles qui donnent lieu à différentes combinaisons de $v$ et $L$.

Les contraintes de coalition liées à l'existence d'unions à préexistantes modélisent des situations dans lesquelles les joueurs sont amenés à s'unir de façon prédéterminées. Elles sont représentées par une partition de l'ensemble des joueurs (par exemple, la figure 1.4). Ces structures sont discutées dans Aumann & Dreze (1974) et Owen (1977). Différentes interprétations de ce modèle sont possibles. Dans Aumann & Dreze (1974), les joueurs appartenant à différentes unions a priori ne peuvent pas coopérer, ce qui n'est pas le cas dans Owen (1977), mais la formation d'une coalition doit être compatible avec les unions. Ainsi, les contraintes coalitionnelles formées par la partition de l'ensemble des joueurs en unions affectent la formation des coalitions. Le modèle des jeux TU avec contraintes de coalition peut être représenté par un triplet $(N, v, W)$, où $(N, v)$ est un jeu TU et $(N, W)$ est une partition de l'ensemble des joueurs représentant les contraintes de coalition. De même que pour les contraintes de communication, ce modèle distingue le jeu de la structure. Aumann & Dreze (1974) et Owen (1977) proposent deux valeurs pour les jeux TU avec des contraintes de coalition et les caractérisent axiomatiquement. Aumann & Dreze (1974) proposent un concept de solution calculé, pour chaque $i \in N$, comme la valeur de Shapley du jeu $(W(i), v^{W(i)})$, où $W(i)$ est une union contenant le joueur $i \in N$, et $v^{W(i)}$ est la restriction de $v$ au domaine $W(i)$. Owen (1977) propose la valeur d'Owen qui peut être considérée comme une procédure en deux étapes. Dans la première étape, un jeu TU entre les unions préexistantes est utilisé pour déterminer le gain total obtenu par chaque union a priori. Dans la seconde, un jeu TU au sein de chaque union a priori est utilisé pour allouer à ses membres le gain total obtenu par l'union a priori dans la première étape. La valeur d'Owen est obtenue en appliquant la valeur de Shapley dans les deux étapes.

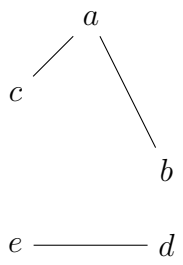



Figure 1.3: $L$

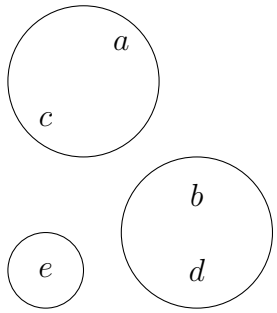



Figure 1.4: $W$

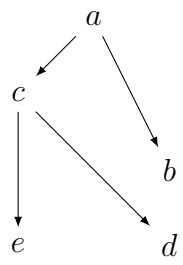



Figure 1.5: $D$

Les structures de permission, modélisées par des graphes dirigés (par exemple, Figure 1.5), furent introduites par Gilles *et al.* (1992). Ces structures décrivent des situations dans lesquelles les joueurs ont besoin de la permission d'autres joueurs avant d'être autorisés à coopérer au sein d'une coalition. Un jeu TU muni d'une structure de permission peut être représenté par un triplet $(N, v, D)$, où $(N, v)$ est un jeu TU et $(N, D)$ est un graphe dirigé représentant la structure de permission. van den Brink & Gilles (1996) introduit la valeur de permission pour de tels jeux. Cette valeur est calculée comme la valeur de Shapley du jeu TU $(N, v^D)$, où $v^D(E)$ mesure la valeur de la plus grande coalition réalisable contenue dans $E \subseteq N$. Dans ce contexte, une coalition est réalisable si elle contient tous les supérieurs des membres de la coalition. Tout comme dans les situations avec contraintes de communication, la structure affecte l'évaluation des coalitions. Plusieurs caractérisations axiomatiques de la valeur de permission ont été proposées par van den Brink & Gilles (1996), van den Brink *et al.* (2015) et van den Brink *et al.* (2018) entre autres.

Enfin, les structures de priorité, modélisées par ordre partiel, furent introduites par Béal *et al.* (2021a). Elles décrivent des situations dans lesquelles certains joueurs ont la priorité sur d'autres joueurs dans le processus d'allocation. Le modèle des jeux TU avec une structure de priorité peut être représenté par un triplet $(N, v, D)$, où $(N, v)$ est un jeu TU et $(N, D)$ est un ordre partiel représentant la structure de priorité. Béal *et al.* (2021a) introduisent la valeur de priorité pour les jeux TU muni d'une structure de priorité. Selon cette valeur, la structure affecte le processus d'allocation, mais n'a pas d'effet sur la formation ou l'évaluation des coalitions. Deux caractérisations axiomatiques de la valeur de priorité sont proposées par Béal *et al.* (2021a). De toute évidence, le modèle des jeux TU avec une structure de priorité est très similaire aux jeux TU avec une structure de permission. Cependant, les deux modèles ont une interprétation et une utilisation différentes de la structure.

Dans cette thèse, nous étudions les jeux multi-choix muni d'une structure. Quelques études ont été menées sur ce sujet. Par exemple, Béal *et al.* (2012) aborde le thème des jeux multi-choix avec des contraintes de communication et caractérise axiomatiquement une solution étendant la solution proposée par Herings *et al.* (2008) pour les jeux TU avec des contraintes de communication. Albizuri (2009) étudie les jeux multi-choix avec des unions préexistantes. L'auteur caractérise axiomatiquement une

solution qui étend la valeur d'Owen (voir Owen (1977)).

Dans les jeux TU, les joueurs n'ont qu'un seul niveau d'activité auquel participer au sein d'une coalition. Par conséquent, la structure est facile à interpréter puisqu'elle établit des relations entre les joueurs. Dans les jeux multi-choix, ces relations sont plus difficiles à déterminer en raison de la multiplicité des niveaux d'activité. Par exemple, considérons une structure de permission. Une relation de permission directe entre deux joueurs est claire si les deux joueurs n'ont que deux choix : 0 ou 1, c'est-à-dire s'ils coopèrent ou non. En effet, supposons que $i \in N$ ait besoin de la permission de $k \in N$ pour coopérer. Si $k$ joue 1, alors $i$ est autorisé à jouer 1. Au contraire, si $k$ joue 0, alors $i$ n'est pas autorisé à jouer 1. Maintenant, supposons que $k$ ait 3 niveaux d'activité, et supposons que $m_i = 2$. Quel niveau d'activité doit jouer $k$ afin de permettre au joueur $i$ de jouer son niveau d'activité 1 ? Il n'y a pas de manière triviale de déterminer un tel niveau. Un scénario possible serait le suivant : $i$ ne peut coopérer que si $k$ coopère. Dans ce cas, $i$ a besoin que $k$ joue son niveau d'activité 1 (ou plus) avant de coopérer lui-même (voir figure 1.6). Un autre serait : $i$ ne peut pas coopérer à moins que $k$ ne coopère à son niveau d'activité maximal. Dans ce cas, $i$ a besoin que $k$ joue son niveau d'activité 3 avant de pouvoir coopérer lui-même (voir figure 1.7). Ceci rend l'étude des jeux multi-choix muni d'une structure plus complexe que les jeux TU muni d'une structure.

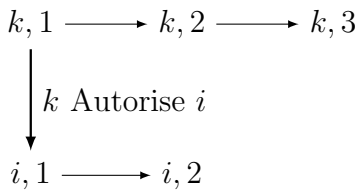



Figure 1.6

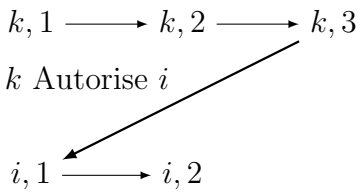



Figure 1.7

Les niveaux d'activité d'un joueur sont supposés linéairement ordonnés : un joueur $k$ ne peut pas jouer 2 s'il ne joue pas 1 au préalable. En un sens, les niveaux d'activité des joueurs font déjà partie d'une structure. Cette idée est illustrée en dessinant des flèches entre les niveaux d'activité de chaque joueur dans les figures 1.6 et 1.7. Un premier objectif de cette thèse est de fournir des moyens pertinents de combiner la structure sur l'ensemble des niveaux d'activité et la structure sur l'ensemble des joueurs en une structure sur l'ensemble des niveaux d'activité (tout comme nous l'avons fait dans la figure 1.6 et 1.7). Nous abordons cette question en considérant les structures de permission et les structures de priorité. Nous étudions comment ces structures peuvent être combinées avec la structure sur l'ensemble des niveaux d'activité, puis nous fournissons des concepts de solution basés sur ces combinaisons. Cette discussion est particulièrement intéressante pour GRDF car elle fournit différentes interprétations sur la façon dont les caractéristiques géographiques du réseau affectent ses caractéristiques économiques.

## 1.6 Indépendance aux demandes supérieures

Des propriétés similaires au principe d'indépendance aux demandes supérieures existent déjà dans la littérature économique. Par exemple, la méthode serial cost sharing pour les problèmes de partage des coûts discrets, introduite par Moulin & Shenker (1992), satisfait à la propriété. Plus récemment, Albizuri *et al.* (2020) ont étudié des solutions pour des problèmes de négociation satisfaisant à une propriété similaire d'indépendance aux demandes supérieures. Dans cette section, nous discutons du principe d'indépendance aux demandes supérieures dans le cadre des jeux multi-choix.

Une solution pour jeux multi-choix suit le principe d'indépendance aux demandes supérieures si le montant alloué à un niveau d'activité d'un joueur est indépendant aux niveaux d'activité supérieurs au sien. Malheureusement, la plupart des concepts de solution pour jeux multi-choix ne vérifient pas ce principe. Ces solutions sont généralement des extensions de la valeur de Shapley au cadre multi-choix. La première extension est introduite par Hsiao & Raghavan (1992). Les auteurs étudient les jeux multi-choix dans lesquels les joueurs partagent tous le même niveau d'activité maximal. Les niveaux d'activité sont pondérés, ce qui permet aux auteurs d'étendre l'idée des valeurs de Shapley pondérées (voir Kalai & Samet (1987)) des jeux TU aux jeux multi-choix. Cette valeur ne vérifie pas le principe d'indépendance aux demandes supérieures. van den Nouweland *et al.* (1995) se focalisent sur la classe complète des jeux multi-choix et proposent une seconde extension de la valeur de Shapley, que l'on écrira $vdN$. Supposons que la grande coalition $m$ se forme étape par étape, selon un certain ordre, en partant de la coalition vide $(0, \ldots, 0)$, ce qui signifie qu'à chaque étape, un joueur augmente son niveau d'activité d'une unité. Ainsi, à chaque étape, la contribution marginale d'un joueur à la coalition correspond à la variation de valeur que subit la coalition lorsque ce joueur augmente son niveau d'activité d'une unité. La valeur $vdN$ alloue à chaque joueur l'espérance de ses contributions marginales, en supposant que la grande coalition puisse se former selon tout ordre admissible, où chaque ordre admissible à la même probabilité de survenir. Calvo & Santos (2000) montrent que la valeur $vdN$ coïncide avec la méthode d'Aumann-Shapley discrète sur la classe des problèmes de partage des coûts discrets. Ainsi, la valeur $vdN$ ne verifie pas le principe d'indépendance aux demandes supérieures. D'autres extensions peuvent être trouvées dans Derks & Peters (1993), Peters & Zank (2005) et Grabisch & Lange (2007). Aucune de ces solutions ne satisfait au principe d'indépendance aux demandes supérieures (voir Chapitre 2). Ce principe est également lié à un concept de stabilité : le Coeur d'un jeu multi-choix introduit par Grabisch & Xie (2007) (le Coeur, en abrégé). Le Coeur est un ensemble de solutions pour jeux multi-choix contenant tous les vecteurs de gains stables. Un vecteur de gain est stable si aucune coalition ne peut obtenir, par elle-même, une meilleure utilité que celle prescrite par le vecteur de gain. Pour qu'un vecteur de gain soit dans le Coeur, il doit nécessairement satisfaire la propriété de multi-efficience. Cette propriété stipule que, pour tout niveau d'activité, si tous les joueurs sont d'accord pour

jouer ce niveau (ou leur niveau d'activité maximal s'ils sont incapables d'atteindre ce niveau), alors ils devraient obtenir la même valeur que celle prescrite par le vecteur de gain dans le Coeur. Ce type d'accord est appelé une coalition synchronisée. Si une valeur satisfait au principe d'indépendance aux demandes supérieures et recouvre également la valeur de la grande coalition, alors elle est multi-efficiente. Cela rend le principe d'indépendance aux demandes supérieures particulièrement désirable.

## 1.7 Péréquation

Le principe de péréquation préconise une égalité de traitement entre deux consommateurs ayant la même demande effective. En termes de jeux multi-choix, cela signifie que deux joueurs ayant le même niveau d'activité maximal doivent recevoir le même gain. De même que pour le principe d'indépendance aux demandes supérieures, aucun concept de solution des jeux multi-choix ne satisfait au principe de péréquation puisque la plupart d'entre eux sont calculés en fonction des contributions marginales des joueurs.

Au sein de la classe des jeux TU, la division égalitaire, qui divise également le gain de la grande coalition entre les joueurs, satisfait clairement au principe de péréquation. Étonnamment, il existe très peu d'études dans la littérature qui étendent la division égalitaire des jeux TU aux jeux multi-choix. A notre connaissance, la seule solution qui étende la division égalitaire est la solution égalitaire multi-choix contrainte introduite par Branzei *et al.* (2014). Nous introduisons une extension de cette valeur satisfaisant au principe de péréquation et au principe d'indépendance aux demandes supérieures dans le chapitre 4.

## 1.8 Aperçu de la thèse

### 1.8.1 Chapitre 2: Préliminaires

Le second chapitre de cette thèse contient des préliminaires sur les jeux TU, sur les jeux TU muni d'une structure et sur les problèmes de partage de coûts discrets. Tout d'abord, nous introduisons des définitions de base sur les jeux TU, ainsi que sur les concepts classiques de solution ainsi que sur leur caractérisation axiomatique. Ensuite, nous présentons deux modèles de jeux TU muni d'une structure : les jeux muni d'une structure de permission (voir Gilles *et al.* (1992)) et les jeux muni d'une structure de priorité (voir Béal *et al.* (2021a)). Ensuite, nous introduisons des définitions sur les jeux multi-choix et discutons des extensions de certain concepts de solution des jeux TU aux jeux multi-choix. Enfin, nous présentons les problèmes de partage de coûts discrets (voir Moulin (1995)). Nous présentons plusieurs concepts de solution pour les problèmes de partage des coûts discrets, tels que la serial cost sharing method (voir Moulin & Shenker (1992)) et la pseudo average method (voir Moulin

(1995)). Nous faisons aussi le parallèle entre les problèmes de partage de coûts et les jeux multi-choix.

### 1.8.2 Chapitre 3: Valeurs pour jeux multi-choix muni d'une structure de permission

Les structures de permission modélisent des situations dans lesquelles certains joueurs ont besoin de la permission d'autres joueur pour coopérer. Il s'avère que l'on peut établir des relations de permission entre les consommateurs de gaz. En effet, historiquement, le réseau de distribution a été construit de manière progressive en fonction des demandes croissantes des consommateurs. Ainsi, certains consommateurs n'auraient jamais été en mesure de demander du gaz si d'autres consommateurs n'avaient pas été connectés au préalable : on peut considérer qu'il s'agit d'une relation de permission.

Ce chapitre traite des jeux multi-choix muni d'une structure de permission (arborescente). Nous étudions comment une structure de permission sur l'ensemble des joueurs peut être combinée avec différents niveaux de participation. Une structure de permission est représentée par un graphe dirigé $D$. van den Brink & Gilles (1996) combinent la fonction caractéristique d'un jeu TU $(N, v)$ avec une structure de permission en une nouvelle fonction $v^D$ (voir la section 1.5). Ce chapitre propose une étude similaire réalisée dans le cadre des jeux multi-choix. Cependant, comme mentionné dans la section 1.5, il n'y a pas de manière triviale d'aggréger une structure de permission avec plusieurs niveaux d'activité. Pour ce faire, nous procédons en deux étapes.

Dans un premier temps, nous introduisons les structures de pal-permission.[3] Ces structures dépeignent des relations de permission entre les niveaux d'activité des joueurs. Par exemple, les graphes dirigés représentés sur la figure 1.6 et la figure 1.7 peuvent être considérés comme des structures de pal-permission. Cette approche généralise le modèle introduit par Gilles *et al.* (1992). Nous introduisons la valeur de pal-permission pour les jeux multi-choix muni d'une structure de pal-permission. Cette valeur est calculée comme la valeur $DP$ (voir Derks & Peters (1993)) du jeu multi-choix $(m, v^Q)$, où $v^Q$ est une combinaison d'un jeu multi-choix $(m, v)$ et d'une structure de pal-permission $Q$. Cette combinaison étend la combinaison $v^D$ proposée par van den Brink & Gilles (1996).

Ensuite, nous étudions les jeux multi-choix muni d'une structure de permission, cette fois ci, définie sur l'ensemble des joueurs. Nous exploitons le fait que l'ensemble des niveaux d'activité de chaque joueur soit totalement ordonné. De nouveaux axiomes sont proposés et nous permettent de caractériser trois nouvelles valeurs pour les jeux multi-choix muni d'une structure de permission. Chacune coïncide avec une valeur de pal-permission pour jeux multi-choix muni d'une structure de pal-permission telle que décrite dans la première étape. Cette approche est intéressante

[3] abréviation de player-activity level permission

pour plusieurs raisons. D'abord, elle distingue a priori une structure de permission entre les joueurs (les relations inter-joueurs) de l'ordre linéaire sur les niveaux d'activité de chaque joueur (les relations intra-joueurs). Puis, l'étude axiomatique met en lumière la manière dont les relations inter-joueurs et intra-joueurs se combinent pour former une structure de pal-permission. L'étude axiomatique fait apparaître des structures de permission très différentes, qui ont des interprétations naturelles mais néanmoins différentes.

Ce chapitre met en lumière les difficultés qui surviennent lorsqu'on essaie de combiner un jeu multi-choix avec une structure sur l'ensemble des joueurs. Notons que l'on ne tient pas compte de l'indépendance aux demandes supérieures ou du principe de péréquation.

### 1.8.3 Chapitre 4: Marginalisme, égalitarisme et efficacité dans les jeux multi-choix

Ce chapitre se concentre sur les jeux multi-choix sans structure. Il aborde le compromis entre le marginalisme et l'égalitarisme, qui est l'une des principales questions dans les problèmes d'allocation économique. Le marginalisme soutient les allocations basées sur la contribution marginale d'un joueur aux coalitions, tandis que l'égalitarisme est en faveur d'une allocation égale de l'utilité. Le principe de péréquation peut être vu comme une forme d'égalitarisme.

Dans le contexte des jeux TU, ce compromis peut être vu comme un arbitrage entre la valeur de Shapley et la division égalitaire puisque ces deux valeurs sont respectivement considérées comme l'incarnation du marginalisme et de l'égalitarisme. Cet arbitrage peut peut se faire par des combinaisons convexes entre la valeur de Shapley et la division égalitaire (voir Joosten (1996), van den Brink *et al.* (2013), Casajus & Huettner (2013), Abe & Nakada (2019) et Béal *et al.* (2021b)). Nous discutons de ce compromis dans le cadre des jeux multi-choix. L'indépendance aux demandes supérieures et la péréquation sont pris en compte dans ce chapitre.

Tout d'abord, nous proposons une extension de la valeur de Shapley, appelée valeur de Shapley multi-choix. Elle est calculée comme suit. Supposons que la grande coalition se forme étape par étape en partant de la coalition vide. À chaque étape, un joueur augmente son niveau d'activité d'une unité, disons de $j$ à $j+1$, mais ce, à condition que tous les autres joueurs (qui sont capables de jouer $j$) aient déjà atteint leur niveau $j$. Ce processus de formation de coalition est un **ordre restreint**. La valeur de Shapley multi-choix attribue à chaque joueur l'espérance de ses contributions marginales à chacun de ses niveau d'activité en supposant que chaque ordre restreint survienne à probabilité égale. Cette valeur se situe dans le Coeur de tout jeu multi-choix super-modulaire. Un jeu multi-choix super-modulaire peut être vu comme un jeu dans lequel les incitations à rejoindre une coalition augmentent au fur et à mesure que la coalition grandit. En outre, le valeur de Shapley multi-choix satisfait au principe d'indépendance aux demandes supérieures.

Ensuite, nous proposons une extension de la division égalitaire, appelée valeur

de division égalitaire multi-choix. Cette valeur divise de manière égale le surplus de gain entre deux coalitions synchronisées consécutives parmi les joueurs capables de jouer les niveaux d'activité requis. Cette valeur n'est pas une extension triviale de la valeur de division égalitaire: c'est une solution égalitaire qui satisfait au principe d'indépendance aux demandes supérieures et au principe de péréquation. Cette valeur est clairement souhaitable puisqu'elle satisfait à tous les principes poursuivis par GRDF.

A notre connaissance, aucun travail antérieur n'a abordé le compromis entre marginalisme et égalitarisme dans le contexte des jeux multi-choix. Dans ce chapitre, nous étudions ce compromis à l'aide de la valeur de Shapley multi-choix et de la valeur de division égalitaire multi-choix. A cette fin, nous proposons les valeurs de Shapley égalitaires multi-choix. Cette famille de valeurs est composée de combinaisons convexes de la valeur de Shapley multi-choix et de la division égalitaire multi-choix. Nous fournissons plusieurs caractérisations axiomatiques de ces nouveaux concepts de solution en utilisant des axiomes classiques ainsi que de nouveaux axiomes pour les jeux multi-choix.

### 1.8.4 Chapitre 5: Valeurs pour jeux multi-choix muni d'une structure de priorité

Une structure de priorité capture les asymétries entre joueurs, celles-ci pouvant refléter des droits, des besoins ou des mérites, etc. Des relations de priorité apparaissent naturellement entre les consommateurs de gaz. Par exemple, il semble raisonnable de donner la priorité à la fourniture d'un hôpital plutôt qu'à celle d'un parc d'attractions. En hiver, il semble raisonnable de donner la priorité à l'approvisionnement des ménages avant celui des équipements publics.

Ce chapitre traite des jeux multi-choix muni d'une structure de priorité. Nous introduisons une valeur pour cette classe de jeux : la valeur de priorité multi-choix. Elle étend la valeur de priorité pour les jeux TU avec une structure de priorité introduite par Béal *et al.* (2021a). Notre valeur peut être vue comme une procédure lexicographique définie comme suit. Pour chaque coalition (multi-choix), choisissons les joueurs ayant les niveaux d'activité les plus élevés dans la coalition. Parmi ces joueurs, sélectionnons ceux qui n'ont pas de supérieurs selon la structure de priorité. Le surplus généré par la coalition, c'est-à-dire son dividende d'Harsanyi, est divisé de manière égale entre les joueurs sélectionnés.

Puisque la valeur de priorité multi-choix discrimine d'abord les joueurs selon leur niveau d'activité, elle satisfait au principe d'indépendance aux demandes supérieures. Cette valeur est la première à combiner un principe poursuivi par GRDF avec une information fournie par le réseau de distribution de gaz. Deux caractérisations axiomatiques de cette valeur sont proposées. L'une est une caractérisation classique qui utilise un axiome d'additivité, la seconde est basée sur un axiome de contributions équilibrées (voir Myerson (1980)).

### 1.8.5 Chapitre 6: Partager les coûts d'un réseau de distribution de gaz

Un problème de partage de coûts dans le cadre de la distribution de gaz peut être représenté par un triplet $(q, C^{Net}, Net)$, où $q$ est le profil des demandes effectives, $Net$ est le réseau de distribution, et $C^{Net}$ est une fonction de coût intégrant des informations de $Net$. Rappelons que $C^{Net}$ peut être vue comme une combinaison d'une fonction de coût $C$ et de $Net$. Dans le Chapitre 1.2, il est préconisé d'étudier le problème $(q, C, Net)$, dans lequel $C$ est indépendant de $Net$, plutôt que $(q, C^{Net})$. Ici, nous nous focaliserons sur le triplet $(q, C^{Net}, Net)$ plutôt que sur $(q, C, Net)$, car la fonction de coût est censée mesurer le coût des canalisations du réseau. Par conséquent, la fonction de coût doit inévitablement prendre en compte $Net$. Cependant, nous nous différencions du cas où l'on ne considère que $(q, C^{Net})$ et gardons $Net$ comme une donnée supplémentaire afin de ne négliger aucune information du réseau.

Nous proposons trois règles distinctes de partage des coûts qui dépendent du réseau et des demandes des consommateurs. À cette fin, un nouveau principe est introduit. Pour approvisionner un consommateur en gaz, toutes les canalisations reliant ce consommateur à la source doivent être mobilisées. En d'autres termes, ce consommateur dépend d'un certain nombre de canalisations. Il semble raisonnable de facturer à ce consommateur les coûts générés par ces canalisations. D'une manière plus générale, les consommateurs ne devraient payer que pour la portion du réseau qu'ils utilisent. Cette idée se nomme : le principe de raccordement.

La première règle est appelée règle de raccordement, et elle satisfait au principe d'indépendance aux demandes supérieures ainsi qu'au principe de raccordement. La deuxième règle s'appelle la règle de péréquation, et elle satisfait au principe d'indépendance aux demandes supérieures ainsi qu'au principe de péréquation. De la même manière que le marginalisme est incompatible avec l'égalitarisme, le principe de raccordement est incompatible avec le principe de péréquation. Pour faire un compromis entre ces deux principes, nous introduisons les règles mixtes qui constituent un arbitrage entre la règle de raccordement et la règle de péréquation. Pour chaque règle, une caractérisation axiomatique est fournie.

La règle de raccordement coïncide avec la valeur de Shapley multi-choix (introduite au Chapitre 4) d'un jeu multi-choix spécifique dérivé du réseau et des demandes des consommateurs. De plus, la règle de raccordement est dans le Coeur de ce jeu multi-choix. De même, la règle de péréquation coïncide avec la division égalitaire multi-choix (introduite au chapitre 4) et les règles Mixtes coïncident avec les valeurs de Shapley égalitaires multi-choix (introduites au chapitre 4).

### 1.8.6 Chapitre 7: Remarques

Il existe deux principes désirables pour GRDF : le principe de péréquation et le principe d'indépendance aux demandes supérieures. L'information fournie par le

réseau de distribution est également importante. Tout cela est étudié de manière progressive au long de la thèse. Le chapitre 3 ne prend en compte que les informations fournies par le réseau de distribution. Le chapitre 4 ne prend en compte que les deux principes désirés par GRDF. Le chapitre 5 ne prend en compte qu'un seul de ces principes et le couple avec les informations fournies par le réseau de distribution. Enfin, le chapitre 6 prend en compte les deux principes ainsi que les informations fournies par le réseau de distribution.

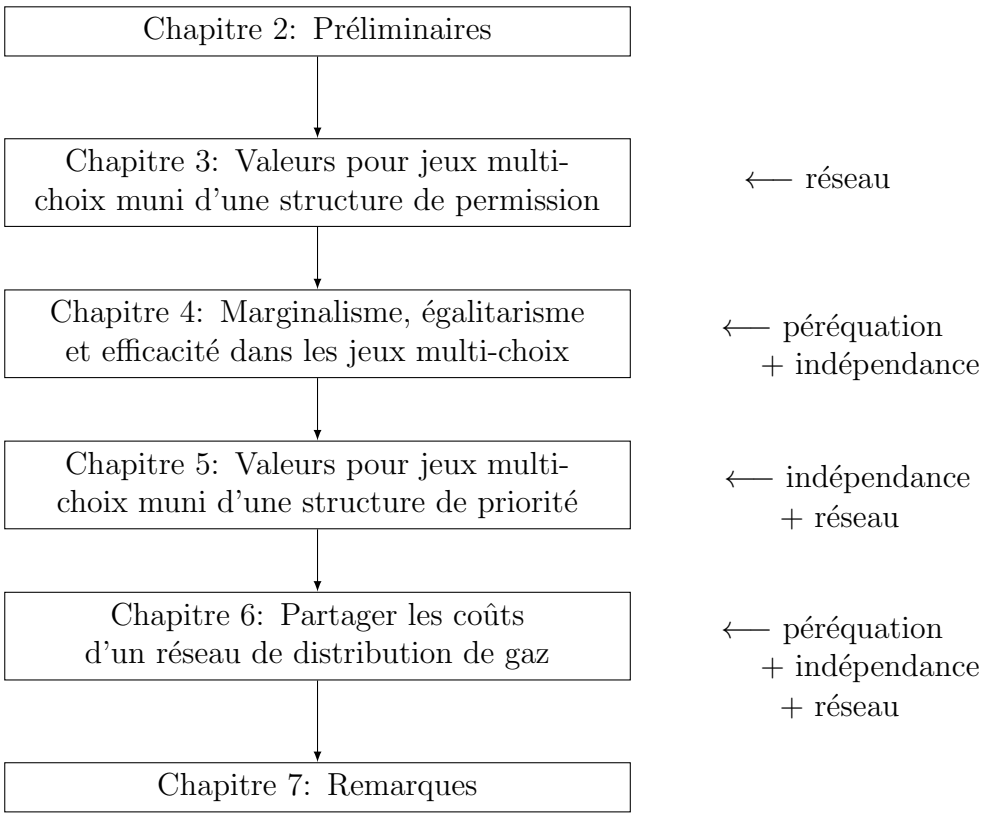



Figure 1.8

Des remarques supplémentaires concernant les travaux de cette thèse sont détaillées au Chapitre 7. Ces remarques mettent en évidence les liens entre les différents chapitres.

- Chapitres 3, 4 et 6 : nous proposons de nouveaux concepts de solution pour les jeux multi-choix muni d'une structure de permission ainsi que pour les jeux multi-choix muni d'une structure de priorité. Ces nouveaux concepts de solution sont définis en utilisant des éléments du chapitre 3 et du chapitre 4. De plus, sous certaines conditions, l'un de ces concepts de solution coïncide avec la règle de connexion du chapitre 6.

- Chapitres 5 et 6 : nous discutons de ce qui se passe si l'on applique la valeur de

priorité multi-choix à un jeu multi-choix dérivé d'un problème de distribution de gaz.

- Chapitres 3, 4 et 5 : nous introduisons les structures de pal-priorité pour les jeux multi-choix et introduisons une valeur sur cette classe de jeux. Nous montrons que cette valeur coïncide, sous certaines conditions, avec la valeur de priorité multi-choix, du chapitre 5, ou avec la valeur de Shapley multi-choix, du chapitre 4.

# Chapter 1: Introduction

## 1.1 Gas distribution in France

Natural gas is a fossil energy, also called primary energy, because it does not come from the transformation of another energy (unlike electricity). Natural gas is used for domestic purposes, but it is also useful in other areas such as industry, agronomy and transportation. Its consumption has been steadily growing in the past few years, mostly because of its environmental qualities: it emits almost no fine particles and is the hydrocarbon that releases the least amount of greenhouse gases into the atmosphere once consumed. Before being delivered to the consumers, natural gas must be extracted from the ground, processed and finally transported to the place of consumption.

In recent years, the production of biogas has become a serious and more ecological alternative to natural gas. Biogas is produced by the fermentation of organic matter in the absence of oxygen. Its production allows to recycle organic wastes while preventing the depletion of the planet's natural resources. Biogas production sites are located relatively close to the places of consumption and it therefore generally requires less effort to be delivered to the consumers.

This thesis is motivated by gas distribution,[1] which is the final step of the gas delivering process from the extraction sites (or biogas production sites) to the consumers. We briefly explain the various steps involved in gas delivering. Gas producers are in charge of extracting gas, which is then transported to border entry points through international pipelines or by LNG tankers. In the case of LNG tankers, the gas must be regasificated from its liquid state to its gas state before any use. The transmission network carry the gas at high pressure through large-diameter steel pipes, from border entry points or LNG regasification sites to the entrance of cities. Contrary to natural gas extraction sites, bio gas production sites are directly connected to the entrance of the cities and towns. Ultimately, the distribution network carry the gas from the entrance of the cities to the final consumer, through smaller diameter pipelines. All the steps of gas delivering are summarized in Figure 1.1.

[1] A particular attention is given to gas distribution in France.

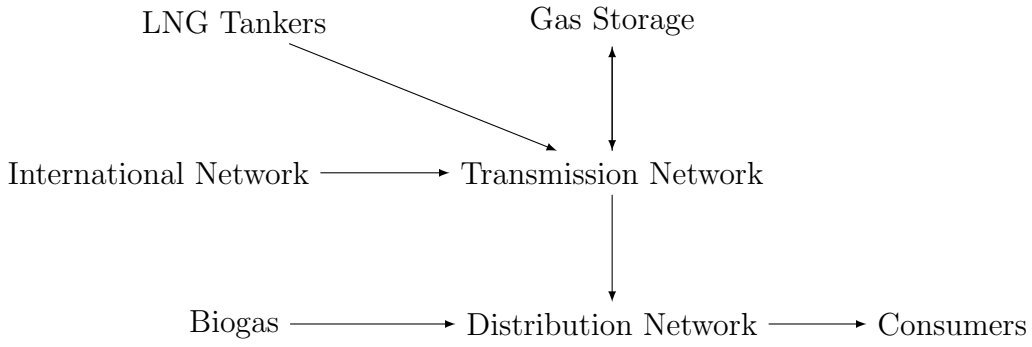



Figure 1.1

A gas distribution network connects consumers to a source in gas through pipelines. The source can either be a transmission network or a biogas production site. A gas distribution network can be illustrated by a rooted tree graph in which the nodes represent the consumers and the arcs represent the pipelines. Note that a gas distribution network can be connected to multiple sources. In this case, a gas distribution network cannot be represented by a rooted tree graph with a unique source. In this thesis, we consider the case where a gas distribution network is connected to only one source (see Figure 1.2 for an example of three gas distribution networks represented by rooted tree graphs).

In France, the energy distribution market was opened to competition on July 1, 2007. It allowed the creation of a single market on an European scale and the possibility for a consumer to freely choose its gas distribution network operator among the fifty or so that currently exist in France. Gas distribution in France is a public service mission. It is carried out under a triplet authority: the State, the local authorities and the Energy Regulation Commission (CRE-French acronym). The State sets the general framework for the energy market by guaranteeing compliance with both the rules of competition and public service missions. Local authorities supervise gas distribution on a smaller scale and ensure its safe running. The CRE ensures that the French energy market runs smoothly and that all gas network operators have the same access conditions to the distribution network. The most prominent gas distribution operator in France is (by far) GRDF (Gaz Réseau Distribution France). This operator manages an over $200,000$ km long gas distribution network with over $10,000$ employees to supply more than 11 millions of customers.

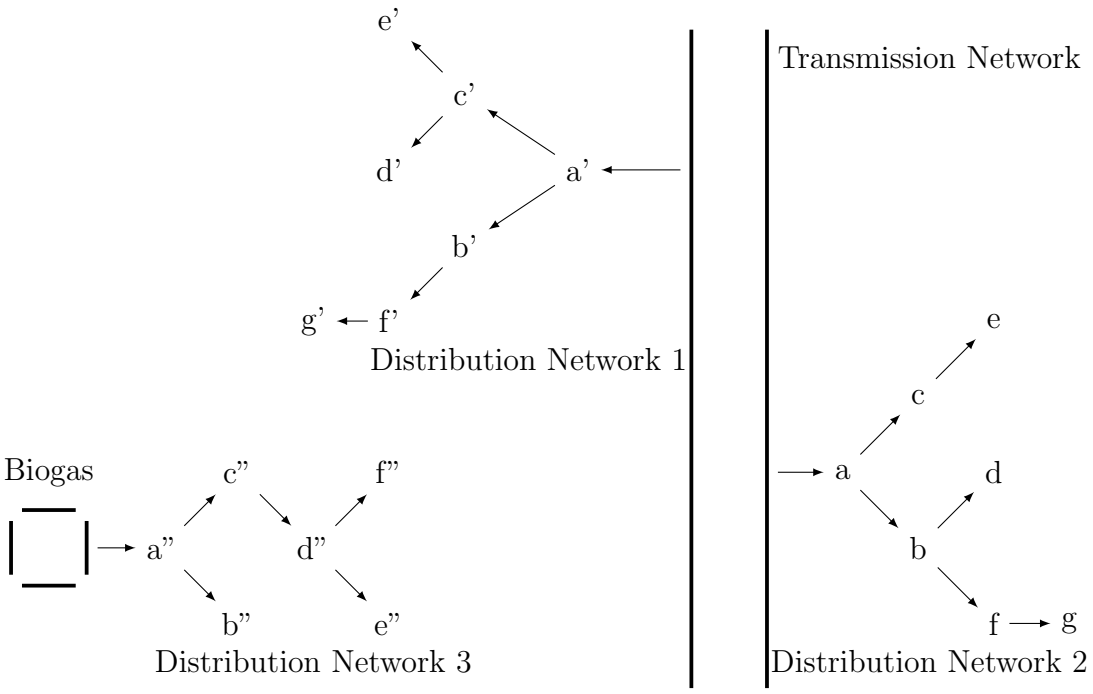



Figure 1.2

## 1.2 The gas distribution problem

In order to carry out its task properly, GRDF is confronted with various costs, some of which are not directly assignable to a given consumer. For instance, there are some costs related to the access to the transmission network and some costs related to the access to biogas production sites. In addition, the maintenance of the network, its monitoring and possible extensions generate additional costs. These costs are called operation costs and are ultimately billed to the consumers. Each consumer signs a distribution contract with GRDF (or another distribution network operator), which specifies two major pieces of information. First, it specifies the category to which the consumer belongs. GRDF classifies the consumers according to a well-established categorization system. The category of a consumer is highly related to its gas usage over the course of a year. In particular, the effective demand of a consumer is the most meaningful information for GRDF, and it determines the category of a consumer. The effective demand of a consumer corresponds to the highest volume in gas a consumer expects to demand over a short period of time. Typically, this corresponds to the amount a household expects to consume during the coldest days of winter. For a transportation company, this corresponds to the period of the year when the company's fleet is the most solicited. For an industry, this corresponds to the period of the year when the production activity is at its greatest level. Without loss of generality, we assume that the set of effective demands coincides with the

categorization system established by GRDF. The distribution contract between a consumer and GRDF specifies the effective demand of the consumer. Second, the contract specifies the amount each consumer has to pay depending on its effective demand. This amount is determined by a distribution rate, which is called "Third-party access to the distribution network" (ATRD - French acronym). The ATRD establishes a grid that specifies the amount the consumers have to pay depending on their demand. GRDF seeks to recover all the operation costs via the ATRD. To that end, GRDF must determine a method to share the operation costs among the consumers. The competitiveness of GRDF significantly depends on how these costs are divided. In particular, a cost sharing method is relevant if it meets the principles retained and the objective pursued by the network operator.

There are two major principles retained by GRDF regarding the cost sharing methods: the Uniformity principle and the Independence of higher demands principle.[2] First, the Uniformity principle simply states that two consumers with the same demands should be charged the same amount regardless of their geographical localization. For instance, there should be no difference in terms of rate applied in rural areas compared to urban areas, although the underlying costs are different. This principle aims to reduce inequality of treatment between different consumers. This principle is highly prevalent in France, not only in the field of gas delivering, but also in electricity delivering, water delivering and other public services (see Fleurbaey & Trannoy (1998)). This principle is even part of French law: "Constitution-article 72-2, révision du 28 mars 2003 [...] the law ensures uniformity mechanisms to promote equality among territories" (author's translation). Second, the Independence of higher demands principle indicates that a consumer should not have to pay for demands higher than its own. This principle allows to avoid situations where the presence of a consumer with a high demand implies additional costs to consumers with lower demands. For example, it seems unjustified to increase the bill of a household whenever a factory with a large gas consumption moves in next door.

Most, if not all, cost sharing methods used by GRDF are solution concepts taken from the standard model of cost sharing problems. In the following, we briefly present the (discrete) Aumann-Shapley method and the (discrete) Shapley-Shubik method, which are two cost sharing methods commonly used by GRDF. To that end, we represent the problem of GRDF as a discrete cost sharing problem (see e.g. Moulin (2003) and Hougaard (2018)).

Fix a discrete finite set $N$ of $n$ gas consumers. Consider that each consumer $i \in N$ has an effective demand $q_i$ declared to the operator. The network operator has no information regarding the other demands that a consumer makes throughout a year, aside from its effective demand. Therefore, each consumer $i \in N$ is endowed with a discrete set of potential demands $\{0, 1, \ldots, q_i\}$ that represents all the potential demands $i$ can make throughout a year. The profile of all effective demands is given by $q = (q_1, \ldots, q_n)$. GRDF's cost sharing problem can be viewed as a couple $(q, C)$,

[2]The Uniformity principle translates into "péréquation" in French.

where the map $C$ measures the cost of operating the distribution network designed to meet any demand profile in which each consumer $i \in N$ demands at most its effective demand $q_i$. Formally, the map $C$ measures the cost $C(s)$ of operating a distribution network designed to meet any profile of demand $s \leq q$. The map $C$ verifies $C(0, \ldots, 0) = 0$ and is a non-decreasing map, meaning that $C(t) \leq C(s)$ for each $t \leq s \leq q$. The cost of operating the gas distribution network is given by $C(q)$. For instance, consider that the set of consumers is given by $N = \{a, b, c, d\}$. The profile of effective demands is given by $q = (2, 1, 1, 1)$, meaning that $a$ has an effective demand of 2 whereas $b, c$ and $d$ have an effective demand of 1. A map $C$ measures the cost of operating a distribution network designed to meet any demand profile $s$ smaller than $q$. For instance, $C$ measures the cost of $(1, 1, 0, 0)$, the profile in which $a$ demands 1, $b$ demands 1 and $c$ and $d$ demand 0. Of course, this scenario is hypothetical and does not correspond to the effective demands. But it is necessary to consider such scenario to use the Aumann-Shapley and the Shapley-Shubik methods. Since $C$ is non-decreasing, the cost $C(1, 1, 0, 0)$ is smaller than the cost $C(1, 1, 1, 0)$, in fact, any demand profile smaller than $q$ has a smaller cost than $C(q)$. Ultimately, GRDF seeks to recover all the operation costs via the ATRD. This means that the rate defined by GRDF should recover $C(q)$, which is equivalent to sharing $C(q)$ among the consumers.

Next, we introduce some prerequisites from cooperative games necessary to understand the Aumann-Shapley and the Shapley-Shubik methods. The standard model of cooperative games is transferable utility games (TU-games for short). TU-games model situations in which some players (possibly consumers) can form coalitions to generate a worth. A TU-game consists of a couple $(N, v)$, where $N$ is a discrete and finite set of players and $v$ is a characteristic function that measures the worth of each coalition of players $E \subseteq N$. A payoff vector for a TU-game assigns a payoff to each player. A single-valued solution on a class of TU-games assigns a unique payoff vector to each game in this class. The Shapley value (see Shapley (1953)) is probably the most prominent single-valued solution for TU-games. It is computed as follows. Assume that the grand coalition $N$, in which all players cooperate, is being formed step by step according to a certain linear order over the player set. At each step, a player joins the coalition and obtains its marginal contribution (corresponding to the variation in worth generated when this player joins the coalition). The Shapley value assigns to each player its expected marginal contribution assuming that each linear order over the player set occurs with equal probability. We now have the material to briefly present the Aumann-Shapley method and the Shapley-Shubik method.

*The Shapley-Shubik method*: given a coalition $E \subseteq N$ of consumers, the method considers the cost associated with the effective demands of those consumers, which is given by $v(E) = C(q_E, 0_{N \setminus E})$, where $(q_E, 0_{N \setminus E})$ is the profile in which consumers in $E$ demand their effective demand and consumers outside of $E$ have a null demand. The Shapley-Shubik method computes the Shapley value of the transferable utility game $(N, v)$. Put differently, the cost share of a consumer is determined by its expected marginal contribution to this game. Observe that the Shapley-Shubik method has no

consideration for the discrete set of demands $\{0, 1, \ldots, q_i\}$ that each consumer $i \in N$ is endowed with. This method assumes that each consumer attains its effective consumption without passing through intermediary consumption levels.

*The Aumann-Shapley method*: given a consumer $i \in N$, each unit demanded by $i$ is viewed as a different consumer. Thus, there is a set of $N^q = \cup_{i \in N} N_i$ consumers, where $|N_i| = q_i$. Each $N_i$ can be viewed as a multi-set that contains $q_i$ of the same unit. When a consumer selects a certain number $d_i$ of units of gas in $N_i$, it is equivalent to saying that $i$ demands $d_i$ units of gas. Given $E \subseteq N^q$, denote by $d(E) = (|E \cap N_i|)_{i \in N}$ the units of gas demanded by consumers in $E$ and denote the cost associated with those demands by $v^q(E) = C(d(E))$. The Shapley value of $(N^q, v)$ allocates to each element in $N^q$ a certain payoff. For each $i \in N$, the sum of all the payoffs of the elements of $N_i$ constitutes the allocation for consumer $i$ provided by the Aumann-Shapley method. Observe that, contrary to the Shapley-Shubik method, the Aumann-Shapley method considers the discrete set of demands $\{0, 1, \ldots, q_i\}$ that each consumer $i \in N$ is endowed with through the set $N^q$. This method assumes that each consumer attains its effective consumption gradually.

Since we consider the framework of gas distribution, two criticisms can be voiced regarding these two methods. First, the principles pursued by GRDF are violated by these methods. By definition of the solutions, the Independence of higher demands is violated. If a consumer increases its demand, then other consumers with lower demands can be potentially impacted. Additionally, the Uniformity principle is also violated by the two solutions. Both methods are based on the marginal contributions of the consumers to some specific TU-games. Therefore, it is unlikely that consumers with the same effective demand will be charged the same amount.

Second, none of these methods consider the distribution network as an independent information within the cost sharing problem. This is because the information provided by the network is often contained in the map that measures the operation costs of GRDF. To be precise, the cost sharing problem of GRDF should be written $(q, C^{Net})$, where the map $C^{Net}$ measures the operation costs by relying on the information provided by the distribution network. This map can be viewed as the combination of two information: the distribution network $Net$ and the economic characteristics of GRDF $C$ (e.g. access costs to the transmission network and to biogas producers, executive activity, contract management, etc.). Applying the Aumann-Shapley method or the Shapley-Shubik method to the problem $(q, C^{Net})$ has its technical advantages but also its drawbacks since $C^{Net}$ compresses a large amount of information. The main drawback lies in its consequences on the axiomatic method, which is central in this thesis. Such a method is used to determine a solution to a class of problems on the basis of its properties (axioms). An axiom takes advantage of the information of the problem to formulate a property. Therefore, if the problem compresses the information provided by the distribution network and the economic characteristics of GRDF, this compression has a direct impact on the relevance of the axioms. We propose to dissociate $Net$ and $C$ as two separate pieces of information, and to consider a new problem denoted by the triplet $(q, C, Net)$. This way,

the distribution network is an information independent from the other information of the problem. From a theoretical point of view, this approach allows for a finer axiomatic study of GRDF's cost sharing problem. This point is detailed in section 1.5. The next section presents the axiomatic method.

## 1.3 The axiomatic method

This section is highly inspired from Thomson (2001). An axiomatic study of a situation in which several agents interact begins with the specification of a class of problems. A problem is given by a set of alternative available to the agents and information about the agents (whether they are players, consumers, voters, etc). Additional information can be provided with the problem (preferences of the agents, a structure over the set of agents, etc). To illustrate the wide range of possible problems, a social choice problem consists of an unstructured set of feasible alternative, together with the preferences of the agents over this set. The problem is then to elect one or several alternative based on the agents' preferences. For bargaining problems and cost allocation problems, the set of alternative consists only of a set of attainable utility vectors. The problem is then to select one subset of these vectors. In bankruptcy problems, there is a given amount of a perfectly divisible good, and each agent claims a certain amount of this good. The problem is then to share the total amount of this good among the agents under the assumption that this amount is not sufficient to satisfy all their claims.

Given a class of problems $D$, a solution on $D$ is a map that associates a non-empty set of alternative to each problem in $D$. A solution is denoted by $f$ and the set of alternative $X$. Therefore, a solution can be written as $f : D \to X$. Solutions can be set-valued in some models and must be single-valued in others. Whether the goal is descriptive or prescriptive, single-valued solutions are preferred, since a solution that makes specific predictions or recommendations is more likely to be useful. However, single-valued solutions can be very hard to characterize, and for many models, set-valued solutions are retained. In bargaining theory and social choice problems, most solution concepts are single-valued solutions. In cooperative games and cost allocation problems, both types of solutions are popular.

An axiomatic study begins with the specification of a class of problems, which is followed by the formulation of a list of desirable properties (axioms) of solutions for problems in this class. The study ends with the description of the set of solutions satisfying various combinations of the axioms. The study should also offer an analysis of the logical independence between the axioms, since it is an effective way to assess their relative power. Moreover, formulating and exploring variants of the axioms is equally useful as it is not rare that the general ideas that inspire them could have been given in slightly different and almost as appealing mathematical formulations. An axiomatic study often results in characterization theorems. These theorems identify a single-valued or a set-valued solution satisfying a given list of

axioms. An axiomatic study may also produce impossibility theorems, stating the incompatibility of a certain list of axioms on a certain class of problems. For practical reasons, the study may begin from the solutions. Although axioms come first conceptually, it is useful to have at our disposal several solution concepts. Indeed, this allows us to assess the strength of the axioms, testing conjectures regarding the compatibility of the axioms and the logical independence between them. We develop the latter point. Given a certain characterization theorem, we say that the axioms are logically independent if by deleting any one of them, the characterization does not holds anymore. The logical independence of the axioms is important since it ensures that the characterization theorem is as general as possible. In practice, to establish the logical independence of one axioms from the others in a characterization result, it is sufficient to exhibit one solution, different from the characterized one, which satisfies all the axioms but the one we are interested in.

## 1.4 Purpose of the thesis

The ultimate objective of this thesis is to provide cost sharing methods to divide GRDF's operation costs among the consumers. In opposition to the Aumann-Shapley and the Shapley-Shubik methods, these methods should consider the distribution network as a separate information from the game and be consistent with the principles pursued by GRDF. By dividing operation costs, GRDF wishes to estalish a rate grid. The basic idea of a rate is to provide a grid that specifies an amount to pay for each potential demand of the consumers. A cost sharing method for GRDF must therefore be adapted to establish a rate grid.

The theory of cooperative games is used to determine relevant methods. TU-games have been extensively studied and applied to cost sharing problems and related situations. However, most solution concepts from TU-games allocate a payoff per player (e.g. the Shapley value, the Equal Division value, see Béal *et al.* (2016), or the Egalitarian Shapley values, see Joosten (1996)) without any details on how this payoff varies according to its demand.[3] From the point of view of GRDF, solution concepts from TU-games are clearly not enough to establish a rate grid. To overcome this drawback, the thesis focuses on an extension of TU-games, known as multi-choice games.

Multi-choice games, introduced by Hsiao & Raghavan (1992) and van den Nouweland (1993), are a natural extension of TU-games in which a player can choose several activity levels at which it can cooperate within a coalition.

Multi-choice games have been successfully applied to economic theory. For instance, Branzei *et al.* (2009) study multi-choice games that arise from market situations with two factions. One faction consists of a group of powerful players with yes-or-no choices and clan behavior. The other faction consists of non-powerful play-

[3]Single-valued solution concepts from discrete cost sharing problems also allocate a unique payoff per player.

ers with several choices regarding the extent at which cooperation with the clan can be achieved; Grabisch & Rusinowska (2010) generalize a yes-no influence model to a multi-choice framework. The authors consider a situation in which some agents are part of a social network. Each agent has an ordered set of possible actions and is influenced by its neighbors in the network when choosing its action; and Techer (2021) addresses the social cost problem, originally introduced by Coase (1960), using multi-choice games. The author studies situations in which one polluter interacts with several potential victims, and aims at negotiating a stable agreement regarding the level of pollution. The polluter has several levels at which it wishes to pollute, whereas the victims can either participate or not in the negotiations.

Let $N$ be the set of players. In a multi-choice game, each player $i \in N$ is allowed to choose several activity levels at which it can cooperate, from 0, meaning that it does not cooperate, up to $m_i$, representing its maximal activity level. The profile $m = (m_1, \ldots, m_n)$ collects all the maximal activity levels of the players. A profile $s \leq m$ is called a (multi-choice) coalition. A multi-choice game is a couple $(m, v)$ where $v$ is the characteristic function that associates to each coalition $s \leq m$ a real worth and verifies $v(0, \ldots, 0) = 0$, where $(0, \ldots, 0)$ is the empty coalition. The subclass of multi-choice games where $m = (1, \ldots, 1)$ is the full class of TU-games with the fixed player set $N$. Moreover, the class of (discrete) cost sharing problems can be viewed as the class of non-decreasing multi-choice games. Therefore, multi-choice games are a more general framework than (discrete) cost sharing problems. For the rest of the thesis, we assume that activity levels and demands are two equivalent concepts. A (multi-choice) payoff vector describes how much each player's payoff varies according to its activity levels, which is suitable to establish a rate grid. A single-valued solution (a value, for short) on a class of multi-choice games assigns a unique payoff vector to each game in this class. A set-valued solution on a class of multi-choice games assigns a (possibly empty) set of payoff vectors to each game in this class.

In this thesis, we introduce new solution concepts for multi-choice games. In particular, we consider situations in which the game is endowed with a structure. A structure takes advantage of the information provided by the distribution network and uses it to establish relationships between the consumers. This way, the solution concepts take into account the information provided by the distribution network. Moreover, some of the solution concepts follow the principles pursued by GRDF: the Uniformity principle and the Independence of higher demands principle. We point out that not all solution concepts proposed in this thesis manage to satisfy the principles pursued by GRDF and take the network into account at the same time.

The rest of this introduction is organized as follows. The interaction between cooperative games and structures is discussed in section 1.5. Solution concepts from multi-choice games and the Independence of higher demands principle are discussed in section 1.6. Additionally, the Uniformity principle is discussed in Section 1.7. Finally, Section 1.8 presents an overview of this thesis and details the contributions of the different chapters.

## 1.5 Games and structures

In cooperative games, it is generally assumed that cooperation among a group of players is not constrained. However, in practice, several structures may have an impact on the formation of coalitions, the evaluation of coalitions or the allocation process. TU-games endowed with a structure on the player set have been introduced to model situations where the formation or evaluation of coalitions depends on communicational, coalitional or hierarchical constraints.

Communication constraints, modeled by an undirected graph (e.g. Figure 1.3), are discussed in Myerson (1977). Players are located on the nodes of an undirected graph where the edges represent the communication channels between pairs of players. A coalition of players can cooperate only if they can communicate through a path that lies within this coalition. The model of TU-games with communication constraints can be represented by a triplet $(N, v, L)$, where $(N, v)$ is a TU-game and $(N, L)$ an undirected graph representing the communication constraints. Observe that this model distinguishes the characteristic function from the structure. As mentioned in Section 1.2, this approach allows to conduct an axiomatic study on a problem $(N, v, L)$ where $(N, v)$ and $(N, L)$ are two different pieces of information. Myerson (1977) axiomatically characterizes a solution concept computed as the Shapley value of the game $(N, v^L)$, where $v^L$ is a characteristic function that combines $v$ and $L$. The worth $v^L(E)$ of a coalition $E \subseteq N$ in the game $(N, v^L)$ is computed as the sum of the worth of each maximal connected coalition (with respect to the communication graph) contained in $E$. Thus, communication constraints affect the evaluation of coalitions.

The way $v$ and $L$ are combined into $v^L$ is entirely determined by the axioms, which means that another set of axioms could lead to a different result. This emphasises one of the theoretical advantages of distinguishing the characteristic function from the structure: there are different possible sets of axioms resulting in different combinations of $v$ and $L$.

Coalitional constraints, where players meet together in coalitions and form a priori unions, are modeled by a partition of the player set (e.g. Figure 1.4). These structures are discussed in Aumann & Dreze (1974) and Owen (1977). Different interpretations of this model are possible. In Aumann & Dreze (1974) players belonging to different a priori unions can not cooperate, whereas in Owen (1977) the formation of a coalition must only be consistent with the a priori unions. Thus, coalitional constraints affect the formation of coalitions. The model of TU-games with coalitional constraints can be represented by a triplet $(N, v, W)$, where $(N, v)$ is a TU-game and $(N, W)$ is a partition of the player set representing the coalitional constraints. Similarly to communication constraints, this model distinguishes the game from the structure. Aumann & Dreze (1974) and Owen (1977) propose two values for TU-games with coalitional constraints and axiomatically characterize them. Aumann & Dreze (1974) propose a solution concept computed, for each $i \in N$, as the Shapley value of the game $(W(i), v^{W(i)})$, where $W(i)$ is the a priori union containing

player $i \in N$ and $v^{W(i)}$ is the restriction of $v$ to the domain $W(i)$. Owen (1977) proposes the Owen value that can be viewed as a two-step procedure. In the first, a TU-game among a priori unions is used to determine the total payoff obtained by each a priori union. In the second, a TU-game within each a priori union is used to allocate the total payoff obtained by the a priori union in the first step to its members. The Owen value is obtained by applying the Shapley value at each step.

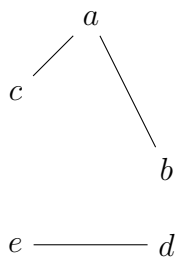



Figure 1.3: $L$

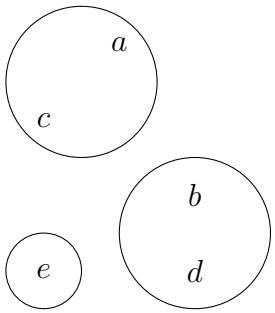



Figure 1.4: $W$

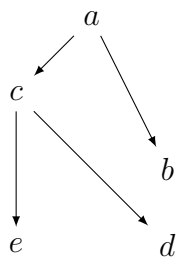



Figure 1.5: $D$

Permission structures, modeled by directed graphs (e.g. Figure 1.5), are introduced by Gilles *et al.* (1992). Such structures describe situations in which players need the permission from other players before they are allowed to cooperate within a coalition. The model of TU-games with a permission structure can be represented by a triplet $(N, v, D)$, where $(N, v)$ is a TU-game and $(N, D)$ is a directed graph representing the permission structure. van den Brink & Gilles (1996) introduce the Permission value for TU-games with a permission structure. This value is computed as the Shapley value of the TU-game $(N, v^D)$, where $v^D(E)$ measures the worth of the largest feasible coalition contained in $E \subseteq N$. In this context, a coalition is feasible if it contains all the superiors of the coalition members. As in situations with communication constraints, the structure affects the evaluation of the coalitions. Several axiomatic characterizations of the Permission value can be found in van den Brink & Gilles (1996), van den Brink *et al.* (2015) and van den Brink *et al.* (2018).

Finally, priority structures, modeled by acyclic directed graphs (e.g. Figure 1.5), are introduced by Béal *et al.* (2021a). They describe situations in which some players have priority over other players in the allocation process. The model of TU-games with a priority structure can be represented by a triplet $(N, v, D)$, where $(N, v)$ is a TU-game and $(N, D)$ is a directed graph representing the priority structure. Béal *et al.* (2021a)introduce the Priority value for TU-games with a priority structure. The structure affects the allocation process: the surplus generated by each coalition is allocated among the players in the coalition that have no superior present within the coalition. Two axiomatic characterizations of the Priority value are proposed by Béal *et al.* (2021a). Obviously, the model of TU-games with a priority structure is very similar to TU-games with a permission structure. However, the two models have a different interpretation and use of the structure.

In this thesis, we study multi-choice games endowed with a structure. A few studies have been conducted on this topic. For instance, Béal *et al.* (2012) consider multi-choice games with communicational constraints and axiomatically characterize a solution that extends the Average tree solution (see Herings *et al.* (2008)) from TU-games with communication constraints. Albizuri (2009) considers multi-choice games with a coalition structure in which players come together in coalitions and form a coalition structure. The author axiomatically characterizes a solution that extends the Owen value (see Owen (1977)) from TU-games with a coalition structure to multi-choice games with a coalition structure.

Observe that, in TU-games, players have only one activity level at which they can participate in a coalition. Therefore, the structure is easy to interpret since it establishes relationships between the participating players. In multi-choice games, these relationships are harder to determine due to the multiplicity of the activity levels. For instance, let us consider a permission structure. A direct permission relationship between two players is clear if the two players only have two choices: 0 or 1, i.e., if they cooperate or not. Indeed, assume that $i \in N$ needs permission from $k \in N$ to cooperate. Then, if $k$ plays 1, then $i$ is authorized to play 1. On the contrary, if $k$ plays 0, then $i$ is not authorized to play 1. Now, assume that $k$ has 3 activity levels, meaning that $m_k = 3$, and assume that $m_i = 2$. What gives player $i$ the permission to play its activity level 1? There is no trivial way to determine which activity level of $k$ is necessary to authorize $i$ to play 1. For instance, one possible scenario is: $i$ cannot cooperate unless $k$ cooperates. In this case, $i$ needs $k$ to play its activity 1 (or above) before cooperating itself (see Figure 1.6). Another possible scenario is: $i$ cannot cooperate unless $k$ cooperates at its maximal activity level. In this case, $i$ needs $k$ to play its activity 3 before cooperating itself (see Figure 1.7). This makes the study of multi-choice games endowed with a structure more complex than TU-games endowed with a structure.

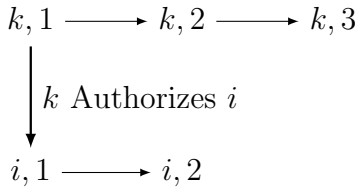



Figure 1.6

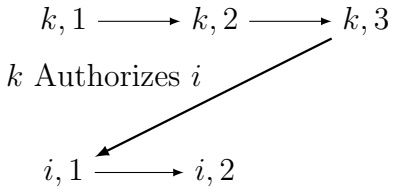



Figure 1.7

Observe that the activity levels of a player are already linearly ordered: a player $k$ cannot play 2 unless it plays 1 beforehand. In a sense, the activity levels of the players are already part of a structure in which they are linearly ordered. This idea is illustrated by drawing arrows between the activity levels of each player in Figure 1.6 and 1.7. A first objective of this thesis is to provide relevant ways to combine the structure over the set of activity levels and the structure over the set of players into one structure over the set of player-activity level pairs (just as we did

in Figure 1.6 and 1.7). This issue is addressed by considering permission structures and priority structures. We study how these structures can be combined with the structure over the set of activity levels, and provide solution concepts based on these aggregations. This discussion is particularly worthwhile for GRDF since it provides different interpretations on how the geographical characteristics of the network affect its economic characteristics.

## 1.6 Independence of higher demands

Properties similar to the Independence of higher demands principle exist in the economic literature. The serial cost sharing method for discrete cost sharing problems, introduced by Moulin & Shenker (1992), satisfies the property of Independence of higher demands. Recently, Albizuri *et al.* (2020) study solutions for bargaining problems that satisfy the independence to higher claims property. In this section, we discuss the Independence of higher demands principle in the framework of multi-choice games.

A solution concept for multi-choice games follows the Independence of higher demands principle if the amount allocated to the activity level of a player is independent from higher activity levels. Unfortunately, most solution concepts from multi-choice games do not satisfy this principle. Values for multi-choice games are usually extensions of the Shapley value. The first extension is introduced by Hsiao & Raghavan (1992). The authors consider multi-choice games in which players all share the same maximal activity level. Weights on the activity levels are used, which allow the authors to extend the idea of the weighted Shapley values (see Kalai & Samet (1987)) from TU-games to multi-choice games. This value does not satisfy the Independence of higher demands principle. van den Nouweland *et al.* (1995) consider the full class of multi-choice games and provide a second extension of the Shapley value, denoted by $vdN$. Assume that the grand coalition $m$ forms step by step, according to a certain order, starting from the empty coalition $(0, \dots, 0)$, meaning that, at each step, a player increases its activity level by one unit. Thus, at each step, a player's marginal contribution to the coalition corresponds to the variation in worth the coalition undergoes when this player increases its activity level by one unit. The $vdN$ value allocates to each player the sum of its expected marginal contributions, assuming that the grand coalition can form according to any admissible order, where each admissible order occurs with the same probability. Calvo & Santos (2000) show that the $vdN$ value coincides with the discrete Aumann-Shapley method on the class of discrete cost sharing problems. This shows that the $vdN$ value does not satisfy the Independence of higher demands principle. Other extensions can be found in Derks & Peters (1993), Peters & Zank (2005) and Grabisch & Lange (2007). None of these solutions satisfy the Independence of higher demands (see Chapter 2). The Independence of higher demands principle is also related to a concept of stability: the Core of a multi-choice game introduced by Grabisch & Xie (2007) (the Core, for

short). The Core is a set-valued solution for multi-choice games that contains all stable payoff vectors. A payoff vector is stable if no coalition can achieve, by itself, a better worth than the one prescribed by the payoff vector. For a payoff vector to be in the Core, it necessarily needs to satisfy the multi-efficiency property. This property states that, for any activity level, If all the players agree on playing this level (or their maximal activity level if they are unable to reach this level), then they achieve the same worth as the one prescribed by the payoff vector in the Core. This kind of agreement is called a synchronized coalition. If a value satisfies the Independence of higher demands principle and also recovers the worth of the grand coalition, then it satisfies multi-efficiency. This makes the Independence of higher demands principle particularly desirable.

## 1.7 Uniformity

The Uniformity principle advocates for an equality of treatment between two consumers with the same effective demand. In terms of multi-choice games, this means that two players with the same maximal activity level should receive the same payoff. Similarly to the Independence of higher demands principle, no solution concept from multi-choice games satisfies the Uniformity principle since most of them are computed according to the contributions of the players to coalitions.

On the class of TU-games, the Equal division value, which equally divides the worth of the grand coalition among the players, clearly satisfies the Uniformity principle. Surprisingly, there are very few studies in the literature that extend the Equal division value from TU-games to multi-choice games. To our knowledge, the only single-valued solution extending the Equal division value from TU-games to multi-choice games is the multi-choice constrained egalitarian solution introduced by Branzei *et al.* (2014). We introduce an extension that satisfies the Uniformity principle and the Independence of higher demands principle in Chapter 4.

## 1.8 Outline of the thesis

### 1.8.1 Chapter 2: Preliminaries

The first chapter of this thesis contains preliminaries on cooperative games (TU and multi-choice games), TU-games endowed with a structure and discrete cost sharing problems. First, basic definitions from TU-games, along with classical solution concepts and their axiomatic characterization are introduced. Then, two models of TU-games with a structure defined on the player set are discussed: games with a permission structure (see Gilles *et al.* (1992)) and games with a priority structure (see Béal *et al.* (2021a)). The two models fit GRDF's cost sharing problem since they both consider a directed graph as an exogenous information. Next, specific definitions from multi-choice games are introduced. Extensions of solution concepts

from TU-games to multi-choice games and their axiomatic characterizations are discussed. Finally, discrete cost sharing problems (see Moulin (1995)), are introduced. Several solution concepts for discrete cost sharing problems such as the discrete serial sharing methods (see Moulin & Shenker (1992)) and the discrete pseudo average method (see Moulin (1995)) are presented. Special attention is given to how solution concepts from discrete cost sharing problems relate to solution concepts from multi-choice games.

### 1.8.2 Chapter 3: Values for multi-choice games with a permission structure

Permission structures model situations in which some players may need permission from other players to cooperate. It turns out that permission relationships between gas consumers arise in the framework of gas distribution. The gas distribution network has a history: it has been constructed in a progressive way according to the increasing demands of the consumers. As such, some consumers would never have been able to demand gas if other consumers were not connected beforehand: this can be seen as a permission relationship.

This chapter focuses on multi-choice games with a permission (tree) structure. We investigate how a permission structure on the player set can be combined with different levels of participation. A permission structure is represented by a directed graph $D$. van den Brink & Gilles (1996) combine the characteristic function of a TU-game $(N, v)$ with a permission structure into a new function $v^D$ (see Section 1.5). This chapter proposes a similar study performed within the framework of multi-choice games. However, as mentioned in Section 1.5, there is no straightforward way to interpret a permission structure if the players have several activity levels. To conduct this study, we proceed in two-steps.

In a first step, pal-permission[4] structures are introduced. These structures describe permission relationships between the activity levels of the players. For instance, the directed graphs depicted in Figure 1.6 and Figure 1.7 can be viewed as pal-permission structures. Pal-permission structures allow us to consider multi-choice games with a pal-permission structure. This approach generalizes in a straightforward way the model introduced by Gilles *et al.* (1992). The pal-Permission value for multi-choice games with a pal-permission structure is introduced. This value is computed as the $DP$ value (see Derks & Peters (1993)) of the multi-choice game $(m, v^Q)$, where $v^Q$ is a combination of a multi-choice game $(m, v)$ and a pal-permission structure $Q$. This combination extends the combination $v^D$ proposed by van den Brink & Gilles (1996) for TU-games with a permission structure.

In a second step, multi-choice games with a permission structure on the player set are considered. We exploit the fact that the set of activity levels of each player is totally ordered. New axioms are proposed and allow us to characterize three

[4]player-activity level permission

new values for multi-choice games with a permission structure. Each one coincides with a pal-Permission value on multi-choice games enriched with a pal-permission structure as described in the first step. They differ in the way the pal-permission structure emerges from the set of axioms. This approach is interesting for several reasons. First, it distinguishes a priori a permission structure between the players (the inter-player relationships) from the linear order on the activity levels of each player (the intra-player relationships). Then, the axiomatic study sheds light on how the inter-player and intra-player relationships are combined to form a pal-permission structure. Quite different pal-permission structures emerge from the axiomatic study, which have natural but different interpretations.

All in all, this chapter sheds light on the additional difficulties that arise when one tries to combine a multi-choice game with a structure on the player set, but does not take into account the Independence of higher demands or the Uniformity principle.

### 1.8.3 Chapter 4: Marginalism, egalitarianism and efficiency in multi-choice games

This chapter focuses on multi-choice games without any structure. This study addresses the trade-off between marginalism and egalitarianism, which is one of the main issues in economic allocation problems. Marginalism supports allocations based on a player's marginal contribution to coalitions, while egalitarianism is in favor of an equal allocation at the expense of the differences between players' marginal contribution to coalitions. The Uniformity principle can be seen as a form of egalitarianism.

In the context of TU-games, this trade-off can be seen as a compromise between the Shapley value and the Equal division value since the two values are often seen as the embodiment of marginalism and egalitarianism, respectively. This compromise can be made by considering convex combinations of the Shapley value and the Equal division value (see Joosten (1996)). These convex combinations of the Shapley value and the Equal division value have been recently studied by van den Brink *et al.* (2013), Casajus & Huettner (2013), Abe & Nakada (2019) and Béal *et al.* (2021b). This chapter investigates the trade-off between marginalism and egalitarianism in the context of multi-choice games. The Independence of higher demands and Uniformity principles are taken into account.

An extension of the Shapley value, called the multi-choice Shapley value, is proposed. This value is computed as follows. Assume that the grand coalition forms step by step starting from the empty coalition. At each step, one player increases its activity by one unit, let us say from $j$ to $j+1$, under the condition that all other players (that are capable to play $j$) have already reached their level $j$. This coalition formation process is called a **restricted order**. The multi-choice Shapley value assigns to each player the sum of its expected marginal contribution assuming that each restricted order occurs with equal probability. This value assigns a payoff vector in the Core, for each super-modular multi-choice game. A super-modular multi-choice

games can be viewed as a game in which the incentives for joining a coalition increase as the coalition grows. Furthermore, this value satisfies the Independence of higher demands principle.

