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## Convex hull peeling

Gauthier Quilan

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Normandie Université

## THÈSE

Pour obtenir le diplôme de doctorat

Spécialité MATHÉMATIQUES

Préparée au sein de l'Université de Rouen Normandie

### Enveloppes convexes pelées

Présentée et soutenue par  
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Thèse soutenue le 15/12/2022  
devant le jury composé de

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# Introduction

En 1864, paraît dans la revue *The Educational Times* un problème de mathématiques proposé par Sylvester [77] et qui s'énonce simplement de la manière suivante : étant donnés quatre points jetés au hasard, quelle est la probabilité qu'ils soient les sommets d'un quadrilatère convexe ? Ce casse-tête, connu de nos jours sous le nom de *problème des quatre points de Sylvester*, convoque à la fois la théorie des probabilités et la géométrie euclidienne, sans pour autant qu'un cadre formel précisant la nature de l'aléa et le lieu dans lequel les points sont jetés soit donné. Il s'inscrit dans une succession de questions de probabilités géométriques qui ont émaillé l'histoire des mathématiques depuis le 18ème siècle, comme par exemple celle liée à l'expérience de l'aiguille de Buffon décrite en 1733 ou le paradoxe de Bertrand cité en 1888.

Ces problèmes ne sont alors considérés que comme des mathématiques récréatives. Il faut attendre le milieu du vingtième siècle pour qu'un contexte théorique se mette en place. Apparaît alors un nouveau domaine des probabilités axé sur l'étude de données spatiales aléatoires et dont la naissance est notamment motivée par d'importants besoins en modélisation en sciences expérimentales (notamment la physique des matériaux, la géologie puis plus tard, les télécommunications et l'analyse d'image). Le nom de géométrie stochastique qui lui est donné est traditionnellement attribué à Kendall et Krickeberg en 1969 (bien que Kendall note que le nom apparaît déjà en 1963 dans un article de percolation de Frisch et Hammersley). Ce domaine résulte de la conjonction de la théorie des processus ponctuels, la théorie des files d'attente, la géométrie convexe et intégrale, la théorie des ensembles aléatoires et la géométrie combinatoire. Citons notamment les deux ouvrages de référence, dus à Stoyan, Kendall, Mecke, Chiu [32] d'une part et à Schneider et Weil [71] d'autre part. Le premier met l'accent sur les questions de modélisation et de simulation tandis que le second se distingue par une approche basée sur les fondements de géométrie convexe et intégrale. Par ailleurs, la statistique spatiale qui vise à développer des méthodes d'inférence pour des données géométriques multidimensionnelles constitue l'indispensable pendant de la géométrie stochastique. Certaines questions probabilistes sont justement motivées par ces applications statistiques, comme c'est par exemple le cas du *convex hull peeling* abordé dans cette thèse.

Le problème des quatre points de Sylvester a conduit les mathématiciens à proposer différentes solutions suivant le modèle probabiliste adopté [57]. De par ses nombreuses extensions possibles, il a surtout été à l'origine d'un sous-domaine de la géométrie stochastique consacré à l'étude des polytopes aléatoires générés comme enveloppes convexes de nuages de points jetés au hasard. Cette thèse s'inscrit exactement dans ce cadre. Plus précisément, étant donné un corps convexe  $K$  de  $\mathbb{R}^d$  d'intérieur non vide, où  $d$  est la dimension supposée au moins égale à 2, on considère un ensemble aléatoire de points  $X \subset K$  et on étudie le polytope  $\text{conv}(X)$  obtenu en prenant l'enveloppe convexe de ces points. Les questions abordées (nombre de points extrémaux et de faces, volumes, qualité de l'approximation polytopale) sont notamment motivées par les applications en analyse de

la complexité moyenne d'algorithmes de géométrie computationnelle et combinatoire [59, 40], en optimisation [19], en biologie [75] ou encore en traitement du signal [39]. Pour plus de détails, on renvoie le lecteur aux chapitres de synthèse dus à Bárány [6], Schneider [69], Reitzner [63] et Hug [48].

Dans cette thèse, l'ensemble aléatoire  $X$  suit la loi d'un processus ponctuel de Poisson dans  $K$ , c'est-à-dire un ensemble de points de  $K$  dont le nombre est poissonnien et tel que conditionnellement à ce nombre, les points sont indépendants et identiquement distribués. En particulier, les nombres de points tombant dans des boréliens disjoints de  $\mathbb{R}^d$  sont indépendants. Cette hypothèse poissonnienne est essentielle à notre étude. La mesure d'intensité d'un tel processus est la mesure qui à un borélien de  $\mathbb{R}^d$  associe le nombre moyen de points du processus dans ce borélien. Celle-ci caractérise la loi du processus ponctuel de Poisson. On considérera en particulier le processus noté  $\mathcal{P}_f$  dont la mesure d'intensité s'écrit  $f(x)dx$  où  $f$  est une fonction mesurable positive de support inclus dans  $K$  et où  $dx$  est la mesure de Lebesgue de  $\mathbb{R}^d$ . Lorsque  $f = \lambda \mathbf{1}_K$  avec  $\lambda > 0$ , on note  $\mathcal{P}_\lambda = \mathcal{P}_f$ . Pour un traitement plus général des processus ponctuels dont les processus de Poisson sont des cas particuliers, on peut se référer notamment à l'ouvrage de Daley et Vere-Jones [34, 35]. Le livre de Last et Penrose [53] se concentre quant à lui sur le processus de Poisson.

Il existe peu de résultats généraux sur la loi de  $\text{conv}(\mathcal{P}_f)$  à  $f$  fixé. Dans cette thèse, nous nous concentrons sur des propriétés asymptotiques de  $\text{conv}(\mathcal{P}_\lambda)$  lorsque le nombre moyen  $\lambda$  de points par unité de volume dans  $K$  tend vers l'infini. En particulier, on s'intéresse aux nombres de sommets, arêtes, faces et plus généralement de  $k$ -faces pour tout  $k \in \{0, \dots, d-1\}$  de la frontière du polytope  $\partial \text{conv}(\mathcal{P}_\lambda)$  ainsi qu'aux volumes intrinsèques, dont le volume est un cas particulier, de  $\text{conv}(\mathcal{P}_\lambda)$ . On note  $f_k(P)$  le nombre de  $k$ -faces d'un polytope  $P$  et  $V_k(P)$  son  $k$ -ième volume intrinsèque. En particulier  $f_0(P)$  est le nombre de sommets de  $P$  et  $V_d$  est le volume dans  $\mathbb{R}^d$ . L'étude du premier moment de ces caractéristiques remonte aux deux articles fondateurs de Rényi et Sulanke datant de 1963 [64] et 1964 [66] et consacrés au cas de la dimension deux. Les auteurs y obtiennent notamment le fait surprenant que la vitesse de croissance du nombre moyen de points extrémaux est polynomiale ou logarithmique suivant la régularité du bord du corps convexe  $K$ . Plus précisément, il existe deux constantes explicites strictement positives  $c_K$  et  $c'_K$  et ne dépendant que de  $K$  telles que

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-1/3} \mathbb{E}[f_0(\text{conv}(\mathcal{P}_\lambda))] = c_K$$

dans le cas où  $K$  est à bord lisse, c'est-à-dire de classe  $\mathcal{C}^2$ , et

$$\lim_{\lambda \rightarrow +\infty} \log^{-1}(\lambda) \mathbb{E}[f_0(\text{conv}(\mathcal{P}_\lambda))] = c'_K$$

lorsque  $K$  est un polygone. La constante  $c_K$  est proportionnelle au périmètre affine de  $K$  lorsque  $K$  est à bord lisse tandis que  $c'_K$  est proportionnelle au nombre de sommets de  $K$  lorsque  $K$  est un polygone. À partir des années 1990, ces résultats sont étendus à l'espérance du nombre de  $k$ -faces en toute dimension [62, 2, 7, 74]. Plus précisément, il existe des constantes explicites strictement positives  $c_{K,k}$  et  $c'_{K,k}$  ne dépendant que de  $K$  et  $k \in \{0, \dots, d-1\}$  telles que si  $K$  est à bord lisse,

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-(d-1)/(d+1)} \mathbb{E}[f_k(\text{conv}(\mathcal{P}_\lambda))] = c_{K,k}$$

et si  $K$  est un polytope,

$$\lim_{\lambda \rightarrow +\infty} \log^{-(d-1)}(\lambda) \mathbb{E}[f_k(\text{conv}(\mathcal{P}_\lambda))] = c'_{K,k}.$$

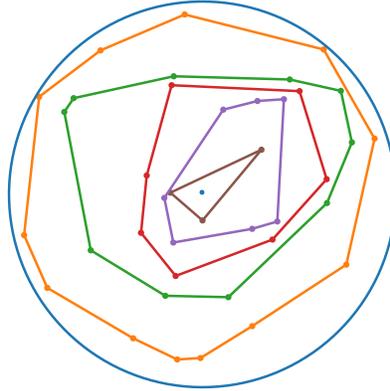


Figure 1 – Exemple de *convex hull peeling* dans un disque.

Des résultats du second ordre (bornes et limite pour la variance, théorème central limite) [62, 61, 11, 12, 24, 25, 27] et des inégalités de concentration [63, 79, 44] sont venus compléter depuis l'analyse asymptotique de ces fonctionnelles combinatoires du polytope aléatoire. Parallèlement, des résultats analogues ont été établis pour les volumes intrinsèques dans le cas où  $K$  est à bord lisse [72, 2, 8]. Dans le cas où  $K$  est un polytope, le problème reste ouvert pour les volumes intrinsèques distincts du volume  $V_d$  [7] et  $V_1$  [67]. Pour plus de détails, nous renvoyons le lecteur à la section 1.3.

Cette thèse porte sur la construction du *convex hull peeling* (qu'on pourrait traduire littéralement par *enveloppe convexe pelée*) qui est une généralisation des polytopes aléatoires décrits précédemment. Plus précisément, le *convex hull peeling* d'un ensemble localement fini  $X$  consiste à prendre l'enveloppe convexe de  $X$ , puis à enlever les points sur le bord de l'enveloppe et ensuite à répéter l'opération jusqu'à ce qu'il ne reste plus de points, voir Figure 1. L'enveloppe prise à l'étape  $n$  est notée  $\text{conv}_n(X)$  avec la convention  $\text{conv}_1(X) = \text{conv}(X)$  et sa frontière  $\partial\text{conv}_n(X)$  est appelée la  $n$ -ième couche du *convex hull peeling* de  $X$ . La procédure de *convex hull peeling* a été introduite par Barnett [13] en 1976. Son but était de donner un sens au caractère "central" d'un point par rapport à un ensemble de données en dimension  $d$ . En effet, le numéro de la couche du *convex hull peeling* auquel appartient un point peut être interprété comme la *profondeur* de ce point à l'intérieur de l'ensemble de données. On s'attend ainsi à ce qu'un point soit d'autant plus central que son numéro de couche est grand. Les applications du *convex hull peeling* concernent surtout les statistiques et en particulier la détection d'anomalie, voir par exemple les articles de Donoho et Gasko [38], Hodge et Austin [47] et Rousseeuw et Struyf [66]. Le *convex hull peeling* a également été utilisé pour la reconnaissance d'ensembles déformés par des projections dans le cadre de la vision par ordinateur [76] ou dans le contexte d'un algorithme de reconnaissance d'empreintes digitales [58].

À notre connaissance, il existe peu de résultats théoriques connus sur le *convex hull peeling*, à l'exception notable essentiellement de deux articles.

Tout d'abord, Dalal [33] obtient en 2004 l'ordre de grandeur de l'espérance du nombre total de couches du *convex hull peeling* de  $\mathcal{P}_\lambda$  dans n'importe quelle région bornée  $K$  de  $\mathbb{R}^d$ , qui n'est pas nécessairement convexe. Plus précisément, pour toute région bornée  $K$ , il existe deux constantes positives  $c_K$  et  $c'_K$  ne dépendant que de  $K$  telles que le nombre total moyen de couches du *convex hull peeling* de  $\mathcal{P}_\lambda$  est compris entre  $c_K \lambda^{2/(d+1)}$  et  $c'_K \lambda^{2/(d+1)}$ . La démonstration repose d'une part

sur le codage par un graphe acyclique orienté déterministe d'une partition bien choisie de la zone d'intérêt et d'autre part sur un résultat probabiliste de Bárány [1] selon lequel le nombre de points sur la frontière de l'enveloppe convexe de  $n$  points uniformes i.i.d. dans un corps convexe quelconque est toujours compris entre  $\log^{d-1}(n)$  et  $n^{(d-1)/(d+1)}$  à constante multiplicative près, c'est-à-dire que les cas des corps convexes lisse et des polytopes sont les deux cas extrêmes. En particulier, un raisonnement heuristique simple permet de vérifier que le résultat de Dalal semble cohérent dans le cas des convexes à bord lisse. En effet il y a asymptotiquement et en moyenne de l'ordre de  $\lambda^{(d-1)/(d+1)}$  points sur le bord de l'enveloppe convexe. En supposant que le nombre moyen de points sur chacune des couches soit du même ordre de grandeur, on retrouve bien un total de  $\lambda$  points en moyenne dans le corps convexe  $K$ . Dans le chapitre 2 de la thèse, nous exploitons la construction de Dalal pour montrer que la famille des variables égales au nombre total de couches renormalisé pour  $\lambda > 0$  est uniformément intégrable.

