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# Asymptotically quasiperiodic solutions for time-dependent Hamiltonians

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Donato Scarcella. Asymptotically quasiperiodic solutions for time-dependent Hamiltonians. Dynamical Systems [math.DS]. Université Paris sciences et lettres, 2022. English. NNT : 2022UPSLD047 . tel-03963784v2

**HAL Id: tel-03963784**

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Submitted on 16 May 2023

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**THÈSE DE DOCTORAT**

**DE L'UNIVERSITÉ PSL**

Préparée à l'Université Paris-Dauphine

# Asymptotically quasiperiodic solutions for time-dependent Hamiltonians

Soutenue par

**Donato SCARCELLA**

Le 08 Décembre 2022

Ecole doctorale n° ED 543

**Sciences de la décision, des  
organisations, de la société et de  
l'échange**

Spécialité

**Mathématiques appliquées et  
applications des mathématiques**

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## Remerciements

En premier lieu, je tiens à remercier mes directeurs de thèse, Abed Bounemoura et Jacques Féjoz. Vous avez choisi un très bon sujet de thèse. Je vous remercie pour avoir partagé votre connaissance et votre expérience. Discuter de mathématiques avec vous est fascinant.

Je suis très reconnaissant envers Massimiliano Berti et Marcel Guardia qui ont accepté d'être rapporteurs de cette thèse. Merci pour votre intérêt pour mon travail, vos questions et commentaires ont été précieux. C'était un plaisir parler de mathématiques avec vous.

Un grand merci à Alain Chenciner et Anna Florio. Vos remarques et questions ont donné lieu à des échanges très intéressantes. Merci de m'avoir fait l'honneur d'accepter de faire partie du jury. Surtout, je tiens à te remercier Anna pour ta disponibilité et ta gentillesse.

Je remercie Patrick Bernard et Raphael Krikorian, je suis honoré de vous avoir parmi mon jury de thèse.

Merci beaucoup au staff de MIDO, l'équipe de CEREMADE et à l'école doctorale. En particulier je tiens à remercier Isabelle Bellier, Anne-Laure Chagnon et César Faivre pour m'être venu au secours dans les démarches administratives.

Un remerciement special à Ryad. Collègue, confident mais surtout ami. Ton soutien a été essentiel pour pouvoir terminer ce travail. Je remercie tous mes collègues. Surtout mes amis Adrien, Jean, Giovanni, Théo, Samuel et Umberto. Merci pour votre bonne humeur.

Merci à Santiago, collègue d'(més)aventures sur les systèmes hamiltoniens.

Un enorme remerciement à Amanda pour m'avoir soutenu et supporté pendant toutes ces années de thèse. Merci d'avoir toujours cru en moi.

Per terminare, ringrazio la mia famiglia per il sostegno e l'affetto.

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## Part I

# Introduction et Résultats

## 1 Introduction

### 1.1 Théorie KAM

La théorie de Kolmogorov-Arnold-Moser (KAM) montre la persistance de solutions quasipériodiques dans les systèmes hamiltoniens presque intégrables. Cette théorie est intéressante, surtout en ce qui concerne les applications dans les problèmes classiques de mécanique céleste, comme par exemple le problème à  $n$ -corps. Nous renvoyons à [Bos86], [Pö01] [DIL01] et [Fé17] pour de très bonnes introductions. Nous notons qu'un système hamiltonien est intégrable si des variables action-angle appropriées existent de telle sorte que le hamiltonien ne dépend que des variables d'action. Nous faisons référence à [Arn13].

Soit  $B \subset \mathbb{R}^n$  une boule centrée à l'origine et  $0 < \varepsilon \ll 1$  un petit paramètre approprié. Nous considérons les hamiltoniens de la forme

$$H : \mathbb{T}^n \times B \longrightarrow \mathbb{R}, \quad H(q, p) = h(p) + \varepsilon f(q, p), \quad (\text{H})$$

où  $p \in B$  sont les actions, alors que  $q \in \mathbb{T}^n$  sont les angles conjugués variant sur le tore  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$ .

Nous associons au hamiltonien  $H$  le suivant système hamiltonien

$$X_H(q, p) = (\partial_p H(q, p), -\partial_q H(q, p)),$$

où  $\partial_p$  et  $\partial_q$  représentent respectivement les dérivées partielles par rapport à  $p$  et  $q$ . Les équations du mouvement prennent la forme suivante

$$\dot{q} = \partial_p H(q, p), \quad \dot{p} = -\partial_q H(q, p),$$

où le point indique la dérivée par rapport au temps  $t$ .

Si  $\varepsilon = 0$ , le hamiltonien  $H$  coïncide avec le hamiltonien non perturbé  $h$ . Dans ce cas, les équations du mouvement sont de telle forme

$$\dot{q} = \omega(p), \quad \dot{p} = 0$$

où  $\omega = \partial_p h(p)$ . Nous pouvons facilement intégrer ce dernier et pour tout  $(q_0, p_0) \in \mathbb{T}^n \times B$

$$q(t) = q_0 + \omega(p_0)t, \quad p(t) = p_0$$

c'est la solution du système précédent. Nous observons que chaque solution donne lieu à un flot linéaire et, à cause de l'identification des coordonnées  $q$  modulo 1, chaque orbite tourne autour du tore invariant

$$\mathcal{T}_{\omega(p_0)} = \mathbb{T}^n \times \{p_0\}$$



avec vecteur fréquence  $\omega(p_0)$  constant. Soit  $\alpha = dp \wedge dq$  la forme symplectique standard associée à l'espace de phase  $\mathbb{T}^n \times B \subset \mathbb{T}^n \times \mathbb{R}^n$ , alors ces tores sont lagrangiens. Cela signifie que leur dimension est  $n = \frac{1}{2} \dim(\mathbb{T}^n \times \mathbb{R}^n)$  et la restriction de la forme symplectique  $\alpha$  à l'espace tangent s'annule.

Plus précisément, dans ce cas particulier, l'espace de phase est feuilleté en une famille à  $n$  paramètres de tores lagrangiens invariants  $\mathcal{T}_{\omega(p_0)}$  supportant des flots linéaires avec vecteurs fréquence constants  $\omega(p_0)$ .

La question suivante s'ensuit naturellement. Parmi cette famille de tores invariants, lesquels persistent si nous considérons une petite perturbation  $\varepsilon f(q, p)$  du hamiltonien non perturbé  $h$ ? Dit autrement, combien de ces tores invariants persistent si  $0 < \varepsilon \ll 1$ ?

A. N. Kolmogorov a été le premier à répondre à cette question et, en 1954, il a prouvé ce qui est maintenant généralement connu comme le théorème KAM [Kol54] (dans le cadre des hamiltoniens analytiques réels). La preuve repose sur deux idées principales. La première consiste à prouver la persistance des tores invariants avec une dynamique quasipériodique dont le vecteur fréquence satisfait certaines conditions arithmétiques. La deuxième concerne la convergence. L'auteur a prouvé ce résultat avec un schéma itératif basé sur l'algorithme de Newton caractérisé par une convergence quadratique.

À cette fin, nous introduisons la définition de vecteur  $(\gamma, \tau)$ -diophantien.

**Définition.** Pour  $\gamma > 0$  et  $\tau > 0$ , un vecteur  $\omega \in \mathbb{R}^n$  est  $(\gamma, \tau)$ -diophantien si

$$|k \cdot \omega| \geq \frac{\gamma}{|k|^\tau} \quad \text{pour tout } k \in \mathbb{Z}^n / \{0\},$$

où  $|k| = |k_1| + \dots + |k_n|$ .

Pour tout  $\gamma > 0$  et  $\tau \geq 0$ , soit  $D_\gamma^\tau$  l'ensemble de tous les vecteurs  $\omega \in \mathbb{R}^n$  qui satisfont cette infinité de conditions. En outre, nous définissons les ensembles de vecteurs suivants

$$D_\gamma = \bigcup_{\tau > 0} D_\gamma^\tau, \quad D^\tau = \bigcup_{\gamma > 0} D_\gamma^\tau.$$

Si  $\tau < n - 1$ , alors  $D^\tau$  est vide. Nous devons assumer  $\tau \geq n - 1$  afin d'avoir  $D^\tau \neq \emptyset$ . De plus, quand  $\tau > n - 1$ , alors  $D^\tau$  est un ensemble de mesure de Lebesgue totale. D'un autre côté,  $\tau = n - 1$  implique que  $D^\tau$  est un ensemble de mesure de Lebesgue nulle mais la dimension de Hausdorff est maximale.

Nous supposons  $n \geq 2$ , parce que les systèmes à un degré de liberté sont toujours intégrables. Soit  $\Omega = \partial_p h(B)$  l'ensemble des fréquences, nous définissons

$$\Omega_\gamma = \{\omega \in \Omega \cap D_\gamma : d(\omega, \partial\Omega) \geq \gamma\}$$

l'ensemble des vecteurs diophantiens avec au moins une distance  $\gamma$  du bord de  $\Omega$ .

**Théorème (KAM).** Soit  $H$  le hamiltonien défini par (H). Nous supposons que  $H$  est analytique réel et l'application fréquence  $\partial_p h : B \rightarrow \Omega$  est un difféomorphisme. Alors il existe une constante  $c > 0$  telle que si

$$\varepsilon < c\gamma^2$$

tous les tores invariants  $\mathcal{T}_\omega$  du système non perturbé avec  $\omega \in \Omega_\gamma$  persistent comme tores lagrangiens invariants, n'étant que légèrement déformés.

Plus précisément, pour tout  $\omega \in \Omega_\gamma$  il existe un plongement analytique  $\Phi_\omega^\varepsilon : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B$  tel que l'image  $\text{Im}\Phi_\omega^\varepsilon = \mathcal{T}_\omega^\varepsilon$  est un tore invariant par  $H$  et  $\Phi_\omega^\varepsilon$  conjugue la restriction de  $X_H$  à  $\mathcal{T}_\omega^\varepsilon$  au champ de vecteurs constant  $\omega$  sur  $\mathbb{T}^n$ . De plus, le déplacement des variables action du tore perturbé par rapport à celui non perturbé on peut l'estimer comme  $\mathcal{O}(\sqrt{\varepsilon})$ .

Cette formulation du théorème KAM est due à Pöschel et nous nous référons à [Pö01] pour la preuve.

En ce qui concerne l'acronyme KAM, en 1963, Arnold a publié une autre démonstration du théorème de Kolmogorov [Arn63a] avec une hypothèse de non-dégénérescence différente pour les systèmes hamiltoniens analytiques réels. La même année, il a appliqué cette théorie aux questions de stabilité pour le problème à  $n$ -corps. Il a démontré l'existence d'un ensemble de mesure positive des conditions initiales dans l'espace de phase donnant lieu à des mouvements quasipériodiques proches à des trajectoires képlériennes non perturbées, circulaires et coplanaires [Arn63b]. En 1962, Moser a démontré un théorème sur la persistance des courbes invariantes pour twist maps du plan [Mos62]. C'était surprenant parce que, en mélangeant les idées de Kolmogorov et Nash, l'auteur a prouvé ce résultat en différentiabilité finie, remplaçant l'hypothèse analytique assumée par Kolmogorov.

La théorie KAM a connu de nombreux développements au cours des années. Pour une preuve détaillée du théorème de Kolmogorov basé sur ses idées, nous renvoyons à [BGGS84]. Plusieurs autres preuves et raffinements sont apparus dans la littérature. Concernant un travail plus récent, celui de Pöschel [Pö01] est remarquable. Tenant compte de l'idée de Moser [Mos67], l'auteur introduit les fréquences  $\omega = \partial_p h$  comme des paramètres indépendants, donnant un énoncé très subtil et une preuve élégante du théorème.

D'autre part, une série de preuves sont données par l'introduction d'un théorème des fonctions implicites adapté dans une échelle d'espace de Banach qui remplace le système itératif inauguré par Kolmogorov. Nous renvoyons aux travaux de Zehnder [Zeh75], [Zeh76], Herman [Hem86] et Féjoz [Fé04].

Salamon et Zehnder [SZ89] proposent une autre approche au théorème KAM en utilisant le formalisme lagrangien au lieu du formalisme hamiltonien afin d'éviter la composition dénombrable de changement de coordonnées.

Nous concluons avec quelques remarques sur les hypothèses du théorème KAM. Tout d'abord, nous savons déjà que ce résultat peut être prouvé pour des hamiltoniens en différentiabilité finie [Mos62]. Il suffit de supposer que le système soit de classe  $C^k$  avec  $k > 2(\tau + 1) > 2n$  (voir [Pö82], [Sal04], [Bou20], [Kou20]).

Nous mentionnons également le théorème KAM dans le cas Gevrey [Pop04] et ultra différentiable [BF21].

Rüssmann a un rôle important dans la théorie KAM avec plusieurs contributions remarquables. Pour ce qui concerne l'hypothèse de non-dégénérescence, dans le cas analytique, il suffit de supposer que localement l'image de l'application fréquence n'est pas contenue dans aucun hyperplan de  $\mathbb{R}^n$  [Rü89]. Par ailleurs, en utilisant des résultats d'approximation très précis de fonctions analytiques par des polynômes, il a amélioré la condition diophantienne sur le vecteur fréquence en supposant la plus générale qui est maintenant appelée condition Bruno-Rüssmann [Rü01]. Dans ce contexte, nous faisons référence au résultat de Bounemoura-

Fischler [BF14]. Ils donnent une preuve du théorème KAM, qui, en utilisant des approximations rationnelles, ne comporte pas des estimations des petits diviseurs et des expansions en séries de Fourier. Il s'agit d'une autre approche à la preuve du théorème KAM où la condition Bruno-Rüssmann mentionnée ci-dessus remplace la condition diophantienne sur le vecteur fréquence.

Maintenant, nous prenons un peu de recul et proposons un problème, résolu par Arnold [Arn61], sur les petites perturbations des champs de vecteurs constants sur le tore (théorème de forme normale d'Arnold pour les champs de vecteurs sur le tore). Soit  $0 < \varepsilon \ll 1$  un paramètre suffisamment petit, nous considérons le champ de vecteurs sur  $\mathbb{T}^n$  suivant

$$Z : \mathbb{T}^n \longrightarrow \mathbb{R}^n, \quad Z(q) = \omega + \varepsilon f(q) \quad (Z)$$

avec  $\omega \in \mathbb{R}^n$ . Il est légitime de se demander s'il existe un difféomorphisme  $\varphi$  de  $\mathbb{T}^n$ ,  $\varphi = \text{id} + v$  avec  $v : \mathbb{T}^n \rightarrow \mathbb{R}^n$ , qui conjugue le champ de vecteurs  $Z$  avec le champ de vecteurs constant  $\omega$ . C'est-à-dire,

$$Z \circ \varphi(q) = \nabla \varphi(q) \omega$$

pour tout  $q \in \mathbb{T}^n$ , où  $\nabla \varphi$  est la différentielle de  $\varphi$ . En général, ce n'est pas possible. Même si  $f$  est constante et  $\varepsilon$  arbitrairement petit, nous ne pouvons pas conjuguer  $\omega + \varepsilon f$  à  $\omega$  avec un difféomorphisme homotope à l'identité. Cela a forcé Arnold à prouver un théorème légèrement différent et plus artificiel.

**Théorème (Arnold).** *Soit  $Z$  comme dans (Z). Nous supposons que  $Z$  est analytique réel et  $\omega \in D_\gamma^\tau$ . Alors, pour  $\varepsilon$  suffisamment petit, il existe un vecteur  $\alpha \in \mathbb{R}^n$  et un difféomorphisme analytique réel  $\varphi : \mathbb{T}^n \rightarrow \mathbb{T}^n$  tel que*

$$(Z + \alpha) \circ \varphi(q) = \nabla \varphi(q) \omega$$

pour tout  $q \in \mathbb{T}^n$ .

Dans la preuve de ce théorème nous retrouvons certains problèmes similaires à ceux du théorème KAM. Alors que dans le théorème KAM nous recherchons un tore invariant avec une dynamique quasipériodique, ici nous cherchons seulement une conjugaison au champ de vecteurs constant  $\omega$ .

## 1.2 Théorie KAM non-autonome

En ce qui concerne cette thèse, nous nous sommes intéressés aux perturbations non autonomes satisfaisant de bonnes propriétés de décroissance lorsque  $t \rightarrow \pm\infty$ . Dans ce cas, nous ne cherchons pas des solutions quasipériodiques, mais des différents types de solutions qui sont encore liées à des solutions quasipériodiques.

Pour être plus précis, introduisons la définition du tore KAM asymptotique analytique. Pour certains  $s > 0$ , nous définissons les domaines complexes suivants

$$\begin{aligned} \mathbb{T}_s^n &:= \{q \in \mathbb{C}^n / \mathbb{Z}^n : |\text{Im}(q)| \leq s\} \\ B_s &:= \{p \in \mathbb{C}^n : |p| \leq s\}, \end{aligned}$$

et pour tout  $v > 0$ , nous introduisons l'intervalle qui suit

$$J_v = [v, +\infty) \subset \mathbb{R}.$$

Pour chaque fonction  $f$  définie sur  $\mathbb{T}_s^n \times B_s \times J_v$  et pour  $t \in J_v$  fixé, soit  $f^t$  la fonction définie sur  $\mathbb{T}_s^n \times B_s$  telle que

$$f^t(q, p) = f(q, p, t).$$

Nous gardons cette notation pour le reste de ce travail, également lorsque  $\mathbb{T}_s^n \times B_s$  est remplacé par  $\mathbb{T}^n \times B$  ou une variété lisse appropriée.

Soit  $\mathcal{P}$  égal à  $\mathbb{T}^n \times B$ ,  $\mathbb{T}^n$  ou à un sous-ensemble ouvert de  $\mathbb{R}^{2n}$ . Nous considérons les champs de vecteurs analytiques réels dépendant du temps  $X^t$  et  $X_0^t$  sur  $\mathcal{P}$ , pour tout  $t \in J_v$ , et un plongement analytique de  $\mathbb{T}^n$  sur  $\mathcal{P}$  tel que

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_s = 0, \quad (1.1)$$

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q) \omega \text{ pour tout } (q, t) \in \mathbb{T}^n \times J_v, \quad (1.2)$$

où  $\omega \in \mathbb{R}^{2n}$  et  $|\cdot|_s$  est la norme analytique (voir Appendix B). Par souci de clarté, nous précisons que  $\partial_q \varphi_0(q) \omega$  est l'élément de  $\mathbb{R}^{2n}$  ayant la composante  $j$  égale à

$$\left( \partial_q \varphi_0(q) \omega \right)_j = \partial_q \varphi_{0,j}(q) \cdot \omega,$$

pour tout  $j = 1, \dots, 2n$ . Autrement dit, nous considérons un champ de vecteurs  $X^t$  convergent asymptotiquement dans le temps vers un champ de vecteurs  $X_0^t$  ayant un tore invariant supportant une dynamique quasipériodique de vecteur fréquence  $\omega$ .

**Définition 1.1.** *Nous supposons que  $(X, X_0, \varphi_0)$  satisfont (1.1) et (1.2). Une famille de plongements analytiques réels  $\varphi^t : \mathbb{T}^n \rightarrow \mathcal{P}$  est un tore KAM (positif) asymptotique analytique associé à  $(X, X_0, \varphi_0)$  s'il existe  $0 < s' \leq s$  et  $v' \geq v \geq 0$  tel que*

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t), \quad (1.3)$$

$$\lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{s'} = 0, \quad (1.4)$$

pour tout  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Lorsque  $\mathcal{P}$  est une variété symplectique avec  $\dim \mathcal{P} = 2n$ , alors nous disons que  $\varphi^t$  est lagrangien si  $\varphi^t(\mathbb{T}^n)$  est lagrangien pour tout  $t$ .

Si nous définissons  $J_v = (-\infty, -v]$  et nous remplaçons (1.1) et (1.4) avec

$$\lim_{t \rightarrow -\infty} |X^t - X_0^t|_s = 0, \quad \lim_{t \rightarrow -\infty} |\varphi^t - \varphi_0|_{s'} = 0,$$

nous obtenons la définition du tore KAM (négatif) asymptotique analytique associé à  $(X, X_0, \varphi_0)$ . Dans ce qui suit, nous discutons le cas pour les temps positifs ( $t \in \mathbb{R}^+$ ). Des résultats similaires sont vrais pour les temps négatifs.

Cette définition est due à M. Canadell et R. de la Llave (see [CdIL15]). Dans cet article, ils utilisent l'expression tore KAM non-autonome. Nous préférons tore

KAM asymptotique pour souligner les propriétés asymptotiques de cette famille de plongements. La définition originale est pour des champs de vecteurs définis sur des variétés lisses. Cependant, dans ce travail nous nous intéressons aux champs de vecteurs hamiltoniens ou aux champs de vecteurs sur le tore. C'est pour cette raison que  $\mathcal{P}$  est égal à  $\mathbb{T}^n \times B$ ,  $\mathbb{T}^n$  ou à un sous-ensemble ouvert de  $\mathbb{R}^{2n}$ . Dans ce qui suit, nous faisons une série de remarques sur la définition précédente toujours due à M. Canadell et R. de la Llave.

Nous observons que nous pouvons réécrire (1.3) en termes du flot de  $X$ . En fait, soit  $\psi_{t_0, X}^t$  le flot au moment  $t$  avec un temps initial  $t_0$  de  $X$ .

**Proposition 1.1.** *Si le flot  $\psi_{t_0, X}^t$  est défini pour tout  $t$ ,  $t_0 \in J_{v'}$ , alors (1.3) est équivalent à*

$$\psi_{t_0, X}^t \circ \varphi^{t_0}(q) = \varphi^t(q + \omega(t - t_0)), \quad (1.5)$$

pour tout  $t$ ,  $t_0 \in J_{v'}$  et  $q \in \mathbb{T}^n$ .

*Démonstration.* Nous nous référons à la Section 3.1. □

Grâce à la proposition précédente, il est facile à voir que (1.3) est trivial.

**Proposition 1.2.** *Si  $\psi_{t_0, X}^t$  est défini pour tout  $t$ ,  $t_0 \in J_{v'}$ , il est toujours possible de trouver une famille de plongements satisfaisant (1.3).*

*Démonstration.* Soit  $\hat{\varphi} : \mathbb{T}^n \rightarrow \mathcal{P}$  un plongement alors, pour tout  $t$ ,  $t_0 \in J_{v'}$  et  $q \in \mathbb{T}^n$ , nous considérons

$$\varphi^t(q) = \psi_{t_0, X}^t \circ \hat{\varphi}(q - \omega(t - t_0)).$$

Il s'agit d'une famille de plongements satisfaisant (1.5). En effet, par la précédente définition de  $\varphi^t$  nous avons que  $\varphi^{t_0}(q) = \hat{\varphi}(q)$  pour tout  $q \in \mathbb{T}^n$ . Ensuite, par construction,  $\varphi^t$  satisfait (1.5) et donc (1.3). □

Une autre conséquence importante de (1.5) est la suivante.

**Proposition 1.3.** *Nous supposons que  $\psi_{t_0, X}^t$  est défini pour tout  $t$ ,  $t_0 \in \mathbb{R}$ . S'il existe un tore KAM asymptotique analytique  $\varphi^t$  défini pour tout  $t \geq v'$ , alors nous pouvons étendre l'ensemble de définition pour tout  $t \in \mathbb{R}$ .*

*Démonstration.* Pour tout  $q \in \mathbb{T}^n$ , nous considérons

$$\phi^t(q) = \begin{cases} \varphi^t(q) & \text{pour tout } t \geq v' \\ \psi_{v', X}^t \circ \varphi^{v'}(q - \omega(t - v')) & \text{pour tout } t \leq v'. \end{cases} \quad (1.6)$$

Il s'agit d'une famille de plongements qui vérifie (1.3) et (1.4). □

Malheureusement, nous ne pouvons déduire aucune information asymptotique pour la famille de plongements (1.6) quand  $t \rightarrow -\infty$ .

Pour ce qui concerne la dynamique associée à un tore KAM asymptotique analytique, nous introduisons la définition de solution asymptotiquement quasipériodique et discutons certaines propriétés de ces orbites.

**Définition 1.2.** Nous supposons que  $(X, X_0, \varphi_0)$  satisfait (1.1) et (1.2). Une courbe intégrale  $g(t)$  de  $X$  est une solution asymptotiquement quasipériodique (positive) associée à  $(X, X_0, \varphi_0)$  s'il existe  $q \in \mathbb{T}^n$  de telle sorte que

$$\lim_{t \rightarrow +\infty} |g(t) - \varphi_0(q + \omega(t - t_0))| = 0.$$

La proposition suivante prouve que si  $\varphi^t$  est un tore KAM asymptotique analytique associé à  $(X, X_0, \varphi_0)$ , alors chaque point initial  $\varphi^{t_0}(q)$  donne lieu à une solution asymptotiquement quasipériodique associée à  $(X, X_0, \varphi_0)$ .

**Proposition 1.4.** Soit  $\varphi^t$  un tore KAM asymptotique analytique associé à  $(X, X_0, \varphi_0)$ . Alors, pour tout  $q \in \mathbb{T}^n$  et  $t_0 \in J_v$ ,

$$g(t) = \psi_{t_0, X}^t \circ \varphi^{t_0}(q)$$

est une solution asymptotiquement quasipériodique associée à  $(X, X_0, \varphi_0)$ .

*Démonstration.* Grâce à (1.5)

$$g(t) = \psi_{t_0, X}^t \circ \varphi^{t_0}(q) = \varphi^t(q + \omega(t - t_0))$$

et donc, par (1.4), nous obtenons ce qu'on voulait démontrer.  $\square$

Nous concluons cette section avec une propriété importante concernant le cas dans lequel  $X$  et  $X_0$  sont des champs de vecteurs hamiltoniens. Soit  $\mathcal{P} = \mathbb{T}^n \times B$  et nous supposons que  $\varphi^t$  est un tore KAM asymptotique analytique associé à  $(X, X_0, \varphi_0)$ . Dans le cas particulier des systèmes hamiltoniens, si le tore invariant  $\varphi_0$  est lagrangien, alors  $\varphi^t$  est lagrangien pour tout  $t$ . Plus concrètement, nous avons la proposition suivante

**Proposition.** Soit  $\varphi^t$  un tore KAM asymptotique analytique associé à  $(X, X_0, \varphi_0)$ . Si  $\varphi_0$  est lagrangien, alors  $\varphi^t$  est lagrangien pour tout  $t \in J_v$ .

M.Canadell et R. de la Llave le prouvent dans le cas discret. Dans la Section 3.1, nous le prouvons dans le cas continu.

Le premier résultat concernant l'existence d'un tore KAM asymptotique analytique associé à des hamiltoniens dépendants du temps est dû à A. Fortunati et S. Wiggins [FW14]. Ils considèrent des hamiltoniens analytiques réels dépendant du temps de la forme suivante

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times \mathbb{R}^+ \rightarrow \mathbb{R}, \\ H(q, p, t) = \omega \cdot p + \underbrace{\frac{1}{2} \Gamma \cdot p^2}_h + \varepsilon f(q, p, t), \\ \omega \in D_\tau^\gamma, \\ \Gamma \text{ est réelle, symétrique et non-singulière,} \\ f \text{ est quadratique en } p, \\ |f^t|_s \leq C e^{-at} \text{ pour tout } t \in \mathbb{R}^+, \end{array} \right. \quad (\text{FW})$$

pour certains  $s > 0$ , une constante positive  $C$  et  $a \in (0, 1)$ . Nous soulignons que  $\Gamma \cdot p^2$  représente le vecteur  $p$  donné deux fois comme argument de la forme bilinéaire symétrique  $\Gamma$ .

Soit  $\varphi_0$  le plongement trivial suivant

$$\varphi_0 : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B, \quad \varphi_0(q) = (q, 0).$$

**Théorème** (Fortunati-Wiggins). *Soit  $H$  comme dans (FW). Alors, si  $\varepsilon$  est suffisamment petit, il existe un tore KAM asymptotique analytique associé à  $(X_H, X_h, \varphi_0)$ .*

Rappelons que  $X_H$  représente le champ de vecteurs hamiltonien associé au hamiltonien  $H$ . Dans la preuve, ils convertissent le hamiltonien  $H$  en un hamiltonien autonome en introduisant  $\eta \in \mathbb{R}$  comme le momentum conjugué à  $\xi = t$ . La preuve repose sur un schéma à la Kolmogorov avec des modifications appropriées. La partie intéressante consiste en la solution de l'équation homologique correspondante, qui est résolue en utilisant des estimations des petits diviseurs et des expansions en séries de Fourier.

Environ un an plus tard, M. Canadell and R. de la Llave [CdLL15] ont publié un résultat qui généralise celui de Fortunati-Wiggins. Dans cet article, ils travaillent en différentiabilité finie. Ils considèrent des champs de vecteurs dépendant du temps qui convergent dans le temps à des champs de vecteurs autonomes ayant un tore invariant supportant une dynamique quasipériodique. À cette fin, introduisons la définition de  $C^\sigma$ -tore KAM asymptotique qui est la version à différentiabilité finie de la Définition 1.1.

Étant donné  $\sigma \geq 0$  et un entier positif  $k \geq 0$ , nous considérons les champs de vecteurs dépendant du temps  $X^t$  et  $X_0^t$  de classe  $C^{\sigma+k}$  sur une variété lisse  $\mathcal{P}$ , pour tout  $t \in J_v$ , et un plongement  $\varphi_0$  de  $\mathbb{T}^n$  sur  $\mathcal{P}$  de classe  $C^\sigma$  tel que

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_{C^{\sigma+k}} = 0, \tag{1.7}$$

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q) \omega \text{ pour tout } (q, t) \in \mathbb{T}^n \times J_v, \tag{1.8}$$

où  $\omega \in \mathbb{R}^n$ ,  $C^\sigma$  est l'espace des fonctions Hölder et  $|\cdot|_{C^\sigma}$  est la norme Hölder (voir Appendix A).

**Définition 1.3.** *Nous supposons que  $(X, X_0, \varphi_0)$  satisfait (1.7) et (1.8). Une famille de  $C^\sigma$ -plongements  $\varphi^t : \mathbb{T}^n \rightarrow \mathcal{P}$  est un  $C^\sigma$ -tore KAM asymptotique associé à  $(X, X_0, \varphi_0)$  s'il existe  $v' \geq v \geq 0$  tel que*

$$\begin{aligned} \lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{C^\sigma} &= 0, \\ X(\varphi(q, t), t) &= \partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t), \end{aligned}$$

pour tout  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Lorsque  $\mathcal{P}$  est une variété symplectique avec  $\dim \mathcal{P} = 2n$ , alors nous disons que  $\varphi^t$  est lagrangien si  $\varphi^t(\mathbb{T}^n)$  est lagrangien pour tout  $t$ .

Toutes les remarques et considérations faites ci-dessus pour ce qui concerne des tores KAM asymptotique analytique restent également vrai pour des  $C^\sigma$ -tores KAM asymptotiques.

Pour revenir au travail de Canadell-de la Llave, soit  $\mathcal{P}$  une variété lisse arbitraire et  $\sigma \geq 1$ . Nous considérons un champ de vecteurs dépendant du temps  $X^t$  de classe  $C^{\sigma+1}$  sur  $\mathcal{P}$ , un champ de vecteurs autonome  $X_0$  de classe  $C^{\sigma+1}$  sur  $\mathcal{P}$  et un plongement de  $\mathbb{T}^n$  sur  $\mathcal{P}$  de classe  $C^\sigma$  tel que

$$\begin{cases} |X^t - X_0|_{C^{\sigma+1}} \leq \varepsilon \lambda^t & \text{pour tout } t \in \mathbb{R}^+ \\ X_0 \circ \varphi_0(q) = \partial_q \varphi_0(q) \omega & \text{pour tout } q \in \mathbb{T}^n \end{cases} \quad (\text{CdL})$$

pour des paramètres positifs appropriés  $\varepsilon$  et  $0 \leq \lambda < 1$ . Soit  $\phi_{X_0}^t$  le flot de  $X_0$  et  $D\phi_{X_0}^t$  son différentiel.

**Théorème** (Canadell-de la Llave). *Soit  $X$  comme dans (CdL). Nous supposons qu'il existe  $\mu > 0$  tel que*

$$|D\phi_{X_0}^d \circ \varphi_0 \cdots D\phi_{X_0}^0 \circ \varphi_0|_{C^0} \leq C\mu^d \quad (1.9)$$

$$\mu^\sigma \lambda < 1 \quad (1.10)$$

pour tout  $d \in \mathbb{N}$  et une constante positive appropriée  $C$ . Alors, pour  $\varepsilon$  suffisamment petit, il existe un  $C^\sigma$ -tore KAM asymptotique associé à  $(X, X_0, \varphi_0)$

Les auteurs considèrent un champ de vecteurs dépendant du temps convergeant, avec une vitesse exponentielle par rapport au temps, vers un champ de vecteurs autonome ayant un tore invariant supportant une dynamique quasipériodique. Ils ont besoin d'un certain contrôle, donné par le paramètre  $\mu$ , sur la dynamique transverse au tore invariant  $\varphi_0$ . Ce paramètre est en compétition avec le paramètre  $\lambda$  qui mesure la vitesse de décroissance du champ de vecteurs dépendant du temps  $X^t$ . Ils démontrent aussi que le  $C^\sigma$ -tore KAM asymptotique trouvé converge avec une vitesse exponentielle par rapport au temps vers le plongement autonome  $\varphi_0$ . Cela signifie que, pour tout  $t \in \mathbb{R}^+$ ,

$$|\varphi^t - \varphi_0|_{C^\sigma} \leq C\lambda^t$$

pour une constante appropriée  $C$  qui peut différer de celle dans l'énoncé du théorème. La preuve repose sur le théorème des fonctions implicites.

Ce théorème généralise celui de Fortunati-Wiggins. Tout d'abord, Fortunati-Wiggins prouvent un théorème concernant des systèmes hamiltoniens dépendant du temps, tandis que Canadell-de la Llave travaillent avec des champs de vecteurs plus généraux sur des variétés lisses. Nous notons que Canadell-de la Llave ne supposent aucune hypothèse de non dégénérescence sur  $X_0$  et aucune condition arithmétique sur le vecteur fréquence  $\omega$ . Par contre, ils maintiennent la décroissance exponentielle par rapport au temps du champ de vecteurs dépendant du temps  $X^t$ . Concernant (1.9) et (1.10), nous avons quelques considérations à faire. Rappelons le hamiltonien  $H$  considéré par Fortunati et Wiggins

$$H : \mathbb{T}^n \times B \times \mathbb{R}^+ \rightarrow \mathbb{R}, \quad H(q, p, t) = \underbrace{\omega \cdot p + \frac{1}{2} \Gamma \cdot p^2}_{h(p)} + \varepsilon f(q, p, t)$$



où  $\Gamma$  est une matrice réelle, symétrique, non-singulière et  $h$  est le hamiltonien non perturbé. Par ailleurs, pour un certain  $s > 0$

$$|f^t|_s \leq C e^{-at}$$

pour tout  $t \in \mathbb{R}^+$ , où  $a$  est un paramètre positif pris conventionnellement plus petit que 1. Soit  $\phi_{X_h}^t$  le flot du hamiltonien  $X_h$  associé au hamiltonien  $h$ , alors pour tout  $(q, p) \in \mathbb{T}^n \times B$

$$\phi_{X_h}^t(q, p) = (q + (\omega + \Gamma p)t, p).$$

Donc, en prenant la différentielle de  $\phi_{X_h}^t$

$$D\phi_{X_h}^t(q, p) = \begin{pmatrix} \text{Id} & \Gamma t \\ 0 & \text{Id} \end{pmatrix}.$$

Cela signifie que, pour tout  $d \in \mathbb{N}$

$$|D\phi_{X_h}^d \circ \varphi_0 \cdots D\phi_{X_h}^0 \circ \varphi_0|_{C^0} \leq C(1 + \dots + d)|\Gamma| = C|\Gamma| \frac{d(d+1)}{2},$$

où  $|\Gamma| = \max_{1 \leq i, j \leq n} |\Gamma_{ij}|$  et  $C$  est une constante qui ne dépend pas de  $d$ . Dans ce cas, nous observons que la norme précédente est contrôlée par le terme  $d(d+1)$  qui, quand  $\mu > 1$ , diverge plus lentement que  $\mu^d$  si  $d \rightarrow +\infty$ . Alors, dans cette situation particulière, nous pouvons remplacer (1.10) avec  $\lambda < 1$ . Cela équivaut à supposer  $a > 0$  dans le résultat de Fortunati-Wiggins. Autrement dit, si nous considérons des perturbations dépendant du temps des systèmes hamiltoniens intégrables, il suffit de supposer que la perturbation décroît avec une vitesse exponentielle par rapport au temps, sans aucune hypothèse sur le taux de décroissance. Cela se traduit en  $a > 0$  dans le théorème de Fortunati-Wiggins et  $\lambda < 1$  dans celui de Canadell-de la Llave.

Cela nous amène à nous poser la question suivante. Est-il possible de relaxer l'hypothèse de décroissance exponentielle par rapport au temps dans le cas de perturbations dépendant du temps des systèmes hamiltoniens intégrables? En d'autres termes, si nous considérons une perturbation dépendant du temps d'un système hamiltonien ayant un tore invariant supportant une dynamique quasipériodique, est-il possible de prouver un résultat similaire lorsque la perturbation décroît avec une vitesse plus faible qu'exponentielle? C'est bien cette question qui a motivé le début de ce travail.

L'intérêt pour ce genre de perturbations n'est pas artificiel. Ces types de systèmes seraient intéressants en astronomie. L'exemple qui a motivé ce travail est celui d'un système planétaire perturbé par une comète provenant et retournant à l'infini, sur une orbite asymptotiquement hyperbolique. Dans la partie suivante, nous présenterons les principaux résultats contenus dans cette thèse.

## 2 Résultats Principaux

Les résultats principaux de cette thèse sont organisés en quatre parties. La première partie, appelée *Asymptotic Motions Converging to Quasiperiodic Dynamics* (voir II), traite certains résultats qui améliorent les théorèmes de Fortunati-Wiggins et Canadell-de la Llave dans le cas des champs de vecteurs hamiltoniens et

champs de vecteurs sur le tore dépendant du temps. La décroissance exponentielle par rapport au temps est relaxée et l'hypothèse de petitesse sur la perturbation est supprimée.

Dans la deuxième partie, *Biasymptotic Motions Converging to Quasiperiodic Dynamics* (voir III), en utilisant une version différente du résultat mentionné ci-dessus, nous prouvons l'existence de solutions biasymptotiquement quasipériodiques pour des champs de vecteurs hamiltoniens dépendant du temps. Plus précisément, nous montrons l'existence de certaines orbites convergeant vers une dynamique quasipériodique caractérisée par un certain vecteur fréquence  $\omega_+ \in \mathbb{R}^n$  dans le futur ( $t \rightarrow +\infty$ ) et vers une dynamique quasipériodique de vecteur fréquence  $\omega_- \in \mathbb{R}^n$  dans le passé ( $t \rightarrow -\infty$ ).

La troisième partie, *Applications to Celestial Mechanics* (voir IV), est consacrée à l'étude de l'exemple d'un système planétaire (le problème à trois corps dans le plan) perturbé par une comète donnée provenant et retournant à l'infini, asymptotiquement le long d'une orbite képlérienne hyperbolique. Nous démontrons qu'il existe des solutions où la comète attire le centre de masse du système planétaire avec une vitesse asymptotiquement nulle quand le temps tend vers  $+\infty$ . En outre, si nous considérons un repère attaché au centre de masse du système planétaire, les mouvements des planètes convergent vers certaines dynamiques qui sont proches (dans un sens que nous préciserons plus tard) de certains mouvements quasipériodiques associés au hamiltonien du problème à trois corps dans le plan, dont l'existence est garantie par [Arn63b]

La partie *Asymptotic Motions Converging to Arbitrary Dynamics* (voir V) conclut cette thèse. Nous prouvons certains résultats qui sont une variation du théorème de Canadell-de la Llave dans les cas particuliers des champs de vecteurs hamiltoniens et champs de vecteurs sur le tore dépendant du temps. Plus précisément, nous considérons des champs de vecteurs dépendant du temps appropriés  $X$  convergeant avec une vitesse exponentielle par rapport au temps vers des champs de vecteurs  $X_0$  ayant un tore invariant  $\varphi_0$  supportant une dynamique quelconque. La nouveauté repose sur le fait que la dynamique sur  $\varphi_0$  est arbitraire (donc, en général, pas quasipériodique). La décroissance exponentielle par rapport au temps est maintenue, mais nous n'avons pas besoin d'hypothèses de petitesse.

Dans ce qui suit, nous examinerons plus en profondeur ce qui a été dit jusqu'à présent. Plus précisément, la dernière partie de ce chapitre est divisée en quatre sections. Chacune d'elles est consacrée à une partie de cette thèse, où nous énoncerons et commenterons chaque résultat contenu dans ce travail.

Nous rappelons que  $B \subset \mathbb{R}^n$  est une boule centrée à l'origine et, pour un certain  $v > 0$ , nous introduisons l'intervalle suivant

$$J_v = [v, +\infty) \subset \mathbb{R}.$$

Pour chaque fonction  $f$  définie sur  $\mathbb{T}^n \times B \times J_v$  et pour  $t \in J_v$  fixé, soit  $f^t$  la fonction définie sur  $\mathbb{T}^n \times B$  tel que

$$f^t(q, p) = f(q, p, t)$$

pour tout  $(q, p) \in \mathbb{T}^n \times B$ . Par ailleurs, pour  $p_0 \in B$  fixé, soit  $f_{p_0}$  la fonction définie sur  $\mathbb{T}^n \times J_v$  tel que

$$f_{p_0}(q, t) = f(q, p_0, t)$$

pour tout  $(q, t) \in \mathbb{T}^n \times J_v$ . Nous gardons cette notation aussi dans le cas où  $\mathbb{T}^n \times B$  est remplacé par un voisinage complexe  $\mathbb{T}_s^n \times B_s$  de  $\mathbb{T}^n \times B$  ou une variété lisse appropriée. De plus, nous utilisons cette notation également dans le cas où  $J_v$  est remplacé par  $\mathbb{R}$ .

## 2.1 Mouvements Asymptotiques Convergeant vers une Dynamique Quasipériodique

Cette section est divisée en deux sous-sections. La raison de cette division est que nous prouvons les mêmes résultats dans le cadre à différentiabilité finie et analytique. Nous sommes intéressés à des perturbations dépendant du temps de certains hamiltoniens non autonomes et des champs de vecteurs constants sur le tore. Plus précisément, nous considérons les hamiltoniens dépendant du temps de la forme

$$H : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R}, \quad H(q, p, t) = h(q, p, t) + f(q, p, t),$$

où  $h$  est dans la  $\omega$ -forme normal de Kolmogorov, avec  $\omega \in \mathbb{R}^n$ . Cela signifie que, pour tout  $(q, t) \in \mathbb{T}^n \times J_0$

$$h(q, 0, t) = c, \quad \partial_p h(q, 0, t) = \omega,$$

avec  $c \in \mathbb{R}$ . Soit  $\varphi_0$  le plongement trivial suivant

$$\varphi_0 : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B, \quad \varphi_0(q) = (q, 0),$$

alors  $\varphi_0$  est un tore lagrangien invariant pour  $X_h$  supportant une dynamique quasipériodique de vecteur fréquence  $\omega$ . Nous rappelons que  $X_h$  est le champ de vecteurs hamiltonien associé au hamiltonien  $h$ . Nous définissons  $\mathcal{K}_\omega$  comme étant l'ensemble des Hamiltoniens  $h : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R}$  sous la  $\omega$ -forme normale de Kolmogorov.

D'autre part, nous considérons les champs de vecteurs sur le tore dépendant du temps  $Z$  de telle sorte que

$$Z : \mathbb{T}^n \times J_0 \rightarrow \mathbb{R}^n, \quad Z(q, t) = \omega + P(q, t)$$

avec  $\omega \in \mathbb{R}^n$ .

### 2.1.1 Cas à Différentiabilité Finie

Nous nous intéressons aux fonctions holderiennes  $C^\sigma$ . Nous nous référons à Appendix A pour une petite introduction. Plus précisément, afin de quantifier la régularité des fonctions, nous introduisons l'espace suivant. Étant donné  $\sigma \geq 0$ ,  $\nu \geq 0$  et un entier positif  $k \geq 0$ , nous avons la définition suivante

**Définition.** Soit  $\bar{\mathcal{S}}_{\sigma, k}^\nu$  l'espace de fonctions  $f$  définies sur  $\mathbb{T}^n \times B \times J_\nu$  tel que, pour tout  $t \in J_\nu$

$$f^t \in C^{\sigma+k}(\mathbb{T}^n \times B) \quad \text{et} \quad \partial_{(q,p)}^i f \in C(\mathbb{T}^n \times B \times J_\nu)$$

pour tout  $0 \leq i \leq k$ .

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Conventionnellement  $f = \partial_{(q,p)}^0 f$ . En d'autres termes,  $f \in \bar{\mathcal{S}}_{\sigma,k}^v$  si  $f^t \in C^{\sigma+k}(\mathbb{T}^n \times B)$ , pour tout  $t \in J_v$ , et  $f$  est continue et ses dérivés partielles par rapport à  $(q,p)$  le sont aussi jusqu'à l'ordre  $k$ . Nous utiliserons cette notation également pour des applications définies sur  $\mathbb{T}^n \times J_v$ . Ceci sera spécifié par le contexte.

Pour mesurer la décroissance par rapport au temps des perturbations, nous introduisons des fonctions positives, décroissantes et intégrables  $\mathbf{u}$  sur  $J_0$  et nous notons

$$\bar{\mathbf{u}}(t) = \int_t^{+\infty} \mathbf{u}(\tau) d\tau$$

pour tout  $t \in J_0$ .

Maintenant, nous avons tout ce dont nous avons besoin pour énoncer le théorème suivant. Étant donné  $\omega \in \mathbb{R}^n$  et  $\sigma \geq 1$ , nous considérons l'hamiltonien dépendant du temps  $H$  de la forme

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times J_0 \longrightarrow \mathbb{R} \\ H(q, p, t) = h(q, p, t) + f(q, p, t), \\ h \in \mathcal{K}_\omega \\ f_0, \partial_p f_0, \partial_p^2 H \in \bar{\mathcal{S}}_{\sigma,2}^0 \\ \sup_{t \in J_0} |f_0^t|_{C^{\sigma+2}} < \infty, \quad \sup_{t \in J_0} |\partial_p^2 H^t|_{C^{\sigma+2}} < \infty, \\ |\partial_q f_0^t|_{C^{\sigma+1}} \leq \mathbf{a}(t), \quad |\partial_p f_0^t|_{C^{\sigma+2}} \leq \mathbf{b}(t) \quad \text{pour tout } t \in J_0, \end{array} \right. \quad (*_A)$$

où  $\mathbf{a}$  et  $\mathbf{b}$  sont des fonctions positives, décroissantes et intégrables sur  $J_0$ . Nous supposons qu'il existe  $v \geq 0$  tel que  $\mathbf{a}$  et  $\mathbf{b}$  satisfont les conditions suivantes

$$\left\{ \begin{array}{l} \bar{\mathbf{a}}(t) \leq \mathbf{b}(t) \\ \bar{\mathbf{a}}(t)\mathbf{b}(t) \leq \mathbf{a}(t)\bar{\mathbf{b}}(t) \end{array} \right. \quad (\#)$$

pour tout  $t \in J_v$ .

**Théorème A.** *Soit  $H$  comme dans  $(*_A)$  avec  $\mathbf{a}$  et  $\mathbf{b}$  qui satisfont  $(\#)$ . Alors, il existe  $\tilde{h} \in \mathcal{K}_\omega$  et un  $C^\sigma$ -tore KAM asymptotique lagrangien  $\varphi^t$  associé à  $(X_H, X_{\tilde{h}}, \varphi_0)$ .*

Nous commençons avec deux exemples des fonctions  $\mathbf{a}$  et  $\mathbf{b}$  qui vérifient  $(\#)$ . Tout d'abord, nous considérons le cas à décroissance exponentielle. Soient  $\mathbf{a}$  et  $\mathbf{b}$  les fonctions suivantes

$$\mathbf{a}(t) = e^{-\lambda_1 t}, \quad \mathbf{b}(t) = \frac{e^{-\lambda_2 t}}{\lambda_1},$$

pour certains paramètres positifs  $\lambda_1 \geq \lambda_2 > 0$ . Nous pouvons vérifier facilement que  $(\#)$  est satisfait pour tout  $t \in J_0$ .

L'exemple suivant est plus intéressant que le précédent. Il s'agit de décroissance polynomiale. Nous considérons

$$\mathbf{a}(t) = \frac{1}{t^{l+1}}, \quad \mathbf{b}(t) = \frac{1}{t^l},$$

pour un paramètre réel positif  $l > 1$ . Ce couple de fonctions satisfait (#) pour tout  $t \in J_1$ .

Le théorème précédent prouve l'existence d'un  $C^\sigma$ -tore KAM asymptotique  $\varphi^t$  de la forme

$$\varphi^t(q) = (q + u^t(q), v^t(q))$$

pour tout  $q \in \mathbb{T}^n$  et  $t$  suffisamment grand, où  $\text{id} + u^t$  est un difféomorphisme du tore pour tout  $t$  fixé. En outre, nous obtenons également des informations sur la décroissance par rapport au temps de  $u$  et  $v$ . Plus précisément

$$|u^t|_{C^\sigma} \leq C\bar{\mathbf{b}}(t), \quad |v^t|_{C^\sigma} \leq C\bar{\mathbf{a}}(t),$$

pour tout  $t$  assez grand et pour une constante appropriée  $C$ .

Ce théorème généralise celui de Canadell-de la Llave dans le cas des champs de vecteurs hamiltoniens dépendant du temps. Concernant la preuve, elle est basée sur le théorème des fonctions implicites. Par ailleurs, une partie importante repose sur la résolution du problème linéaire associé. Étant donné  $\sigma \geq 0$ ,  $\nu \geq 0$  et  $\omega \in \mathbb{R}^n$ , il consiste dans la solution de l'équation suivante d'inconnu  $\varkappa : \mathbb{T}^n \times J_\nu \rightarrow \mathbb{R}$

$$\begin{cases} \omega \cdot \partial_q \varkappa(q, t) + \partial_t \varkappa(q, t) = g(q, t), \\ g \in \bar{\mathcal{S}}_{\sigma, 0}^\nu, \\ |g^t|_{C^\sigma} \leq \mathbf{g}(t), \quad \text{for all } t \in J_\nu, \end{cases} \quad (HE_A)$$

où  $\mathbf{g}(t)$  est une fonction positive, décroissante et intégrable sur  $J_\nu$  et  $g : \mathbb{T}^n \times J_\nu \rightarrow \mathbb{R}$  est donnée. Nous pouvons facilement trouver une solution à l'équation précédente par intégration grâce à un changement de coordonnées approprié qui rectifie la dynamique sur le tore. Cette solution  $\varkappa$  existe et satisfait

$$|\varkappa^t|_{C^\sigma} \leq \bar{\mathbf{g}}(t) \quad \text{pour tout } t \in J_\nu.$$

Dans un premier lieu, nous observons que  $\varkappa$  a la même régularité que  $g$  (elle est de classe  $C^\sigma$ ). Si  $\mathbf{g}(t) = e^{-\lambda t}$  avec  $\lambda > 0$  (comme dans le théorème de Canadell-de la Llave), alors  $\bar{\mathbf{g}}(t) = \frac{1}{\lambda} e^{-\lambda t}$ . Par conséquent, dans ce cas particulier, la solution  $\varkappa$  décroît avec une vitesse exponentielle et le même taux de décroissance  $\lambda$  de  $g$ . Maintenant, si nous considérons  $\mathbf{g}(t) = \frac{1}{t^l}$  avec  $l > 1$ , alors  $\varkappa$  décroît comme  $\frac{1}{t^{l-1}}$  ( $\bar{\mathbf{g}}(t) = \frac{1}{l-1} \frac{1}{t^{l-1}}$ ). Dans ce cas, nous perdons en décroissance par rapport au temps.

En retournant à la preuve du Théorème A, nous considérons  $\mathbf{a}(t) = \frac{1}{t^{l+1}}$  et  $\mathbf{b}(t) = \frac{1}{t^l}$  avec  $l > 1$ . Nous savons que (#) sont satisfaits. Le problème de perte de décroissance par rapport au temps est résolu dans la partie non linéaire. Les solutions de l'équation homologique, caractérisées par une lente décroissance dans le temps (c'est-à-dire  $\frac{1}{t^l}$  et  $\frac{1}{t^{l-1}}$ ), sont toujours multipliées par d'autres termes satisfaisant de bonnes propriétés de décroissance ( $\frac{1}{t^{l+1}}$  ou  $\frac{1}{t^l}$ ). Donc, grâce à une série de multiplications, nous pouvons regagner ce que nous perdons en matière de décroissance par rapport au temps. De même que dans le cas général ( $\mathbf{a}(t)$  et  $\mathbf{b}(t)$  sont comme en (#)). Ici, l'hypothèse (#) assure que cette série de multiplications dans la partie non linéaire résolve la perte de décroissance par rapport au temps.

Contrairement au théorème de Canadell-de la Llave, nous n'avons pas besoin d'assumer une condition de petitesse sur la perturbation. Nous introduisons  $\nu' \geq 0$  afin d'avoir

$$|\partial_q f_0^t|_{C^{\sigma+1}}, \quad |\partial_p f_0^t|_{C^{\sigma+2}}$$

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suffisamment petit pour tout  $t \geq v'$ . Par conséquent, nous étudions  $H$  pour tout  $t \geq v'$  et nous prouvons l'existence d'un tore KAM asymptotique associé à  $(X_H, X_{\tilde{h}}, \varphi_0)$  défini pour tout  $t \geq v'$ . Alors, grâce à la Proposition 1.3, nous pouvons étendre le domaine de définition pour tout  $t \in J_0$ .

Pour ce qui concerne les perturbations dépendant du temps des champs de vecteurs constants sur le tore, étant donné  $\sigma \geq 1$  et  $\omega \in \mathbb{R}^n$ , nous considérons le champ de vecteurs dépendant du temps suivant

$$\begin{cases} Z : \mathbb{T}^n \times J_0 \longrightarrow \mathbb{R}^n \\ Z(q, t) = \omega + P(q, t) \\ P \in \bar{\mathcal{S}}_{\sigma, 1}^0, \\ |P^t|_{C^{\sigma+1}} \leq \mathbf{P}(t) \quad \text{pour tout } t \in J_0, \end{cases} \quad (Z_A)$$

où  $\mathbf{P}$  est une fonction positive, décroissante et intégrable sur  $J_0$ .

**Corollaire A.** *Soit  $Z$  comme dans  $(Z_A)$ . Alors, il existe un  $C^\sigma$ -tore KAM asymptotique  $\psi^t$  associé à  $(Z, \omega, \text{Id})$ .*

De façon similaire à comment nous avons procédé dans le Théorème A, nous prouvons le résultat ci-dessus. Nous concluons cette partie en observant que l'hypothèse d'intégrabilité sur  $\mathbf{P}$  est optimale. Étant donné  $\hat{Z}$  le champ de vecteurs dépendant du temps défini sur  $\mathbb{T}^1 \times J_0$  de la forme

$$\hat{Z}(q, t) = \omega + \hat{P}(t),$$

où  $\omega \in \mathbb{R}$  et  $\hat{P}(t) > 0$  pour tout  $t > 0$ . Nous supposons que

$$\int_{t_0}^{+\infty} \hat{P}(\tau) d\tau = +\infty,$$

pour tout  $t_0 \geq 0$ . Soit  $\psi_{t_0, X}^t$  le flot au moment  $t$  avec un temps initial  $t_0$  de  $\hat{Z}$ . Alors, pour tout  $q \in \mathbb{T}^n$  et  $t > 0$

$$\psi_{t_0, \hat{Z}}^{t_0+t}(q) = q + \omega t + \int_{t_0}^{t_0+t} \hat{P}(\tau) d\tau$$

et donc, pour tout  $q \in \mathbb{T}^n$

$$\left| \psi_{t_0, \hat{Z}}^{t_0+t}(q) - q - \omega t \right| = \int_{t_0}^{t_0+t} \hat{P}(\tau) d\tau.$$

Par conséquent, en prenant la limite pour  $t \rightarrow +\infty$ , le côté droit de l'égalité précédent diverge à  $+\infty$ . Cela signifie qu'il n'existe pas une solution asymptotiquement quasipériodique associée à  $(\hat{Z}, \omega, \text{Id})$  et donc, grâce à la Proposition 1.4, il n'existe pas un  $C^\sigma$ -tore KAM asymptotique associé à  $(\hat{Z}, \omega, \text{Id})$ .

### 2.1.2 Cas Analytique Réel

Comme mentionné ci-dessus, nous présentons la version analytique réelle des résultats précédents. Étant donné  $s > 0$  et  $v \geq 0$ , nous introduisons l'espace de fonctions suivant

**Définition.** Soit  $\mathcal{A}_s^v$  l'espace des fonctions  $f$  définies sur  $\mathbb{T}_s^n \times B_s \times J_v$  tel que  $f \in C(\mathbb{T}_s^n \times B_s \times J_v)$  et, pour tout  $t \in J_v$ ,  $f^t$  est analytique réel sur  $\mathbb{T}_s^n \times B_s$ .

Nous utilisons cette notation aussi pour des applications définies sur  $\mathbb{T}_s^n \times J_v$ . Étant donné  $\omega \in \mathbb{R}^n$  et un paramètre réel positif  $s_0 > 0$ , nous considérons le hamiltonien dépendant du temps  $H$  suivant

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times J_0 \longrightarrow \mathbb{R} \\ H(q, p, t) = h(q, p, t) + f(q, p, t), \\ h \in \mathcal{K}_\omega \\ h, f \in \mathcal{A}_{s_0}^0 \\ \sup_{t \in J_0} |f_0^t|_{s_0} < \infty, \quad \sup_{t \in J_0} |\partial_p^2 H^t|_{s_0} < \infty \\ |\partial_q f_0^t|_{s_0} \leq \mathbf{a}(t), \quad |\partial_p f_0^t|_{s_0} \leq \mathbf{b}(t), \quad \text{pour tout } t \in J_0 \end{array} \right. \quad (*_B)$$

où  $\mathbf{a}$ ,  $\mathbf{b}$  sont des fonctions positives, décroissantes et intégrables sur  $J_0$ .

**Théorème B.** Soit  $H$  comme dans  $(*_B)$  avec  $\mathbf{a}$  et  $\mathbf{b}$  qui satisfont  $(\#)$ . Alors, il existe  $\tilde{h} \in \mathcal{K}_\omega$  et un tore KAM asymptotique analytique lagrangien  $\varphi^t$  associé à  $(X_H, X_{\tilde{h}}, \varphi_0)$ .

Comme dans Théorème A, nous prouvons l'existence d'un tore KAM asymptotique analytique  $\varphi^t$  de la forme

$$\varphi^t(q) = (q + u^t(q), v^t(q))$$

pour tout  $q \in \mathbb{T}^n$  et  $t$  suffisamment grand, où, pour chaque  $t$  fixé,  $\text{id} + u^t$  est un difféomorphisme du tore. En outre, pour une constante appropriée  $C$

$$|u^t|_{\frac{s}{4}} \leq C\bar{\mathbf{b}}(t), \quad |v^t|_{\frac{s}{4}} \leq C\bar{\mathbf{a}}(t),$$

pour tout  $t$  suffisamment grand. La preuve de ce théorème est essentiellement la même que celle du Théorème A avec des petites modifications. De même que le cas à différentiabilité finie, nous résolvons le problème linéaire associé par un changement de coordonnées approprié qui dépend du flot linéaire sur le tore généré par le champ de vecteurs constant  $\omega$ . En outre, si  $t$  est réel, nous pouvons résoudre l'équation homologique sans réduire le domaine où la solution est analytique. Pour cette raison, nous trouvons un tore KAM asymptotique analytique  $\varphi^t$  qui est juste continu par rapport au temps.

Encore une fois, nous prouvons un résultat analogue pour des perturbations analytiques réelles dépendant du temps des champs de vecteurs constants sur le tore. Soit  $Z$  le champ de vecteur non autonome sur  $\mathbb{T}^n \times J_0$  de la forme

$$\left\{ \begin{array}{l} Z : \mathbb{T}^n \times J_0 \longrightarrow \mathbb{R}^n, \\ Z(q, t) = \omega + P(q, t) \\ P \in \mathcal{A}_{s_0}^0, \\ |P^t|_{s_0} \leq \mathbf{P}(t) \quad \text{pour tout } t \in J_0 \end{array} \right. \quad (Z_B)$$

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où  $\omega \in \mathbb{R}^n$  et  $0 < s_0 < 1$ . Nous supposons que  $\mathbf{P}$  est une fonction positive, décroissante et intégrable sur  $J_0$ .

**Corollaire B.** *Soit  $Z$  comme dans  $(Z_B)$ . Alors il existe un tore KAM asymptotique analytique  $\psi^t$  associé à  $(Z, \omega, \text{Id})$ .*

Nous démontrons ce résultat comme conséquence du Théorème B.

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Ici, nous nous intéressons aux systèmes non autonomes  $X^t$  définis pour tout  $t \in \mathbb{R}$ . En particulier, cette section vise à prouver l'existence de solutions définies pour tout  $t \in \mathbb{R}$  et convergeant vers certaines solutions quasipériodiques dans le futur ( $t \rightarrow +\infty$ ) et dans le passé ( $t \rightarrow -\infty$ ). Pour cette raison, introduisons la définition de  $C^\sigma$ -tore KAM biasymptotique.

Étant donné  $\sigma \geq 0$  et un entier positif  $k \geq 0$ , nous considérons les champs de vecteurs dépendant du temps  $X^t, X_{0,+}^t, X_{0,-}^t$  de classe  $C^{\sigma+k}$  sur  $\mathbb{T}^n \times B$ , pour tout  $t \in \mathbb{R}$ , et des plongements  $\varphi_{0,+}, \varphi_{0,-}$  de  $\mathbb{T}^n$  sur  $\mathbb{T}^n \times B$  de classe  $C^\sigma$  tel que

$$\lim_{t \rightarrow \pm\infty} |X^t - X_{0,\pm}^t|_{C^{\sigma+k}} = 0, \quad (2.1)$$

$$X_{0,\pm}(\varphi_{0,\pm}(q), t) = \partial_q \varphi_{0,\pm}(q) \omega_{\pm} \text{ pour tout } (q, t) \in \mathbb{T}^n \times \mathbb{R}, \quad (2.2)$$

où  $\omega_+, \omega_- \in \mathbb{R}^n$ .

**Définition 2.1.** *Nous supposons que  $(X, X_{0,\pm}, \varphi_{0,\pm})$  satisfont (2.1) et (2.2). Pour tout  $t \in \mathbb{R}$ , une famille continue de  $C^\sigma$ -plongements  $\varphi^t : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B$  est un  $C^\sigma$ -tore KAM biasymptotique associé à  $(X, X_{0,\pm}, \varphi_{0,\pm})$  s'il existe  $v \geq 0$  tel que*

$$\lim_{t \rightarrow \pm\infty} |\varphi^t - \varphi_{0,\pm}|_{C^\sigma} = 0,$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) \omega_+ + \partial_t \varphi(q, t), \text{ pour tout } (q, t) \in \mathbb{T}^n \times (v, +\infty)$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) \omega_- + \partial_t \varphi(q, t), \text{ pour tout } (q, t) \in \mathbb{T}^n \times (-\infty, -v).$$

De plus, nous disons que  $\varphi^t$  est lagrangien si  $\varphi^t(\mathbb{T}^n)$  est lagrangien pour tout  $t$ .

Malheureusement, nous ne sommes pas en mesure de prouver l'existence de  $C^\sigma$ -tores KAM biasymptotiques sans faire l'hypothèse que  $\omega_+ = \omega_- = \omega$  et que les hamiltoniens  $H : \mathbb{T}^n \times B \times \mathbb{R} \rightarrow \mathbb{R}$  satisfont des symétries très fortes. Plus précisément, nous devons supposer que

$$H(q + \omega t, p, t) = -H(q - \omega t, p, -t)$$

pour tout  $(q, p, t) \in \mathbb{T}^n \times B \times \mathbb{R}$  et un certain  $\omega \in \mathbb{R}^n$ . Alors, nous pouvons démontrer l'existence d'un  $C^\sigma$ -tore KAM biasymptotique avec  $\omega_+ = \omega_- = \omega$ .

Dans ce qui suit, nous démontrerons des résultats plus faibles. Pour cette raison, nous avons la définition suivante.



**Définition 2.2.** Nous supposons que  $(X, X_{0,\pm}, \varphi_{0,\pm})$  satisfait (2.1) et (2.2). Une courbe intégrale  $g(t)$  de  $X$  est une solution *biasymptotiquement quasipériodique* associée à  $(X, X_{0,\pm}, \varphi_{0,\pm})$  s'il existe  $q_-, q_+ \in \mathbb{T}^n$  de telle sorte que

$$\lim_{t \rightarrow \pm\infty} |g(t) - \varphi_{0,\pm}(q_{\pm} + \omega_{\pm}t)| = 0. \quad (2.3)$$

Donc, nous ne prouverons pas l'existence de  $C^\sigma$ -tores KAM biasymptotiques mais plutôt des solutions biasymptotiquement quasipériodiques associées à certains systèmes hamiltoniens dépendant du temps.

Dans ce qui suit, nous analyserons deux cas. Dans le premier, nous considérons des perturbations dépendant du temps des hamiltoniens intégrables. Le second concerne des perturbations dépendant du temps des hamiltoniens ayant un large (au sens de la mesure) sous-ensemble des tores invariants.

Tout d'abord, nous rappelons que, pour tout  $(p, t) \in B \times \mathbb{R}$  fixé, nous considérons  $f_p^t$  comme la fonction définie sur  $\mathbb{T}^n$  telle que

$$f_p^t(q) = f(q, p, t)$$

pour tout  $q \in \mathbb{T}^n$ .

### 2.2.1 Cas Intégrable

Étant donné un paramètre réel positif  $\sigma \geq 1$ , nous considérons l'espace de fonctions suivant.

**Définition.** Soit  $\mathcal{B}_\sigma$  l'espace des fonctions  $f$  définies sur  $\mathbb{T}^n \times B \times \mathbb{R}$  tel que  $f, \partial_p f \in C(\mathbb{T}^n \times B \times \mathbb{R})$  et  $f_p^t \in C^\sigma(\mathbb{T}^n)$  pour tout  $(p, t) \in B \times \mathbb{R}$ .

Nous introduisons une norme spéciale qui joue un rôle central dans cette partie. Pour tout  $f \in \mathcal{B}_\sigma$  and  $l > 1$ , nous définissons

$$|f|_{\sigma,l} = \sup_{(p,t) \in B \times \mathbb{R}} |f_p^t|_{C^\sigma} (1 + |t|^l) + \sup_{(p,t) \in B \times \mathbb{R}} |(\partial_p f)_p^t|_{C^0} (1 + |t|^{l-1}). \quad (2.4)$$

**Définition.** Étant donné  $\sigma \geq 1$  et un entier positif  $k \geq 0$ , nous définissons  $\bar{\mathcal{B}}_{\sigma,k}$  comme l'espace des fonctions  $f$  tel que

$$f \in \mathcal{B}_{\sigma+k}, \text{ et } \partial_q^i f \in \mathcal{B}_{\sigma+k-i}$$

pour tout  $0 \leq i \leq k$ .

Dans la définition précédente,  $\partial_q^i f$  indique les dérivés partielles d'ordre  $i$  par rapport aux variables  $q = (q_1, \dots, q_n)$  de  $f$ . Nous utilisons la convention  $\partial_q^0 f = f$ . Évidemment, nous avons  $\bar{\mathcal{B}}_{\sigma,0} = \mathcal{B}_\sigma$ . Pour tout  $f \in \bar{\mathcal{B}}_{\sigma,k}$  et  $l > 1$ , nous concluons cette partie avec l'introduction de la norme suivante

$$\|f\|_{\sigma,k,l} = \max_{0 \leq i \leq k} |\partial_q^i f|_{\sigma+k-i,l}.$$

Soit  $B_r \subset \mathbb{R}^n$  une boule centrée à l'origine de rayon  $r > 0$ . Si une fonction  $f$  définie sur  $\mathbb{T}^n \times B_r \times \mathbb{R}$  appartient à  $\mathcal{B}_\sigma$ , nous considérons que  $f$  satisfait les propriétés dans la définition précédente avec  $B$  remplacée par  $B_r$ .

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Étant donné  $\sigma \geq 1$ ,  $l > 1$  et  $0 < \varepsilon < 1$ , nous considérons le hamiltonien suivant

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B_1 \times \mathbb{R} \longrightarrow \mathbb{R} \\ H(q, p, t) = h(p) + f(q, p, t) \\ f, \partial_p f \in \bar{\mathcal{B}}_{\sigma, 2}, \\ |f|_{\sigma+2, 0} + \|\partial_q f\|_{\sigma, 1, l+2} + \|\partial_p f\|_{\sigma, 2, l+1} < \varepsilon, \\ \partial_p^2 H^t \in C^{\sigma+2}(\mathbb{T}^n \times B_1) \text{ pour tout } t \in \mathbb{R} \text{ fixé} \\ \partial_{qp}^i (\partial_p^2 H) \in C(\mathbb{T}^n \times B_1 \times \mathbb{R}) \text{ pour tout } 0 \leq i \leq 3. \\ \sup_{t \in \mathbb{R}} |\partial_p^2 H^t|_{C^{\sigma+2}} < \infty. \end{array} \right. \quad (*_C)$$

Pour chaque  $p \in B_1$ , nous considérons le plongement trivial suivant

$$\varphi_{0,p} : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B_1, \quad \varphi_{0,p}(q) = (q, p).$$

**Théorème C.** *Soit  $H$  comme dans  $(*_C)$ . Alors il existe un hamiltonien dépendant du temps  $\tilde{h}$  tel que, si  $\varepsilon$  est suffisamment petit, pour tout  $(q, p) \in \mathbb{T}^n \times B_{\frac{1}{2}}$  il existe  $p_-, p_+ \in B_1$  et une solution biasymptotiquement quasipériodique  $g(t)$  associée à  $(X_H, X_{\tilde{h}}, \varphi_{0,p_{\pm}})$  tel que  $g(0) = (q, p)$ .*

En ce qui concerne la régularité de  $H$ , nous soulignons que si nous prenons l'hypothèse plus forte

$$f \in C^{\sigma+3}(\mathbb{T}^n \times B_1 \times \mathbb{R}) \text{ et } \partial_p^2 H \in C^{\sigma+2}(\mathbb{T}^n \times B_1 \times \mathbb{R}),$$

alors les hypothèses de régularité du théorème précédent sont satisfaites.

Pour chaque  $p \in B_{\frac{3}{4}}$ , nous prouvons l'existence d'un  $C^\sigma$ -tore KAM asymptotique  $\varphi_{+,p}^t$  associé à  $(X_H, X_{\tilde{h}}, \varphi_{0,p})$  défini pour tout  $t \geq 0$ . En plus, soit  $\varphi_+^t(q, p) = \varphi_{+,p}^t(q)$  pour tout  $(q, p) \in \mathbb{T}^n \times B_{\frac{3}{4}}$ , nous vérifions que

$$\sup_{t \geq 0} |\varphi_+^t - \text{Id}|_{C^1} < C_0 \varepsilon, \quad (2.5)$$

pour une constante appropriée  $C_0$  qui dépend de  $n, l, \sup_{t \in \mathbb{R}} |\partial_p^2 H^t|_{C^{\sigma+2}}$  et  $|\partial_p h|_{C^1}$ . De la même manière, pour tout  $t \leq 0$ , il existe une famille de  $C^\sigma$ -tores KAM asymptotiques  $\varphi_-^t$  définie sur  $\mathbb{T}^n \times B_{\frac{3}{4}}$  qui satisfont

$$\sup_{t \leq 0} |\varphi_-^t - \text{Id}|_{C^1} < C_0 \varepsilon. \quad (2.6)$$

Les estimations précédentes, (2.5) et (2.6), sont les clés pour prouver que

$$\mathbb{T}^n \times B_{\frac{1}{2}} \subset \varphi_{\pm}^0(\mathbb{T}^n \times B_{\frac{3}{4}}).$$

Ceci conclut la preuve. En effet, si nous nous rappelons des propriétés des  $C^\sigma$ -tores KAM asymptotique, chaque  $(q, p) \in \mathbb{T}^n \times B_{\frac{1}{2}}$  donne lieu à des solutions biasymptotiquement quasipériodiques associées à  $H$ .

Comparé au Théorème A, nous supposons ici une décroissance par rapport au temps plus forte. Cela est dû au contrôle sur les variables  $p$  obtenu par (2.5) et (2.6). Par contre, ce théorème est perturbatif parce que nous avons besoin que  $|\varphi_{\pm}^t - \text{Id}|_{C^1}$  soit petit pour  $t = 0$  et pas seulement pour  $|t|$  grand.

### 2.2.2 Cas Presque Intégrable

Soit  $A \subset \mathbb{R}^n$  un ensemble fermé et  $E$  égal à  $\mathbb{T}^n \times B$  ou  $\mathbb{T}^n$ , pour chaque fonction  $f : E \times A \rightarrow \mathbb{R}$  nous définissons la norme suivante

$$|f|_{L(A)} = \sup_{z \in B} \left( \sup_{x, y \in A, x \neq y} \frac{|f(z, x) - f(z, y)|}{|x - y|} \right) + |f|_{C^0}.$$

Étant donné un paramètre positif réel  $\sigma \geq 1$  et  $A \subset \mathbb{R}^n$ , nous avons la définition qui suit.

**Définition.** Soit  $\mathcal{D}_\sigma$  l'espace de fonctions  $f$  définies sur  $\mathbb{T}^n \times A \times \mathbb{R}$  tel que  $f \in C(\mathbb{T}^n \times A \times \mathbb{R})$  et  $f_p^t \in C^\sigma(\mathbb{T}^n)$  pour tout  $(p, t) \in A \times \mathbb{R}$ .

Comme dans le cas précédent, nous définissons la norme suivante. Pour tout  $f \in \mathcal{D}_\sigma$  et  $l \geq 1$ ,

$$|f|_{\sigma, l, L(A)} = \sup_{(p, t) \in A \times \mathbb{R}} |f_p^t|_{C^\sigma} (1 + |t|^l) + \sup_{t \in \mathbb{R}} |f^t|_{L(A)} (1 + |t|^{l-1}). \quad (2.7)$$

Soit  $f \in \mathcal{B}_\sigma$  (où  $\mathcal{B}_\sigma$  est l'espace défini précédemment). Pour chaque  $A \subset B$  fermé, évidemment  $f \in \mathcal{D}_\sigma$  et si  $|f|_{\sigma, l} < \infty$  alors

$$|f|_{\sigma, l, L(A)} \leq |f|_{\sigma, l},$$

où  $|\cdot|_{\sigma, l}$  est la norme définie par (2.4).

**Définition.** Étant donné  $\sigma \geq 1$  et un entier positif  $k \geq 0$ , nous définissons  $\bar{\mathcal{D}}_{\sigma, k}$  comme l'espace de fonctions  $f$  tel que

$$f \in \mathcal{D}_{\sigma+k}, \text{ et } \partial_q^i f \in \mathcal{D}_{\sigma+k-i}$$

pour tout  $0 \leq i \leq k$ .

De plus, pour tout  $f \in \bar{\mathcal{D}}_{\sigma, k}$  et  $l > 1$ , nous considérons la norme qui suit

$$\|f\|_{\sigma, k, l, L(A)} = \max_{0 \leq i \leq k} |\partial_q^i f|_{\sigma+k-i, l, L(A)},$$

Dans ce cas aussi, le hamiltonien suivant et ses composants sont des fonctions définies sur  $\mathbb{T}^n \times B_r \times \mathbb{R}$ , pour certains  $r > 0$ . Ensuite, si une fonction  $f$  appartient à  $\mathcal{D}_\sigma$ , nous considérons que  $f$  satisfait la régularité demandée dans la définition précédente avec  $A$  remplacé par  $B_r$ .

Ici, nous considérons le cas où le hamiltonien intégrable  $h(p)$  est remplacé par un hamiltonien autonome ayant un large ensemble des tores invariants.

Étant donné  $\sigma \geq 1$ ,  $l > 1$ ,  $0 < \varepsilon < 1$  et  $\mu > 0$ . Nous considérons le hamiltonien suivant

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B_1 \times \mathbb{R} \longrightarrow \mathbb{R} \\ H(q, p, t) = h(p) + R(q, p) + f(q, p, t) \\ f, \partial_p f \in \bar{\mathcal{D}}_{\sigma, 2}, \\ |f|_{\sigma+2, 0, L(B_1)} + \|\partial_q f\|_{\sigma, 1, l+2, L(B_1)} + \|\partial_p f\|_{\sigma, 2, l+1, L(B_1)} < \varepsilon, \\ D \subset B_1, \quad \text{Leb}(B_1 \setminus D) < \mu, \\ R \in C^2(\mathbb{T}^n \times B_1) \\ R(q, p) = \partial_p R(q, p) = 0 \text{ pour tout } (q, p) \in \mathbb{T}^n \times D, \\ \partial_p^2 H^t \in C^{\sigma+2}(\mathbb{T}^n \times B_1) \text{ pour tout } t \in \mathbb{R} \text{ fixé} \\ \partial_{qp}^i (\partial_p^2 H) \in C(\mathbb{T}^n \times B_1 \times \mathbb{R}) \text{ pour tout } 0 \leq i \leq 2. \\ \sup_{t \in \mathbb{R}} |\partial_p^2 H^t|_{C^{\sigma+2}} < \infty, \end{array} \right. \quad (*_D)$$

**Théorème D.** *Soit  $H$  comme dans  $(*_D)$ . Alors il existe un hamiltonien dépendant du temps  $\tilde{h}$  tel que, si  $\varepsilon$  est suffisamment petit, nous avons l'existence d'un ensemble  $\mathcal{W} \subset \mathbb{T}^n \times B_1$  de telle sorte que, pour tout  $(q, p) \in \mathcal{W}$ , il existe  $p_-, p_+ \in D$  et une solution biasymptotiquement quasipériodique  $g(t)$  associée à  $(X_H, X_{\tilde{h}}, \varphi_{0, p_{\pm}})$  tel que  $g(0) = (q, p)$ . De plus, nous avons*

$$\text{Leb}((\mathbb{T}^n \times B_1) \setminus \mathcal{W}) \leq 4\mu.$$

L'hypothèse sur le hamiltonien précédent  $H$ , concernant la partie autonome  $h + R$ , n'est pas artificielle. Pöschel, dans son travail [Pö82], considère une petite perturbation  $H_1$  et  $C^\infty$  d'un hamiltonien intégrable analytique réel non-dégénéré  $H_0$  de la forme

$$H : \mathbb{T}^n \times B \longrightarrow \mathbb{R}, \quad H(q, p) = H_0(p) + H_1(q, p).$$

L'auteur démontre l'existence d'un symplectomorphisme  $\phi$  de classe  $C^\infty$  tel que

$$H \circ \phi : \mathbb{T}^n \times B \longrightarrow \mathbb{R}, \quad H(q, p) = h(p) + R(q, p),$$

où  $h$  est proche de  $H_0$  et le jet infini de  $R$  est nul sur  $\mathbb{T}^n \times D_\gamma$ . Dans ce travail,  $D_\gamma$  est un sous-ensemble approprié de  $B$  tel que

$$\text{Leb}(B \setminus D_\gamma) \leq C\gamma^2,$$

pour une certaine constante  $C$ . En faisant référence à cet article de Pöschel, dans le Théorème D, nous prenons  $D = D_\gamma$  et  $\mu = C\gamma^2$ .

Concernant la preuve du Théorème D, nous définissons  $D' \subset D$  comme les éléments de  $D$  suffisamment loin du bord de  $B_1$ . Ensuite, pour chaque  $p \in D'$ , nous démontrons l'existence d'un  $C^\sigma$ -tore KAM asymptotique  $\varphi_{+, p}^t$  (resp.  $\varphi_{-, p}^t$ ) associé à  $(X_H, X_{\tilde{h}}, \varphi_{0, p})$  défini pour tout  $t \geq 0$  (resp.  $t \leq 0$ ). En posant  $\varphi_\pm^t(q, p) = \varphi_{\pm, p}^t(q)$  pour tout  $(q, p) \in \mathbb{T}^n \times D'$ , nous vérifions que

$$\sup_{t \geq 0} |\varphi_+^t - \text{Id}|_{L, \mathbb{T}^n \times D'} < C_0 \varepsilon, \quad \sup_{t \leq 0} |\varphi_-^t - \text{Id}|_{L, \mathbb{T}^n \times D'} < C_0 \varepsilon \quad (2.8)$$

pour une constante appropriée  $C_0$  qui dépend de  $n, l, \sup_{t \in \mathbb{R}} |\partial_p^2 H^t|_{C^{\sigma+2}}$  et  $|\partial_p h|_{L(D)}$ .  
 Maintenant, soit  $\mathcal{W}$  le suivant sous-ensemble de  $\mathbb{T}^n \times B_1$

$$\mathcal{W} = \varphi_+^0(\mathbb{T}^n \times D') \cap \varphi_-^0(\mathbb{T}^n \times D').$$

Donc, la conclusion du théorème précédent est due à (2.8) et  $\text{Leb}(B_1 \setminus D) < \mu$ .

### 2.3 Applications à la Mécanique Céleste

Dans cette partie, nous étudions l'existence d'orbites pour un système planétaire (problème à trois corps dans le plan) perturbé par une comète donnée, venant et revenant à l'infini asymptotiquement le long d'une orbite képlérienne hyperbolique. Sur un espace de phase approprié, nous considérons le hamiltonien  $H = H_0 + H_c$ , où  $H_0$  joue le rôle du hamiltonien du problème à trois corps dans le plan et  $H_c$  l'interaction avec la comète donnée. Malheureusement, la perturbation  $H_c$  ne satisfait pas de bonnes propriétés de décroissance lorsque  $t \rightarrow +\infty$ . Dans une norme appropriée,  $|H_c^t| \sim \frac{1}{t^l}$  avec  $l = 2$ , alors que nous avons besoin de  $l > 2$  pour appliquer le Théorème A. Pour résoudre ce problème, nous sommes obligés de démontrer un autre théorème abstrait qui est une version plus faible du Théorème A.

Nous introduisons les définitions de  $C^\sigma$ -cylindre faiblement asymptotique et de solution faiblement asymptotiquement quasipériodique. À cette fin, nous définissons  $B \subset \mathbb{R}^{n+m}$  comme une boule centrée à l'origine et nous notons  $q \in \mathbb{T}^n \times \mathbb{R}^m$  et  $p \in B$ . Soit  $\mathcal{P}$  égal à  $\mathbb{T}^n \times \mathbb{R}^m \times B$  ou à un sous-ensemble ouvert de  $\mathbb{R}^{2(n+m)}$  et  $J = [1, +\infty) \subset \mathbb{R}$ .

Étant donné  $\sigma \geq 0$  et un entier positif  $k \geq 0$ , nous considérons les champs de vecteurs dépendant du temps  $X^t, X_0^t$  de classe  $C^{\sigma+k}$  sur  $\mathcal{P}$ , pour tout  $t \in J$ , un plongement  $\varphi_0$  de  $\mathbb{T}^n \times \mathbb{R}^m$  sur  $\mathcal{P}$  de classe  $C^\sigma$  et un champ de vecteurs dépendant du temps  $\gamma^t$  de classe  $C^\sigma$  sur  $\mathbb{T}^n \times \mathbb{R}^m$ , pour tout  $t \in J$ , tel que

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_{C^{\sigma+k}} = 0, \quad (2.9)$$

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q)(\bar{\omega} + \gamma(q, t)) \text{ pour tout } (q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J, \quad (2.10)$$

$$\lim_{t \rightarrow +\infty} |\gamma^t|_{C^\sigma} = 0, \quad (2.11)$$

où  $\bar{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$  avec  $\omega \in \mathbb{R}^n$ .