Then, an extension of the Equal division value, called the multi-choice Equal division value is proposed. This value equally divides the surplus in worth between two consecutive synchronized coalitions among the players able to play the required activity levels. Thus, this value is not a trivial extension of the Equal division value: it is an egalitarian solution that satisfies the Independence of higher demands principle and the Uniformity principle. This value is clearly desirable since it satisfies all the principles pursued by GRDF.

To our knowledge, no previous work has addressed the trade-off between marginalism and egalitarianism in the context of multi-choice games. This trade-off is made by compromising between the multi-choice Shapley value and the multi-choice Equal division value. To that end, the multi-choice Egalitarian Shapley values for multi-choice games are introduced. This family of values is composed of convex combinations of the multi-choice Shapley value and the multi-choice Equal division value. We provide several axiomatic characterizations of these new solution concepts, by invoking classical axioms as well as new axioms for multi-choice games.

### 1.8.4 Chapter 5: Values for multi-choice games with a priority structure

A priority structure captures asymmetries between players which may reflect exogenous rights, different needs or merit, and so on. Priority relationships naturally arise between GRDF's consumers. For instance, it seems reasonable to prioritize the supply of a hospital rather than the supply of an amusement park. During winter, it seems reasonable to prioritize the supply of households before public facilities.

This chapter considers multi-choice games with a priority structure. A value for multi-choice games with a priority structure is introduced. This value is called the multi-choice Priority value, and it extends the Priority value for TU-games with a priority structure introduced by Béal *et al.* (2021a). Our value can be viewed as a lexicographic procedure defined as follows. For each (multi-choice) coalition, choose the players with the highest activity levels in the coalition. Among these players, select the players with no superiors in the priority structure. The surplus generated by the coalition, i.e., its Harsanyi dividend is equally divided among the selected players.

Regarding the Priority value for TU-games with a priority structure, the surplus generated by a coalition is shared according to one criterion: the inter-player relationships given by the priority structure. Our extension of this value shares the surplus generated by a coalition according to two criteria: the inter-player relationships given by the priority structure and the intra-player relationships given by the linear order of the activity levels. First, players are discriminated according to their activity level, then the remaining players are selected according to their position in

the directed graph.

Since the multi-choice Priority value first discriminates the players according to their activity level, it satisfies the Independence of higher demands principle. This result is the first result that combines a principle pursued by GRDF with an information provided by the gas distribution network. Two axiomatic characterizations of this value are proposed. One is a classical characterization that uses an additivity axiom, the second is based on a balanced contributions axiom (see Myerson (1980)).

### 1.8.5 Chapter 6: Sharing the cost of a gas distribution network

This chapter introduces the class of gas distribution cost sharing problems (gas distribution problems for short). A gas distribution problem can be represented by a triplet $(q, C^{Net}, Net)$, where $q$ is the profile of effective demands of some consumers, $Net$ is the distribution network, and $C^{Net}$ is a cost function that uses the information provided by $Net$. Recall that $C^{Net}$ can be viewed as a combination of a cost function $C$ and $Net$. In Section 1.2, it is advocated that considering the problem $(q, C, Net)$, in which $C$ is independent from $Net$, instead of $(q, C^{Net})$, is more desirable. In this chapter, the triplet $(q, C^{Net}, Net)$ is considered instead of $(q, C, Net)$, because the cost function is supposed to measure the cost of the pipelines in the network. Therefore, the cost function inevitably has to take $Net$ into account. However, we differ from the case where one considers $(q, C^{Net})$ and keep $Net$ as a separate data, so that we do not neglect any information from the gas distribution network.

We define solution concepts on the class of gas distribution problems that associate a payoff to each demand of each consumer, and call them cost sharing rules. Three distinct rules that depend on the network and the demands of the consumers are proposed. To that end, a new principle is introduced. Consider a gas distribution network in which all consumers, except one, have a null demand in gas. To supply this consumer with gas, all the pipelines connecting this consumer to the source must be involved. In other words, this consumer depends on a number of pipelines in addition to the one to which it is the tail. It should be reasonable to charge this consumer for the costs of operating these pipelines. In a more general way, consumers should pay for the portion of the network they use. This idea is the Connection principle.

The first rule is called Connection rule, and it satisfies the Independence of higher demands principle and the Connection principle. The second rule is called the Uniform rule, and it satisfies the Independence of higher demands principle and the Uniformity principle. In the same way that marginalism is incompatible with egalitarianism, the Connection principle is incompatible with the Uniformity principle. To make a trade-off between these two principles, the Mixed rules, which compromise between the Connection rule and the Uniform rule, are proposed. For each rule, an axiomatic characterization is provided.

The Connection rule coincides with the multi-choice Shapley value (introduced

in Chapter 4) of a specific multi-choice game derived from the network and the demands of the consumers. Moreover, the Connection rule is in the Core of this specific multi-choice game. Similarly, the Uniform rule coincides with the multi-choice Equal division value (introduced in Chapter 4) and the Mixed rules coincide with the multi-choice Egalitarian Shapley values (introduced in Chapter 4).

### 1.8.6 Chapter 7: Concluding remarks

Two major principles are pursued by GRDF: the Uniformity principle and the Independence of higher demands principle. The information provided by the distribution network is also important. These considerations are studied in a progressive way throughout the thesis. Chapter 3 only considers the information provided by the distribution network. Chapter 4 only considers the two principles pursued by GRDF. Chapter 5 only considers one of these principles and the information provided by the distribution network. Finally, Chapter 6 considers the two principles and the information provided by the distribution network.

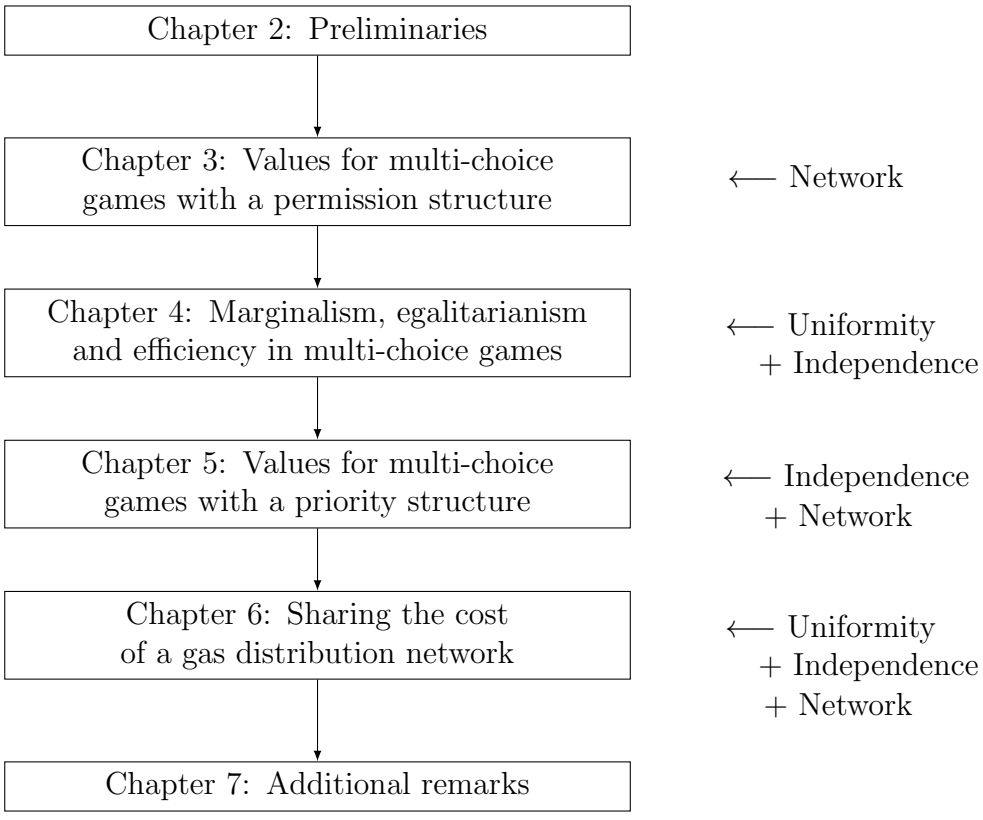



Figure 1.8

The sixth chapter of this thesis provides additional remarks. These remarks highlight the links between the different chapters.

- Chapter 3, 4 and 6: new solution concepts for multi-choice games with a pal-permission structure and multi-choice games with a permission structure are proposed. These new solution concepts are defined using elements from Chapter 3 and Chapter 4. Then, observe that, under certain conditions, one of these solution concepts coincides with the Connection rule from Chapter 6.

- Chapter 5 and 6: we discuss what happens if one applies the multi-choice priority value to a multi-choice game derived from a gas distribution problem.

- Chapter 3, 4 and 5: we introduce pal-priority structures for multi-choice games. Such structures can be understood as priority structures defined over the set of player-activity level pairs. We introduce a value for multi-choice games with a pal-priority structure, and show that it coincides with the multi-choice Priority value, from Chapter 5, and the multi-choice Shapley value, from Chapter 4, under certain conditions.

# Chapter 2: Preliminaries

This chapter provides preliminary definitions and notation for the thesis and is organized as follows. The first section presents transferable utility games, solution concepts for such games and their axiomatic characterizations. The second section presents the model of transferable utility games with a permission structure as introduced by Gilles *et al.* (1992). The Permission value and two of its axiomatic characterizations are discussed. Then, the third section presents the model of transferable utility games with a priority structure as introduced by Béal *et al.* (2021a). The Priority value and two of its axiomatic characterizations are discussed. The fourth section presents multi-choice games as introduced by Hsiao & Raghavan (1992) and van den Nouweland (1993). Extensions of solution concepts from TU-games to multi-choice games and their axiomatic characterizations are discussed. Finally, discrete cost sharing problems in the sense of Moulin (1995) are introduced. Several solution concepts for discrete cost sharing problems are presented. Special attention is given to how solution concepts from discrete cost sharing problems relate to solution concepts from multi-choice games.

## 2.1 TU-games

A situation in which a group of players can obtain certain payoffs by cooperation can be described by a cooperative game with transferable utility, or simply a TU-game. TU-games are appropriate tools to address many economic allocation problems. For instance, Littlechild & Owen (1973) apply TU-games to address the allocation of aircraft landing fees problem. Another application is proposed by Ambec & Sprumont (2002). The authors use TU-games to model river sharing problems that deal with the fair allocation of water among a set of players. As a final example, Ni & Wang (2007) model the polluted river problem with TU-games. In such setting, a set of players, located along a polluted river, seek to determine a way to share the cost of cleaning the river. This section introduces the model of TU-games along with some definitions. We discuss several solution concepts on this class along with their axiomatic characterizations.

### 2.1.1 Basic definitions

Let $N \subset \mathbb{N}$ be a non-empty and finite set of players. Each subset $E \in 2^N$ is referred to as a **coalition** of cooperating players. The **grand coalition** $N$ represents a situation in which all players cooperate. Coalition $\emptyset$ represents a situation in which no player cooperates, it is called the **empty coalition**. For each $E \in 2^N$, the integer $|E| \in \mathbb{N}$ denotes the cardinality of coalition $E$.

**Definition 1 (Transferable utility game).** *A transferable utility game, or simply a TU-game, is a couple $(N, v)$ consisting of a finite player set $N \subset \mathbb{N}$ and a characteristic function $v : 2^N \to \mathbb{R}$, with the convention that $v(\emptyset) = 0$.*

The real number $v(E)$ can be interpreted as the worth the players in $E$ generate when they cooperate. The class of TU-games with the finite set of players $N \subset \mathbb{N}$ is denoted by $\mathbb{G}^N$. The class of all TU-games with a finite set of players is denoted by $\mathbb{G} = \cup_{N \subset \mathbb{N}} \mathbb{G}^N$.

A TU-game is **non-negative** if each coalition generates a non-negative worth in this game. Formally, $(N, v) \in \mathbb{G}$ is non-negative if

$$\forall E \in 2^N, \quad v(E) \geq 0.$$

In some TU-games, a larger coalition always implies a higher worth. Such TU-games are called **monotonic** TU-games. Formally, $(N, v) \in \mathbb{G}$ is monotonic if

$$\forall E, T \in 2^N, \quad \big[E \subseteq T\big] \implies \big[v(E) \leq v(T)\big].$$

A monotonic game is non-negative by the convention $v(\emptyset) = 0$.

The sub-class of monotonic TU-games is denoted by $G_m$, and verifies $G_m \subseteq \mathbb{G}$.

In some situations, cooperation is useless because the players do not manage to generate any surplus through cooperation. An **additive** TU-game models this idea. In such game, the worth of a coalition is equal to the sum of each coalition member's marginal contribution to the empty coalition. Formally, $(N, v) \in \mathbb{G}$ is additive if

$$\forall E \in 2^N, \quad v(E) = \sum_{i \in E} v(\{i\}).$$

On the contrary, in some situations, cooperation is clearly beneficial to players. In particular, two coalitions of players may have an interest in merging in the sense that the worth they would generate together outperforms the sum of their worth. **Super-additive** TU-games model this idea.

A TU-game $(N, v) \in \mathbb{G}$ is **super-additive** if

$$\forall E, T \in 2^N, E \cap T = \emptyset, \quad v(E) + v(T) \leq v(E \cup T).$$

A stronger requirement than super-additivity is super-modularity. A TU-game $(N, v) \in \mathbb{G}$ is **super-modular** if

$$\forall E, T \in 2^N, \quad v(E) + v(T) \leq v(E \cup T) + v(E \cap T).$$

The class of super-modular games is denoted by $G_{sm} \subseteq \mathbb{G}$. If a game is super-modular, then it is super-additive by the convention $v(\emptyset) = 0$. Similarly, a TU-game $(N, v) \in \mathbb{G}$ is **sub-modular** if

$$\forall E, T \in 2^N, \quad v(E) + v(T) \geq v(E \cup T) + v(E \cap T).$$

In a TU-game, different types of players can be distinguished. In particular, these types can be defined according to the marginal contributions of the players to the coalitions. Take any $(N, v) \in \mathbb{G}$. The **marginal contribution** of a player $i \in N$ to a coalition $E \in 2^{N \setminus \{i\}}$ in $(N, v)$ is defined as

$$v(E \cup \{i\}) - v(E).$$

Note that, for each $(N, v) \in G_m$, the marginal contribution of each player is always non-negative. Two players $i, j \in N$ are **equal** in $(N, v)$ if they have the same marginal contributions to coalitions. Formally, such players are defined as

$$\forall E \in 2^{N \setminus \{i,j\}}, \quad v(E \cup \{i\}) = v(E \cup \{j\}).$$

A player $i \in N$ is a **veto player** in $(N, v)$ if a non-null worth cannot be achieved without its cooperation in a coalition. Formally, such player is defined as

$$\forall E \in 2^{N \setminus \{i\}}, \quad v(E) = 0.$$

Two veto players both have the same marginal contributions to coalitions since their cooperation is necessary to generate a non-null worth. Thus, two veto players are equal, but the converse is not true.

A player $i \in N$ is a **null player** in $(N, v)$ if it has null marginal contributions to coalitions. Formally, such player is defined as

$$\forall E \in 2^{N \setminus \{i\}}, \quad v(E \cup \{i\}) = v(E).$$

A player $i \in N$ is a **nullifying player** in $(N, v)$ if its cooperation within a coalition implies a null worth for this coalition. Formally, such player is defined as

$$\forall E \in 2^N : E \ni i, \quad v(E) = 0.$$

Take any $\mathbb{G}^N \subseteq \mathbb{G}$ and any $(N, v) \in \mathbb{G}^N$. Since $2^N$ is a finite set, a characteristic function $v : 2^N \to \mathbb{R}$ can be described by the vector $(v(E))_{E \in 2^N} \in \mathbb{R}^{2^{|N|}}$. Therefore, the class $\mathbb{G}^N$ is a linear sub-space of $\mathbb{R}^{2^{|N|}}$. It follows that $(N, v)$ can be expressed as a linear combination of $2^{|N|} - 1$ linearly independent games, since $v(\emptyset) = 0$. In his original chapter, Shapley (Shapley (1953)) identifies a salient basis: the basis formed by unanimity games. For any non-empty coalition $E \in 2^N$, the unanimity game $u_E$ with ruling coalition $E$ is defined as

$$u_E(T) = \begin{cases} 1 \text{ if } E \subseteq T, \\ 0 \text{ otherwise.} \end{cases} \tag{2.1}$$

Therefore, for each $N$, and each $(N, v) \in \mathbb{G}^N$, there exists a unique decomposition of $v$ in terms of unanimity games

$$v = \sum_{E \in 2^N \setminus \{\emptyset\}} \Delta_v(E) u_E. \tag{2.2}$$

The coordinate $\Delta_v(E)$ is the Harsanyi dividend (see Harsanyi (1958)) of a coalition $E \in 2^N$. This dividend reflects the net surplus generated by $E$. Formally, Harsanyi dividends are defined as

$$\Delta_v(E) = v(E) - \sum_{T \subset E} \Delta_v(T). \tag{2.3}$$

**Example 1.** *Consider a TU-game $(N, v) \in \mathbb{G}$ and take two players $i, j \in N$. The surplus of the coalition $\{i, j\}$ is the difference between the worth of the coalition minus the dividends of all its sub-coalition $\{i\}$ and $\{j\}$. The dividend of a singleton is equal to the worth of the singleton. Formally, the dividend of coalition $\{i, j\}$ is $\Delta_v(\{i, j\}) = v(\{i, j\}) - v(\{i\}) - v(\{j\})$.*

### 2.1.2 Solution concepts

One of the basic issues in the theory of cooperative TU-games is as follows: "If the grand coalition forms, how to divide its worth among the players". This issue is addressed through solution concepts for TU-games. In a TU-game $(N, v) \in \mathbb{G}$, each player $i \in N$ may receive a **payoff**. A payoff vector $x \in \mathbb{R}^{|N|}$ is a $|N|$-dimensional vector that assigns a payoff $x_i \in \mathbb{R}$ to each player $i \in N$.

Take any $(N, v) \in \mathbb{G}$. A payoff vector $x \in \mathbb{R}^{|N|}$ is **efficient** if it redistributes the worth of the grand coalition among the players, formally

$$\sum_{i \in N} x_i = v(N). \tag{2.4}$$

A payoff vector $x \in \mathbb{R}^{|N|}$ is **individual rational** if the payoff of each player is higher than its stand-alone worth in the game, formally

$$x_i \geq v(\{i\}). \tag{2.5}$$

A payoff vector $x \in \mathbb{R}^{|N|}$ is **coalitionally rational** if no coalition of players has an interest to split off from the grand coalition. Formally

$$\forall E \in 2^N, \quad \sum_{i \in E} x_i \geq v(E). \tag{2.6}$$

A single-valued solution on $\mathbb{G}$ (a **value** for short) is a map $f$ that assigns a unique payoff vector $f(N, v)$ to each $(N, v) \in \mathbb{G}$. A **set-valued** solution on $\mathbb{G}$ is a map $F$ that assigns a (possibly empty) set of payoff vectors $F(N, v)$ to each $(N, v) \in \mathbb{G}$.

There are several well-known solution concepts for TU-games that have given rise to many studies.

The Core (see Gillies (1953) and Shapley (1955)) is probably the most well-known set-valued solution for TU-games. This solution concept is often viewed as the set of stable payoff vectors since each payoff vector in this set is efficient and coalitionally rational.

**Definition 2 (Core).** *For each $(N, v) \in \mathbb{G}$, the Core $C(N, v)$ of $(N, v)$ is defined as the set of payoff vectors satisfying both* (2.4) *and* (2.6).

Another well-known set-valued solution for TU-games is the Weber set (Weber (1988)). To define this solution, denote the set of all (linear) orders on $N$ by $\Theta_N$. Assume that the grand coalition forms step by step, starting from the empty coalition. At each step, a player enters the coalition. The players enter according to an order $\theta \in \Theta_N$. The step $\theta(i)$ is the step at which $i \in N$ enters. When a player enters, it contributes positively or negatively to the worth of the coalition. The **marginal vector** of $(N, v)$ with respect to the order $\theta$ collects all the marginal contributions of the players when they enter the coalition according to $\theta$. This vector is denoted by $m_v^\theta$ and is defined as

$$m_v^\theta = v(E^{\theta,i} \cup \{i\}) - v(E^{\theta,i}), \tag{2.7}$$

where $E^{\theta,i} = \{j \in N : \theta(j) < \theta(i)\}$ is the set of predecessors of $i \in N$ with respect to $\theta$. The Weber set is defined as the convex hull of the set of all $n!$ marginal vectors. Observe that this set is always non-empty.

**Definition 3 (Weber set).** *For each $(N, v) \in \mathbb{G}$, the Weber set is defined as*

$$W(N, v) = conv\{m_v^\theta : \theta \in \Theta_N\}.$$

**Proposition 1 (Weber (1988), Derks (1992)).** *For each $(N, v) \in \mathbb{G}$,*

$$C(N, v) \subseteq W(N, v).$$

In a super-modular TU-game, cooperation is clearly beneficial to players. Therefore, no coalition of players has an interest to split off from the grand coalition. The next result, proved by Shapley (1971), indicates that the Core and the Weber set coincide on the class of super-modular TU-games. This directly implies that the Core is non-empty on the class of super-modular TU-games.

**Proposition 2 (Shapley (1971)).** *For each $(N, v) \in G_{sm}$,*

$$W(N, v) = C(N, v).$$

The Shapley value (see Shapley (1953)) is probably the most prominent single-valued solution for TU-games. This value admits a number of different expressions, but we only retain two of them since the others are not useful for the rest of the thesis.

**Definition 4 (Shapley value).** *Take any $(N, v) \in \mathbb{G}$. The Shapley value is the average of all marginal vectors, meaning that the payoff of each player corresponds to its average marginal contribution to coalitions. Formally, the Shapley value can be written as*

$$\forall i \in N, \quad Sh_i(N, v) = \frac{1}{n!} \sum_{\theta \in \Theta_N} m_v^\theta. \tag{2.8}$$

*The Shapley value can be defined using Harsanyi dividends. The value divides the dividend of each coalition equally among its members. Therefore, for each $(N, v) \in \mathbb{G}$, the Shapley value can be written as*

$$\forall i \in N, \quad Sh_i(N, v) = \sum_{E \in 2^N : E \ni i} \frac{\Delta_v(E)}{|E|}. \tag{2.9}$$

By definition, the Shapley value is always in the Weber set of any game. Therefore, this value is always in the Core of super-modular games.

The Shapley value is often seen as an embodiment of marginalism. On the contrary, the Equal division value ensures an identical payoff to each player by dividing the worth of the grand coalition equally among the players. Naturally, this value is often seen as an embodiment of egalitarianism.

**Definition 5 (Equal division value).** *For each $(N, v) \in \mathbb{G}$, the Equal division value is defined as*

$$\forall i \in N, \quad ED_i(N, v) = \frac{v(N)}{n}. \tag{2.10}$$

The Equal division value is part of a larger family of values called (non-negative) Weighted division values, which are introduced in Béal *et al.* (2016). A value in this class divides the worth of the grand coalition among the players according to certain weights. Let $\Omega = \{\omega \in \mathbb{R}_+^\infty | (\omega_i)_{i \in \mathbb{N}}\}$ be an infinite set of strictly positive weights.

**Definition 6 (Weighted division values).** *Take any $\omega \in \Omega$. For each $(N, v) \in \mathbb{G}$, the Weighted division value $WED^\omega$ is defined as*

$$\forall i \in N, \quad WED_i^\omega(N, v) = \frac{\omega_i}{\sum_{j \in N} \omega_j} v(N). \tag{2.11}$$

For any weights $\omega \in \Omega$ such that $\omega_i = \omega_j$ for any two distinct players $i, j \in N$, the Weighted division value $WED^\omega$ coincides with the Equal division value on any game in $\mathbb{G}$.

One of the main issues in economic allocation problems is the trade-off between marginalism and egalitarianism. Marginalism supports allocations based on a player's marginal contributions, while egalitarianism is in favor of an equal allocation

at the expense of the differences between players' marginal contributions. In the context of TU-games, this trade-off can be seen as a compromise between the Shapley value and the Equal division value. This compromise can be made by considering convex combinations between the Shapley value and the Equal division value (see Joosten (1996)). These convex combinations of the Shapley value and the Equal division value have been recently studied by van den Brink *et al.* (2013), Casajus & Huettner (2013), Abe & Nakada (2019) and Béal *et al.* (2021b). These combinations form a set-valued solution called Egalitarian Shapley values and denoted by $ESh$.

**Definition 7 (Egalitarian Shapley values).** *Take any* $\alpha \in [0,1]$. *For each* $(N,v) \in \mathbb{G}$, *the Egalitarian Shapley value* $ESh^\alpha$ *is defined as*

$$\forall i \in N, \quad ESh_i^\alpha(N,v) = \alpha Sh_i(N,v) + (1-\alpha)ED_i(N,v). \tag{2.12}$$

### 2.1.3 Axiomatic characterizations

This section presents and discusses classical axioms for solution concepts on $\mathbb{G}$. Combining these axioms, we present at least one axiomatic characterization for each value introduced in the last section. Consider a value $f$ on $\mathbb{G}$.

The first axiom is ubiquitous in almost every axiomatic characterization, as it transcribes the simple idea that the total worth of the grand coalition should be entirely allocated among the players.

**Axiom 1 (Efficiency).** *For each* $(N,v) \in \mathbb{G}$,

$$\sum_{i \in N} f_i(N,v) = v(N). \tag{2.13}$$

Note that Efficiency only makes sense under the assumption that the grand coalition forms. There exists some axioms that play a similar role than Efficiency but do not take this assumption for granted. For instance, Pérez-Castrillo & Wettstein (2001) and Béal *et al.* (2021b) study the Cohesive efficiency axiom. This axiom states that the sum of the payoffs should be equal to the maximal total worth that the players are able to achieve by organizing themselves into a partition.

When dealing with a problem, it can be useful to divide this problem into several smaller problems. By doing so, it is desirable that the sum of the outcomes of the smaller problems is equal to the outcome of the original problem. The next axiom translates this idea to the framework of TU-games. It is frequently used in many axiomatic characterizations. This axiom states that the solution is additive with respect to TU-games with the same player set.

**Axiom 2 (Additivity).** *For each* $(N,v),(N,w) \in \mathbb{G}$,

$$f(N,v+w) = f(N,v) + f(N,w).$$

A slightly stronger axiom is also common in the literature. This axiom states that the solution is linear with respect to TU-games with the same player set.

**Axiom 3 (Linearity).** *For each* $(N, v), (N, w) \in \mathbb{G}$ *and* $\lambda \in \mathbb{R}$*,*

$$f(N, v + \lambda w) = f(N, v) + \lambda f(N, w).$$

Obviously,

$$\text{Linearity} \implies \text{Additivity}$$

If a player is unproductive or causes harm to others, it seems reasonable to penalize them accordingly. The next axiom, introduced by Shubik (1962), indicates that if a player never contributes to the worth of the coalitions, then it should not receive anything from the value.

**Axiom 4 (Null player property).** *For each* $(N, v) \in \mathbb{G}$*, if a player* $i \in N$ *is a null player, then*

$$f_i(N, v) = 0.$$

A player can harm the other players in a coalition if its cooperation nullifies the worth of the coalition. van den Brink (2007) introduces an axiom that advocates for a null payoff to any nullifying player.

**Axiom 5 (Nullifying player property).** *For each* $(N, v) \in \mathbb{G}$*, if a player* $i \in N$ *is a nullifying player, then*

$$f_i(N, v) = 0.$$

When the worth of the grand coalition is non-negative, it is reasonable that any player ends up with a non-negative payoff. In particular, null players need not to receive a negative payoff. Since they do not harm any the players, they should obtain non-negative payoffs. Casajus & Huettner (2013) models this idea and proposes an axiom that guarantees a non-negative payoff to null players whenever the worth of the grand coalition is non-negative.

**Axiom 6 (Null player in a productive environment).** *For each* $(N, v) \in \mathbb{G}$ *such that* $v(N) \geq 0$ *and each null player* $i \in N$ *in* $(N, v)$*,*

$$f_i(N, v) \geq 0.$$

If a value satisfies the Null player property, then if satisfies Null player in a productive environment, but the converse is not true.

$$\text{Null player property} \implies \text{Null player in a productive environment}$$

In some situations, it may be desirable that the payoff of a player does not depend on exogenous factors. Shapley (1953) proposes an axiom ensuring anonymity between the players. This means that the payoff of a player does not depend on its label $i \in N$. A preliminary definition is necessary to introduce the axiom. Take any order $\theta \in \Theta_N$. The game $(N, \theta v)$ is defined as

$$\forall E \in 2^N, \quad \theta v(\cup_{i \in E}\{\theta(i)\}) = v(E).$$

In this game, each player in $E$ has a new label but keeps the same marginal contributions to coalitions.

**Axiom 7 (Anonymity).** *For each $(N, v) \in \mathbb{G}$ and each order $\theta$ on $N$,*

$$f_i(N, v) = f_{\theta(i)}(N, \theta v).$$

Alternatively, the payoff of a player may depends on its marginal contributions to coalitions. Maschler & Peleg (1966) introduce an axiom that requires players to obtain no lower payoffs than less productive players.

**Axiom 8 (Desirability).** *For each $(N, v) \in \mathbb{G}$, if for two distinct players $i, j \in N$ it holds that $v(E \cup \{i\}) \geq v(E \cup \{j\})$, for each $E \in 2^{N \setminus \{i,j\}}$, then*

$$f_i(N, v) \geq f_j(N, v). \tag{2.14}$$

In case two players have the same marginal contribution to coalitions. Desirability implies that these two players receive the same payoff. Shubik (1962) proposes an axiom that formalizes this idea.

**Axiom 9 (Equal treatment of equals).** *For each $(N, v) \in \mathbb{G}$, if two distinct players $i, j \in N$ are equals, then*

$$f_i(N, v) = f_j(N, v).$$

If a value satisfies Anonymity, then it satisfies Equal treatment of equals. The converse is not true. Similarly, if a value satisfies Desirability then it satisfies Equal treatment of equals. The converse is also not true.

$$\begin{aligned} \text{Anonymity} &\implies \text{Equal treatment of equals} \\ \text{Desirability} &\implies \text{Equal treatment of equals} \end{aligned}$$

Two veto players both have the same marginal contributions to coalitions since their presence is necessary to generate a non-null worth. Equal treatment of equals indicates that such players should receive the same payoffs. Béal & Navarro (2020) propose an axiom that formalizes this idea.

**Axiom 10 (Veto players property).** *For each* $(N, v) \in \mathbb{G}$ *and two distinct veto players* $i, j \in N$*,*

$$f_i(N, v) = f_j(N, v).$$

If a value satisfies Equal treatment of equals, then it satisfies the Veto players property. The converse is not true.

$$\text{Equal treatment of equals} \implies \text{Veto players property}$$

The same payoff for two equal players can be viewed as a strong requirement, especially if exogenous factors may justify different payoffs for such equal players. The next axiom follows this idea and proposes a qualitative version of Equal treatment of equals. This axiom relaxes Equal treatment of equals and states that two equal players may receive a payoff of the same sign (see Casajus (2018)). Define the sign of a real number as

$$\forall x \in \mathbb{R}, \quad sign(x) = \begin{cases} +1 & \text{if } x > 0, \\ -1 & \text{if } x < 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Axiom 11 (Sign symmetry).** *For each* $(N, v) \in \mathbb{G}$ *and each* $i, j \in N$ *such that* $i$ *and* $j$ *are equal,*

$$sign(f_i(N, v)) = sign(f_j(N, v)).$$

Clearly, if a value satisfies Equal treatment of equals, then it satisfies Sign symmetry.

$$\text{Equal treatment of equals} \implies \text{Sign symmetry}$$

Some axioms indicate how the payoff of players may vary when some inputs of the problem vary. If a player leaves the game, this may affect other players' payoffs. For instance, **Null player out** introduced by Derks & Haller (1999) states that if a null player leaves the game, then the payoff of the other players remain unchanged. This axiom will be formally introduced later in Section 2.3.2. Myerson (1980) introduces an axiom stating that for any two players, the amount that each player would gain or lose by the other's withdrawal from the game should be equal.

**Axiom 12 (Balanced contributions).** *For each* $(N, v) \in \mathbb{G}$ *and each* $i, j \in N$*,*

$$f_i(N, v) - f_i(N \setminus \{j\}, v) = f_j(N, v) - f_j(N \setminus \{i\}, v).$$

Casajus (2017) proposes the **Weak balanced contributions** axiom, which relaxes Balanced contributions. This axiom requires that the amount that each player would gain or lose by the other's withdrawal from the game should be of the same sign.

Some axioms assume that the characteristic function may vary. van den Brink (2002) introduces a fairness axiom for TU-games that makes such assumption. This axiom states that if one adds a game $(N, v) \in \mathbb{G}$ to another game $(N, w) \in \mathbb{G}$ in which two players $i$ and $j$ are equal, then the payoffs of the players $i$ and $j$ change by the same amount.

**Axiom 13 (Fairness).** *If $i, j \in N$ are equal in $(N, w) \in \mathbb{G}$, then*

$$\forall (N, v) \in \mathbb{G}, \quad f_i(v + w) - f_i(v) = f_j(v + w) - f_j(v).$$

Casajus (2011) shows that Fairness is equivalent to the **Differential marginality** axiom. Differential marginality compares two different games. It indicates that the difference between two players' payoffs is completely determined by the difference between their productivities. Then, **Weak Differential marginality** is introduced by Casajus & Yokote (2017). This axiom indicates that the payoffs of the two players vary in the same direction if the difference between their productivities is the same is both situations.

If a value satisfies Equal treatment of equals and Additivity, then it satisfies Fairness. The converse is not true. Moreover, if a value satisfies the Null player property and Fairness, then it satisfies Equal treatment of equals. Again, the converse is not true. Clearly, these implications still holds if one replaces Fairness by Differential marginality.

$$\text{Equal treat of eq} + \text{Additivity} \implies \text{Fairness/Diff marginality}$$
$$\text{Null player prop} + \text{Fairness/Diff marginality} \implies \text{Equal treat of eq}$$

As a reminder, Desirability compares the payoffs of two players with different productivities in a unique situation. Young (1985) proposes an axiom that compares the payoffs of a single player in two different situations. This axiom guarantees that a player whose marginal contributions weakly increases does not end up with a lower payoff.

**Axiom 14 (Strong monotonicity).** *For each $(N, v), (N, w) \in \mathbb{G}$ and each $i \in N$ such that*

$$\forall E \in 2^{N \setminus \{i\}}, \quad v(E \cup \{i\}) - v(E) \geq w(E \cup \{i\}) - w(E),$$

*it holds that*

$$f_i(N, v) \geq f_i(N, w).$$

Requiring that the payoff of a player does not decrease if its marginal contributions do not decrease (as required by Strong monotonicity) irrespective of what is to be allocated is a very strong requirement. Indeed, this property may imply a decrease in payoff for other players if the total worth to be shared does not increase significantly. The next axiom requires that the worth of the grand coalition does not decrease on top of the original conditions stated in Strong monotonicity (see van den Brink *et al.* (2013)).

**Axiom 15 (Weak monotonicity).** *For each $(N, v), (N, w) \in \mathbb{G}$ such that $v(N) \geq w(N)$, and each $i \in N$ such that*

$$\forall E \in 2^{N \setminus \{i\}}, \quad v(E \cup \{i\}) - v(E) \geq w(E \cup \{i\}) - w(E),$$

*it holds that*

$$f_i(N, v) \geq f_i(N, w). \tag{2.15}$$

If a value satisfies Strong monotonicity, then it satisfies Weak monotonicity. The converse is not true.

$$\text{Strong monotonicity} \implies \text{Weak monotonicity}$$

Strong monotonicity also implies **Marginality** (see Young (1985)), which states that if the marginal contribution of a player does not change from one game to another, so does its payoff in these games.

We now have the material to present some axiomatic characterizations of the Shapley value, the Equal division value, the Egalitarian Shapley values and the Weighted division values. This list of characterizations is not exhaustive, but each characterization will prove to be useful for the rest of this thesis.

**Theorem 1 (Shubik (1962)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, Additivity, the Null player property and Equal treatment of equals if and only if $f = Sh$.*

This characterization still holds if one replaces Equal treatment of equals in the axioms of Theorem 1 by the Veto players property (see Béal & Navarro (2020)) or by Sign symmetry (see Casajus (2018)).

Additivity and Linearity are often criticized for their lack of economic interpretation, and are often accused of only being of technical usefulness. Several studies proposed axiomatic characterization of the Shapley value without relying on these axioms. Myerson (1980) proposes an axiomatic characterization of the Shapley value that relies on only two axioms, none of them being Additivity or Linearity.

**Theorem 2 (Myerson (1980)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency and Balanced contributions if and only if $f = Sh$.*

Young (1985) proposes another characterization by replacing Additivity and the Null player property by Strong monotonicity in Theorem 1.

**Theorem 3 (Young (1985)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, Equal treatment of equals and Strong monotonicity if and only if $f = Sh$.*

It is possible to replace Strong monotonicity by Marginality in Theorem 3 (see Young (1985) and Pintér (2015)). Casajus (2018) shows that one can relax Equal treatment of equals in Theorem 3 into Sign symmetry without affecting the result.

**Theorem 4 (Casajus (2018)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, Sign symmetry and Strong monotonicity if and only if $f = Sh$.*

Another approach that does not rely on Additivity/Linearity is proposed by van den Brink (2002). The author shows that Fairness captures enough information to characterize the Shapley value in presence of Efficiency and the Null player property.

**Theorem 5 (van den Brink (2002)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, the Null player property and Fairness if and only if $f = Sh$.*

Since Fairness is equivalent to Differential marginality, one can substitute Fairness by Differential marginality in Theorem 5. Additionally, Casajus & Yokote (2017) show that the characterization still holds if one weakens Differential marginality into Weak Differential marginality. Finally, Casajus (2017) shows that Efficiency, Weak marginal contribution and Weak Differential marginality are enough to characterize the Shapley value.

Efficiency seems to be mandatory to characterize the Shapley value, but it is in fact possible to do without. For instance, Einy & Haimanko (2011) manages to characterize the Shapley value without Efficiency. Casajus (2014) drops both Efficiency and Additivity and still manages to characterize the Shapley value. Finally, Béal *et al.* (2015) provides an axiomatic characterization of the Shapley value that does not rely on Efficiency, Additivity or even Equal Treatment of Equals.

The characterization of the Shapley value provided by Shubik (1962) can be slightly modified to characterize other values. van den Brink (2007) provides an axiomatic characterization of the Equal division value by substituting the Null player property by the Nullifying player property.

**Theorem 6 (van den Brink (2007)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, Additivity, the Nullifying player property and Equal treatment of equals if and only if $f = ED$.*

Béal *et al.* (2016) show that one can substitute Equal treatment of equals in Theorem 6 by the Null player in a productive environment axiom and obtain a characterization of the Weighted division values.

**Theorem 7 (Béal *et al.* (2016)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, Linearity, the Nullifying player property and Null player in a productive environment if and only if $f = WED^{\omega}$, for some $\omega \in \Omega$.*

Casajus (2018) points out that, for each $\omega \in \Omega$, a Weighted division value satisfies Sign symmetry and the Nullifying player property. Casajus & Huettner (2013) replace Linearity and the Nullifying player property in Theorem 7 by Linearity and Desirability to characterize Egalitarian Shapley values.

**Theorem 8 (Casajus & Huettner (2013)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, Additivity, Desirability and Null player in a productive environment if and only if $f = ESh^{\alpha}$, for some $\alpha \in [0, 1]$.*

At last, we present a second axiomatic characterization of the Egalitarian Shapley values proposed by van den Brink *et al.* (2013).

**Theorem 9 (van den Brink *et al.* (2013)).** *A solution $f$ on $\mathbb{G}$ satisfies Efficiency, Linearity, Anonymity and Weak monotonicity if and only if $f = ESh^{\alpha}$, for some $\alpha \in [0, 1]$.*

## 2.2 TU-games with a permission structure

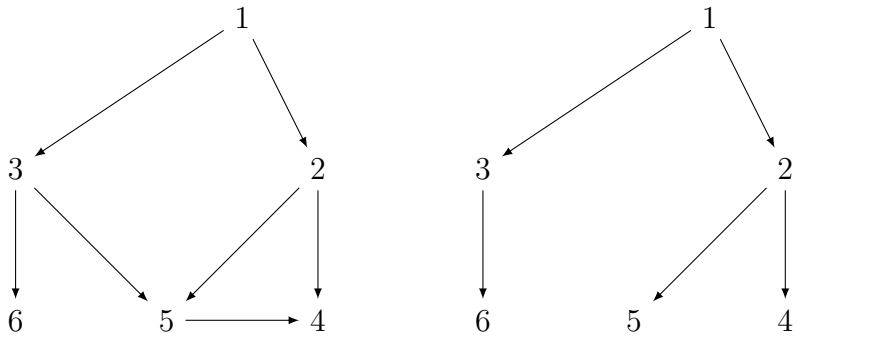
This section presents TU-games with a permission structure as introduced by Gilles *et al.* (1992). This class of games describes situations in which the players are part of a permission structure that is represented by a directed graph. In a permission structure, there are players that need permission from other players before they are allowed to cooperate.

Various assumptions can be made about how a permission structure affects cooperation. In the conjunctive approach, it is assumed that every player needs permission from all its predecessors before it is allowed to cooperate. Alternatively, in the disjunctive approach, it is assumed that each player needs permission from at least one of its predecessors before it is allowed to cooperate with other players. Depending on the approach, a coalition can be evaluated in different manners. Two major solution concepts exist, which take into account two different manners a coalition can be evaluated in a game with a permission structure. van den Brink & Gilles (1996) introduce the permission value for TU-games with a permission structure by adopting the conjunctive approach. van den Brink (1997) conduct a similar study but adopt the disjunctive approach. This section describes the model of TU-games with a permission structure. It introduces the (conjunctive) permission value, and provides two axiomatic characterizations of this value.

### 2.2.1 The model

A **permission structure** is a couple $(N, P)$, where $N \subset \mathbb{N}$ is a finite set of players and $P$ is a map $P : N \to 2^N$. When no confusion arises, we simply denote a permission structure $(N, P)$ by $P$. The relationship $j \in P(i)$ means that $j$ needs the permission from $i$ to cooperate. Equivalently, denote $i \in P^{-1}(j)$ if and only if $j \in P(i)$. In this case, player $i$ is the **predecessor** of $j$, whereas $j$ is the **successor** of $i$. The **trivial** structure $P^0$ is such that, for each $i \in N$, $P^0(i) = \emptyset$. The transitive closure of a permission structure $P$ is a permission structure $\hat{P}$ such that, for each $i \in N$, we have $j \in \hat{P}(i)$ if and only if there exists a path $i = h_1, h_2, \ldots, h_k = j$ such that $h_k \in P(h_{k-1}), \ldots, h_2 \in P(h_1)$. The players in $\hat{P}(i)$ are called the **subordinates** of $i$ in $P$, and the players in $\hat{P}^{-1}(i) := \{j \in N : i \in \hat{P}(j)\}$ are called the **superiors** of $i$ in $P$. A **cycle** occurs in the structure when there exists a path $i = h_1, h_2, \ldots, h_k = i$ such that $h_k \in P(h_{k-1}), \ldots, h_2 \in P(h_1)$. A permission structure is **acyclic** if no cycle occurs in it.

The class of acyclic permission structures on a finite set of players is denoted by $\mathbb{P}$. A **permission tree structure** is an acyclic permission structure in which each player has at most one predecessor and only one player has no predecessor. Such a player is referred to as the root of the tree. The subclass of permission tree structures is denoted by $\mathbb{P}_T \subseteq \mathbb{P}$.




(a) Acyclic structure (b) Tree structure with root 1

In a permission structure, cooperation between the players is restricted. A player cannot cooperate unless some of its superior(s) also cooperate. Various assumptions can be made about how a permission structure affects cooperation. A distinction is made between the conjunctive approach and the disjunctive approach. In the conjunctive approach, as developed in Gilles *et al.* (1992) and van den Brink & Gilles (1996), a player needs permission from all its predecessors (If any) in the permission structure. On the other hand, in the disjunctive approach, a player needs permission from at least one of its predecessors (If any). Both approaches coincide on permission tree structures since each player has at most one predecessor.

A TU-game with a permission structure on $N \subset \mathbb{N}$ is a triplet $(N, v, P)$, where $(N, v) \in \mathbb{G}$ is a TU-game, and $(N, P) \in \mathbb{P}$ is a (acyclic) permission structure. The class of TU-games with an acyclic permission structure is denoted by $\mathbb{GP}$, and the class of TU-games with a permission tree structure is denoted by $\mathbb{GP}_T$.

Essentially, one can think of $v$ as representing the economic possibilities open to every coalition in $N$. Thus $v(E)$ represents the amount of utility, which coalition $E \in 2^N$ could normally obtain if no permission structure was imposed on the game. Due to the presence of the permission structure, all coalition can form, by only a subset of coalitions can efficiently cooperate. Such coalitions are the ones containing all the superiors of their members. They are called feasible coalitions.

**Definition 8 (Feasible coalitions).** *Take any $(N, P) \in \mathbb{P}$. The set of feasible coalitions is defined as*

$$\mathrm{A}_P := \big\{ E \in 2^N : P^{-1}(i) \subseteq E \text{ for each } i \in E \big\}. \tag{2.16}$$

The feasible coalitions are essentially the only worth generating coalitions within a game with a permission structure. The definition shows explicitly that indeed all superiors of the players in an feasible coalition are also members of that coalition.

**Proposition 3 (Gilles *et al.* (1992)).** *Take any* $(N, P) \in \mathbb{P}$. *The set of feasible coalitions* $\mathrm{A}_P$ *verifies the following properties*

- $\emptyset \in \mathrm{A}_P$,
- $N \in \mathrm{A}_P$,
- *If* $E, F \in \mathrm{A}_P$, *then* $E \cup F \in \mathrm{A}_P$ *and* $E \cap F \in \mathrm{A}_P$.

Even if acoalition is not feasible, there may be one or several worth generating sub-coalition(s) contained within it. The largest of these sub-coalitions is called the sovereign part of the original coalition. Put differently, the sovereign part of a coalition is the payoff generating part of a coalition. Additionally, to any coalition, it is possible to find one or several feasible super-coalition(s) containing it. The smallest of these super-coalition(s) is called the authorizing part of the original coalition. Put differently, the authorizing part of a coalition is the smallest coalition that contains all the players necessary to make the original coalition feasible.

**Definition 9 (Sovereign and authorizing parts).** *Take any* $(N, P) \in \mathbb{P}$ *and* $E \in 2^N$. *The sovereign part of* $E$ *according to* $P$ *is the largest feasible subset of* $E$ *defined as*

$$\sigma(E) := \bigcup_{\{F \in \mathrm{A}_P : F \subseteq E\}} F.$$

*The authorizing set of* $E \in 2^N$ *according to* $P$ *is the the smallest feasible set containing* $E$ *defined as*

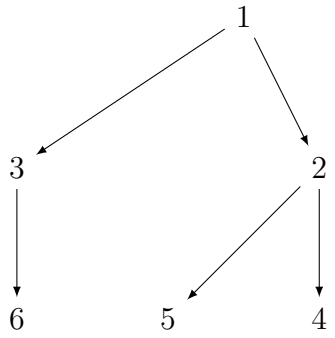
$$\alpha(E) := \bigcap_{\{F \in \mathrm{A}_P : E \subseteq F\}} F.$$

The sovereign of the authorizing operators are well defined by Proposition 3.

**Example 2.** *Consider the set of player* $N = \{1, 2, 3, 4, 5, 6\}$. *Define the following permission structure* $P$ *as follows*

$$\begin{aligned} P :1 &\longrightarrow \{2, 3\} \\ 2 &\longrightarrow \{4, 5\} \\ 3 &\longrightarrow \{6\} \\ 4 &\longrightarrow \emptyset \\ 5 &\longrightarrow \emptyset \\ 6 &\longrightarrow \emptyset \end{aligned} \qquad \begin{aligned} P^{-1} :1 &\longrightarrow \emptyset \\ 2 &\longrightarrow \{1\} \\ 3 &\longrightarrow \{1\} \\ 4 &\longrightarrow \{2\} \\ 5 &\longrightarrow \{2\} \\ 6 &\longrightarrow \{3\} \end{aligned}$$

*Here, players* 2 *and* 3 *are successors of* 1*. Player* 4 *and* 5 *are successors of* 2 *and subordinates of* 1*.*




*We have* $\sigma(\{2,3,4,5\}) = \emptyset$ *because there is no feasible subset of* $\{2,3,4,5\}$*,* $\sigma(\{1,2,4,6\}) = \{1,2,4\}$ *because* $\{1,2,4\}$ *is the largest feasible subset of* $\{1,2,4,6\}$ *and* $\alpha(\{1,2,4,6\}) = \{1,2,3,4,6\}$ *because* $\{1,2,3,4,6\}$ *is the smallest feasible superset of* $\{1,2,4,6\}$*.*

Some properties of the sovereing and the authorizing sets, proved in Gilles *et al.* (1992), are introduced.

**Proposition 4 (Gilles *et al.* (1992)).** *Take any* $(N,P) \in \mathbb{P}$*. For and each* $E, F \in 2^N$*, it holds that:*

- $\sigma(E) \cup \sigma(T) \leq \sigma(E \cup T)$*;*
- $\sigma(E) \cap \sigma(T) = \sigma(E \cap T)$*;*
- $\alpha(E) \cup \alpha(T) = \alpha(E \cup T)$*;*
- $\alpha(E \cap T) \leq \alpha(E) \cap \alpha(T)$*.*

### 2.2.2 The Permission value

In a TU-game with a permission structure $(N,v,P) \in \mathbb{GP}$, the characteristic function can be combined with the permission structure into a new characteristic function. This new function, describes all possibilities open to the players embedded in $P$, given their potential marginal contributions to coalitions as described by the game $v$. The resulting characteristic function is called the restriction of $v$ by $P$ and is denoted by $\mathcal{R}_P(v)$. Take any $(N,v,P) \in \mathbb{GP}$. The restriction of $v$ by $P$ is defined as

$$\mathcal{R}_P(v) = v \circ \sigma. \tag{2.17}$$

The restriction $\mathcal{R}_P$ is linear. Indeed, for $\lambda \in \mathbb{R}$ and $(N,v),(N,w) \in \mathbb{G}$,

$$\begin{aligned}\mathcal{R}_P(\lambda v + w) &= (\lambda v + w)(\sigma(E)) \\ &= \lambda v(\sigma(E)) + w(\sigma(E)) \\ &= \lambda \mathcal{R}_P(v) + \mathcal{R}_P(w).\end{aligned}$$

**Proposition 5.** *Take any* $(N, v, P) \in \mathbb{GP}$. *For each,* $E \in 2^N, E \neq \emptyset$,

$$\mathcal{R}_P(u_E) = u_{\alpha(E)}. \tag{2.18}$$

*Proof.* Take any $T \in 2^N$, and let us show that $\mathcal{R}_P(u_E)(T) = u_{\alpha(E)}(T)$. One has

$$\begin{aligned}\mathcal{R}_P(u_E)(T) &= u_E(\sigma(T)) \\ &= \begin{cases} 1 & \sigma(T) \supseteq E, \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Let us show that

$$E \subseteq \sigma(T) \iff \alpha(E) \subseteq T.$$

First, since $\alpha(E) = \bigcap_{\{T \in \mathrm{A}_P : E \subseteq T\}} T$,

$$E \subseteq \sigma(T) \implies \alpha(E) \subseteq \sigma(T).$$

Then, since $\sigma(T) \subseteq T$,

$$E \subseteq \sigma(T) \implies \alpha(E) \subseteq T.$$

Second, since $\sigma(T) = \bigcup_{\{E \in \mathrm{A}_P : E \subseteq T\}} E$,

$$\alpha(E) \subseteq T \implies \alpha(E) \subseteq \sigma(T).$$

However, since $E \subseteq \alpha(E)$, it holds that $E \subseteq \sigma(T)$. This shows that

$$E \subseteq \sigma(T) \iff \alpha(E) \subseteq T. \tag{2.19}$$

By (2.19), for each $T \in 2^N$,

$$\mathcal{R}_P(u_E)(T) = \begin{cases} 1 & \alpha(E) \subseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

This is equivalent to say that $\mathcal{R}_P(u_E) = u_{\alpha(E)}$. ■

From the linearity of $\mathcal{R}_P(v) : 2^N \to \mathbb{R}$ and Proposition 5, we provide an alternative expression of the conjunctive restriction of $v$ by $P$.

$$\begin{aligned}\mathcal{R}_P(v) &= \mathcal{R}_P\left(\sum_{E \in 2^N} \Delta_v(E) u_E\right) \\ &= \sum_{E \in 2^N} \Delta_v(E) \mathcal{R}_P(u_E) \\ &= \sum_{E \in 2^N} \Delta_v(E) u_{\alpha(E)}. \end{aligned} \tag{2.20}$$

The next result provides some inheritance properties from $(N, v)$ to $(N, \mathcal{R}_P(v))$, which are proved in Gilles *et al.* (1992).

**Proposition 6 (Gilles *et al.* (1992)).** *Take any* $(N, v, P) \in \mathbb{GP}$.

- *If* $(N, v)$ *is monotonic, then* $(N, \mathcal{R}_P(v))$ *is monotonic.*
- *If* $(N, v)$ *is super-modular, then* $(N, \mathcal{R}_P(v))$ *is super-modular.*

The (conjunctive) permission value, introduced by van den Brink & Gilles (1996), of a TU-game with a permission structure $(N, v, P) \in \mathbb{GP}$ is computed as the Shapley value of $(N, \mathcal{R}_P(v)) \in \mathbb{G}$.

**Definition 10 (Permission value, van den Brink & Gilles (1996)).** *The Permission value* $Pm$ *is defined, for each* $(N, v, P) \in \mathbb{GP}$, *as*

$$Pm(N, v, S) = Sh(N, \mathcal{R}_P(v)). \tag{2.21}$$

If one considers a TU-game $(N, v, P^0) \in \mathbb{GP}$ with the trivial permission structure $P^0$, then it is easy to see that $\sigma(E) = E$ for each $E \in 2^N$, and thus $\mathcal{R}_{P^0}(v) = v$. Thus, the conjunctive permission value is a generalization of the Shapley value for TU-games. One can express the conjunctive permission value in terms of Harsanyi dividends.

Take any $(N, v, P) \in \mathbb{GP}$. For each $i \in N$ and each $E \in 2^N$, by (2.9), it holds that

$$Sh_i(N, u_{\alpha(E)}) = \begin{cases} \dfrac{1}{|\alpha(E)|} & i \in \alpha(E), \\ 0 & \text{otherwise.} \end{cases}$$

Since the Shapley value satisfies Additivity, it follows that, for each $i \in N$,

$$\begin{aligned} Sh_i(N, \mathcal{R}_P(v)) &= \sum_{E \in 2^N} \Delta_v(E) Sh_i(N, u_{\alpha(E)}) \\ &= \sum_{\substack{E \in 2^N \\ \alpha(E) \ni i}} \frac{\Delta_v(E)}{|\alpha(E)|}. \end{aligned}$$

Thus, the Permission value $Pm$ can be re-written, for each $(N, v, P) \in \mathbb{GP}$, as

$$\forall i \in N, \quad Pm_i(N, v, S) = \sum_{\substack{E \in 2^N \\ \alpha(E) \ni i}} \frac{\Delta_v(E)}{|\alpha(E)|}.$$

This value distributes the surplus generated by each coalition equally among the authorizing part of the coalition.

### 2.2.3 Axiomatic characterizations

This section introduces two axiomatic characterizations of the permission value. The first characterization works on the class of TU-games with an acyclic permission structure, whereas the second axiomatic characterization works on the class of TU-games with a permission tree structure. Consider a value $f$ for TU-games on $\mathbb{GP}$.

In a TU-game with a permission structure, an **inessential player** is a null player such that each of its subordinates in the permission structure is also a null player. For instance, consider a firm in which the hierarchical organization of the employees is represented by a permission structure. An inessential player can be viewed as an employee going on strike with all its subordinates. This group of strikers no longer makes any contribution to the firm. Clearly, an inessential player has null marginal contributions to coalitions, but can also be viewed as responsible for its subordinates' null marginal contributions. It seems reasonable to penalize such an inessential player. van den Brink & Gilles (1996) introduces an axiom requiring that an inessential players should obtain a null payoff.

**Axiom 16 (Inessential player property).** *For each* $(N, v, P) \in \mathbb{GP}$ *and each inessential player* $i \in N$*,*

$$f_i(N, v, P) = 0. \tag{2.22}$$

If a value satisfies the Inessential player property, then it satisfies the Null player property.

$$\text{Inessential player property} \implies \text{Null player property}$$

Many economic decision situations may be described by monotonic TU-games. Since the total amount to be shared is non-negative in a monotonic game, and the players are part of a permission structure it seems reasonable to require that a player $i \in N$ obtains at least as much as any of its successors in the permission structure. For instance, it seems natural that an executive, in a firm, earns more than any of its subordinates if the firm produces good results. The class of monotonic TU-games with an acyclic permission structure is denoted by $\mathbb{G}_m\mathbb{P}$.

**Axiom 17 (Structural monotonicity).** *For each* $(N, v, P) \in \mathbb{G}_m\mathbb{P}$ *and each player* $i \in N$ *such that* $P(i) \neq \emptyset$*,*

$$f_i(N, v, P) \geq \max_{j \in P(i)} f_j(N, v, P).$$

Suppose that a player is veto for any coalition to obtain any positive payoff in a monotone TU-game. Then, regardless of its position in the permission structure, this player can always guarantee that the other players earn nothing by refusing any cooperation. Therefore it seems fair that such a veto player gets at least as much as any other player.

**Axiom 18 (Veto player dominance).** *For each* $(N, v, P) \in \mathbb{G}_m\mathbb{P}$ *and each veto player* $i \in N$*,*

$$\forall j \in N, \quad f_i(N, v, P) \geq f_j(N, v, P).$$

**Theorem 10 (van den Brink & Gilles (1996)).** *A value* $f$ *on* $\mathbb{GP}$ *satisfies Efficiency, Additivity, the Inessential player property, Structural monotonicity and Veto player dominance if and only if* $f = Pm$*.*

Just like the Shapley value, it is possible to axiomatically characterize the Permission value without relying on Additivity on the sub-class $\mathbb{GP}_T$. Take any two players $i, j \in N$ such that $j \in P(i)$. Define the associated game $(v_{-ij}, P)$ in which $i$ is enforcing its power over $j$ in the sense that it does not allow player $j$ and all its subordinates to cooperate. This game is formally defined as

$$\forall E \in 2^N \setminus \{\emptyset\}, \quad v_{-ij}(E) = v(E \setminus (\hat{P}(j) \cup \{j\})).$$

In $(N, v_{-ij}, P)$, all subordinates of $i$ are null players. If a player $i \in N$ enforces its power over player $j \in P(i)$, the change in payoffs of player $i$ is equal to the change in payoff to the player $j$.

**Axiom 19 (Permission fairness).** *For each* $(N, v, P) \in \mathbb{GP}_T$ *and each* $\{i, j\} \in 2^N$ *such that* $j \in P(i)$*,*

$$f_i(N, v, P) - f_i(v_{-ij}, P) = f_j(N, v, P) - f_j(v_{-ij}, P). \tag{2.23}$$

**Theorem 11 (van den Brink *et al.* (2015))).** *A value* $f$ *on* $\mathbb{GP}_T$ *satisfies Efficiency, the Inessential player property and Permission fairness if and only if* $f = Pm$*.*

For further details on the axiomatic characterizations of the permission value we refer to van den Brink & Gilles (1996), van den Brink (1997), van den Brink *et al.* (2015), van den Brink (2017), and van den Brink *et al.* (2018).

## 2.3 TU-games with a priority structure

TU-games with a priority structure were originally introduced by Béal *et al.* (2021a). A priority structure can be viewed as a hierarchical organization of the players where some players have priority over other players. In the introduction, we mentioned that priority structures can be modeled by a directed graph. We adopt an equivalent approach and model priority structure by a partial order on the player set to make the distinction with permission structures clearer. Béal *et al.* (2021a) introduce the Priority value for TU-games with a priority structure. This value shares the Harsanyi dividend of each coalition equally among the subset of its members over whom no

other player in the coalition has priority. Contrary to the previous model where the permission structure influences the evaluation of the coalitions through a restricted game, the priority structure influences the allocation process. This section describes the model of TU-games with a priority structure, it introduces the Priority value, and provides two axiomatic characterizations of this value.

### 2.3.1 The model

Take any $N \subset \mathbb{N}$. A binary relation on $N$ is some relation $\succeq$ where, for each $i, j \in N$, the statement $i \succeq j$ is either true of false.

**Definition 11.** *A binary relation $\succeq$ on $N \subset \mathbb{N}$ is*

- *reflexive if $i \succeq i$ for each $i \in N$;*
- *antisymmetric if $i \succeq j$ and $j \succeq i$ then $i = j$ for each $i, j \in N$;*
- *transitive if $i \succeq j$ and $j \succeq k$ then $i \succeq k$ for each $i, j, k \in N$;*
- *complete if $i \succeq j$ or $j \succeq i$ for each $i, j \in N$.*

A **partially ordered set**, or simply a poset, on $N$ is a reflexive, antisymmetric and transitive binary relation on $N$. A priority structure on $N$ can be represented by a poset $(N, \succeq)$ on the player set $N$. The relation $i \succeq j$ means that $i$ has priority over $j$. The poset $(N, \succeq^0)$ containing no priority relations among pair of distinct players is called the trivial poset. If $\succeq$ is a complete binary relation, then the priority structure is said to be **linear**. The class of all posets on a finite set of players is denoted by $\mathbb{S}$. The class of all linear posets on a finite set of players is denoted by $\mathbb{S}_L$.

A poset $(N, \succeq)$ gives rise to the asymmetric binary relation $(N, \succ)$: $i \succ j$ if $i \succeq j$ and $i \neq j$. Two distinct players $i$ and $j$ are **incomparable** in $(N, \succeq)$ if neither $i \succeq j$ nor $j \succeq i$. For each player $i \in N$, define the **priority group** of $i$, denoted by $\uparrow_{\succ} i$, as the set of players having priority over $i$ in $(N, \succeq)$

$$\uparrow_{\succ} i = \{j \in N : j \succ i\},$$

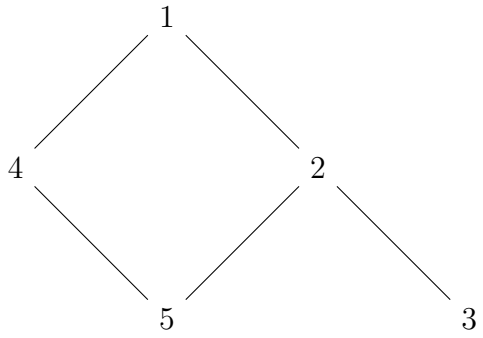
and the set players over which $i$ has priority in $(N, \succeq)$ as

$$\downarrow_{\succ} i = \{j \in N : i \succ j\}.$$

For each nonempty $E \in 2^N$, the **sub-poset** $(E, \succeq^E)$ of $(N, \succeq)$ induced by $E$ is defined as follows: for each $i \in E$ and $j \in E$, $i \succeq^E j$ if $i \succeq j$. We will also use the notation $(E, \succeq)$ instead of $(E, \succeq^E)$. A player $i \in N$ is a priority player in $(E, \succeq)$ if, for $j \in E$, the relation $j \succeq i$ implies $i = j$. The non-empty subset of **priority players** in $(E, \succeq)$ is defined as

$$Z(E, \succeq) = \{i \in E : \forall j \in E \text{ such that } j \succeq i, j = i\}.$$

**Example 3.** *Consider the set of player* $N = \{1, 2, 3, 4, 5\}$. *Consider the priority structure described by the following directed graph*




*The set of priority players of the coalition* $\{2, 3, 4, 5\}$ *is given by*

$$T(\{2, 3, 4, 5\}, \succeq) = \{2, 4\}.$$

*The only priority player in* $N$ *is* $T(N, \succeq) = \{1\}$. *The priority group of player* 5 *is given by*

$$\uparrow_{\succ} \{5\} = \{1, 2, 4\}.$$

*The set of players over which player* 2 *has priority is given by*

$$\downarrow_{\succ} \{2\} = \{3, 5\}.$$

A TU-game with a priority structure is a triplet $(N, v, \succeq)$ such that $(N, v) \in \mathbb{G}$ and $(N, \succeq) \in \mathbb{S}$. The class of TU-games with a priority structure is denoted by $\mathbb{GS}$. Béal *et al.* (2021a) introduce the Priority value on $\mathbb{GS}$. This value divides the Harsanyi dividend of each coalition among the priority players of this coalition.

**Definition 12 (Priority value, Béal *et al.* (2021a)).** *The Priority value* $Pr$ *is defined, for each* $(N, v, \succeq) \in \mathbb{GS}$, *as*

$$\forall i \in N, \quad Pr_i(N, v, \succeq) = \sum_{\substack{E \in 2^N \\ Z(E, \succeq) \ni i}} \frac{\Delta_v(E)}{|Z(E, \succeq)|}. \tag{2.24}$$

The Priority value extends the Shapley value (Shapley (1953)). Indeed the two values coincide when $(N, \succeq)$ is the trivial poset.

### 2.3.2 Axiomatic Characterizations

This section presents the two axiomatic characterizations of the Priority value introduced by Béal *et al.* (2021a). Let $f$ be an allocation rule for TU-games with a priority structure.

If a null player leaves a game, it seems natural that the payoff of each remaining player should not get impacted. Derks & Haller (1999) formalize this idea via the following axiom.

**Axiom 20 (Null player out).** *For each* $(N, v, \succeq) \in \mathbb{GS}$ *and each* $j \in N$ *such that* $j$ *is a null player in* $(N, v, \succeq)$*,*

$$\forall i \in N \setminus j, \quad f_i(N, v, \succeq) = f_i(N \setminus j, v, \succeq).$$

If a value satisfies Efficiency and Null player out, then it satisfies the Null player property.

$$\text{Efficiency + Null player out} \implies \text{Null player property}$$

The next axiom is a weak version of the Veto players property. It requires that two veto players with the same priority group are treated equally, no matter over which set of players they have priority.

**Axiom 21 (Veto players with equal priority group property).** *For each* $(N, v, \succeq) \in \mathbb{GS}$ *and each* $i, j \in N$ *such that* $i$ *and* $j$ *are veto players and* $\uparrow_{\succ i} = \uparrow_{\succ j}$*,*

$$f_i(N, v, \succeq) = f_j(N, v, \succeq).$$

If a value satisfies the Veto players property, then it satisfies the Veto players with equal priority group property.

$$\text{Veto players prop} \implies \text{Veto players with equal priority group prop}$$

If a player is dominated by a veto player, its payoff possibilities are doubly blocked by the fact that another player both has priority over it and vetoes any creation of worth.

**Axiom 22 (Veto and priority player property).** *For each* $(N, v, \succeq) \in \mathbb{GS}$ *and each* $j \in N$ *such that* $j$ *is a veto player,*

$$\forall i \in \downarrow_{\succ j}, \quad f_i(N, v, \succeq) = 0.$$

The next axiom expresses an idea close to the Veto and priority player property. It states that removing a player from both the game and the priority structure does not affect the payoff of the players over which this player has priority.

**Axiom 23 (Priority player out).** *For each* $(N, v, \succeq) \in \mathbb{GS}$ *and each* $j \in N$*,*

$$\forall i \in \downarrow_{\succ j}, \quad f_i(N, v, \succeq) = f_i(N \setminus \{j\}, v, \succeq).$$

The next axiom indicates that adding a priority relation between two incomparable players $i$ and $j$, in the sense that $i$ has now priority over $j$, does not affect the players payoffs, including $j$, if player $i$ and the players in the priority group of $i$ are null players. This means that a player does not care that a group of unproductive players have priority over it, and that this local change in the priority structure

should have no impact on the other players. In a sense here, the lack of productivity takes precedence over priorities.

Given a poset $(N, \succeq)$ and two distinct players $i, j \in N$ such that $j \in N \setminus \uparrow_{\succ i}$, define the poset $(N, \succeq_{i\to j})$ as follows

$$\forall k, l \in N, \quad l \succeq_{i\to j} k \iff \begin{cases} l \in \{i\} \cup (\uparrow_{\succ i}) \text{ and } k \in \{j\} \cup (\downarrow_{\succ j}), \\ l \succeq k \text{ otherwise.} \end{cases}$$

The poset $(N, \succeq_{i\to j})$ is called the **elementary extension** of $(N, \succeq)$ with respect to $\{i, j\}$. In this poset, player $i$ has now priority on player $j$.

**Axiom 24 (Invariance to unproductive priority extension).** *For each* $(N, v, \succeq) \in \mathbb{GS}$ *and each* $i, j \in N$ *such that* $i$ *and each* $k \in \uparrow_{\succ i}$ *are null players in* $(N, v)$*,*

$$f(N, v, \succeq) = f(N, v, \succeq_{i\to j}).$$

**Theorem 12 (Béal *et al.* (2021a)).** *A value* $f$ *on* $\mathbb{GS}$ *satisfies Efficiency, Additivity, Priority player out, the Veto players with equal priority group property and Invariance to unproductive priority extension if and only if* $f = Pr$*.*

Another characterization of the Priority value can be obtained by substituting Priority player out and Invariance to unproductive priority extension, by Null player out and the Veto and priority player property.

**Theorem 13 (Béal *et al.* (2021a)).** *A value* $f$ *on* $\mathbb{GS}$ *satisfies Efficiency, Additivity, Null player out, the Veto players with equal priority group property and the Veto and priority player property if and only if* $f = Pr$*.*

## 2.4 Multi-choice games

Multi-choice games, introduced by Hsiao & Raghavan (1992) and van den Nouweland (1993), are a natural extension of TU-games in which each player can choose several activity levels at which it can cooperate within a coalition. This section presents the model of multi-choice games and introduces some definitions. Solution concepts and axiomatic characterization of some of these concepts are introduced.

### 2.4.1 The model

Let $N = \{1, \ldots, n\}$, be a fixed[1] **set of players** and fix an upper-bound $K \in \mathbb{N}$ for activity levels. Each player $i \in N$ has a finite set of pairwise distinct **activity levels** $M_i := \{0, \ldots, m_i\}$ such that $m_i \leq K$. For each player $i \in N$, the set $M_i$ is linearly ordered from the lowest activity level 0 ($i$ does not cooperate) to the maximal

[1]The set of players is fixed but we allow the activity levels to vary.

activity level $m_i$. Denote by $Q(j) \in 2^N$ the set of **players able** to play the activity level $j \leq K$. Formally, for each $j \leq K$, the set $Q(j)$ is defined as

$$Q(j) = \Big\{ i \in N : m_i \geq j \Big\}.$$

Without loss of generality, assume that $Q(1) = N$. Let $\mathcal{M}$ be the cartesian product $\prod_{i \in N} M_i$. Each element $s = (s_1, \ldots, s_n) \in \mathcal{M}$ specifies a participation profile for players and is referred to as a (multi-choice) coalition. So, a coalition indicates each player's activity level. Then, $m = (m_1, \ldots, m_n) \in \mathcal{M}$ is the greatest participation profile, which plays the role of the grand coalition, whereas $\vec{0} = (0, \ldots, 0)$ plays the role of the empty coalition. For each $s \in \mathcal{M}$, denote by $(s_{-i}, k)$ the coalition where all players except $i$ play at levels defined in $s$ while $i$ plays at $k \in M_i$. For each $s \in \mathcal{M}$, denote by $s^T = \max_{i \in N} s_i$ the highest activity level player by a player within $s$. The set $\mathcal{M}$ endowed with the usual binary relation $\leq$ on $\mathbb{R}^n$ induces a (complete) lattice with greatest element $m$ and least element $\vec{0}$. For any two coalitions $a, b \in \mathcal{M}$, $a \vee b$ and $a \wedge b$ denote their least upper bound and their greatest lower bound over $\mathcal{M}$, respectively. We use the notation $M_i^+ = M_i \setminus \{0\}$ for each $i \in N$ and $M^+ = \bigcup_{i \in N} \big( \{i\} \times M_i^+ \big)$. A pair $(i, j) \in M^+$ represents a player and one of its activity levels.