Le deuxième travail consacré au *convex hull peeling* et qu'il est pertinent de mentionner dans le cadre de cette thèse remonte à 2020 et est dû à Calder et Smart [22]. Dans cet article, les travaux de Dalal sont étendus de multiples façons. Tout d'abord, le processus de Poisson peut avoir pour intensité n'importe quelle mesure  $\lambda f(x)dx$  où  $f$  est une fonction continue strictement positive sur un corps convexe  $K$ . Ensuite, une valeur limite presque sûre est obtenue pour le nombre total de couches. Surtout, ils étendent cette convergence à un cadre fonctionnel. Plus précisément, ils définissent une fonction  $h_\lambda$  qui, à chaque point du convexe mère  $K$ , associe le nombre d'enveloppes du *convex hull peeling* de  $\mathcal{P}_{\lambda f}$  qui contiennent ce point. Cette fonction, appelée fonction de hauteur (ou profondeur) convexe, admet une limite uniforme presque sûre explicite, c'est-à-dire que presque sûrement,

$$\lambda^{-\frac{2}{d+1}} h_\lambda \xrightarrow{\text{unif}} \alpha h \quad (1)$$

où  $\alpha > 0$  est une constante positive qui ne dépend que de la dimension  $d$  et  $h$  est l'unique solution de viscosité de l'équation aux dérivées partielles

$$\begin{cases} \langle Dh, \text{cof}(-D^2h)Dh \rangle = f^2 \text{ dans int}(K) \\ h = 0 \text{ sur } \partial K \end{cases},$$

avec les notations  $\langle \cdot, \cdot \rangle$  et  $\text{cof}$  désignant le produit scalaire usuel de  $\mathbb{R}^d$  et la comatrice respectivement. Dans la mesure où le maximum de  $h_\lambda$  est le nombre total de couches, les auteurs obtiennent comme corollaire la convergence presque sûre et en espérance du nombre total de couches renormalisé par  $\lambda^{2/(d+1)}$  vers  $\alpha \max_K h$ . L'originalité de leur travail repose notamment sur une interprétation de la procédure de *peeling* en termes de théorie des jeux, une étude des solutions de l'EDP ci-dessus et de leur approximation par morceaux et dans la transformation locale de la construction du *convex hull peeling* en ce qu'ils appellent un *semiconvex peeling* et qui n'est rien d'autre que le *peeling* parabolique décrit dans notre chapitre 3.

Nous adoptons dans cette thèse un point de vue complémentaire de celui de Dalal et Calder et Smart. Au lieu de compter le nombre de couches, nous fixons le numéro  $n$  d'une couche, indépendamment de  $\lambda$  et étudions ensuite les propriétés de la  $n$ -ième enveloppe convexe dans le même esprit que la littérature existante sur les polytopes aléatoires. Plus précisément, nous cherchons à obtenir des asymptotiques de moments pour le nombre de points sur la  $n$ -ième couche, puis pour le nombre de  $k$ -faces et les volumes intrinsèques de cette couche. Nous nous sommes concentrés sur deux cas précis : lorsque  $K$  est la boule unité  $\mathbb{B}^d$  de  $\mathbb{R}^d$  et lorsque  $K$  est un polytope simple.

Le chapitre 3 est consacré aux résultats inédits obtenus dans le cas  $K = \mathbb{B}^d$ . Nous désignons par  $N_{n,k,\lambda}$  le nombre de  $k$ -faces de la  $n$ -ième couche du *convex hull peeling* de  $\mathcal{P}_\lambda$  pour  $k \in \{0, \dots, d-1\}$

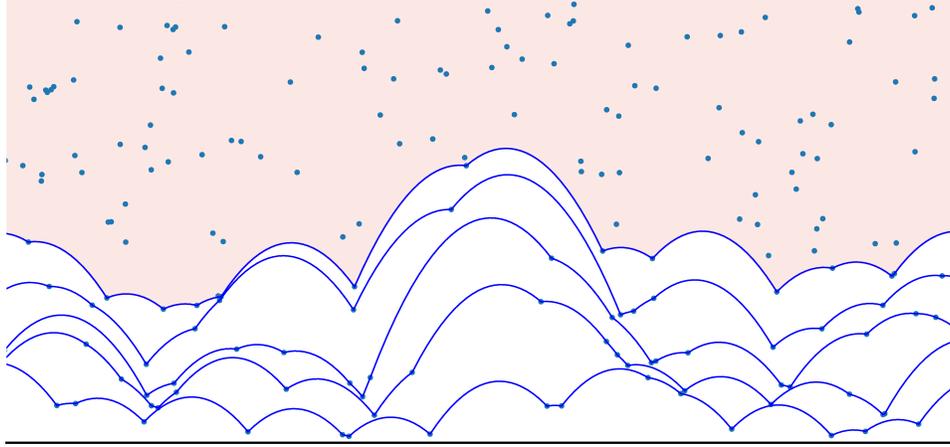


Figure 2 – Premières couches d'un *peeling* parabolique, avec en rose la 5-ième enveloppe parabolique.

et par  $V_{n,k,\lambda}$  le  $k$ -ième volume intrinsèque défaut  $V_k(\mathbb{B}^d) - V_k(\text{conv}_n(\mathcal{P}_\lambda))$  pour  $k \in \{1, \dots, d\}$ . Nous obtenons alors les résultats suivants:

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-\frac{d-1}{d+1}} \mathbb{E}[N_{n,k,\lambda}] = C_{n,k,d} \text{ et } \lim_{\lambda \rightarrow +\infty} \lambda^{-\frac{d-1}{d+1}} \text{Var}[N_{n,k,\lambda}] = C'_{n,k,d}$$

et

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\frac{2}{d+1}} \mathbb{E}[V_{n,k,\lambda}] = C_{V,n,k,d} \text{ et } \lim_{\lambda \rightarrow +\infty} \lambda^{\frac{d+3}{d+1}} \text{Var}[V_{n,k,\lambda}] = C'_{V,n,k,d}$$

où  $C_{n,k,d}$ ,  $C'_{n,k,d}$ ,  $C_{V,n,k,d}$  et  $C'_{V,n,k,d}$  sont des constantes strictement positives ne dépendant que de  $n$ ,  $k$  et  $d$ . On peut remarquer en particulier que l'ordre de grandeur du nombre moyen de  $k$ -faces (et également du volume intrinsèque défaut d'ordre  $k$ ) reste le même pour la  $n$ -ième couche que pour la toute première. Nous démontrons par ailleurs des théorèmes centraux limite pour les variables  $N_{n,k,\lambda}$  et  $V_{n,k,\lambda}$ .

Les démonstrations de ces résultats sont assez différentes des articles précédents sur le *convex hull peeling*. Nous nous appuyons en grande partie sur les techniques développés par Calka, Schreiber et Yukich [24] pour étudier la première enveloppe convexe. La première étape consiste à réécrire les variables considérées,  $N_{n,k,\lambda}$  et  $V_{n,k,\lambda}$ , comme la somme sur les points poissonniens de *scores* qui sont des variables aléatoires dépendant du point poissonnien choisi et de l'ensemble du processus. Ce score décrit la contribution d'un point donné à la variable globale  $N_{n,k,\lambda}$  ou  $V_{n,k,\lambda}$ . Comme les premières couches ont tendance à s'agglutiner près du bord de  $\mathbb{B}^d$  lorsque  $\lambda$  devient grand, on choisit dans une seconde étape d'appliquer un changement d'échelle adapté. Ceci nous conduit à considérer un autre modèle de *peeling* dit parabolique dans un demi-espace  $\mathbb{R}^{d-1} \times \mathbb{R}_+$  où le rôle joué par les hyperplans dans la construction du *convex hull peeling* classique de départ est à présent tenu par des paraboloides, voir Figure 2. En suivant la stratégie de [24], tout l'enjeu est alors de montrer la stabilisation des scores dans ce nouveau modèle, c'est-à-dire de vérifier qu'avec grande probabilité qui tend exponentiellement vite vers 1, la détermination du score d'un point donné ne dépend que de l'intersection du processus avec un voisinage de ce point. Ceci est la difficulté majeure de ce chapitre et la principale nouveauté par rapport à la littérature existante. En effet, la

construction des couches du *peeling* nécessite la connaissance globale de tout l'ensemble de points de  $\mathcal{P}_\lambda$  et de tout l'historique des couches qui précèdent. De fait, dès que  $n \geq 2$ , il est nettement plus délicat d'établir un bon critère permettant de décider si un point donné est sur la  $n$ -ième couche. Notamment, le seul critère que l'on connaisse pour déterminer si un point est sur la  $n$ -ième couche est un critère récursif qui rappelle l'interprétation en termes de théorie des jeux exploitée par Calder et Smart. Cela nous pousse à faire une démonstration par récurrence de la stabilisation qui nécessite par ailleurs d'estimer la position en hauteur de chacune des couches. L'autre point délicat de ce chapitre est la justification de la stricte positivité des espérances et variances limites de  $N_{n,k,\lambda}$  et  $V_{n,k,\lambda}$ . Pour ce faire, nous procédons à la construction d'une configuration de points déterministe idéale, représentée par un arbre, et nous la perturbons ensuite de manière aléatoire.

Le chapitre 4 porte sur nos résultats inédits dans le cadre d'un convexe-mère  $K$  qui est lui-même un polytope simple. En notant de nouveau  $N_{n,k,\lambda}$  pour le nombre de  $k$ -faces du *convex hull peeling* de  $\mathcal{P}_\lambda$ , on obtient

$$\lim_{\lambda \rightarrow +\infty} \log(\lambda)^{-(d-1)} \mathbb{E}[N_{n,k,\lambda}] = C_{n,k,d} \text{ et } \lim_{\lambda \rightarrow +\infty} \log(\lambda)^{-(d-1)} \text{Var}[N_{n,k,\lambda}] = C'_{n,k,d}$$

où  $C_{n,k,d}$  et  $C'_{n,k,d}$  sont des constantes positives ne dépendant que de  $n$ ,  $k$  et  $d$ . Là encore, à constante multiplicative près, le comportement asymptotique de l'espérance et de la variance est le même pour la  $n$ -ième couche que pour la première.

Une partie des outils du chapitre 3 se transposent assez bien au cas où  $K$  est un polytope simple. En effet, on peut à nouveau utiliser une décomposition en somme de scores de  $N_{n,k,\lambda}$  puis un changement d'échelle. Le modèle transformé est alors un *peeling* où le rôle des hyperplans est joué par des grains ressemblant à des cônes. Dans les faits, malgré quelques complications techniques, on peut montrer la stabilisation de façon relativement similaire à celle utilisée dans le cas de la boule unité. En revanche, on fait face à un nouveau problème : le changement d'échelle utilisé ne peut fonctionner qu'en se limitant au voisinage d'un sommet de  $K$  et il n'existe pas de changement d'échelle global. Cela nous oblige à montrer deux faits importants, d'une part que les contributions des sommes de scores au voisinage de chacun des sommets de  $K$  sont additives, que l'on considère l'espérance ou la variance, et d'autre part que la contribution des scores de points suffisamment éloignés des sommets de  $K$  est négligeable. La démonstration de ces deux faits nécessite d'introduire des objets géométriques spécifiques, en particulier les corps flottants qui sont des sous-ensembles convexes de  $K$  destinés à approcher avec grande probabilité les enveloppes convexes successives du *peeling*. En reprenant des idées développées par Bárány et Reitzner dans le cas de la première enveloppe [12], nous montrons que les couches successives du *peeling* sont contenues dans une zone étroite près de la frontière de  $K$ , délimitée par deux corps flottants convenablement calibrés. Ce résultat dit de *sandwiching* constitue l'une des principales nouveautés de ce chapitre. Sa démonstration repose notamment sur un résultat Bárány [5] sur la probabilité pour  $n$  points d'être en position convexe. La fin de la démonstration adapte aux enveloppes successives du *peeling* la plupart des arguments utilisés par Calka et Yukich [27] dans le cas de la première enveloppe et inspirés par [12], à savoir une partition explicite du *sandwich* et la construction d'un graphe de dépendance.

La suite de cette thèse, rédigée en anglais, s'organise comme suit.

- Le premier chapitre rappelle quelques résultats de géométrie stochastique et de géométrie convexe. Nous rappelons tout d'abord les définitions et principales propriétés des processus

ponctuels, processus de Poisson,  $k$ -faces et volumes intrinsèques. Le chapitre se concentre ensuite sur la description des principaux résultats connus sur les polytopes aléatoires. Nous insistons en particulier sur les techniques de démonstrations utilisées par [24] et [27] car elles sont réutilisées dans les chapitres 3 et 4.

- Dans le chapitre 2, nous donnons quelques définitions et résultats généraux sur le *convex hull peeling* déterministe, puis nous présentons de manière précise les résultats de Dalal [33] dans le cadre aléatoire. Nous en profitons pour déduire des arguments de Dalal un résultat d’uniforme intégrabilité, ce qui constitue une première nouveauté dans cette thèse. Nous décrivons ensuite la contribution de Calder et Smart [22] en détaillant en particulier la convergence en espérance du nombre total de couches.
- Le chapitre 3 est dédié à l’étude des premières couches du *convex hull peeling* d’un processus de Poisson d’intensité  $\lambda dx$  dans la boule unité de  $\mathbb{R}^d$ . Il s’agit d’un article soumis pour publication et comprenant une partie des nouvelles contributions de cette thèse. On y obtient une limite pour l’espérance et la variance renormalisées du nombre de  $k$ -faces ainsi que de tous les volumes intrinsèques pour les premières couches du *convex hull peeling*. On montre par ailleurs que les limites obtenues sont non nulles et on établit enfin un théorème central limite pour chacune de ces quantités.
- Enfin le chapitre 4 concerne le cas où le convexe mère  $K$  est un polytope simple. Il est constitué d’un article destiné à être soumis et incluant la seconde partie des nouvelles contributions de la thèse. Y sont obtenues les limites de l’espérance et de la variance renormalisées du nombre de  $k$ -faces des premières couches du *convex hull peeling*. Un résultat intermédiaire intéressant en soi porte sur la localisation entre deux corps flottants des premières couches.

Les chapitres 2, 3 et 4 se concluent par une sélection de problèmes ouverts et de perspectives engendrées par notre travail.

# Chapter 1

## Elements of convex and stochastic geometry

A general idea in the construction of numerous models of stochastic geometry consists in first defining random points and then building geometric structures from these points according to a particular deterministic rule. This is the case for random geometric graphs [56], random tessellations [71, Chapter 10] or continuum percolation [55] for instance. The models of random polytopes under study in this thesis also fit into this category.

This chapter aims at introducing the useful definitions and known results from both stochastic and convex geometry. More precisely, we focus on the following objects: point processes, convex bodies and random polytopes.

### 1.1 Point processes

This section is dedicated to reminders on random locally finite sets of points, that we call points processes. A special attention is given to Poisson point processes which satisfies a property of non-interaction between the points. In this regard, they are the most convenient to handle and as such are the ones that are used in the following chapters.

A point process is the basic object from which we build our random structures. It is a random set of points which is locally finite almost surely, i.e. with probability 1, there is a finite number of points in its intersection with any compact set. Such a model can be defined in any locally compact space with a countable base, see e.g. [71, Chapter 3], or even more general spaces, see [53, Chapter 2] but for our purposes, we restrict ourselves to simple points processes in  $\mathbb{R}^d$ .