En d'autres termes,  $X^t - X_0^t$  converge vers zéro lorsque  $t \rightarrow +\infty$ . De plus, le champ de vecteurs  $X_0$  possède un cylindre invariant  $\varphi_0$  et la restriction de  $X_0$  est conjuguée au champ de vecteurs non-autonome  $\bar{\omega} + \gamma$ , qui est un champ de vecteurs dépendant du temps convergeant vers  $\bar{\omega}$  lorsque  $t \rightarrow +\infty$ .

**Définition 2.3.** *Nous supposons que  $(X, X_0, \varphi_0)$  satisfait (2.9), (2.10) et (2.11). Une famille de  $C^\sigma$ -plongements  $\varphi^t : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathcal{P}$  est un  $C^\sigma$ -cylindre faiblement asymptotique associé à  $(X, X_0, \varphi_0)$  s'il existe un champ de vecteurs dépendant du temps  $\Gamma^t$  de classe  $C^\sigma$  sur  $\mathbb{T}^n \times \mathbb{R}^m$ , pour tout  $t \in J$ , tel que*

$$\lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{C^\sigma} = 0, \quad (2.12)$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t)(\bar{\omega} + \Gamma(q, t)) + \partial_t \varphi(q, t), \quad (2.13)$$

$$\lim_{t \rightarrow +\infty} |\Gamma^t|_{C^\sigma} = 0, \quad (2.14)$$

pour tout  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ . De plus,  $\varphi^t$  est lagrangien si  $\varphi^t(\mathbb{T}^n \times \mathbb{R}^m)$  est lagrangien pour tout  $t$ .

Cette définition est une version plus générale de celle du  $C^\sigma$ -tore KAM asymptotique (Définition 1.3). En effet, en prenant  $m = 0$ ,  $\gamma \equiv 0$  et  $\Gamma \equiv 0$ , nous obtenons la Définition 1.3. Par ailleurs, nous observons que si  $m = 0$  et  $|\Gamma^t|_{C^{\sigma+1}}$  est intégrable sur  $J$ , alors, grâce au Corollaire A, on peut prouver l'existence d'un  $C^\sigma$ -tore KAM asymptotique associé à  $(X, X_0, \varphi_0)$ .

Ici,  $\mathcal{P}$  est égal à  $\mathbb{T}^n \times \mathbb{R}^m \times B$  ou à un sous-ensemble ouvert de  $\mathbb{R}^{2(n+m)}$ . Par contre, nous nous intéressons à une famille de cylindres plongés et non à une famille de tores plongés. Ceci est dû à l'exemple en mécanique céleste que nous étudierons dans la section suivante.

Contrairement à la définition de  $C^\sigma$ -tore KAM asymptotique, nous recherchons ici des familles  $\varphi^t$  de plongements de classe  $C^\sigma$  définis pour tout  $t \in J$  et non seulement  $t$  grand. Toutefois, il ne s'agit pas d'une différence substantielle. Nous verrons que, comme pour des  $C^\sigma$ -tores KAM asymptotique, si  $\varphi^t$  est un  $C^\sigma$ -cylindre faiblement asymptotique défini pour tout  $t$  suffisamment grand, alors nous pouvons étendre l'ensemble de définition pour tout  $t \in \mathbb{R}$ .

Nous avons une série de propriétés en commun avec les  $C^\sigma$ -tores KAM asymptotiques. Dans ce cas également, nous pouvons réécrire (2.13) en termes du flot de  $X$ . En fait, soit  $\psi_{t_0, X}^t$  le flot au moment  $t$  avec un temps initial  $t_0$  de  $X$  et  $\psi_{t_0, \bar{\omega} + \Gamma}^t$  le flot au moment  $t$  avec un temps initial  $t_0$  de  $\bar{\omega} + \Gamma$ . Nous supposons que  $\psi_{t_0, X}^t$  et  $\psi_{t_0, \bar{\omega} + \Gamma}^t$  sont définis pour tout  $t, t_0 \in J$ . Alors, (2.13) est équivalent à

$$\psi_{t_0, X}^t \circ \varphi^{t_0} = \varphi^t \circ \psi_{t_0, \bar{\omega} + \Gamma}^t \quad (2.15)$$

pour tout  $t, t_0 \in J$ . Nous faisons deux remarques importantes par la suite. La première est que (2.13) est triviale. Il est toujours possible de trouver une famille des plongements satisfaisant cette équation. Soit  $\hat{\varphi} : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathcal{P}$  un plongement, alors, pour tout  $t, t_0 \in J$

$$\varphi^t = \psi_{t_0, X}^t \circ \hat{\varphi} \circ \psi_{t_0, \bar{\omega} + \Gamma}^{t_0}$$

est une famille des plongements qui satisfont (2.15) et donc (2.13).

Si  $\psi_{t_0, X}^t$  est défini pour tout  $t, t_0 \in \mathbb{R}$  et s'il existe un  $C^\sigma$ -cylindre faiblement asymptotique  $\varphi^t$  défini pour tout  $t \geq 1$ , alors nous pouvons étendre l'ensemble de définition pour tout  $t \in \mathbb{R}$ . Plus précisément, nous définissons

$$\phi^t = \begin{cases} \varphi^t & \text{pour tout } t \geq 1 \\ \psi_{1, X}^t \circ \varphi^1 \circ \psi_{t, \bar{\omega} + \Gamma}^1 & \text{pour tout } t \leq 1. \end{cases}$$

Nous observons que  $\phi^t$  est un  $C^\sigma$ -cylindre faiblement asymptotique défini pour tout  $t \in \mathbb{R}$ .

Maintenant, afin de donner quelques informations concernant la dynamique associée à un  $C^\sigma$ -cylindre faiblement asymptotique, nous introduisons la définition de solution faiblement asymptotiquement quasipériodique.

**Définition 2.4.** *Nous supposons que  $(X, X_0, \varphi_0)$  satisfont (2.9), (2.10) et (2.11). Une courbe intégrale  $g(t)$  de  $X$  est une solution faiblement asymptotiquement*

quasipériodique associée à  $(X, X_0, \varphi_0)$  s'il existe un champ de vecteurs dépendant du temps  $\Gamma : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m}$  et  $q \in \mathbb{T}^n \times \mathbb{R}^m$  tel que

$$\lim_{t \rightarrow +\infty} |g(t) - \varphi_0 \circ \psi_{t_0, \bar{\omega} + \Gamma}^t(q)| = 0.$$

Nous observons que ce type d'orbites de  $X$  ne convergent pas vers les mouvements associés à  $X_0$  sur  $\varphi_0$  mais vers la dynamique sur  $\varphi_0$  générée par le champ de vecteurs dépendant du temps  $\bar{\omega} + \Gamma$ , d'où le terme faiblement. Évidemment, en prenant  $m = 0$ ,  $\gamma \equiv 0$  et  $\Gamma \equiv 0$ , on obtient la définition de solution asymptotiquement quasipériodique (Définition 1.2). De manière similaire à la Définition 2.3, si  $m = 0$  et  $|\Gamma^t|_{C^{\sigma+1}}$  est intégrable sur  $J$ , alors on peut prouver l'existence de solutions asymptotiquement quasipériodiques associées à  $(X, X_0, \varphi_0)$ .

Comme on peut s'y attendre, nous avons la proposition suivante, dont la preuve est similaire à la Proposition 1.4.

**Proposition 2.1.** *Soit  $\varphi^t$  un  $C^\sigma$ -cylindre faiblement asymptotique associé à  $(X, X_0, \varphi_0)$ . Alors, pour tout  $q \in \mathbb{T}^n \times \mathbb{R}^m$  et  $t_0 \in J$ ,*

$$g(t) = \psi_{t_0, X}^t \circ \varphi^{t_0}(q)$$

*est une solution faiblement asymptotiquement quasipériodique  $(X, X_0, \varphi_0)$ .*

Cette proposition, et notamment ces orbites, jouent un rôle important en particulier dans la partie consacrée au hamiltonien du problème à trois corps plus comète dans le plan.

Nous divisons ce qui suit en deux sous-parties. La première est consacrée à l'énoncé d'une version plus faible du Théorème A, tandis que la seconde est dédiée aux applications en mécanique céleste.

### 2.3.1 Le Théorème Abstrait

Nous commençons par l'introduction de l'espace de fonctions suivant. Étant donné un paramètre réel positif  $\sigma \geq 0$ , nous avons la définition suivante

**Définition.** *Soit  $\mathcal{S}_\sigma$  l'espace de fonctions  $f$  définies sur  $\mathbb{T}^n \times \mathbb{R}^m \times B \times J$  tel que  $f^t \in C^\sigma(\mathbb{T}^n \times \mathbb{R}^m \times B)$  pour tout  $t \in J$  fixé et  $\partial_{(q,p)}^i f \in C(\mathbb{T}^n \times \mathbb{R}^m \times B \times J)$  pour tout  $0 \leq i \leq [\sigma]$ .*

Dans la définition précédente,  $\partial_{(q,p)}^i$  représente les dérivés partiels par rapport à  $(q, p)$  d'ordre  $i$  et  $[\sigma]$  est la partie entière de  $\sigma$ . Conventionnellement  $\partial_{(q,p)}^0 f = f$ . Comme d'habitude, nous utilisons cette notation également pour les fonctions définies sur  $\mathbb{T}^n \times \mathbb{R}^m \times J$ . Ceci sera spécifié par le contexte. Contrairement au Théorème A, nous considérons des fonctions qui sont continues et dont les dérivés partielles par rapport à  $(q, p)$  le sont également jusqu'à l'ordre  $[\sigma]$ . En outre, pour tout  $f \in \mathcal{S}_\sigma$  et un paramètre positif  $l > 0$ , nous introduisons la norme suivante

$$|f|_{\sigma, l} = \sup_{t \in J} |f^t|_{C^\sigma} t^l.$$

Maintenant, soient  $s, \lambda, \rho, \beta$  et  $\alpha$  des paramètres positifs qui satisfont les conditions suivantes

$$\begin{cases} 1 \leq \rho < \lambda < s, \\ s > \max \left\{ \frac{\alpha}{\alpha-1}, \lambda + \frac{\alpha}{\beta-1} \right\}, & 1 < \beta < 2, \quad \alpha > 1, \\ \lambda > \frac{2\beta}{2-\beta}, \quad \rho < \frac{\lambda-\beta}{\beta^2}. \end{cases} \quad (\#_E)$$

Étant donné  $\omega \in \mathbb{R}^n$  et des paramètres positifs réels  $\delta > 0$  et  $\varepsilon > 0$ , nous considérons le hamiltonien dépendant du temps suivant

$$\begin{cases} H : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \longrightarrow \mathbb{R} \\ H(q, p, t) = \omega \cdot p + a(q, t) + \underbrace{(b_0(q, t) + b_r(q, t))}_{b(q,t)} \cdot p + m(q, p, t) \cdot p^2 \\ a, b_0, b_r, \partial_p^2 H \in \mathcal{S}_{s+1} \\ |b_0|_{2,1} < \delta, \quad |b_0|_{s+1,1} < \infty, \\ |a|_{\lambda+1,0} + |\partial_q a|_{\lambda,2} < \varepsilon, \quad |b_r|_{\lambda+1,1} < \varepsilon, \\ |a|_{s+1,0} + |\partial_q a|_{s,2} < \infty, \quad |b_r|_{s+1,1} < \infty, \quad |\partial_p^2 H|_{s+1,0} < \infty. \end{cases} \quad (*_E)$$

Soit  $\varphi_0$  le plongement trivial suivant

$$\varphi_0 : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m \times B, \quad \varphi_0(q) = (q, 0).$$

De plus, nous considérons le hamiltonien  $\tilde{h} : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \rightarrow \mathbb{R}$  tel que

$$\tilde{h}(q, p, t) = \omega \cdot p + m(q, p, t) \cdot p^2. \quad (2.16)$$

**Théorème E.** *Soit  $H$  comme dans  $(*_E)$ . Alors, pour  $\delta$  suffisamment petit et pour  $\varepsilon$  assez petit par rapport à  $\delta$ , il existe un  $C^\rho$ -cylindre faiblement asymptotique associé à  $(X_H, X_{\tilde{h}}, \varphi_0)$ .*

Ici,  $b_0$  joue le rôle de  $\gamma$  dans la Définition 2.3. De plus, nous observons que les termes perturbatifs  $\partial_q a$  et  $b_r$  (avec des normes appropriées) décroissent par rapport au temps comme  $\frac{1}{t^2}$  et  $\frac{1}{t}$ , respectivement. Même en prenant  $b_0 \equiv 0$  et  $m = 0$ , le hamiltonien  $H$  ne satisfait pas les hypothèses du Théorème A. L'idée de trouver des solutions localisées dans l'espace de phase pour ces types de systèmes donne lieu au théorème précédent. Cependant, le prix à payer est une conclusion plus faible que celle du Théorème A parce que nous perdons quelques informations sur la dynamique des orbites trouvées. De plus, à cause de la décroissance plus lente par rapport au temps des termes perturbatifs, nous avons besoin d'un hamiltonien  $H$  plus régulier et des conditions de petitesse sur les termes perturbatifs.

La preuve repose sur une version du théorème de Nash-Moser due à Zehnder [Zeh75]. Les conditions  $(\#_E)$  sont une conséquence de ce théorème (Theorem 6.1). Par souci de clarté, nous donnerons un exemple de paramètres satisfaisant  $(\#_E)$ .

$$s > \max \left\{ 7, \lambda + \frac{7}{3} \right\}, \quad \beta = \frac{3}{2}, \quad \alpha = \frac{7}{6}, \quad \lambda > 6, \quad \rho < \frac{4}{9}\lambda - \frac{2}{3}.$$



Notamment, nous pouvons prendre  $\rho$  suffisamment grand au prix de prendre  $s$  et  $\lambda$  assez grands. Zehnder dans son travail a prouvé que l'ordre minimum requis par  $s$ , afin de satisfaire  $(\#_E)$ , est  $s \geq 8$  (Corollaire 6.1). Cela se traduit par

$$s \geq 8, \quad \beta = 1 + \frac{7}{3s}, \quad \alpha = \frac{7}{6}, \quad \lambda = 2 + \frac{14}{s}, \quad \rho = 1$$

pour les autres paramètres.

Concernant la preuve du Théorème E, nous commençons par étudier le problème linéaire associé. Étant donné  $\sigma \geq 1$ ,  $\mu > 0$  et  $\omega \in \mathbb{R}^n$ , cela consiste à résoudre l'équation suivante d'inconnu  $\varkappa : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m}$

$$\begin{cases} \partial_q \varkappa(q, t) (\bar{\omega} + f(q, t)) + \partial_t \varkappa(q, t) + g(q, t) \varkappa(q, t) = z(q, t) \\ f, g, z \in \mathcal{S}_\sigma, \\ |f|_{1,1} \leq \mu \quad |g|_{1,1} \leq \mu, \quad |z|_{\sigma,2} < \infty, \\ |f|_{\sigma,1} < \infty \quad |g|_{\sigma,1} < \infty \end{cases} \quad (HE_E)$$

avec  $\bar{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$ . Les fonctions  $f : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^n$ ,  $z : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^n$  et  $g : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow M_{n+m}$  sont données, où  $M_{n+m}$  est l'ensemble des matrices de dimension  $(n+m)$ .

Si  $\mu = 0$ , celle-ci prend la forme beaucoup plus simple

$$\begin{cases} \partial_q \varkappa(q, t) \bar{\omega} + \partial_t \varkappa(q, t) = z(q, t) \\ z \in \mathcal{S}_\sigma, \quad |z|_{\sigma,2} < \infty. \end{cases}$$

Quand  $m = 0$ , il s'agit du problème linéaire résolu dans la preuve du Théorème A. Cependant, également dans le cas  $m \neq 0$ , suivant les lignes de la preuve du Théorème A, une solution de l'équation précédente existe et satisfait  $|\varkappa|_{\sigma,1} \leq |z|_{\sigma,2}$ .

En revenant au cas général  $\mu \geq 0$ ,  $(HE_E)$  semble plus compliqué que ce dernier. Nous devons supposer que  $\mu$  soit suffisamment petit pour pouvoir le résoudre. Plus précisément, si  $\mu$  est assez petit, nous trouvons une solution de  $(HE_E)$  par intégration grâce à un changement de coordonnées approprié  $\phi$  (qui consiste à rectifier le flot sur le cylindre). La partie la plus laborieuse consiste à trouver une bonne estimation de la solution, qui vérifient

$$|\varkappa|_{\sigma,1} \leq C(\sigma) \frac{|z|_{\sigma,2}}{1 - c_\sigma^\varkappa \mu} + C(\sigma) \frac{|f|_{\sigma,1} + |g|_{\sigma,1}}{(1 - c_\sigma^\varkappa \mu)^2} |z|_{1,2}$$

pour des constantes appropriées  $C(\sigma)$  et  $c_\sigma^\varkappa$  dépendant de  $\sigma$ . Dans ce cas également, la solution  $\varkappa$  est moins régulière (en termes de décroissance par rapport au temps) que  $z$ . Comme dans le Théorème A, nous résolvons ce problème dans la partie non linéaire. Les solutions des équations homologues sont toujours multipliées par d'autres termes satisfaisant de bonnes propriétés de décroissance dont la multiplication nous permet de retrouver ce que nous perdons en termes de décroissance par rapport au temps. Malheureusement, nous avons une perte de dérivés à cause des nouveaux termes  $\gamma$  et  $\Gamma$  (les deux sont égaux à zéro dans le Théorème A). Pour cette raison, nous ne sommes pas en mesure de prouver ce théorème par

le théorème des fonctions implicites, mais nous devons faire recours à une des variantes du théorème de Nash-Moser.

Cette preuve ne marche pas si nous travaillons avec des hamiltoniennes analytiques réels parce que le changement de coordonnées  $\phi$ , utilisé pour résoudre l'équation homologique, dépend du flot de  $\bar{\omega} + f$ . Le terme  $f$  est responsable d'une perte de régularité pour la solution de  $(HE_E)$  qui s'ajoute à la perte de décroissance par rapport au temps, ce qui fait que notre preuve est sans espoir de fonctionner.

Cette démonstration ne marche pas dans le cas des hamiltoniens  $C^\infty$  parce que plus  $\sigma$  est grand et plus nous devons prendre  $\mu$  petit afin de résoudre  $(HE_E)$ .

Nous concluons cette partie avec une remarque concernant l'hypothèse de petitesse sur les termes perturbatifs. Il semble raisonnable de penser que cette hypothèse n'est pas essentielle et, en raisonnant comme dans le Théorème A, on devrait être en mesure de la supprimer. Par contre, l'exemple d'un système planétaire perturbé par une comète donnée est un problème perturbatif. Donc, le Théorème E suffit pour l'application.

### 2.3.2 Le Problème à Trois Corps plus Comète

Nous considérons trois points de masses fixes  $m_0, m_1$  et  $m_2$  subissant une attraction gravitationnelle dans le plan et une comète de masse fixe  $m_c$ . La comète provient et retourne à l'infini le long d'une orbite képlérienne hyperbolique. Nous supposons que le mouvement de la comète est une fonction donnée  $c(t)$  et que seul le système planétaire est influencé par la comète. Nous supposons

$$|c(t)| \rightarrow_{t \rightarrow +\infty} \infty, \quad \frac{d}{dt}|c(t)| \rightarrow_{t \rightarrow +\infty} v > 0.$$

Si la comète est sur une orbite képlérienne hyperbolique,  $\frac{d}{dt}c(t)$  lui-même a une limite. Mais nous n'utiliserons pas cette hypothèse plus forte.

Étant donné  $0 < \varepsilon < 1$  et  $J = [1, +\infty)$  l'espace de phase est l'espace

$$\left\{ ((x_i, y_i)_{0 \leq i \leq 2}, t) \in (\mathbb{R}^2 \times \mathbb{R}^{2*})^3 \times J \mid \begin{array}{l} \forall 0 \leq i < j \leq 2, x_i \neq x_j \\ \frac{|x_i|}{|c(t)|} < \varepsilon \end{array} \right\}$$

de covecteurs de quantité de mouvement  $(y_0, y_1, y_2)$  et vecteurs de position  $(x_0, x_1, x_2)$  de chaque corps. Le hamiltonien du problème à trois corps plus la comète dans le plan (P3BP+C de l'anglais "planar three body problem plus comet") est

$$H(x, y, t) = \underbrace{\sum_{i=0}^2 \frac{|y_i|^2}{2m_i} - \sum_{0 \leq i < j \leq 2} G \frac{m_i m_j}{|x_i - x_j|}}_{H_0(x, y)} - \underbrace{\sum_{i=0}^2 G \frac{m_i m_c}{|x_i - c(t)|}}_{H_c(x, t)},$$

où  $G$  est la constante universelle de gravitation que nous pouvons supposer égale à 1. Comme mentionné précédemment,  $H$  est la somme du hamiltonien du problème à trois corps dans le plan  $H_0$  et du hamiltonien  $H_c$  qui est responsable de l'interaction avec la comète.

Soit  $\varphi_0$  une famille à 1-paramètre de tores invariants pour  $H_0$  supportant une dynamique quasipériodique avec quatre fréquences et  $\psi_{t_0, H}^t$  le flot au moment  $t$  avec un temps initial  $t_0$  de  $H$ .

**Théorème F.** *Soit  $H$  le hamiltonien de  $P3BP+C$ . Alors, si  $|c(1)|$  et  $v$  sont suffisamment grands par rapport à  $\varepsilon$ , pour  $\varepsilon$  assez petit, il existe un sous-ensemble ouvert  $\mathcal{W} \subset (\mathbb{R}^2 \times \mathbb{R}^{2*})^3$  tel que, pour tout  $x \in \mathcal{W}$ ,  $\psi_{1, H}^t(x)$  est une solution faiblement asymptotiquement quasipériodique associée à  $(X_H, X_{H_0}, \varphi_0)$ .*

Concernant  $\varphi_0$ , en 1963, Arnold a prouvé l'existence de solutions quasipériodiques pour le hamiltonien du problème à trois corps dans le plan [Arn63b]. Ici, nous suivons Féjóz [Fé02], qui fournit des solutions plus générales. Dans un repère rotatif, l'auteur prouve l'existence d'orbites quasipériodiques à trois fréquences pour le hamiltonien du problème à trois corps dans le plan. Avant la réduction symplectique par la symétrie des rotations, ces mouvements quasipériodiques ont une fréquence supplémentaire, qui est la vitesse angulaire de la rotation simultanée de trois ellipses. De plus, avant la réduction symplectique par la symétrie des translations, chacune de ces tores invariants se traduit par une famille à 1-paramètre de tores invariants paramétrés par le centre de masse du système planétaire.

La preuve du Théorème F repose sur [Fé02] et le Théorème E. La première partie est dédiée au hamiltonien  $H_0$ . Nous commençons par introduire un changement symplectique linéaire de coordonnées  $\phi_0$ . Dans ces nouvelles variables  $\{(X_i, Y_i)\}_{i=0,1,2}$ , nous pouvons diviser le hamiltonien  $H_0$  du problème à trois corps dans le plan comme suit

$$H_0 \circ \phi_0(X, Y) = \frac{|Y_0|^2}{2M} + K(X_1, X_2, Y_1, Y_2), \quad (2.17)$$

où  $K$  est le hamiltonien du problème à trois corps dans le plan après la réduction par la symétrie des translations,  $X_0$  est le centre de masse du système planétaire,  $Y_0$  est le moment linéaire du système planétaire et  $M = m_0 + m_1 + m_2$ .

Comme mentionné précédemment,  $K$  possède un tore invariant supportant une dynamique quasipériodique avec quatre fréquences. Grâce à (2.17), ce tore invariant se traduit en une famille à 1-paramètre de tores invariants paramétrés par le centre de masse du système planétaire.

La seconde partie de la preuve est dédiée à la perturbation  $H_c$ . Nous prouvons que dans un certain sous-ensemble de l'espace de phase, où nous nous attendons à ce que les orbites d'intérêt aient lieu, la perturbation  $H_c$  satisfait de bonnes propriétés de décroissance lorsque  $t \rightarrow +\infty$ , mais pas toutes les hypothèses du Théorème E. Afin de résoudre ce problème, nous introduisons une extension lisse appropriée  $\tilde{H}$  du hamiltonien  $H$  et nous prouvons l'existence d'un  $C^1$ -cylindre faiblement asymptotique pour cette extension. Donc, nous concluons la preuve du Théorème F vérifiant l'existence d'un sous-ensemble ouvert de points initiaux donnant lieu à des solutions faiblement asymptotiquement quasipériodiques pour  $X_{\tilde{H}}$ , qui sont également des orbites pour  $X_H$ .

Soit  $\omega \in \mathbb{R}^4$  la fréquence d'une solution quasipériodique pour le hamiltonien du problème à trois corps dans le plan. Ce résultat (Théorème F) garantit l'existence d'orbites où le centre de masse du système planétaire est attiré par la comète avec une vitesse asymptotiquement nulle quand le temps tend à  $+\infty$ . De plus, dans

un repère de référence attaché au centre de masse du système planétaire, les mouvements des planètes convergent vers certaines dynamiques qui sont conjuguées à des orbites générées par une petite perturbation dépendant du temps du champ de vecteurs constant sur le tore  $\omega$ .

## 2.4 Mouvements Asymptotiques Convergeant vers une Dynamique Arbitraire

Nous considérons des champs de vecteurs dépendant du temps convergeant avec une vitesse exponentielle par rapport au temps vers des champs de vecteurs ayant un tore invariant supportant une dynamique arbitraire. Nous prouvons des résultats qui sont des variations du théorème de Canadell-de la Llave dans le cas particulier des champs de vecteurs hamiltoniens et des champs de vecteurs sur le tore.

Étant donné  $B \subset \mathbb{R}^n$  une boule centrée à l'origine et  $\mathcal{P}$  égal à  $\mathbb{T}^n \times B$  ou à  $\mathbb{T}^n$ . Pour  $v > 0$ , on rappelle que  $J_v = [v, +\infty) \subset \mathbb{R}$ .

Commençons par la définition de  $C^\sigma$ -tore asymptotique. Étant donné  $\sigma \geq 0$ ,  $v \geq 0$  et d'un entier positif  $k \geq 0$ , nous considérons les champs de vecteurs dépendant du temps  $X^t$  et  $X_0^t$  de classe  $C^{\sigma+k}$  sur  $\mathcal{P}$ , pour tout  $t \in J_v$ , un plongement  $\varphi_0$  de  $\mathbb{T}^n$  sur  $\mathcal{P}$  de classe  $C^\sigma$  et un champ de vecteurs sur le tore  $W$  de classe  $C^\sigma$  tel que

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_{C^{\sigma+k}} = 0, \quad (2.18)$$

$$X(\varphi_0(q), t) = \partial_q \varphi_0(q) W(q) \text{ pour tout } (q, t) \in \mathbb{T}^n \times J_v. \quad (2.19)$$

Cela signifie que  $X^t - X_0^t$  converge vers zéro lorsque  $t \rightarrow +\infty$ . De plus, le champ de vecteurs  $X_0$  possède un tore invariant  $\varphi_0$  supportant une dynamique générée par le champ de vecteurs autonome  $W$ .

**Définition 2.5.** *Nous supposons que  $(X, X_0, \varphi_0, W)$  satisfont (2.18) et (2.19). Une famille de  $C^\sigma$ -plongements  $\varphi^t : \mathbb{T}^n \rightarrow \mathcal{P}$  est un  $C^\sigma$ -tore asymptotique associé à  $(X, X_0, \varphi_0, W)$  s'il existe  $v' \geq v \geq 0$  tel que*

$$\lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{C^\sigma} = 0, \quad (2.20)$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) W(q) + \partial_t \varphi(q, t), \quad (2.21)$$

pour tout  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Lorsque  $\dim \mathcal{P} = 2n$ , alors nous disons que  $\varphi^t$  est lagrangien si  $\varphi^t(\mathbb{T}^n)$  est lagrangien pour tout  $t \in J_{v'}$ .

Contrairement à la définition de  $C^\sigma$ -tore KAM asymptotique (Définition 1.3), ici, nous ne supposons pas  $W$  constant. En effet, si  $W(q) \equiv \text{cst}$ , nous obtenons la définition de  $C^\sigma$ -tore KAM asymptotique.

En ce qui concerne certaines propriétés des  $C^\sigma$ -tores asymptotiques, si  $X$  est complet, alors, de la même manière que pour des  $C^\sigma$ -tores KAM asymptotiques, nous pouvons réécrire (2.21) en termes du flot de  $X$ . De plus, (2.21) est trivial et si  $\varphi^t$  est un  $C^\sigma$ -tore asymptotique défini pour tout  $t \geq v'$ , alors nous pouvons étendre l'ensemble de définition pour tout  $t \in \mathbb{R}$ . Par ailleurs, comme on peut s'y attendre, concernant la dynamique dans la projection sur l'espace de phase  $\mathcal{P}$ , les

trajectoires convergent asymptotiquement par rapport au temps vers les orbites de  $X_0$  générées par  $W$  sur le tore invariant  $\varphi_0$ .

Étant donné  $\sigma \geq 0$ ,  $\nu \geq 0$  et un entier positif  $k \geq 0$ , nous rappelons la définition suivante

**Definition.** Soit  $\bar{\mathcal{S}}_{\sigma,k}^\nu$  l'espace de fonctions  $f$  définies sur  $\mathbb{T}^n \times B \times J_\nu$  tel que, pour tout  $t \in J_\nu$

$$f^t \in C^{\sigma+k}(\mathbb{T}^n \times B) \quad \text{et} \quad \partial_{(q,p)}^i f \in C(\mathbb{T}^n \times B \times J_\nu)$$

pour tout  $0 \leq i \leq k$ .

Maintenant, nous considérons le hamiltonien suivant  $h : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R}$  tel que, pour tout  $(q, t) \in \mathbb{T}^n \times J_0$ ,

$$h(q, 0, t) = c, \quad \partial_p h(q, 0, t) = W(q) \tag{2.22}$$

pour certains  $c \in \mathbb{R}$  et  $W \in C^{\sigma+2}(\mathbb{T}^n)$ . Soit  $\mathcal{K}_W$  l'ensemble des hamiltoniens  $h : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R}$  qui satisfont (2.22).

Nous observons que, pour tout  $h \in \mathcal{K}_W$ , le plongement trivial  $\varphi_0$  donné par

$$\varphi_0 : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B, \quad \varphi_0(q) = (q, 0),$$

est un tore invariant pour  $X_h$  et le champ de vecteurs restreint est  $W$ .

Étant donné  $\sigma \geq 1$  et  $\lambda \geq 0$ , nous considérons le hamiltonien  $H$  suivant

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R} \\ H(q, p, t) = h(q, p, t) + f(q, p, t) \\ h \in \mathcal{K}_W, \quad W \in C^{\sigma+2}(\mathbb{T}^n), \\ f_0, \partial_p f_0, \partial_p^2 H \in \bar{\mathcal{S}}_{\sigma,2}^0, \\ |f_0|_{\sigma+2,0}^0 + |\partial_q f_0|_{\sigma+1,\lambda}^0 < \infty, \quad |\partial_p f_0|_{\sigma+2,\lambda}^0 < \infty \\ |\partial_p^2 H|_{\sigma+2,0}^0 < \infty. \end{array} \right. \tag{*G}$$

**Théorème G.** Soit  $H$  comme dans  $(*G)$ . Alors, il existe un hamiltonien  $\tilde{h} \in \mathcal{K}_W$  et une constante  $C(\sigma)$  qui dépend de  $\sigma$  tel que si

$$\lambda > C(\sigma) |\partial_q W|_{C^0}, \tag{\lambda}$$

il existe un  $C^\sigma$ -tore asymptotique lagrangien associé à  $(X_H, X_{\tilde{h}}, \varphi_0, W)$ .

Contrairement au théorème de Canadell-de la Llave, ici, le champ de vecteurs  $W$  n'est pas constant. De plus, nous observons que si  $W \equiv \text{cst}$  nous obtenons  $\lambda > 0$ , ce qui est l'hypothèse de Canadell-de la Llave dans le cas des systèmes hamiltoniens. Évidemment, le Théorème A prouve que dans ce cas nous n'avons pas besoin de décroissance exponentielle.

Nous démontrons l'existence d'un  $C^\sigma$ -tore asymptotique  $\varphi^t$  de la forme

$$\varphi^t(q) = (q + u^t(q), v^t(q))$$

pour tout  $q \in \mathbb{T}^n$  et  $t$  suffisamment grand, où  $\text{id} + u^t$  est un difféomorphisme du tore pour chaque  $t$  fixé. Concernant  $u$  et  $v$ , nous avons

$$|u^t|_{C^\sigma} \leq C e^{-\lambda t}, \quad |v^t|_{C^\sigma} \leq C e^{-\lambda t},$$

pour tout  $t$  assez grand et pour une constante appropriée  $C$ .

La preuve est essentiellement la même que celle du Théorème A. La principale différence repose sur la solution du problème linéaire associé. Résoudre ce problème, dans ce cas, est plus compliqué. Étant donné  $\sigma \geq 1$ ,  $\lambda > 0$  et  $\nu \geq 0$ , il consiste à résoudre l'équation suivante d'inconnu  $\varkappa : \mathbb{T}^n \times J_\nu \rightarrow \mathbb{R}^n$

$$\begin{cases} \partial_q \varkappa(q, t) W(q) + \partial_t \varkappa(q, t) + \partial_q W(q) \varkappa(q, t) = z(q, t) \\ W \in C^{\sigma+1}(\mathbb{T}^n), \quad z \in \bar{\mathcal{S}}_{\sigma, 0}^\nu, \\ \sup_{t \in J_\nu} |z^t|_{C^\sigma} e^{\lambda t} < \infty. \end{cases} \quad (HE_G)$$

Quand  $W(q) \equiv W \in \mathbb{R}^n$  est constante, l'expression précédente prend la forme suivante plus simple

$$\begin{cases} \partial_q \varkappa(q, t) W + \partial_t \varkappa(q, t) = z(q, t) \\ z \in \bar{\mathcal{S}}_{\sigma, 0}^\nu, \\ \sup_{t \in J_\nu} |z^t|_{C^\sigma} e^{\lambda t} < \infty. \end{cases}$$

Ce système est un cas particulier de  $(HE_A)$  et nous savons déjà comment le résoudre. En ce qui concerne le cas général, nous pouvons trouver une solution pour  $(HE_G)$  par intégration grâce à un changement de coordonnées approprié, dépendant du flot de  $W$ , qui rectifie la dynamique sur le tore. Plus précisément, nous prouvons l'existence d'une constante  $c_\sigma^\varkappa$ , dépendant de  $\sigma$ , de telle sorte que si

$$\lambda > c_\sigma^\varkappa |\partial_q W|_{C^0},$$

alors une solution de  $(HE_G)$  existe et

$$\begin{aligned} \sup_{t \in J_\nu} |\varkappa^t|_{C^\sigma} e^{\lambda t} &\leq C(\sigma) \left( \frac{1}{\lambda - c_\sigma^\varkappa |\partial_q W|_{C^0}} \right) \sup_{t \in J_\nu} |z^t|_{C^\sigma} e^{\lambda t} \\ &+ C(\sigma) \left( \frac{|\partial_q W|_{C^\sigma}}{(\lambda - c_\sigma^\varkappa |\partial_q W|_{C^0})^2} + \frac{|\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}}{(\lambda - c_\sigma^\varkappa |\partial_q W|_{C^0})^3} \right) \sup_{t \in J_\nu} |z^t|_{C^\sigma} e^{\lambda t} \end{aligned}$$

pour une constante  $C(\sigma)$  qui dépend de  $\sigma$ . Contrairement au Théorème A, la solution  $\varkappa$  a la même régularité que  $z$ .

De même que pour le Théorème A, nous n'avons pas besoin que la perturbation soit petite pour tout  $t \in J_0$ . Nous recherchons plutôt un  $C^\sigma$ -tore asymptotique associé à  $(X_H, X_{\tilde{h}}, \varphi_0, W)$  défini pour tout  $t$  suffisamment grand tel que  $|\partial_q f_0^t|_{C^{\sigma+1}}$  et  $|\partial_p f_0^t|_{C^{\sigma+2}}$  soient suffisamment petits.

Comme le Théorème E, cette preuve ne marche pas dans le cadre analytique. En effet, le changement de coordonnées utilisé dans la solution de l'équation homologique  $(HE_G)$  dépend du flot de  $W$ .

Cette preuve ne fonctionne pas non plus dans le cas des hamiltoniens  $C^\infty$  parce que plus  $\sigma$  est grand, plus nous devons prendre  $\lambda$  grand afin de résoudre  $(HE_G)$ .

Nous avons également un résultat concernant les perturbations dépendant du temps des champs de vecteurs autonomes sur le tore. Étant donné  $\sigma \geq 1$ , soit  $Z$  un champ de vecteurs non autonome sur  $\mathbb{T}^n \times J_0$  de la forme

$$\begin{cases} Z(q, t) = W(q) + P(q, t) \\ W \in C^{\sigma+1}(\mathbb{T}^n), \quad P \in \bar{\mathcal{S}}_{\sigma,1}^0, \\ |P|_{\sigma+1,\lambda}^0 < \infty. \end{cases} \quad (Z_C)$$

**Corollaire C.** *Soit  $Z$  comme dans  $(Z_C)$ . Alors, il existe une constante  $C(\sigma)$  dépendant de  $\sigma$  telle que si*

$$\lambda > C(\sigma) |\partial_q W|_{C^0},$$

*alors il existe un  $C^\sigma$ -tore asymptotique  $\psi^t$  associé à  $(Z, W, \text{Id}, W)$ .*

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## Part II

# Asymptotic Motions Converging to Quasiperiodic Dynamics

This part is devoted to a series of results which improve those of Fortunati-Wiggins [FW14] and Canadell-de la Llave [CdLL15] in the case of time-dependent Hamiltonian vector fields and time-dependent vector fields on the torus. The decision to work with these systems has allowed us to relax the exponential decay in time in view of the possible applications in celestial mechanics.

We do not need the perturbation to be small for all times but just for times large enough. This led us to prove the existence of a  $C^\sigma$ -asymptotic KAM torus defined for all  $t$  sufficiently large.

This part is divided into two chapters. First, we prove the existence of a  $C^\sigma$ -asymptotic KAM torus for finite differentiable time-dependent Hamiltonian vector fields and for time-dependent vector fields on the torus. Then, in the second chapter, we prove the same results for real analytic systems.

## 3 Finitely Differentiable Case

This chapter is divided into four sections. We begin with a part where we recall and explain the definition of  $C^\sigma$ -asymptotic KAM torus (Section 3.1). We continue with a section (Section 3.2) dedicated to the main results of this chapter (Theorem A and Corollary A). The last sections contain the proofs of these results (Sections 3.3 and 3.4)

### 3.1 $C^\sigma$ -asymptotic KAM torus

We begin with the definition of  $C^\sigma$ -asymptotic KAM torus. Let  $B \subset \mathbb{R}^n$  be a ball centred at the origin,  $\mathcal{P}$  equal to  $\mathbb{T}^n \times B$  or  $\mathbb{T}^n$  and, for all  $v \geq 0$ ,  $J_v = [v, +\infty) \subset \mathbb{R}$ .

Given  $\sigma \geq 0$ ,  $v \geq 0$  and a positive integer  $k \geq 0$ , we consider time-dependent vector fields  $X^t$  and  $X_0^t$  of class  $C^{\sigma+k}$  on  $\mathcal{P}$ , for all  $t \in J_v$ , and an embedding  $\varphi_0$  from  $\mathbb{T}^n$  to  $\mathcal{P}$  of class  $C^\sigma$  such that

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_{C^{\sigma+k}} = 0, \quad (3.1)$$

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q) \omega \text{ for all } (q, t) \in \mathbb{T}^n \times J_v, \quad (3.2)$$

where  $\omega \in \mathbb{R}^n$ .

**Definition** (Définition 1.3). *We assume that  $(X, X_0, \varphi_0)$  satisfy (3.1) and (3.2). A family of  $C^\sigma$  embeddings  $\varphi^t : \mathbb{T}^n \rightarrow \mathcal{P}$  is a  $C^\sigma$ -asymptotic KAM torus associated to  $(X, X_0, \varphi_0)$  if there exists  $v' \geq v \geq 0$  such that*

$$\lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{C^\sigma} = 0, \quad (3.3)$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t), \quad (3.4)$$



for all  $(q, t) \in \mathbb{T}^n \times J_{\nu'}$ . When  $\dim \mathcal{P} = 2n$ , then we say that  $\varphi^t$  is Lagrangian if  $\varphi^t(\mathbb{T}^n)$  is Lagrangian for all  $t$ .

For all  $q \in \mathbb{T}^n$  and  $t, t_0 \in J_{\nu'}$ , let  $\psi_{t_0, X}^t$  be the flow at time  $t$  with initial time  $t_0$  of  $X$  and  $\psi_{t_0, \omega}^t(q) = q + \omega(t - t_0)$ . We assume that  $\psi_{t_0, X}^t$  is defined for all  $t, t_0 \in J_{\nu'}$ . Then, we can rewrite (3.4) in terms of the flow of  $X$ . That is to say, that (3.4) is equivalent to

$$\psi_{t_0, X}^t \circ \varphi^{t_0}(q) = \varphi^t \circ \psi_{t_0, \omega}^t(q) \quad (3.5)$$

for all  $q \in \mathbb{T}^n$  and  $t, t_0 \in J_{\nu'}$ . For the sake of clarity, we prove it.

**Proposition** (Proposition 1.1). *If the flow  $\psi_{t_0, X}^t$  is defined for all  $t, t_0 \in J_{\nu'}$ , then (3.4) is equivalent to (3.5).*

*Proof.* In this proof, we denote the time dependence by indexes. We assume (3.4) and we prove (3.5). For fixed  $t_0$ , let  $\varphi^{-t_0}$  be the inverse map of  $\varphi^{t_0}$ . It suffices to show that  $\psi_{t_0, X}^t$  and  $\varphi^t \circ \psi_{t_0, \omega}^t \circ \varphi^{-t_0}$  verify the same differential equation. For all  $x \in \varphi^{t_0}(\mathbb{T}^n)$

$$\begin{aligned} \frac{d}{dt} (\varphi^t \circ \psi_{t_0, \omega}^t \circ \varphi^{-t_0}(x)) &= \partial_q \varphi^t (\psi_{t_0, \omega}^t (\varphi^{-t_0}(x))) \psi_{t_0, \omega}^t (\varphi^{-t_0}(x)) \\ &\quad + \partial_t \varphi^t (\psi_{t_0, \omega}^t (\varphi^{-t_0}(x))) \\ &= \partial_q \varphi^t (\psi_{t_0, \omega}^t (\varphi^{-t_0}(x))) \omega + \partial_t \varphi^t (\psi_{t_0, \omega}^t (\varphi^{-t_0}(x))) \\ &= X^t \circ \varphi^t \circ \psi_{t_0, \omega}^t \circ \varphi^{-t_0}(x), \end{aligned}$$

where  $\dot{\psi}_{t_0, \omega}^t$  stands for the derivative with respect to  $t$  of  $\psi_{t_0, \omega}^t$ , it is obviously equal to  $\omega$ . The last equality is a consequence of (3.4). This concludes the first part of the proof.

Now, we assume (3.5) and we prove (3.4). We fix  $t_0 \in J_{\nu'}$ , for all  $t \in J_{\nu'}$  and  $x \in \varphi^{t_0}(\mathbb{T}^n)$

$$\frac{d}{dt} (\varphi^t \circ \psi_{t_0, \omega}^t \circ \varphi^{-t_0}(x)) = \dot{\psi}_{t_0, X}^t(x) = X^t \circ \psi_{t_0, X}^t(x) = X^t \circ \varphi^t \circ \psi_{t_0, \omega}^t \circ \varphi^{-t_0}(x).$$

On the other hand, by the chain rule

$$\frac{d}{dt} (\varphi^t \circ \psi_{t_0, \omega}^t \circ \varphi^{-t_0}(x)) = \partial_q \varphi^t (\psi_{t_0, \omega}^t (\varphi^{-t_0}(x))) \omega + \partial_t \varphi^t (\psi_{t_0, \omega}^t (\varphi^{-t_0}(x))).$$

We know that  $\varphi^{t_0}$  is an embedding, then there exists  $q \in \mathbb{T}^n$  such that  $\varphi^{t_0}(q) = x$ . Thanks to the above equations

$$X^t \circ \varphi^t \circ \psi_{t_0, \omega}^t(q) = \partial_q \varphi^t (\psi_{t_0, \omega}^t(q)) \omega + \partial_t \varphi^t (\psi_{t_0, \omega}^t(q))$$

for all  $q \in \mathbb{T}^n$  and for all  $t \in J_{\nu'}$ . Letting  $t = t_0$  we have the claim.  $\square$

We observe that (3.4) is trivial and if  $\varphi^t$  is a  $C^\sigma$ -asymptotic KAM torus defined for all  $t \geq \nu'$ , then we can extend the set of definition for all  $t \in \mathbb{R}$ .

Concerning the dynamics associated to a  $C^\sigma$ -asymptotic KAM torus, we recall the definition of asymptotically quasiperiodic solution and discuss some properties of these motions.

**Definition 3.1.** We assume that  $(X, X_0, \varphi_0)$  satisfy (3.1) and (3.2). An integral curve  $g(t)$  of  $X$  is an asymptotically quasiperiodic solution associated to  $(X, X_0, \varphi_0)$  if there exists  $q \in \mathbb{T}^n$  in such a way that

$$\lim_{t \rightarrow +\infty} |g(t) - \varphi_0 \circ \psi_{t_0, \omega}^t(q)| = 0.$$

As mentioned in the introduction of this work, the following proposition provides an evident link between a  $C^\sigma$ -asymptotic KAM torus and the above-mentioned motions.

**Proposition 3.1.** Let  $\varphi^t$  be a  $C^\sigma$ -asymptotic KAM torus associated to  $(X, X_0, \varphi_0)$ . Then, for all  $q \in \mathbb{T}^n$  and  $t_0 \in J_{v'}$ ,

$$g(t) = \psi_{t_0, X}^t \circ \varphi^{t_0}(q)$$

is an asymptotically quasiperiodic solution associated to  $(X, X_0, \varphi_0)$ .

Let  $X, X_0$  and  $\varphi_0$  in such a way that  $(X, X_0, \varphi_0)$  satisfy (3.1) and (3.2). We consider the case when  $X$  and  $X_0$  are Hamiltonian vector fields. Letting  $\mathcal{P} = \mathbb{T}^n \times B$ , we assume that, for all  $t \in J_{v'}$ ,  $\varphi^t$  is a  $C^\sigma$ -asymptotic KAM torus associated to  $(X, X_0, \varphi_0)$ . We prove the following property mentioned in the introduction of this thesis. Canadell-de la Llave prove it in the discrete case. Here, we prove it in the continuous case.

**Proposition 3.2.** Let  $\varphi^t$  be a  $C^\sigma$ -asymptotic KAM torus associated to  $(X, X_0, \varphi_0)$ . If  $\varphi_0(\mathbb{T}^n)$  is Lagrangian, then  $\varphi^t(\mathbb{T}^n)$  is Lagrangian for all  $t \in J_{v'}$ .

*Proof.* Let  $\alpha = dp \wedge dq$  be the standard symplectic form on  $(q, p) \in \mathbb{T}^n \times B$ . For all fixed  $t, t_0 \in J_{v'}$ , the map  $\psi_{t_0, X}^t$  is a symplectomorphism. This means that  $(\psi_{t_0, X}^t)^* \alpha = \alpha$  for all fixed  $t, t_0 \in J_{v'}$ . Since (3.5), for all  $t_0 \in J_{v'}$  and  $t \geq 0$

$$\psi_{t_0, X}^{t_0+t} \circ \varphi^{t_0} = \varphi^{t_0+t} \circ \psi_{t_0, \omega}^{t_0+t} \tag{3.6}$$

and, taking the pull-back with respect to the standard form  $\alpha$  on both sides of the latter, we obtain

$$(\varphi^{t_0})^* (\psi_{t_0, X}^{t_0+t})^* \alpha = (\psi_{t_0, \omega}^{t_0+t})^* (\varphi^{t_0+t})^* \alpha.$$

We know that  $\psi_{t_0, X}^{t_0+t}$  is symplectic then, replacing  $(\psi_{t_0, X}^{t_0+t})^* \alpha = \alpha$  on the left hand side of the above equation, we have

$$(\varphi^{t_0})^* \alpha = (\psi_{t_0, \omega}^{t_0+t})^* (\varphi^{t_0+t})^* \alpha.$$

We want to prove that for all  $q \in \mathbb{T}^n$ ,  $((\varphi^{t_0})^* \alpha)_q = 0$ , where  $((\varphi^{t_0})^* \alpha)_q$  stands for the symplectic form calculated on  $q \in \mathbb{T}^n$ . The idea consists in verifying that, for all fixed  $q \in \mathbb{T}^n$ , the limit when  $t \rightarrow +\infty$  on the right-hand side of the above equation converges to zero. Then, taking the limit for  $t \rightarrow +\infty$  on both sides of the latter, we have the claim.

We introduce the following notation

$$\varphi^t(q) = (U^t(q), V^t(q)), \quad \varphi_0(q) = (U_0(q), V_0(q))$$

for suitable families of functions  $U^t, V^t : \mathbb{T}^n \rightarrow \mathbb{R}^n$  and  $U_0, V_0 : \mathbb{T}^n \rightarrow \mathbb{R}^n$ . One can see that, for all  $q \in \mathbb{T}^n$

$$\left( (\psi_{t_0, \omega}^{t_0+t})^* (\varphi^{t_0+t})^* \alpha \right)_q = \sum_{1 \leq i < j \leq n} \alpha_{ij}^{t_0+t}(q) dq_i \wedge dq_j,$$

where for all  $1 \leq i < j \leq n$

$$\alpha_{i,j}^{t_0+t}(q) = (\partial_{q_i} V^{t_0+t} \cdot \partial_{q_j} U^{t_0+t} - \partial_{q_j} V^{t_0+t} \cdot \partial_{q_i} U^{t_0+t}) \circ \psi_{t_0, \omega}^{t_0+t}(q).$$

We observe that, for all  $q \in \mathbb{T}^n$  and for all fixed  $1 \leq i < j \leq n$

$$\partial_{q_i} V_0(q) \cdot \partial_{q_j} U_0(q) - \partial_{q_j} V_0(q) \cdot \partial_{q_i} U_0(q) = 0$$

because  $\varphi_0$  is Lagrangian. Then,

$$\begin{aligned} |\alpha_{i,j}^{t_0+t}|_{C^0} &\leq |\partial_{q_i} V^{t_0+t} \cdot \partial_{q_j} U^{t_0+t} - \partial_{q_j} V^{t_0+t} \cdot \partial_{q_i} U^{t_0+t}|_{C^0} \\ &= |(\partial_{q_i} V^{t_0+t} \cdot \partial_{q_j} U^{t_0+t} - \partial_{q_j} V^{t_0+t} \cdot \partial_{q_i} U^{t_0+t}) - (\partial_{q_i} V_0 \cdot \partial_{q_j} U_0 - \partial_{q_j} V_0 \cdot \partial_{q_i} U_0)|_{C^0} \\ &\leq |\partial_{q_i} V^{t_0+t} \cdot \partial_{q_j} U^{t_0+t} - \partial_{q_i} V_0 \cdot \partial_{q_j} U_0|_{C^0} + |\partial_{q_j} V^{t_0+t} \cdot \partial_{q_i} U^{t_0+t} - \partial_{q_j} V_0 \cdot \partial_{q_i} U_0|_{C^0}, \end{aligned}$$

and we can estimate each term in the last line of the latter by

$$|V^{t_0+t}|_{C^1} |U^{t_0+t} - U_0|_{C^1} + |U_0|_{C^1} |V^{t_0+t} - V_0|_{C^1}$$

multiplied by a suitable constant  $C$ . This concludes the proof of this proposition because, by (3.3), the latter converges to 0 if  $t \rightarrow +\infty$ .  $\square$

## 3.2 Results

This section is dedicated to the main results of this chapter. First, we need to introduce some notations. Given positive real parameters  $\sigma \geq 0$  and  $v \geq 0$ , we have the following definition

**Definition 3.2.** *Let  $\mathcal{S}_\sigma^v$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times B \times J_v$  such that  $f \in C(\mathbb{T}^n \times B \times J_v)$  and, for all  $t \in J_v$ ,  $f^t \in C^\sigma(\mathbb{T}^n \times B)$ .*

We use this notation also for functions defined on  $\mathbb{T}^n \times J_v$ , this will be specified by the context. Furthermore, for a positive integer  $k \geq 0$ , we have the following space of functions

**Definition 3.3.** *Let  $\bar{\mathcal{S}}_{\sigma,k}^v$  be the space of functions  $f$  such that*

$$f \in \mathcal{S}_{\sigma+k}^v \quad \text{and} \quad \partial_{(q,p)}^i f \in \mathcal{S}_{\sigma+k-i}^v$$

for all  $0 \leq i \leq k$ .

We conventionally let  $f = \partial_{(q,p)}^0 f$ . In other words,  $f \in \bar{\mathcal{S}}_{\sigma,k}^v$  if  $f \in \mathcal{S}_{\sigma+k}^v$  and  $\partial_{(q,p)}^i f \in C(\mathbb{T}^n \times B \times J_v)$  for all  $0 \leq i \leq k$ . That is,  $f^t \in C^{\sigma+k}(\mathbb{T}^n \times B)$  for all  $t \in J_v$  and the partial derivatives of  $f$  with respect to  $(q, p)$  are continuous until the order  $k$ . It is straightforward to verify that  $\bar{\mathcal{S}}_{\sigma,0}^v = \mathcal{S}_\sigma^v$ .

In order to measure the decay in time of the perturbations, we introduce positive, decreasing, integrable functions  $\mathbf{u}$  on  $J_0$  and we denote

$$\bar{\mathbf{u}}(t) = \int_t^{+\infty} \mathbf{u}(\tau) d\tau$$

for all  $t \in J_0$ .

We recall that  $\mathcal{K}_\omega$  is the set of the Hamiltonians in  $\omega$ -Kolmogorov normal form and we consider the following trivial embedding

$$\varphi_0 : \mathbb{T}^n \longrightarrow \mathbb{T}^n \times B, \quad \varphi_0(q) = (q, 0).$$

Given  $\omega \in \mathbb{R}^n$  and  $\sigma \geq 1$ , we consider a time-dependent Hamiltonian  $H$  of the form

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times J_0 \longrightarrow \mathbb{R} \\ H(q, p, t) = h(q, p, t) + f(q, p, t), \\ h \in \mathcal{K}_\omega \\ f_0, \partial_p f_0, \partial_p^2 H \in \bar{\mathcal{S}}_{\sigma, 2}^0 \\ \sup_{t \in J_0} |f_0^t|_{C^{\sigma+2}} < \infty, \quad \sup_{t \in J_0} |\partial_p^2 H^t|_{C^{\sigma+2}} < \infty, \\ |\partial_q f_0^t|_{C^{\sigma+1}} \leq \mathbf{a}(t), \quad |\partial_p f_0^t|_{C^{\sigma+2}} \leq \mathbf{b}(t) \quad \text{for all } t \in J_0, \end{array} \right. \quad (*_A)$$

where  $\mathbf{a}, \mathbf{b}$  are positive, decreasing, integrable functions on  $J_0$ .

We assume that there exists  $v \geq 0$ , such that  $\mathbf{a}$  and  $\mathbf{b}$  satisfy the following conditions

$$\left\{ \begin{array}{l} \bar{\mathbf{a}}(t) \leq \mathbf{b}(t) \\ \bar{\mathbf{a}}(t)\mathbf{b}(t) \leq \mathbf{a}(t)\bar{\mathbf{b}}(t) \end{array} \right. \quad (\#)$$

for all  $t \in J_v$ .

**Theorem A.** *Let  $H$  be as in  $(*_A)$  with  $\mathbf{a}$  and  $\mathbf{b}$  satisfying  $(\#)$ . Then, there exist  $\tilde{h} \in \mathcal{K}_\omega$  and a Lagrangian  $C^\sigma$ -asymptotic KAM torus  $\varphi^t$  associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ .*

About time-dependent perturbations of constant vector fields on the torus, given  $\sigma \geq 1$  and  $\omega \in \mathbb{R}^n$ , we consider the following time-dependent vector field

$$\left\{ \begin{array}{l} Z : \mathbb{T}^n \times J_0 \longrightarrow \mathbb{R}^n \\ Z(q, t) = \omega + P(q, t) \\ P \in \bar{\mathcal{S}}_{\sigma, 1}^0, \\ |P^t|_{C^{\sigma+1}} \leq \mathbf{P}(t) \quad \text{for all } t \in J_0, \end{array} \right. \quad (Z_A)$$

where  $\mathbf{P}$  is a positive, decreasing, integrable function on  $J_0$ .

**Corollary A.** *Let  $Z$  be as in  $(Z_A)$ . Then, there exists a  $C^\sigma$ -asymptotic KAM torus  $\psi^t$  associated to  $(Z, \omega, \text{Id})$ .*

### 3.3 Proof of Theorem A

This section is devoted to the proof of Theorem A. To this end, we expand the Hamiltonian  $H$  in  $(*_A)$  in a small neighbourhood of  $0 \in B$ ,

$$\begin{aligned} h(q, p, t) &= h(q, 0, t) + \partial_p h(q, 0, t) \cdot p + \int_0^1 (1 - \tau) \partial_p^2 h(q, \tau p, t) d\tau \cdot p^2 \\ f(q, p, t) &= f(q, 0, t) + \partial_p f(q, 0, t) \cdot p + \int_0^1 (1 - \tau) \partial_p^2 f(q, \tau p, t) d\tau \cdot p^2, \end{aligned}$$

we can assume without loss of generality that  $h(q, 0, t) = 0$  for all  $(q, t) \in \mathbb{T}^n \times J_0$ . Letting

$$\begin{aligned} \omega &= \partial_p h(q, 0, t) \\ a(q, t) &= f(q, 0, t) \\ b(q, t) &= \partial_p f(q, 0, t) \\ m(q, p, t) &= \int_0^1 (1 - \tau) (\partial_p^2 h(q, \tau p, t) + \partial_p^2 f(q, \tau p, t)) d\tau \\ &= \int_0^1 (1 - \tau) \partial_p^2 H(q, \tau p, t) d\tau, \end{aligned}$$

for a positive real parameter  $\Upsilon \geq 1$ , we can rewrite the Hamiltonian  $H$  in the following form

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times J_0 \longrightarrow \mathbb{R} \\ H(q, p, t) = \omega \cdot p + a(q, t) + b(q, t) \cdot p + m(q, p, t) \cdot p^2, \\ a, b \in \bar{\mathcal{S}}_{\sigma, 2}^0, \\ \sup_{t \in J_0} |a^t|_{C^{\sigma+2}} < \infty, \quad \sup_{t \in J_0} |\partial_p^2 H^t|_{C^{\sigma+2}} \leq \Upsilon, \\ |\partial_q a^t|_{C^{\sigma+1}} \leq \mathbf{a}(t), \quad |b^t|_{C^{\sigma+2}} \leq \mathbf{b}(t), \quad \text{for all } t \in J_0 \end{array} \right. \quad (**_A)$$

where  $\partial_p^2 H \in \bar{\mathcal{S}}_{\sigma, 2}^0$  and  $\mathbf{a}(t), \mathbf{b}(t)$  are the functions introduced in  $(*_A)$  satisfying  $(\#)$ . This Hamiltonian is our new starting point. Furthermore, let  $\tilde{h}$  be the following Hamiltonian

$$\tilde{h}(q, p, t) = h(q, p, t) + \int_0^1 (1 - \tau) \partial_p^2 f(q, \tau p, t) d\tau \cdot p^2$$

for all  $(q, p, t) \in \mathbb{T}^n \times B \times J_0$ . Obviously  $\tilde{h} \in \mathcal{K}_\omega$ . Moreover,  $X_H$  and  $X_{\tilde{h}}$  verify (3.1).

#### 3.3.1 Outline of the Proof of Theorem A

We are looking for a  $C^\sigma$ -asymptotic KAM torus  $\varphi^t$  associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ , where  $H$  is the Hamiltonian in  $(**_A)$ ,  $\tilde{h}$  is the Hamiltonian previously defined and  $\varphi_0$  is the trivial embedding  $\varphi_0 : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B$ ,  $\varphi_0(q) = (q, 0)$ . More specifically, for given  $H$ , we are searching for  $v' \geq 0$  sufficiently large and suitable functions  $u, v : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n$  such that

$$\varphi(q, t) = (q + u(q, t), v(q, t))$$

and in such a way that  $\varphi$ ,  $u$  and  $v$  satisfy the following conditions

$$X_H(\varphi(q, t), t) - \partial_q \varphi(q, t) \omega - \partial_t \varphi(q, t) = 0, \quad (3.7)$$

$$\lim_{t \rightarrow +\infty} |u^t|_{C^\sigma} = 0, \quad \lim_{t \rightarrow +\infty} |v^t|_{C^\sigma} = 0, \quad (3.8)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . The parameter  $v'$  is free and it will be chosen large enough in Lemma 3.5 below.

The proof rests on the implicit function theorem. To this end, we need to introduce a suitable functional  $\mathcal{F}$  given by (3.7). We consider

$$\bar{m}(q, p, t)p = \left( \int_0^1 \partial_p^2 H(q, \tau p, t) d\tau \right) p = \partial_p \left( m(q, p, t) \cdot p^2 \right).$$

This is well defined because

$$\begin{aligned} \partial_p \left( m(q, p, t) \cdot p^2 \right) &= \partial_p \left( \int_0^1 (1 - \tau) \partial_p^2 H(q, \tau p, t) d\tau \cdot p^2 \right) \\ &= \partial_p \left( \int_0^1 (p - \xi) \partial_p^2 H(q, \xi, t) d\xi \right) \\ &= \int_0^1 \partial_p^2 H(q, \xi, t) d\xi = \left( \int_0^1 \partial_p^2 H(q, \tau p, t) d\tau \right) p, \end{aligned}$$

where the second equality of the latter is due to the change of variables  $\xi = \tau p$ . Going back to the definition of the functional  $\mathcal{F}$ , we observe that the Hamiltonian system associated to the Hamiltonian  $H$  is equal to

$$X_H(q, p, t) = \left( \begin{array}{c} \omega + b(q, t) + \bar{m}(q, p, t)p \\ -\partial_q a(q, t) - \partial_q b(q, t)p - \partial_q m(q, p, t)p^2 \end{array} \right),$$

where we recall that  $H$  is the Hamiltonian defined by (\*\*<sub>A</sub>). We introduce

$$\tilde{\varphi}(q, t) = (q + u(q, t), v(q, t), t), \quad \tilde{u}(q, t) = (q + u(q, t), t),$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Composing the Hamiltonian system  $X_H$  with  $\tilde{\varphi}$ , we can write  $X_H \circ \tilde{\varphi}$  in the following form

$$X_H \circ \tilde{\varphi}(q, t) = \left( \begin{array}{c} \omega + b \circ \tilde{u}(q, t) + \bar{m} \circ \tilde{\varphi}(q, t)v(q, t) \\ -\partial_q a \circ \tilde{u}(q, t) - \partial_q b \circ \tilde{u}(q, t)v(q, t) - \partial_q m \circ \tilde{\varphi}(q, t) \cdot v(q, t)^2 \end{array} \right)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$  and moreover,

$$\partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t) = \left( \begin{array}{c} \omega + \partial_q u(q, t) \omega + \partial_t u(q, t) \\ \partial_q v(q, t) \omega + \partial_t v(q, t) \end{array} \right)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . We define

$$\nabla u(q, t) \Omega = \partial_q u(q, t) \omega + \partial_t u(q, t), \quad \nabla v(q, t) \Omega = \partial_q v(q, t) \omega + \partial_t v(q, t)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Then, we can rewrite (3.7) in the following form

$$\left( \begin{array}{c} b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi})v - (\nabla u) \Omega \\ -\partial_q a \circ \tilde{u} - (\partial_q b \circ \tilde{u})v - (\partial_q m \circ \tilde{\varphi}) \cdot v^2 - (\nabla v) \Omega \end{array} \right) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (3.9)$$

This is composed of sums and products of functions defined on  $(q, t) \in \mathbb{T}^n \times J_{v'}$ , we have omitted the arguments  $(q, t)$  in order to achieve a more elegant form. We keep this notation for the rest of this proof. Over suitable Banach spaces, that we will specify later, let  $\mathcal{F}$  be the following functional

$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \Omega, \\ F_2(a, b, m, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q b \circ \tilde{u}) v + (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v) \Omega. \end{aligned}$$

The latter is obtained by (3.9) and we observe that for all  $m$  and  $\bar{m}$ ,

$$\mathcal{F}(0, 0, m, \bar{m}, 0, 0) = 0.$$

We can reformulate our problem in the following form. For fixed  $m$  and  $\bar{m}$  in a suitable Banach space and for  $(a, b)$  sufficiently close to  $(0, 0)$ , we are looking for some functions  $u, v$  in such a way that  $\mathcal{F}(a, b, m, \bar{m}, u, v) = 0$  and the asymptotic conditions (3.8) are satisfied.

Concerning the associated linearized problem, the differential of  $\mathcal{F}$  with respect to the variables  $(u, v)$  calculated on  $(0, 0, m, \bar{m}, 0, 0)$  is equal to

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (\bar{m}_0 \hat{v} - (\nabla \hat{u}) \Omega, (\nabla \hat{v}) \Omega)$$

where, in according to the notation previously introduced, for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$  we let  $\bar{m}_0(q, t) = \bar{m}(q, 0, t)$ .

The proof of this theorem is a straightforward application of the implicit function theorem if we assume the following norm

$$\sup_{t \in J_0} |a^t|_{C^{\sigma+2}} + \sup_{t \in J_0} \frac{|\partial_q a^t|_{C^{\sigma+1}}}{\mathbf{a}(t)}, \quad \sup_{t \in J_0} \frac{|b^t|_{C^{\sigma+2}}}{\mathbf{b}(t)},$$

to be sufficiently small. To avoid this smallness assumption, we study the problem from another point of view. We are looking for a  $C^\sigma$ -asymptotic KAM torus defined for  $t$  sufficiently large in such a way

$$|\partial_q a^t|_{C^{\sigma+1}}, \quad |\partial_q b^t|_{C^{\sigma+1}}$$

are sufficiently small. It suffices for proving the existence of functions  $(u, v)$  satisfying (3.7) and (3.8).

The following four sections are devoted to the proof of Theorem A. In the first, we introduce suitable Banach spaces on which the previous functional is defined. The second is dedicated to solving the homological equation, which is the main tool to prove that  $D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)$  is invertible. In the penultimate section, we verify that  $\mathcal{F}$  is well defined and satisfies the hypotheses of the implicit function theorem. Finally, the last section concludes the proof of this theorem.

### 3.3.2 Preliminary Settings

Given positive real parameters  $\sigma \geq 0$ ,  $v \geq 0$  and a positive integer  $k \geq 0$ , we recall that  $\mathcal{S}_\sigma^v$  and  $\bar{\mathcal{S}}_{\sigma,k}^v$  are respectively the spaces of functions defined by Definition 3.2 and Definition 3.3. We introduce the following norm that we will widely use in the rest of this section. For every  $f \in \mathcal{S}_\sigma^v$  and for a positive real function  $\mathbf{u}(t)$  defined on  $J_v$ , we let

$$|f|_{\sigma,\mathbf{u}}^v = \sup_{t \in J_v} \frac{|f^t|_{C^\sigma}}{\mathbf{u}(t)}. \quad (3.10)$$

Furthermore, we recall that for all positive, decreasing, integrable functions  $\mathbf{u}$  on  $J_v$ , we let  $\bar{\mathbf{u}}$  be

$$\bar{\mathbf{u}}(t) = \int_t^{+\infty} \mathbf{u}(\tau) d\tau$$

for all  $t \in J_v$ .

Now, let  $\sigma \geq 1$ ,  $v \geq 0$  and  $\Upsilon \geq 1$  be the positive parameters introduced in (\*\*<sub>A</sub>) and (#). For  $v' \geq v \geq 0$  that will be chosen later, we consider the following Banach spaces  $(\mathcal{A}, |\cdot|)$ ,  $(\mathcal{B}, |\cdot|)$ ,  $(\mathcal{U}, |\cdot|)$ ,  $(\mathcal{V}, |\cdot|)$ ,  $(\mathcal{Z}, |\cdot|)$  and  $(\mathcal{G}, |\cdot|)$  (see Appendix C)

$$\begin{aligned} \mathcal{A} &= \left\{ a : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R} \mid a \in \bar{\mathcal{S}}_{\sigma,2}^{v'} \text{ and } |a| = |a|_{\sigma+2,1}^{v'} + |\partial_q a|_{\sigma+1,\mathbf{a}}^{v'} < \infty \right\} \\ \mathcal{B} &= \left\{ b : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid b \in \bar{\mathcal{S}}_{\sigma,2}^{v'}, \text{ and } |b| = |b|_{\sigma+2,\mathbf{b}}^{v'} < \infty \right\} \\ \mathcal{U} &= \left\{ u : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid u, (\nabla u) \Omega \in \mathcal{S}_\sigma^{v'} \right. \\ &\quad \left. \text{and } |u| = \max\{|u|_{\sigma,\bar{\mathbf{b}}}^{v'}, |(\nabla u) \Omega|_{\sigma,\mathbf{b}}^{v'}\} < \infty \right\} \\ \mathcal{V} &= \left\{ v : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid v, (\nabla v) \Omega \in \mathcal{S}_\sigma^{v'} \right. \\ &\quad \left. \text{and } |v| = \max\{|v|_{\sigma,\bar{\mathbf{a}}}^{v'}, |(\nabla v) \Omega|_{\sigma,\mathbf{a}}^{v'}\} < \infty \right\} \\ \mathcal{Z} &= \left\{ z : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid z \in \mathcal{S}_\sigma^{v'}, \text{ and } |z| = |z|_{\sigma,\mathbf{b}}^{v'} < \infty \right\} \\ \mathcal{G} &= \left\{ g : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R} \mid g \in \mathcal{S}_\sigma^{v'} \text{ and } |g| = |g|_{\sigma,\mathbf{a}}^{v'} < \infty \right\} \end{aligned}$$

where, in the definition of  $\mathcal{A}$ , the norm  $|a|_{\sigma+2,1}^{v'} = \sup_{t \in J_{v'}} |a^t|_{C^{\sigma+2}}$ . This means that 1 stands for the function identically equal to 1 for all  $t \in J_{v'}$ . Let  $M_n$  be the set of the  $n$ -dimensional matrices. We introduce another Banach space  $(\mathcal{M}, |\cdot|)$  in such a way that

$$\mathcal{M} = \left\{ m : \mathbb{T}^n \times B \times J_{v'} \rightarrow M_n \mid m \in \bar{\mathcal{S}}_{\sigma,2}^{v'} \text{ and } |m| = |m|_{\sigma+2,1}^{v'} \leq \Upsilon \right\}$$

where  $\Upsilon$  is the positive parameter in (\*\*<sub>A</sub>). Now, we have everything we need to define more precisely the functional  $\mathcal{F}$  introduced in the previous section. Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{Z} \times \mathcal{G}$$



$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \Omega, \\ F_2(a, b, m, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q b \circ \tilde{u}) v + (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v) \Omega. \end{aligned}$$

### 3.3.3 Homological Equation

Given  $\sigma \geq 0$ ,  $v \geq 0$  and  $\omega \in \mathbb{R}^n$ , in this section, we solve the following equation for the unknown  $\varkappa : \mathbb{T}^n \times J_v \rightarrow \mathbb{R}$

$$\begin{cases} \omega \cdot \partial_q \varkappa(q, t) + \partial_t \varkappa(q, t) = g(q, t), \\ g \in \mathcal{S}_\sigma^v \quad |g|_{\sigma, \mathbf{g}}^v < \infty \end{cases} \quad (HE_A)$$

where  $\mathbf{g}(t)$  is a positive, decreasing, integrable function on  $J_v$  and  $g : \mathbb{T}^n \times J_v \rightarrow \mathbb{R}$  is given.

**Lemma 3.1 (Homological Equation).** *There exists a unique solution  $\varkappa \in \mathcal{S}_\sigma^v$  of  $(HE_A)$  such that*

$$\lim_{t \rightarrow +\infty} |\varkappa^t|_{C^0} = 0. \quad (3.11)$$

Moreover,

$$|\varkappa|_{\sigma, \bar{\mathbf{g}}}^v \leq |g|_{\sigma, \mathbf{g}}^v.$$

*Proof. Existence:* Let us define the following transformation

$$h : \mathbb{T}^n \times J_v \rightarrow \mathbb{T}^n \times J_v, \quad h(q, t) = (q - \omega t, t),$$

that is the key to solve the homological equation.

We claim that it is enough to prove the first part of this lemma for the much simpler equation

$$\partial_t \kappa = g(q + \omega t, t). \quad (3.12)$$

As a matter of fact, if  $\kappa$  is a solution of the latter satisfying the asymptotic condition (3.11), then  $\varkappa = \kappa \circ h$  is a solution of  $(HE_A)$  satisfying the same asymptotic condition and viceversa. For the sake of clarity, we prove this claim. Let  $\varkappa$  be a solution of  $(HE_A)$  satisfying the asymptotic condition (3.11), then

$$\partial_t(\varkappa \circ h^{-1}) = \partial_q \varkappa \circ h^{-1} \cdot \omega + \partial_t \varkappa \circ h^{-1} = g \circ h^{-1},$$

where the last equality is due to  $(HE_A)$ . This implies that  $\kappa = \varkappa \circ h^{-1}$  is a solution of (3.12) and by

$$|\kappa^t|_{C^0} = |(\varkappa \circ h^{-1})^t|_{C^0} \leq |\varkappa^t|_{C^0}$$

$\kappa = \varkappa \circ h^{-1}$  satisfies the asymptotic condition because  $\varkappa$  does. Viceversa, let  $\kappa$  be a solution of (3.12) satisfying the asymptotic condition (3.11), then

$$\partial_q(\kappa \circ h) \cdot \omega + \partial_t(\kappa \circ h) = \partial_q \kappa \circ h \cdot \omega - \partial_q \kappa \circ h \cdot \omega + \partial_t \kappa \circ h = g.$$

By (3.12), we have the last equality of the latter and hence  $\kappa \circ h$  is a solution of  $(HE_A)$ . Moreover, thanks to

$$|\varkappa^t|_{C^0} = |(\kappa \circ h)^t|_{C^0} \leq |\kappa^t|_{C^0}$$

$\varkappa = \kappa \circ h$  satisfies the asymptotic condition (3.11). This proves the claim.

For all  $q \in \mathbb{T}^n$  a solution of (3.12) exists and

$$\kappa(q, t) = e(q) + \int_v^t g(q + \omega\tau, \tau) d\tau$$

with a function  $e$  defined on the torus. We have to choose  $e$  in such a way that  $\kappa$  satisfies the following asymptotic condition for all fixed  $q \in \mathbb{T}^n$

$$0 = \lim_{t \rightarrow +\infty} \kappa(q, t) = e(q) + \int_v^{+\infty} g(q + \omega\tau, \tau) d\tau.$$

There is only one possible choice for  $e$ , that is

$$e(q) = - \int_v^{+\infty} g(q + \omega\tau, \tau) d\tau.$$

This implies that

$$\kappa(q, t) = - \int_t^{+\infty} g(q + \omega\tau, \tau) d\tau$$

is the solution of (3.12) we are looking for. Therefore,  $e$  is well defined, indeed

$$\left| \int_v^{+\infty} g(q + \omega\tau, \tau) d\tau \right| \leq |g|_{\sigma, \mathbf{g}}^v \int_v^{+\infty} \mathbf{g}(\tau) d\tau = |g|_{\sigma, \mathbf{g}}^v \bar{\mathbf{g}}(v) < \infty.$$

Moreover,

$$|\kappa^t|_{C^0} \leq \int_t^{+\infty} |g^\tau|_{C^0} d\tau \leq |g|_{\sigma, \mathbf{g}}^v \int_t^{+\infty} \mathbf{g}(\tau) d\tau = |g|_{\sigma, \mathbf{g}}^v \bar{\mathbf{g}}(t),$$

since  $\bar{\mathbf{g}}(t)$  converges to 0 when  $t \rightarrow +\infty$ , taking the limit for  $t \rightarrow +\infty$  on both sides of the latter, we have that  $|\kappa^t|_{C^0} \rightarrow 0$  when  $t \rightarrow +\infty$ . This concludes the first part of the proof because

$$\varkappa(q, t) = \kappa \circ h(q, t) = - \int_t^{+\infty} g(q + \omega(\tau - t), \tau) d\tau$$

is the unique solution of  $(HE_A)$  satisfying (3.11) that we are looking for.

*Regularity and Estimates:* We observe that  $g \in \mathcal{S}_\sigma^v$  implies  $\kappa \in \mathcal{S}_\sigma^v$  and hence  $\varkappa = \kappa \circ h \in \mathcal{S}_\sigma^v$ . Moreover, for all fixed  $t \in J_v$

$$|\varkappa^t|_{C^\sigma} \leq |g|_{\sigma, \mathbf{g}}^v \bar{\mathbf{g}}(t).$$

Multiplying both sides of the latter by  $\frac{1}{\bar{\mathbf{g}}(t)}$  and taking the sup for all  $t \in J_v$ , we prove the second part of this lemma.  $\square$

### 3.3.4 Regularity of $\mathcal{F}$

We recall the definition of the functional  $\mathcal{F}$ ,

$$\mathcal{F} : \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{Z} \times \mathcal{G}$$

$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi})v - (\nabla u) \Omega, \\ F_2(a, b, m, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q b \circ \tilde{u})v + (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v) \Omega. \end{aligned}$$

We verify that the functional  $\mathcal{F}$  satisfies the hypothesis of the implicit function theorem. The proof consists of three lemmas. We prove that  $\mathcal{F}$  is well defined,  $\mathcal{F}$  is differentiable with respect to the components  $(u, v)$  and this differential calculated on  $(0, 0, m, \bar{m}, 0, 0)$  is invertible. In what follows, we will widely use the properties contained in Proposition A.2 (see Appendix A). For this reason, we recall it. Let  $D$  be equal to  $\mathbb{T}^n$  or  $\mathbb{T}^n \times B$ . To avoid a flow of constants, let  $C(\cdot)$  be constants depending on  $n$  and the other parameters in brackets. On the other hand,  $C$  stands for constants depending only on  $n$ .

**Proposition** (Proposition A.2). *We consider  $f, g \in C^\sigma(D)$  and  $\sigma \geq 0$ .*

1. *For all  $\beta \in \mathbb{N}^n$ , if  $|\beta| + s = \sigma$  then  $\left| \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} f \right|_{C^s} \leq |f|_{C^\sigma}$ .*
2.  $|fg|_{C^\sigma} \leq C(\sigma) (|f|_{C^0}|g|_{C^\sigma} + |f|_{C^\sigma}|g|_{C^0})$ .

*Now we consider composite functions. Let  $z$  be defined on  $D_1 \subset \mathbb{R}^n$  and takes its values on  $D_2 \subset \mathbb{R}^n$  where  $f$  is defined.*

*If  $\sigma < 1$ ,  $f \in C^1(D_2)$ ,  $z \in C^\sigma(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

3.  $|f \circ z|_{C^\sigma} \leq C(|f|_{C^1}|z|_{C^\sigma} + |f|_{C^0})$ .

*If  $\sigma < 1$ ,  $f \in C^\sigma(D_2)$ ,  $z \in C^1(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

4.  $|f \circ z|_{C^\sigma} \leq C(|f|_{C^\sigma}|\nabla z|_{C^0}^\sigma + |f|_{C^0})$ .

*If  $\sigma \geq 1$  and  $f \in C^\sigma(D_2)$ ,  $z \in C^\sigma(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

5.  $|f \circ z|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma}|\nabla z|_{C^0}^\sigma + |f|_{C^1}|\nabla z|_{C^{\sigma-1}} + |f|_{C^0})$ .

Upon choosing  $v'$  large enough, we can assume  $\mathbf{b}(v') \leq 1$  and  $\bar{\mathbf{b}}(v') \leq 1$ . We will take a stronger restriction after. Therefore, for all  $(u, v) \in \mathcal{U} \times \mathcal{V}$ , thanks to (#), we have the following estimates

$$|u^t|_{C^\sigma} \leq |u|_{\sigma, \bar{\mathbf{b}}}, \quad |v^t|_{C^\sigma} \leq |v|_{\sigma, \bar{\mathbf{a}}} \tag{3.13}$$

for all  $t \in J_{v'}$ .