A (cooperative) **multi-choice game** on $N$ is a couple $(m, v)$ where $v : \mathcal{M} \longrightarrow \mathbb{R}$ is a characteristic function such that $v(\vec{0}) = 0$, that specifies the worth $v(s)$ when players participate at profile $s$. Take any $q \in \mathbb{R}^n$ such that $q_i \leq K$ for each $i \in N$. Denote by $\mathcal{G}^q$ the set of multi-choice games on $N$ with the grand coalition $q$. Denote by $\mathcal{G} = \cup_{q \leq (K, \ldots, K)} \mathcal{G}^q$ the set of all multi-choice games on $N$ with $K$ as an upper bound for activity levels. Notice that TU-games can be viewed as a subclass of multi-choice games satisfying $m = (1, \ldots, 1)$. Several sub-class of multi-choice games are identified. To that end, let us introduce a few definitions.

A multi-choice game $(m, v) \in \mathcal{G}$ is the **null game** if each coalition has a null worth. Formally, such game is defined as

$$\forall s \in \mathcal{M}, \quad v(s) = 0.$$

A multi-choice game $(m, v) \in \mathcal{G}$ is **non-negative** if each coalition has a non-negative worth. Formally, such game is defined as

$$\forall s \in \mathcal{M}, \quad v(s) \geq 0.$$

A multi-choice game $(m, v) \in \mathcal{G}$ is **monotonic** If an increase in activity levels always leads to an increase in worth. Formally, such game is defined as

$$\forall s, t \in \mathcal{M} : s \leq t, \quad v(s) \leq v(t).$$

The class of monotonic multi-choice games is denoted by $\mathcal{G}_m \subseteq \mathcal{G}$. A null game is a non-negative game and a monotonic game is a non-negative game due to the fact that $v(\vec{0}) = 0$.

A multi-choice game $(m, v) \in \mathcal{G}$ is **super-modular** if

$$\forall s, t \in \mathcal{M}, \quad v(s \vee t) + v(s \wedge t) \geq v(s) + v(t).$$

On the contrary, a multi-choice game $(m, v) \in \mathcal{G}$ is **sub-modular** if

$$\forall s, t \in \mathcal{M}, \quad v(s \vee t) + v(s \wedge t) \leq v(s) + v(t).$$

The classes of super-modular and sub-modular multi-choice games are denoted by $\mathcal{G}^{spm} \subseteq \mathcal{G}$ and $\mathcal{G}^{sbm} \subseteq \mathcal{G}$ respectively.

Take any $(m, v) \in \mathcal{G}$ and any coalition $s \in \mathcal{M}$ such that $s_i = j-1$. The **marginal contribution** of player $i$ for its activity level $j$ (or simply the marginal contribution of the pair $(i, j)$) to the coalition $s$ is defined as

$$v(s + e_i) - v(s).$$

This corresponds to the variation in worth incurred by coalition $s$ following an increase in activity of player $i$ by one unit.

A pair $(i, j) \in M^+$ is a **weakly null pair** in $(m, v) \in \mathcal{G}$ if it has null marginal contribution to coalitions.

A pair $(i, j) \in M^+$ is a **null pair** in $(m, v) \in \mathcal{G}$ if the marginal contribution of each pair $(i, j')$ such that $j \leq j' \leq m_i$, to each coalition is null. Put differently, $(i, j)$ is a null pair if player $i$ generates no worth from its activity level $j$ up to its maximal activity level $m_i$. Formally, $(i, j) \in M^+$ is a null pair if

$$\forall s \in \mathcal{M}, \forall j \leq l \leq m_i, \quad v(s_{-i}, l) = v(s_{-i}, j-1). \tag{2.25}$$

In the null game, each pair is a null pair. Moreover, a null pair is also a weakly null pair. The converse is not true. Two distinct pairs featuring the same activity level are **equal** if they have the same marginal contribution to coalitions. Formally, $(i, j), (i', j) \in M^+$ are equal pairs if

$$\forall s \in \mathcal{M} : s_i = s_{i'} = j-1, \quad v(s + e_i) = v(s + e_{i'}). \tag{2.26}$$

Observe that two null pairs in a game are equal.

A pair $(i, j) \in M^+$ is a **veto pair** in $(m, v) \in \mathcal{G}$ If any coalition in which player $i$ plays strictly less than $j$ has a null worth. Formally, $(i, j) \in M^+$ is a veto pair if

$$\forall s \in \mathcal{M} : s_i < j, \quad v(s) = 0. \tag{2.27}$$

By definition, if $(i, j)$ is a veto pair, then each pair $(i, j')$ such that $j' < j$, is also a veto pair. Two distinct veto pairs featuring the same activity level are equal pairs.

Among the veto pairs, we identify the decisive pairs. If there exist several veto pairs featuring the same player, then the pair featuring the highest activity level among these pairs is called a **decisive pair**. Formally, a pair $(i, j) \in M^+$ is a

decisive pair in $(m, v) \in \mathcal{G}$ if it is a veto pair and there is at least one coalition $s$ with non-null worth and such that $s_i = j$. Formally, $(i, j) \in M^+$ is a decisive pair if

$$\begin{aligned} &\forall s \in \mathcal{M} : s_i < j, v(s) = 0 \quad \text{and} \\ &\exists t \in \mathcal{M} : t_i = j, v(t) \neq 0. \end{aligned} \tag{2.28}$$

For $s \in \mathcal{M}$, we introduce the set of **top pairs** $T(s)$ containing players playing the highest activity levels in $s$. Formally, the set of top pairs in $s$ is defined as

$$\forall s \in \mathcal{M}, \quad T(s) = \Big\{(i, s_i) \in M^+ : s_i \geq s_k, \ \forall k \in N\Big\}. \tag{2.29}$$

For $(m, v) \in \mathcal{G}$, define the sub-game $(t, v^t) \in \mathcal{G}$, induced by $t \in \mathcal{M}$, as

$$\forall s \in \mathcal{M}, \quad v^t(s) = \begin{cases} v(s) & \text{if } s \leq t, \\ 0 & \text{otherwise.} \end{cases} \tag{2.30}$$

When no confusion arises, we simply denote the sub-game $(t, v^t)$ of $(m, v)$ by $(t, v)$. The sub-game $(t, v)$ corresponds to a cooperative situation in which each player $i \in N$ can play at most the level $t_i$, where $t_i \leq m_i$. In other words, this describes a situation where the maximal activity level of some players have been reduced.

Take any $t \in \mathcal{M}$, $t \neq \vec{0}$. An analogue of an unanimity TU-game in the multi-choice setting is the concept of **minimal effort game** $(m, u_t) \in \mathcal{G}$ defined as

$$\forall s \in \mathcal{M}, \quad u_t(s) = \begin{cases} 1 & \text{if } s \geq t, \\ 0 & \text{otherwise.} \end{cases} \tag{2.31}$$

In a minimal effort game $(m, u_t)$, each player needs to achieve at least the activity level specified by coalition $t$ to generate a non-null worth. Observe that any game $(s, u_t) \in \mathcal{G}$ such that $s < t$, is the null game.

**Remark 1.** *Take any $t \in \mathcal{M}$, $t \neq \vec{0}$. Each pair $(i, j) \in M^+$ such that $j > t_i$, is a null pair in $(m, u_t)$. Let $(i, j), (i', j) \in M^+$ be two distinct pairs such that $j \leq t_i$ and $j \leq t_{i'}$. Both pairs are decisive in $(m, u_t)$.*

For each $q \leq (K, \ldots, K)$, Hsiao & Raghavan (1992) show that $\{u_t\}_{t \leq q}$ are linearly independent and thus constitute a basis for $\mathcal{G}^q \subseteq \mathcal{G}$. Therefore, for each $(m, v) \in \mathcal{G}^q$,

$$v = \sum_{t \leq m} \Delta_v(t) u_t, \quad \text{where} \quad \Delta_v(t) = v(t) - \sum_{s \leq t, s \neq t} \Delta_v(s). \tag{2.32}$$

For each $t \in \mathcal{M}$, $\Delta_v(t)$ is the (multi-choice) Harsanyi dividend of $t$. Similarly to TU-games, the Harsanyi dividend of a multi-choice coalition can be viewed as the net surplus generated by a coalition.

Consider $t \in \mathcal{M}$, $t \neq \vec{0}$. The **Dirac game** $(m, \delta_t) \in \mathcal{G}$, induced by $t$, is defined as

$$\forall s \in \mathcal{M}, \quad \delta_t(s) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{otherwise.} \end{cases} \tag{2.33}$$

**Remark 2.** *Take any $t \in \mathcal{M}$, $t \neq \vec{0}$. Each pair $(i,j) \in M^+$ such that $j > t_i + 1$ is a null pair in $(m, \delta_t)$. If there exists two distinct players $i, i' \in N$ such that $t_i = t_{i'}$, then $(i, t_i)$ and $(i', t_{i'})$ are equal in $(m, \delta_t)$.*

For each multi-choice game $(m, v) \in \mathcal{G}$, the characteristic function $v$ admits a unique linear decomposition in terms of Dirac games as follows

$$v = \sum_{t \leq m} v(t)\delta_t. \tag{2.34}$$

### 2.4.2 Solution concepts

Solution concepts for multi-choice games differs from solution concepts from TU-games. The later allocates a collection of payoffs to each player, whereas the former allocates a collection of payoffs to each activity level of each player. Therefore, in multi-choice games, a player receives as many collection of payoffs as it has activity levels. A **payoff vector** for a game $(m, v) \in \mathcal{G}$ is an element $x \in \mathbb{R}^{|M^+|}$, where $x_{ij} \in \mathbb{R}$ is the payoff received by the pair $(i,j) \in M^+$. A set-valued solution on $\mathcal{G}$ is a map $F$ that assigns a collection of payoff vectors $F(m,v)$ to each $(m,v) \in \mathcal{G}$. A **value** $f$ is a single-valued solution on $\mathcal{G}$ that assigns a unique payoff vector $f(m,v)$ to each $(m,v) \in \mathcal{G}$. This section presents single-valued solution and set-valued solutions for multi-choice games. Regarding single-valued solutions, extensions of the Shapley value and the Equal division value from TU-games to multi-choice games are presented. As for set-valued solutions, we present extensions of the Core and the Weber set from TU-games to multi-choice games.

#### Extensions of the Shapley value

This section presents three extensions of the Shapley value from TU-games to multi-choice games. These are the extensions proposed by Derks & Peters (1993), van den Nouweland *et al.* (1995) and Peters & Zank (2005) respectively.[2] Each extension admits an expression in terms of marginal contributions and in terms of Harsanyi dividends.

A first approach is proposed by Derks & Peters (1993) and studied by Klijn *et al.* (1999). An order over $M^+$ can be represented by a map

$$\theta_{DP} : M^+ \to \{1, \ldots, \sum_{i \in N} m_i\}. \tag{2.35}$$

The integer $\theta_{DP}(i,j)$ indicates the position at which the pair $(i,j)$ is ordered in $\theta_{DP}$. Denote by $\Theta_{DP}$ the set of all orders over $M^+$ whose cardinality is given by $(\sum_{i \in N} m_i)!$. The subset $\theta_{DP}^{-1}(\{1, \ldots, k\})$ of $M^+$, which is the subset of pairs ordered

[2] Additional extensions of the Shapley value from TU-games to multi-choice games can be found in Hsiao & Raghavan (1992) and Grabisch & Lange (2007).

up to $k$ steps, is denoted by $S^{\theta_{DP},k}$. Additionally, define the map $\sigma$ that assigns to each subset $S \subseteq M^+$ the maximal feasible coalition $\sigma(S) \in \mathcal{M}$. Formally, this map is defined, for $S \subseteq M^+$, as

$$\begin{aligned} &\sigma(S) = (t_1, \ldots, t_n), \\ \text{where} \quad &t_i = \begin{cases} \max\{l \in M_i^+ : (i,1), \ldots, (i,l) \in S\} & \text{if } (i,1) \in S, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

In other words, $t_i$ is the highest level that can be observed for a pair in $S$ featuring player $i$, given that all the pairs featuring player $i$ from $(i,1)$ to $(i,t_i-1)$ are also in $S$. The marginal vector $\eta^{\theta_{DP}}(m,v)$ corresponding to $\theta_{DP} \in \Theta_{DP}$ is defined as

$$\forall (i,j) \in M^+, \quad \eta_{ij}^{\theta_{DP}}(m,v) = v(\sigma(S^{\theta_{DP},\theta_{DP}(i,j)})) - v(\sigma(S^{\theta_{DP},\theta_{DP}(i,j)-1})).$$

Observe that $v \circ \sigma$ can be interpreted as the restriction of $v$ by $\sigma$. Then, each $\eta_{ij}^{\theta_{DP}}(m,v)$ represents the marginal contribution of $(i,j)$ to the coalition $\sigma(S^{\theta_{DP},\theta_{DP}(i,j)-1})$. The Derks Peters value, or simply the $DP$ value, proposed by Derks & Peters (1993), assigns to each pair its expected marginal contribution to the restriction of $v$, by assuming that each order $\theta_{DP} \in \Theta_{DP}$ over the set of pairs occurs with equal probability.

**Definition 13 (Derks & Peters (1993)).** *For each $(m,v) \in \mathcal{G}$, the $DP$ value is defined as*

$$\forall (i,j) \in M^+, \quad DP_{ij}(m,v) = \frac{1}{(\sum_{i \in N} m_i)!} \sum_{\theta_{DP} \in \Theta_{DP}} \eta_{ij}^{\theta_{DP}}(m,v). \tag{2.36}$$

The $DP$ value admits an expression in terms of Harsanyi dividends. Define the set

$$\forall s \in \mathcal{M}, \quad B(s) = \{(i,j) \in M^+ : j \leq s_i\}$$

as an interpretation of the **support** of a coalition $s$. For each $(m,v) \in \mathcal{G}$, an alternative expression of the $DP$ value is given by

$$\forall (i,j) \in M^+, \quad DP_{ij}(m,v) = \sum_{\substack{s \leq m \\ (i,j) \in B(s)}} \frac{\Delta_v(s)}{|B(s)|}. \tag{2.37}$$

The $DP$ value allocates to each pair $(i,j) \in M^+$ a share of the dividend of each coalition $s$ such that $(i,j)$ is in the support of $s$.

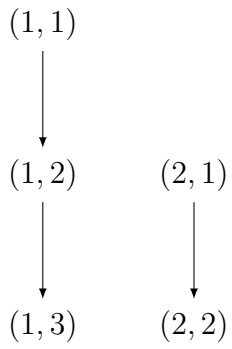
**Remark 3.** *Take any $(m,v) \in \mathcal{G}$. Let us interpret $M^+$ as a set of players, meaning that each pair is viewed as a player. Take any characteristic function $w : 2^{M^+} \to \mathbb{R}$ satisfying*

$$\forall S \subseteq M^+, \quad w(S) = v(\sigma(S)).$$

*The couple $(M^+, w)$ forms a TU-game. Observe that $w(M^+) = v(m)$. There exists a unique permission structure over $M^+$, denoted by $(M^+, P)$, which captures the linear order of the activity levels. This permission structure is such that each pair $(i, j) \in M^+$, $j > 1$, needs permission from the pairs $(i, j-1), (i, j-2), \ldots, (i, 1) \in M^+$. For instance, take*

$$M^+ = \{(1,1), (1,2), (1,3), (2,1), (2,2)\}.$$

*The permission structure over $M^+$ can be illustrated by*




*In this structure, the pair $(1, 2)$ needs the permission from $(1, 1)$, $(1, 3)$ needs the permission from $(1, 2)$ and $(1, 1)$. The DP value of a multi-choice game $(m, v) \in \mathcal{G}$ can be viewed as the Permission value of the TU-game with a permission structure $(M^+, w, P)$.*

Another approach is proposed by van den Nouweland *et al.* (1995). The authors suppose that the grand coalition forms step by step, starting from the empty coalition and where in each step, the level of one of the player increases by one unit. Compared to Derks & Peters (1993) and Klijn *et al.* (1999), van den Nouweland *et al.* (1995) consider the subset of **admissible orders** on $M^+$ such that no pair $(i, j) \in M^+$ is ordered before a pair $(i, l)$ featuring the same player and such that $l < j$. Formally, an admissible order for the game $(m, v)$ is a map $\theta_{vdN} : M^+ \to \{1, \ldots, \sum_{i \in N} m_i\}$ such that $\theta_{vdN}(i, j-1) < \theta_{vdN}(i, j)$ for each $(i, j) \in M^+$. Denote by $\Theta_{vdN} \subseteq \Theta_{DP}$ the set of all admissible orders on $M^+$ whose cardinality is given by

$$\frac{(\sum_{i \in N} m_i)!}{\prod_{i \in N}(m_i!)}.$$

Let $\theta_{vdN} \in \Theta_{vdN}$ be an admissible order and $h \in \{1, \ldots, \sum_{i \in N} m_i\}$. We denote by $s^{\theta_{vdN}, h}$ the coalition formed after step $h$. Formally, it is defined as

$$\forall i \in N, \quad s_i^{\theta_{vdN}, h} = \max\{j \in M_i : \theta_{vdN}(i, j) \leq h\} \cup \{0\}.$$

The marginal vector $\eta^{\theta_{vdN}}(m, v)$ corresponding to $\theta_{vdN} \in \Theta_{vdN}$ is defined as

$$\forall (i, j) \in M^+, \quad \eta_{ij}^{\theta_{vdN}}(m, v) = v(s^{\theta_{vdN}, \theta_{vdN}(i,j)}) - v(s^{\theta_{vdN}, \theta_{vdN}(i,j)-1}). \tag{2.38}$$

Each $\eta^{\theta_{vdN}}(m,v)$ represents the marginal contribution of $(i,j)$ to the coalition $s^{\theta_{vdN},\theta_{vdN}(i,j)-1}$. The value $vdN$ proposed by van den Nouweland *et al.* (1995) assigns to each pair its expected marginal contribution assuming that all admissible orders over the set of pairs occur with equal probability. Contrary to the $DP$ value, the game is not restricted by the operator $\sigma$. Instead, the set of orders is reduced to admissible orders in $\Theta_{vdN}$.

**Definition 14 (van den Nouweland (1993)).** *For each $(m,v) \in \mathcal{G}$, the vdN value is defined as*

$$\forall (i,j) \in M^+, \quad vdN_{ij}(m,v) = \frac{\prod_{i\in N}(m_i!)}{(\sum_{i\in N} m_i)!} \sum_{\theta_{vdN}\in\Theta_{vdN}} \eta_{ij}^{\theta_{vdN}}(m,v).$$

The $vdN$ value admits an expression in terms of Harsanyi dividends provided by Faigle & Kern (1992). Define $h_{ij}(s)$ as the **hierarchical strength** of a pair $(i,j) \in M^+$ in the coalition $s$, $s_i = j$. The hierarchical strength of $(i,j)$ in $s$ represents the average number of admissible orders such that all the pairs in $B(s)$, except $(i,j)$, are ordered prior to step $|B(s)|-1$ and $(i,j)$ is ordered last among the pairs in $B(s)$. Formally, the hierarchical strength of a pair $(i,j) \in M^+$ in a coalition $s$, in which $s_i = j$, is defined as

$$h_{ij}(s) = \frac{1}{|\Theta_{vdN}|} |\{\theta_{vdN} \in \Theta_{vdN} : \theta_{vdN}(i,j) = |B(s)|\}|.$$

In addition, define

$$\forall s \in \mathcal{M}, \quad C(s) = \{(i,j) \in M^+ : j = s_i\} \tag{2.39}$$

as an interpretation of the **support** of coalition $s$. Observe that $C(s) \subseteq B(s)$. For each $(m,v) \in \mathcal{G}$, an alternative expression of the $vdN$ value is given by

$$\forall (i,j) \in M^+, \quad vdN_{ij}(m,v) = \sum_{\substack{s\leq m \\ (i,j)\in C(s)}} h_{ij}(s)\Delta_v(s). \tag{2.40}$$

This value allocates to each pair a share of a dividend $\Delta_v(s)$ proportional to its hierarchical strength.

Finally, we consider the value proposed by Peters & Zank (2005). We refer to this value as the $PZ$ value. The PZ value allows for a probabilistic interpretation closely related to the Shapley value. Consider $(i,j) \in M^+$. The $PZ$ value of $(i,j)$ is computed as follows. Assume that the coalition $(m_{-i}, j)$ is being formed in $n$ steps starting from the coalition $(\vec{0}_{-i}, j-1)$, with respect to a certain order over the player set. Let $h$, $h \leq n$, be the step at which $i$ is ordered. At each step $t \neq h$, a player $k \neq i$ is ordered and immediately plays its maximum activity level. At step $h$, player $i$ is ordered and increases its activity level from $j-1$ to $j$. The $PZ$ value assigns

to $(i,j)$ its expected marginal contribution assuming that each order over the player set occurs with equal probability.

Formally, define $Y_{ij}$ the set of coalitions in which $i$ plays $j-1$ while the other players play either 0 or their maximal activity level. Formally, it is defined as

$$\forall (i,j) \in M^+, \quad Y_{ij} = \{s \in \mathcal{M} : s_i = j-1 \text{ and } s_k = \{0, m_k\} \text{ for each } k \neq i\}.$$

For each $s \in Y_{ij}$, denote by

$$z(s) = |\{k \in N : k \neq i, s_k = 0\}|$$

the number of players that do not cooperate in $s$.

**Definition 15 (Peters & Zank (2005)).** *For each $(m,v) \in \mathcal{G}$, the $PZ$ value is defined as*

$$\forall (i,j) \in M^+, \quad PZ_{ij}(m,v) = \sum_{s \in Y_{ij}} \frac{z(s)!(n-z(s)-1)!}{n!} \Big( v(s+e_i) - v(s) \Big).$$

In contrast to the $DP$ and $vdN$ values, the $PZ$ value only uses a restricted subset of information provided by the game $(m,v)$. In order to compute this value, it suffices to know the worth of those coalitions where only one player is allowed to play at an activity level different from 0 or its maximum activity level. The $PZ$ value admits an expression in terms of Harsanyi dividends. For each $(m,v) \in \mathcal{G}$, an alternative expression of the $PZ$ value is given by

$$\forall (i,j) \in M^+, \quad PZ_{ij}(m,v) = \sum_{\substack{s \leq m \\ (i,j) \in C(s)}} \frac{\Delta_v(s)}{|C(s)|}. \tag{2.41}$$

The $PZ$ value allocates to each pair $(i,j) \in M^+$ an equal share of the dividend of each coalition $s$ such that $(i,j)$ is in the support $C(s)$ of $s$. Put differently, each dividend $\Delta_v(s)$ is equally shared among the players that play a non-null activity level in $s$. This allocation of the dividend differs from Derks & Peters (1993) and van den Nouweland *et al.* (1995) since the division of the dividend does not depend on the level of activity of the players within the coalition.

A new extension of the Shapley value from TU-games to multi-choice games is introduced in Chapter 4. It turns out that this value satisfies the Independence of higher demands principle (if one interprets activity levels as demands), which is not the case of the $DP$, $vdN$ or $PZ$ values.

### Extensions of the Equal division value

This section briefly presents some possible extensions of the Equal division value from TU-games to multi-choice games. The $d$-value proposed by Branzei *et al.* (2014).

This value generalizes the constrained egalitarian solution (see Dutta & Ray (1989)) from TU-games to super-modular multi-choice games. The $d$-value is computed with an algorithm that searches for a solution as egalitarian as possible, while taking into account the activity levels of the players. Not all activity levels of the same player will receive the same payoff, but similar activity levels of different players will be treated as equally as possible.

We now present two reasonable extensions of the Equal division value from TU-games to multi-choice games. The first extension divides the worth of the grand coalition equally among the pairs in $M^+$. This value is defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i,j) \in M^+, \quad ED^1_{ij}(m,v) = \frac{v(m)}{B(m)}. \tag{2.42}$$

Obviously, the player's total payoff increases with respect to their maximal activity level. The second extension can be defined as a two-steps procedure. This value divides the worth of the grand coalition equally among the players, then the amount allocated to each player is divided equally among their activity level. This value is defined, for each $(m, v) \in \mathcal{G}$, as

$$\forall (i,j) \in M^+, \quad ED^2_{ij}(m,v) = \frac{v(m)}{n \times m_i}.$$

Another extension of the Equal division value from TU-games to multi-choice games is introduced in Chapter 4. It turns out that this value satisfies the Independence of higher demands principle (if one interprets activity levels as demands).

### Extensions of the Core and the Weber set

This section presents two extensions of the Core and the Weber set from TU-games to multi-choice games proposed by Grabisch & Xie (2007) and van den Nouweland *et al.* (1995) respectively. We discuss these solution concepts and point out that the Weber set and the Core proposed by Grabisch & Xie (2007) coincide on the class of super-modular multi-choice games.

**Definition 16 (Core, Grabisch & Xie (2007)).** *For each $(m, v) \in \mathcal{G}$, the Core of $(m, v)$ is denoted by $\mathcal{C}(m, v)$. A payoff vector $x$ is in $\mathcal{C}(m, v)$ if*

$$\begin{cases} \forall s \in \mathcal{M}, \quad \displaystyle\sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \geq v(s) & (2.43) \\ \forall h \leq \max\limits_{k \in N} m_k, \quad \displaystyle\sum_{i \in N} \sum_{j=1}^{h \wedge m_i} x_{ij} = v((h \wedge m_i)_{i \in N}). & (2.44) \end{cases}$$

The first Core condition (2.43) states that no coalition can achieve, by itself, a better outcome than the one prescribed by the payoff vectors in the Core. Observe

that, on the class of multi-choice games such that $m = (1, \ldots, 1)$, condition (2.43) coincides with the coalitional rationality condition for TU-games (2.6). Next, assume that all players agree on forming a coalition in which everyone plays the same activity level, let us say $h$. Players unable to cooperate at such level play their maximal activity level. We call such coalition an $h$-**synchronized coalition**. The second condition (2.44) states that an $h$-synchronized coalition achieves the same outcome than the one prescribed by the payoff vectors. On the class of multi-choice games such that $m = (1, \ldots, 1)$, condition (2.44) coincides with the efficiency Core condition (2.4) for TU-games.

Grabisch & Xie (2007) also propose an extension of the Weber set from TU-games to multi-choice games. To do so, let us introduce **restricted orders** over $M^+$. These orders are such that no pair $(i, j) \in M^+$ is ordered before a pair $(i', j') \in M^+$ containing a strictly lower activity level $j' < j$. Formally, a restricted order over the set of pairs is a map

$$\theta_{GX} : M^+ \to \{1, \ldots, \sum_{i \in N} m_i\} \tag{2.45}$$

defined as

$$\forall (i, j), (i', j') \in M^+, \quad \big[j < j'\big] \implies \big[\theta_{GX}(i, j) < \theta_{GX}(i', j')\big].$$

Denote by $\Theta_{GX}$ the set of all restricted orders over the set of pairs. Obviously, $\Theta_{GX} \subseteq \Theta_{vdN}$. The number of restricted orders over the set of pairs is given by

$$\prod_{j \leq \max\limits_{k \in N} m_k} |Q(j)|!$$

Let $\theta_{GX} \in \Theta_{GX}$ be a restricted order and $h \in \{1, \ldots, \sum_{i \in N} m_i\}$. We denote by $s^{\theta_{GX}, h}$ the coalition formed after step $h$. Formally, it is defined as

$$\forall i \in N, \quad s_i^{\theta_{GX}, h} = \max\big\{j \in M_i : \theta_{GX}(i, j) \leq h\big\} \cup \{0\}. \tag{2.46}$$

We use the convention $s^{\theta_{GX}, 0} = \vec{0}$. For each $\theta_{GX} \in \Theta_{GX}$, the marginal vector $\eta^{\theta_{GX}}(m, v)$ is defined as

$$\forall (i, j) \in M^+, \quad \eta_{ij}^{\theta_{GX}}(m, v) = v(s^{\theta_{GX}, \theta_{GX}(i,j)}) - v(s^{\theta_{GX}, \theta_{GX}(i,j)-1}).$$

Each $\eta_{ij}^{\theta_{GX}}(m, v)$ represents the marginal contribution of $(i, j)$ to the coalition $s^{\theta_{GX}, \theta_{GX}(i,j)-1}$ formed after $\theta_{GX}(i, j) - 1$ steps according to the restricted order $\theta_{GX}$.

**Definition 17 (Grabisch & Xie (2007)).** *For each $(m, v) \in \mathcal{G}$, the Weber set $\mathcal{W}$ of $(m, v)$ is the convex hull of all marginal vectors computed from restricted orders. This set is defined as*

$$\mathcal{W}(m, v) = co(\{\eta^{\theta_{GX}}(m, v) \,|\, \theta_{GX} \in \Theta_{GX}\}).$$

**Proposition 7 (Grabisch & Xie (2007)).** *The Weber set coincides with the Core on the class of super-modular multi-choice games.*

van den Nouweland *et al.* (1995) propose alternative extensions of the Core and the Weber set from TU-games to multi-choice games.

**Definition 18 (van den Nouweland *et al.* (1995)).** *For each $(m, v) \in \mathcal{G}$, the Core of $(m, v)$ is denoted by $C(m, v)$. A payoff vector $x$ is in $C(m, v)$ if and only if it satisfies these three conditions*

$$\begin{cases} \forall s \in \mathcal{M}, \quad \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \geq v(s) & (2.47) \\ \sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} = v(m) & (2.48) \\ \forall i \in N, \forall j \in M_i \setminus \{0\}, \quad x_{ij} \geq v(je^i) - v((j-1)e^i). & (2.49) \end{cases}$$

Condition (2.47) is the same condition than (2.43). Condition (2.48) states that the sum of the payoffs is equal to the worth of the grand coalition. Clearly, this condition extends the Efficiency condition of the Core for TU-games (see 2.4). This condition can be viewed as a particular case of condition (2.44), where $h = \max_{k \in N} m_k$. Condition (2.49) states that the payoff of any player $i$ for any of its activity level $j$ is at least equal to the increase in worth that player $i$ can obtain when it works alone and changes its activity from level $j-1$ to level $j$. This condition extends the individual rationality condition (2.5) from TU-games to multi-choice games.

**Definition 19 (van den Nouweland *et al.* (1995)).** *For each $(m, v) \in \mathcal{G}$, the Weber set* W *of $(m, v)$ is the convex hull of all marginal vectors computed from admissible orders* (2.38)*. This set is defined as*

$$\mathrm{W}(m, v) = co(\{\eta^{\theta_{vdN}}(m, v) \,|\, \theta_N \in \theta_{vdN}\}).$$

Contrary to the extensions of the Core and the Weber set proposed by Grabisch & Xie (2007), the extensions of the Core and the Weber set proposed by van den Nouweland *et al.* (1995) do not coincide on the class of super-modular multi-choice games. This is why we prefer to focus on the extensions provided by Grabisch & Xie (2007).

### 2.4.3 Axiomatic characterizations

This section presents axiomatic characterizations of the $DP$, $vdN$ and $PZ$ values for multi-choice games. Several axioms for multi-choice games are discussed, some of them being straightforward extensions of axioms from TU-games to multi-choice games.[3]

[3]Certain names of the axioms are identical to Section 2.1.3 due to their normative similarities. In the following, when using such names, we refer to the multi-choice version of these axioms.

Take a value $f$ on $\mathcal{G}$. The first axiom is ubiquitous in almost every axiomatic characterization, as it translates the simple idea that the total worth of the grand coalition should be entirely allocated among the pairs in $M^+$. Clearly, this axiom extends Efficiency from TU-games to multi-choice games.

**Axiom 25 (Efficiency).** *For each* $(m, v) \in \mathcal{G}$,

$$\sum_{i \in N} \sum_{j \leq q_i} f_{ij}(m, v) = v(m).$$

The next axiom is also frequently used in many axiomatic characterizations. This axiom states that the solution is additive with respect to multi-choice games. This means that there is no difference between considering two problems separately or at the same time. This axiom extends Additivity from TU-games to multi-choice games.

**Axiom 26 (Additivity).** *For each* $(m, v), (m, w) \in \mathcal{G}$,

$$f(m, v + w) = f(m, v) + f(m, w).$$

If a player is unproductive at a certain activity level, it seems reasonable to penalize it accordingly. An unproductive activity level of a player can be modeled by a weakly null pair, which has null marginal contributions to coalitions. Peters & Zank (2005) introduce an axiom indicating that weakly null pairs should obtain a null payoff.

**Axiom 27 (Weak null pair property).** *For each* $(m, v) \in \mathcal{G}$, *if* $(i, j) \in M^+$ *is a weakly null pair, then*

$$f_{ij}(m, v) = 0.$$

If a player stops being productive starting from a certain activity level up to its maximal activity level, it seems reasonable to penalize it accordingly. This situation can be modeled by a null pair that describes a player and one of its activity level from which it stops being productive. Klijn *et al.* (1999) introduce a stronger version of the Weak null pair property indicating that any null pair should obtain a null payoff.

**Axiom 28 (Null pair property).** *For each* $(m, v) \in \mathcal{G}$, *if* $(i, j) \in M^+$ *is a null pair, then*

$$f_{ij}(m, v) = 0. \tag{2.50}$$

The Null pair property implies that not only a null pair $(i, j)$ receives a null payoff, but each of the pairs $(i, j')$ such that $j' > j$ also receives a null payoff, since they are also null pair themselves. Clearly, if a value satisfies the Weak null pair property,

then it satisfies the Null pair property. The Null pair property and the Weak null pair property both boil down to the Null player property on the class of multi-choice games with $m = (1, \ldots, 1)$.

$$\text{Weak null pair property} \implies \text{Null pair property}$$

A veto pair represents a player's activity level that is necessary for a coalition to generate a non-null worth. Since all veto pairs are all equally necessary, it seems reasonable to allocate them the same amount. van den Nouweland (1993) introduce an axiom stating that two veto pairs should receive the same payoffs. This axiom extends the Veto players property from TU-games to multi-choice games.

**Axiom 29 (Veto pair property).** *For each $(m, v) \in \mathcal{G}$ and each $(i, j), (i', j') \in M^+$ two veto pairs,*

$$f_{ij}(m, v) = f_{i'j'}(m, v).$$

Anonymity for solutions of TU-games indicates that the payoff of the players does not depend on their label, meaning that one can permute the labels of two players without changing the final allocation. Peters & Zank (2005) proposes a straightforward extension of Anonymity from TU-games to multi-choice games. To permute the labels of two players it is necessary that these players have the same number of activity levels. Therefore, this axiom only holds on the class of multi-choice games in which all the players have the same maximal activity level. The sub-class of multi-choice games such that all players have the **same maximal activity level** is denoted by $\overline{\mathcal{G}}$.

**Axiom 30 (Anonymity).** *For each $(m, v) \in \overline{\mathcal{G}}$, each $t \in \mathcal{M}$ and each order $\theta \in \Theta_N$, we define $\theta t$ as $\theta t_{\theta(i)} = t_i$ for each $i \in N$, and $\theta v$ as $\theta v(\theta t) = v(t)$. Then, it holds that*

$$f_{ij}(m, v) = f_{\theta(i)j}(m, \theta v).$$

If Anonymity for TU-games implies Equal treatment for equals, which in turn implies the Veto players property for TU-games, this is no longer true for Anonymity for multi-choice games.

If a player decides to stop playing above a certain activity level, this can potentially affects the payoffs of its remaining activity levels. Klijn *et al.* (1999) introduce an axiom stating that if aplayer reduces its maximal activity level by one unit, then the variation in payoff incurred by each of its remaining activity levels is equal to the payoff it received for this maximal activity level.[4]

[4]This axiom is called the Equal loss property in Klijn *et al.* (1999).

**Axiom 31 (Intra-balanced contributions).** *For each* $(m, v) \in \mathcal{G}$ *and each* $(i, j) \in M^+$ *such that* $j \neq m_i$,

$$f_{ij}(m, v) - f_{ij}(m - e_i, v) = f_{im_i}(m, v).$$

Klijn *et al.* (1999) propose an axiom indicating that for any two players, the amount that each player would gain or lose for its maximal activity level if the other's reduces its maximal activity level by one unit should be equal.

**Axiom 32 (Upper balanced contributions).** *For each* $(m, v) \in \mathcal{G}$ *and any* $i, i' \in N$,

$$f_{im_i}(m, v) - f_{im_i}(m - e_{i'}, v) = f_{i'm_{i'}}(m, v) - f_{i'm_{i'}}(m - e_i, v).$$

Tang *et al.* (2019) propose an axiom that compares the payoffs of two veto pairs. This axiom is inspired from the Hierarchical strength axiom introduced by Faigle & Kern (1992) for TU-games with precedence constrains. Take any $(m, v) \in \mathcal{G}$. To introduce the axiom, define a **veto coalition** $t \in \mathcal{M}$ as

$$\forall s \in \mathcal{M}, \quad v(s) = \begin{cases} v(t) & \text{if } s \geq t, \\ 0 & \text{otherwise.} \end{cases}$$

A veto coalition for a game $(m, v) \in \mathcal{G}$, is a profile that collects all the minimal activity levels required to generate a non-null worth. For instance, in a minimal effort game $(m, u_t)$, $t$ is a veto coalition. Observe that a veto coalition $t$ always verifies $t > (0, \ldots, 0)$.

**Axiom 33 (Hierarchical symmetry).** *Take any* $(m, v) \in \mathcal{G}$. *For each veto coalition* $t \in \mathcal{M}$ *and each pairs* $(i, t_i), (i', t_{i'}) \in M^+$,

$$\sum_{\substack{s \geq t \\ s_{i'} = t_{i'}}} h_{i't_{i'}} f_{it_i}(m, v) = \sum_{\substack{s \geq t \\ s_i = t_i}} h_{it_i} f_{i't_{i'}}(m, v).$$

The last axiom indicates that if two activity levels of the same player have the same marginal contributions, then it seems reasonable that they obtain the same payoff.

**Axiom 34 (Intra symmetry).** *For each* $(m, v) \in \mathcal{G}$ *and each* $(i, j), (i, j') \in M^+$ *such that*

$$\forall s \in \mathcal{M}, \quad v(s_{-i}, j) - v(s_{-i}, j - 1) = v(s_{-i}, j') - v(s_{-i}, j' - 1),$$

*we have*

$$f_{ij}(m, v) = f_{ij'}(m, v).$$

Klijn *et al.* (1999) provide an axiomatic characterization of the $DP$ value conceptually close to the characterization of the Shapley value provided by Béal & Navarro (2020) (see Theorem 1).

**Theorem 14 (Klijn *et al.* (1999)).** *A value $f$ on $\mathcal{G}$ satisfies Efficiency, Additivity, the Veto pair property and the Null pair property if and only if $f = DP$.*

Klijn *et al.* (1999) propose another characterization of the $DP$ value conceptually close to the characterization of the Shapley value provided by Myerson (1980).

**Theorem 15 (Klijn *et al.* (1999)).** *A value $f$ on $\mathcal{G}$ satisfies Efficiency, Intra-balanced contribution, and Upper balanced contributions if and only if $f = DP$.*

Tang *et al.* (2019) propose an axiomatic characterization of the $vdN$ value, which is conceptually close to the characterization of the Shapley value for TU-games with precedence constrains proposed by Faigle & Kern (1992). Another characterization, conceptually close to the characterization of the Shapley value provided by Myerson (1980), is proposed by Calvo & Santos (2000). Here, we limit ourselves to the characterization provided by Tang *et al.* (2019).

**Theorem 16 (Tang *et al.* (2019)).** *A value $f$ on $\mathcal{G}$ satisfies Efficiency, Additivity, the Weak null pair property and Hierarchical symmetry if and only if $f = vdN$.*

One can replace Hierarchical symmetry in Theorem 16 by Anonymity and Intra symmetry to obtain a characterization of the $PZ$ value on $\overline{\mathcal{G}}$. Such a characterization is proposed by Peters & Zank (2005).

**Theorem 17 (Peters & Zank (2005)).** *A value $f$ on $\overline{\mathcal{G}}$ satisfies Efficiency, Additivity, Anonymity, the Weak null pair property and Intra symmetry if and only if $f = PZ$.*

### 2.4.4 Consistent discrete cost sharing methods

This section presents the class of discrete cost sharing problems (see Moulin (1995), Albizuri *et al.* (2003), Sprumont (2005) and Bahel & Trudeau (2013) to cite a few) and discusses several solution concepts on this class of problems called **discrete cost sharing methods**.

Fix $N = \{1, \dots, n\}$ a set of $n$ different consumers that demand a certain commodity.[5] A discrete cost sharing problem is a couple $(q, C)$, where $q = (q_1, \dots, q_n)$. Fix $K \in \mathbb{N}$ an upper bound for demands. Each $q_i \leq K$ represents the **effective demand** of $i \in N$. A demand profile $s \leq q$ describes a situation in which each consumer $i \in N$ demands a quantity $s_i \leq q_i$ of commodity. The cost function

$$C : \prod_{i \in N} \{0, 1, \dots, q_i\} \to \mathbb{R}_+$$

[5] In the original discrete cost sharing model, the set $N$ refers to goods instead of consumers. Without loss of generality, we stick to the term *consumer* to fit in the framework of the thesis.

is a non-decreasing map verifying $C(\vec{0}) = 0$ for each $s \leq q$. The real $C(s)$ represents the cost of satisfying the demand profile $s$. The total cost to be shared is given by $C(q)$. Clearly, $q$ can be interpreted as a vector of maximal activity levels and $C$ can be interpreted as a non-decreasing characteristic function. Therefore, the class of discrete cost sharing problems is the class of monotonic multi-choice games $\mathcal{G}_m$.

In the cost sharing literature, a method on $\mathcal{G}_m$ is a map $g$ that associates to each problem $(q, C) \in \mathcal{G}_m$ a vector $g(q, C) \in \mathbb{R}^n_+$ satisfying the **budget balanced** condition, i.e,

$$\sum_{i \in N} g_i(q, C) = C(q).$$

Take any $(q, C) \in \mathcal{G}_m$. A cost sharing method on $\mathcal{G}_m$ differs from a value on $\mathbb{G}$, for the following reasons:

- a discrete cost sharing method is a map from $\mathcal{G}_m$ to $\mathbb{R}^{|M^+|}$, whereas a value is a map from $\mathcal{G}$ to $\mathbb{R}^{|M^+|}$. Put differently, a discrete cost sharing method allocates a payoff to each consumer (player), whereas a value allocates a payoff to each demand of each player;

- by definition, a discrete cost sharing method satisfies the budget balanced condition, which is equivalent to say that to satisfies Efficiency. On the contrary, a value does not necessarily satisfy Efficiency.

Despite their differences, it is possible to establish some links between discrete cost sharing methods and values on $\mathcal{G}_m$. Two types of links are detailed: the consistency and the weak consistency.

A value $f$ on $\mathbb{G}$ is **consistent** with a method $g$ on $\mathcal{G}_m$ if

$$\forall (q, C) \in \mathcal{G}_m, \forall i \in N, \quad g_i(q, C) - g_i(q - e_i, C) = f_{iq_i}(q, C).$$

In other words, $f$ describes the variation in cost share a consumer undergoes when its demand increases by one unit whereas $g$ describes the total cost share allocated to each consumer. We say that a value $f$ on $\mathbb{G}$ is **weakly consistent** with a method $g$ on $\mathcal{G}_m$ if

$$\forall (q, C) \in \mathcal{G}_m, \forall i \in N, \quad g_i(q, C) = \sum_{j \in M_i^+} f_{ij}(q, C).$$

Obviously, if a value $f$ is consistent with a method $g$ on $\mathcal{G}_m$, then it is also weakly consistent with $g$. The converse is not true.

Three well known discrete cost sharing methods are the Aumann-Shapley method (see Sprumont (2005)), the Pseudo Average method (see Moulin (1995)) and the Serial cost sharing method (see Moulin (1995)). Each of these methods is an additive method, in the sense that, for each $(q, C), (q, C') \in \mathcal{G}_m$,

$$g(q, C) + g(q, C') = g(q, C + C').$$

In the following, we provide a mathematical expression of these methods and point out if the method is consistent/weakly consistent with an existing value for multi-choice games.

Since a discrete cost sharing problem is a multi-choice game, then by (2.32),

$$\forall (q, C) \in \mathcal{G}_m, \quad C = \sum_{s \leq q} \Delta_C(s) u_s.$$

For a certain $s \in \mathcal{M}$ such that $\Delta_C(s)$ is negative, $(q, \Delta_C(s)u_s)$ is a non-monotonic game and so is not a discrete cost sharing problem. To get around this drawback, let us re-writte the cost function $C$ as

$$C = \sum_{\substack{s \leq q \\ \Delta_C(s) \geq 0}} \Delta_C(s) u_s + \sum_{\substack{s \leq q \\ \Delta_C(s) < 0}} \Delta_C(s) u_s$$

$$C = \sum_{\substack{s \leq q \\ \Delta_C(s) \geq 0}} \Delta_C(s) u_s - \sum_{\substack{s \leq q \\ \Delta_C(s) < 0}} (-\Delta_C(s) u_s).$$

With this re-writing, one can express any cost function with monotonic games only. Formally, for any additive discrete cost sharing method $g$,

$$g(q, C) = \sum_{\substack{s \leq q \\ \Delta_C(s) \geq 0}} g(q, \Delta_C(s) u_s) - \sum_{\substack{s \leq q \\ \Delta_C(s) < 0}} g(q, -\Delta_C(s) u_s). \tag{2.51}$$

For the sake of clarity, we define discrete cost sharing methods on a minimal effort game $(q, u_s)$. One can always retrieve the expression of the method on $(q, C)$ by (2.51).

The discrete Aumann-Shapley method, originally introduced in Moulin (1995), has been defined, in the introduction, in terms of Shapley values. This method has an expression in terms of hierarchical strengths (see (2.40)) when applied to a minimal effort game.

**Definition 20 (Aumann-Shapley method, Moulin (1995)).** *For each $(q, u_s) \in \mathcal{G}_m$, where $s \leq q$, the (discrete) Aumann-Shapley method is defined as*

$$\forall i \in N, \quad AS_i(q, u_s) = h_{is_i}(s).$$

Recall that the hierarchical strength of a pair $(i, j) \in M^+$ in $s \leq q$ represents the average number of admissible orders over $M^+$ such that all the pairs in $B(s) = \{(k, l) \in M^+ : l \leq s_k\}$, except $(i, j)$, are ordered prior to step $|B(s)| - 1$ and $(i, j)$ is ordered last. Formally, the hierarchical strength of a pair $(i, j) \in M^+$ in $s$ is defined as

$$h_{ij}(s) = \frac{1}{|\Theta_{vdN}|} |\{\theta_{vdN} \in \Theta_{vdN} : \theta_{vdN}(i, j) = |B(s)|\}|.$$

Clearly, the higher the demand $s_i$ of $i$ within $s$, the more admissible orders satisfying in which $(i, j)$ is ordered last. Therefore, the higher the demand of $i$ within $s$, the higher its cost share in $(q, u_s)$.

**Proposition 8 (Calvo & Santos (2000)).** *The vdN value is weakly consistent with the discrete Aumann-Shapley method on $\mathcal{G}_m$.*

The (discrete) Pseudo-average method is introduced by Moulin (1995). For a minimal effort game $(q, u_s)$, this method allocates to each consumer a cost share proportional to its demand $s_i$.

**Definition 21 (Pseudo-average method, Moulin (1995)).** *For each $(q, u_s) \in \mathcal{G}_m$, where $s \leq q$, the (discrete) Pseudo-average method is defined as*

$$\forall i \in N, \quad PA_i(q, u_s) = \frac{s_i}{|B(s)|}.$$

Similarly to the Aumann-Shapley method, the Pseudo-average method allocates an amount to each player that increases with respect to the player's demand within $s$.

**Proposition 9 (Albizuri *et al.* (2003)).** *The DP value is weakly consistent with the pseudo average cost method on $\mathcal{G}_m$.*

To conclude this section, we introduce the (discrete) Serial cost sharing method (see Moulin & Shenker (1992) and Moulin (1995)), which proves useful for Chapter 4. For a minimal effort game $(q, u_s)$, this method allocates the total cost among the consumers with the highest demands in $s$. For each profile $s \in \mathcal{M}$, define the set of players with the highest demands within the profile by

$$Y(s) = \{i \in N : s_i \geq s_{i'}, \forall i' \in N\}.$$

**Definition 22 (Serial cost sharing method, Moulin (1995)).** *For each $(q, u_s) \in \mathcal{G}_m$, where $s \leq q$, the (discrete) Serial cost sharing is defined as*

$$\forall i \in N, \quad SCS_i(q, u_s) = \begin{cases} \dfrac{1}{|Y(s)|} & \text{if } i \in Y(s), \\ 0 & \text{otherwise.} \end{cases} \tag{2.52}$$

A new extension of the Shapley value from TU-games to multi-choice games is introduced in Chapter 4. We will show that this value is consistent with the Serial cost sharing method on $\mathcal{G}_m$.

# Chapter 3: Values for Multi-Choice Games with a Permission (tree) Structure

## 3.1 Introduction

Permission structures model situations in which some players may need permission from other players to cooperate (see Section 2.2). It turns out that permission relationships between gas consumers can arise in the context of gas distribution. Indeed, the gas distribution network has a history: it was built progressively according to the growing demands of consumers. Thus, some consumers would never have been able to demand gas if other consumers had not been connected beforehand: this can be considered as a permission relationship.

For example, let us consider Figure 3.1 representing a distribution (tree) network managed by GRDF. Consumer $a$ is located at the root of this network, which means that $a$ was the first consumer that GRDF connected to the network. In particular, GRDF determined that it was worth connecting $a$ to the network given the investment required to connect it.

Consumers $b$ and $c$ were able to take advantage of $a$'s connection to connect themselves to the distribution network. Indeed, in the absence of $a$, the investments required for their connection may have been too great. Consequently, GRDF may have decided to not connect them directly to the root. Since $a$ is connected, GRDF determined that it was worth connecting $b$ and $c$ to the network since the only investment required is an additional connection to $a$ for both of them.

This situation can be represented by permission relationship: $b$ and $c$ need permission from $a$ to be connected. Similarly, $f$ needs permission from $c$, $d$ needs permission from $b$ and $e$ needs permission from $b$. All in all, a permission structure can be extracted from the information provided by the distribution network.

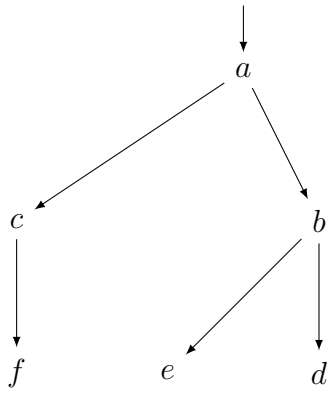



Figure 3.1: A distribution network.

This chapter takes advantage of the information provided by the distribution network to enrich the model of multi-choice games with a permission (tree) structure. No consideration is given to the principles pursued by GRDF. Instead, we focus on the information provided by the distribution network and investigate how to combine a permission structure with a multi-choice game to derive relevant solution concepts.

A few studies have enriched the model of multi-choice games with a structure (see Section 1.5). In this study, we investigate how a permission structure on the player set can be combined with different levels of participation. There is no straightforward way to combine these two aspects. We propose to deal with this problem in two steps that correspond to two approaches describing two different but closely related models. In a first approach, we consider multi-choice games enriched with a permission structure defined on the set of player-activity levels pairs. In a second approach, we consider multi-choice games with a permission structure defined on the set of players.

The first approach consists in defining a permission structure on the set of player-activity level pairs (a pal-permission structure henceforth) and not on the player set as in Section 2.2. The underlying idea is that a player $i$ making a certain activity level $j$ may need another player $k$ to make a certain activity level $l$ beforehand to cooperate. This approach generalizes in a straightforward way the model of TU-games with a permission structure (see Section 2.2). We proceed in several steps. First, we define a set of feasible coalitions with respect to a pal-permission structure and show that this set forms a sub-lattice of the set of coalitions. Second, we define the sovereign and authorizing parts of a coalition $s$ as its largest feasible sub-coalition and its smallest feasible super-coalition, respectively. Third, we introduce a restricted multi-choice game given by the worth of the sovereign part of the coalitions. Finally, we define the Pal-permission value as the $DP$ value (see (2.37)) applied to the above restricted multi-choice game. This value generalizes both the $DP$ value for multi-choice games and the Permission value (2.21) for TU-games with a permission structure.

The second approach keeps a permission structure on the player set, as in Section

2.2, and combines it with a multi-choice game. In this second approach, it is always considered that the players are part of a permission structure but we also exploit the fact that the set of activity levels of each player is totally ordered. This leads us to consider that there are two types of permission structures: the first one operates between players and the second one operates between the activity levels of each player. Indeed, one considers that the activity levels are part of a permission structure in the sense that choosing the activity level $j$ requires having made or being able to make the activity level $j-1$. In other words, players gradually deploy their activity levels, from the lowest to the highest level.

We then propose several axioms allowing us to characterize three new values for multi-choice games with a permission structure. Each one coincides with a Pal-permission value on multi-choice games enriched with a specific pal-permission structure as described in the first approach. They differ in the way that the pal-permission structure emerges from the set of axioms:

- for the first value, the pal-permission structure is such that a player is allowed to cooperate at any activity level if and only if its superiors in the hierarchy cooperate at their maximal level;

- for the second one, the pal-permission structure is such that a player is allowed to cooperate at any activity level if its superiors already participate (meaning that each superior is deploying at least its first activity level);

- for the third one, a player can cooperate at some activity level $j$ if, and only if, its superiors cooperate at least at the activity level $j$. In this case, it should be noted that even if the permission structure on the player set is a rooted tree, the induced pal-permission structure is not a rooted tree.

These three axiomatic characterizations are inspired from the axiomatic characterization of the Permission value for TU-games with a permission tree structure recently provided by van den Brink *et al.* (2015) (see Theorem 11). Our approach is interesting for several reasons. First, it distinguishes a priori a permission structure between the players from a permission structure on the activity levels of each player. Then, the axiomatic study sheds light on how these two types of permission structures are combined into one pal-permission structure on the player-activity level pairs, taking into account the inter and intra players relationships. Quite different pal-permission structures emerge, each one of them having its own natural interpretation.

All in all, this chapter sheds light on the additional difficulties that arise when one tries to combine a multi-choice game with a structure on the player set. This chapter makes use of the information provided by the distribution network, but has no consideration for the Independence of higher demands or the Uniformity principle.

The chapter is organized as follows. Section 3.2 introduces pal-permission structures. Section 3.3 deals with the set of feasible coalitions and its properties as well as the sovereign and authorizing parts of a coalition. Section 3.4 introduces the

Pal-permission value. Section 3.5 investigates the model of multi-choice games with a permission structure on the player set and how a pal-permission structure can emerge from such model. Section 3.6 introduces the above three values for multi-choice games with a permission tree structure. Section 3.7 contains the axiomatic results. Section 3.8 concludes the chapter.

## 3.2 Pal-permission structures

In the context of multi-choice games, we consider situations in which some players cannot cooperate at a certain activity level until some other players have reached a certain activity level. To model such situations, we introduce **player-activity level permission structures** (pal-permission structures henceforth) that extend permission structures from $N$ to $M^+$.

Take any nonempty set of player-activity level pairs $M^+$. A pal-permission structure on $M^+$ is a map

$$P^+ : M^+ \longrightarrow 2^{M^+}$$

such that the relationship $(k,l) \in P^+(i,j)$ indicates that player $k$ cannot cooperate at activity level $l$ until $i$ has reached its activity level $j$. In this case, the pair $(k,l) \in M^+$ is a successor of the pair $(i,j) \in M^+$. Moreover, $(i,j) \in (P^+)^{-1}(k,l)$ indicates that the pair $(i,j)$ is a predecessor of the pair $(k,l)$. The map $P^+$ is assumed to be asymmetric, that is,

$$\forall i \in N, \forall (i,j) \in M_i^+, \quad (i,j) \notin P^+(i,j).$$

The notation $\hat{P}^+$ stands for the transitive closure of $P^+$, the set of superiors and subordinates of a player are defined accordingly.

Because player's $i$ set of activity levels $M_i$ is totally ordered from the null activity level 0 to the greatest activity level $m_i$, the set of pairs $\{i\} \times M_i^+$ can be also totally ordered as follows: $(i,1) < (i,2) < \ldots < (i,m_i)$. From this point of view, we obtain a pal-permission structure between $i$'s activity levels. The **trivial pal-permission structure** $P^{+,0}$ on $M^+$ is the pal-permission structure modeling this idea. In $P^{+,0}$, a player cannot cooperate at its activity level $j$ if it has not reached its activity level $j-1$ and no pair formed by two player-activity level pairs involving distinct players are in relation. Formally,

$$P^{+,0} : M^+ \longrightarrow 2^{M^+}$$

is defined as

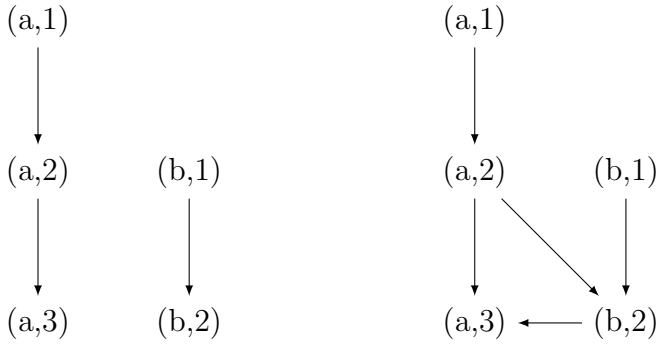
$$\forall i \in N, \forall j \in M_i^+ \setminus \{m_i\}, \quad (i,j+1) \in P^{+,0}(i,j).$$

Take any two pal-permission structures $P^+$ and $Q^+$. We say that $P^+$ **extends** $Q^+$ if

$$\forall (i,j),(k,l) \in M^+, \quad \big[(i,j) \in Q^+(k,l)\big] \Longrightarrow \big[(i,j) \in P^+(k,l)\big].$$

Finally, we say that a pal-permission structure $P^+$ is **cohesive with respect to** $P^{+,0}$ or simply cohesive if it extends $P^{+,0}$. Thus, in a cohesive pal-permission structure, a player cannot cooperate at its activity level $j$ if it has not reached its activity level $j-1$.

**Example 4.** *Consider the player set $N = \{a, b\}$ and the vector of maximal activity levels $m = (3, 2)$ where $m_a = 3$ and $m_b = 2$. Consider the pal-permission structures $P^{+,0}$ and $P^+$ such that $P^+$ extends $P^{+,0}$. We represent the two pal-permission structures by the following digraphs*




(a) $P^{+,0}$ (b) $P^+$

*In $P^{+,0}$, a needs to achieve its activity levels 1 and 2 before deploying its activity level 3. In $P^+$, a not only has to achieve its activity levels 1 and 2 to deploy its activity level 3, but also needs that b achieves its activity levels 1 and 2.*

In the following, we only consider the set of cohesive and acyclic pal-permission structures that we denote by $\mathcal{P}$.

## 3.3 Feasible multi-choice coalitions

A multi-choice game $(m, v) \in \mathcal{G}$ enriched with a **pal-permission structure** $P^+ \in \mathcal{P}$ is the triplet $(m, v, P^+)$. The class of **multi-choice games with a pal-permission structure** is denoted by $\mathcal{GP}$. A pal-permission structure $P^+ \in \mathcal{P}$ influences the possibilities of cooperation in a multi-choice game in the following way. Let $(k, l)$ be a predecessor of $(i, j)$ with respect to $P^+$ and $s$ be a coalition such that $s_i = j$. In order for player $i$ to be able to to cooperate at activity level $j$, player $k$ must have an activity level of at least $l$ in $s$, that is, $s_k \geq l$. In this sense, the player-activity level pair $(i, j)$ needs the permission of the player-activity level pair $(k, l)$ to cooperate

in $s$. A coalition $s$ is **feasible** if each player-activity level pair has the permission of all its superiors in $s$. This way of introducing the influence of the permission relationships in a coalition is a natural generalization of the conjunctive approach from TU-games to multi-choice games (see 2.16).

**Definition 23 (Feasible coalitions).** *Take any $m \leq (K, \ldots, K)$. Take any pal-permission structure $P^+ \in \mathcal{P}$ on $M^+$. The set of feasible coalitions $\mathcal{F}(m, P^+)$ is defined as*

$$\mathcal{F}(m, P^+) = \left\{ s \leq m : s_i > 0 \implies \forall (k, l) \in (\hat{P}^+)^{-1}(i, s_i), s_k \geq l) \right\} \tag{3.1}$$

**Example 5.** *Consider again Example 4 where the pal-permission structure $P^+$ is given by Figure (b). A coalition $s$ is of the form $(s_a, s_b)$, where $s_a \in \{0, 1, 3\}$ is the activity level of $a$ and $s_b \in \{0, 1, 2\}$ is the activity level of $b$. Then, the set of feasible coalitions is defined as*

$$\mathcal{F}(m, P^+) = \mathcal{M} \setminus \{(0, 2), (3, 0), (3, 1)\}.$$

*For instance, consider the coalition $(s_a, s_b) = (0, 2)$. This coalition is not feasible because $b$ needs $a$ to play an activity level $s_a \geq 1$ to cooperate at level 2, which is not the case in coalition $(0, 2)$. On the other hand, coalition $(s_a, s_b) = (0, 1)$ is feasible because $b$ does not need the permission of any activity level of $a$ to be able to cooperate at activity level 1.*

The next proposition establishes that the set of feasible coalitions forms a (finite) lattice whose greatest element is the grand coalition and the least element is the null coalition.

**Proposition 10.** *Take any $(m, v, P^+) \in \mathcal{GP}$. The set $\mathcal{F}(m, P^+)$ satisfies the following properties*

*(i)* $\vec{0} \in \mathcal{F}(m, P^+)$;

*(ii)* $m \in \mathcal{F}(m, P^+)$;

*(iii) for each $s, t \in \mathcal{F}(m, P^+)$, it holds that $s \wedge t$ and $s \vee t$ are still elements of $\mathcal{F}(m, P^+)$.*

*Proof.* Take any $(m, v, P^+) \in \mathcal{GP}$.

(i) Consider the null coalition $s = \vec{0}$. Because there is no $i \in N$ such that $s_i > 0$, by definition of $\mathcal{F}(m, P^+)$ conclude that $\vec{0} \in \mathcal{F}(m, P^+)$.

(ii) Consider the grand coalition $m$. Because each coordinate of $m$ is maximal, by definition of $\mathcal{F}(m, P^+)$ conclude that $m \in \mathcal{F}(m, P^+)$.

(iii) Consider two feasible coalitions $s, t \in \mathcal{F}(m, P^+)$ and set $a = s \wedge t$ and $b = s \vee t$. Let us show that $a, b \in \mathcal{F}(m, P^+)$.

First, we show that $a \in \mathcal{F}(m, P^+)$. We proceed by contradiction. Suppose that $a \notin \mathcal{F}(m, P^+)$. By definition, there exist $i, k \in N$ such that $a_i > 0$, $a_k < l$, and $(k, l) \in (\hat{P}^+)^{-1}(i, a_i)$. Observe that for such a player $i$, it holds that either $a_i = s_i$ or $a_i = t_i$. Without loss of generality, suppose that $a_i = s_i \leq t_i$. In the same way, either $a_k = s_k$ or $a_k = t_k$.

- If $a_k = s_k$, then $s_k < l$. Because $a_i = s_i$, it follows that $(k, l) \in (\hat{P}^+)^{-1}(i, s_i)$ and so $s \notin \mathcal{F}(m, P^+)$, which is a contradiction.
- If $a_k = t_k$, then $t_k < l$. Because $a_i = s_i \leq t_i$ and $(k, l) \in (\hat{P}^+)^{-1}(i, s_i)$, by consistency of $P^+$ we also have $(k, l) \in (\hat{P}^+)^{-1}(i, t_i)$. Thus,$t \notin \mathcal{F}(m, P^+)$, which is a contradiction.

All in all, we have shown that $a \in \mathcal{F}(m, P^+)$. The proof that $b \in \mathcal{F}(m, P^+)$ is similar and so is left to the reader.

■

Clearly, Proposition 10 extends Proposition 3 from TU-games with a permission structure to multi-choice games with a pal-permission structure.

Because $\mathcal{F}(m, P^+)$ is a finite sub-lattice of $\mathcal{M}$ for each coalition $s$, we can define the supremum of the feasible coalitions smaller than $s$ and the infimum of the feasible coalitions larger than $s$. In the first case, we obtain the greatest feasible coalition $\sigma(s) \in \mathcal{F}(m, P^+)$ such that $\sigma(s) \leq s$; in the second case, we obtain the smallest feasible coalition $\alpha(s) \in \mathcal{F}(m, P^+)$ such that $\alpha(s) \geq s_i$ for each $i \in N$. To paraphrase Definition 9, $\sigma(s)$ is the **sovereign part** of $s$ and $\alpha(s)$ is the **authorizing part** of $s$.

**Definition 24 (Sovereign and authorizing parts).** *Take any $(m, v, P^+) \in \mathcal{GP}$. Take any pal-permission structure $P^+$ on $M^+$ and any multi-choice coalition $s \in \mathcal{M}$. The sovereign and authorizing parts of $s$ with respect to $P^+$ are respectively given by*

$$\sigma(s) = \bigvee_{\substack{t \in \mathcal{F}(m, P^+) \\ t \leq s}} t, \qquad \alpha(s) = \bigwedge_{\substack{t \in \mathcal{F}(m, P^+) \\ t \geq s}} t. \tag{3.2}$$

Obviously, for each $t \in \mathcal{F}(m, P^+)$, it holds that $\sigma(t) = \alpha(t) = t$; in particular, $\sigma(m) = \alpha(m) = m$. Finally, for any $s, t \in \mathcal{M}$ such that $s \leq t$, it holds that $\sigma(s) \leq \sigma(t)$ and $\alpha(s) \leq \alpha(t)$.

To close this section, the following proposition provides interesting properties of the sovereign and authorizing parts of a multi-choice coalition. This proposition echoes Proposition 4 for coalitions of players.

**Proposition 11.** *Take any $(m, v, P^+) \in \mathcal{GP}$. For any pal-permission structure $P^+$ on $M^+$ and any pair of coalitions $s, t \in \mathcal{M}$, it holds that:*

*(i)* $\sigma(s) \vee \sigma(t) \leq \sigma(s \vee t)$;

*(ii)* $\sigma(s) \wedge \sigma(t) = \sigma(s \wedge t)$;

*(iii)* $\alpha(s) \vee \alpha(t) = \alpha(s \vee t)$;

*(iv)* $\alpha(s \wedge t) \leq \alpha(s) \wedge \alpha(t)$.

*Proof.* Take any pal-permission structure $P^+$ on $M^+$ and any pair of coalitions $s, t \in \mathcal{M}$.

(i) By definition, $s \leq s \vee t$ and $t \leq s \vee t$ and so $\sigma(s) \leq \sigma(s \vee t)$ and $\sigma(t) \leq \sigma(s \vee t)$. By definition of the supremum,

$$\sigma(s) \vee \sigma(t) \leq \sigma(s \vee t) \vee \sigma(s \vee t) = \sigma(s \vee t).$$

(ii) By definition, $t \geq s \wedge t$ and $s \geq s \wedge t$ and so $\sigma(s) \geq \sigma(s \wedge t)$ and $\sigma(t) \geq \sigma(s \wedge t)$. This implies that $\sigma(s) \wedge \sigma(t) \geq \sigma(s \wedge t)$. On the other hand, by definition of $\sigma$, $\sigma(s) \leq s$ and $\sigma(t) \leq t$. Thus, we obtain

$$\begin{aligned} & \sigma(s) \wedge \sigma(t) \leq s \wedge t \\ \iff & \sigma(\sigma(s) \wedge \sigma(t)) \leq \sigma(s \wedge t) \\ \iff & \sigma(s) \wedge \sigma(t) \leq \sigma(s \wedge t), \end{aligned}$$

where the equality

$$\sigma(\sigma(s) \wedge \sigma(t)) = \sigma(s) \wedge \sigma(t)$$

comes from the fact that $\sigma(s) \wedge \sigma(t) \in \mathcal{F}(m, P^+)$ by Proposition 10. Conclude that $\sigma(s) \wedge \sigma(t) = \sigma(s \wedge t)$.

(iii) By definition of $\alpha$, $\alpha(s) \geq s$ and $\alpha(t) \geq t$, from which we get:

$$\begin{aligned} & \alpha(s) \vee \alpha(t) \geq s \vee t \\ \iff & \alpha(\alpha(s) \vee \alpha(t)) \geq \alpha(s \vee t) \\ \iff & \alpha(s) \vee \alpha(t) \geq \alpha(s \vee t) \end{aligned}$$

On the other hand, $s \leq s \vee t$, which implies that $\alpha(s) \leq \alpha(s \vee t)$. By definition, $\alpha(t) \leq \alpha(s \vee t)$. It implies that $\alpha(t) \vee \alpha(s) \leq \alpha(s \vee t)$. From the above arguments, we arrive at $\alpha(s) \vee \alpha(t) = \alpha(s \vee t)$.

(iv) The fact that $s \wedge t \leq s$ and $s \wedge t \leq t$ implies that $\alpha(s \wedge t) \leq \alpha(s)$ and $\alpha(s \wedge t) \leq \alpha(t)$, and so $\alpha(s \wedge t) \leq \alpha(s) \wedge \alpha(t)$.

■

## 3.4 Values for multi-choice games with a pal-permission structure

In a multi-choice game with a pal-permission structure $(m, v, P^+)$, the multi-choice game $(m, v)$ describes the possibilities of cooperation regardless of the restrictions induced by the pal-permission structure. To take these restrictions into account, a new multi-choice game $(m, \mathcal{R}_{P^+}(v)) \in \mathcal{G}$ is constructed. This game takes into account both the cooperation possibilities represented by $(m, v)$ and the permission relationships represented by $P^+$. The resulting multi-choice game $(m, \mathcal{R}_{P^+}(v))$ is the (conjunctive) restriction of the multi-choice game of $(m, v)$ induced by the pal-permission structure $P^+$. Then, the $DP$ value (see (2.37)) is applied to $(m, \mathcal{R}_{P^+}(v))$. This allows to define a value for multi-choice games with a pal-permission structure. Because the $DP$ value is a generalization of the Shapley value, and the Permission value is the Shapley value of a TU-game restricted by a permission structure, it follows that the $DP$ value applied to $(m, \mathcal{R}_{P^+}(v))$ can be viewed as a possible generalization of the Permission value from TU-games with a permission structure to multi-choice games with a pal-permission structure.

Take any $(m, v, P^+) \in \mathcal{GP}$. The **restriction** of $(m, v)$ on $P^+$ is the multi-choice game $(m, \mathcal{R}_{P^+}(v))$ where the characteristic function $\mathcal{R}_{P^+}(v)$ is defined as

$$\mathcal{R}_{P^+}(v) = v \circ \sigma. \tag{3.3}$$

Thus, the worth $\mathcal{R}_{P^+}(v)(s)$ of the multi-choice coalition $s$ is the worth $v(\sigma(s))$ of its sovereign part. In other words, the mapping $\sigma$ indicates how $s$ is evaluated: the sovereign part with respect to $P^+$, that is, the greatest feasible coalition smaller than $s$, generates a worth $v(\sigma(s))$ when $s$ forms. Clearly, this restriction extends the restriction of a TU-game by a permission structure (see (2.17)) to the framework of multi-choice games.