A simple point process can be introduced as a random measure  $\nu$  which is the sum of Dirac measures at distincts points of  $\mathbb{R}^d$  or alternatively as a locally finite closed set. Let us present below both points of view. For sake of brevity, we omit the word *simple* in the subsequent lines although it is implicit. For further details on point processes, we refer to both [71, Chapter 3] and [53, Chapter 2].

### 1.1.1 Generalities

**Definition 1.1.1** (Counting measure). We call counting measure on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  a measure  $\nu$  on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$  that is locally finite (i.e. finite on any compact set) with values in  $\mathbb{N} \cup \{+\infty\}$  and such that for all  $x \in \mathbb{R}^d$  we have  $\nu(\{x\}) \in \{0, 1\}$ .

In particular, when  $\nu$  is a counting measure, its support  $\text{supp}(\nu)$  is given by

$$\text{supp}(\nu) = \{x \in \mathbb{R}^d : \nu(\{x\}) = 1\}.$$

We write  $\mathbf{N}$  for the set of all the counting measures on  $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ . We endow it with the smallest  $\sigma$ -algebra  $\mathcal{N}$  which guarantees the measurability of the functions

$$\begin{cases} \mathbf{N} & \rightarrow \mathbb{N} \cup \{+\infty\} \\ \nu & \mapsto \nu(A) \end{cases}, \quad A \in \mathcal{B}(\mathbb{R}^d).$$

**Definition 1.1.2** (Point process). A point process is a random variable with values in  $\mathbf{N}$ .

The denomination of point process is justified by the following property which states that a point process can be written as a sum of Dirac measures at distinct random points.

**Proposition 1.1.3.** *There exist measurable functions  $(\zeta_i)_{i \in \mathbb{N}}$  from  $\mathbf{N}$  to  $\mathbb{R}^d$  such that for all  $\nu \in \mathbf{N}$*

$$\nu = \sum_{i=1}^{\nu(\mathbb{R}^d)} \delta_{\zeta_i(\nu)}.$$

The alternative way of defining a point process consists in considering a random closed set which is locally finite almost surely. We describe this approach below. We denote by  $\mathcal{F}_{loc}$  the set of all closed locally finite sets of  $\mathbb{R}^d$ . We endow it with the smallest sigma algebra that makes the functions  $\begin{cases} \mathcal{F}_{loc} & \rightarrow \mathbb{N} \cup \{\infty\} \\ \varphi & \mapsto \text{card}(\varphi \cap A) \end{cases}$  measurable for all  $A \in \mathcal{B}(\mathbb{R}^d)$ .

The idea stated in Proposition 1.1.4 below consists in identifying a point process defined as a random measure with the set of all the points where the measure is one, i.e. its support.

**Proposition 1.1.4.** *The function*

$$\begin{cases} \mathbf{N} & \rightarrow \mathcal{F}_{loc} \\ \nu & \mapsto \text{supp}(\nu) \end{cases}$$

*is a one-to-one measurable function with measurable inverse.*

The so-called intensity of a point process which is defined below plays the role of a characteristic of order one, in the spirit of the expectation of a real random variable.

**Definition 1.1.5** (Intensity). Let  $\eta$  be a point process. We call intensity measure of  $\eta$  the measure  $\mu$  defined by

$$\mu(A) := \mathbb{E}[\eta(A)]$$

for all  $A \in \mathcal{B}(\mathbb{R}^d)$ .

For sake of simplicity, we assume from now on that the intensity measure of the considered point processes are locally finite. Definition 1.1.5 combined with the monotone class theorem for functions yields the following useful formula. Using the identification between a point process  $\eta$  and its support  $\text{supp}(\eta)$ , we generally write  $\sum_{x \in \nu} f(x)$  for the random variable  $\int_{\mathbb{R}^d} f d\nu$  for any measurable function  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ .

**Proposition 1.1.6** (Campbell formula). *Let  $\eta$  be a point process with intensity measure  $\mu$ . Then for all measurable  $f : \mathbb{R}^d \rightarrow \mathbb{R}_+$ ,  $\sum_{x \in \eta} f(x) = \int_{\mathbb{R}^d} f d\eta$  is a measurable function and*

$$\mathbb{E}[\sum_{x \in \eta} f(x)] = \mathbb{E} \left[ \int_{\mathbb{R}^d} f d\eta \right] = \int_{\mathbb{R}^d} f d\mu.$$

The moment measures that are defined now must be understood as analogues for a point process of higher order moments of a real random variable. It can be proved that they satisfy generalized Campbell formulas.

**Definition 1.1.7** (*m*-th moment measure and factorial *m*-th moment measure). Let  $\eta$  be a point process. The *m*-th moment measure (resp. *m*-th factorial moment measure) is the intensity measure of the point process  $\eta^m$  (resp. of the point process  $\eta_{\neq}^m$  defined as  $\eta_{\neq}^m := \sum_{\substack{(x_1, \dots, x_m) \in \eta^m \\ x_i \neq x_j, \forall i \neq j}} \delta_{(x_1, \dots, x_m)}$ ).

## 1.1.2 Poisson point processes

The particular point processes that we focus on in this thesis are the Poisson point processes. They are point processes such that the number of points in disjoint subsets of  $\mathbb{R}^d$  are independent, which means in other words that the points do not interact. Moreover, they are the only point processes with non-atomic intensity measures that satisfy this independence property, see e.g. [53, Theorems 6.10 and 6.12].

The number of points of a Poisson point process in a Borel subset  $A$  of  $\mathbb{R}^d$  is Poisson distributed with mean  $\mu(A)$  where  $\mu$  is its intensity measure and is non-atomic. In fact, any of the two properties, independence property or Poisson property, is enough to characterize the Poisson point processes among all point processes under the assumption that the intensity measure is non-atomic.

The independence property enjoyed by Poisson point processes is crucial for the study of the random structures that we build in this thesis. An important consequence is Mecke's formula, that we recall in Theorem 1.1.10 and that is used extensively in Chapters 3 and 4.

**Definition 1.1.8** (Poisson point processes). For any locally finite non-atomic measure  $\mu$ , there exists a point process  $\eta$  with intensity measure  $\mu$  which satisfies the following two conditions.

- For all pairwise disjoint sets  $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R}^d)$  the random variables  $(\text{card}(\eta \cap A_i))_i$  are independent.
- For all  $A \in \mathcal{B}(\mathbb{R}^d)$  such that  $\mu(A) < \infty$ ,  $\text{card}(\eta \cap A)$  is a Poisson random variable with mean  $\mu(A)$ .

Such a point process is unique in distribution and is called the Poisson point process with intensity measure  $\mu$ .

When  $f$  is a non-negative measurable function, we denote by  $\mathcal{P}_f$  the Poisson point process with intensity measure  $f(x)dx$ . With a slight abuse, we keep the same notation when  $f$  is defined on a

compact set  $K$  of  $\mathbb{R}^d$  and the Lebesgue measure is restricted to  $K$ . The context should clear out any confusion.

A Poisson point process with intensity  $\lambda dx$  for some  $\lambda > 0$  is called a homogeneous or stationary Poisson point process.

As stated in Proposition 1.1.9, the intersection of a Poisson point process with a fixed compact set may be realized explicitly, which provides an easy method of simulation.

**Proposition 1.1.9.** *Let  $\eta$  be a Poisson point process with intensity measure  $\mu$  on a compact set  $K$ . Then  $\eta$  has the same distribution as the point process  $\sum_{i=1}^N \delta_{X_i}$  where  $N \sim \mathcal{P}(\mu(K))$  and  $(X_i)_{i=1}^\infty$  are i.i.d. random variables independent from  $N$  with distribution  $\mu/\mu(K)$  on  $K$ .*

*As a consequence for any  $m \in \mathbb{N}$ , the conditional distribution  $\mathbb{P}(\eta \cap K \in \cdot \mid \text{card}(\eta \cap K) = m)$  is the distribution of a set constituted with  $m$  i.i.d. random variables with distribution  $\mu/\mu(K)$  in  $K$ .*

The following formula known in the literature as Mecke's formula extends Campbell's formula as it calculates the expectation of a measurable functional of both a point and the whole point process. It characterizes the Poisson point processes among all point processes. Furthermore and unsurprisingly for those accustomed to deal with Poisson point processes, the use of Mecke's formula turns out to be decisive in many places in Chapters 3 and 4.

**Theorem 1.1.10** (Mecke). *Let  $m \in \mathbb{N}^*$ ,  $\eta$  be a Poisson point process with intensity  $\mu$  and  $f : (\mathbb{R}^d)^m \times \mathbf{N} \rightarrow \mathbb{R}_+$  a measurable function. Then we have*

$$\mathbb{E} \left[ \sum_{(x_1, \dots, x_m) \in \eta_{\neq}^m} f(x_1, \dots, x_m, \eta) \right] = \int_{(\mathbb{R}^d)^m} \mathbb{E}[f(x_1, \dots, x_m, \eta \cup \{x_1\} \cup \dots \cup \{x_m\})] d\mu^m.$$

The following proposition provides a canonical way to build a Poisson point process with intensity  $f(x)dx$  when  $f$  is a continuous function from  $\mathbb{R}^d$  to  $\mathbb{R}_+$ . One of its benefits is the possibility to define two Poisson point processes on the same probability space with one included in the other.

**Proposition 1.1.11.** *Let  $\mathcal{P}$  be a Poisson point process with intensity  $\mathbb{1}_{\mathbb{R}^d \times ]0, +\infty[}(x)dx$  in  $\mathbb{R}^{d+1}$  and  $0 \leq f \leq g$  two measurable functions. The point process*

$$\mathcal{P}_f := \{x \in \mathbb{R}^d : \exists y \in ]0, f(x)[ \text{ s.t. } (x, y) \in \eta\}$$

*has the distribution of a Poisson point process with intensity  $f(x)dx$  in  $\mathbb{R}^d$ . In particular  $\mathcal{P}_f \subset \mathcal{P}_g$ .*

Incidentally, since the Poisson distribution plays a central role in our work, we state here a useful concentration inequality satisfied by any Poisson random variable. We are aware of numerous such estimates, see e.g. [56, Lemma 1.2] or [20, Section 2.2] but we opted for a simple upper bound which is proved in the online course [29].

**Proposition 1.1.12.** *Let  $\lambda > 0$  and  $X \sim \mathcal{P}(\lambda)$ . For all  $x > 0$  we have*

$$\mathbb{P}(|X - \lambda| \geq x) \leq 2 \exp\left(-\frac{x^2}{2(\lambda + x)}\right).$$

## 1.2 Convex and integral geometry

For a given Poisson point process, we aim at describing its convex hull peeling. We postpone the details of this construction to Chapter 2 and just state here that it involves the study of convex hulls of random points which are then random polytopes. Defining and studying the useful characteristics of these polytopes requires some knowledge of convex and integral geometry. In this section, we recall the notions of polytope,  $k$ -dimensional face and intrinsic volume and we conclude with a particular integral formula that is needed further on. For further details including the precise definitions of Hausdorff distance, Hausdorff measures and Haar measures, we refer to [71, Parts II & IV], [70, Chapters 1 to 4] and [16, Chapter 12].

### 1.2.1 Polytopes

In this section, we collect a few basic definitions related to the polytopes of  $\mathbb{R}^d$ .

**Definition 1.2.1** (Polytope). A polytope  $K$  is a compact set obtained as the intersection of a finite number of closed half-spaces of  $\mathbb{R}^d$ . It can be written uniquely as  $K = \bigcap_{i=1}^n R_i$  where  $n$  is minimal and  $R_i$  are closed half-spaces of  $\mathbb{R}^d$  and well-determined up to a permutation.

The combinatorial structure of a polytope is described through a collection of  $(d + 1)$  integer quantities, called the numbers of  $k$ -dimensional faces,  $0 \leq k \leq d$ , or  $k$ -faces in abbreviated form. They turn out to be some of our main objects of study in Chapters 3 and 4. In particular, for any  $d$ -dimensional polytope, a 0-face is also called a vertex, a 1-face an edge and a  $(d - 1)$ -face a facet.

**Definition 1.2.2** ( $k$ -faces). Let  $K = \bigcap_{i=1}^n R_i$  be a polytope with non-empty interior. The sets  $\partial R_i \cap K$  are called facets, or  $(d - 1)$ -faces of  $K$ . A  $k$ -face of  $K$  is a facet of a  $(k + 1)$ -face of  $K$  seen as a polytope with non-empty interior in a  $(k + 1)$ -dimensional affine subspace of  $\mathbb{R}^d$  ( $k \in \{0, \dots, d - 2\}$ ). We write  $f_k(K)$  for the number of  $k$ -faces of  $K$ .

We introduce below the notion of simple polytope which is needed in Chapter 4.

**Definition 1.2.3** (Simple polytope). A polytope is said to be simple if each of his vertices is included in exactly  $d$  facets.

### 1.2.2 Intrinsic volumes

Since Chapter 3 deals in part with the defect intrinsic volumes of the first convex hulls of the convex hull peeling in the ball, we introduce now the notion of intrinsic volumes. The definition of the intrinsic volumes is linked to the following formula, called Steiner's formula. We denote by  $\text{Vol}_d$  the  $d$ -dimensional Lebesgue measure and write  $\kappa_d = \text{Vol}_d(\mathbb{B}^d)$  where  $\mathbb{B}^d$  is the  $d$ -dimensional Euclidean unit ball.

**Theorem 1.2.4** (Steiner formula). *There exists non-negative functions  $V_0, \dots, V_d$  such that for any convex body  $K \subset \mathbb{R}^d$  and any  $\varepsilon > 0$*

$$\text{Vol}_d(K + \varepsilon \mathbb{B}^d) = \sum_{j=0}^d \varepsilon^{d-j} \kappa_{d-j} V_j(K).$$

**Definition 1.2.5** (Intrinsic volumes). For any  $j \in \{0, \dots, d\}$ , the function  $V_j$  of Theorem 1.2.4 defined on the set of all convex bodies of  $\mathbb{R}^d$  is called the  $j^{\text{th}}$  intrinsic volume.

Some of these intrinsic volumes can be linked to more intuitive geometric quantities. For a convex body  $K$  of  $\mathbb{R}^d$ , we have the following properties.

- $V_0(K)$  is equal to the Euler characteristic of  $K$ , i.e. is always equal to 1.
- $V_1(K) = \frac{d\kappa_d}{2\kappa_{d-1}}b(K)$  where  $b(K)$  is the mean width of  $K$ .
- $V_{d-1}(K) = \frac{1}{2}\mathcal{H}^{d-1}(\partial K)$  where  $\mathcal{H}^{d-1}$  is the  $(d-1)$ -dimensional Hausdorff measure.
- $V_d$  coincides with the  $d$ -dimensional Lebesgue measure in dimension  $d$ .