**Lemma 3.2.**  *$\mathcal{F}$  is well defined.*

*Proof.* We begin by proving that for all  $(b, \bar{m}, u, v) \in \mathcal{B} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V}$ ,  $F_1(b, \bar{m}, u, v) \in \mathcal{Z}$ . Concerning the regularity  $F_1(b, \bar{m}, u, v)$  is the sum of functions in  $\mathcal{S}_\sigma^{v'}$  then  $F_1(b, \bar{m}, u, v) \in \mathcal{S}_\sigma^{v'}$ . We have to find an upper bound for  $|F_1(b, \bar{m}, u, v)|_{\sigma, \mathbf{b}}^{v'}$ , then for all  $t \in J_{v'}$

$$\frac{|F_1(b, \bar{m}, u, v)^t|_{C^\sigma}}{\mathbf{b}(t)} \leq \frac{|(b \circ \tilde{u})^t|_{C^\sigma}}{\mathbf{b}(t)} + \frac{|(\bar{m} \circ \tilde{\varphi})^t v^t|_{C^\sigma}}{\mathbf{b}(t)} + \frac{|(\nabla u^t) \Omega|_{C^\sigma}}{\mathbf{b}(t)}. \quad (3.14)$$

Now, we have to estimate each member on the right-hand side of the latter. The last one is obviously bounded, we have to estimate the others. Concerning the member in the middle of the previous inequality,

$$\begin{aligned} \frac{|(\bar{m} \circ \tilde{\varphi})^t v^t|_{C^\sigma}}{\mathbf{b}(t)} &\leq C(\sigma) |(\bar{m} \circ \tilde{\varphi})^t|_{C^\sigma} \frac{|v^t|_{C^\sigma}}{\mathbf{b}(t)} \\ &\leq C(\sigma) \Upsilon \frac{|v^t|_{C^\sigma}}{\mathbf{b}(t)} (1 + |\partial_q \varphi^t|_{C^0} + |\partial_q \varphi^t|_{C^{\sigma-1}}) \\ &\leq C(\sigma) \Upsilon \frac{|v^t|_{C^\sigma}}{\mathbf{a}(t)} \left(1 + \left(1 + |u|_{\sigma, \bar{\mathbf{b}}}^{v'}\right)^\sigma + |u|_{\sigma, \bar{\mathbf{b}}}^{v'} + |v|_{\sigma, \bar{\mathbf{a}}}^{v'} + \left(|v|_{\sigma, \bar{\mathbf{a}}}^{v'}\right)^\sigma\right) \\ &\leq C(\sigma) \Upsilon |v|_{\sigma, \bar{\mathbf{a}}}^{v'} \left(1 + \left(1 + |u|_{\sigma, \bar{\mathbf{b}}}^{v'}\right)^\sigma + |u|_{\sigma, \bar{\mathbf{b}}}^{v'} + |v|_{\sigma, \bar{\mathbf{a}}}^{v'} + \left(|v|_{\sigma, \bar{\mathbf{a}}}^{v'}\right)^\sigma\right) \end{aligned}$$

for all  $t \in J_{v'}$ . We observe that the first and the second line of the latter are due, respectively, to properties 2. and 5. of Proposition A.2. The third line is a consequence of the first condition in (#), the form of  $\varphi^t$  and (3.13). The last inequality follows by the definition of the norm  $|\cdot|_{\sigma, \bar{\mathbf{a}}}^{v'}$  (see (3.10)). Similarly to the previous case, by Proposition A.2 and (3.13)

$$\begin{aligned} \frac{|(b \circ \tilde{u})^t|_{C^\sigma}}{\mathbf{b}(t)} &\leq C(\sigma) \frac{|b^t|_{C^\sigma}}{\mathbf{b}(t)} (1 + |\partial_q u^t|_{C^{\sigma-1}} + (1 + |\partial_q u^t|_{C^0})^\sigma) \\ &\leq C(\sigma) |b|_{\sigma+2, \mathbf{b}}^{v'} \left(1 + |u|_{\sigma, \bar{\mathbf{b}}}^{v'} + \left(1 + |u|_{\sigma, \bar{\mathbf{b}}}^{v'}\right)^\sigma\right) \end{aligned}$$

for all  $t \in J_{v'}$ . Taking the sup for all  $t \in J_{v'}$  on the left-hand side of the above estimates, we prove the existence of an upper bound for the first two terms on the right-hand side of (3.14). This implies that  $|F_1(b, \bar{m}, u, v)|_{\sigma, \mathbf{b}}^{v'} < \infty$  and hence, for all  $(b, \bar{m}, u, v) \in \mathcal{B} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V}$ ,  $F_1(b, \bar{m}, u, v) \in \mathcal{Z}$ . Similarly, for all  $(a, b, m, u, v) \in \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V}$ ,  $F_2(a, b, m, u, v) \in \mathcal{G}$ . We point out that in this case we use both conditions in (#).  $\square$

This proves that  $\mathcal{F}$  is well defined, moreover, one can prove that it is continuous. As we mentioned before, in the following lemma, we show that  $\mathcal{F}$  is differentiable with respect to the variables  $(u, v)$ . Let  $D_{(u,v)} \mathcal{F}$  be the differential with respect to  $(u, v)$

**Lemma 3.3.**  $\mathcal{F}$  is differentiable with respect to  $(u, v)$  with

$$\begin{aligned}
 D_{(u,v)}F_1(b, \bar{m}, u, v)(\hat{u}, \hat{v}) &= D_u F_1(b, \bar{m}, u, v)\hat{u} + D_v F_1(b, \bar{m}, u, v)\hat{v} \\
 &= (\partial_q b \circ \tilde{u})\hat{u} + v^T (\partial_q \bar{m} \circ \tilde{\varphi})\hat{u} + v^T (\partial_p \bar{m} \circ \tilde{\varphi})\hat{v} \\
 &\quad + (\bar{m} \circ \tilde{\varphi})\hat{v} - (\nabla \hat{u})\Omega \\
 D_{(u,v)}F_2(a, b, m, u, v)(\hat{u}, \hat{v}) &= D_u F_2(a, b, m, u, v)\hat{u} + D_v F_2(a, b, m, u, v)\hat{v} \\
 &= (\partial_q^2 a \circ \tilde{u})\hat{u} + v^T (\partial_q^2 b \circ \tilde{u})\hat{u} + (v^T)^2 (\partial_q^2 m \circ \tilde{\varphi})\hat{u} \\
 &\quad + (\partial_q b \circ \tilde{u})\hat{v} + (v^T)^2 (\partial_{pq}^2 m \circ \tilde{\varphi})\hat{v} + 2v^T (\partial_q m \circ \tilde{\varphi})\hat{v} \\
 &\quad + (\nabla \hat{v})\Omega,
 \end{aligned}$$

where  $T$  stands for the transpose of a vector and  $D_u, D_v$  are respectively the differentials with respect to  $u$  and  $v$ .

*Proof.* We begin by proving that  $F_1$  is differentiable with respect to  $u$  with

$$D_u F_1(b, \bar{m}, u, v)\hat{u} = (\partial_q b \circ \tilde{u})\hat{u} + v^T (\partial_q \bar{m} \circ \tilde{\varphi})\hat{u} - (\nabla \hat{u})\Omega.$$

First, let us introduce the following notation

$$\begin{aligned}
 \tilde{u}(q, t) + \tau \hat{u}(q, t) &= (q + u(q, t) + \tau \hat{u}(q, t), t) \\
 \tilde{\varphi}_{+\tau \hat{u}}(q, t) &= (q + u(q, t) + \tau \hat{u}(q, t), v(q, t), t),
 \end{aligned}$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$  and  $\tau \in (0, 1)$ . Now, thanks to Taylor's formula

$$\begin{aligned}
 &F_1(b, \bar{m}, u + \hat{u}, v) - F_1(b, \bar{m}, u, v) - (\partial_q b \circ \tilde{u})\hat{u} - v^T (\partial_q \bar{m} \circ \tilde{\varphi})\hat{u} + (\nabla \hat{u})\Omega \\
 &= \left( \int_0^1 \partial_q b \circ (\tilde{u} + \tau \hat{u}) - \partial_q b \circ \tilde{u} d\tau \right) \hat{u} + v^T \left( \int_0^1 \partial_q \bar{m} \circ \tilde{\varphi}_{+\tau \hat{u}} - \partial_q \bar{m} \circ \tilde{\varphi} d\tau \right) \hat{u}.
 \end{aligned}$$

This implies the claim because, for all fixed  $t \in J_{v'}$ , by the property 2. of Proposition A.2, the first condition in (#) and (3.13)

$$\begin{aligned}
 &\left| \frac{\left( F_1(b, \bar{m}, u + \hat{u}, v) - F_1(b, \bar{m}, u, v) - (\partial_q b \circ \tilde{u})\hat{u} - v^T (\partial_q \bar{m} \circ \tilde{\varphi})\hat{u} + (\nabla_{\theta t} \hat{u})\Omega \right)^t}{\mathbf{b}(t)} \right|_{C^\sigma} \\
 &\leq C(\sigma) \left( \int_0^1 \frac{|\partial_q b \circ (\tilde{u} + \tau \hat{u}) - \partial_q b \circ \tilde{u}|_{C^\sigma}^t}{\mathbf{b}(t)} d\tau \right) |\hat{u}^t|_{C^\sigma} \\
 &\quad + C(\sigma) \frac{|v^t|_{C^\sigma}}{\mathbf{b}(t)} \left( \int_0^1 |\partial_q \bar{m} \circ \tilde{\varphi}_{+\tau \hat{u}} - \partial_q \bar{m} \circ \tilde{\varphi}|_{C^\sigma}^t d\tau \right) |\hat{u}^t|_{C^\sigma} \\
 &\leq C(\sigma) \left( \int_0^1 |\partial_q b \circ (\tilde{u} + \tau \hat{u}) - \partial_q b \circ \tilde{u}|_{\sigma, \mathbf{b}}^{v'} d\tau \right) |\hat{u}|_{\sigma, \mathbf{b}}^{v'} \\
 &\quad + C(\sigma) |v|_{\sigma, \bar{\mathbf{a}}}^{v'} \left( \int_0^1 |\partial_q \bar{m} \circ \tilde{\varphi}_{+\tau \hat{u}} - \partial_q \bar{m} \circ \tilde{\varphi}|_{\sigma, 1}^{v'} d\tau \right) |\hat{u}|_{\sigma, \bar{\mathbf{b}}}^{v'}
 \end{aligned}$$

and, by the regularity of  $b$  and  $\bar{m}$ , we have the claim. Similarly  $F_1$  is differentiable with respect to  $v$  and  $F_2$  is differentiable with respect  $(u, v)$ .  $\square$

This shows that  $\mathcal{F}$  is differentiable with respect to the variables  $(u, v)$ . Furthermore, one can show that  $D_{(u,v)}\mathcal{F}$  is continuous. This differential calculated on  $(0, 0, m, \bar{m}, 0, 0)$  is equal to

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (\bar{m}_0\hat{v} - (\nabla\hat{u})\Omega, (\nabla\hat{v})\Omega). \quad (3.15)$$

In the following lemma, we verify that for all fixed  $m, \bar{m} \in \mathcal{M}$ , the latter is invertible.

**Lemma 3.4.** *For all  $(z, g) \in \mathcal{Z} \times \mathcal{G}$  there exists a unique  $(\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V}$  such that*

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (z, g).$$

Moreover, there exists a suitable constant  $\bar{C}$  such that

$$|\hat{u}| \leq \bar{C}\Upsilon|g|_{\sigma,\mathbf{a}}^{v'} + |z|_{\sigma,\mathbf{b}}^{v'}, \quad |\hat{v}| \leq |g|_{\sigma,\mathbf{a}}^{v'}, \quad (3.16)$$

where we recall that  $|\hat{u}| = \max\{|\hat{u}|_{\sigma,\bar{\mathbf{b}}}^{v'}, |(\nabla\hat{u})\Omega|_{\sigma,\mathbf{b}}^{v'}\}$  and  $|\hat{v}| = \max\{|\hat{v}|_{\sigma,\bar{\mathbf{a}}}^{v'}, |(\nabla\hat{v})\Omega|_{\sigma,\mathbf{a}}^{v'}\}$ .

*Proof.* The proof of this lemma rests on Lemma 3.1. By (3.15), we can reformulate the problem in the following form. Given  $(z, g) \in \mathcal{Z} \times \mathcal{G}$ , we are looking for the unique solution  $(\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V}$  of the following system

$$\begin{cases} (\nabla\hat{u})\Omega = \bar{m}_0\hat{v} - z \\ (\nabla\hat{v})\Omega = g. \end{cases} \quad (3.17)$$

By Lemma 3.1, the unique solution  $\hat{v}$  for the last equation of the latter system exists and satisfies

$$|\hat{v}|_{\sigma,\bar{\mathbf{a}}}^{v'} \leq |g|_{\sigma,\mathbf{a}}^{v'}.$$

Moreover, by  $|(\nabla\hat{v})\Omega|_{\sigma,\mathbf{a}}^{v'} = |g|_{\sigma,\mathbf{a}}^{v'}$ , we have the second estimate in (3.16)

$$|\hat{v}| = \max\{|\hat{v}|_{\sigma,\bar{\mathbf{a}}}^{v'}, |(\nabla\hat{v})\Omega|_{\sigma,\mathbf{a}}^{v'}\} \leq |g|_{\sigma,\mathbf{a}}^{v'}. \quad (3.18)$$

Now, it remains to solve the first equation of (3.17) where  $\hat{v}$  is known. For all fixed  $t \in J_{v'}$  and thanks to property 2. of Proposition A.2, the first condition of (#) and (3.18)

$$\begin{aligned} \frac{|(\bar{m}_0\hat{v} - z)^t|_{C^\sigma}}{\mathbf{b}(t)} &\leq C\Upsilon \frac{|\hat{v}^t|_{C^\sigma}}{\mathbf{b}(t)} + \frac{|z^t|_{C^\sigma}}{\mathbf{b}(t)} \leq C\Upsilon \frac{|\hat{v}^t|_{C^\sigma}}{\bar{\mathbf{a}}(t)} + \frac{|z^t|_{C^\sigma}}{\mathbf{b}(t)} \\ &\leq C\Upsilon|\hat{v}|_{\sigma,\bar{\mathbf{a}}}^{v'} + |z|_{\sigma,\mathbf{b}}^{v'} \leq C\Upsilon|g|_{\sigma,\mathbf{a}}^{v'} + |z|_{\sigma,\mathbf{b}}^{v'}, \end{aligned}$$

for a suitable constant  $C$ . Taking the sup for all  $t \in J_{v'}$  on the left-hand side of the latter, we obtain

$$|\bar{m}_0\hat{v} - z|_{\sigma,\mathbf{b}}^{v'} \leq C\Upsilon|g|_{\sigma,\mathbf{a}}^{v'} + |z|_{\sigma,\mathbf{b}}^{v'}$$

and hence

$$|(\nabla\hat{u})\Omega|_{\sigma,\mathbf{b}}^{v'} = |\bar{m}_0\hat{v} - z|_{\sigma,\mathbf{b}}^{v'} \leq C\Upsilon|g|_{\sigma,\mathbf{a}}^{v'} + |z|_{\sigma,\mathbf{b}}^{v'}.$$

Thanks to Lemma 3.1 the unique solution  $\hat{u}$  of the first equation of (3.17) exists verifying

$$|\hat{u}|_{\sigma,\bar{\mathbf{b}}}^{v'} \leq |\bar{m}_0\hat{v} - z|_{\sigma,\mathbf{b}(t)}^{v'} \leq C\Upsilon|g|_{\sigma,\mathbf{a}}^{v'} + |z|_{\sigma,\mathbf{b}}^{v'}.$$

This concludes the proof of this lemma with  $\bar{C} = C$  because

$$|\hat{u}| = \max\{|\hat{u}|_{\sigma,\bar{\mathbf{b}}}^{v'}, |(\nabla\hat{u})\Omega|_{\sigma,\mathbf{b}}^{v'}\} \leq C\Upsilon|g|_{\sigma,\mathbf{a}}^{v'} + |z|_{\sigma,\mathbf{b}}^{v'}.$$

□

### 3.3.5 $C^\sigma$ -asymptotic KAM torus

In the previous section, we proved that the functional  $\mathcal{F}$  satisfies the hypotheses of the implicit function theorem. Here, we prove the existence of a  $C^\sigma$ -asymptotic KAM torus associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$  and we conclude the proof of Theorem A.

Let  $x = (a, b)$ , where  $a$  and  $b$  are those defined by (\*\* $A$ ). Obviously  $(a, b) \in \mathcal{A} \times \mathcal{B}$  and

$$|\partial_q a|_{\sigma+1, \mathbf{a}}^{v'} \leq 1, \quad |b|_{\sigma+2, \mathbf{b}}^{v'} \leq 1. \quad (3.19)$$

We introduce the Banach space  $(\mathcal{Y}, |\cdot|)$  where  $\mathcal{Y} = \mathcal{U} \times \mathcal{V}$  and for all  $y = (u, v) \in \mathcal{Y}$ ,  $|y| = \max\{|u|, |v|\}$ . Let  $m, \bar{m} \in \mathcal{M}$  be as in (\*\* $A$ ) and we consider

$$\mathcal{F}(x, m, \bar{m}, y) = D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)y + \mathcal{R}(x, m, \bar{m}, y).$$

The aim is to find  $y \in \mathcal{Y}$  in such a way that

$$\mathcal{F}(x, m, \bar{m}, y) = 0,$$

where we recall that we have fixed  $x, m$  and  $\bar{m}$ . This is equivalent to find  $y \in \mathcal{Y}$  such that

$$\begin{aligned} y &= -D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1}\mathcal{R}(x, m, \bar{m}, y) \\ &= y - D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1}\mathcal{F}(x, m, \bar{m}, y). \end{aligned}$$

This is well defined because we have already proved that  $D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)$  is invertible (see Lemma 3.4). To this end, we introduce the following functional

$$\mathcal{L}(x, m, \bar{m}, \cdot) : \mathcal{Y} \longrightarrow \mathcal{Y}$$

in such a way that

$$\mathcal{L}(x, m, \bar{m}, y) = y - D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1}\mathcal{F}(x, m, \bar{m}, y). \quad (\mathcal{L})$$

This is well defined and, by the regularity of  $\mathcal{F}$ , we deduce that  $\mathcal{L}$  is continuous, differentiable with respect to  $y = (u, v)$  with differential  $D_y\mathcal{L}$  continuous. The proof is reduced to find a fixed point of the latter. For this purpose, we introduce the following lemma.

**Lemma 3.5.** *There exists  $v'$  large enough with respect to  $n, \sigma, \Upsilon$  and  $\mathbf{b}$ , such that, for all  $y_*, y \in \mathcal{Y}$  with  $|y_*| \leq 1$ ,*

$$|D_y\mathcal{L}(x, m, \bar{m}, y_*)y| \leq \frac{1}{2}|y|. \quad (3.20)$$

*Proof.* The proof relies on Lemma 3.4. By  $(\mathcal{L})$ , for all  $y_*, y \in \mathcal{Y}$

$$\begin{aligned} &D_y\mathcal{L}(x, m, \bar{m}, y_*)y \\ &= D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1} \left( D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0) - D_{(u,v)}\mathcal{F}(x, m, \bar{m}, y_*) \right) y. \end{aligned}$$

We can reformulate this problem in terms of estimating the unique solution  $\hat{y} = (\hat{u}, \hat{v}) \in \mathcal{Y}$  of the following system

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)\hat{y} = \left( D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0) - D_{(u,v)}\mathcal{F}(x, m, \bar{m}, y_*) \right) y. \quad (3.21)$$

Now, it suffices to estimate the right-hand side of the latter and apply Lemma 3.4. First, let us introduce the following notation. We observe that  $y_* = (u_*, v_*) \in \mathcal{Y}$  and, for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ , we let

$$\tilde{u}_*(q, t) = (q + u_*(q, t), t), \quad \tilde{\varphi}_*(q, t) = (q + u_*(q, t), v_*(q, t), t).$$

By Lemma 3.3, the right-hand side of (3.21) is equal to

$$\left( D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0) - D_{(u,v)}\mathcal{F}(x, m, \bar{m}, y_*) \right) y = \begin{pmatrix} \bar{m}_0 v - (\nabla u) \Omega - D_{(u,v)}F_1(b, \bar{m}, y_*) y \\ (\nabla v) \Omega - D_{(u,v)}F_2(x, m, y_*) y \end{pmatrix}$$

where

$$\begin{aligned} \bar{m}_0 v - (\nabla u) \Omega - D_{(u,v)}F_1(b, \bar{m}, y_*) y &= (\bar{m}_0 - \bar{m} \circ \tilde{\varphi}_*) v - (\partial_q b \circ \tilde{u}_*) u \\ &\quad - v_*^T (\partial_q \bar{m} \circ \tilde{\varphi}_*) u - v_*^T (\partial_p \bar{m} \circ \tilde{\varphi}_*) v \\ (\nabla v) \Omega - D_{(u,v)}F_2(x, m, y_*) y &= -(\partial_q^2 a \circ \tilde{u}_*) u - v_*^T (\partial_q^2 b \circ \tilde{u}_*) u \\ &\quad - (v_*^T)^2 (\partial_q^2 m \circ \tilde{\varphi}_*) u - (\partial_q b \circ \tilde{u}_*) v \\ &\quad - (v_*^T)^2 (\partial_{pq}^2 m \circ \tilde{\varphi}_*) v - 2v_*^T (\partial_q m \circ \tilde{\varphi}_*) v. \end{aligned}$$

Thanks to property 2. of Proposition A.2, we can estimate the first member on the left-hand side of the latter as follows

$$\begin{aligned} \left| (\bar{m}_0 v - (\nabla u) \Omega - D_{(u,v)}F_1(b, \bar{m}, y_*) y)^t \right|_{C^\sigma} &\leq C(\sigma) \left( \left| (\bar{m}_0^t - \bar{m} \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} \right. \\ &\quad + \left| (\partial_q b \circ \tilde{u}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} \\ &\quad + |v_*^t|_{C^\sigma} \left| (\partial_q \bar{m} \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} \\ &\quad \left. + |v_*^t|_{C^\sigma} \left| (\partial_p \bar{m} \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} \right) \end{aligned}$$

for all  $t \in J_{v'}$ . We point out that  $|y_*| = \max\{|u_*|, |v_*|\} \leq 1$  and we find an upper bound for each member on the right-hand side of the previous inequality. For all  $t \in J_{v'}$

$$\begin{aligned} \left| (\bar{m}_0^t - \bar{m} \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} &\leq C(\sigma) \left( \left| \partial_q \bar{m}^t(\text{id} + \tau u_*, \tau v_*) u_*^t \right|_{C^\sigma} \right. \\ &\quad \left. + \left| \partial_p \bar{m}^t(\text{id} + \tau u_*, \tau v_*) v_*^t \right|_{C^\sigma} \right) |v^t|_{C^\sigma} \\ &\leq C(\sigma) \Upsilon (1 + \bar{\mathbf{b}}(v') + \bar{\mathbf{a}}(v')) |u_*^t|_{C^\sigma} |v^t|_{C^\sigma} \\ &\quad + C(\sigma) \Upsilon (1 + \bar{\mathbf{b}}(v') + \bar{\mathbf{a}}(v')) |v_*^t|_{C^\sigma} |v^t|_{C^\sigma} \\ &\leq C(\sigma) \Upsilon (|u_*| \bar{\mathbf{b}}(t) + |v_*| \bar{\mathbf{a}}(t)) |v| \bar{\mathbf{a}}(t) \\ &\leq C(\sigma) \Upsilon \bar{\mathbf{b}}(v') |y| \mathbf{b}(t) + C(\sigma) \Upsilon \mathbf{b}(v') |y| \mathbf{b}(t) \end{aligned}$$

The first line of the latter is a consequence of the mean value theorem for a suitable  $\tau \in [0, 1]$ . Concerning the second inequality, it is due to properties 2. and 5. of Proposition A.2. Moreover, we use also that, thanks to (#) and for  $v'$  large enough, we may assume  $\bar{\mathbf{b}}(v') \leq 1$  and  $\bar{\mathbf{a}}(v') \leq \mathbf{b}(v') \leq 1$ . In the penultimate line on the right-hand side of the previous inequalities, we apply the following estimate  $1 + \bar{\mathbf{b}}(v') + \bar{\mathbf{a}}(v') \leq 3$  for all  $t \in J_{v'}$ . In the last line, we use the first condition in (#).



Similarly to the previous case, thanks to property 5. of Proposition A.2, the first condition in (#), (3.19) and  $\bar{\mathbf{b}}(v') \leq 1$ ,  $\bar{\mathbf{a}}(v') \leq 1$ , we obtain

$$\begin{aligned}
 |(\partial_q b \circ \tilde{u}_*)^t|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |b|_{\sigma+2, \mathbf{b}}^{v'} \mathbf{b}(t) (1 + \bar{\mathbf{b}}(v')) |u| \bar{\mathbf{b}}(t) \\
 &\leq C(\sigma) \bar{\mathbf{b}}(v') |y| \mathbf{b}(t) \\
 |v_*^t|_{C^\sigma} |(\partial_q \bar{m} \circ \tilde{\varphi}_*)^t|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |v_*| \bar{\mathbf{a}}(t) \Upsilon (1 + \bar{\mathbf{b}}(v') + \bar{\mathbf{a}}(v')) |u| \bar{\mathbf{b}}(t) \\
 &\leq C(\sigma) \Upsilon \bar{\mathbf{b}}(v') |y| \mathbf{b}(t) \\
 |v_*^t|_{C^\sigma} |(\partial_p \bar{m} \circ \tilde{\varphi}_*)^t|_{C^\sigma} |v^t|_{C^\sigma} &\leq C(\sigma) |v_*| \bar{\mathbf{a}}(t) \Upsilon (1 + \bar{\mathbf{b}}(v') + \bar{\mathbf{a}}(v')) |v| \bar{\mathbf{a}}(t) \\
 &\leq C(\sigma) \Upsilon \mathbf{b}(v') |y| \mathbf{b}(t),
 \end{aligned}$$

for all  $t \in J_{v'}$ . Now, for  $v'$  large enough, the previous estimates imply

$$\left| (\bar{m}_0 v - (\nabla u) \Omega - D_{(u,v)} F_1(b, \bar{m}, y_*) y)^t \right|_{C^\sigma} \leq \frac{1}{4} |y| \mathbf{b}(t)$$

for all  $t \in J_{v'}$ . Multiplying both sides of the latter by  $\frac{1}{\mathbf{b}(t)}$  and taking the sup for all  $t \in J_{v'}$ , we obtain

$$|\bar{m}_0 v - (\nabla u) \Omega - D_{(u,v)} F_1(b, \bar{m}, y_*) y|_{\sigma, \mathbf{b}}^{v'} \leq \frac{1}{4} |y|. \quad (3.22)$$

Similarly to the previous case,

$$\begin{aligned}
 \left| ((\nabla v) \Omega - D_{(u,v)} F_2(x, m, y_*) y)^t \right|_{C^\sigma} &\leq C(\sigma) \left( \left| (\partial_q^2 a \circ \tilde{u}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} \right. \\
 &\quad + |v_*^t|_{C^\sigma} \left| (\partial_q^2 b \circ \tilde{u}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} \\
 &\quad + |v_*^t|_{C^\sigma}^2 \left| (\partial_q^2 m \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} \\
 &\quad + \left| (\partial_q b \circ \tilde{u}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} \\
 &\quad + |v_*^t|_{C^\sigma}^2 \left| (\partial_{pq}^2 m \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} \\
 &\quad \left. + |v_*^t|_{C^\sigma} \left| (\partial_q m \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} \right),
 \end{aligned}$$

for all  $t \in J_{v'}$ . Therefore, we have to estimate each member on the right-hand side of the latter. We begin with the element in the second line. For all  $t \in J_{v'}$

$$\begin{aligned}
 |v_*^t|_{C^\sigma} \left| (\partial_q^2 b \circ \tilde{u}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |v_*^t|_{C^\sigma} |b|_{\sigma+2, \mathbf{b}}^{v'} \mathbf{b}(t) (1 + \bar{\mathbf{b}}(v')) |u^t|_{C^\sigma} \\
 &\leq C(\sigma) \bar{\mathbf{a}}(t) |b|_{\sigma+2, \mathbf{b}}^{v'} \mathbf{b}(t) |u| \bar{\mathbf{b}}(t) \\
 &\leq C(\sigma) \bar{\mathbf{b}}(v')^2 |y| \mathbf{a}(t).
 \end{aligned}$$

The first line of the above estimate is due to property 5. of Proposition A.2 and  $\bar{\mathbf{b}}(v') \leq 1$ . In the second line we use  $|v_*^t|_{C^\sigma} \leq \bar{\mathbf{a}}(t)$  and  $|u^t|_{C^\sigma} \leq |u| \bar{\mathbf{b}}(t)$  for all  $t \in J_{v'}$ . The last inequality is a consequence of the second condition of (#).

Thanks to property 5. of Proposition A.2, (#), (3.19) and  $\bar{\mathbf{b}}(v') \leq 1$ ,  $\bar{\mathbf{a}}(v') \leq 1$ ,

in the same way we have

$$\begin{aligned}
\left| (\partial_q^2 a \circ \tilde{u}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |\partial_q a|_{\sigma+1, \mathbf{a}}^{v'} \mathbf{a}(t) |u| \bar{\mathbf{b}}(t) \\
&\leq C(\sigma) \bar{\mathbf{b}}(v') |y| \mathbf{a}(t), \\
|v_*^t|_{C^\sigma}^2 \left| (\partial_q^2 m \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |v_*|_{\sigma}^2 \bar{\mathbf{a}}(t)^2 \Upsilon |u| \bar{\mathbf{b}}(t) \\
&\leq C(\sigma) \Upsilon \bar{\mathbf{a}}(t) \mathbf{b}(t) \bar{\mathbf{b}}(t) |u| \\
&\leq C(\sigma) \Upsilon \bar{\mathbf{b}}(v')^2 |y| \mathbf{a}(t) \\
\left| (\partial_q b \circ \tilde{u}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} &\leq C(\sigma) |b|_{\sigma+2, \mathbf{b}}^{v'} \mathbf{b}(t) |v| \bar{\mathbf{a}}(t) \\
&\leq C(\sigma) \bar{\mathbf{b}}(v') |y| \mathbf{a}(t) \\
|v_*^t|_{C^\sigma}^2 \left| (\partial_{pq}^2 m \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} &\leq C(\sigma) |v_*|_{\sigma}^2 \bar{\mathbf{a}}(t)^2 \Upsilon |v| \bar{\mathbf{a}}(t) \\
&\leq C(\sigma) \Upsilon \mathbf{b}(t)^2 |v| \bar{\mathbf{a}}(t) \\
&\leq C(\sigma) \Upsilon \mathbf{b}(v') \bar{\mathbf{b}}(v') |y| \mathbf{a}(t) \\
|v_*^t|_{C^\sigma} \left| (\partial_q m \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} &\leq C(\sigma) |v_*|_{\sigma} \bar{\mathbf{a}}(t) \Upsilon |v| \bar{\mathbf{a}}(t) \\
&\leq C(\sigma) \Upsilon \mathbf{b}(t) |v| \bar{\mathbf{a}}(t) \\
&\leq C(\sigma) \Upsilon \bar{\mathbf{b}}(v') |y| \mathbf{a}(t)
\end{aligned}$$

for all  $t \in J_{v'}$ . Then, for  $v'$  large enough

$$\left| ((\nabla v) \Omega - D_{(u,v)} F_2(x, m, y_*) y)^t \right|_{C^\sigma} \leq \frac{1}{4\bar{C}\Upsilon} |y| \mathbf{a}(t)$$

for all  $t \in J_{v'}$ . We recall that  $\bar{C}$  is the constant introduced in Lemma 3.4. Multiplying both sides of the latter by  $\frac{1}{\mathbf{a}(t)}$  and taking the sup for all  $t \in J_{v'}$ , we obtain

$$\left| ((\nabla v) \Omega - D_{(u,v)} F_2(x, m, y_*) y) \right|_{\sigma, \mathbf{a}}^{v'} \leq \frac{1}{4\bar{C}\Upsilon} |y|. \quad (3.23)$$

This concludes the proof of this lemma because, thanks to Lemma 3.4, the unique solution of (3.21) exists and by (3.22), (3.23)

$$\begin{aligned}
|\hat{u}| &\leq \left| \bar{m}_0 v - (\nabla u) \Omega - D_{(u,v)} F_1(b, \bar{m}, y_*) y \right|_{\sigma, \mathbf{b}}^{v'} \\
&\quad + \bar{C}\Upsilon \left| ((\nabla v) \Omega - D_{(u,v)} F_2(x, m, y_*) y) \right|_{\sigma, \mathbf{a}}^{v'} \leq \frac{1}{2} |y| \\
|\hat{v}| &\leq \left| ((\nabla v) \Omega - D_{(u,v)} F_2(x, m, y_*) y) \right|_{\sigma, \mathbf{a}}^{v'} \leq \frac{1}{4\bar{C}\Upsilon} |y| \leq \frac{1}{2} |y|.
\end{aligned}$$

□

We observe that the choice of the constant 1 in the ball  $|y_*| \leq 1$  is completely arbitrary. One can choose another threshold provided to take  $v'$  sufficiently large.

Now, the previous lemma proves that  $\mathcal{L}(x, m, \bar{m}, \cdot)$  is a contraction of a complete subset of  $\mathcal{Y}$ . Then, there exists a unique fixed point  $y \in \mathcal{Y}$  with  $|y| \leq 1$ . This concludes the proof of Theorem A.

### 3.4 Proof of Corollary A

The proof is essentially the same as that of Theorem A. Because of that, we will not give all the details. However, we will provide the necessary elements to reconstruct the proof.

We are looking for a  $C^\sigma$ -asymptotic KAM torus  $\psi^t$  associated to  $(Z, \omega, \text{Id})$ , where  $Z$  is the vector field defined by  $(Z_A)$ . This means that, for given  $Z$ , we are searching for  $v' \geq 0$  sufficiently large and a suitable function  $u : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n$  such that

$$\psi(q, t) = q + u(q, t)$$

and in addition,  $\psi$  and  $u$  satisfy

$$Z(\psi(q, t), t) - \partial_q \psi(q, t)\omega - \partial_t \psi(q, t) = 0, \quad (3.24)$$

$$\lim_{t \rightarrow +\infty} |u^t|_{C^\sigma} = 0. \quad (3.25)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . We will choose  $v'$  sufficiently large in Lemma 3.6. Similarly to the proof of Theorem A, we introduce a suitable functional  $\mathcal{F}$  given by (3.24). To this end, we define

$$\tilde{\psi}(q, t) = (q + u(q, t), t),$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . The composition of  $Z$  with  $\tilde{\psi}$  is equal to

$$Z \circ \tilde{\psi}(q, t) = \omega + P \circ \tilde{\psi}(q, t)$$

and

$$\partial_q \psi(q, t)\omega + \partial_t \psi(q, t) = \omega + \partial_q u(q, t)\omega + \partial_t u(q, t)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . We recall the notation introduced in the previous section

$$\nabla u(q, t)\Omega = \partial_q u(q, t)\omega + \partial_t u(q, t)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Then, we can rewrite (3.24) in the following form

$$P \circ \tilde{\psi} - (\nabla u)\Omega = 0. \quad (3.26)$$

This is the sum of functions defined on  $(q, t) \in \mathbb{T}^n \times J_{v'}$ , we have omitted the arguments  $(q, t)$  in order to achieve a more elegant form.

Before the introduction of the functional  $\mathcal{F}$ , let  $v \geq 0$  and  $\sigma \geq 1$  be the positive parameters defined in Corollary A. For  $v' \geq v \geq 0$  that will be chosen later, we introduce the following Banach spaces  $(\mathcal{P}, |\cdot|)$ ,  $(\mathcal{U}, |\cdot|)$  and  $(\mathcal{Z}, |\cdot|)$

$$\begin{aligned} \mathcal{P} &= \left\{ P : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid P \in \bar{\mathcal{S}}_{\sigma, 1}^{v'}, \text{ and } |P| = |P|_{\sigma+1, \mathbf{P}}^{v'} < \infty \right\} \\ \mathcal{U} &= \left\{ u : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid u, (\nabla u)\Omega \in \mathcal{S}_\sigma^{v'} \right. \\ &\quad \left. \text{and } |u| = \max\{|u|_{\sigma, \mathbf{P}}^{v'}, |(\nabla u)\Omega|_{\sigma, \mathbf{P}}^{v'}\} < \infty \right\} \\ \mathcal{Z} &= \left\{ z : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid z \in \mathcal{S}_\sigma^{v'}, \text{ and } |z| = |z|_{\sigma, \mathbf{P}}^{v'} < \infty \right\} \end{aligned}$$

Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{P} \times \mathcal{U} \longrightarrow \mathcal{Z}$$

$$\mathcal{F}(P, u) = P \circ \tilde{\psi} - (\nabla u) \Omega.$$

This is obtained by (3.26) and we observe that

$$\mathcal{F}(0, 0) = 0.$$

We can reformulate our problem in the following form. For  $P \in \mathcal{P}$  sufficiently close to 0, we are looking for  $u \in \mathcal{U}$  in such a way that  $\mathcal{F}(P, u) = 0$ .

Regarding the differential of  $\mathcal{F}$  with respect to the variable  $u$  calculated in  $(0, 0)$ , this is equal to

$$D_u \mathcal{F}(0, 0) \hat{u} = -(\nabla \hat{u}) \Omega.$$

The functional  $\mathcal{F}$  is well defined, continuous, differentiable with respect to  $u$  with  $D_u \mathcal{F}(P, u)$  continuous. Moreover, as a straightforward consequence of Lemma 3.1,  $D_u \mathcal{F}(0, 0)$  is invertible. Then,  $\mathcal{F}$  satisfies the hypotheses of the implicit function theorem.

Now, similarly to the proof of Theorem A, we fix  $P$  as in Corollary A and we introduce the following functional

$$\mathcal{L}(P, \cdot) : \mathcal{U} \longrightarrow \mathcal{U}$$

in such a way that

$$\mathcal{L}(P, u) = u - D_u \mathcal{F}(0, 0)^{-1} \mathcal{F}(P, u).$$

We recall that  $P$  is fixed and the proof of Corollary A is reduced to find a fixed point of the latter. To this end, we have the following lemma

**Lemma 3.6.** *There exists  $v'$  large enough with respect to  $n$ ,  $\sigma$  and  $\mathbf{P}$ , such that, for all  $u_*, u \in \mathcal{U}$  with  $|u_*| \leq 1$ ,*

$$|D_u \mathcal{L}(P, u_*) u| \leq \frac{1}{2} |u|.$$

Therefore,  $\mathcal{L}(P, \cdot)$  is a contraction of a complete subset of  $\mathcal{P}$  and this concludes the proof of Corollary A.

## 4 Real Analytic Case

This chapter contains the analytic version of the results proved in the previous chapter. However, in order to simplify the reading and the comparison with the previous results, we keep almost the same structure.

This chapter is divided into three sections. We begin by introducing the definition of analytic asymptotic KAM torus (Section 4.1). The second section (Section 4.2) is dedicated to stating and analysing the main results (Theorem B and Corollary B). This section contains the proof of the result concerning time-dependent perturbations of constant vector fields on the torus. We dedicate the last (Sections 4.3) to the proof of the theorem about time-dependent Hamiltonian vector fields. The proof of this result is essentially the same as that contained in the previous chapter of this work. Here, we pay special attention to the modifications in the proof due to the different classes of functions.

### 4.1 Analytic Asymptotic KAM Torus

For some  $s > 0$ , we define the following complex domains

$$\begin{aligned}\mathbb{T}_s^n &:= \{q \in \mathbb{C}^n / \mathbb{Z}^n : |\operatorname{Im}(q)| < s\}, \\ B_s &:= \{p \in \mathbb{C}^n : |p| < s\}.\end{aligned}$$

We recall the definition of analytic asymptotic KAM torus. Let  $\mathcal{P}$  be equal to  $\mathbb{T}^n \times B$  or  $\mathbb{T}^n$ . We consider time-dependent real analytic vector fields  $X^t$  and  $X_0^t$  on  $\mathcal{P}$ , for all  $t \in J_v$ , and a real analytic embedding  $\varphi_0$  from  $\mathbb{T}^n$  to  $\mathcal{P}$  such that

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_s = 0 \quad (4.1)$$

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q) \omega, \text{ for all } (q, t) \in \mathbb{T}^n \times J_v, \quad (4.2)$$

where  $\omega \in \mathbb{R}^n$  and  $|\cdot|_s$  is the analytic norm (see Appendix B).

**Definition** (Définition 1.1). *We assume that  $(X, X_0, \varphi_0)$  satisfy (4.1) and (4.2). A family of real analytic embeddings  $\varphi^t : \mathbb{T}^n \rightarrow \mathcal{P}$  is an analytic asymptotic KAM torus associated to  $(X, X_0, \varphi_0)$  if there exist  $0 < s' \leq s$  and  $v' \geq v \geq 0$  such that*

$$\lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{s'} = 0, \quad (4.3)$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t) \omega + \partial_t \varphi(q, t), \quad (4.4)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . When  $\dim \mathcal{P} = 2n$ , then we say that  $\varphi^t$  is Lagrangian if  $\varphi^t(\mathbb{T}^n)$  is Lagrangian for all  $t$ .

We conclude this section with the introduction of a suitable space of functions. For some  $s > 0$  and  $v \geq 0$ , we have the following definition

**Definition 4.1.** *Let  $\mathcal{A}_s^v$  be the space of functions  $f$  defined on  $\mathbb{T}_s^n \times B_s \times J_v$  such that  $f \in C(\mathbb{T}_s^n \times B_s \times J_v)$  and, for all  $t \in J_v$ ,  $f^t$  is real analytic on  $\mathbb{T}_s^n \times B_s$ .*

We use this notation also for functions defined on  $\mathbb{T}_s^n \times J_v$ . For all  $k \in \mathbb{Z}^{2n}$  with  $|k| \geq 1$ , we let

$$\partial_{(q,p)}^k = \partial_{q_1}^{k_1} \dots \partial_{q_n}^{k_n} \partial_{p_1}^{k_{n+1}} \dots \partial_{p_n}^{k_{2n}}$$

where  $|k| = |k_1| + \dots + |k_{2n}|$ . The following proposition is about an important property concerning each  $f \in \mathcal{A}_s^v$ , which we will widely use in the rest of this chapter.

**Proposition 4.1.** *Let  $f \in \mathcal{A}_s^v$ , then, for all  $k \in \mathbb{Z}^{2n}$  with  $|k| \geq 1$ ,  $\partial_{(q,p)}^k f \in \mathcal{A}_{s'}^v$  for all  $0 < s' < s$ .*

*Proof.* For all  $t \in J_v$ ,  $\partial_{(q,p)}^k f^t$  is real analytic on  $\mathbb{T}_s^n \times B_s$  (see [Rud87]) and hence on  $\mathbb{T}_{s'}^n \times B_{s'}$ . It remains to prove that  $\partial_{(q,p)}^k f \in C(\mathbb{T}_{s'}^n \times B_{s'} \times J_v)$ . For all  $(q_1, p_1, t_1), (q_2, p_2, t_2) \in \mathbb{T}_{s'}^n \times B_{s'} \times J_v$ , by Cauchy's inequality

$$\begin{aligned} |\partial_{(q,p)}^k f(q_1, p_1, t_1) - \partial_{(q,p)}^k f(q_2, p_2, t_2)| &\leq |\partial_{(q,p)}^k f(q_1, p_1, t_1) - \partial_{(q,p)}^k f(q_1, p_1, t_2)| \\ &\quad + |\partial_{(q,p)}^k f(q_1, p_1, t_2) - \partial_{(q,p)}^k f(q_2, p_2, t_2)| \\ &\leq \frac{k_1! \dots k_{2n}!}{(s - s')^{|k|}} |f^{t_1} - f^{t_2}|_s \\ &\quad + |\partial_{(q,p)}^k f^{t_2}(q_1, p_1) - \partial_{(q,p)}^k f^{t_2}(q_2, p_2)| \end{aligned}$$

and hence by the continuity of  $f$  with respect to  $t$  and the continuity of  $\partial_{(q,p)}^k f$  with respect to  $(q, p)$ , we have the claim.  $\square$

## 4.2 Results

Given  $\omega \in \mathbb{R}^n$  and positive real parameters  $s_0 > 0$ , we consider the following time-dependent Hamiltonian  $H$

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times J_0 \longrightarrow \mathbb{R} \\ H(q, p, t) = h(q, p, t) + f(q, p, t), \\ h \in \mathcal{K}_\omega \\ h, f \in \mathcal{A}_{s_0}^0 \\ \sup_{t \in J_0} |f_0^t|_{s_0} < \infty, \quad \sup_{t \in J_0} |\partial_p^2 H^t|_{s_0} < \infty \\ |\partial_q f_0^t|_{s_0} \leq \mathbf{a}(t), \quad |\partial_p f_0^t|_{s_0} \leq \mathbf{b}(t), \quad \text{for all } t \in J_0 \end{array} \right. \quad (*_B)$$

where  $\mathbf{a}, \mathbf{b}$  are positive, decreasing, integrable functions on  $J_0$ .

We assume that there exists  $\nu \geq 0$  such that

$$\left\{ \begin{array}{l} \bar{\mathbf{a}}(t) \leq \mathbf{b}(t) \\ \bar{\mathbf{a}}(t)\mathbf{b}(t) \leq \mathbf{a}(t)\bar{\mathbf{b}}(t) \end{array} \right. \quad (\#)$$

for all  $t \in J_v$ . Let  $\varphi_0$  be the following trivial embedding  $\varphi_0 : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B$ ,  $\varphi_0(q) = (q, 0)$ , we have the following theorem.

**Theorem B.** *Let  $H$  be as in  $(*_B)$  with  $\mathbf{a}$  and  $\mathbf{b}$  satisfying  $(\#)$ . Then, there exist  $\tilde{h} \in \mathcal{K}_\omega$  and a Lagrangian analytic asymptotic KAM torus  $\varphi^t$  associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ .*

As a consequence of the above theorem, we have the following result concerning time-dependent perturbations of constant vector fields on the torus. Let  $Z$  be a non-autonomous vector field on  $\mathbb{T}^n \times J_0$  of the form

$$\begin{cases} Z : \mathbb{T}^n \times J_0 \longrightarrow \mathbb{R}^n, \\ Z(q, t) = \omega + P(q, t) \\ P \in \mathcal{A}_{s_0}^0, \\ |P^t|_{s_0} \leq \mathbf{P}(t) \quad \text{for all } t \in J_0 \end{cases} \quad (Z_B)$$

where  $\omega \in \mathbb{R}^n$  and  $0 < s_0 < 1$ . We assume that  $\mathbf{P}$  is a positive, decreasing, integrable function on  $J_0$ .

**Corollary B.** *Let  $Z$  be as in  $(Z_B)$ . Then, there exists an analytic asymptotic KAM torus  $\psi^t$  associated to  $(Z, \omega, \text{Id})$ .*

*Proof.* The proof is a straightforward application of Theorem B. Let  $\tilde{h}(p) = \omega \cdot p$ , we consider the Hamiltonian  $H$  defined on  $\mathbb{T}^n \times B \times J_0$  of the form

$$H(q, p, t) = \omega \cdot p + P(q, t) \cdot p.$$

The latter satisfies the hypotheses of Theorem B. Then, there exists an analytic asymptotic KAM torus  $\varphi^t$  associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ , where  $\varphi_0$  is the trivial embedding previously introduced. Moreover,  $\varphi^t = (\text{id} + u^t, v^t)$  and, for all fixed  $t$ ,  $\text{id} + u^t$  is a diffeomorphism of the torus. This concludes the proof of this theorem with  $\psi^t = \text{id} + u^t$ .  $\square$

### 4.3 Proof of Theorem B

We expand the Hamiltonian  $H$  in  $(*_B)$  in a small neighbourhood of  $0 \in B$ . Then, thanks to Proposition 4.1 and for a positive parameter  $\Upsilon \geq 1$ , we can rewrite the Hamiltonian  $H$  in the following form

$$\begin{cases} H : \mathbb{T}^n \times B \times J_0 \longrightarrow \mathbb{R} \\ H(q, p, t) = \omega \cdot p + a(q, t) + b(q, t) \cdot p + m(q, p, t) \cdot p^2, \\ a, b, \partial_p^2 H \in \mathcal{A}_s^0, \\ \sup_{t \in J_0} |a^t|_s < \infty, \quad \sup_{t \in J_0} |\partial_p^2 H^t|_s \leq \Upsilon, \\ |\partial_q a^t|_s \leq \mathbf{a}(t), \quad |b^t|_s \leq \mathbf{b}(t), \quad \text{for all } t \in J_0 \end{cases} \quad (**_B)$$

where  $s = \frac{s_0}{2}$  and  $\mathbf{a}(t)$ ,  $\mathbf{b}(t)$  are the functions introduced in  $(*_B)$  satisfying  $(\#)$ .

We consider the following Hamiltonian

$$\tilde{h}(q, p, t) = h(q, p, t) + \int_0^1 (1 - \tau) \partial_p^2 f(q, \tau p, t) d\tau \cdot p^2.$$

for all  $(q, p, t) \in \mathbb{T}^n \times B \times J_0$ . It is obvious that  $\tilde{h} \in \mathcal{K}_\omega$  and  $X_H, X_{\tilde{h}}$  satisfy (4.1).

### 4.3.1 Outline of the Proof of Theorem B

We are looking for an analytic asymptotic KAM torus  $\varphi^t$  associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ , where  $H$  is the Hamiltonian in  $(**_B)$ ,  $\tilde{h}$  is the Hamiltonian previously defined and  $\varphi_0$  the trivial embedding  $\varphi_0 : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B$ ,  $\varphi_0(q) = (q, 0)$ . More specifically, for given  $H$ , we are searching for  $v' \geq 0$  sufficiently large and suitable functions  $u, v : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n$  such that

$$\varphi(q, t) = (q + u(q, t), v(q, t))$$

and in such a way that  $\varphi, u$  and  $v$  satisfy

$$X_H(\varphi(q, t), t) - \partial_q \varphi(q, t) \omega - \partial_t \varphi(q, t) = 0, \quad (4.5)$$

$$\lim_{t \rightarrow +\infty} |u^t|_{\frac{s}{2}} = 0, \quad \lim_{t \rightarrow +\infty} |v^t|_{\frac{s}{2}} = 0, \quad (4.6)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Similarly to Theorem A, we will choose  $v'$  large enough in Lemma 4.5 ( $v'$  will be already required large in Lemma 4.2).

The proof of this theorem rests on the implicit function theorem. For this reason, we begin by defining a suitable functional  $\mathcal{F}$  given by (3.7). Similarly to the proof of Theorem A, we can rewrite (4.5) in the following form

$$\begin{pmatrix} b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \Omega \\ -\partial_q a \circ \tilde{u} - (\partial_q b \circ \tilde{u}) v - (\partial_q m \circ \tilde{\varphi}) \cdot v^2 - (\nabla v) \Omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (4.7)$$

Thanks to the latter, over suitable Banach spaces that we will specify later, we define the functional  $\mathcal{F}$  in such a way that

$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \Omega, \\ F_2(a, b, m, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q b \circ \tilde{u}) v + (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v) \Omega. \end{aligned}$$

Moreover, we observe that for all  $m$  and  $\bar{m}$ ,

$$\mathcal{F}(0, 0, m, \bar{m}, 0, 0) = 0.$$

We reformulate our problem in the following form. For fixed  $m$  and  $\bar{m}$  in a suitable Banach space and for  $(a, b)$  sufficiently close to  $(0, 0)$ , we are looking for some functions  $u, v$  in such a way that  $\mathcal{F}(a, b, m, \bar{m}, u, v) = 0$  and the asymptotic conditions (4.6) are satisfied.

We can see that, the differential of  $\mathcal{F}$  with respect to the variables  $(u, v)$  calculated in  $(0, 0, m, \bar{m}, 0, 0)$  is equal to

$$D_{(u,v)} \mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (\bar{m}_0 \hat{v} - (\nabla \hat{u}) \Omega, (\nabla \hat{v}) \Omega),$$

where, for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ ,  $\bar{m}_0(q, t) = \bar{m}(q, 0, t)$ .

Also in this case, the proof of Theorem B is developed in the following four sections. In the first, we introduce suitable Banach spaces on which the previous functional is defined. These Banach spaces are the analytical version of those used in the proof of Theorem A. The second is dedicated to solving the homological equation. In the third section, we verify that  $\mathcal{F}$  is well defined and satisfies the hypotheses of the implicit function theorem. In the last section, we conclude the proof of this theorem.



### 4.3.2 Preliminary Settings

Given  $s > 0$  and  $v \geq 0$ , for every  $f \in \mathcal{A}_s^v$  and for positive real functions  $\mathbf{u}(t)$  defined on  $J_v$ , we introduce the following norm

$$|f|_{s,\mathbf{u}}^v = \sup_{t \in J_v} \frac{|f^t|_s}{\mathbf{u}(t)},$$

where  $|\cdot|_s$  is the analytic norm and  $\mathcal{A}_s^v$  is the space of functions of Definition 4.1. It is the analytic version of the norm defined in the finitely differentiable case by (3.10).

Let  $v \geq 0$ ,  $s > 0$  and  $\Upsilon \geq 1$  be the positive parameters introduced by (\*\*<sub>B</sub>) and (#). For  $v' \geq v \geq 0$  that will be chosen later, we consider the following Banach spaces  $(\mathcal{A}, |\cdot|)$ ,  $(\mathcal{B}, |\cdot|)$ ,  $(\mathcal{U}, |\cdot|)$ ,  $(\mathcal{V}, |\cdot|)$ ,  $(\mathcal{Z}, |\cdot|)$ ,  $(\mathcal{G}, |\cdot|)$  and  $(\mathcal{M}, |\cdot|)$ .

$$\begin{aligned} \mathcal{A} &= \left\{ a : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R} \mid a \in \mathcal{A}_s^{v'} \text{ and } |a| = |a|_{s,1}^{v'} + |\partial_q a|_{s,\mathbf{a}}^{v'} < \infty \right\} \\ \mathcal{B} &= \left\{ b : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid b \in \mathcal{A}_s^{v'}, \text{ and } |b| = |b|_{s,\mathbf{b}}^{v'} < \infty \right\} \\ \mathcal{U} &= \left\{ u : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid u, (\nabla u) \Omega \in \mathcal{A}_{\frac{s}{2}}^{v'} \right. \\ &\quad \left. \text{and } |u| = \max\{|u|_{\frac{s}{2},\bar{\mathbf{b}}}^{v'}, |(\nabla u) \Omega|_{\frac{s}{2},\mathbf{b}}^{v'}\} < \infty \right\} \\ \mathcal{V} &= \left\{ v : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid v, (\nabla v) \Omega \in \mathcal{A}_{\frac{s}{2}}^{v'} \right. \\ &\quad \left. \text{and } |v| = \max\{|v|_{\frac{s}{2},\bar{\mathbf{a}}}^{v'}, |(\nabla v) \Omega|_{\frac{s}{2},\mathbf{a}}^{v'}\} < \infty \right\} \\ \mathcal{Z} &= \left\{ z : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid z \in \mathcal{A}_{\frac{s}{2}}^{v'}, \text{ and } |z| = |z|_{\frac{s}{2},\mathbf{b}}^{v'} < \infty \right\} \\ \mathcal{G} &= \left\{ g : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R} \mid g \in \mathcal{A}_{\frac{s}{2}}^{v'} \text{ and } |g| = |g|_{\frac{s}{2},\mathbf{a}}^{v'} < \infty \right\} \\ \mathcal{M} &= \left\{ m : \mathbb{T}^n \times B \times J_{v'} \rightarrow M_n \mid m \in \mathcal{A}_s^{v'} \text{ and } |m| = |m|_{s,1}^{v'} \leq \Upsilon \right\} \end{aligned}$$

The risk of mixing the Banach space  $\mathcal{A}$  with the space of functions  $\mathcal{A}_s^v$  is small. Concerning  $\mathcal{A}$ , we have that  $|a|_{s,1}^{v'} = \sup_{t \in J_{v'}} |a^t|_s$ , similarly for  $\mathcal{M}$ . Regarding the last Banach space  $\mathcal{M}$ , we recall that  $M_n$  is the set of  $n$ -dimensional matrices. These are the analytic version of the Banach spaces introduced in the finitely differentiable case (see Section 3.3.2).

Contrary to the previous chapter, we have to define the functional  $\mathcal{F}$  on a suitable subspace  $\mathcal{X}$  of  $\mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V}$ . This is because we have to control the domain of analyticity of the components of  $\mathcal{F}$ . Let  $\mathcal{X}$  be equal to

$$\begin{aligned} \mathcal{X} &= \{(a, b, m, \bar{m}, u, v) \in \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V} : \\ &\quad |\partial_q a|_{s,\mathbf{a}}^{v'} \leq 1, |b| \leq 1, |m| \leq \Upsilon, |\bar{m}| \leq \Upsilon, |u| \leq 1, |v| \leq 1\}. \end{aligned}$$

Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{X} \longrightarrow \mathcal{Z} \times \mathcal{G}$$

$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \Omega, \\ F_2(a, b, m, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q b \circ \tilde{u}) v + (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v) \Omega. \end{aligned}$$

### 4.3.3 Homological Equation

Given  $s > 0$ ,  $v \geq 0$  and  $\omega \in \mathbb{R}^n$ , we are looking for a solution of the following equation for the unknown  $\varkappa : \mathbb{T}_s^n \times J_v \rightarrow \mathbb{R}$

$$\begin{cases} \omega \cdot \partial_q \varkappa(q, t) + \partial_t \varkappa(q, t) = g(q, t), \\ g \in \mathcal{A}_s^v, \quad |g|_{s, \mathbf{g}}^v < \infty \end{cases} \quad (HE_B)$$

where  $\mathbf{g}(t)$  is a positive, decreasing, integrable function on  $J_v$  and  $g : \mathbb{T}_s^n \times J_v \rightarrow \mathbb{R}$  is given.

**Lemma 4.1 (Homological Equation).** *There exists a unique solution  $\varkappa \in \mathcal{A}_s^v$  of (HE<sub>B</sub>) such that*

$$\lim_{t \rightarrow +\infty} |\varkappa^t|_{C^0} = 0.$$

Moreover,

$$|\varkappa|_{s, \bar{\mathbf{g}}}^v \leq |g|_{s, \mathbf{g}}^v.$$

*Proof.* The proof of this lemma is essentially the same as that of Lemma 3.1.

*Existence:* Let us define the following transformation

$$h(q, t) = (q - \omega t, t),$$

then  $h : \mathbb{T}_s^n \times J_v \rightarrow \mathbb{T}_s^n \times J_v$  because  $\omega \in \mathbb{R}^n$ ,  $t \in J_v \subset \mathbb{R}$  and thus  $q - \omega t \in \mathbb{T}_s^n$  if and only if  $q \in \mathbb{T}_s^n$ . The fact that  $t$  is real and not complex is of fundamental importance. This ensures that the latter transformation is well defined.

It is enough to prove the first part of this lemma for the much simpler equation

$$\partial_t \kappa = g(q + \omega t, t).$$

The unique solution of the above equation satisfying the asymptotic condition is

$$\kappa(q, t) = - \int_t^{+\infty} g(q + \omega \tau, \tau) d\tau$$

and hence

$$\varkappa(q, t) = \kappa \circ h(q, t) = - \int_t^{+\infty} g(q + \omega(\tau - t), \tau) d\tau$$

is the unique solution of (HE<sub>B</sub>) that we are looking for.

*Regularity and Estimates:*  $g \in \mathcal{A}_s^v$  implies  $\kappa \in \mathcal{A}_s^v$  and hence  $\varkappa = \kappa \circ h \in \mathcal{A}_s^v$ . Moreover, for all fixed  $t \in J_v$

$$|\varkappa^t|_s \leq |g|_{s, \mathbf{g}}^v \bar{\mathbf{g}}(t).$$

We prove the second part of this lemma by multiplying both sides of the latter by  $\frac{1}{\bar{\mathbf{g}}(t)}$  and taking the sup for all  $t \in J_v$ .  $\square$

### 4.3.4 Regularity of $\mathcal{F}$

Here, we verify that the functional  $\mathcal{F}$  is well defined and satisfies the hypotheses of the implicit function theorem. We begin with a quantitative lemma fundamental to verify that  $\mathcal{F}$  is well defined. In the second part of this section, we state, without proving, that  $\mathcal{F}$  is differentiable with respect to the variables  $(u, v)$  and this differential calculated in  $(0, 0, m, \bar{m}, 0, 0)$  is invertible. The proofs are essentially the same as that in Section 3.3.4. The following lemma imposes the first restriction on  $v'$ . We will take a stronger one after.

**Lemma 4.2.** *For  $v'$  large enough with respect to  $s$  and  $\mathbf{b}$ , if  $(u, v) \in \mathcal{U} \times \mathcal{V}$  satisfies the following estimates  $|u| \leq 1$  and  $|v| \leq 1$ , then*

$$\sup_{t \in J_{v'}} |u^t|_{\frac{s}{2}} \leq \frac{s}{8}, \quad \sup_{t \in J_{v'}} |v^t|_{\frac{s}{2}} \leq \frac{s}{8}.$$

*Proof.* If  $|u| \leq 1$  and  $|v| \leq 1$ , then by (#)

$$|u^t|_{\frac{s}{2}} \leq \bar{\mathbf{b}}(t) \leq \bar{\mathbf{b}}(v'), \quad |v^t|_{\frac{s}{2}} \leq \bar{\mathbf{a}}(t) \leq \mathbf{b}(t) \leq \mathbf{b}(v').$$

for all  $t \in J_{v'}$ . Now, for  $v'$  large enough, we have the claim.  $\square$

**Lemma 4.3.**  *$\mathcal{F}$  is well defined.*

*Proof.* We prove that, for all  $(a, b, m, \bar{m}, u, v) \in \mathcal{X}$ ,  $F_1(b, \bar{m}, u, v) \in \mathcal{Z}$ . By the estimates in Lemma 4.2, the compositions  $b \circ \tilde{u}(q, t)$  and  $\bar{m} \circ \tilde{\varphi}(q, t)$  are well defined for all  $(q, t) \in \mathbb{T}_{\frac{s}{2}}^n \times J_{v'}$ . Concerning the regularity, thanks to Proposition 4.1,  $F_1(b, \bar{m}, u, v) \in \mathcal{A}_{\frac{s}{2}}^{v'}$ . It remains to prove that  $|F_1(b, \bar{m}, u, v)|_{\frac{s}{2}, \mathbf{b}}^{v'} < \infty$ . First, let us remind that

$$|F_1(b, \bar{m}, u, v)|_{\frac{s}{2}, \mathbf{b}}^{v'} = \sup_{t \in J_{v'}} \frac{|F_1(b, \bar{m}, u, v)^t|_{\frac{s}{2}}}{\mathbf{b}(t)}.$$

Moreover, for all  $(a, b, m, \bar{m}, u, v) \in \mathcal{X}$ , we have the following estimates

$$\begin{aligned} |\partial_q a|_{s, \mathbf{a}}^{v'} &\leq 1, \quad |b| = |b|_{s, \mathbf{b}}^{v'} \leq 1, \quad |m| = |m|_{s, 1}^{v'} \leq \Upsilon, \quad |\bar{m}| = |\bar{m}|_{s, 1}^{v'} \leq \Upsilon, \\ |u| &= \max\{|u|_{\frac{s}{2}, \bar{\mathbf{b}}}^{v'}, |(\nabla u) \Omega|_{\frac{s}{2}, \mathbf{b}}^{v'}\} \leq 1, \quad |v| = \max\{|v|_{\frac{s}{2}, \bar{\mathbf{a}}}^{v'}, |(\nabla v) \Omega|_{\frac{s}{2}, \mathbf{a}}^{v'}\} \leq 1. \end{aligned}$$

Then, for all  $t \in J_{v'}$

$$\frac{|F_1(b, \bar{m}, u, v)^t|_{\frac{s}{2}}}{\mathbf{b}(t)} \leq \frac{|(b \circ \tilde{u})^t|_{\frac{s}{2}}}{\mathbf{b}(t)} + \frac{|(\bar{m} \circ \tilde{\varphi})^t v^t|_{\frac{s}{2}}}{\mathbf{b}(t)} + \frac{|(\nabla u^t) \Omega|_{\frac{s}{2}}}{\mathbf{b}(t)}. \quad (4.8)$$

We have to estimate each member on the right-hand side of the latter. The last one is obviously bounded, we have to find an upper bound for the others. Thanks to Lemma 4.2, (#) and the properties contained in Appendix B

$$\begin{aligned} \frac{|(b \circ \tilde{u})^t|_{\frac{s}{2}}}{\mathbf{b}(t)} &\leq \frac{|b^t|_s}{\mathbf{b}(t)} \leq |b| \leq 1, \\ \frac{|(\bar{m} \circ \tilde{\varphi})^t v^t|_{\frac{s}{2}}}{\mathbf{b}(t)} &\leq C \frac{|(\bar{m} \circ \tilde{\varphi})^t|_{\frac{s}{2}}}{\mathbf{b}(t)} \frac{|v^t|_{\frac{s}{2}}}{\mathbf{b}(t)} \leq C |\bar{m}^t|_s \frac{|v^t|_{\frac{s}{2}}}{\bar{\mathbf{a}}(t)} \leq C \Upsilon |v| \leq C \Upsilon, \end{aligned}$$

for all  $t \in J_{v'}$  and a suitable constant depending on  $n$ . As a consequence of the latter, taking the sup for all  $t \in J_{v'}$  on the left-hand side of (4.8), we prove  $|F_1(b, \bar{m}, u, v)|_{\frac{v'}{2}, \mathbf{b}} < \infty$ . Similarly, for all  $(a, b, m, \bar{m}, u, v) \in \mathcal{X}$ ,  $F_2(a, b, m, u, v) \in \mathcal{G}$ .  $\square$

We proved the previous lemma because  $\mathcal{F}$  defined on  $\mathcal{X}$  translates into a slightly different proof from that of Lemma 3.2.

Furthermore, one can prove that  $\mathcal{F}$  is continuous and, similarly to the proof of Theorem A,  $\mathcal{F}$  is differentiable with respect to the variables  $(u, v)$  and this differential  $D_{(u,v)}\mathcal{F}(a, b, m, \bar{m}, u, v)$  is continuous. Moreover, for all fixed  $m, \bar{m} \in \mathcal{M}$ ,  $D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)$  is invertible. More specifically, we have the following lemma

**Lemma 4.4.** *For all  $(z, g) \in \mathcal{Z} \times \mathcal{G}$  there exists a unique  $(\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V}$  such that*

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (z, g).$$

Moreover, for a suitable constant  $\bar{C}$

$$|\hat{u}| \leq \bar{C}\Upsilon |g|_{\frac{v'}{2}, \mathbf{a}} + |z|_{\frac{v'}{2}, \mathbf{b}}, \quad |\hat{v}| \leq |g|_{\frac{v'}{2}, \mathbf{a}},$$

where we recall that  $|\hat{v}| = \max\{|\hat{v}|_{\frac{v'}{2}, \bar{\mathbf{a}}}, |(\nabla \hat{v})\Omega|_{\frac{v'}{2}, \mathbf{a}}\}$  and  $|\hat{u}| = \max\{|\hat{u}|_{\frac{v'}{2}, \bar{\mathbf{b}}}, |(\nabla \hat{u})\Omega|_{\frac{v'}{2}, \mathbf{b}}\}$ .

*Proof.* The proof is essentially the same as that of Lemma 3.4. It relies on Lemma 4.1.  $\square$

### 4.3.5 Analytic Asymptotic KAM Torus

This part is extremely similar to that of the differentiable case (see Section 3.3.5). Let  $a$  and  $b$  be the functions introduced by  $(**_B)$ . It is straightforward to verify that  $(a, b) \in \mathcal{A} \times \mathcal{B}$  and

$$|\partial_q a|_{s, \mathbf{a}}^{v'} \leq 1, \quad |b|_{s, \mathbf{b}}^{v'} \leq 1. \quad (4.9)$$

We introduce the Banach space  $(\mathcal{Y}, |\cdot|)$ , such that  $\mathcal{Y} = \mathcal{U} \times \mathcal{V}$  and for all  $y = (u, v) \in \mathcal{Y}$ ,  $|y| = \max\{|u|, |v|\}$ . Following the lines of the differentiable case (Section (3.3.5)), we fix  $m, \bar{m} \in \mathcal{M}$  as in  $(**_B)$ , we let  $x = (a, b)$  and we introduce the following functional

$$\mathcal{L}(x, m, \bar{m}, \cdot) : \mathcal{Y} \longrightarrow \mathcal{Y}$$

in such a way that

$$\mathcal{L}(x, m, \bar{m}, y) = y - D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1}\mathcal{F}(x, m, \bar{m}, y). \quad (\mathcal{L})$$

This is well defined and, by the regularity of  $\mathcal{F}$ , we deduce that  $\mathcal{L}$  is continuous, differentiable with respect to  $y = (u, v)$  with differential  $D_y\mathcal{L}$  continuous. The proof is reduced to find a fixed point of the latter. For this purpose, we state the following lemma, which is the analytic version of Lemma 3.5.

**Lemma 4.5.** *There exists  $v'$  large enough with respect to  $s$ ,  $\Upsilon$  and  $\mathbf{b}$ , such that, for all  $y_*, y \in \mathcal{Y}$  with  $|y_*| \leq 1$ ,*

$$|D_y \mathcal{L}(x, m, \bar{m}, y_*)y| \leq \frac{1}{2}|y|.$$

*Proof.* The proof of this lemma is extremely similar to that of Lemma 3.5. For this reason, it is omitted.  $\square$

The above lemma proves that  $\mathcal{L}(x, m, \bar{m}, \cdot)$  is a contraction and this concludes the proof of Theorem B.

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## Part III

# Biasymptotic Motions Converging to Quasiperiodic Dynamics

## 5 Biasymptotic Motions for Time Dependent Hamiltonians

In this chapter, we prove the existence of biasymptotically quasiperiodic solutions for time-dependent Hamiltonian vector fields. We consider two different cases. In the first, we work with time-dependent perturbations of integrable Hamiltonians and in the second with time-dependent perturbations of autonomous Hamiltonians having a large (in the sense of measure) subset of invariant tori. The proofs rely on different versions of Theorem A. We need to require more conditions, with respect to Theorem A, to obtain more information.

In the integrable case, we prove the existence of biasymptotically quasiperiodic solutions for every initial condition. Concerning the other case, when the unperturbed Hamiltonian has a large subset of invariant tori, we show the existence of a large subset of initial conditions giving rise to biasymptotically quasiperiodic solutions.

This chapter has seven sections. We begin by stating and explaining the definition of biasymptotically quasiperiodic solutions (Section 5.1). Afterwards, we continue with a part dedicated to the introduction of the result about time-dependent perturbations of integrable Hamiltonians (Section 5.2). The following two sections are dedicated to the proof of this theorem (Section 5.3 and Section 5.4). Then, in the fifth section, we state the result concerning time-dependent perturbations of autonomous Hamiltonians having a large (in the sense of measure) subset of invariant tori (Section 5.5). The proof of this theorem is contained in the last sections (Section 5.6 and Section 5.7)

### 5.1 Biasymptotically quasiperiodic solutions

In the previous chapters, we focused on non-autonomous Hamiltonian systems defined for positive times. Here, we consider time-dependent Hamiltonian systems defined for all  $t \in \mathbb{R}$ . According to that, it makes sense to distinguish between positive  $C^\sigma$ -asymptotic KAM tori and negative  $C^\sigma$ -asymptotic KAM tori, as well as introduce the definition of  $C^\sigma$ -biasymptotic KAM torus (we refer to the introduction of this thesis for this series of definitions). Unfortunately, we are not able to prove the existence of a  $C^\sigma$ -biasymptotic KAM torus. But, we prove some weaker results concerning the existence of biasymptotically quasiperiodic solutions associated with suitable time-dependent Hamiltonian systems. To this end, we recall the following definition.

Given  $\sigma \geq 0$  and a positive integer  $k \geq 0$ , we consider time-dependent vector fields  $X^t$ ,  $X_{0,+}^t$ ,  $X_{0,-}^t$  of class  $C^{\sigma+k}$  on  $\mathbb{T}^n \times B$ , for all  $t \in \mathbb{R}$ , and embeddings  $\varphi_{0,+}$ ,

$\varphi_{0,-}$  from  $\mathbb{T}^n$  to  $\mathbb{T}^n \times B$  of class  $C^\sigma$  such that

$$\lim_{t \rightarrow \pm\infty} |X^t - X_{0,\pm}^t|_{C^{\sigma+k}} = 0, \quad (5.1)$$

$$X_{0,\pm}(\varphi_{0,\pm}(q), t) = \partial_q \varphi_{0,\pm}(q) \omega_\pm, \quad \text{for all } (q, t) \in \mathbb{T}^n \times \mathbb{R} \quad (5.2)$$

where  $\omega_+, \omega_- \in \mathbb{R}^n$ .

**Definition** (Définition 2.2). *We assume that  $(X, X_{0,\pm}, \varphi_{0,\pm})$  satisfy (5.1) and (5.2). An integral curve  $g(t)$  of  $X$  is a biasymptotically quasiperiodic solution associated to  $(X, X_{0,\pm}, \varphi_{0,\pm})$  if there exist  $q_-, q_+ \in \mathbb{T}^n$  in such a way that*

$$\lim_{t \rightarrow \pm\infty} |g(t) - \varphi_{0,\pm}(q_\pm + \omega_\pm t)| = 0. \quad (5.3)$$

Roughly speaking, a biasymptotically quasiperiodic solution  $g$  is an orbit of the time-dependent vector field  $X$ , which converges to a quasiperiodic motion of frequency vector  $\omega_+ \in \mathbb{R}^n$  in the future and to a quasiperiodic motion of frequency vector  $\omega_- \in \mathbb{R}^n$  in the past. Therefore, we conclude this section with an obvious property concerning these motions. Let  $X, X_{0,+}, X_{0,-}, \varphi_{0,+}$  and  $\varphi_{0,-}$  be as in the previous definition

**Proposition 5.1.** *We assume the existence of a positive  $C^\sigma$ -asymptotic KAM torus  $\varphi_+^t$  associated to  $(X, X_{0,+}, \varphi_{0,+})$ , a negative  $C^\sigma$ -asymptotic KAM torus  $\varphi_-^t$  associated to  $(X, X_{0,-}, \varphi_{0,-})$  and  $q_+, q_- \in \mathbb{T}^n$  in such a way that*

$$\varphi_+^0(q_+) = \varphi_-^0(q_-).$$

Then, letting

$$g(t) = \begin{cases} \varphi_+^t(q_+ + \omega_+ t) & \text{for all } t \geq 0 \\ \varphi_-^t(q_- + \omega_- t) & \text{for all } t \leq 0, \end{cases}$$

$g$  is a biasymptotically quasiperiodic solution associated to  $(X, X_{0,\pm}, \varphi_{0,\pm})$ .

## 5.2 Result (Integrable Case)

We recall that, for every function  $f$  defined on  $\mathbb{T}^n \times B \times \mathbb{R}$  and, for fixed  $t \in \mathbb{R}$ , we let  $f^t$  be the function defined on  $\mathbb{T}^n \times B$  in such a way that

$$f^t(q, p) = f(q, p, t).$$

In addition, for all fixed  $p_0 \in B$ , we let  $f_{p_0}$  be the function defined on  $\mathbb{T}^n \times \mathbb{R}$  such that

$$f_{p_0}(q, t) = f(q, p_0, t).$$

Given a positive real parameter  $\sigma \geq 1$ , we have the following definition.

**Definition 5.1.** *Let  $\mathcal{B}_\sigma$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times B \times \mathbb{R}$  such that  $f, \partial_p f \in C(\mathbb{T}^n \times B \times \mathbb{R})$  and  $f_p^t \in C^\sigma(\mathbb{T}^n)$  for all  $(p, t) \in B \times \mathbb{R}$ .*

For all  $f \in \mathcal{B}_\sigma$  and  $l \geq 1$ , we define

$$|f|_{\sigma,l} = \sup_{(p,t) \in B \times \mathbb{R}} |f_p^t|_{C^\sigma} (1 + |t|^l) + \sup_{(p,t) \in B \times \mathbb{R}} |(\partial_p f)_p^t|_{C^0} (1 + |t|^{l-1}), \quad (5.4)$$

$$|f|_{\sigma,0} = \sup_{(p,t) \in B \times \mathbb{R}} |f_p^t|_{C^\sigma} + \sup_{(p,t) \in B \times \mathbb{R}} |(\partial_p f)_p^t|_{C^0}. \quad (5.5)$$

In what follows, we provide a series of properties of the previous norms that we will widely use in the rest of this chapter. Nevertheless, first, we recall that  $C(\cdot)$  stands for constants depending on  $n$  and the other parameters in brackets. On the other hand,  $C$  stands for constants depending only on  $n$ .

**Proposition 5.2.** *Given  $\sigma \geq 1$ , for all  $f, g \in \mathcal{B}_\sigma$  and positive  $l, m \geq 1$*

a.  $|f|_{\sigma,l} \leq |f|_{s,l}$  for all  $1 \leq \sigma \leq s$ ,

b.  $|f|_{\sigma,l} \leq C(l, m) |f|_{\sigma,l+m}$

c.  $|fg|_{\sigma,l+m} \leq C(\sigma) (|f|_{0,l} |g|_{\sigma,m} + |f|_{\sigma,l} |g|_{0,m})$ .

Moreover, we consider  $\tilde{g} \in \mathcal{B}_\sigma$  such that, for all  $(q, p, t) \in \mathbb{T}^n \times B \times \mathbb{R}$ ,  $\tilde{g}(q, p, t) = (g(q, p, t), p, t)$ . Then  $f \circ \tilde{g} \in \mathcal{B}_\sigma$  and

d.  $|f \circ \tilde{g}|_{\sigma,l+m} \leq C(\sigma) (|f|_{\sigma,l} |g|_{1,m}^\sigma + |f|_{1,l} |g|_{\sigma,m} + |f|_{0,l+m})$ .

Before the proof, we observe that the previous properties are still verified when  $l = m = 0$  or only one of the two parameters  $l$  and  $m$  is zero.

*Proof.* The proof rests on Proposition A.2 (see Appendix A). Properties a. and b. are obvious. Then, we verify the others.

c. For all fixed  $(p, t) \in B \times \mathbb{R}$ , by property 2. of Proposition A.2

$$\begin{aligned} |f_p^t g_p^t|_{C^\sigma} (1 + |t|^{l+m}) &\leq C(\sigma) (|f_p^t|_{C^0} |g_p^t|_{C^\sigma} + |f_p^t|_{C^\sigma} |g_p^t|_{C^0}) (1 + |t|^l) (1 + |t|^m) \\ &\leq C(\sigma) (|f_p^t|_{C^0} (1 + |t|^l) |g_p^t|_{C^\sigma} (1 + |t|^m) \\ &\quad + |f_p^t|_{C^\sigma} (1 + |t|^l) |g_p^t|_{C^0} (1 + |t|^m)) \\ &\leq C(\sigma) (|f|_{0,l} |g|_{\sigma,m} + |f|_{\sigma,l} |g|_{0,m}) \end{aligned}$$

where in the second line we use  $(1 + |t|^{l+m}) \leq (1 + |t|^l) (1 + |t|^m)$ . Taking the sup for all  $(p, t) \in B \times \mathbb{R}$  on the left-hand side of the latter, we obtain

$$\sup_{(p,t) \in B \times \mathbb{R}} |f_p^t g_p^t|_{C^\sigma} (1 + |t|^{l+m}) \leq C(\sigma) (|f|_{0,l} |g|_{\sigma,m} + |f|_{\sigma,l} |g|_{0,m}).$$

It remains to prove that the second term of the norm (see the right-hand side of (5.4)) also satisfies the same estimate. For all fixed  $(p, t) \in B \times \mathbb{R}$

$$\begin{aligned} |(\partial_p (fg))_p^t|_{C^0} (1 + |t|^{l+m-1}) &= |(\partial_p f)_p^t g_p^t + f_p^t (\partial_p g)_p^t|_{C^0} (1 + |t|^{l+m-1}) \\ &\leq (|(\partial_p f)_p^t g_p^t|_{C^0} + |f_p^t (\partial_p g)_p^t|_{C^0}) (1 + |t|^{l+m-1}) \\ &\leq C |(\partial_p f)_p^t|_{C^0} (1 + |t|^{l-1}) |g_p^t|_{C^0} (1 + |t|^m) \\ &\quad + C |(\partial_p g)_p^t|_{C^0} (1 + |t|^{m-1}) |f_p^t|_{C^0} (1 + |t|^l) \\ &\leq C (|f|_{0,l} |g|_{\sigma,m} + |f|_{\sigma,l} |g|_{0,m}) \end{aligned}$$



and taking the sup for all  $(p, t) \in B \times \mathbb{R}$  on the left-hand side of the latter, we prove the estimate.

d. For all fixed  $(p, t) \in B \times \mathbb{R}$  and thanks to property 5. of Proposition A.2

$$\begin{aligned}
 |f_p^t \circ g_p^t|_{C^\sigma} (1 + |t|^{l+m}) &\leq C(\sigma) \left( |f_p^t|_{C^\sigma} |g_p^t|_{C^1}^\sigma + |f_p^t|_{C^1} |g_p^t|_{C^\sigma} + |f_p^t|_{C^0} \right) (1 + |t|^{l+m}) \\
 &\leq C(\sigma) \left( |f_p^t|_{C^\sigma} (1 + |t|^l) |g_p^t|_{C^1}^\sigma (1 + |t|^m)^\sigma \right. \\
 &\quad + |f_p^t|_{C^1} (1 + |t|^l) |g_p^t|_{C^\sigma} (1 + |t|^m) \\
 &\quad \left. + |f_p^t|_{C^0} (1 + |t|^{l+m}) \right) \\
 &\leq C(\sigma) (|f|_{\sigma,l} |g|_{1,m}^\sigma + |f|_{1,l} |g|_{\sigma,m} + |f|_{0,l+m})
 \end{aligned}$$

where in the second line we use  $(1 + |t|^m) \leq (1 + |t|^m)^\sigma$ . Taking the sup for all  $(p, t) \in B_1 \times \mathbb{R}$  on the left-hand side of the latter,

$$\sup_{(p,t) \in B \times \mathbb{R}} |f_p^t \circ g_p^t|_{C^\sigma} (1 + |t|^{l+m}) \leq C(\sigma) (|f|_{\sigma,l} |g|_{1,m}^\sigma + |f|_{1,l} |g|_{\sigma,m} + |f|_{0,l+m}).$$

Concerning the second term of the norm (see (5.4)), for all fixed  $(p, t) \in B \times \mathbb{R}$

$$\begin{aligned}
 \left| (\partial_p (f \circ \tilde{g}))_p^t \right|_{C^0} (1 + |t|^{l+m-1}) &= \left| (\partial_q f)_p^t \circ g_p^t (\partial_p g)_p^t \right|_{C^0} (1 + |t|^{l+m-1}) \\
 &\quad + \left| (\partial_p f)_p^t \circ g_p^t \right|_{C^0} (1 + |t|^{l+m-1}) \\
 &\leq C \left| (\partial_q f)_p^t \right|_{C^0} (1 + |t|^l) \left| (\partial_p g)_p^t \right|_{C^0} (1 + |t|^{m-1}) \\
 &\quad + \left| (\partial_p f)_p^t \right|_{C^0} (1 + |t|^{l+m-1}) \\
 &\leq C (|f|_{1,l} |g|_{\sigma,m} + |f|_{0,l+m}).
 \end{aligned}$$

Taking the sup over  $(p, t) \in B \times \mathbb{R}$  on the left-hand side of the latter,

$$\sup_{(p,t) \in B \times \mathbb{R}} \left| (\partial_p (f \circ \tilde{g}))_p^t \right|_{C^0} (1 + |t|^{l+m-1}) \leq C (|f|_{1,l} |g|_{\sigma,m} + |f|_{0,l+m}).$$

This concludes the proof of this lemma. □

Now, we recall the following space of functions.

**Definition 5.2.** *Given  $\sigma \geq 1$  and an integer  $k \geq 0$ , we define  $\bar{\mathcal{B}}_{\sigma,k}$  the space of functions  $f$  such that*

$$f \in \mathcal{B}_{\sigma+k}, \quad \text{and} \quad \partial_q^i f \in \mathcal{B}_{\sigma+k-i}$$

for all  $0 \leq i \leq k$ .

We observe that  $\mathcal{B}_\sigma$  is the space of functions introduced by Definition 5.1. Furthermore, for all  $f \in \bar{\mathcal{B}}_{\sigma,k}$  and  $l > 1$ , we consider the following norm

$$\|f\|_{\sigma,k,l} = \max_{0 \leq i \leq k} |\partial_q^i f|_{\sigma+k-i,l}, \tag{5.6}$$

where  $|\cdot|_{\sigma,l}$  is the norm defined by (5.4)



**Theorem 5.1.** *Let  $H$  be as in  $(\star)$ . Then, if  $\varepsilon$  is sufficiently small with respect to  $n, l, \Upsilon$  and  $|\omega|_{C^1}$ , for all fixed  $p_0 \in B_{\frac{3}{4}}$ , there exists a positive  $C^\sigma$ -asymptotic KAM torus  $\psi_{+p_0}^t$  and a negative  $C^\sigma$ -asymptotic KAM torus  $\psi_{-p_0}^t$  associated to  $(X_{H_{p_0}}, X_{\tilde{h}_{p_0}}, \psi_{0,p_0})$ . Moreover, letting*

$$\psi_{\pm}^t : \mathbb{T}^n \times B_{\frac{3}{4}} \longrightarrow \mathbb{T}^n \times B_{\frac{1}{4}}, \quad \psi_{\pm}^t(q, p_0) = \psi_{\pm, p_0}^t(q),$$

*there exists a constant  $C_0$  depending on  $n, l, \Upsilon$  and  $|\omega|_{C^1}$  such that*

$$\sup_{t \geq 0} |\psi_{+}^t - \psi_0|_{C^1} < C_0 \varepsilon, \quad \sup_{t \leq 0} |\psi_{-}^t - \psi_0|_{C^1} < C_0 \varepsilon. \quad (5.8)$$

### 5.3 Proof of Theorem C assuming Theorem 5.1

In this section, we assume Theorem 5.1 and we deduce Theorem C. First, we introduce the following well-known property.