If $(m, v)$ is monotonic, then $(m, \mathcal{R}_{P^+}(v))$ is also monotonic. Indeed, if $s \leq t$ implies $v(s) \leq v(t)$, then $\sigma(s) \leq \sigma(t)$ and so $\mathcal{R}_{P^+}(v)(s) \leq \mathcal{R}_{P^+}(v)(t)$. Fix the set of multi-choice coalitions $\mathcal{M}$. The operator $\mathcal{R}_{P^+}$ is linear in $v$ on $\mathcal{M}$, that is, $\mathcal{R}_{P^+}(v + \gamma w) = \mathcal{R}_{P^+}(v) + \gamma \mathcal{R}_{P^+}(w)$ for each $(m, v)$, $(m, w) \in \mathcal{G}$ and each $\gamma \in \mathbb{R}$.

**Proposition 12.** *For each $(m, u_s, P^+) \in \mathcal{GP}$, $s \in \mathcal{M}$ such that $s \neq (0, \ldots, 0)$*

$$\mathcal{R}_{P^+}(u_s) = u_{\alpha(s)}.$$

*Proof.* Take any $(m, u_s, P^+) \in \mathcal{GP}$ and any $s \in \mathcal{M}$ such that $s \neq (0, \ldots, 0)$. By definition of $\mathcal{R}_{P^+}(u_s)$,

$$\forall t \in \mathcal{M}, \quad \mathcal{R}_{P^+}(u_s)(t) = u_s(\sigma(t)) = \begin{cases} 1 & \text{if } \sigma(t) \geq s, \\ 0 & \text{otherwise.} \end{cases}$$

First, let us show the following equivalence

$$\sigma(t) \geq s \iff t \geq \alpha(s).$$

- Suppose that $\sigma(t) \geq s$. We have $\alpha(\sigma(t)) = \sigma(t) \geq \alpha(s)$. Because $t \geq \sigma(t)$, we obtain $t \geq \alpha(s)$;
- Reciprocally, if $t \geq \alpha(s)$, then $\sigma(t) \geq \sigma(\alpha(s)) = \alpha(s) \geq s$.

Thus, we obtain that:

$$\forall t \in \mathcal{M}, \quad \mathcal{R}_{P^+}(u_s)(t) = \begin{cases} 1 & \text{if } t \geq \alpha(s), \\ 0 & \text{otherwise,} \end{cases}$$

so that $\mathcal{R}_{P^+}(u_s) = u_{\alpha(s)}$, as desired. ■

By Proposition 12 and the linearity of the operator $\mathcal{R}_{P^+}$, the characteristic function $\mathcal{R}_{P^+}(v)$ can we rewritten as

$$\begin{aligned} \mathcal{R}_{P^+}(v) &= \mathcal{R}_{P^+}\left(\sum_{s\in\mathcal{M}} \Delta_v(s) u_s\right) && (3.4) \\ &= \sum_{s\in\mathcal{M}} \Delta_v(s) \mathcal{R}_{P^+}(u_s) && (3.5) \\ &= \sum_{s\in\mathcal{M}} \Delta_v(s) u_{\alpha(s)}. && (3.6) \end{aligned}$$

We introduce a new value for multi-choice games with a pal permission structure, called the **Pal-permission value**, which allocates the worth of the grand coalition according to the $DP$ value applied to the restricted game induced by the pal-permission structure.

**Definition 25 (Pal-permission value).** *The Pal-permission value* $\Upsilon$ *is defined, for each* $(m, v, P^+) \in \mathcal{GP}$*, as*

$$\forall (i,j) \in M^+, \quad \Upsilon_{ij}(m, v, P^+) = DP_{ij}(m, \mathcal{R}_{P^+}(v)). \tag{3.7}$$

From (2.32), (2.31), and the fact that the $DP$ value is linear (see (2.36)), $\Upsilon$ can be rewritten as

$$\begin{aligned} \forall (i,j) \in M^+, \quad \Upsilon_{ij}(m, v, P^+) &= \sum_{\substack{s\leq m \\ (i,j)\in B(\alpha(s))}} \frac{\Delta_v(s)}{|B(\alpha(s))|} \\ &= \sum_{\substack{s\leq m \\ j\leq \alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N} \alpha_h(s)}. && (3.8) \end{aligned}$$

The Pal-permission value $\Upsilon$ distributes the dividend of each coalition $s$ equally among the admissible pairs of the authorizing part $\alpha(s)$ of $s$. It follows that $\Upsilon$ is a generalization of both the $DP$ value (see (2.37)) and the Permission value (see (2.21)). Indeed, in case $(m, v)$ is a TU-game, i.e., $m = (1, \ldots, 1)$, the Pal-permission value boils down to the Permission value. In case $P^+ = P^{+,0}$, the Pal-permission value boils down to the $DP$ value.

## 3.5 Pal-permission structures induced by a permission tree structure

Consider the situation where the players are part of a permission structure and the sets of activity levels are totally ordered from the lowest activity level to the greatest activity level. In such a situation, several pal-permission structures can be created from the permission structure on the player set and the sets of activity levels. This amounts to say that the relations between the players, the inter-individual relationships, and the relations between the activity levels of each player, the intra-individual relationships, are a priori two independent objects. This means that they can be combined to obtain a single hierarchical organization represented by a pal-permission structure. This leads to the following definition.

**Definition 26 (Induced pal-permission structures).** *A pal-permission structure induced by a permission tree structure $P \in \mathbb{P}_T$ and the vector of maximal activity levels $m$ is a cohesive and acyclic pal-permission structure $P^+ \in \mathcal{P}$ constructed from the relationships given by $P$ and the set of the activity levels obtained from $m$.*

We focus the study on three particular induced pal-permission structures that we detail below. For a permission tree structure $P \in \mathbb{P}_T$ and any profile of maximal activity levels $m$:

- the **m-permission structure** on $M^+$ induced by $P$ and $m$, is the cohesive pal-permission tree structure in which a player can cooperate only if all its superiors in $P$ participate by choosing their maximal activity level. The resulting structure is denoted by $P_m^+$. Formally, $P_m^+ \in \mathcal{P}$ is defined as

$$\forall i, j \in N, \quad \big[j \in P(i)\big] \iff \big[(j,1) \in P_m^+(i, m_i)\big];$$

- the **1-permission structure** on $M^+$ induced by $P$ and $m$ is the cohesive pal-permission tree structure in which a player can cooperate only if all its superiors in $P$ participate. Such a structure is denoted by $P_1^+$. Formally, $P_1^+ \in \mathcal{P}$ is defined as

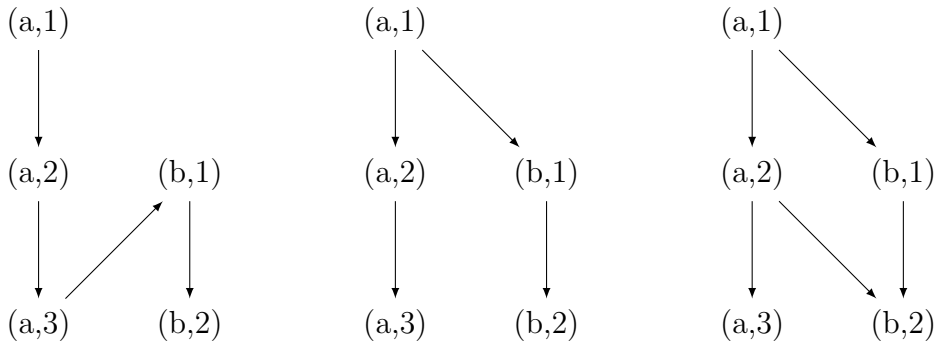
$$\forall i, j \in N, \quad \big[j \in P(i)\big] \iff \big[(j,1) \in P_1^+(i, 1)\big];$$

- the **Full-permission structure** on $M^+$ induced by $P$ and $m$ is the cohesive and acyclic pal-permission structure in which a player can cooperate at a certain activity level $l$ if its superiors in $P$ participate at level $l$ or more. We assume that, for each $j \in P(i)$, $m_j \leq m_i$. This assumption leads to a specific class of games which will be detailed later (in Section 4.3). Formally, $P_F^+ \in \mathcal{P}$ is defined as

$$\forall i, j \in N, \forall l \in M_j, \quad \big[j \in P(i)\big] \iff \big[(j,l) \in P_F^+(i, l)\big].$$

Observe that, for each $P \in \mathbb{P}_T$ and each $m$, the structures $P_m^+$, $P_1^+$ and $P_F^+$ are uniquely determined and are acyclic pal-permission structures $\mathcal{P}$ by construction.

**Example 6.** *Consider once again Example 4, where the two players are $\{a, b\}$ and the vector of maximal activity levels is $m = (3, 2)$. Consider the permission tree structure $P \in \mathbb{P}_T$ such that $b \in P(a)$. The induced pal-permission structures $P_m^+$, $P_1^+$ and $P_F^+$ are represented by figures $(a)$, $(b)$ and $(c)$, respectively.*




(a) $P_m^+$ (b) $P_1^+$ (c) $P_F^+$

Observe that the induced pal-permission structures $P_m^+$ and $P_1^+$ are always trees, whereas $P_F^+$ is clearly not a tree.

## 3.6 Three new values for multi-choice games with a permission structure

Let $\mathcal{GP}$ be the class of games $(m, v, P)$ where the players in $N$ play the multi-choice game $(m, v) \in \mathcal{G}$ and are organized according to the permission structure $P \in \mathbb{P}$ on $N$. The class of multi-choice games with a permission tree structure is denoted by $\mathcal{GP}_T$. Clearly, it holds that $\mathcal{GP}_T \subseteq \mathcal{GP}$.

We consider three possible values on $\mathcal{GP}_T$. Each of these values are computed using the Pal-permission value $\Upsilon$ for multi-choice games with a pal permission structure. The first one consists in applying the Pal-permission value on multi-choice games with the m-permission structure. The second one consists in applying the Pal-permission value on multi-choice games with the 1-permission structure. And the last one consists in applying the Pal-permission value on multi-choice games with the Full-permission structure.

Proceeding in this way, the inter-individual relationships represented by $P$ and the intra-individual relationships represented by $m$ and the linear order of the activity levels are combined into a pal-permission structure ($P_m^+$, $P_1^+$, or $P_F^+$) on $M^+$. Then, the allocation of the payoffs are given by the $DP$ value of the restricted game induced by the corresponding pal-permission structure. This leads to the following definition.

**Definition 27 (Values for multi-choice games with a permission structure).**

- *The value* $f^{(+,m)}$ *on* $\mathcal{GP}_T$ *is defined as*

$$\begin{aligned} f^{(+,m)}(m,v,P) &= \Upsilon(m,v,P_m^+) \\ &= DP(m,\mathcal{R}_{P_m^+}(v)). \end{aligned} \tag{3.9}$$

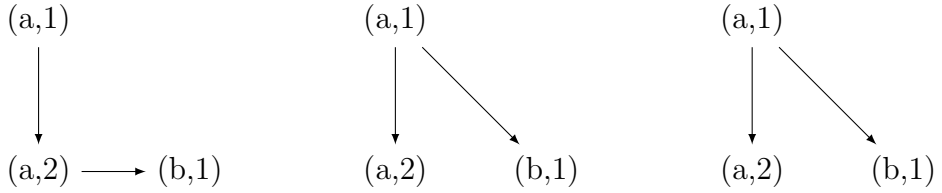
- *The value* $f^{(+,1)}$ *on* $\mathcal{GP}_T$ *is defined as*

$$\begin{aligned} f^{(+,1)}(m,v,P) &= \Upsilon(m,v,P_1^+) \\ &= DP(m,\mathcal{R}_{P_1^+}(v)). \end{aligned} \tag{3.10}$$

- *The value* $f^{(+,F)}$ *on* $\mathcal{GP}_T$ *is defined as*

$$\begin{aligned} f^{(+,F)}(m,v,P) &= \Upsilon(m,v,P_F^+) \\ &= DP(m,\mathcal{R}_{P_F^+}(v)). \end{aligned} \tag{3.11}$$

**Example 7.** *Consider the permission structure of Example 6. For the sake of simplicity, assume that* $m_a = 2$ *and* $m_b = 1$ *so that* $m = (2,1)$*. The induced pal-permission structures* $P_m^+$*,* $P_1^+$ *and* $P_F^+$ *are represented by figures* $(a)$*,* $(b)$ *and* $(c)$*, respectively.*




(a) $P_m^+$ (b) $P_1^+$ (c) $P_F^+$

*The multi-choice game* $(m,v)$ *is given by the following table*

| $(s_a,s_b)$ | $(0,0)$ | $(1,0)$ | $(2,0)$ | $(0,1)$ | $(1,1)$ | $(2,1)$ |
|---|---|---|---|---|---|---|
| $v(s)$ | 0 | 1 | 4 | 1 | 3 | 5 |

*The three restricted games corresponding to each induced pal-permission structure are as follows*

| $(s_F,s_W)$ | $(0,0)$ | $(1,0)$ | $(2,0)$ | $(0,1)$ | $(1,1)$ | $(2,1)$ |
|---|---|---|---|---|---|---|
| $\mathcal{R}_{P_m^+}(v)(s)$ | 0 | 1 | 4 | 0 | 1 | 5 |
| $\mathcal{R}_{P_1^+}(v)(s)$ | 0 | 1 | 4 | 0 | 3 | 5 |
| $\mathcal{R}_{P_F^+}(v)(s)$ | 0 | 1 | 4 | 0 | 3 | 5 |

*The worth* $\mathcal{R}_{P_m^+}(v)(1,1) = v(1,0) = 1$ *because* $b$ *is unable to achieve activity level* 1 *if* $a$ *has not reached activity level* 2*. It follows that* $\sigma(1,1) = (1,0)$ *under the m-permission structure. Thus, we obtain*

$$\mathcal{R}_{P_m^+}(v)(1,1) = v(\sigma(1,1)) = v(1,0) = 1.$$

*Similarly,* $\mathcal{R}_{P_1^+}(v)(0,1) = 0$ *because* $b$ *does not have the permission of* $a$ *to cooperate at level* 1 *if* $a$ *is inactive. It follows that* $\sigma(0,1) = (0,0)$ *under the 1-permission structure. Thus, we obtain*

$$\mathcal{R}_{P_1^+}(v)(0,1) = v(\sigma(0,1)) = v(0,0) = 0.$$

*In this example,* $\mathcal{R}_{P_1^+}(v)$ *and* $\mathcal{R}_{P_F^+}(v)$ *coincide because* $P_1^+$ *and* $P_F^+$ *coincide in this particular case. Finally, we compute* $f^{(+,m)}$*,* $f^{(+,1)}$*,* $f^{(+,F)}$ *and the DP value.*

| $(i,j)$ | $(a,1)$ | $(a,2)$ | $(b,1)$ |
|---|---|---|---|
| $f^{(+,m)}(m,v,P)$ | 3.17 | 1.17 | 0.67 |
| $f^{(+,1)}(m,v,P)$ | 2.83 | 1.83 | 0.33 |
| $f^{(+,F)}(m,v,P)$ | 2.83 | 1.83 | 0.33 |
| $DP(m,v)$ | 2.67 | 1.17 | 1.17 |

*As expected* $f^{(+,m)}$*,* $f^{(+,1)}$ *and* $f^{(+,F)}$ *allocate a greater payoff to the root* $(a,1)$ *than the DP value.*

## 3.7 Axiomatic characterizations

This section provides comparable axiomatic characterizations of $f^{(+,m)}$, $f^{(+,1)}$, and $f^{(+,F)}$. These axiomatic characterizations preserve the principles of Efficiency, Permission fairness, and the Inessential player property contained in Theorem 11 for TU-games with a permission tree structure. However, four types of modifications need to be incorporated due to the multi-choice nature of the game and the two types of permission relationships:

- two, rather than one, principles of permission fairness are introduced. The first one concerns inter-individual relationships, the second one concerns intra-individual relationships;

- the above two principles of permission fairness no longer concern the players but the player-activity level pairs;

- the notion of inessential player must be extended to that of inessential pair;

- because the values $f^{(+,m)}$, $f^{(+,1)}$ and $f^{(+,F)}$ are built on different pal-permission structures, $P_m^+$, $P_1^+$ and $P_F^+$ respectively, each of them have different interpretation of permission fairness and of an inessential pair.

### 3.7.1 Characterization of $f^{(+,m)}$

Suppose that a pair $(i,j) \in M^+$ is null in a game $(m,v,P) \in \mathcal{GP}_T$. Assume first that $P$ coincides with the trivial permission structure $S^0$. Then, player $i$ does not need anyone's permission to participate at level $j$, and the principle incorporated in the Null pair property (see 2.50) applies. But, if $P$ differs from the trivial structure $S^0$, although the pair $(i,j)$ is a null pair, it might be the case that other players need $(i,j)$'s permission to fully cooperate. In that case, it seems no longer reasonable to assume that the pair $(i,j)$ gets a zero payoff. The Null pair property is still applicable for the case in which $(i,j)$ is a null pair and all pairs $(k,l)$ are null pairs in $(m,v)$, where $k$ is a subordinate of $i$. Formally, a pair $(i,j) \in M^+$ is a **m-inessential pair** if it is a null pair in $(m,v)$ and if, for each of its subordinates $k \in \hat{P}(i)$, it holds that $(k,1)$ is a null pair in $(m,v,P)$. The set of all m-inessential pairs in $(m,v,P)$ is denoted by $IL^m(m,v,P)$. The following axiom indicates that each m-inessential pair gets a null payoff.

**Axiom 35 (m-Inessential pair property).** *For each $(m,v,P) \in \mathcal{GP}_T$, if $(i,j) \in IL^m(m,v,P)$, then*

$$f_{ij}(m,v,P) = 0.$$

The m-Inessential pair property implies the Null pair property (see 2.50).

$$\text{m-Inessential pair property} \implies \text{Null pair property}$$

Also, the m-Inessential pair property extends the Inessential player property (see 2.22) from TU-games with a permission structure to multi-choice games with a permission structure.

The following two axioms apply the principle of permission fairness into two directions. Before introducing these axioms, we need a definition. Take any $(m,v,P) \in \mathcal{GP}_T$. For each pair $(i,j') \in M^+$, define the associated $(\mathbf{i},\mathbf{j'})$**-game** $(m, v_m^{-(i,j')}, P) \in \mathcal{GP}_T$ where in $v_m^{-(i,j')}$, player $i$ stops any activity from level $j'$ onwards, and its subordinates are no longer active. For each $s \in \mathcal{M}$, the worth $v_m^{-(i,j')}(s)$ is defined as

$$v_m^{-(i,j')}(s) = v(\bar{s}), \qquad \text{where } \bar{s}_k = \begin{cases} j'-1 & \text{if } k = i \text{ and } s_k \geq j', \\ 0 & \text{if } k \in \hat{P}(i), \\ s_k & \text{otherwise.} \end{cases}$$

By construction, for each $k \in \hat{P}(i)$ and each $l \in M_k^+$,

$$(k,l) \in IL^m(m, v_m^{-(i,j')}, P).$$

Similarly, for each $j' \leq j \leq m_i$,

$$(i,j) \in IL^m(m, v_m^{-(i,j')}, P).$$

The first axiom of permission fairness indicates that the changes in payoff of the pairs $(i,j) \in M_i^+$, for $j \leq j'$ are equal.

**Axiom 36 (Intra m-fairness).** *For each* $(m, v, P) \in \mathcal{GP}_T$ *and each* $(i, j), (i, j') \in M^+$ *such that* $j' \geq j$,

$$f_{ij}(m, v, P) - f_{ij}(m, v_m^{-(i,j')}, P) = f_{ij'}(m, v, P) - f_{ij'}(m, v_m^{-(i,j')}, P).$$

The second axiom of permission fairness considers the payoff variation of the first activity level of a player $k$ and the payoff variation of the maximal activity level of its superior $i$ when player $k$ and all its subordinates do not participate anymore. This axiom requires an equal payoff variation for the pairs $(k, 1)$ and $(i, m_i)$.

**Axiom 37 (Inter m-fairness).** *For each* $(m, v, P) \in \mathcal{GP}_T$ *and each* $(k, 1) \in M^+$, *where* $k$ *is different from the root of* $P$, *and* $i \in P^{-1}(k)$,

$$f_{im_i}(m, v, P) - f_{im_i}(m, v_m^{-(k,1)}, P) = f_{k1}(m, v, P) - f_{k1}(m, v_m^{-(k,1)}, P).$$

Inter m-fairness extends Permission fairness (see 2.23) from TU-games with a permission structure to multi-choice games with a permission structure.

In a pal-permission structure $P_m^+$, a coalition $s \in \mathcal{M}$ is feasible in the sense of (3.1) if the superiors of each active player have reached their maximal activity level. From this observation, the set of feasible coalitions $\mathcal{F}(m, P_m^+)$ is given by

$$\mathcal{F}(m, P_m^+) := \left\{ s \in \mathcal{M} : \left[ s_i > 0 \text{ and } k \in \hat{P}^{-1}(i) \right] \implies \left[ s_k = m_k \right] \right\}, \tag{3.12}$$

from which we deduce the following proposition.

**Proposition 13.** *The sovereign part* $\sigma(s)$ *and the authorizing part* $\alpha(s)$ *of a coalition* $s$ *with respect to* $P_m^+$ *are defined as*[1]

$$\text{- } \forall i \in N, \quad \sigma_i(s) = \begin{cases} 0 & \text{if there exists } k \in \hat{P}^{-1}(i) \text{ such that } s_k < m_k, \\ s_i & \text{otherwise.} \end{cases}$$

$$\text{- } \forall i \in N, \quad \alpha_i(s) = \begin{cases} m_i & \text{if there exists } k \in \hat{P}(i) \text{ such that } s_k > 0, \\ s_i & \text{otherwise.} \end{cases}$$

*Proof.* The proof follows directly from (3.12) and (3.2). ■

**Example 8.** *In Example 6, the set of feasible coalitions under* $P_m^+$ *consists of all coalitions such that if* $s_b > 0$, *then* $s_a = 3$. *For instance, coalition* $(s_a, s_b) = (2, 2)$ *is not feasible since* $s_b = 2 > 0$ *and* $s_a = 2 < 3$. *In this case,* $\sigma(2, 2) = (2, 0)$ *because* $\sigma_b(2, 2) = 0$ *for the above mentioned reason. On the other hand,* $\alpha(2, 2) = (3, 2)$ *since* $\alpha_a(2, 2) = 3$.

[1]We continue to denote the sovereign part and the authorizing part of $s$ by $\sigma(s)$ and $\alpha(s)$ with the understanding that the underlying pal-permission structure is $P_m^+$.

The next proposition identifies a subset of coalitions whose worth is unchanged in the associated $(k,l)$-game $(m, v_m^{-(k,l)}, P)$. From this proposition, we deduce that the Harsanyi dividends of these coalitions coincide in $(m,v,P)$ and $(m, v_m^{-(k,l)}, P)$.

**Proposition 14.** *For each $(m,v,P) \in \mathcal{GP}_T$, each $(k,l) \in M^+$ and each coalition $s \in \mathcal{M}$ such that $\alpha_k(s) < l$,*

$$v_m^{-(k,l)}(s) = v(s).$$

*Proof.* By definition of $v_m^{-(k,l)}(s)$,

$$v_m^{-(k,l)}(s) = v(\bar{s}),$$

where

$$\bar{s}_h = \begin{cases} l-1 & \text{if } h = k \text{ and } s_h \geq l, \\ 0 & \text{if } h \in \hat{P}(k), \\ s_h & \text{otherwise.} \end{cases}$$

First, by Proposition 13, $\alpha_k(s) < l \leq m_i$ and $h \in \hat{P}(k)$ imply that $s_h = 0$, so that $s_h = \bar{s}_h = 0$. Second, by definition of the authorizing part of a coalition, $s_k \leq \alpha_k(s)$. Thus, by definition of $v_m^{-(k,l)}$, $\bar{s}_k = s_k$. Third, by definition of $v_m^{-(k,l)}$, we also have $\bar{s}_h = s_h$ for each other player $h \notin \hat{P}(k) \cup \{k\}$. Conclude that $\bar{s} = s$ and so $v_m^{-(k,l)}(s) = v(\bar{s}) = v(s)$. ■

**Proposition 15.** *For each $(m,v,P) \in \mathcal{GP}_T$ and each $s \in \mathcal{M}$ such that $\alpha_k(s) < l$,*

$$\Delta_{v_m^{-(k,l)}}(s) = \Delta_v(s).$$

*Proof.* The proof follows from Proposition 14 and the recursive definition of the Harsanyi dividend (see (2.3)). ■

**Theorem 18.** *A value $f$ on $\mathcal{GP}_T$ satisfies Efficiency, the m-Inessential pair property, Intra m-fairness and Inter m-fairness if and only if $f = f^{(+,m)}$.*

*Proof.* First, we show that $f^{(+,m)}$ satisfies all the axioms of the statement of Theorem 18. Take any $(m,v,P) \in \mathcal{GP}_T$. The permission structure $P$ induces the unique m-permission structure $P_m^+ \in \mathcal{P}$ and thus a unique game $(m,v,P_m^+) \in \mathcal{GP}$.

**Efficiency**: By definition of $f^{(+,m)}$,

$$f^{(+,m)}(m,v,P) = DP(m, \mathcal{R}_{P_m^+}(v)) = DP(m, v \circ \sigma).$$

From Section 2.4.2, we know that the $DP$ value is an efficient value, and from (3.2), we know that $\sigma(m) = m$. Thus, we obtain

$$\sum_{i\in N}\sum_{j=1}^{m_i} f_{ij}^{(+,m)}(m,v,P) = \sum_{i\in N}\sum_{j=1}^{m_i} DP_{ij}(v\circ\sigma) = v(\sigma(m)) = v(m),$$

which proves that $f^{(+,m)}$ satisfies Efficiency.

**m-Inessential pair property**: From Section 2.4.2, we know that the $DP$ value satisfies the Null pair property, that is, each null pair obtains a null payoff in a multi-choice game. Thus, by definition of $f^{(+,m)}$, it is sufficient to prove that an m-inessential pair in $(m,v,P)$ is a null pair in $(m,\mathcal{R}_{P_m^+}(v))$. Take any m-inessential pair $(i,j) \in IL^m(m,v,P)$ and any coalitions $s, s' \in \mathcal{M}$ such that $s_i = j-1$ and $s' = (s_{-i}, l)$, $j \le l \le m_i$. Because $s'_k = s_k$ for each $k \in N \setminus \hat{P}(i)$, it follows that $\sigma_k(s') = \sigma_k(s)$. For each $k \in \hat{P}(i) \cup \{i\}$, it holds that $\sigma_k(s') \ge \sigma_k(s)$. In particular, the superiors of player $i$ play the same activity level in $s$ and $s'$. Thus, if $\sigma_i(s) = 0$, then $\sigma_i(s') = 0$, and so $v(\sigma(s')) = v(\sigma(s))$. If $\sigma_i(s) = j-1$, then $\sigma_i(s') = l \ge j$. Because $(i,j)$ is an m-inessential pair, we still have $v(\sigma(s')) = v(\sigma(s))$. It follows that $(i,j)$ is a null pair in $\mathcal{R}_{P_m^+}(v)$, and so

$$f_{ij}^{(+,m)}(m,v,P) = DP_{ij}(m,\mathcal{R}_{P_m^+}(v)) = 0,$$

which shows that $f^{(+,m)}$ satisfies the m-Inessential pair property.

**Intra m-fairness**: Take any two pairs $(i,j),(i,j') \in M^+$ such that $j < m_i$ and $j' \ge j$. By definition,

$$\begin{aligned}
&f_{ij}^{(+,m)}(m,v,P) - f_{ij}^{(+,m)}(m,v_m^{-(i,j')},P)\\
&= \sum_{\substack{s\le m\\ j\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ j\le\alpha_i(s)}} \frac{\Delta_{v_m^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= \sum_{\substack{s\le m\\ j\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ j\le\alpha_i(s)\\ j'>\alpha_i(s)}} \frac{\Delta_{v_m^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ j\le\alpha_i(s)\\ j'\le\alpha_i(s)}} \frac{\Delta_{v_m^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)}
\end{aligned}$$

By Proposition 15, $\Delta_{v_m^{-(i,j')}}(s) = \Delta_v(s)$ when $\alpha_i(s) < j'$. Thus, we obtain

$$\begin{aligned}
&f_{ij}^{(+,m)}(m,v,P) - f_{ij}^{(+,m)}(m,v_m^{-(i,j')},P) \\
&= \sum_{\substack{s\le m\\ j\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ j\le\alpha_i(s)\\ j'>\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ j\le\alpha_i(s)\\ j'\le\alpha_i(s)}} \frac{\Delta_{v_m^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= \sum_{\substack{s\le m\\ j\le\alpha_i(s)\\ j'\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ j'\le\alpha_i(s)}} \frac{\Delta_{v_m^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= \sum_{\substack{s\le m\\ j'\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ j'\le\alpha_i(s)}} \frac{\Delta_{v_m^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= f_{ij'}^{(+,m)}(m,v,P) - f_{ij'}^{(+,m)}(m,v_m^{-(i,j')},P),
\end{aligned}$$

which shows that $f^{(+,m)}$ satisfies Intra m-fairness.

**Inter m-fairness**: Take any $k, i \in N$ such that $i \in P^{-1}(k)$. By definition,

$$\begin{aligned}
&f_{im_i}^{(+,m)}(m,v,P) - f_{im_i}^{(+,m)}(m,v_m^{-(k,1)},P) \\
&= \sum_{\substack{s\le m\\ m_i=\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ m_i=\alpha_i(s)}} \frac{\Delta_{v_m^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= \sum_{\substack{s\le m\\ m_i=\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ m_i=\alpha_i(s)\\ \alpha_k(s)=0}} \frac{\Delta_{v_m^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ m_i=\alpha_i(s)\\ 1\le\alpha_k(s)}} \frac{\Delta_{v_m^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)}
\end{aligned}$$

By Proposition 15, $\Delta_{v_m^{-(k,1)}}(s) = \Delta_v(s)$ when $\alpha_k(s) = 0$. Thus, we obtain

$$\begin{aligned}
&f_{im_i}^{(+,m)}(m,v,P) - f_{im_i}^{(+,m)}(v_m^{-(k,1)},P) \\
&= \sum_{\substack{s\le m\\ m_i=\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ m_i=\alpha_i(s)\\ \alpha_k(s)=0}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ m_i=\alpha_i(s)\\ 1\le\alpha_k(s)}} \frac{\Delta_{v_m^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= \sum_{\substack{s\le m\\ m_i=\alpha_i(s)\\ 1\le\alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m\\ m_i=\alpha_i(s)\\ 1\le\alpha_k(s)}} \frac{\Delta_{v_m^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)}.
\end{aligned}$$

Recall that, for each $s \in \mathcal{M}$, $\alpha(s)$ is a feasible coalition. By definition of a feasible coalition, if $\alpha_k(s) \geq 1 > 0$ then $\alpha_i(s) = m_i$ because $k \in P(i)$. Thus, we obtain

$$\begin{aligned}
&f^{(+,m)}_{im_i}(m, v, P) - f^{(+,m)}_{im_i}(v_m^{-(k,1)}, P) \\
&= \sum_{\substack{s \leq m \\ 1 \leq \alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h \in N} \alpha_h(s)} - \sum_{\substack{s \leq m \\ 1 \leq \alpha_k(s)}} \frac{\Delta_{v_m^{-(k,1)}}(s)}{\sum_{h \in N} \alpha_h(s)} \\
&= f^{(+,m)}_{k1}(m, v, P) - f^{(+,m)}_{k1}(m, v_m^{-(k,1)}, P),
\end{aligned}$$

which shows that $f^{(+,m)}$ satisfies Inter m-fairness. In particular, $f^{(+,m)}_{k1}(m, v_m^{-(k,1)}, P) = 0$, because $(k, 1)$ is an m-inessential pair in $v_m^{-(k,1)}$.

To complete the proof, it remains to show that there is at most one value $f$ satisfying Efficiency, the m-Inessential pair property, Intra m-fairness and Inter m-fairness. So, take any value $f$ that satisfies the above four axioms and take any $(m, v, P) \in \mathcal{GP}_T$. Without loss of generality, suppose that $2 \in S(1)$ and 1 is the root of the permission tree. To show uniqueness, we proceed by (descending) induction over the cardinality of $IL^m(m, v, P)$.

**Initialization**: If $|IL^m(m, v, P)| = \sum_{i \in N} m_i$, then each activity level of each player is an m-inessential pair. By the m-Inessential pair property, for each $(i, j) \in M^+$,

$$f_{ij}(m, v, P) = 0.$$

If $|IL^m(m, v, P)| = \sum_{i \in N} m_i - 1$, then each activity level of each player is an m-inessential pair, except the first activity level of the top player, that is, $(1, 1) \notin IL^m(m, v, P)$. Indeed, if $(1, 1) \in IL^m(m, v, P)$ then, by definition of a m-essential pair, any pair in $M^+$ is a null pair, which leads to a contradiction. By Efficiency and the m-Inessential pair property,

$$f_{11}(m, v, P) = v(m) \text{ and } f_{ij}(m, v, P) = 0$$

for each pair $(i, j) \neq (1, 1)$. So, $f$ is uniquely determined.

**Induction hypothesis**: Suppose that the statement is true for any $(m, v, P) \in \mathcal{GP}_T$ such that $|IL^m(m, v, P)| = I + 1$ where $I + 1 \leq \sum_{i \in N} m_i - 1$.

**Induction step**: Take any $(m, v, P) \in \mathcal{GP}_T$ such that $|IL^m(m, v, P)| = I$.

For any pair $(i, j) \in IL^m(m, v, P)$, the m-Inessential pair property implies

$$f_{ij}(m, v, P) = 0.$$

For any two pairs $(i, j), (i, j + 1) \in M_i^+ \setminus IL^m(m, v, P)$ such that $i \in N$ and $j < m_i$, Intra m-fairness implies

$$f_{ij}(m,v,P) - f_{i(j+1)}(m,v,P) = f_{ij}(m,v_m^{-(i,j+1)},P) - f_{i(j+1)}(m,v_m^{-(i,j+1)},P). \quad (3.13)$$

Next, for any two pairs $(i,m_i),(k,1) \in M^+ \setminus IL^m(m,v,P)$ such that $k \in P(i)$, Inter m-fairness implies

$$f_{im_i}(m,v,P) - f_{i1}(m,v_m,P) = f_{im_i}(m,v_m^{-(k,1)},P) - f_{k1}(m,v_m^{-(k,1)},P). \quad (3.14)$$

Using Intra m-fairness and Inter m-fairness and the fact that $P$ is a permission tree structure, we can generate $\sum_{i\in N} m_i - |IL^m(m,v,P)| - 1$ linear equations. By Efficiency,

$$\sum_{(i,j)\in M^+\setminus IL^m(m,v,P)} f_{ij}(m,v,P) = v(m). \quad (3.15)$$

We obtain a system of $\sum_{i\in N} m_i - |IL^m(m,v,P)|$ linear equations of the following form

$$\begin{cases} f_{11}(m,v,P) - f_{12}(m,v,P) = f_{11}(m,v_m^{(1,2)},P) - f_{12}(m,v_m^{-(1,2)},P) \\ f_{12}(m,v,P) - f_{13}(m,v,P) = f_{12}(m,v_m^{(1,3)},P) - f_{13}(m,v_m^{-(1,3)},P) \\ \dots \\ f_{1(m_1-1)}(m,v,P) - f_{1m_1}(m,v,P) = \\ f_{1(m_1-1)}(m,v_m^{(1,m_1)},P) - f_{1(m_1-1)}(m,v_m^{-(1,(m_1-1))},P) \\ f_{1m_1}(m,v,P) - f_{21}(m,v,P) = f_{1m_1}(m,v_m^{-(2,1)},P) - f_{21}(m,v_m^{-(2,1)},P) \\ f_{21}(m,v,P) - f_{22}(m,v,P) = f_{21}(m,v_m^{(2,2)},P) - f_{22}(m,v_m^{-(2,2)},P) \\ \dots \\ \sum_{(i,j)\in M^+\setminus IL^m(m,v,P)} f_{ij}(m,v,P) = v(m). \end{cases}$$

Recall that $(k,1)$ and $(i,j+1)$ are m-inessential pairs in $v_m^{-(k,1)}$ and $v_m^{-(i,j+1)}$ respectively, but are not m-inessential pairs in $(m,v)$. Thus,
$|IL^m(m,v_m^{-(k,1)},P)| \geq I+1$ and $|IL^m(v_m^{-(i,j+1)},P)| \geq I+1$. By the induction hypothesis,
$f(m,v_m^{-(k,1)},P)$ and $f(m,v_m^{-(i,j+1)},P)$ are uniquely determined. Therefore, all the unknowns of the linear system formed by (3.13), (3.14), and (3.15) are located on the left hand side side of the equations.

Let us show that this system of $\sum_{i\in N} m_i - |IL^m(m,v,P)|$ linear equations with $\sum_{i\in N} m_i - |IL^m(m,v,P)|$ unknowns leads to a unique solution. It admits a unique solution if and only if

$$Ax = b$$

admits a unique solution $x \in \mathbb{R}^{\sum_{i\in N} m_i - |IL^m(m,v,P)|}$, where $A$ is a matrix of size

$$\big(\sum_{i\in N} m_i - |IL^m(m,v,P)|\big) \times \big(\sum_{i\in N} m_i - |IL^m(m,v,P)|\big)$$

and $b$ a real vector of size

$$\sum_{i\in N} m_i - |IL^m(m,v,P)|,$$

respectively given by

$$A = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 & 0 \\ 0 & 1 & -1 & \dots & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & \dots & 1 & -1 & 0 \\ 0 & 0 & \dots & 0 & 1 & -1 \\ 1 & 1 & 1 & \dots & 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} f_{11}(m, v_m^{(1,2)}, P) - f_{12}(m, v_m^{-(1,2)}, P) \\ f_{12}(m, v_m^{(1,3)}, P) - f_{13}(m, v_m^{-(1,3)}, P) \\ \dots \\ f_{1m_1}(m, v_m^{-(2,1)}, P) - f_{21}(m, v_m^{-(2,1)}, P) \\ \dots \\ v(m) \end{pmatrix}$$

The equation $Ax = b$ admits a unique solution $x = A^{-1}b$ if and only if $A$ is invertible, i.e., the matrix $A$ is full rank.

We show that $A$ is full rank. We prove such statement by showing the linear independence of $A$'s columns. Let us denote by $a_i$ the $i$-th column of $A$ such that $A = (a_1, \dots, a_n)$. Suppose that there exists $\lambda \in \mathbb{R}^{\sum_{i\in N} m_i - |IL^m(m,v,P)|}$ such that

$$\lambda_1 a_1 + \lambda_2 a_2 + \ldots + \lambda_{\sum_{i\in N} m_i - |IL^m(m,v,P)|} a_{\sum_{i\in N} m_i - |IL^m(m,v,P)|} = 0.$$

First, one can infer from the $\sum_{i\in N} m_i - |IL^m(m,v,P)| - 1$ first rows of $A$ that

$$\lambda_1 = \lambda_2 = \ldots = \lambda_{\sum_{i\in N} m_i - |IL^m(m,v,P)|}.$$

Then, the last row of $A$ indicates that

$$\lambda_1 + \lambda_2 + \ldots + \lambda_{\sum_{i\in N} m_i - |IL^m(m,v,P)|} = 0.$$

Hence $\lambda_1 = \lambda_2 = \ldots = \lambda_{\sum_{i\in N} m_i - |IL^m(m,v,P)|} = 0$, meaning the columns of $A$ are linearly independent which is equivalent to say that $A$ is a full rank matrix. Hence, the linear system of equations leads to a unique solution. This completes the proof of the induction step.

Conclude that $f^{(+,m)}$ is the unique value on $\mathcal{GP}_T$ satisfying Efficency, the m-Inessential pair property, Intra m-fairness and Inter m-fairness, as desired.

■

The four axioms of the statement of Theorem 18 are logically independent, as shown by the following alternative solutions.

- The value $f$ given, for each $(m, v, P) \in \mathcal{GP}_T$, by $f(m, v, P) = \vec{0}$, satisfies all the axioms except Efficiency.

- We use an extension of the Equal division value from TU-games to multi-choice games (see (2.42)). Fix a real number $\alpha \in ]0, 1[$. The value $f^\alpha$ given, for each $(m, v, P) \in \mathcal{GP}_T$, by

$$\forall i \in N, j \in M_i, \quad f_{ij}^\alpha(m, v, P) = \alpha ED_{ij}^1(m, v) + (1 - \alpha) f_{ij}^{(+,m)}(m, v, P),$$

  satisfies all the axioms except the m-Inessential pair property.

- Take any $(m, v, P) \in \mathcal{GP}_T$. Recall that

$$\forall s \in \mathcal{M}, \quad C(s) = \{(i, j) \in M^+ : j = s_i\}. \tag{3.16}$$

  Define, for each $s \in \mathcal{M}$, the set of pairs

$$W(s) = C(s) \cup \{(i, m_i) \in M^+ : (P(i), 1) \in C(s)\}.$$

  The value $f$ given by

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, P) = \sum_{\substack{s \leq m \\ (i,j) \in W(s)}} \frac{\Delta_v(s)}{|W(s)|},$$

  satisfies all the axioms except Intra m-fairness.

- The value $f$ given, for each $(m, v, P) \in \mathcal{GP}_T$, by $f(m, v, P) = DP(m, v)$, satisfies all the axioms except Inter m-fairness. Indeed, Inter m-fairness indicates that the payoff of a pair $(i, m_i)$ varies if the subordinates of $i$ do no longer participate. This implies that the pair $(i, m_i)$ may have a non-null payoff before or after these players stop participating, even if $(i, m_i)$ is a null pair. However, the $DP$ value satisfies the Null pair property, which contradicts this observation.

### 3.7.2 Characterization of $f^{(+,1)}$

The reasoning is similar to the previous characterization, except that the concepts of inessential players and fairness properties must be adjusted according to the pal-permission structure $P^{+,1}$.

Here, a player is considered to be inessential in two cases. First, if it and each of its subordinates do not generate any worth at any level of activity. Second, if there is an activity level greater than 1 at which its productivity falls to zero. Formally, a pair $(i, j) \in M^+$ is a **1-inessential pair** in $(m, v, P) \in \mathcal{GP}_T$ in the following two cases:

- either $j = 1$ and $(i, j)$ is a null pair in $(m, v, P)$ and for each $k \in \hat{P}(i)$, $(k, 1)$ is also a null pair in $(m, v, P)$;

- or $j > 1$ and $(i, j)$ is a null pair in $(m, v, P)$.

The set of 1-inessential pairs in $(m, v, P)$ is denoted by $IL^1(m, v, P)$. Observe that $IL^m(m, v, P) \subseteq IL^1(m, v, P)$. This leads to a new inessential pair property, weaker than the m-Inessential pair property.

**Axiom 38 (1-Inessential pair property).** *For each $(m, v, P) \in \mathcal{GP}_T$, if $(i, j) \in IL^1(m, v, P)$, then*

$$f_{ij}(m, v, P) = 0.$$

The 1-Inessential pair property implies the Null pair property (see 2.50).

$$\text{1-Inessential pair property} \implies \text{Null pair property}$$

Also, the 1-Inessential pair property extends the Inessential player property (see 2.22) from TU-games with a permission structure to multi-choice games with a permission structure.

Next, we provide two new axioms of intra and inter fairness, which are based on similar principles as the previous ones except that the associated game from which they are defined is different. Precisely, take any $(m, v, P) \in \mathcal{GP}_T$. For each pair $(i, j') \in M^+$, define the associated **$(\mathbf{i}, \mathbf{j}')$-game** $(m, v_1^{-(i,j')}, P) \in \mathcal{GP}_T$ where in $v_1^{-(i,j')}$, player $i$ stops its activity from level $j'$ onward. If $j' > 1$, the other players are not affected. If $j = 1$, then $i$ and its subordinates are no longer active. For each $s \in \mathcal{M}$,

$$v_1^{-(i,j')}(s) = v(\tilde{s}) \qquad \text{where } \tilde{s}_k = \begin{cases} j' - 1 & \text{if } k = i \text{ and } s_k \geq j', \\ 0 & \text{if } k \in \hat{P}(i) \text{ and } j' = 1, \\ s_k & \text{otherwise.} \end{cases}$$

In $(m, v_1^{-(i,j)}, P)$ two cases occur: either $j' = 1$, and in this case any pair $(k, l)$ such that $k \in \hat{P}(i) \cup \{i\}$ becomes an 1-inessential pair; or $j' > 1$, and in this case, for each $j \geq j'$, the pair $(i, j)$ becomes 1-inessential.

**Axiom 39 (Intra 1-fairness).** *For each $(m, v, P) \in \mathcal{GP}_T$ and each $(i, j), (i, j') \in M^+$ such that $j' \geq j$,*

$$f_{ij}(m, v, P) - f_{ij}(m, v_1^{-(i,j')}, P) = f_{ij'}(m, v, P) - f_{ij'}(m, v_1^{-(i,j')}, P).$$

**Axiom 40 (Inter 1-fairness).** *For each $(m, v, P) \in \mathcal{GP}_T$, each $k \in N$, each $(k, 1) \in M^+$ and each $i \in P^{-1}(k)$,*

$$f_{i1}(m, v, P) - f_{i1}(m, v_1^{-(k,1)}, P) = f_{k1}(m, v, P) - f_{k1}(m, v_1^{-(k,1)}, P).$$

Contrary to Intra m-fairness, which compares the variations in payoffs of $(k,1)$ and $(i,m_i)$, Inter 1-fairness compares the variations in payoffs of $(k,1)$ and $(i,1)$.

Inter 1-fairness extends Permission fairness (see 2.23) from TU-games with a permission structure to multi-choice games with a permission structure.

In a pal-permission structure $P_1^+$, a coalition $s \in \mathcal{M}$ is feasible if the superiors of each active player are active as well. From this observation the set of feasible coalitions $\mathcal{F}(m, P_1^+)$ is given by

$$\mathcal{F}(m, P_1^+) = \left\{ s \in \mathcal{M} : \big[ s_i > 0 \text{ and } k \in P^{-1}(i) \big] \implies \big[ s_k \geq 1 \big] \right\}. \tag{3.17}$$

**Example 9.** *From Example 6, the coalition $(s_a, s_b) = (2,1)$ is a feasible coalition in $P_1^+$ (see Figure (b)). In this situation, b cannot cooperate until a makes at least its first activity level. When the coalition $(2,1)$ forms, it is understood that a undertakes its activity level 1 before reaching level 2. In this sense, both pairs $(a,1)$ and $(a,2)$ are to be considered, and because $(b,1) \in P_1^+(a,1)$, b is allowed to make its activity level 1. In other words, although $(a,2)$ is not a predecessor of $(b,1)$ in $P_1^+$, we consider that if level 2 is reached by a then level 1 must have been reached as well.*

From the definition of feasible coalitions, we deduce the following proposition.

**Proposition 16.** *The sovereign part $\sigma(s)$ and the authorizing part $\alpha(s)$ associated with a coalition $s \in \mathcal{M}$ in $P_1^+$ are defined as*

$$\forall i \in N, \quad \sigma_i(s) = \begin{cases} 0 & \text{if } \exists k \in \hat{P}^{-1}(i) \text{ such that } s_k = 0, \\ s_i & \text{otherwise.} \end{cases}$$

$$\forall i \in N, \quad \alpha_i(s) = \begin{cases} 1 & \text{if } \exists k \in \hat{P}(i) \text{ such that } s_k > 0 \text{ and } s_i = 0, \\ s_i & \text{otherwise.} \end{cases}$$

*Proof.* The proof of the proposition follows directly from (3.17) and (3.2). ■

**Example 10.** *Continuation of Example 9. The set of feasible coalitions under $P_1^+$ consists of all coalitions such that if $s_b > 0$, then $s_a \geq 1$. For instance, coalition $(s_a, s_b) = (0,1)$ is not feasible since $s_b = 1 > 0$ and $s_a = 0 < 1$.*

The next proposition identifies a subset of coalitions whose worth is unchanged in the associated $(k,l)$-game $(m, v_1^{-(k,l)}, P)$. From this proposition, we obtain that the Harsanyi dividends of these coalitions coincide in $(m, v, P)$ and $(m, v_1^{-(k,l)}, P)$.

**Proposition 17.** *For each $(m, v, P) \in \mathcal{GP}_T$ and each coalition $s \in \mathcal{M}$ such that $\alpha_k(s) < l$,*

$$v_1^{-(k,l)}(s) = v(s).$$

*Proof.* By definition of $v_1^{-(k,l)}$,

$$v_1^{-(k,l)}(s) = v(\tilde{s}),$$

where

$$\tilde{s}_h = \begin{cases} l-1 & \text{if } h = k \text{ and } s_h \geq l, \\ 0 & \text{if } h \in \hat{P}(k) \text{ and } l = 1, \\ s_h & \text{otherwise.} \end{cases}$$

By definition of the authorizing part of $s$ and hypothesis, $s_k \leq \alpha_k(s) < l$. Thus, $s_k < l$ and so $\tilde{s}_k = s_k$. For each $h \in \hat{P}(k)$, if $l = 1$, then $\alpha_k(s) = 0 = s_k$. Thus, by Proposition 16, $s_h = 0$ and, by definition, $\tilde{s}_h = 0$, so that $s_h = \tilde{s}_h$. For any $h \notin \hat{P}(k) \cup \{k\}$, by definition, $\tilde{s}_k = s_k$. All in all, we get $v_1^{-(k,l)}(s) = v(\tilde{s}) = v(s)$. ■

**Proposition 18.** *For each $(m, v, P) \in \mathcal{GP}_T$ and each $s \in \mathcal{M}$ such that $\alpha_k(s) < l$,*

$$\Delta_{v_1^{-(k,l)}(s)} = \Delta_v(s).$$

*Proof.* The proof follows directly from Proposition 17 and the recursive definition of the Harsanyi dividend (see (2.3)). ■

**Theorem 19.** *A value $f$ on $\mathcal{GP}_T$ satisfies Efficiency, the 1-Inessential pair property, Intra 1-fairness and Inter 1-fairness if and only if $f = f^{(+,1)}$.*

*Proof.* We only show that $f^{(+,1)}$ satisfies the 1-Inessential pair property, Intra 1-fairness and Inter 1-fairness. The rest of the proof is similar to the proof of Theorem 18, and so is omitted. Take any $(m, v, P) \in \mathcal{GP}_T$. The permission structure $P$ induces the unique 1-permission structure $P_1^+ \in \mathcal{P}$ and thus a unique game $(m, v, P_1^+) \in \mathcal{GP}$.

**1-Inessential pair property**: to show that $f^{(+,1)}$ satisfies the axiom it is sufficient to show that any 1-inessential pair in $(m, v, P)$ is a null pair in $(m, \mathcal{R}_{P_1^+}(v))$. Take any pair $(i, j) \in IL^1(m, v, P)$. Take any $s \in \mathcal{M}$ such that $s_i = j - 1$, and consider the coalition $s' = (s_{-i}, l)$ such that $j \leq l \leq m_i$. We distinguish two cases:

- if $j = 1$, then $(i, 1) \in IL^1(m, v, P)$ means that $(i, 1)$ is a null pair as well as each pair $(k, 1)$, $k \in \hat{P}(i)$. Next, by definition of the sovereign part of a coalition, $s' \geq s$ implies $\sigma(s') \geq \sigma(s)$. By Proposition 16, for each $k \notin \hat{P}(i)$, $\sigma_k(s') = \sigma_k(s)$. For $k \in \hat{P}(i) \cup \{i\}$, $\sigma_k(s') \geq \sigma_k(s)$. But, because $(i, 1) \in IL^1(m, v, P)$, we obtain that $v(\sigma(s')) = v(\sigma(s))$;

- if $j > 1$, then $(i, j) \in IL^1(m, v, P)$ means that the pair $(i, j)$ is a null level pair. By Proposition 16, only the sovereign part of $(i, j)$'s subordinates can be affected by a deviation initiated by $i$, and, by definition of $P_1^+$, these subordinates are precisely the pairs $(i, j+1), \ldots, (i, m_i)$. Because $j > 1$, the sovereign part of $(i, j)$'s subordinates are not affected by such a deviation. Finally, because $(i, j)$ is a null level pair, it follows that $v(\sigma(s')) = v(\sigma(s))$.

Thus, $(i,j)$ is a null pair in $(m, \mathcal{R}_{P_1^+}(v))$. By definition of $f^{(+,1)}$ and the Null pair property,

$$f_{ij}^{(+,1)}(m,v,P) = DP_{ij}(m, \mathcal{R}_{P_1^+}(v)) = 0,$$

which shows that $f^{(+,1)}$ satisfies the 1-Inessential pair property.

**Intra 1-fairness**: Take any two $(i,j),(i,j') \in M^+$ such that $j < m_i$ and $j' > j$. By definition of $f^{(+,1)}$,

$$\begin{aligned}
&f_{ij}^{(+,1)}(m,v,P) - f_{ij}^{(+,1)}(m, v_1^{-(i,j')}, P) \\
&= \sum_{\substack{s\le m \\ j\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m \\ j\le\alpha_i(s)}} \frac{\Delta_{v_1^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= \sum_{\substack{s\le m \\ j\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m \\ j\le\alpha_i(s) \\ j'>\alpha_i(s)}} \frac{\Delta_{v_1^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m \\ j\le\alpha_i(s) \\ j'\le\alpha_i(s)}} \frac{\Delta_{v_1^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)}.
\end{aligned}$$

By Proposition 18, $\Delta_{v_1^{-(i,j')}} = \Delta_v(s)$ when $\alpha_i(s) < j'$. Thus, we obtain

$$\begin{aligned}
&f_{ij}^{(+,1)}(m,v,P) - f_{ij}^{(+,1)}(m, v_1^{-(i,j')}, P) \\
&= \sum_{\substack{s\le m \\ j\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m \\ j\le\alpha_i(s) \\ j'>\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m \\ j\le\alpha_i(s) \\ j'\le\alpha_i(s)}} \frac{\Delta_{v_1^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= \sum_{\substack{s\le m \\ j\le\alpha_i(s) \\ j'\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m \\ j\le\alpha_i(s) \\ j'\le\alpha_i(s)}} \frac{\Delta_{v_1^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= \sum_{\substack{s\le m \\ j'\le\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\le m \\ j'\le\alpha_i(s)}} \frac{\Delta_{v_1^{-(i,j')}}(s)}{\sum_{h\in N}\alpha_h(s)} \\
&= f_{ij'}^{(+,1)}(m,v,P) - f_{ij'}^{(+,1)}(m, v_1^{-(i,j')}, P),
\end{aligned}$$

which shows that $f^{(+,1)}$ satisfies Intra 1-fairness. Note that $f_{ij'}^{(+,1)}(m, v_1^{-(i,j')}, P) = 0$ because $(i,j')$ is an 1-inessential pair in $(m, v_1^{-(i,j')}, P)$.

**Inter 1-fairness**: Take any $k, i \in N$ such that $i \in P^{-1}(k)$. It holds that

$$
\begin{aligned}
&f_{i1}^{(+,1)}(m,v,P) - f_{i1}^{(+,1)}(m,v_1^{-(k,1)},P)\\
&= \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)}} \frac{\Delta_{v_1^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)\\ 1>\alpha_k(s)}} \frac{\Delta_{v_1^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)\\ 1\leq\alpha_k(s)}} \frac{\Delta_{v_1^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)}.
\end{aligned}
$$

By Proposition 18, note that $\Delta_{v_1^{-(k,1)}(s)} = \Delta_v(s)$ when $\alpha_k(s) < 1$. Thus, we obtain

$$
\begin{aligned}
&f_{i1}^{(+,1)}(m,v,P) - f_{i1}^{(+,1)}(m,v_1^{-(k,1)},P)\\
&= \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)\\ 1>\alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)\\ 1\leq\alpha_k(s)}} \frac{\Delta_{v_1^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)\\ 1\leq\alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_i(s)\\ 1\leq\alpha_k(s)}} \frac{\Delta_{v_1^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= \sum_{\substack{s\leq m\\ 1\leq\alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_k(s)}} \frac{\Delta_{v_1^{-(k,1)}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= f_{k1}^{(+,1)}(m,v,P) - f_{k1}^{(+,1)}(m,v_1^{-(k,1)},P),
\end{aligned}
$$

where the third equality follows from the fact that $\alpha_k(s) \geq 1$ implies $\alpha_i(s) \geq 1$ for $i \in \hat{P}^{-1}(k)$. This shows that $f^{(+,1)}$ satisfies Inter 1-fairness. Observe that $f_{k1}^{(+,1)}(m,v_1^{-(k,1)},P) = 0$, because $(k,1)$ is an 1-inessential pair in $(m,v_1^{-(k,1)},P)$. ■

The four axioms of the statement of Theorem 19 are logically independent. One can find examples showing this independence that are close to the examples proposed for the logical independence of Theorem 18. Therefore, we will omit the demonstration of the logical independence for Theorem 19.

### 3.7.3 Characterization of $f^{(+,F)}$

The characterization of $f^{(+,F)}$ slightly differs from the previous ones. There are two main changes:

- first, the axiomatic characterization cannot be stated on the full class $\mathcal{GP}_T$. Due to the definition of the F-permission structures (see Section 3.5) we need

to restrict the study to situations where, for each multi-choice game $(m, v)$ and each permission tree structure $P$, the following holds:

$$\forall i, j \in N, j \in P(i) \implies m_j \leq m_i.$$

The class of games satisfying this assumption is denoted by $\overline{\mathcal{GP}_T}$. This class of games contains the games in which each player has the same set of activity levels. The reason for this change is that the F-permission structure conveys the idea that each player should be able to supervise its subordinates. Thus, we expect each player to be able to achieve at least the same activity levels as its subordinates;

- second, we will no long apply the principle of permission fairness into two directions. Instead, a single axiom, generalizing the axiom of permission fairness, is proposed. As in the previous sections, the notion of inessential players is adjusted in the same way as in the previous sections.

Here, a player is considered to be inessential if there is an activity level at which its productivity and the productivity of its subordinates fall to zero. Formally, a pair $(i, j) \in M^+$ is a **F-inessential pair** in $(m, v, P) \in \overline{\mathcal{GP}_T}$ if it is a null pair in $(m, v, P)$ and if, for each $k \in \hat{P}(i)$, $(k, j)$ is also a null pair in $(m, v, P)$. The set of F-inessential pairs in $(m, v, P)$ is denoted by $IL^F(m, v, P)$. Observe that $IL^m(m, v, P) \subseteq IL^F(m, v, P)$. This leads to a new inessential pair property, weaker than the m-Inessential pair property.

**Axiom 41 (F-inessential pair property).** *For each* $(m, v, P) \in \overline{\mathcal{GP}_T}$, *if* $(i, j) \in IL^F(m, v, P)$, *then*

$$f_{ij}(m, v, P) = 0.$$

The F-Inessential pair property implies the Null pair property (see 2.50).

$$\text{F-Inessential pair property} \implies \text{Null pair property}$$

Also, the F-Inessential pair property extends the Inessential player property (see 2.22) from TU-games with a permission structure to multi-choice games with a permission structure.

Next, we provide one last axiom of permission fairness. Instead of dealing with permission fairness within and between players, the axiom compares the variations in payoffs of each pair with the variations in payoffs of the root of $P_F^+$ (despite not being a tree, the pal-permission structure $P_F^+$ still has a unique root). Without loss generality, we suppose that $(1, 1)$ is the root of $M^+$. Before introducing the axiom, we need a definition. Take any $(m, v, P) \in \overline{\mathcal{GP}_T}$. For each pair $(i, j') \in M^+$, define the associated $(\mathbf{i}, \mathbf{j'})-$**game** $(m, v_F^{-(i,j')}, P) \in \overline{\mathcal{GP}_T}$ where in $v_F^{-(i,j')}$, player $i$ and its

subordinates stop any activity from level $j'$ onward. For each $s \in \mathcal{M}$, the worth $v_F^{-(i,j')}$ is defined as

$$v_F^{-(i,j')}(s) = v(\hat{s}), \quad \text{where } \hat{s}_k = \begin{cases} j'-1 & \text{if } s_k \geq j' \text{ for } k \in \hat{P}(i) \cup \{i\}, \\ s_k & \text{otherwise.} \end{cases}$$

By construction, for each $k \in \hat{P}(i) \cup \{i\}$ and each $l \in M_k^+$ such that $l \geq j'$, $(k,l) \in IL^F(m, v_F^{-(i,j')}, P)$. The next axiom of permission fairness indicates that the changes in payoffs of the pair $(1,1)$ is equal to the changes in payoffs of the pair $(i,j')$.

**Axiom 42 (F-fairness).** *Take any $(m,v,P) \in \mathcal{GP}_T$, for each $(i,j') \in M^+ \setminus \{(1,1)\}$,*

$$f_{(1,1)}(m,v,P) - f_{(1,1)}(m, v_F^{-(i,j')}, P) = f_{(i,j')}(m,v,P) - f_{(i,j')}(m, v_F^{-(i,j')}, P).$$

In a pal-permission structure $P_F^+$, a coalition $s \in \mathcal{M}$ is feasible in the sense of (3.1) if each player is active at a higher level than its subordinates. From this observation, the set of feasible coalitions according to $P_F^+$ is denoted by $\mathcal{F}(m, P_F^+)$ and given by

$$\mathcal{F}(m, P_F^+) = \left\{ s \in \mathcal{M} : \left[ s_i > 0 \text{ and } k \in \hat{P}^{-1}(i) \right] \implies \left[ s_k \geq s_i \right] \right\}, \tag{3.18}$$

from which we deduce the following proposition.

**Proposition 19.** *The sovereign part $\sigma(s)$ and the authorizing part $\alpha(s)$ of a coalition $s$ in $P_F^+$ are defined as*

- $\forall i \in N, \quad \sigma_i(s) = \min_{k \in \hat{P}^{-1}(i) \cup \{i\}} s_k$;
- $\forall i \in N, \quad \alpha_i(s) = \max_{k \in \hat{P}(i) \cup \{i\}} s_k$.

*Proof.* The proof of the Proposition 19 follows directly from (3.18) and (3.2). ■

**Example 11.** *Continuation of Example 9. The set of feasible coalitions under $P_F^+$ consists of all coalitions such that if $s_a \geq s_b$. For instance, coalition $(s_a, s_b) = (0,1)$ is not feasible since $s_b = 1 > 0$ and $s_a = 0 < 1$.*

As in the previous sections, we introduce a proposition stating that the Harsanyi dividends of a certain subset of coalitions coincide in $(m,v,P)$ and $(m, v^{-(k,l)}, P)$.

**Proposition 20.** *For each $(m,v,P) \in \overline{\mathcal{GP}_T}$, $s \in \mathcal{M}$, $k \in N$ and $l \in M_k^+$ such that $\alpha_k(s) < l$,*

$$\Delta_{v_F^{-(k,l)}}(s) = \Delta_v(s).$$

We have the material to state the main result of this section.

**Theorem 20.** *A value $f$ satisfies Efficiency, the F-inessential pair property and F-fairness on $\overline{\mathcal{GP}_T}$ if and only if $f = f^{(+,F)}$.*

*Proof.* We show that $f^{(+,F)}$ satisfies the F-inessential pair property and the F-fairness. Efficiency follows in the same way as in the previous theorems. Take any F-permission structure $P_F^+ \in \mathcal{P}$ and the game $(m, v, P_F^+) \in \overline{\mathcal{GP}_T}$.

**F-inessential pair property**: It is sufficient to show that an F-inessential pair in $(m, v, P)$ is a null pair in $(m, \mathcal{R}_{P_F^+}(v))$. Take any pair $(i, j) \in IL^F(m, v, P)$. Take any $s \in \mathcal{M}$ such that $s_i = j - 1$, and take any coalition $s' = (s_{-i}, l)$ such that $j \leq l \leq m_i$. By definition of the sovereign part of a coalition, $s' \geq s$ implies $\sigma(s') \geq \sigma(s)$. By Proposition 19, for each $k \in \hat{P}^{-1}(i)$, $\sigma_k(s') = \sigma_k(s)$, and if, for $k \in \hat{P}(i) \cup \{i\}$, $\sigma_k(s') > \sigma_k(s)$ then $\sigma_k(s') = l$. But, because $(i, j) \in IL^F(m, v, P)$, we obtain $v(\sigma(s')) = v(\sigma(s))$.

Thus, $(i, j)$ is a null pair in $(m, \mathcal{R}_{P_F^+}(v))$. By definition of $f^{(+,F)}$ and the Null pair property,

$$f_{ij}^{(+,F)}(m, v, P) = DP_{ij}(m, \mathcal{R}_{P_F^+}(v)) = 0,$$

which shows that $f^{(+,F)}$ satisfies the F-inessential pair property.

**F-fairness**: Take any $(k, l) \in M^+ \setminus \{(1, 1)\}$. By definition of $f^{(+,F)}$,

$$\begin{aligned}
&f_{11}^{(+,F)}(m, v, P) - f_{11}^{(+,F)}(m, v_F^{-(k,l)}, P) \\
&= \sum_{\substack{s \leq m \\ 1 \leq \alpha_1(s)}} \frac{\Delta_v(s)}{\sum_{h \in N} \alpha_h(s)} - \sum_{\substack{s \leq m \\ 1 \leq \alpha_1(s)}} \frac{\Delta_{v_F^{-(k,l)}}(s)}{\sum_{h \in N} \alpha_h(s)} \\
&= \sum_{\substack{s \leq m \\ 1 \leq \alpha_1(s)}} \frac{\Delta_v(s)}{\sum_{h \in N} \alpha_h(s)} - \sum_{\substack{s \leq m \\ 1 \leq \alpha_1(s) \\ \alpha_k(s) < l}} \frac{\Delta_{v_F^{-(k,l)}}(s)}{\sum_{h \in N} \alpha_h(s)} - \sum_{\substack{s \leq m \\ 1 \leq \alpha_1(s) \\ l \leq \alpha_k(s)}} \frac{\Delta_{v_F^{-(k,l)}}(s)}{\sum_{h \in N} \alpha_h(s)}
\end{aligned}$$

By Proposition 20, $\Delta_{v_F^{-(k,l)}}(s) = \Delta_v(s)$ when $\alpha_k(s) < l$. Thus, we obtain

$$\begin{aligned}
&f_{11}^{(+,F)}(m,v,P) - f_{11}^{(+,F)}(m,v_F^{-(k,l)},P)\\
&= \sum_{\substack{s\leq m\\ 1\leq\alpha_1(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_1(s)\\ l>\alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_1(s)\\ l\leq\alpha_k(s)}} \frac{\Delta_{v^{-(k,l)}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= \sum_{\substack{s\leq m\\ 1\leq\alpha_1(s)\\ l\leq\alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ 1\leq\alpha_1(s)\\ l\leq\alpha_k(s)}} \frac{\Delta_{v_F^{-(k,l)}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= \sum_{\substack{s\leq m\\ l\leq\alpha_k(s)}} \frac{\Delta_v(s)}{\sum_{h\in N}\alpha_h(s)} - \sum_{\substack{s\leq m\\ l\leq\alpha_k(s)}} \frac{\Delta_{v_F^{-(k,l)}}(s)}{\sum_{h\in N}\alpha_h(s)}\\
&= f_{kl}^{(+,F)}(m,v,P) - f_{kl}^{(+,F)}(m,v_F^{-(k,l)},P),
\end{aligned}$$

where the third equality follows from Proposition 19. Conclude that $f_{kl}^{(+,F)}$ satisfies F-fairness. Note that $f_{kl}^{(+,F)}(m,v_F^{-(k,l)},P) = 0$ because $(k,l)$ is an F-inessential pair in $(m,v_F^{-(k,l)},P)$.

The uniqueness part of the proof is similar to the proof of Theorem 18. The only difference lies in the fact that the linear system of equations arises from F-fairness and Efficiency, instead of two axioms of permission fairness and Efficiency as in the previous proofs. Indeed, remark the F-fairness generates $\sum_{i\in N} m_i - 1$ linear equations and Efficiency generates one linear equation. We obtain a linear system of $\sum_{i\in N} m_i$ equations with $\sum_{i\in N} m_i$ unknows. Proceeding as in the proof of Theorem 18, it is easy to show that this system is linearly independent. ■

The three axioms of the statement of Theorem 20 are logically independent, as shown by the following alternative solutions.

- The value $f$ given, for each $(m,v,P) \in \mathcal{GP}_T$, by $f(m,v,P) = \vec{0}$, satisfies all the axioms except Efficiency.

- We use an extension of the Equal division value from TU-games to multi-choice games (see (2.42)). Fix a real number $\alpha \in ]0,1[$. The value $f^\alpha$ given, for each $(m,v,P) \in \mathcal{GP}_T$, by

$$\forall i \in N, j \in M_i, \quad f_{ij}^\alpha(m,v,P) = \alpha ED_{ij}^1(m,v) + (1-\alpha) f_{ij}^{(+,F)}(m,v,P),$$

  satisfies all the axioms except the F-inessential pair property.

- The value $f$ given, for each $(m, v, P) \in \mathcal{GP}_T$, by $f(m, v, P) = DP(m, v)$, satisfies all the axioms except F-fairness. Indeed, F-fairness indicates that the payoff of the pair $(1, 1)$ varies if a player and all of its subordinates no longer participate above a certain activity level. This implies that the pair $(1, 1)$ may receive a non-null payoff before or after these players stop participating, even if $(1, 1)$ is a null pair. However, the $DP$ value satisfies the Null pair property, which contradicts this observation.

## 3.8 Conclusion

This chapter takes advantage of the information provided by the distribution network to combine multi-choice games and permission tree structures. We have proposed two approaches to combine the intra-player relationships provided by the permission structure with the intra-player relationships provided by the game. This led us to the definition of three specific pal-permission structures from which we have constructed three new values. Finally, we have provided comparable axiomatic characterizations of these values.

This chapter sheds light on the additional difficulties that arise when one tries to combine a multi-choice game with a structure on the player set. It makes use of the information provided by the distribution network, but has no consideration for the Independence of higher demands or the Uniformity principle.

# Chapter 4: Marginalism, Egalitarianism and Efficiency in Multi-Choice Games

## 4.1 Introduction

Contrary to Chapter 3, this chapter focuses on the Independence of higher demands principle and the Uniformity principle. It will disregard the information provided by the distribution network.

This chapter addresses the trade-off between marginalism and egalitarianism, which is one of the main issues in economic allocation problems. Marginalism supports allocations based on a player's marginal contributions to coalitions, whereas egalitarianism is in favor of an equal allocation at the expense of the differences between the players' marginal contributions to coalitions. Observe that the Uniformity principle can be seen as a form of egalitarianism. In the context of TU-games, the trade-off between marginalism and egalitarianism can be seen as a compromise between the Shapley value and the Equal division value since the two values are often seen as the embodiment of marginalism and egalitarianism, respectively. This compromise can be made by considering convex combinations of the Shapley value and the Equal division value (see Section 2.1.2).