In Proposition 1.2.6 below, we state in particular that the intrinsic volumes are additive, i.e. they belong to the class of valuations on convex bodies, see e.g. [70, Chapter 6]. The famous Hadwiger's theorem shows that conversely, any valuation under continuity and rigid motion invariance assumptions can be decomposed into a linear combination of intrinsic volumes.

**Proposition 1.2.6.** *The intrinsic volumes are additive in the sense that for all  $j$  and convex bodies  $K$  and  $L$*

$$V_j(K \cup L) + V_j(K \cap L) = V_j(K) + V_j(L).$$

*Furthermore the intrinsic volumes are invariant under rigid motions, continuous with respect to Hausdorff distance, nonnegative, monotone under set inclusion, and locally bounded.*

The classical Cauchy-Crofton formula states that up to a multiplicative constant, the intrinsic volume  $V_1$  of a convex body  $K$  in  $\mathbb{R}^2$  is equal to the mean of the diameter of  $K$  in all directions and also equal to the length of the boundary of  $K$ . Kubota's formula which is recalled in Theorem 1.2.7 below extends this identity to further dimension, i.e. it expresses the intrinsic volume  $V_k(K)$  for any convex body  $K$  of  $\mathbb{R}^d$ ,  $0 \leq k \leq d$  as the mean over a certain canonical measure of the  $k$ -th dimensional Lebesgue measure of the projection of  $K$  onto all  $k$ -dimensional subspaces. Kubota's formula proves to be an important tool that is used in Chapter 3 for the study of the defect intrinsic volumes of the  $k$ -th layer of the convex hull peeling in the unit ball. We write  $K|L$  for the orthogonal projection of  $K$  onto  $L$  and  $\nu_k$  for the normalized Haar measure on the  $k$ -th Grassmanian  $G(d, k)$  of  $\mathbb{R}^d$ .

**Theorem 1.2.7** (Kubota's formula). *For any convex compact set  $K$  of  $\mathbb{R}^d$  we have*

$$V_j(K) = \frac{d! \kappa_d}{j! \kappa_j (d-j)! \kappa_{d-j}} \int_{G(d, j)} V_j(K|L) d\nu_j(L).$$

### 1.3 Random polytopes

In this section we discuss random polytopes generated as the convex hull of a Poisson point process. This is the first step of the construction of the convex hull peeling, where we then remove the extreme points and take the convex hull of the remaining points repeatedly until there is no remaining point. However before even thinking of the convex hull peeling, we wish to emphasize here that several models of convex hulls of random sets of points have been studied extensively. For precise surveys, we refer to [63], [6] and [48].

To the best of our knowledge, the list of available non-asymptotic results is limited to Efron's formula which relates the mean number of extreme points to the mean volume of the convex hull [41], its extensions to higher moments due notably to Buchta [21], as well as Wendel's formula for the probability that the origin is included in the convex hull of a symmetric input [81]. Very recently, by considering random angles in simplices, Kabluchko derived explicit formulas for the expected

so-called  $f$ -vector of a random polytope, constituted with the number of  $k$ -faces for  $0 \leq k \leq d$  [49, 50].

Still, it is commonly considered as very delicate to derive distributional results on such random polytopes. We have chosen to focus here on asymptotic results when the size of the input goes to infinity. We concentrate on the following functionals : number of  $k$ -faces and  $k$ -dimensional intrinsic volumes,  $0 \leq k \leq d$ . The general problem of obtaining the limit of the average number of vertices of the convex hull of  $n$  i.i.d. random points was originally solved in the case of uniform points in a smooth convex body or in a polygon in dimension 2 by Rényi and Sulanke in their seminal works [64, 65]. In particular, they identified the growth rate of the expected number of vertices to be  $n^{1/3}$  in a smooth convex body and  $\log(n)$  in a polygon up to multiplicative constants.

In this section we start by defining the floating bodies and giving some useful results that connects them to the study of random polytopes generated as convex hulls of Poisson point processes. Then, we propose a selection of several historical results on random polytopes. Some of these results concern in fact convex hulls generated by a deterministic number of i.i.d. random points, i.e. a so-called binomial point process, but the Poisson case can be deduced from them by a classical use of Bayes' formula combined with a concentration result of the Poisson distribution. We omit this technical step and state the results for a Poisson input even when they are written originally for a binomial point process. Note that the reverse procedure known as *de-Poissonization* is infamous for being much more intricate in general. We separate the case of random inputs in smooth convex bodies with a special focus on the unit ball and the case of random inputs in polytopes. Incidentally, we describe with more details the results that we intend to extend to the subsequent layers of the convex hull peeling in Chapters 3 and 4 and we take this opportunity to introduce essential tools that are needed for the study of the convex hull peeling. Throughout the whole section,  $\mathcal{P}_\lambda$  denotes a Poisson point process of intensity  $\lambda dx$  where  $dx$  is the  $d$ -dimensional Lebesgue measure.

### 1.3.1 Floating bodies

First of all, we need to introduce two deterministic geometric constructions, namely the floating bodies and the Macbeath regions, as they play an important role in the sequel, notably in Subsection 1.3.3 and in Chapter 4.

**Definition 1.3.1** (Floating body). Let  $K$  be a convex body in  $\mathbb{R}^d$ . We define the function  $v : K \rightarrow \mathbb{R}$  for every  $z \in K$  by

$$v(z) := \min\{\text{Vol}_d(K \cap H) : H \text{ halfspace containing } z\}.$$

The floating body of parameter  $t$  is the level set

$$K(v \geq t) = \{z \in K : v(z) \geq t\}.$$

It is a convex set and its counterpart  $K(v \leq t) = \{z \in K : v(z) \leq t\}$  is called the wet part at order  $t$ .

Let us give some intuition to explain this designation. When we put a volume  $t$  of water in a cube in dimension 3, the wet part is the region of the cube that can be put under the water by rotating the cube in some direction. The floating body of parameter  $t$  is the part of the cube that is never under water, no matter how the cube is rotated.

Bárány and Larman were the first to make the connection between floating bodies and the study of random polytopes. In [20] they prove that for any convex body  $K$  and i.i.d. random variables

$(X_1, \dots, X_n)$  uniformly distributed in  $K$ , the expected volume of  $K \setminus \text{conv}(X_1, \dots, X_n)$  is close to the volume of the wet part of order  $\frac{1}{n}$ . More precisely we have the following result.

**Theorem 1.3.2.** *Let  $K$  be a convex body in  $\mathbb{R}^d$  and  $X_1, \dots, X_n$  be  $n$  i.i.d. uniformly distributed random variables in  $K$ . There exists constants  $c_1, c_2 > 0$  and  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$*

$$c_1 \text{Vol}_d(K(v \leq \frac{1}{n})) \leq \mathbb{E}[\text{Vol}_d(K \setminus \text{conv}(X_1, \dots, X_n))] \leq c_2 \text{Vol}_d(K(v \leq \frac{1}{n})).$$

Throughout the thesis, unless noted otherwise,  $c, c', c_1, c_2 \dots$  denote positive constants which only depend on dimension  $d$  (and possibly  $n$  and  $k$  in Chapters 2, 3 and 4) and whose value may change at each occurrence.

Theorem 1.3.2 can be used to derive the asymptotic rate when  $n$  goes to infinity of the expected volume of  $K \setminus \text{conv}(X_1, \dots, X_n)$  and subsequently of  $f_0(\text{conv}(X_1, \dots, X_n))$  by Efron's formula.

The essential ingredient of the proof of Theorem 1.3.2 is the economic cap covering theorem that they introduce in the same paper. It is also a key element of the proof of the sandwiching result, see Theorem 1.3.5, that is used in [27] and that we extend to the layers of the convex hull peeling in Chapter 4. Before stating the economic cap covering theorem, we need to define the Macbeath regions.

**Definition 1.3.3** (Macbeath regions). Let  $K$  be a convex body. The Macbeath region, or M-region for short, with center  $z \in \mathbb{R}^d$  and factor  $\lambda > 0$  is  $M(z, \lambda) = M_K(z, \lambda) := z + \lambda[(K - z) \cap (z - K)]$ .

Next we need some notations for the caps of  $K$ . Let  $h_K$  be the support function of  $K$ . It is defined for any  $u \in \mathbb{S}^{d-1}$  by  $h_K(u) := \max\{\langle u, x \rangle : x \in K\}$  where  $\mathbb{S}^{d-1}$  denotes the unit sphere of  $\mathbb{R}^d$  and  $\langle \cdot, \cdot \rangle$  is the usual scalar product in  $\mathbb{R}^d$ . Any cap of  $K$ , i.e. any intersection of  $K$  with a half-space of  $\mathbb{R}^d$  can be written as

$$C = \{x \in \mathbb{R}^d : \langle u, x \rangle \geq h_K(u) - t\}$$

for some  $u \in \mathbb{S}^{d-1}$  and  $t \geq 0$ . For any  $\gamma > 0$  we write

$$C^\gamma := \{x \in \mathbb{R}^d : \langle u, x \rangle \geq h_K(u) - \gamma t\}.$$

We call minimal cap of  $z$  a cap that contains  $z$  denoted by  $C(z)$  that is such that  $\text{Vol}_d(C(z)) = v(z)$ . It is not necessarily unique.

Let  $s_0 := (2d)^{-2d}$ . For any  $s \in [0, s_0]$  we choose a maximal set of points  $z_1(s), \dots, z_{m(s)}(s)$  on  $K(v = s)$  having pairwise disjoint M-regions  $M(z_i, \frac{1}{2})$ . We call such a system saturated and notice that it is not necessarily unique. We write  $K'_i(s) = M(z_i, \frac{1}{2}) \cap C(z_i)$  and  $K_i(s) := C^6(z_i)$ . In this context we have

**Theorem 1.3.4** (Economic cap covering). *Let  $K$  be a convex body with volume 1. For all  $s \in [0, s_0]$*

1.  $\cup_{i=1}^s K'_i(s) \subset K(v \leq s) \subset \cup_{i=1}^{m(s)} K_i(s)$ ,
2.  $s \leq \text{Vol}_d(K_i(s)) \leq 6^d s$  for all  $i = 1, \dots, m(s)$ ,
3.  $(6d)^{-d} s \leq \text{Vol}_d(K'_i(s)) \leq 2^{-d} s$  for all  $i = 1, \dots, m(s)$ .

Theorem 1.3.2 has been later improved by Bárány and Reitzner in [11] in the case where  $K$  is a polytope and with a Poisson input  $\mathcal{P}_\lambda \cap K$ . They obtain that the boundary of  $\text{conv}(\mathcal{P}_\lambda \cap K)$  can in fact be sandwiched between two floating bodies with high probability, see Theorem 1.3.5

below. This general fact is used in particular for deriving asymptotic bounds for the variance of the number of  $k$ -faces and of the volume of the convex hull of  $\mathcal{P}_\lambda \cap K$  and for obtaining a CLT.

Let us write  $s = s(\lambda) := \frac{1}{\lambda \log^{4d^2+d-1}(\lambda)}$ ,  $T = T(\lambda) := \alpha \frac{\log \log(\lambda)}{\lambda}$  with  $\alpha = (6d)^d(4d^2 + d - 1)$  and  $T^* = d6^dT$ . We define  $\mathcal{A}(s, T^*, K) := K(v \geq s) \setminus K(v \geq T^*)$ .

**Theorem 1.3.5** (Sandwiching). *When  $K$  is a polytope with volume 1, there exists  $c > 0$  such that for  $\lambda$  large enough,*

$$\mathbb{P}(\partial \text{conv}(\mathcal{P}_\lambda \cap K) \subset \mathcal{A}(s, T^*, K)) \geq 1 - c \ln^{-4d^2}(\lambda).$$

### 1.3.2 Random polytopes in the unit ball and in smooth convex bodies

This subsection is devoted to the study of random polytopes generated by an homogeneous Poisson point process when the ambient space  $K$  is a smooth convex body, i.e. its boundary is at least  $\mathcal{C}^2$ -regular with a bounded Gaussian curvature from below and from above. Let us make an overview of the known results about each of our functionals of interest, i.e. expectation asymptotics, variance asymptotics, central limit theorems and concentration estimates.

As we stated before, the study of this case started with [64], a paper of Rényi and Sulanke where they find the expectation of the number of vertices in dimension 2. In [65], they also find the expectation of the defect volume between  $K$  and the convex hull of the random point set. Through Efron's formula, the study of these two quantities are intertwined in the works that come afterwards. Following Wieacker's work when  $K$  is the unit ball [82], Bárány finds the asymptotics for both the defect volume and the number of vertices in [2] in the case of a  $\mathcal{C}^3$  convex body. The asymptotics for the number of vertices is given by

$$\lim_{\lambda \rightarrow \infty} \frac{\mathbb{E}[f_0(\text{conv}(\mathcal{P}_\lambda \cap K))]}{\lambda^{(d-1)/(d+1)}} = c(d) \text{as}(K) \quad (1.1)$$

where  $c(d) > 0$  is a constant that only depends on the dimension and  $\text{as}(K)$  is the so-called affine surface area of  $K$ . It is defined by

$$\text{as}(K) := \int_{\partial K} \kappa^{1/(d+1)}(z) dz \quad (1.2)$$

where for  $z \in \partial K$ ,  $\kappa(z)$  is the Gaussian curvature of  $\partial K$  at point  $z$  and  $dz$  denotes the surface measure on  $\partial K$ . An extension to  $\mathcal{C}^2$  convex bodies and the exact value of the constant  $c(d)$  are provided by Schütt in [74]. In [62], Reitzner generalizes the previous results and obtains the limit of the expectation of the number of  $k$ -faces for any  $k \in \{0, \dots, d-1\}$ . The formula (1.1) then becomes

$$\lim_{\lambda \rightarrow \infty} \frac{\mathbb{E}[f_k(\text{conv}(\mathcal{P}_\lambda \cap K))]}{\lambda^{(d-1)/(d+1)}} = c(d, k) \text{as}(K) \quad (1.3)$$

where again  $c(d, k) > 0$  only depends on the dimension and  $k$ . In particular we notice that the dependence on  $k$  is only found in the constant and not in the exponent of  $\lambda$ . Moreover, the intrinsic volumes have also been studied. Rényi and Sulanke derived the asymptotic mean of the perimeter of the convex hull in dimension two in [65]. In dimension  $d$ , the expected mean width was found by Schneider and Wieacker in [72]. The above-mentioned result of Bárány in [2] was actually extended in the same paper to all of the intrinsic volumes, i.e. for any  $k \in \{0, \dots, (d-1)\}$ ,

$$\lim_{\lambda \rightarrow \infty} \lambda^{2/(d+1)} (\mathbb{E}[V_k(K)] - \mathbb{E}[V_k(\text{conv}(\mathcal{P}_\lambda \cap K))]) = c(d, k, K). \quad (1.4)$$

Note that again, the order of magnitude  $\lambda^{-2/(d+1)}$  does not depend on the dimension of the intrinsic volume that is considered.