**Proposition 5.3.** *Given  $r > 0$  and  $0 < \epsilon < \delta < r$ , let  $\phi$  be a map*

$$\phi : B_r \longrightarrow B_{r+\delta}$$

*of class  $C^1$  such that  $|\phi - \text{Id}|_{C^1} < \epsilon$ . Then, for  $\epsilon$  small enough,  $\phi$  is a diffeomorphism onto its image and*

$$B_{r-\delta} \subset \phi(B_r).$$

For all  $p_0 \in B_{\frac{3}{4}}$ , we let  $p = p_0 + I$  and we expand the Hamiltonian  $(*_C)$  around  $p_0$  so that

$$\begin{aligned} h(p) &= h(p_0) + \partial_p h(p_0) \cdot I + \int_0^1 (1 - \tau) \partial_p^2 h(p_\tau) d\tau \cdot I^2 \\ f(q, p, t) &= f(q, p_0, t) + \partial_p f(q, p_0, t) \cdot I + \int_0^1 (1 - \tau) \partial_p^2 f(q, p_\tau, t) d\tau \cdot I^2 \end{aligned}$$

where  $p_\tau = p_0 + \tau I$  and  $I \in B_{\frac{1}{4}}$ . For any  $p_0 \in B_{\frac{3}{4}}$ , we define

$$\begin{aligned} e(p_0) &= h(p_0) \\ \omega(p_0) &= \partial_p h(p_0) \\ a(q, t; p_0) &= f(q, p_0, t) \\ b(q, t; p_0) &= \partial_p f(q, p_0, t) \\ m(q, I, t; p_0) &= \int_0^1 (1 - \tau) (\partial_p^2 h(p_\tau) + \partial_p^2 f(q, p_\tau, t)) d\tau \\ &= \int_0^1 (1 - \tau) \partial_p^2 H(q, p_\tau, t) d\tau, \end{aligned}$$

for all  $(q, I, t) \in \mathbb{T}^n \times B_{\frac{1}{4}} \times \mathbb{R}$ . Writing  $\theta$  instead of  $q$  for the angular variables, we can rewrite  $H$  in the following form as a family of Hamiltonians parametrized by  $p_0 \in B_{\frac{3}{4}}$ ,

$$\begin{aligned} H : \mathbb{T}^n \times B_{\frac{1}{4}} \times \mathbb{R} \times B_{\frac{3}{4}} &\longrightarrow \mathbb{R} \\ H(\theta, I, t; p_0) &= e(p_0) + \omega(p_0) \cdot I \\ &\quad + a(\theta, t; p_0) + b(\theta, t; p_0) \cdot I + m(\theta, I, t; p_0) \cdot I^2. \end{aligned} \quad (5.9)$$

We consider the following family of Hamiltonians

$$\tilde{h}(\theta, I, t; p_0) = e(p_0) + \omega(p_0) \cdot I + m(\theta, I, t; p_0) \cdot I^2$$

for all  $(\theta, I, t; p_0) \in \mathbb{T}^n \times B_{\frac{1}{4}} \times \mathbb{R} \times B_{\frac{3}{4}}$ . For each fixed  $p_0 \in B_{\frac{3}{4}}$ ,  $X_H$  and  $X_{\tilde{h}}$  satisfy (5.1). Furthermore, for all  $p_0 \in B_{\frac{3}{4}}$ , it is obvious that  $\tilde{h}$  has an invariant torus supporting quasiperiodic dynamics.

It is straightforward to verify that the Hamiltonian  $H$  defined by (5.9) satisfies the hypotheses of Theorem 5.1. Then, there exists a family of positive  $C^\sigma$ -asymptotic KAM tori

$$\psi_+^t : \mathbb{T}^n \times B_{\frac{3}{4}} \longrightarrow \mathbb{T}^n \times B_{\frac{1}{4}}$$

associated to  $(X_H, X_{\tilde{h}}, \psi_0)$  and a family of negative  $C^\sigma$ -asymptotic KAM tori

$$\psi_-^t : \mathbb{T}^n \times B_{\frac{3}{4}} \longrightarrow \mathbb{T}^n \times B_{\frac{1}{4}}$$

associated to  $(X_H, X_{\tilde{h}}, \psi_0)$ , where  $\psi_0$  is the family of trivial embeddings introduced by (5.7). Moreover, we have

$$\sup_{t \geq 0} |\psi_+^t - \psi_0|_{C^1} < C_0 \varepsilon, \quad \sup_{t \leq 0} |\psi_-^t - \psi_0|_{C^1} < C_0 \varepsilon,$$

where  $C_0$  is a constant depending on  $n, l, \Upsilon$  and  $|\omega|_{C^1}$ .

Therefore, by the latter, there exist  $u_\pm^t, v_\pm^t : \mathbb{T}^n \times B_{\frac{3}{4}} \rightarrow \mathbb{R}^n$  such that we can rewrite  $\psi_+^t$  and  $\psi_-^t$  in the following form

$$\psi_\pm^t(\theta, p_0) = (\theta + u_\pm^t(\theta, p_0), v_\pm^t(\theta, p_0))$$

for all  $(\theta, p_0) \in \mathbb{T}^n \times B_{\frac{3}{4}}$  with

$$\sup_{t \geq 0} |u_+^t|_{C^1} < C_0 \varepsilon, \quad \sup_{t \geq 0} |v_+^t|_{C^1} < C_0 \varepsilon, \quad \sup_{t \leq 0} |u_-^t|_{C^1} < C_0 \varepsilon, \quad \sup_{t \leq 0} |v_-^t|_{C^1} < C_0 \varepsilon.$$

By construction, an orbit  $(\theta(t), I(t))$  for the previous Hamiltonian at the parameter value  $p_0 \in B_{\frac{3}{4}}$  translates into a trajectory  $(q(t), p(t)) = (\theta(t), p_0 + I(t))$  for the Hamiltonian in  $(q, p)$ -coordinates. Then, letting

$$\varphi_0 : \mathbb{T}^n \times B_{\frac{3}{4}} \longrightarrow \mathbb{T}^n \times B_1, \quad \varphi_0(q, p_0) = (q, p_0), \quad (5.10)$$

the following family of maps

$$\begin{aligned} \varphi_\pm^t &: \mathbb{T}^n \times B_{\frac{3}{4}} \longrightarrow \mathbb{T}^n \times B_1, \\ \varphi_\pm^t(q, p_0) &= (q + u_\pm^t(q, p_0), p_0 + v_\pm^t(q, p_0)) \end{aligned}$$

is a family of positive (resp. negative)  $C^\sigma$ -asymptotic KAM tori associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ . In other words, for all  $p_0 \in B_{\frac{3}{4}}$ ,  $\varphi_{+p_0}^t$  (resp.  $\varphi_{-p_0}^t$ ) is a positive (resp. negative)  $C^\sigma$ -asymptotic KAM torus associated to  $(X_H, X_{\tilde{h}}, \varphi_{0,p_0})$ . Thanks to Proposition 5.3,

$$\mathbb{T}^n \times B_{\frac{1}{2}} \subset \varphi_\pm^0(\mathbb{T}^n \times B_{\frac{3}{4}}).$$

This concludes the proof of the theorem because, for all  $(q, p_0) \in \mathbb{T}^n \times B_{\frac{1}{2}}$ , there exist  $(q_+, p_{0+}), (q_-, p_{0-}) \in \mathbb{T}^n \times B_{\frac{3}{4}}$  such that

$$\varphi_+^0(q_+, p_{0+}) = (q, p_0) = \varphi_-^0(q_-, p_{0-}).$$

Then, by Proposition 5.1 there exists a biasymptotically quasiperiodic solution  $g(t)$  associated to  $(X_H, X_{\tilde{h}}, \varphi_{0,p_{0\pm}})$  such that  $g(0) = (q, p_0)$ .

## 5.4 Proof of Theorem 5.1

The proof of Theorem 5.1 is essentially the same as Theorem A with suitable modifications, especially in the section dedicated to the homological equation. In the third chapter of this thesis, we show the existence of a positive  $C^\sigma$ -asymptotic KAM torus for a suitable time-dependent Hamiltonian. Here, the Hamiltonian in  $(\star)$  consists of a family of Hamiltonians parametrized by  $p_0 \in B_{\frac{3}{4}}$ . This section aims to prove the existence of a positive  $C^\sigma$ -asymptotic KAM torus for each  $p_0 \in B_{\frac{3}{4}}$ . Similarly, we have the claim concerning the existence of a family of negative  $C^\sigma$ -asymptotic KAM tori. Therefore, we need these families to satisfy (5.8). This control on the variables  $(\theta, p_0)$  is the reason why we need to assume a stronger time decay for  $a$  and  $b$  in  $(\star)$  than that of Theorem A.

### 5.4.1 Outline of the proof of Theorem 5.1

We are looking for a family of positive  $C^\sigma$ -asymptotic KAM tori  $\psi^t$  associated to  $(X_H, X_{\tilde{h}}, \psi_0)$ , where we drop the subscript  $+$  in order to obtain a more elegant form. Here,  $H$  is the Hamiltonian in  $(\star)$  and  $\psi_0$  is the following family of trivial embeddings

$$\psi_0 : \mathbb{T}^n \times B_{\frac{3}{4}} \rightarrow \mathbb{T}^n \times B_{\frac{1}{4}}, \quad \psi_0(\theta, p_0) = (\theta, 0).$$

More specifically, we are looking for  $u, v : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R}^n$  such that

$$\psi(\theta, t; p_0) = (\theta + u(\theta, t; p_0), v(\theta, t; p_0))$$

and, for all fixed  $p_0 \in B_{\frac{3}{4}}$ ,  $\psi$ ,  $u$  and  $v$  satisfy the following conditions

$$X_H(\psi(\theta, t; p_0), t; p_0) - \partial_\theta \psi(\theta, t; p_0) \omega(p_0) - \partial_t \psi(\theta, t; p_0) = 0 \quad (5.11)$$

$$\lim_{t \rightarrow +\infty} |u_{p_0}^t|_{C^\sigma} = 0, \quad \lim_{t \rightarrow +\infty} |v_{p_0}^t|_{C^\sigma} = 0, \quad (5.12)$$

for all  $(\theta, t) \in \mathbb{T}^n \times \mathbb{R}^+$ .

As mentioned before, this proof relies on the implicit function theorem. As we did in the previous chapters, we introduce a suitable functional  $\mathcal{F}$  given by (5.11). To this end, we recall the following definitions. For all  $(\theta, I, t; p_0) \in \mathbb{T}^n \times B_{\frac{1}{4}} \times \mathbb{R}^+ \times B_{\frac{3}{4}}$ ,

$$\bar{m}(\theta, I, t; p_0) I = \left( \int_0^1 \partial_p^2 H(\theta, p_0 + \tau I, t) d\tau \right) I = \partial_I \left( m(\theta, I, t; p_0) \cdot I^2 \right),$$

$$\tilde{\psi}(\theta, t; p_0) = (\psi(\theta, t; p_0), t; p_0) = (\theta + u(\theta, t; p_0), v(\theta, t; p_0), t; p_0),$$

$$\tilde{u}(\theta, t; p_0) = (\theta + u(\theta, t; p_0), t; p_0)$$

$$\nabla_{\theta t} u(\theta, t; p_0) \Omega(p_0) = \partial_\theta u(\theta, t; p_0) \omega(p_0) + \partial_t u(\theta, t; p_0),$$

$$\nabla_{\theta t} v(\theta, t; p_0) \Omega(p_0) = \partial_\theta v(\theta, t; p_0) \omega(p_0) + \partial_t v(\theta, t; p_0).$$

Similarly to the proof of Theorem A, we can rewrite (5.11) in the following form (see Section 3.3.1)

$$\begin{pmatrix} b \circ \tilde{u} + \left( \bar{m} \circ \tilde{\psi} \right) v - (\nabla_{\theta t} u) \Omega \\ -\partial_\theta a \circ \tilde{u} - (\partial_\theta b \circ \tilde{u}) v - \left( \partial_\theta m \circ \tilde{\psi} \right) \cdot v^2 - (\nabla_{\theta t} v) \Omega \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (5.13)$$

This is composed by sums and products of functions defined of  $(\theta, t; p_0) \in \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}}$ , we have omitted the arguments  $(\theta, t; p_0)$  in order to achieve a more elegant form. Over suitable Banach spaces, that we will specify later, let  $\mathcal{F}$  be the following functional

$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v)),$$

with

$$\begin{aligned} F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + \left( \bar{m} \circ \tilde{\psi} \right) v - (\nabla_{\theta t} u) \Omega, \\ F_2(a, b, m, u, v) &= \partial_{\theta} a \circ \tilde{u} + (\partial_{\theta} b \circ \tilde{u}) v + \left( \partial_{\theta} m \circ \tilde{\psi} \right) \cdot v^2 + (\nabla_{\theta t} v) \Omega \end{aligned}$$

which is obtained by (5.13). We can reformulate our problem in the following form. For fixed  $m$  and  $\bar{m}$  in a suitable Banach space and for  $(a, b)$  sufficiently close to  $(0, 0)$ , we are looking for  $u, v : \mathbb{T}^n \times \mathbb{R}^+ \times B_1 \rightarrow \mathbb{R}^n$  satisfying (5.12) such that  $\mathcal{F}(a, b, m, \bar{m}, u, v) = 0$ .

Here, contrary to Theorem A, we can not get rid of the smallness assumption over  $a$  and  $b$  (see  $(\star)$ ). This is because we are looking for solutions  $u$  and  $v$  with  $|u^t|_{C^1}$  and  $|v^t|_{C^1}$  sufficiently small for all  $t \in \mathbb{R}^+$  and not only for  $t$  sufficiently large.

Concerning the associated linearized problem, the differential of  $\mathcal{F}$  with respect to the variables  $(u, v)$  calculated in  $(0, 0, m, \bar{m}, 0, 0)$  is equal to

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (\bar{m}_0 \hat{v} - (\nabla_{\theta t} \hat{u}) \Omega, (\nabla_{\theta t} \hat{v}) \Omega),$$

where, for all  $(\theta, t, p_0) \in \mathbb{T}^n \times B_{\frac{1}{4}} \times B_{\frac{3}{4}}$ , we let  $\bar{m}_0(\theta, t; p_0) = \bar{m}_0(\theta, 0, t; p_0)$ . As one can expect, this differential is invertible.

In the following four sections we prove Theorem 5.1. First, we introduce suitable Banach spaces on which the functional  $\mathcal{F}$  is defined. In the second section, we solve the homological equation, which is the key to prove that the latter operator is invertible. In the third section, we verify that  $\mathcal{F}$  satisfies the hypotheses of the implicit function theorem. Finally, we verify the estimates and we conclude the proof of the theorem.

### 5.4.2 Preliminary Settings

We begin this section by introducing some notations, spaces of functions and suitable norms. Given  $\sigma \geq 1$ , we have the following definition

**Definition 5.3.** Let  $\mathcal{B}_{\sigma}^+$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}}$  such that  $f, \partial_{p_0} f \in C(\mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}})$  and  $f_{p_0}^t \in C^{\sigma}(\mathbb{T}^n)$  for all  $(t, p_0) \in \mathbb{R}^+ \times B_{\frac{3}{4}}$ .

For all  $f \in \mathcal{B}_{\sigma}^+$  and  $l > 1$ , we define the following norms

$$\begin{aligned} |f|_{\sigma, l}^+ &= \sup_{(t, p_0) \in \mathbb{R}^+ \times B_{\frac{3}{4}}} |f_{p_0}^t|_{C^{\sigma}} (1 + t^l) + \sup_{(t, p_0) \in \mathbb{R}^+ \times B_{\frac{3}{4}}} |(\partial_{p_0} f)_{p_0}^t|_{C^0} (1 + t^{l-1}), \\ |f|_{\sigma, 0}^+ &= \sup_{(t, p_0) \in \mathbb{R}^+ \times B_{\frac{3}{4}}} |f_{p_0}^t|_{C^{\sigma}} + \sup_{(t, p_0) \in \mathbb{R}^+ \times B_{\frac{3}{4}}} |(\partial_{p_0} f)_{p_0}^t|_{C^0}. \end{aligned}$$

These norms satisfy the properties in Proposition 5.2. As one can expect, we define the following subset of  $\mathcal{B}_{\sigma}^+$

**Definition 5.4.** Given  $\sigma \geq 1$  and an integer  $k \geq 0$ , we define  $\bar{\mathcal{B}}_{\sigma,k}^+$  the space of functions  $f$  such that

$$f \in \mathcal{B}_{\sigma+k}^+, \text{ and } \partial_q^i f \in \mathcal{B}_{\sigma+k-i}^+$$

for all  $0 \leq i \leq k$ .

We conclude this part of settings with the following norm. For all  $f \in \bar{\mathcal{B}}_{\sigma,k}^+$  and  $l > 1$ , we define

$$\|f\|_{\sigma,k,l}^+ = \max_{0 \leq i \leq k} |\partial_q^i f|_{\sigma+k-i,l}^+.$$

Now, let  $\sigma \geq 1$  and  $l > 1$  be the positive parameters introduced by  $(\star)$ . We consider the following Banach spaces  $(\mathcal{A}, |\cdot|)$ ,  $(\mathcal{B}, |\cdot|)$ ,  $(\mathcal{U}, |\cdot|)$ ,  $(\mathcal{V}, |\cdot|)$ ,  $(\mathcal{Z}, |\cdot|)$  and  $(\mathcal{G}, |\cdot|)$

$$\begin{aligned} \mathcal{A} &= \left\{ a : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R} \mid a \in \bar{\mathcal{B}}_{\sigma,2}^+ \text{ and } |a| = |a|_{\sigma+2,0}^+ + \|\partial_\theta a\|_{\sigma,1,l+2}^+ < \infty \right\} \\ \mathcal{B} &= \left\{ b : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R}^n \mid b \in \bar{\mathcal{B}}_{\sigma,2}^+, \text{ and } |b| = \|b\|_{\sigma,2,l+1}^+ < \infty \right\} \\ \mathcal{U} &= \left\{ u : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R}^n \mid u, (\nabla_{\theta t} u) \Omega \in \mathcal{B}_\sigma^+ \right. \\ &\quad \left. \text{and } |u| = \max\{|u|_{\sigma,l}^+, |(\nabla_{\theta t} u) \Omega|_{\sigma,l+1}^+\} < \infty \right\} \\ \mathcal{V} &= \left\{ v : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R}^n \mid v, (\nabla_{\theta t} v) \Omega \in \mathcal{B}_\sigma^+ \right. \\ &\quad \left. \text{and } |v| = \max\{|v|_{\sigma,l+1}^+, |(\nabla_{\theta t} v) \Omega|_{\sigma,l+2}^+\} < \infty \right\} \\ \mathcal{Z} &= \left\{ z : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R}^n \mid z \in \mathcal{B}_\sigma^+, \text{ and } |z| = |z|_{\sigma,l+1}^+ < \infty \right\} \\ \mathcal{G} &= \left\{ g : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R}^n \mid g \in \mathcal{B}_\sigma^+, \text{ and } |g| = |g|_{\sigma,l+2}^+ < \infty \right\} \end{aligned}$$

Similarly to what we did in Appendix C, it is straightforward to verify that the previous normed spaces are Banach spaces. Let  $M_n$  be the set of the  $n$ -dimensional matrices and  $\Upsilon \geq 1$  the positive parameter in  $(\star)$ . We introduce another Banach space  $(\mathcal{M}, |\cdot|)$ , such that

$$\begin{aligned} \mathcal{M} &= \left\{ m : \mathbb{T}^n \times B_{\frac{1}{4}} \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow M_n \mid \partial_{\theta I p_0}^i m \in C(\mathbb{T}^n \times B_{\frac{1}{4}} \times \mathbb{R}^+ \times B_{\frac{3}{4}}) \right. \\ &\quad \left. \text{for all } 0 \leq i \leq 3, m^t \in C^{\sigma+2}(\mathbb{T}^n \times B_{\frac{1}{4}} \times B_{\frac{3}{4}}) \text{ for all fixed } t \in \mathbb{R}^+ \right. \\ &\quad \left. \text{and } |m| = \sup_{t \in \mathbb{R}^+} |m^t|_{C^{\sigma+2}} \leq \Upsilon \right\} \end{aligned}$$

Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{Z} \times \mathcal{G}$$

$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + \left( \bar{m} \circ \tilde{\psi} \right) v - (\nabla_{\theta t} u) \Omega, \\ F_2(a, b, m, u, v) &= \partial_\theta a \circ \tilde{u} + (\partial_\theta b \circ \tilde{u}) v + \left( \partial_\theta m \circ \tilde{\psi} \right) \cdot v^2 + (\nabla_{\theta t} v) \Omega. \end{aligned}$$

The following section is devoted to the solution of the homological equation. Meanwhile, in the last part of the proof, we verify that  $\mathcal{F}$  is well defined and it satisfies the hypotheses of the implicit function theorem.

### 5.4.3 Homological equation

Before analyzing the homological equation, let us prove the following estimates

**Lemma 5.1.** *Given  $m > 1$ ,*

$$\int_t^{+\infty} \frac{1}{1+\tau^m} d\tau \leq \frac{C(m)}{1+t^{m-1}}, \quad \int_t^{+\infty} \frac{\tau-t}{1+\tau^{m+1}} d\tau \leq \frac{C(m)}{1+t^{m-1}} \quad (5.14)$$

for all  $t \geq 0$  and some constants  $C(m)$  depending on  $m$ .

*Proof.* We define the following function  $f_m : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that

$$f_m(t) = (1+t^{m-1}) \int_t^{+\infty} \frac{1}{1+\tau^m} d\tau.$$

It is straightforward to verify that  $f$  is continuous. We will prove the existence of a constant  $C(m)$ , depending on  $m$ , such that  $f_m(t) \leq C(m)$  for all  $t \geq 0$ , which implies the first estimate in (5.14). It suffices to prove that there exists  $\lim_{t \rightarrow +\infty} f_m(t)$  and it is finite. Thanks to l'Hôpital's rule

$$\lim_{t \rightarrow +\infty} f_m(t) = \lim_{t \rightarrow +\infty} \frac{\frac{d}{dt} \int_t^{+\infty} \frac{1}{1+\tau^m} d\tau}{\frac{d}{dt} \frac{1}{1+t^{m-1}}} = \lim_{t \rightarrow +\infty} \frac{(1+t^{m-1})^2}{(m-1)t^{m-2}(1+t^m)} = \frac{1}{m-1}.$$

Concerning the second inequality in (5.14), similarly to the previous case, we define the following function  $g_m : \mathbb{R}^+ \rightarrow \mathbb{R}$  such that

$$g_m(t) = (1+t^{m-1}) \int_t^{+\infty} \frac{\tau-t}{1+\tau^{m+1}} d\tau.$$

One can see that  $g_m$  is continuous. We have to verify that there exists  $\lim_{t \rightarrow +\infty} g_m(t)$  and it is finite. Applying l'Hôpital's rule twice

$$\begin{aligned} \lim_{t \rightarrow +\infty} g_m(t) &= \lim_{t \rightarrow +\infty} \frac{\frac{d}{dt} \int_t^{+\infty} \frac{\tau-t}{1+\tau^{m+1}} d\tau}{\frac{d}{dt} \frac{1}{1+t^{m-1}}} = \lim_{t \rightarrow +\infty} \frac{\int_t^{+\infty} \frac{1}{1+\tau^m} d\tau}{\frac{(m-1)t^{m-2}}{(1+t^{m-1})^2}} = \lim_{t \rightarrow +\infty} \frac{\frac{d}{dt} \int_t^{+\infty} \frac{1}{1+\tau^m} d\tau}{\frac{d}{dt} \frac{(m-1)t^{m-2}}{(1+t^{m-1})^2}} \\ &= \lim_{t \rightarrow +\infty} \frac{(1+t^{m-1})^4}{(1+t^{m+1})t^{3m-5}h_m(t)} \end{aligned}$$

where

$$h_m(t) = (m-1) \left( 2(m-1) \left( \frac{1}{t^{m-1}} + 1 \right) - (m-2) \left( \frac{1}{t^{m-1}} + 1 \right)^2 \right).$$

Then, by the latter

$$\lim_{t \rightarrow +\infty} g_m(t) = \lim_{t \rightarrow +\infty} \frac{(1+t^{m-1})^4}{(1+t^{m+1})t^{3m-5}h_m(t)} = \lim_{t \rightarrow +\infty} \frac{\left(\frac{1}{t^{m-1}} + 1\right)^4}{\left(\frac{1}{t^{m+1}} + 1\right)h_m(t)} = \frac{1}{m(m-1)}.$$

□



Given  $\sigma \geq 1$  and  $l > 1$ , we consider the following equation in the unknown  $\varkappa : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{R}$

$$\begin{cases} \omega(p_0) \cdot \partial_q \varkappa(\theta, t; p_0) + \partial_t \varkappa(\theta, t; p_0) = g(\theta, t; p_0), \\ g \in \mathcal{B}_\sigma^+, \quad |g|_{\sigma, l+1}^+ < \infty, \\ \omega : B_{\frac{3}{4}} \rightarrow \mathbb{R}^n, \quad \omega \in C^1(B_{\frac{3}{4}}). \end{cases} \quad (HE_C)$$

**Lemma 5.2 (Homological Equation).** *There exists a unique solution  $\varkappa \in \mathcal{B}_\sigma^+$  of  $(HE_C)$  such that, for all fixed  $p_0 \in B_{\frac{3}{4}}$ ,*

$$\lim_{t \rightarrow +\infty} |\varkappa_{p_0}^t|_{C^0} = 0.$$

Moreover,

$$|\varkappa|_{\sigma, l}^+ \leq C |g|_{\sigma, l+1}^+ \quad (5.15)$$

for a suitable constant  $C$  depending on  $n, l$  and  $|\omega|_{C^1}$ .

*Proof. Existence:* We sketch this first part because it is similar to Lemma 3.1 (see Section 3.3.3). Let us define the following transformation

$$h : \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}} \rightarrow \mathbb{T}^n \times \mathbb{R}^+ \times B_{\frac{3}{4}}, \quad h(\theta, t; p_0) = (\theta - \omega(p_0)t, t; p_0).$$

It is enough to prove the first part of this lemma for the much simpler equation

$$\partial_t \kappa(\theta, t; p_0) = g(\theta + \omega(p_0)t, t; p_0).$$

The unique solution of the above equation satisfying the asymptotic condition is equal to

$$\kappa(\theta, t; p_0) = - \int_t^{+\infty} g(\theta + \omega(p_0)\tau, \tau; p_0) d\tau$$

and hence composing  $k$  with  $h$

$$\varkappa(\theta, t; p_0) = \kappa \circ h(\theta, t; p_0) = - \int_t^{+\infty} g(\theta + \omega(p_0)(\tau - t), \tau, p_0) d\tau \quad (5.16)$$

is the unique solution of  $(HE_C)$  that we are looking for.

*Regularity and Estimates:*  $g \in \mathcal{B}_\sigma^+$  implies  $\kappa \in \mathcal{B}_\sigma^+$  and thus  $\varkappa = \kappa \circ h \in \mathcal{B}_\sigma^+$ . Now, we have to verify the estimate (5.15). By (5.16) and Lemma 5.1

$$|\varkappa_{p_0}^t|_{C^\sigma} \leq \int_t^{+\infty} |g_{p_0}^\tau|_{C^\sigma} d\tau \leq |g|_{\sigma, l+1}^+ \int_t^{+\infty} \frac{1}{1 + \tau^{l+1}} d\tau \leq C(l) \frac{|g|_{\sigma, l+1}^+}{1 + t^l},$$

for all fixed  $(t, p_0) \in \mathbb{R}^+ \times B_{\frac{3}{4}}$ . Multiplying both sides of the latter by  $1 + t^l$  and taking the sup for all  $\mathbb{R}^+ \times B_{\frac{3}{4}}$ , we obtain

$$\sup_{(t, p_0) \in \mathbb{R}^+ \times B_{\frac{3}{4}}} |\varkappa_{p_0}^t|_{C^\sigma} (1 + t^l) \leq C(l) |g|_{\sigma, l+1}^+.$$

It remains to estimate the second member of the norm  $|\cdot|_{\sigma,l}^+$ . The partial derivative of  $\varkappa$  with respect to  $p_0$  is equal to

$$\begin{aligned}\partial_{p_0}\varkappa(\theta, t; p_0) &= - \int_t^{+\infty} \partial_{p_0}\omega(p_0)\partial_\theta g(\theta + \omega(p_0)(\tau - t), \tau; p_0)(\tau - t)d\tau \\ &\quad - \int_t^{+\infty} \partial_{p_0}g(\theta + \omega(p_0)(\tau - t), \tau; p_0)d\tau.\end{aligned}$$

Then, thanks to Lemma 5.1, we can estimate the norm  $C^0$  on the left-hand side of the latter as follows

$$\begin{aligned}|\partial_{p_0}\varkappa_{p_0}^t|_{C^0} &\leq \int_t^{+\infty} |g_{p_0}^\tau|_{C^1}|\omega|_{C^1}(\tau - t) + |\partial_{p_0}g^\tau|_{C^0}d\tau \\ &\leq |g|_{1,l+1}^+|\omega|_{C^1} \int_t^{+\infty} \frac{(\tau - t)}{1 + \tau^{l+1}}d\tau + |g|_{1,l+1}^+ \int_t^{+\infty} \frac{1}{1 + \tau^l}d\tau, \\ &\leq C(l, |\omega|_{C^1}) \frac{|g|_{1,l+1}^+}{1 + t^{l-1}}.\end{aligned}$$

Similarly to the previous case, by multiplying both sides of the previous inequality by  $1 + t^{l-1}$  and taking the sup for all  $\mathbb{R}^+ \times B_{\frac{3}{4}}$ , we have

$$\sup_{(t, p_0) \in \mathbb{R}^+ \times B_1} |\partial_{p_0}\varkappa_{p_0}^t|_{C^0}(1 + t^{l-1}) \leq C(l, |\omega|_{C^1})|g|_{1,l+1}^+.$$

Moreover, reminding that  $|g|_{1,l+1}^+ \leq |g|_{\sigma,l+1}^+$ , we conclude the proof of this lemma.  $\square$

#### 5.4.4 Regularity of $\mathcal{F}$

The content of this section is essentially the same as that of Theorem A (see Section 3.3.4). By Proposition 5.2,  $\mathcal{F}$  is well defined, continuous, differentiable with respect to the variables  $(u, v)$  and this differential  $D_{(u,v)}\mathcal{F}$  is continuous. As we have already seen,  $D_{(u,v)}\mathcal{F}$  calculated in  $(0, 0, m, \bar{m}, 0, 0)$  is equal to

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (\bar{m}_0\hat{v} - (\nabla_{\theta t}\hat{u})\Omega, (\nabla_{\theta t}\hat{v})\Omega). \quad (5.17)$$

It remains to verify that, for all fixed  $m, \bar{m} \in \mathcal{M}$ , the latter is invertible.

**Lemma 5.3.** *For all  $(z, g) \in \mathcal{Z} \times \mathcal{G}$  there exists a unique  $(\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V}$  such that*

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (z, g).$$

Moreover, for a suitable constant  $\bar{C}$  depending on  $n, l$  and  $|\omega|_{C^1}$

$$|\hat{u}| \leq \bar{C} \left( |\bar{m}_0|_{\sigma,0}^+ |g|_{\sigma,l+2}^+ + |z|_{\sigma,l+1}^+ \right), \quad |\hat{v}| \leq \bar{C} |g|_{\sigma,l+1}^+, \quad (5.18)$$

where we recall that  $|u| = \max\{|u|_{\sigma,l}^+, |(\nabla_{\theta t}u)\Omega|_{\sigma,l+1}^+\}$   
and  $|v| = \max\{|v|_{\sigma,l+1}^+, |(\nabla_{\theta t}v)\Omega|_{\sigma,l+2}^+\}$ .

*Proof.* The key of the proof is Lemma 5.2. By (5.17), the proof consists in searching for the unique solution to the following system

$$\begin{cases} \bar{m}_0 \hat{v} - (\nabla_{\theta t} \hat{u}) \Omega = z \\ (\nabla_{\theta t} \hat{v}) \Omega = g. \end{cases} \quad (5.19)$$

Thanks to Lemma 5.2, a unique solution  $\hat{v}$  for the last equation of the above system exists and

$$|\hat{v}|_{\sigma, l+1}^+ \leq C(|\omega|_{C^1}, l) |g|_{\sigma, l+2}^+. \quad (5.20)$$

Moreover, by  $|(\nabla_{\theta t} \hat{v}) \Omega|_{\sigma, l+2}^+ = |g|_{\sigma, l+2}^+$ , we obtain the following estimate

$$|\hat{v}| = \max\{|\hat{v}|_{\sigma, l+1}^+, |(\nabla_{\theta t} \hat{v}) \Omega|_{\sigma, l+2}^+\} \leq C(|\omega|_{C^1}, l) |g|_{\sigma, l+1}^+.$$

Now, it remains to solve the first equation of the system (5.20) where  $\hat{v}$  is known. We can rewrite this equation in the following form

$$(\nabla_{\theta t} \hat{u}) \Omega = \bar{m}_0 \hat{v} - z. \quad (5.21)$$

By Proposition 5.2 and (5.20), we can estimate the norm  $|\cdot|_{\sigma, l+1}$  on the right-hand side of the latter as follows

$$|\bar{m}_0 \hat{v} - z|_{\sigma, l+1}^+ \leq |\bar{m}_0|_{\sigma, 0}^+ |\hat{v}|_{\sigma, l+1}^+ + |z|_{\sigma, l+1}^+ \leq C(|\omega|_{C^1}, l) |\bar{m}_0|_{\sigma, 0}^+ |g|_{\sigma, l+2}^+ + |z|_{\sigma, l+1}^+.$$

This implies

$$|(\nabla_{\theta t} \hat{u}) \Omega|_{\sigma, l+1}^+ = |\bar{m}_0 \hat{v} - z|_{\sigma, l+1}^+ \leq C(|\omega|_{C^1}, l) |\bar{m}_0|_{\sigma, 0}^+ |g|_{\sigma, l+2}^+ + |z|_{\sigma, l+1}^+$$

and thanks to Lemma 5.2, a unique solution of (5.21) exists satisfying

$$|\hat{u}|_{\sigma, l}^+ \leq C(|\omega|_{C^1}, l) \left( |\bar{m}_0|_{\sigma, 0}^+ |g|_{\sigma, l+2}^+ + |z|_{\sigma, l+1}^+ \right).$$

This concludes the proof of this lemma because

$$|\hat{u}| = \max\{|\hat{u}|_{\sigma, l}^+, |(\nabla_{\theta t} \hat{u}) \Omega|_{\sigma, l+1}^+\} \leq C(|\omega|_{C^1}, l) \left( |\bar{m}_0|_{\sigma, 0}^+ |g|_{\sigma, l+2}^+ + |z|_{\sigma, l+1}^+ \right).$$

□

#### 5.4.5 Families of $C^\sigma$ -asymptotic tori

The functional  $\mathcal{F}$  satisfies the hypotheses of the implicit function theorem. Then, for  $\varepsilon$  small enough, there exists a family of positive  $C^\sigma$ -asymptotic KAM tori

$$\psi_+^t : \mathbb{T}^n \times B_{\frac{3}{4}} \longrightarrow \mathbb{T}^n \times B_{\frac{1}{4}}$$

associated to  $(X_H, X_{\tilde{h}}, \psi_0)$ , where  $\psi_0$  is the following family of trivial embeddings

$$\psi_0 : \mathbb{T}^n \times B_{\frac{3}{4}} \rightarrow \mathbb{T}^n \times B_{\frac{1}{4}}, \quad \psi_0(\theta, p_0) = (\theta, 0).$$

Similarly, for negative times. It remains to verify the estimates (5.8). Let us state a quantitative version of the implicit function theorem.

**Theorem 5.2** (Implicit Function Theorem). *Let  $(\mathcal{X}, |\cdot|)$ ,  $(\mathcal{Y}, |\cdot|)$  and  $(\mathcal{Z}, |\cdot|)$  be Banach spaces. For some  $(x_0, y_0) \in \mathcal{X} \times \mathcal{Y}$  and  $\varepsilon, \mu > 0$ , we introduce the following spaces*

$$\mathcal{X}_0 = \{x \in \mathcal{X} : |x - x_0| \leq \varepsilon\}, \quad \mathcal{Y}_0 = \{y \in \mathcal{Y} : |y - y_0| \leq \mu\}.$$

We assume that

$$\mathcal{F} : \mathcal{X}_0 \times \mathcal{Y}_0 \longrightarrow \mathcal{Z}$$

is continuous and has the property that  $D_y \mathcal{F}$  exists and is continuous at each point of  $\mathcal{X}_0 \times \mathcal{Y}_0$ . Moreover,  $D_y \mathcal{F}(x_0, y_0)$  is invertible and

$$\sup_{x \in \mathcal{X}_0} |D_y \mathcal{F}(x_0, y_0)^{-1} \mathcal{F}(x, y_0)| \leq \frac{\mu}{2} \quad (5.22)$$

$$\sup_{(x,y) \in \mathcal{X}_0 \times \mathcal{Y}_0} \|\text{Id} - D_y \mathcal{F}(x_0, y_0)^{-1} D_y \mathcal{F}(x, y)\| \leq \frac{1}{2} \quad (5.23)$$

where  $\text{Id} \in M_n$  is the identity matrix and  $\|\cdot\|$  stands for the operator norm. Then there exists a unique  $g \in C(\mathcal{X}_0, \mathcal{Y}_0)$  such that

$$g(x_0) = y_0 \quad \text{and} \quad \mathcal{F}(g(x), x) = 0$$

for all  $x \in \mathcal{X}_0$ .

*Proof.* We refer to [Chi97]. □

We observe that condition (5.22) tells us how to choose  $\mu$  as a function of  $\varepsilon$ . Then (5.23) determines a threshold for  $\varepsilon$ . In order to conclude the proof of Theorem 5.1, we have to establish the relation between  $\varepsilon$  and  $\mu$ .

**Lemma 5.4.** *The estimates (5.8) are satisfied.*

*Proof.* The proof relies on Lemma 5.3 and (5.22). For all fixed  $\bar{m}, m \in \mathcal{M}$  and for all  $(a, b) \in \mathcal{A} \times \mathcal{B}$ ,

$$D_{(u,v)} \mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1} \mathcal{F}(a, b, m, \bar{m}, 0, 0) = D_{(u,v)} \mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1} \begin{pmatrix} b \\ \partial_{\theta} a \end{pmatrix}.$$

We want to estimate the right-hand side of the latter. Therefore, we can reformulate this problem in terms of estimating the unique solution  $(\hat{u}, \hat{v}) \in \mathcal{Y}$  of the following system

$$D_{(u,v)} \mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = \begin{pmatrix} b \\ \partial_{\theta} a \end{pmatrix}.$$

By Lemma 5.3, this solution exists and

$$\begin{aligned} |\hat{u}| &\leq \bar{C} (\Upsilon |\partial_{\theta} a|_{\sigma, l+2}^+ + |b|_{\sigma, l+1}^+) \leq \bar{C} \Upsilon \varepsilon \\ |\hat{v}| &\leq \bar{C} |\partial_{\theta} a|_{\sigma, l+2}^+ \leq \bar{C} \varepsilon, \end{aligned}$$

where  $\bar{C}$  is a constant depending on  $n, l$  and  $|\omega|_{C^1}$ . Using the notation of Theorem 5.2, by (5.18), we can choose  $\mu = 2\bar{C}\Upsilon\varepsilon$ .

Now, we observe that for all fixed  $\bar{m}, m \in \mathcal{M}$

$$\text{Id} - D_{(u,v)} \mathcal{F}(0, 0, m, \bar{m}, 0, 0)^{-1} D_{(u,v)} \mathcal{F}(a, b, m, \bar{m}, u, v)$$

is continuous with respect to  $(a, b, u, v) \in \mathcal{A} \times \mathcal{B} \times \mathcal{U} \times \mathcal{V}$ . Then, thanks to Lemma 5.3, there exists  $\varepsilon_0$  depending on  $n, l, \Upsilon$  and  $|\omega|_{C^1}$  such that, for all  $\varepsilon \leq \varepsilon_0$ , (5.23) is satisfied. □

## 5.5 Result (Near Integrable Case)

Let  $A \subset \mathbb{R}^n$  be a closed set and  $E$  equal to  $\mathbb{T}^n$  or  $\mathbb{T}^n \times B$ , for every function  $f : E \times A \subset \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  we recall the definition of the following norm

$$|f|_{L(A)} = \sup_{z \in B} \left( \sup_{x, y \in A, x \neq y} \frac{|f(z, x) - f(z, y)|}{|x - y|} \right) + |f|_{C^0}.$$

We consider a real parameter  $\sigma \geq 1$  and  $A \subset \mathbb{R}^n$ .

**Definition 5.5.** Let  $\mathcal{D}_\sigma$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times A \times \mathbb{R}$  such that  $f \in C(\mathbb{T}^n \times A \times \mathbb{R})$  and  $f_p^t \in C^\sigma(\mathbb{T}^n)$  for all  $(p, t) \in A \times \mathbb{R}$ .

For all  $f \in \mathcal{D}_\sigma$  and  $l \geq 1$ , we define

$$|f|_{\sigma, l, L(A)} = \sup_{(p, t) \in A \times \mathbb{R}} |f_p^t|_{C^\sigma} (1 + |t|^l) + \sup_{t \in \mathbb{R}} |f^t|_{L(A)} (1 + |t|^{l-1}), \quad (5.24)$$

$$|f|_{\sigma, 0, L(A)} = \sup_{(p, t) \in A \times \mathbb{R}} |f_p^t|_{C^\sigma} + \sup_{t \in \mathbb{R}} |f^t|_{L(A)}. \quad (5.25)$$

Similarly to the previous case, the following proposition contains several properties of the previous norms.

**Proposition 5.4.** Given  $\sigma \geq 1$ , for all  $f, g \in \mathcal{D}_\sigma$  and positive  $l, m \geq 1$

a.  $|f|_{\sigma, l, L(A)} \leq |f|_{s, l, L(A)}$  for all  $1 \leq \sigma \leq s$ ,

b.  $|f|_{\sigma, l, L(A)} \leq C(l, m) |f|_{\sigma, l+m, L(A)}$

c.  $|fg|_{\sigma, l+m, L(A)} \leq C(\sigma) (|f|_{0, l, L(A)} |g|_{\sigma, m, L(A)} + |f|_{\sigma, l, L(A)} |g|_{0, m, L(A)})$ .

Moreover, we consider  $\tilde{g} \in \mathcal{D}_\sigma$  such that, for all  $(q, p, t) \in \mathbb{T}^n \times A \times \mathbb{R}$ ,  $\tilde{g}(q, p, t) = (g(q, p, t), p, t)$ . Then  $f \circ \tilde{g} \in \mathcal{D}_\sigma$  and

d.  $|f \circ \tilde{g}|_{\sigma, l+m, L(A)} \leq C(\sigma) (|f|_{\sigma, l, L(A)} |g|_{1, m, L(A)}^\sigma + |f|_{1, l, L(A)} |g|_{\sigma, m, L(A)} + |f|_{0, l+m, L(A)})$ .

The previous properties are still verified when  $l = m = 0$  or only one of the two parameters  $l$  and  $m$  is zero.

*Proof.* The proof is quite similar to that of Proposition 5.2. The first two properties a. and b. are obvious. Hence, we prove c. and d.

c. Similarly to Proposition 5.2, one has

$$\sup_{(p, t) \in A \times \mathbb{R}} |f_p^t g_p^t|_{C^\sigma} (1 + |t|^{l+m}) \leq C(\sigma) (|f|_{0, l, L(A)} |g|_{\sigma, m, L(A)} + |f|_{\sigma, l, L(A)} |g|_{0, m, L(A)}).$$

We want to prove that the same estimate is also verified for the second term of the norm (see the right-hand side of (5.24)). For all fixed  $t \in \mathbb{R}$  and  $x, y \in A$  such that  $x \neq y$

$$\begin{aligned} & \frac{|f^t(q, x) g^t(q, x) - f^t(q, y) g^t(q, y)|}{|x - y|} (1 + |t|^{l+m-1}) \\ &= \frac{|f^t(q, x) g^t(q, x) - f^t(q, y) g^t(q, x) + f^t(q, y) g^t(q, x) - f^t(q, y) g^t(q, y)|}{|x - y|} (1 + |t|^{l+m-1}) \\ &\leq C (|f^t|_{L(A)} (1 + |t|^{l-1}) |g_x^t|_{C^0} (1 + |t|^m) + |g^t|_{L(A)} (1 + |t|^{m-1}) |f_y^t|_{C^0} (1 + |t|^l)) \\ &\leq C (|f|_{0, l, L(A)} |g|_{\sigma, m, L(A)} + |f|_{\sigma, l, L(A)} |g|_{0, m, L(A)}). \end{aligned}$$

Taking the sup for all  $q \in \mathbb{T}^n$  and  $x, y \in A$  with  $x \neq y$  on the left-hand side of the latter and then for all  $t \in \mathbb{R}$ , we prove *c*.

*d*. Also in this case, similarly to Proposition 5.2, one has

$$\begin{aligned} \sup_{(p,t) \in A \times \mathbb{R}} |f_p^t \circ g_p^t|_{C^\sigma} (1 + |t|^{l+m}) &\leq C(\sigma) |f|_{\sigma,l,L(A)} |g|_{1,m,L(A)}^\sigma \\ &+ C(\sigma) (|f|_{1,l,L(A)} |g|_{\sigma,m,L(A)} + |f|_{0,l+m,L(A)}). \end{aligned}$$

Now, we estimate the second term of the norm (see (5.24)). For all fixed  $t \in \mathbb{R}$  and  $x, y \in A$  such that  $x \neq y$

$$\begin{aligned} &\frac{|f^t(g^t(q, x), x) - f^t(g^t(q, y), y)|}{|x - y|} (1 + |t|^{l+m-1}) \\ &= \frac{|f^t(g^t(q, x), x) - f^t(g^t(q, x), y) + f^t(g^t(q, x), y) - f^t(g^t(q, y), y)|}{|x - y|} (1 + |t|^{l+m-1}) \\ &\leq \frac{|f^t(g^t(q, x), x) - f^t(g^t(q, x), y)|}{|x - y|} (1 + |t|^{l+m-1}) \\ &+ \frac{|f^t(g^t(q, x), y) - f^t(g^t(q, y), y)|}{|x - y|} (1 + |t|^{l+m-1}) \\ &\leq C (|f^t|_{L(A)} (1 + |t|^{l+m-1}) + |f_y^t|_{C^1} (1 + |t|^l) |g^t|_{L(A)} (1 + |t|^{m-1})) \\ &\leq C (|f|_{0,l+m,L(A)} + |f|_{1,l,L(A)} |g|_{\sigma,m,L(A)}), \end{aligned}$$

where we used  $|f^t(g^t(q, x), y) - f^t(g^t(q, y), y)| \leq |(\partial_q f)_y^t|_{C^0} |g^t|_{L(A)} |x - y|$ . Taking the sup for all  $q \in \mathbb{T}^n$  and  $x, y \in A$  with  $x \neq y$  on the left-hand side of the latter and then for all  $t \in \mathbb{R}$ , we conclude the proof of this lemma.  $\square$

Now, let us define the following space of functions.

**Definition 5.6.** Given  $\sigma \geq 1$  and an integer  $k \geq 0$ , we define  $\bar{\mathcal{D}}_{\sigma,k}$  the space of functions  $f$  such that

$$f \in \mathcal{D}_{\sigma+k}, \quad \text{and} \quad \partial_q^i f \in \mathcal{D}_{\sigma+k-i}$$

for all  $0 \leq i \leq k$ .

Furthermore, for all  $f \in \bar{\mathcal{D}}_{\sigma,k}$  and  $l > 1$ , we consider the following norm

$$\|f\|_{\sigma,k,l,L(A)} = \max_{0 \leq i \leq k} |\partial_q^i f|_{\sigma+k-i,l,L(A)}, \quad (5.26)$$

where  $|\cdot|_{\sigma,l,L(A)}$  is the norm defined by (5.24)

Now, we state the main theorem of this section. Here, the Hamiltonian and its components are functions defined on  $\mathbb{T}^n \times B_r \times \mathbb{R}$ , for some  $r > 0$ . Then, if a function  $f$  belongs to  $\mathcal{D}_\sigma$ , we consider that  $f$  satisfies the properties in Definition 5.1 with  $A$  replaced by  $B_r$ . Let  $\sigma \geq 1$ ,  $\Upsilon \geq 1$ ,  $l > 1$  and  $0 < \varepsilon < 1$ . We consider

the following Hamiltonian

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B_1 \times \mathbb{R} \longrightarrow \mathbb{R} \\ H(q, p, t) = h(p) + R(q, p) + f(q, p, t) \\ f, \partial_p f \in \bar{\mathcal{D}}_{\sigma, 2}, \\ |f|_{\sigma+2, 0, L(B_1)} + \|\partial_q f\|_{\sigma, 1, l+2, L(B_1)} + \|\partial_p f\|_{\sigma, 2, l+1, L(B_1)} < \varepsilon, \\ D \subset B_1, \quad \text{Leb}(B_1 \setminus D) < \mu, \\ R \in C^2(\mathbb{T}^n \times B_1) \\ R(q, p) = \partial_p R(q, p) = 0 \text{ for all } (q, p) \in \mathbb{T}^n \times D, \\ \partial_p^2 H^t \in C^{\sigma+2}(\mathbb{T}^n \times B_1) \text{ for all fixed } t \in \mathbb{R} \\ \partial_{qp}^i (\partial_p^2 H) \in C(\mathbb{T}^n \times B_1 \times \mathbb{R}) \text{ for all } 0 \leq i \leq 2. \\ \sup_{t \in \mathbb{R}} |\partial_p^2 H^t|_{C^{\sigma+2}} < \Upsilon, \end{array} \right. \quad (*_D)$$

**Theorem D.** *Let  $H$  be as in  $(*_D)$ . Then, there exists a time-dependent Hamiltonian  $\tilde{h}$  such that, for  $\varepsilon$  small enough with respect to  $n, l, \Upsilon, |\partial_p h|_{L(D)}$  and  $\mu$ , we have the existence of a set  $\mathcal{W} \subset \mathbb{T}^n \times B_1$  in such a way that, for all  $(q, p) \in \mathcal{W}$ , there exist  $p_-, p_+ \in D$  and a biasymptotically quasiperiodic solution  $g$  associated to  $(X_H, X_{\tilde{h}}, \varphi_{0, p_{\pm}})$  such that  $g(0) = (q, p)$ . Moreover,*

$$\text{Leb}((\mathbb{T}^n \times B_1) \setminus \mathcal{W}) \leq 4\mu.$$

Also in this case, we are going to deduce the latter from another theorem. Let  $\sigma \geq 1, \Upsilon \geq 1, l > 1, 0 < \varepsilon < 1, \mu > 0$  and  $0 < \delta < 1$  with  $\delta \leq \mu$ . We consider the following family of Hamiltonians

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B_\delta \times \mathbb{R} \times D' \longrightarrow \mathbb{R} \\ H(\theta, I, t; p_0) = e(p_0) + \omega(p_0) \cdot I \\ \quad \quad \quad + a(\theta, t; p_0) + b(\theta, t; p_0) \cdot I + m(\theta, I, t; p_0) \cdot I^2 \\ \omega \in C(D'), \quad |\omega|_{L(D')} < \infty \\ a, b \in \bar{\mathcal{D}}_{\sigma, 2}, \\ |a|_{\sigma+2, 0, L(D')} + \|\partial_\theta a\|_{\sigma, 1, l+2, L(D')} < \varepsilon, \quad \|b\|_{\sigma, 2, l+1, L(D')} < \varepsilon, \\ \partial_I^2 H_{p_0}^t \in C^{\sigma+2}(\mathbb{T}^n \times B_\delta) \text{ for all fixed } (t, p_0) \in \mathbb{R} \times D' \\ \partial_{\theta I}^i (\partial_I^2 H) \in C(\mathbb{T}^n \times B_\delta \times \mathbb{R} \times D') \text{ for all } 0 \leq i \leq 2. \\ \sup_{(t, p_0) \in \mathbb{R} \times D'} |\partial_I^2 H_{p_0}^t|_{C^{\sigma+2}} \leq \Upsilon, \\ \sup_{0 \leq i \leq 2} \left( \sup_{t \in \mathbb{R}} |\partial_{\theta I}^i (\partial_I^2 H^t)|_{L(D')} \right) \leq \Upsilon. \end{array} \right. \quad (**)$$

We define the following family of trivial embeddings

$$\psi_0 : \mathbb{T}^n \times D' \longrightarrow \mathbb{T}^n \times B_\delta, \quad \psi_0(\theta, p_0) = (\theta, 0)$$

and we consider the following family Hamiltonians  $\tilde{h} : \mathbb{T}^n \times B_\delta \times \mathbb{R} \times D' \rightarrow \mathbb{R}$  such that

$$\tilde{h}(\theta, I, t; p_0) = e(p_0) + \omega(p_0) \cdot I + m(\theta, I, t; p_0) \cdot I^2.$$

**Theorem 5.3.** *Let  $H$  be as in  $(\star\star)$ . Then, if  $\varepsilon$  is sufficiently small with respect to  $n, l, \Upsilon$  and  $|\omega|_{L(D')}$ , for all fixed  $p_0 \in D'$ , there exists a positive  $C^\sigma$ -asymptotic KAM torus  $\psi_{+p_0}^t$  and a negative  $C^\sigma$ -asymptotic KAM torus  $\psi_{-p_0}^t$  associated to  $(X_{H_{p_0}}, X_{\tilde{h}_{p_0}}, \psi_{0,p_0})$ . Moreover, letting*

$$\psi_{\pm}^t : \mathbb{T}^n \times D' \longrightarrow \mathbb{T}^n \times B_\delta, \quad \psi_{\pm}^t(q, p_0) = \psi_{\pm, p_0}^t(q),$$

there exists a constant  $C_0$  depending on  $n, l, \Upsilon$  and  $|\omega|_{L(D')}$  such that

$$\sup_{t \geq 0} |\psi_{+}^t - \psi_0|_{L(\mathbb{T}^n \times D')} < C_0 \varepsilon, \quad \sup_{t \leq 0} |\psi_{-}^t - \psi_0|_{L(\mathbb{T}^n \times D')} < C_0 \varepsilon. \quad (5.27)$$

## 5.6 Proof of Theorem D assuming Theorem 5.3

Here, we assume Theorem 5.3 and we prove Theorem D. The following well-known property is the Lipschitz version of Proposition 5.3

**Proposition 5.5.** *Given  $r > 0$  and  $0 < \varepsilon < \delta < r$ , let  $\phi$  be a Lipschitz map*

$$\phi : B_r \longrightarrow B_{r+\delta}$$

such that  $|\phi - \text{Id}|_{L(B_r)} < \varepsilon$ . Then, for  $\varepsilon$  small enough,  $\phi$  is a diffeomorphism onto its image and

$$B_{r-\delta} \subset \phi(B_{r+\delta}).$$

We define  $\delta = 2C_0\varepsilon$ , where  $C_0$  is the constant introduced in Theorem 5.3. Therefore, we consider

$$D' = B_{1-\delta} \cap D.$$

Now, for all  $p_0 \in D'$ , we let  $p = p_0 + I$ . Similarly to the proof of Theorem C, we expand the Hamiltonian  $H$  in  $(*_D)$  around  $p_0$  in such a way that

$$\begin{aligned} h(p) &= h(p_0) + \partial_p h(p_0) \cdot I + \int_0^1 (1-\tau) \partial_p^2 h(p_\tau) d\tau \cdot I^2 \\ R(q, p) &= R(q, p_0) + \partial_p R(q, p_0) \cdot I + \int_0^1 (1-\tau) \partial_p^2 R(q, p_\tau, t) d\tau \cdot I^2 \\ &= \int_0^1 (1-\tau) \partial_p^2 R(q, p_\tau, t) d\tau \cdot I^2 \\ f(q, p, t) &= f(q, p_0, t) + \partial_p f(q, p_0, t) \cdot I + \int_0^1 (1-\tau) \partial_p^2 f(q, p_\tau, t) d\tau \cdot I^2. \end{aligned}$$

where  $p_\tau = p_0 + \tau I$  and  $I \in B_\delta$ . For all  $p_0 \in D'$ , we define

$$\begin{aligned} e(p_0) &= h(p_0) \\ \omega(p_0) &= \partial_p h(p_0) \\ a(q, t; p_0) &= f(q, p_0, t) \\ b(q, t; p_0) &= \partial_p f(q, p_0, t) \\ m(q, I, t; p_0) &= \int_0^1 (1-\tau) (\partial_p^2 h(p_\tau) + \partial_p^2 R(q, p_\tau, t) + \partial_p^2 f(q, p_\tau, t)) d\tau \\ &= \int_0^1 (1-\tau) \partial_p^2 H(q, p_\tau, t) d\tau \end{aligned}$$



for all  $(q, I, t) \in \mathbb{T}^n \times B_\delta \times \mathbb{R}$ . Writing  $\theta$  instead of  $q$  for the angular variables, we rewrite the Hamiltonian  $H$ , restricted to  $\mathbb{T}^n \times B_\delta \times \mathbb{R} \times D'$ , in the following form as a family of Hamiltonians parametrized by  $p_0 \in D'$ ,

$$H : \mathbb{T}^n \times B_\delta \times \mathbb{R} \times D' \longrightarrow \mathbb{R} \quad (5.28)$$

such that

$$H(\theta, I, t; p_0) = e(p_0) + \omega(p_0) \cdot I + a(\theta, t; p_0) + b(\theta, t; p_0) \cdot I + m(\theta, I, t; p_0) \cdot I^2.$$

As in the proof of Theorem C, for all  $(\theta, I, t; p_0) \in \mathbb{T}^n \times B_\delta \times \mathbb{R} \times D'$ , let  $\tilde{h}$  be the following family of Hamiltonians

$$\tilde{h}(\theta, I, t; p_0) = e(p_0) + \omega(p_0) \cdot I + m(\theta, I, t; p_0) \cdot I^2.$$

Obviously, for each fixed  $p_0 \in D'$ ,  $X_H$  and  $X_{\tilde{h}}$  satisfy (5.1) and, for all  $p_0 \in D'$ ,  $\tilde{h}$  has an invariant torus supporting quasiperiodic dynamics.

The Hamiltonian  $H$  in (5.28) satisfies the hypotheses of Theorem 5.3. Then, similarly to the proof of Theorem C, there exist  $u_\pm^t, v_\pm^t : \mathbb{T}^n \times D' \rightarrow \mathbb{R}^n$  such that, in the  $(q, p)$ -coordinates, the following family of embeddings

$$\begin{aligned} \varphi_+^t &: \mathbb{T}^n \times D' \longrightarrow \mathbb{T}^n \times B_1, \\ \varphi_+^t(q, p_0) &= (q + u_+^t(q, p_0), p_0 + v_+^t(q, p_0)) \end{aligned}$$

is a family of positive  $C^\sigma$ -asymptotic KAM tori associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$  and

$$\begin{aligned} \varphi_-^t &: \mathbb{T}^n \times D' \longrightarrow \mathbb{T}^n \times B_1, \\ \varphi_-^t(q, p_0) &= (q + u_-^t(q, p_0), p_0 + v_-^t(q, p_0)) \end{aligned}$$

is a family of negative  $C^\sigma$ -asymptotic KAM tori associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ , where  $\varphi_0$  is the family of trivial embeddings defined by (5.10). Moreover,

$$\begin{aligned} \sup_{t \geq 0} |u_+^t|_{L(\mathbb{T}^n \times D')} &< C_0 \varepsilon, & \sup_{t \geq 0} |v_+^t|_{L(\mathbb{T}^n \times D')} &< C_0 \varepsilon, \\ \sup_{t \leq 0} |u_-^t|_{L(\mathbb{T}^n \times D')} &< C_0 \varepsilon, & \sup_{t \leq 0} |v_-^t|_{L(\mathbb{T}^n \times D')} &< C_0 \varepsilon, \end{aligned}$$

where  $C_0$  is a constant depending on  $n, l, \Upsilon$  and  $|\omega|_{L(D')}$ .

Now, there exist  $\tilde{u}_\pm^t, \tilde{v}_\pm^t : \mathbb{T}^n \times B_{1-\delta} \rightarrow \mathbb{R}^n$  such that  $\tilde{u}_\pm^t, \tilde{v}_\pm^t$  extend  $u_\pm^t, v_\pm^t$  without affecting their Lipschitz constant. This means that,

$$\begin{aligned} \tilde{u}_\pm^t|_{\mathbb{T}^n \times D'} &= u_\pm^t, & \tilde{v}_\pm^t|_{\mathbb{T}^n \times D'} &= v_\pm^t, \\ \sup_{t \geq 0} |\tilde{u}_+^t|_{L(\mathbb{T}^n \times B_{1-\delta})} &= \sup_{t \geq 0} |u_+^t|_{L(\mathbb{T}^n \times D')} & \sup_{t \geq 0} |\tilde{v}_+^t|_{L(\mathbb{T}^n \times B_{1-\delta})} &= \sup_{t \geq 0} |v_+^t|_{L(\mathbb{T}^n \times D')}. \end{aligned}$$

Therefore, letting

$$\begin{aligned} \tilde{\varphi}_\pm^t &: \mathbb{T}^n \times B_{1-\delta} \longrightarrow \mathbb{T}^n \times B_1, \\ \tilde{\varphi}_\pm^t(q, p_0) &= (q + \tilde{u}_\pm^t(q, p_0), p_0 + \tilde{v}_\pm^t(q, p_0)) \end{aligned}$$

we have

$$\sup_{t \geq 0} |\tilde{\varphi}_{\pm}^t - \text{Id}|_{L(\mathbb{T}^n \times B_{1-\delta})} = \sup_{t \leq 0} |\varphi_{\pm}^t - \text{Id}|_{L(\mathbb{T}^n \times D')} < C_0 \varepsilon.$$

Now, we recall that  $\text{Leb}(B_1 \setminus D) < \mu$ ,

$$\begin{aligned} \text{Leb}(\tilde{\varphi}_{\pm}^0(\mathbb{T}^n \times B_{1-\delta}) \setminus \varphi_{\pm}^0(\mathbb{T}^n \times D')) &= \text{Leb}(\tilde{\varphi}_{\pm}^0(\mathbb{T}^n \times B_{1-\delta}) \setminus \tilde{\varphi}_{\pm}^0(\mathbb{T}^n \times D')) \\ &\leq C_{\varepsilon} \text{Leb}(B_{1-\delta} \setminus D') \\ &= C_{\varepsilon} \text{Leb}(B_{1-\delta} \setminus (B_{1-\delta} \cap D)) \\ &\leq C_{\varepsilon} \text{Leb}(B_1 \setminus D) < C_{\varepsilon} \mu \end{aligned} \quad (5.29)$$

where  $C_{\varepsilon}$  is a constant converging to 1 if  $\varepsilon \rightarrow 0$ .

We observe that

$$\begin{aligned} (\mathbb{T}^n \times B_1) \setminus \varphi_{\pm}^0(\mathbb{T}^n \times D') &= (\mathbb{T}^n \times B_1) \setminus \tilde{\varphi}_{\pm}^0(\mathbb{T}^n \times B_{1-\delta}) \\ &\quad + \tilde{\varphi}_{\pm}^0(\mathbb{T}^n \times B_{1-\delta}) \setminus \varphi_{\pm}^0(\mathbb{T}^n \times D'). \end{aligned} \quad (5.30)$$

Thanks to Proposition 5.5 and the special form of  $\tilde{\varphi}_{\pm}^0$

$$\mathbb{T}^n \times B_{1-2\delta} \subset \tilde{\varphi}_{\pm}^0(\mathbb{T}^n \times B_{1-\delta}).$$

Then, by the latter, (5.29) and (5.30)

$$\begin{aligned} \text{Leb}((\mathbb{T}^n \times B_1) \setminus \varphi_{\pm}^0(\mathbb{T}^n \times D')) &= \text{Leb}((\mathbb{T}^n \times B_1) \setminus \tilde{\varphi}_{\pm}^0(\mathbb{T}^n \times B_{1-\delta})) + C_{\varepsilon} \mu \\ &\leq \text{Leb}((\mathbb{T}^n \times B_1) \setminus (\mathbb{T}^n \times B_{1-2\delta})) + C_{\varepsilon} \mu \\ &\leq 2\mu \end{aligned} \quad (5.31)$$

for  $\varepsilon$  sufficiently small. Now, let us introduce the following set

$$\mathcal{W} = \varphi_+^0(\mathbb{T}^n \times D') \cap \varphi_-^0(\mathbb{T}^n \times D')$$

and by (5.31)

$$\begin{aligned} \text{Leb}((\mathbb{T}^n \times B_1) \setminus \mathcal{W}) &\leq \text{Leb}((\mathbb{T}^n \times B_1) \setminus \varphi_+^0(\mathbb{T}^n \times D')) \\ &\quad + \text{Leb}((\mathbb{T}^n \times B_1) \setminus \varphi_-^0(\mathbb{T}^n \times D')) \leq 4\mu. \end{aligned}$$

This concludes the proof of this theorem because, for all  $(q, p_0) \in \mathcal{W} = \varphi_+^0(\mathbb{T}^n \times D') \cap \varphi_-^0(\mathbb{T}^n \times D')$ , there exist  $(q_+, p_{0+}), (q_-, p_{0-}) \in \mathbb{T}^n \times D'$  such that

$$\varphi_+^0(q_+, p_{0+}) = (q, p_0) = \varphi_-^0(q_-, p_{0-}).$$

Therefore, by Proposition 5.1, there exists a biasymptotically quasiperiodic solution  $g(t)$  associated to  $(X_H, X_{\tilde{h}}, \varphi_{0, p_{0\pm}})$  such that  $g(0) = (q, p_0)$ , where  $\varphi_0$  is the family of trivial embeddings defined by (5.10).

## 5.7 Proof of Theorem 5.3

The proof of this theorem is the same as Theorem 5.1. However, we have some obvious differences in the estimation of the solution of the homological equation. Here, we prove the existence of a family of positive  $C^\sigma$ -asymptotic KAM tori  $\psi_+^t$  parametrized by  $p_0 \in D'$ . Similarly, we have the claim concerning negative times. In what follows, we drop the subscript  $+$  to obtain a more elegant form.

We are looking for  $u, v : \mathbb{T}^n \times D' \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$  such that letting

$$\psi(\theta, t; p_0) = (\theta + u(\theta, t; p_0), v(\theta, t; p_0)),$$

$\psi, u$  and  $v$  satisfy the following conditions

$$X_H(\psi(\theta, t; p_0), t; p_0) - \partial_\theta \psi(\theta, t; p_0) \omega(p_0) - \partial_t \psi(\theta, t; p_0) = 0 \quad (5.32)$$

$$\lim_{t \rightarrow +\infty} |u_{p_0}^t|_{C^\sigma} = 0, \quad \lim_{t \rightarrow +\infty} |v_{p_0}^t|_{C^\sigma} = 0, \quad (5.33)$$

To this end, given  $\sigma \geq 1$ , let us introduce the following definitions

**Definition 5.7.** Let  $\mathcal{D}_\sigma^+$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times \mathbb{R}^+ \times D'$  such that  $f \in C(\mathbb{T}^n \times \mathbb{R}^+ \times D')$  and  $f_p^t \in C^\sigma(\mathbb{T}^n)$  for all  $(t, p) \in \mathbb{R}^+ \times D'$ .

For all  $f \in \mathcal{D}_\sigma^+$  and  $l > 1$ , we define the following norms

$$\begin{aligned} |f|_{\sigma, l, L(D')} &= \sup_{(t, p) \in \mathbb{R}^+ \times D'} |f_p^t|_{C^\sigma} (1 + |t|^l) + \sup_{t \in \mathbb{R}^+} |f^t|_{L(D')} (1 + |t|^{l-1}), \\ |f|_{\sigma, 0, L(D')} &= \sup_{(t, p) \in \mathbb{R}^+ \times D'} |f_p^t|_{C^\sigma} + \sup_{t \in \mathbb{R}^+} |f^t|_{L(D')}. \end{aligned}$$

These norms satisfy the properties in Proposition 5.4. As one can expect, we define the following subset of  $\mathcal{D}_\sigma^+$

**Definition 5.8.** Given  $\sigma \geq 1$  and an integer  $k \geq 0$ , we define  $\bar{\mathcal{D}}_{\sigma, k}^+$  the space of functions  $f$  such that

$$f \in \mathcal{D}_{\sigma+k}^+, \quad \text{and} \quad \partial_q^i f \in \mathcal{D}_{\sigma+k-i}^+$$

for all  $0 \leq i \leq k$ .

Therefore, for all  $f \in \bar{\mathcal{D}}_{\sigma, k}^+$  and  $l > 1$ , we define

$$\|f\|_{\sigma, k, l, L(D')}^+ = \max_{0 \leq i \leq k} |\partial_q^i f|_{\sigma+k-i, l, L(D')}^+.$$

Now, let  $\sigma \geq 1$  and  $l > 1$  be the positive parameters introduced by  $(\star\star)$ . We define the following Banach spaces  $(\mathcal{A}, |\cdot|)$ ,  $(\mathcal{B}, |\cdot|)$ ,  $(\mathcal{U}, |\cdot|)$ ,  $(\mathcal{V}, |\cdot|)$ ,  $(\mathcal{Z}, |\cdot|)$  and  $(\mathcal{G}, |\cdot|)$

$$\begin{aligned}
 \mathcal{A} &= \left\{ a : \mathbb{T}^n \times \mathbb{R}^+ \times D' \rightarrow \mathbb{R} \mid a \in \bar{\mathcal{D}}_{\sigma,2}^+ \right. \\
 &\quad \left. \text{and } |a| = |a|_{\sigma+2,0,L(D')}^+ + \|\partial_{\theta} a\|_{\sigma,1,l+2,L(D')}^+ < \infty \right\} \\
 \mathcal{B} &= \left\{ b : \mathbb{T}^n \times \mathbb{R}^+ \times D' \rightarrow \mathbb{R}^n \mid b \in \bar{\mathcal{D}}_{\sigma,2}^+, \text{ and } |b| = \|b\|_{\sigma,2,l+1,L(D')}^+ < \infty \right\} \\
 \mathcal{U} &= \left\{ u : \mathbb{T}^n \times \mathbb{R}^+ \times D' \rightarrow \mathbb{R}^n \mid u, (\nabla_{\theta t} u) \Omega \in \mathcal{D}_{\sigma}^+ \right. \\
 &\quad \left. \text{and } |u| = \max\{|u|_{\sigma,l,L(D')}^+, |(\nabla_{\theta t} u) \Omega|_{\sigma,l+1,L(D')}^+\} < \infty \right\} \\
 \mathcal{V} &= \left\{ v : \mathbb{T}^n \times \mathbb{R}^+ \times D' \rightarrow \mathbb{R}^n \mid v, (\nabla_{\theta t} v) \Omega \in \mathcal{D}_{\sigma}^+ \right. \\
 &\quad \left. \text{and } |v| = \max\{|v|_{\sigma,l+1,L(D')}^+, |(\nabla_{\theta t} v) \Omega|_{\sigma,l+2,L(D')}^+\} < \infty \right\} \\
 \mathcal{Z} &= \left\{ z : \mathbb{T}^n \times \mathbb{R}^+ \times D' \rightarrow \mathbb{R}^n \mid z \in \mathcal{D}_{\sigma}^+, \text{ and } |z| = |z|_{\sigma,l+1,L(D')}^+ < \infty \right\} \\
 \mathcal{G} &= \left\{ g : \mathbb{T}^n \times \mathbb{R}^+ \times D' \rightarrow \mathbb{R}^n \mid g \in \mathcal{D}_{\sigma}^+, \text{ and } |g| = |g|_{\sigma,l+2,L(D')}^+ < \infty \right\}
 \end{aligned}$$

Similarly to what we did in Appendix C, one can prove that the previous normed spaces are Banach spaces. Let  $M_n$  be the set of the  $n$ -dimensional matrices and  $\Upsilon \geq 1$  the positive parameter in (\*\*). We introduce another Banach space  $(\mathcal{M}, |\cdot|)$ , such that

$$\begin{aligned}
 \mathcal{M} &= \left\{ m : \mathbb{T}^n \times B_{\delta} \times \mathbb{R}^+ \times D' \rightarrow M_n \mid \partial_{\theta I}^i m \in C(\mathbb{T}^n \times B_{\delta} \times \mathbb{R}^+ \times D') \right. \\
 &\quad \left. \text{for all } 0 \leq i \leq 2, m_{p_0}^t \in C^{\sigma+2}(\mathbb{T}^n \times B_{\delta}) \text{ for all fixed } (t, p_0) \in \mathbb{R}^+ \times D' \right. \\
 &\quad \left. \text{and } |m| = \sup_{(t,p_0) \in \mathbb{R}^+ \times D'} |m_{p_0}^t|_{C^{\sigma+2}} + \sup_{0 \leq i \leq 2} \left( \sup_{t \in \mathbb{R}^+} |\partial_{\theta I}^i m^t|_{L(D')} \right) \leq 2\Upsilon \right\}
 \end{aligned}$$

Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{Z} \times \mathcal{G}$$

$$\mathcal{F}(a, b, m, \bar{m}, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, u, v))$$

with

$$\begin{aligned}
 F_1(b, \bar{m}, u, v) &= b \circ \tilde{u} + \left( \bar{m} \circ \tilde{\psi} \right) v - (\nabla_{\theta t} u) \Omega, \\
 F_2(a, b, m, u, v) &= \partial_{\theta} a \circ \tilde{u} + (\partial_{\theta} b \circ \tilde{u}) v + \left( \partial_{\theta} m \circ \tilde{\psi} \right) \cdot v^2 + (\nabla_{\theta t} v) \Omega.
 \end{aligned}$$

It is obtained by (5.32). For fixed  $m, \bar{m} \in \mathcal{M}$  and for  $(a, b)$  sufficiently close to  $(0, 0)$ , we are looking for  $(u, v) \in \mathcal{U} \times \mathcal{V}$  satisfying (5.33) such that  $\mathcal{F}(a, b, m, \bar{m}, u, v) = 0$ . Following the lines of the proof of Theorem 5.1, one can prove that  $\mathcal{F}$  is well defined, continuous, differentiable with respect to the variables  $(u, v)$  with  $D_{(u,v)} \mathcal{F}$  continuous. Moreover, for all fixed  $m, \bar{m} \in \mathcal{M}$

$$D_{(u,v)} \mathcal{F}(0, 0, m, \bar{m}, 0, 0)(\hat{u}, \hat{v}) = (\bar{m}_0 \hat{v} - (\nabla_{\theta t} \hat{u}) \Omega, (\nabla_{\theta t} \hat{v}) \Omega).$$

is invertible. The proof relies on the solution of the following homological equation. Given  $\sigma \geq 1$  and  $l > 1$ , we consider the following equation in the unknown  $\varkappa : \mathbb{T}^n \times \mathbb{R}^+ \times D' \rightarrow \mathbb{R}$

$$\begin{cases} \omega(p_0) \cdot \partial_q \varkappa(\theta, t; p_0) + \partial_t \varkappa(\theta, t; p_0) = g(\theta, t; p_0), \\ g \in \mathcal{D}_\sigma^+, \quad |g|_{\sigma, l+1, L(D')}^+ < \infty, \\ \omega : D' \rightarrow \mathbb{R}^n, \\ \omega \in C(D'), \quad |\omega|_{L(D')} < \infty. \end{cases} \quad (HE_D)$$

**Lemma 5.5 (Homological Equation).** *There exists a unique solution  $\varkappa \in \mathcal{D}_\sigma^+$  of  $(HE_D)$  such that, for all fixed  $p_0 \in D$ ,*

$$\lim_{t \rightarrow +\infty} |\varkappa_{p_0}^t|_{C^0} = 0. \quad (5.34)$$

Moreover,

$$|\varkappa|_{\sigma, l, L(D')}^+ \leq C |g|_{\sigma, l+1, L(D')}^+$$

for a suitable constant  $C$  depending on  $n$ ,  $l$  and  $|\omega|_{L(D')}$ .

*Proof.* We know that

$$\varkappa(\theta, t; p_0) = - \int_t^{+\infty} g(\theta + \omega(p_0)(\tau - t), \tau; p_0) d\tau$$

is the unique solution of  $(HE_D)$  satisfying (5.34). Concerning the estimates, similarly to the proof of Lemma 5.2, we have

$$\sup_{(t, p_0) \in \mathbb{R}^+ \times D'} |\varkappa_{p_0}^t|_{C^\sigma} (1 + t^l) \leq C(l) |g|_{\sigma, l+1, L(D')}^+.$$

It remains to estimate the second member of the norm. By Lemma 5.1, for all  $(\theta, t, p_{0_1}), (\theta, t, p_{0_2}) \in \mathbb{T}^n \times \mathbb{R}^+ \times D'$  with  $p_{0_1} \neq p_{0_2}$ ,

$$\begin{aligned} & \frac{|\varkappa(\theta, t; p_{0_1}) - \varkappa(\theta, t; p_{0_2})|}{|p_{0_1} - p_{0_2}|} \\ & \leq \int_t^{+\infty} \frac{|g(\theta + \omega(p_{0_2})(\tau - t), t, p_{0_2}) - g(\theta + \omega(p_{0_2})(\tau - t), t, p_{0_1})|}{|p_{0_1} - p_{0_2}|} d\tau \\ & \leq \int_t^{+\infty} \frac{|g(\theta + \omega(p_{0_2})(\tau - t), t, p_{0_1}) - g(\theta + \omega(p_{0_1})(\tau - t), t, p_{0_1})|}{|p_{0_1} - p_{0_2}|} d\tau \\ & \leq C \left( \sup_{t \in \mathbb{R}^+} |g^t|_{L(D')} (1 + t^l) \right) \int_t^{+\infty} \frac{1}{1 + \tau^l} d\tau \\ & + C \left( \sup_{(t, p_0) \in \mathbb{R}^+ \times D'} |g_p^t|_{C^1} (1 + t^{l+1}) \right) |\omega|_{C^1} \int_t^{+\infty} \frac{\tau - t}{1 + \tau^{l+1}} d\tau, \\ & \leq C(l, |\omega|_{C^1}) \frac{|g|_{1, l+1, L(D)}^+}{1 + t^{l-1}}. \end{aligned}$$

Taking the sup for all  $\theta \in \mathbb{T}^n$ ,  $p_{0_1}, p_{0_2} \in D'$  with  $p_{0_1} \neq p_{0_2}$ , and then for all  $t \in \mathbb{R}^+$  on the left hand side of the latter, we conclude the proof of this lemma.  $\square$

We proved that the functional  $\mathcal{F}$  satisfies the hypotheses of the implicit function theorem. Therefore, following the lines of the proof of Theorem 5.1, we conclude the proof of Theorem 5.3.

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## Part IV

# Applications to Celestial Mechanics

We consider the example of a planetary system (planar three-body problem) perturbed by a given comet coming from and going back to infinity, asymptotically along a hyperbolic Keplerian orbit. We prove the existence of orbits in such a way that the center of mass of the planetary system is attracted by the comet with zero asymptotic velocity when  $t \rightarrow +\infty$ . Moreover, in a frame of reference attached to the center of mass of the planetary system, the motion of the planets converges to some dynamics that are close (in the sense that we will recall later) to suitable quasiperiodic solutions associated to the Hamiltonian of the planar three-body problem.

The Hamiltonian of the planar three-body problem plus comet (P3BP+C) is  $H = H_0 + H_c$ , where  $H_0$  is the Hamiltonian of the planar three-body problem and  $H_c$  is the perturbation given by the interaction of the planets with the comet. On a suitable subset of the phase space, the Hamiltonian  $H_c$  does not satisfy good decay properties. This makes Theorem A useless for this particular case. For this reason, we begin this part by proving another abstract theorem, a weaker version of Theorem A. This is the content of the following chapter. The consecutive chapter is devoted to the application in celestial mechanics.

## 6 The Abstract Theorem

This chapter is divided into three sections. The first is devoted to the definition of  $C^\sigma$ -weakly asymptotic cylinder (Section 6.1). The above-mentioned weak theorem is introduced in the second section (Section 6.2) and the last section contains its proof (Section 6.3).

### 6.1 $C^\sigma$ -weakly asymptotic cylinder

Let  $B \subset \mathbb{R}^{n+m}$  be a ball centred at the origin. We denote  $q \in \mathbb{T}^n \times \mathbb{R}^m$  and  $p \in B$ . Let  $\mathcal{P}$  be equal to  $\mathbb{T}^n \times \mathbb{R}^m \times B$  or an open subset of  $\mathbb{R}^{2(n+m)}$  and  $J = [1, +\infty) \subset \mathbb{R}$  be a real interval.

Given  $\sigma \geq 0$  and a positive integer  $k \geq 0$ , we consider time-dependent vector fields  $X^t, X_0^t$  of class  $C^{\sigma+k}$  on  $\mathcal{P}$ , for all fixed  $t \in J$ , an embedding  $\varphi_0$  from  $\mathbb{T}^n \times \mathbb{R}^m$  to  $\mathcal{P}$  of class  $C^\sigma$  and a time-dependent vector field  $\gamma^t$  of class  $C^\sigma$  on  $\mathbb{T}^n \times \mathbb{R}^m$ , for all fixed  $t \in J$ , such that

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_{C^{\sigma+k}} = 0, \quad (6.1)$$

$$X_0(\varphi_0(q), t) = \partial_q \varphi_0(q)(\bar{\omega} + \gamma(q, t)) \text{ for all } (q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J, \quad (6.2)$$

$$\lim_{t \rightarrow +\infty} |\gamma^t|_{C^\sigma} = 0, \quad (6.3)$$

where  $\bar{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$  with  $\omega \in \mathbb{R}^n$ .

**Definition** (Définition 2.3). *We assume that  $(X, X_0, \varphi_0)$  satisfy (6.1), (6.2) and (6.3). A family of  $C^\sigma$  embeddings  $\varphi^t : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathcal{P}$  is a  $C^\sigma$ -weakly asymptotic cylinder associated to  $(X, X_0, \varphi_0)$  if there exists a time-dependent vector field  $\Gamma^t$  of class  $C^\sigma$  on  $\mathbb{T}^n \times \mathbb{R}^m$ , for all fixed  $t$ , such that*

$$\lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{C^\sigma} = 0, \quad (6.4)$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t)(\bar{\omega} + \Gamma(q, t)) + \partial_t \varphi(q, t), \quad (6.5)$$

$$\lim_{t \rightarrow +\infty} |\Gamma^t|_{C^\sigma} = 0, \quad (6.6)$$

for all  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ . Moreover,  $\varphi$  is Lagrangian if  $\varphi^t(\mathbb{T}^n \times \mathbb{R}^m)$  is Lagrangian for all  $t \in J$ .

We observe that taking  $m = 0$ ,  $\gamma \equiv 0$  and  $\Gamma \equiv 0$ , we obtain Definition 1.3. As one can expect, all the properties and considerations discussed for  $C^\sigma$ -asymptotic KAM tori remain true for  $C^\sigma$ -weakly asymptotic cylinders with suitable modifications. We can rewrite (6.5) in terms of the flow of  $X$ . Let  $\psi_{t_0, X}^t$  and  $\psi_{t_0, \bar{\omega} + \Gamma}^t$  be the flow at time  $t$  with initial time  $t_0$  of  $X$  and  $\bar{\omega} + \Gamma$ , respectively. We assume that  $\psi_{t_0, X}^t$  and  $\psi_{t_0, \bar{\omega} + \Gamma}^t$  are defined for all  $t, t_0 \in J$ . Then, (6.5) is equivalent to

$$\psi_{t_0, X}^t \circ \varphi^{t_0} = \varphi^t \circ \psi_{t_0, \bar{\omega} + \Gamma}^t. \quad (6.7)$$

Moreover, we can always find a family of embeddings satisfying (6.5). Furthermore, if  $\varphi^t$  is a  $C^\sigma$ -weakly asymptotic cylinder defined for all  $t$  large, we can extend the set of definition for all  $t \in \mathbb{R}$ . Regarding the dynamics, we recall the definition of weakly asymptotically quasiperiodic solution.