In multi-choice games, several solution concepts were inspired by the Shapley value, the Equal division value, the Core and the Weber set. In Chapter 2, the extensions of the Core from TU-games to multi-choice games by Grabisch & Xie (2007) (see Definition 16) and van den Nouweland *et al.* (1995) (see Definition 18) are presented. Grabisch & Xie (2007) show that their extension of the Core and the Weber set both coincide on the class of super-modular multi-choice games (see Proposition 7). It should be observed that this property does not hold for the extension of the Core and the Weber set provided by van den Nouweland *et al.* (1995), since, for each multi-choice game, their extension of the Weber set is strictly included in their extension of the Core. For this reason, we prefer to consider the Core and the Weber set as introduced by Grabisch & Xie (2007) (or simply the Core and the Weber set afterwards). Precisely, we focus on a necessary condition for a

payoff vector to be in the Core (see Definition 16). This condition, called **Multi-efficiency**, extends Efficiency from TU-game to multi-choice games. As a reminder, Multi-efficiency is based on the concept of synchronized coalition (see (2.44)). A payoff vector is Multi-efficient if for each level $j$, the sum of the payoffs of all players for their activity levels up to $j$ is equal to the worth of the $j$-synchronized coalition. We say that a solution on multi-choice games satisfies Multi-efficiency if it assigns a multi-efficient payoff vector to each game in this class.

In this chapter, we introduce a new axiom for multi-choice games related to Multi-efficiency and the Independence of higher demands principle: Independence of higher activity levels. This axiom ensures that the payoff distributed to a player's activity level is independent from higher activity levels. In particular, Independence of higher activity levels protects players with lower activity levels from being influenced by players with higher activity levels. We show that if a value satisfies Independence of higher activity levels and Efficiency, then it satisfies Multi-efficiency. Therefore, Multi-efficiency can be seen as a desirable axiom for multi-choice games. First, it is implied by two desirable axioms for multi-choice games, one of them being a direct translation of the Independence of higher demands principle. Second, from a technical point of view, it is a necessary condition to be in the Core. However, none of the previously introduced single-valued solutions for multi-choice games (see Section 2.4.2) satisfies Multi-efficiency.[1] For this reason, we propose several solution concepts for multi-choice games satisfying Multi-efficiency. This allows us to discuss the trade-off between marginalism and egalitarianism by means of a compromise between multi-efficient solutions. To that end, we first study a multi-efficient extension of the Shapley value, which we call the **multi-choice Shapley value**. This value is computed as follows. Assume that the grand coalition forms step by step starting from the empty coalition, in which no player participates at all. At each step, one player increases its activity by one unit according to a restricted order (see (2.35)). The marginal contribution of a player for an activity level to a coalition is the variation in worth that is created when that player reaches that particular level from the level just below. The multi-choice Shapley value assigns to each activity level of each player its expected marginal contribution assuming that each restricted order occurs with equal probability. This value is the centroid of the Weber set and therefore belongs to the Core of super-modular multi-choice games. As an additional remark, we show that the multi-choice Shapley value is consistent with the discrete serial cost sharing method for discrete cost sharing problems (see Definition 22).

Then, we introduce the **multi-choice Equal division value**: it divides the variation in worth between two consecutive synchronized coalitions (e.g. the $j$-synchronized and the $(j+1)$-synchronized coalitions) equally among the players able to play the required activity levels. This value applies the Uniformity principle since it allocates the same amount to any two players with the same maximal activity level.

[1]The list of single valued solutions provided in Section 2.4.2 is not exhaustive, but, to our knowledge, no multi-efficient value for multi-choice games exist in the literature.

To our knowledge, no previous work has addressed the trade-off between marginalism and egalitarianism in the context of multi-choice games. We address this trade-off by compromising between the multi-choice Shapley value and the multi-choice Equal division value. To that end, we introduce the **multi-choice Egalitarian Shapley values** for multi-choice games. This family of values is composed of convex combinations of the multi-choice Shapley value and the multi-choice Equal division value. Obviously, the multi-choice Egalitarian Shapley values are multi-efficient. Since we consider multi-choice games, we can define a specific convex combination at each activity level. This allows for different types of compromise, depending on the activity level.

We provide several axiomatic characterizations of these new multi-efficient solution concepts. To that end, we introduce Sign symmetry for equal pairs, which is an extension of Sign symmetry from TU-games to multi-choice games. Additionally, we propose Equal treatment for equal pairs, which strengthens Sign symmetry for equal pairs. Furthermore, we introduce Weak monotonicity, which relaxes Strong monotonicity. Combining classical and new axioms for multi-choice games, we provide two characterizations of the multi-choice Shapley value, one that relies on Additivity (Theorem 21) and another one that does not (Theorem 22). Furthermore, we show that the multi-choice Shapley value admits an expression in terms of Harsanyi dividends (see (4.23)). Next, we provide an axiomatic characterization of the multi-choice Equal Division value (Theorem 23). Finally, we provide an axiomatic characterization of the Egalitarian Shapley values (Theorem 24).

The rest of the chapter is organized as follows. Multi-efficient solution concepts are introduced in Section 4.2. Section 4.2.1 introduces Multi-efficiency. Subsection 4.2.2 introduces the multi-choice Shapley value, Section 4.2.3 introduces the multi-choice Equal Division value and Section 4.2.4 introduces the multi-choice Egalitarian Shapley values. Axiomatic characterizations of these solution concepts are introduced in Section 4.3. Additional remarks regarding the multi-choice Shapley value and its relationship with the serial cost sharing method are made in Section 4.4. Finally, Section 4.5 concludes the chapter.

## 4.2 Multi-efficient solution concepts

In this section, we discuss a necessary condition for a payoff vector to be in the Core of multi-choice games (see Definition 16), which we call **Multi-efficiency**. We propose new multi-efficient solution concepts for multi-choice games. We first provide a new extension of the Shapley value (see Definition 4) from TU-games to multi-choice games. Next, we provide new extensions of the Equal division value and the Egalitarian Shapley values from TU-games to multi-choice games.

### 4.2.1 Multi-efficiency

Recall that the Core of a multi-choice game $(m, v) \in \mathcal{G}$ (see Definition 16) is denoted by $\mathcal{C}(m, v)$ and defined as the set of payoff vector $x$ satisfying

$$\forall s \in \mathcal{M}, \quad \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} \geq v(s) \tag{4.1}$$

$$\text{and} \quad \forall l \leq \max_{k \in N} m_k, \quad \sum_{i \in N} \sum_{j=1}^{l \wedge m_i} x_{ij} = v((l \wedge m_i)_{i \in N}). \tag{4.2}$$

Equation (4.2) is the Multi-efficiency condition. Let us translate this condition into an axiom for solutions on $\mathcal{G}$. Let $f$ be a solution on $\mathcal{G}$.

**Axiom 43 (Multi-efficiency).** *For each $(m, v) \in \mathcal{G}$,*

$$\forall l \leq \max_{k \in N} m_k, \quad \sum_{i \in N} \sum_{j=1}^{l \wedge m_i} f_{ij}(m, v) = v((l \wedge m_i)_{i \in N}). \tag{4.3}$$

**Remark 4.** *For each $(m, v) \in \mathcal{G}$, (4.3) can be re-written as*

$$\begin{aligned} &\forall l \leq \max_{k \in N} m_k, \\ &\sum_{i \in Q(l)} f_{il}(m, v) = v((l \wedge m_k)_{k \in N}) - v(((l-1) \wedge m_k)_{k \in N}). \end{aligned} \tag{4.4}$$

*The sum of the payoffs of all pairs $(i, l)$ containing activity level $l$ is equal to the surplus generated between the $l$-synchronized coalition and the $(l-1)$-synchronized coalition.*

### 4.2.2 The multi-choice Shapley value

In this section, we define the multi-choice Shapley value, which is a multi-efficient value that extends the Shapley value from TU-games to multi-choice games.

Assume that the grand coalition $m$ forms according to a restricted order (see (2.45)) over the set of pairs. The multi-choice Shapley value assigns to each pair $(i, j) \in M^+$ its expected marginal contribution, assuming that each restricted order over the set of pairs occurs with equal probability.

**Definition 28 (Multi-choice Shapley value).** *For each $(m, v) \in \mathcal{G}$, the multi-choice Shapley value $\varphi$ is defined as*

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \frac{1}{\prod_{j \leq \max\limits_{k \in N} m_k} |Q(j)|!} \sum_{\theta_{GX} \in \Theta_{GX}} \eta_{ij}^{\theta_{GX}}(m, v). \tag{4.5}$$

Whenever $m = (1, \ldots, 1)$, this value coincides with the Shapley value on TU-games.

Recall that, for each $(m, v) \in \mathcal{G}$, the Weber set $W$ is the convex hull of all marginal vectors defined as

$$\mathcal{W}(m, v) = co(\{\eta^{\theta_{GX}}(m, v) \,|\, \theta_{GX} \in \Theta_{GX}\}).$$

The multi-choice Shapley value is the centroid of the Weber set. By Proposition 7, the Weber set coincides with the Core on the class of super-modular multi-choice games. Therefore, the following result holds.

**Proposition 21.** *For each super-modular multi-choice game $(m, v) \in \mathcal{G}$,*

$$\varphi(m, v) \in \mathcal{C}(m, v).$$

The next results states that the multi-choice Shapley value admits an alternative expression which requires less orders over the set of pairs to be computed. For each $j \leq \max_{k \in N} m_k$, denote by

$$M^{+,j} = \{(i, j) \in M^+ : i \in Q(j)\}$$

the subset of pairs containing the activity level $j$. An order over $M^{+,j}$ is given by

$$\theta_j : M^{+,j} \to \{1, \ldots, |Q(j)|\}.$$

Denote by $\Theta_j$ the set of all orders over $M^{+,j}$. These orders can also be interpreted as orders over the set of players in $Q(j)$. For each $\theta_j \in \Theta_j$ and $h \in \{0, \ldots, |Q(j)|\}$, define $s^{\theta_j, h}$ as

$$\forall i \in N, \quad s_i^{\theta_j, h} = \begin{cases} j & \text{if } \; i \in Q(j) \;\text{ and }\; \theta_j(i, j) \leq h, \\ j - 1 & \text{if } \; i \in Q(j) \;\text{ and }\; \theta_j(i, j) > h, \\ m_i & \text{if } \; i \notin Q(j). \end{cases} \tag{4.6}$$

Observe that

$$s^{\theta_j, |Q(j)|} = (j \wedge m_k)_{k \in N} \quad \text{and} \quad s^{\theta_j, 0} = ((j - 1) \wedge m_k)_{k \in N}.$$

The coalition $s^{\theta_j, h} \in \mathcal{M}$ represents a situation in which each player able to play at $j$ and ordered prior to step $h$, with respect to $\theta_j$, participates at its activity level $j$, whereas each player able to play $j$ but not ordered prior to step $h$, with respect to $\theta_j$, participates at its activity level $j - 1$. Players unable to play $j$ participate at their maximal activity level.

**Proposition 22.** *For each $(m, v) \in \mathcal{G}$, the multi-choice Shapley value $\varphi$ admits an alternative expression given by*

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \frac{1}{|Q(j)|!} \sum_{\theta_j \in \Theta_j} \Big[ v(s^{\theta_j, \theta_j(i,j)}) - v(s^{\theta_j, \theta_j(i,j)-1}) \Big]. \tag{4.7}$$

*Proof.* We show that the multi-choice Shapley value admits an alternative expression given by (4.7). Observe that there are $|\Theta_l| = |Q(l)|!$ ways to order the pairs in $M^{+,l}$ for each $l \leq \max_{k\in N} m_k$. Additionally, there are

$$\prod_{l<j} |Q(l)|!$$

ways to order the pairs in $M^{+,1}$, then the pairs in $M^{+,2}$, and so on, until the pairs in $M^{+,j-1}$. Similarly, there are

$$\prod_{l>j} |Q(l)|!$$

ways to order the pairs in $M^{+,j+1}$, then the pairs in $M^{+,j+2}$, and so on. Observe that, for each $\theta_{GX} \in \Theta_{GX}$, there exists exactly one order $\theta_j \in \Theta_j$ such that $s^{\theta_{GX},\theta_{GX}(i,j)} = s^{\theta_j,\theta_j(i,j)}$. Additionally, for each $\theta_j \in \Theta_j$, there are

$$\prod_{l<j} |Q(l)|! \times \prod_{l>j} |Q(l)|!$$

orders $\theta_{GX} \in \Theta_{GX}$ such that $s^{\theta_{GX},\theta_{GX}(i,j)} = s^{\theta_j,\theta_j(i,j)}$. It follows that, for each $(m, v) \in \mathcal{G}$ and $(i, j) \in M^+$,

$$\begin{aligned}
&\varphi_{ij}(m, v) \\
&= \frac{1}{\prod_{l\leq \max\limits_{k\in N} m_k} |Q(l)|!} \sum_{\theta_{GX}\in\Theta_{GX}} \Big[v\big(s^{\theta_{GX},\theta_{GX}(i,j)}\big) - v\big(s^{\theta_{GX},\theta_{GX}(i,j)-1}\big)\Big] \\
&= \frac{(\prod_{l<j} |Q(l)|!)(\prod_{l>j} |Q(l)|!)}{\prod_{\leq \max\limits_{k\in N} m_k} |Q(l)|!} \sum_{\theta_j\in\Theta_j} \Big[v\big(s^{\theta_j,\theta_j(i,j)}\big) - v\big(s^{\theta_j,\theta_j(i,j)-1}\big)\Big].
\end{aligned}$$

The first line comes from the definition of the multi-choice Shapley value, the second line follows from (4.6) and the fact that there are

$$\prod_{l<j} |Q(l)|! \times \prod_{l>j} |Q(l)|!$$

orders $\theta_{GX} \in \Theta_{GX}$ such that $s^{\theta_{GX},\theta_{GX}(i,j)} = s^{\theta_j,\theta_j(i,j)}$ for each $\theta_j \in \Theta_j$. Simplifying the expression, we obtain the desired result

$$\forall (i, j) \in M^+, \quad \varphi_{ij}(m, v) = \frac{1}{|Q(j)|!} \sum_{\theta_j\in\Theta_j} \Big[v\big(s^{\theta_j,\theta_j(i,j)}\big) - v\big(s^{\theta_j,\theta_j(i,j)-1}\big)\Big].$$

■

In the sequel, we will retain expression (4.7) of the multi-choice Shapley value.

### 4.2.3 The multi-choice Equal division value

In this section, we propose a new multi-efficient value that extends the Equal division value from TU-games to multi-choice games. This value is referred to as the multi-choice Equal division value. The multi-choice Equal division value divides the surplus generated between two consecutive synchronized coalitions (4.4) equally among the pairs containing the activity level on which the players in the larger of the two coalitions are synchronized.

**Definition 29 (Multi-choice Equal division value).** *For each $(m, v) \in \mathcal{G}$, the multi-choice Equal division value $\xi$ is defined as*

$$\forall (i,j) \in M^+,$$
$$\xi_{ij}(m,v) = \frac{1}{|Q(j)|}\Big[v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}))\Big]. \tag{4.8}$$

Whenever $m = (1, \ldots, 1)$, the multi-choice Equal division value boils down to the Equal division value on TU-games (see (2.10)). The Multi-choice Equal division value allocates the same total payoff to any two players sharing the same maximal activity level. Therefore, this value is in line with the Uniformity principle.

### 4.2.4 The multi-choice Egalitarian-Shapley values

In this section, we propose a trade-off between marginalism and egalitarianism by considering convex combinations of the multi-choice Shapley value and the multi-choice Equal division value.

**Definition 30 (Multi-choice Egalitarian Shapley value).** *Let $\alpha = \{\alpha^j\}_{1\leq j\leq K}$ be a parameter system such that $\alpha^j \in [0,1]$ for each $1 \leq j \leq K$. For each $(m, v) \in \mathcal{G}$, the multi-choice Egalitarian Shapley value $\chi^\alpha$ is defined as*

$$\forall (i,j) \in M^+, \quad \chi^\alpha_{ij}(m,v) = \alpha^j \varphi_{ij}(m,v) + (1-\alpha^j)\xi_{ij}(m,v). \tag{4.9}$$

Whenever $m = (1, \ldots, 1)$, these values boil down to the Egalitarian Shapley values on TU-games (see (2.12)). We illustrate the possibilities offered by multiple convex combinations through an example.

**Example 12.** *Consider $(m, v) \in \mathcal{G}$ and $i \in N$ such that $m_i = 3$. Consider an Egalitarian Shapley value defined by $\alpha^1 = 0.2$, $\alpha^2 = 0.5$ and $\alpha^3 = 0.8$. The payoff $\chi^\alpha_{ij}$ will be closer to the multi-choice Equal division value if $j = 1$ and closer to the multi-choice Shapley value if $j = 3$. Thus, egalitarianism is progressively overtaken by marginalism as the activity level increases. This is due to the fact that $\alpha^1 < \alpha^2 < \alpha^3$. Depending on the parameter system, a multi-choice Egalitarian Shapley value operates different compromises between egalitarianism and marginalism for different activity levels. These differences can be progressive as it is the case in this example.*

## 4.3 Axiomatic characterizations

In this section, we discuss new and classical axioms for multi-choice games. We also provide axiomatic characterizations of each solution introduced in Section 4.2.

### 4.3.1 Characterizations of the multi-choice Shapley value

We provide two axiomatic characterizations of the multi-choice Shapley value. The first characterization relies on Linearity, whereas the second does not. We also provide an expression of the multi-choice Shapley value in terms of Harsanyi dividends. Let $f$ be a value on multi-choice games.

The next axiom adapts the Independence of higher demands principle to the framework of multi-choice games. It requires that if the maximal activity level of each player reduces to a certain level, then the payoff of each player for this activity level remains unchanged.

**Axiom 44 (Independence of higher activity levels).** *For each* $(m, v) \in \mathcal{G}$,

$$\forall (i,j) \in M^+, \quad f_{ij}(m,v) = f_{ij}((j \wedge m_k)_{k\in N}, v). \tag{4.10}$$

It turns out that Independence of higher activity levels combined with Efficiency implies Multi-efficiency.

**Proposition 23.** *If a value $f$ on $\mathcal{G}$ satisfies Efficiency and Independence of higher activity levels, then it satisfies Multi-efficiency.*

*Proof.* Let $(m, v) \in \mathcal{G}$, $l \leq \max_{k\in N} m_k$ and $f$ a value satisfying Efficiency and Independence of higher activity levels. Consider the sub-game $((l \wedge m_k)_{k\in N}, v)$. By Efficiency, it holds that

$$\sum_{i\in N} \sum_{j=1}^{l\wedge m_i} f_{ij}((l \wedge m_k)_{k\in N}, v) = v((l \wedge m_k)_{k\in N}). \tag{4.11}$$

By Independence of higher activity levels,

$$\sum_{i\in N} \sum_{j=1}^{h\wedge m_i} f_{ij}((l \wedge m_k)_{k\in N}, v) = \sum_{i\in N} \sum_{j=1}^{l\wedge m_i} f_{ij}(m, v). \tag{4.12}$$

Combining (4.11) with (4.12), we obtain the desired result. ■

**Remark 5.** *The converse of Proposition 23 is not true. Indeed, consider the value $d$ defined for each $(m, v) \in \mathcal{G}$ and each $(i, j) \in M^+$ as*

$$d_{ij}(m,v) = \begin{cases} \dfrac{v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N})}{|\{h \in N : m_h \geq m_k, \forall k \in N\}|} & \text{if } m_i \geq m_k, \forall k \in N, \\ 0 & \text{otherwise.} \end{cases}$$

*The value $d$ satisfies Multi-efficiency, but does not verify Independence of higher activity levels. To see this, consider $N = \{1,2,3\}$ and $(m,v) \in \mathcal{G}$ such that $m = (3,2,3)$. Observe that $d_{1,1}((2,2,2),v) = \frac{1}{3}v(1,1,1) \neq d_{1,1}(m,v)$ which shows that $d$ violates Independence of higher activity levels.*

The next two axioms compare the payoffs of equal pairs (see (2.26) for the definition of equal pairs). First, we introduce the Equal treatment for equal pairs axiom, which states that two equal pairs should receive the same payoff. We also suggest a relaxation of Equal treatment for equal pairs into Sign symmetry for equal pairs. This new axiom states that two equal pairs should receive a payoff of the same sign.

**Axiom 45 (Equal treatment for equal pairs).** *For each $(m,v) \in \mathcal{G}$ and two distinct equal pairs $(i,j)(i',j) \in M^+$,*

$$f_{ij}(m,v) = f_{i'j}(m,v).$$

Whenever $m = (1,\ldots,1)$, Equal treatment for equal pairs boils down to the classical axiom of Equal treatment for equal for TU-games.

**Axiom 46 (Sign symmetry for equal pairs).** *For each $(m,v) \in \mathcal{G}$ and two distinct equal pairs $(i,j),(i',j) \in M^+$,*

$$sign(f_{ij}(m,v)) = sign(f_{i'j}(m,v)).$$

Whenever $m = (1,\ldots,1)$, Sign symmetry for equal pairs boils down to the Sign symmetry axiom for TU-games.

Equal treatment for equal pairs $\implies$ Sign symmetry for equal pairs

We have the material to provide a first axiomatic characterization of the multi-choice Shapley value.

**Theorem 21.** *A value $f$ on $\mathcal{G}$ satisfies Efficiency, Independence of higher activity levels, Linearity, Sign symmetry for equal pairs and the Null pair property if and only if $f = \varphi$.*

*Proof.* The proof is divided in two-steps.

**Step 1**: we show that $\varphi$ satisfies all the axioms of the statement of Theorem 21.

**Efficiency:** For each $(m,v) \in \mathcal{G}$,

$$\begin{aligned}
&\sum_{i\in N}\sum_{j\in M_i^+}\varphi_{ij}(m,v)\\
&=\sum_{j\leq \max_{k\in N} m_k}\sum_{i\in Q(j)}\varphi_{ij}(m,v)\\
&\overset{(4.7)}{=}\sum_{j\leq \max_{k\in N} m_k}\frac{1}{|Q(j)|!}\sum_{\theta_j\in\Theta_j}\sum_{i\in Q(j)}\Big[v(s^{\theta_j,\theta_j(i,j)})-v(s^{\theta_j,\theta_j(i,j)-1})\Big].
\end{aligned}$$

Observe that, for each $\theta_j \in \Theta_j$,

$$\sum_{i \in Q(j)} \Big[ v(s^{\theta_j, \theta_j(i,j)}) - v(s^{\theta_j, \theta_j(i,j)-1}) \Big] = v(s^{\theta_j, |Q(j)|}) - v(s^{\theta_j, 0}).$$

By (4.6), for each $\theta_j \in \Theta_j$,

$$s^{\theta_j, |Q(j)|} = (j \wedge m_k)_{k \in N}, \quad \text{and} \quad s^{\theta_j, 0} = ((j-1) \wedge m_k)_{k \in N}.$$

It follows that

$$\begin{aligned} &\sum_{i \in N} \sum_{j \in M_i^+} \varphi_{ij}(m, v) \\ &= \sum_{j \leq \max\limits_{k \in N} m_k} \frac{1}{|Q(j)|!} \sum_{\theta_j \in \Theta_j} \Big[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \Big]. \end{aligned} \tag{4.13}$$

Since the quantity $v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})$ is independent from any order $\theta_j \in \Theta_j$, it follows that it is summed as many times in (4.13) as there are orders in $\Theta_j$. Therefore, it holds that

$$\begin{aligned} &\sum_{i \in N} \sum_{j \in M_i^+} \varphi_{ij}(m, v) \\ &= \sum_{j \leq \max\limits_{k \in N} m_k} \frac{1}{|Q(j)|!} Q(j)! \Big[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N} \Big] \\ &= \sum_{j \leq \max\limits_{k \in N} m_k} \Big[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \Big] \\ &= v(m), \end{aligned}$$

which shows that the value satisfies Efficiency.

**Other axioms:** By definition of the multi-choice Shapley value (see (4.7)), the payoff of a pair is independent from any activities different from the activity level contained in this pair. Therefore, we have that $\varphi$ satisfies Independence of higher activity levels. Linearity follows directly from (4.7). By definition of equal pairs (see (2.26)), $\varphi$ satisfies Equal treatment of equal pairs, which implies that $\varphi$ satisfies Sign symmetry for equal pairs. By definition of null pairs (see (2.25)), $\varphi$ satisfies the Null pair property. This concludes Step 1.

**Step 2**: To complete the proof, it remains to show that there is at most one value satisfying all the axioms of the statement of Theorem 21. Take any $f$ satisfying all the axioms of the statement of Theorem 21. Consider any $(m, v) \in \mathcal{G}$. We know that

each multi-choice game admits a unique linear decomposition in terms of minimal effort games $\{u_s\}_{s\in\mathcal{M}}$. Consider $s \in \mathcal{M}$ such that $s \neq \overrightarrow{0}$. The set of top pairs $T(s)$ (see (2.29)) can be re-written as

$$T(s) = \{(i, s^T) \in M^{+,s^T} : s_i = s^T\},$$

where $s^T = \max_{i\in N} s_i$. Let us show that $f(m, u_s)$ is uniquely determined. We divide this Step 2 into several sub-steps.

**Step 2.1.** Let us show that, for each $(i, j) \in M^+$ such that $j \neq s^T$, $f_{ij}(m, u_s)$ is uniquely determined.

**Step 2.1.1.** If $j < s^T$, then $(j \wedge m_k)_{k\in N} \not\geq s$. It follows that $((j \wedge m_k)_{k\in N}, u_s)$ is the null game since $u_s(t) = 0$ for each $t \leq (j \wedge m_k)_{k\in N}$. Recall that each pair is a null pair (see (2.25)) in the null game. By Independence of higher activity levels and the Null pair property, for each $(i, j) \in M^+$ such that $j < s^T$, we obtain

$$f_{ij}(m, u_s) = f_{ij}((j \wedge m_k)_{k\in N}, u_s) = 0.$$

**Step 2.1.2.** If $j > s^T$ then, by definition of a minimal effort game (see (2.31)), $(i, j)$ is a null pair in $(m, u_s)$. By the Null pair property, for each $(i, j) \in M^+$ such that $j > s^T$,

$$f_{ij}(m, u_s) = 0.$$

We have shown that $f_{ij}(m, u_s) = 0$, and so is uniquely determined, for each $(i, j) \in M^+$ such that $j \neq s^T$.

**Step 2.2.** We now show that, for each pair $(i, j) \in M^+$ such that $j = s^T$, i.e., each pair $(i, s^T) \in M^{+,s^T}$, $f_{is^T}(m, u_s)$ is uniquely determined. To that end, consider the game $(m, w) \in \mathcal{G}$ defined as

$$\forall t \leq m, \quad w(t) = u_s(t) - \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m, u_s) u_{(0_{-i},s^T)}(t). \tag{4.14}$$

**Step 2.2.1.** We show that

$$\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w) = 0.$$

We consider pairs in $M^{+,s^T}$. By definition of $M^{+,s^T}$, observe that

$$\sum_{i\in Q(s^T)} f_{is^T}(m, w) = \sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m, w).$$

We have that any pair $(i, s^T) \in M^{+,s^T}$ is either in $T(s)$ or not. Since $f$ satisfies Efficiency and Independence of higher activity levels, by Proposition 23, $f$ also satisfies Multi-efficiency. Therefore,

$$\begin{aligned}\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m,w) =& w((s^T \wedge m_k)_{k\in N}) - w(((s^T-1)\wedge m_k)_{k\in N})\\ \overset{(4.14)}{=}& u_s((s^T \wedge m_k)_{k\in N})\\ &- \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m,u_s)u_{(0_{-i},s^T)}((s^T \wedge m_k)_{k\in N})\\ &-u_s(((s^T-1)\wedge m_k)_{k\in N})\\ &+ \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m,u_s)u_{(0_{-i},s^T)}(((s^T-1)\wedge m_k)_{k\in N}).\end{aligned} \tag{4.15}$$

Observe that $((s^T \wedge m_k)_{k\in N}) \geq s \geq ((0_{-i}, s^T)$, $(((s^T-1)\wedge m_k)_{k\in N}) \ngeq s$ and $(((s^T-1)\wedge m_k)_{k\in N}) \ngeq (0_{-i}, s^T)$, where $(i, s^T) \in T(s)$. By definition of a minimal effort game (2.31),

$$\begin{aligned}&u_s((s^T \wedge m_k)_{k\in N}) = 1,\\ &\text{and,} \quad \forall (i,s^T) \in M^{+,s^T}, \quad u_{(0_{-i},s^T)}((s^T \wedge m_k)_{k\in N}) = 1.\\ &u_s(((s^T-1)\wedge m_k)_{k\in N}) = 0,\\ &\text{and,} \quad \forall (i,s^T) \in M^{+,s^T}, \quad u_{(0_{-i},s^T)}(((s^T-1)\wedge m_k)_{k\in N}) = 0.\end{aligned}$$

It follows that (4.15) becomes

$$\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m,w) = 1 - \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m,u_s) - 0 + 0. \tag{4.16}$$

Observe that, since $(i, s^T) \notin T(s)$ if and only if $s^T > s_i$, then each $(i, s^T) \notin T(s)$ is also a null pair in $(m, u_s)$. Since $\varphi$ satisfies the Null pair property, $\varphi_{is^T}(m, u_s) = 0$ for each $(i, s^T) \notin T(s)$. Since $\varphi$ satisfies Efficiency, Independence of higher activity levels, by Proposition 23 the value satisfies Multi-efficiency. Therefore,

$$\begin{aligned}\sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m,u_s) &= \sum_{(i,s^T)\in T(s)} \varphi_{is^T}(m,u_s) + \sum_{(i,s^T)\notin T(s)} \varphi_{is^T}(m,u_s)\\ &= \sum_{(i,s^T)\in M^{+,s^T}} \varphi_{is^T}(m,u_s)\\ &= u_s((s^T \wedge m_k)_{k\in N})\\ &= 1.\end{aligned}$$

Therefore, (4.16) becomes

$$\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m,w) = 1-1=0, \tag{4.17}$$

which concludes Step 2.2.1.

**Step 2.2.2.** We show that, for each $(i,s^T)\in M^{+,s^T}$,

$$f_{is^T}(m,w)=0.$$

We know that each pair $(i,s^T)\notin T(s)$ is a null pair in $(m,u_s)$. Moreover, each pair $(i,s^T)\notin T(s)$ is a null pair in each $(m,u_{(\vec{0}_{-i'},s^T)})$, $(i',s^T)\in T(s)$. Indeed, in $(m,u_{(\vec{0}_{-i'},s^T)})$, $(i',s^T)$ is the only productive pair and all other pairs are null pairs. It follows that each pair $(i,s^T)\notin T(s)$ is a null pair in $(m,w)$. By the Null pair property, for each $(i,s^T)\notin T(s)$,

$$f_{is^T}(m,w)=0. \tag{4.18}$$

It follows that

$$\begin{aligned}\sum_{(i,s^T)\in M^{+,s^T}} f_{is^T}(m,w) &= \sum_{(i,s^T)\in T(s)} f_{is^T}(m,w) + \sum_{(i,s^T)\notin T(s)} f_{is^T}(m,w)\\ &= \sum_{(i,s^T)\in T(s)} f_{is^T}(m,w) + 0\\ &\overset{(4.17)}{=} 0.\end{aligned} \tag{4.19}$$

To complete the proof of Step 2.2.2, it remains to show that if there exist two distinct pairs $(i,s^T),(i',s^T)\in T(s)$, then these pairs are equal. By definition of a minimal effort game (see (2.31)), two distinct pairs $(i,s^T),(i',s^T)\in T(s)$ are equal in $(m,u_s)$. Since $\varphi$ satisfies Equal treatment for equal pairs, it follows that $\varphi_{is^T}(m,u_s)=\varphi_{i's^T}(m,u_s)$. By definition of a minimal effort game, for each $t\in\mathcal{M}$ such that $t_i=t_{i'}=s^T-1$,

$$u_{(0_{-i},s^T)}(t)=u_{(0_{-i'},s^T)}(t)=0, \tag{4.20}$$

$$\text{and}\quad u_{(0_{-i},s^T)}(t+e_i)=u_{(0_{-i'},s^T)}(t+e_{i'})=1. \tag{4.21}$$

Therefore, for each $t \in \mathcal{M}$ such that $t_i = t_{i'} = s^T - 1$,

$$\begin{aligned}
&\sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m,u_s)u_{(0_{-k},s^T)}(t+e_i) \\
&= \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m,u_s)u_{(0_{-k},s^T)}(t) + \varphi_{is^T}(m,u_s) \\
&= \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m,u_s)u_{(0_{-k},s^T)}(t) + \varphi_{i's^T}(m,u_s) \\
&= \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m,u_s)u_{(0_{-k},s^T)}(t+e_{i'}),
\end{aligned}$$

where the first equality and the third equality follow from (4.20) and (4.21), and the second equality follows from $\varphi_{is^T}(m,u_s) = \varphi_{i's^T}(m,u_s)$ by Equal treatment for equal pairs. It follows that

$$w(t+e_i) = w(t+e_{i'}),$$

for each $t \in \mathcal{M}$ such that $t_i = t_{i'} = s^T - 1$, showing that $(i,s^T),(i',s^T) \in T(s)$ are equal pairs in $(m,w)$. By Sign symmetry for equal pairs, $sign(f_{is^T}(m,w)) = sign(f_{i's^T}(m,w))$. It follows from (4.19) that, for each $(i,s^T) \in T(s)$,

$$f_{is^T}(m,w) = 0. \tag{4.22}$$

Combining (4.18) with (4.22), the proof of Step 2.2.2 is complete.

**Step 2.2.3.** We show that for each $(i,s^T) \in M^{+,s^T}$,

$$f_{is^T}(m,u_s) = \varphi_{is^T}(m,u_s).$$

By (4.14), (4.22) and Linearity, for each $(i,s^T) \in M^{+,s^T}$,

$$\begin{aligned}
& f_{is^T}(m,w) = f_{is^T}(m,u_s) - f_{is^T}\Big(m, \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m,u_s)u_{(0_{-k},s^T)}\Big) \\
\iff\ & f_{is^T}(m,u_s) \overset{(4.22)}{=} f_{is^T}\Big(m, \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m,u_s)u_{(0_{-k},s^T)}\Big) \\
& \qquad = \sum_{(k,s^T)\in T(s)} \varphi_{ks^T}(m,u_s) f_{is^T}\Big(m, u_{(0_{-k},s^T)}\Big).
\end{aligned}$$

Additionally, by the Null pair property and Multi-efficiency,

$$f_{is^T}(m,u_{(0_{-i},s^T)}) = 1$$

since $(i, s^T)$ is the only productive pair in $(m, u_{(0_{-i}, s^T)})$. Therefore, for each $(i, s^T) \in M^{+,s^T}$,

$$\varphi_{ks^T}(m, u_s) f_{is^T}\Big(m, u_{(0_{-k}, s^T)}\Big) = \begin{cases} \varphi_{ks^T}(m, u_s) & \text{if } k = i, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that, for each $(i, s^T) \in M^{+,s^T}$,

$$f_{is^T}(m, u_s) = \varphi_{is^T}(m, u_s),$$

therefore $f_{is^T}(m, u_s)$ is uniquely determined. This concludes Step 2.2.3.

From Step 2.1 and Step 2.2, we conclude that $f(m, u_s)$ is uniquely determined. By Linearity, $f(m, v)$ is uniquely determined, which concludes the proof of Theorem 21. ■

The five axioms of the statement of Theorem 21 are logically independent, as shown by the following alternative solutions.

- The value $f$ given, for each $(m, v) \in \mathcal{G}$, by $f(m, v) = \vec{0}$ satisfies all the axioms except Efficiency.

- The value $f$ given, for each $(m, v) \in \mathcal{G}$, by

$$f_{ij}(m, v) = \begin{cases} \varphi_{ij}(m, v) + \dfrac{\Delta_{((2 \wedge m_k)_{k \in N})}(v)}{|Q(1)|} & \text{if } j = 1 \text{ and } m^T > 1, \\ \varphi_{ij}(m, v) - \dfrac{\Delta_{((2 \wedge m_k)_{k \in N})}(v)}{|Q(2)|} & \text{if } j = 2 \text{ and } m^T > 1, \\ \varphi_{ij}(m, v) & \text{otherwise,} \end{cases}$$

  satisfies all the axioms except Independence of higher activity levels.

- The value $f$ given, for each $(m, v) \in \mathcal{G}$, by

$$\forall (i, j) \in M^+,$$
$$f_{ij}(m, v) = \sum_{\substack{s \leq m \\ (i,j) \in T(s)}} \frac{(v(((j-1) \wedge m_h)_{h \in N} + e_i)^2) + 1}{\sum_{(k, s_k) \in T(s)} ((v(((j-1) \wedge m_h)_{h \in N} + e_k)^2) + 1} \Delta_v(s),$$

  satisfies all the axioms except Linearity.

- The multi-choice Equal division value $\xi$ satisfies all the axioms except the Null pair property.

- Take any $(m, v) \in \mathcal{G}$ and fix any arbitrary integer $\beta_{ij} \in \{1, 2\}$ for each $(i, j) \in M^+$. The value $f^\beta$ given by

$$\forall (i,j) \in M^+, \quad f_{ij}^{\beta}(m,v) = \sum_{\substack{s \leq m \\ (i,j) \in T(s)}} \frac{\beta_{ij}}{\sum_{(k,l) \in T(s)} \beta_{kl}} \Delta_v(s),$$

satisfies all the axioms except Sign symmetry for equal pairs.

By Theorem 21, we provide another alternative expression of the multi-choice Shapley value in terms of Harsanyi dividends.

**Corollary 1.** *The multi-choice Shapley value admits an alternative expression in terms of Harsanyi dividends. For each game $(m, v) \in \mathcal{G}$, the value is defined as*

$$\forall (i,j) \in M^+, \quad \varphi_{ij}(m,v) = \sum_{\substack{s \leq m \\ (i,j) \in T(s)}} \frac{\Delta_v(s)}{|T(s)|}. \tag{4.23}$$

*Proof.* By the the proof of Theorem 21, $\varphi$ satisfies Efficiency, Linearity, Independence of higher activity levels, the Null pair property and Equal treatment for equal pairs. Consider $(m, u_s) \in \mathcal{G}$, $s \in \mathcal{M}$ such that $s \neq \overrightarrow{0}$. Similarly to (4.18), for each $(i, j) \notin T(s)$, the Null pair property and Multi-efficiency imply

$$\varphi_{ij}(m, u_s) = 0. \tag{4.24}$$

All pairs in $T(s)$ are equal pairs in $(m, u_s)$. Thus, by Equal treatment for equal pairs,

$$\varphi_{is^T}(m, u_s) = \ldots = \varphi_{i's^T}(m, u_s). \tag{4.25}$$

By Efficiency and Linearity, we obtain the desired result. ■

We provide a second axiomatic characterization of the multi-choice Shapley value without resorting to Linearity. In line with Young (1985) and Casajus (2018) (see Theorem 3 and 4), we use a Strong monotonicity axiom. This axiom states that, if the marginal contributions to coalitions of a pair increase from a game $(m, w)$ to another game $(m, v)$, then the payoff of this pair also increases.

**Axiom 47 (Strong monotonicity.).** *For each $(m, v), (m, w) \in \mathcal{G}$, each $(i, j) \in M^+$ and each $s \in \mathcal{M}$ such that $s_i = j - 1$,*

$$v(s + e_i) - v(s) \geq w(s + e_i) - w(s),$$

*it holds that*

$$f_{ij}(m, v) \geq f_{ij}(m, w).$$

Whenever $m = (1, \ldots, 1)$, Strong monotonicity boils down to the axiom of Strong monotonicity for TU-games introduced by Young (1985). We have the material to provide a second axiomatic characterization of the multi-choice Shapley value.

**Theorem 22.** *A value $f$ on $\mathcal{G}$ satisfies Efficiency, Independence of higher activity levels, Strong monotonicity and Sign symmetry for equal pairs if and only if $f = \varphi$.*

*Proof.* From Theorem 21, we know that $\varphi$ satisfies Efficiency, Independence of higher activity levels and Sign symmetry for equal pairs. By definition (see (4.7)), the multi-choice Shapley value satisfies Strong monotonicity.

Next, we show that $\varphi$ is the unique value satisfying all the axioms of the statement of Theorem 22. Take any $f$ satisfying all the axioms of the statement of Theorem 22 and consider any $(m, v) \in \mathcal{G}$. Recall that $(m, v) \in \mathcal{G}$ can be rewritten as $(m, \sum_{t \in M} \Delta_v(t) u_t)$. We define the set of coalitions for which the Harsanyi dividend is non-null as

$$\mathcal{T}(v) = \{t \in \mathcal{M} \mid \Delta_v(t) \neq 0\}.$$

By induction on the cardinality of $\mathcal{T}(v)$, we show that

$$f(m, v) = \varphi(m, v).$$

**Initialization**: If $|\mathcal{T}(v)| = 0$, then each Harsanyi dividend is null. The only game $(m, v) \in \mathcal{G}$ such that $|\mathcal{T}(v)| = 0$ is the null game. Recall that

$$M^{+,j} = \{(i, j) \in M^+ : i \in Q(j)\}.$$

Since $f$ satisfies Efficiency and Independence of higher activity levels, by Proposition 23, it satisfies Multi-efficiency. It follows that, for each $j \leq \max_{k \in N} m_k$,

$$\begin{aligned} \sum_{(i,j) \in M^{+,j}} f_{ij}(m, v) =& v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \\ =& 0. \end{aligned} \tag{4.26}$$

Recall that any two distinct pairs $(i, j), (i', j) \in M^{+,j}$ are equal in the null game $(m, v)$. Therefore, by Sign symmetry for equal pairs,

$$sign(f_{ij}(m, v)) = sign(f_{i'j}(m, v)). \tag{4.27}$$

Combining (4.26) and (4.27), for each $j \leq \max_{k \in N} m_k$ and each $(i, j) \in M^{+,j}$, we obtain

$$f_{ij}(m, v) = 0.$$

Recall also that each pair is a null pair in the null game. Since $\varphi$ satisfies the Null pair property, for each $j \leq \max_{k \in N} m_k$ and each $(i,j) \in M^{+,j}$,

$$\varphi_{ij}(m,v) = 0 = f_{ij}(m,v).$$

This concludes the initialization.

**Hypothesis**: Fix $r \in \mathbb{N}$ such that $r < |\mathcal{M}| - 1$. We assume that, for each $(m,v) \in \mathcal{G}$ such that $|\mathcal{T}(v)| \leq r$,

$$f(m,v) = \varphi(m,v).$$

**Induction**: Consider any $(m,v) \in \mathcal{G}$ such that $|\mathcal{T}(v)| = r+1$. Let us show that

$$f(m,v) = \varphi(m,v).$$

We define the minimum coalition of the set $\mathcal{T}(v)$ as

$$p = \bigwedge_{t \in \mathcal{T}(v)} t.$$

Two cases can be distinguished. First, assume that $p \neq \overrightarrow{0}$. Consider any pair $(i,j) \in M^+$ such that $j > p_i$. By definition of $p$, there exists a $t \in \mathcal{T}(v)$ such that $j > t_i$. For such $t$, consider the game $(m, v - \Delta_v(t)u_t)$. By definition of a minimal effort game (2.31) and (4.4), $(i,j)$ is a null pair in $(m, \Delta_v(t)u_t)$. Therefore $(i,j)$ has the same marginal contributions in $(m,v)$ and in $(m, v - \Delta_v(t)u_t)$. Moreover, observe that

$$|\mathcal{T}(v)| > |\mathcal{T}(v - \Delta_v(t)u_t)|.$$

Therefore,

$$r \geq |\mathcal{T}(v - \Delta_v(t)u_t)|$$

By the induction hypothesis and Strong monotonicity, for each $(i,j) \in M^+$ such that $j > p_i$,

$$\begin{aligned} f_{ij}(m,v) &= f_{ij}(m, v - \Delta_v(t)u_t) \\ &\overset{Hyp}{=} \varphi_{ij}(m, v - \Delta_v(t)u_t) \\ &= \varphi_{ij}(m,v). \end{aligned} \tag{4.28}$$

Next, we assume that $p = \overrightarrow{0}$. For each $(i,j) \in M^+$, there exists a $t \in \mathcal{T}(v)$ such that $j > t_i$. In this case, (4.28) holds for each $(i,j) \in M^+$ and the proof is complete.

It remains to show that, if $p \neq \overrightarrow{0}$, then for each $(i,j) \in M^+$ such that $j \leq p_i$,

$$f_{ij}(m,v) = \varphi_{ij}(m,v).$$

We proceed in two-steps.

**Step 1.** We define the game $(m, w) \in \mathcal{G}$ as

$$w = v - \sum_{\substack{(i,j)\in M^+ \\ j\leq p_i}} \varphi_{ij}(m, v)u_{(0_{-i},j)}, \tag{4.29}$$

and we show that, for each $(i, l) \in M^+$ such that $l \leq p_i$,

$$f_{il}(m, w) = 0. \tag{4.30}$$

**Step 1.1.** To that end, we show that

$$\sum_{\substack{(i,l)\in M^{+,l} \\ l\leq p_i}} f_{il}(m, w) = 0.$$

By Proposition 23, $f$ satisfies Multi-efficiency. By Multi-efficiency and (4.29), for each $l \leq \max_{k\in N} m_k$,

$$\begin{aligned}
\sum_{(i,l)\in M^{+,l}} f_{il}(m, w) =& w((l \wedge m_k)_{k\in N}) - w((l - 1 \wedge m_k)_{k\in N} \\
\iff \sum_{\substack{(i,l)\in M^{+,l} \\ l\leq p_i}} f_{il}(m, w) =& w((l \wedge m_k)_{k\in N}) - w((l - 1 \wedge m_k)_{k\in N} \\
&- \sum_{\substack{(i,l)\in M^{+,l} \\ l> p_i}} f_{il}(m, w) \\
&\overset{(4.29)}{=} v((l \wedge m_k)_{k\in N}) - v((l - 1 \wedge m_k)_{k\in N} \\
&\quad - \sum_{\substack{(i,j)\in M^+ \\ j\leq p_i}} \varphi_{ij}(m, v)u_{(0_{-i},j)}((l \wedge m_k)_{k\in N}) \\
&\quad + \sum_{\substack{(i,j)\in M^+ \\ j\leq p_i}} \varphi_{ij}(m, v)u_{(0_{-i},j)}((l - 1 \wedge m_k)_{k\in N}) \\
&\quad - \sum_{\substack{(i,l)\in M^{+,l} \\ l> p_i}} f_{il}(m, w).
\end{aligned} \tag{4.31}$$

Before proceeding further into the computation of (4.31), observe that

$$
\begin{aligned}
&- \sum_{\substack{(i,j)\in M^+\\ j\le p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N})\\
&+ \sum_{\substack{(i,j)\in M^+\\ j\le p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N})\\
=&- \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j<l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N})\\
&- \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j=l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N})\\
&- \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j>l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N})\\
&+ \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j<l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N})\\
&+ \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j\ge l}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N})
\end{aligned}
$$

By definition, $p_i \le m_i$ for each $i \in N$. For each $i \in N$ and $j \le p_i \le m_i$,

$$
u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) = \begin{cases} 1 & \text{if } j \le (l \wedge m_i),\\ 0 & \text{otherwise.}\end{cases}
$$

It follows that

$$
\begin{aligned}
&- \sum_{\substack{(i,j)\in M^+\\ j\le p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l\wedge m_k)_{k\in N}) && (4.32)\\
&+ \sum_{\substack{(i,j)\in M^+\\ j\le p_i}} \varphi_{ij}(m,v)u_{(0_{-i},j)}((l-1\wedge m_k)_{k\in N})\\
=&- \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j<l}} \varphi_{ij}(m,v) - \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j=l}} \varphi_{ij}(m,v) + \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j<l}} \varphi_{ij}(m,v)\\
=&- \sum_{\substack{(i,j)\in M^+\\ j\le p_i\\ j=l}} \varphi_{ij}(m,v) = - \sum_{\substack{(i,l)\in M^{+,l}\\ l\le p_i}} \varphi_{il}(m,v). && (4.33)
\end{aligned}
$$

By (4.32), (4.31) becomes

$$\sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} f_{il}(m,w) = v((l\wedge m_k)_{k\in N}) - v((l-1\wedge m_k)_{k\in N} - \sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} \varphi_{il}(m,v) - \sum_{\substack{(i,l)\in M^{+,l}\\ l> p_i}} f_{il}(m,w). \tag{4.34}$$

By (4.28), for each $(i,l)\in M^{+,l}$ such that $l>p_i$,

$$f_{il}(m,w) = \varphi_{il}(m,w). \tag{4.35}$$

Combining (4.34) and (4.35), we obtain

$$\sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} f_{il}(m,w) = v((l\wedge m_k)_{k\in N}) - v((l-1\wedge m_k)_{k\in N} - \sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} \varphi_{il}(m,v) - \sum_{\substack{(i,l)\in M^{+,l}\\ l> p_i}} \varphi_{il}(m,w). \tag{4.36}$$

Moreover, each $(i,l)\in M^{+,l}$ such that $l>p_i$, is a null pair in $(m,u_{0_{-i},j})$, where $(i,j)\in M^+$ is such that $j\leq p_i$. By definition of $(m,w)$ (see (4.29)), it follows that each pair $(i,l)$ such that $l>p_i$, has the same marginal contributions in $(m,w)$ and in $(m,v)$. Since $\varphi$ satisfies Strong monotonicity, for each $(i,l)\in M^{+,l}$ such that $l>p_i$,

$$\varphi_{il}(m,w) = \varphi_{il}(m,v). \tag{4.37}$$

Combining (4.36), (4.37) and the fact that $\varphi$ satisfies Multi-efficiency, we obtain

$$\sum_{\substack{(i,l)\in M^{+,l}\\ l\leq p_i}} f_{il}(m,w) = v((l\wedge m_k)_{k\in N}) - v((l-1\wedge m_k)_{k\in N} \tag{4.38}$$

$$- \sum_{(i,l)\in M^{+,l}} \varphi_{il}(m,v)$$

$$= 0. \tag{4.39}$$

This concludes Step 1.1.

**Step 1.2.** We show that all the pairs $(i,l)\in M^{+,l}$ such that $l\leq p_i$, are equal in $(m,w)$.

By definition of $M^{+,l}$, it holds that $l\geq 1$. Consider two pairs $(i,l),(i',l)\in M^{+,l}$ such that $l\leq p_i$ and $l\leq p_{i'}$. Since $p=\bigwedge_{t\in\mathcal{T}(v)} t$, each $t\in\mathcal{T}(v)$ verifies $t_i\geq l$ and

$t_{i'} \geq l$. In other words, for each $s \in \mathcal{M}$ such that $s_i < l$ or $s_{i'} < l$, $\Delta_v(s) = 0$. Therefore, for each $s \in \mathcal{M}$ such that $s_i = s_{i'} = l - 1$,

$$v(s + e_i) = v(s + e_{i'}) = 0. \tag{4.40}$$

Therefore, $(i, l)$ and $(i', l)$ are equal in $(m, v)$. Since $\varphi$ satisfies Equal treatment for equal pairs and by (4.29), for each $s \in \mathcal{M}$ such that $s_i = s_{i'} = l - 1$,

$$\begin{aligned}
w(s + e_i) \overset{(4.29)}{=} & v(s + e_i) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h}} \varphi_{hj}(m, v) u_{(0_{-h}, j)}(s + e_i) \\
= & v(s + e_i) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h \\ h \neq i, i'}} \varphi_{hj}(m, v) u_{(0_{-h}, j)}(s + e_i) \\
& - \sum_{\substack{j \leq p_{i'} \\ j \leq l-1}} \varphi_{i'j}(m, v) u_{(0_{-i'}, j)}(s + e_i) \\
& - \sum_{\substack{j \leq p_i \\ j \leq l-1}} \varphi_{ij}(m, v) u_{(0_{-i}, j)}(s + e_i) - \varphi_{il}(m, v) u_{(0_{-i}, l)}(s + e_i) \\
= & v(s + e_i) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h \\ h \neq i, i'}} \varphi_{hj}(m, v) u_{(0_{-h}, j)}(s + e_i) \\
& - \sum_{\substack{j \leq p_{i'} \\ j \leq l-1}} \varphi_{i'j}(m, v) u_{(0_{-i'}, j)}(s + e_i) \\
& - \sum_{\substack{j \leq p_i \\ j \leq l-1}} \varphi_{ij}(m, v) u_{(0_{-i}, j)}(s + e_i) - \varphi_{il}(m, v) \\
= & v(s + e_{i'}) - \sum_{\substack{(h,j) \in M^+ \\ j \leq p_h \\ h \neq i, i'}} \varphi_{hj}(m, v) u_{(0_{-h}, j)}(s + e_{i'}) \\
& - \sum_{\substack{j \leq p_{i'} \\ j \leq l-1}} \varphi_{i'j}(m, v) u_{(0_{-i'}, j)}(s + e_{i'}) \\
& - \sum_{\substack{j \leq p_i \\ j \leq l-1}} \varphi_{ij}(m, v) u_{(0_{-i}, j)}(s + e_{i'}) - \varphi_{i'l}(m, v) \\
\overset{(4.29)}{=} & w(s + e_{i'}).
\end{aligned}$$

Therefore, two pairs $(i, l)$ and $(i', l)$ such that $l \leq p_i$ and $l \leq p_{i'}$, are equal in $(m, w)$. This concludes Step 1.2

By Sign symmetry for equal pairs,

$$sign(f_{il}(m,w)) = sign(f_{i'l}(m,w)). \tag{4.41}$$

Combining (4.38) and (4.41), for each $(i,l) \in M^+$ such that $l \leq p_i$, we obtain

$$f_{il}(m,w) = 0,$$

which concludes Step 1.

**Step 2.** For each $(i,j) \in M^+$ such that $0 < j \leq p_i$, we define the game $(m, w^{ij}) \in \mathcal{G}$ as

$$w^{ij} = v - \varphi_{ij}(m,v)u_{(0_{-i},j)}. \tag{4.42}$$

In this step, we first show that, for each $(i,j) \in M^+$ such that $j \leq p_i$,

$$\varphi_{ij}(m,v) = f_{ij}(m,v) - f_{ij}(m,w^{ij}).$$

The game $(m, w^{ij})$ is defined in such a way that the pair $(i,j)$ has the same marginal contributions in $(m,w)$ and in $(m,w^{ij})$. Indeed, observe that the pair $(i,j)$ has null marginal contributions to coalition in each game $(m, u_{(0_{-i'},j')})$ such that $i' \neq i$ or $i' = i$ and $j' \neq j$. Therefore, by Strong monotonicity, for each $(i,j) \in M^+$ such that $j \leq p_i$,

$$f_{ij}(m,w) = f_{ij}(m,w^{ij}). \tag{4.43}$$

Additionally, by Multi-efficiency, (4.42) and the definition of a minimal effort game (see (2.31)),

$$\begin{aligned}
\sum_{(k,j)\in M^{+,j}} f_{kj}(m,w^{ij}) =& w^{ij}((j \wedge m_k)_{k\in N}) - w^{ij}(((j-1) \wedge m_k)_{k\in N}) \\
\overset{(4.42)}{=}& v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) \\
& - \varphi_{ij}(m,v)u_{(0_{-i},j)}((j \wedge m_k)_{k\in N}) \\
& + \varphi_{ij}(m,v)u_{(0_{-i},j)}(((j-1) \wedge m_k)_{k\in N}) \\
\overset{(2.31)}{=}& v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) \\
& - \varphi_{ij}(m,v).
\end{aligned} \tag{4.44}$$

Each pair in $M^{+,j} \setminus \{(i,j)\}$ is null in $(m, u_{0_{-i},j})$. Therefore, by (4.42), each pair in $M^{+,j} \setminus \{(i,j)\}$ has the same marginal contributions in $(m,w^{ij})$ and in $(m,v)$. It follows that, by Strong monotonicity, each pair in $M^{+,j} \setminus \{(i,j)\}$ receives the same

payoff in $(m, w^{ij})$ and in $(m, v)$. Thus,

$$\begin{aligned}\sum_{(k,j)\in M^{+,j}} f_{kj}(m, w^{ij}) &= \sum_{\substack{(k,j)\in M^{+,j}\\ k\neq i}} f_{kj}(m, w^{ij}) + f_{ij}(m, w^{ij})\\ &= \sum_{\substack{(k,j)\in M^{+,j}\\ k\neq i}} f_{kj}(m, v) + f_{ij}(m, w^{ij})\\ &= v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N})\\ &\quad - f_{ij}(m, v) + f_{ij}(m, w^{ij}). \end{aligned} \tag{4.45}$$

Combining (4.44) and (4.45), for each $(i, j) \in M^+$ such that $j \leq p_i$, we obtain

$$\varphi_{ij}(m, v) = f_{ij}(m, v) - f_{ij}(m, w^{ij}), \tag{4.46}$$

which concludes Step 2.

We have the material to conclude the proof of the Induction step. By (4.43), $f_{ij}(m, w^{ij}) = f_{ij}(m, w)$ and by (4.30), $f_{ij}(m, w) = 0$ for each $(i, j) \in M^+$ such that $j \leq p_i$. By (4.46), for each $(i, j) \in M^+$ such that $j \leq p_i$,

$$f_{ij}(m, v) = \varphi_{ij}(m, v).$$

Therefore, for each $(m, v) \in \mathcal{G}$ and each $(i, j) \in M^+$,

$$f_{ij}(m, v) = \varphi_{ij}(m, v).$$

The proof of the theorem is complete. ■

The four axioms of the statement of Theorem 22 are logically independent, as shown by the following alternative solutions.

- The value $f$ given by $f(m, v) = \vec{0}$ for each $(m, v) \in \mathcal{G}$ satisfies all the axioms except Efficiency.

- The value $f$ given, for each $(m, v) \in \mathcal{G}$, by

$$\forall (i, j) \in M^+,$$

$$f_{ij}(m, v) = \begin{cases} \varphi_{ij}(m, v) + \dfrac{m^T}{|Q(1)|} & \text{if } j = 1 \text{ and } m^T > 1,\\ \varphi_{ij}(m, v) - \dfrac{m^T}{|Q(2)|} & \text{if } j = 2 \text{ and } m^T > 1,\\ \varphi_{ij}(m, v) & \text{otherwise.} \end{cases}$$

  satisfies all the axioms except Independence of higher activity levels.

- The multi-choice Equal division value $\xi$ satisfies all the axioms except Strong monotonicity.

- The value $f$ given, for each $(m, v) \in \mathcal{G}$, by

$$\forall (i,j) \in M^+, \quad f_{ij}(m,v) = \begin{cases} \varphi_{ij}(m,v) + 1 & \text{if } (i,j) = (1,1), \\ \varphi_{ij}(m,v) - 1 & \text{if } (i,j) = (n,1), \\ \varphi_{ij}(m,v) & \text{otherwise,} \end{cases}$$

satisfies all the axioms except Sign symmetry for equal pairs.

### 4.3.2 Characterization of the multi-choice Equal Division value

To characterize the multi-choice Equal Division value, we introduce the Sign preservation axiom. This axiom is a stronger version of Sign symmetry for equal pairs. It requires that two pairs featuring the same activity level receive a payoff of the same sign.

**Axiom 48 (Sign preservation).** *For each $(m, v) \in \mathcal{G}$ and each $(i, j), (i', j) \in M^+$,*

$$sign(f_{ij}(m,v)) = sign(f_{i'j}(m,v)).$$

Sign preservation $\implies$ Sign symmetry for equal pairs.

We have the material to provide an axiomatic characterization of the multi-choice Equal Division value.

**Theorem 23.** *A value $f$ on $\mathcal{G}$ satisfies Efficiency, Linearity, Independence of higher activity levels, Sign preservation and Equal treatment of equal pairs if and only if $f = \xi$.*

*Proof.* First, let us show that $\xi$ satisfies all the axioms of the statement of Theorem 23.

**Efficiency:** For each $(m, v) \in \mathcal{G}$,

$$\begin{aligned} &\sum_{i \in N} \sum_{j \in M_i^+} \xi_{ij}(m,v) \\ &= \sum_{j \leq \max\limits_{k \in N} m_k} \sum_{i \in Q(j)} \frac{1}{|Q(j)|} \Big[ v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \Big] \\ &= \sum_{j \leq \max\limits_{k \in N} m_k} v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}) \\ &= v(m). \end{aligned}$$

This shows that the value satisfies Efficiency.

**Other axioms:** By definition of $\xi$ (see (29)), the payoff of a pair does not depends on activity levels different from the one contained in this pair. Therefore, $\xi$ satisfies Independence of higher activity levels. Linearity, Sign preservation and Equal treatment of equal pairs are direct consequences from the definition of $\xi$.

Next, we show the uniqueness of the solution. Let $f$ be a value satisfying all the axioms of the statement of Theorem 23. We know that each characteristic function $v$ admits a linear decomposition in terms of Dirac games. By Linearity, for each $(m, v) \in \mathcal{G}$,

$$f(m, v) = \sum_{s \leq m} v(s) f(m, \delta_s).$$

For each $s \in \mathcal{M}$, we show that

$$f(m, \delta_s) = \xi_{ij}(m, \delta_s).$$

We consider several cases.

**Case 1.** Suppose that $s \in \mathcal{M}$ is not a synchronized coalition, that is $s \neq ((l \wedge m_k)_{k \in N})$ for each $l \leq \max_{k \in N} m_k$. Since $f$ satisfies Efficiency, and Independence of higher activity levels, by Proposition 23 it satisfies Multi-efficiency. Therefore, by Multi-efficiency, for each $j \leq \max_{k \in N} m_k$,

$$\sum_{(i,j) \in M^{+,j}} f_{ij}(m, \delta_s) = \delta_s((j \wedge m_k)_{k \in N}) - \delta_s(((j-1) \wedge m_k)_{k \in N}).$$

Since $s \neq ((j \wedge m_k)_{k \in N})$ and $s \neq (((j-1) \wedge m_k)_{k \in N})$, by definition of a Dirac game,

$$\sum_{(i,j) \in M^{+,j}} f_{ij}(m, \delta_s) = 0. \tag{4.47}$$

Since $\delta_s((j \wedge m_k)_{k \in N}) - \delta_s(((j-1) \wedge m_k)_{k \in N}) = 0$, by Sign preservation and (4.47), for each $(i, j) \in M^{+,j}$,

$$f_{ij}(m, \delta_s) = 0 = \xi_{ij}(m, \delta_s).$$

**Case 2.** Suppose that $s \in \mathcal{M}$ is a synchronized coalition, that is $s = (l \wedge m_k)_{k \in N}$, where $l \leq \max_{k \in N} m_k$. Take any activity level $j$ such that $j < l$. By Multi-efficiency,

$$\sum_{(i,j) \in M^{+,j}} f_{ij}(m, \delta_s) = \delta_s((j \wedge m_k)_{k \in N}) - \delta_s(((j-1) \wedge m_k)_{k \in N}).$$

Since $s \neq ((j \wedge m_k)_{k\in N})$ and $s \neq (((j-1) \wedge m_k)_{k\in N})$, by definition of a Dirac game,

$$\sum_{(i,j)\in M^{+,j}} f_{ij}(m, \delta_s) = 0.$$

Since $\delta_s((j \wedge m_k)_{k\in N}) - \delta_s(((j-1) \wedge m_k)_{k\in N}) = 0$, by Sign preservation and (4.47), for each pair $(i, j) \in M^{+,j}$ such that $j < l$,

$$f_{ij}(m, \delta_s) = 0 = \xi_{ij}(m, \delta_s).$$

**Case 3.** Suppose that $s \in \mathcal{M}$ is a synchronized coalition such that $s = (l \wedge m_k)_{k\in N}$, where $l \leq \max_{k\in N} m_k$. Similarly to Case 1 and 2, for each $(i, j) \in M^+$ such that $j > l + 1$,

$$f_{ij}(m, \delta_s) = 0 = \xi_{ij}(m, \delta_s).$$

**Case 4.** Suppose that $s \in \mathcal{M}$ is a synchronized coalition such that $s = (l \wedge m_k)_{k\in N}$, where $l \leq \max_{k\in N} m_k$. Consider the pairs $(i, j) \in M^+$ such that $j = l$, that is the pairs in $M^{+,l}$. By Multi-efficiency and the definition of a Dirac game,

$$\begin{aligned}\sum_{(i,l)\in M^{+,l}} f_{il}(m, \delta_s) &= \delta_s((l \wedge m_k)_{k\in N}) - \delta_s((l-1 \wedge m_k)_{k\in N}) \\ &= 1. \end{aligned} \tag{4.48}$$

Two distinct pairs $(i, l), (i', l) \in M^{+,l}$ are equal in $(m, \delta_s)$. Therefore, by Equal treatment of equal pairs,

$$f_{il}(m, \delta_s) = f_{i'l}(m, \delta_s). \tag{4.49}$$

From (4.48) and (4.49), it follows that for each $(i, l) \in M^{+,l}$,

$$f_{il}(m, \delta_s) = \frac{1}{|Q(l)|} = \xi_{il}(m, \delta_s).$$

**Case 5.** Consider $s \in \mathcal{M}$ such that $s = (l \wedge m_k)_{k\in N}$, where $l < \max_{k\in N} m_k$. Consider the pairs $(i, l+1) \in M^{+,l+1}$. By Multi-efficiency and the definition of a Dirac game,

$$\begin{aligned}\sum_{(i,l+1)\in M^{+,l+1}} f_{i(l+1)}(m, \delta_s) &= \delta_s(((l+1) \wedge m_k)_{k\in N}) - \delta_s((l \wedge m_k)_{k\in N}) \\ &= 0 - 1 \\ &= -1.\end{aligned}$$

Similarly to Case 4, for each $(i, l+1) \in M^{+,l+1}$,

$$f_{i(l+1)}(m, \delta_s) = -\frac{1}{|Q(l+1)|} = \xi_{i(l+1)}(m, \delta_s).$$

Therefore, for each $s \in \mathcal{M}$, $f(m, \delta_s) = \xi(m, \delta_s)$. By Linearity, we conclude the proof of Theorem 23. ■

The five axioms of the statement of Theorem 23 are logically independent, as shown by the following alternative solutions.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by $f(m,v) = \vec{0}$ satisfies all the axioms except Efficiency.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by

$$\forall (i,j) \in M^+,$$
$$f_{ij}(m,v) = \frac{1}{|Q(j)|} \sum_{k \geq j} \frac{v((k \wedge m_h)_{h \in N}) - v(((k-1) \wedge m_h)_{h \in N})}{k}$$

  satisfies all the axioms except Independence of higher activity levels. Observe that this value extends the Equal division value from TU-games to multi-choice games.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by

$$\forall (i,j) \in M^+,$$
$$f_{ij}(m,v) = \frac{(v(((j-1) \wedge m_h)_{h \in N} + e_i)^2) + 1}{\sum\limits_{k \in Q(k)} ((v(((j-1) \wedge m_h)_{h \in N} + e_k)^2) + 1} \times \Big[v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N}\Big],$$

  satisfies all the axioms except Linearity.

- The multi-choice Shapley value $\varphi$ satisfies all the axioms except Sign preservation.

- Take any $(m,v) \in \mathcal{G}$ and fix any arbitrary integer $\beta^{ij} \in \{1,2\}$ for each $(i,j) \in M^+$. The value $f^\beta$ given by

$$\forall (i,j) \in M^+,$$
$$f^\beta_{ij}(m,v) = \frac{\beta^{ij}}{\sum_{k \in Q(j)} \beta^{kj}} \Big[v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})\Big],$$

  satisfies all the axioms except Equal treatment for equal pairs.

### 4.3.3 Characterization of the multi-choice Egalitarian Shapley values

In this subsection we provide an axiomatic characterization of the multi-choice Egalitarian Shapley values. To that end, we introduce two new axioms.

A multi-efficient value shares the surplus generated between two consecutive synchronized coalitions among the pairs containing the required activity level (see (4.4)). This surplus can eventually be negative. Requiring that the payoff of a pair varies according to its marginal contributions to coalitions regardless of the surplus to be shared is then a strong requirement in Strong monotonicity. On the contrary, it seems reasonable that the payoff of a pair, let us say $(i,j) \in M^{+,j}$, does not decrease from one game, let us say $(m,v) \in \mathcal{G}$, to another, let us say $(m,w) \in \mathcal{G}$, if the surplus generated between the $j$-synchronized coalition and the $(j-1)$-synchronized coalition does not decrease from $(m,v)$ to $(m,w)$. The next axiom is a weaker version of Strong monotonicity which requires that the surplus generated between two synchronized coalitions should not decrease from one game to another.

**Axiom 49 (Weak monotonicity).** *For each $(m,v),(m,w) \in \mathcal{G}$ and each $(i,j) \in M^+$ such that*

$$\begin{aligned} &v((j \wedge m_k)_{k\in N}) - v(((j-1) \wedge m_k)_{k\in N}) \\ &\geq \\ &w((j \wedge m_k)_{k\in N}) - w(((j-1) \wedge m_k)_{k\in N}), \end{aligned}$$

*and for each $s \in \mathcal{M}$ such that $s_i = j-1$,*

$$v(s+e_i) - v(s) \geq w(s+e_i) - w(s),$$

*then,*

$$f_{ij}(m,v) \geq f_{ij}(m,w).$$

Whenever $m = (1,\ldots,1)$, Weak monotonicity boils down to the axiom of Weak monotonicity for TU-games (see (2.15)). Obviously, Strong monotonicity implies Weak monotonicity.

$$\text{Strong monotonicity} \implies \text{Weak monotonicity}$$

Consider $(m,v) \in \mathcal{G}$ and two distinct pairs $(i,j),(i',j) \in M^{+,j}$. We say that the pair $(i,j)$ is more **desirable** than the pair $(i',j)$ in $(m,v)$ if its has better marginal contributions to coalitions. Formally, $(i,j)$ is more **desirable** than $(i',j)$ if for each $s \in \mathcal{M}$ such that $s_i = s_{i'} = j-1$,

$$v(s+e_i) \geq v(s+e_{i'}).$$

The next axiom requires that a pair receives a greater payoff than other less desirable pairs.

**Axiom 50 (Level desirability).** *For each $(m,v) \in \mathcal{G}$ and two distinct pairs $(i,j),(i',j) \in M^+$ such that $(i,j)$ is more desirable than $(i',j)$ in $(m,v)$,*

$$f_{ij}(m,v) \geq f_{i'j}(m,v).$$

Whenever $m = (1, \ldots, 1)$, Level desirability boils down to the axiom of desirability for TU-games (see (2.14)). We have the material to provide a characterization of the multi-choice Egalitarian Shapley values.

**Remark 6.** *By definition, Level desirability implies Equal treatment for equal pairs. If $(m, v) \in \mathcal{G}$ is the null game, then Multi-efficiency and Level desirability imply $f_{ij}(m, v) = 0$ for each $(i, j) \in M^+$.*

**Theorem 24.** *A solution $f$ on $\mathcal{G}$ satisfies Efficiency, Linearity, Independence of higher activity levels, Weak monotonicity and Level desirability if and only if $f = \chi^\alpha$, for some parameter system $\alpha$.*

*Proof.* Consider any parameter system $\alpha$. By definition and the fact that multi-choice Egalitarian Shapley values are convex combinations of the multi-choice Shapley value and the multi-choice Equal division value (see (4.9)), $\chi^\alpha$ satisfies all the axioms of the statement of Theorem 24.