Moments of higher order have been investigated as well. In [60] and then in [61], Reitzner proves asymptotic lower and upper bounds for the variance of the number of  $k$ -faces and for the defect volume. The order of magnitude of the variance is shown to be the same as for the expectation. This result was extended to a limit by Calka, Schreiber and Yukich in [24], but only when the convex body  $K$  is the unit ball. It was later shown for any  $\mathcal{C}^3$  convex body by Calka and Yukich in [27]. For the number of  $k$ -faces we have

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}[f_k(\text{conv}(\mathcal{P}_\lambda \cap K))]}{\lambda^{(d-1)/(d+1)}} = c'(d, k, K) \quad (1.5)$$

with  $c'(d, k, K) > 0$ . However, the knowledge on the intrinsic volumes, save for  $V_d$ , is more limited. Lower and upper bounds were obtained by Bárány, Fodor and Vigh in [8], in particular the order of magnitude of the variance of the defect intrinsic volumes is  $\lambda^{(d+3)/(d+1)}$  up to a multiplicative constant for any  $k$ . Furthermore as opposed to the number of  $k$ -faces it differs from the order of magnitude of the expectation by a factor  $\lambda$ . Calka, Shreiber and Yukich also give a precise limit in [24] but only in the case of  $K = \mathbb{B}^d$  where we have

$$\lim_{\lambda \rightarrow \infty} \lambda^{(d+3)/(d+1)} \text{Var}[V_k(K) - V_k(\text{conv}(\mathcal{P}_\lambda \cap \mathbb{B}^d))] = c_{V_k}(d, \mathbb{B}^d).$$

To the best of our knowledge the problem is still open in a general smooth convex body  $K$ .

Next, we mention central limit theorems (CLT) in the literature. In the case of a general smooth convex body  $K$  and in dimension  $d$ , the first central limit theorems were found for the number of  $k$ -faces and for the volume by Reitzner in the Poisson case [61] and Vu in the binomial case [79]. In [73], Schreiber and Yukich strengthen this to a multivariate central limit theorem in the unit ball and Calka Shreiber and Yukich prove a central limit theorem for the number of  $k$ -faces and all of the intrinsic volumes in the ball as well as a functional central limit theorem for the defect volume [24]. More recently, the central limit theorem of [24] has been extended by Lachièze-Rey, Schulte and Yukich in [52]. Their result is valid in any smooth convex body and the rate of convergence provided by [24] is improved.

Finally, we mention several results in the flavor of large deviation (LD) estimates. Using the Hoeffding-Azuma inequality, Reitzner provided general concentrations inequalities for the number of  $k$ -faces and the defect volume in his survey [63]. Thanks to the so-called *divide and conquer martingale* method, more precise and possibly optimal estimates were obtained by Vu in [80] in the particular cases of the volume and the number of vertices. Finally, Grote and Thäle recently proved moderate deviations for the number of  $k$ -faces and intrinsic volumes in the context of the unit ball in [44].

All the references cited above are summarized in the following table.

|              | $f_0$                                    | $f_k$                         | $V_d$                               | $V_k$                        |
|--------------|--|-------------------------------|-------------------------------------|------------------------------|
| $\mathbb{E}$ | [64] for $d = 2$ , [2, 74]               | [62]                          | [65] for $d = 2$ , [2, 74]          | [72] for $V_1$ , [2]         |
| Var          | [60, 61], [24] for $\mathbb{B}^d$ , [25] |                               |                                     | [8], [24] for $\mathbb{B}^d$ |
| CLT          | [61], [79] [73, 24], [52]                |                               |                                     | [24], [52]                   |
| LD           | [63], [80], [44] for $\mathbb{B}^d$      | [63], [44] for $\mathbb{B}^d$ | [63], [80], [44] for $\mathbb{B}^d$ | [44] for $\mathbb{B}^d$      |

We concentrate below on the description of [24] which deals with limit theory for random polytopes in the  $d$ -dimensional unit ball  $\mathbb{B}^d$  and is the main inspiration for Chapter 3 and even Chapter 4 to

some extent. For  $k \in \{0, \dots, d-1\}$  we write  $N_{1,k,\lambda}(\mathbb{B}^d)$  for the number of  $k$ -faces of the convex hull of  $\mathcal{P}_\lambda \cap \mathbb{B}^d$ . Our notation which might seem a bit intricate to this point is in fact tailored for the future introduction of the consecutive layers of the convex hull peeling from Chapter 2 on. Similarly for  $k \in \{1, \dots, d\}$  we denote by  $V_{1,k,\lambda}(\mathbb{B}^d)$  the  $k$ -th defect intrinsic volume of the convex hull of  $\mathcal{P}_\lambda \cap \mathbb{B}^d$ , i.e. the difference between the  $k$ -th intrinsic volumes of the unit ball and of the convex hull of  $\mathcal{P}_\lambda \cap \mathbb{B}^d$ .

In this context, [24] develops a method which allows to recover both the expectation asymptotics and central limit theorem for  $N_{1,k,\lambda}(\mathbb{B}^d)$  and provides in addition the existence of a limiting variance. In particular, both the expectation and variance grow like  $\lambda^{(d-1)/(d+1)}$  up to a multiplicative constant. This is stated in Theorem 1.3.6 below.

**Theorem 1.3.6.** *For any  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$  there exist  $C_{1,k,d}(\mathbb{B}^d), C'_{1,k,d}(\mathbb{B}^d) \in (0, \infty)$  such that*

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-\frac{d-1}{d+1}} \mathbb{E}[N_{1,k,\lambda}(\mathbb{B}^d)] = C_{1,k,d}(\mathbb{B}^d) \text{ and } \lim_{\lambda \rightarrow +\infty} \lambda^{-\frac{d-1}{d+1}} \text{Var}[N_{1,k,\lambda}(\mathbb{B}^d)] = C'_{1,k,d}(\mathbb{B}^d).$$

Moreover, when  $\lambda \rightarrow \infty$ , we have

$$\sup_t \left| \mathbb{P} \left( \frac{N_{1,k,\lambda} - \mathbb{E}[N_{1,k,\lambda}]}{\sqrt{\text{Var}[N_{1,k,\lambda}]}} \leq t \right) - \mathbb{P}(\mathcal{N}(0, 1) \leq t) \right| = O \left( \lambda^{-\frac{d-1}{2(d+1)}} (\log \lambda)^{3d+1} \right).$$

Furthermore, similar results are derived for the defect intrinsic volumes. This time, the expectation grows like  $\lambda^{2/(d+1)}$  while the variance grows like  $\lambda^{(d+3)/(d+1)}$ .

**Theorem 1.3.7.** *For any  $n \geq 1$  and  $k \in \{1, \dots, d\}$  there exist  $C_{V,1,k,d}, C'_{V,1,k,d} \in (0, \infty)$  such that*

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\frac{2}{d+1}} \mathbb{E}[V_{1,k,\lambda}] = C_{V,1,k,d} \text{ and } \lim_{\lambda \rightarrow +\infty} \lambda^{\frac{d+3}{d+1}} \text{Var}[V_{1,k,\lambda}] = C'_{V,1,k,d}.$$

Moreover, when  $\lambda \rightarrow \infty$ , we have

$$\sup_t \left| \mathbb{P} \left( \frac{V_{1,k,\lambda} - \mathbb{E}[V_{1,k,\lambda}]}{\sqrt{\text{Var}[V_{1,k,\lambda}]}} \leq t \right) - \mathbb{P}(\mathcal{N}(0, 1) \leq t) \right| = O \left( \lambda^{-\frac{d-1}{2(d+1)}} (\log \lambda)^{3d+1} \right).$$

One of the goals of this thesis is to extend the two previous theorems to the next layers of the convex peeling of  $\mathcal{P}_\lambda \cap \mathbb{B}^d$ , see Chapter 3. That is why we pave the way for the methods from Chapter 3 by sketching now the main ideas of the proofs of Theorems 1.3.6 and 1.3.7. For sake of simplicity, we focus on the case of  $\mathbb{E}[N_{1,k,\lambda}(\mathbb{B}^d)]$  as the overall plan of the proof is similar in the case of the  $n$ -th layer of the convex hull peeling.

**Scores.** The first idea is to write  $N_{1,k,\lambda}(\mathbb{B}^d)$  as a sum of a functional taken over each of the points, that represents the contribution of the point to the number of  $k$ -faces of the convex hull. More precisely, for each point  $x \in \mathbb{B}^d$  we write

$$\xi_{1,k}(x, \mathcal{P}_\lambda \cap \mathbb{B}^d) := \begin{cases} \frac{1}{k+1} \text{card}(\mathcal{F}_{1,k}(x, \mathcal{P}_\lambda \cap \mathbb{B}^d \cup \{x\})) & \text{if } x \in \partial \text{conv}(\mathcal{P}_\lambda \cap \mathbb{B}^d \cup \{x\}) \\ 0 & \text{otherwise} \end{cases}$$

where  $\mathcal{F}_{1,k}(x, \mathcal{P}_\lambda \cap \mathbb{B}^d)$  denotes the set of all the  $k$ -faces of  $\text{conv}(\mathcal{P}_\lambda \cap \mathbb{B}^d \cup \{x\})$  that contain  $x$ . This allows us to write

$$N_{1,k,\lambda} = \sum_{x \in \mathcal{P}_\lambda \cap \mathbb{B}^d} \xi_{1,k}(x, \mathcal{P}_\lambda \cap \mathbb{B}^d).$$

The renormalization  $\frac{1}{k+1}$  in the definition of the score comes from the fact that each  $k$ -face contains exactly  $(k+1)$  points almost surely. One of the benefits of this rewriting is that it has the right shape to use Mecke's theorem, recalled in Theorem 1.1.10. The application of this theorem gives

$$\mathbb{E}[N_{1,k,\lambda}(\mathbb{B}^d)] = \lambda \int_{\mathbb{B}^d} \mathbb{E}[\xi_{1,k}(x, \mathcal{P}_\lambda \cap \mathbb{B}^d)] dx. \quad (1.6)$$

**Scaling transform.** The next important step is to use a scaling transform. The need for rescaling is roughly due to the fact that when  $\lambda$  goes to infinity, there are so many points close to the boundary of  $\mathbb{B}^d$  that it is hard to distinguish the convex hull from  $\mathbb{B}^d$  itself. The aim is then to send the model to a new one, where each unit volume would contain a bounded number of points while preserving the convex hull structure to some extent. The only possible scaling transform which satisfies these requirements is defined as follows. We write  $T_{e_d}$  for the tangent space of  $\mathbb{S}^{d-1}$  at point  $e_d = (0, 0, \dots, 0, 1)$ . The exponential map  $\exp_{e_d} : T_{e_d} \cong \mathbb{R}^{d-1} \rightarrow \mathbb{S}^{d-1}$  sends a vector  $v$  of  $T_{e_d}$  to the vector  $u$  that lies at the end of the geodesic of  $\mathbb{S}^{d-1}$  of length  $\|v\|$  that starts at  $e_d$  with direction  $v$ . The function  $\exp_{e_d}$  induces a one-to-one map between  $B_{d-1}(\pi)$  and  $\mathbb{S}^{d-1} \setminus \{-e_d\}$  of inverse  $\exp^{-1}$  where  $B_l(r)$  denotes the open ball centered at 0 of radius  $r$  in  $\mathbb{R}^l$ . This lets us define a one-to-one map  $T^{(\lambda)}$  between  $\mathbb{B}^d \setminus [0, -e_d]$  and  $W_\lambda := \lambda^{\frac{1}{d+1}} B_{d-1}(\pi) \times [0, \lambda^{\frac{2}{d+1}}]$  by

$$T^{(\lambda)}(x) := \left( \lambda^{\frac{1}{d+1}} \exp^{-1} \left( \frac{x}{\|x\|} \right), \lambda^{\frac{2}{d+1}} (1 - \|x\|) \right) \quad (1.7)$$

for all  $x \in \mathbb{B}^d \setminus [0, -e_d]$ .

The scaling transform  $T^{(\lambda)}$  maps the Poisson point process  $\mathcal{P}_\lambda \cap \mathbb{B}^d$  to a Poisson point process  $\mathcal{P}^{(\lambda)}$  on  $W_\lambda$  that converges to a Poisson point process  $\mathcal{P}$  whose intensity is simply the Lebesgue measure on  $\mathbb{R}^{d-1} \times \mathbb{R}_+$ . This fulfills the goal of having roughly one point per unit volume. The convex hull structure is also preserved but with a different geometry. The intersections of  $\mathbb{B}^d$  with half-spaces of  $\mathbb{R}^d$ , which we call caps are transformed into a set that converge to a translate of the paraboloid  $\Pi^\downarrow$  given by

$$\Pi^\downarrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h < -\frac{\|v\|^2}{2}\}.$$

As the convex hull of  $\mathcal{P}_\lambda \cap \mathbb{B}^d$  is the complement of all the caps that do not contain any point of  $\mathcal{P}_\lambda \cap \mathbb{B}^d$ ,  $\text{conv}(\mathcal{P}_\lambda \cap \mathbb{B}^d)$  is mapped asymptotically to the complement of all the translates of  $\Pi^\downarrow$  that do not contain any point of  $\mathcal{P}$ . We are then led to study this limit set that we call the parabolic hull process related to  $\mathcal{P}$ , see Figure 1.1.