**Definition** (Définition 2.4). *We assume that  $(X, X_0, \varphi_0)$  satisfy (6.1), (6.2) and (6.3). An integral curve  $g(t)$  of  $X$  is a weakly asymptotically quasiperiodic solution associated to  $(X, X_0, \varphi_0)$  if there exist a time-dependent vector field  $\Gamma : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m}$  and  $q \in \mathbb{T}^n \times \mathbb{R}^m$  such that*

$$\lim_{t \rightarrow +\infty} |g(t) - \varphi_0 \circ \psi_{t_0, \bar{\omega} + \Gamma}^t(q)| = 0.$$

The latter is a weakly version of the definition of asymptotically quasiperiodic solutions (Definition 3.1). Indeed, taking  $m = 0$ ,  $\gamma \equiv 0$  and  $\Gamma \equiv 0$ , we obtain Definition 3.1. We conclude by recalling the following proposition.

**Proposition** (Proposition 2.1). *Let  $\varphi^t$  be a  $C^\sigma$ -weakly asymptotic cylinder associated to  $(X, X_0, \varphi_0)$ . Then, for all  $q \in \mathbb{T}^n \times \mathbb{R}^m$  and  $t_0 \in J$ ,*

$$g(t) = \psi_{t_0, X}^t \circ \varphi^{t_0}(q)$$

is a weakly asymptotically quasiperiodic solution associated to  $(X, X_0, \varphi_0)$ .

## 6.2 Result

Given a positive real parameter  $\sigma \geq 0$ , we have the following definition

**Definition 6.1.** Let  $\mathcal{S}_\sigma$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times \mathbb{R}^m \times B \times J$  such that  $f^t \in C^\sigma(\mathbb{T}^n \times \mathbb{R}^m \times B)$ , for all fixed  $t \in J$ , and  $\partial_{(q,p)}^i f \in C(\mathbb{T}^n \times \mathbb{R}^m \times B \times J)$  for all  $0 \leq i \leq [\sigma]$ .

We point out that  $\partial_{(q,p)}^i$  stands for the partial derivatives with respect to  $(q, p)$  of order  $i$  and  $[\sigma]$  for the integer part of  $\sigma$ . Given  $\sigma \geq 0$  and  $l \geq 0$ , for every  $f \in \mathcal{S}_\sigma$ , we introduce the following norm

$$|f|_{\sigma,l} = \sup_{t \in J} |f^t|_{C^\sigma} t^l. \quad (6.8)$$

The following proposition proves some properties of this norm.

**Proposition 6.1.** We consider  $f, g \in \mathcal{S}_\sigma$  and positive  $l, m > 0$ .

- a. For all  $\beta \in \mathbb{N}^{2n}$ , if  $|\beta| + s \leq \sigma$ , then  $\left| \frac{\partial^{|\beta|}}{\partial q_1^{\beta_1} \dots \partial q_n^{\beta_n} \partial p_1^{\beta_{n+1}} \dots \partial p_n^{\beta_{2n}}} f \right|_{s,l} \leq |f|_{\sigma,l}$ ,
- b.  $|f|_{\sigma,l} \leq |f|_{\sigma,l+m}$ ,
- c.  $|fg|_{\sigma,l+m} \leq C(\sigma) (|f|_{0,l} |g|_{\sigma,m} + |f|_{\sigma,l} |g|_{0,m})$ .

If  $\sigma \geq 1$  and  $f, z \in \mathcal{S}_\sigma$  then  $f \circ z \in \mathcal{S}_\sigma$ ,

- d.  $|f \circ z|_{\sigma,l+m} \leq C(\sigma) (|f|_{\sigma,l} |\nabla z|_{0,m}^\sigma + |f|_{1,l} |\nabla z|_{\sigma-1,m} + |f|_{0,l+m})$ .

*Proof.* The proof consists of a straightforward application of Proposition A.2 (see Appendix A). Properties a. and b. are obvious, we verify the others

c.

$$\begin{aligned} |fg|_{\sigma,l+m} &= \sup_{t \in J} |f^t g^t|_{C^\sigma} t^{l+m} \leq C(\sigma) \sup_{t \in J} (|f^t|_{C^0} |g^t|_{C^\sigma} + |f^t|_{C^\sigma} |g^t|_{C^0}) t^{l+m} \\ &\leq C(\sigma) \sup_{t \in J} (|f^t|_{C^0} t^l |g^t|_{C^\sigma} t^m + |f^t|_{C^\sigma} t^l |g^t|_{C^0} t^m) \\ &\leq C(\sigma) (|f|_{0,l} |g|_{\sigma,m} + |f|_{\sigma,l} |g|_{0,m}) \end{aligned}$$

d.

$$\begin{aligned} |f \circ z|_{\sigma,l+m} &= \sup_{t \in J} |f^t \circ z^t|_{C^\sigma} t^{l+m} \\ &\leq C(\sigma) \sup_{t \in J} (|f^t|_{C^\sigma} |\nabla z^t|_{C^0}^\sigma + |f^t|_{C^1} |\nabla z^t|_{C^{\sigma-1}} + |f|_{C^0}) t^{l+m} \\ &\leq C(\sigma) \sup_{t \in J} (|f^t|_{C^\sigma} t^l |\nabla z^t|_{C^0}^\sigma t^{\sigma m} t^{(1-\sigma)m} + |f^t|_{C^1} t^l |\nabla z^t|_{C^{\sigma-1}} t^m + |f|_{C^0} t^{l+m}) \\ &\leq C(\sigma) (|f|_{\sigma,l} |\nabla z|_{0,m}^\sigma + |f|_{1,l} |\nabla z|_{\sigma-1,l} + |f|_{0,l+m}) \end{aligned}$$

notice that  $t \geq 1$  and  $1 - \sigma \leq 0$  imply  $t^{(1-\sigma)m} \leq 1$ . □



Now, we state the main theorem that we shall prove in this chapter. Let  $s, \lambda, \rho, \beta$  and  $\alpha$  be positive parameters satisfying the following conditions

$$\begin{cases} 1 \leq \rho < \lambda < s, \\ s > \max \left\{ \frac{\alpha}{\alpha-1}, \lambda + \frac{\alpha}{\beta-1} \right\}, & 1 < \beta < 2, \quad \alpha > 1, \\ \lambda > \frac{2\beta}{2-\beta}, \quad \rho < \frac{\lambda-\beta}{\beta^2}. \end{cases} \quad (\#_E)$$

Given  $\omega \in \mathbb{R}^n$  and real positive parameters  $\delta > 0, \varepsilon > 0$  and  $\Upsilon \geq 1$ , we consider the following time-dependent Hamiltonian

$$\begin{cases} H : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \longrightarrow \mathbb{R} \\ H(q, p, t) = \omega \cdot p + a(q, t) + \underbrace{(b_0(q, t) + b_r(q, t))}_{b(q, t)} \cdot p + m(q, p, t) \cdot p^2 \\ a, b_0, b_r, \partial_p^2 H \in \mathcal{S}_{s+1} \\ |b_0|_{2,1} < \delta, \quad |b_0|_{s+1,1} < \Upsilon, \\ |a|_{\lambda+1,0} + |\partial_q a|_{\lambda,2} < \varepsilon, \quad |b_r|_{\lambda+1,1} < \varepsilon, \\ |a|_{s+1,0} + |\partial_q a|_{s,2} < \Upsilon, \quad |b_r|_{s+1,1} < \Upsilon, \quad |\partial_p^2 H|_{s+1,0} \leq \Upsilon, \end{cases} \quad (*_E)$$

Let  $\varphi_0$  be the following trivial embedding  $\varphi_0 : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m \times B, \varphi_0(q) = (q, 0)$ . Furthermore, we consider the Hamiltonian  $\tilde{h} : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \rightarrow \mathbb{R}$  such that

$$\tilde{h}(q, p, t) = \omega \cdot p + m(q, p, t) \cdot p^2. \quad (6.9)$$

**Theorem E.** *Let  $H$  be as in  $(*_E)$  and we assume that  $s, \lambda, \rho, \beta$  and  $\alpha$  satisfy  $(\#_E)$ . Then, for  $\delta$  small enough with respect to  $s$ , there exists  $\varepsilon_0$ , depending on  $\delta, s, \lambda, \beta, \alpha$  and  $\Upsilon$ , such that for all  $\varepsilon \leq \varepsilon_0$  there exists a  $C^\rho$ -weakly asymptotic cylinder associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ .*

## 6.3 Proof of Theorem E

### 6.3.1 The Nash-Moser Theorem (Zehnder)

The proof of Theorem E relies on a version of the Nash-Moser theorem proved by Zehnder (see [Zeh75]). For the sake of clarity, we dedicate this section to explaining this result.

We consider a one-parameter family of Banach spaces  $\{(\mathcal{X}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  and we assume that for all  $0 \leq \sigma' \leq \sigma < \infty$

$$\mathcal{X}^0 \supseteq \mathcal{X}^{\sigma'} \supseteq \mathcal{X}^\sigma \supseteq \mathcal{X}^\infty = \bigcap_{\sigma \geq 0} \mathcal{X}^\sigma$$

$$|x|_{\sigma'} \leq |x|_\sigma$$

for all  $x \in \mathcal{X}^\sigma$ . Now, we introduce the definition of  $C^\infty$ -smoothing, which plays a very important role in the proof of the Theorem.

**Definition 6.2.** A  $C^\infty$ -smoothing in  $\{(\mathcal{X}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  is a one-parameter family  $\{S_\tau\}_{\tau > 0}$  of linear mappings  $S_\tau : \mathcal{X}^0 \rightarrow \mathcal{X}^\infty$  together with constants  $C(m, d)$ , for positive integers  $m$  and  $d$ , satisfying the following conditions:

$$|S_\tau x|_m \leq \tau^{m-d} C(m, d) |x|_d \quad (\text{S1})$$

for all  $x \in \mathcal{X}^d$  and  $0 \leq d \leq m$ ,

$$|(S_\tau - 1)x|_d \leq \tau^{-(m-d)} C(m, d) |x|_m \quad (\text{S2})$$

for all  $x \in \mathcal{X}^m$  and  $0 \leq d \leq m$ .

The existence of a  $C^\infty$ -smoothing implies the following well-known convexity property.

**Lemma 6.1.** We assume that  $\{(\mathcal{X}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  has a  $C^\infty$ -smoothing. Then, for all  $0 \leq \lambda_1 \leq \lambda_2$ ,  $\alpha \in [0, 1]$  and  $x \in \mathcal{X}^{\lambda_2}$ ,

$$|x|_\lambda \leq C(\alpha, \lambda_1, \lambda_2) |x|_{\lambda_1}^{1-\alpha} |x|_{\lambda_2}^\alpha$$

with  $\lambda = (1 - \alpha)\lambda_1 + \alpha\lambda_2$ .

*Proof.* We refer to [Zeh75] for the proof. □

In what follows, we report the hypotheses of the Zehnder theorem. We consider three one-parameter families of Banach spaces  $\{(\mathcal{X}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ ,  $\{(\mathcal{V}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  and  $\{(\mathcal{Z}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  each with a  $C^\infty$ -smoothing, which we denote by the same letter  $\{S_\tau\}_{\tau > 0}$ . Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{X}^0 \times \mathcal{V}^0 \longrightarrow \mathcal{Z}^0$$

and we assume that

$$\mathcal{F}(u_0, v_0) = 0$$

for some  $(u_0, v_0) \in \mathcal{X}^0 \times \mathcal{V}^0$ . Given a positive parameter  $0 < \zeta \leq 1$ , we define

$$\mathcal{O}_\zeta^\sigma = \{(x, v) \in \mathcal{X}^\sigma \times \mathcal{V}^\sigma : |x - x_0|_\sigma, |v - v_0|_\sigma < \zeta\} \quad (6.10)$$

and we consider  $\mathcal{F} : \mathcal{O}_\zeta^0 \rightarrow \mathcal{Z}^0$  to be continuous. In his work, Zehnder takes  $\zeta = 1$ . We will see that this does not really change the proof. But, as we shall see in the proof of Theorem E, we need  $\zeta$  to satisfy a suitable smallness condition to define the right inverse of our functional  $\mathcal{F}$ .

For given  $x \in \mathcal{X}^0 \cap \mathcal{O}_\zeta^0$ , the aim of the Zehnder theorem is to solve the equation  $\mathcal{F}(x, v) = 0$  assuming  $x$  sufficiently close to  $x_0$ . The author makes the following hypotheses.

### Hypotheses H.1-H.4

*H.1 Smoothness:* We assume that  $\mathcal{F}(x, \cdot) : \mathcal{V}^0 \rightarrow \mathcal{Z}^0$  is two times differentiable with the uniform estimate

$$|D_v \mathcal{F}(x, v)|_0, |D_v^2 \mathcal{F}(x, v)|_0 \leq C$$

for all  $(x, v) \in \mathcal{O}_\zeta^0$  and for some constant  $C \geq 1$ , where  $D_v$  is the differential with respect to the second component.

*H.2  $\mathcal{F}$  is uniformly Lipschitz in  $\mathcal{X}^0$ :* For all  $(x_1, v), (x_2, v) \in \mathcal{O}_\zeta^0$ ,

$$|\mathcal{F}(x_1, v) - \mathcal{F}(x_2, v)|_0 \leq C|x_1 - x_2|_0.$$

*H.3 Existence of a right-inverse of loss  $\gamma$ ,  $1 \leq \gamma < s$  ( $s$  will be specified later):* For every  $(x, v) \in \mathcal{O}_\zeta^\gamma$  there exists a linear map  $\eta(x, v) : \mathcal{Z}^\gamma \rightarrow \mathcal{V}^0$  such that, for all  $z \in \mathcal{Z}^\gamma$ ,

$$\begin{aligned} D_v \mathcal{F}(x, v) \circ \eta(x, v)z &= z \\ |\eta(x, v)z|_0 &\leq C|z|_\gamma. \end{aligned} \tag{\eta 1}$$

Moreover, for all  $\gamma \leq \sigma \leq s$ , if  $(x, v) \in \mathcal{O}_\zeta^\gamma \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)$ , then the linear map  $\eta : \mathcal{Z}^\sigma \rightarrow \mathcal{V}^{\sigma-\gamma}$  is well defined and if  $|x - x_0|_\sigma, |v - v_0|_\sigma \leq K$ , then

$$|\eta(x, v)\mathcal{F}(x, v)|_{\sigma-\gamma} \leq C(\sigma)K. \tag{\eta 2}$$

*H.4 Order:* The triple  $(\mathcal{F}, x_0, v_0)$  is of order  $s$ ,  $s > \gamma \geq 1$ . Here, Zehnder uses the following

**Definition 6.3.**  *$(\mathcal{F}, x_0, v_0)$  is called of order  $s$ ,  $1 \leq s < \infty$ , if the following three conditions are satisfied:*

1.  $(x_0, v_0) \in \mathcal{X}^s \times \mathcal{V}^s$ ,
2.  $\mathcal{F}(\mathcal{O}_\zeta^0 \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)) \subset \mathcal{Z}^\sigma$ ,  $1 \leq \sigma \leq s$
3. *there exist constants  $C(\sigma)$ ,  $1 \leq \sigma \leq s$ , such that if  $(x, v) \in (\mathcal{X}^\sigma \times \mathcal{V}^\sigma) \cap \mathcal{O}_\zeta^1$  satisfies  $|x - x_0|_\sigma, |v - v_0|_\sigma \leq K$  then*

$$|\mathcal{F}(x, v)|_\sigma \leq C(\sigma)K.$$

Zehnder, in his paper, assumes the existence of an approximate right-inverse. The reason is that, in his works [Zeh75] and [Zeh76], he wants to apply generalized implicit function theorems to solve some small divisor problems. In particular, to prove Arnold's normal form theorem for vector fields on the torus and the KAM theorem. In the proof of the previous theorems, the author defines a functional  $\mathcal{F}$  which does not admit a right-inverse but just an approximate right-inverse. Here, we do not have this problem; hence we prefer to write H.4 in this form.

**Theorem 6.1** (Zehnder). *Let  $\alpha, \beta, \lambda, \rho, \gamma$  and  $s$  be positive real numbers satisfying the following set of inequalities:*

$$1 < \beta < 2, \quad 1 < \alpha, \quad 1 \leq \gamma \leq \rho < \lambda < s, \tag{6.11}$$

$$\lambda > \max\left\{\frac{2\beta\gamma}{2-\beta}, \beta(\gamma + \rho\beta)\right\} \tag{6.12}$$

$$s > \max\left\{\frac{\alpha\gamma}{\alpha-1}, \lambda + \frac{\alpha\gamma}{\beta-1}\right\}. \tag{6.13}$$

Let  $(\mathcal{F}, x_0, v_0)$  be of order  $s$  and satisfy H.1-H.4 with a loss of  $\gamma$ . Then there exists  $\varepsilon_0$ , depending on  $\alpha, \beta, \lambda, \gamma, s$  and  $\zeta$ , such that for all  $\varepsilon \leq \varepsilon_0$  we have the existence of an open neighborhood  $\mathcal{D}^\lambda \subset \mathcal{X}^\lambda$  of  $x_0$ ,  $\mathcal{D}^\lambda = \{x \in \mathcal{X}^\lambda : |x - x_0|_\lambda \leq \varepsilon\}$  and a mapping  $\psi : \mathcal{D}^\lambda \rightarrow \mathcal{V}^\rho$  such that

$$\begin{aligned}\mathcal{F}(x, \psi(x)) &= 0, \quad x \in \mathcal{D}^\lambda \\ |\psi(x) - v_0|_\rho &\leq \zeta,\end{aligned}$$

where  $\zeta$  is the positive parameter defined by (6.10).

*Proof.* The statement of the theorem is slightly different from the original. As mentioned before, Zehnder considers  $\zeta = 1$ . For the sake of clarity, in what follows, we will report the parts that differ from the original proof.

The proof uses an iteration technique similar to the Newton algorithm modified by a double  $C^\infty$ -smoothing. The first is introduced in  $\mathcal{V}^0$  to regain the loss of derivatives  $\gamma$  at each iteration step. The second approximates elements in  $\mathcal{D}^\lambda \subset \mathcal{X}^\lambda$  with smoother ones to keep the loss of regularity minimal in the  $\mathcal{X}^0$  space. Zehnder makes great use of Lemma 6.1, he estimates the lowest norms  $|\cdot|_0$  very carefully to keep them down and the highest norms  $|\cdot|_s$  are left to grow. Then, he uses the aforementioned Lemma 6.1 to estimate the intermediate norms.

Let  $\varepsilon = v\varepsilon_0$  for some  $0 < v \leq 1$  and a sufficiently small parameter  $\varepsilon_0$  to be determined later. We define

$$\mathcal{D}^\lambda = \{x \in \mathcal{X}^\lambda : |x - x_0|_\lambda \leq v\varepsilon_0\}.$$

Following the lines of the proof of Zehnder, we define a sequence  $\{\phi_j\}_{j \geq 0}$  of linear mapping  $\phi_j : \mathcal{D}^\lambda \rightarrow \mathcal{X}^\infty$  in such a way that,

$$\begin{aligned}\phi_0(x) &= x_0 \\ \phi_j(x) - x_0 &= S_{\tau_j}(x - x_0),\end{aligned}$$

for all  $j \geq 1$ , where  $\tau_j = Q^{\beta j}$  for some  $Q > 1$  sufficiently large to be chosen later. Since  $\beta > 1$  then  $\tau_j \rightarrow +\infty$  if  $j \rightarrow +\infty$ . By the latter, we write  $\phi_j(x) - x$  in the following form

$$\phi_j(x) - x = (S_{\tau_j} - 1)(x - x_0).$$

Thanks to (S2), for all  $0 \leq \mu < \lambda$

$$|\phi_j(x) - x|_\mu = |(S_{\tau_j} - 1)(x - x_0)|_\mu \leq \tau_j^{\mu - \lambda} C(\lambda, \mu) |x - x_0|_\lambda \leq \tau_j^{\mu - \lambda} C(\lambda, \mu) v\varepsilon_0$$

and taking the limit for  $j \rightarrow +\infty$ , we have  $\lim_{j \rightarrow +\infty} |\phi_j(x) - x|_\mu = 0$  for all  $0 \leq \mu < \lambda$ . We construct inductively a sequence of mapping  $\{\psi_j\}_{j \geq 0}$ ,  $\psi_j : \mathcal{D}^\lambda \rightarrow \mathcal{V}^\infty$  such that

$$\begin{aligned}\psi_0(x) &= v_0 \\ \psi_{j+1}(x) - \psi_j(x) &= S_{t_{j+1}} \eta(\phi_{j+1}(x), \psi_j(x)) \mathcal{F}(\phi_{j+1}(x), \psi_j(x))\end{aligned}$$

for all  $j \geq 0$ , with  $t_j = \tau_j^\alpha = Q^{\alpha\beta j}$ . We use two different rates of approximations,  $S_{\tau_j}$  and  $S_{t_j}$ , for the family of Banach spaces  $\{(\mathcal{X}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  and  $\{(\mathcal{V}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ , respectively. We shall show by induction that, if  $\varepsilon_0$  is sufficiently small with respect to  $\alpha, \beta, \lambda, \gamma, s$  and  $\zeta$ , and  $x \in \mathcal{X}^\lambda$  satisfies  $|x - x_0|_\lambda \leq v\varepsilon_0$ , then the following statements  $S(d)$  hold for  $d \geq 1$ :

$S(d, 1)$   $(\phi_d(x), \psi_d(x)) \in \mathcal{O}_\zeta^\gamma \cap (\mathcal{X}^\infty \times \mathcal{V}^\infty)$  and  $|\mathcal{F}(\phi_d(x), \psi_d(x))|_0 \leq \frac{1}{2}vQ^{-\lambda\beta^d}$

$S(d, 2)$   $|\psi_d(x) - \psi_{d-1}(x)|_0 \leq CvQ^{-(\lambda-\beta\gamma)\beta^{d-1}}$ ,

$S(d, 3)$   $|\psi_d(x) - \psi_{d-1}(x)|_s \leq vQ^{(s-\lambda)\beta^{d+1}}$ ,

for a suitable constant  $C$ . The only difference compared to the proof of Zehnder is in  $S(d, 1)$ , where he considers  $\zeta = 1$ . In what follows, we verify only the first part of  $S(d, 1)$ . The rest follows from the original proof.

We introduce the abbreviated notation

$$x_j = \phi_j(x) \quad \text{and} \quad v_j = \psi_j(x).$$

We claim that  $x_j \in \mathcal{O}_\zeta^\gamma \cap \mathcal{X}^\infty$  if  $\varepsilon_0$  is sufficiently small. We remind that  $1 \leq \gamma < \lambda$ , then, by the definition of  $x_j$  and (S1),

$$|x_j - x_0|_\gamma = |S_{\tau_j}(x - x_0)|_\gamma \leq C|x - x_0|_\gamma \leq C|x - x_0|_\lambda$$

for a suitable constant  $C$ . Letting  $C\varepsilon_0 < \zeta$ , we have the claim.

Zehnder proves the totality of the three statements  $S(d)$  by induction. Letting  $d = 1$ ,  $S(1)$  follows from the smallness condition by choosing  $\varepsilon_0$  sufficiently small with respect to  $\alpha, \beta, \lambda, \gamma, s$  and  $\zeta$ . Now, we assume  $S(d)$  for  $1 \leq d \leq j$  and we prove  $S(j+1)$ . We do not provide any details concerning the proofs of  $S(j+1, 2)$  and  $S(j+1, 3)$  because they coincide with the original proof. We verify that  $|v_{j+1} - v_0|_\gamma < \zeta$ . By  $S(j+1, 2)$ ,  $S(j+1, 3)$  and Lemma 6.1

$$|v_{j+1} - v_j|_\gamma \leq C(\gamma, s)|v_{j+1} - v_j|_0^{1-\frac{\gamma}{s}}|v_{j+1} - v_j|_s^{\frac{\gamma}{s}} \leq vCQ^{-\xi\beta^j}$$

for a suitable constant  $C$  and with

$$\begin{aligned} \xi &= \lambda - \beta(\gamma + \beta\gamma) + \frac{\gamma}{s}(\lambda(\beta^2 - 1) + \beta\gamma) \\ &> \lambda - \beta(\gamma + \beta\gamma) > \lambda - \beta(\gamma + \beta\rho) \end{aligned}$$

where we have used  $\gamma \leq \rho$  and  $\beta > 1$ . This means that,

$$|v_{j+1} - v_0|_\gamma \leq \sum_{d=0}^j |v_{d+1} - v_d|_\gamma \leq Cv \sum_{d \geq 0} Q^{-\xi\beta^d} < \zeta.$$

for  $Q$  large enough. □

Zehnder distinguishes between the order of  $(\mathcal{F}, x_0, v_0)$  and the smoothness assumption quantified by  $\lambda$ . In his work, Zehnder states that the minimal order for which we can apply the previous theorem, and thus the minimal order which assures the convergence of the algorithm, is  $s \geq 8\gamma$ , where we point out that  $s < \infty$ . Moreover, concerning the corresponding minimal smoothness assumption for  $\lambda$ , one has  $3\gamma < \lambda < 4\gamma$ .

**Corollary 6.1.** *For all  $s \geq 8\gamma$ , the following holds: let  $\lambda(s) = 2\gamma + 14\frac{\gamma^2}{s}$ , there exists in  $\mathcal{X}^{\lambda(s)}$  a neighborhood  $\mathcal{D}^{\lambda(s)} = \{x \in \mathcal{X}^{\lambda(s)} : |x - x_0|_{\lambda(s)} \leq \varepsilon(s, \zeta)\}$  and a mapping  $\psi_s : \mathcal{D}^{\lambda(s)} \rightarrow \mathcal{V}^\gamma$  such that, for all  $x \in \mathcal{D}^{\lambda(s)}$ ,*

$$\begin{aligned} \mathcal{F}(x, \psi_s(x)) &= 0, \\ |\psi_s(x) - v_0|_\gamma &\leq \zeta. \end{aligned}$$

*Proof.* Taking  $\alpha = \frac{7}{6}$ ,  $\beta = 1 + \frac{7\gamma}{3s}$ ,  $\rho = \gamma$ , and  $\lambda = 2\gamma + 14\frac{\gamma^2}{s}$ , these numbers satisfy the inequalities (6.11) - (6.13) if  $s \geq 8\gamma$  and hence the result follows from Theorem 6.1.  $\square$

### 6.3.2 Outline of the Proof of Theorem E

We are looking for a  $C^\rho$ -weakly asymptotic cylinder  $\varphi$  associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ , where  $H$  is the Hamiltonian defined by  $(*_E)$ ,  $\tilde{h}$  the Hamiltonian in (6.9) and  $\varphi_0$  the trivial embedding  $\varphi_0 : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m \times B$ ,  $\varphi_0(q) = (q, 0)$ . More concretely, for given  $H$ , we are searching for  $v, \Gamma : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m}$  such that for all  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$

$$\varphi(q, t) = (q, v(q, t))$$

and in such a way that  $\varphi, v$  and  $\Gamma$  satisfy

$$\begin{aligned} X_H(\varphi(q, t), t) - \partial_q \varphi(q, t)(\bar{\omega} + \Gamma(q, t)) - \partial_t \varphi(q, t) &= 0, \\ \lim_{t \rightarrow +\infty} |v^t|_{C^\rho} = 0, \quad \lim_{t \rightarrow +\infty} |\Gamma^t|_{C^\rho} &= 0, \end{aligned} \tag{6.14}$$

for all  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ . The vector  $\bar{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$  where  $\omega \in \mathbb{R}^n$  is the frequency vector introduced by  $(*_E)$ . As mentioned above, the proof rests on Theorem 6.1. To this end, we need to introduce a suitable functional  $\mathcal{F}$  given by (6.14). First, we recall that

$$\bar{m}(q, p, t)p = \left( \int_0^1 \partial_p^2 H(q, \tau p, t) d\tau \right) p = \partial_p \left( m(q, p, t) \cdot p^2 \right).$$

However, concerning the definition of the functional  $\mathcal{F}$ , the Hamiltonian system associated to the Hamiltonian  $H$  is equal to

$$X_H(q, p, t) = \begin{pmatrix} \bar{\omega} + b(q, t) + \bar{m}(q, p, t)p \\ -\partial_q a(q, t) - \partial_q b(q, t)p - \partial_q m(q, p, t)p^2 \end{pmatrix},$$

where  $H$  is the Hamiltonian defined by  $(*_E)$ . Let

$$\tilde{\varphi}(q, t) = (q, v(q, t), t)$$

for all  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ , then  $X_H \circ \tilde{\varphi}$  takes the following form

$$X_H \circ \tilde{\varphi}(q, t) = \begin{pmatrix} \bar{\omega} + b(q, t) + (\bar{m} \circ \tilde{\varphi}(q, t))v(q, t) \\ -\partial_q a(q, t) - \partial_q b(q, t)v(q, t) - \partial_q m \circ \tilde{\varphi}(q, t)v^2(q, t) \end{pmatrix}$$

for all  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ . Moreover, for all  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ ,

$$\partial_q \varphi(q, t) (\bar{\omega} + \Gamma(q, t)) + \partial_t \varphi(q, t) = \begin{pmatrix} \bar{\omega} + \Gamma(q, t) \\ \partial_q v(q, t) (\omega + \Gamma(q, t)) + \partial_t v(q, t) \end{pmatrix},$$

and hence, we can rewrite (6.14) in the following form

$$\begin{pmatrix} \Gamma - b - (\bar{m} \circ \tilde{\varphi}) v \\ \partial_q a + (\partial_q b) v + \partial_q m \circ \tilde{\varphi} \cdot v^2 - \partial_q v (\omega + \Gamma) - \partial_t v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (6.15)$$

This is composed of sums and products of functions defined on  $(q, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ . We have omitted the arguments  $(q, t)$ . We keep this notation for the rest of the proof. We define

$$(\nabla v) \bar{\Omega} = (\partial_q v) \bar{\omega} + \partial_t v$$

and over suitable Banach spaces that we will specify later, let  $\mathcal{F}$  be the following functional

$$\mathcal{F}(a, b, m, \bar{m}, v) = (\nabla v) \bar{\Omega} + \partial_q v (b + (\bar{m} \circ \tilde{\varphi}) v) + \partial_q a + (\partial_q b) v + \partial_q m \circ \tilde{\varphi} \cdot v^2.$$

This is obtained by the second equation of (6.15), where we have replaced  $\Gamma$  with  $b + (\bar{m} \circ \tilde{\varphi}) v$ . This is our starting point. We observe that, for all  $b, m$  and  $\bar{m}$ ,

$$\mathcal{F}(0, b, m, \bar{m}, 0) = 0.$$

Therefore, we can reformulate our problem in the following form. For fixed  $m$  and  $\bar{m}$ , for a suitable  $b_0$  and for  $(a, b)$  sufficiently close to  $(0, b_0)$ , we are looking for a function  $v : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m}$  in such a way that  $\mathcal{F}(a, b, m, \bar{m}, v) = 0$  and  $\lim_{t \rightarrow +\infty} |v^t|_{C^\rho} = 0$ . Once we have  $v$ , we let  $\Gamma = b + (\bar{m} \circ \tilde{\varphi}) v$  and this concludes the proof.

As one might expect, because of the term  $\partial_q v (b + (\bar{m} \circ \tilde{\varphi}) v)$ , which does not appear in the proof of Theorem A, we are not able to prove Theorem E with the implicit function theorem.

The most technical part of the proof consists in showing that the differential of  $\mathcal{F}$  with respect to the variable  $v$  admits a right-inverse. Let

$$\begin{aligned} f &= b + (\bar{m} \circ \tilde{\varphi}) v, \\ g &= \partial_q b + \partial_q v (\partial_p \bar{m} \circ \tilde{\varphi}) v + \partial_q v (\bar{m} \circ \tilde{\varphi}) + v^T (\partial_{pq}^2 m \circ \tilde{\varphi}) v + 2 (\partial_q m \circ \tilde{\varphi}) v, \end{aligned}$$

where  $T$  denotes the transpose. Over suitable Banach spaces, the differential of  $\mathcal{F}$  with respect to the variable  $v$  is equal to

$$D_v \mathcal{F}(a, b, m, v) \hat{v} = (\nabla \hat{v}) \bar{\Omega} + (\partial_q \hat{v}) f + g \hat{v}.$$

We will see that, assuming  $f$  and  $g$  sufficiently small, we are able to find a right inverse of the latter.

The proof of Theorem E is split up into the following four sections. The first is dedicated to introducing the special one-parameter families of Banach spaces on which the functional  $\mathcal{F}$  is defined. The second is about the solution of the homological equation. In other words, we prove the existence of a right-inverse for the latter differential. In the third section, we verify that the functional  $\mathcal{F}$  satisfies the hypotheses of the Nash-Moser theorem proved by Zehnder (Theorem 6.1) and in the last section, we conclude the proof.

### 6.3.3 Preliminary Settings

Given  $\sigma \geq 0$ , we recall the following definition

**Definition.** Let  $\mathcal{S}_\sigma$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times \mathbb{R}^m \times B \times J$  such that  $f^t \in C^\sigma(\mathbb{T}^n \times \mathbb{R}^m \times B)$  for all fixed  $t \in J$  and  $\partial_{(q,p)}^i f \in C(\mathbb{T}^n \times \mathbb{R}^m \times B \times J)$  for all  $0 \leq i \leq [\sigma]$ .

We use this notation also for functions defined on  $\mathbb{T}^n \times \mathbb{R}^m \times J$ . Given  $\sigma \geq 0$  and  $l \geq 0$ , for every  $f \in \mathcal{S}_\sigma$ , we recall the definition of the following norm

$$|f|_{\sigma,l} = \sup_{t \in J} |f^t|_{C^\sigma} t^l.$$

Some properties of this norm are contained in Proposition 6.1.

Now, we consider the following families of Banach spaces  $\{(\mathcal{A}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ ,  $\{(\mathcal{B}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ ,  $\{(\mathcal{V}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ ,  $\{(\mathcal{Z}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  (see Appendix C) such that, for all  $\sigma \geq 0$ ,

$$\begin{aligned} \mathcal{A}^\sigma &= \left\{ a : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R} \mid a \in \mathcal{S}_{\sigma+1} \text{ and } |a|_\sigma = |a|_{\sigma+1,0} + |\partial_q a|_{\sigma,2} < \infty \right\} \\ \mathcal{B}^\sigma &= \left\{ b : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid b \in \mathcal{S}_{\sigma+1} \text{ and } |b|_\sigma = |b|_{\sigma+1,1} < \infty \right\} \\ \mathcal{V}^\sigma &= \left\{ v : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid v \in \mathcal{S}_{\sigma+1}, (\nabla v) \bar{\Omega} \in \mathcal{S}_\sigma \right. \\ &\quad \left. \text{and } |v|_\sigma = \max\{|v|_{\sigma+1,1}, |(\nabla v) \bar{\Omega}|_{\sigma,2}\} < \infty \right\} \\ \mathcal{Z}^\sigma &= \left\{ z : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid z \in \mathcal{S}_\sigma, \text{ and } |z|_\sigma = |z|_{\sigma,2} < \infty \right\} \end{aligned}$$

It is straightforward to verify that, for all  $0 \leq \sigma' \leq \sigma < \infty$ ,

$$\mathcal{A}^0 \supseteq \mathcal{A}^{\sigma'} \supseteq \mathcal{A}^\sigma \supseteq \mathcal{A}^\infty = \bigcap_{\sigma \geq 0} \mathcal{A}^\sigma, \quad \mathcal{B}^0 \supseteq \mathcal{B}^{\sigma'} \supseteq \mathcal{B}^\sigma \supseteq \mathcal{B}^\infty = \bigcap_{\sigma \geq 0} \mathcal{B}^\sigma,$$

$$|a|_{\sigma'} \leq |a|_\sigma \quad |b|_{\sigma'} \leq |b|_\sigma$$

$$\mathcal{V}^0 \supseteq \mathcal{V}^{\sigma'} \supseteq \mathcal{V}^\sigma \supseteq \mathcal{V}^\infty = \bigcap_{\sigma \geq 0} \mathcal{V}^\sigma, \quad \mathcal{Z}^0 \supseteq \mathcal{Z}^{\sigma'} \supseteq \mathcal{Z}^\sigma \supseteq \mathcal{Z}^\infty = \bigcap_{\sigma \geq 0} \mathcal{Z}^\sigma,$$

$$|v|_{\sigma'} \leq |v|_\sigma \quad |z|_{\sigma'} \leq |z|_\sigma$$

for all  $a \in \mathcal{A}_\sigma$ ,  $b \in \mathcal{B}_\sigma$ ,  $v \in \mathcal{V}_\sigma$  and  $z \in \mathcal{Z}_\sigma$ .

This part aims to prove the existence of a  $C^\infty$ -smoothing, see Definition 6.2, for these families of Banach spaces. This is not surprising because the behaviour of these norms is very similar to that of the Hölder norms.

**Lemma 6.2.** *There exists a  $C^\infty$ -smoothing for the latter families of Banach spaces.*

*Proof.* We begin by proving the existence of a  $C^\infty$ -smoothing for the family of Banach spaces  $\{(\mathcal{Z}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ . Following the lines of [Zeh75], we take a function



$\tilde{s} \in C_0^\infty(\mathbb{R}^{n+m})$  vanishing outside a compact set and identically equal to 1 in a neighbourhood of 0. Let  $s$  be its Fourier transform then, for all  $z \in \mathcal{S}_0$ ,

$$S_\tau z(q, t) = \frac{1}{\tau^{n+m}} \int_{\mathbb{R}^{n+m}} s\left(\frac{q - \vartheta}{\tau}\right) z(\vartheta, t) d\vartheta.$$

For all fixed  $t \in J$ ,  $S_\tau z^t \in C^\infty(\mathbb{T}^n \times \mathbb{R}^m) = \bigcap_{\sigma \geq 0} C^\sigma(\mathbb{T}^n \times \mathbb{R}^m)$  (see [Zeh75]). Now, we verify that  $\partial_q^i (S_\tau z) \in C(\mathbb{T}^n \times \mathbb{R}^m \times J)$  for all  $i \geq 0$ . We observe that, for every  $m > 0$  and  $p > 0$ , there exists a constant  $C(m, p) > 0$  such that

$$|\partial^m s(x)| \leq C(m, p)(1 + |x|)^{-p},$$

where  $\partial^m$  stands for partial derivatives of order  $m$ . The claim is a consequence of the regularity of  $z$  and the latter. Indeed, for all  $(q_1, t_1), (q_2, t_2) \in \mathbb{T}^n \times \mathbb{R}^m \times J$  and  $i \geq 0$ ,

$$\begin{aligned} & \left| \partial_q^i (S_\tau z)(q_1, t_1) - \partial_q^i (S_\tau z)(q_2, t_2) \right| \\ &= \left| \frac{1}{\tau^{n+m+i}} \int_{\mathbb{R}^{n+m}} \partial^i s\left(\frac{q_1 - \vartheta}{\tau}\right) z(\vartheta, t_1) d\vartheta - \frac{1}{\tau^{n+m+i}} \int_{\mathbb{R}^{n+m}} \partial^i s\left(\frac{q_2 - \vartheta}{\tau}\right) z(\vartheta, t_2) d\vartheta \right| \\ &= \frac{1}{\tau^i} \left| \int_{\mathbb{R}^{n+m}} \partial^i s(\rho) (z(q_1 - \rho\tau, t_1) - z(q_2 - \rho\tau, t_2)) d\rho \right| \\ &\leq \frac{1}{\tau^i} \int_{\mathbb{R}^{n+m}} |\partial^i s(\rho)| |(z(q_1 - \rho\tau, t_1) - z(q_2 - \rho\tau, t_2))| d\rho \end{aligned}$$

where  $|\cdot|$  stands for the standard Euclidean norm and, in the last line of the latter, we did the following change of coordinates  $\frac{q_i - \vartheta}{\tau} = \rho$  for  $i = 1, 2$ . This implies the claim.

It remains to prove (S1) and (S2). For all  $z \in \mathcal{Z}^d$ ,  $0 \leq d \leq m$  and fixed  $t \in J$

$$|S_\tau z^t|_{C^m} \leq \tau^{m-d} C(m, d) |z^t|_{C^d}$$

(always look at [Zeh75]), then

$$|S_\tau z|_m = \sup_{t \in J} |S_\tau z^t|_{C^m} t^2 \leq \tau^{m-d} C(m, d) \sup_{t \in J} |z^t|_{C^d} t^2 = \tau^{m-d} C(m, d) |z|_d$$

and (S1) is verified. For all  $z \in \mathcal{Z}^m$ ,  $0 \leq d \leq m$  and fixed  $t \in J$

$$|(S_\tau - 1)z^t|_{C^d} \leq \tau^{-(m-d)} C(m, d) |z^t|_{C^m}$$

(always see [Zeh75]), then

$$\begin{aligned} |(S_\tau - 1)z|_d &= \sup_{t \in J} |(S_\tau - 1)z^t|_{C^d} t^2 \leq \tau^{-(m-d)} C(m, d) \sup_{t \in J} |z^t|_{C^m} t^2 \\ &\leq \tau^{-(m-d)} C(m, d) |z|_m \end{aligned}$$

and (S2) is also verified. This implies the existence of a  $C^\infty$ -smoothing for  $\{(\mathcal{Z}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ . Remembering that  $S_\tau$  commutes with partial differential operators, similarly, we have the claim for the family of Banach spaces  $\{(\mathcal{A}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  and  $\{(\mathcal{B}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ .

It remains to prove the existence of a  $C^\infty$ -smoothing for  $\{(\mathcal{V}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$ . Similarly to the previous case, because of  $S_\tau$  commutes with partial differential operators,  $S_\tau : \mathcal{V}^0 \rightarrow \mathcal{V}^\infty$ .

Now, we verify (S1) and (S2). We begin by remembering that, for all  $v \in \mathcal{V}^\sigma$

$$|v|_\sigma = \max\{|v|_{\sigma+1,1}, |(\nabla v) \bar{\Omega}|_{\sigma,2}\}.$$

Similarly to the previous case, for all  $v \in \mathcal{V}^d$ ,  $0 \leq d \leq m$  and fixed  $t \in J$

$$|S_\tau v^t|_{C^{m+1}} \leq \tau^{m-d} C(m, d) |v^t|_{C^{d+1}},$$

which implies

$$|S_\tau v|_{m+1,1} = \sup_{t \in J} |S_\tau v^t|_{C^{m+1}} t \leq \tau^{m-d} C(m, d) \sup_{t \in J} |v^t|_{C^{d+1}} t \leq \tau^{m-d} C(m, d) |v|_d.$$

Noting that  $S_\tau$  commutes with partial differential operators, for fixed  $t \in J$ ,

$$|\nabla (S_\tau v^t) \bar{\Omega}|_{C^m} = |S_\tau (\nabla v^t \bar{\Omega})|_{C^m} \leq \tau^{m-d} C(m, d) |\nabla v^t \bar{\Omega}|_{C^d}.$$

Multiplying both sides of the latter by  $t^2$  and taking the sup for all  $t \in J$

$$\begin{aligned} |\nabla (S_\tau v) \bar{\Omega}|_{m,2} &= \sup_{t \in J} |\nabla (S_\tau v^t) \bar{\Omega}|_{C^m} t^2 \leq \tau^{m-d} C(m, d) \sup_{t \in J} |\nabla v^t \bar{\Omega}|_{C^d} t^2 \\ &\leq \tau^{m-d} C(m, d) |v|_d. \end{aligned}$$

This implies (S1) because

$$|S_\tau v|_m = \max\{|S_\tau v|_{m+1,1}, |\nabla (S_\tau v) \bar{\Omega}|_{m,2}\} \leq \tau^{m-d} C(m, d) |v|_d.$$

Concerning (S2), for all  $v \in \mathcal{V}^m$ ,  $0 \leq d \leq m$  and fixed  $t \in J$ ,

$$|(S_\tau - 1)v^t|_{C^{d+1}} \leq \tau^{-(m-d)} C(m, d) |v^t|_{C^{m+1}}.$$

Multiplying both sides by  $t$  and taking the sup for all  $t \in J$ ,

$$\begin{aligned} |(S_\tau - 1)v|_{d+1,1} &= \sup_{t \in J} |(S_\tau - 1)v^t|_{C^{d+1}} t \leq \tau^{-(m-d)} C(m, d) \sup_{t \in J} |v^t|_{C^{m+1}} t \\ &\leq \tau^{-(m-d)} C(m, d) |v|_{m+1,1} \leq \tau^{-(m-d)} C(m, d) |v|_m. \end{aligned}$$

The operator  $S_\tau$  commutes with partial differential operators, then for fixed  $t \in J$ ,

$$|\nabla ((S_\tau - 1)v^t) \bar{\Omega}|_{C^d} = |(S_\tau - 1) (\nabla v^t \bar{\Omega})|_{C^d} \leq \tau^{-(m-d)} C(m, d) |\nabla v^t \bar{\Omega}|_{C^m}$$

and hence

$$\begin{aligned} |\nabla ((S_\tau - 1)v) \bar{\Omega}|_{d,2} &= \sup_{t \in J} |\nabla ((S_\tau - 1)v^t) \bar{\Omega}|_{C^d} t^2 \\ &\leq \tau^{-(m-d)} C(m, d) \sup_{t \in J} |\nabla v^t \bar{\Omega}|_{C^m} t^2 \\ &\leq \tau^{-(m-d)} C(m, d) |\nabla v \bar{\Omega}|_{m,2} \leq \tau^{-(m-d)} C(m, d) |v|_m. \end{aligned}$$

This concludes the proof of this lemma because

$$\begin{aligned} |(S_\tau - 1)v|_d &= \max\{|(S_\tau - 1)v|_{d+1,1}, |\nabla ((S_\tau - 1)v) \bar{\Omega}|_{d,2}\} \\ &\leq \tau^{-(m-d)} C(m, d) |v|_m. \end{aligned}$$

□

### 6.3.4 Homological Equation

We introduce some fundamental Gronwall-type inequalities that we widely use in this section.

**Proposition 6.2.** *Let  $J$  be an interval in  $\mathbb{R}$ ,  $t_0 \in J$ , and  $a, b, u \in C(J)$  continuous positive functions. If we assume that*

$$u(t) \leq a(t) + \left| \int_{t_0}^t b(s)u(s)ds \right|, \quad \forall t \in J$$

then it follows that

$$u(t) \leq a(t) + \left| \int_{t_0}^t a(s)b(s)e^{\left| \int_s^t b(\tau)d\tau \right|} ds \right|, \quad \forall t \in J. \quad (6.16)$$

If  $a$  is a monotone increasing function and we assume that

$$u(t) \leq a(t) + \int_{t_0}^t b(s)u(s)ds \quad \forall t \geq t_0,$$

then we obtain the estimate

$$u(t) \leq a(t)e^{\int_{t_0}^t b(s)ds}, \quad \forall t \geq t_0. \quad (6.17)$$

*Proof.* We refer to [Ama90] for the proof of (6.16). Concerning the other, by (6.16)

$$u(t) \leq a(t) + \int_{t_0}^t a(s)b(s)e^{\int_s^t b(\tau)d\tau} ds$$

for all  $t \geq t_0$ . Therefore,  $a$  is a monotone increasing function, then

$$u(t) \leq a(t) \left( 1 + \int_{t_0}^t b(s)e^{\int_s^t b(\tau)d\tau} ds \right) = a(t)e^{\int_{t_0}^t b(s)ds}$$

for all  $t \geq t_0$ . □

Given  $\sigma \geq 1$ ,  $\mu > 0$  and  $\omega \in \mathbb{R}^n$ , this section is devoted to the solution of the following equation for the unknown  $\varkappa : \mathbb{T}^n \times \mathbb{R}^m \times J \longrightarrow \mathbb{R}^{n+m}$

$$\begin{cases} \partial_q \varkappa(q, t) \underbrace{(\bar{\omega} + f(q, t))}_{F(q, t)} + \partial_t \varkappa(q, t) + g(q, t)\varkappa(q, t) = z(q, t) \\ f \in \bar{\mathcal{S}}_{\sigma+1,1}, \quad g, z \in \mathcal{S}_\sigma, \\ |f|_{1,1} \leq \mu \quad |g|_{1,1} \leq \mu, \quad |z|_{\sigma,2} < \infty, \\ |f|_{\sigma,1} < \infty \quad |g|_{\sigma,1} < \infty \end{cases} \quad (HE_E)$$

where  $\bar{\omega} = (\omega, 0) \in \mathbb{R}^{n+m}$  and  $|\cdot|_{\sigma,l}$  is the norm defined by (6.8). The functions  $f : \mathbb{T}^n \times \mathbb{R}^m \times J \longrightarrow \mathbb{R}^{n+m}$ ,  $z : \mathbb{T}^n \times \mathbb{R}^m \times J \longrightarrow \mathbb{R}^{n+m}$  and  $g : \mathbb{T}^n \times \mathbb{R}^m \times J \longrightarrow M_{n+m}$  are given, where  $M_{n+m}$  is the set of  $(n+m)$ -dimensional matrices. For the sake of

clarity, we recall that  $\partial_q \varkappa(q, t) (\omega + f(q, t))$  is a vector in  $\mathbb{R}^{n+m}$  with  $j$  component equal to

$$\left( \partial_q \varkappa(q, t) (\bar{\omega} + f(q, t)) \right)_j = \partial_q \varkappa_j(q, t) \cdot (\bar{\omega} + f(q, t))$$

for all  $1 \leq j \leq n + m$ .

If  $\mu = 0$ , the problem takes the easier form

$$\begin{cases} \partial_q \varkappa(q, t) \bar{\omega} + \partial_t \varkappa(q, t) = z(q, t) \\ z \in \mathcal{S}_\sigma, \quad |z|_{\sigma, 2} < \infty. \end{cases} \quad (6.18)$$

When  $m = 0$ , this is the homological equation we have already studied in the third chapter of this thesis (Lemma 3.1). Letting  $m \neq 0$ , following the lines of the proof of Lemma 3.1, we have the following lemma

**Lemma.** *There exists a unique solution  $\varkappa \in \mathcal{S}_\sigma$  of (6.18) such that*

$$\lim_{t \rightarrow +\infty} |\varkappa^t|_{C^0} = 0.$$

Moreover,

$$|\varkappa|_{\sigma, 1} \leq |z|_{\sigma, 2}.$$

We begin with several estimates. We define  $\psi_{t_0}^t$  as the flow at time  $t$  with initial time  $t_0$  of  $F(q, t)$ . We continue to use the notation previously introduced where  $C(\cdot)$  means constants depending on  $n, m$  and we indicate in brackets the other parameters on which these constants depend. On the other hand,  $C$  stands for constants depending only on  $n + m$ . We recall that  $\sigma \geq 1$  and we have the following estimates.

**Lemma 6.3.** *For all  $t, t_0 \in J$ , if  $t \geq t_0$*

$$|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \leq C(\sigma) \left( 1 + |f|_{\sigma, 1} \ln \left( \frac{t}{t_0} \right) \right) \left( \frac{t}{t_0} \right)^{c_\sigma \mu}, \quad (6.19)$$

whereas if  $t \leq t_0$

$$|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \leq C(\sigma) \left( 1 + |f|_{\sigma, 1} \ln \left( \frac{t_0}{t} \right) \right) \left( \frac{t_0}{t} \right)^{c_\sigma \mu}. \quad (6.20)$$

with a positive constant  $c_\sigma$  depending on  $n + m$  and  $\sigma$ .

Before the proof of this lemma, we observe that when  $\sigma = 1$  and  $t \geq t_0$ , by (6.19), we have

$$\begin{aligned} |\partial_q \psi_{t_0}^t|_{C^0} &\leq C \left( 1 + |f|_{1, 1} \ln \left( \frac{t}{t_0} \right) \right) \left( \frac{t}{t_0} \right)^{c_1 \mu} \\ &\leq C \left( 1 + \ln \left( \frac{t}{t_0} \right)^\mu \right) \left( \frac{t}{t_0} \right)^{c_1 \mu} \leq C \left( \frac{t}{t_0} \right)^{\bar{c}_1 \mu} \end{aligned}$$

for a suitable constant  $C$  depending on  $n + m$  and  $\bar{c}_1 > c_1$ . Similarly, when  $t_0 \geq t$ ,  $|\partial_q \psi_{t_0}^t|_{C^0} \leq C \left( \frac{t_0}{t} \right)^{\bar{c}_1 \mu}$ .

*Proof.* For all  $q \in \mathbb{T}^n \times \mathbb{R}^m$ , we can write  $\psi_{t_0}^t$  in the following form

$$\psi_{t_0}^t(q) = q + \int_{t_0}^t F^\tau \circ \psi_{t_0}^\tau(q) d\tau$$

as a consequence of the fundamental theorem of calculus. Taking the derivative with respect to  $q$

$$\partial_q \psi_{t_0}^t(q) = \text{Id} + \int_{t_0}^t \partial_q (f^\tau \circ \psi_{t_0}^\tau(q)) d\tau,$$

where  $\text{Id}$  stands for the identity matrix. We assume  $t \geq t_0$ , and then the norm  $C^{\sigma-1}$  of the left-hand side of the latter can be estimated as follows

$$|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \leq 1 + \int_{t_0}^t |f^\tau \circ \psi_{t_0}^\tau|_{C^\sigma} d\tau. \quad (6.21)$$

*Case  $\sigma = 1$ .* By Proposition A.2

$$|\partial_q \psi_{t_0}^t|_{C^0} \leq 1 + C \int_{t_0}^t |f^\tau|_{C^0} d\tau + C \int_{t_0}^t |f^\tau|_{C^1} |\partial_q \psi_{t_0}^\tau|_{C^0} d\tau$$

for a suitable constant  $C$ . We can easily estimate the first integral of the latter

$$\int_{t_0}^t |f^\tau|_{C^0} d\tau \leq \mu \int_{t_0}^t \frac{1}{\tau} d\tau = \ln \left( \frac{t}{t_0} \right)^\mu$$

and hence

$$|\partial_q \psi_{t_0}^t|_{C^0} \leq 1 + \ln \left( \frac{t}{t_0} \right)^{C\mu} + C \int_{t_0}^t |f^\tau|_{C^1} |\partial_q \psi_{t_0}^\tau|_{C^0} d\tau.$$

We know that  $|f|_{1,1} \leq \mu$  and thus by Gronwall's inequalities (6.17)

$$\begin{aligned} |\partial_q \psi_{t_0}^t|_{C^0} &\leq \left( 1 + \ln \left( \frac{t}{t_0} \right)^{C\mu} \right) e^{C \int_{t_0}^t |f^\tau|_{C^1} d\tau} \\ &\leq \left( 1 + \ln \left( \frac{t}{t_0} \right)^{C\mu} \right) e^{\ln \left( \frac{t}{t_0} \right)^{C\mu}} \leq C \left( \frac{t}{t_0} \right)^{\bar{c}_1 \mu} \end{aligned}$$

for a suitable constant  $\bar{c}_1 \geq 1$ .

*Case  $\sigma > 1$ .* Similarly to the previous case, by Proposition A.2

$$|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} \leq 1 + C(\sigma) \left( \int_{t_0}^t |f^\tau|_{C^0} d\tau + \int_{t_0}^t |f^\tau|_{C^\sigma} |\partial_q \psi_{t_0}^\tau|_{C^0}^\sigma d\tau + \int_{t_0}^t |f^\tau|_{C^1} |\partial_q \psi_{t_0}^\tau|_{C^{\sigma-1}} d\tau \right).$$

We have to estimate the first two integrals. We have already calculated the first one,

$$\begin{aligned} \int_{t_0}^t |f^\tau|_{C^\sigma} |\partial_q \psi_{t_0}^\tau|_{C^0}^\sigma d\tau &\leq C(\sigma) \int_{t_0}^t \frac{|f|_{\sigma,1}}{\tau} \left( \frac{\tau}{t_0} \right)^{\bar{c}_1 \sigma \mu} d\tau \\ &\leq C(\sigma) |f|_{\sigma,1} \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu} \int_{t_0}^t \tau^{-1} d\tau \\ &= C(\sigma) |f|_{\sigma,1} \ln \left( \frac{t}{t_0} \right) \left( \frac{t}{t_0} \right)^{\bar{c}_1 \sigma \mu}. \end{aligned}$$

In the second line of the latter, rather than directly integrating the member in the first line, we prefer using the trivial inequality  $\left(\frac{\tau}{t_0}\right)^{\bar{c}_1\sigma\mu} \leq \left(\frac{t}{t_0}\right)^{\bar{c}_1\sigma\mu}$  to avoid the term  $\frac{1}{\mu}$  because we do not assume that  $\mu$  is not zero. Thus, we can estimate  $|\partial_q \psi_{t_0}^t|_{C^{\sigma-1}}$  as follows

$$\begin{aligned} |\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} &\leq 1 + C(\sigma) \ln \left(\frac{t}{t_0}\right)^\mu + C(\sigma) |f|_{\sigma,1} \ln \left(\frac{t}{t_0}\right) \left(\frac{t}{t_0}\right)^{\bar{c}_1\sigma\mu} \\ &\quad + C(\sigma) \int_{t_0}^t |f^\tau|_{C^1} |\partial_q \psi_{t_0}^\tau|_{C^\sigma} d\tau. \end{aligned}$$

Furthermore, thanks to Gronwall's inequality (6.17)

$$\begin{aligned} |\partial_q \psi_{t_0}^t|_{C^{\sigma-1}} &\leq \left(1 + C(\sigma) \ln \left(\frac{t}{t_0}\right)^\mu + C(\sigma) |f|_{\sigma,1} \ln \left(\frac{t}{t_0}\right) \left(\frac{t}{t_0}\right)^{\bar{c}_1\sigma\mu}\right) e^{C(\sigma) \int_{t_0}^t |f^\tau|_{C^1} d\tau} \\ &\leq \left(1 + C(\sigma) \ln \left(\frac{t}{t_0}\right)^\mu + C(\sigma) |f|_{\sigma,1} \ln \left(\frac{t}{t_0}\right) \left(\frac{t}{t_0}\right)^{\bar{c}_1\sigma\mu}\right) e^{\ln \left(\frac{t}{t_0}\right)^{C(\sigma)\mu}} \\ &\leq C(\sigma) \left(\left(\frac{t}{t_0}\right)^\mu + |f|_{\sigma,1} \ln \left(\frac{t}{t_0}\right) \left(\frac{t}{t_0}\right)^{\bar{c}_1\sigma\mu}\right) \left(\frac{t}{t_0}\right)^{C(\sigma)\mu} \\ &\leq C(\sigma) \left(1 + |f|_{\sigma,1} \ln \left(\frac{t}{t_0}\right)\right) \left(\frac{t}{t_0}\right)^{c_\sigma\mu}, \end{aligned}$$

for a suitable constant  $c_\sigma \geq \bar{c}_1\sigma$ . This concludes the proof when  $t \geq t_0$ . Similarly, we have the other case.  $\square$

We can see that the constant  $c_\sigma \geq c_1\sigma$ . This means that  $c_\sigma$  goes to infinity when  $\sigma \rightarrow \infty$ .

We consider  $R : \mathbb{T}^n \times \mathbb{R}^m \times J \times J \rightarrow M_{n+m}$ , where  $M_{n+m}$  is the set of the  $(n+m)$ -dimensional matrices. For all  $(q, \tau, t) \in \mathbb{T}^n \times \mathbb{R}^m \times J \times J$ , let  $R(q, t, \tau) = \{r_{ij}(q, t, \tau)\}_{1 \leq i, j \leq n+m}$ . We define the following family of norms

$$|R_\tau^t|_{C^s} = \max_{1 \leq i, j \leq n} |r_{ij}(q, t, \tau)|_{C^s},$$

for positive real parameters  $s \geq 0$ . We consider the following system

$$\begin{cases} \dot{R}(q, t, \tau) = -g(\psi_{t_0}^t(q), t) R(q, t, \tau) \\ R(q, \tau, \tau) = \text{Id} \end{cases} \quad (\text{R})$$

where  $g$  is introduced in  $(HE_E)$ . In what follows, for fixed  $t, \tau \in J$ , we denote  $R_\tau^t(q) = R(q, t, \tau)$ . The following lemma is dedicated to studying the latter system and providing proper estimations of the solutions.

**Lemma 6.4.** *The latter system admits a unique solution. Moreover, for all  $\tau, t \in J$  with  $\tau \geq t$ , letting  $\tilde{R}(q, t, \tau) = R(\psi_\tau^{t_0}(q), t, \tau)$ , we have the following estimates*

$$|R_\tau^t|_{C^0} \leq \left(\frac{\tau}{t}\right)^{c_0^R \mu} \quad (6.22)$$

$$|\tilde{R}_\tau^t|_{C^\sigma} \leq C(\sigma) \left(1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left(\frac{\tau}{t}\right)\right) \left(\frac{\tau}{t}\right)^{c_\sigma^R \mu} \quad (6.23)$$

with a positive constant  $c_\sigma^R$  depending on  $n$  and  $\sigma$ , where  $\sigma$  and  $\mu$  are those defined by (HEE).

Also in this case, before the proof, we observe that when  $\sigma = 1$ , by (6.23), we have the following estimate

$$\begin{aligned} |\tilde{R}_\tau^t|_{C^1} &\leq C \left(1 + (|f|_{1,1} + |g|_{1,1}) \ln \left(\frac{\tau}{t}\right)\right) \left(\frac{\tau}{t}\right)^{c_1^R \mu} \\ &\leq C \left(1 + \ln \left(\frac{\tau}{t}\right)^\mu\right) \left(\frac{\tau}{t}\right)^{c_1 \mu} \leq C \left(\frac{\tau}{t}\right)^{\bar{c}_1^R \mu} \end{aligned}$$

for a suitable constant  $C$  depending on  $n + m$  and  $\bar{c}_1^R > c_1^R$ .

*Proof.* For all  $q \in \mathbb{T}^n \times \mathbb{R}^m$ , by the theorem of existence and uniqueness, a unique solution of (R) exists. It remains to prove the estimates. We begin with the first and then we verify the other. Similarly to the proof of the previous lemma, by the fundamental theorem of calculus, we can write  $R$  in the following form

$$R_\tau^t(q) = \text{Id} + \int_t^\tau (g^s \circ \psi_{t_0}^s(q)) R_\tau^s(q) ds \quad (6.24)$$

for all  $q \in \mathbb{T}^n \times \mathbb{R}^m$  and  $t, \tau \in J$  with  $\tau \geq t$ . Taking the norm  $C^0$  on the left-hand side of the latter, we obtain

$$|R_\tau^t|_{C^0} \leq 1 + \int_t^\tau |g^s \circ \psi_{t_0}^s R_\tau^s|_{C^0} ds \leq 1 + C \int_t^\tau |g^s|_{C^0} |R_\tau^s|_{C^0} ds,$$

for a suitable constant  $C$  depending on  $n + m$ . Thanks to Gronwall's inequality (6.17) and remembering that  $|g|_{1,1} \leq \mu$ ,

$$|R_\tau^t|_{C^0} \leq e^{C\mu \int_t^\tau \frac{1}{s} ds} \leq \left(\frac{\tau}{t}\right)^{C\mu}.$$

Hence, letting  $c_0^R = C$ , (6.22) is proved. It remains to prove (6.23). The composition of  $R_\tau^t$  with  $\psi_\tau^{t_0}(q)$  is

$$R_\tau^t \circ \psi_\tau^{t_0}(q) = \tilde{R}_\tau^t(q) = \text{Id} + \int_t^\tau (g^s \circ \psi_\tau^s(q)) \tilde{R}_\tau^s(q) ds \quad (6.25)$$

for all  $q \in \mathbb{T}^n \times \mathbb{R}^m$  and  $t, \tau \in J$  with  $\tau \geq t$ . For  $\sigma \geq 1$ , we can estimate the  $C^\sigma$  norm of the right-hand side of the latter as follows

$$|\tilde{R}_\tau^t|_{C^\sigma} \leq 1 + \int_t^\tau |(g^s \circ \psi_\tau^s) \tilde{R}_\tau^s|_{C^\sigma} ds. \quad (6.26)$$

First of all, we estimate the norm into the integral. To this end, we use the properties in Proposition A.2.

$$\begin{aligned} |(g^s \circ \psi_\tau^s) \tilde{R}_\tau^s|_{C^\sigma} &\leq C(\sigma) \left( |g^s \circ \psi_\tau^s|_{C^\sigma} |\tilde{R}_\tau^s|_{C^0} + |g^s \circ \psi_\tau^s|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} \right) \\ |g^s \circ \psi_\tau^s|_{C^\sigma} &\leq C(\sigma) (|g^s|_{C^\sigma} |\partial_q \psi_\tau^s|_{C^0}^\sigma + |g^s|_{C^1} |\partial_q \psi_\tau^s|_{C^{\sigma-1}} + |g^s|_{C^0}) \end{aligned}$$

and, thanks to the latter and (6.26), we can estimate  $|\tilde{R}_\tau^t|_{C^\sigma}$  in the following way

$$\begin{aligned} |\tilde{R}_\tau^t|_{C^\sigma} &\leq 1 + C(\sigma) \int_t^\tau |g^s|_{C^\sigma} |\partial_q \psi_\tau^s|_{C^0}^\sigma |\tilde{R}_\tau^s|_{C^0} ds + C(\sigma) \int_t^\tau |g^s|_{C^1} |\partial_q \psi_\tau^s|_{C^{\sigma-1}} |\tilde{R}_\tau^s|_{C^0} ds \\ &\quad + C(\sigma) \int_t^\tau |g^s|_{C^0} |\tilde{R}_\tau^s|_{C^0} ds + C(\sigma) \int_t^\tau |g^s|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds. \end{aligned}$$

Now, by (6.22) and Lemma 6.3, we can find upper bounds for the first three integrals on the right-hand side of the previous inequality

$$\begin{aligned} \int_t^\tau |g^s|_{C^\sigma} |\partial_q \psi_\tau^s|_{C^0}^\sigma |\tilde{R}_\tau^s|_{C^0} ds &\leq C(\sigma) |g|_{\sigma,1} \int_t^\tau s^{-1} \left(\frac{\tau}{s}\right)^{\bar{c}_1 \sigma \mu} \left(\frac{\tau}{s}\right)^{c_0^R \mu} ds \\ &\leq C(\sigma) |g|_{\sigma,1} \left(\frac{\tau}{t}\right)^{(\bar{c}_1 \sigma + c_0^R) \mu} \int_t^\tau s^{-1} ds \\ &= C(\sigma) |g|_{\sigma,1} \ln \left(\frac{\tau}{t}\right) \left(\frac{\tau}{t}\right)^{(\bar{c}_1 \sigma + c_0^R) \mu} \\ \int_t^\tau |g^s|_{C^1} |\partial_q \psi_\tau^s|_{C^{\sigma-1}} |\tilde{R}_\tau^s|_{C^0} ds &\leq C(\sigma) \int_t^\tau \frac{|g|_{1,1}}{s} \left(1 + |f|_{\sigma,1} \ln \left(\frac{\tau}{s}\right)\right) \left(\frac{\tau}{s}\right)^{(c_\sigma + c_0^R) \mu} ds \\ &\leq C(\sigma) \mu \int_t^\tau \frac{1}{s} \left(\frac{\tau}{s}\right)^{(c_\sigma + c_0^R) \mu} ds \\ &\quad + C(\sigma) |f|_{\sigma,1} \mu \ln \left(\frac{\tau}{t}\right) \int_t^\tau \frac{1}{s} \left(\frac{\tau}{s}\right)^{(c_\sigma + c_0^R) \mu} ds \\ &= \frac{C(\sigma)}{c_\sigma + c_0^R} \left( \left(\frac{\tau}{t}\right)^{(c_\sigma + c_0^R) \mu} - 1 \right) \\ &\quad + \frac{C(\sigma)}{c_\sigma + c_0^R} |f|_{\sigma,1} \ln \left(\frac{\tau}{t}\right) \left( \left(\frac{\tau}{t}\right)^{(c_\sigma + c_0^R) \mu} - 1 \right) \\ \int_t^\tau |g^s|_{C^0} |\tilde{R}_\tau^s|_{C^0} ds &\leq \mu \int_t^\tau s^{-1} \left(\frac{\tau}{s}\right)^{c_0^R \mu} ds \\ &\leq \frac{1}{c_0^R} \left( \left(\frac{\tau}{t}\right)^{c_0^R \mu} - 1 \right). \end{aligned}$$

Similarly to the previous lemma, in the second line of the latter, we use the trivial estimate  $\left(\frac{\tau}{s}\right)^{(\bar{c}_1 \sigma + c_0^R) \mu} \leq \left(\frac{\tau}{t}\right)^{(\bar{c}_1 \sigma + c_0^R) \mu}$ .

Therefore, remembering that  $\bar{c}_1 \sigma \leq c_\sigma$ ,

$$\begin{aligned} |\tilde{R}_\tau^t|_{C^\sigma} &\leq 1 + C(\sigma) |g|_{\sigma,1} \ln \left(\frac{\tau}{t}\right) \left(\frac{\tau}{t}\right)^{(\bar{c}_1 \sigma + c_0^R) \mu} + C(\sigma) \left( \left(\frac{\tau}{t}\right)^{(c_\sigma + c_0^R) \mu} - 1 \right) \\ &\quad + C(\sigma) |f|_{\sigma,1} \ln \left(\frac{\tau}{t}\right) \left( \left(\frac{\tau}{t}\right)^{(c_\sigma + c_0^R) \mu} - 1 \right) + C(\sigma) \left( \left(\frac{\tau}{t}\right)^{c_0^R \mu} - 1 \right) \\ &\quad + C(\sigma) \int_t^\tau |g^s|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds \\ &\leq C(\sigma) \left(1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left(\frac{\tau}{t}\right)\right) \left(\frac{\tau}{t}\right)^{(c_\sigma + c_0^R) \mu} \\ &\quad + \left| C(\sigma) \int_t^\tau |g^s|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds \right|. \end{aligned}$$



Now, let

$$a(t) = C(\sigma) \left( 1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left( \frac{\tau}{t} \right) \right) \left( \frac{\tau}{t} \right)^{(c_\sigma + c_R^0)\mu},$$

we can rewrite the latter as follows

$$|\tilde{R}_\tau^t|_{C^\sigma} \leq a(t) + \left| C(\sigma) \int_\tau^t |g^s|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds \right|.$$

We observe that  $a$  is a monotone decreasing function, and thus by the more general inequality (6.16)

$$\begin{aligned} |\tilde{R}_\tau^t|_{C^\sigma} &\leq a(t) + C(\sigma) \left| \int_\tau^t a(s) |g^s|_{C^0} e^{C(\sigma) \int_s^t |g^\delta|_{C^0} d\delta} ds \right| \\ &\leq a(t) + C(\sigma) a(t) \int_t^\tau \frac{\mu}{s} e^{\ln(\frac{s}{t})^{C(\sigma)\mu}} ds = a(t) + C(\sigma) a(t) \int_t^\tau \frac{\mu}{s} \left( \frac{s}{t} \right)^{C(\sigma)\mu} ds \\ &= a(t) \left( 1 + C(\sigma) \left( \left( \frac{\tau}{t} \right)^{C(\sigma)\mu} - 1 \right) \right) \leq C(\sigma) a(t) \left( \frac{\tau}{t} \right)^{C(\sigma)\mu} \\ &\leq C(\sigma) \left( 1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln \left( \frac{\tau}{t} \right) \right) \left( \frac{\tau}{t} \right)^{c_\sigma^R \mu} \end{aligned}$$

for a suitable  $c_\sigma^R \geq c_\sigma + c_R^0$ .  $\square$

As for  $c_\sigma$  of the previous lemma, we note that the constant  $c_\sigma^R$  goes to infinity if  $\sigma \rightarrow \infty$ . To solve the homological equation, we must counter the growth of  $c_\sigma^R$  and  $c_\sigma$  taking  $\mu$  sufficiently small. It will be clear from the following lemma.

**Lemma 6.5 (Homological equation).** *There exists a solution  $\varkappa \in \mathcal{S}_\sigma$ ,  $(\nabla \varkappa) \bar{\Omega} \in \mathcal{S}_{\sigma-1}$  of  $(HE_E)$ . Moreover, letting  $c_\sigma^\varkappa = \max\{c_0^R + c_\sigma, c_\sigma^R + \bar{c}_1\sigma, \bar{c}_1^R + c_\sigma\}$ , if*

$$\mu < \frac{1}{c_\sigma^\varkappa} \quad (6.27)$$

then

$$|\varkappa|_{\sigma,1} \leq C(\sigma) \frac{|z|_{\sigma,2}}{1 - c_\sigma^\varkappa \mu} + C(\sigma) \frac{|f|_{\sigma,1} + |g|_{\sigma,1}}{(1 - c_\sigma^\varkappa \mu)^2} |z|_{1,2}. \quad (6.28)$$

*Proof. Existence:* For fixed  $t_0 \in J$ , let us define the following transformation

$$\begin{aligned} h : \mathbb{T}^n \times \mathbb{R}^m \times J &\longrightarrow \mathbb{T}^n \times \mathbb{R}^m \times J \\ h(q, t) &= (\psi_{t_0}^t(q), t) \end{aligned}$$

where  $\psi_{t_0}^t$  is the flow at time  $t$  with initial time  $t_0$  of  $F(q, t)$  previously defined. We claim that it is enough to prove the first part of this lemma for the much simpler equation

$$\partial_t \kappa(q, t) + g \circ h^{-1}(q, t) \kappa(q, t) = z \circ h^{-1}(q, t). \quad (6.29)$$

If  $\kappa$  is a solution of the latter, then  $\varkappa = \kappa \circ h$  is a solution of  $(HE_E)$  and viceversa. For the sake of clarity, we prove this claim. Let  $\varkappa$  be a solution of  $(HE_E)$ ,

$$\begin{aligned} \partial_t (\varkappa \circ h^{-1}) + (g \circ h^{-1}) (\varkappa \circ h^{-1}) &= (\partial_q \varkappa \circ h^{-1}) \partial_t \psi_{t_0}^t + \partial_t \varkappa \circ h^{-1} \\ &+ (g \circ h^{-1}) (\varkappa \circ h^{-1}) \\ &= (\partial_q \varkappa \circ h^{-1}) (F \circ h^{-1}) + \partial_t \varkappa \circ h^{-1} \\ &+ (g \circ h^{-1}) (\varkappa \circ h^{-1}) \\ &= ((\partial_q \varkappa) F + \partial_t \varkappa + g \varkappa) \circ h^{-1} = z \circ h^{-1}. \end{aligned}$$

Since  $(HE_E)$ , we have the last equality of the latter. This implies that  $\varkappa \circ h^{-1}$  is a solution for (6.29). Let us first show that  $\partial_q \psi_t^{t_0} F + \partial_t \psi_t^{t_0} = 0$ . We consider the following trivial equality

$$\psi_{t_0}^t \circ \psi_t^{t_0}(q) = q, \quad (6.30)$$

for all  $t, t_0 \in J$  and  $q \in \mathbb{T}^n$ . Differentiating both sides of the latter with respect to the variable  $q \in \mathbb{T}^n$ , we obtain

$$\partial_q \psi_{t_0}^t \circ \psi_t^{t_0}(q) \partial_q \psi_t^{t_0}(q) = \text{Id}$$

and by the above equation

$$\partial_q \psi_t^{t_0}(q) = (\partial_q \psi_{t_0}^t \circ \psi_t^{t_0}(q))^{-1}. \quad (6.31)$$

Taking the derivative with respect to  $t$  on both sides of (6.30)

$$0 = \frac{d}{dt} (\psi_{t_0}^t \circ \psi_t^{t_0}(q)) = \partial_q \psi_{t_0}^t \circ \psi_t^{t_0}(q) \partial_t \psi_t^{t_0}(q) + \partial_t \psi_{t_0}^t \circ \psi_t^{t_0}(q).$$

Hence,  $\partial_t \psi_t^{t_0}(q)$  is equal to

$$\partial_t \psi_t^{t_0}(q) = - (\partial_q \psi_{t_0}^t \circ \psi_t^{t_0}(q))^{-1} \partial_t \psi_{t_0}^t \circ \psi_t^{t_0}(q). \quad (6.32)$$

Therefore, thanks to (6.31) and (6.32), we can rewrite  $\partial_q \psi_t^{t_0} F + \partial_t \psi_t^{t_0} = 0$  in the following form

$$\partial_q \psi_t^{t_0}(q) F(q, t) + \partial_t \psi_t^{t_0}(q) = (\partial_q \psi_{t_0}^t \circ \psi_t^{t_0}(q))^{-1} (F(q, t) - \partial_t \psi_{t_0}^t \circ \psi_t^{t_0}(q))$$

for all  $t, t_0 \in J$  and  $q \in \mathbb{T}^n$ . This implies the claim because

$$F(q, t) - \partial_t \psi_{t_0}^t \circ \psi_t^{t_0}(q) = F(q, t) - F(\psi_{t_0}^t \circ \psi_t^{t_0}(q), t) = 0$$

for all  $t, t_0 \in J$  and  $q \in \mathbb{T}^n$ .

Now, let  $\kappa$  be a solution of (6.29)

$$\begin{aligned} \partial_q (\kappa \circ h) F + \partial_t (\kappa \circ h) + g(\kappa \circ h) &= (\partial_q \kappa \circ h) (\partial_q \psi_t^{t_0} F + \partial_t \psi_t^{t_0}) + \partial_t \kappa \circ h \\ &+ g(\kappa \circ h) \\ &= \partial_t \kappa \circ h + g(\kappa \circ h) = z. \end{aligned}$$

Hence  $\kappa \circ h$  is a solution of  $(HE_E)$ , where the last equality of the latter is a consequence of (6.29).

Now, we can reduce the proof of the first part of this lemma by studying the existence of a solution for the easier equation (6.29). For all  $(q, t_0) \in \mathbb{T}^n \times \mathbb{R}^m \times J$ , let  $R(q, t, t_0)$  be the unique solution of (R). Then, a solution  $\kappa$  of the above equation exists and

$$\begin{aligned} \kappa(q, t) &= R(q, t, t_0) e(q) - \int_{t_0}^t R(q, t, \tau) z \circ h^{-1}(q, \tau) d\tau \\ &= R(q, t, t_0) \left( e(q) - \int_{t_0}^t R(q, t_0, \tau) z \circ h^{-1}(q, \tau) d\tau \right) \end{aligned}$$

with a function  $e$  defined on  $\mathbb{T}^n \times \mathbb{R}^m$ .

*Estimates:* We choose  $e$  in such a way that

$$e(q) = \int_{t_0}^{+\infty} R(q, t_0, \tau) z \circ h^{-1}(q, \tau) d\tau.$$

It is well defined because by Lemma 6.4 and (6.27),

$$\begin{aligned} \left| \int_{t_0}^{+\infty} R(q, t_0, \tau) z \circ h^{-1}(q, \tau) d\tau \right| &\leq C \int_{t_0}^{+\infty} |R_\tau^{t_0}|_{C^0} |z^\tau|_{C^0} d\tau \\ &\leq C \int_{t_0}^{+\infty} \left( \frac{\tau}{t_0} \right)^{c_0^R \mu} \frac{|z|_{0,2}}{\tau^2} d\tau \\ &= C \frac{|z|_{0,2}}{t_0^{c_0^R \mu}} \int_{t_0}^{+\infty} \tau^{c_0^R \mu - 2} d\tau \\ &= C \frac{|z|_{0,2}}{1 - c_0^R \mu t_0}. \end{aligned}$$

Furthermore,

$$\begin{aligned} \varkappa(q, t) &= \kappa \circ h(q, t) = - \int_t^{+\infty} R_\tau^t \circ \psi_t^{t_0}(q) z^\tau \circ \psi_t^\tau(q) d\tau \\ &= - \int_t^{+\infty} R_\tau^t \circ \psi_\tau^{t_0} \circ \psi_t^\tau(q) z^\tau \circ \psi_t^\tau(q) d\tau = - \int_t^{+\infty} \tilde{R}_\tau^t \circ \psi_t^\tau(q) z^\tau \circ \psi_t^\tau(q) d\tau \end{aligned}$$

is the solution of  $(HE_E)$  we are looking for.