Next, we show that the multi-choice Egalitarian Shapley values are the only values satisfying all the axioms of the statement of Theorem 24. Consider a value $f$ satisfying all the axioms of the statement of Theorem 24. To prove the uniqueness part, we show that, for each $(m, v) \in \mathcal{G}$, there exists a parameter system $\alpha$ such that

$$f(m, v) = \chi^\alpha(m, v).$$

By Linearity, for each $(m, v) \in \mathcal{G}$,

$$f(m, v) = \sum_{t \leq m} \Delta_t(v) f(m, u_t).$$

For each $t \in \mathcal{M}$, we introduce the notation $t^T = \max_{i \in N} t_i$. Take any $1 \leq l \leq m^T$. Let us show that $f$ can be written, for each $(m, u_t)$ such that $t^T = l$, as

$$\forall (i, j) \in M^+,$$
$$f_{ij}(m, u_t) = \begin{cases} \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l)\xi_{il}(m, u_t) & \text{if } j = l, \\ 0 & \text{otherwise,} \end{cases}$$

for some $0 \leq \alpha^l \leq 1$. To that end, consider all pairs $(i, j) \in M^+$ such that $j < l$. By Independence of higher activity levels,

$$f_{ij}(m, u_t) = f_{ij}((j \wedge m_k)_{k \in N}, u_t).$$

Since $((j \wedge m_k)_{k \in N}, u_t)$ is the null game, by Remark 6, for each $(i, j) \in M^+$ such that $j < l$,

$$f_{ij}(m, u_t) = 0. \tag{4.50}$$

Consider all pairs $(i,j) \in M^+$ such that $j > l$. These pairs are null pairs in $(m, u_t)$ and thus are equal. From Remark 6 and Multi-efficiency, for each $(i,j) \in M^+$ such that $j > l$,

$$f_{ij}(m, u_t) = 0. \tag{4.51}$$

Now, consider all pairs $(i,l) \in M^+$ such that $i \in Q(l)$. We show that $f$ can be written, for each $(m, u_t)$ such that $t^T = l$, as

$$\forall (i,l) \in M^+, \quad f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l)\xi_{il}(m, u_t),$$

for some $0 \leq \alpha^l \leq 1$. We proceed by induction on $q^t(l)$ the number of players that play $l$ in coalition $t$.

**Initialization**: Take any minimal effort game $(m, u_t) \in \mathcal{G}$ such that $t^T = l$ and $q^t(l) = 1$. In such game, there is exactly one player, let us say $k \in N$, that plays the activity level $l$ in $t$.

Before proceeding further into the initialization step, we must show the following claim. There exists a unique $c^l \in \mathbb{R}$ such that, for any minimal effort game $(m, u_t) \in \mathcal{G}$, $t^T = l$, $q^t(l) = 1$, it holds that

$$\forall (i,l) \in M^+, i \neq k, \quad f_{il}(m, u_t) = c^l, \tag{4.52}$$

where $k$ refers to the only player that plays the activity level $l$ in $t$. This means that all the pairs $(i,l) \in M^+$, $i \neq k$, receive the same payoff in $(m, u_t)$ and that this payoff does not depend on the player $k$ that plays the activity level $l$ in $t$. To this end, we distinguish three exclusive cases.

**Case 1.** If $|Q(m^T)| \geq 3$, then there is at least three players in $Q(l)$. Consider any three distinct players in $Q(l)$ denoted by $k$, $i$ and $i'$. Consider the three coalitions $t, t', t'' \in \mathcal{M}$ defined as

$$\begin{aligned}
&t_k = l, \quad \textit{and} \quad \forall h \in N \setminus \{k\}, \quad t_h < l, \\
&t'_i = l, \quad \textit{and} \quad \forall h \in N \setminus \{i\}, \quad t'_h < l, \\
&t''_{i'} = l, \quad \textit{and} \quad \forall h \in N \setminus \{i'\}, \quad t''_h < l.
\end{aligned}$$

Consider the minimal effort games $(m, u_t)$, $(m, u_{t'})$ and $(m, u_{t''})$. Observe that:

- $(i,l) \in M^+$ is a null pair in $(m, u_t)$ and $(m, u_{t''})$, and it is the only non null pair in $(m, u_{t'})$;
- $(i',l) \in M^+$ is a null pair in $(m, u_t)$ and $(m, u_{t'})$, and it is the only non null pair in $(m, u_{t''})$;
- $(k,l) \in M^+$ is a null pair in $(m, u_{t'})$ and $(m, u_{t''})$, and it is the only non null pair in $(m, u_t)$;

- $(i,l),(i',l) \in M^+$ are equal pairs in $(m,u_t)$;
- $(i,l),(k,l) \in M^+$ are equal pairs in $(m,u_{t''})$;
- $(i',l),(k,l) \in M^+$ are equal pairs in $(m,u_{t'})$.

Observe that

$$\begin{aligned} & u_t((l \wedge m_h)_{h\in N}) = u_{t'}((l \wedge m_h)_{h\in N}) \\ & \qquad\qquad\qquad = u_{t''}((l \wedge m_h)_{h\in N}) \\ \text{and} \quad & u_t(((l-1) \wedge m_h)_{h\in N}) = u_{t'}(((l-1) \wedge m_h)_{h\in N}) \\ & \qquad\qquad\qquad = u_{t''}(((l-1) \wedge m_h)_{h\in N}). \end{aligned} \tag{4.53}$$

By (4.53), the fact that $(i,l) \in M^+$ is a null pair in $(m,u_t)$ and $(m,u_{t''})$, and Weak monotonicity, one obtains

$$f_{il}(m,u_t) = f_{il}(m,u_{t''}). \tag{4.54}$$

Similarly,

$$\begin{aligned} & f_{i'l}(m,u_t) = f_{i'l}(m,u_{t'}) \\ \text{and} \quad & f_{kl}(m,u_{t'}) = f_{kl}(m,u_{t''}). \end{aligned} \tag{4.55}$$

By the fact that $(i,l),(i',l) \in M^+$ are equal pairs in $(m,u_t)$ and Level desirability, one obtains

$$f_{il}(m,u_t) = f_{i'l}(m,u_t). \tag{4.56}$$

Similarly,

$$\begin{aligned} & f_{i'l}(m,u_{t'}) = f_{kl}(m,u_{t'}) \\ \text{and} \quad & f_{il}(m,u_{t''}) = f_{kl}(m,u_{t''}). \end{aligned} \tag{4.57}$$

Combining (4.54), (4.55), (4.56) and (4.57), one obtains

$$\begin{aligned} f_{il}(m,u_t) &\overset{(4.54)}{=} f_{il}(m,u_{t''}) \\ &\overset{(4.57)}{=} f_{kl}(m,u_{t''}) \\ &\overset{(4.55)}{=} f_{kl}(m,u_{t'}) \\ &\overset{(4.57)}{=} f_{i'l}(m,u_{t'}) \\ &\overset{(4.55)}{=} f_{i'l}(m,u_t) \\ &= c^l, \end{aligned}$$

for some $c^l \in \mathbb{R}$.

We have shown that there exists a unique $c^l \in \mathbb{R}$ such that, for any minimal effort game $(m, u_t) \in \mathcal{G}$, $|Q(m^T)| \geq 3$, $t^T = l$, $q^t(l) = 1$, (4.52) holds.

**Case 2.** If $|Q(m^T)| = 2$, then there is at least two players in $Q(l)$. If there are at least three players in $Q(l)$, then the proof is identical to the one in Case 1. Therefore, let us assume that $Q(l) = 2$. Let us call the players in $Q(l)$ by $i$ and $k$. Consider the two coalitions $t, t' \in \mathcal{M}$ defined as

$$\begin{aligned} &t_i = l, \quad \text{and} \quad \forall h \in N \setminus \{i\}, \quad t_h < l, \\ &t'_k = l, \quad \text{and} \quad \forall h \in N \setminus \{k\}, \quad t'_h < l. \end{aligned}$$

Consider the minimal games $(m, u_t) \in \mathcal{G}$, $(m, u_{t'}) \in \mathcal{G}$, and $(m, u_t + u_{t'}) \in \mathcal{G}$. Observe that $(i, l), (k, l) \in M^+$ are equal pairs in $(m, u_t + u_{t'})$. Therefore, by Level desirability, it holds that

$$f_{il}(m, u_t + u_{t'}) = f_{kl}(m, u_t + u_{t'}). \tag{4.58}$$

By Linearity, (4.58) becomes

$$\begin{aligned} & f_{il}(m, u_t) + f_{il}(m, u_{t'}) = f_{kl}(m, u_t) + f_{kl}(m, u_{t'}) \\ \iff \quad & f_{il}(m, u_t) = f_{kl}(m, u_t) + f_{kl}(m, u_{t'}) - f_{il}(m, u_{t'}). \end{aligned} \tag{4.59}$$

Since $f$ satisfies Efficiency and Independence of higher activity, by Proposition 23, $f$ satisfies Multi-efficiency. Since $i$ and $k$ are the only two players in $Q(l)$, by Multi-efficiency, it holds that

$$\begin{aligned} & f_{il}(m, u_t) + f_{kl}(m, u_t) = 1 \\ \text{and} \quad & f_{il}(m, u_{t'}) + f_{kl}(m, u_{t'}) = 1. \end{aligned}$$

It follows that

$$f_{il}(m, u_t) + f_{kl}(m, u_t) = f_{il}(m, u_{t'}) + f_{kl}(m, u_{t'}) \tag{4.60}$$

Combining (4.59) and (4.60), one obtains

$$\begin{aligned} & f_{kl}(m, u_t) + f_{kl}(m, u_{t'}) - f_{il}(m, u_{t'}) + f_{kl}(m, u_t) \\ & = f_{il}(m, u_{t'}) + f_{kl}(m, u_{t'}) \\ \iff \quad & f_{kl}(m, u_t) - f_{il}(m, u_{t'}) + f_{kl}(m, u_t) = f_{il}(m, u_{t'}) \\ \iff \quad & f_{kl}(m, u_t) = f_{il}(m, u_{t'}) = c^l, \end{aligned}$$

for some $c^l \in \mathbb{R}$.

We have shown that there exists a unique $c^l \in \mathbb{R}$ such that, for any minimal effort game $(m, u_t) \in \mathcal{G}$, $|Q(m^T)| = 2$, $t^T = l$, $q^t(l) = 1$, (4.52) holds.

**Case 3.** Finally, assume that $|Q(m^T)| = 1$. If there is only one player in $Q(l)$, then there is nothing to show. If there are two players in $Q(l)$, then (4.52) holds according to Case 2. If there three players or more in $Q(l)$, then (4.52) holds according to Case 1.

We have shown that there exists a unique $c^l \in \mathbb{R}$ such that, for any minimal effort game $(m, u_t) \in \mathcal{G}$, $t^T = l$ and $q^t(l) = 1$, (4.52) holds.

Next, take any minimal effort game $(m, u_t) \in \mathcal{G}$ such that $t^T = l$ and $q^t(l) = 1$. The pair $(k, l)$ is the only non-null pair in $(m, u_t)$ featuring the activity level $l$ and $u_t(m) \geq 0$. By (4.52), Multi-efficiency and the fact that null pairs featuring the activity levels $l$ are equal pairs in $(m, u_t)$, it holds that

$$f_{kl}(m, u_t) = 1 - (|Q(l)| - 1)c^l.$$

Define $\alpha^l$ as

$$\alpha^l = 1 - c^l|Q(l)|,$$

so that one obtains

$$c^l = \frac{1 - \alpha^l}{|Q(l)|}.$$

Now, we show that $\alpha^l \leq 1$. By Remark 6, each pair receives a zero payoff in the null game. Observe that each pair in $M^+$ has better marginal contributions to coalitions in $(m, u_t)$ than in the null game. Moreover, it holds that

$$u_t((l \wedge m_k)_{k \in N}) - u_t(((l-1) \wedge m_k)_{k \in N}) \geq 0.$$

Thus, by Weak monotonicity, $f_{il}(m, u_t) \geq 0$ for each $(i, l) \in M^+$. It follows that

$$c^l = \frac{1 - \alpha^l}{|Q(l)|} \geq 0 \implies \alpha^l \leq 1.$$

Therefore, for each $(i, l) \in M^+$,

$$f_{il}(m, u_t) = \begin{cases} \dfrac{1 - \alpha^l}{|Q(l)|} & \text{if } j = l \text{ and } i \neq k, \\ \\ \dfrac{1 - \alpha^l}{|Q(l)|} + \alpha^l & \text{if } j = l \text{ and } i = k, \end{cases} \tag{4.61}$$

for some $0 \leq \alpha^l \leq 1$. Observe that, for each $(i, l) \in M^+$,

$$\xi_{il}(m, u_t) = \frac{1}{|Q(l)|}, \quad \varphi_{il}(m, u_t) = \begin{cases} 0 & \text{if } i \neq k, \\ 1 & \text{if } i = k. \end{cases}$$

Comparing $\xi_{il}(m, u_t)$ and $\varphi_{il}(m, u_t)$ with (4.61), one obtains

$$\forall (i,l) \in M^+, \quad f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l)\xi_{il}(m, u_t).$$

for some $0 \leq \alpha^l \leq 1$. This concludes the initialization step.

**Hypothesis**: Consider $r \in \mathbb{N}$ such that $1 \leq r < |Q(l)|$. Consider any $t$ such that $q^t(l) = r$. In this case, there are $r$ players that play $l$ in $t$. Assume that

$$\forall (i,l) \in M^+, \quad f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1 - \alpha^l)\xi_{il}(m, u_t).$$

**Induction**: Consider any $t$ such that $q^t(l) = r + 1$. Let $s = t - e_h$, for some $k \in N$ such that $t_k = x$. Obviously, it holds that $q^s(l) = W$. Recall that $(i,l) \notin T(t)$ if $t_i < l$. Observe that if $(i,l) \notin T(t)$ then $(i,l) \notin T(s)$. If $(i,l) \notin T(t)$ then $(i,l)$ is a null pair in $(m, u_t)$ and is also a null pair in $(m, u_s)$. Therefore, each $(i,l) \notin T(t)$ has the same marginal contributions in both games $(m, u_t)$ and $(m, u_s)$. Moreover, it holds that

$$\begin{aligned} &u_t((l \wedge m_h)_{h \in N}) - u_t(((l-1) \wedge m_h)_{h \in N}) \\ &= u_s((l \wedge m_h)_{h \in N}) - u_s(((l-1) \wedge m_h)_{h \in N}). \end{aligned}$$

Then by double application of Weak monotonicity, the induction hypothesis and by definition of $\varphi$ and $\xi$, for each $(i,l) \notin T(t)$,

$$\begin{aligned} f_{il}(m, u_t) &= f_{il}(m, u_s) \\ &\overset{Hyp}{=} \alpha^l \varphi_{il}(m, u_s) + (1 - \alpha^l)\xi_{il}(m, u_s) \\ &= \frac{(1 - \alpha^l)}{|Q(l)|}. \end{aligned} \tag{4.62}$$

By Multi-efficiency, (4.62) and the definition of a minimal effort game,

$$\begin{aligned} \sum_{(i,l) \in T(t)} f_{il}(m, u_t) =& u_t((l \wedge m_h)_{h \in N}) - u_t(((l-1) \wedge m_h)_{h \in N}) \\ &- \sum_{(i,l) \notin T(t)} f_{il}(m, u_t) \\ =& 1 - 0 - (|Q(l)| - |T(t)|)\frac{1 - \alpha^l}{|Q(l)|}. \end{aligned} \tag{4.63}$$

Additionally, any two distinct pairs $(i,l), (i',l) \in M^+$ such that $(i,l), (i',l) \in T(t)$, are equal in $(m, u_t)$. By Remark 6 and by Level desirability, for each $(i,l) \in T(t)$,

$$f_{il}(m, u_t) = c',$$

for some $c' \in \mathbb{R}$. It follows that

$$\sum_{(i,l)\in T(t)} f_{il}(m, u_t) = |T(t)|c'. \tag{4.64}$$

Therefore, combining (4.63) and (4.64), for each $(i, l) \in T(t)$, we obtain

$$c' = \frac{1 - (|Q(l)| - |T(t)|)\dfrac{1-\alpha^l}{|Q(l)|}}{|T(t)|}.$$

It follows that, for each $(i, l) \in T(t)$,

$$\begin{aligned} f_{il}(m, u_t) &= \frac{1 - (|Q(l)| - |T(t)|)\dfrac{1-\alpha^l}{|Q(l)|}}{|T(t)|} \\ &= \frac{\alpha^l}{|T(t)|} + \frac{1-\alpha^l}{|Q(l)|} \\ &= \alpha^l \varphi_{il}(m, u_t) + (1-\alpha^l)\xi_{il}(m, u_t). \end{aligned} \tag{4.65}$$

Combining (4.62) and (4.65), if $t^T = l$, then for each $(i, l) \in M^+$,

$$f_{il}(m, u_t) = \alpha^l \varphi_{il}(m, u_t) + (1-\alpha^l)\xi_{il}(m, u_t).$$

This concludes the induction step.

We have shown that there exists a parameter system $\alpha$ such that $f$ can be written, for each $(m, u_t)$ such that $t^T = l$, as

$$\forall (i, j) \in M^+, \quad f_{ij}(m, u_t) = \begin{cases} \alpha^l \varphi_{il}(m, u_t) + (1-\alpha^l)\xi_{il}(m, u_t) & \text{if } j = l, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of multi-choice Egalitarian Shapley values (see (4.9)), for such a parameter systems $\alpha$, there is a $\chi^\alpha$ such that, for each $(m, u_t)$,

$$f(m, u_t) = \chi^\alpha(m, u_t).$$

We conclude by Linearity that there exists a parameter system $\alpha$ such that, for each $(m, v) \in \mathcal{G}$,

$$f(m, v) = \chi^\alpha(m, v).$$

This concludes the proof of the theorem. ■

The five axioms of the statement of Theorem 24 are logically independent, as shown by the following alternative solutions.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by $f(m,v) = \vec{0}$ satisfies all the axioms except Efficiency.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by

$$\forall (i,j) \in M^+,$$
$$f_{ij}(m,v) = \frac{\max_{k\in N} m_k}{\sum_{k\in N} m_k}\varphi(m,v) + \Big(1 - \frac{\max_{k\in N} m_k}{\sum_{k\in N} m_k}\Big)\xi(m,v),$$

satisfies all the axioms except Independence of higher activity levels.

- The following value $f$ on $\mathcal{G}$ satisfies all the axioms except Linearity. Take any $(m,v) \in \mathcal{G}$. Denote the difference in worth between the grand coalition and the $(m_T-1)$-synchronized coalition by

$$V = v(m) - v(((m_T - 1) \wedge m_k)_{k\in N}.$$

The value $f$ is defined as follows:

· if $|Q(m^T)| \neq 2$, then $f(m,v) = \varphi(m,v)$;

· if $|Q(m^T)| = 2$, then there are two players $i$ and $i'$ in $Q(m^T)$. For each $(k,j) \in M^+$ such that $j \neq m_T$,

$$f_{kj}(m,v) = \varphi_{kj}(m,v),$$

and for the two remaining pairs $(i,m_T),(i',m_T) \in M^+$, it holds that

$$(f_{im_T}(m,v), f_{i'm_T}(m,v)) =$$
$$\begin{cases} (\varphi_{im_T}(m,v), \varphi_{i'm_T}(m,v)) & \text{if } \varphi_{im_T}(m,v) \geq 0 \text{ and } \varphi_{i'm_T}(m,v) \geq 0, \\ (0,V) & \text{if } \varphi_{im_T}(m,v) < 0, \varphi_{i'm_T}(m,v) > 0 \text{ and } V \geq 0, \\ (V,0) & \text{if } \varphi_{im_T}(m,v) < 0, \varphi_{i'm_T}(m,v) > 0 \text{ and } V < 0, \\ (\varphi_{im_T}(m,v), \varphi_{i'm_T}(m,v)) & \text{if } \varphi_{im_T}(m,v) \leq 0 \text{ and } \varphi_{i'm_T}(m,v) \leq 0, \\ (0,V) & \text{if } \varphi_{im_T}(m,v) > 0, \varphi_{i'm_T}(m,v) < 0 \text{ and } V \leq 0, \\ (V,0) & \text{if } \varphi_{im_T}(m,v) > 0, \varphi_{i'm_T}(m,v) < 0 \text{ and } V > 0. \end{cases}$$

Let us illustrate this value with a simple example. Consider the two games $(m,v),(m,w) \in \mathcal{G}$, where $Q(m^T) = \{1,2\}$, and such that

$$V = v(m) - v(((m_T - 1) \wedge m_k)_{k\in N} > 0$$
$$W = w(m) - w(((m_T - 1) \wedge m_k)_{k\in N} \leq 0$$
$$V + W > 0$$
$$\varphi_{1m_T}(m,v) > 0 \quad \text{and} \quad \varphi_{1m_T}(m,w) > 0$$
$$\varphi_{2m_T}(m,v) < 0 \quad \text{and} \quad \varphi_{2m_T}(m,w) < 0.$$

According to the value $f$, it holds that

$$(f_{1m_T}(m,v), f_{2m_T}(m,v)) = (V,0)$$
$$(f_{1m_T}(m,w), f_{2m_T}(m,w)) = (0,W)$$
$$(f_{1m_T}(m,v+w), f_{2m_T}(m,v+w)) = (V+W,0).$$

Obviously, $(V+W,0) \neq (V,0)+(0,W)$. This shows that $f$ does not satisfy Linearity.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by

$$\begin{aligned}\forall (i,j) \in M^+,&\\ f_{ij}(m,v) =& v(((j-1)\wedge m_k)_{k\in N}) + e_i)\\ &+ \frac{(v((j\wedge m_k)_{k\in N}) - v(((j-1)\wedge m_k)_{k\in N}))}{|Q(j)|}\\ &- \frac{\sum_{k\in Q(j)} v(((j-1)\wedge m_k)_{k\in N}) + e_k)}{|Q(j)|},\end{aligned}$$

satisfies all the axioms except Weak monotonicity. Observe that this value extends the Equal surplus division from TU-games to multi-choice games.

- Take any $(m,v) \in \mathcal{G}$ and fix any arbitrary integer $\beta^{ij} \in \{1,2\}$ for each $(i,j) \in M^+$. The value $f^\beta$ given by

$$\forall (i,j) \in M^+,$$
$$f^\beta_{ij}(m,v) = \frac{\beta^{ij}}{\sum_{k\in Q(j)} \beta^{kj}} \Big[ v((j\wedge m_k)_{k\in N}) - v(((j-1)\wedge m_k)_{k\in N}) \Big],$$

satisfies all the axioms except Level desirability.

## 4.4 Additional remarks

In this last section, we formulate two remarks regarding the solutions introduced in this chapter. First, we discuss the relationship between the multi-choice Shapley value and the discrete serial cost sharing method introduced by Moulin & Shenker (1992) for discrete cost sharing problems (see Definition 22). Second, we discuss a potential application of the multi-choice Egalitarian Shapley values.

Fix $N = \{1,\dots,n\}$ a set of $n$ different goods produced in indivisible units. Recall that a discrete cost sharing problem can be expressed by a couple $(q,C)$, where $q = (q_1,\dots,q_n)$. Each $q_i \in \mathbb{N}$ represents the demand in good $i$, and $C$ is a non decreasing real-valued function on $\prod_{i\in N}\{0,1,\dots,q_i\}$ such that $C(\overrightarrow{0}) = 0$. The total cost to be shared is given by $C(q)$. As mentioned in Section 2.4.4, the class of discrete

cost sharing problems is the class of monotonic multi-choice games $\mathcal{G}_m$. Recall that a method on $\mathcal{G}_m$ is a map $g$ that associates to each problem $(q, C) \in \mathcal{G}_m$ a vector $g(q, C) \in \mathbb{R}^n$ satisfying the budget balanced condition, i.e., $\sum_{i \in N} g_i(q, C) = C(q)$. A popular cost sharing method for cost sharing problems is the **discrete serial cost sharing method** (see Definition 22) introduced by Moulin & Shenker (1992).

We provide an alternative expression of the discrete serial cost sharing method proposed by Albizuri *et al.* (2003). To that end, we define a specific TU-game. Consider $(q, C) \in \mathrm{C}$ and $j \leq m^T$. Define the TU-game $(Q(j), w_j^{(q,C)})$ as

$$\begin{array}{l} \forall E \subseteq Q(j), \\ w_j^{(q,C)}(E) = C\Big(((j-1) \wedge q_k)_{k \in N} + e_E\Big) - C\Big(((j-1) \wedge q_k)_{k \in N}\Big). \end{array}$$

The worth $w_j^{(q,C)}(E)$ can be interpreted as the additional costs generated when each player in $E$ increases its activity level (demand) from $j-1$ to $j$ while all the other players play either the activity level $j-1$ or their maximal feasible activity level if they are unable to do so. Albizuri *et al.* (2003) show that the discrete serial cost sharing admits the following expression

$$\forall i \in N, \quad SCS_i(q, C) = \sum_{j=1}^{q_i} Sh_i\Big(Q(j), w_j^{(q,C)}\Big). \tag{4.66}$$

**Proposition 24.** *The multi-choice Shapley value is consistent with the discrete serial cost sharing method proposed by Moulin & Shenker (1992).*

*Proof.* Observe that, for each $i \in N$ and each $j < q_i$,

$$\forall E \subseteq Q(j), \quad w_j^{(q,C)}(E) = w_j^{(q-e_i,C)}(E).$$

Therefore,

$$\forall i \in N, j < q_i, \quad Sh_i\Big(N, w_j^{(q,C)}\Big) = Sh_i\Big(N, w_j^{(q-e_i,C)}\Big). \tag{4.67}$$

Additionally, recall that, for each $j \leq m^T$, the set of orders $\Theta_j$ over $M^{+,j}$ can be interpreted as the set of orders over the set of players in $Q(j)$. An order over $Q(j)$ is a map $\theta_j^N : Q(j) \to \{1, \ldots, |Q(j)|\}$. We denote by $\overline{Q}(j)$ the set of of orders over $Q(j)$. Consider an order $\theta_j^N \in \overline{Q}_j$ and $h \in \{1, \ldots, |Q(j)|\}$. Recall that, for each $B \in 2^N$, the vector $e_B \in \mathbb{R}^{|A|}$ is defined as $(e_B)_i = 1$ if $i \in B$ and $(e_B)_i = 0$ otherwise. We denote by

$$((j-1) \wedge q_k)_{k \in N} + e_{E^{\theta_j^N, h}}$$

the coalition in which each player in $Q(j)$ ordered prior to step $h$ with respect to $\theta_j^N$, participates at its activity level $j$, whereas each player in $Q(j)$ ordered after

step $h$ with respect to $\theta_j^N$, participates at its activity level $j-1$. Each player not in $Q(j)$ participates at its maximal activity level. Obviously, this coalition coincides with $s^{\theta_j,h}$, where $\theta_j$ is the counterpart of $\theta_j^N$ among the orders in $\Theta_j$. We use the convention

$$((j-1)\wedge q_k)_{k\in N}+e_{E^{\theta_j^N,0}}=((j-1)\wedge q_k)_{k\in N}.$$

Consider an order $\theta_j^N\in\overline{Q}_j$. For each $i\in Q(j)$, we denote by

$$\begin{aligned}\mu_i^{\theta_j^N}(q,C)=&C\big(((j-1)\wedge q_k)_{k\in N}+e_{E^{\theta_j^N,\theta_j^N(i)}}\big)\\&-C\big(((j-1)\wedge q_k)_{k\in N}+e_{E^{\theta_j^N,\theta_j^N(i)-1}}\big),\end{aligned}\tag{4.68}$$

the marginal contribution of player $i$ for its activity level $j$ with respect to the order $\theta_j^N$. By (4.6), (4.7) and (4.68), for each $(q,C)\in\mathrm{C}$, the multi-choice Shapley value can be re-written as

$$\forall(i,j)\in M^{+,j},\quad \varphi_{ij}(q,C)=\frac{1}{|Q(j)|!}\sum_{\theta_j^N\in\overline{Q}_j}\mu_i^{\theta_j^N}(q,C).$$

By definition of the Shapley value for TU-games (see Shapley (1953)), for each $j\leq m^T$,

$$\forall i\in Q(j),\quad Sh_i\Big(N,w_j^{(q,C)}\Big)=\frac{1}{|Q(j)|!}\sum_{\theta_j^N\in\overline{Q}_j}\mu_i^{\theta_j^N}(q,C)=\varphi_{ij}(q,C).\tag{4.69}$$

It follows that the multi-choice Shapley value is consistent with the discrete serial cost sharing method since, for each $i\in N$,

$$\begin{aligned}SCS_i(q,C)-SCS_i(q-e_i,C)=&\sum_{j=1}^{q_i}Sh_i\Big(N,w_j^{(q,C)}\Big)\\&-\sum_{j=1}^{q_i-1}Sh_i\Big(N,w_j^{(q-e_i,C)}\Big)\\=&Sh_i\Big(N,w_{q_i}^{(q,C)}\Big)\\=&\varphi_{iq_i}(q,C),\end{aligned}$$

where the second equality follows from (4.66) and (4.67), and the third equality follows from (4.69). ■

To conclude this section, we discuss a potential application of the multi-choice Egalitarian Shapley values. Consider a wage assignment problem in a firm as discussed in Abe & Nakada (2019). In a firm, each worker may receive a base salary in

addition to a reward for its contribution to the firm. This wage assignment may be more secure than an assignment without a base salary given the possibility that employees cannot contribute due to raising children, for instance. Such assignment can obviously be viewed as a compromise between marginalism and egalitarianism. Abe & Nakada (2019) point out that the wage may be affected by exogenous variables independent of one's contributions, such as seniority or educational background. A way to address this problem is to model it with a multi-choice game and endow each employee with a maximal activity level representing its seniority or education background. In this case, if we assume that the assignment of an employee is equal to the total payoff she receives by a multi-choice Egalitarian Shapley value, then the base salary a worker receive corresponds to the egalitarian part of the value. Observe that, the base salary of an employee will increase with respect to her seniority or education. In addition, this increase depends on the parameter system used for the computation of the value. For instance, one could select a parameter system that operates a progressive compromise between marginalism and egalitarianism.

## 4.5 Conclusion

This chapter focuses on the principles pursued by GRDF and the doctrines that are driving the solution concepts for cooperative games. Consequently, it disregards the information provided by the distribution network.

We introduced: the multi-choice Shapley value and the multi-choice Equal division value. The two values satisfy Independence of higher activity levels, which is a translation of the Independence of higher demands principle to the framework of multi-choice games. We show that the combination of Independence of higher activity levels and Efficiency implies Multi-efficiency, which is Core necessary condition. Additionally, the multi-choice Equal division value is a solution in line with the Uniformity principle.

Finally, we introduce the multi-choice Egalitarian Shapley values for multi-choice games. These values are computed as the convex combination of the multi-choice Shapley value and the multi-choice Equal division value.

Two axiomatic characterizations of the multi-choice Shapley value are provided. Following the first characterization, we show that the multi-choice Shapley value admits an expression in terms of Harsanyi dividends. Finally, one axiomatic characterization of the Equal division value and one axiomatic characterization of the Egalitarian Shapley values are provided.

# Chapter 5: Values for Multi-Choice Games with a Priority Structure

## 5.1 Introduction

The two previous chapters deal with the principles pursued by GRDF and the information provided by the distribution network respectively. This chapter aims to address both issues at the same time. Contrary to Chapter 3, a permission structure is not extracted from the information provided by the distribution network. Instead, we extract a priority structure, which naturally arises in the context of gas distribution. For instance, it seems reasonable to prioritize the supply of a hospital rather than the supply of an amusement park. Additionally, contrary to Chapter 4, which considers both the Uniformity principle and the Independence of higher demands principle, this chapter only focuses on the latter.

In the framework of multi-choice games, the use of a priority structure on the player set brings additional possibilities that would not have been possible in the framework of TU-games. Similarly to Chapter 2, the set of activity levels is linearly ordered. Consequently, it provides an intra-player information, whereas the priority structure provides an inter-player information. Depending on the cooperative situation, these two pieces of information can be useful to define a relevant allocation process. In this chapter, a lexicographic order defined over the set of player-activity level pairs is defined. Consider any two player-activity level pairs $(i, j)$ and $(i', j')$. The two players $i$ and $i'$ are part of a priority structure. The activity levels of the players are assumed to be the most significant criterion. If one of the two pairs has a higher position than the second in the lexicographic order, then it features a higher activity level than the second pair. If the two activity levels are equal, then the pair featuring the player that has priority over the second player has a higher position than the other pair in the lexicographic order.

In line with this lexicographic order, a generalization of the Priority value from TU-games to multi-choice games is introduced: the **multi-choice Priority value**. This value equally divides the net surplus of each coalition among the player(s) in the coalition with the highest position in the lexicographic order. Such allocation process especially makes sense when resources are scarce, and trade-offs have to be

made among different criteria.

We provide two axiomatic characterizations of the multi-choice Priority value. Both characterizations rely on known axioms for multi-choice games: Efficiency, Additivity and Independence of higher activity levels. In addition, we use new axioms that take into account the priority structure. The two characterizations differ in these new axioms.

As our first main result, we show that the multi-choice Priority value is characterized by the combination of Efficiency, Additivity, Independence of higher activity levels and the next two axioms (see Theorem 25). The first axiom is called: Priority relation for the same maximal activity level. This axiom considers two players with the same maximal activity level and comparable in terms of priority. This axiom expresses the idea that the payoff possibilities of a player are blocked by any other player with higher priority, even if they both have the same maximal activity level. In terms of gas distribution, this axioms holds a nice interpretation: the supply possibilities in gas of a consumer are blocked if there is another consumer with the same demand in gas and who has priority over it. For instance, the supply of a hospital should be prioritized over the supply of an amusement park, even if both of them have the same demand in gas. The second axiom is called: Balanced contributions for the same prevailing group. This axiom relies on a fairness requirement for indistinguishable players regarding both the priority structure and their maximal activity level. It indicates that such players affect each other payoff in the same manner.

The second characterization of the multi-choice Priority value consists of a set of axioms with similar interpretations than the axioms of the statement of Theorem 21. It invokes Independence of null pairs: removing a player's maximal activity level does not alter the payoffs of the remaining player's activity level if this player is unproductive in each coalition in which it plays its maximal activity level. Yet, the multi-choice Priority value departs from the multi-choice Shapley value in one important aspect. Contrary to the latter, it does not satisfy Equal treatment for equal pairs: two players that contribute the same amount to each coalition in which they play the same activity level should receive the same payoff for this activity level. In the presence of priority relations, this axiom becomes very strong. Therefore, it is replaced by two new axioms that take into account the priority structure. Both axioms deal with decisive players. A player is decisive if each coalition in which it does not play its maximal activity level generates a null worth. This notion generalizes the notion of a veto player from TU-games to multi-choice games. In terms of gas distribution, a decisive player can be viewed as a consumer without whom the distribution network would never have existed. For instance, some consumers with very large demand in gas can be held responsible for the creation of the distribution network. Indeed, in some cases, a distribution network is created for the sole purpose to supply a single consumer with an important demand in gas. Subsequently, this allowed other (smaller) consumers to be connected to this distribution network. All in all, the decisive gas consumers can be viewed as the necessary condition for a gas distribution network to exist. Consider two decisive players with the same maximal

activity level. The axiom of Priority relation for decisive players requires that the payoff of the first decisive player is zero if the second decisive player has priority over it. This axiom has a strong meaning in terms of gas distribution since it ensures a null payoff to any decisive gas consumer as long as another consumer with the same demand has priority over it. Second, the axiom of Decisive players with equal prevailing group property deals with decisive players that are incomparable in terms of both maximal activity level and priority. It requires that such players should be treated equally for their maximal activity level. Therefore, this axiom is a relaxation of Equal treatment for equal pairs. As our second main result, we show that the multi-choice Priority value is the unique value that satisfies Efficiency, Additivity, Independence of higher activity levels, Independence of null pairs, Priority relation for decisive players and the Decisive players with equal prevailing group property (see Theorem 26).

Finally, we consider priority structures in which the set of players can be partitioned into several priority classes. Precisely, each class contains incomparable players that have priority over each player in the next class. In such priority relations structured by classes, we show that the multi-choice Priority value can be interpreted as a sequential procedure involving specific TU-games. Consider a given activity level that is played by at least one player and a priority class $p$. Consider all players in this priority class which are able to play the required activity level. The payoff obtained by these players correspond to their marginal contribution to the coalition formed by all players over which this class has priority (and that are able to play the required activity level). The final payoff of each player in this priority class for the required activity level is the Shapley value applied to a TU-game on the subset of players in the priority class $p$, which are able to play the required activity level (Theorem 27).

The rest of the chapter is organized as follows. Section 5.2 introduces the axioms and proceeds to the axiomatic study. Section 5.3 discusses multi-choice games with a priority relation structured by classes. Finally, Section 5.4 concludes the chapter.

## 5.2 Axiomatic study

The class of multi-choice games with a priority structure is denoted by $\mathcal{GS}$. There are several ways to consider the influence of a priority structure on multi-choice games. Here, this issue is approached through an axiomatic study. Denote by $f$ a value on $\mathcal{GS}$.

**Remark 7.** *If $f$ satisfies Independence of higher activity levels, then for each $(m, v, \succeq) \in \mathcal{GS}$ and each $i, i' \in N$ such that $m_i = m_{i'}$,*

$$\forall j < m_i, \quad f_{ij}(m, v, \succeq) = f_{ij}(m - e_{i'}, v, \succeq).$$

### 5.2.1 Balanced contributions

This section introduces axioms that compare players with the same maximal activity level. Consider two players $i$ and $i'$ such that $m_i = m_{i'}$ and $i \succ i'$. This axiom states that the payoff of $i'$'s maximal activity level remains unchanged if player $i$'s maximal activity level decreases by one unit.

**Axiom 51 (Priority relation for the same maximal activity level).** *For each $(m, v, \succeq) \in \mathcal{GS}$, and each $i, i' \in N$ such that $m_i = m_{i'}$ and $i' \succ i$,*

$$f_{im_i}(m, v, \succeq) = f_{im_i}(m - e_{i'}, v, \succeq).$$

To introduce the next axiom, a definition is needed. A player $i$ **prevails** on another player $i'$ if it has priority over $i'$ and its maximal activity level $m_i$ is not lower than $m_{i'}$. Take any $(m, v, \succeq) \in \mathcal{GS}$. For each $i \in N$, the set of players that prevail on $i$ is defined as

$$L_i(m, v, \succeq) = \uparrow_{\succ} i \cap \big\{k \in N \setminus \{i\} : m_k \geq m_i\big\}.$$

The next axiom is inspired by the well known Balanced contributions axiom introduced by Myerson (1980) for TU-games (see Axiom 12). Originally, Balanced contributions asserts that for any two players, the amount that each player gains or loses by the other player's withdrawal from the game should be equal. It expresses a fairness requirement according to which two players affect each other payoff in the same way. There are several generalizations of this axiom from TU-games to multi-choice games that translate this fairness condition in case a player gets access to (or looses) an additional activity level (see Theorem 15). However, because of the exogenous asymmetries between players, it seems more reasonable that such condition applies with respect to the priority structure. In particular, the next axiom relies on the idea that two players affect each other payoff in the same manner if they have the same maximal activity level and the same prevailing group.

**Axiom 52 (Balanced contributions for the same prevailing group).** *For each $(m, v, \succeq) \in \mathcal{GS}$, and each $i, i' \in N$ such that $i \neq i'$, $m_i = m_{i'}$ and $L_i(m, v, \succeq) = L_{i'}(m, v, \succeq)$,*

$$f_{im_i}(m, v, \succeq) - f_{im_i}(m - e_{i'}, v, \succeq) = f_{i'm_{i'}}(m, v, \succeq) - f_{i'm_{i'}}(m - e_i, v, \succeq).$$

**Proposition 25.** *There is at most one value on $\mathcal{GS}$ that satisfies Efficiency, Independence of higher activity levels, Priority relation for the same maximal activity level and Balanced contributions for the same prevailing group.*

*Proof.* Let $f$ be a value on $\mathcal{GS}$ that satisfies all the axioms from the statement of Proposition 25. Let us show that, for each $(m, v, \succeq) \in \mathcal{GS}$, $f(m, v, \succeq)$ is uniquely determined. We proceed by induction on the number of activity levels in a game

given by $\sum_{i\in N} m_i$ for each $(m,v)\in\mathcal{GS}$.

**Initialization**: Take any $(m,v,\succeq)\in\mathcal{GS}$ such that $\sum_{i\in N} m_i = 1$. There exists a unique $i\in N$ such that $m_i = 1$ and $m_k = 0$ for each $k\neq i$. By Efficiency and by definition of a payoff vector, each payoff vector verifies $f_{i1}(m,v,\succeq) = v(m)$. This shows that $f(m,v,\succeq)$ is uniquely determined.

**Induction hypothesis**: Assume that $f(m,v,\succeq)$ is uniquely determined for each $(m,v,\succeq)\in\mathcal{GS}$ such that

$$\sum_{i\in N} m_i = W \geq 1.$$

**Induction step**: Take any $(m,v,\succeq)\in\mathcal{GS}$ such that

$$\sum_{i\in N} m_i = W + 1.$$

Let us show that $f(m,v,\succeq)$ is uniquely determined. Recall that the highest maximal activity level in $(m,v,\succeq)$ is denoted by

$$m^T = \max_{k\in N} m_k,$$

and denote the set of players having access to $m^T$ by

$$Q(m^T) = \{i\in N \mid m_i = m^T\}.$$

By Independence of higher activity levels, for each player $i\in N\setminus Q(m^T)$ and for each $j\leq m_i$,

$$f_{ij}(m,v,\succeq) = f_{ij}((m_k\wedge j)_{k\in N},v,\succeq). \tag{5.1}$$

Observe that there is less than $W$ activity levels in $(m-e_i,v,\succeq)$. Therefore, by the induction hypothesis, $f_{ij}((m_k\wedge j)_{k\in N},v,\succeq)$ is uniquely determined. Thus, by (5.1), $f_{ij}(m,v,\succeq)$ is uniquely determined, for each $i\in N\setminus Q(m^T)$ and for each $j\leq m_i$.

Then, take any $i\in Q(m^T)$. Assume that $|Q(m^T)| = 1$. It follows that $m_k < m_i$ for each $k\in N$. Denote by $M^+_{-e_i} = M^+\setminus\{(i,m_i)\}$ the set of pairs associated with the game $((m_k\wedge(m_i-1))_{k\in N},v,\succeq) = (m-e_i,v,\succeq)$. By Independence of higher activity levels,

$$\begin{aligned} &\forall k\in N, j<m_i \quad f_{kj}(m,v,\succeq) = f_{kj}(m-e_i,v,\succeq)\\ \Longrightarrow &\sum_{(k,j)\in M^+_{-e_i}} f_{kj}(m-e_i,v,\succeq) = \sum_{(k,j)\in M^+_{-e_i}} f_{kj}(m,v,\succeq). \end{aligned} \tag{5.2}$$

By Efficiency and (5.2),

$$\begin{aligned}
&\sum_{(k,j)\in M^+_{-e_i}} f_{kj}(m-e_i,v,\succeq) = v(m-e_i)\\
\Longrightarrow\ &\sum_{(k,j)\in M^+_{-e_i}} f_{kj}(m,v,\succeq) = v(m-e_i)\\
\Longrightarrow\ &f_{im_i}(m,v,\succeq) = \sum_{(k,j)\in M^+} f_{kj}(m,v,\succeq)\\
&\qquad\qquad - \sum_{(k,j)\in M^+_{-e_i}} f_{kj}(m,v,\succeq)\\
&\qquad\qquad = v(m)-v(m-e_i).
\end{aligned}$$

This shows that $f_{im_i}(m,v,\succeq)$ is uniquely determined.

Now, assume that $|Q(m^T)|>1$. Consider the subposet $(Q(m^T),\succeq^{Q(m^T)})$ of $(N,\succeq)$ and the priority group $\uparrow_{\succ^{Q(m^T)}} k$ of a player $k\in Q(m^T)$. Two separate cases are considered.

**Case 1.** Take any $k\in Q(m^T)$ such that $\uparrow_{\succ^{Q(m^T)}} k \neq \emptyset$. Then, there is at least one $i\in Q(m^T)$ such that $i\succ k$. By Priority relation for the same maximal activity level,

$$f_{k,m_k}(m,v,\succeq) = f_{k,m_k}(m-e_i,v,\succeq). \tag{5.3}$$

Observe that there is a total of $W$ activity levels in $(m-e_i,v,\succeq)$. By the induction hypothesis, $f_{k,m_k}(m-e_i,v,\succeq)$ is uniquely determined. Therefore, by (5.3), $f_{k,m_k}(m,v,\succeq)$ is uniquely determined.

**Case 2.** Take any $i\in Q(m^T)$ such that $\uparrow_{\succ^{Q(m^T)}} i = \emptyset$. Define

$$\overline{Q}(m^T) = \{i\in Q(m^T) \mid \uparrow_{\succ^{Q(m^T)}} i = \emptyset\}$$

the set of players with the highest maximal activity level and with an empty priority group in $(Q(m^T),\succeq)$. Denote by $|\overline{Q}(m^T)|$ the number of players in $\overline{Q}(m^T)$. By definition, for each $i\in\overline{Q}(m^T)$, $L_i(m,v,\succeq)=\emptyset$. If $|\overline{Q}(m^T)|>1$, then by Balanced contributions for the same prevailing group,

$$\begin{aligned}
&\forall i,i'\in\overline{Q}(m^T),\\
&f_{im_i}(m,v,\succeq) - f_{i'm_{i'}}(m,v,\succeq) = f_{im_i}(m-e_{i'},v,\succeq) - f_{i'm_{i'}}(m-e_i,v,\succeq).
\end{aligned} \tag{5.4}$$

Observe that there is a total of $W$ activity levels in $(m-e_i,v,\succeq)$ and $(m-e_{i'},v,\succeq)$. The right hand-side of (5.4) is uniquely determined by the induction hypothesis. Applying the same argument for each player in $\overline{Q}(m^T)$ one generates $\binom{|\overline{Q}(m^T)|}{2}$

equations of type (5.4). Moreover, by Efficiency

$$\begin{aligned}\sum_{(i,j)\in M^+} f_{ij}(m,v,\succeq) =& \sum_{i\in N}\sum_{j=1}^{m_i} f_{ij}(m,v,\succeq)\\ =& v(m)\\ =& \sum_{i\in N}\sum_{j=1}^{m_i-1} f_{ij}(m,v,\succeq)\\ &+ \sum_{k\notin Q(m^T)} f_{km_k}(m,v,\succeq) + \sum_{i\in Q(m^T)} f_{im_i}(m,v,\succeq).\end{aligned}$$

This can also be written as

$$\begin{aligned}\sum_{i\in \overline{Q}(m^T)} f_{im_i}(m,v,\succeq) =& v(m) - \sum_{i\in N}\sum_{j=1}^{m_i-1} f_{ij}(m,v,\succeq)\\ &- \sum_{k\notin Q(m^T)} f_{km_k}(m,v,\succeq) - \sum_{\substack{k\in Q(m^T)\\ k\notin \overline{Q}(m^T)}} f_{km_k}(m,v,\succeq). \quad (5.5)\end{aligned}$$

Let $b$ denote the right-hand side of (5.5). At this step, it has been shown that:

- $f_{ij}(m,v,\succeq)$ is uniquely determined for each $j < m_i$;
- $f_{km_k}(m,v,\succeq)$ is uniquely determined for each $k \notin Q(m^T)$;
- $f_{km_k}(m,v,\succeq)$ is uniquely determined for each $k \in Q(m^T)$ and $k \notin \overline{Q}(m^T)$.

Therefore, $b$ is uniquely determined. From (5.5) and (5.4), one generates the system of $\binom{|\overline{Q}(m^T)|}{2} + 1$ equations with $Q(m^T)$ unknowns

$$(A) = \begin{cases} \displaystyle\sum_{i\in\overline{Q}(m^T)} f_{im_i}(m,v,\succeq) = b,\\ \forall i,i' \in N_m, \quad f_{im_i}(m,v,\succeq) - f_{i'm_{i'}}(m,v,\succeq)\\ \qquad\qquad = f_{im_i}(m-e_{i'},v,\succeq) - f_{i'm_{i'}}(m-e_i,v,\succeq).\end{cases}$$

Without loss of generality, relabel each player in $\overline{Q}(m^T)$ by $i_1, i_2, \ldots, i_{|\overline{Q}(m^T)|}$. By considering the sequence of players $(i_1, i_2, \ldots, i_{|\overline{Q}(m^T)|})$, one obtains the subsystem of $|\overline{Q}(m^T)|$ equations with $|\overline{Q}(m^T)|$ unknowns

$$\begin{cases} \displaystyle\sum_{i\in\overline{Q}(m^T)} f_{im_i}(m,v,\succeq) = b,\\ \forall k \in \{1,\ldots|\overline{Q}(m^T)|-1\},\\ f_{i_k m_{i_k}}(m,v,\succeq) - f_{i_{k+1}m_{i_{k+1}}}(m,v,\succeq)\\ = f_{i_k m_{i_k}}(m-e_{i_{k+1}},v,\succeq) - f_{i_{k+1}m_{i_{k+1}}}(m-e_{i_k},v,\succeq).\end{cases}$$

It is straightforward to verify that these $|\overline{Q}(m^T)|$ equations are linearly independent. Therefore, the subsystem of $(A)$ admits a unique solution. We conclude that the original system $(A)$ admits at most one solution. This allows to conclude the induction step. The proof of the theorem is completed.

■

In many economic decision problems, a decision maker has to choose an option within a finite set of possible alternative. Often, these alternative are defined according to a finite number of criteria (see Svenson (1979)). A popular decision making process is to select the alternative according to a lexicographic order based on these criteria (see Fishburn (1974)). Put simply, one classifies the criteria according to their relative significance. Then, the alternative that performs the best on the first criterion is selected. If more than one alternative is selected, then the second criterion is used to tiebreak the alternative. This process continues until there is only one alternative left, or no criterion left.

We introduce the **multi-choice Priority value** for multi-choice games with a priority structure that follows a lexicographic allocation process. The activity levels of the players are assumed to be the most significant criterion. The multi-choice Priority value allocates the surplus of each coalition among a subset of players within this coalition. These players are chosen according to a lexicographic decision process that selects the players with the highest activity level in the coalition. In case more than one player is selected, the process sorts the players according to their position in the priority structure.

Precisely, we derive a lexicographic partial order $(M^+, \succeq^*)$ from the total order on the set of activity levels $(M_i^+, \geq)$ of the players $i \in N$ and the priority structure $(N, \succeq)$.

**Definition 31.** *Given a multi-choice game with a priority structure $(m, v, \succeq) \in \mathcal{GS}$, define the lexicographic partial order $\succeq^*$ on the set of pairs $M^+$ as*

$$\begin{aligned} &\forall (i,j), (i'j') \in M^+, \\ &(i,j) \succeq^* (i',j') \iff \Big[ [j > j'] \text{ or } [j = j' \text{ and } i \succeq i'] \Big]. \end{aligned} \tag{5.6}$$

We now have the material to define the multi-choice Priority value, $\Gamma$. For each $s \in \mathcal{M}$, the value assigns to a pair $(i, s_i) \in C(s)$ a share of the Harsanyi dividend $\Delta_v(s)$ if no other pair in $C(s)$ has a higher position in the lexicographic order induced by the coalition.

**Definition 32 (Multi-choice Priority value).** *For each $(m, v, \succeq) \in \mathcal{GS}$, the multi-choice Priority value $\Gamma$ is defined as*

$$\forall (i,j) \in M^+, \quad \Gamma_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ j = s_i \\ (i,j) \in Z(C(s), \succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s), \succeq^*)|}, \tag{5.7}$$

*where for each $s \in \mathcal{M}$, $Z(C(s), \succeq^*)$ is the set of maximal elements of the sub-poset $(C(s), \succeq^*)$, and $C(s)$ is the set of pairs featuring players and their non-null activity level in $s$ (see (2.39)).*

Whenever $m = (1, \ldots, 1)$, this value coincides with the Priority value for TU-games with a priority structure (2.24). Whenever the priority structure is a trivial poset $(N, \succeq^0)$, this value coincides with the multi-choice Shapley value (4.5).

The next theorem provides an axiomatic characterization of the value $\Gamma$ on the class of multi-choice games with a priority structure.

**Theorem 25.** *A value $f$ on $\mathcal{GS}$ satisfies Efficiency, Independence of higher activity levels, Priority relation for the same maximal activity level and Balanced contributions for the same prevailing group if and only if $f = \Gamma$.*

*Proof.* By Proposition 25, we know that there is at most one value that satisfies all the axioms from the statement of Theorem 25. It remains to show that the value $\Gamma$ satisfies these axioms.

**Efficiency:** Take any $(m, v, \succeq) \in \mathcal{GS}$. By definition of $\Gamma$, the Harsanyi dividend of a coalition $s \in \mathcal{M}$ is equally allocated among the pairs in $Z(C(s), \succeq^*)$. Therefore, it holds that

$$\begin{aligned}\sum_{(i,j)\in M^+} \Gamma_{ij}(m, v, \succeq) &= \sum_{s\leq m} \sum_{(i,j)\in Z(C(s),\succeq^*)} \frac{\Delta_v(s)}{|Z(C(s), \succeq^*)|} \\ &= \sum_{s\leq m} \Delta_v(s) \\ &= v(m).\end{aligned}$$

This shows that the value $\Gamma$ satisfies Efficiency.

**Independence of higher activity levels:** Take any $(m, v, \succeq) \in \mathcal{GS}$ and any pair $(i, j) \in M^+$ such that $j < m_i$. By definition of $\Gamma$, the payoff $\Gamma_{ij}(m, v, \succeq)$ only depends on the coalitions $s \in \mathcal{M}$ verifying $s_k \leq j$ for each $k \in N$. Thus, it is straightforward to see that the value $\Gamma$ satisfies Independence of higher activity levels.

**Priority relation for the same maximal activity level:** Take any $(m, v, \succeq) \in \mathcal{GS}$, any $s \in \mathcal{M}$, and any two players $i, i' \in N$ such that $m_i = m_{i'}$ and $i' \succ i$. If $(i, m_i) \in Z(C(s), \succeq^*)$, then we necessarily have $s_{i'} < m_{i'}$, i.e., $s \leq m - e_{i'}$. Therefore,

it holds that

$$\begin{aligned}\Gamma_{im_i}(m,v,\succeq) &= \sum_{\substack{s\leq m\\ m_i=s_i\\ (i,m_i)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}\\ &= \sum_{\substack{s\leq m-e_{i'}\\ m_i=s_i\\ (i,m_i)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}\\ &= \Gamma_{im_i}(m-e_{i'},v,\succeq).\end{aligned}$$

This shows that the value $\Gamma$ satisfies Priority relation for the same maximal activity level.

**Balanced contributions for the same prevailing group:** Take any $(m,v,\succeq)\in \mathcal{GS}$ and any two players $i,i'\in N$ such that $m_i=m_{i'}$ and $L_i(m,v,\succeq)=L_{i'}(m,v,\succeq)$. Observe that, for each $s\in\mathcal{M}$, if $(i,m_i),(i,m_{i'})\in C(s)$, then

$$(i,m_i)\in Z(C(s),\succeq^*) \iff (i',m_{i'})\in Z(C(s),\succeq^*). \tag{5.8}$$

Let us compute the payoff of the pair $(i,m_i)$. By definition of $\Gamma$,

$$\begin{aligned}\Gamma_{im_i}(m,v,\succeq) &= \sum_{\substack{s\leq m\\ (i,m_i)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}\\ &= \sum_{\substack{s\leq m\\ (i,m_i)\in Z(C(s),\succeq^*)\\ s_{i'}<m_{i'}}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|} + \sum_{\substack{s\leq m\\ (i,m_i)\in Z(C(s),\succeq^*)\\ s_{i'}=m_{i'}}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}.\end{aligned}$$

By (5.8),

$$\begin{aligned}\Gamma_{im_i}(m,v,\succeq) &= \sum_{\substack{s\leq m\\ (i,m_i)\in Z(C(s),\succeq^*)\\ s_{i'}<m_{i'}}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}\\ &\quad + \sum_{\substack{s\leq m\\ (i,m_i),(i',m_{i'})\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}.\end{aligned}$$

Moreover,

$$\begin{aligned}\Gamma_{im_i}((m-e_{i'}),v,\succeq) &= \sum_{\substack{s\leq (m-e_{i'})\\ (i,m_i)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}\\ &= \sum_{\substack{s\leq m\\ (i,m_i)\in Z(C(s),\succeq^*)\\ s_{i'}<m_{i'}}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}.\end{aligned}$$

The same reasoning holds for $(i', m_{i'})$, from which one concludes that

$$\Gamma_{im_i}(m, v, \succeq) - \Gamma_{im_i}((m - e_{i'}), v, \succeq) = \sum_{\substack{s \leq m \\ (i,m_i),(i',m_{i'}) \in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s), \succeq^*)|}$$
$$= \Gamma_{i'm_{i'}}(m, v, \succeq) - \Gamma_{i'm_{i'}}((m - e_i), v, \succeq),$$

as desired. This concludes the proof of the theorem. ■

The four axioms of the statement of Theorem 25 are logically independent, as shown by the following alternative solutions.

- The value $f$ given, for each $(m, v) \in \mathcal{G}$, by $f(m, v) = \vec{0}$ satisfies all the axioms except Efficiency.
- The value $f$ given, for each $(m, v) \in \mathcal{G}$, by

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i \geq j > 0 \\ (i,s_i) \in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s), \succeq^*)| \sum_{(k,s_k) \in Z(C(s),\succeq^*)} s_k},$$

  satisfies all the axioms except Independence of higher activity levels.
- The multi-choice Shapley value satisfies all the axioms except Priority relation for the same maximal activity level.
- Take any $(m, v) \in \mathcal{G}$ and fix any arbitrary integer $\beta_{k,l} \in \{1, 2\}$ for each $(k, l) \in M^+$. The value $f^\beta$ given by

$$\forall (i, j) \in M^+, \quad f^\beta_{ij}(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ (i,s_i) \in Z(C(s),\succeq^*)}} \frac{\beta_{i,j}}{\sum_{(i',j') \in Z(C(s),\succeq^*)} \beta_{i'j'}} \Delta_v(s),$$

  satisfies all the axioms except Balanced contributions for the same prevailing group.

### 5.2.2 Decisive players and priority relations

This section provides a second characterization of the value $\Gamma$ for multi-choice games with a priority structure. The next two axioms are generalizations of standard axioms for multi-choice games to multi-choice games with a priority structure.

**Axiom 53 (Null game).** *If $(m, v, \succeq) \in \mathcal{GS}$ is the null game, then*

$$\forall (i, j) \in M^+, \quad f_{ij}(m, v, \succeq) = 0.$$

**Axiom 54 (Independence of null pairs).** *For each* $(m, v, \succeq) \in \mathcal{GS}$ *and each null pair* $(i, j) \in M^+$,

$$\forall (i', j') \in M^+ \setminus \{(i, m_i)\}, \quad f_{i'j'}(m, v, \succeq) = f_{i'j'}(m - e_i, v, \succeq).$$

By successive applications of Independence of null pairs and Efficiency one obtains the following result.

**Proposition 26.** *If a value* $f$ *on* $\mathcal{GS}$ *satisfies Efficiency and Independence of null pairs, then it satisfies the Null pair property.*

The next two axioms relate to decisive players with the same maximal activity level. A player $i \in N$ is decisive if any coalition in which it does not play its maximal activity level generates zero worth. Put differently, $i \in N$ is decisive if the pair $(i, m_i)$ is a decisive pair. Let $s \in \mathcal{M}$ be a coalition such that $s_i = j - 1$. The surplus $v(s + e_i) - v(s)$ refers to the marginal contribution of $i$ for its activity level $j$ to the coalition $s$. It is straightforward to see that two decisive players $i, i'$ such that $m_i = m_{i'}$ have the same marginal contributions to coalitions, i.e., for each $s$ where $s_i = s_{i'} = m_i - 1$ and $s_k \leq m_k, k \neq i, i'$, it holds that $v(s+e_i) = v(s+e_{i'}) = 0$.[1] In this case, the pairs $(i, m_i)$ and $(i', m_{i'})$ are equals. Equal treatment of equal pairs imposes that equal pairs receive the same payoff. However, two decisive players may differ according to their position in the priority structure. Priority relation for decisive players aims to accommodate this difference. It requires that the payoff of a decisive player is zero if the other decisive player has priority over it.

**Axiom 55 (Priority relation for decisive players).** *For each* $(m, v, \succeq) \in \mathcal{GS}$*, for each* $i, i' \in N$ *decisive players such that* $m_{i'} = m_i$ *and* $i' \succ i$,

$$f_{im_i}(m, v, \succeq) = 0.$$

Whenever two decisive players are indistinguishable in terms of both maximal activity levels and priority, there is no reason to treat them differently. The next axiom relies on this idea and weakens the Equal treatment of equal pairs by considering decisive players with the same prevailing group and maximal activity level.

**Axiom 56 (Decisive players with equal prevailing group property).** *For each* $(m, v, \succeq) \in \mathcal{GS}$*, for each pair of decisive players* $i, i' \in N$ *such that* $m_i = m_{i'}$ *and* $L_i(m, v, \succeq) = L_{i'}(m, v, \succeq)$,

$$f_{im_i}(m, v, \succeq) = f_{im_{i'}}(m, v, \succeq).$$

Interestingly, the combination of Priority relation for the same maximal activity level and Null game implies Priority relation for decisive players. Moreover, the combination of Balanced contributions for the same prevailing group and Null game implies the Decisive players with equal prevailing group property.

[1] This observation can be extended by considering two decisive players with different maximal activity levels.

**Proposition 27.** *On $\mathcal{GS}$, the two following properties hold:*

a. *Priority relation for the same maximal activity level and Null game imply Priority relation for decisive players;*

b. *Balanced contributions for the same prevailing group and Null game imply the Decisive players with equal prevailing group property.*

*Proof.* We prove both points of the proposition separately.

**Point a:** Let $f$ be a value on $\mathcal{GS}$ that satisfies Priority relation for the same maximal activity level and Null game. Take any $(m, v, \succeq) \in \mathcal{GS}$ and any decisive player $i' \in N$. By Priority relation for the same maximal activity level, for each decisive player $i \in N$ such that $m_{i'} = m_i$ and $i' \succeq i$,

$$f_{im_i}(m, v, \succeq) = f_{im_i}(m - e_{i'}, v, \succeq).$$

Since $(i', m_{i'})$ is decisive in $(m, v, \succeq)$, the sub-game $(m - e_{i'}, v, \succeq)$ is a null game. Therefore, by Null game,

$$f_{im_i}(m, v, \succeq) = f_{im_i}(m - e_{i'}, v, \succeq) = 0.$$

This shows that $f$ satisfies Priority relation for decisive players.

**Point b:** Let $f$ be a value on $\mathcal{GS}$ that satisfies Balanced contributions for the same prevailing group and Null game. Take any $(m, v, \succeq) \in \mathcal{GS}$ and any two decisive players $i, i' \in N$ such that $m_i = m_{i'}$ and $L_i(m, v, \succeq) = L_{i'}(m, v, \succeq)$. By Balanced contributions for the same prevailing group,

$$f_{im_i}(m, v, \succeq) - f_{im_i}(m - e_{i'}, v, \succeq) = f_{i'm_{i'}}(m, v, \succeq) - f_{i'm_{i'}}(m - e_i, v, \succeq).$$

Since $(i, m_i)$ and $(i', m_{i'})$ are decisive in $(m, v, \succeq)$, it holds that $(m - e_{i'}, v, \succeq)$ and $(m - e_i, v, \succeq)$ are null games. By Null game,

$$f_{im_i}(m - e_{i'}, v, \succeq) = f_{i'm_{i'}}(m - e_i, v, \succeq) = 0.$$

Thus, we conclude that

$$f_{im_i}(m, v, \succeq) = f_{i'm_{i'}}(m, v, \succeq).$$

This shows that $f$ satisfies the Decisive players with equal prevailing group property. ■

The next result provides a second axiomatic characterization of $\Gamma$ on the class of multi-choice games with a priority structure.

**Theorem 26.** *A value $f$ on $\mathcal{GS}$ satisfies Efficiency, Additivity, Independence of higher activity levels, Independence of null pairs, Priority relation for decisive players and the Decisive players with equal prevailing group property if and only if $f = \Gamma$.*

*Proof.* By Theorem 25, we know that the value $\Gamma$ satisfies Efficiency and Independence of higher activity levels. Since $\Gamma$ is a weighted sum of the Harsanyi dividend, it satisfies Additivity. Observe that, for each null pair $(i, j) \in M^+$ and each $s \in \mathcal{M}$ such that $s_i \geq j$, it holds that $\Delta_v(s) = 0$. From this observation, it follows that $\Gamma$ satisfies Independence of null pairs. Since Additivity implies Null game, by Proposition 27, $\Gamma$ satisfies Priority relation for decisive players and the Decisive players with equal prevailing group property.

It remains to show the uniqueness of the value. Let $f$ be a value on $\mathcal{GS}$ that satisfies all the axioms from the statement of Theorem 26. Let us show that $f$ is uniquely determined. Take any $(m, v, \succeq) \in \mathcal{GS}$. By Additivity, it is enough to show that, for each $t \in \mathcal{M}$, $f(m, \Delta_v(t)u_t, \succeq)$ is uniquely determined. Take any $t \in \mathcal{M}$, recall that the highest activity level in $t$ is denoted by $t^T$.

Each pair $(i, j) \in M^+$ such that $j > t_i$, is a null pair in $(m, \Delta_v(t)u_t, \succeq)$. Since $f$ satisfies Efficiency and Independence of null pairs, by Proposition 26 it satisfies the Null pair property. Therefore, for each $(i, j) \in M^+$ such that $j > t_i$, $f_{ij}(m, \Delta_v(t)u_t, \succeq) = 0$. By successive applications of Independence of null pairs, for each $(i, j) \in M^+$ such that $j \leq t_i$, we obtain

$$f_{ij}(m, \Delta_v(t)u_t, \succeq) = f_{ij}(t, \Delta_v(t)u_t, \succeq).$$

Take any pair $(i, j) \in M^+$ such that $j < t^T$. By Independence of higher activity levels,

$$f_{ij}(t, \Delta_v(t)u_t, \succeq) = f_{ij}((t_k \wedge j)_{k\in N}, \Delta_v(t)u_t, \succeq).$$

Then, by definition of a minimal effort game (see Definition 2.31) and by the fact that Additivity implies Null game,

$$\begin{aligned} f_{ij}((t_k \wedge j)_{k\in N}, \Delta_v(t)u_t, \succeq) &= 0 \\ &= f_{ij}(t, \Delta_v(t)u_t, \succeq). \end{aligned}$$

Observe that any $i \in Q(t^T)$ is a decisive player in the game $(t, \Delta_v(t)u_t, \succeq)$. Take any pair $(i, j) \in M^+$ such that $j = t_i = t^T$, i.e., $i \in Q(t^T)$. Two cases are distinguished. First if $L_i(t, \Delta_v(t)u_t, \succeq) \neq \emptyset$. Since $i$'s maximal activity level in $(t, \Delta_v(t)u_t, \succeq)$ is $t^T$, $i' \in L_i(t, \Delta_v(t)u_t, \succeq)$ implies $i' \succ i$ and $t_{i'} = t^T$. Thus, there exists another player $i' \in Q(t^T)$ such that $i' \succ i$. By Priority relation for decisive players,

$$f_{ij}(t, \Delta_v(t)u_t, \succeq) = 0.$$

Now suppose that $L_i(t, \Delta_v(t)u_t, \succeq) = \emptyset$. Consider the set of pairs

$$M^+_{L(t)} = \{(i,j) \in M^+ \mid i \in Q(t^T),\ L_i(t, \Delta_v(t)u_t, \succeq) = \emptyset\}.$$

At this step, we know that $f_{ij}(m, \Delta_v(t)u_t, \succeq) = 0$ for each $(i,j) \in M^+ \setminus M^+_{L(t)}$. If $|M^+_{L(t)}| = 1$, then $f_{ij}(m, \Delta_v(u_t), \succeq)$, for $(i,j) \in M^+_{L(t)}$, is uniquely determined by Efficiency. If $|M^+_{L(t)}| > 1$, then by Efficiency,

$$\sum_{(i,j)\in M^+} f_{ij}(m, \Delta_v(t)u_t, \succeq) = \sum_{(i,j)\in M^+_{L(t)}} f_{ij}(t, \Delta_v(t)u_t, \succeq) = \Delta_v(t). \tag{5.9}$$

By the Decisive players with equal prevailing group property,

$$\forall (i,j), (i',j') \in M^+_{L(t)}, \quad f_{ij}(t, \Delta_v(t)u_t, \succeq) = f_{i'j'}(t, \Delta_v(t)u_t, \succeq) = c, \tag{5.10}$$

for some $c \in \mathbb{R}$. From (5.10) and (5.9), $f_{ij}(m, \Delta_v(t)u_t, \succeq)$ is uniquely determined for each $(i,j) \in M^+_{L(t)}$. Additivity allows to conclude the proof.

■

The six axioms of the statement of Theorem 26 are logically independent, as shown by the following alternative solutions.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by $f(m,v) = \vec{0}$ satisfies all the axioms except Efficiency.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by

$$\forall (i,j) \in M^+,$$
$$f_{ij}(m, v, \succeq) = \sum_{\substack{s \le m \\ s_i \ge j > 0 \\ (i,s_i)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)| \sum_{(k,s_k)\in Z(C(s),\succeq^*)} s_k},$$

  satisfies all the axioms except Independence of higher activity levels.

- The value $f$ given, for each $(m,v) \in \mathcal{G}$, by

$$\forall (i,j) \in M^+,$$
$$f_{ij}(m, v, \succeq) = \sum_{\substack{s \le m \\ s_i = j \\ (i,j)\in Z(C(s),\succeq^*)}} \frac{(v(0_{-i}, j))^2 + 1}{\sum_{(k,j)\in Z(C(s),\succeq^*)} (v(0_{-k}, j))^2 + 1} \Delta_v(s),$$

  satisfies all the axioms except Additivity.

- Take any $(m, v) \in \mathcal{G}$. For the sake of presentation, we use the notation

$$\forall j \leq m^T, \quad ZC(j) = Z(C((j \wedge m_k)_{k \in N}), \succeq^*).$$

The value $f$ given by

$$f_{ij}(m, v, \succeq) = \begin{cases} \dfrac{v((j \wedge m_k)_{k \in N}) - v(((j-1) \wedge m_k)_{k \in N})}{|ZC(j)|} & \text{if } (i,j) \in ZC(j), \\ 0 & \text{otherwise,} \end{cases}$$

satisfies all the axioms except Independence of null pairs. Indeed, this value allocates to each pair $(i, j) \in ZC(j)$ an equal share of the difference between the worth of $(j \wedge m_k)_{k \in N}$ and $((j-1) \wedge m_k)_{k \in N}$. Take any pair $(i, j) \in ZC(j)$ and assume that it is a null pair. If player $i$ reduces its maximal activity level to $j-1$, this may impact the payoff of the other pairs in $ZC(j)$, which contradicts Independence of null pairs.

- The multi-choice Shapley value satisfies all the axioms except Priority relation for decisive players.

- Take any $(m, v) \in \mathcal{G}$ and fix any arbitrary integer $\beta_{k,l} \in \{1, 2\}$ for each $(k, l) \in M^+$. The value $f^\beta$ given by

$$\forall (i,j) \in M^+, \quad f_{ij}^\beta(m, v, \succeq) = \sum_{\substack{s \leq m \\ s_i = j \\ (i,s_i) \in Z(C(s), \succeq^*)}} \frac{\beta_{i,j}}{\sum_{(i',j') \in Z(C(s), \succeq^*)} \beta_{i'j'}} \Delta_v(s),$$

satisfies all the axioms except the Decisive players with equal prevailing group property.

## 5.3 Priority relations structured by classes

This section presents situations in which the set of players can be partitioned into several priority classes $(N_1, \ldots, N_q)$. Each priority class contains incomparable players that have priority over each player in the next class. Such priority structures generalize the linear priority structures. Formally,

$$\forall i, i' \in N, \quad [i \succ i'] \iff [i \in N_p, i' \in N_{p'} \implies p < p'].$$

The set of players over which player $i \in N_p$ has priority is

$$\downarrow_\succ i = \bigcup_{p' > p} N_{p'}.$$

In the following, $N_{>p}$ stands for $\bigcup_{p' > p} N_{p'}$.