In practice, using a spherical change of coordinates, the rotation invariance of  $\mathcal{P}_\lambda$  and the change of variables given by the scaling transform  $T^{(\lambda)}$ , we deduce from (1.6) that

$$\mathbb{E}[N_{1,k,\lambda}] = \lambda^{\frac{d-1}{d+1}} \int_{\mathbb{S}^{d-1}} \int_0^{\lambda^{\frac{2}{d+1}}} \mathbb{E}[\xi_{1,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})] (1 - \lambda^{-\frac{2}{d+1}} h)^{d-1} dh d\sigma_{d-1}(u) \quad (1.8)$$

where  $\xi_{1,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}) = \xi_{1,k}([T^{(\lambda)}]^{-1}(0, h), \mathcal{P}_\lambda \cap \mathbb{B}^d)$  is the score in the rescaled model. We have the right scaling in front of the integral as  $\mathbb{E}[N_{1,k,\lambda}(\mathbb{B}^d)]$  is supposed to be asymptotically  $\lambda^{(d-1)/(d+1)}$  times a constant. It remains to show that the integral converges to a finite limite. To do so, we wish to use Lebesgue's dominated convergence theorem and have then to overcome two obstacles.

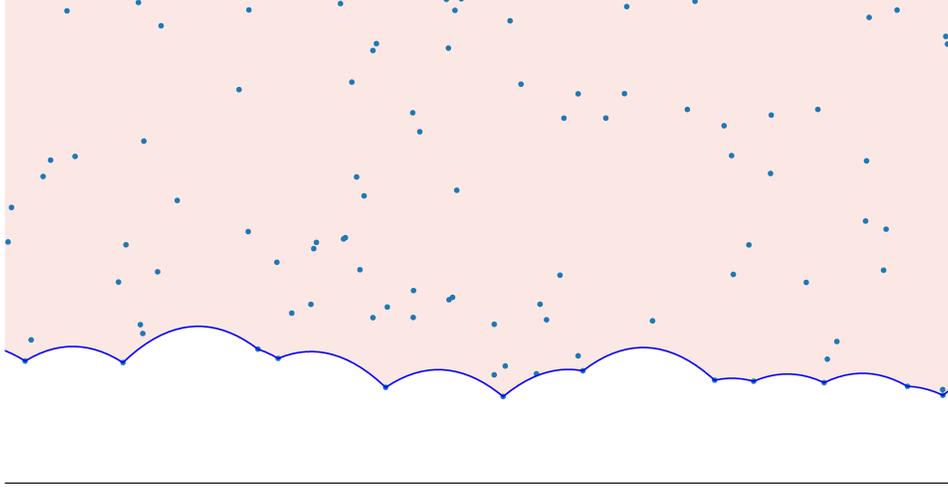


Figure 1.1 – In blue, the boundary of the parabolic hull process, in pink the image of the convex hull by the rescaling.

First, we need to prove that  $\mathbb{E}[\xi_{1,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})]$  converges almost surely to  $\mathbb{E}[\xi_{1,k}^{(\infty)}((0, h), \mathcal{P})]$  where  $\xi_{1,k}^{(\infty)}((0, h), \mathcal{P})$  is the equivalent of the score in the limit model, i.e.

$$\xi_{n,k}^{(\infty)}((0, h), \mathcal{P}) := \begin{cases} \frac{1}{k+1} \text{card}(\mathcal{F}_{n,k}^{(\infty)}((0, h), \mathcal{P})) & \text{if } (0, h) \in \partial\Phi_n^{(\infty)}(\mathcal{P} \cup \{(0, h)\}) \\ 0 & \text{otherwise} \end{cases} \quad (1.9)$$

where  $\mathcal{F}_{n,k}^{(\infty)}((0, h), \mathcal{P})$  is the set of all the parabolic  $k$ -faces of  $\Phi(\mathcal{P})$  that contain  $(0, h)$ . Here, we omit the precise definition of a parabolic  $k$ -face which is the analogue of a  $k$ -face in the context of parabolic convexity. The second obstacle consists in showing that  $\xi_{1,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})$  is dominated by an integrable function of  $h$ . Both of these two problems are solved by a result of stabilization of the scores.

**Stabilization.** The idea for proving the convergence of the scores consists in using a version of the continuous mapping theorem, namely [17, Theorem 5.5]. Thanks to the convergence of  $\mathcal{P}^{(\lambda)} = T^{(\lambda)}(\mathcal{P}_\lambda \cap \mathbb{B}^d)$  to  $\mathcal{P}$  and of the image of a cap to a translate of the paraboloid  $\Pi^\downarrow$ , the theorem can be applied save for one thing: we need to be in a compact set, which is not the case as  $W_\lambda$  goes to  $\mathbb{R}^{d-1} \times \mathbb{R}_+$  when  $\lambda$  goes to infinity. In order to circumvent that issue, we observe the crucial fact that it is not needed to know the whole point process  $\mathcal{P}^{(\lambda)}$  to determine if a point is extreme or not and the number of  $k$ -faces that possibly contain that point. The knowledge of the intersection of the point process with a region around the point which is not too large should suffice to determine its status, i.e. its score.

More precisely, for  $(v, h) \in W_\lambda$  and  $r > 0$  we write  $\xi_{1,k,[r]}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)}) := \xi_{1,k}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)} \cap C_v(r))$  where  $C_v(r)$  is the vertical cylinder  $B_{d-1}(v, r) \times \mathbb{R}_+$ ,  $B_{d-1}(v, r)$  being the ball with center  $v$  and radius  $r$  in dimension  $(d - 1)$ . Basically,  $\xi_{1,k,[r]}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})$  represents the score of  $(v, h)$  when

the whole process  $\mathcal{P}^{(\lambda)}$  is replaced by the subset of points of  $\mathcal{P}^{(\lambda)}$  that are in a vertical cylinder of radius  $r$  around  $(v, h)$ . The radius of stabilization is defined by  $R_{1,k}^{(\lambda)}(v, h) := \inf\{R > 0 : \xi_{1,k}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)}) = \xi_{1,k,[r]}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)}) \forall r \geq R\}$ . We say that the score stabilizes in width, if this radius of stabilization has a sub-exponential tail. This is proved to be the case in [24].

Let us sketch the proof of this result in the particular case  $\lambda = \infty$ ,  $v = (0, h)$  and  $k = 0$ , i.e. for the score  $\xi_{1,0}^{(\infty)}((0, h), \mathcal{P})$  equal to 1 if  $(0, h)$  is extreme in  $\mathcal{P}$  and 0 if not. Technical details aside, it is sufficient to prove that for any  $r > 0$ ,

$$\mathbb{P}(\xi_{1,0}^{(\infty)}((0, h), \mathcal{P}) \neq \xi_{1,0,[r]}^{(\infty)}((0, h), \mathcal{P})) \leq c_1 \exp(-c_2 r^{c_3}) \quad (1.10)$$

with positive constants  $c_1, c_2, c_3$ . In the base model before rescaling, a point  $x$  is extreme if there is a half-space, whose bounding hyperplane contains  $x$  that does not contain any other point of the point process. In the rescaled (limit) model, this criterion becomes: a point  $(v, h)$  is extreme if there is a translate of  $\Pi^\downarrow$  going through  $(v, h)$  that contains no other point of  $\mathcal{P}$ . When the point process is restricted to  $C_0(r)$  it is the same but this translate of  $\Pi^\downarrow$  has to contain no point of  $\mathcal{P} \cap C_0(r)$ . Thus we see that the only possibility to have  $\xi_{1,0}^{(\infty)}((0, h), \mathcal{P}) \neq \xi_{1,0,[r]}^{(\infty)}((0, h), \mathcal{P})$  is the case where  $(0, h)$  is not extreme for the whole process and extreme for  $\mathcal{P} \cap C_0(r)$ . This implies the existence of a translate of  $\Pi^\downarrow$ , denoted by  $\Pi^\downarrow(v_1, h_1) := (v_1, h_1) + \Pi^\downarrow$ , that goes through  $(0, h)$ , contains no point of  $\mathcal{P} \cap C(r)$  but contains at least one point of  $\mathcal{P}$ . For this to be possible,  $\Pi^\downarrow(v_1, h_1)$  can not be entirely contained in  $C_0(r)$ . Thus, given the equation of  $\Pi^\downarrow$ , the height of this paraboloid must be larger than  $cr^2$  for some  $c > 0$ . The volume of such a paraboloid intersected with  $C_0(r)$  is at least  $cr^{d+1}$ . This area does not contain a point of  $\mathcal{P}^{(\lambda)}$  with probability smaller than  $c_1 \exp(-c_2 r^{d+1})$  by definition of a Poisson point process. This provides the required estimate (1.10) for the stabilization in width.

In the same spirit, it is possible to show a so-called stabilization in height which states that with high probability, only the knowledge of the point process below a certain height is needed to recover the score. Combining both the stabilization in width and stabilization in height, we obtain that with a probability exponentially close to 1, the score of  $(0, h)$  is the same when the process is restricted to a compact set around  $(0, h)$ . This is enough to use [17, Theorem 5.5] and obtain the convergence of  $\mathbb{E}[\xi_{1,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})]$  to  $\mathbb{E}[\xi_{1,k}^{(\infty)}((0, h), \mathcal{P})]$ . The stabilization in width and in height also imply that the scores of  $(0, h)$  are uniformly bounded in  $L^p$  for all  $p \geq 1$  by a function that decays exponentially fast in  $h$ . In conclusion, the stabilization implies that the conditions to apply Lebesgue's dominated convergence theorem are verified. As a consequence, the integral in (1.6) converges, and the asymptotic result for the expectation follows.

### 1.3.3 Random polytopes in a polytope

The present subsection concerns random polytopes defined as the convex hull of a random input in a polytope. As we did in the previous section for the case when the mother body  $K$  is smooth, we survey below a selection of known results when  $K$  is a polytope.

Again, the study started with Rényi and Sulanke in [64] and [65]. They obtained a limit for the expected number of vertices when  $K$  is any convex polygon as well as the expected defect area and perimeter in dimension 2 in the particular case of the square. These results were extended to any dimension by Bárány and Buchta in [7] who obtained the asymptotic expected defect volume and recovered the asymptotic expected number of vertices through Efron's formula. Later, Reitzner

found a limit for the expected number of  $k$ -faces for any  $k \in \{0, \dots, (d-1)\}$  in [62]

$$\lim_{\lambda \rightarrow \infty} \frac{\mathbb{E}[f_k(\text{conv}(\mathcal{P}_\lambda \cap K))]}{\log^{d-1}(\lambda)} = c(d, k)T(K)$$

where  $T(K)$  is the number of towers of  $K$ , a tower being an increasing chain  $F_0 \subset F_1 \dots \subset F_{d-1}$  of  $i$ -faces  $F_i$  of  $K$ . As in the smooth case, the growth rate  $\log^{d-1}(\lambda)$  does not depend on  $k$ . In a polytope, the problem of finding the asymptotics for the intrinsic volumes is mostly open, save for the two cases of  $V_1$  and  $V_d$  treated by Schneider [67] and by Bárány and Buchta [7] respectively:

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda \mathbb{E}[V_d(K \setminus \text{conv}(\mathcal{P}_\lambda \cap K))]}{\log^{d-1}(\lambda)} = c_d T(K)$$

and

$$\lim_{\lambda \rightarrow \infty} \lambda^{1/d} \mathbb{E}[V_1(K) - V_1(\text{conv}(\mathcal{P}_\lambda))] = c'_d(K).$$

Next, regarding second-order estimates, lower and upper bounds of the variance of the number of  $k$ -faces and of the defect volume were found by Bárány and Reitzner in [11] and [12]. A precise limit was obtained by Calka and Yukich in [27] but only in the case of a simple polytope. They found that

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}[f_k(\text{conv}(\mathcal{P}_\lambda \cap K))]}{\log^{d-1}(\lambda)} = c'(d, k)f_0(K)$$

while the variance of the defect volume is given by

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda^2 \text{Var}[V_d(K \setminus \text{conv}(\mathcal{P}_\lambda \cap K))]}{\log^{d-1}(\lambda)} = c''_d f_0(K).$$

The variance of the defect intrinsic volumes  $V_i(K) - V_i(\text{conv}(\mathcal{P}_\lambda))$  is unknown for  $2 \leq i \leq d-1$ .

In [12], a central limit theorem has been proved by Bárány and Reitzner for the number of  $k$ -faces and the defect volume. Here again, to the best of our knowledge, the question is still open for the other intrinsic volumes.

Finally, let us conclude this overview with some concentration results. A concentration inequality for the number of  $k$ -faces and for the defect volume has been derived by Reitzner in his survey [63] and in the particular case of the defect volume, the inequality has been sharpened by Vu [80]. As expected, no such result is known in the case of the other intrinsic volumes.

The references given in this overview can be summarized by the following table.

|     | $f_0$                             | $f_k$ | $V_d$                  | $V_k$                |
|-----|-----------------------------------|-------|------------------------|----------------------|
| E   | [64] for $d = 2$ , [7]            | [62]  | [65] for $d = 2$ , [7] | open, [67] for $V_1$ |
| Var | [11, 12], [27] in the simple case |       |                        | open                 |
| CLT | [12]                              |       |                        | open                 |
| LD  | [63]                              |       | [80]                   | open                 |

Now let us describe in more detail the result of [27] where  $K$  is a simple polytope. This is the main inspiration for the results of Chapter 4. Actually, the results of [27] share some techniques with the case of the unit ball that was studied in [24]. In particular, a scaling transform and a result of stabilization are of great help in this case as well. However, it does not work as smoothly as for the ball. Extra tools from Subsection 1.3.1 are used to tackle these problems. We recall below the main results of [27] and explain the strategy of proof.

**Theorem 1.3.8.** *For all  $k \in \{0, 1, \dots, d-1\}$ , there exists a constant  $F_{k,d} \in (0, \infty)$ , such that*

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}[f_k(\text{conv}(\mathcal{P}_\lambda))]}{\log^{d-1}(\lambda)} = F_{k,d} f_0(K).$$

*There exists a constant  $c_{V_d} \in (0, \infty)$  such that*

$$\lim_{\lambda \rightarrow \infty} \frac{\lambda^2 \text{Var}[V_d(\text{conv}(\mathcal{P}_\lambda))]}{\log^{d-1}(\lambda)} = c_{V_d} f_0(K).$$

The proof relies on the one hand on the techniques of scaling and stabilization from [24] and on the other hand on the techniques of economic cap covering and sandwiching introduced by Bárány and Larman [10] and Bárány and Reitzner [11] respectively. This strategy is the backbone of the proofs of the results of Chapter 4 where we partly extend Theorem 1.3.8 to the subsequent layers of the convex hull peeling.

**Scaling transform and stabilization.** As in the case of the unit ball, we would like to use a rescaling in order to obtain a model that is easier to analyze and use stabilization results for deriving the asymptotics. This strategy is not as easily applicable here. The main problem is that there is no global rescaling that fits the situation. However, it is possible to use this strategy when we restrict ourselves to a region close to a vertex of  $K$ .