The estimate (6.28) is a consequence of Proposition A.2, the two previous lemmas and (6.27). For all  $t \in J$  and by Proposition A.2, we can estimate  $|\varkappa^t|_{C^\sigma}$  as follows

$$|\varkappa^t|_{C^\sigma} \leq C(\sigma) \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau \circ \psi_t^\tau|_{C^\sigma} + |\tilde{R}_\tau^t \circ \psi_t^\tau|_{C^\sigma} |z^\tau|_{C^0} d\tau.$$

Moreover,

$$\begin{aligned} |z^\tau \circ \psi_t^\tau|_{C^\sigma} &\leq C(\sigma) (|z^\tau|_{C^\sigma} |\partial_q \psi_t^\tau|_{C^0}^\sigma + |z^\tau|_{C^1} |\partial_q \psi_t^\tau|_{C^{\sigma-1}} + |z^\tau|_{C^0}) \\ |\tilde{R}_\tau^t \circ \psi_t^\tau|_{C^\sigma} &\leq C(\sigma) \left( |\tilde{R}_\tau^t|_{C^\sigma} |\partial_q \psi_t^\tau|_{C^0}^\sigma + |\tilde{R}_\tau^t|_{C^1} |\partial_q \psi_t^\tau|_{C^{\sigma-1}} + |R_\tau^t|_{C^0} \right) \end{aligned}$$

and replacing the latter into the above integral

$$\begin{aligned} |\varkappa^t|_{C^\sigma} &\leq C(\sigma) \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} |\partial_q \psi_t^\tau|_{C^0}^\sigma d\tau + C(\sigma) \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^1} |\partial_q \psi_t^\tau|_{C^{\sigma-1}} d\tau \\ &+ C(\sigma) \int_t^{+\infty} |\tilde{R}_\tau^t|_{C^\sigma} |\partial_q \psi_t^\tau|_{C^0}^\sigma |z^\tau|_{C^0} d\tau + C(\sigma) \int_t^{+\infty} |\tilde{R}_\tau^t|_{C^1} |\partial_q \psi_t^\tau|_{C^{\sigma-1}} |z^\tau|_{C^0} d\tau \\ &+ C(\sigma) \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^0} d\tau. \end{aligned}$$

Now, we have to estimate each integral on the right-hand side of the latter. First, we observe that, for all  $t \in J$  and  $x < 1$

$$\int_t^{+\infty} \tau^{x-2} \ln\left(\frac{\tau}{t}\right) d\tau = \frac{1}{1-x} \int_t^{+\infty} \tau^{x-2} d\tau.$$

It is obtained by integrating by part. Then, using Lemma 6.3, Lemma 6.4, (6.27) and the latter

$$\begin{aligned} \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} |\partial_q \psi_t^\tau|_{C^0}^\sigma d\tau &\leq C(\sigma) \int_t^{+\infty} \frac{|z|_{\sigma,2}}{\tau^2} \left(\frac{\tau}{t}\right)^{(c_0^R + \bar{c}_1\sigma)\mu} d\tau \\ &= C(\sigma) \frac{|z|_{\sigma,2}}{t^{(c_0^R + \bar{c}_1\sigma)\mu}} \int_t^{+\infty} \tau^{(c_0^R + \bar{c}_1\sigma)\mu-2} d\tau \\ &= C(\sigma) \frac{|z|_{\sigma,2}}{1 - (c_0^R + \bar{c}_1\sigma)\mu} \frac{1}{t} \\ \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^1} |\partial_q \psi_t^\tau|_{C^{\sigma-1}} d\tau &\leq C(\sigma) \int_t^{+\infty} \frac{|z|_{1,2}}{\tau^2} \left(1 + |f|_{\sigma,1} \ln\left(\frac{\tau}{t}\right)\right) \left(\frac{\tau}{t}\right)^{(c_0^R + c_\sigma)\mu} d\tau \\ &= C(\sigma) \int_t^{+\infty} \frac{|z|_{1,2}}{\tau^2} \left(\frac{\tau}{t}\right)^{(c_0^R + c_\sigma)\mu} d\tau \\ &\quad + C(\sigma) \int_t^{+\infty} \frac{|z|_{1,2}}{\tau^2} |f|_{\sigma,1} \ln\left(\frac{\tau}{t}\right) \left(\frac{\tau}{t}\right)^{(c_0^R + c_\sigma)\mu} d\tau \\ &= C(\sigma) \frac{|z|_{1,2}}{1 - (c_0^R + c_\sigma)\mu} \frac{1}{t} \\ &\quad + C(\sigma) \frac{|z|_{1,2} |f|_{\sigma,1}}{1 - (c_0^R + c_\sigma)\mu} \frac{1}{t^{(c_0^R + c_\sigma)\mu}} \int_t^{+\infty} \tau^{(c_0^R + c_\sigma)\mu-2} d\tau \\ &= C(\sigma) \frac{|z|_{1,2}}{1 - (c_0^R + c_\sigma)\mu} \frac{1}{t} + C(\sigma) \frac{|z|_{1,2} |f|_{\sigma,1}}{(1 - (c_0^R + c_\sigma)\mu)^2} \frac{1}{t} \\ \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^0} d\tau &\leq C \int_t^{+\infty} \frac{|z|_{0,2}}{\tau^2} \left(\frac{\tau}{t}\right)^{c_0^R \mu} d\tau = C \frac{|z|_{0,2}}{1 - c_0^R \mu} \frac{1}{t} \\ \int_t^{+\infty} |\tilde{R}_\tau^t|_{C^\sigma} |\partial_q \psi_t^\tau|_{C^0}^\sigma |z^\tau|_{C^0} d\tau &\leq C(\sigma) \int_t^{+\infty} \left(1 + (|f|_{\sigma,1} + |g|_{\sigma,1}) \ln\left(\frac{\tau}{t}\right)\right) \frac{|z|_{0,2}}{\tau^2} \left(\frac{\tau}{t}\right)^{(c_\sigma^R + \bar{c}_1\sigma)\mu} d\tau \\ &= C(\sigma) \int_t^{+\infty} \frac{|z|_{0,2}}{\tau^2} \left(\frac{\tau}{t}\right)^{(c_\sigma^R + \bar{c}_1\sigma)\mu} d\tau \\ &\quad + C(\sigma) (|f|_{\sigma,1} + |g|_{\sigma,1}) \int_t^{+\infty} \ln\left(\frac{\tau}{t}\right) \frac{|z|_{0,2}}{\tau^2} \left(\frac{\tau}{t}\right)^{(c_\sigma^R + \bar{c}_1\sigma)\mu} d\tau \\ &= C(\sigma) \frac{|z|_{0,2}}{1 - (c_\sigma^R + \bar{c}_1\sigma)\mu} \frac{1}{t} + C(\sigma) \frac{|z|_{0,2} (|f|_{\sigma,1} + |g|_{\sigma,1})}{(1 - (c_\sigma^R + \bar{c}_1\sigma)\mu)^2} \frac{1}{t} \end{aligned}$$

$$\begin{aligned}
 \int_t^{+\infty} |\tilde{R}_\tau|_{C^1} |\partial_q \psi_t^\tau|_{C^{\sigma-1}} |z^\tau|_{C^0} d\tau &\leq C(\sigma) \int_t^{+\infty} \frac{|z|_{0,2}}{\tau^2} \left(1 + |f|_{\sigma,1} \ln \left(\frac{\tau}{t}\right)\right) \left(\frac{\tau}{t}\right)^{(\bar{c}_1^R + c_\sigma)\mu} d\tau \\
 &= C(\sigma) \int_t^{+\infty} \frac{|z|_{0,2}}{\tau^2} \left(\frac{\tau}{t}\right)^{(\bar{c}_1^R + c_\sigma)\mu} d\tau \\
 &+ C(\sigma) \int_t^{+\infty} \frac{|z|_{0,2}}{\tau^2} |f|_{\sigma,1} \ln \left(\frac{\tau}{t}\right) \left(\frac{\tau}{t}\right)^{(\bar{c}_1^R + c_\sigma)\mu} d\tau \\
 &= C(\sigma) \frac{|z|_{0,2}}{1 - (\bar{c}_1^R + c_\sigma)\mu} \frac{1}{t} + C(\sigma) \frac{|z|_{0,2} |f|_{\sigma,1}}{(1 - (\bar{c}_1^R + c_\sigma)\mu)^2} \frac{1}{t}
 \end{aligned}$$

Then, thanks to the latter

$$\begin{aligned}
 |z^t|_{C^\sigma} &\leq C(\sigma) \left( \frac{|z|_{\sigma,2}}{1 - (c_0^R + \bar{c}_1\sigma)\mu} + \frac{|z|_{1,2}}{1 - (c_0^R + c_\sigma)\mu} + \frac{|z|_{1,2} |f|_{\sigma,1}}{(1 - (c_0^R + c_\sigma)\mu)^2} \right. \\
 &+ \frac{|z|_{0,2}}{1 - c_0^R \mu} + \frac{|z|_{0,2}}{1 - (c_\sigma^R + \bar{c}_1\sigma)\mu} + \frac{|z|_{0,2} (|f|_{\sigma,1} + |g|_{\sigma,1})}{(1 - (c_\sigma^R + \bar{c}_1\sigma)\mu)^2} \\
 &\left. + \frac{|z|_{0,2}}{1 - (\bar{c}_1^R + c_\sigma)\mu} + \frac{|z|_{0,2} |f|_{\sigma,1}}{(1 - (\bar{c}_1^R + c_\sigma)\mu)^2} \right) \\
 &\leq C(\sigma) \frac{|z|_{\sigma,2}}{1 - c_\sigma^\chi \mu} + C(\sigma) \frac{(|f|_{\sigma,1} + |g|_{\sigma,1})}{(1 - c_\sigma^\chi \mu)^2} |z|_{1,2}
 \end{aligned}$$

for all  $t \in J$ , where we recall that  $c_\sigma^\chi = \max\{c_0^R + c_\sigma, c_\sigma^R + \bar{c}_1\sigma, \bar{c}_1^R + c_\sigma\}$ . Taking the sup for all  $t \in J$  on the left-hand side of the latter, we conclude the proof.  $\square$

### 6.3.5 Regularity of $\mathcal{F}$

We begin this part by reminding the following families of Banach spaces introduced in Section 6.3.3  $\{(\mathcal{A}^\sigma, |\cdot|_\sigma)_{\sigma \geq 0}, \{(\mathcal{B}^\sigma, |\cdot|_\sigma)_{\sigma \geq 0}, \{(\mathcal{V}^\sigma, |\cdot|_\sigma)_{\sigma \geq 0}, \{(\mathcal{Z}_d^\sigma, |\cdot|_\sigma)_{\sigma \geq 0}$  such that, for all  $\sigma \geq 0$ ,

$$\begin{aligned}
 \mathcal{A}^\sigma &= \left\{ a : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R} \mid a \in \mathcal{S}_{\sigma+1} \text{ and } |a|_\sigma = |a|_{\sigma+1,0} + |\partial_q a|_{\sigma,2} < \infty \right\} \\
 \mathcal{B}^\sigma &= \left\{ b : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid b \in \mathcal{S}_{\sigma+1} \text{ and } |b|_\sigma = |b|_{\sigma+1,1} < \infty \right\} \\
 \mathcal{V}^\sigma &= \left\{ v : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid v \in \mathcal{S}_{\sigma+1}, (\nabla v) \bar{\Omega} \in \mathcal{S}_\sigma \right. \\
 &\quad \left. \text{and } |v|_\sigma = \max\{|v|_{\sigma+1,1}, |(\nabla v) \bar{\Omega}|_{\sigma,2}\} < \infty \right\} \\
 \mathcal{Z}^\sigma &= \left\{ z : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m} \mid z \in \mathcal{S}_\sigma, \text{ and } |z|_\sigma = |z|_{\sigma,2} < \infty \right\}
 \end{aligned}$$

where  $(\nabla v) \bar{\Omega} = (\partial_q v) \bar{\omega} + \partial_t v$ . Let  $s$  and  $\Upsilon$  be the positive parameters introduced by  $(*_E)$ , we define the following Banach space

$$\mathcal{M} = \left\{ m : \mathbb{T}^n \times \mathbb{R}^m \times B \times J \rightarrow M_{n+m} \mid m \in \mathcal{S}_{s+1} \text{ and } |m| = |m|_{s+1,0} \leq \Upsilon \right\}$$

where  $M_{n+m}$  is the set of  $(n+m)$ -dimensional matrices. We consider an additional family of Banach spaces  $\{(\mathcal{X}^\sigma, |\cdot|_\sigma)\}_{\sigma \geq 0}$  such that, for all  $\sigma \geq 0$ ,  $\mathcal{X}^\sigma = \mathcal{A}^\sigma \times \mathcal{B}^\sigma$  and for all  $x \in \mathcal{X}^\sigma$ ,  $|x|_\sigma = \max\{|a|_\sigma, |\bar{b}|_\sigma\}$ . Now, we have everything we need to define the functional  $\mathcal{F}$  more precisely. Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{X}^0 \times \mathcal{M} \times \mathcal{M} \times \mathcal{V}^0 \longrightarrow \mathcal{Z}^0$$

$$\mathcal{F}(x, m, \bar{m}, v) = (\nabla v) \bar{\Omega} + \partial_q v (b + (\bar{m} \circ \tilde{\varphi}) v) + \partial_q a + (\partial_q b) v + \partial_q m \circ \tilde{\varphi} \cdot v^2.$$

Thanks to Proposition 6.1, it is straightforward to verify that  $\mathcal{F}$  is well defined. We observe that, for all  $(b, m, \bar{m}) \in \mathcal{B}^0 \times \mathcal{M} \times \mathcal{M}$ , letting  $x^0 = (0, b)$ ,

$$\mathcal{F}(x^0, m, \bar{m}, 0) = 0.$$

Let  $\delta$  and  $b_0$  be as in  $(*_E)$ . Obviously,  $b_0 \in \mathcal{B}^s \subset \mathcal{B}^0$ . Moreover, we recall that  $b_0$  satisfies the following estimate

$$|b_0|_1 < \delta.$$

We fix  $x_0 = (0, b_0)$ . For all  $\sigma \geq 0$  and for a suitable parameter  $0 < \zeta < 1$ , that we will choose sufficiently small with respect to  $\delta$  in Lemma 6.6, we define the following subset of  $\mathcal{X}^\sigma \times \mathcal{V}^\sigma$

$$\mathcal{O}_\zeta^\sigma = \{(x, v) \in \mathcal{X}^\sigma \times \mathcal{V}^\sigma : |x - x_0|_\sigma, |v|_\sigma < \zeta\} \subset \mathcal{X}^\sigma \times \mathcal{V}^\sigma.$$

Let  $m, \bar{m} \in \mathcal{M}$  be as in  $(*_E)$ , we consider the following functional

$$\mathcal{F}_{m, \bar{m}} : \mathcal{O}_\zeta^0 \longrightarrow \mathcal{Z}^0$$

such that, for all  $(x, v) \in \mathcal{O}_\zeta^0$ ,

$$\mathcal{F}_{m, \bar{m}}(x, v) = \mathcal{F}(x, m, \bar{m}, v).$$

It is well defined, continuous and  $\mathcal{F}_{m, \bar{m}}(x_0, 0) = 0$ , where  $x_0 = (0, b_0)$  is the element of  $\mathcal{X}^0$  previously introduced. This section aims to prove that  $\mathcal{F}_{m, \bar{m}}$  satisfies hypotheses H.1-H.4 of Theorem 6.1. Then, Zehnder's Theorem ensures the existence of a  $C^\rho$ -weakly asymptotic cylinder associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ .

In the proof of the following lemma, we widely use the properties contained in Proposition 6.1 without specifying them each time.

**Lemma 6.6.**  $\mathcal{F}_{m, \bar{m}}$  satisfies hypotheses H.1-H.4 of Theorem 6.1.

*Proof. H.1. Smoothness:*  $\mathcal{F}_{m, \bar{m}}(x, \cdot) : \mathcal{V}^0 \longrightarrow \mathcal{Z}^0$  is two times differentiable with respect to the variable  $v$  and

$$\begin{aligned} D_v \mathcal{F}_{m, \bar{m}}(x, v) \hat{v} &= (\nabla \hat{v}) \bar{\Omega} + (\partial_q \hat{v}) f + g \hat{v}, \\ D_v^2 \mathcal{F}_{m, \bar{m}}(x, v) (\hat{v}, \tilde{v}) &= (\partial_q \hat{v}) \tilde{f} \tilde{v} + (\partial_q \tilde{v}) \hat{f} \hat{v} + \tilde{v}^T \tilde{g} \hat{v}, \end{aligned}$$

with

$$\begin{aligned}
f &= b + (\bar{m} \circ \tilde{\varphi}) v, \\
g &= \partial_q b + \partial_q v (\partial_p \bar{m} \circ \tilde{\varphi}) v + \partial_q v (\bar{m} \circ \tilde{\varphi}) + v^T (\partial_{pq}^2 m \circ \tilde{\varphi}) v + 2 (\partial_q m \circ \tilde{\varphi}) v, \\
\tilde{f} &= \bar{m} \circ \tilde{\varphi} + (\partial_p \bar{m} \circ \tilde{\varphi}) v, \\
\hat{f} &= (\partial_p \bar{m} \circ \tilde{\varphi}) v + \bar{m} \circ \tilde{\varphi}, \\
\tilde{g} &= \partial_q v (\partial_p^2 m \circ \tilde{\varphi}) v + 2 \partial_q v (\partial_p \bar{m} \circ \tilde{\varphi}) + v^T (\partial_p^2 \partial_q m \circ \tilde{\varphi}) v \\
&\quad + 4 (\partial_{pq}^2 m \circ \tilde{\varphi}) v + 2 \partial_q m \circ \tilde{\varphi},
\end{aligned}$$

where  $T$  denotes the transpose. We have to verify that

$$|D_v \mathcal{F}_{m, \bar{m}}(x, v)|_0, |D_v^2 \mathcal{F}_{m, \bar{m}}(x, v)|_0 \leq \mathcal{C}$$

for all  $(x, v) \in \mathcal{O}_\zeta^0$  and some  $\mathcal{C} \geq 1$ .

In the latter, we consider the norm corresponding to the Banach space  $(\mathcal{Z}^0, |\cdot|_0)$ . Referring to the notation introduced by (6.8), this norm coincides with  $|\cdot|_{0,2}$ . For all  $\hat{v} \in \mathcal{V}^0$ ,

$$\begin{aligned}
|D_v \mathcal{F}_{m, \bar{m}}(x, v) \hat{v}|_{0,2} &\leq |(\nabla \hat{v}) \bar{\Omega}|_{0,2} + C (|f|_{0,1} |\hat{v}|_{1,1} + |g|_{0,1} |\hat{v}|_{0,1}) \\
&\leq |\hat{v}|_0 (1 + C (|f|_{0,1} + |g|_{0,1})),
\end{aligned}$$

where we recall that  $|\hat{v}|_0 = \max\{|\hat{v}|_{1,1}, |(\nabla \hat{v}) \Omega|_{0,2}\}$ . Moreover, for all  $(x, v) \in \mathcal{O}_\zeta^0$  and  $m \in \mathcal{M}$ ,

$$\begin{aligned}
|f|_{0,1} &\leq C (|b|_{0,1} + |v|_{0,1} |\bar{m}|_{0,0}) \leq C (|b|_0 + |v|_0 |\bar{m}|_{0,0}), \\
&\leq C (|b|_0 + |b - b_0|_0) + C (|v|_0 |\bar{m}|_{0,0}) \leq C(\delta + \zeta) + C\Upsilon\zeta \\
|g|_{0,1} &\leq C (|b|_{1,1} + |v|_{0,1} |\bar{m}|_{1,0} |v|_{1,1} + |v|_{1,1} |\bar{m}|_{0,0} + (|v|_{0,1})^2 |\bar{m}|_{2,0} + |v|_{0,1} |m|_{1,0}), \\
&\leq C (|b|_0 + (|v|_0 + |v|_0^2) \Upsilon) \leq C(\delta + \zeta) + C\Upsilon\zeta.
\end{aligned}$$

This implies the claim for  $D_v \mathcal{F}_{m, \bar{m}}(x, v)$ . Similarly, we have the claim for  $D_v^2 \mathcal{F}_{m, \bar{m}}(x, v)$ .

*H.2.  $\mathcal{F}_{m, \bar{m}}$  is uniformly Lipschitz in  $\mathcal{X}^0$ :* For all  $(x_1, v), (x_2, v) \in \mathcal{O}_\zeta^0$ , remembering that  $|x|_0 = \max\{|a|_0, |b|_0\}$ ,

$$\begin{aligned}
|\mathcal{F}_{m, \bar{m}}(x_1, v) - \mathcal{F}_{m, \bar{m}}(x_2, v)|_{0,2} &= |\partial_q v (b_1 - b_2) + (\partial_q a_1 - \partial_q a_2) + (\partial_q b_1 - \partial_q b_2) v|_{0,2} \\
&\leq C |b_1 - b_2|_{0,1} |v|_{1,1} \\
&\quad + C (|\partial_q a_1 - \partial_q a_2|_{0,2} + |b_1 - b_2|_{1,1} |v|_{0,1}) \\
&\leq C (|b_1 - b_2|_0 |v|_0 + |a_1 - a_2|_0 + |b_1 - b_2|_0 |v|_0) \\
&\leq C(1 + \zeta) |x_1 - x_2|_0,
\end{aligned}$$

which proves H.2. Now, we verify H.4 before H.3.

*H.4. Order:* The first two conditions of Definition 6.3 are satisfied, meaning  $(x_0, 0) \in \mathcal{X}^s \times \mathcal{V}^s$  and  $\mathcal{F}_{m, \bar{m}}(\mathcal{O}_\zeta^0 \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)) \subset \mathcal{Z}^\sigma$  for all  $1 \leq \sigma \leq s$ . We verify the tame estimate.

For all  $1 \leq \sigma \leq s$  and  $(x, v) \in \mathcal{O}_\zeta^1 \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)$ , we rewrite the functional  $\mathcal{F}_{m, \bar{m}}$  in the following form

$$\begin{aligned} \mathcal{F}_{m, \bar{m}}(x, v) &= (\nabla v) \bar{\Omega} + \partial_q v (b_0 + (b - b_0) + (\bar{m} \circ \tilde{\varphi}) v) + \partial_q a \\ &\quad + (\partial_q b_0) v + \partial_q (b - b_0) v + \partial_q m \circ \tilde{\varphi} \cdot v^2. \end{aligned}$$

We assume  $|x - x_0|_\sigma, |v|_\sigma \leq K$ , then

$$\begin{aligned} |\mathcal{F}_{m, \bar{m}}(x, v)|_{\sigma, 2} &\leq |(\nabla v) \bar{\Omega}|_{\sigma, 2} + |(\partial_q v) b_0|_{\sigma, 2} + |(\partial_q v) (b - b_0)|_{\sigma, 2} + |\partial_q v (\bar{m} \circ \tilde{\varphi}) v|_{\sigma, 2} \\ &\quad + |a|_{\sigma, 2} + |(\partial_q b_0) v|_{\sigma, 2} + |\partial_q (b - b_0) v|_{\sigma, 2} + |\partial_q m \circ \tilde{\varphi} \cdot v^2|_{\sigma, 2}. \end{aligned}$$

We have to estimate each term on the right-hand side of the latter. The terms  $|a|_{\sigma, 2}$  and  $|(\nabla v) \bar{\Omega}|_{\sigma, 2}$  are bounded by  $K$ . We estimate the others

$$\begin{aligned} |b_0 (\partial_q v)|_{\sigma, 2} &\leq C(\sigma) |b_0|_{s+1, 1} |v|_{\sigma+1, 1} \leq C(\sigma) |b_0|_s |v|_\sigma \leq C(\sigma) |b_0|_s K \\ |(\partial_q v) (b - b_0)|_{\sigma, 2} &\leq C(\sigma) (|\partial_q v|_{0, 1} |b - b_0|_{\sigma, 1} + |\partial_q v|_{\sigma, 1} |b - b_0|_{0, 1}) \\ &\leq C(\sigma) (|v|_{1, 1} |b - b_0|_{\sigma, 1} + |v|_{\sigma+1, 1} |b - b_0|_{0, 1}) \\ &\leq C(\sigma) (|v|_0 |b - b_0|_\sigma + |v|_\sigma |b - b_0|_0) \\ &\leq C(\sigma) \zeta K \leq C(\sigma) K \\ |\partial_q v (\bar{m} \circ \tilde{\varphi}) v|_{\sigma, 2} &\leq C(\sigma) (|\partial_q v (\bar{m} \circ \tilde{\varphi})|_{\sigma, 1} |v|_{0, 1} + |\partial_q v (\bar{m} \circ \tilde{\varphi})|_{0, 1} |v|_{\sigma, 1}) \\ &\leq C(\sigma) |v|_0 (|\bar{m} \circ \tilde{\varphi}|_{\sigma, 0} |v|_0 + |\bar{m}|_{0, 0} |v|_\sigma) + C(\sigma) |v|_\sigma |\bar{m}|_{0, 0} |v|_0 \\ &\leq C(\sigma) |v|_0 |\bar{m}|_{\sigma, 0} (1 + |v|_0^\sigma + |v|_\sigma) |v|_0 + C(\sigma) \zeta \Upsilon |v|_\sigma \\ &\leq C(\sigma) \Upsilon K \\ |(\partial_q b_0) v|_{\sigma, 2} &\leq C(\sigma) |b_0|_{s+1, 1} |v|_{\sigma, 1} \leq C(\sigma) |b_0|_s K \\ |\partial_q (b - b_0) v|_{\sigma, 2} &\leq C(\sigma) (|v|_{0, 1} |b - b_0|_{\sigma+1, 1} + |v|_{\sigma, 1} |b - b_0|_{1, 1}) \\ &\leq C(\sigma) (|v|_0 |b - b_0|_{\sigma, 1} + |v|_\sigma |b - b_0|_0) \\ &\leq C(\sigma) \zeta K \leq C(\sigma) K \\ |\partial_q m \circ \tilde{\varphi} \cdot v^2|_{\sigma, 2} &\leq C(\sigma) (|\partial_q m \circ \tilde{\varphi}|_{\sigma, 1} |v|_{\sigma, 1} + |\partial_q m \circ \tilde{\varphi}|_{\sigma, 1} |v|_{0, 1}) \\ &\leq C(\sigma) |m|_{1, 0} |v|_0 |v|_\sigma \\ &\quad + C(\sigma) |v|_0 (|\partial_q m \circ \tilde{\varphi}|_{\sigma, 0} |v|_{0, 1} + |\partial_q m \circ \tilde{\varphi}|_{0, 0} |v|_{\sigma, 1}) \\ &\leq C(\sigma) \zeta \Upsilon K + C(\sigma) |v|_0 |m|_{\sigma+1, 0} (1 + |v|_0^\sigma + |v|_\sigma) |v|_0 \\ &\leq C(\sigma) \Upsilon K. \end{aligned}$$

Therefore, H.4 is satisfied. Now, we fix  $\delta < \frac{1}{c_s^X}$ , where  $s$  is the positive parameter defined by  $(*_E)$  and  $c_s^X$  is the constant in Lemma 6.5. Furthermore, we choose  $\zeta$  depending on  $\delta$  in such a way that

$$\delta + C \Upsilon \zeta < \frac{1}{c_s^X} \quad (\delta \zeta)$$

for a suitable constant  $C$  depending on  $n + m$ . This hypothesis is crucial if we want to define a right-inverse of  $\mathcal{F}$ .

*H.3. Existence of a right-inverse of loss 1:* In this part, we prove that for all  $(x, v) \in \mathcal{O}_\zeta^1 \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)$  with  $1 \leq \sigma \leq s$ , a right-inverse of loss 1 is well



defined. This means that, for all  $(x, v) \in \mathcal{O}_\zeta^1 \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)$ , there exists a liner map  $\eta_{m, \bar{m}} : \mathcal{Z}^\sigma \rightarrow \mathcal{V}^{\sigma-1}$  such that  $D_v \mathcal{F}_{m, \bar{m}}(x, v) \eta_{m, \bar{m}}(x, v) z = z$  for all  $z \in \mathcal{Z}^\sigma$ . In other words, for all  $z \in \mathcal{Z}^\sigma$ , we have to solve the following equation in the unknown  $\hat{v}$

$$D_v \mathcal{F}_{m, \bar{m}}(x, v) \hat{v} = (\nabla \hat{v}) \bar{\Omega} + (\partial_q \hat{v}) f + g \hat{v} = z, \quad (6.33)$$

where  $f$  and  $g$  are defined at the beginning of the proof of this lemma. If

$$|f|_{1,1} \leq \delta + C\Upsilon\zeta, \quad |g|_{1,1} \leq \delta + C\Upsilon\zeta, \quad (6.34)$$

thanks to Lemma 6.5 and  $(\delta\zeta)$ , a solution to the above equation exists. It remains to verify the estimate on  $|f|_{1,1}$  and  $|g|_{1,1}$ .

For all  $1 \leq \sigma \leq s$  and  $(x, v) \in \mathcal{O}_\zeta^1$ ,

$$\begin{aligned} |f|_{\sigma,1} &\leq |b|_{\sigma,1} + |(\bar{m} \circ \tilde{\varphi}) v|_{\sigma,1} \leq |b|_{\sigma,1} + C(\sigma) (|\bar{m} \circ \tilde{\varphi}|_{\sigma,0} |v|_{0,1} + |\bar{m} \circ \tilde{\varphi}|_{0,0} |v|_{\sigma,1}) \\ &\leq |b|_{\sigma-1} + C(\sigma) (\Upsilon (1 + |v|_{1,1}^\sigma + |v|_{\sigma,1}) |v|_{0,1} + \Upsilon |v|_{\sigma,1}) \\ &\leq |b|_{\sigma-1} + C(\sigma) \Upsilon |v|_{\sigma-1} \\ |g|_{\sigma,1} &\leq |b|_{\sigma+1,1} + |\partial_q v (\partial_q \bar{m} \circ \tilde{\varphi}) v|_{\sigma,1} + |\partial_q v (\bar{m} \circ \tilde{\varphi})|_{\sigma,1} + |v^T (\partial_{pq}^2 \bar{m} \circ \tilde{\varphi}) v|_{\sigma,1} \\ &\quad + |(\partial_q m \circ \tilde{\varphi}) v|_{\sigma,1} \\ &\leq |b|_\sigma + C(\sigma) \Upsilon |v|_\sigma. \end{aligned}$$

Taking  $\sigma = 1$ , we obtain

$$\begin{aligned} |f|_{1,1} &\leq |b|_0 + C\Upsilon |v|_0 \leq |b|_0 + |b - b_0|_0 + C\Upsilon |v|_0 \\ &\leq (\delta + \zeta) + C\Upsilon\zeta \leq \delta + C\Upsilon\zeta \end{aligned} \quad (6.35)$$

$$\begin{aligned} |g|_{1,1} &\leq |b|_1 + C(\sigma) \Upsilon |v|_1 \leq |b_0|_1 + |b - b_0|_1 + C\Upsilon |v|_0, \\ &\leq (\delta + \zeta) + C\Upsilon\zeta \leq \delta + C\Upsilon\zeta \end{aligned} \quad (6.36)$$

This implies the claim. The second part of this proof is dedicated to verifying  $(\eta 1)$  and  $(\eta 2)$ . In what follows, we drop the indexes  $m, \bar{m}$  from  $\mathcal{F}$  and  $\eta$  to achieve a more elegant proof.

For all  $(x, v) \in \mathcal{O}_\zeta^1$  and  $z \in \mathcal{Z}^1$

$$|\eta(x, v) z|_0 = \max\{|\eta(x, v) z|_{1,1}, |\nabla (\eta(x, v) z) \bar{\Omega}|_{0,2}\}.$$

By Lemma 6.5 (more specifically (6.28)),  $(\delta\zeta)$  and the previous estimates concerning  $|f|_{1,1}$  and  $|g|_{1,1}$

$$|\eta(x, v) z|_{1,1} \leq C(\delta, \zeta) |z|_{1,2} = C(\delta, \zeta) |z|_1.$$

Moreover, as a consequence of (6.33), the latter estimate, (6.35), (6.36) and  $(\delta\zeta)$

$$\begin{aligned} |\nabla (\eta(x, v) z) \bar{\Omega}|_{0,2} &= |z - \partial_q (\eta(x, v) z) f - g (\eta(x, v) z)|_{0,2} \\ &\leq |z|_0 + C |f|_{0,1} |\eta(x, v) z|_{1,1} + C |g|_{0,1} |\eta(x, v) z|_{0,1} \\ &\leq |z|_0 + |\eta(x, v) z|_{1,1} \leq C(\delta, \zeta) |z|_1. \end{aligned}$$

This implies  $(\eta 1)$  because

$$|\eta(x, v) z|_0 = \max\{|\eta(x, v) z|_{1,1}, |\nabla (\eta(x, v) z) \bar{\Omega}|_{0,2}\} \leq C(\delta, \zeta) |z|_1.$$

Concerning  $(\eta 2)$ , for all  $1 \leq \sigma \leq s$  and  $(x, v) \in \mathcal{O}_\zeta^1 \cap (\mathcal{X}^\sigma \times \mathcal{V}^\sigma)$ , we assume  $|x - x_0|_\sigma, |v|_\sigma \leq K$  and we recall that

$$|\eta(x, v)\mathcal{F}(x, v)|_{\sigma-1} = \max\{|\eta(x, v)\mathcal{F}(x, v)|_{\sigma,1}, |\nabla(\eta(x, v)\mathcal{F}(x, v))\bar{\Omega}|_{\sigma-1,2}\}.$$

We shall prove that the two norms on the right-hand side of the latter are smaller or equal to  $K$  multiplied by a suitable constant.

We consider the estimates of  $|f|_{\sigma,1}$  and  $|g|_{\sigma,1}$  calculated above

$$\begin{aligned} |f|_{\sigma,1} &\leq |b|_{\sigma-1} + C(\sigma)\Upsilon|v|_{\sigma-1} \\ &\leq |b_0|_{\sigma-1} + |b - b_0|_{\sigma-1} + C(\sigma)\Upsilon|v|_{\sigma-1} \leq |b_0|_s + C(\sigma)\Upsilon K \\ |g|_{\sigma,1} &\leq |b|_\sigma + C(\sigma)\Upsilon|v|_\sigma, \\ &\leq |b_0|_\sigma + |b - b_0|_\sigma + C(\sigma)\Upsilon|v|_\sigma \leq |b_0|_s + C(\sigma)\Upsilon K. \end{aligned}$$

Furthermore, by (6.34) and  $(\delta\zeta)$

$$|f|_{0,1} \leq 1, \quad |g|_{0,1} \leq 1.$$

Moreover, thanks to H.4 and  $(\delta\zeta)$

$$|\mathcal{F}(x, v)|_{1,2} \leq C\Upsilon\zeta \leq 1, \quad |\mathcal{F}(x, v)|_{\sigma,2} \leq C(\sigma)\Upsilon K.$$

Now, by (6.28) and the above estimates

$$\begin{aligned} |\eta(x, v)\mathcal{F}(x, v)|_{\sigma,1} &\leq C(\sigma, \Upsilon, \delta, \zeta)|\mathcal{F}(x, v)|_{\sigma,2} \\ &\quad + C(\sigma, \Upsilon, \delta, \zeta)(|f|_{\sigma,1} + |g|_{\sigma,1})|\mathcal{F}(x, v)|_{1,2} \\ &\leq C(\sigma, \Upsilon, \delta, \zeta)|\mathcal{F}(x, v)|_{\sigma,2} \\ &\quad + C(\sigma, \Upsilon, \delta, \zeta)(|b_0|_s + C(\sigma)\Upsilon K)|\mathcal{F}(x, v)|_{1,2} \\ &\leq C(\sigma, \Upsilon, \delta, \zeta)(1 + |b_0|_s)|\mathcal{F}(x, v)|_{\sigma,2} \\ &\quad + C(\sigma, \Upsilon, \delta, \zeta)K|\mathcal{F}(x, v)|_{1,2} \leq C(\sigma, |b_0|_s, \Upsilon, \delta, \zeta)K \end{aligned}$$

and by (6.33)

$$\begin{aligned} |\nabla(\eta(x, v)\mathcal{F}(x, v))\bar{\Omega}|_{\sigma-1,2} &= |\mathcal{F}(x, v) - \partial_q(\eta(x, v)\mathcal{F}(x, v))f - g(\eta(x, v)\mathcal{F}(x, v))|_{\sigma-1,2} \\ &\leq |\mathcal{F}(x, v)|_{\sigma,2} + |\partial_q(\eta(x, v)\mathcal{F}(x, v))f|_{\sigma-1,2} \\ &\quad + |g(\eta(x, v)\mathcal{F}(x, v))|_{\sigma,2}. \end{aligned}$$

It remains to prove that each term on the right-hand side of the latter can be estimated by  $K$  multiplied by a suitable constant. By H.4, this is true for the first term  $|\mathcal{F}(x, v)|_{\sigma,2}$ . We will prove it for the others using the estimates previously verified

$$\begin{aligned} |\partial_q(\eta(x, v)\mathcal{F}(x, v))f|_{\sigma-1,2} &\leq C(\sigma)(|\eta(x, v)\mathcal{F}(x, v)|_{\sigma,1}|f|_{0,1} + |\eta(x, v)\mathcal{F}(x, v)|_{1,1}|f|_{\sigma,1}) \\ &\leq C(\sigma, |b_0|_s, \Upsilon, \delta, \zeta)K|f|_{0,1} + C(\delta, \zeta)|\mathcal{F}(u, v)|_{1,2}|f|_{\sigma,1} \\ &\leq C(\sigma, |b_0|_s, \Upsilon, \delta, \zeta)K \end{aligned}$$

and similarly

$$\begin{aligned}
|g(\eta(x, v)\mathcal{F}(x, v))|_{\sigma, 2} &\leq C(\sigma) (|g|_{0,1}|\eta(x, v)\mathcal{F}(x, v)|_{\sigma, 1} + |g|_{\sigma, 1}|\eta(x, v)\mathcal{F}(x, v)|_{0,1}) \\
&\leq C(\sigma) (|\eta(x, v)\mathcal{F}(x, v)|_{\sigma, 1} + |g|_{\sigma, 1}C(\delta, \zeta)|\mathcal{F}(x, v)|_{1,2}) \\
&\leq C(\sigma, |b_0|_s, \Upsilon, \delta, \zeta)K + C(\sigma, \delta, \zeta) (|b_0|_s + C(\sigma)\Upsilon K) |\mathcal{F}(x, v)|_{1,2} \\
&\leq C(\sigma, |b_0|_s, \Upsilon, \delta, \zeta)K + C(\sigma, \delta, \zeta)|b_0|_s|\mathcal{F}(x, v)|_{\sigma, 2} \\
&\leq C(\sigma, |b_0|_s, \Upsilon, \delta, \zeta)K.
\end{aligned}$$

This concludes the proof of H.3 and also of this lemma.  $\square$

### 6.3.6 $C^\rho$ -Weakly Asymptotic Cylinder

We proved that, for fixed  $m, \bar{m} \in \mathcal{M}$  as in  $(*_E)$ , the functional  $\mathcal{F}_{m, \bar{m}}$  satisfies the hypotheses of Theorem 6.1. Then, there exists  $v : \mathbb{T}^n \times \mathbb{R}^m \times J \rightarrow \mathbb{R}^{n+m}$  such that

$$\varphi^t = (\text{id}, v^t)$$

is a  $C^\rho$ -weakly asymptotic cylinder associated to  $(X_H, X_{\tilde{h}}, \varphi_0)$ . We recall that  $H$  is the Hamiltonian defined by  $(*_E)$ ,  $\tilde{h}$  is the Hamiltonian in (6.9) and  $\varphi_0$  is the trivial embedding  $\varphi_0 : \mathbb{T}^n \times \mathbb{R}^m \rightarrow \mathbb{T}^n \times \mathbb{R}^m \times B^{n+m}$ ,  $\varphi_0(q) = (q, 0)$ . Moreover, letting  $\Gamma = b + (\bar{m} \circ \tilde{\varphi})v$  (see Section 6.3.2),

$$\begin{aligned}
|v|_{\rho, 1} &\leq \zeta, \\
|\Gamma|_{\rho, 1} &\leq |b|_{\rho, 1} + |(\bar{m} \circ \tilde{\varphi})v|_{\rho, 1} \\
&\leq |b_0|_{\rho, 1} + |b - b_0|_{\rho, 1} + |(\bar{m} \circ \tilde{\varphi})v|_{\rho, 1} \\
&\leq |b_0|_{\rho, 1} + |b - b_0|_{\lambda, 1} + |(\bar{m} \circ \tilde{\varphi})v|_{\rho, 1} \leq |b_0|_{\rho, 1} + \varepsilon + C\Upsilon\zeta
\end{aligned}$$

for a positive constant  $C \geq 1$  depending on  $n + m$ . It remains to prove that  $\varphi^t$  is Lagrangian for all  $t \in J$ . First, we observe that

$$|\Gamma|_{1,1} \leq |b_0|_{1,1} + \varepsilon + C\Upsilon\zeta \leq \delta + \varepsilon + C\Upsilon\zeta.$$

Letting  $\psi_{t_0, \bar{\omega} + \Gamma}^t$  be the flow at time  $t$  with initial time  $t_0$  of  $\bar{\omega} + \Gamma$ , we have the following proposition

**Proposition 6.3.** *For all  $t, t_0 \in J$ , if  $t \geq t_0$*

$$|\partial_q \psi_{t_0, \bar{\omega} + \Gamma}^t|_{C^0} \leq C \left( \frac{t}{t_0} \right)^{C(\delta + \varepsilon + C\Upsilon\zeta)},$$

for a suitable positive constant  $C$ .

*Proof.* Similarly to the proof of Lemma 6.3 we have the claim. We observe that the constant  $C$  in this proposition may differ from that in the estimate of  $|\Gamma|_{1,1}$  previously calculated.  $\square$

Let  $\psi_{t_0, H}^t$  be the flow at times  $t$  with initial time  $t_0$  of  $H$ , the following lemma concludes the proof of Theorem E.

**Lemma 6.7.**  $\varphi^{t_0}$  is Lagrangian for all  $t_0 \in J$

*Proof.* Let  $\alpha = dp \wedge dq$  be the standard symplectic form associated to  $(q, p) \in \mathbb{T}^n \times \mathbb{R}^m \times B^n \times B^m$ . For all fixed  $t, t_0 \in J$ , the flow  $\psi_{t_0, H}^t$  is a symplectomorphisms. This means that, for all fixed  $t, t_0 \in J$ ,  $(\psi_{t_0}^t)^* \alpha = \alpha$ . By (6.7),

$$\psi_{t_0, H}^{t_0+t} \circ \varphi^{t_0} = \varphi^{t_0+t} \circ \psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t} \quad (6.37)$$

and taking the pull-back with respect to the standard form  $\alpha$  on both sides of the latter, we obtain

$$(\varphi^{t_0})^*(\psi_{t_0, H}^{t_0+t})^* \alpha = (\psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t})^*(\varphi^{t_0+t})^* \alpha.$$

We remind that  $\psi_{t_0, H}^{t_0+t}$  is symplectic, then letting  $(\psi_{t_0}^{t_0+t})^* \alpha = \alpha$  on the left-hand side of the above equation, we have

$$(\varphi^{t_0})^* \alpha = (\psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t})^*(\varphi^{t_0+t})^* \alpha.$$

We want to prove that, for all  $q \in \mathbb{T}^n \times \mathbb{R}^m$ ,  $((\varphi^{t_0})^* \alpha)_q = 0$ , where  $((\varphi^{t_0})^* \alpha)_q$  stands for the symplectic form calculated on  $q \in \mathbb{T}^n \times \mathbb{R}^m$ . The idea is to prove that, for all  $q \in \mathbb{T}^n \times \mathbb{R}^m$ , the limit for  $t \rightarrow +\infty$  on the right-hand side of the above equation converges to zero. We know that  $\varphi^{t_0+t}(q) = (q, v^{t_0+t}(q))$ , then for all  $q \in \mathbb{T}^n \times \mathbb{R}^m$

$$((\psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t})^*(\varphi^{t_0+t})^* \alpha)_q = \sum_{1 \leq i < j \leq n+m} \sum_{1 \leq k < d \leq n+m} \alpha_{i,j,k,d}^t(q) dq_k \wedge dq_d$$

where

$$\begin{aligned} \alpha_{i,j,k,d}^t(q) &= (\partial_{q_i} v_j^{t_0+t} \circ \psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t}(q) - \partial_{q_j} v_i^{t_0+t} \circ \psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t}(q)) \\ &\quad \times (\partial_{q_k} \psi_{t_0, \bar{\omega}+\Gamma, i}^{t_0+t}(q) \partial_{q_d} \psi_{t_0, \bar{\omega}+\Gamma, j}^{t_0+t}(q) - \partial_{q_d} \psi_{t_0, \bar{\omega}+\Gamma, i}^{t_0+t}(q) \partial_{q_k} \psi_{t_0, \bar{\omega}+\Gamma, j}^{t_0+t}(q)). \end{aligned}$$

In the latter  $\times$  stands for the usual multiplication in  $\mathbb{R}$ . Then, for  $t > 0$  and fixed  $1 \leq i < j \leq n+m, 1 \leq k < d \leq n+m$ , by Proposition 6.3

$$\begin{aligned} |\alpha_{i,j,k,d}^t|_{C^0} &\leq |\partial_{q_i} v_j^{t_0+t} \circ \psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t} - \partial_{q_j} v_i^{t_0+t} \circ \psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t}|_{C^0} \\ &\quad \times |\partial_{q_k} \psi_{t_0, \bar{\omega}+\Gamma, i}^{t_0+t} \partial_{q_d} \psi_{t_0, \bar{\omega}+\Gamma, j}^{t_0+t} - \partial_{q_d} \psi_{t_0, \bar{\omega}+\Gamma, i}^{t_0+t} \partial_{q_k} \psi_{t_0, \bar{\omega}+\Gamma, j}^{t_0+t}|_{C^0} \\ &\leq \left( |\partial_{q_i} v_j^{t_0+t}|_{C^0} + |\partial_{q_j} v_i^{t_0+t}|_{C^0} \right) \\ &\quad \times \left( |\partial_{q_k} \psi_{t_0, \bar{\omega}+\Gamma, i}^{t_0+t}|_{C^0} |\partial_{q_d} \psi_{t_0, \bar{\omega}+\Gamma, j}^{t_0+t}|_{C^0} + |\partial_{q_d} \psi_{t_0, \bar{\omega}+\Gamma, i}^{t_0+t}|_{C^0} |\partial_{q_k} \psi_{t_0, \bar{\omega}+\Gamma, j}^{t_0+t}|_{C^0} \right) \\ &\leq C |v^{t_0+t}|_{C^1} (|\partial_q \psi_{t_0, \bar{\omega}+\Gamma}^{t_0+t}|_{C^0})^2 \leq C \frac{\zeta}{t_0+t} \left( \frac{t_0+t}{t_0} \right)^{C(\delta+\varepsilon+C\Upsilon\zeta)} \end{aligned}$$

for a suitable constant  $C \geq 1$ . Thanks to  $(\delta\zeta)$  and for  $\varepsilon$  small enough, taking the limit for  $t \rightarrow +\infty$  on both sides of the latter, the term in the last line converges to zero. This concludes the proof of this lemma.  $\square$

## 7 The Three-Body Problem plus Comet

This chapter studies the existence of suitable motions for a planetary system (planar three-body problem) perturbed by a given comet coming from and going back to infinity, asymptotically to a hyperbolic Keplerian orbit.

In [Arn63b] Arnold proves the existence of quasiperiodic motions for the Hamiltonian of the planar three-body problem. In this work, we follow the setting of Féjoz [Fé02], which provides more general solutions. In a rotating frame of reference, the author proves the existence of quasiperiodic orbits with three frequencies for the Hamiltonian of the planar three-body problem. Before the symplectic reduction by the symmetry of rotations, these quasiperiodic motions have one additional frequency, which is the angular speed of the simultaneous rotation of the three ellipses. Furthermore, before the symplectic reduction by the symmetry of translations, each of these invariant tori translates into a 1-parameter family of invariant tori parametrized by the center of mass of the planetary system.

In a neighbourhood of these quasiperiodic orbits, we want to perturb the Hamiltonian of the planar three-body problem with a time-dependent Hamiltonian that quantifies the interaction of the planets with the given comet. In this chapter, we prove the existence of orbits which are close (in a sense that we will specify later) to the quasiperiodic motions associated with the Hamiltonian of the planar three-body problem. The proof relies on Theorem E proved in the previous chapter.

### 7.1 Reduced Problem and Result

Consider three points of fixed masses  $m_0$ ,  $m_1$  and  $m_2$  undergoing gravitational attraction in the plane and a comet of fixed mass  $m_c$ . The comet comes from and goes to infinity, along a hyperbolic Keplerian orbit. We assume that the motion of the comet is a given smooth function  $c(t)$  and that only the planetary system is influenced by the comet. We assume,

$$|c(t)| \rightarrow_{t \rightarrow +\infty} \infty, \quad \frac{d}{dt}|c(t)| \rightarrow_{t \rightarrow +\infty} v > 0.$$

By the latter, there exists  $t_0 \gg 0$  such that

$$\frac{v}{2} \leq \frac{d}{dt}|c(t)| \leq 2v \tag{7.1}$$

for all  $t \geq t_0$ . At the risk of replacing  $t$  by  $t + t_0 - 1$ , we can take  $t_0 = 1$ .

Let  $J = [1, +\infty)$ , for a positive parameter  $0 < \varepsilon < 1$ , the phase space is the space

$$\left\{ ((x_i, y_i)_{0 \leq i \leq 2}, t) \in (\mathbb{R}^2 \times \mathbb{R}^{2*})^3 \times J \mid \begin{array}{l} \forall 0 \leq i < j \leq 2, x_i \neq x_j \\ \frac{|x_i|}{|c(t)|} < \varepsilon \end{array} \right\} \tag{7.2}$$

of linear momentum covectors  $(y_0, y_1, y_2)$  and position vectors  $(x_0, x_1, x_2)$  of each body. The Hamiltonian of the planar three-body problem plus comet (P3BP+C)

is

$$H(x, y, t) = \underbrace{\sum_{i=0}^2 \frac{|y_i|^2}{2m_i}}_{H_0(x,y)} - G \underbrace{\sum_{0 \leq i < j \leq 2} \frac{m_i m_j}{|x_i - x_j|}}_{H_0(x,y)} - G \underbrace{\sum_{i=0}^2 \frac{m_i m_c}{|x_i - c(t)|}}_{H_c(x,t)},$$

where  $G$  is the universal constant of gravitation that we may suppose equal to 1. It is the sum of the Hamiltonian of the planar three-body problem  $H_0$  and the Hamiltonian of the interaction with the comet  $H_c$ . Let  $\varphi_0$  be a 1-parameter family of invariant tori for  $H_0$  supporting quasiperiodic dynamics with four frequencies and  $\psi_{t_0, H}^t$  be the flow at time  $t$  with initial time  $t_0$  of  $H$

**Theorem F.** *Let  $H$  be the Hamiltonian of the P3BP+C. Then, there exist constants  $C(H_0, \varphi_0)$  depending on  $H_0$ ,  $\varphi_0$  and  $C(H)$  depending on  $H$  such that if*

$$|c(1)| > \frac{C(H_0, \varphi_0)}{\varepsilon}, \quad v > \frac{C(H)}{\varepsilon}, \quad (\text{cv})$$

for  $\varepsilon$  small enough, there exists an open subset  $\mathcal{W} \subset (\mathbb{R}^2 \times \mathbb{R}^{2*})^3$  such that, for all  $x \in \mathcal{W}$ ,  $\psi_{1, H}^t(x)$  is a weakly asymptotically quasiperiodic solution associated to  $(X_H, X_{H_0}, \varphi_0)$ .

As mentioned before, we observe that the existence of  $\varphi_0$  is guaranteed by [F 02]. Furthermore, we will see that the constants in (cv) are specified in Proposition 7.1, Lemma 7.3 and Lemma 7.9.

Before going into the details of the proof, we will prove the following property

**Proposition 7.1.** *If*

$$|c(1)| > \frac{1}{\varepsilon}, \quad v > \frac{2}{\varepsilon}, \quad (7.3)$$

then

$$\sup_{t \geq 1} \frac{t}{|c(t)|} < \varepsilon. \quad (7.4)$$

*Proof.* By (7.1) and the fundamental theorem of calculus

$$|c(1)| + \frac{v}{2}(t-1) \leq |c(t)| \leq |c(1)| + 2v(t-1) \quad (7.5)$$

for all  $t \geq 1$ . Thanks to the latter, we can estimate  $\frac{t}{|c(t)|}$  as follows

$$\frac{t}{|c(t)|} \leq \frac{1 + (t-1)}{|c(1)| + \frac{v}{2}(t-1)},$$

for all  $t \geq 1$ . Thanks to (7.3), the claim is true for  $t = 1$ . Now, we suppose that there exists  $t_0 > 1$  such that

$$\frac{1 + (t_0 - 1)}{|c(1)| + \frac{v}{2}(t_0 - 1)} \geq \varepsilon.$$

We can rewrite the latter in the following form

$$1 - \varepsilon|c(1)| \geq \left(\frac{\varepsilon}{2}v - 1\right)(t_0 - 1)$$

and this is a contradiction because, by (7.3),  $1 - \varepsilon|c(1)| < 0$  and  $\left(\frac{\varepsilon}{2}v - 1\right)(t_0 - 1) > 0$ .  $\square$

## 7.2 Proof of Theorem F

### 7.2.1 Quasiperiodic Motions in the Planar Three-Body Problem

This part is dedicated to a very brief introduction to the work of Féjóz [Fé02] concerning the existence, in a rotating frame of reference, of quasiperiodic motions with three frequencies for the Hamiltonian of the planar three-body problem. This result is an important element for the proof of Theorem F.

In this work, the author splits the dynamic into two parts: a fast, called Keplerian dynamic, and a slow, called secular dynamic. The first describes the motion of the bodies along three ellipses as if each body undergoes the attraction of only one fictitious center of attraction. The slow dynamic describes how the mutual attraction of each planet deforms these Keplerian ellipses. There is a natural splitting

$$H_0 = H_{\text{Kep}} + H_{\text{per}}$$

of the Hamiltonian when one uses the Jacobi coordinates.

The author defines the perturbing region contained in the direct product of the phase and parameter spaces. In this region, the Hamiltonian of the planar three-body problem is  $C^k$ -close to the dynamically degenerate Hamiltonian of two decoupled two-body problems. In order to get rid of the degeneracy of the Kepler Hamiltonian  $H_{\text{Kep}}$ , in a suitable open subset of the above-mentioned perturbing region, he introduces the secular system. It is obtained by an averaging process. It consists in averaging along the Keplerian ellipses parametrized by the mean anomalies  $\lambda_1$  and  $\lambda_2$  of the two fictitious Kepler problems where the Keplerian frequencies are non-resonant. We thus obtain an integrable approximation.

After the reduction by the symmetry of rotation and far from elliptic singularities, the phase space of the secular Hamiltonian contains a positive measure of Lagrangian diophantine invariant tori.

The claim relies on a sophisticated version of KAM theorem, which is proved using a normal form theorem due to Herman.

*Jacobi's Splitting and Keplerian Dynamics.* Let three points of masses  $m_0$ ,  $m_1$  and  $m_2$  undergoing gravitational attraction in the plane. We consider the following phase space

$$\left\{ (x_i, y_i)_{0 \leq i \leq 2} \in (\mathbb{R}^2 \times \mathbb{R}^{2*})^3 \mid 0 \leq i < j \leq 2, x_i \neq x_j \right\}.$$

The Hamiltonian of the planar three-body problem is

$$H_0(x, y) = \sum_{i=0}^2 \frac{|y_i|^2}{2m_i} - \sum_{0 \leq i < j \leq 2} \frac{m_i m_j}{|x_i - x_j|}.$$

In order to carry out the reduction by the symmetry of translations, he chooses the following symplectic change of variables

$$\begin{cases} X_0 = x_0 \\ X_1 = x_1 - x_0 \\ X_2 = x_2 - \sigma_0 x_0 - \sigma_1 x_1 \end{cases} \quad \begin{cases} Y_0 = y_0 + y_1 + y_2 \\ Y_1 = y_1 + \sigma_1 y_2 \\ Y_2 = y_2 \end{cases}$$

where  $\sigma_0 = \frac{m_0}{m_0+m_1}$  and  $\sigma_1 = \frac{m_1}{m_0+m_1}$ . The coordinates  $\{(X_i, Y_i)\}_{i=0,1,2}$  are the well-known Jacobi coordinates. In these variables, the Hamiltonian  $H_0$  does not depend on  $X_0$  (because of the symmetry by translations). This means that, considering the frame of reference attached to the center of mass and if  $X_2 \neq 0$ , the system is then described by the 4 Jacobi coordinates  $\{(X_i, Y_i)\}_{i=1,2}$ . The reduced Hamiltonian can be written as

$$H_0 = H_{Kep} + H_{per},$$

where  $H_{Kep}$  is the degenerate Hamiltonian of two decoupled two-body problems and  $H_{per}$  is the perturbation. More specifically,  $H_{Kep}$  is the completely integrable Hamiltonian of two fictitious bodies which revolve along ellipses around a fixed center of attraction without mutual interaction.

Now, we introduce some notations concerning the Keplerian dynamics. This part is necessary in order to be able to define the above-mentioned perturbing region. For the  $i$ th fictitious body, with  $i = 1$  or  $2$ , the mean longitude will be designated by  $\lambda_i$ , the semi-major axis by  $a_i$ , the eccentricity by  $e_i$ , the "centricity"  $\sqrt{1 - e_i^2}$  by  $\epsilon_i$ , the argument of the pericenter by  $g_i$ , the mean motion by  $v_i$  and the difference of the arguments of the pericenter by  $g = g_1 - g_2$ . We also introduce the well-known Poincaré coordinates  $(\Lambda_i, \lambda_i, \xi_i, \eta_i)$ , where we refer for example to the notes of A. Chenciner and J. Laskar [Che89, Las89] or the work of J. Féjóz [Fé13].

*Perturbing Region.* To measure how close the outer ellipse is from the inner ellipses when they are in opposition, the author defines

$$\Delta = \max_{(\lambda_1, \lambda_2, g) \in \mathbb{T}^3} \max\{\sigma_0, \sigma_1\} \frac{|X_1|}{|X_2|} = \max\{\sigma_0, \sigma_1\} \frac{a_1(1 + e_1)}{a_2(1 - e_2)}.$$

He assumes that  $\Delta < 1$ . This means that the outer ellipse does not meet the other two, whatever the difference  $g$  of the arguments of the pericenters. Moreover, the eccentricity  $e_2$  of the outer ellipse cannot be arbitrarily close to 1. He also assumes that the eccentricity of the inner ellipses is upper bounded from 1.

Let  $\mathfrak{P}$  be the reduced symplectically by translations phase space and  $\mathfrak{M}$  be the space described by the three mass parameters  $m_0, m_1$  and  $m_2$ .

**Definition 7.1.** For a positive parameter  $\delta$  and a non negative integer  $k$ , the perturbing region  $\Pi_\delta^k$  of parameters  $\delta$  and  $k$  is the open subset of  $\mathfrak{P} \times \mathfrak{M}$  defined by the following inequality

$$\max \left\{ \frac{m_2}{M_1} \left( \frac{a_1}{a_2} \right)^{\frac{3}{2}}, \frac{\mu_1 \sqrt{M}}{M_1^{\frac{3}{2}}} \left( \frac{a_1}{a_2} \right)^2 \right\} \frac{1}{\epsilon_2^{3(2+k)} (1 - \Delta)^{2k+1}} < \delta, \quad (7.6)$$

where  $M_1 = m_0 + m_1$ ,  $M = m_0 + m_1 + m_2$  and  $\mu_1 = \frac{m_0 m_1}{m_0 + m_1}$ .

Féjóz writes in his work that this inequality is not optimal and the given powers are not meaningful. He justifies this definition by proving that, inside the perturbing region, the perturbing function is  $\delta$ -small in a suitable  $C^k$ -norm. Another remark is that  $\frac{1}{\epsilon_2^{3(2+k)}}$  prevents the outer body from getting too close from collisions with the fictitious center of attraction ( $X_2 = 0$ ) and the factor



$\frac{1}{(1-\Delta)^{2k+1}}$  prevents the two outer bodies from getting too close from each other ( $x_2 = x_0$  or  $x_1$ ). For the proof of Theorem F, where the masses are fixed, note that the inequality (7.6) may be satisfied merely by assuming that  $\frac{a_1}{a_2} \ll 1$  and  $e_2 \leq \text{Cst} < 1$ . This is the so-called lunar or hierarchical regime.

If  $z_1$  and  $z_2$  are two quantities, let  $\check{z} = \min\{z_1, z_2\}$ . When  $\delta \rightarrow 0$ , let

$$\tilde{\Pi}_\delta^k = (\Pi_\delta^k \times \mathbb{R}^2) \cap \{\check{\lambda} = \mathcal{O}(\check{\lambda}_0)\} \cap \{\check{v} = \mathcal{O}(\check{v}_0)\}$$

be open sets of  $\Pi_\delta^k \times \mathbb{R}^2$ , where  $(\check{\lambda}_0, \check{v}_0)$  stands for coordinates of  $\mathbb{R}^2$ . These open sets can be thought of as fiber bundles over the parameter space  $\mathfrak{M} \times \mathbb{R}^2$ . The additional parameters  $(\check{\lambda}_0, \check{v}_0)$  are meant to localize the particular region on which we focus in the phase space.

*Elimination of Fast Angles.* In order to get rid of the degeneracy of the Keplerian Hamiltonian and hence apply the well-known KAM theorem, the secular Hamiltonian is introduced. Let  $d$  and  $k$  be suitable positive integers. On a suitable open set  $\dot{\Pi}_\delta^k$  of  $\tilde{\Pi}_\delta^k$ , the author proves the existence of a  $C^\infty$ -symplectomorphism  $\phi^d$  which is  $\delta$ -close to the identity in a suitable  $C^k$ -norm. The Hamiltonian  $H_0 \circ \phi^d$  can be split as follows

$$H_0 \circ \phi^d = H_\pi^d + H_{\text{comp}}^d,$$

where  $H_{\text{comp}}^d$  is of size  $\mathcal{O}(\delta^{d+1})$  on  $\dot{\Pi}_\delta^{k+d(\tau+4)}$ . The secular Hamiltonian is  $H_\pi^d$ , which is Pöschel integrable. It can be split into an integrable part  $H_{\text{int}}^d$  and a resonant part  $H_{\text{res}}^d$  of size  $\mathcal{O}(\delta)$ . The infinite jet of  $H_{\text{res}}^d$  vanishes along a suitable Cantor set.

In what follows, we only give an idea of the proof. It consists of a finite iterating procedure. The symplectomorphism  $\phi^d$  is the composition of  $d$  time-one maps  $\psi^1$  of a sufficiently small (on a suitable  $C^k$  norm) autonomous Hamiltonian vector field. To this end, let  $F$  be a Hamiltonian to be determined. We denote  $X_F$  as its vector field and  $\psi^t$  as its flow. To describe the transformed Hamiltonian  $H \circ \psi^1$ , we recall that for a function  $K$

$$\frac{d}{dt}K \circ \psi^t = \{K, F\} \circ \psi^t,$$

where the right-hand side of the latter stands for the Poisson bracket of  $K$  and  $F$  evaluated at  $\psi^t$ . Coming back to the work of Féjóz, he defines the (first order) complementary part  $H_{\text{comp}}^1$  of  $H$  by the equality

$$H_0 \circ \phi^1 = H_{\text{Kep}} + (H_{\text{per}} + \{H_{\text{Kep}}, F\}) + H_{\text{comp}}^1.$$

We remind that we need to eliminate the fast angles from  $H_{\text{per}}$  modulo a complementary part previously defined. Letting

$$\langle H_{\text{per}} \rangle = \frac{1}{4\pi^2} \int_{\mathbb{T}^2} H_{\text{per}} d\lambda_1 d\lambda_2$$

be the average of  $H_{\text{per}}$ , this is equivalent to solve the following homological equation in the unknown  $F$

$$\{F, H_{\text{Kep}}\} = H_{\text{per}} - \langle H_{\text{per}} \rangle.$$

This partial differential equation has a unique solution on a suitable Cantor set of diophantine tori. The Hamiltonian  $F$  is of class Whitney- $C^\infty$  and thus can be extended into a  $C^\infty$ -function which is rotating-invariant. Then, the Hamiltonian  $H_0$  is conjugated to

$$H_0 \circ \phi^1 = H_{\text{Kep}} + \langle H_{\text{per}} \rangle + H_{\text{comp}}^1.$$

Iterating by induction, the claim is proved.

*Secular Dynamics* As mentioned before, in a rotating frame of reference, Féjóz proves the existence of three-dimensional invariant tori supporting quasiperiodic dynamics for the Hamiltonian of the planar three-body problem. This means that, before the symplectic reduction by the symmetry of rotations, these quasiperiodic motions have one additional frequency, namely the angular speed of the simultaneous rotation of the three ellipses. More specifically, the author proves the following theorem

**Theorem 7.1.** *In a rotating frame of reference, there are integers  $k \geq 1$  and  $d \geq 1$  and real numbers  $\delta > 0$  and  $\tau \geq 1$  such that inside the perturbing region  $\dot{\Pi}_\delta^{k+d(\tau+4)}$  a positive measure of quasiperiodic Lagrangian tori of  $H_\pi^d$  survive in the dynamics of the planar three-body problem.<sup>1</sup>*

Far from elliptic singularities and in the rotating frame of reference, the secular Hamiltonian  $H_\pi^d$  possesses a family of Lagrangian invariant tori supporting diophantine quasiperiodic dynamics. As mentioned before, the proof of this theorem is a consequence of a sophisticated KAM theorem proved by using a normal form theorem due to Herman.

Introducing secular dynamics is the key to get rid of the degeneracy of the Hamiltonian of two decoupled two-body problems. Furthermore, investigating the secular dynamics allows us to find a series of solutions for the Hamiltonian of the planar three-body problem.

In what follows, we will not provide other information concerning the proof of the above theorem. Instead, we will only use Theorem 7.1 as a blackbox to prove the existence of weakly asymptotically quasiperiodic solutions associated to the Hamiltonian of the planar three-body problem plus comet (see Theorem F).

### 7.2.2 Outline of the proof of Theorem F

Our proof relies on [Fé02] and Theorem E proved in the previous chapter. We observe that we do not prove the existence of a weakly asymptotic cylinder but only a set of initial points giving rise to weakly asymptotically solutions associated to the Hamiltonian  $H$  of the planar three-body problem plus comet.

The proof is divided into five parts. The first two concern the Hamiltonian of the planar three-body problem  $H_0$ . In the first part, which we call *Splitting*, we introduce a linear symplectic change of variable  $\phi_0$ . Letting  $(X_i, Y_i)_{i=0,1,2}$  be the new

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<sup>1</sup>Moreover, for any fixed masses, the set of quasiperiodic Lagrangian tori has positive Lebesgue measure.

variables, which should not be confused with the Jacobi coordinates introduced in the previous section, we can split the Hamiltonian  $H_0$  in such a way that

$$H_0 \circ \phi_0(X, Y) = \frac{|Y_0|^2}{2M} + K(X_1, X_2, Y_1, Y_2),$$

where  $K$  is the Hamiltonian of the planar three-body problem after the reduction by the symmetry of translations,  $X_0$  is the center of mass of the planetary system,  $Y_0$  is the linear momentum of the planetary system and  $M = m_0 + m_1 + m_2$ .

We call the second part *Quasiperiodic Dynamics Associated to  $K$* . Here, Theorem 7.1 ensures the existence of Lagrangian four-dimensional invariant tori in the phase space after the symplectic reduction by the symmetry of translations. As mentioned before, F ejoz proves the existence of quasiperiodic solutions with three frequencies for the Hamiltonian of the planar three-body problem in a rotating frame of reference. The additional frequency is given by the angular speed of the simultaneous rotations of the three ellipses. Therefore, we prove the existence of a symplectic change of variables  $\phi_F$  in such a way that

$$K \circ \phi_F : \mathbb{T}^4 \times B^4 \rightarrow \mathbb{R}, \quad K \circ \phi_F(\theta, r) = c + \omega \cdot r + R_0(\theta, r) \cdot r^2$$

where  $c \in \mathbb{R}$ ,  $\omega \in \mathbb{R}^4$  and  $R_0(\theta, r) \cdot r^2$  stands for the vector given twice as an argument of the symmetric bilinear form  $R_0(\theta, r)$ .

We lift the above symplectic transformation  $\phi_F(\theta, r)$  to a symplectic transformation  $\tilde{\phi}_F(\theta, \xi, r, \eta)$  defined on  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2$ , for some balls  $B^4 \subset \mathbb{R}^4$  and  $B^2 \subset \mathbb{R}^2$  of small radius, such that

$$\tilde{\phi}_F(\theta, \xi, r, \eta) = (\xi, \eta, \phi_F(\theta, r)),$$

where  $\xi = X_0$  and  $\eta = Y_0$ . Letting  $\phi = \phi_0 \circ \tilde{\phi}_F$ , we can rewrite the Hamiltonian of the planar three-body problem  $H_0$  in the following form

$$\begin{aligned} H_0 \circ \phi : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 &\longrightarrow \mathbb{R}, \\ H_0 \circ \phi(\theta, \xi, r, \eta) &= c + \omega \cdot r + R_0(\theta, r) \cdot r^2 + \frac{|\eta|^2}{2M}. \end{aligned}$$

The third part of the proof is devoted to the perturbing function and, for this reason, we call it *Perturbing Function*. We introduce a suitable open subset  $\mathcal{U}$  of  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$  characterized by the orbits of  $H$  that stay sufficiently far from the comet. Letting

$$\tilde{\phi}(\theta, \xi, r, \eta, t) = (\phi(\theta, \xi, r, \eta), t),$$

$H_c \circ \tilde{\phi} : \mathcal{U} \rightarrow \mathbb{R}$  is well defined and satisfies good time-dependent estimates. Unfortunately, the Hamiltonian  $H \circ \tilde{\phi} = H_0 \circ \phi + H_c \circ \tilde{\phi} : \mathcal{U} \rightarrow \mathbb{R}$  does not satisfy the hypotheses of Theorem E. This is because  $H \circ \tilde{\phi}$  is not defined in the whole phase space  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$  but only on a subset. To solve this problem, in the fourth part called *Smooth Extension of the Perturbing Function*, we introduce the Hamiltonian  $\tilde{H}$ ,

$$\tilde{H} : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \longrightarrow \mathbb{R}$$

in such a way that  $\tilde{H}$  coincides with  $H \circ \tilde{\phi}$  on a suitable subset  $\mathcal{U}_{\frac{1}{2}}$  of  $\mathcal{U}$  where one expects the motions to take place, and  $\tilde{H}$  satisfies the same estimates of  $H \circ \tilde{\phi}$  outside  $\mathcal{U}_{\frac{1}{2}}$ . We will see that  $\tilde{H} \circ \tilde{\phi}$  satisfies the hypotheses of Theorem E. Letting  $\varphi_0$  be the following trivial embedding

$$\varphi_0 : \mathbb{T}^4 \times \mathbb{R}^2 \longrightarrow \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2, \quad \varphi_0(\theta, \xi, r, \eta) = (\theta, \xi, 0, 0),$$

we prove the existence of a  $C^1$ -weakly asymptotic cylinder  $\varphi^t$  associated to  $(X_{\tilde{H}}, X_{H_0 \circ \phi}, \varphi_0)$ .

In the last part, called *Weakly Asymptotically Quasiperiodic Solutions*, we conclude the proof of Theorem F. We define

$$B_1^2/2 = \left\{ \xi \in \mathbb{R}^2 : |\xi| < \frac{\varepsilon}{6} |c(1)| \right\} \subset \mathbb{R}^2,$$

where  $|c(1)|$  stands for the distance of the comet from the origin when  $t = 1$ . Therefore, we show that, for all  $z \in \varphi^1(\mathbb{T}^4 \times B_1^2/2)$ ,

$$\psi_{1, \tilde{H}}^t(z) \in \mathcal{U}_{\frac{1}{2}}$$

for all  $t \in J$ . This is a consequence of (cv),  $|c(t)| \sim vt$  and  $|\xi(t)| = |X_0(t)| \sim \ln t$ . Then, by the definition of  $\tilde{H}$ , for all  $z \in \varphi^1(\mathbb{T}^4 \times B_1^2/2)$ ,

$$\psi_{1, \tilde{H}}^t(z) = \psi_{1, H \circ \tilde{\phi}}^t(z)$$

for all  $t \in J$ . Now, because of  $\phi$  is symplectic, we conclude the proof verifying that for all  $w \in \mathcal{W} = \phi \circ \varphi^1(\mathbb{T}^4 \times B_1^2/2)$ ,

$$\psi_{1, H}^t(w)$$

is a weakly asymptotically quasiperiodic solution associated to  $(X_H, X_{H_0}, \phi \circ \varphi_0)$ .

### 7.2.3 Splitting

The phase space and the Hamiltonian of the planar three-body problem are respectively

$$\left\{ (x_i, y_i)_{0 \leq i \leq 2} \in (\mathbb{R}^2 \times \mathbb{R}^{2*})^3 \mid 0 \leq i < j \leq 2, x_i \neq x_j \right\},$$

and

$$H_0(x, y) = \sum_{i=0}^2 \frac{|y_i|^2}{2m_i} - \sum_{0 \leq i < j \leq 2} \frac{m_i m_j}{|x_i - x_j|}.$$

We would like to split the dynamics into the absolute motion of the center of mass and the relative motion of the three bodies. For this purpose, let us introduce the following linear symplectic change of coordinates  $\phi_0$

$$\begin{cases} X_0 = \frac{m_0}{M} x_0 + \frac{m_1}{M} x_1 + \frac{m_2}{M} x_2 \\ X_1 = x_0 - x_1 \\ X_2 = x_0 - x_2 \end{cases} \quad \begin{cases} Y_0 = y_0 + y_1 + y_2 \\ Y_1 = \frac{m_1}{M} y_0 - \frac{m_0+m_2}{M} y_1 + \frac{m_1}{M} y_2 \\ Y_2 = \frac{m_2}{M} y_0 + \frac{m_2}{M} y_1 - \frac{m_0+m_1}{M} y_2 \end{cases} \quad (7.7)$$

where  $M = m_0 + m_1 + m_2$ . The left-hand side of the latter recalls the well-known heliocentric coordinates (see for exemple [Las89]). We will see that, in these new coordinates, the Hamiltonian  $H$  is split into a Hamiltonian depending on  $Y_0$  and a Hamiltonian depending on the variables  $\{X_i, Y_i\}_{i=1,2}$  (see (7.8)). In these new coordinates, if  $X_1 \neq 0$ ,  $X_2 \neq 0$  and  $X_1 \neq X_2$ , the Hamiltonian of the planar three-body problem  $H_0$  is equal to

$$H_0 \circ \phi_0(X, Y) = \frac{|Y_0|^2}{2M} + \underbrace{\left( \frac{|Y_1|^2}{2\mu_1} - \frac{m_0 m_1}{|X_1|} \right) + \left( \frac{|Y_2|^2}{2\mu_2} - \frac{m_0 m_2}{|X_2|} \right) + \left( \frac{Y_1 \cdot Y_2}{m_0} - \frac{m_1 m_2}{|X_2 - X_1|} \right)}_{K(X_1, X_2, Y_1, Y_2)} \quad (7.8)$$

where  $\mu_1 = \frac{m_0 m_1}{m_0 + m_1}$  and  $\mu_2 = \frac{m_0 m_2}{m_0 + m_2}$ . We observe that  $H_0$  is the sum of two independent Hamiltonians. The first is responsible for the motion of the center of mass  $X_0$  and the linear momentum  $Y_0$ , the Hamiltonian  $K$  plays the role of the Hamiltonian of the planar three-body problem after the reduction by the symmetry of translations.

#### 7.2.4 Quasiperiodic Dynamics Associated with $K$

For suitable integers  $k \geq 1$ ,  $d \geq 1$  and real numbers  $\delta > 0$  and  $\tau \geq 1$ , inside the perturbing region  $\dot{\Pi}_\delta^{k+d(\tau+4)}$  (see Definition 7.1), Theorem 7.1 proves the existence of three-dimensional invariant tori for the Hamiltonian of the planar three-body problem  $K$  in a rotating frame of reference. As mentioned above, these quasiperiodic motions have one additional frequency before the symplectic reduction by the symmetry of rotations. This frequency plays the role of the angular speed of the simultaneous rotation of the three ellipses. Here, we fix  $m_0$ ,  $m_1$  and  $m_2$  as in Theorem F and we introduce the slice

$$\dot{\Pi}_{\delta, m}^{k+d(\tau+4)} = \dot{\Pi}_\delta^{k+d(\tau+4)} \Big|_{m_0, m_1, m_2} \subset \mathfrak{P}$$

where  $\mathfrak{P}$  is the phase space after the symplectic reduction by the symmetry by translations. In other words,  $\dot{\Pi}_{\delta, m}^{k+d(\tau+4)}$  is the subset of  $\mathfrak{P}$  obtained by  $\dot{\Pi}_\delta^{k+d(\tau+4)}$  once we have fixed the masses  $m_0$ ,  $m_1$  and  $m_2$ .

We begin this second section with the following lemma concerning the dynamics of the Hamiltonian of the planar three-body problem  $K$  with the frame of reference attached to the center of mass.

**Lemma 7.1.** *There exists a symplectic transformation  $\phi_F$  defined on  $\mathbb{T}^4 \times B^4$ , where  $B^4$  is a 4-dimensional ball with some small unspecified radius, with  $\phi_F(\mathbb{T}^4 \times B^4) \subset \mathfrak{P}$  such that the Hamiltonian  $K \circ \phi_F : \mathbb{T}^4 \times B^4 \rightarrow \mathbb{R}$  can be written in the following form*

$$K \circ \phi_F(\theta, r) = c + \omega \cdot r + R_0(\theta, r) \cdot r^2$$

for some  $c \in \mathbb{R}$  and  $\omega \in \mathbb{R}^4$ .

*Proof.* By Theorem 7.1, there exists a 4-dimensional Lagrangian invariant torus  $\mathcal{T} \subset \dot{\Pi}_{\delta, m}^{k+d(\tau+4)}$  for  $K$  supporting quasiperiodic dynamics. These tori form a set of positive Lebesgue measure, but we use only one such torus.

Due to the Weinstein Lagrangian neighbourhood theorem (see e.g. McDuff-Salamon [MS17]), there exists a neighbourhood  $N(\mathcal{T})$  of  $\mathcal{T}$  and a symplectomorphism

$$\phi_F : \mathbb{T}^4 \times B^4 \longrightarrow N(\mathcal{T}) \quad \text{such that} \quad \varphi(\mathbb{T}^4 \times \{0\}) = \mathcal{T}.$$

We observe that  $\phi_F(\mathbb{T}^4 \times \{0\}) = \mathcal{T}$  is a Lagrangian invariant torus for  $K$ . Hence

$$K \circ \phi_F(\theta, 0) = c$$

for all  $\theta \in \mathbb{T}^4$  and a suitable constant  $c \in \mathbb{R}$ . Moreover,  $\phi_F(\mathbb{T}^4 \times \{0\}) = \mathcal{T}$  support a quasiperiodic dynamics with some frequency vector  $\omega \in \mathbb{R}^4$ , so

$$\partial_r (K \circ \phi_F)(\theta, 0) = \omega.$$

□

In order to obtain suitable coordinates for the Hamiltonian  $H_0$  of the planar three-body problem, we need to lift the symplectic change of variables  $\phi_F$  introduced in the previous lemma. Let  $\tilde{\phi}_F(\theta, \xi, r, \eta)$  be the symplectic transformation defined on  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2$  in such a way that

$$\tilde{\phi}_F(\theta, \xi, r, \eta) = (\xi, \eta, \phi_F(\theta, r)) \tag{7.9}$$

with  $\xi = X_0$  and  $\eta = Y_0$ . We recall that  $(X_0, Y_0)$  are respectively the center of mass and the linear momentum of the planetary system defined in the previous section (see (7.7)). Letting

$$\phi = \phi_0 \circ \tilde{\phi}_F, \tag{7.10}$$

we have the following lemma

**Lemma 7.2.** *We can write the Hamiltonian of the planar three-body problem  $H_0$  in the following form*

$$\begin{aligned} H_0 \circ \phi : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 &\longrightarrow \mathbb{R}, \\ H_0 \circ \phi(\theta, \xi, r, \eta) &= c + \omega \cdot r + R_0(\theta, r) \cdot r^2 + \frac{|\eta|^2}{2M}. \end{aligned}$$

*Proof.* The proof of this lemma is a consequence of (7.8) and Lemma 7.1. □

However, if the planetary system is composed of only two bodies, we can explicitly construct the change of coordinates  $\phi$  (see Appendix D).

We conclude this part dedicated to the Hamiltonian  $H_0$  with a property that plays an important role in the following section. First, we observe that, for all  $i = 0, 1$

$$\sup_{(\theta, r) \in \mathbb{T}^4 \times B^4} |X_i(\theta, r)| < \infty$$

where  $X_i$  are the coordinates defined in (7.7). This is because, for all  $i = 1, 2$ ,  $X_i$ , as a function of  $(\theta, r) \in \mathbb{T}^4 \times B^4$ , is continuous and 1- periodic with respect to  $\theta_j$  for all  $0 \leq j \leq 4$ .

**Lemma 7.3.** *We assume that*

$$|c(1)| > \max_{i=1,2} \left( \sup_{(\theta,r) \in \mathbb{T}^4 \times B^4} |X_i(\theta, r)| \right) \frac{3}{\varepsilon} \quad (7.11)$$

where  $\varepsilon$  is the positive real parameter introduced in Section 7.1. Then, for all  $(\theta, r) \in \mathbb{T}^4 \times B^4$  and  $i = 1, 2$

$$\frac{|X_i(\theta, r)|}{|c(t)|} < \frac{\varepsilon}{3}$$

for all  $t \geq 1$ .

*Proof.* The proof is a straightforward computation. For all  $i = 1, 2$

$$\frac{|X_i(\theta, r)|}{|c(t)|} \leq \frac{\sup_{(\theta,r) \in \mathbb{T}^4 \times B^4} |X_i(\theta, r)|}{|c(1)|} < \frac{\varepsilon}{3}$$

for all  $t \geq 1$ . Therefore, thanks to (7.11), we conclude the proof of this lemma.  $\square$

### 7.2.5 Perturbing Function

We begin this section with the introduction of a suitable neighbourhood  $\mathcal{U}$  of  $\mathbb{T}^4 \times \{0\} \times B^6 \times J \subset \mathbb{T}^4 \times \mathbb{R}^2 \times B^6 \times J$  where we expect the motions of interest to take place.

For all fixed  $t \in J$ , we define

$$B_t^2 = \left\{ \xi \in \mathbb{R}^2 : \frac{|\xi|}{|c(t)|} < \frac{\varepsilon}{3} \right\}. \quad (7.12)$$

We consider  $\mathcal{U}$  as the following subset of  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$ ,

$$\mathcal{U} = \bigcup_{t \in J} (\mathbb{T}^4 \times B_t^2 \times B^4 \times B^2 \times \{t\}). \quad (\mathcal{U})$$

Let

$$\tilde{\phi}(\theta, \xi, r, \eta, t) = (\phi(\theta, \xi, r, \eta), t), \quad (7.13)$$

where  $\phi$  is the symplectic transformation defined by (7.10). This part aims to prove that the Hamiltonian  $H_c \circ \tilde{\phi} : \mathcal{U} \rightarrow \mathbb{R}$  satisfies suitable estimates. First, let us show that  $H_c \circ \tilde{\phi} : \mathcal{U} \rightarrow \mathbb{R}$  is well defined

**Lemma 7.4.** *For all  $t \in J$  and  $(\theta, \xi, r, \eta) \in \mathbb{T}^4 \times B_t^2 \times B^4 \times B^2$ ,*

$$\frac{|x_i(\theta, \xi, r)|}{|c(t)|} < \varepsilon$$

for all  $i = 0, 1, 2$ .

*Proof.* The proof is a straightforward computation. Because of (7.7) and Lemma 7.1, we can rewrite the cartesian coordinates  $(x_0, x_1, x_2)$  as follows

$$\begin{cases} x_0(\theta, \xi, r) = \xi + \frac{m_1}{M} X_1(\theta, r) + \frac{m_2}{M} X_2(\theta, r) \\ x_1(\theta, \xi, r) = \xi - \frac{m_0+m_2}{M} X_1(\theta, r) + \frac{m_2}{M} X_2(\theta, r) \\ x_2(\theta, \xi, r) = \xi + \frac{m_1}{M} X_1(\theta, r) - \frac{m_0+m_1}{M} X_2(\theta, r), \end{cases} \quad (7.14)$$

for all  $t \in J$  and  $(\theta, \xi, r, \eta) \in \mathbb{T}^4 \times B_t^2 \times B^4 \times B^2$ . By the latter, (U) and Lemma 7.3, for all  $t \in J$  and  $(\theta, \xi, r, \eta) \in \mathbb{T}^4 \times B_t^2 \times B^4 \times B^2$

$$\begin{aligned} \frac{|x_0(\theta, \xi, r)|}{|c(t)|} &\leq \frac{|\xi + \frac{m_1}{M} X_1(\theta, r) + \frac{m_2}{M} X_2(\theta, r)|}{|c(t)|} \\ &\leq \frac{|\xi|}{|c(t)|} + \frac{m_1}{M} \frac{|X_1(\theta, r)|}{|c(t)|} + \frac{m_2}{M} \frac{|X_2(\theta, r)|}{|c(t)|} \\ &\leq \frac{|\xi|}{|c(t)|} + \frac{|X_1(\theta, r)|}{|c(t)|} + \frac{|X_2(\theta, r)|}{|c(t)|} < \varepsilon \end{aligned}$$

for all  $t \in J$ . We recall that  $M = m_0 + m_1 + m_2$ , which implies  $\frac{m_i}{M} \leq 1$  for all  $0 \leq i \leq 2$ . Similarly, we have the claim for  $x_1(\theta, \xi, r)$  and  $x_2(\theta, \xi, r)$ .  $\square$

The previous lemma ensures that  $H_c \circ \tilde{\phi}$  is well defined on  $\mathcal{U}$ . Moreover, by (7.14), it is straightforward to verify that  $H_c \circ \tilde{\phi}$  does not depend on the variable  $\eta$ . The following lemma provides time-dependent estimations of  $H_c$ .