**Example 13.** *Let $N = \{1, 2, 3, 4, 5, 6, 7, 8\}$ be the set of players and consider $(N, \succeq)$ a priority structured by classes. Suppose that the set of players is partitioned into three cells: $N_1 = \{4, 5, 6\}$, $N_2 = \{2, 3\}$ and $N_3 = \{1, 7, 8\}$. Each player in $N_1$ has priority over players in $N_2 \cup N_3$, and each player in $N_2$ has priority over players in $N_3$. Figure 2 represents the Hasse diagram of the poset $(N, \succeq)$.*

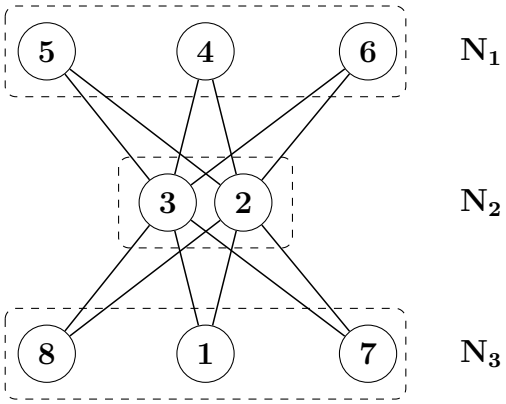



*Figure 2: Hasse diagram of $(N, \succeq)$ structured by classes.*

■

### 5.3.1 The multi-choice Priority value as a sequential allocation process

Consider the class of multi-choice games with priority relations structured by classes. We define a sequential allocation process denoted by $f$. First consider an activity level $j \leq \max_{i \in N} m_i$ and the set of players that are able to play this activity level, i.e., the set $Q(j)$. Then, consider the priority class $p \in \{1, \ldots, q\}$, and players in $N_p$ that are able to play $j$, i.e., the subset of players $N_p \cap Q(j)$. Finally, one assigns to each pair $(i, j) \in M^+$ such that $i \in N_p$, the Shapley value obtained by $i$ in a TU-game on $N_p \cap Q(j)$. This sequential allocation process is detailed by the following procedure.

**Allocation process for multi-choice games with a priority relation structured by classes**

1: **for** $j = 1$ **to** $\max_{i \in N} m_i$ **do**
2:
3: **for** $t = 1$ **to** $q$ **do**
4: $p = q - t + 1$
5: **if** $N_p \cap Q(j) \neq \emptyset$ **then**
6:

$$\begin{aligned} & \forall E \subseteq N_p \cap Q(j), \\ w_j^p(E) = & v\Big(((j-1) \wedge m_k)_{k \in N} + e_{(E \cup N_{>p}) \cap Q(j)}\Big) \\ & - v\Big(((j-1) \wedge m_k)_{k \in N} + e_{N_{>p} \cap Q(j)}\Big) \\ & \forall i \in N_p \cap Q(j), \\ & f_{ij}(m, v, \succeq) = Sh_i(N_p \cap Q(j), w_j^P) \end{aligned} \tag{5.11}$$

7: **else**
8:
9: **end if**
10: **end for**
11: **end for**

Line 1 highlights that the procedure applies from the smallest activity level $j = 1$ to the highest activity level in the multi-choice game. Similarly, line 3 emphasizes that for each level of activity, the procedure applies from the last priority class $N_q$ to the first priority class $N_1$. Line 6 defines the TU-game $(N_p \cap Q(j), w_j^p)$, whose interpretation is the following. Assume that all players in $N_p \cap Q(j)$ play the activity level $j$. These players obtain their marginal contribution to the coalition in which all players in $N_{>p} \cap Q(j)$ play $j$, while all other players play $j-1$ (or their maximal activity level). To determine how this contribution is allocated, one defines the TU-game $(N_p \cap Q(j), w_j^p)$ as in (5.11). For each coalition $E \subseteq N_p \cap Q(j)$, the worth $w_j^p(E)$ corresponds to the surplus generated (in the game $(m, v, \succeq)$) when each player in $E$ increases its activity level from $j-1$ to $j$. For an illustration of the procedure, see Example 14 below.

**Remark 8.** *Observe that, for each priority class $p \in \{1, \ldots, q\}$ the TU-game $(N_p \cap Q(j), w_j^p)$, as defined in (5.11), is the sub-game of the TU-game $(Q(j), w_j)$ where*

$$\forall E \subseteq Q(j), \quad w_j(E) = v\Big(((j-1) \wedge m_k)_{k \in N} + e_E\Big) - v\Big(((j-1) \wedge m_k)_{k \in N}\Big). \tag{5.12}$$

*One can interpret $w_j(E)$ as follows. Suppose that each player outside coalition $E$ plays its activity level $j-1$ or its maximal activity level (if it is unable to play $j-1$).*

*Then, the worth $w_j(E)$ corresponds to the surplus generated (in the game $(m, v, \succeq)$) when each player in $E$ increases its activity level from $j-1$ to $j$.*[2]

*Let $E \subseteq Q(j)$, by definition of the Harsanyi dividends given by (2.32),*

$$w_j(E) = \sum_{t \leq ((j-1)\wedge m_k)_{k\in N}+e_E} \Delta_v(t) - \sum_{t \leq ((j-1)\wedge m_k)_{k\in N}} \Delta_v(t),$$

*that is*

$$\forall E \subseteq Q(j), \quad w_j(E) = \sum_{\substack{t \leq ((j-1)\wedge m_k)_{k\in N}+e_E \\ t \not\leq ((j-1)\wedge m_k)_{k\in N}}} \Delta_v(t). \tag{5.13}$$

*It should be observed that this sum takes in arguments each coalition $t \leq ((j-1)\wedge m_k)_{k\in N}+e_E$ in which at least one player in $E$ plays activity level $j$. This observation will be useful to prove the next results.*

**Example 14.** *To illustrate the allocation process, consider $(m, v, \succeq) \in \mathcal{GS}$ such that $(N, \succeq)$ is the priority structured by classes given in Example 13. Let $m_7 = 1$, $m_1 = m_2 = m_6 = 2$, $m_3 = m_4 = m_5 = m_8 = 3$. It holds that $Q(1) = N$, $Q(2) = N \setminus \{7\}$ and $Q(3) = \{3, 4, 5, 8\}$. Apply the sequential procedure to determine the payoffs of the pairs in $M^+$.*

1. *Consider the activity level $j = 1$, and the subset of players $Q(1) = N$.*

1.1. *Consider the priority class $p = 3$ and players in $N_3 \cap Q(1) = \{1, 7, 8\} = N_3$. All players in this subset cooperate at activity level $j = 1$. The contribution $v(e_{N_3}) - v(\vec{0})$ is allocated among all players in $N_3$ according to the TU-game $(N_3, w_1^3)$ defined as,*

$$\forall E \subseteq N_3, \quad w_1^3(E) = v(e_E).$$

*Then, assign $Sh_i(N_3, w_1^3)$ to each player in $i \in N_3$, which corresponds to their payoff for their activity level $j = 1$.*

1.2. *Consider the priority class $p = 2$ and players in $N_2 \cap Q(1) = \{3, 2\} = N_2$. Define the TU-game $(N_2, w_1^2)$ as*

$$\forall E \subseteq N_2, \quad w_1^2(E) = v(e_E + e_{N_3}) - v(e_{N_3}).$$

*Then, assign $Sh_i(N_2, w_1^2)$ to each player $i \in N_2$, which corresponds to their payoff for their activity level $j = 1$.*

[2] One can also find a similar construction of TU-games in the context of cost sharing problems (for instance, see Albizuri *et al.* (2003)).

*1.3. The procedure is applied similarly. Thus, one defines the TU-game* $(N_1, w_1^1)$ *as*

$$\forall E \subseteq N_1, \quad w_1^1(E) = v(e_E + e_{N_2 \cup N_3}) - v(e_{N_2 \cup N_3}).$$

*Then, each player* $i \in N_1$ *receives* $Sh_i(N_1, w_1^1)$ *which corresponds to their payoff for their activity level* $j = 1$.

*The total worth allocated to players in* $Q(1)$ *is*

$$v(e_{N_1} + e_{N_2 \cup N_3}) + v(e_{N_2 \cup N_3}) + v(e_{N_3}) - v(e_{N_2 \cap N_3}) - v(e_{N_3}) = v(1, \ldots, 1).$$

*2. Consider the activity level* $j = 2$ *and the subset of players* $Q(2)$.

*2.1. Consider the priority class* $p = 3$ *and players in* $N_3 \cap Q(2)$. *Then, define the TU-game* $(N_3 \cap Q(2), w_2^3)$ *as*

$$\forall E \in 2_3^N \cap Q(2), \quad w_2^3(E) = v((1, \ldots, 1) + e_E) - v((1, \ldots, 1)).$$

*Then, assigns* $Sh_i(N_3 \cap Q(2), w_1^3)$ *to each player* $i \in N_3 \cap Q(2)$.

*The next steps are applied similarly.* ■

We now show that the multi-choice Priority value applied to a multi-choice game with a priority structure organized by classes coincides with the Shapley value of a certain TU-game. First, we need an intermediary result.

The following proposition establishes that the Harsanyi dividend of a coalition $E \subseteq Q(j)$ of the intermediary TU-game $(Q(j), w_j)$, defined as (5.12), has a relevant expression in terms of the Harsanyi dividends of multi-choice coalitions (of the multi-choice game $(m, v, \succeq)$). Specifically, the Harsanyi dividend of a coalition $E \subseteq Q(j)$ is the sum of the Harsanyi dividend of the multi-choice coalitions $t$ such that for each $i \notin E$, $t_i \leq (j-1) \wedge m_i$ and for each $i \in E$, $t_i = j$.

**Proposition 28.** *For each TU-game* $(Q(j), w_j)$ *as defined by* (5.12)*, it holds that*

$$\forall E \subseteq Q(j), \quad \Delta_{w_j}(E) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E: \\ \forall i \in E, t_i = j}} \Delta_v(t).$$

*Proof.* Let $(Q(j), w_j)$ be the TU-game derived from $(m, v, \succeq) \in \mathcal{GS}$ and the activity level $j \leq \max_{i \in N} m_i$. The proof proceeds by induction on the size of coalitions $E \subseteq Q(j)$.

**Initialization**: Take any coalition $E \subseteq Q(j)$ such that $|E| = 1$. Let say that $E = \{i\} \subseteq Q(j)$. By definition of the Harsanyi dividend of a TU-game,

$$\begin{aligned} \Delta_{w_j}(\{i\}) &= w_j(\{i\}) \\ &= \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_i \\ t \not\leq ((j-1) \wedge m_k)_{k \in N}}} \Delta_v(t). \end{aligned}$$

where the first equality comes from the definition of the Harsanyi dividend in TU-games and the definition of $w_j$; and the second equality comes from (5.13). Observe that the set of coalitions $t \leq ((j-1) \wedge m_k)_{k \in N} + e_i$ and $t \not\leq ((j-1) \wedge m_k)_{k \in N}$ is equal to the set of coalitions $t \leq ((j-1) \wedge m_k)_{k \in N} + e_i$ such that $t_i = j$. Therefore, we obtain

$$\Delta_{w_j}(\{i\}) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_i \\ t_i = j}} \Delta_v(t),$$

the desired result.

**Induction hypothesis**: Assume that for each $E \subseteq Q(j)$ such that $|E| = r$ with $1 \leq r < |Q(j)|$,

$$\Delta_{w_j}(E) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E: \\ \forall i \in E: t_i = j}} \Delta_v(t).$$

**Induction step**: Take any coalition $E \subseteq Q(j)$ such that $|E| = r + 1$. By definition of the Harsanyi dividend, it holds that

$$\Delta_{w_j}(E) = w_j(E) - \sum_{T \subset E} \Delta_{w_j}(T).$$

Observe that, for each $T \subset E$, it holds that $|T| \leq r$. By (5.13) and the induction hypothesis, it follows that

$$\Delta_{w_j}(E) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ t \not\leq ((j-1) \wedge m_k)_{k \in N}}} \Delta_v(t) - \sum_{T \subset E} \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_T \\ \forall i \in T: t_i = j}} \Delta_v(t). \tag{5.14}$$

Note that, each coalition $t \leq ((j-1) \wedge m_k)_{k \in N} + e_E$ and $t \not\leq ((j-1) \wedge m_k)_{k \in N}$ is such that

- for each player $i \in N \setminus E$, $t_i \leq (j-1) \wedge m_i$,
- for some players in $E$, $t_i = j$.

Consider a coalition $t \leq ((j-1) \wedge m_k)_{k \in N} + e_E$ and $t \not\leq ((j-1) \wedge m_k)_{k \in N}$. Denote by $T \subseteq E$, the subset of players such that $t_i = j$. Summing over all coalitions $t$ of this form, one obtains

$$\sum_{T \subseteq E} \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_T \\ \forall i \in T: t_i = j}} \Delta_v(t) = \sum_{\substack{t \leq ((j-1) \wedge m_k)_{k \in N} + e_E \\ t \not\leq ((j-1) \wedge m_k)_{k \in N}}} \Delta_v(t).$$

From this observation, (5.14) now becomes

$$\begin{aligned}\Delta_{w_j}(E) &= \sum_{T\subseteq E} \sum_{\substack{t\leq((j-1)\wedge m_k)_{k\in N}+e_T \\ \forall i\in T: t_i=j}} \Delta_v(t) - \sum_{T\subset E} \sum_{\substack{t\leq((j-1)\wedge m_k)_{k\in N}+e_T \\ \forall i\in T: t_i=j}} \Delta_v(t) \\ &= \sum_{\substack{t\leq((j-1)\wedge m_k)_{k\in N}+e_E \\ \forall i\in E: t_i=j}} \Delta_v(t).\end{aligned}$$

This concludes the induction step and completes the proof. ■

The main result of this section establishes that, on the class of multi-choice games with a priority structure organized by classes, the multi-choice Priority value assigns to each pair $(i,j)\in M^+$ a payoff resulting from the above sequential procedure: the payoff assigned to an activity level $j$ of the player $i\in N_p$ coincides with the Shapley value obtained by $i$ in the TU-game $(N_p\cap Q(j), w_j^p)$.

**Theorem 27.** *For each $(m,v,\succeq)\in\mathcal{GS}$, where $(N,\succeq)$ is a priority structured by the classes $(N_1,\ldots,N_q)$, for each priority class $p\in\{1,\ldots,q\}$, and for each pair $(i,j)\in M^+$ where $i\in(N_p\cap Q(j))$, it holds that*

$$\Gamma_{ij}(m,v,\succeq) = Sh_i((N_p\cap Q(j)), w_j^p),$$

*where the TU-game $(N_p\cap Q(j), w_j^p)$ is defined as* (5.11).

*Proof.* Take any multi-choice game with priority structure $(m,v,\succeq)\in\mathcal{G}_P$, where $(N,\succeq)$ is structured by the classes $(N_1,\ldots,N_q)$. Take any class $p$, any activity level $j\leq\max_{i\in N} m_i$ and consider the TU-game $(N_p\cap Q(j), w_j^p)$ given by (5.11). Adapting an intermediary result provided by Béal *et al.* (2021a),[3] one can show that the Harsanyi dividends of the TU-game $(N_p\cap Q(j), w_j^p)$ are given by

$$\forall E\subseteq N_p\cap Q(j),\quad \Delta_{w_j^p}(E) = \sum_{T\subseteq N_{>p}\cap Q(j)} \Delta_{w_j}(T\cup E).$$

From, Proposition 28, one can write

$$\forall E\subseteq N_p\cap Q(j),\quad \Delta_{w_j^p}(E) = \sum_{T\subseteq N_{>p}\cap Q(j)} \sum_{\substack{t\leq((j-1)\wedge m_k)_{k\in N}+e_{E\cup T} \\ \forall i\in E\cup T: t_i=j}} \Delta_v(t). \tag{5.15}$$

Next, take any priority class $p\in\{1,\ldots,q\}$ and any pair $(i,j)\in M^+$ such that $i\in N_p$. The payoff of the pair $(i,j)$ through the value $\Gamma$ is

$$\Gamma_{ij}(m,v,\succeq) = \sum_{\substack{s\leq m \\ s_i=j \\ (i,j)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}.$$

[3]The reader is referred to the proof of Theorem 3 in Béal *et al.* (2021a).

Observe that, if apair $(i',j') \in C(s)$ is such that either $i' \neq i$ and $j' > j$ or $i' = i$ and $j' > j$, then we necessarily have $(i,j) \notin Z(C(s), \succeq^*)$. From this observation, $\Gamma_{ij}(m, v, \succeq)$ can be written as

$$\begin{aligned}\Gamma_{ij}(m,v,\succeq) &= \sum_{\substack{s \leq (j \wedge m_k)_{k\in N} \\ s_i = j \\ (i,j)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|} \\ &= \sum_{\substack{s \leq ((j-1)\wedge m_k)_{k\in N} + e_{Q(j)} \\ s_i = j \\ (i,j)\in Z(C(s),\succeq^*)}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}.\end{aligned}$$

Because $i \in N_p$, for each coalition $s \leq ((j-1) \wedge m_k)_{k\in N} + e_{Q(j)}$ such that $(i,j) \in Z(C(s), \succeq^*)$, it holds that

- $s_i = j$
- $\forall k \in N \setminus Q(j)$: $s_k \leq m_k < j$;
- $\forall k \in N_{p'} \cap Q(j)$, where $p' > p$: $s_k < j$;
- $\forall k \in (N_p \cup N_{>p}) \cap Q(j)$: $s_k \leq j$.

These four points imply that the payoff of the pair $(i,j)$ is obtained by summing the Harsanyi dividends of $(m, v, \succeq)$ over the multi-choice coalitions $s \leq ((j-1) \wedge m_k)_{k\in N} + e_{(N_p \cup N_{>p}) \cap Q(j)}$ such that $s_i = j$. Moreover, by summing over such multi-choice coalitions, one considers all sub-coalitions of players $T \subseteq (N_p \cup N_{>p}) \cap Q(j)$ containing player $i$. Thus, we obtain

$$\Gamma_{ij}(m,v,\succeq) = \sum_{\substack{E \subseteq N_p \cap Q(j) \\ i \in E}} \sum_{T \subseteq N_{>p} \cap Q(j)} \sum_{\substack{s \leq ((j-1)\wedge m_k)_{k\in N} + e_{E\cup T} \\ \forall k \in E \cup T : s_k = j}} \frac{\Delta_v(s)}{|Z(C(s),\succeq^*)|}.$$

It should be observed that $|Z(C(s), \succeq^*)|$ does not depend on the coalitions $T \subseteq N_{>p} \cap Q(j)$. More precisely, it can be shown that, for each coalition $s \leq ((j-1) \wedge m_k)_{k\in N} + e_{E\cup T}$ such that $\forall k \in E \cup T$ where $s_k = j$, one have $|Z(C(s), \succeq^*)| = |E|$. Therefore, it holds that

$$\begin{aligned}\Gamma_{ij}(m,v,\succeq) &= \sum_{\substack{E \subseteq N_p \cap Q(j) \\ i \in E}} \frac{1}{|E|} \sum_{T \subseteq N_{>p} \cap Q(j)} \sum_{\substack{s \leq ((j-1)\wedge m_k)_{k\in N} + e_{E\cup T} \\ \forall k \in E \cup T : s_k = j}} \Delta_v(s) \\ &= \sum_{\substack{E \subseteq N_p \cap Q(j) \\ i \in E}} \frac{1}{|E|} \Delta_{w_j^p}(E) \\ &= Sh_i((N_p \cap Q(j)), w_j^p),\end{aligned}$$

where the second equality comes from (5.15), and the third equality comes from the definition of the Shapley value for TU-games. This completes the proof of the theorem. ■

## 5.4 Conclusion

This chapter provides an axiomatic study on the class of multi-choice games with a priority structure. A new value for this class is introduced and characterized: the multi-choice Priority value. This value takes advantage of a lexicographic partial order resulting from a specific combination of the ordered sets of activity levels of the players with the priority structure. The axiomatic approach allows to endogenously determine the partial lexicographic order on the set of pairs (composed of players and their activity levels). It can be pointed out that many other combinations are possible. In particular, several relaxations/alternative forms of the axioms would lead to different combinations, and thus, to different values.

This chapters takes advantage of the information provided by the distribution network by extracting a priority structure from it. Moreover, this chapter has consideration for the principles pursued by GRDF since the multi-choice Priority value is in line with the Independence of higher demands principle. However, this chapter fails at considering the Uniformity principle.

# Chapter 6: Sharing the Cost of a gas Distribution Network

## 6.1 Introduction

To convey out its task properly, a gas distribution network operator is confronted with various operation costs, some of which are not directly assignable to a given consumer. Usually, a network operator recovers these costs by sharing them among the consumers. A cost sharing is relevant if it meets the principles selected and the objective pursued by the network operator.

In our case, the main principles pursued by GRDF are the Independence of higher demands principle and the Uniformity principle. In this chapter, we consider these two principles along with a third one that makes sense in the context of gas distribution. Moreover, we take the distribution network into account in the model and in the cost sharing rules developed in this chapter.

The gas distribution network is represented by a finite set $N \subset \mathbb{N}$ of $n$ consumers and a rooted tree graph $D$. The nodes of the graph represent the consumers, the root represents the source in gas and the arcs of the graph represent the pipelines making up the network. An integer in $N$ refers to both a consumer and the pipeline having this consumer at its tail since each node has at most one direct predecessor in a rooted tree graph.

Each consumer $i \in N$ has an **effective demand** $q_i \leq K$, where $K \in \mathbb{N}$ is an upper bound for effective demands. This effective demand corresponds to the highest daily volume that this consumer expects to achieve in a year. For instance, the effective demand of a regular household is often determined by its consumption during winter. The effective demand is communicated to the operator in advance in order to design the network accordingly. The profile of all effective demands is given by $q = (q_1, \ldots, q_n)$. Without loss of generality, for each $i \in N$, assume that $q_n \geq q_i$. Each consumer $i \in N$ is endowed with a discrete set $\{0, 1, \ldots, q_i\}$ that describes its effective demand $q_i$ and each of the **demands** smaller than $q_i$. This set can be interpreted as the set of all demands available to $i$, since $i$ is not required to demand $q_i$ all year round.

The network operator must be able to satisfy any effective demand at any time. Therefore, we assume that each pipeline is designed to satisfy the highest effective demand of the consumers located at the tail and downstream of that pipeline (its **highest downstream demand** for short). Formally, a pipeline with consumer $i$ located at its tail is designed to meet its highest downstream demand denoted by $\overline{q}_i$, and is given by

$$\overline{q}_i = \max_{k \in \{i\} \cup \{\text{downstream consumers of } i\}} q_k.$$

There are alternative to this approach to pipeline design that we will not consider for our study. Each of these approaches, including this one, has its advantages and its drawbacks. We will discuss these alternative approaches at the end of the chapter.

A non-decreasing cost function $C : N \times \{0, \ldots, q_n\} \to \mathbb{R}_+$ computes the cost of any pipeline of any capacity. From this function, the **total cost** of operating the network is computed as the sum of the costs of all pipelines, assuming that each pipeline is designed to meet its highest downstream demand.

The problem is then to determine a relevant rule to share the total cost among the consumers. This problem is called the cost sharing problem of gas distribution (the gas distribution problem for short) and is denoted by $(N, q, C, D)$. A **cost sharing rule** is a map $f$ that associates to each gas distribution problem $(N, q, C, D)$ a profile

$$f(N, q, C, D) \in \mathbb{R}_+^{\sum_{i \in N} q_i}.$$

This profile specifies a positive cost share $f_{ij}(N, q, C, D)$ to each demand $j \in \{1, \ldots, q_i\}$ of each consumer $i$. This allows us to have a detailed overview of how much a consumer must pay to increase its demand by one unit or more. A cost sharing rule is particularly useful if the operator wants to derive a rate schedule from it.

Subsequently, a consumer $i \in N$ with an effective demand $q_i$ is charged with the total cost share

$$\sum_{j \in \{1, \ldots, q_i\}} f_{ij}(N, q, C, D).$$

Obviously, a consumer's total cost share is non-decreasing in its effective demand. This property relates to the axiom of (demand) monotonicity discussed in Moulin & Shenker (1992) and Moulin (1995). A cost sharing rule satisfies the budget balanced condition, meaning that it recovers the total cost of operating the network.

We adopt a normative approach to determine relevant rules. Three principles are considered: the Independence of higher demands principle, the Uniformity principle and the Connection principle. The first two principles have been extensively discussed in the previous chapters. In particular, we saw that the Uniformity principle can be seen as a form of egalitarianism. The Connection principle has been mentioned in the introduction of the thesis (see Section 1.8.5). Below we provide additional details.

Consider a gas distribution network in which all the consumers, except one, have a null demand in gas. To supply this consumer with gas, all the pipelines connecting this consumer to the source must be involved. In other words, this consumer depends on a number of pipelines in addition to the one to which it is the tail. It should be reasonable to charge this consumer for the costs of operating these pipelines. In a more general way, a consumer should pay for the portion of the network it uses. This idea is the Connection principle. This principle is in the same spirit than the upstream responsibility principle that comes from the unlimited territorial integrity theory for polluted river problems. In short, the upstream responsibility principle states that agents, located on a polluted river network, should pay for the cost of cleaning the portion of the river that connect them to the source (see Dong *et al.* (2012) for details).

We define three cost sharing rules on the basis of these principles. Recall that a cost sharing rule charges a cost share to each demand of each consumer. To clearly present our cost sharing rules, assume that a pipeline is built, step by step, by increasing its capacity to satisfy a demand from 0 up to its highest downstream demand. At each step, an incremental cost is generated. This allows us to understand how each incremental cost is shared among the demands of the consumers depending on the cost sharing rule.

The first cost sharing rule is called the **Connection rule** and is computed as follows. Consider any pipeline. Assume that each consumer located downstream of this pipeline has an effective demand of at most 1 unit. Then, this pipeline is designed to meet a demand of 1 unit. The Connection rule shares the cost of this pipeline equally among the first unit of demand of all downstream consumers. Next, assume that each consumer located downstream of this pipeline now has an effective demand of at most 2 units. Then, upgrade this pipeline in order to meet a demand of 2. This upgrade generates a positive incremental cost. The Connection rule shares this incremental cost equally among the second unit of demand of all downstream consumers whose demand is at least 2 units. This procedure continues until the demand for which this pipeline is designed, i.e., its higher downstream demand is reached and is applied to each pipeline. Obviously, this cost sharing rule recovers the total cost of the network and respects the Connection principle. It should be pointed out that the Connection rule also respects the Independence of higher demands principle. Indeed, a consumer will never have to pay for upgrades with the purpose of meeting demands higher than its own effective demand. This cost sharing rule shares some similarities with the Downstream equal sharing solution, introduced by Dong *et al.* (2012) for polluted river problems. However, contrary to the Downstream equal sharing solution, the Connection rule takes into account the demands of the consumers.

The second cost sharing rule is called the Uniform rule and is computed in a similar manner to the Connection rule. Consider any pipeline. Assume that this pipeline is designed to meet a demand of 1 unit. The Uniform rule shares the cost of this pipeline equally among the first unit of demand of all the consumers regardless of

their position on the network. Then, assume that this pipeline is upgraded to meet a demand of 2 units. This upgrade generates a positive incremental cost. The Uniform rule shares this incremental cost equally among the second unit of demand of all the consumers whose demand is at least 2 units. This procedure continues until the demand for which this pipeline is designed (i.e., its higher downstream demand) is reached and is applied to each pipeline. Obviously, this cost sharing rule recovers the total cost of the network and respects the uniform principle and the Independence of higher demands principle.

In the same way that we reach a compromise between marginalism and egalitarianism in Chapter 4, we try to reach a compromise between the Connection principle and the Uniformity principle through a new family of cost sharing rules. Each cost sharing rule in this family achieves a compromise by means of convex combinations between the Connection rule and the Uniform rule. These cost sharing rules are called the Mixed rules and are computed in a similar manner to the two other cost sharing rules. Consider any pipeline. Assume that this pipeline is designed to meet a demand of 1 unit. The Mixed rule charges to the first unit of demand of each consumer a share of the cost of this pipeline. This share is computed as a convex combination between the share that would have been allocated by the Connection rule and the share that would have been allocated by the Uniform rule. In other words, the amount charged to the consumers for their first unit of demand results from a compromise between the Connection principle and the uniform principle. Then, assume that this pipeline is upgraded in order to meet a demand of 2 units. The incremental cost generated when this pipeline is upgraded is shared among the second unit of demand of all the consumers whose demand is at least 2 units. This share is computed as a convex combination (possibly different from the first combination) between the share that would have been allocated by the Connection rule and the share that would have been allocated by the Uniform rule. In other words, the amount charged to the consumers for their second unit of demand results from a compromise between the Connection principle and the Uniformity principle. This second compromise being potentially different from the first compromise. This procedure continues until the demand for which this pipeline is designed (i.e., its higher downstream demand) is reached and is applied to each pipeline. The Mixed rules allow for different compromises depending on the level of demand. For instance, it is possible to operate a compromise in favor of the Uniformity principle for the demand 1, in which case, each consumer will be charged an amount relatively similar to other consumers for their first demand. Meanwhile, it is possible to operate a compromise in favor of the Connection principle for the demand 2, in which case, each consumer with a demand of at least 2 will be charged an amount that greatly depends on its position on the network for its second demand.

We axiomatically characterize these three cost sharing rules. To do so, we formalize the three principles into axioms for cost sharing rules. Moreover, we provide additional axioms that do not necessarily fit into the idea of the three principles but can still be viewed as desirable properties for cost sharing rules. We formally detail

all the axioms in Section 6.3.

Finally, we establish a link between the cost sharing rules introduced in this chapter and the values for multi-choice games introduced in Chapter 4. From any gas distribution problem $(N, q, C, D)$, one can derive the gas distribution game $(N, q, v^{C,D})$. In this game, the player set represents the set of consumers, and the activity levels represent the demands of the consumers. The worth of a coalition $s$ corresponds to the cost of the gas distribution network $(N, s, C, D)$ designed to meet the demands of the coalition $s$. We show that the Connection rule of a gas distribution problem corresponds to the multi-choice Shapley value of the corresponding gas distribution game. Similarly, we show that the Uniform rule corresponds to the multi-choice Equal division value and the Mixed rules to the multi-choice Egalitarian Shapley values. Moreover, we show that, for each gas distribution problem, the multi-choice Shapley value of the corresponding gas distribution game is in the Core of the gas distribution game. A related study by van den Brink *et al.* (2018) considers a polluted river problem and derives a special TU-game with a permission structure from the problem. In particular, the authors show that the permission value (see Section 2.2) of this special TU-game coincides with the Downstream equal sharing solution applied to the original polluted river problem. Additionally, several other papers address the cooperative aspect of gas-related problems. For instance, Massol & Tchung-Ming (2010) consider cooperation among liquefied gas suppliers and Csercsik *et al.* (2019) partition function form games to model transfer profit as externalities.

The rest of this chapter is organized as follows. After dealing with preliminaries on the gas distribution framework in Section 6.2.1, we define gas distribution problems in Section 6.2.2. In Section 6.3, we define above mentioned rules and provide their axiomatic characterization. Then, we show how solution concepts for multi-choice games relate to our rules in Section 6.4. Section 6.5 concludes.

## 6.2 The model

### 6.2.1 Notation and Definitions

Fix $N = \{1, 2, \ldots, n\}$, $n \geq 2$ a finite set of **consumers**. These consumers are connected to a source in gas $\mathcal{S}$ through **pipelines**. The consumers and the pipelines form a fixed **gas distribution network**, which is modeled by a rooted tree graph. A rooted tree graph is a couple $(N \cup \mathcal{S}, D)$ where

$$D : N \cup \mathcal{S} \longrightarrow 2^{N \cup \mathcal{S}}$$

is a map representing the connections (pipelines) between the nodes. The relationship $i' \in D(i)$ means that $i' \in N$ is supplied right after consumer $i$. Equivalently, we denote by $i \in D^{-1}(i')$ if and only if $i' \in D(i)$. In this case, consumer $i$ is the **predecessor** of $i'$ while $i'$ is the **successor** of $i$. Put differently, $i$ is the **head** of the pipeline and $i'$ is the **tail** of the pipeline.

The transitive closure of a directed graph $D$ is a directed graph $\hat{D}$ such that, for each $i \in N$, $i' \in \hat{D}(i)$ if and only if there exists a path

$$i = h_1, h_2, \ldots, h_k = i', \quad \text{such that} \quad h_k \in D(h_{k-1}), \ldots, h_2 \in D(h_1).$$

The consumers in $\hat{D}(i)$ are called the **subordinates** of $i \in N$ in $D$, and the consumers in $\hat{D}^{-1}(i) := \{i' \in N : i \in \hat{D}(i')\}$ are called the **superiors** of $i \in N$ in $D$. Similarly, the set $\hat{D}(E)$ represents the subordinates of the players in $E \in 2^N$ and $\hat{D}^{-1}(E)$ represents the superiors of the players in $E \in 2^N$. A **cycle** occurs in a graph $(N \cup \mathcal{S}, D)$ if there exists a path

$$i = h_1, h_2, \ldots, h_k = i, \quad \text{such that} \quad h_k \in D(h_{k-1}), \ldots, h_2 \in D(h_1).$$

A rooted tree graph is an acyclic directed graph in which each node has at most one predecessor and only one node (the source) has no predecessor, i.e., $D^{-1}(\mathcal{S}) = \emptyset$. For each $i \in N$, we call **pipeline i** the pipeline having consumer $i$ as a tail. Consumer $i \in N$ and its subordinates are called the **downstream consumers** of pipeline $i$. If no confusion arises, we simply denote the gas distribution network by $D$.

Each consumer $i \in N$ has several **levels of demand** in gas that vary throughout a year. A consumer's **effective demand** $q_i \in \mathbb{N}_*$, refers to the highest daily volume that a consumer $i \in N$ expects to demand in a year. Fix $K \in \mathbb{N}$ an upper bound for effective demands, such that $q_i \leq K$ for each $i \in N$. We model the set of all available demands of any consumer $i \in N$ by the discrete set $M_i = \{0, \ldots, q_i\}$. We denote by

$$Q(j) = \{i \in N : q_i \geq j\}$$

the set of consumers with an effective demand of at least $j$. The **profile of effective demands** is denoted by $q = (q_1, \ldots, q_n)$. Without loss of generality, assume that $q_n \geq q_i$ for each $i \in N$.

A **cost function** is a map

$$C : N \times \{0, \ldots, K\} \to \mathbb{R}_+$$

that measures the cost of any pipeline when designed to meet any demand. Formally, for each $i \in N$ and $j \in \{1, \ldots, K\}$, $C(i, j)$ represents the cost of operating the pipeline $i$ when it is designed to meet a demand of $j$. We use the convention $C(i, 0) = 0$ for each $i \in N$. We assume that each map

$$\forall i \in N, \quad C(i, .) : \{1, \ldots, K\} \to \mathbb{R}_+$$

is non-decreasing: the larger the demand, the larger is the pipeline's capacity, which leads to higher costs. On the other hand, for any $j \in \{1, \ldots, K\}$ and any two $i, i' \in N$, we do not necessarily have $C(i, j) = C(i', j)$. Indeed, the cost of operating

a pipeline may differ depending on the pipeline. These differences are due to exogenous features of the gas distribution network such as the length or the geographical location.

Equivalently, a cost function can be expressed as a matrix of incremental costs. We denote by $A^C_{ij}$ the **incremental cost** of pipeline $i$ for a demand $j$. The incremental cost $A^C_{ij}$ represents the increase in cost of pipeline $i$ when it is upgraded by one unit to meet a demand of $j$ instead of $j-1$. Formally, it is defined as

$$\forall i \in N, \forall j \in \{1, \dots, K\}, \quad A^C_{ij} = C(i,j) - C(i,j-1).$$

Obviously, $A^C_{ij} \geq 0$ for each $i \in N$ and each $j \in \{1, \dots, K\}$, since $C(i,.)$ is a non-decreasing map. The $(n, K)$ **matrix of incremental costs** collects all the incremental costs, and is denoted by $A^C$. Take any $i \in N$ and any $j \leq K$. The **unit cost matrix** $I^{ij}$ is the $(n, K)$ matrix defined as

$$\forall k \in N, l \leq K, \quad I^{ij}_{kl} = \begin{cases} 1 & \text{if } k = i, l = j, \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

In such matrix, only one pipeline $i$ generates a non-null incremental cost for a certain demand $j$. A matrix of incremental costs $A^C$ can be expressed as the linear combination of $n \times K$ unit cost matrices:

$$A^C = \sum_{i \in N} \sum_{j \leq K} A^C_{ij} I^{ij}. \tag{6.2}$$

Alternatively, the $(n, q_n)$ sub-matrix $A^{C,q_n}$ collects all the incremental costs generated by demands lower or equal than $q_n$. Such sub-matrix represents a situation where the highest effective demand of the consumers in $N$ is $q_n$. Similarly, a sub-matrix $I^{ij,q_n}$ is a $(n, q_n)$ matrix where only one pipeline $i \in N$ generates a non-null incremental cost for a certain demand $j \leq q_n$. In the following, we are only interested in the incremental costs contained within $A^{C,q_n}$. Thus, we are going to focus on the sub-matrix $A^{C,q_n}$ instead of the whole matrix $A^C$. For the sake of clarity, we keep the notation $A^C$ to refer to $A^{C,q_n}$. Similarly, for each $i \in N$ and each $j \leq q_n$, we keep the notation $I^{ij}$ to refer to $I^{ij,q_n}$.

### 6.2.2 The gas distribution problem

We propose an approach to compute the total cost of operating a gas distribution network from $N$, $q$, $D$ and $C$. Alternative approaches are discussed at the end of this chapter.

Assume that each pipeline is always designed to meet the highest effective demand of its downstream consumers. Then, for each $i \in N$, the cost of operating pipeline $i$ is given by

$$C(i, \overline{q}_i), \quad \text{where} \quad \overline{q}_i = \max_{k \in \hat{D}(i) \cup \{i\}} q_k. \tag{6.3}$$

Equivalently, observe that

$$\forall i \in N, \quad C(i, \overline{q}_i) = \sum_{j \in \{1,\dots,\overline{q}_i\}} A^C_{ij}.$$

For each $i \in N$, one may have $\overline{q}_i > q_i$. This is due to the fact that the highest effective demand of the downstream consumers of $i$ may be strictly greater than the effective demand of $i$. We compute the **total cost** of operating the gas distribution network as the sum of the costs of all the pipelines designed as in (6.3), which is given by

$$\sum_{i \in N} C(i, \overline{q}_i), \quad \text{or equivalently} \quad \sum_{i \in N} \sum_{j \in \{1,\dots,\overline{q}_i\}} A^C_{ij}. \tag{6.4}$$

**Definition 33 (The cost sharing problem of gas distribution).** *A cost sharing problem of gas distribution (a gas distribution problem for short) is denoted by $(N, q, C, D)$, or $(q, A^C)$ for short, since $N$ and $D$ are fixed and $C$ and $A^C$ are equivalent. The problem is to determine a way to share the total cost among consumers based on their demands and location on the distribution network. The set of gas distribution problems is denoted by* GDP.

We address gas distribution problems by defining **cost sharing rules** (rules for short). A **rule** on GDP is a map

$$f : \text{GDP} \to \bigcup_{q \leq (K,\dots,K)} \mathbb{R}_+^{\sum_{i \in N} q_i},$$

that describes how much each consumer is charged for each of its demand units. Take any $(q, A^C) \in \text{GDP}$. A rule $f$ applied on $(q, A^C) \in \text{GDP}$ assigns a non-negative real number $f_{ij}(q, A^C) \in \mathbb{R}_+$ to each demand $j \in \{1, \dots, q_i\}$ of each consumer $i \in N$. This number represents the amount that a consumer must pay for its demand $j$, assuming that it already pays a certain amount for its demand $j-1$, its demand $j-2$, and so on. A rule is particularly useful if the gas network operator wants to derive a rate schedule from it. A rule satisfies the **budged balanced** condition, which states that a rule recovers the total cost of operating the gas distribution network. Formally, the budget balanced condition is given by

$$\begin{aligned} \forall (q, A^C) \in \text{GDP}, \quad \sum_{i \in N} \sum_{j \in \{1,\dots,q_i\}} f_{ij}(q, A^C) &= \sum_{i \in N} C(i, \overline{q}_i) \\ &= \sum_{i \in N} \sum_{j \in \{1,\dots,\overline{q}_i\}} A^C_{ij}. \end{aligned} \tag{6.5}$$

**Example 15.** *Consider the set of consumers $N = \{a, b, c, d\}$, the upper bound for effective demands is $K = 3$, and the source in gas is $\mathcal{S}$. In this example, we consider the following tree graph $D$ representing a gas distribution network.*

$$D(\mathcal{S}) = \{a\}, D(a) = \{b, c\}, D(c) = \{d\}, D(b) = D(d) = \emptyset.$$

*Consider the profile of effective demands $q = (1, 3, 1, 2)$. We introduce the lengths of the pipelines, which is one of the possible exogenous features of a gas distribution network. The profile $L = (1, 1, 2, 1)$ collects the length of each pipeline. The network, the profile of effective demands and the profile of lengths are described in Figure 1.*

*Consider the cost function*

$$\begin{aligned} C : \{a, b, c, d\} \times \{1, 2, 3\} &\to \mathbb{R}_+ \\ (i, j) &\mapsto L_i + j, \end{aligned}$$

*which specifies the cost of each pipeline when designed to meet a certain demand. The cost of a pipeline is computed as the sum of its length and the highest downstream demand it needs to meet. This cost function is obviously overly simplified but it is consistent with the facts since the cost of a pipeline is increasing with respect to both parameters.*

*Consider pipeline $c$. This pipeline is designed to meet its downstream effective demand. In this case, the downstream effective demand of pipeline $c$ is 2 since $d \in D(c)$, $q_d = 2$ and $\max_{k \in \hat{D}(c) \cup c} q_k = q_d$. Thus, this pipeline costs*

$$\begin{aligned} C(c, q_d) &= L_c + q_d \\ &= 4. \end{aligned}$$

*Applying the same reasoning to each pipeline, we obtain the cost of operating each pipeline. These costs are represented in Figure 2. The total cost of operating this gas distribution network is given by*

$$\begin{aligned} \sum_{i \in N} C(i, \overline{q}_i) &= C(a, q_b) + C(b, q_b) + C(c, q_d) + C(d, q_d) \\ &= 15. \end{aligned}$$

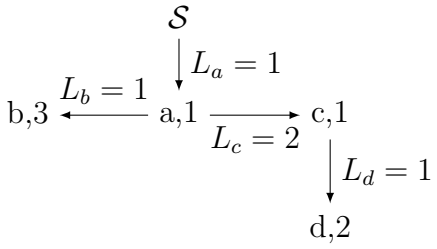



Figure 6.1

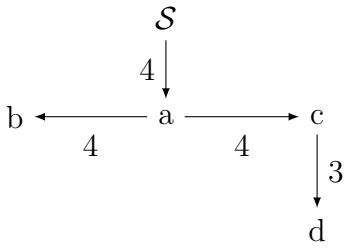



Figure 6.2

## 6.3 Rules and their axiomatic characterizations

This section is devoted to the study of the Connection rule, the Uniform rule and the Mixed rules. We provide an axiomatic characterization for each one of these rules.

### 6.3.1 The Connection rule

We define a rule on gas distribution problems that satisfies the Connection principle and the Independence of higher demands principle. This rule is called the Connection rule, it ensures that each consumer pays a fair share of the portion of the network that connects it to the source. Formally, for each pipeline $k$ and each demand $j$ such that some downstream consumers of pipeline $k$ are in $Q(j)$, the Connection rule shares the incremental cost $A^C_{kj}$ equally among the downstream consumers of pipeline $k$ that are in $Q(j)$.

**Definition 34 (Connection rule).** *The Connection rule $\Psi$ is defined, for each $(q, A^C) \in \text{GDP}$, as*

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Psi_{ij}(q, A^C) = \sum_{k \in \hat{D}^{-1}(i) \cup \{i\}} \frac{A^C_{kj}}{|(\hat{D}(k) \cup \{k\}) \cap Q(j)|}. \tag{6.6}$$

**Proposition 29.** *The Connection rule satisfies the budget balanced condition.*

*Proof.* Take any $(q, A^C) \in \text{GDP}$ and any incremental cost $A^C_{kj}$, where $k \in N$ and $j \leq q_n$. By Definition 34, $A^C_{kj}$ is taken into account in the computation of the Connection rule if and only if the set of consumers $(\hat{D}(k) \cup \{k\}) \cap Q(j)$ is non-empty. The Connection rule divides $A^C_{kj}$ equally among the consumers in $(\hat{D}(k) \cup \{k\}) \cap Q(j)$ for their level of demand $j$, which implies that the rule recovers $A^C_{kj}$ entirely.

The set $A^C(\Psi)$ of incremental costs taken into account in the computation of the Connection rule can be defined as

$$A^C(\Psi) = \{A^C_{kj} : (\hat{D}(k) \cup \{k\}) \cap Q(j) \neq \emptyset\}.$$

Alternatively, the set $A^C(Tot)$ of incremental costs taken into account in the computation of the total cost (6.4) can be defined as

$$A^C(Tot) = \{A^C_{kj} : j \leq \overline{q}_k\}.$$

To prove Proposition 29, let us show that

$$A^C(\Psi) = A^C(Tot). \tag{6.7}$$

First, let us show that $A^C(\Psi) \subseteq A^C(Tot)$. Take any $A^C_{kj} \in A^C(\Psi)$. If $(\hat{D}(k) \cup \{k\}) \cap Q(j) \neq \emptyset$, then there is at least one downstream consumer of pipeline $k$ with

an effective demand of at least $j$. Put differently, this means that $\overline{q}_k \geq j$. This shows that $A^C(\Psi) \subseteq A^C(Tot)$.

Next, let us show that $A^C(Tot) \subseteq A^C(\Psi)$. Take any $A^C_{kj} \in A^C(Tot)$. By definition of $A^C(Tot)$, it holds that $j \leq \overline{q}_k$. Therefore, there is at least one downstream consumer of pipeline $k$ with an effective demand of at least $j$. It follows that $(\hat{D}(k) \cup \{k\}) \cap Q(j) \neq \emptyset$. This shows that $A^C(Tot) \subseteq A^C(\Psi)$.

Since (6.7) holds, and since the Connection rule recovers the entirety of the incremental costs in $A^C(\Psi)$, it holds that

$$\begin{aligned}\sum_{i\in N}\sum_{j\in\{1,\dots,q_i\}} \Psi_{ij}(q, A^C) &= \sum_{A\in A^C(\Psi)} A \\ &= \sum_{A\in A^C(Tot)} A \\ &= \sum_{k\in N}\sum_{j\in\{1,\dots,\overline{q}_k\}} A^C_{kj}.\end{aligned}$$

The proof of Proposition 29 is complete. ■

The Connection rule shares some similarities with the Downstream equal sharing solution introduced by Dong *et al.* (2012) for polluted river problems. Both solutions share the costs generated by an arc of the graph among downstream consumers. The difference between the two solutions lies in the fact that gas distribution problems consider situations in which agents can have several demands, which is not the case in the model studied by Dong *et al.* (2012). We provide an axiomatic characterization of the Connection rule. Let $f$ be a rule on GDP. The costs of operating a distribution network can be divided into two categories called: the construction costs and the maintenance costs. The first axiom, Weak linearity, ensures that there is no difference whether the consumers share the two costs separately or together. If the costs are expressed in USD and we want the rule to be expressed in EUR, then Weak linearity ensures that there is no difference between converting the currencies before or after the application of the rule.

**Axiom 57 (Weak linearity).** *For each* $(q, A^C), (q, A^{C'}) \in \text{GDP}$ *and each* $\beta \in \mathbb{R}_+$,

$$\forall i \in N, j \leq q_i, \quad f_{ij}(q, A^C + \beta A^{C'}) = f_{ij}(q, A^C) + \beta f_{ij}(q, A^{C'}).$$

The next axiom is a formal expression of the Independence of higher demands principle. It ensures that the amount charged to a consumer for a certain demand is independent of any higher demand. This axiom compares two situations. The first situation is a gas distribution problem. The second situation is another gas distribution problem similar to the first one, except that it is no longer possible for consumers to demand more than a certain quantity, let us say $l$. This axiom requires that the amount charged to any consumer for any demand below $l$ remains the same

in both situations. Clearly, this axiom is an adaptation of Independence of higher activity level (see Axiom 44) to the framework of gas distribution problems.

**Axiom 58 (Independence of higher demands).** *For each* $(q, A^C) \in$ GDP *and each* $l \leq q_n$*,*

$$\forall i \in Q(l), j \leq l, \quad f_{ij}(q, A^C) = f_{ij}((l \wedge q_k)_{k\in N}, A^C).$$

The next result states that if a rule satisfies Independence of higher demands, then the sum of all cost shares charged to consumers in $Q(j)$ for their demand $j$ is equal to the sum of all incremental costs, generated by the pipelines, for a demand $j$, that connect the consumers in $Q(j)$ to the source.

**Proposition 30.** *If a rule* $f$ *on* GDP *satisfies Independence of higher demands then, for each* $(q, A^C) \in$ GDP*,*

$$\forall l \leq q_n, \quad \sum_{k\in Q(l)} f_{kl}(q, A^C) = \sum_{i\in \hat{D}^{-1}(Q(l))\cup Q(l)} A^C_{il}.$$

*Proof.* Consider a rule $f$ on GDP that satisfies Independence of higher demands. By the Budget Balanced condition (see (6.5)), for each $(q, A^C) \in$ GDP,

$$\begin{aligned} &\sum_{k\in N}\sum_{j\in\{1,\dots,q_k\}} f_{kj}(q, A^C) = \sum_{i\in N}\sum_{j\in\{1,\dots,\overline{q}_i\}} A^C_{ij} \\ \iff \quad &\sum_{j\in\{1,\dots,q_n\}}\sum_{k\in Q(j)} f_{kj}(q, A^C) = \sum_{i\in N}\sum_{j\in\{1,\dots,\overline{q}_i\}} A^C_{ij}. \end{aligned} \tag{6.8}$$

Recall that, for each $i \in N$, $\overline{q}_i = \max_{h\in\hat{D}(i)\cup\{i\}} q_h$. Observe that, for each $l \leq q_n$,

$$\overline{((l \wedge q)_{k\in N})}_i = l \wedge \overline{q}_i. \tag{6.9}$$

Indeed, for each $l \leq q_n$ and $i \in N$,

$$\begin{aligned} \overline{((l \wedge q)_{k\in N})}_i &= \max_{h\in\hat{D}(i)\cup\{i\}} ((l \wedge q)_{k\in N})_h \\ &= \max_{h\in\hat{D}(i)\cup\{i\}} (l \wedge q_h) \\ &= \begin{cases} l & \text{if } \max_{h\in\hat{D}(i)\cup\{i\}} q_h \geq l \\ \max_{h\in\hat{D}(i)\cup\{i\}} q_h & \text{otherwise.} \end{cases} \\ &= l \wedge \overline{q}_i. \end{aligned}$$

By Independence of higher demands (IHD), (6.8) and (6.9), for each $l \leq q_n$,

$$\begin{aligned} \sum_{j\in\{1,\dots,l\}}\sum_{k\in Q(j)} f_{kj}(q, A^C) &\overset{(IHD)}{=} \sum_{j\in\{1,\dots,l\}}\sum_{i\in Q(j)} f_{ij}((l \wedge q_k)_{k\in N}, A^C) \\ &\overset{(6.8)}{=} \sum_{i\in N}\sum_{j\in\{1,\dots,\overline{((l\wedge q)_{k\in N})}_i\}} A^C_{ij} \\ &\overset{(6.9)}{=} \sum_{i\in N}\sum_{j\in\{1,\dots,l\wedge\overline{q}_i\}} A^C_{ij}. \end{aligned}$$

Similarly, it holds that

$$\sum_{j\in\{1,\dots,l-1\}}\sum_{k\in Q(j)} f_{kj}(q,A^C) = \sum_{j\in\{1,\dots,l-1\}}\sum_{i\in Q(j)} f_{ij}(((l-1)\wedge q_k)_{k\in N},A^C)$$
$$= \sum_{i\in N}\sum_{j\in\{1,\dots,(l-1)\wedge\overline{q}_i\}} A^C_{ij}.$$

Therefore, for each $l \leq q_n$,

$$\sum_{k\in Q(l)} f_{kl}(q,A^C) = \sum_{j\in\{1,\dots,l\}}\sum_{i\in Q(j)} f_{ij}(q,A^C) - \sum_{j\in\{1,\dots,l-1\}}\sum_{i\in Q(j)} f_{ij}(q,A^C)$$
$$= \sum_{i\in N}\sum_{j\in\{1,\dots,l\wedge\overline{q}_i\}} A^C_{ij} - \sum_{i\in N}\sum_{j\in\{1,\dots,(l-1)\wedge\overline{q}_i\}} A^C_{ij}$$
$$= \sum_{i\in N}\left[\sum_{j\in\{1,\dots,l\wedge\overline{q}_i\}} A^C_{ij} - \sum_{j\in\{1,\dots,(l-1)\wedge\overline{q}_i\}} A^C_{ij}\right].$$

To conclude this proof, observe that, for each $i \in N$ such that $l\wedge\overline{q}_i = (l-1)\wedge\overline{q}_i$, i.e., $\overline{q}_i \leq l-1$,

$$\sum_{j\in\{1,\dots,l\wedge\overline{q}_i\}} A^C_{ij} - \sum_{j\in\{1,\dots,(l-1)\wedge\overline{q}_i\}} A^C_{ij} = 0.$$

On the contrary, for each $i \in N$ such that $l\wedge\overline{q}_i > (l-1)\wedge\overline{q}_i$, i.e., $\overline{q}_i \geq l$),

$$\sum_{j\in\{1,\dots,l\wedge\overline{q}_i\}} A^C_{ij} - \sum_{j\in\{1,\dots,(l-1)\wedge\overline{q}_i\}} A^C_{ij} = A^C_{il}.$$

Observe that $\overline{q}_i \geq l$ if and only if $i \in \hat{D}^{-1}(Q(l)) \cup Q(j)$. Therefore, for each $l \leq q_n$, we obtain the desired result

$$\sum_{k\in Q(l)} f_{kl}(q,A^C) = \sum_{i\in\hat{D}^{-1}(Q(l))\cup Q(l)} A^C_{il}.$$

■

We say that a pipeline is **irrelevant** to a consumer if it is not a pipeline that helps to connect this consumer to the source. The next axiom is in line with the Connection principle. It ensures that a consumer is not charged for the costs generated by irrelevant pipelines. Formally, the axiom is formulated using unit cost matrices (see (6.1)). Take any $(q, I^{ij}) \in$ GDP. Recall that the unit cost matrix $I^{ij}$ possesses a unique non-null incremental cost. The next axiom states that the consumers for whom this incremental cost is irrelevant should pay nothing in $(q, I^{ij})$.

**Axiom 59 (Independence of irrelevant costs).** *For each* $(q, I^{ij}) \in$ GDP*,*

$$\forall h \in Q(j), h \notin (\hat{D}(i)\cup\{i\}), \quad f_{hj}(q,I^{ij}) = 0.$$

The next axiom ensures that the rule treats each downstream consumer equally regarding a given upstream cost. Take any $(q, I^{ij}) \in \text{GDP}$. This axiom states that any two distinct consumers located downstream of pipeline $i$ are charged the same amount for their demand $j$ in $(q, I^{ij})$.

**Axiom 60 (Downstream symmetry).** *For each* $(q, I^{ij}) \in \text{GDP}$,

$$\forall h, h' \in [\hat{D}(i) \cup \{i\}] \cap Q(j), \quad f_{hj}(q, I^{ij}) = f_{h'j}(q, I^{ij}).$$

We have the material to characterize the Connection rule.

**Theorem 28.** *A rule* $f$ *on* GDP *satisfies Weak linearity, Independence of higher demands, Independence of irrelevant costs and Downstream symmetry if and only if* $f = \Psi$.

*Proof.* First, we show that $\Psi$ satisfies the axioms of the statement of Theorem 28.

**Weak linearity:** The proof is straightforward and so is omitted.

**Independence of higher demands:** The proof follows directly from the definition of the Connection rule. The cost share charged to a consumer $i$ for a demand $j$ is computed using only the incremental costs generated by the demand $j$ of the consumers. Therefore, we directly have that $\Psi$ satisfies the axiom.

**Independence of irrelevant costs:** The proof follows directly from the definition of the Connection rule. The cost share charged to a consumer $i$ for a demand $j$ is computed using only the incremental costs generated by the pipelines that connect $i$ to the source. Therefore, the incremental costs generated by the pipelines irrelevant to $i$ are not taken into account by the Connection rule in the computation of the cost share of $i$ for its demand $j$.

**Downstream symmetry:** Take any $(q, I^{ij}) \in \text{GDP}$. For each $h, h' \in [\hat{D}(i) \cup \{i\}] \cap Q(j)$, it holds that

$$\begin{aligned}
\Psi_{hj}(q, I^{ij}) &= \sum_{k \in \hat{D}^{-1}(h) \cup h} \frac{I^{ij}_{kj}}{|(\hat{D}(k) \cup \{k\}) \cap Q(j)|} \\
&= \frac{1}{|(\hat{D}(k) \cup \{k\}) \cap Q(j)|}, \\
\text{and} \quad \Psi_{h'j}(q, I^{ij}) &= \sum_{k \in \hat{D}^{-1}(h') \cup l} \frac{I^{ij}_{kj}}{|(\hat{D}(k) \cup \{k\}) \cap Q(j)|} \\
&= \frac{1}{|(\hat{D}(k) \cup \{k\}) \cap Q(j)|},
\end{aligned}$$

which shows that $\Psi$ satisfies the axiom.

Next, we show that the Connection rule is the only rule on GDP that satisfies all the axioms of the statement of Theorem 28. Take any $(q, A^C) \in \text{GDP}$. Let $f$ be a rule that satisfies all the axioms of the statement of Theorem 28 on GDP. Let us show that $f(q, A^C)$ is uniquely determined. To do so, we focus on unit cost matrices, since each problem $(q, A^C) \in \text{GDP}$ can be expressed as a non-negative linear combination of unit cost matrices (see (6.2)), $f$ satisfies Weak linearity and each incremental cost is non-negative.

Take any $I^{ij}$ such that $j \leq q_n$ and $i \notin \hat{D}^{-1}(Q(j)) \cup Q(j)$. By the budget balanced condition, it holds that

$$\sum_{k \in N} \sum_{l \in \{1,\dots,q_k\}} f_{kl}(q, I^{ij}) = \sum_{k \in N} \sum_{l \in \{1,\dots,\overline{q}_k\}} I^{ij}_{kl}. \tag{6.10}$$

By (6.1), it holds that $I^{ij}_{kl} \neq 0$ if $k = i$ and $l = j$. However, since

$$i \notin \hat{D}^{-1}(Q(j)) \cup Q(j),$$

it holds that

$$\overline{q}_i = \max_{h \in \hat{D}(i) \cup \{i\}} q_h < j.$$

Therefore, there is no $k \in N$ and $l \in \{1, \dots, \overline{q}_k\}$ such that $I^{ij}_{kl} \neq 0$. Therefore, (6.10) becomes

$$\sum_{k \in N} \sum_{l \in \{1,\dots,q_k\}} f_{kl}(q, I^{ij}) = 0.$$

Since the cost share charged to each demand of each consumer is always non-negative, it follows that, for each $k \in N$ and each $l \in \{1, \dots, q_k\}$,

$$f_{kl}(q, I^{ij}) = 0.$$

Therefore, $f(q, I^{ij})$ is uniquely determined.

Now, consider $I^{ij}$ such that $j \leq q_n$ and $i \in \hat{D}^{-1}(Q(j)) \cup Q(j)$. By Independence of irrelevant costs, for each $k \in Q(j)$ such that $k \notin \hat{D}(i) \cup \{i\}$,

$$f_{kj}(q, I^{ij}) = 0. \tag{6.11}$$

By Downstream symmetry, for each $k, h \in Q(j)$ such that $k, h \in \hat{D}(i) \cup \{i\}$,

$$f_{kj}(q, I^{ij}) = f_{hj}(q, I^{ij}). \tag{6.12}$$

By Proposition 30,

$$\sum_{k\in Q(j)} f_{kj}(q, I^{ij}) = \sum_{k\in \hat{D}^{-1}(Q(j))\cup Q(j)} I^{ij}_{kj}$$
$$= 1. \tag{6.13}$$

Combining (6.11), (6.12) and (6.13), for each $k \in (\hat{D}(i) \cup \{i\}) \cap Q(j)$, we obtain

$$f_{kj}(q, I^{ij}) = \frac{1}{|(\hat{D}(i) \cup \{i\}) \cap Q(j)|}. \tag{6.14}$$

By the budget balanced condition,

$$\sum_{k\in N} \sum_{l\in\{1,\ldots,q_k\}} f_{kl}(q, I^{ij}) = 1. \tag{6.15}$$

Combining (6.13) and (6.15), we obtain

$$\sum_{k\in N} \sum_{\substack{l\in\{1,\ldots,q_k\}\\ l\neq j}} f_{kl}(q, I^{ij}) = \sum_{k\in N} \sum_{l\in\{1,\ldots,q_k\}} f_{kl}(q, I^{ij}) - \sum_{k\in Q(j)} f_{kj}(q, I^{ij})$$
$$= 1 - 1$$
$$= 0.$$

Since the cost share charged to each demand of each consumer is always non-negative, for each $l \neq j$ and each $k \in Q(l)$, it holds that

$$f_{kl}(q, I^{ij}) = 0. \tag{6.16}$$

Combining (6.11), (6.14) and (6.16), for each $l \leq q_k$ and $k \in Q(l)$, we obtain

$$f_{kl}(q, I^{ij}) = \begin{cases} \dfrac{1}{|(\hat{D}(i) \cup \{i\}) \cap Q(j)|} & \text{if } k \in \hat{D}(i) \cup \{i\} \text{ and } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $f(q, I^{ij})$ is uniquely determined. By (6.2) and by Weak linearity we conclude that $f(q, A^C)$ is uniquely determined. The proof of Theorem 28 is complete. ■

The four axioms of the statement of Theorem 28 are logically independent, as shown by the following alternative solutions.

- Take any $(q, A^C) \in$ GDP and fix any arbitrary integer $\beta_{i'} \in \{1, 2\}$ for each $i' \in N$. The rule $f^\beta$ given by

$$\forall j \leq q_n, \forall i \in Q(j),$$
$$f^\beta_{ij}(q, A^C) = \begin{cases} \Psi_{ij}(q, A^C) & \text{if } |\{A^C_{kj} \neq 0, k \in Q(j)\}| \leq 1, \\ \displaystyle\sum_{k\in \hat{D}^{-1}(i)\cup\{i\}} \frac{\beta_i}{\displaystyle\sum_{i'\in Q(j)} \beta_{i'}} \times A^C_{kj} & \text{otherwise,} \end{cases}$$

  satisfies all the axioms except Weak linearity.

- The rule $f$ given, for each $(q, A^C) \in \text{GDP}$, by

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}(q, A^C) = \sum_{k \in \hat{D}^{-1}(i) \cup \{i\}} \sum_{l \geq j} \frac{A^C_{kl}}{|(\hat{D}(k) \cup \{k\})| \times l}.$$

  satisfies all the axioms except Independence of higher demands.

- The Uniform rule (introduced in the next section, see Definition 35) satisfies all the axioms except Independence of irrelevant costs.

- Take any $(q, A^C) \in \text{GDP}$ and fix any arbitrary integer $\beta_{i'} \in \{1, 2\}$ for each $i' \in N$. The rule $f^\beta$ given by

$$\begin{aligned} &\forall j \leq q_n, \forall i \in Q(j), \\ &f^\beta_{ij}(q, A^C) = \sum_{k \in \hat{D}^{-1}(i) \cup \{i\}} \frac{\beta_i}{\sum_{x \in Q(j)} \beta_x} \times A^C_{kj}, \end{aligned}$$

  satisfies all the axioms except Downstream symmetry.

### 6.3.2 The Uniform rule

We define a rule on gas distribution problems that follows the Independence of higher demands principle and the Uniformity principle. This rule is called the Uniform rule, it ensures that two consumers are charged with the same amount for the same demand, regardless of their location on the distribution network. Formally, take any $j \leq q_n$. The Uniform rule shares each incremental cost $A^C_{kj}$, where $k \in \hat{D}^{-1}(Q(j)) \cup Q(j)$, equally among each consumer in $Q(j)$, even if some of the consumers in $Q(j)$ are not downstream consumers of pipeline $k$.

**Definition 35 (Uniform rule).** *The Uniform rule $\Omega$ is defined, for each $(q, A^C) \in$ GDP, as*

$$\forall j \leq q_n, \forall i \in Q(j), \quad \Omega_{ij}(q, A^C) = \frac{1}{|Q(j)|} \sum_{k \in \hat{D}^{-1}(Q(j)) \cup Q(j)} A^C_{kj}. \tag{6.17}$$

**Proposition 31.** *The Uniform rule satisfies the budget balanced condition.*

*Proof.* The proof is similar to the proof of Proposition 29, and so is omitted.

■

Observe that, for each $j \in \{0, \dots, q_n\}$ and each $i \in Q(j)$, $\Omega_{ij}(q, A^C)$ does not depend on the position of $i$ in the gas distribution network. Observe that the Uniform rule satisfies Weak linearity and Downstream symmetry.

We introduce a new axiom for rules on gas distribution problems. Let $f$ be a rule on GDP. Assume that one or several pipelines generate additional costs due to exogenous reasons (incident, natural disaster, etc), which leads to an increase of the incremental costs. This axiom ensures that additional costs do not increase the inequalities between the cost shares of the consumers. Formally, consider any demand $j$. This axiom compares the difference in cost share between two consumers in $Q(j)$ before and after the costs increase. In particular, it compares the difference between the highest cost share charged to a consumer for its demand $j$ with the lowest cost share charged to a consumer for its demand $j$. This axiom states that if the incremental costs increase, then this difference does not increase.

**Axiom 61 (Non-increasing inequalities).** *For each* $(q, A^C), (q, A^{C'}) \in$ GDP *such that* $A^{C'}_{ij} \geq A^C_{ij}$, *each* $i \in N$ *and each* $j \leq q_n$,

$$\begin{aligned} \forall j \in \{1, \dots, q_n\}, \quad & \max_{i \in Q(j)} f_{ij}(q, A^{C'}) - \min_{i \in Q(j)} f_{ij}(q, A^{C'}) \\ & \leq \max_{i \in Q(j)} f_{ij}(q, A^C) - \min_{i \in Q(j)} f_{ij}(q, A^C). \end{aligned}$$

We have the material to characterize the Uniform rule.

**Theorem 29.** *A rule* $f$ *on* GDP *satisfies Independence of higher demands and Non-increasing inequalities if and only if* $f = \Omega$.

*Proof.* First, we show that $\Omega$ satisfies the axioms of the statement of Theorem 29.

**Independence of higher demands:** The proof follows directly from the definition of the Uniform rule. The cost share charged to a consumer $i$ for a demand $j$ is computed using only the incremental costs generated by the demand $j$ of the consumers. Therefore, we directly have that $\Omega$ satisfies the axiom.

**Non-increasing inequalities:** The proof is straightforward since each consumer is charged with the same amount for a given demand.

Next, we show that the Uniform rule is the only rule on GDP that satisfies all the axioms of the statement of Theorem 29. Consider $(q, A^C) \in$ GDP and a rule $f$ that satisfies all the axioms of the statement of Theorem 29 on GDP. Let us show that $f(q, A^C)$ is uniquely determined. By Proposition 30, for each $j \leq q_n$,

$$\sum_{i \in Q(j)} f_{ij}(q, A^C) = \sum_{i \in \hat{D}^{-1}(Q(j)) \cup Q(j)} A^C_{ij}. \tag{6.18}$$

Consider the matrix of incremental costs $0^C$ in which each incremental cost is null. By definition of a rule, each cost share is non-negative. Thus, by (6.18),

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}(q, 0^C) = 0.$$

Observe that the difference between any two cost shares is always null. Therefore, by Non-increasing inequalities and the non-negativity of the cost shares, we obtain

$$\forall j \leq q_n, \forall i, i' \in Q(j), \quad f_{ij}(q, A^C) = f_{i'j}(q, A^C). \tag{6.19}$$

Combining (6.18) and (6.19), $f(q, A^C)$ is uniquely determined, which concludes the proof of Theorem 29. ■

The two axioms of the statement of Theorem 29 are logically independent, as shown by the following alternative solutions.

- The rule $f$ given, for each $(q, A^C) \in \text{GDP}$, by

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}(q, A^C) = \frac{1}{\sum_{k \in N} q_k} \sum_{i \in N} \sum_{j \in \{1, \ldots, \overline{q}_i\}} A^C_{ij}$$

  satisfies Non-increasing inequalities but does not satisfy Independence of higher demands.

- The Connection rule satisfies Independence of higher demands but does not satisfy Non-increasing inequalities.

### 6.3.3 The Mixed rules

Observe that the Uniform rule does not satisfy Independence of irrelevant costs. On the other hand, the Connection rule does not satisfy Non-increasing inequalities. This testifies that the Uniformity principle and the Connection principle are clearly incompatible. However, both can be highly desirable in the context of gas distribution problems. For this reason, we propose a trade-off between the two principles by compromising between the Connection rule and the Uniform rule. We define a class of rules on gas distribution problems that follow the Independence of higher demands principle and compromise between the Uniformity principle and the Connection principle. A rule in this class is called a Mixed rule and is computed as convex combinations of the Connection rule and the Uniform rule. Moreover, the Mixed rules allow for different compromises between the Uniformity principle and the Connection principle, depending on the level of demand.

**Definition 36 (Mixed rules).** *Let $\alpha = \{\alpha^j\}_{1 \leq j \leq K}$ be a parameter system such that $\alpha^j \in [0, 1]$ for each $1 \leq j \leq K$. The $\alpha$-Mixed rule $\mu^\alpha$ is defined, for each $(q, A^C) \in \text{GDP}$, as*

$$\forall j \leq q_n, \forall i \in Q(j), \quad \mu^\alpha_{ij}(q, A^C) = \alpha^j \Psi_{ij}(q, A^C) + (1 - \alpha^j)\Omega_{ij}(q, A^C).$$

An $\alpha$-Mixed rule operates convex combinations between the Connection rule and the Uniform rule. For each demand $j \in \{1, \ldots, q_n\}$, a consumer $i \in Q(j)$ is charged a cost share $\mu_{ij}^{\alpha}(q, A^C)$ lying between $\Psi_{ij}(q, A^C)$ and $\Omega_{ij}(q, A^C)$. This cost share is determined by the parameter $\alpha^j$. For instance, pretend that $\alpha^j$ is closer to 1. This leads to a cost share closer to $\Psi_{ij}(q, A^C)$. On the other hand, if $\alpha^j$ is closer to 0, then this cost share will be closer to $\Omega_{ij}(q, A^C)$.

**Proposition 32.** *The Mixed rules all satisfy the bugdet balanced condition.*

*Proof.* The proof follows directly from Proposition 29 and Proposition 31.

■

To characterize the Mixed rules, we introduce three new axioms. The first axiom describes how an irrelevant cost impacts the cost share of two distinct consumers. The second axiom describes how the cost share of a consumer is impacted by two distinct irrelevant costs. Finally, the last axiom describes the impact of a given cost on two consumers. This cost being irrelevant for one of them, and relevant for the other.