As the polytope  $K$  is assumed to be simple, everything works in the same way near each vertex and we can assume that this vertex is 0 and that  $Q_0 = [0, \delta_0]^d$  is included in  $K$  for some  $\delta_0$  that goes to 0 with  $\lambda$ . Let  $V = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \sum_i x_i = 0\}$ . For  $v \in V$ , we denote by  $l_i(v)$  the  $i$ -th coordinate of  $v$  in the standard basis with respect to  $\mathbb{R}^d$  and  $l(v) := (l_1(v), \dots, l_d(v))$ . We denote by  $p_V$  the orthogonal projection onto  $V$  and for any function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and  $x \in \mathbb{R}^d$  we write  $f(x) := (f(x_1), \dots, f(x_d)) \in \mathbb{R}^d$ . Now we define the transformation  $T^{(\lambda)}$  as

$$T^{(\lambda)} : \begin{cases} (0, \infty)^d & \longrightarrow V \times \mathbb{R} \\ (z_1, \dots, z_d) & \longmapsto (p_V(\log(z)), \frac{1}{d}(\log(\lambda) + \sum_i \log(z_i))) \end{cases} \quad (1.11)$$

It is possible to prove that  $\mathcal{P}_\lambda \cap Q_0$  is mapped by  $T^{(\lambda)}$  to  $\mathcal{P}^{(\lambda)} = \mathcal{P} \cap W_\lambda$  where  $\mathcal{P}$  is a Poisson point process with intensity  $\sqrt{d}e^{dh}dv dh$  on  $\mathbb{R}^{d-1} \times \mathbb{R}$  and where

$$W_\lambda := \{(v, h) \in V \times \mathbb{R} : -l_i(v) + \log^{1/d}(\lambda)\delta_0, 1 \leq i \leq d\}.$$

Moreover, on the event of the sandwiching from Theorem 1.3.5,  $T^{(\lambda)}$  maps the facets to portions of cone-like grains  $\Pi^{(\lambda)}(v, h)$  which converge to an explicit limit shape when  $\lambda$  goes to infinity. Even if the situation differs from the unit ball case in the intensity of the rescaled point process and in the replacement of paraboloids by cone-like grains, the ideas of the case of the unit ball are still working up to technical modifications. In particular, we can prove stabilization results and obtain a limit for the variance of the sum of the scores in  $\mathcal{P}_\lambda \cap Q_0$ , which is  $c \log^{d-1}(\lambda)$ . Two problems remain, i.e. we need to prove that the variance is additive over the vertices of  $K$  and that the contribution of the Poisson points far from the vertices is negligible.

**Flat part.** Another consequence of Theorem 1.3.5 on the sandwiching is that only points included in the region between two floating bodies actually matter for the construction of the convex hull. The part of this region that is far from the vertices is what we call the flat part. Careful estimations of the volume of the flat part show that the number of points in the flat part and a

combinatorial result on the maximal number of faces of the convex hull of  $n$  points [54] show that the number of faces in the flat part is of order  $(\log \log(\lambda))^c$  for some  $c > 0$  with high probability. As a consequence, the variance coming from the flat part is negligible compared to the order of magnitude of  $\log^{d-1}(\lambda)$  faces close to each vertex.

**Additivity of the scores close to a vertex.** The last missing ingredient is the additivity of the variances of the scores over each of the vertices. Since this is undoubtedly the most technical step of the proof, we will not delve too much into the details for the time being. The idea, borrowed from [11], consists roughly in using the Macbeath regions that appear in the economic cap covering theorem, see Theorem 1.3.4, to partition the sandwich of Theorem 1.3.5. We call supersets the elements of the partition, as they are sets that contain a half-Macbeath region of the economic cap covering. These supersets are the basis for building a so-called dependency graph used for showing the CLT in [11]. The novelty introduced in [27] is the explicit calculation of the Macbeath regions of the economic cap covering in a fixed cube that contains  $Q_0$ . This induces an explicit description of each superset and bounds for its diameter. In turn, the supersets in the vicinity of different vertices of  $K$  are proved to be disconnected in the dependency graph and consequently, their contribution in the sum of scores are independent. This explains why the variance is additive over the vertices of  $K$ .

Combining the three ingredients above in [27], they obtain subsequently an explicit limiting variance of size  $c \log^{d-1}(\lambda)$  for the sum of the scores in the vicinity of each vertex  $K$ , then the negligibility of the contribution of the part of the convex hull far from the vertices of  $K$  and finally the additivity of the variance over the vertices.

### 1.3.4 Bounds in a general convex body

In this subsection, we aim at stating general bounds for the mean number of  $k$ -faces when  $K$  is any convex body. It is proved that the two previous cases, smooth and polytope, yield the two extreme cases for the expected number of  $k$ -faces. In other words, the case when  $K$  is a polytope (resp. is smooth) corresponds to the minimal (resp. maximal) mean number up to a multiplicative constant. This result is interesting in itself and is also used by Dalal to find a lower bound for the expected number of layers in the convex hull peeling of i.i.d. random points, see Lemma 2.2.1.

We state below a partial version of [1, Corollary 3].

**Theorem 1.3.9.** *There exists constants  $c_1, c_2 > 0$  such that for any convex body  $K$  in  $\mathbb{R}^d$  and for any  $k \in \{0, \dots, d-1\}$ , writing  $K_n$  for the convex hull of  $n$  i.i.d. uniformly distributed points in  $K$  we have*

$$c_1 \log^{d-1}(n) \leq \mathbb{E}[f_k(K_n)] \leq c_2 n^{(d-1)/(d+1)}.$$

## Chapter 2

# Convex hull peeling

This chapter presents the central object of study in this thesis, namely the so-called *convex hull peeling*. Its iterative construction extends the previously described models of random polytopes: we start by taking the convex hull of the whole point process and then repeatedly take the convex hull of the points that were not extreme at the previous step until no point remains. The boundary of the convex hull obtained at step  $n$  is called the  $n$ -th layer.

The convex hull peeling was first introduced by Barnett in [13] as a way to order multivariate data and give a meaning to how central a point is with respect to a dataset. Indeed, the layer number of a point can be interpreted as the *depth* of that point with respect to the input and we expect it to be all the larger, the more central the point is. The convex hull peeling has then been used in robust statistics and outlier detection, see [38, 47, 66]. It fits into a list of classical techniques for ordering multivariate data, including half-space depth, simplicial depth or zonoid depth, see e.g. [30] for an overview on these techniques.

The theoretical literature on the topic is scarce. Computer scientists have notably investigated the complexity of the peeling in a deterministic setting. In particular, Chazelle [31] provides an optimal algorithm for any set of  $n$  points in dimension 2 to compute the complete set of layers of the convex hull peeling. It runs in  $O(n \log(n))$  in time and  $O(n)$  in space. As far as we know, there is no extension of this algorithm to any dimension. When the points are placed on the integer  $n \times n$  grid, Har-Peled and Lidický [45] prove that the number of layers is bounded from below and from above by  $n^{4/3}$  up to a multiplicative constant.

In this thesis, we consider the convex hull peeling of a random set and investigate several random variables, including the total number of layers or the limit shape and some geometric and combinatorial functionals of a fixed layer.

In this context, one of the first works that we are aware of is due to Davydov, Nagaev and Philippe [36]. In the spirit of stable random vectors, they opt for a Poisson input with an intensity measure which explodes at the origin, i.e. of type  $\theta \times \nu$  in spherical coordinates where  $\theta$  is a measure on  $(0, \infty)$  with density  $\alpha r^{-\alpha-1}$  for fixed  $\alpha > 0$  and  $\nu$  is a measure on the unit sphere of  $\mathbb{R}^d$  with a finite support which generates all of  $\mathbb{R}^d$ . This setting guarantees that the number of layers is infinite almost surely and they exhibit a limit shape for the  $n$ -th layer when  $n$  goes to infinity which is described as the convex hull of the renormalized vectors in the support of  $\nu$ .

In the more classical case of a Poisson point process with bounded intensity, which includes in particular the uniform input in a fixed convex body that we consider in Chapters 3 and 4, we

describe below two of the main existing papers. Before doing so, we start in a first section with a few general properties of the deterministic convex hull peeling, including the monotonicity of the total number of layers with respect to the inclusion.

The next section is devoted to Dalal's work [33] which shows that the total number of layers of  $n$  i.i.d. points uniformly distributed in some bounded region of  $\mathbb{R}^d$  behaves like  $n^{2/(d+1)}$  up to a multiplicative constant. We propose here a slightly modified account of his results. We rewrite a Poissonized version of his original proof, i.e. we replace the input by the Poisson point process with intensity  $\lambda$  times the Lebesgue measure in the bounded region. We also take this opportunity to show that Dalal's method leads to an original extra result, i.e. the uniform integrability of the sequence of renormalized numbers of layers for  $\lambda > 0$ .

In the last section, we focus on the very recent work due to Calder and Smart which extends Dalal's result in multiple ways. In particular, they consider the peeling of a general Poisson point process in a convex body and adopt a functional point of view. In other words, they interpret the total number of layers as the maximum of a so-called convex height function which associates to any point the number of consecutive convex hulls containing that point and they show that the renormalized convex height function converges uniformly almost surely to the viscosity solution of a certain non-linear PDE.

## 2.1 Definitions, general properties

In this section,  $X$  denotes a finite closed set of  $\mathbb{R}^d$ .

**Definition 2.1.1** (Convex hull peeling, layers). We write  $\text{conv}_1(X) := \text{conv}(X)$  and by induction for any  $n \geq 1$ ,  $\text{conv}_{n+1}(X) := \text{conv}(\text{int}(\text{conv}_n(X)) \cap X)$ . The boundary  $\partial\text{conv}_n(X)$  of the  $n$ -th convex hull is called the  $n$ -th *layer* of the convex hull peeling of  $X$ .

The words *peeling* and *layers* were chosen by analogy with the peeling of an onion. We often abbreviate *convex hull peeling* to only *peeling*. For any  $x \in \mathbb{R}^d$  we write

$$\ell_X(x) := \max\{n : x \in \text{conv}_n(X \cup \{x\})\}.$$

We call  $\ell_X(x)$  the layer number of the point  $x$  with respect to the peeling of  $X$ .

We provide below a few properties of the convex hull peeling, namely the invariance under affine transformation, a suitable criterion for being on the  $n$ -th layer and the monotonicity of the layer number. All of them will be continuously used in the sequel.

As affine transformations preserve the convexity of sets, it also preserves the convex hull peeling in the following sense.

**Proposition 2.1.2.** *For any affine transformation  $f$  and any  $n \geq 1$ , we have  $f(\text{conv}_n(X)) = \text{conv}_n(f(X))$ .*

One of the main issues when studying the convex hull peeling is that we do not know of an easy criterion for determining the layer number of a point without the knowledge of all the preceding layers. The only criterion that we propose below is recursive. Incidentally, this explains why we need to use an induction reasoning in Chapters 3 and 4 when proving stabilization.

We start by recalling the base case, namely the criterion for being included in the boundary of the first convex hull. For any unit vector  $u \in \mathbb{S}^{d-1}$  and any point  $x \in \mathbb{R}^d$ , we introduce the following half-space  $H_{x,u}^+ = \{y \in \mathbb{R}^d : \langle y - x, u \rangle > 0\}$ .

**Lemma 2.1.3.** *Let  $x \in \mathbb{R}^d$ , we have  $(\ell_X(x) = 1) \iff (\exists u \in \mathbb{S}^{d-1} \text{ s.t. } X \cap H_{x,u}^+ = \emptyset)$ .*

Below, we extend recursively this criterion to the subsequent layers.

**Lemma 2.1.4.** *Let  $x \in \mathbb{R}^d$  and  $n \geq 1$ . Then we have the two following equivalences.*

(i)  $(\ell_X(x) \geq n) \iff (\forall u \in \mathbb{S}^{d-1}, X \cap H_{x,u}^+ \not\subset \cup_{i=1}^{n-2} \partial \text{conv}_i(X))$ .

(ii)  $(\ell_X(x) \leq n) \iff (\exists u \in \mathbb{S}^{d-1} \text{ s.t. } X \cap H_{x,u}^+ \subset \cup_{i=1}^{n-1} \partial \text{conv}_i(X))$ .

*Proof.* (i) Let us assume that  $\ell_X(x) \geq n$  and fix  $u \in \mathbb{S}^{d-1}$ . If we had  $X \cap H_{x,u}^+ \subset \cup_{i=1}^{n-2} \partial \text{conv}_i(X)$ , this half-space would not intersect  $X$  after the removal of the first  $(n-2)$  layers, which means that the layer number of  $x$  would be at most  $(n-1)$  according to Lemma 2.1.3. Consequently, the first implication in (i) holds.

Conversely, let us assume that  $X \cap H_{x,u}^+ \not\subset \cup_{i=1}^{n-2} \partial \text{conv}_i(X)$  for any  $u \in \mathbb{S}^{d-1}$ . Then once the first  $(n-2)$  layers are removed,  $H_{x,u}^+$  has to meet the remainder of  $X$  for any  $u \in \mathbb{S}^{d-1}$ . Thanks to Lemma 2.1.3, this implies that  $x$  is not an extreme point of  $X \setminus (\cup_{i=1}^{n-2} \partial \text{conv}_i(X))$  and thus  $\ell_X(x) \geq n$ .

(ii) If  $\ell_X(x) = m \leq n$ ,  $x$  is extreme when we remove the first  $(m-1)$  layers. Thus, let  $u \in \mathbb{S}^{d-1}$  be such that  $H_{x,u}^+$  does not contain any point of  $X$  after the removal of the first  $(m-1)$  layers. This implies that

$$X \cap H_{x,u}^+ \subset \cup_{i=1}^{m-1} \partial \text{conv}_i(X) \subset \cup_{i=1}^{n-1} \partial \text{conv}_i(X).$$

Conversely, if we assume that  $X \cap H_{x,u}^+ \subset \cup_{i=1}^{n-1} \partial \text{conv}_i(X)$ , for some  $u \in \mathbb{S}^{d-1}$ , either  $x$  belongs to the first  $(n-1)$  layers or it is extreme once the first  $(n-1)$  layers are removed because of Lemma 2.1.3. Thus  $\ell_X(x) \leq n$ .  $\square$

The following result means that the layer number of any point is increasing with  $X$ . In [33], Dalal proves the monotonicity of the total number of layers and the proof of Proposition 2.1.5 is strongly inspired by his ideas. Calder and Smart in [22] also state a similar fact but in the context of a rescaled parabolic model, see Section 2.3.