**Lemma 7.5.** *For all  $k \in \mathbb{Z}$  with  $k \geq 0$  and a suitable constant  $C(k)$  depending on  $k$*

$$\begin{aligned} \sup_{t \geq 1} |H_c^t|_{C^k} &< C(k) M m_c \varepsilon, \\ \sup_{t \geq 1} |\partial_x H_c^t|_{C^k} t^2 &< C(k) M m_c \varepsilon. \end{aligned}$$

For all  $t \in J$ , the above norms  $|\cdot|_{C^k}$  are taken on  $\phi(\mathbb{T}^4 \times B_t^2 \times B^2 \times B^4)$ , where  $\phi$  is defined by (7.10).

*Proof.* For all  $t \in J$  and  $(\theta, \xi, r, \eta) \in \mathbb{T}^4 \times B_t^2 \times B^2 \times B^4$ , let us rewrite the Hamiltonian  $H_c \circ \tilde{\phi}$  in the following form

$$H_c \circ \tilde{\phi}(\theta, \xi, r, \eta, t) = \sum_{i=0}^2 \frac{m_i m_c}{|x_i(\theta, \xi, r) - c(t)|}.$$

For the rest of this proof,  $x_i = x_i(\theta, \xi, r)$  for all  $0 \leq i \leq 2$ . We drop the coordinates  $(\theta, \xi, r)$  to obtain a more elegant form.

Using Legendre polynomials, we have that

$$\begin{aligned} \frac{1}{|x_i - c(t)|} &= \frac{1}{|c(t)|} \sum_{n \geq 0} \mathcal{P}_n(\cos \widehat{x_i c(t)}) \left( \frac{|x_i|}{|c(t)|} \right)^n \\ &= \frac{1}{|c(t)|} \left( 1 + \sum_{n \geq 1} \mathcal{P}_n(\cos \widehat{x_i c(t)}) \left( \frac{|x_i|}{|c(t)|} \right)^n \right) \end{aligned} \quad (7.15)$$



for all  $0 \leq i \leq 2$ , where  $\mathcal{P}_n$  means the  $n$ th Legendre polynomial. This expansion hold if  $\frac{|x_i|}{|c(t)|} < 1$  and this prerequisite is verified by Lemma 7.4.

Now, we can conclude the proof of this lemma. We recall that  $C(\cdot)$  means constants depending on some parameters indicated in brackets. Therefore, thanks to (7.15), Lemma 7.4 and Proposition 7.1, for all  $t \in J$  and  $(x, y) = (x_0, x_1, x_2, y_0, y_1, y_2) \in \phi(\mathbb{T}^4 \times B_t^2 \times B^4 \times B^2)$ ,

$$\begin{aligned} |H_c^t(x)| &= \sum_{i=0}^2 \frac{m_i m_c}{|x_i - c(t)|} \\ &= \sum_{i=0}^2 \frac{m_i m_c}{|c(t)|} \left( 1 + \sum_{n \geq 1} \mathcal{P}_n(\cos \widehat{x_i c(t)}) \left( \frac{|x_i|}{|c(t)|} \right)^n \right) \\ &\leq \sum_{i=0}^2 \frac{m_i m_c t}{|c(t)|} \left( 1 + \sum_{n \geq 1} \varepsilon^n \right) < CMm_c \varepsilon. \end{aligned}$$

In the last line,  $t \geq 1$  implies  $\sup_{t \geq 1} \frac{1}{|c(t)|} \leq \sup_{t \geq 1} \frac{t}{|c(t)|}$  and, letting  $0 < \varepsilon \leq \frac{1}{2}$ , we can estimate  $(1 + \sum_{n \geq 1} \varepsilon^n)$  with a suitable constant  $C$ . Taking the sup on  $\phi(\mathbb{T}^4 \times B_t^2 \times B^4 \times B^2)$  and then for all  $t \in J$  on the left-hand side of the latter, we obtain

$$\sup_{t \geq 1} |H_c^t|_{C^0} < CMm_c \varepsilon. \quad (7.16)$$

Now, for all  $0 \leq i \leq 2$ ,  $k \geq 1$ ,  $t \in J$  and  $(x, y) = (x_0, x_1, x_2, y_0, y_1, y_2) \in \phi(\mathbb{T}^4 \times B_t^2 \times B^4 \times B^2)$

$$\begin{aligned} |\partial_{x_i}^k H(x)| t^2 &\leq C(k) \frac{m_i m_c t^2}{|x_i - c(t)|^{k+1}} \\ &= C(k) m_i m_c \frac{t^2}{|c(t)|^{k+1}} \left( 1 + \sum_{n \geq 1} \mathcal{P}_n(\cos \widehat{x_i c(t)}) \left( \frac{|x_i|}{|c(t)|} \right)^n \right)^{k+1} \\ &\leq C(k) m_i m_c \frac{t^{k+1}}{|c(t)|^{k+1}} \left( 1 + \sum_{n \geq 1} \mathcal{P}_n(\cos \widehat{x_i c(t)}) \left( \frac{|x_i|}{|c(t)|} \right)^n \right)^{k+1} \\ &\leq C(k) \varepsilon^{k+1} m_i m_c \left( 1 + \sum_{n \geq 1} \varepsilon^n \right)^{k+1} \leq \varepsilon^{k+1} C(k) m_i m_c, \\ &< \varepsilon^{k+1} C(k) M m_c \end{aligned}$$

where  $t \geq 1$  and  $k + 1 \geq 2$  imply  $t^2 \leq t^{k+1}$  in the third line. Similarly to the previous case, because of  $0 < \varepsilon \leq \frac{1}{2}$ , we can estimate  $(1 + \sum_{n \geq 1} \varepsilon^n)^{k+1}$  by a suitable constant depending on  $k$ . Similarly to the previous case, taking the max for all  $0 \leq i \leq 2$  on the left-hand side of the latter we have

$$\sup_{t \geq 1} |\partial_x^k H_c^t|_{C^0} t^2 < C(k) M m_c \varepsilon^{k+1}. \quad (7.17)$$

Now, thanks to (7.16), (7.17) and remembering the definition of Hölder's norm (A.1), we conclude the proof of this lemma.  $\square$

### 7.2.6 Smooth Extension of the Perturbing Function

This section is dedicated to the introduction of a suitable smooth extension of  $H_c \circ \tilde{\phi}$ . First, we consider the following subset of  $\mathcal{U}$ . For all fixed  $t \in J$ , we define

$$B_t^2/2 = \{\xi \in \mathbb{R}^2 : \frac{|\xi|}{|c(t)|} < \frac{\varepsilon}{6}\} \subset B_t^2. \quad (7.18)$$

Let  $\mathcal{U}_{\frac{1}{2}}$  be the following subset of  $\mathcal{U}$ ,

$$\mathcal{U}_{\frac{1}{2}} = \bigcup_{t \in J} (\mathbb{T}^4 \times (B_t^2/2) \times B^4 \times B^2 \times \{t\}) \subset \mathcal{U}. \quad (\mathcal{U}_{\frac{1}{2}})$$

**Lemma 7.6.** *There exists*

$$H_{ex} : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \longrightarrow \mathbb{R}$$

such that  $H_{ex}$  does not depend on  $\eta$  and

$$H_{ex}|_{\mathcal{U}_{\frac{1}{2}}} = H_c \circ \tilde{\phi}. \quad (7.19)$$

Moreover, If  $|c(1)| > \frac{1}{\varepsilon}$  and  $v > \frac{2}{\varepsilon}$ , then for all  $k \in \mathbb{Z}$  with  $k \geq 0$  and some constants  $C(k)$  depending on  $k$ , we have the following estimates

$$\sup_{t \geq 1} |H_{ex}^t|_{C^k} < C(k) M m_c \varepsilon, \quad (7.20)$$

$$\sup_{t \geq 1} |\partial_{(\theta \xi r)} H_{ex}^t|_{C^k} t^2 < C(k) M m_c \varepsilon, \quad (7.21)$$

where the previous norms are taken on  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2$ .

Before the proof, we have some comments. First, for the sake of clarity, (7.19) means that, for all  $(\theta, \xi, r, \eta, t) \in \mathcal{U}_{\frac{1}{2}}$ ,  $H_{ex}(\theta, \xi, r, \eta, t) = H_c \circ \tilde{\phi}(\theta, \xi, r, \eta, t)$ . Furthermore, we observe that the constants in the last estimates may differ from those in Lemma 7.5 and they depend on the chosen extension.

*Proof.* For all fixed  $t \in J$ , we consider the following family of functions

$$\begin{cases} \rho^t : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \\ \rho^t \in C_0^\infty(\mathbb{R}^2), \\ \rho^t(\xi) = \xi \text{ for all } |\xi| \leq \varepsilon \frac{|c(t)|}{6}, \\ \rho^t(\xi) = 0 \text{ for all } |\xi| \geq \varepsilon \frac{|c(t)|}{3}. \end{cases}$$

Hence, we define the following map

$$\begin{aligned} \pi : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J &\longrightarrow \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \\ \pi(\theta, \xi, r, \eta, t) &= (\theta, \rho^t(\xi), r, \eta, t). \end{aligned}$$

It is straightforward to verify that

$$\begin{aligned} \pi(\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J) &\subset \mathcal{U} \\ \pi|_{\mathcal{U}_{\frac{1}{2}}} &= \text{Id}. \end{aligned}$$

Now, we define

$$H_{ex} = H_c \circ \tilde{\phi} \circ \pi. \quad (7.22)$$

Obviously, (7.19) is satisfied because  $\pi|_{\mathcal{U}_{\frac{1}{2}}} = \text{Id}$ . This proves the first part of this lemma. Concerning the second part, we observe that, for all fixed  $t \in J$  and  $k \in \mathbb{Z}$  with  $k \geq 0$

$$|\rho^t|_{C^k} \leq \frac{C(k)}{(\varepsilon|c(t)|)^k} \leq C(k) \left( \frac{t}{\varepsilon|c(t)|} \right)^k \leq C(k)$$

for a suitable constant  $C(k)$  depending on  $k$ . We point out that the second inequality of the previous estimate is due to  $t \geq 1$  and the last is a consequence of Proposition 7.1. Thanks to the latter, one can see that

$$\sup_{t \in J} |\nabla \pi^t|_{C^k} < \infty.$$

Now, we have everything we need to prove the second part of this lemma. We begin with the case  $k = 0$ . For all  $t \in J$ , we observe that by (7.22) and Lemma 7.5

$$\begin{aligned} |H_{ex}^t|_{C^0(\mathbb{T}^4 \times \mathbb{R}^2 \times B^6)} &= \left| \left( H_c \circ \tilde{\phi} \circ \pi \right)^t \right|_{C^0(\mathbb{T}^4 \times \mathbb{R}^2 \times B^6)} \\ &\leq |H_c^t|_{C^0(\phi(\mathbb{T}^4 \times B_t^2 \times B^6))} \leq CMm_c\varepsilon \end{aligned}$$

where  $B_t^2$  is defined by (7.12). For the sake of clarity, we have specified the domain where the Hölder norms are taken. Now, concerning the case  $k \geq 1$ , for all  $t \in J$

$$\begin{aligned} |H_{ex}^t|_{C^k(\mathbb{T}^4 \times \mathbb{R}^2 \times B^6)} &= \left| \left( H_c \circ \tilde{\phi} \circ \pi \right)^t \right|_{C^k(\mathbb{T}^4 \times \mathbb{R}^2 \times B^6)} \\ &\leq |H_c^t|_{C^k(\phi(\mathbb{T}^4 \times B_t^2 \times B^6))} \\ &\quad + |H_c^t|_{C^k(\phi(\mathbb{T}^4 \times B_t^2 \times B^6))} |\nabla \pi^t|_{C^0(\mathbb{T}^4 \times \mathbb{R}^2 \times B^6)}^k \\ &\quad + |H_c^t|_{C^k(\phi(\mathbb{T}^4 \times B_t^2 \times B^6))} |\nabla \pi^t|_{C^{k-1}(\mathbb{T}^4 \times \mathbb{R}^2 \times B^6)} \\ &\leq C \left( k, \sup_{t \in J} |\nabla \pi^t|_{C^{k-1}(\mathbb{T}^4 \times \mathbb{R}^2 \times B^6)} \right) Mm_c\varepsilon. \end{aligned}$$

We note that the first inequality (second line) is due to property 5. of Proposition A.2, while the last line is a consequence of Lemma 7.5. This proves (7.20). Similarly, one can prove (7.21).  $\square$

The rest of this section is devoted to showing that the Hamiltonian

$$\begin{aligned} \tilde{H} : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 &\longrightarrow \mathbb{R}, \\ \tilde{H} &= H_0 \circ \phi + H_{ex} \end{aligned}$$

satisfies the hypothesis of Theorem E, where  $H_0$  is the Hamiltonian of the planar three-body problem. Obviously, by the previous lemma

$$\tilde{H}|_{\mathcal{U}_{\frac{1}{2}}} = H \circ \tilde{\phi}. \quad (7.23)$$

First, let us recall some definitions introduced at the beginning of this chapter. Let  $\sigma \geq 0$  be a positive real parameter

**Definition.** Let  $\mathcal{S}_\sigma$  be the space of functions  $f$  defined on  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J$  such that  $f^t \in C^\sigma(\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2)$  for all fixed  $t \in J$  and  $\partial_{(q,p)}^i f \in C(\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J)$  for all  $0 \leq i \leq [\sigma]$ .

For every  $f \in \mathcal{S}_\sigma$ , we define the following norm

$$|f|_{\sigma,l} = \sup_{t \in J} |f^t|_{C^\sigma} t^l$$

for a real positive parameter  $l$ . Obviously, for all  $t \in J$ , the above norm  $|\cdot|_{C^\sigma}$  is taken on  $\mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2$ .

**Lemma 7.7.** We can rewrite  $H_{ex}$  in the following form

$$H_{ex}(\theta, \xi, r, t) = a(\theta, \xi, t) + b(\theta, \xi, t) \cdot r + R_c(\theta, \xi, r, t) \cdot r^2, \quad (7.24)$$

for all  $(\theta, \xi, r, t) \in \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times J$ . Moreover, for all  $k \in \mathbb{Z}$  with  $k \geq 1$

$$\begin{aligned} |a|_{k+1,0} + |\partial_{(\theta,\xi)} a|_{k,2} &< C(k) M m_c \varepsilon, \\ |b|_{k+1,2} &< C(k) M m_c \varepsilon, \\ |\partial_r^2 H_{ex}|_{k+1,2} &< C(k) M m_c \varepsilon, \end{aligned}$$

for some constants  $C$  depending on  $k$ .

*Proof.* Expanding  $H_{ex}$  in a small neighborhood of  $r = 0$

$$\begin{aligned} H_{ex}(\theta, \xi, r, \eta, t) &= H_{ex}(\theta, \xi, 0, t) + \partial_r H_{ex}(\theta, \xi, 0, t) \cdot r \\ &+ \int_0^1 (1 - \tau) \partial_r^2 H_{ex}(\theta, \xi, \tau r, t) d\tau \cdot r^2 \end{aligned}$$

and letting

$$\begin{aligned} a(\theta, \xi, t) &= H_{ex}(\theta, \xi, 0, t), \\ b(\theta, \xi, t) &= \partial_r H_{ex}(\theta, \xi, 0, t)(\theta, \xi, 0, \eta, t), \\ R_c(\theta, \xi, r, t) &= \int_0^1 (1 - \tau) \partial_r^2 H_{ex}(\theta, \xi, \tau r, t) d\tau \end{aligned}$$

we prove the first part of this lemma. The second part is a straightforward consequence of Lemma 7.6. More specifically,

$$\begin{aligned} |a|_{k+1,0} &\leq \sup_{t \geq 1} |H_{ex}^t|_{C^{k+1}}, \\ |\partial_{(\theta,\xi)} a|_{k,2} &\leq \sup_{t \geq 1} |\partial_{(\theta,\xi)} H_{ex}^t|_{C^{k+1}} t^2, \\ |b|_{k+1,2} &\leq \sup_{t \geq 1} |\partial_r H_{ex}^t|_{C^{k+1}} t^2, \\ |\partial_r^2 H_{ex}|_{k+1,2} &\leq \sup_{t \geq 1} |\partial_r^2 H_{ex}^t|_{C^{k+1}} t^2 \end{aligned}$$

and thanks to (7.20) and (7.21) we conclude the proof of this lemma.  $\square$

Summarizing the contents of the previous sections, we conclude this part with the following lemma

**Lemma 7.8.** *We can rewrite  $\tilde{H}$  in the following form*

$$\tilde{H} : \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2 \times J \longrightarrow \mathbb{R}$$

$$\tilde{H}(\theta, \xi, r, \eta, t) = c + \omega \cdot r + a(\theta, \xi, t) + b(\theta, \xi, t) \cdot r + m(\theta, \xi, r, t) \cdot \begin{pmatrix} r \\ \eta \end{pmatrix}^2$$

where

$$m(\theta, \xi, r, t) \cdot \begin{pmatrix} r \\ \eta \end{pmatrix}^2 = R_c(\theta, \xi, r, t) \cdot r^2 + R_0(\theta, r) \cdot r^2 + \frac{|\eta|^2}{2M}.$$

We note that  $c$ ,  $\omega$  and  $R_0$  are defined in Lemma 7.1, while  $a$ ,  $b$  and  $R_c$  are introduced in Lemma 7.7 and  $M = m_0 + m_1 + m_2$ . Moreover, for all  $k \in \mathbb{Z}$  with  $k \geq 1$ ,

$$|a|_{k+1,0} + |\partial_{(\theta,\xi)} a|_{k,2} < C(k)Mm_c\varepsilon, \quad (7.25)$$

$$|b|_{k+1,2} < C(k)Mm_c\varepsilon, \quad (7.26)$$

$$\left| \partial_{(r,\eta)}^2 \tilde{H} \right|_{k+1,0} < C(k, M, m_c, \varepsilon, |\partial_{(x,y)} H_0 \circ \phi|_{C^{k+2}}, |\nabla \phi|_{C^{k+2}}), \quad (7.27)$$

for some constants  $C(k)$  depending on  $k$  and  $C(k, M, m_c, \varepsilon, |\partial_{(x,y)} H_0 \circ \phi|_{C^{k+2}}, |\nabla \phi|_{C^{k+2}})$  depending on  $k, M, m_c, |\partial_{(x,y)} H_0 \circ \phi|_{C^{k+2}}$  and  $|\nabla \phi|_{C^{k+2}}$ .

*Proof.* The first part of this lemma is due to Lemme 7.2 and Lemma 7.7. Concerning the estimates, the first two are proved in Lemma 7.7. It remains to prove the last one. First, we claim that for all  $k \in \mathbb{Z}$  with  $k \geq 1$ , the symplectic change of coordinates  $\phi$  satisfies

$$|\nabla \phi|_{C^k} < \infty, \quad (7.28)$$

where  $\nabla \phi = (\nabla \phi_1, \dots, \nabla \phi_8)$  is the transposed of the Jacobian of  $\phi$ . We will use this notation also for  $\nabla \phi_0$  and  $\nabla \tilde{\phi}_F$ . By Proposition A.2 concerning the properties of the Hölder norms and recalling that  $\phi = \phi_0 \circ \tilde{\phi}_F$  (see (7.10))

$$\begin{aligned} |\nabla \phi|_{C^k} &= \left| \nabla \left( \phi_0 \circ \tilde{\phi}_F \right) \right|_{C^k} = \left| \nabla \phi_0 \circ \tilde{\phi}_F \left( \nabla \tilde{\phi}_F \right)^T \right|_{C^k} \\ &\leq C(k) |\nabla \phi_0|_{C^k} |\nabla \tilde{\phi}_F|_{C^k} \left( |\nabla \tilde{\phi}_F|_{C^{k-1}} + |\nabla \tilde{\phi}_F|_{C^0}^k + 1 \right), \end{aligned}$$

where  $T$  stands for the transpose of a matrix. We know that  $\phi_0$  is a linear transformation (see (7.7)), then  $|\nabla \phi_0|_{C^k} < \infty$ . On the other hand,  $\tilde{\phi}_F$  is  $C^\infty$ -function, 1-periodic with respect to  $\theta_i$  for all  $0 \leq i \leq 4$ , the variables  $r$  vary on a bounded subspace and it is the identity with respect to  $(\xi, \eta)$  (see Lemma 7.1 and (7.9)). This implies  $|\nabla \tilde{\phi}_F|_{C^k} < \infty$  and hence the claim.

Now, concerning the inequality (7.27), by Lemma 7.7

$$|\partial_r^2 H_{ex}|_{k+1,2} < C(k)Mm_c\varepsilon.$$

Now, we have to estimate  $|\partial_r^2 (H_0 \circ \phi)|_{k+1,0}$ . By the chain rule, (7.28) and properties 2. and 5. of Proposition A.2

$$\begin{aligned} |\partial_r^2 (H_0 \circ \phi)|_{C^{k+1}} &\leq C (|\partial_{(x,y)} H_0 \circ \phi|_{C^{k+1}} |\partial_r^2 \phi|_{C^{k+1}} + |\partial_{(x,y)}^2 H_0 \circ \phi|_{C^{k+1}} |\partial_r \phi|_{C^{k+1}}^2) \\ &\leq C(k, |\nabla \phi|_{C^{k+2}}) |\partial_{(x,y)} H_0 \circ \phi|_{C^{k+2}}. \end{aligned}$$

We observe that  $\partial_{(x,y)} H_0 \circ \phi$  does not depend on  $\xi$ . Moreover, it is  $C^\infty$ , 1-periodic with respect to  $\theta_i$  for all  $0 \leq i \leq 4$  and the variable  $(r, \eta)$  vary on  $B^4 \times B^2$  that is bounded. This implies  $|\partial_{(x,y)} H_0 \circ \phi|_{C^{k+2}} < \infty$ .

Now summarizing the previous estimates we obtain

$$\begin{aligned} \left| \partial_{(r,\eta)}^2 \tilde{H} \right|_{k+1,0} &\leq |\partial_r^2 H_{ex}|_{k+1,0} + |\partial_r^2 (H_0 \circ \phi)|_{k+1,0} \\ &\leq C(k, M, m_c, \varepsilon, |\partial_{(x,y)} H_0 \circ \phi|_{C^{k+2}}, |\nabla \phi|_{C^{k+2}}). \end{aligned}$$

□

### 7.2.7 Weakly Asymptotically Quasiperiodic Solutions

In the previous section, for  $k$  sufficiently large, Lemma 7.8 ensures that  $\tilde{H}$  satisfies the hypotheses of Theorem E. Then, letting  $\varphi_0 : \mathbb{T}^4 \times \mathbb{R}^2 \rightarrow \mathbb{T}^4 \times \mathbb{R}^2 \times B^4 \times B^2$  be the trivial embedding  $\varphi_0(\theta, \xi) = (\theta, \xi, 0, 0)$ , for  $\varepsilon$  small enough, there exist  $v_1 : \mathbb{T}^4 \times \mathbb{R}^2 \times J \rightarrow \mathbb{R}^4$  and  $v_2 : \mathbb{T}^4 \times \mathbb{R}^2 \times J \rightarrow \mathbb{R}^2$  such that

$$\varphi^t(\theta, \xi) = (\theta, \xi, v_1^t(\theta, \xi), v_2^t(\theta, \xi)) \quad (7.29)$$

is a  $C^1$ -weakly asymptotic cylinder associated to  $(X_{\tilde{H}}, X_{H_0 \circ \phi}, \varphi_0)$ . This means that there exist  $\Gamma_1 : \mathbb{T}^4 \times \mathbb{R}^2 \times J \rightarrow \mathbb{R}^4$  and  $\Gamma_2 : \mathbb{T}^4 \times \mathbb{R}^2 \times J \rightarrow \mathbb{R}^2$  such that, letting  $\Gamma = (\Gamma_1, \Gamma_2)$  and  $v = (v_1, v_2)$ , for all  $(\theta, \xi, t) \in \mathbb{T}^4 \times \mathbb{R}^2 \times J$ ,

$$X_{\tilde{H} \circ \bar{\varphi}}(\varphi(\theta, \xi, t), t) = \partial_{(\theta, \xi)} \varphi(\theta, \xi, t) (\bar{\omega} + \Gamma(\theta, \xi, t)) + \partial_t \varphi(\theta, \xi, t), \quad (7.30)$$

$$|v|_{1,1} \leq \zeta, \quad |\Gamma|_{1,1} \leq \varepsilon + \bar{C} \zeta \quad (7.31)$$

for a suitable constant  $\bar{C}$  and a small parameter  $0 < \zeta \leq 1$  depending on  $k$  and  $\left| \partial_{(r,\eta)}^2 \tilde{H} \right|_{k+1,0}$ .

We recall that  $B_t^2/2$  is defined in (7.18). Letting  $\psi_{1,H}^t$  be the flow at time  $t$  with initial time 1 of  $H$ , we have the following lemma.

**Lemma 7.9.** *We assume*

$$v > \frac{12(1 + \bar{C})}{\varepsilon}. \quad (7.32)$$

*Then, for all  $w \in \mathcal{W} = \phi \circ \varphi^1(\mathbb{T}^4 \times (B_1^2/2))$ ,  $\psi_{1,H}^t(w)$  is a weakly asymptotically quasiperiodic solution associated to  $(X_H, X_{H_0}, \phi \circ \varphi_0)$ .*

*Proof.* Let  $\psi_{1,\tilde{H}}^t$  be the flow at time  $t$  with initial time 1 of  $\tilde{H}$ . For all  $(\theta, \xi) \in \mathbb{T}^4 \times (B_1^2/2)$ , we define

$$(\theta_1^t(\theta, \xi), \xi_1^t(\theta, \xi), r_1^t(\theta, \xi), \eta_1^t(\theta, \xi)) = \psi_{1,\tilde{H}}^t \circ \varphi^1(\theta, \xi).$$

We begin by proving that, for all  $(\theta, \xi) \in \mathbb{T}^4 \times (B_1^2/2)$ ,  $\psi_{1, \tilde{H}}^t \circ \varphi^1(\theta, \xi) \in \mathcal{U}_{\frac{1}{2}}$  for all  $t \in J$ , where  $\mathcal{U}_{\frac{1}{2}}$  is defined in  $(\mathcal{U}_{\frac{1}{2}})$ . This is equivalent to show that, for all  $(\theta, \xi) \in \mathbb{T}^4 \times (B_1^2/2)$

$$(r_1^t(\theta, \xi), \eta_1^t(\theta, \xi)) \in B^4 \times B^2, \quad \frac{|\xi_1^t(\theta, \xi)|}{|c(t)|} < \frac{\varepsilon}{6} \quad (7.33)$$

for all  $t \in J$ . Let  $\psi_{1, \bar{\omega} + \Gamma}^t$  be the flow at time  $t$  with initial time 1 of  $\bar{\omega} + \Gamma$ . We know that (7.30) is equivalent to

$$\psi_{1, \tilde{H}}^t \circ \varphi^1 = \varphi^t \circ \psi_{1, \bar{\omega} + \Gamma}^t \quad (7.34)$$

for all  $t \in J$ .

Now, for all  $t \in J$ ,  $\varphi^t$  is the identity with respect to  $(\theta, \xi)$  (see (7.29)). Then, by (7.34)

$$(\theta_1^t(\theta, \xi), \xi_1^t(\theta, \xi), r_1^t(\theta, \xi), \eta_1^t(\theta, \xi)) = \psi_{1, \tilde{H}}^t \circ \varphi^1(\theta, \xi) = \varphi^t \circ \psi_{1, \bar{\omega} + \Gamma}^t(\theta, \xi)$$

for all  $t \in J$ . Hence, thanks to the special form of  $\varphi^t$

$$r_1^t(\theta, \xi) = v_1^t(\theta_1^t(\theta, \xi), \xi_1^t(\theta, \xi)), \quad \eta_1^t(\theta, \xi) = v_2^t(\theta_1^t(\theta, \xi), \xi_1^t(\theta, \xi))$$

for all  $t \in J$ . Then, by (7.31)

$$|r_1^t(\theta, \xi)| \leq |v_1^t|_{C^1} \leq \frac{\zeta}{t} \leq \zeta, \quad |\eta_1^t(\theta, \xi)| \leq |v_2^t|_{C^1} \leq \frac{\zeta}{t} \leq \zeta$$

for all  $t \in J$ . This proves the first part of (7.33). Concerning the second part, thanks to (7.29) and (7.34),  $\xi_1^t(\theta, \xi)$  is the unique solution of the following system

$$\begin{cases} \dot{\xi}_1^t(\theta, \xi) = \Gamma_2(\theta_1^t(\theta, \xi), \xi_1^t(\theta, \xi), t) \\ \xi_1^1(\theta, \xi) = \xi \end{cases}$$

where  $\dot{\xi}_1^t(\theta, \xi)$  stands for the derivative with respect to  $t$  of  $\xi_1^t(\theta, \xi)$ ,  $\Gamma_2$  is defined by (7.30) and  $\xi \in (B_1^2/2)$ . Now, thanks to (7.31) and the latter

$$|\xi_1^t(\theta, \xi)| \leq |\xi_1^1(\theta, \xi)| + \int_1^t |\Gamma^\tau|_{C^1} d\tau \leq |\xi| + (\varepsilon + \bar{C}\zeta) \ln t \leq |\xi| + (1 + \bar{C}) \ln t$$

for all  $t \in J$ . We claim that

$$\frac{|\xi_1^t(\theta, \xi)|}{|c(t)|} \leq \frac{|\xi| + (1 + \bar{C}) \ln t}{|c(1)| + \frac{v}{2}(t-1)} < \frac{\varepsilon}{6}$$

for all  $t \in J$ . It is true for  $t = 1$ . If we suppose the existence of  $t_0 > 1$  in such a way that

$$\frac{|\xi| + (1 + \bar{C}) \ln t_0}{|c(1)| + \frac{v}{2}(t_0 - 1)} \geq \frac{\varepsilon}{6}.$$

We can rewrite the latter in the following way

$$|\xi| - \frac{\varepsilon}{6}|c(1)| \geq \frac{\varepsilon}{12}v(t_0 - 1) - (1 + \bar{C}) \ln t_0$$

we observe that this is a contradiction because  $\xi \in (B_1^2/2)$  implies  $|\xi| - \frac{\varepsilon}{6}|c(1)| < 0$  and by (7.32) one can prove  $\frac{\varepsilon}{12}v(t-1) - (1+\bar{C})\ln t > 0$  for all  $t \in J$ . This concludes the proof of (7.33). Then, for all  $(\theta, \xi) \in (\mathbb{T}^4 \times (B_1^2/2))$ ,

$$\psi_{1, \tilde{H}}^t \circ \varphi^1(\theta, \xi) \in \mathcal{U}_{\frac{1}{2}}$$

for all  $t \in J$ . Hence, thanks to Lemma 7.6, for all  $(\theta, \xi) \in (\mathbb{T}^4 \times (B_1^2/2))$ ,

$$\psi_{1, \tilde{H}}^t \circ \varphi^1(\theta, \xi) = \psi_{1, H \circ \tilde{\phi}}^t \circ \varphi^1(\theta, \xi)$$

for all  $t \in J$ .

Now, for all  $w \in \mathcal{W} = \phi \circ \varphi^1(\mathbb{T}^4 \times (B_1^2/2))$  there exists  $(\theta, \xi) \in \mathbb{T}^4 \times (B_1^2/2)$  such that  $w = \phi \circ \varphi^1(\theta, \xi)$  and, because of  $\phi$  is symplectic, by (7.34) and the latter, we can rewrite  $\psi_{1, H}^t(w)$  in the following way

$$\begin{aligned} \psi_{1, H}^t(w) &= \psi_{1, H}^t \circ \phi \circ \varphi^1(\theta, \xi) = \phi \circ \psi_{1, H \circ \tilde{\phi}}^t \circ \varphi^1(\theta, \xi) = \phi \circ \psi_{1, \tilde{H}}^t \circ \varphi^1(\theta, \xi) \\ &= \phi \circ \varphi^t \circ \psi_{1, \tilde{\omega} + \Gamma}^t(\theta, \xi). \end{aligned}$$

Moreover, for all  $t \in J$

$$\begin{aligned} |\psi_{1, H}^t(w) - \phi \circ \varphi_0 \circ \psi_{1, \tilde{\omega} + \Gamma}^t(\theta, \xi)| &\leq |\phi \circ \varphi^t \circ \psi_{1, \tilde{\omega} + \Gamma}^t(\theta, \xi) - \phi \circ \varphi_0 \circ \psi_{1, \tilde{\omega} + \Gamma}^t(\theta, \xi)| \\ &\leq |\phi \circ \varphi^t - \phi \circ \varphi_0|_{C^0} \\ &\leq C |\nabla \phi|_{C^1} |\varphi^t - \varphi_0|_{C^0} \end{aligned}$$

for a suitable constant  $C$ . Therefore, reminding that  $|\nabla \phi|_{C^1} < \infty$  and taking the limit for  $t \rightarrow +\infty$ , thanks to (7.31), we conclude the proof of this lemma.  $\square$

This concludes the proof of Theorem F





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## Part V

# Asymptotic Motions Converging to Arbitrary Dynamics

This last chapter contains a variation of the result of Canadell-de la Llave [CdLL15] for time-dependent Hamiltonian vector fields and time-dependent vector fields on the torus. In the first case, we consider a time-dependent Hamiltonian vector field  $X$  converging exponentially fast in time to a Hamiltonian vector field  $X_0$  having an invariant torus  $\varphi_0$  supporting arbitrary dynamics generated by the vector field  $W$ . Unlike [CdLL15], we do not assume the dynamics associated with  $X_0$  on  $\varphi_0$  to be quasiperiodic. This situation already appears in Part IV, where the dynamics associated with  $X_0$  on  $\varphi_0$  are generated by time-dependent perturbations of constant vector fields on the torus.

Here, we maintain the exponential decay but do not assume any smallness condition. We prove the existence of a  $C^\sigma$ -asymptotic torus associated to  $(X, X_0, \varphi_0, W)$  and thus the existence of solutions that asymptotically converge to the arbitrary dynamics associated to  $X_0$  on  $\varphi_0$ . Similarly, we have an analogous result concerning the case of time-dependent vector fields on the torus. The proofs are essentially the same as those of Theorem A and Corollary A. They rely on the implicit function theorem, where we look for a  $C^\sigma$ -asymptotic torus defined for all  $t$  sufficiently large. Unlike the above-mentioned theorem, the solution of the associated homological equation is more complicated.

## 8 Asymptotic Motions for Time Dependent Hamiltonians

This chapter is divided into four sections. The first (Section 8.1) contains the definition of  $C^\sigma$ -asymptotic torus. The above mentioned results (Theorem G and Corollary C) are stated in the second section (Section 8.2). The last two (Section 8.3 and Section 8.4) contain the proofs.

### 8.1 $C^\sigma$ -Asymptotic Torus

We recall the definition of  $C^\sigma$ -asymptotic torus that generalizes that of  $C^\sigma$ -asymptotic KAM torus (see Definition 1.3). Let  $B \subset \mathbb{R}^n$  be a ball centred at the origin,  $\mathcal{P}$  be equal to  $\mathbb{T}^n$  or  $\mathbb{T}^n \times B$  and, for all  $v \geq 0$ ,  $J_v = [v, +\infty) \subset \mathbb{R}$ .

Given  $\sigma \geq 0$ ,  $v \geq 0$  and a positive integer  $k \geq 0$ , we consider time-dependent vector fields  $X^t$  and  $X_0^t$  of class  $C^{\sigma+k}$  on  $\mathcal{P}$ , for all  $t \in J_v$ , an embedding  $\varphi_0$  from  $\mathbb{T}^n$  to  $\mathcal{P}$  of class  $C^\sigma$  and a vector field on the torus  $W$  of class  $C^\sigma$  such that

$$\lim_{t \rightarrow +\infty} |X^t - X_0^t|_{C^{\sigma+k}} = 0, \quad (8.1)$$

$$X(\varphi_0(q), t) = \partial_q \varphi_0(q) W(q) \text{ for all } (q, t) \in \mathbb{T}^n \times J_v. \quad (8.2)$$

**Definition** (Définition 2.5). *We assume that  $(X, X_0, \varphi_0, W)$  satisfy (8.1) and (8.2). A family of  $C^\sigma$  embeddings  $\varphi^t : \mathbb{T}^n \rightarrow \mathcal{P}$  is a  $C^\sigma$ -asymptotic torus associated to  $(X, X_0, \varphi_0, W)$  if there exists  $v' \geq v \geq 0$  such that*

$$\lim_{t \rightarrow +\infty} |\varphi^t - \varphi_0|_{C^\sigma} = 0, \quad (8.3)$$

$$X(\varphi(q, t), t) = \partial_q \varphi(q, t)W(q) + \partial_t \varphi(q, t), \quad (8.4)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . When  $\dim \mathcal{P} = 2n$ , then  $\varphi^t$  is Lagrangian if  $\varphi^t(\mathbb{T}^n)$  is Lagrangian for all  $t$ .

First, we observe that if  $W(q) \equiv \text{cst}$ , then we obtain Définition 1.3. As one can expect, we can rewrite (8.4) in terms of the flow of  $X$ . Let  $\psi_{t_0, X}^t$  and  $\psi_{t_0, W}^t$  be the flow at time  $t$  with initial time  $t_0$  of  $X$  and  $W$ , respectively. We assume that  $\psi_{t_0, X}^t$  is defined for all  $t, t_0 \in J_{v'}$ . Then (8.4) is equivalent to

$$\psi_{t_0, X}^t \circ \varphi^{t_0} = \varphi^t \circ \psi_{t_0, W}^t \quad (8.5)$$

for all  $t, t_0 \in J_{v'}$ . By the latter, (8.4) is trivial and if there exists a  $C^\sigma$ -asymptotic torus  $\varphi^t$  defined for all  $t$  large, then we can extend the set of definition for all  $t \in \mathbb{R}$ .

## 8.2 Results

As usual, we begin with some definitions

**Definition.** *Let  $\mathcal{S}_\sigma^v$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times B \times J_v$  such that  $f \in C(\mathbb{T}^n \times B \times J_v)$  and, for all  $t \in J_v$ ,  $f^t \in C^\sigma(\mathbb{T}^n \times B)$ .*

We use this notation also for functions defined on  $\mathbb{T}^n \times J_v$ . For every  $f \in \mathcal{S}_\sigma^v$  and for fixed  $\lambda \geq 0$ , we define the following norm

$$|f|_{\sigma, \lambda}^v = \sup_{t \in J_v} |f^t|_{C^\sigma} e^{\lambda t}. \quad (8.6)$$

We conclude this first part with the following proposition, which contains a series of properties of the previous norm. As one can expect, there is a significant similarity with those enumerated in Proposition A.2 (see Appendix A).

**Proposition 8.1.** *For all  $f, g \in \mathcal{S}_\sigma^v$  and positive parameters  $m, d \geq 1$ , we have the following properties.*

a. *For all  $\beta \in \mathbb{N}^{2n}$ , if  $|\beta| + r \leq \sigma$  then* 
$$\left| \frac{\partial^{|\beta|}}{\partial q_1^{\beta_1} \dots \partial q_n^{\beta_n} \partial p_1^{\beta_{n+1}} \dots \partial p_n^{\beta_{2n}}} f \right|_{r, \lambda}^v \leq |f|_{\sigma, \lambda}^v$$

b.  $|f|_{\sigma, \lambda}^v \leq |f|_{\sigma, k\lambda}^v$

c.  $|fg|_{\sigma, d\lambda + m\lambda}^v \leq C(\sigma) (|f|_{0, d\lambda}^v |g|_{\sigma, m\lambda}^v + |f|_{\sigma, d\lambda}^v |g|_{0, m\lambda}^v).$

Given  $\sigma \geq 1$ , for all  $f, z \in \mathcal{S}_\sigma^v$  then  $f \circ z \in \mathcal{S}_\sigma^v$

$$d. |f \circ z|_{\sigma, k\lambda+m\lambda}^v \leq C(\sigma) (|f|_{\sigma, k\lambda}^v (|\nabla z|_{0, m\lambda}^v)^\sigma + |f|_{1, k\lambda}^v |\nabla z|_{\sigma-1, m\lambda}^v + |f|_{0, k\lambda+m\lambda}^v),$$

*Proof.* The proof is a straightforward application of Proposition A.2. Properties a. and b. are obvious. We verify the others.

c.

$$\begin{aligned} |fg|_{\sigma, k\lambda+m\lambda}^v &= \sup_{t \in J_v} |f^t g^t|_{C^\sigma} e^{(k\lambda+m\lambda)t} \\ &\leq C(\sigma) \sup_{t \in J_v} (|f^t|_{C^0} |g^t|_{C^\sigma} + |f^t|_{C^\sigma} |g^t|_{C^0}) e^{(k\lambda+m\lambda)t} \\ &\leq C(\sigma) \sup_{t \in J_v} (|f^t|_{C^0} e^{k\lambda t} |g^t|_{C^\sigma} e^{m\lambda t} + |f^t|_{C^\sigma} e^{k\lambda t} |g^t|_{C^0} e^{m\lambda t}) \\ &\leq C(\sigma) (|f|_{0, k\lambda}^v |g|_{\sigma, m\lambda}^v + |f|_{\sigma, k\lambda}^v |g|_{0, m\lambda}^v) \end{aligned}$$

d.

$$\begin{aligned} |f \circ z|_{\sigma, k\lambda+m\lambda}^v &= \sup_{t \in J_v} |f^t \circ z^t|_{C^\sigma} e^{(k\lambda+m\lambda)t} \\ &\leq C(\sigma) \sup_{t \in J_v} (|f^t|_{C^\sigma} e^{k\lambda t} (|\nabla z^t|_{C^0} e^{m\lambda t})^\sigma e^{(1-\sigma)m\lambda t}) \\ &\quad + C(\sigma) \sup_{t \in J_v} (|f^t|_{C^1} e^{k\lambda t} |\nabla z^t|_{C^{\sigma-1}} e^{m\lambda t} + |f|_{C^0} e^{(k\lambda+m\lambda)t}) \\ &\leq C(\sigma) (|f|_{\sigma, k\lambda}^v (|\nabla z|_{0, m\lambda}^v)^\sigma + |f|_{1, k\lambda}^v |\nabla z|_{\sigma-1, m\lambda}^v + |f|_{0, k\lambda+m\lambda}^v) \end{aligned}$$

where we observe that if  $t \geq 0$  and  $\sigma \geq 1$  then  $e^{(1-\sigma)m\lambda t} \leq 1$ .  $\square$

Given  $\sigma, v \geq 0$  and an integer  $k \geq 0$ , we conclude this part of set-up by reminding the following definition

**Definition.** Let  $\bar{\mathcal{S}}_{\sigma, k}^v$  be the space of functions  $f$  such that

$$f \in \mathcal{S}_{\sigma+k}^v \text{ and } \partial_{qp}^i f \in \mathcal{S}_{\sigma+k-i}^v$$

for all  $0 \leq i \leq k$ .

In other words, we recall that this is the space of functions  $f \in \mathcal{S}_{\sigma+k}^v$  with partial derivatives with respect to  $(q, p)$  continuous until the order  $k$ .

Let  $W \in C^{\sigma+2}(\mathbb{T}^n)$ , we recall that  $\mathcal{K}_W$  is the set of the Hamiltonians  $h : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R}$  such that, for all  $(q, t) \in \mathbb{T}^n \times J_0$ ,

$$h(q, 0, t) = c, \quad \partial_p h(q, 0, t) = W(q)$$

for some  $c \in \mathbb{R}$ . Therefore, for all  $h \in \mathcal{K}_W$ , the trivial embedding  $\varphi_0$  given by

$$\varphi_0 : \mathbb{T}^n \rightarrow \mathbb{T}^n \times B, \quad \varphi_0(q) = (q, 0),$$

is an invariant torus for  $X_h$  and the restricted vector field is  $W$ .

Given  $\sigma \geq 1$  and  $\lambda \geq 0$ , let  $H$  be the Hamiltonian of the following form

$$\left\{ \begin{array}{l} H : \mathbb{T}^n \times B \times J_0 \rightarrow \mathbb{R} \\ H(q, p, t) = h(q, p, t) + f(q, p, t) \\ h \in \mathcal{K}_W, \quad W \in C^{\sigma+2}(\mathbb{T}^n), \\ f_0, \partial_p f_0, \partial_p^2 H \in \bar{\mathcal{S}}_{\sigma, 2}^0, \\ |f_0|_{\sigma+2, 0}^0 + |\partial_q f_0|_{\sigma+1, \lambda}^0 < \infty, \quad |\partial_p f_0|_{\sigma+2, \lambda}^0 < \infty \\ |\partial_p^2 H|_{\sigma+2, 0}^0 < \infty \end{array} \right. \quad (*G)$$

**Theorem G.** *Let  $H$  be as in  $(*_G)$ . Then, there exists a Hamiltonian  $\tilde{h} \in \mathcal{K}_W$  and a constant  $C(\sigma)$  depending on  $\sigma$  such that if*

$$\lambda > C(\sigma)|\partial_q W|_{C^0}, \quad (\#_G)$$

*there exists a Lagrangian  $C^\sigma$ -asymptotic torus associated to  $(X_H, X_{\tilde{h}}, \varphi_0, W)$ .*

We will see that  $(\#_G)$  plays a crucial role in the section dedicated to the solution of the homological equation. Furthermore, we will see that the previous constant  $C(\sigma)$  is defined in Lemma 8.3

Therefore, we have the following result concerning time-dependent vector fields on the torus. Given  $\sigma \geq 1$ , let  $Z$  be a non-autonomous vector field on  $\mathbb{T}^n \times J_0$  of the form

$$\begin{cases} Z(q, t) = W(q) + P(q, t) \\ W \in C^{\sigma+1}(\mathbb{T}^n), \quad P \in \bar{\mathcal{S}}_{\sigma,1}^0, \\ |P|_{\sigma+1,\lambda}^0 < \infty. \end{cases} \quad (Z_C)$$

**Corollary C.** *Let  $Z$  be as in  $(Z_C)$ . Then, there exists a constant  $C(\sigma)$  depending on  $\sigma$  such that if*

$$\lambda > C(\sigma)|\partial_q W|_{C^0},$$

*there exists a  $C^\sigma$ -asymptotic torus  $\psi^t$  associated to  $(Z, W, \text{Id}, W)$ .*

### 8.3 Proof of Theorem G

As in Theorem A, the proof rests on the implicit function theorem. As mentioned before, it is essentially the same as that of Theorem A. Following the ideas in the third chapter of this thesis, we are looking for a  $C^\sigma$ -asymptotic torus defined for  $t$  sufficiently large so that the perturbative terms are small enough. To this end, by expanding the Hamiltonian  $H$  in  $(*_G)$  in a small neighbourhood of  $0 \in B$ , we can rewrite  $H$  in the following form

$$\begin{cases} H : \mathbb{T}^n \times B \times J_0 \longrightarrow \mathbb{R} \\ H(q, p, t) = W(q) \cdot p + a(q, t) + b(q, t) \cdot p + m(q, p, t) \cdot p^2, \\ a, b, \partial_p^2 H \in \bar{\mathcal{S}}_{\sigma,2}^0, \quad W \in C^{\sigma+2} \\ |a|_{\sigma+2,0}^0 + |\partial_q a|_{\sigma+1,\lambda}^0 \leq \Upsilon, \quad |b|_{\sigma+2,\lambda}^0 \leq \Upsilon \\ |\partial_p^2 H|_{\sigma+2,0}^0 \leq \Upsilon \end{cases} \quad (**_G)$$

for a suitable  $\Upsilon \geq 1$ . In the latter, we consider  $h(q, 0, t) = 0$  for all  $(q, t) \in \mathbb{T}^n \times J_0$ . We can do it without loss of generality. Let  $\tilde{h}$  be the following Hamiltonian

$$\tilde{h}(q, p, t) = W(q) \cdot p + m(q, p, t) \cdot p^2$$

for all  $(q, p, t) \in \mathbb{T}^n \times B \times J_0$ . Obviously  $X_H$  and  $X_{\tilde{h}}$  satisfy (8.1).

### 8.3.1 Outline of the Proof of Theorem G

We are looking for a  $C^\sigma$ -asymptotic torus  $\varphi^t$  associated to  $(X_H, X_{\tilde{h}}, \varphi_0, W)$ . More concretely, for given  $H$ , we will choose  $v' \geq 0$  sufficiently large and we search for some functions  $u, v : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n$  such that

$$\varphi(q, t) = (q + u(q, t), v(q, t)),$$

and to satisfy the following conditions

$$X_H(\varphi(q, t), t) - \partial_q \varphi(q, t)W(q) - \partial_t \varphi(q, t) = 0, \quad (8.7)$$

$$\lim_{t \rightarrow +\infty} |u^t|_{C^\sigma} = 0, \quad \lim_{t \rightarrow +\infty} |v^t|_{C^\sigma} = 0. \quad (8.8)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . The parameter  $v'$  is free and we will fix it large enough in Lemma 8.5.

As expected, we introduce a suitable functional  $\mathcal{F}$  given by (8.7). To this end, we define

$$\begin{aligned} \bar{m}(q, p, t)p &= \left( \int_0^1 \partial_p^2 H(q, \tau p, t) d\tau \right) p = \partial_p (m(q, p, t) \cdot p^2), \\ \tilde{\varphi}(q, t) &= (q + u(q, t), v(q, t), t), \\ \tilde{u}(q, t) &= (q + u(q, t), t), \\ \nabla u(q, t)\bar{W}(q) &= \partial_q u(q, t)W(q) + \partial_t u(q, t), \\ \nabla u(q, t)\bar{W}(q) &= \partial_q u(q, t)W(q) + \partial_t u(q, t). \end{aligned}$$

for all  $(q, p, t) \in \mathbb{T}^n \times J_{v'}$ . Similarly to Section 3.3.1, we define the following functional

$$\mathcal{F}(a, b, m, \bar{m}, W, u, v) = (F_1(b, \bar{m}, W, u, v), F_2(a, b, m, W, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, W, u, v) &= W \circ (\text{id} + u) - W + b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi})v - (\nabla u)\bar{W} \\ , F_2(a, b, m, W, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q W \circ (\text{id} + u) + \partial_q b \circ \tilde{u})v \\ &\quad + (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v)\bar{W}. \end{aligned}$$

It is defined over suitable Banach spaces, which we will specify later. We observe that for all  $m, \bar{m}$  and  $W$ ,

$$\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0) = 0.$$

As one can expect, we reformulate this problem in these terms. For fixed  $m, \bar{m}$  and  $W$  in suitable Banach spaces and for  $(a, b)$  sufficiently close to  $(0, 0)$ , we are looking for some functions  $u, v$  such that  $\mathcal{F}(a, b, m, \bar{m}, W, u, v) = 0$  and in order to satisfy the asymptotic conditions (8.8).

Concerning the associated linearized problem, the differential of  $\mathcal{F}$  with respect to the variables  $(u, v)$  calculated in  $(0, 0, m, \bar{m}, W, 0, 0)$  is equal to

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0)(\hat{u}, \hat{v}) = \begin{pmatrix} \partial_q W \hat{u} - (\nabla \hat{u})\bar{W} + \bar{m}_0 \hat{v} \\ \partial_q W \hat{v} + (\nabla \hat{v})\bar{W} \end{pmatrix}.$$

The proof preserves the same structure as that of Theorem A.

### 8.3.2 Preliminary Settings

Let  $\sigma$ ,  $\lambda$  and  $\Upsilon$  be the positive parameters introduced by  $(*_G)$ . For  $v' \geq 0$  that we will specify later, we consider the following Banach spaces  $(\mathcal{A}, |\cdot|)$ ,  $(\mathcal{B}, |\cdot|)$ ,  $(\mathcal{U}, |\cdot|)$ ,  $(\mathcal{V}, |\cdot|)$ ,  $(\mathcal{Z}, |\cdot|)$  and  $(\mathcal{G}, |\cdot|)$  (see Appendix C)

$$\begin{aligned} \mathcal{A} &= \left\{ a : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R} \mid a \in \bar{S}_{\sigma,2}^{v'} \text{ and } |a| = |a|_{\sigma+2,0}^{v'} + |\partial_q a|_{\sigma+1,\lambda}^{v'} < \infty \right\} \\ \mathcal{B} &= \left\{ b : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid b \in \bar{S}_{\sigma,2}^{v'}, \text{ and } |b| = |b|_{\sigma+2,\lambda}^{v'} < \infty \right\} \\ \mathcal{U} &= \left\{ u : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid u, (\nabla u) \bar{W} \in S_{\sigma}^{v'} \right. \\ &\quad \left. \text{and } |u| = \max\{|u|_{\sigma,\lambda}^{v'}, |(\nabla u) \bar{W}|_{\sigma,\lambda}^{v'}\} < \infty \right\} \\ \mathcal{V} &= \left\{ v : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid v, (\nabla v) \bar{W} \in S_{\sigma}^{v'} \right. \\ &\quad \left. \text{and } |v| = \max\{|v|_{\sigma,\lambda}^{v'}, |(\nabla v) \bar{W}|_{\sigma,\lambda}^{v'}\} < \infty \right\} \\ \mathcal{Z} &= \left\{ z : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid z \in S_{\sigma}^{v'}, \text{ and } |z| = |z|_{\sigma,\lambda}^{v'} < \infty \right\} \\ \mathcal{G} &= \left\{ g : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R} \mid g \in S_{\sigma}^{v'} \text{ and } |g| = |g|_{\sigma,\lambda}^{v'} < \infty \right\} \end{aligned}$$

where we recall that the norm  $|\cdot|_{\sigma,\lambda}^{v'}$  is defined by (8.6). Let  $M_n$  be the set of the  $n$ -dimensional matrices. We introduce two other Banach spaces  $(\mathcal{M}, |\cdot|)$  and  $(\mathcal{W}, |\cdot|)$

$$\begin{aligned} \mathcal{M} &= \left\{ m : \mathbb{T}^n \times B \times J_{v'} \rightarrow M_n \mid m \in \bar{S}_{\sigma,2}^{v'} \text{ and } |m| = |m|_{\sigma+2,1}^{v'} \leq \Upsilon \right\} \\ \mathcal{W} &= \left\{ W : \mathbb{T}^n \rightarrow \mathbb{R}^n \mid W \in C^{\sigma+2}(\mathbb{T}^n) \text{ and } |W| = |W|_{C^{\sigma+2}} < \infty \right\}. \end{aligned}$$

where  $\Upsilon$  is the positive parameter in  $(*_G)$ . Now, we can correctly introduce the previous functional  $\mathcal{F}$ . Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{W} \times \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{Z} \times \mathcal{G}$$

$$\mathcal{F}(a, b, m, \bar{m}, W, u, v) = (F_1(b, \bar{m}, W, u, v), F_2(a, b, m, W, u, v))$$

with

$$\begin{aligned} F_1(b, \bar{m}, W, u, v) &= W \circ (\text{id} + u) - W + b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \bar{W} \\ , F_2(a, b, m, W, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q W \circ (\text{id} + u) + \partial_q b \circ \tilde{u}) v \\ &\quad + (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v) \bar{W}. \end{aligned}$$

### 8.3.3 Homological Equation

We begin this section by reminding Proposition 6.2 concerning some fundamental Gronwall-type inequalities.

**Proposition** (Proposition 6.2). *Let  $J$  be an interval in  $\mathbb{R}$ ,  $t_0 \in J$ , and  $a, b, u \in C(J)$  continuous positive functions. If we assume that*

$$u(t) \leq a(t) + \left| \int_{t_0}^t b(s)u(s)ds \right|, \quad \forall t \in J$$

then it follows that

$$u(t) \leq a(t) + \left| \int_{t_0}^t a(s)b(s)e^{\left| \int_s^t b(\tau)d\tau \right|} ds \right|, \quad \forall t \in J. \quad (8.9)$$

If  $a$  is a monotone increasing function and we assume that

$$u(t) \leq a(t) + \int_{t_0}^t b(s)u(s)ds \quad \forall t \geq t_0,$$

then we obtain the estimate

$$u(t) \leq a(t)e^{\int_{t_0}^t b(s)ds}, \quad \forall t \geq t_0. \quad (8.10)$$

Given  $\sigma \geq 1$ ,  $\lambda > 0$  and  $v \geq 0$ , this section aims to solve the following equation for the unknown  $\varkappa : \mathbb{T}^n \times J_v \rightarrow \mathbb{R}^n$

$$\begin{cases} \partial_q \varkappa(q, t)W(q) + \partial_t \varkappa(q, t) \pm \partial_q W(q)\varkappa(q, t) = z(q, t) \\ W \in C^{\sigma+1}(\mathbb{T}^n), \quad z \in \mathcal{S}_\sigma^v, \\ |z|_{\sigma, \lambda}^v < \infty. \end{cases} \quad (HE_G)$$

If  $W(q) \equiv W \in \mathbb{R}^n$  is constant, then the latter translates into the following easier problem

$$\begin{cases} \partial_q \varkappa(q, t)W + \partial_t \varkappa(q, t) = z(q, t) \\ z \in \mathcal{S}_\sigma^v, \quad |z|_{\sigma, \lambda}^v < \infty. \end{cases}$$

In the third chapter of this thesis (Section 3.3.3), we proved the existence of a unique solution to the above system satisfying suitable asymptotic conditions (see Lemma 3.1).

We observe that  $(HE_G)$  is quite similar to the homological equation solved in the sixth chapter of this thesis (Lemma 6.5). The proof resembles that in Section 8.3.3 with suitable modifications. Therefore, we begin by proving several estimates. Let  $\phi_W^t$  be the flow at time  $t$  of  $W(q)$ . As usual,  $C(\cdot)$  stands for constants depending on  $n$  and the other parameters into brackets. On the other hand,  $C$  means constants depending on  $n$ .

**Lemma 8.1.** *For all  $t \in \mathbb{R}$*

$$|\partial_q \phi_W^t|_{C^{\sigma-1}} \leq C(\sigma) (1 + |\partial_q W|_{C^{\sigma-1}}|t|) e^{c_\sigma |\partial_q W|_{C^0}|t|}, \quad (8.11)$$

with a positive constant  $c_\sigma \geq 1$  depending on  $n$  and  $\sigma$ .

By (8.11), we note that when  $\sigma = 1$  and  $t \in \mathbb{R}$

$$|\partial_q \phi_W^t|_{C^0} \leq C(1 + |\partial_q W|_{C^0}|t|) e^{c_1 |\partial_q W|_{C^0}|t|} \leq C e^{\bar{c}_1 |\partial_q W|_{C^0}|t|}$$

for a suitable  $\bar{c}_1 > c_1$ .



*Proof.* The proof is quite similar to that of Lemma 6.3 (see Section 6.3.4). By the fundamental theorem of calculus, we can write  $\phi_W^t$  in the following form

$$\phi_W^t(q) = q + \int_0^t W \circ \phi_W^\tau(q) d\tau.$$

Therefore, taking the derivative with respect to  $q$

$$\partial_q \phi_W^t(q) = \text{Id} + \int_0^t \partial_q W \circ \phi_W^\tau(q) \partial_q \phi_W^\tau(q) d\tau,$$

where Id stands for the identity matrix. We assume  $t \geq 0$ . Then, we can estimate the norm  $C^{\sigma-1}$  of the left-hand side of the latter as follows

$$|\partial_q \phi_W^t|_{C^{\sigma-1}} \leq 1 + \int_0^t |\partial_q W \circ \phi_W^\tau \partial_q \phi_W^\tau|_{C^{\sigma-1}} d\tau. \quad (8.12)$$

*Case  $\sigma = 1$ .* By Proposition A.2

$$|\partial_q \phi_W^t|_{C^0} \leq 1 + C \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^0} d\tau,$$

for a suitable constant  $C$ . Then, thanks to (8.10)

$$|\partial_q \phi_W^t|_{C^0} \leq e^{\bar{c}_1 |\partial_q W|_{C^0} t} \quad (8.13)$$

for a suitable constant  $\bar{c}_1 \geq 1$ .

It remains to verify (8.11) when  $\sigma > 1$ . By Proposition (A.2), we can estimate the norm on the right-hand side of (8.12) as follows

$$|\partial_q W \circ \phi_W^\tau \partial_q \phi_W^\tau|_{C^{\sigma-1}} \leq C(\sigma) (|\partial_q W \circ \phi_W^\tau|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0} + |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}}).$$

Hence, we can rewrite (8.11) in the following form

$$\begin{aligned} |\partial_q \phi_W^t|_{C^{\sigma-1}} &\leq 1 + C(\sigma) \int_0^t |\partial_q W \circ \phi_W^\tau|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0} d\tau \\ &\quad + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}} d\tau. \end{aligned} \quad (8.14)$$

Unlike the proof of Lemma 6.5, we need to treat cases  $1 < \sigma < 2$  and  $\sigma \geq 2$  separately.

*Case  $1 < \sigma < 2$ .* Thanks to Proposition (A.2),

$$|\partial_q W \circ \phi_W^\tau|_{C^{\sigma-1}} \leq C(\sigma) (|\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^{\sigma-1} + |\partial_q W|_{C^0}).$$

Replacing the latter into (8.14), we can rewrite it in the following way

$$\begin{aligned} |\partial_q \phi_W^t|_{C^{\sigma-1}} &\leq 1 + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^0} d\tau \\ &\quad + C(\sigma) \int_0^t |\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^\sigma d\tau + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}} d\tau. \end{aligned}$$

Now, (8.13) allows us to find an upper bound for the first two integrals on the right-hand side of the latter

$$\begin{aligned} \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^0} d\tau &\leq |\partial_q W|_{C^0} \int_0^t e^{\bar{c}_1 |\partial_q W|_{C^0} \tau} d\tau = \frac{e^{\bar{c}_1 |\partial_q W|_{C^0} t} - 1}{\bar{c}_1} \\ \int_0^t |\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^\sigma d\tau &\leq |\partial_q W|_{C^{\sigma-1}} \int_0^t e^{c_1 \sigma |\partial_q W|_{C^0} \tau} d\tau \leq |\partial_q W|_{C^{\sigma-1}} t e^{\bar{c}_1 \sigma |\partial_q W|_{C^0} t}. \end{aligned}$$

In the second line of the latter, rather than calculating the integral, we prefer using the trivial estimate  $e^{c_1 \sigma |\partial_q W|_{C^0} \tau} \leq e^{c_1 \sigma |\partial_q W|_{C^0} t}$  to avoid a division by  $|\partial_q W|_{C^0}$  since we do not assume it is not zero.

Hence, we can estimate  $|\partial_q \phi_W^t|_{C^{\sigma-1}}$  as follows

$$\begin{aligned} |\partial_q \phi_W^t|_{C^{\sigma-1}} &\leq 1 + C(\sigma) \frac{e^{\bar{c}_1 |\partial_q W|_{C^0} t} - 1}{\bar{c}_1} + C(\sigma) |\partial_q W|_{C^{\sigma-1}} t e^{\bar{c}_1 \sigma |\partial_q W|_{C^0} t} \\ &\quad + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}}, d\tau, \\ &\leq C(\sigma) (1 + |\partial_q W|_{C^{\sigma-1}} t) e^{\bar{c}_1 \sigma |\partial_q W|_{C^0} t} + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}}, d\tau. \end{aligned}$$

Then, thanks to the Gronwall inequality (8.10)

$$\begin{aligned} |\partial_q \phi_W^t|_{C^{\sigma-1}} &\leq C(\sigma) (1 + |\partial_q W|_{C^{\sigma-1}} t) e^{\bar{c}_1 \sigma |\partial_q W|_{C^0} t} e^{C(\sigma) \int_0^t |\partial_q W|_{C^0} d\tau} \\ &\leq C(\sigma) (1 + |\partial_q W|_{C^{\sigma-1}} t) e^{c_\sigma |\partial_q W|_{C^0} t} \end{aligned}$$

for a suitable constant  $c_\sigma \geq \bar{c}_1 \sigma$ . This concludes the proof for the case  $1 < \sigma < 2$ . The general case  $\sigma > 2$  is quite similar to the previous one. The main difference lies in the estimation of  $|\partial_q W \circ \phi_W^\tau|_{C^{\sigma-1}}$ .

*Case  $\sigma > 2$ .* By Proposition A.2,

$$|\partial_q W \circ \phi_W^\tau|_{C^{\sigma-1}} \leq C(\sigma) \left( |\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^{\sigma-1} + |\partial_q W|_{C^1} |\partial_q \phi_W^\tau|_{C^{\sigma-2}} + |\partial_q W|_{C^0} \right)$$

and replacing the latter into (8.14), we can estimate  $|\partial_q \phi_W^t|_{C^{\sigma-1}}$  as follows

$$\begin{aligned} |\partial_q \phi_W^t|_{C^{\sigma-1}} &\leq 1 + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^0} d\tau + C(\sigma) \int_0^t |\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^\sigma d\tau \\ &\quad + C(\sigma) \int_0^t |\partial_q W|_{C^1} |\partial_q \phi_W^\tau|_{C^{\sigma-2}} |\partial_q \phi_W^\tau|_{C^0} d\tau \\ &\quad + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}} d\tau. \end{aligned}$$

We have already estimated the first two integrals on the right-hand side of the latter. It remains the integral in the second line. By the convexity property of the Hölder norms (Proposition A.1), for all fixed  $\tau$

$$|\partial_q W|_{C^1} |\partial_q \phi_W^\tau|_{C^{\sigma-2}} \leq C(\sigma) \left( |\partial_q W|_{C^0}^{\frac{\sigma-2}{\sigma-1}} |\partial_q W|_{C^{\sigma-1}}^{\frac{1}{\sigma-1}} \right) \left( |\partial_q \phi_W^\tau|_{C^0}^{\frac{1}{\sigma-1}} |\partial_q \phi_W^\tau|_{C^{\sigma-1}}^{\frac{\sigma-2}{\sigma-1}} \right)$$

and hence

$$|\partial_q W|_{C^1} |\partial_q \phi_W^\tau|_{C^{\sigma-2}} |\partial_q \phi_W^\tau|_{C^0} \leq C(\sigma) (|\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}})^{\frac{\sigma-2}{\sigma-1}} (|\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^\sigma)^{\frac{1}{\sigma-1}}.$$

From  $a^\lambda b^{1-\lambda} \leq C(a+b)$  for  $0 < \lambda < 1$ , we have that

$$|\partial_q W|_{C^1} |\partial_q \phi_W^\tau|_{C^{\sigma-2}} |\partial_q \phi_W^\tau|_{C^0} \leq C(\sigma) (|\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}} + |\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^\sigma).$$

Furthermore, replacing the latter in the previous estimate of  $|\partial_q \phi_W^t|_{C^{\sigma-1}}$ , we obtain

$$\begin{aligned} |\partial_q \phi_W^t|_{C^{\sigma-1}} &\leq 1 + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^0} d\tau \\ &\quad + C(\sigma) \int_0^t |\partial_q W|_{C^{\sigma-1}} |\partial_q \phi_W^\tau|_{C^0}^\sigma d\tau + C(\sigma) \int_0^t |\partial_q W|_{C^0} |\partial_q \phi_W^\tau|_{C^{\sigma-1}} d\tau. \end{aligned}$$

Now, similarly to the previous case ( $1 < \sigma < 2$ ), we conclude the proof of (8.11) also in this general case.

Similarly, we have the claim when  $t \leq 0$ . □

Now, we consider  $R : \mathbb{T}^n \times J_v \times J_v \rightarrow M_n$ , where  $M_n$  is the set of the  $n$ -dimensional matrices. For all  $(q, \tau, t) \in \mathbb{T}^n \times J_v \times J_v$ ,  $R(q, t, \tau)$  is the matrix having elements equal to  $r_{ij}(q, t, \tau)$  for all  $1 \leq i, j \leq n$ . In other words,  $R(q, t, \tau) = \{r_{ij}(q, t, \tau)\}_{1 \leq i, j \leq n}$ . We define the following family of norms

$$|R_\tau^t|_{C^s} = \max_{1 \leq i, j \leq n} |r_{ij}(q, t, \tau)|_{C^s},$$

for positive real parameters  $s \geq 0$ . We consider the following system that plays an important role in the solution of the homological equation ( $HE_G$ )

$$\begin{cases} \dot{R}(q, t, \tau) = \mp \partial_q W \circ \phi_W^t(q) R(q, t, \tau) \\ R(q, \tau, \tau) = \text{Id}. \end{cases} \quad (\text{R})$$

where  $W$  is defined in ( $HE_G$ ). For all fixed  $\tau, t \in J_v$ , in what follows we denote  $R_\tau^t(q) = R(q, t, \tau)$ .

**Lemma 8.2.** *The latter system admits a unique solution. Moreover, for all  $\tau, t \in J_v$  with  $\tau \geq t$ , letting  $\tilde{R}(q, t, \tau) = R(\phi_W^{-\tau}(q), t, \tau)$ , we have the following estimates*

$$|R_\tau^t|_{C^0} \leq e^{c_0^R |\partial_q W|_{C^0} (\tau-t)} \quad (8.15)$$

$$\begin{aligned} |\tilde{R}_\tau^t|_{C^\sigma} &\leq C(\sigma) (1 + |\partial_q W|_{C^\sigma} (\tau-t)) e^{c_\sigma^R |\partial_q W|_{C^0} (\tau-t)} \\ &\quad + C(\sigma) |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} (\tau-t)^2 e^{c_\sigma^R |\partial_q W|_{C^0} (\tau-t)} \end{aligned} \quad (8.16)$$

with positive constants  $c_0^R > 0$  and  $c_\sigma^R \geq c_\sigma$ . We point out that  $c_\sigma$  is the positive constant introduced in the previous lemma.

Before the proof, we observe that when  $\sigma = 1$ , thanks to (8.16),

$$\begin{aligned} |\tilde{R}_\tau^t|_{C^1} &\leq C (1 + |\partial_q W|_{C^1} (\tau-t) + |\partial_q W|_{C^1} |\partial_q W|_{C^0} (\tau-t)^2) e^{c_1^R |\partial_q W|_{C^0} (\tau-t)} \\ &\leq C (1 + |\partial_q W|_{C^1} (\tau-t)) e^{c_1^R |\partial_q W|_{C^0} (\tau-t)} \\ &\quad + C |\partial_q W|_{C^1} (\tau-t) e^{|\partial_q W|_{C^0} (\tau-t)} e^{c_1^R |\partial_q W|_{C^0} (\tau-t)} \\ &\leq C (1 + |\partial_q W|_{C^1} (\tau-t)) e^{\bar{c}_1^R |\partial_q W|_{C^0} (\tau-t)} \end{aligned}$$

for a suitable  $\bar{c}_1^R > c_1^R$ .

*Proof.* We prove this lemma in the case  $\dot{R}(q, t, \tau) = \partial_q W \circ \phi_W^t(q) R(q, t, \tau)$ . The other case ( $\dot{R}(q, t, \tau) = -\partial_q W \circ \phi_W^t(q) R(q, t, \tau)$ ) can be proved similarly. For all  $q \in \mathbb{T}^n$ , a unique solution of (R) exists by the theorem of existence and uniqueness. It remains to prove the estimates.

By the fundamental theorem of calculus, we can write  $R$  as follows

$$R_\tau^t(q) = \text{Id} - \int_t^\tau \partial_q W \circ \phi_W^s(q) R_\tau^s(q) ds. \quad (8.17)$$

for all  $q \in \mathbb{T}^n$  and  $t, \tau \in J_\nu$  with  $\tau \geq t$ . Then, thanks to the latter, we can estimate  $|R_\tau^t|_{C^0}$  in the following way

$$|R_\tau^t|_{C^0} \leq 1 + C \int_t^\tau |\partial_q W|_{C^0} |R_\tau^s|_{C^0} ds.$$

Therefore, by the Gronwall inequality (8.10), we have

$$|R_\tau^t|_{C^0} \leq e^{\int_t^\tau C |\partial_q W|_{C^0} ds} \leq e^{c_R^0 |\partial_q W|_{C^0} (\tau-t)}$$

for a suitable positive constant  $c_0^R$ . This concludes the proof of (8.15). Now, we prove (8.16). By (8.17), we can write  $\tilde{R}_\tau^t$  in the following form

$$\tilde{R}_\tau^t(q) = R_\tau^t \circ \phi_W^{-\tau}(q) = \text{Id} - \int_t^\tau \partial_q W \circ \phi_W^{s-\tau}(q) \tilde{R}_\tau^s(q) ds.$$

Hence, we can estimate  $|\tilde{R}_\tau^t|_{C^\sigma}$  in such a way that

$$|\tilde{R}_\tau^t|_{C^\sigma} \leq 1 + \int_t^\tau |\partial_q W \circ \phi_W^{s-\tau} \tilde{R}_\tau^s|_{C^\sigma} ds. \quad (8.18)$$

We will estimate the norm into the integral on the right-hand side of the latter using Proposition A.2 and (8.15). The claim is a consequence of the Gronwall inequality (8.9).

As a consequence of Proposition A.2,

$$\begin{aligned} |\partial_q W \circ \phi_W^{s-\tau} \tilde{R}_\tau^s|_{C^\sigma} &\leq C(\sigma) \left( |\partial_q W \circ \phi_W^{s-\tau}|_{C^\sigma} |R_\tau^s|_{C^0} + |\partial_q W|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} \right) \\ |\partial_q W \circ \phi_W^{s-\tau}|_{C^\sigma} &\leq C(\sigma) \left( |\partial_q W|_{C^\sigma} |\partial_q \phi_W^{s-\tau}|_{C^0}^\sigma + |\partial_q W|_{C^1} |\partial_q \phi_W^{s-\tau}|_{C^{\sigma-1}} + |\partial_q W|_{C^0} \right) \end{aligned}$$

and replacing the latter into (8.18)

$$\begin{aligned} |\tilde{R}_\tau^t|_{C^\sigma} &\leq 1 + C(\sigma) \int_t^\tau |\partial_q W|_{C^\sigma} |\partial_q \phi_W^{s-\tau}|_{C^0}^\sigma |R_\tau^s|_{C^0} ds \\ &\quad + C(\sigma) \int_t^\tau |\partial_q W|_{C^1} |\partial_q \phi_W^{s-\tau}|_{C^{\sigma-1}} |R_\tau^s|_{C^0} ds \\ &\quad + C(\sigma) \int_t^\tau |\partial_q W|_{C^0} |R_\tau^s|_{C^0} ds + C(\sigma) \int_t^\tau |\partial_q W|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds. \end{aligned}$$

Now, by (8.15) and Lemma 8.1, we can estimate the first three integrals on the right-hand side of the latter

$$\begin{aligned}
 \int_t^\tau |\partial_q W|_{C^\sigma} |\partial_q \phi_W^{s-\tau}|_{C^0} |R_\tau^s|_{C^0} ds &\leq C(\sigma) |\partial_q W|_{C^\sigma} \int_t^\tau e^{\bar{c}_1 \sigma |\partial_q W|_{C^0}(\tau-s)} e^{c_0^R |\partial_q W|_{C^0}(\tau-s)} ds \\
 &\leq C(\sigma) |\partial_q W|_{C^\sigma} (\tau-t) e^{(\bar{c}_1 \sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 \int_t^\tau |\partial_q W|_{C^1} |\partial_q \phi_W^{s-\tau}|_{C^{\sigma-1}} |R_\tau^s|_{C^0} ds &\leq C(\sigma) |\partial_q W|_{C^1} \int_t^\tau e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-s)} ds \\
 &\quad + C(\sigma) |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} \int_t^\tau (\tau-s) e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-s)} ds \\
 &\leq C(\sigma) |\partial_q W|_{C^1} (\tau-t) e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 &\quad + C(\sigma) |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} (\tau-t)^2 e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 \int_t^\tau |\partial_q W|_{C^0} |R_\tau^s|_{C^0} ds &\leq |\partial_q W|_{C^0} \int_t^\tau e^{c_0^R |\partial_q W|_{C^0}(\tau-s)} ds \\
 &= \frac{1}{c_0^R} \left( e^{c_0^R |\partial_q W|_{C^0}(\tau-t)} - 1 \right).
 \end{aligned}$$

Similarly to the previous lemma, in the first two integrals on the left-hand side of the latter, we use some trivial inequalities to avoid the division by  $|\partial_q W|_{C^0}$ .

Then, by the above estimations and remembering that  $\bar{c}_1 \sigma \leq c_\sigma$

$$\begin{aligned}
 |\tilde{R}_\tau^t|_{C^\sigma} &\leq 1 + C(\sigma) \left( e^{c_0^R |\partial_q W|_{C^0}(\tau-t)} - 1 \right) + C(\sigma) |\partial_q W|_{C^\sigma} (\tau-t) e^{(\bar{c}_1 \sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 &\quad + C(\sigma) |\partial_q W|_{C^1} (\tau-t) e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 &\quad + C(\sigma) |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} (\tau-t)^2 e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 &\quad + C(\sigma) \int_t^\tau |\partial_q W|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds \\
 &\leq C(\sigma) (1 + |\partial_q W|_{C^\sigma} (\tau-t)) e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 &\quad + C(\sigma) |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} (\tau-t)^2 e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 &\quad + \left| C(\sigma) \int_\tau^t |\partial_q W|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds \right|.
 \end{aligned}$$

Similarly to the proof of Lemma 6.4, we define the following function

$$\begin{aligned}
 a(t) &= C(\sigma) (1 + |\partial_q W|_{C^\sigma} (\tau-t)) e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)} \\
 &\quad + C(\sigma) |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} (\tau-t)^2 e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0}(\tau-t)}
 \end{aligned}$$

and we rewrite the latter in the following way

$$|\tilde{R}_\tau^t|_{C^\sigma} \leq a(t) + \left| C(\sigma) \int_\tau^t |\partial_q W|_{C^0} |\tilde{R}_\tau^s|_{C^\sigma} ds \right|.$$

However, it is straightforward to verify that  $a$  is a monotone decreasing func-

tion. Hence, by the more general inequality (6.16)

$$\begin{aligned}
 |\tilde{R}_\tau^t|_{C^\sigma} &\leq a(t) + C(\sigma) \left| \int_\tau^t a(s) |\partial_q W|_{C^0} e^{C(\sigma) \int_s^t |\partial_q W|_{C^0} d\delta} ds \right| \\
 &\leq a(t) \left( 1 + C(\sigma) |\partial_q W|_{C^0} \int_t^\tau e^{C(\sigma) |\partial_q W|_{C^0} (s-t)} ds \right) \\
 &= a(t) (1 + (e^{C(\sigma) |\partial_q W|_{C^0} (\tau-t)} - 1)) \\
 &\leq a(t) (1 + e^{C(\sigma) |\partial_q W|_{C^0} (\tau-t)}) \\
 &\leq C(\sigma) (1 + |\partial_q W|_{C^\sigma} (\tau-t)) e^{c_\sigma^R |\partial_q W|_{C^0} (\tau-t)} \\
 &\quad + C(\sigma) |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} (\tau-t)^2 e^{c_\sigma^R |\partial_q W|_{C^0} (\tau-t)}
 \end{aligned}$$

for a suitable constant  $c_\sigma^R \geq c_\sigma + c_0^R$ .  $\square$

We observe that the constant  $c_\sigma^R$ , as for  $c_\sigma$ , goes to infinity if  $\sigma \rightarrow \infty$ . This means that, in order to solve the homological equation, we must counter the growth of  $c_\sigma^R$  and  $c_\sigma$  assuming  $\lambda$  sufficiently large.

**Lemma 8.3 (Homological equation).** *There exists a solution  $\varkappa$ ,  $(\nabla \varkappa) \bar{W} \in \mathcal{S}_\sigma^v$  of  $(HE_G)$ . Moreover, letting  $c_\sigma^\varkappa = \max\{c_\sigma + c_0^R, c_\sigma^R + c_0^R, \bar{c}_1^R + c_\sigma\}$ , if*

$$\lambda > c_\sigma^\varkappa |\partial_q W|_{C^0} \quad (8.19)$$

then,

$$\begin{aligned}
 |\varkappa|_{\sigma, \lambda}^v &\leq C(\sigma) \left( \frac{1}{\lambda - c_\sigma^\varkappa |\partial_q W|_{C^0}} \right) |z|_{\sigma, \lambda}^v \\
 &\quad + C(\sigma) \left( \frac{|\partial_q W|_{C^\sigma}}{(\lambda - c_\sigma^\varkappa |\partial_q W|_{C^0})^2} + \frac{|\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}}{(\lambda - c_\sigma^\varkappa |\partial_q W|_{C^0})^3} \right) |z|_{\sigma, \lambda}^v.
 \end{aligned} \quad (8.20)$$

*Proof. Existence:* Let us define the following transformation

$$\begin{aligned}
 h : \mathbb{T}^n \times J_v &\longrightarrow \mathbb{T}^n \times J_v \\
 h(q, t) &= (\phi_W^{-t}(q), t)
 \end{aligned}$$

where  $\phi_W^t$  is the flow of  $W$  previously introduced. We claim that it suffices to prove the first part of this lemma for the much easier equation

$$\partial_t \kappa(q, t) \pm \partial_q W \circ \phi_W^t(q) \kappa(q, t) = z \circ h^{-1}(q, t). \quad (8.21)$$

If  $\kappa$  is a solution of the latter, then  $\varkappa = \kappa \circ h$  is a solution of  $(HE_G)$  and viceversa. We prove this claim. Let  $\varkappa$  be a solution of  $(HE_G)$ ,

$$\begin{aligned}
 \partial_t (\varkappa \circ h^{-1}) \pm (\partial_q W \circ \phi_W^t) (\varkappa \circ h^{-1}) &= (\partial_q \varkappa \circ h^{-1}) \dot{\phi}_W^t + \partial_t \varkappa \circ h^{-1} \pm (\partial_q W \circ \phi_W^t) (\varkappa \circ h^{-1}) \\
 &= (\partial_q \varkappa \circ h^{-1}) (W \circ \phi_W^t) + \partial_t \varkappa \circ h^{-1} \\
 &\quad \pm (\partial_q W \circ \phi_W^t) (\varkappa \circ h^{-1}) \\
 &= z \circ h^{-1}
 \end{aligned}$$

where  $\dot{\phi}_W^t$  stands for the derivative of  $\phi_W^t$  with respect to  $t$ . It is obviously equal to  $W \circ \phi_W^t$ . Furthermore, the last equality is a consequence of  $(HE_G)$ . This proves

that  $\varkappa \circ h^{-1}$  is a solution of (8.21). Let us first show that  $\partial_q \phi_W^{-t} W = \dot{\phi}_W^{-t}$ . We know that  $\phi_W^t$  is the flow of  $W$ , then  $\dot{\phi}_W^{-t} = W \circ \phi_W^{-t}$ . However, for all  $(q, t) \in \mathbb{T}^n \times J_v$ ,

$$W \circ \phi_W^{-t}(q) = \partial_q \phi_W^{-t}(q) W(q).$$

This is because the pull-back of  $W$  by  $\phi_W^{-t}$  is equal to  $W$ . In others words,  $(\phi_W^{-t})^* W = W$  where  $(\phi_W^{-t})^* W = (\partial_q \phi_W^{-t})^{-1} W \circ \phi_W^{-t}$ . Then,  $\partial_q \phi_W^{-t} W = \dot{\phi}_W^{-t}$ .

Now, let  $\kappa$  be a solution of (8.21), then

$$\begin{aligned} \partial_q (\kappa \circ h) W + \partial_t (\kappa \circ h) \pm \partial_q W (\kappa \circ h) &= (\partial_q \kappa \circ h) \partial_q \phi_W^{-t} W - (\partial_q \kappa \circ h) \dot{\phi}_W^{-t} \\ &+ \partial_t \kappa \circ h \pm \partial_q W (\kappa \circ h) \\ &= \partial_t \kappa \circ h \pm \partial_q W (\kappa \circ h) = z. \end{aligned}$$

Hence  $\kappa \circ h$  is a solution of  $(HE_G)$ , where the last equality of the latter is a consequence of (8.21). This proves the claim.