Let $f$ be a rule on GDP. The first axiom states that any two distinct consumers are equally impacted by irrelevant costs, without specifying the extent of this impact. Take any $(q, I^{ij}) \in \text{GDP}$. Recall that a unit cost matrix $I^{ij}$ possesses a unique non-null incremental cost. The next axiom states that any two distinct consumers for whom this incremental cost is irrelevant should pay the same amount in $(q, I^{ij})$.

**Axiom 62 (Equal impact of irrelevant costs).** *For each* $(q, I^{ij}) \in \text{GDP}$*,*

$$\forall h, h' \in Q(j), h, h' \notin \hat{D}(i) \cup \{i\}, \quad f_{hj}(q, I^{ij}) = f_{h'j}(q, I^{ij}).$$

Observe that both the Connection rule and the Uniform rule satisfy Equal impact of irrelevant costs. Moreover, Independence of irrelevant costs implies Equal impact of irrelevant costs.

$$\text{Indep of irrelevant costs} \implies \text{Equal impact of irrelevant costs}$$

The next axiom states that no matter where an irrelevant cost is located, the impact of that cost on a consumer, for whom that cost is irrelevant, remains the same. Take any two distinct problems $(q, I^{ij}), (q, I^{i'j}) \in \text{GDP}$. Assume that each of the two unit cost matrices features a unique non-null incremental cost that can be considered irrelevant for a given consumer. The axiom states that the consumer for whom these incremental costs are not relevant should pay the same amount in both problems.

**Axiom 63 (Location independence of irrelevant costs).** *For each* $(q, I^{ij}), (q, I^{i'j}) \in \text{GDP}$*,*

$$\forall h \in Q(j), h \notin \left[\hat{D}(i) \cup \{i\}\right] \cup \left[\hat{D}(i') \cup \{i'\}\right], \quad f_{hj}(q, I^{ij}) = f_{hj}(q, I^{i'j}).$$

Observe that both the Connection rule and the Uniform rule satisfy Location independence of irrelevant costs. Moreover, Independence of irrelevant costs implies Location independence of irrelevant costs.

$$\text{Indep of irrelevant costs} \implies \text{Location indep of irrelevant costs}$$

Similarly to Equal impact of irrelevant costs, the third axiom compares the cost share charged to the consumers on the basis of the consumer's geographical localization. Take any problem $(q, I^{ij}) \in \text{GDP}$. A unit cost matrix $I^{ij}$ possesses a unique non-null incremental cost. The axiom states that any consumer located downstream of the cost generating pipeline should not pay less than any other consumer for whom this cost is irrelevant.

**Axiom 64 (Fairness).** *For each* $(q, I^{ij}) \in \text{GDP}$*,*

$$\begin{aligned}&\forall k \in [\hat{D}(i) \cup \{i\}] \cap Q(j),\\ &\forall h \in Q(j), h \notin \hat{D}(i) \cup \{i\}, \quad f_{kj}(q, I^{ij}) \geq f_{hj}(q, I^{ij}).\end{aligned}$$

Observe that both the Connection rule and the Uniform rule satisfy Fairness. We have the material to characterize the Mixed rules.

**Theorem 30.** *A rule on* GDP *satisfies Weak linearity, Independence of higher demands, Equal impact of irrelevant costs, Location independence of irrelevant costs, Fairness and Downstream symmetry if and only if* $f = \mu^{\alpha}$*, for some parameter system* $\alpha$*.*

*Proof.* Take any $\alpha$ parameter system and consider the Mixed rule $\mu^{\alpha}$. By definition, and the fact that $\mu^{\alpha}$ is computed as a convex combination of $\Psi$ and $\Omega$, $\mu^{\alpha}$ satisfies all the axioms of the statement of Theorem 30.

It remains to show that the Mixed rules are the only rules that satisfy all the axioms of the statement of Theorem 30. Consider $(q, A^C) \in \text{GDP}$ and $f$ on GDP a rule that satisfies all the axioms of the statement of Theorem 30.

Take any $I^{ij}$ such that $j \leq q_n$ and $i \notin \hat{D}^{-1}(Q(j)) \cup Q(j)$. In the same way as the proof of Theorem 28, by the budget balanced condition,

$$\begin{aligned}\sum_{k \in N} \sum_{l \in \{1,\ldots,q_k\}} f_{kl}(q, I^{ij}) &= \sum_{k \in N} \sum_{l \in \{1,\ldots,\bar{q}_k\}} I^{ij}_{kl}\\ &= 0.\end{aligned}$$

Since the cost share charged to each demand of each consumer is always non-negative, it follows that, for each $k \in N$ and each $l \in \{1, \ldots, q_k\}$,

$$f_{kl}(q, I^{ij}) = 0.$$

Therefore, $f(q, I^{ij})$ is uniquely determined whenever $j \leq q_n$ and $i \notin \hat{D}^{-1}(Q(j)) \cup Q(j)$.

Now, take any $I^{ij}$ such that $j \leq q_n$ and $i \in \hat{D}^{-1}(Q(j)) \cup Q(j)$. By Equal impact of irrelevant costs, for each $h, h' \in Q(j)$ such that $h, h' \notin \hat{D}(i) \cup \{i\}$,

$$\begin{aligned} f_{hj}(q, I^{ij}) &= f_{h'j}(q, I^{ij}) \\ &= Y, \end{aligned} \tag{6.20}$$

for some $Y \geq 0$. By Fairness and Downstream symmetry, for each $h \in (\hat{D}(i) \cup \{i\}) \cap Q(j)$,

$$\begin{aligned} & f_{hj}(q, I^{ij}) \geq Y \\ \iff \quad & f_{hj}(q, I^{ij}) = Y + W, \end{aligned} \tag{6.21}$$

for some $W \geq 0$. By Independence of higher demands and Proposition 30,

$$\sum_{h \in Q(j)} f_{hj}(q, I^{ij}) = 1. \tag{6.22}$$

Observe that $Y \leq \frac{1}{Q(j)}$. By 6.21 and (6.22), it follows that

$$\begin{aligned} & |(\hat{D}(i) \cup \{i\}) \cap Q(j)|(Y + W) + |Q(j) \setminus ((\hat{D}(i) \cup \{i\}) \cap Q(j))|Y = 1 \\ \iff \quad & |(\hat{D}(i) \cup \{i\}) \cap Q(j)|W = 1 - |Q(j)|Y \\ \implies \quad & 0 \leq |(\hat{D}(i) \cup \{i\}) \cap Q(j)|W \leq 1 \\ \iff \quad & 0 \leq W \leq \frac{1}{|(\hat{D}(i) \cup \{i\}) \cap Q(j)|} \\ \iff \quad & W = \frac{\alpha^j}{|(\hat{D}(i) \cup \{i\}) \cap Q(j)|}, \end{aligned} \tag{6.23}$$

for some $0 \leq \alpha^j \leq 1$. Combining (6.20), (6.21),(6.22) and (6.23), we obtain

$$\begin{aligned} & |Q(j)|Y + \alpha^j = 1 \\ \iff \quad & Y = \frac{1 - \alpha^j}{|Q(j)|}. \end{aligned} \tag{6.24}$$

Combining (6.21) and (6.24), for each $h \in (\hat{D}(i) \cup \{i\}) \cap Q(j)$, we obtain

$$f_{hj}(q, I^{ij}) = \frac{1 - \alpha^j}{|Q(j)|} + \frac{\alpha^j}{|(\hat{D}(i) \cup \{i\}) \cap Q(j)|}. \tag{6.25}$$

By (6.22) and the budget balanced condition, for each $l \neq j$ and $h \in Q(l)$,

$$f_{hl}(q, I^{ij}) = 0. \tag{6.26}$$

Finally, combining (6.24),(6.25) and (6.26), we obtain

$$\forall l \leq q_n, h \in Q(l),$$
$$f_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ \dfrac{1-\alpha^j}{|Q(j)|} & \text{if } l = j \text{ and } h \notin \hat{D}(i) \cup \{i\}, \\ \dfrac{1-\alpha^j}{|Q(j)|} + \dfrac{\alpha^j}{|(\hat{D}(i) \cup \{i\}) \cap Q(j)|} & \text{if } l = j \text{ and } h \in \hat{D}(i) \cup \{i\}, \end{cases}$$

for some $0 \leq \alpha^j \leq 1$. However, observe that

$$\forall l \leq q_n, h \in Q(l),$$
$$\Psi_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ 0 & \text{if } l = j \text{ and } h \notin \hat{D}(i) \cup \{i\}, \\ \dfrac{1}{|(\hat{D}(i) \cup \{i\}) \cap Q(j)|} & \text{if } l = j \text{ and } h \in \hat{D}(i) \cup \{i\}. \end{cases}$$

and

$$\Omega_{hl}(q, I^{ij}) = \begin{cases} 0 & \text{if } l \neq j, \\ \dfrac{1}{|Q(j)|} & \text{if } l = j \text{ and } h \notin \hat{D}(i) \cup \{i\}, \\ \dfrac{1}{|Q(j)|} & \text{if } l = j \text{ and } h \in \hat{D}(i) \cup \{i\}. \end{cases}$$

Thus, for each $h \in N$ and $l \leq q_h$,

$$f_{hl}(q, I^{ij}) = \alpha^l \Psi_{hl}(q, I^{ij}) + (1-\alpha^l)\Omega_{hl}(q, I^{ij}), \tag{6.27}$$

for some parameter system $\alpha$ such that $\alpha^j$ is the one determined above. Observe that, for any $I^{i'j}$ such that $i' \in \hat{D}^{-1}(Q(j)) \cup Q(j)$ and $i' \neq i$, Equal impact of irrelevant costs and Location independence of irrelevant costs ensure that 6.21 still holds. It follows that (6.27) still holds with the same parameter system $\alpha$.

By Weak linearity, we conclude that $f(q, A^C) = \mu^\alpha(q, A^C)$. The proof of the theorem is complete. ■

The six axioms of the statement of Theorem 30 are logically independent, as shown by the following alternative solutions.

- The rule $f$ given, for each $(q, A^C) \in \text{GDP}$, by

$$\forall j \leq q_n, \forall i \in Q(j),$$
$$f_{ij}(q, A^C) = \begin{cases} \Omega_{ij}(q, A^C) & \text{if } \sum_{i \in N} \sum_{j \in \{1, \ldots, \overline{q}_i\}} A^C_{ij} \leq 10, \\ \Psi_{ij}(q, A^C) & \text{otherwise.} \end{cases}$$

satisfies all the axioms except Weak linearity.

- The rule $f$ given, for each $(q, A^C) \in \text{GDP}$, by

$$\forall j \leq q_n, \forall i \in Q(j), \quad f_{ij}(q, A^C) = \frac{1}{\sum_{k \in N} q_k} \sum_{i \in N} \sum_{j \in \{1,\dots,\bar{q}_i\}} A^C_{ij}$$

satisfies all the axioms except Independence of higher demands.

To properly explain the next two alternative solutions, we introduce the following claim. Take any $(q, A^C) \in \text{GDP}$. Denote by $A^C(Tot)$ the set of incremental costs taken into account in the computation of the total cost (6.4). Take any sequence $(f^{ij})_{ij:A^C_{ij} \in A^C(Tot)}$ of $|A^C(Tot)|$ rules on GDP. Then, the map $f$ defined as

$$f(q, A^C) = \sum_{A^C_{ij} \in A^C(Tot)} f^{ij}(q, I^{ij} A^C_{ij}) \tag{6.28}$$

is also a rule on GDP. Indeed, by the properties of the sum, it holds that $f$ assigns a positive cost share to each demand of each consumer. Moreover, by (6.4), $f$ satisfies the budget balanced condition.

In the next two alternative solutions, we define rules computed in a similar manner than (6.28). This will be useful to understand how the rule treats each incremental cost individually.

- Take any $(q, A^C) \in \text{GDP}$ and any $h \in N$. Let us focus on a given $(q, I^{k1} A^C_{k1})$. Recall that $Q(1) = N$. Define the cost share $\Psi^{-h}_{i1}(q, I^{k1} A^C_{k1})$ of a consumer $i \in N$ for its demand unit 1 by

$$\Psi^{-h}_{i1}(q, I^{k1} A^C_{k1}) = \begin{cases} \dfrac{A^C_{k1}}{N} & \text{if} \quad i \notin \hat{D}(k) \cup \{k\} \text{ and } i = h, \\ 0 & \text{if} \quad i \notin \hat{D}(k) \cup \{k\} \text{ and } i \neq h, \\ A^C_{k1} \dfrac{1 - \frac{1}{N}}{|\hat{D}(i) \cup \{i\}|} & \text{if} \quad i \in \hat{D}(k) \cup \{k\} \text{ and } h \notin \hat{D}(k) \cup \{k\}, \\ \Psi_{ij}(q, I^{k1} A^C_{k1}) & \text{otherwise.} \end{cases}$$

This cost share behaves just like the Connection rule, but it gives special treatment to $h$ by allocating it a non-null share for irrelevant costs. Indeed, for each $i \in N$, the cost share allocated by the Connection rule in $(q, I^{k1} A^C_{k1})$ is given by

$$\Psi_{i1}(q, I^{k1} A^C_{k1}) = \begin{cases} 0 & \text{if} \quad i \notin \hat{D}(k) \cup \{k\}, \\ \dfrac{A^C_{k1}}{\hat{D}(i) \cup \{i\}} & \text{if} \quad i \in \hat{D}(k) \cup \{k\}. \end{cases}$$

The main difference between the two costs shares $\Psi^{-h}_{i1}(q, I^{k1} A^C_{k1})$ and $\Psi_{i1}(q, I^{k1} A^C_{k1})$ is that the former allocates $\frac{A^C_{k1}}{N}$ to $h$ whenever $A^C_{k1}$ is an irrelevant cost for $h$, whereas the later allocates it a null share. In case $A^C_{k1}$ is a relevant cost for $h$, then the two cost shares coincide.

Observe that

$$\sum_{i\in N} \Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C) = A_{k1}^C. \tag{6.29}$$

Indeed, if $h \in \hat{D}(k) \cup \{k\}$, then $\Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C) = \Psi_{i1}(q, I^{k1}A_{k1}^C)$. By the fact that $\Psi$ satisfies Independence of higher demands and by Proposition 30, it follows that (6.29) holds. If $h \notin \hat{D}(k) \cup \{k\}$, then

$$\begin{aligned}\sum_{i\in N} \Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C) &= \frac{A_{k1}^C}{N} + |\hat{D}(i) \cup \{i\}|A_{k1}^C \frac{1-\frac{1}{N}}{|\hat{D}(i) \cup \{i\}|} \\ &= A_{k1}^C,\end{aligned}$$

which shows that (6.29) holds.

Using the rule $\Psi^{-h}$, let us define another rule denoted by $f$. For each demand level $j > 1$ and each consumer $i \in Q(j)$, $f_{ij}(q, A^C) = \Psi_{ij}(q, A^C)$. As for the demand level 1, $f$ is defined by

$$\forall i \in N, \quad f_{i1}(q, A^C) = \sum_{k\in N} \Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C).$$

We show that $f$ satisfies the budget balanced condition. By (6.29), it holds that

$$\begin{aligned}\sum_{i\in N} f_{i1}(q, A^C) &= \sum_{k\in N}\sum_{i\in N} \Psi_{i1}^{-h}(q, I^{k1}A_{k1}^C) \\ &= \sum_{k\in N} A_{k1}^C.\end{aligned}$$

This means that $f$ recovers all the incremental costs generated by the first demands of the consumers, and share them entirely among the first demand unit of the consumers. Since $f$ coincides with the Connection rule for any level of demand above 1, by definition of the Connection rule, we can conclude that $f$ satisfies the budget balanced condition. The rule $f$ satisfies all the axioms except Equal impact of irrelevant costs since consumer $h$ is getting a special treatment for its first level of demand.

- Take any $(q, A^C) \in \text{GDP}$. For each $k \in N$, fix any parameter $\alpha^k \in [0, 1]$. The rule $f^\alpha$ is defined, for each $(q, A^C) \in \text{GDP}$, as

$$\begin{aligned}&\forall j \le q_n, \forall i \in Q(j), \\ &f_{ij}^\alpha(q, A^C) \\ &= \sum_{k\in \hat{D}^{-1}(Q(j))\cup Q(j)} \Big[\alpha^k \Psi_{ij}(q, I^{kj}A_{kj}^C) + (1-\alpha^k)\Omega_{ij}(q, I^{kj}A_{kj}^C)\Big].\end{aligned}$$

This rule operates different compromises between the connection principle and the uniformity principle for each pipeline of the network. For instance, pipeline $k \in N$ may be attributed a parameter $\alpha^k = 1$, in which case the rule $f^\alpha$ will share the incremental costs generated by $k$ according to the connection principle. Alternatively, pipeline $k' \in N$ may be attributed a parameter $\alpha^{k'} = 0$, in which case the rule $f^\alpha$ will share the incremental costs generated by $k'$ according to the uniformity principle.

We show that $f^\alpha$ satisfies the budget balanced condition. Pick any $j \leq q_n$. By definition of the Connection rule and the Uniform rule, it holds that

$$\begin{aligned}
&\sum_{i \in Q(j)} f^\alpha_{ij}(q, A^C) \\
&= \sum_{k \in \hat{D}^{-1}(Q(j)) \cup Q(j)} \Big[\alpha^k \sum_{i \in Q(j)} \Psi_{ij}(q, I^{kj} A^C_{kj}) + (1 - \alpha^k) \sum_{i \in Q(j)} \Omega_{ij}(q, I^{kj} A^C_{kj})\Big] \\
&= \sum_{k \in \hat{D}^{-1}(Q(j)) \cup Q(j)} \Big[\alpha^k A^C_{kj} + (1 - \alpha^k) A^C_{kj}\Big] \\
&= \sum_{k \in \hat{D}^{-1}(Q(j)) \cup Q(j)} A^C_{kj}.
\end{aligned}$$

We obtain the desired result

$$\begin{aligned}
\sum_{j \leq q_n} \sum_{i \in Q(j)} f^\alpha_{ij}(q, A^C) &= \sum_{j \leq q_n} \sum_{k \in \hat{D}^{-1}(Q(j)) \cup Q(j)} A^C_{kj} \\
&= \sum_{k \in N} \sum_{j \leq \bar{q}_k} A^C_{kj}.
\end{aligned}$$

The rule $f^\alpha$ satisfies all the axioms except Location independence of irrelevant costs.

- The rule $f$ is defined, for each $(q, A^C) \in \text{GDP}$, as

$$\begin{aligned}
&\forall j \leq q_n, \forall i \in Q(j), \\
&f_{ij}(q, A^C) = \Big[\Big(\frac{1}{|Q(j)|^2} + \frac{1}{|Q(j)|}\Big) \sum_{k \in \hat{D}^{-1}(Q(j)) \cup Q(j)} A^C_{kj}\Big] - \frac{\Psi_{ij}(q, A^C)}{|Q(j)|}.
\end{aligned}$$

To leave no room for doubt, let us show that this rule satisfies the budget balanced condition. Take any $(q, A^C) \in \text{GDP}$ and any $j \leq q_n$. Since $\Psi$ satisfies

Independence of higher demands, by Proposition 30, it holds that

$$\begin{aligned}\sum_{i\in Q(j)} f_{ij}(q,A^C) =&|Q(j)|\Big[\Big(\frac{1}{|Q(j)|^2}+\frac{1}{|Q(j)|}\Big)\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} A^C_{kj}\Big]\\ &-\frac{1}{|Q(j)|}\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} A^C_{kj}\\ =&\frac{1}{|Q(j)|}\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} A^C_{kj}+\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} A^C_{kj}\\ &-\frac{1}{|Q(j)|}\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} A^C_{kj}\\ =&\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} A^C_{kj}.\end{aligned}$$

Therefore, we obtain the desired result

$$\begin{aligned}\sum_{j\le q_n}\sum_{i\in Q(j)} f_{ij}(q,A^C) &= \sum_{j\le q_n}\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} A^C_{kj}\\ &=\sum_{k\in N}\sum_{j\le\overline{q}_k} A^C_{kj}.\end{aligned}$$

This rule satisfies all the axioms except Fairness. Indeed, for any $(q,I^{ij})\in$ GDP such that $i\in Q(j)$, it holds that

$$\begin{aligned}f_{ij}(q,I^{ij}) &= \Big[\Big(\frac{1}{|Q(j)|^2}+\frac{1}{|Q(j)|}\Big)\sum_{k\in\hat{D}^{-1}(Q(j))\cup Q(j)} I^{ij}_{kj}\Big]-\frac{\Psi_{ij}(q,I^{ij})}{|Q(j)|}\\ &=\frac{1}{|Q(j)|^2}+\frac{1}{|Q(j)|}-\frac{1}{|Q(j)|}\frac{1}{(\hat{D}(i)\cup\{i\})\cap Q(j)}.\end{aligned}$$

Moreover, for each $k\in Q(j)$ such that $k\notin\hat{D}(i)\cup\{i\}$, it holds that

$$f_{kj}(q,I^{ij})=\frac{1}{|Q(j)|^2}+\frac{1}{|Q(j)|}.$$

The desired result holds since

$$f_{kj}(q,I^{ij})>f_{ij}(q,I^{ij}).$$

- Take any $(q,A^C)\in$ GDP. For each $k\in N$ and each $j\le q_k$, define the vector $\beta(kj)$ as

$$\forall i\in N,\quad \beta_{ij}(kj)=\begin{cases}20 & \text{if } i=k,\\ 10 & \text{if } i\in\hat{D}(k),\\ 1 & \text{otherwise.}\end{cases}$$

The rule $f^{\beta}$ given by

$$\forall j \leq q_n, \forall i \in Q(j),$$
$$f^{\beta}_{ij}(q, A^C) = \sum_{k \in \hat{D}^{-1}(i) \cup \{i\}} \frac{\beta_{ij}(kj)}{\sum_{i' \in Q(j)} \beta_{i'j}(kj)} \times A^C_{kj},$$

satisfies all the axioms except Downstream symmetry.

## 6.4 Rules and multi-choice games

For each gas distribution problem, we derive a specific multi-choice game, which we call the gas distribution game. Using this gas distribution game, we study how our rules relate to the solution concepts for multi-choice games introduced in Section 4.2. In particular, we show that the Connection rule is a **stable rule**, in the sense that consumers have an interest in becoming customers of the operator.

### 6.4.1 The gas distribution game

A gas distribution game, derived from a gas distribution problem $(q, A^C)$, measures the total cost of each gas distribution problem $(s, A^C)$ in which each consumer $i \in N$ has an effective demand of $s_i \leq q_i$. Put differently, a gas distribution game is a collection of all the total costs that could be generated by gas distribution networks smaller than the one they are derived from.

**Definition 37 (The gas distribution game).** *For each $(q, A^C) \in$ GDP, its associated gas distribution (multi-choice) game $(q, v^{C,D})$ is defined as*

$$\begin{aligned} \forall s \leq q, \quad v^{C,D}(s) &= \sum_{i \in N} C(i, \overline{s}_i) \\ &= \sum_{i \in N} \sum_{j \in \{1, \ldots, \overline{s}_i\}} A^C_{ij} \end{aligned} \tag{6.30}$$

*where* $\forall i \in N, \quad \overline{s}_i = \max_{k \in \hat{D}(i) \cup \{i\}} s_k.$

In a gas distribution game, the players represent the consumers and the activity levels represent the demands of the consumers. The worth $v^{C,D}(s)$ represents the total cost associated with the gas distribution problem $(s, A^C)$ in which each consumer $i \in N$ has an effective demand of $s_i$. Obviously, $v^{C,D}(q)$ coincides with the total cost of operating the gas distribution network and $v^{C,D}(0, \ldots, 0) = 0$ since $C(i, 0) = 0$ for each $i \in N$. Since $C(i, .)$ is non-decreasing, for each $i \in N$, it follows that a gas distribution game is monotonic, meaning that for each $(q, A^C) \in$ GDP,

$$\forall s \geq t, \quad v^{C,D}(s) \geq v^{C,D}(t).$$

### 6.4.2 Rules and solution concepts for multi-choice games

For each gas distribution problem, the Connection rule applied to the problem coincides with the multi-choice Shapley value applied to the gas distribution game associated with the problem. To prove this, we need an intermediary result. To that end, consider a gas distribution problem $(q, A^C) \in \text{GDP}$ and its associated gas distribution game $(q, v^{C,D})$. For each $j \leq q_n$, define the TU-game $(N, w_j^{(q,v^{C,D})})$ as

$$\begin{aligned}\forall E \in 2^N, \quad w_j^{(q,v^{C,D})}(E) =& v^{C,D}\Big[\big(\sum_{k\in N}(j-1)e_k + \sum_{k\in E} e_k\big) \wedge q\Big] \\ & - v^{C,D}\Big[\big(\sum_{k\in N}(j-1)e_k\big) \wedge q\Big].\end{aligned}$$

The worth $w_j^{(m,v^{C,D})}(E)$ can be interpreted as the surplus in cost generated in $v^{C,D}$ when a group of players $E$ decide to increase their activity level from $j-1$ to $j$ while all the other players play the activity level $j-1$ or their maximal feasible activity level if they are unable to play $j-1$. The next proposition is already proved on the class of monotonic multi-choice games (see 4.69).

**Proposition 33.** *For each gas distribution problem $(q, A^C) \in \text{GDP}$ and its associated gas distribution game $(q, v^{C,D})$, it holds that*

$$\forall (i,j) \in M^+, \quad \varphi_{ij}(q, v^{C,D}) = Sh_i(N, w_j^{(q,v^{C,D})}). \tag{6.31}$$

**Theorem 31.** *For each gas distribution problem $(q, A^C) \in \text{GDP}$ and its associated gas distribution game $(q, v^{C,D})$, it holds that*

$$\varphi(q, v^{C,D}) = \Psi(q, A^C). \tag{6.32}$$

*Proof.* Consider $(q, A^C) \in \text{GDP}$ and $j \leq q_n$. For each $k \in N$, define the TU-game $(Q(j), R^k)$ as

$$\forall E \subseteq Q(j), \quad R^k(E) = \begin{cases} 1 & \text{if } E \cap (\hat{D}(k) \cup \{k\}) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

By definition of the Shapley value,

$$\forall i \in N, \quad Sh_i(Q(j), R^k) = \begin{cases} \dfrac{1}{|(\hat{D}(k) \cup \{k\}) \cap Q(j)|} & \text{if } i \in \hat{D}(k) \cup \{k\}, \\ 0 & \text{otherwise.} \end{cases} \tag{6.33}$$

Consider $(q, v^{C,D}) \in \mathcal{G}$, the gas distribution game associated with $(q, A^C)$. Consider the TU-game $(Q(j), w^{(q,v^{C,D})})$ and observe that, for each $E \subseteq Q(j)$,

$$\begin{aligned}
w_j^{(q,v^{C,D})}(E) =& v^{C,D}\left[\left(\sum_{k\in N}(j-1)e_k + \sum_{k\in E} e_k\right) \wedge q\right] \\
& - v^{C,D}\left[\left(\sum_{k\in N}(j-1)e_k\right) \wedge q\right] \\
=& \sum_{i\in N}\left[C\left(i, \max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k + \sum_{k\in E} e_k\right) \wedge q\right]_h\right)\right. \\
& \left. - C\left(i, \max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k\right) \wedge q\right]_h\right)\right] \\
=& \sum_{i\in \hat{D}^{-1}(E)\cup E}\left[C\left(i, \max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k + \sum_{k\in E} e_k\right) \wedge q\right]_h\right)\right. \\
& \left. - C\left(i, \max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k\right) \wedge q\right]_h\right)\right] \\
& + \sum_{i\notin \hat{D}^{-1}(E)\cup E}\left[C\left(i, \max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k + \sum_{k\in E} e_k\right) \wedge q\right]_h\right)\right. \\
& \left. - C\left(i, \max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k\right) \wedge q\right]_h\right)\right]. \qquad (6.34)
\end{aligned}$$

For each $i \in \hat{D}^{-1}(E) \cup E$, there exists at least one $h \in \hat{D}(i) \cup \{i\}$ such that $h \in E \subseteq Q(j)$. Therefore, for each $i \in \hat{D}^{-1}(E) \cup E$,

$$\max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k + \sum_{k\in E} e_k\right) \wedge q\right]_h = j. \qquad (6.35)$$

On the other hand,

$$\max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k\right) \wedge q\right]_h = j-1. \qquad (6.36)$$

For each $i \notin \hat{D}^{-1}(E) \cup E$, there is no $h \in \hat{D}(i) \cup \{i\}$ such that $h \in E$. Therefore, for each $i \notin \hat{D}^{-1}(E) \cup E$,

$$\max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k + \sum_{k\in E} e_k\right) \wedge q\right]_h = \max_{h\in \hat{D}(i)\cup\{i\}}\left[\left(\sum_{k\in N}(j-1)e_k\right) \wedge q\right]_h. \qquad (6.37)$$

For the sake of clarity, for each $i \notin \hat{D}^{-1}(E) \cup E$, we use the notation

$$K_i = \max_{h \in \hat{D}(i) \cup \{i\}} \left[ \Big( \sum_{k \in N} (j-1)e_k \Big) \wedge q \right]_h .$$

Therefore, by (6.35), (6.36) and (6.37), (6.34) becomes

$$\begin{aligned} w_j^{(q,v^{C,D})}(E) &= \sum_{i \in \hat{D}^{-1}(E) \cup E} \Big[ C(i,j) - C(i,j-1) \Big] \\ &\quad + \sum_{i \notin \hat{D}^{-1}(E) \cup E} \Big[ C(i,K_i) - C(i,K_i) \Big] \\ &= \sum_{i \in \hat{D}^{-1}(E) \cup E} A_{ij}^C + 0 \\ &= \sum_{i \in N} A_{ij}^C \times R^i(E). \end{aligned}$$

By Proposition 33, for each $i \in N$ and $j \in \{1, \ldots, q_i\}$,

$$\begin{aligned} \varphi_{ij}(q, v^{C,D}) &= Sh_i \Big( Q(j), \sum_{l \in N} A_{lj}^C \times R^l \Big) \\ &= \sum_{l \in N} A_{lj}^C \times Sh_i\big(Q(j), R^l\big) \\ &= \sum_{l \in \hat{D}^{-1}(i) \cup \{i\}} \frac{A_{lj}^C}{|(\hat{D}(l) \cup l) \cap Q(j)|} \\ &= \Psi_{ij}(q, A^C). \end{aligned}$$

The second equality follows from the Linearity of the Shapley value and the third equality follows from (6.33). This concludes the proof of the theorem. ■

The next result states that, for each gas distribution problem, the Uniform rule applied to the gas distribution problem coincides with the multi-choice Equal division value applied to the gas distribution game associated with the problem.

**Theorem 32.** *For each gas distribution problem* $(q, A^C) \in$ GDP *and its associated gas distribution game* $(q, v^{C,D})$*, it holds that*

$$\xi(q, v^{C,D}) = \Omega(q, A^C). \tag{6.38}$$

*Proof.* The proof of the theorem follows directly from (6.30) and Definition 29. ■

The next result states that, for each gas distribution problem, the set of Mixed rules applied to the problem coincides with the set of multi-choice Egalitarian Shapley values applied to the gas distribution game associated with the problem.

**Theorem 33.** *Let $\alpha = \{\alpha^j\}_{1\leq j\leq K}$ be a parameter system such that $\alpha^j \in [0,1]$ for each $1 \leq j \leq K$. For each gas distribution problem $(q, A^C) \in$ GDP and its associated gas distribution game $(q, v^{C,D})$, it holds that*

$$\chi^{\alpha}(q, v^{C,D}) = \mu^{\alpha}(q, A^C). \tag{6.39}$$

*Proof.* The proof of the theorem follows directly from Theorem 31 and Theorem 32. ■

We introduce the definition of the Core of a gas distribution game. This definition is highly inspired from the definition of the Core of multi-choice games (see Definition 16).

**Definition 38 (The Core).** *The Core of a gas distribution game $(q, v^{C,D})$, denoted by $\mathcal{C}(q, v^{C,D})$, is the set of payoff vectors $x \in \mathbb{R}^{\sum_{i\in N} q_i}$ defined as*

$$\begin{cases} \forall s \leq q, \quad \displaystyle\sum_{i\in N}\sum_{j=1}^{s_i} x_{ij} \leq v^{C,D}(s), & (6.40) \\ \forall l \leq q_n, \quad \displaystyle\sum_{i\in N}\sum_{j=1}^{l\wedge q_i} x_{ij} = v^{C,D}((l \wedge q_i)_{i\in N}). & (6.41) \end{cases}$$

If we re-interpret the Core conditions in terms of rule for gas distribution problems, condition (6.40) ensures that consumers always pay less than the cost of supplying themselves in gas and (6.41) states that if a group of consumers, which synchronize their demands in the sense of a synchronized coalition, decide to supply themselves without resorting to the network operator, then they should pay the same amount as they would have been charged by the operator. Thus, the Core can be viewed as the set of **stable rules**, in the sense that consumers have an interest in becoming customers of the operator. We point out that condition (2.43) is the opposite of the original definition of the Core of a multi-choice game (see Definition 16). This difference lies in the fact that the worth of a coalition and the payoffs both represent some costs. In this context, it is more desirable, for a coalition of consumer, to pay less than their actual worth. For each gas distribution problem, the multi-choice Shapley value applied to a gas distribution game is always in the Core. Therefore, the Connection rule is a stable rule.

**Theorem 34.** *For each gas distribution problem $(q, A^C) \in$ GDP and its associated gas distribution game $(q, v^{C,D})$, it holds that*

$$\varphi(q, v^{C,D}) \in \mathcal{C}(q, v^{C,D}). \tag{6.42}$$

*Proof.* In Chapter 4, we show that the multi-choice Shapley value belongs to the Core of super-modular games. Since the game reflects costs in this framework, the multi-choice Shapley value belongs to the Core of sub-modular (cost) games. Therefore, it

suffice to show that gas distribution games are sub-modular games to prove Theorem 34. Consider a gas distribution game $(q, v^{C,D}) \in \mathcal{G}$ associated with a gas distribution problem $(q, A^C) \in \mathrm{GDP}$. For each $s, t \in \mathcal{M}$,

$$\begin{aligned} v^{C,D}(t) + v^{C,D}(s) =& \sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} s_k) \\ &+ \sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} t_k), \\ \text{and} \quad v^{C,D}(s \vee t) + v^{C,D}(s \wedge t) =& \sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} (s_k \vee t_k)) \\ &+ \sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} s_k \wedge t_k)). \end{aligned}$$

Take any $i \in N$. Without loss of generality, assume that

$$\max_{k \in \hat{D}(i) \cup \{i\}} s_k \geq \max_{k \in \hat{D}(i) \cup \{i\}} t_k.$$

Then, on the one hand,

$$\begin{aligned} &\max_{k \in \hat{D}(i) \cup \{i\}} (s_k \vee t_k) = \max_{k \in \hat{D}(i) \cup \{i\}} s_k \\ \Longrightarrow\ & C(i, \max_{k \in \hat{D}(i) \cup \{i\}} (s_k \vee t_k)) = C(i, \max_{k \in \hat{D}(i) \cup \{i\}} s_k). \end{aligned}$$

On the other hand, since $C$ is non-decreasing, it holds that

$$\begin{aligned} &\forall k \in \hat{D}(i) \cup \{i\}, \quad s_k \wedge t_k \leq t_k \\ \Longleftrightarrow\ & \max_{k \in \hat{D}(i) \cup \{i\}} (s_k \wedge t_k) \leq \max_{k \in \hat{D}(i) \cup \{i\}} t_k \\ \Longrightarrow\ & C(i, \max_{k \in \hat{D}(i) \cup \{i\}} (s_k \wedge t_k)) \leq C(i, \max_{k \in \hat{D}(i) \cup \{i\}} t_k). \end{aligned}$$

This shows that, for each $i \in N$,

$$\begin{aligned} &\sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} (s_k \vee t_k)) + \sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} (s_k \wedge t_k)) \\ &\leq \sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} s_k) + \sum_{i \in N} C(i, \max_{k \in \hat{D}(i) \cup \{i\}} t_k), \end{aligned}$$

and so, we obtain the desired result

$$v^{C,D}(s \vee t) + v^{C,D}(s \wedge t) \leq v^{C,D}(s) + v^{C,D}(t).$$

■

## 6.5 Conclusion

In this chapter, we define gas distribution problems and provide three solutions concepts for these problems. Applying the Connection principle and the Independence of higher demands principle, we propose the Connection rule. Applying the Uniformity principle and the Independence of higher demands principle, we propose the Uniform rule. In order to make a trade-off between the Connection principle and the Uniformity principle, we propose the Mixed rules. We provide an axiomatic characterization of each one of these rules. Additionally, we show that these rules coincide with solution concepts from multi-choice games. In particular, we show that the Connection rule applied to a gas distribution problem belongs to the Core of a specific multi-choice game derived from this problem.

Throughout this chapter, we have assumed that each pipeline is designed to meet its highest downstream demand. Let us call this approach the **optimistic approach** to pipeline design. The main drawback of this approach is that a pipeline can only satisfy a few effective demands at a time. In particular, the highest downstream demand of a pipeline already saturates its capacity, preventing any other demand from being satisfied at the same time. However, this approach has its advantages since it can be implemented at a low cost while ensuring a minimal service to the consumers.

An alternative approach is the **pessimistic approach** to pipeline design: the network operator must be able to satisfy all the effective demands at any time. Therefore, each pipeline should be designed to meet the sum of all of its downstream effective demands. Unlike the previous approach, this approach ensures a flawless service to the consumers. The main drawback of this approach is the resulting total cost of operating the network. Indeed, a network designed this way would be possibly too huge and too expensive to be operational.

The two approaches have their advantages and their drawbacks. In practice, a gas distribution network is rarely design according to the optimistic or the pessimistic approach. It may be interesting to investigate a new approach that compromises between these two. Such an approach would be more consistent with the facts since it makes the network more capable of handling multiple effective demands at the same time, while limiting the total cost of the network.

# Chapter 7: Concluding Remarks

To conclude, we propose an overview of the results obtained in this thesis. This allows us to discuss the connections between the chapters, and to show that some solution concepts introduced in this thesis are interlinked. A diagram to explicit these connections concludes this chapter.

## 7.1 Overview

Chapter 3 focuses on multi-choice games with a permission (tree) structure. It investigates how a permission structure on the player set can be combined with different levels of participation. To conduct this study, pal-permission structures are introduced, allowing us to define the pal-permission value for multi-choice games with a pal-permission structure. The pal-permission value is computed as the $DP$ value of a game restricted by a pal-permission structure. Alternatively, three solution concepts for multi-choice games with a permission structure are defined. Each of these solutions is computed as the pal-permission value of a multi-choice game with a specific pal-permission structure, which is derived from the original permission structure and the participation levels of the players. An axiomatic characterization is provided for each of these three solutions. This chapter sheds light on the additional difficulties that arise when one tries to combine a multi-choice game with a structure on the player set, but does not take into account the Independence of higher demands or the Uniformity principle.

Chapter 4 focuses on multi-choice games without consideration for any structure. It addresses the trade-off between marginalism and egalitarianism, which is one of the main issues in economic allocation problems. The multi-choice Shapley value, the multi-choice Equal division value and the multi-choice Egalitarian Shapley values are introduced. Each of these solution concepts satisfies Independence of higher activity levels, which clearly is an equivalent formulation of the Independence of higher demands principle adapted to the class of multi-choice games. Additionally, the multi-choice Equal division value is in line with the Uniformity principle. At least one axiomatic characterization is provided for each of these solution concepts. This chapter puts into perspective the principles pursued by GRDF by showing that they are not always compatible with other desirable properties. We should observe that

this chapter does not consider the information provided by the distribution network.

Chapter 5 focuses on multi-choice games with a priority structure. The multi-choice Priority value is introduced, which is defined according to a lexicographic procedure on two criteria: the activity levels and the position of the players in the priority structure. Contrary to the solution concepts introduced in Chapter 3, the multi-choice Priority value satisfies Independence of higher activity levels. This value combines a principle pursued by GRDF with an information provided by the gas distribution network. Two axiomatic characterizations of this value are provided.

Chapter 6 proposes cost sharing rules to determine relevant distribution rates. We resort to three principles: (i) the Independence of higher demands principle, (ii) the Connection principle and (iii) the Uniformity principle. Applying (i) and (ii), we derive the Connection rule and applying (i) and (iii), we derive the Uniform rule. In order to make a trade-off between (ii) and (iii), we propose the Mixed rules. For each cost sharing rule, an axiomatic characterization is provided. Then, we show that the Connection rule coincides with the multi-choice Shapley value of a specific multi-choice game derived from the network and the demands of the consumers. Moreover, the Connection rule is in the Core of this specific multi-choice game. Finally, we show that the Uniform rule coincides with the multi-choice Equal division value and the Mixed rules coincide with the multi-choice Egalitarian Shapley values. Similarly to Chapter 4, this chapter puts into perspective the principles pursued by GRDF in the framework of gas distribution problems. This chapter is the only chapter in this thesis that considers both the information provided by the distribution network and the two principles retained by GRDF.

## 7.2 Discussion on Chapters 2, 3 and 5

In this section, new solution concepts for multi-choice games with a pal-permission structure and multi-choice games with a permission structure are proposed. These new solution concepts are defined using elements from Chapter 3 and Chapter 4. Then, we observe that one of these solution concepts coincides with the Connection rule from Chapter 6, under certain conditions.

In Chapter 3, the Pal-permission value $\Upsilon$ for multi-choice games with a pal-permission structure is computed by applying the $DP$ value to a restricted game induced by a pal-permission structure (see (3.7)). Instead of applying the $DP$ value, one can apply the multi-choice Shapley value, from Chapter 5, to the restricted game. This results in a new solution concept for multi-choice games with a pal-permission structure, denoted by $\upsilon$, and defined, for each $(m, v, P^+) \in \mathcal{GP}$, as

$$\begin{aligned}\forall (i,j) \in M^+, \quad \upsilon_{ij}(m, v, P^+) &= \varphi_{ij}(m, \mathcal{R}_{P^+}(v)) \\ &= \sum_{\substack{s \leq m \\ (i,j) \in T(\alpha(s))}} \frac{\Delta_v(s)}{|T(\alpha(s))|}.\end{aligned}$$

This value extends the Permission value from TU-games with a permission structure and the multi-choice Shapley value from multi-choice games to multi-choice games with a pal-permission structure.

Similarly to Definition 27, three new solution concepts for multi-choice games with a permission (tree) structure can be defined. These solutions are computed from $v$ and the $m$-permission structure, the 1-permission structure and the $F$-permission structure from Chapter 2 (see Section 3.5).

- The value $g^{(+,m)}$ on $\mathcal{GP}_T$ is defined as

$$g^{(+,m)}(m, v, P) = v(m, v, P_m^+).$$

- The value $g^{(+,1)}$ on $\mathcal{GP}_T$ is defined as

$$g^{(+,1)}(m, v, P) = v(m, v, P_1^+).$$

- The value $g^{(+,F)}$ on $\mathcal{GP}_T$ is defined as

$$g^{(+,F)}(m, v, P) = v(m, v, P_F^+).$$

Ni & Wang (2007) and Dong *et al.* (2012) study a situation where a river is polluted. To consume the water, the river must be cleaned. When the river passes through several different countries, a natural question is how should the cleaning costs be shared among the countries. An extreme solution is that each country only pays for the cleaning cost at its own region. However, if upstream countries are also partly responsible for the pollution at a certain river segment, then it seems reasonable that upstream countries share their downstream countries' pollution cost. This idea resulted into the Upstream equal sharing method (UES). On the other hand, since downstream countries benefit from upstream countries cleaning the river, it might be reasonable that downstream countries contribute in the cleaning cost of upstream countries. This idea resulted into the Downstream equal sharing method (DES). van den Brink *et al.* (2018) show that the UES and DES methods can be obtained as the (conjunctive) permission value of an associated TU-game with a permission structure. The permission structure is opposite to the river structure. Meanwhile, the associated TU-game, called the **stand-alone game**, is an additive game that associates to each coalition of countries the sum of their cleaning costs.

A study similar to van den Brink *et al.* (2018) can be conducted to show that the Connection rule can be computed as a value of a specific multi-choice game with a permission structure. Consider a gas distribution network $D$ and a finite set of consumers $N$. Assume that each consumer has an effective demand smaller than its subordinates in the distribution network. Let us define an associated permission structure $P_D$ that indicates opposite relationships to the ones described by the distribution network.

**Example 16.** *For instance, consider a situation with six consumers in a distribution network $D$. The gas distribution network and its associated permission structure $P_D$ are given by*

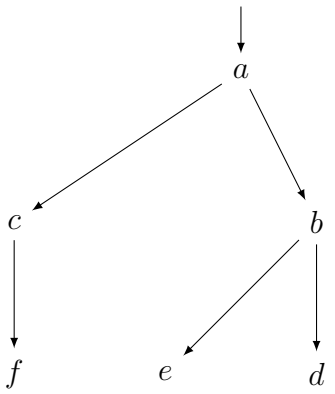



Figure 7.1: $D$

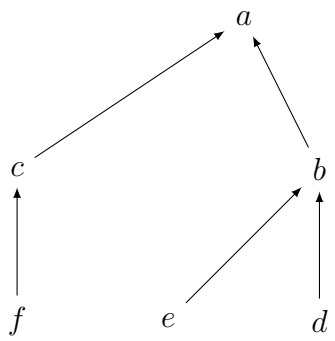



Figure 7.2: $P_D$

Thus, in such associated permission structure $P_D$, each consumer has an effective demand greater than its subordinates.

Consider a gas distribution problem $(q, A^C) \in \text{GDP}$, where $A^C$ is a matrix of incremental costs induced by a cost function $C : \Pi_{i \in N}\{0, 1, \ldots, q_i\} \to \mathbb{R}$. From $(q, A^C) \in \text{GDP}$, one can derive a multi-choice game with a permission structure $(q, C_{SA}, P_D)$, $C(\vec{0}) = 0$, where $C_{SA} : \mathcal{M} \to \mathbb{R}$ is a characteristic function different from $v^{C,P}$, and conceptually close to the stand-alone TU-game used by van den Brink *et al.* (2018). For each $(q, A^C) \in \text{GDP}$, this game verifies

$$\Psi(q, A^C) = g^{(+,F)}(q, C_{SA}, P_D). \tag{7.1}$$

Observe that $P_D$ is not a (rooted) permission tree. Therefore, the conjunctive approach and the disjunctive approach to coalition restriction do not coincide anymore. Then, (7.1) only holds if one defines $\mathcal{R}_{P_F^+}$ as a conjunctive restriction operator.

## 7.3 Discussion on Chapters 4 and 5

In Chapter 6, we show that the Connection rule applied to a gas distribution problem coincides with the multi-choice Shapley value applied to a multi-choice game derived from the gas distribution problem.

Take any gas distribution problem $(q, A^C) \in \text{GDP}$ and derive the multi-choice game, as in (6.30), $(q, v^{C,P})$ from it. Endow $(q, A^C)$ and $(q, v^{C,P})$ with a priority structure $(N, \succeq)$ on the player set. The priority structure can model the priority needs of certain consumers (heating for households, hospital, etc). The triplet $(q, v^{C,P}, \succeq)$ can be viewed as a monotonic multi-choice game with a priority structure. Therefore,

one can apply the multi-choice Priority value to $(q, v^{C,P}, \succeq)$, which results in a rule $\gamma$ for gas distribution problems with a priority structure. Formally, the rule is defined as

$$\begin{aligned}\forall (i,j) \in M^+, \quad \gamma(q, C, \succeq) &= \Gamma_{ij}(q, v^{C,P}, \succeq) \\ &= \sum_{\substack{s \leq q \\ (i,j) \in H(C(s), \succeq^*)}} \frac{\Delta_{v^{C,P}}(s)}{|H(C(s), \succeq^*)|}.\end{aligned}$$

This rule divides the surplus generated by each coalition of gas consumers according to the lexicographic partial order $(M^+, \succeq^*)$ introduced in Chapter 5. This rule differs from the Connection rule since exogenous asymmetries between the players are taken into account. In this case, the information provided by the gas distribution network is taken into account two times. It is used to define the game $(q, v^{C,P})$, and also affect the allocation process through a priority structure $(N, \succeq)$. Similarly, a number of rules for gas distribution problems can be defined in this way.

## 7.4 Discussion on Chapters 2,3 and 4

Chapter 3 considers pal-permission structures, which are structures defined over the set of pairs player-activity levels, whereas Chapter 5 has very little consideration for structures defined over the set of pairs, excepted the lexicographic partial order over the set of pairs.

This section introduces pal-priority structures for multi-choice games. Such structures can be understood as priority structures defined over the set of player-activity level pairs. We introduce a value for multi-choice games with a pal-priority structure, and show that it coincides with the multi-choice Priority value, from Chapter 5, and the multi-choice Shapley value, from Chapter 4, under certain conditions.

Take any $(m, v) \in \mathcal{G}$. A pal-priority structure on $M^+$ represents priority relationships between the pairs formed by a player and one of its activity levels. It reflects the fact that some activity levels of a player may have priority over some activity levels of other players in the allocation process. Formally, a pal-priority structure on $M^+$ is a poset $(M^+, \succeq^+)$. The class of pal-priority structures is denoted by $\mathcal{S}$.

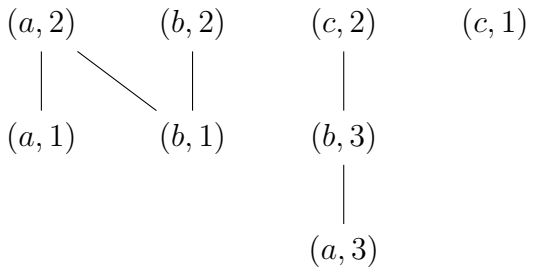
Take any $A \subseteq M^+$. Denote by $H(A, \succeq^+)$ the non-empty subset of priority pairs in $(A, \succeq^+)$ defined as

$$H(A, \succeq^+) = \Big\{(i,j) \in A : \big[(i',j') \succeq^+ (i,j)\big] \implies \big[(i,j) = (i',j')\big]\Big\}.$$

**Example 17.** *Let $N = \{a, b, c\}$ and $m = (3, 3, 2)$, we have*

$$M^+ = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2)\}.$$

*Take the pal-priority structure $(M^+, \succeq^+)$ illustrated by the following Hasse diagram*




*In this case,*

$$H(\{(a,1),(a,2),(b,1),(b,2)\},\succeq^{+}) = \big\{(a,2),(b,2)\big\};$$
$$H(\{(a,3),(b,3),(c,2)\},\succeq^{+}) = \big\{(c,2)\big\};$$
$$H(M^{+},\succeq^{+}) = \big\{(a,2),(b,2),(c,1),(c,2)\big\}.$$

A multi-choice game on $N$ with a pal-priority structure is a triplet $(m,v,\succeq^{+})$. The class of multi-choice games with a pal-priority structure is denoted by $\mathcal{GS}$. We introduce a new value on $\mathcal{GS}$ denoted by $\Lambda$, and defined for each $(m,v,\succeq^{+}) \in \mathcal{GS}$ as

$$\forall (i,j) \in M^{+}, \quad \Lambda_{ij}(m,v,\succeq^{+}) = \sum_{\substack{s \leq m \\ (i,j) \in H(B(s),\succeq^{+})}} \frac{\Delta_{v}(s)}{|H(B(s),\succeq^{+})|}. \tag{7.2}$$

This value extends the Priority value from TU-games with a priority structure and the $DP$ value from multi-choice games to multi-choice games with a pal-priority structure.

Observe that the lexicographic partial order $(M^{+},\succeq^{*})$ (see Definition 31) is a pal-priority structure. Therefore, for each $(m,v,\succeq) \in \mathcal{GS}$, one can derive the unique pal-priority structure $(m,v,\succeq^{*}) \in \mathcal{GS}$, which verifies

$$\forall (i,j) \in M^{+}, \quad \Gamma_{ij}(m,v,\succeq) = \Lambda_{ij}(m,v,\succeq^{*}).$$

Take any $(m,v) \in \mathcal{G}$. Define the pal-priority structure $(M^{+},\succeq^{Sh})$ that prioritizes the pairs with the highest activity levels in $(m,v) \in \mathcal{G}$ by

$$\forall (i,j),(k,l) \in M^{+}, \quad (i,j) \succeq^{Sh} (k,l) \iff j > l.$$

Put simply, a pair has priority over another pair in $(M^{+},\succeq^{Sh})$ if and only if the former features a strictly greater activity level than the latter. Clearly, to each multi-choice game $(m,v) \in \mathcal{G}$, one can associates a unique $(M^{+},\succeq^{Sh})$. Observe that $(M^{+},\succeq^{Sh})$ verifies

$$\forall s \in \mathcal{M}, \quad H(B(s),\succeq^{Sh}) = T(s). \tag{7.3}$$

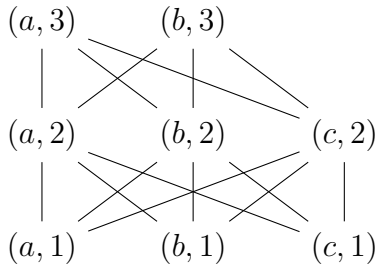
By (4.23) and (7.3), for each $(m,v) \in \mathcal{G}$, it holds that

$$\forall (i,j) \in M^{+}, \quad \varphi_{ij}(m,v) = \Lambda_{ij}(m,v,\succeq^{Sh}).$$

**Example 18.** *Let* $N = \{a, b, c\}$ *and* $m = (3, 3, 2)$*. The set of pairs is given by*

$$M^+ = \{(a,1), (a,2), (a,3), (b,1), (b,2), (b,3), (c,1), (c,2)\}.$$

*The pal-priority structure* $(M^+, \succeq^{Sh})$ *is illustrated by the following digraph.*




*If* $s = (1, 2, 2)$*, then*

$$B(s) = \{(a,1), (a,2), (b,1), (b,2), (c,1), (c,2)\}$$

*and*

$$H(B(s), \succeq^{Sh}) = \big\{(b,2), (c,2)\big\} = T(s).$$

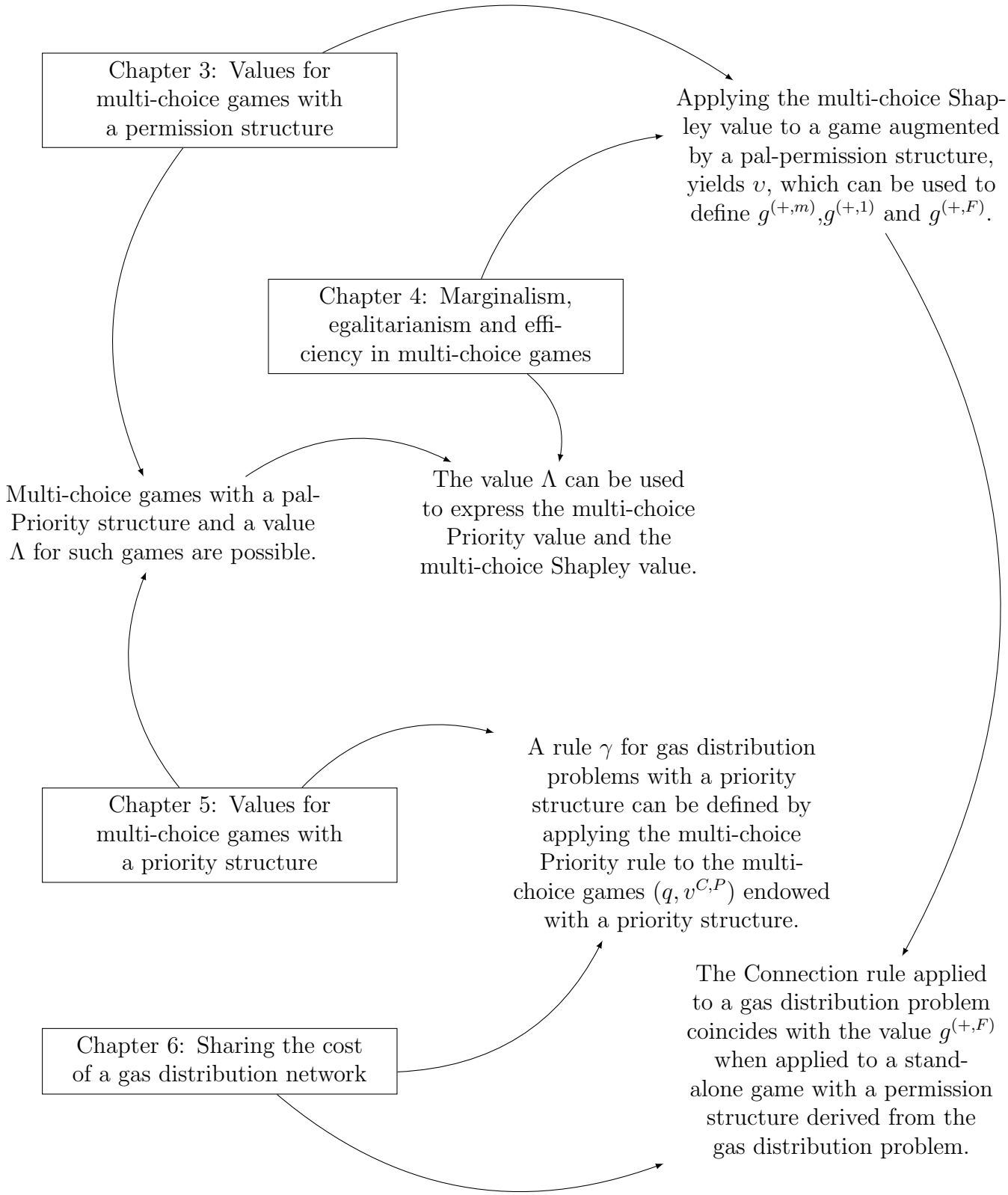
*If* $s = (3, 2, 2)$*, then*

$$B(s) = \{(a,1), (a,2), (a,3), (b,1), (b,2), (c,1), (c,2)\}$$

*and*

$$H(B(s), \succeq^{Sh}) = \big\{(a,3)\big\} = T(s).$$

All the elements discussed in this conclusion are illustrated by the following figure.


Chapter 3: Values for multi-choice games with a permission structure
Applying the multi-choice Shapley value to a game augmented by a pal-permission structure, yields $v$, which can be used to define $g^{(+,m)}$,$g^{(+,1)}$ and $g^{(+,F)}$.
Chapter 4: Marginalism, egalitarianism and efficiency in multi-choice games
Multi-choice games with a pal-Priority structure and a value $\Lambda$ for such games are possible.
The value $\Lambda$ can be used to express the multi-choice Priority value and the multi-choice Shapley value.
A rule $\gamma$ for gas distribution problems with a priority structure can be defined by applying the multi-choice Priority rule to the multi-choice games $(q, v^{C,P})$ endowed with a priority structure.
Chapter 5: Values for multi-choice games with a priority structure
The Connection rule applied to a gas distribution problem coincides with the value $g^{(+,F)}$ when applied to a stand-alone game with a permission structure derived from the gas distribution problem.
Chapter 6: Sharing the cost of a gas distribution network

# Bibliography


Abe, Takaaki, & Nakada, Satoshi. 2019. The weighted-egalitarian Shapley values. *Social Choice and Welfare*, **52**(2), 197–213.

Albizuri, M Josune. 2009. The multichoice coalition value. *International Journal of Game Theory*, **172**(1), 363.

Albizuri, M Josune, Santos, Juan Carlos, & Zarzuelo, José Manuel. 2003. On the serial cost sharing rule. *International Journal of Game Theory*, **31**(3), 437–446.

Albizuri, M Josune, Dietzenbacher, BJ, & Zarzuelo, José M. 2020. Bargaining with independence of higher or irrelevant claims. *Journal of Mathematical Economics*, **91**, 11–17.

Ambec, Stefan, & Sprumont, Yves. 2002. Sharing a river. *Journal of Economic Theory*, **107**(2), 453–462.

Aumann, Robert J, & Dreze, Jacques H. 1974. Cooperative games with coalition structures. *International Journal of Game Theory*, **3**(4), 217–237.

Bahel, Eric, & Trudeau, Christian. 2013. A discrete cost sharing model with technological cooperation. *International Journal of Game Theory*, **42**(2), 439–460.

Béal, S., Ferrières, S., & Solal, P. 2021a. The priority value for cooperative games with a priority structure. *International Journal of Game Theory*, https://doi.org/10.1007/s00182–021–00799–5.

Béal, Sylvain, & Navarro, Florian. 2020. Necessary versus equal players in axiomatic studies. *Operations Research Letters*, **48**(3), 385–391.

Béal, Sylvain, Lardon, Aymeric, Rémila, Eric, & Solal, Philippe. 2012. The average tree solution for multi-choice forest games. *Annals of Operations Research*, **196**(1), 27–51.

Béal, Sylvain, Rémila, Eric, & Solal, Philippe. 2015. Axioms of invariance for TU-games. *International Journal of Game Theory*, **44**(4), 891–902.

Béal, Sylvain, Casajus, André, Huettner, Frank, Rémila, Eric, & Solal, Philippe. 2016. Characterizations of weighted and equal division values. *Theory and Decision*, **80**(4), 649–667.

Béal, Sylvain, Casajus, André, Rémila, Eric, & Solal, Philippe. 2021b. Cohesive efficiency in TU-games: axiomatizations of variants of the Shapley value, egalitarian values and their convex combinations. *Annals of Operations Research*, **302**(1), 23–47.

Branzei, R, Llorca, N, Sánchez-Soriano, Joaquín, & Tijs, Stef. 2014. A constrained egalitarian solution for convex multi-choice games. *TOP*, **22**(3), 860–874.

Branzei, Rodica, Llorca, Natividad, Sánchez-Soriano, Joaquín, & Tijs, Stef. 2009. Multi-choice clan games and their core. *TOP*, **17**(1), 123–138.

Calvo, Emilio, & Santos, Juan Carlos. 2000. A value for multichoice games. *Mathematical Social Sciences*, **40**(3), 341–354.

Casajus, André. 2011. Differential marginality, van den Brink fairness, and the Shapley value. *Theory and Decision*, **71**(2), 163–174.

Casajus, André. 2014. The Shapley value without efficiency and additivity. *Mathematical Social Sciences*, **68**, 1–4.

Casajus, André. 2017. Weakly balanced contributions and solutions for cooperative games. *Operations Research Letters*, **45**(6), 616–619.

Casajus, André. 2018. Sign symmetry vs symmetry: Young's characterization of the Shapley value revisited. *Economics Letters*, **169**, 59–62.

Casajus, André, & Huettner, Frank. 2013. Null players, solidarity, and the egalitarian Shapley values. *Journal of Mathematical Economics*, **49**(1), 58–61.

Casajus, André, & Yokote, Koji. 2017. Weak differential marginality and the Shapley value. *Journal of Economic Theory*, **167**, 274–284.

Coase, Ronald H. 1960. The problem of social cost. *Journal of Law and Economics, The University of Chicago Press*, **3**, 1–44.

Csercsik, Dávid, Hubert, Franz, Sziklai, Balázs R, & Kóczy, László Á. 2019. Modeling transfer profits as externalities in a cooperative game-theoretic model of natural gas networks. *Energy Economics*, **80**, 355–365.

Derks, Jean, & Peters, Hans. 1993. A Shapley value for games with restricted coalitions. *International Journal of Game Theory*, **21**(4), 351–360.

Derks, Jean JM, & Haller, Hans H. 1999. Null players out? Linear values for games with variable supports. *International Game Theory Review*, **1**(03n04), 301–314.

Derks, JJM. 1992. A short proof of the inclusion of the core in the Weber set. *International Journal of Game Theory*, **21**(2), 149–150.

Dong, Baomin, Ni, Debing, & Wang, Yuntong. 2012. Sharing a polluted river network. *Environmental and Resource Economics*, **53**(3), 367–387.

Dutta, Bhaskar, & Ray, Debraj. 1989. A concept of egalitarianism under participation constraints. *Econometrica*, 615–635.

Einy, Ezra, & Haimanko, Ori. 2011. Characterization of the Shapley–Shubik power index without the efficiency axiom. *Games and Economic Behavior*, **73**(2), 615–621.

Faigle, Ulrich, & Kern, Walter. 1992. The Shapley value for cooperative games under precedence constraints. *International Journal of Game Theory*, **21**(3), 249–266.

Fishburn, Peter C. 1974. Lexicographic orders, utilities and decision rules: A survey. *Management Science*, **20**(11), 1442–1471.

Fleurbaey, Marc, & Trannoy, Alain. 1998. La péréquation territoriale en question. *FLUX Cahiers scientifiques internationaux Réseaux et Territoires*, **14**(31), 91–98.

Gilles, Robert P, Owen, Guillermo, & van den Brink, René. 1992. Games with permission structures: the conjunctive approach. *International Journal of Game Theory*, **20**(3), 277–293.

Gillies, Donald. 1953. Some theorems on n-person games. *Ph. D. Dissertation, Princeton University, Department of Mathematics*.

Grabisch, Michel, & Lange, Fabien. 2007. Games on lattices, multichoice games and the Shapley value: a new approach. *Mathematical Methods of Operations Research*, **65**(1), 153–167.

Grabisch, Michel, & Rusinowska, Agnieszka. 2010. A model of influence with an ordered set of possible actions. *Theory and Decision*, **69**(4), 635–656.

Grabisch, Michel, & Xie, Lijue. 2007. A new approach to the core and Weber set of multichoice games. *Mathematical Methods of Operations Research*, **66**(3), 491–512.

Harsanyi, John C. 1958. A bargaining model for the cooperative n-person game. *Stanford University*.

Herings, Jean-Jacques, Van Der Laan, Gerard, & Talman, Dolf. 2008. The average tree solution for cycle-free graph games. *Games and Economic Behavior*, **62**(1), 77–92.

Hougaard, Jens Leth. 2018. *Allocation in networks*. MIT Press.

Hsiao, C-R, & Raghavan, TES. 1992. Monotonicity and dummy free property for multi-choice cooperative games. *International Journal of Game Theory*, **21**(3), 301–312.

Joosten, Reinoud Anna Maria Gerardus. 1996. Dynamics, equilibria, and values. *PhD Thesis, Universiteit Maastricht*.

Kalai, Ehud, & Samet, Dov. 1987. On weighted Shapley values. *International Journal of Game Theory*, **16**(3), 205–222.

Klijn, Flip, Slikker, Marco, & Zarzuelo, Jose. 1999. Characterizations of a multi-choice value. *International Journal of Game Theory*, **28**(4), 521–532.

Littlechild, Stephen C, & Owen, Guillermo. 1973. A simple expression for the Shapley value in a special case. *Management Science*, **20**(3), 370–372.

Maschler, Michael, & Peleg, Bezalel. 1966. A characterization, existence proof and dimension bounds for the kernel of a game. *Pacific Journal of Mathematics*, **18**(2), 289–328.

Massol, Olivier, & Tchung-Ming, Stéphane. 2010. Cooperation among liquefied natural gas suppliers: Is rationalization the sole objective? *Energy Economics*, **32**(4), 933–947.

Moulin, Hervé. 1995. On additive methods to share joint costs. *The Japanese Economic Review*, **46**(4), 303–332.

Moulin, Hervé. 2003. *Fair division and collective welfare*. MIT press.

Moulin, Hervé, & Shenker, Scott. 1992. Serial cost sharing. *Econometrica*, **60**, 1009–1037.

Myerson, Roger B. 1977. Graphs and cooperation in games. *Mathematics of Operations Research*, **2**(3), 225–229.

Myerson, Roger B. 1980. Conference structures and fair allocation rules. *International Journal of Game Theory*, **9**(3), 169–182.

Ni, Debing, & Wang, Yuntong. 2007. Sharing a polluted river. *Games and Economic Behavior*, **60**(1), 176–186.

Owen, Guilliermo. 1977. Values of games with a priori unions. *Mathematical economics and game theory*, 76–88.

Pérez-Castrillo, David, & Wettstein, David. 2001. Bidding for the surplus: a non-cooperative approach to the Shapley value. *Journal of Economic Theory*, **100**(2), 274–294.

Peters, Hans, & Zank, Horst. 2005. The egalitarian solution for multichoice games. *International Journal of Game Theory*, **137**(1), 399–409.

Pintér, Miklós. 2015. Young's axiomatization of the Shapley value: a new proof. *International Journal of Game Theory*, **235**(1), 665–673.

Shapley, Lloyd S. 1953. A value for n-person games. *Contributions to the Theory of Games*, **2**(28), 307–317.

Shapley, Lloyd S. 1955. *Markets as cooperative games.* Tech. rept. RAND CORP SANTA MONICA CA.

Shapley, Lloyd S. 1971. Cores of convex games. *International Journal of Game Theory*, **1**(1), 11–26.

Shubik, Martin. 1962. Incentives, decentralized control, the assignment of joint costs and internal pricing. *Management Science*, **8**(3), 325–343.

Sprumont, Yves. 2005. On the discrete version of the Aumann–Shapley cost-sharing method. *Econometrica*, **73**(5), 1693–1712.

Svenson, Ola. 1979. Process descriptions of decision making. *Organizational Behavior and Human Performance*, **23**(1), 86–112.

Tang, Jie, Meng, Fanyong, & Zhang, Qiang. 2019. Characterizations of a Shapley value for multichoice games. *International Journal of General Systems*, **48**(2), 186–209.

Techer, Kevin. 2021. Stable agreements through liability rules: A multi-choice game approach to the social cost problem. *Mathematical Social Sciences*, **111**, 77–88.

Thomson, William. 2001. On the axiomatic method and its recent applications to game theory and resource allocation. *Social Choice and Welfare*, **18**(2), 327–386.

van den Brink, René. 1997. An axiomatization of the disjunctive permission value for games with a permission structure. *International Journal of Game Theory*, **26**(1), 27–43.

van den Brink, René. 2002. An axiomatization of the Shapley value using a fairness property. *International Journal of Game Theory*, **30**(3), 309–319.

van den Brink, René. 2007. Null or nullifying players: the difference between the Shapley value and equal division solutions. *Journal of Economic Theory*, **136**(1), 767–775.

van den Brink, René. 2017. Games with a permission structure-A survey on generalizations and applications. *TOP*, **25**(1), 1–33.

van den Brink, René, & Gilles, Robert P. 1996. Axiomatizations of the conjunctive permission value for games with permission structures. *Games and Economic Behavior*, **12**(1), 113–126.

van den Brink, René, Funaki, Yukihiko, & Ju, Yuan. 2013. Reconciling marginalism with egalitarianism: consistency, monotonicity, and implementation of egalitarian Shapley values. *Social Choice and Welfare*, **40**(3), 693–714.

van den Brink, René, Herings, Jean-Jacques, Van Der Laan, Gerard, & Talman, AJJ. 2015. The average tree permission value for games with a permission tree. *Economic Theory*, **58**(1), 99–123.

van den Brink, René, He, Simin, & Huang, Jia-Ping. 2018. Polluted river problems and games with a permission structure. *Games and Economic Behavior*, **108**, 182–205.

van den Nouweland, Anne, Tijs, Stef, Potters, J, & Zarzuelo, J. 1995. Cores and related solution concepts for multi-choice games. *Zeitschrift für Operations Research*, **41**(3), 289–311.

van den Nouweland, Catharina Gerardina Anna. 1993. Games and graphs in economic situations. Tilburg University.

Weber, Robert J. 1988. Probabilistic values for games. *The Shapley Value. Essays in Honor of Lloyd S. Shapley*, 101–119.

Young, H Peyton. 1985. Monotonic solutions of cooperative games. *International Journal of Game Theory*, **14**(2), 65–72.