For all  $q \in \mathbb{T}^n$ , let  $R(q, t, v)$  be the unique solution of (R). For all  $(q, t) \in \mathbb{T}^n \times J_v$  a solution  $\kappa$  of (8.21) exists and

$$\begin{aligned} \kappa(q, t) &= R(q, t, v) e(q) - \int_v^t R(q, t, \tau) z \circ h^{-1}(q, \tau) d\tau \\ &= R(q, t, v) \left( e(q) - \int_v^t R(q, v, \tau) z \circ h^{-1}(q, \tau) d\tau \right) \end{aligned}$$

where  $e$  is a function defined on the torus.

*Estimates:* We choose  $e$  equal to

$$e(q) = \int_v^{+\infty} R(q, v, \tau) z \circ h^{-1}(q, \tau) d\tau$$

for all  $q \in \mathbb{T}^n$ . It is well defined because, by Lemma 8.2 and  $(\#_G)$ ,

$$\begin{aligned} \left| \int_v^{+\infty} R(q, v, \tau) z \circ h^{-1}(q, \tau) d\tau \right| &\leq C \int_v^{+\infty} |R_\tau^v|_{C^0} |z^\tau|_{C^0} d\tau \\ &\leq C |z|_{0, \lambda}^v \int_v^{+\infty} e^{(c_0^R |\partial_q W|_{C^0} - \lambda) s} ds \\ &= \frac{C |z|_{0, \lambda}^v}{\lambda - c_0^R |\partial_q W|_{C^0}} e^{(c_0^R |\partial_q W|_{C^0} - \lambda) v} \end{aligned}$$

Therefore, for all  $(q, t) \in \mathbb{T}^n \times J_v$ ,

$$\begin{aligned} \varkappa(q, t) &= \kappa \circ h(q, t) = - \int_t^{+\infty} R_\tau^t \circ \phi_W^{-t}(q) z^\tau \circ \phi_W^{\tau-t}(q) d\tau \\ &= - \int_t^{+\infty} R_\tau^t \circ \phi_W^{-\tau} \circ \phi_W^{\tau-t}(q) z^\tau \circ \phi_W^{\tau-t}(q) d\tau \\ &= - \int_t^{+\infty} \tilde{R}_\tau^t \circ \phi_W^{\tau-t}(q) z^\tau \circ \phi_W^{\tau-t}(q) d\tau \end{aligned}$$

is the solution of  $(HE_G)$  we are looking for.

The estimate (8.20) is a consequence of Proposition A.2, Lemma 8.1, Lemma 8.2 and (8.19). For all fixed  $t \in J_v$ , by Proposition A.2, we can estimate  $|z^t|_{C^\sigma}$  as follows

$$|z^t|_{C^\sigma} \leq C(\sigma) \int_t^{+\infty} |\tilde{R}_\tau^t \circ \phi_W^{\tau-t}|_{C^\sigma} |z^\tau|_{C^0} + |R_\tau^t|_{C^0} |z^\tau \circ \phi_W^{\tau-t}|_{C^\sigma} d\tau.$$

Always using Proposition A.2

$$\begin{aligned} |z^\tau \circ \phi_W^{\tau-t}|_{C^\sigma} &\leq C(\sigma) |z^\tau|_{C^\sigma} (|\partial_q \phi_W^{\tau-t}|_{C^0}^\sigma + |\partial_q \phi_W^{\tau-t}|_{C^{\sigma-1}} + 1) \\ |\tilde{R}_\tau^t \circ \phi_W^{\tau-t}|_{C^\sigma} &\leq C(\sigma) \left( |\tilde{R}_\tau^t|_{C^\sigma} |\partial_q \phi_W^{\tau-t}|_{C^0}^\sigma + |\tilde{R}_\tau^t|_{C^1} |\partial_q \phi_W^{\tau-t}|_{C^{\sigma-1}} + |R_\tau^t|_{C^0} \right) \end{aligned}$$

and replacing the latter into the above integral

$$\begin{aligned} |z^t|_{C^\sigma} &\leq C(\sigma) \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} |\partial_q \phi_W^{\tau-t}|_{C^0}^\sigma d\tau + C(\sigma) \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} |\partial_q \phi_W^{\tau-t}|_{C^{\sigma-1}} d\tau \\ &+ C(\sigma) \int_t^{+\infty} |\tilde{R}_\tau^t|_{C^\sigma} |\partial_q \phi_W^{\tau-t}|_{C^0}^\sigma |z^\tau|_{C^\sigma} d\tau + C(\sigma) \int_t^{+\infty} |\tilde{R}_\tau^t|_{C^1} |\partial_q \phi_W^{\tau-t}|_{C^{\sigma-1}} |z^\tau|_{C^\sigma} d\tau \\ &+ C(\sigma) \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} d\tau. \end{aligned}$$

It remains to estimate each integral on the right-hand side of the latter. But, first, we note that for all  $t \in J_v$  and  $x < 0$

$$\int_t^{+\infty} e^{x\tau} (\tau - t) d\tau = \frac{e^{xt}}{x^2}, \quad \int_t^{+\infty} e^{x\tau} (\tau - t)^2 d\tau = -2 \frac{e^{xt}}{x^3}$$

where the latter is obtained by integrating by part. Now, thanks to Lemma 8.1, Lemma 8.2, (8.19) and the latter

$$\begin{aligned} &\int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} |\partial_q \phi_W^{\tau-t}|_{C^0}^\sigma d\tau \\ &\leq C(\sigma) |z|_{\sigma,\lambda}^v \int_t^{+\infty} e^{(\bar{c}_1\sigma + c_0^R) |\partial_q W|_{C^0} (\tau-t)} e^{-\lambda\tau} d\tau = C(\sigma) \frac{|z|_{\sigma,\lambda}^v}{\lambda - (\bar{c}_1\sigma + c_0^R) |\partial_q W|_{C^0}} e^{\lambda t} \\ &\int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} |\partial_q \phi_W^{\tau-t}|_{C^{\sigma-1}} d\tau \\ &\leq C(\sigma) |z|_{\sigma,\lambda}^v \int_t^{+\infty} (1 + |\partial_q W|_{C^{\sigma-1}} (\tau - t)) e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0} (\tau-t)} e^{-\lambda\tau} d\tau \\ &= C(\sigma) |z|_{\sigma,\lambda}^v \int_t^{+\infty} e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0} (\tau-t)} e^{-\lambda\tau} d\tau \\ &+ C(\sigma) |z|_{\sigma,\lambda}^v |\partial_q W|_{C^{\sigma-1}} \int_t^{+\infty} e^{(c_\sigma + c_0^R) |\partial_q W|_{C^0} (\tau-t)} e^{-\lambda\tau} (\tau - t) d\tau \\ &= C(\sigma) \frac{|z|_{\sigma,\lambda}^v}{\lambda - (c_\sigma + c_0^R) |\partial_q W|_{C^0}} e^{\lambda t} + C(\sigma) \frac{|z|_{\sigma,\lambda}^v |\partial_q W|_{C^{\sigma-1}}}{(\lambda - (c_\sigma + c_0^R) |\partial_q W|_{C^0})^2} e^{\lambda t} \end{aligned}$$



$$\begin{aligned}
 & \int_t^{+\infty} |\tilde{R}_\tau^t|_{C^\sigma} |\partial_q \phi_W^{\tau-t}|_{C^0}^\sigma |z^\tau|_{C^\sigma} d\tau \\
 & \leq C(\sigma) |z|_{\sigma,\lambda}^v \int_t^{+\infty} (1 + |\partial_q W|_{C^\sigma}(\tau-t)) e^{(c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau \\
 & + C(\sigma) |z|_{\sigma,\lambda}^v \int_t^{+\infty} |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}(\tau-t)^2 e^{(c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau \\
 & = C(\sigma) \frac{|z|_{\sigma,\lambda}^v}{\lambda - (c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0}} e^{\lambda t} + C(\sigma) \frac{|z|_{\sigma,\lambda}^v |\partial_q W|_{C^\sigma}}{(\lambda - (c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0})^2} e^{\lambda t} \\
 & + C(\sigma) \frac{|z|_{\sigma,\lambda}^v |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}}{(\lambda - (c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0})^3} e^{\lambda t}
 \end{aligned}$$

$$\begin{aligned}
 & \int_t^{+\infty} |\tilde{R}_\tau^t|_{C^1} |\partial_q \phi_W^{\tau-t}|_{C^{\sigma-1}} |z^\tau|_{C^\sigma} d\tau \\
 & \leq C(\sigma) |z|_{\sigma,\lambda}^v \int_t^{+\infty} (1 + |\partial_q W|_{C^1}(\tau-t)) (1 + |\partial_q W|_{C^{\sigma-1}}(\tau-t)) e^{(\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau \\
 & \leq C(\sigma) |z|_{\sigma,\lambda}^v \int_t^{+\infty} e^{(\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau \\
 & + C(\sigma) |z|_{\sigma,\lambda}^v |\partial_q W|_{C^1} \int_t^{+\infty} (\tau-t) e^{(\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau \\
 & + C(\sigma) |z|_{\sigma,\lambda}^v |\partial_q W|_{C^{\sigma-1}} \int_t^{+\infty} (\tau-t) e^{(\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau \\
 & + C(\sigma) |z|_{\sigma,\lambda}^v |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}} \int_t^{+\infty} (\tau-t)^2 e^{(\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau \\
 & = C(\sigma) \frac{|z|_{\sigma,\lambda}^v}{\lambda - (\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0}} e^{\lambda t} + C(\sigma) \frac{|z|_{\sigma,\lambda}^v (|\partial_q W|_{C^1} + |\partial_q W|_{C^{\sigma-1}})}{(\lambda - (\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0})^2} e^{\lambda t} \\
 & + C(\sigma) \frac{|z|_{\sigma,\lambda}^v |\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}}{(\lambda - (\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0})^3} e^{\lambda t}
 \end{aligned}$$

$$\begin{aligned}
 & \int_t^{+\infty} |R_\tau^t|_{C^0} |z^\tau|_{C^\sigma} d\tau \\
 & \leq C |z|_{\sigma,\lambda}^v \int_t^{+\infty} e^{c_0^R |\partial_q W|_{C^0}(\tau-t)} e^{-\lambda \tau} d\tau = C(\sigma) \frac{|z|_{\sigma,\lambda}^v}{\lambda - c_0^R |\partial_q W|_{C^0}} e^{\lambda t}.
 \end{aligned}$$

Now, we remind that  $c_\sigma^z = \max\{c_\sigma + c_0^R, c_\sigma^R + c_0^R, \bar{c}_1^R + c_\sigma\}$ . Hence, thanks to the

latter

$$\begin{aligned}
 |z^t|_{C^\sigma} e^{\lambda t} &\leq C(\sigma) \left( \frac{1}{\lambda - (\bar{c}_1 \sigma + c_0^R) |\partial_q W|_{C^0}} + \frac{1}{\lambda - (c_\sigma + c_0^R) |\partial_q W|_{C^0}} \right. \\
 &+ \frac{|\partial_q W|_{C^{\sigma-1}}}{(\lambda - (c_\sigma + c_0^R) |\partial_q W|_{C^0})^2} + \frac{1}{\lambda - (c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0}} \\
 &+ \frac{|\partial_q W|_{C^\sigma}}{(\lambda - (c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0})^2} + \frac{|\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}}{(\lambda - (c_\sigma^R + \bar{c}_1 \sigma) |\partial_q W|_{C^0})^3} \\
 &+ \frac{1}{\lambda - (\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0}} + \frac{|\partial_q W|_{C^1} + |\partial_q W|_{C^{\sigma-1}}}{(\lambda - (\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0})^2} \\
 &+ \left. \frac{|\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}}{(\lambda - (\bar{c}_1^R + c_\sigma) |\partial_q W|_{C^0})^3} + \frac{1}{\lambda - c_0^R |\partial_q W|_{C^0}} \right) |z|_{\sigma, \lambda}^v \\
 &\leq C(\sigma) \left( \frac{1}{\lambda - c_\sigma^z |\partial_q W|_{C^0}} + \frac{|\partial_q W|_{C^\sigma}}{(\lambda - c_\sigma^z |\partial_q W|_{C^0})^2} + \frac{|\partial_q W|_{C^1} |\partial_q W|_{C^{\sigma-1}}}{(\lambda - c_\sigma^z |\partial_q W|_{C^0})^3} \right) |z|_{\sigma, \lambda}^v
 \end{aligned}$$

for all  $t \in J_v$ . Furthermore, taking the sup for all  $t \in J_v$  on the left-hand side of the latter, we conclude the proof.  $\square$

We observe that we do not find a unique solution to the homological equation in this case, unlike when  $W$  is constant. That is why we prove only the existence of a right inverse of the differential of  $\mathcal{F}$  introduced in Section 8.3.1.

The following two sections conclude the proof of Theorem G. As mentioned before, the proof is quite similar to that of Theorem A.

### 8.3.4 Regularity of $\mathcal{F}$

We begin this section by reminding the definition of the functional  $\mathcal{F}$ ,

$$\mathcal{F} : \mathcal{A} \times \mathcal{B} \times \mathcal{M} \times \mathcal{M} \times \mathcal{W} \times \mathcal{U} \times \mathcal{V} \longrightarrow \mathcal{Z} \times \mathcal{G}$$

$$\mathcal{F}(a, b, m, \bar{m}, W, u, v) = (F_1(b, \bar{m}, u, v), F_2(a, b, m, W, u, v))$$

with

$$\begin{aligned}
 F_1(b, \bar{m}, W, u, v) &= W \circ (\text{id} + u) - W + b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \bar{W} \\
 &= \left( \int_0^1 \partial_q W(\text{id} + \tau u) d\tau \right) u + b \circ \tilde{u} + (\bar{m} \circ \tilde{\varphi}) v - (\nabla u) \bar{W} \\
 F_2(a, b, m, W, u, v) &= \partial_q a \circ \tilde{u} + (\partial_q W \circ (\text{id} + u) + \partial_q b \circ \tilde{u}) v \\
 &+ (\partial_q m \circ \tilde{\varphi}) \cdot v^2 + (\nabla v) \bar{W},
 \end{aligned}$$

where the second line of the latter is a consequence of the Taylor formula. Thanks to Proposition 8.1, the functional  $\mathcal{F}$  is well defined and continuous. Moreover,  $\mathcal{F}$

is differentiable with respect to the components  $(u, v)$ , with

$$\begin{aligned}
 D_{(u,v)}F_1(b, \bar{m}, W, u, v)(\hat{u}, \hat{v}) &= D_u F_1(b, \bar{m}, W, u, v)\hat{u} + D_v F_1(b, \bar{m}, W, u, v)\hat{v} \\
 &= (\partial_q W \circ (\text{id} + u) + \partial_q b \circ \tilde{u})\hat{u} + v^T (\partial_q \bar{m} \circ \tilde{\varphi})\hat{u} \\
 &\quad + v^T (\partial_p \bar{m} \circ \tilde{\varphi})\hat{v} + (\bar{m} \circ \tilde{\varphi})\hat{v} - (\nabla \hat{u})\bar{W} \\
 D_{(u,v)}F_2(a, b, m, W, u, v)(\hat{u}, \hat{v}) &= D_u F_2(a, b, m, W, u, v)\hat{u} + D_v F_2(a, b, m, W, u, v)\hat{v} \\
 &= (\partial_q^2 a \circ \tilde{u})\hat{u} + v^T (\partial_q^2 W \circ (\text{id} + u) + \partial_q^2 b \circ \tilde{u})\hat{u} \\
 &\quad + (v^T)^2 (\partial_q^2 m \circ \tilde{\varphi})\hat{u} + (\partial_q W \circ (\text{id} + u) + \partial_q b \circ \tilde{u})\hat{v} \\
 &\quad + (v^T)^2 (\partial_{pq}^2 m \circ \tilde{\varphi})\hat{v} + 2v^T (\partial_q m \circ \tilde{\varphi})\hat{v} \\
 &\quad + (\nabla \hat{v})\bar{W},
 \end{aligned}$$

where  $T$  stands for transpose. These differentials are continuous. Furthermore, the differential  $D_{(u,v)}\mathcal{F}$  calculated in  $(0, 0, m, \bar{m}, W, 0, 0)$  is equal to

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0)(\hat{u}, \hat{v}) = \begin{pmatrix} \partial_q W \hat{u} - (\nabla \hat{u})\bar{W} + \bar{m}_0 \hat{v} \\ \partial_q W \hat{v} + (\nabla \hat{v})\bar{W} \end{pmatrix}, \quad (8.22)$$

where for all  $(q, t) \in \mathbb{T}^n \times J_v$ , we let  $\bar{m}_0(q, t) = \bar{m}(q, 0, t)$ . The following lemma proves that, for all fixed  $m, \bar{m} \in \mathcal{M}$  and  $W \in \mathcal{W}$ ,  $D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0)$  admits a right inverse.

**Lemma 8.4.** *For all  $(z, g) \in \mathcal{Z} \times \mathcal{G}$ , there exists  $(\hat{u}, \hat{v}) \in \mathcal{U} \times \mathcal{V}$  such that*

$$D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0)(\hat{u}, \hat{v}) = (z, g). \quad (8.23)$$

Moreover, for a suitable constant  $\bar{C}$  depending on  $\sigma, \Upsilon, \lambda, |\partial_q W|_{C^0}$  and  $|\partial_q W|_{C^\sigma}$

$$|\hat{u}| \leq \bar{C} \left( |g|_{\sigma, \lambda}^{v'} + |z|_{\sigma, \lambda}^{v'} \right), \quad |\hat{v}| \leq \bar{C} |g|_{\sigma, \lambda}^{v'}$$

where, we recall that  $|\hat{u}| = \max\{|\hat{u}|_{\sigma, \lambda}^{v'}, |(\nabla \hat{u})\bar{W}|_{\sigma, \lambda}^{v'}\}$  and  $|\hat{v}| = \max\{|\hat{v}|_{\sigma, \lambda}^{v'}, |(\nabla \hat{v})\bar{W}|_{\sigma, \lambda}^{v'}\}$ .

*Proof.* The proof of this lemma relies on Lemma 8.3. Thanks to (8.22), we can rewrite equation (8.23) in terms of the following system in the unknown  $(\hat{u}, \hat{v})$

$$\begin{cases} (\nabla \hat{u})\bar{W} - \partial_q W \hat{u} = \bar{m}_0 \hat{v} - z \\ (\nabla \hat{v})\bar{W} + \partial_q W \hat{v} = g. \end{cases} \quad (8.24)$$

These equations are decoupled, and hence we can study them separately. We begin by solving the last one. Then, we replace the found solution  $\hat{v}$  in the first equation, which now can be solved, and we conclude the proof of this lemma.

By Lemma 8.3, a solution  $\hat{v}$  of the second equation of the above system exists and satisfies

$$|\hat{v}|_{\sigma, \lambda}^{v'} \leq C(\sigma, \lambda, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma}) |g|_{\sigma, \lambda}^{v'}.$$

Moreover, thanks to Proposition 8.1, (8.24) and the latter

$$\begin{aligned}
 |(\nabla \hat{u})\bar{W}|_{\sigma, \lambda}^{v'} &= |g - \partial_q W \hat{v}|_{\sigma, \lambda}^{v'} \leq |g|_{\sigma, \lambda}^{v'} + C(\sigma) |\partial_q W|_{C^\sigma} |\hat{v}|_{\sigma, \lambda}^{v'} \\
 &\leq C(\sigma, \lambda, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma}) |g|_{\sigma, \lambda}^{v'}.
 \end{aligned}$$

As a consequence of the previous estimates, we obtain

$$|\hat{v}| = \max\{|\hat{v}|_{\sigma,\lambda}^{v'}, |(\nabla\hat{v})\bar{W}|_{\sigma,\lambda}^{v'}\} \leq C(\sigma, \lambda, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma})|g|_{\sigma,\lambda}^{v'},$$

which proves the first estimate of this lemma. Now, we can solve the first equation of (8.24) where  $\hat{v}$  is known. Thanks to Proposition 8.1 and the previous estimate

$$\begin{aligned} |\bar{m}_0\hat{v} - z|_{\sigma,\lambda}^{v'} &\leq C(\sigma)\Upsilon|\hat{v}|_{\sigma,\lambda}^{v'} + |z|_{\sigma,\lambda}^{v'} \leq C(\sigma, \lambda, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma})\Upsilon|g|_{\sigma,\lambda}^{v'} + |z|_{\sigma,\lambda}^{v'} \\ &\leq C(\sigma, \lambda, \Upsilon, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma}) \left( |g|_{\sigma,\lambda}^{v'} + |z|_{\sigma,\lambda}^{v'} \right). \end{aligned}$$

By Lemma 8.3, a solution of the first equation of (8.24) exists and we have that

$$\begin{aligned} |\hat{u}|_{\sigma,\lambda}^{v'} &\leq C(\sigma, \lambda, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma})|\bar{m}_0\hat{v} - z|_{\sigma,\lambda}^{v'} \\ &\leq C(\sigma, \lambda, \Upsilon, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma}) \left( |g|_{\sigma,\lambda}^{v'} + |z|_{\sigma,\lambda}^{v'} \right). \end{aligned}$$

It remains to estimate  $|(\nabla\hat{u})\bar{W}|_{\sigma,\lambda}^{v'}$  to conclude the proof of this lemma. This is a consequence of the previous estimates and (8.24)

$$\begin{aligned} |(\nabla\hat{u})\bar{W}|_{\sigma,\lambda}^{v'} &= |\bar{m}_0\hat{v} - z + \partial_q W\hat{u}|_{\sigma,\lambda}^{v'} \leq |\bar{m}_0\hat{v} - z|_{\sigma,\lambda}^{v'} + C(\sigma)|\partial_q W|_{C^\sigma}|\hat{u}|_{\sigma,\lambda}^{v'} \\ &\leq C(\sigma, \lambda, \Upsilon, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma}) \left( |g|_{\sigma,\lambda}^{v'} + |z|_{\sigma,\lambda}^{v'} \right) \end{aligned}$$

and thus

$$|\hat{u}| = \max\{|\hat{u}|_{\sigma,\lambda}^{v'}, |(\nabla\hat{u})\bar{W}|_{\sigma,\lambda}^{v'}\} \leq C(\sigma, \lambda, \Upsilon, |\partial_q W|_{C^0}, |\partial_q W|_{C^\sigma}) \left( |g|_{\sigma,\lambda}^{v'} + |z|_{\sigma,\lambda}^{v'} \right).$$

□

### 8.3.5 $C^\sigma$ -Asymptotic Torus

Following Section 3.3.5, this part is devoted to proving the existence of a  $C^\sigma$ -asymptotic torus associated to  $(X_H, X_{\tilde{h}}, \varphi_0, W)$ . To this end, we fix  $x = (a, b)$ , where  $a$  and  $b$  are those defined by  $(**_G)$ . It is straightforward to verify that  $(a, b) \in \mathcal{A} \times \mathcal{B}$ . Moreover, we define the following Banach space  $(\mathcal{Y}, |\cdot|)$  such that  $\mathcal{Y} = \mathcal{U} \times \mathcal{V}$  and, for all  $y = (u, v) \in \mathcal{Y}$ ,  $|y| = \max\{|u|, |v|\}$ . Let  $m, \bar{m} \in \mathcal{M}$  and  $W \in \mathcal{W}$  be as in  $(**_A)$ , we rewrite  $\mathcal{F}$  in the following form

$$\mathcal{F}(x, m, \bar{m}, W, y) = D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0)y + \mathcal{R}(x, m, \bar{m}, y). \quad (8.25)$$

For fixed  $x, m, \bar{m}$  and  $W$ , the purpose of this section is to find  $y \in \mathcal{Y}$  such that

$$\mathcal{F}(x, m, \bar{m}, W, y) = 0.$$

Let  $\eta(m, \bar{m}, W)$  be the right inverse of  $D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0)$  whose existence is guaranteed by Lemma 8.4. Therefore, we are looking for  $y \in \mathcal{Y}$  in such a way that

$$y = y - \eta(m, \bar{m}, W)\mathcal{F}(x, m, \bar{m}, W, y).$$

To this end, we define the following functional

$$\mathcal{L}(x, m, \bar{m}, W, \cdot) : \mathcal{Y} \longrightarrow \mathcal{Y}$$

where

$$\mathcal{L}(x, m, \bar{m}, W, y) = y - \eta(m, \bar{m}, W)\mathcal{F}(x, m, \bar{m}, W, y). \quad (\mathcal{L})$$

As one can expect, it is well defined, and by the regularity of  $\mathcal{F}$ , we deduce that  $\mathcal{L}$  is continuous and differentiable with respect to  $y = (u, v)$  with differential  $D_y\mathcal{L}$  continuous. The proof is reduced to find a fixed point of the latter.

In what follows, we will widely use Proposition A.2 (see Appendix A). Then, we recall it. Let  $D$  be equal to  $\mathbb{T}^n$  or  $\mathbb{T}^n \times B$ .

**Proposition** (Proposition A.2). *We consider  $f, g \in C^\sigma(D)$  and  $\sigma \geq 0$ .*

1. *For all  $\beta \in \mathbb{N}^n$ , if  $|\beta| + s = \sigma$  then  $\left| \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} f \right|_{C^s} \leq |f|_{C^\sigma}$ .*

2.  $|fg|_{C^\sigma} \leq C(\sigma) (|f|_{C^0}|g|_{C^\sigma} + |f|_{C^\sigma}|g|_{C^0})$ .

Now we consider composite functions. Let  $z$  be defined on  $D_1 \subset \mathbb{R}^n$  and takes its values on  $D_2 \subset \mathbb{R}^n$  where  $f$  is defined.

*If  $\sigma < 1$ ,  $f \in C^1(D_2)$ ,  $z \in C^\sigma(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

3.  $|f \circ z|_{C^\sigma} \leq C(|f|_{C^1}|z|_{C^\sigma} + |f|_{C^0})$ .

*If  $\sigma < 1$ ,  $f \in C^\sigma(D_2)$ ,  $z \in C^1(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

4.  $|f \circ z|_{C^\sigma} \leq C(|f|_{C^\sigma}|\nabla z|_{C^0}^\sigma + |f|_{C^0})$ .

*If  $\sigma \geq 1$  and  $f \in C^\sigma(D_2)$ ,  $z \in C^\sigma(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

5.  $|f \circ z|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma}|\nabla z|_{C^0}^\sigma + |f|_{C^1}|\nabla z|_{C^{\sigma-1}} + |f|_{C^0})$ .

The following lemma is the main tool to conclude the proof of Theorem G.

**Lemma 8.5.** *There exists  $v'$  large enough with respect to  $n, \sigma, \lambda, |\partial_q W|_{C^{\sigma+1}}$  and  $\Upsilon$ , such that, for all  $y_*, y \in \mathcal{Y}$  with  $|y_*| \leq 1$ ,*

$$|D_y\mathcal{L}(x, m, \bar{m}, W, y_*)y| \leq \frac{1}{2}|y|. \quad (8.26)$$

*Proof.* The proof of this lemma is similar to that of Lemma 3.5 with some modifications due to the different homological equations. As one can expect, the proof rests on Lemma 8.4. By  $(\mathcal{L})$ , for all  $y, y^* \in \mathcal{Y}$

$$D_y\mathcal{L}(x, m, \bar{m}, W, y_*)y = (\text{Id} - \eta(m, \bar{m}, W)D_{(u,v)}\mathcal{F}(x, m, \bar{m}, W, y_*))y.$$

We can reformulate the problem in terms of estimating the solution  $\hat{y} = (\hat{u}, \hat{v}) \in \mathcal{Y}$  of the following system

$$\begin{aligned} & D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0)\hat{y} \\ & = \left( D_{(u,v)}\mathcal{F}(0, 0, m, \bar{m}, W, 0, 0) - D_{(u,v)}\mathcal{F}(x, m, \bar{m}, W, y_*) \right) y. \end{aligned} \quad (8.27)$$

In what follows, we estimate the right-hand side of the latter. Then, by Lemma 8.4, we conclude the proof. We point out that  $y_* = (u_*, v_*) \in \mathcal{Y}$  and for all  $(q, t) \in \mathbb{T}^m \times J_{v'}$ , we let

$$\tilde{u}_*(q, t) = (q + u_*(q, t), t), \quad \tilde{\varphi}_*(q, t) = (q + u_*(q, t), v_*(q, t), t).$$

Thanks to (8.22), we can rewrite the right-hand side of (8.27) in the following form

$$\begin{pmatrix} \partial_q W u - (\nabla u) \bar{W} + \bar{m}_0 v - D_{(u,v)} F_1(b, \bar{m}, W, y_*) y \\ \partial_q W v + (\nabla v) \bar{W} - D_{(u,v)} F_2(x, m, W, y_*) y \end{pmatrix}$$

(see Section 8.3.4), moreover

$$\begin{aligned} \partial_q W u - (\nabla u) \bar{W} + \bar{m}_0 v - D_{(u,v)} F_1(b, \bar{m}, W, y_*) y &= (\bar{m}_0 - \bar{m} \circ \tilde{\varphi}_*) v - (\partial_q b \circ \tilde{u}_*) u \\ &+ (\partial_q W - \partial_q W \circ (\text{id} + u_*)) u \\ &- v_*^T (\partial_q \bar{m} \circ \tilde{\varphi}_*) u - v_*^T (\partial_p \bar{m} \circ \tilde{\varphi}_*) v \\ \partial_q W v + (\nabla v) \bar{W} - D_{(u,v)} F_2(x, m, W, y_*) y &= (\partial_q W - \partial_q W \circ (\text{id} + u_*)) v \\ &- (\partial_q^2 a \circ \tilde{u}_*) u \\ &- v_*^T (\partial_q^2 W \circ (\text{id} + u_*) + \partial_q^2 b \circ \tilde{u}_*) u \\ &- (v_*^T)^2 (\partial_q^2 m \circ \tilde{\varphi}_*) u - (\partial_q b \circ \tilde{u}_*) v \\ &- (v_*^T)^2 (\partial_{pq}^2 m \circ \tilde{\varphi}_*) v \\ &- 2v_*^T (\partial_q m \circ \tilde{\varphi}_*) v. \end{aligned}$$

Now, thanks to property 2. of Proposition A.2,

$$\begin{aligned} \left| (\partial_q W u - (\nabla u) \bar{W} + \bar{m}_0 v - D_{(u,v)} F_1(b, \bar{m}, W, y_*) y)^t \right|_{C^\sigma} &\leq C(\sigma) \left( \left| (\bar{m}_0^t - \bar{m} \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} \right. \\ &+ \left| (\partial_q b \circ \tilde{u}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} \\ &+ \left| (\partial_q W - \partial_q W \circ (\text{id} + u_*))^t \right|_{C^\sigma} |u^t|_{C^\sigma} \\ &+ |v_*^t|_{C^\sigma} \left| (\partial_q \bar{m} \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |u^t|_{C^\sigma} \\ &\left. + |v_*^t|_{C^\sigma} \left| (\partial_p \bar{m} \circ \tilde{\varphi}_*)^t \right|_{C^\sigma} |v^t|_{C^\sigma} \right) \end{aligned}$$

for all  $t \in J_{v'}$ . We have to estimate each term on the right-hand side of the latter. We begin with the third because it is the only term that does not appear in the proof of Lemma 3.5.

$$\begin{aligned} \left| (\partial_q W - \partial_q W \circ (\text{id} + u_*))^t \right|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |\partial_q^2 W \circ (\text{id} + \tau u_*)^t u_*^t|_{C^\sigma} |u^t|_{C^\sigma} \\ &\leq C(\sigma) |\partial_q W|_{C^{\sigma+1}} |y_*| e^{-\lambda t} \left( 1 + (1 + |\partial_q u_*^t|_{C^0})^\sigma + |\partial_q u_*^t|_{C^{\sigma-1}} \right) |y| e^{-\lambda t} \\ &\leq C(\sigma) |\partial_q W|_{C^{\sigma+1}} e^{-\lambda v'} |y| e^{-\lambda t} \end{aligned}$$

for all  $t \in J_{v'}$ . The first line of the latter is due to the mean value theorem for a suitable  $\tau \in [0, 1]$ . In the second line, we use properties 2. and 5. of Proposition A.2. The last line is due to  $|y_*| \leq 1$ . Similarly to the previous case and to the estimates in Lemma 3.5, thanks to the mean value theorem, properties 2. and 5.

of Proposition A.2 and  $|y_*| \leq 1$ , we obtain

$$\begin{aligned}
 |(\bar{m}_0 - \bar{m} \circ \tilde{\varphi}_*)^t|_{C^\sigma} |v^t|_{C^\sigma} &\leq C(\sigma) \left( |\partial_q \bar{m}^t(\text{id} + \tau u_*, \tau v_*) u_*^t|_{C^\sigma} \right. \\
 &\quad \left. + |\partial_p \bar{m}^t(\text{id} + \tau u_*, \tau v_*) v_*^t|_{C^\sigma} \right) |v^t|_{C^\sigma} \\
 &\leq C(\sigma) \Upsilon (|u_*^t|_{C^\sigma} + |v_*^t|_{C^\sigma}) |v^t|_{C^\sigma} \\
 &\leq C(\sigma) \Upsilon e^{-\lambda v'} |y| e^{-\lambda t} \\
 |(\partial_q b \circ \tilde{u}_*)^t|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |b|_{\sigma+2, \lambda}^{v_*} e^{-\lambda v'} |y| e^{-\lambda t} \\
 &\leq C(\sigma) \Upsilon e^{-\lambda t} |y| e^{-\lambda t} \\
 |v_*^t|_{C^\sigma} |(\partial_q \bar{m} \circ \tilde{\varphi}_*)^t|_{C^\sigma} |u^t|_{C^\sigma} &\leq C(\sigma) |y_*| e^{-\lambda v'} \Upsilon |y| e^{-\lambda t} \\
 |v_*^t|_{C^\sigma} |(\partial_p \bar{m} \circ \tilde{\varphi}_*)^t|_{C^\sigma} |v^t|_{C^\sigma} &\leq C(\sigma) |y_*| e^{-\lambda v'} \Upsilon |y| e^{-\lambda t}
 \end{aligned}$$

for all  $t \in J_{v'}$ . Therefore, for  $v'$  large enough, the above estimates imply

$$\left| (\partial_q W u - (\nabla u) \bar{W} + \bar{m}_0 v - D_{(u,v)} F_1(b, \bar{m}, W, y_*) y)^t \right|_{C^\sigma} \leq \frac{1}{4\bar{C}} |y| e^{-\lambda t}$$

for all  $t \in J_{v'}$ . We point out that  $\bar{C}$  is the constant introduced in Lemma 8.4. Multiplying both sides of the latter by  $e^{\lambda t}$  and taking the sup for all  $t \in J_{v'}$ , we obtain

$$|\partial_q W u - (\nabla u) \bar{W} + \bar{m}_0 v - D_{(u,v)} F_1(b, \bar{m}, W, y_*) y|_{\sigma, \lambda}^{v'} \leq \frac{1}{4\bar{C}} |y|.$$

Similarly to the previous case, for  $v'$  large enough, we have

$$|\partial_q W v + (\nabla v) \bar{W} - D_{(u,v)} F_2(x, m, W, y_*) y|_{\sigma, \lambda}^{v'} \leq \frac{1}{4\bar{C}} |y|.$$

This concludes the proof of this lemma. Thanks to Lemma 8.4, a solution  $\hat{y} \in \mathcal{Y}$  of (8.27) exists satisfying

$$\begin{aligned}
 |\hat{u}| &\leq \bar{C} \left( |\partial_q W u - (\nabla u) \bar{W} + \bar{m}_0 v - D_{(u,v)} F_1(b, \bar{m}, W, y_*) y|_{\sigma, \lambda}^{v'} \right. \\
 &\quad \left. + |\partial_q W u - (\nabla u) \bar{W} + \bar{m}_0 v - D_{(u,v)} F_1(b, \bar{m}, W, y_*) y|_{\sigma, \lambda}^{v'} \right) \leq \frac{1}{2} |y| \\
 |\hat{v}| &\leq \bar{C} |\partial_q W v + (\nabla v) \bar{W} - D_{(u,v)} F_2(b, \bar{m}, W, y_*) y|_{\sigma, \lambda}^{v'} \leq \frac{1}{4} |y|
 \end{aligned}$$

and hence

$$|D_y \mathcal{L}(x, m, \bar{m}, W, y_*) y| \leq \frac{1}{2} |y|.$$

□

We proved that  $\mathcal{L}(x, m, \bar{m}, \cdot)$  is a contraction of a compact subset of  $\mathcal{Y}$ . Then, there exists a unique fixed point  $y \in \mathcal{Y}$  with  $|y| \leq 1$ .

More specifically, there exists  $(u, v) \in \mathcal{U} \times \mathcal{V}$  such that, for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$

$$\varphi^t(q) = (q + u(q, t), v(q, t))$$

is a  $C^\sigma$ -asymptotic torus associated to  $(X_H, X_{\tilde{h}}, \varphi_0, W)$ . We conclude the proof by verifying that  $\varphi^t$  is a Lagrangian  $C^\sigma$ -asymptotic torus.

**Lemma 8.6.**  $\varphi^{t_0}$  is Lagrangian for all  $t_0 \in J_{v'}$ .

*Proof.* Let  $\alpha = dp \wedge dq$  be the standard symplectic form associated to  $(q, p) \in \mathbb{T}^n \times B$ . Similarly to Proposition 3.2, by (8.5)

$$(\varphi^{t_0})^* \alpha = (\psi_{t_0, W}^{t_0+t})^* (\varphi^{t_0+t})^* \alpha$$

for all fixed  $t_0 \in J_{v'}$  and  $t \geq 0$ . We want to prove that, for all  $q \in \mathbb{T}^n \times \mathbb{R}^m$ ,  $((\varphi^{t_0})^* \alpha)_q = 0$ . The proof rests on the same ideas of Proposition 3.2 and Lemma 6.7.

We observe that, for all  $q \in \mathbb{T}^n$ , we can rewrite the right-hand side of the latter as follows

$$((\psi_{t_0, W}^{t_0+t})^* (\varphi^{t_0+t})^* \alpha)_q = \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < d \leq n} \alpha_{i,j,k,d}^t(q) dq_k \wedge dq_d$$

where

$$\begin{aligned} \alpha_{i,j,k,d}^t(q) &= (\partial_{q_i} v^{t_0+t} \cdot \partial_{q_j} (\text{id} + u^t) - \partial_{q_j} v^{t_0+t} \cdot \partial_{q_i} (\text{id} + u^t)) \circ \psi_{t_0, \bar{\omega} + \Gamma}^{t_0+t}(q) \\ &\times (\partial_{q_k} \psi_{t_0, W, i}^{t_0+t}(q) \partial_{q_d} \psi_{t_0, W, j}^{t_0+t}(q) - \partial_{q_d} \psi_{t_0, W, i}^{t_0+t}(q) \partial_{q_k} \psi_{t_0, W, j}^{t_0+t}(q)). \end{aligned}$$

and  $\times$  stands for the usual multiplication in  $\mathbb{R}$ . Then, for fixed  $1 \leq i < j \leq n$ ,  $1 \leq k < d \leq n$ , by Lemma 8.1

$$\begin{aligned} |\alpha_{i,j,k,d}^t|_{C^0} &\leq |(\partial_{q_i} v^{t_0+t} \cdot \partial_{q_j} (\text{Id} + u^{t_0+t}) - \partial_{q_j} v^{t_0+t} \cdot \partial_{q_i} (\text{Id} + u^{t_0+t})) \circ \psi_{t_0, W}^{t_0+t}(q)|_{C^0} \\ &\times |\partial_{q_k} \psi_{t_0, W, i}^{t_0+t} \partial_{q_d} \psi_{t_0, W, j}^{t_0+t} - \partial_{q_d} \psi_{t_0, W, i}^{t_0+t} \partial_{q_k} \psi_{t_0, W, j}^{t_0+t}|_{C^0} \\ &\leq |\partial_{q_i} v^{t_0+t} \cdot \partial_{q_j} (\text{Id} + u^{t_0+t}) - \partial_{q_j} v^{t_0+t} \cdot \partial_{q_i} (\text{Id} + u^{t_0+t})|_{C^0} \\ &\times \left( |\partial_{q_k} \psi_{t_0, W, i}^{t_0+t}|_{C^0} |\partial_{q_d} \psi_{t_0, W, j}^{t_0+t}|_{C^0} + |\partial_{q_d} \psi_{t_0, W, i}^{t_0+t}|_{C^0} |\partial_{q_k} \psi_{t_0, W, j}^{t_0+t}|_{C^0} \right) \\ &= |\partial_{q_i} v_j^{t_0+t} + \partial_{q_i} v^{t_0+t} \cdot \partial_{q_j} u^{t_0+t} - \partial_{q_j} v_i^{t_0+t} - \partial_{q_j} v^{t_0+t} \cdot \partial_{q_i} u^{t_0+t}|_{C^0} \\ &\times \left( |\partial_{q_k} \psi_{t_0, W, i}^{t_0+t}|_{C^0} |\partial_{q_d} \psi_{t_0, W, j}^{t_0+t}|_{C^0} + |\partial_{q_d} \psi_{t_0, W, i}^{t_0+t}|_{C^0} |\partial_{q_k} \psi_{t_0, W, j}^{t_0+t}|_{C^0} \right) \\ &\leq C |\partial_q v^{t_0+t}|_{C^0} (1 + |\partial_q u^{t_0+t}|_{C^0}) |\partial_q \psi_{t_0, W}^{t_0+t}|_{C^0}^2 \\ &\leq C |v^{t_0+t}|_{C^1} (1 + |u^{t_0+t}|_{C^1}) |\partial_q \psi_{t_0, W}^{t_0+t}|_{C^0}^2 \leq C e^{-\lambda(t_0+t)} e^{2\bar{c}_1 |\partial_q W|_{C^0} t} \end{aligned}$$

for a suitable constant  $C \geq 1$ . Thanks to  $(\#_G)$ , taking the limit for  $t \rightarrow +\infty$  on both sides of the latter, the term in the last line converges to zero. This concludes the proof of this lemma.  $\square$

## 8.4 Proof of Corollary C

The proof is quite similar to that of Theorem G and Corollary A. We are looking for a  $C^\sigma$ -asymptotic torus  $\psi^t$  associated to  $(Z, W, \text{Id}, W)$ . More specifically, for given  $Z$ , we are searching for  $v' \geq 0$  sufficiently large and a suitable function  $u : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n$  such that

$$\psi(q, t) = q + u(q, t)$$



and in addition  $\psi$  and  $u$  satisfy

$$Z(\psi(q, t), t) - \partial_q \psi(q, t)W(q) - \partial_t \psi(q, t) = 0, \quad (8.28)$$

$$\lim_{t \rightarrow +\infty} |u^t|_{C^\sigma} = 0 \quad (8.29)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . We introduce a suitable functional  $\mathcal{F}$  given by (8.28). To this end, we define

$$\tilde{\psi}(q, t) = (q + u(q, t), t),$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . The composition of  $Z$  with  $\tilde{\psi}$  is equal to

$$Z \circ \tilde{\psi}(q, t) = W \circ \psi(q, t) + P \circ \tilde{\psi}(q, t)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ , moreover

$$\partial_q \psi(q, t)W(q) + \partial_t \psi(q, t) = W(q) + \partial_q u(q, t)W(q) + \partial_t u(q, t)$$

for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$ . Then, we can rewrite (8.28) as follows

$$P \circ \tilde{\psi} + (W \circ \psi - W) - (\nabla u) \bar{W} = 0. \quad (8.30)$$

This is the sum of functions defined for all  $(q, t) \in \mathbb{T}^n \times J_{v'}$  or  $q \in \mathbb{T}^n$ . As usual, we have omitted the arguments  $(q, t)$  and  $q$  in order to achieve a more elegant form. For the sake of clarity, we recall that, during the proof of Theorem G, we have introduced the following notation

$$(\nabla u) \bar{W} = (\partial_q u) W + \partial_t u.$$

Before the introduction of the functional  $\mathcal{F}$ , letting  $\sigma \geq 1$  be the positive parameter defined in Corollary C. For a suitable positive parameter  $v' \geq 0$  that we will choose large enough in Lemma 8.7, we introduce the following Banach spaces  $(\mathcal{P}, |\cdot|)$ ,  $(\mathcal{U}, |\cdot|)$ ,  $(\mathcal{Z}, |\cdot|)$  and  $(\mathcal{W}, |\cdot|)$

$$\begin{aligned} \mathcal{P} &= \left\{ P : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid P \in \bar{\mathcal{S}}_{\sigma,1}^{v'}, \text{ and } |P| = |P|_{\sigma+1,\lambda}^{v'} < \infty \right\} \\ \mathcal{U} &= \left\{ u : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid u, (\nabla u) \Omega \in \mathcal{S}_\sigma^{v'} \right. \\ &\quad \left. \text{and } |u| = \max\{|u|_{\sigma,\lambda}^{v'}, |(\nabla u) \Omega|_{\sigma,\lambda}^{v'}\} < \infty \right\} \\ \mathcal{Z} &= \left\{ z : \mathbb{T}^n \times J_{v'} \rightarrow \mathbb{R}^n \mid z \in \mathcal{S}_\sigma^{v'}, \text{ and } |z| = |z|_{\sigma,\lambda}^{v'} < \infty \right\} \\ \mathcal{W} &= \left\{ W : \mathbb{T}^n \rightarrow \mathbb{R}^n \mid W \in C^{\sigma+1}(\mathbb{T}^n) \text{ and } |W| = |W|_{C^{\sigma+1}} < \infty \right\}. \end{aligned}$$

Now, thanks to (8.30) and the previous Banach spaces, we have everything we need to introduce the functional  $\mathcal{F}$ . Let  $\mathcal{F}$  be the following functional

$$\mathcal{F} : \mathcal{P} \times \mathcal{W} \times \mathcal{U} \longrightarrow \mathcal{Z}$$

$$\mathcal{F}(P, W, u) = P \circ \tilde{\psi} + (W \circ \psi - W) - (\nabla u) \bar{W}.$$

We observe that for all  $W \in \mathcal{W}$

$$\mathcal{F}(0, W, 0) = 0.$$

Therefore, we can reformulate our problem in the following form. We fix  $W \in \mathcal{W}$  and for  $P \in \mathcal{P}$  sufficiently close to 0, we are looking for  $u \in \mathcal{U}$  in such a way that  $\mathcal{F}(P, W, u) = 0$ .

Concerning the associated linearized problem, the differential of  $\mathcal{F}$  with respect to the variable  $u$  calculated in  $(0, W, 0)$  is equal to

$$D_u \mathcal{F}(0, W, 0) \hat{u} = \partial_q W \hat{u} - (\nabla \hat{u}) \bar{W}.$$

The functional  $\mathcal{F}$  is well defined, continuous, differentiable with respect to the coordinate  $u$  with  $D_u \mathcal{F}(P, W, u)$  continuous. Moreover, by Lemma 8.3, for all fixed  $W \in \mathcal{W}$ ,  $D_u \mathcal{F}(0, W, 0)$  admits a right inverse  $\eta(W)$ . Then,  $\mathcal{F}$  satisfies the hypotheses of the implicit function theorem.

We fix  $P \in \mathcal{P}$  and  $W \in \mathcal{W}$  as in Corollary C and we define the following functional

$$\mathcal{L}(P, W, \cdot) : \mathcal{U} \longrightarrow \mathcal{U}$$

in such a way that

$$\mathcal{L}(P, W, u) = u - \eta(W) \mathcal{F}(P, W, u).$$

The proof of Corollary C is reduced to find a fixed point of the latter. Similarly to the proof of Lemma 8.5, we have the following lemma

**Lemma 8.7.** *There exists  $\nu'$  large enough with respect to  $n, \sigma, \lambda$  and  $|\partial_q W|_{C^\sigma}$ , such that, for all  $u_*, u \in \mathcal{U}$  with  $|u_*| \leq 1$ ,*

$$|D_u \mathcal{L}(P, W, u_*) u| \leq \frac{1}{2} |u|.$$

This concludes the proof of Corollary C.

## A Hölder classes of functions

This part is dedicated to a very brief introduction to Hölder classes of functions  $C^\sigma$ . Let  $D$  be equal to  $\mathbb{T}^n \times B$ ,  $\mathbb{T}^n$  or an open subset of  $\mathbb{R}^n$ . For integers  $k \geq 0$ , we denote by  $C^k(D)$  the spaces of functions  $f : D \rightarrow \mathbb{R}$  with continuous partial derivatives  $\partial^\alpha f \in C^0(D)$  for all  $\alpha \in \mathbb{N}^n$  with  $|\alpha| = \alpha_1 + \dots + \alpha_n \leq k$ . We define the norm

$$|f|_{C^k} = \sup_{|\alpha| \leq k} |\partial^\alpha f|_{C^0},$$

where  $|\partial^\alpha f|_{C^0} = \sup_{x \in D} |\partial^\alpha f(x)|$  denotes the sup norm. For  $\sigma = k + \mu$ , with  $k \in \mathbb{Z}$ ,  $k \geq 0$  and  $0 < \mu < 1$ , the Hölder spaces  $C^\sigma(D)$  are the spaces of functions  $f \in C^k(D)$  such that  $|f|_{C^\sigma} < \infty$ , where

$$|f|_{C^\sigma} = \sup_{|\alpha| \leq k} |\partial^\alpha f|_{C^0} + \sup_{|\alpha|=k} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|}{|x - y|^\mu}. \quad (\text{A.1})$$

In the case of functions  $f = (f_1, \dots, f_n)$  with values in  $\mathbb{R}^n$ , we set  $|f|_{C^\sigma} = \max_{1 \leq i \leq n} |f_i|_{C^\sigma}$ . Moreover, in agreement with the convention made above, if  $M = \{m_{ij}\}_{1 \leq i, j \leq n}$  is a  $n \times n$  matrix, we set  $|M|_{C^\sigma} = \max_{1 \leq i, j \leq n} |m_{ij}|_{C^\sigma}$ .

In what follows, we have some properties of these norms that we widely use in this thesis. First, we recall that  $C(\cdot)$  stands for constants depending on  $n$  and other parameters into brackets.

**Proposition A.1.** *For all  $f \in C^{\sigma_1}(\mathbb{R}^n)$ , then*

$$|f|_{C^\sigma}^{\sigma_1 - \sigma_0} \leq C(\sigma_1) |f|_{C^{\sigma_0}}^{\sigma_1 - \sigma} |f|_{C^{\sigma_1}}^{\sigma - \sigma_0} \quad \text{for all } 0 \leq \sigma_0 \leq \sigma \leq \sigma_1.$$

*Proof.* We refer to [Hö76] for the proof. □

This is a fundamental property that plays a substantial role in this thesis. Furthermore, we have the following Proposition.

**Proposition A.2.** *We consider  $f, g \in C^\sigma(D)$  and  $\sigma \geq 0$ .*

1. *For all  $\beta \in \mathbb{N}^n$ , if  $|\beta| + s = \sigma$  then  $\left| \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} f \right|_{C^s} \leq |f|_{C^\sigma}$ .*
2.  $|fg|_{C^\sigma} \leq C(\sigma) (|f|_{C^0} |g|_{C^\sigma} + |f|_{C^\sigma} |g|_{C^0})$ .

*Now we consider composite functions. Let  $z$  be defined on  $D_1 \subset \mathbb{R}^n$  and takes its values on  $D_2 \subset \mathbb{R}^n$  where  $f$  is defined.*

*If  $\sigma < 1$ ,  $f \in C^1(D_2)$ ,  $z \in C^\sigma(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

3.  $|f \circ z|_{C^\sigma} \leq C(|f|_{C^1} |z|_{C^\sigma} + |f|_{C^0})$ .

*If  $\sigma < 1$ ,  $f \in C^\sigma(D_2)$ ,  $z \in C^1(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

4.  $|f \circ z|_{C^\sigma} \leq C(|f|_{C^\sigma} |\nabla z|_{C^0}^\sigma + |f|_{C^0})$ .

*If  $\sigma \geq 1$  and  $f \in C^\sigma(D_2)$ ,  $z \in C^\sigma(D_1)$  then  $f \circ z \in C^\sigma(D_1)$*

$$5. |f \circ z|_{C^\sigma} \leq C(\sigma) (|f|_{C^\sigma} |\nabla z|_{C^0}^\sigma + |f|_{C^1} |\nabla z|_{C^{\sigma-1}} + |f|_{C^0}).$$

*Proof.* The proofs of the properties contained in this proposition are similar to those in [Hö76]. The first is obvious. For the second, we refer to [Hö76]. Properties 3. and 4. are quite trivial. We prove the last property. By (A.1),

$$|f \circ z|_{C^\sigma} \leq |f|_{C^0} + |(\nabla f \circ z)^T \nabla z|_{C^{\sigma-1}}, \quad (\text{A.2})$$

where  $T$  stands for the transpose and  $(\nabla f \circ z)^T \nabla z$  is the vector having  $i$  component equal to  $((\nabla f \circ z)^T \nabla z)_i = \nabla f \circ z \cdot \partial_{x_i} z$ . Thanks to the property 2.

$$\begin{aligned} |f \circ z|_{C^\sigma} &\leq |f|_{C^0} + |(\nabla f \circ z)^T \nabla z|_{C^{\sigma-1}} \\ &\leq |f|_{C^0} + C(\sigma) |\nabla f \circ z|_{C^{\sigma-1}} |\nabla z|_{C^0} + C(\sigma) |\nabla f \circ z|_{C^0} |\nabla z|_{C^{\sigma-1}}. \end{aligned}$$

The last term of the latter is bounded by  $|f|_{C^1} |\nabla z|_{C^{\sigma-1}}$ , it remains to estimate  $|\nabla f \circ z|_{C^{\sigma-1}} |\nabla z|_{C^0}$ . If  $\sigma \leq 2$ ,  $|\nabla f \circ z|_{C^{\sigma-1}} \leq |f|_{C^\sigma} |\nabla z|_{C^0}^{\sigma-1} + |f|_{C^1}$  thanks to 4.. Then

$$\begin{aligned} |\nabla f \circ z|_{C^{\sigma-1}} |\nabla z|_{C^0} &\leq C(\sigma) (|f|_{C^\sigma} |\nabla z|_{C^0}^\sigma + |f|_{C^1} |\nabla z|_{C^0}) \\ &\leq C(\sigma) (|f|_{C^\sigma} |\nabla z|_{C^0}^\sigma + |f|_{C^1} |\nabla z|_{C^{\sigma-1}}), \end{aligned}$$

whence the property holds in this case. If  $\sigma > 2$ , assuming that 5. is already proven for  $\sigma - 1$ , we find

$$|\nabla f \circ z|_{C^{\sigma-1}} |\nabla z|_{C^0} \leq C(\sigma) (|\nabla f|_{C^{\sigma-1}} |\nabla z|_{C^0}^\sigma + |f|_{C^2} |\nabla z|_{C^{\sigma-2}} |\nabla z|_{C^0} + |f|_{C^1} |\nabla z|_{C^0}).$$

It remains to find a good estimate for the central term. By Proposition A.1

$$\begin{aligned} |f|_{C^2} |\nabla z|_{C^{\sigma-2}} |\nabla z|_{C^0} &\leq C(\sigma) \left( |f|_{C^1}^{\frac{\sigma-2}{\sigma-1}} |f|_{C^\sigma}^{\frac{1}{\sigma-1}} \right) \left( |\nabla z|_{C^0}^{\frac{1}{\sigma-1}} |\nabla z|_{C^{\sigma-1}}^{\frac{\sigma-2}{\sigma-1}} \right) |\nabla z|_{C^0} \\ &\leq C(\sigma) (|f|_{C^1} |\nabla z|_{C^{\sigma-1}})^{\frac{\sigma-2}{\sigma-1}} (|f|_{C^\sigma} |\nabla z|_{C^0}^\sigma)^{\frac{1}{\sigma-1}}, \end{aligned}$$

since  $a^\lambda b^{1-\lambda} \leq C(a+b)$  for  $0 < \lambda < 1$ , we have the claim.  $\square$

## B Real analytic classes of functions

This section will collect some well-known facts about real analytic functions. For some  $s > 0$ , we begin with the introduction of complex domains

$$\begin{aligned} \mathbb{T}_s^n &:= \{q \in \mathbb{C}^n / \mathbb{Z}^n : |\operatorname{Im}(q)| \leq s\} \\ B_s &:= \{p \in \mathbb{C}^n : |p| \leq s\}, \end{aligned}$$

with  $\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n$  and  $B \subset \mathbb{R}^n$  a sufficiently large neighborhood of the origin. Let  $D$  be equal to  $\mathbb{T}^n \times B$  or  $\mathbb{T}^n$  and we consider a real analytic function in a neighborhood of  $D$

$$f : D \rightarrow \mathbb{R}.$$

Let  $D_s$  be equal to  $\mathbb{T}_s^n \times B_s$  or  $\mathbb{T}_s^n$ , for a suitable small  $s$ . It is known that  $f$  extends to a function

$$f : D_s \rightarrow \mathbb{C}$$

that is real, holomorphic and bounded. We define the following norm

$$|f|_s = \sup_{z \in D_s} |f(z)|.$$

In the case of vector valued functions  $f = (f_1, \dots, f_n)$  with values in  $\mathbb{C}^n$ , we set  $|f|_s = \max_i |f_i|_s$ . Moreover, if  $C = \{C_{ij}\}_{1 \leq i, j \leq n}$  is a  $n \times n$  matrix, we let  $|C|_s = \max_{ij} |C_{ij}|_s$ . We define  $\mathcal{A}_s$  as the space of such functions. The rest of this section is devoted to a series of general well-known properties.

**Proposition B.1.** *Let  $f, g \in \mathcal{A}_s$ , then the product  $fg \in \mathcal{A}_s$  and*

$$|fg|_s \leq |f|_s |g|_s.$$

*Let  $f \in \mathcal{A}_s$  and  $0 \leq \sigma \leq s$ . Then  $\partial_x f \in \mathcal{A}_s$  and we have*

$$|\partial_x f|_{s-\sigma} \leq \frac{1}{\sigma} |f|_s.$$

*Let  $f \in \mathcal{A}_s$ ,  $0 \leq \sigma \leq s$  and  $\phi \in \mathcal{A}_{s-\sigma}$  such that  $\phi : D_{s-\sigma} \rightarrow D_s$ . Then  $f \circ \phi \in \mathcal{A}_{s-\sigma}$  and*

$$|f \circ \phi|_{s-\sigma} \leq |f|_s.$$

## C Banach spaces

Here, we prove that the normed spaces, introduced in the third chapter of this thesis, are Banach spaces. Similarly, we have the claim for those defined in the other chapters of this work. First, however, for clarity, let us remind some definitions.

For a positive parameter  $v \geq 0$ , we define a real interval  $J_v = [v, +\infty) \subset \mathbb{R}$ . For every function  $f$  defined on  $\mathbb{T}^n \times J_v$  and for fixed  $t \in J_v$ , we let  $f^t$  be the function defined on  $\mathbb{T}^n$  in such a way that

$$f^t(q) = f(q, t).$$

Given  $\sigma \geq 0$ , we have the following definition

**Definition.** *Let  $\mathcal{S}_\sigma^v$  be the space of functions  $f$  defined on  $\mathbb{T}^n \times J_v$  such that  $f \in C(\mathbb{T}^n \times J_v)$  and for all  $t \in J_v$ ,  $f^t \in C^\sigma(\mathbb{T}^n)$ .*

For all  $f \in \mathcal{S}_\sigma^v$  and a positive real function  $\mathbf{u}(t)$  defined on  $J_v$ , we recall the definition of the following norm

$$|f|_{\sigma, \mathbf{u}}^v = \sup_{t \in J_v} \frac{|f^t|_{C^\sigma}}{\mathbf{u}(t)}.$$

We conclude this first part with the following subset of  $\mathcal{S}_\sigma^v$ .

**Definition.** *Given  $\sigma, v \geq 0$  and an integer  $k \geq 0$ , we let  $\bar{\mathcal{S}}_{\sigma, k}^v$  be the space of functions  $f$  such that*

$$f \in \mathcal{S}_{\sigma+k}^v \quad \text{and} \quad \partial_{(q,p)}^i f \in \mathcal{S}_{\sigma+k-i}^v$$

*for all  $0 \leq i \leq k$ .*

Now, let  $\mathbf{b}$  be a positive, decreasing, integrable function on  $J_v$ . We define

$$\bar{\mathbf{b}}(t) = \int_t^{+\infty} \mathbf{b}(\tau) d\tau.$$

We may assume

$$\mathbf{b}(t) \leq 1, \quad \bar{\mathbf{b}}(t) \leq 1$$

for all  $t \in J_v$ . For fixed  $\sigma \geq 1$  and an integer  $k \geq 0$ , we consider the following spaces  $(\mathcal{G}, |\cdot|)$  and  $(\mathcal{W}, \|\cdot\|)$  such that

$$\begin{aligned} \mathcal{G} &= \{g : \mathbb{T}^n \times J_v \rightarrow \mathbb{R} : g \in \bar{\mathcal{S}}_{\sigma,k}^v \text{ and } |g| = |g|_{\sigma+k,\mathbf{b}}^v < \infty\} \\ \mathcal{W}^\sigma &= \left\{ w : \mathbb{T}^n \times J_v \rightarrow \mathbb{R}^n : w, (\nabla w) \Omega \in \mathcal{S}_\sigma^v \right. \\ &\quad \left. \text{and } \|w\|_\sigma = \max\{|w|_{\sigma,\bar{\mathbf{b}}}^v, |(\nabla w) \Omega|_{\sigma,\mathbf{b}}^v\} < \infty \right\} \end{aligned}$$

We recall that, for all  $(q, t) \in \mathbb{T}^n \times J_v$ ,  $\nabla w(q, t) \Omega = \partial_q w(q, t) \omega + \partial_t w(q, t)$  with  $\omega \in \mathbb{R}^n$ . We prove that these spaces are complete. We begin with the first. Let  $\{g_d\}_{d \geq 0} \subset \mathcal{G}$  be a Cauchy sequence. This means that, for all  $\varepsilon > 0$  there exists  $D \in \mathbb{N}$  such that for all  $d, m \geq D$ ,  $|g_d - g_m|_{\sigma+k,\mathbf{b}}^v < \varepsilon$ . For all fixed  $t \in J_v$ , we claim that the sequence  $\{g_d^t\}_{d \geq 0}$  contained in the Banach space  $(C^{\sigma+k}(\mathbb{T}^n), |\cdot|_{C^{\sigma+k}})$  is a Cauchy sequence. This is because for all fixed  $t \in J_v$

$$|g_d^t - g_m^t|_{C^{\sigma+k}} \leq \frac{|g_d^t - g_m^t|_{C^{\sigma+k}}}{\mathbf{b}(t)} \leq |g_d - g_m|_{\sigma+k,\mathbf{b}}^v.$$

Then, for all fixed  $t \in J_v$ , there exists  $g^t \in C^{\sigma+k}$  such that

$$\lim_{d \rightarrow +\infty} |g_d^t - g^t|_{C^{\sigma+k}} = 0.$$

We have to verify that  $g \in \mathcal{G}$  (that is  $g \in \bar{\mathcal{S}}_{\sigma,k}^v$  and  $|g|_{\sigma+k,\mathbf{b}}^v < \infty$ ) and  $\lim_{d \rightarrow +\infty} |g_d - g|_{\sigma+k,\mathbf{b}}^v = 0$ .

We prove that  $g \in \bar{\mathcal{S}}_{\sigma,k}^v$ . Obviously, for all fixed  $t \in J_v$ ,  $g^t \in C^{\sigma+k}(\mathbb{T}^n)$ . It remains to verify that  $\partial_q^i g \in C(\mathbb{T}^n \times J_v)$  for all  $0 \leq i \leq k$ . For all  $(q_1, t_1), (q_2, t_2) \in \mathbb{T}^n \times J_v$

$$\begin{aligned} |\partial_q^i g^{t_1}(q_1) - \partial_q^i g^{t_2}(q_2)| &\leq |\partial_q^i g_d^{t_1}(q_1) - \partial_q^i g^{t_1}(q_1)| + |\partial_q^i g_d^{t_1}(q_1) - \partial_q^i g_d^{t_2}(q_2)| \\ &\quad + |\partial_q^i g_d^{t_2}(q_2) - \partial_q^i g^{t_2}(q_2)| \end{aligned}$$

for all  $0 \leq i \leq k$ . Now, for all  $\varepsilon > 0$  there exists  $D \in \mathbb{N}$  such that, for all  $d \geq D$ , the first and the last term on the right-hand side of the latter are smaller than  $\frac{\varepsilon}{3}$ . This is because, for all fixed  $t \in J_v$ ,  $g_d^t$  converges to  $g^t$  in the norm  $C^{\sigma+k}$ . Concerning the second term, we know that  $g_d \in \bar{\mathcal{S}}_{\sigma,k}^v$ . Hence, by the definition of  $\bar{\mathcal{S}}_{\sigma,k}^v$ ,  $\partial_q^i g_d \in C(\mathbb{T}^n \times J_v)$  for all  $0 \leq i \leq k$ . Then, there exists  $\delta > 0$  such that if  $|(q_1, t_1) - (q_2, t_2)| < \delta$  also the second term on the right-hand side of the latter is smaller than  $\frac{\varepsilon}{3}$ . This proves the claim.

We prove that  $\lim_{d \rightarrow +\infty} |g_d - g|_{\sigma+k, \mathbf{b}}^v = 0$ . Let  $g_{k_d}$  be a subsequence of  $g_d$  such that

$$|g_{k_{d+1}} - g_{k_d}|_{\sigma+k, \mathbf{b}}^v < \left(\frac{1}{2}\right)^d$$

for all  $d \in \mathbb{N}$ . We claim that it suffices to prove the above property for  $g_{k_d}$ . Indeed, we assume that  $\lim_{d \rightarrow +\infty} |g_{k_d} - g|_{\sigma+k, \mathbf{b}}^v = 0$ . Then, for all  $d \in \mathbb{N}$

$$|g_d - g|_{\sigma+k, \mathbf{b}}^v \leq |g_d - g_{k_d}|_{\sigma+k, \mathbf{b}}^v + |g_{k_d} - g|_{\sigma+k, \mathbf{b}}^v.$$

Therefore, for all  $\varepsilon > 0$ , there exists  $D \in \mathbb{N}$  such that,  $|g_d - g_{k_d}|_{\sigma+k, \mathbf{b}}^v < \frac{\varepsilon}{2}$  and  $|g_{k_d} - g|_{\sigma+k, \mathbf{b}}^v < \frac{\varepsilon}{2}$  for all  $d \geq D$ . Because  $g_d$  is a Cauchy sequence, we have the first inequality. The second follows because we assumed that  $g_{k_d}$  converges to  $g$  in the norm  $|\cdot|_{\sigma+k, \mathbf{b}}^v$ . This implies  $\lim_{d \rightarrow +\infty} |g_d - g|_{\sigma+k, \mathbf{b}}^v = 0$  and hence the claim.

Now, for all fixed  $t \in J_v$

$$\frac{|g_{k_d}^t - g^t|_{C^{\sigma+k}}}{\mathbf{b}(t)} \leq \sum_{i=d}^{+\infty} \frac{|g_{k_i}^t - g_{k_{i+1}}^t|_{C^{\sigma+k}}}{\mathbf{b}(t)} \leq \sum_{i=d}^{+\infty} |g_{k_i} - g_{k_{i+1}}|_{\sigma+k, \mathbf{b}}^v \leq \sum_{i=d}^{+\infty} \left(\frac{1}{2}\right)^i = 2 \left(\frac{1}{2}\right)^d$$

and hence taking the sup for all  $t \in J_v$ , we obtain

$$|g_{k_d} - g|_{\sigma+k, \mathbf{b}}^v \leq 2 \left(\frac{1}{2}\right)^d.$$

Then, for every  $\varepsilon > 0$  there exists  $D \in \mathbb{N}$  such that  $|g_{k_d} - g|_{\sigma+k, \mathbf{b}}^v < \varepsilon$  for all  $d \geq D$ .

We prove that  $|g|_{\sigma+k, \mathbf{b}}^v < \infty$ . For all  $d \in \mathbb{N}$  we can estimate  $|g|_{\sigma+k, \mathbf{b}}^v$  as follows

$$|g|_{\sigma+k, \mathbf{b}}^v \leq |g_d - g|_{\sigma+k, \mathbf{b}}^v + |g_d|_{\sigma+k, \mathbf{b}}^v.$$

For  $d$  sufficiently large,  $|g_d - g|_{\sigma+k, \mathbf{b}}^v < \infty$  because  $\lim_{d \rightarrow +\infty} |g_d - g|_{\sigma+k, \mathbf{b}}^v = 0$ . Moreover,  $|g_d|_{\sigma+k, \mathbf{b}}^v < \infty$  because  $g_d \in \mathcal{G}$ . Then  $(\mathcal{G}, |\cdot|)$  is a Banach space.

In the second part of this section we prove that  $(\mathcal{W}, \|\cdot\|)$  is a Banach space. Let  $\{w_d\}_{d \geq 0} \subset \mathcal{W}$  be a Cauchy sequence. Similarly to the previous case, there exist  $w \in \mathcal{S}_\sigma^v$  and  $f \in \mathcal{S}_\sigma^v$  such that

$$\lim_{d \rightarrow +\infty} |w_d - w|_{\sigma, \bar{\mathbf{b}}}^v = 0, \quad \lim_{d \rightarrow +\infty} |(\nabla w_d) \Omega - f|_{\sigma, \mathbf{b}}^v = 0. \quad (\text{C.1})$$

We have to verify that  $\nabla w(q, t) \Omega = f(q, t)$  for all  $(q, t) \in \mathbb{T}^n \times J_v$ . Let us denote  $z = (q, t)$  and we remind that  $\Omega = (\omega, 1)$ . We will prove that for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that

$$\left| \frac{w(z + \tau \Omega) - w(z)}{\tau} - f(z) \right| < \varepsilon$$

for all  $|\tau| < \delta$ . Thanks to the triangle inequality

$$\begin{aligned} \left| \frac{w(z + \tau\Omega) - w(z)}{\tau} - f(z) \right| &< \left| \frac{w_d(z + \tau\Omega) - w_d(z)}{\tau} - \frac{w(z + \tau\Omega) - w(z)}{\tau} \right| \\ &+ \left| \frac{w_d(z + \tau\Omega) - w_d(z)}{\tau} - \nabla w_d(z)\Omega \right| \\ &+ |\nabla w_d(z)\Omega - f(z)|. \end{aligned}$$

By (C.1) there exists  $D > 0$ , depending on  $\varepsilon$  and  $\tau$ , such that the first and the third terms on the right-hand side of the latter are smaller than  $\frac{\varepsilon}{3}$  for all  $d \geq D$ . Now, thanks to Taylor's formula, we can rewrite the second term on the right-hand side of the latter as follows

$$\left| \frac{w_d(z + \tau\Omega) - w_d(z)}{\tau} - \nabla w_d(z)\Omega \right| = \left| \int_0^1 \nabla w_d(z + \tau s\Omega)\Omega - \nabla w_d(z)\Omega ds \right|$$

and using the triangle inequality

$$\begin{aligned} \left| \int_0^1 \nabla w_d(z + \tau s\Omega)\Omega - \nabla w_d(z)\Omega ds \right| &\leq \int_0^1 |\nabla w_d(z + \tau s\Omega)\Omega - f(z + \tau s\Omega)| ds \\ &+ \int_0^1 |f(z + \tau s\Omega) - f(z)| ds \\ &+ \int_0^1 |f(z) - \nabla w_d(z)\Omega| ds. \end{aligned}$$

We know that  $f$  is continuous, then there exists  $\delta$  such that for all  $|\tau| < \delta$  the second term on the right-hand side of the latter is smaller than  $\frac{\varepsilon}{9}$ . Since the uniform convergence of  $(\nabla w_d)\Omega$  there exists  $D > 0$ , depending on  $\varepsilon$  and  $\tau$ , such that the first and the third terms on the right-hand side of the latter are smaller than  $\frac{\varepsilon}{9}$ . This concludes the proof.

## D Planar two body problem plus comet

We consider the Hamiltonian of the planar two-body problem  $H_0$ . We verify that, in this case, we can explicitly construct the symplectic transformation  $\phi$  of Section 7.2.4. We consider two points of masses  $m_0$  and  $m_1$  undergoing gravitational attraction in the plane. The phase space is the space

$$\left\{ (x_j, y_j)_{0 \leq j \leq 1} \in (\mathbb{R}^2 \times \mathbb{R}^{2*})^2 \mid x_0 \neq y_1 \right\}$$

of linear momentum covectors  $(y_0, y_1)$  and position vectors  $(x_0, x_1)$  of each body. The Hamiltonian of the planar two-body problem is equal to

$$H_0(x, y) = \sum_{i=0}^1 \frac{|y_i|^2}{2m_i} - \frac{m_0 m_1}{|x_0 - x_1|}.$$



We introduce the following symplectic change of variable  $\phi_0$  defined by

$$\begin{cases} X_0 = \sigma_0 x_0 + \sigma_1 x_1 \\ X_1 = x_0 - x_1 \end{cases} \quad \begin{cases} Y_0 = y_0 + y_1 \\ Y_1 = \sigma_1 y_0 - \sigma_0 y_1 \end{cases}$$

where  $1/\sigma_0 = 1 + m_1/m_0$  and  $1/\sigma_1 = 1 + m_0/m_1$ . It is symplectic because

$$\begin{aligned} dY_1 \wedge dX_1 + dY_0 \wedge dX_0 &= d(\sigma_1 y_0 - \sigma_0 y_1) \wedge d(x_0 - x_1) \\ &+ d(y_0 + y_1) \wedge d(\sigma_0 x_0 + \sigma_1 x_1) \\ &= \sigma_1 dy_0 \wedge dx_0 - \sigma_1 dy_0 \wedge dx_1 - \sigma_0 dy_1 \wedge dx_0 + \sigma_0 dy_1 \wedge dx_1 \\ &+ \sigma_0 dy_0 \wedge dx_0 + \sigma_1 dy_0 \wedge dx_1 + \sigma_0 dy_1 \wedge dx_0 + \sigma_1 dy_1 \wedge dx_1 \\ &= dy_0 \wedge dx_0 + dy_1 \wedge dx_1. \end{aligned}$$

In the new coordinates, the Hamiltonian of the planar two-body problem is equal to

$$H_0 \circ \phi_0(X, Y) = \frac{|Y_0|^2}{2M} + \frac{|Y_1|^2}{2\mu} - \frac{\mu M}{|X_1|}$$

with  $M = m_0 + m_1$  and  $1/\mu = 1/m_0 + 1/m_1$ . Now, we introduce polar coordinates  $X_1 = x_0 - x_1 = r(\cos \varphi, \sin \varphi)$ . The variables  $(r, \varphi)$  do not depend on the center of mass  $X_0$ . This suggests the introduction of the following change of coordinates

$$\text{Pol} : \mathbb{R}^{+*} \times \mathbb{T} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^{2*} \times \mathbb{R}^2, \quad \text{Pol}(r, \varphi, X_0) = (r \cos \varphi, r \sin \varphi, X_0).$$

Letting  $(X_0, Y_0) = (\xi, \eta)$ , in order to get a symplectic transformation, one is led to the symplectic map

$$\phi_{pc} : \mathbb{R}^{+*} \times \mathbb{R} \times \mathbb{T} \times \mathbb{R} \times \mathbb{R}^4 \longrightarrow \mathbb{R}^{2*} \times \mathbb{R}^2 \times \mathbb{R}^4,$$

$$\phi_{pc}((r, R), (\varphi, \Phi), (\xi, \eta)) = \left( (r \cos \varphi, r \sin \varphi), \left( R \cos \varphi - \frac{\Phi}{r} \sin \varphi, R \sin \varphi + \frac{\Phi}{r} \cos \varphi \right), (\xi, \eta) \right).$$

There are many references concerning the two-body problem and these kinds of symplectic transformations. We propose the work of Féjóz [Fé13]. In the new variables

$$H_0 \circ \phi_0 \circ \phi_{pc}(r, R, \varphi, \Phi, \xi, \eta) = \frac{|\eta|^2}{2M} + \frac{R^2}{2\mu} + \underbrace{\frac{\Phi^2}{2\mu r^2}}_{V_{eff}(r; \Phi)} - \frac{\mu M}{r}. \quad (\text{D.1})$$

The variables  $\varphi$  and  $\xi$  are cyclic (this means that  $\partial_\varphi (H_0 \circ \phi_0 \circ \phi_{pc}) = 0$  and  $\partial_\xi (H_0 \circ \phi_0 \circ \phi_{pc}) = 0$ , so that  $\Phi = \text{cst}$ ,  $\eta = \text{cst}$ ). Then,  $H_0 \circ \phi_0 \circ \phi_{pc}$  is one degree-of-freedom and thus integrable.

Now, we let

$$\tilde{H}_0(r, R, \varphi, \Phi) = H_0 \circ \phi_0 \circ \phi_{pc}(r, R, \varphi, \Phi, \xi, \eta) - \frac{|\eta|^2}{2M}.$$

Drawing from the work of Chenciner [Che89] or that of Celletti-Chierchia [CC07] about the variables of Delaunay, there exists a symplectic change of coordinates  $\phi_D$  defined on

$$\left\{ (l, L, g, G) \in (\mathbb{T} \times \mathbb{R}^{+*})^2 \mid 0 < G < L \right\}$$

such that

$$\tilde{H}_0 \circ \phi_D(l, L, g, G) = -\frac{\mu^3 M^2}{2L^2} =: h(L).$$

The angles  $(l, g)$  are respectively the mean anomaly and the argument of the perihelion of the fictitious body  $x_0 - x_1$ . Let  $a$  be the semi-major axis and  $e$  the eccentricity, then

$$L = \mu\sqrt{Ma}, \quad G = \mu\sqrt{Ma(1 - e^2)}.$$

The symplectic transformation  $\phi_D$  is generated by

$$\begin{aligned} \mathcal{S}_0(L, G, r, \varphi) &= \sqrt{2\mu} \int_{r_-(h(L))}^r \sqrt{h(L) - V_{eff}(\rho; G)} d\rho + G\varphi \\ &= \int_{r_-(h(L))}^r \sqrt{-\frac{\mu^4 M^2}{L^2} + \frac{2\mu^2 M}{\rho} - \frac{G^2}{\rho^2}} d\rho + G\varphi \end{aligned}$$

where  $V_{eff}$  is the effective potential defined in (D.1) and  $r_-(h(L))$  is a root of the equation  $h(L) - V_{eff}(\rho; G) = 0$ . The previous generating function  $\mathcal{S}_0$  is independent of  $\eta$ . In particular, the symplectic change of variables  $\phi_D$  is defined by the following conditions

$$R = \frac{\partial \mathcal{S}_0}{\partial r}, \quad \Phi = \frac{\partial \mathcal{S}_0}{\partial \varphi}, \quad l = \frac{\partial \mathcal{S}_0}{\partial L}, \quad g = \frac{\partial \mathcal{S}_0}{\partial G}.$$

Going back to the Hamiltonian  $H_0 \circ \phi_0 \circ \phi_{pc}$ , we define

$$\mathcal{S}(L, G, \eta, r, \varphi, \xi) = \mathcal{S}_0(L, G, r, \varphi) + \xi \cdot \eta.$$

It is the generating function of a symplectic transformation  $\tilde{\phi}_D$  defined on

$$\left\{ (l, L, g, G, \xi, \eta) \in (\mathbb{T} \times \mathbb{R}^{+*})^2 \times \mathbb{R}^2 \times \mathbb{R}^{2*} \mid 0 < G < L \right\}$$

by

$$R = \frac{\partial \mathcal{S}}{\partial r}, \quad \Phi = \frac{\partial \mathcal{S}}{\partial \varphi}, \quad \eta = \frac{\partial \mathcal{S}}{\partial \xi}, \quad l = \frac{\partial \mathcal{S}}{\partial L}, \quad g = \frac{\partial \mathcal{S}}{\partial G}, \quad \xi = \frac{\partial \mathcal{S}}{\partial \eta}.$$

Moreover, in these new variables, the Hamiltonian takes the following form

$$H_0 \circ \underbrace{\phi_0 \circ \phi_{pc} \circ \tilde{\phi}_D}_{\phi}(l, L, g, G, \xi, \eta) = -\frac{\mu^3 M^2}{2L^2} + \frac{|\eta|^2}{2M}. \quad (\text{D.2})$$

This concludes this slight digression.

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## RÉSUMÉ

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Dans cette thèse, nous nous intéressons aux systèmes hamiltoniens dépendant du temps. Plus précisément, nous étudions l'existence d'orbites convergeant, quand  $t \rightarrow \pm\infty$ , vers des solutions quasipériodiques. Après la première partie consacrée à l'introduction, les résultats de cette thèse sont divisés en quatre parties.

Dans la première partie, nous analysons quand ce genre d'orbites existent pour des hamiltoniens dépendant du temps convergeant asymptotiquement dans le temps ( $t \rightarrow +\infty$ ) vers des hamiltoniens ayant un tore invariant supportant des solutions quasipériodiques.

Une deuxième partie est consacrée à l'étude des conditions d'existence des orbites définies pour tous les temps et convergeant asymptotiquement dans le temps vers certaines solutions quasipériodiques dans le passé ( $t \rightarrow -\infty$ ) et dans le futur ( $t \rightarrow +\infty$ ).

La troisième partie est dédiée aux applications en mécanique céleste. Nous considérons l'exemple d'un système planétaire perturbé par une comète venant et revenant à l'infini asymptotiquement le long d'une orbite képlérienne hyperbolique. Ici, le hamiltonien qui décrit ce système ne satisfait pas des bonnes propriétés de décroissance par rapport au temps. Par conséquent, les résultats des parties précédentes ne s'appliquent pas. Donc, nous sommes obligés de prouver un autre théorème abstrait pour montrer l'existence d'orbites appropriées associées à ce système. Nous concluons cette thèse en étudiant des hamiltoniens dépendant du temps convergeant asymptotiquement par rapport au temps ( $t \rightarrow +\infty$ ) vers des hamiltoniens ayant un tore invariant supportant une dynamique arbitraire. Ici, nous prouvons l'existence de certaines orbites convergeant asymptotiquement dans le temps ( $t \rightarrow +\infty$ ) vers la dynamique arbitraire associée au tore invariant précédent.

## MOTS CLÉS

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Système dynamique, tores KAM, dépendance du temps

## ABSTRACT

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In this thesis, we are interested in time-dependent Hamiltonian systems. More specifically, we study the existence of orbits converging, when  $t \rightarrow \pm\infty$ , to some quasiperiodic solutions. After the first part dedicated to the introduction, the results of this thesis are divided into four parts.

In the first part, we analyze when the above-mentioned orbits exist for time-dependent Hamiltonians converging asymptotically in time  $t \rightarrow +\infty$  to Hamiltonians having an invariant torus supporting quasiperiodic solutions. A second part is devoted to studying the conditions of existence of orbits defined for all times and converging asymptotically in time to some quasiperiodic solutions in the past  $t \rightarrow -\infty$  and in the future  $t \rightarrow +\infty$ .

The third part is dedicated to the applications in celestial mechanics. We consider the example of a planetary system perturbed by a comet coming from and going back to infinity asymptotically along a hyperbolic keplerian orbit. Here, the Hamiltonian which describes this system does not satisfy good time decay properties. Hence, the results of the previous parts do not apply. Therefore, we are forced to prove another abstract theorem to show the existence of suitable orbits associated with this system.

We conclude this thesis by studying time-dependent Hamiltonians converging asymptotically in time  $t \rightarrow +\infty$  to Hamiltonians having an invariant torus supporting arbitrary dynamics. Here, we prove the existence of some orbits converging asymptotically in time  $t \rightarrow +\infty$  to the arbitrary dynamics associated with the above-mentioned invariant torus.

## KEYWORDS

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Dynamical systems, KAM tori, time-dependence