**Proposition 2.1.5.** *If  $x \in \mathbb{R}^d$  and  $X \subset Y$  are two finite closed sets of  $\mathbb{R}^d$  then*

$$\ell_X(x) \leq \ell_Y(x) \leq \ell_X(x) + \text{card}(Y \setminus X).$$

*Proof.* Without loss of generality, we can assume that  $Y = X \cup \{z\}$  where  $z \notin X$ . Let  $L(x)$  be the  $\ell_X(x)$ -th convex hull of the convex hull peeling of  $X \cup \{x\}$ . If  $z$  belongs to  $L(x)$ , then  $\ell_{X \cup \{z\}}(x)$  is equal to  $\ell_X(x)$ . We now assume that  $z$  is outside of  $L(x)$ , meaning that  $\ell_{X \cup \{z\}}(z) \leq \ell_X(x)$ . Denoting by  $n$  the layer number  $\ell_{X \cup \{z\}}(z)$ , we notice that  $z$  belongs to  $\partial \text{conv}_n(X \cup \{x, z\})$ . Moreover,  $\partial \text{conv}_n(X \cup \{x, z\}) \cap (X \cup \{x\}) \subset \partial \text{conv}_n(X \cup \{x\})$  and the points from  $X \cup \{x\}$  in  $\partial \text{conv}_n(X \cup \{x, z\}) \setminus \partial \text{conv}_n(X \cup \{x\})$  will belong to the next layer, i.e.  $\partial \text{conv}_{n+1}(X \cup \{x, z\})$ . By induction, this implies that the layer number of each point of  $X \cup \{x\}$  in  $\text{conv}_{n+1}(X \cup \{x, z\})$  is either unchanged or increased by one. We apply this remark to the particular point  $x$ .  $\square$

In particular the maximum of the function  $\ell_X$  over  $\mathbb{R}^d$ , denoted by  $\max \ell_X$ , is either the total number of layers of the convex hull peeling of  $X$  or that quantity plus one. This subtlety is obviously irrelevant in the asymptotic context that we consider further on.

## 2.2 Number of layers

The goal of this section is to present the known asymptotic results about the total number of layers of the convex hull peeling. These results come from [33]. However we present here a Poissonized version of these results as it is more consistent with the results of the upcoming section and chapters. We also give a slightly enhanced version of the upper bound of Dalal that implies that the renormalized number of layers is uniformly integrable. The lower bound is going to be improved similarly but is not needed in the sequel. This refinement is new, even if it does not change much of the proof and is required to justify properly Corollary 2.3.4 in the next section. In this subsection we write  $\ell_\lambda$  for  $\ell_{\mathcal{P}_\lambda}$ , where  $\mathcal{P}_\lambda$  is a homogeneous Poisson point process that is going to be restricted to a subset of  $\mathbb{R}^d$  depending on the context.

### 2.2.1 Lower bound

We start with a lower bound for the number of layers in any ball of  $\mathbb{R}^d$ . This is a Poissonized version of [33, Lemma 4.1]. Roughly, the idea in the case of uniform i.i.d. points is that, thanks to Theorem 1.3.9, each layer can have at most  $cn^{(d-1)/(d+1)}$  points and thus there must be at least  $cn^{2/(d+1)}$  layer to recover all of the  $n$  points. Even though there is no new idea in the Poissonized version, the proof is a bit more technical as we need to condition on the number of points of  $\mathcal{P}_\lambda$  to use Theorem 1.3.9 properly.

**Lemma 2.2.1.** *Let  $\mathcal{P}_\lambda$  be a Poisson point process with intensity  $\lambda$  times the Lebesgue measure in a  $d$ -dimensional ball and  $p \geq 1$ . Then*

$$\mathbb{E}[(\max \ell_\lambda)^p] = \Omega(\lambda^{\frac{2p}{d+1}})$$

where  $f(\lambda) = \Omega(g(\lambda))$  means that for  $\lambda$  large enough,  $f(\lambda) \geq cg(\lambda)$  for some positive constant  $c$ .

*Proof.* It is enough to consider the case  $\text{Vol}_d(K) = 1$  as the general case is deduced from it by replacing  $\lambda$  with  $\lambda \text{Vol}_d(K)$  everywhere.

Let us write  $n_i := \text{card}(\partial \text{conv}_i(\mathcal{P}_\lambda))$ . We recall that the sequence  $(n_i)_{i \geq 1}$  is eventually zero. Let us estimate  $\mathbb{P}(\max \ell_\lambda < t)$  for any constant  $t$ . First we have

$$\mathbb{P}(\max \ell_\lambda < t) \leq \mathbb{P}(\max \ell_\lambda \leq t) = \mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda)\right).$$

Then we condition on the possible values of  $\mathcal{P}_\lambda$  and split the sum with the idea that we are interested in the part where  $\text{card}(\mathcal{P}_\lambda)$  is not too far from  $\lambda$  and that the other parts are small because of the

concentration of a Poisson variable close to its expectation.

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda)\right) &= \sum_{p=0}^{\infty} \mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda), \text{card}(\mathcal{P}_\lambda) = p\right) \\
&= \sum_{p=0}^{\lceil \lambda/2 \rceil} \mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda), \text{card}(\mathcal{P}_\lambda) = p\right) \\
&\quad + \sum_{\lceil \lambda/2 \rceil + 1}^{2\lceil \lambda \rceil} \mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda), \text{card}(\mathcal{P}_\lambda) = p\right) \\
&\quad + \sum_{2\lceil \lambda \rceil + 1}^{\infty} \mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda), \text{card}(\mathcal{P}_\lambda) = p\right).
\end{aligned}$$

Using Proposition 1.1.12, the first and third term in the right hand side are upper bounded by  $c_1 \exp(-c_2 \lambda)$  so we deduce

$$\mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda)\right) \leq \sum_{\lceil \lambda/2 \rceil + 1}^{2\lceil \lambda \rceil} \mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda), \text{card}(\mathcal{P}_\lambda) = p\right) + c_1 \exp(-c_2 \lambda). \quad (2.1)$$

Now we try to find an upper bound for first term in the right hand side of this inequality.

Let us write  $n_{i,p}$  for the number of points on layer  $i$  in the convex hull peeling of  $p$  i.i.d. uniformly distributed points in  $K$ . Using Markov's inequality and the fact that conditional on  $\text{card}(\mathcal{P}_\lambda) = p$ ,  $\mathcal{P}_\lambda$  has the distribution of  $p$  i.i.d. uniformly distributed points in  $K$ , see Proposition 1.1.9, we get for  $i$  between  $\lfloor \lambda/2 \rfloor$  and  $2\lceil \lambda \rceil$

$$\begin{aligned}
\mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda), \text{card}(\mathcal{P}_\lambda) = p\right) &= \mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_{i,p} \geq p\right) \mathbb{P}(\text{card}(\mathcal{P}_\lambda) = p) \\
&\leq \frac{c}{\lambda} \sum_{i=1}^{\lfloor t \rfloor} \mathbb{E}[n_{i,p}] \mathbb{P}(\text{card}(\mathcal{P}_\lambda) = p).
\end{aligned}$$

Conditional on the construction of the  $(i-1)$  first layers, the remaining points are i.i.d. uniformly distributed in the  $i^{\text{th}}$  convex hull which, combined with Theorem 1.3.9, implies that  $\mathbb{E}[n_{i,p}] \leq c\lambda^{(d-1)/(d+1)}$ . Thus

$$\mathbb{P}\left(\sum_{i=1}^{\lfloor t \rfloor} n_i \geq \text{card}(\mathcal{P}_\lambda), \text{card}(\mathcal{P}_\lambda) = p\right) \leq \frac{ct}{\lambda} \lambda^{(d-1)/(d+1)} \mathbb{P}(\text{card}(\mathcal{P}_\lambda) = p). \quad (2.2)$$

Inserting this into (2.1), we obtain in the end

$$\mathbb{P}(\max \ell_\lambda < t) \leq \frac{cdt}{\lambda} \lambda^{(d-1)/(d+1)}.$$

Consequently, Markov's inequality gives

$$\mathbb{E}[\max \ell_\lambda] \geq t \mathbb{P}(\max \ell_\lambda \geq t) \geq t \left(1 - \frac{cdt}{\lambda} \lambda^{(d-1)/(d+1)}\right).$$

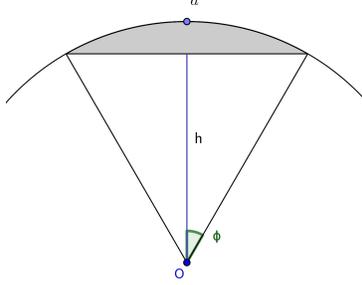


Figure 2.1 – The area in grey is the cap  $C_\varphi(a)$  with center  $a$  and angle  $\varphi$ ,  $h$  is the height of the cap.

The particular choice  $t = \frac{\lambda}{2c_d} \lambda^{-(d-1)/(d+1)}$  yields

$$\mathbb{E}[\max \ell_\lambda] \geq t/2 \geq c\lambda^{2/(d+1)}.$$

□

With the previous lemma we are able to provide an asymptotic lower bound for the total number of layers in the convex hull peeling of any Poisson point process with intensity measure  $\lambda dx$  in a bounded region  $A$  of  $\mathbb{R}^d$  with a non empty interior. The region  $A$  does not even need to be convex.

**Proposition 2.2.2.** *Let  $\mathcal{P}_\lambda$  be a Poisson point process with intensity  $\lambda$  times the Lebesgue measure in a bounded region  $A$  of  $\mathbb{R}^d$  with a non empty interior and let  $p \geq 1$ . Then*

$$\mathbb{E}[(\max \ell_\lambda)^p] = \Omega(\lambda^{2p/(d+1)}).$$

*Proof.* Let  $B$  be a ball included in  $A$ , which is possible because  $A$  has non empty interior. As  $B \subset A$ , Proposition 2.1.5 implies that

$$\max \ell_\lambda = \max \ell_{\mathcal{P}_\lambda} \geq \max \ell_{\mathcal{P}_\lambda \cap B}.$$

Now Lemma 2.2.1 tells us that  $\mathbb{E}[(\max \ell_{\mathcal{P}_\lambda \cap B})^p] = \Omega(\lambda^{2p/(d+1)})$ . combining these two facts yields the result. □

## 2.2.2 Upper bound

We now provide an upper bound for the total number of layers to the power  $p \geq 1$ . This is a much more challenging task. We mostly follow the proof of [33, Theorem 5.1] with some adjustments to deal with the power  $p$ . Beforehand, let us introduce a notation for the caps that are used in the proof. We denote by  $R$  the radius of the ball of volume 1 and by  $C_\varphi(a)$  the spherical cap of  $B(R)$  with center  $a$  and angle  $\varphi$ , see Figure 2.1.

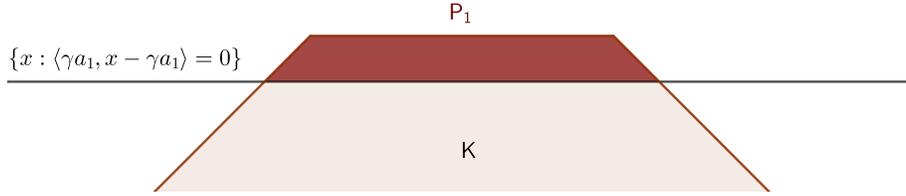


Figure 2.2 –  $P_1$  for  $m = 8$ . We have taken  $\gamma = 1 - 1/m$  instead of  $1 - 1/m^2$  to make the figure clearer.

**Proposition 2.2.3.** *Let  $\mathcal{P}_\lambda$  be a Poisson point process with intensity measure  $\lambda$  times the Lebesgue measure in a closed ball  $B(x, r)$  and  $p \geq 1$ . We have*

$$\mathbb{E}[(\max \ell_\lambda)^p] = O(\lambda^{2p/(d+1)})$$

where  $f(\lambda) = O(g(\lambda))$  means that for  $\lambda$  large enough,  $f(\lambda) \leq cg(\lambda)$  for some positive constant  $c$ .

*Proof.* By translation invariance and scaling invariance, see Proposition 2.1.2, we can prove the result in the case of the unit volume ball centered at 0 and of radius  $R$  without loss of generality. Thanks to Propositions 1.1.11 and 2.1.5, it is enough to show the estimate when  $m := \lambda^{1/(d+1)}$  is an integer.

Let us consider the packing number

$$N := \max\{p \in \mathbb{N} : \exists a_1, \dots, a_p \in \mathbb{R}\mathbb{S}^{d-1} \text{ s.t. } C_\varphi(a_i) \cap C_\varphi(a_j) = \emptyset \forall i \neq j\} \quad (2.3)$$

where  $\varphi := \frac{1}{m}$ . We denote by  $a_1, \dots, a_N$  the points in  $\mathbb{R}\mathbb{S}^{d-1}$  which realize the optimal packing. We then define the polytope  $K$  induced by the polar hyperplanes at points  $a_1, \dots, a_N$ , i.e.

$$K := \bigcap_{i=1}^N \{x \in \mathbb{R}^d : \langle a_i, x - a_i \rangle \leq 0\}.$$

This polytope which includes  $\mathbb{R}\mathbb{B}^d$  will be the base of our construction. Since  $N$  is an increasing function of  $\lambda$ , we assert that  $K$  is a suitable approximation of  $\mathbb{R}\mathbb{B}^d$ . For any  $1 \leq i \leq N$  we define

$$P_i := K \cap \{x : \langle \gamma a_i, x - \gamma a_i \rangle \geq 0\}$$

where  $\gamma := 1 - \frac{1}{m^2}$ , see Figure 2.2.

We now define homothetic copies of  $K$  in the following way. Let  $K_0 := K$  and for  $p = 0, \dots, m^2$ , let  $K_k := \gamma^k K_0$ . Similarly, we define homothetic copies of  $P_i$ , i.e.  $P(i, k) := \gamma^{k-1} P_i$  for all  $i = 1, \dots, N$  and  $k = 1, \dots, m^2$ . Finally, for  $i = 1, \dots, N$  and  $k \in 0, \dots, m^2$ , we introduce the half-spaces

$$H(i, k) := \{x : \langle \gamma^k a_i, x - \gamma^k a_i \rangle > 0\}.$$

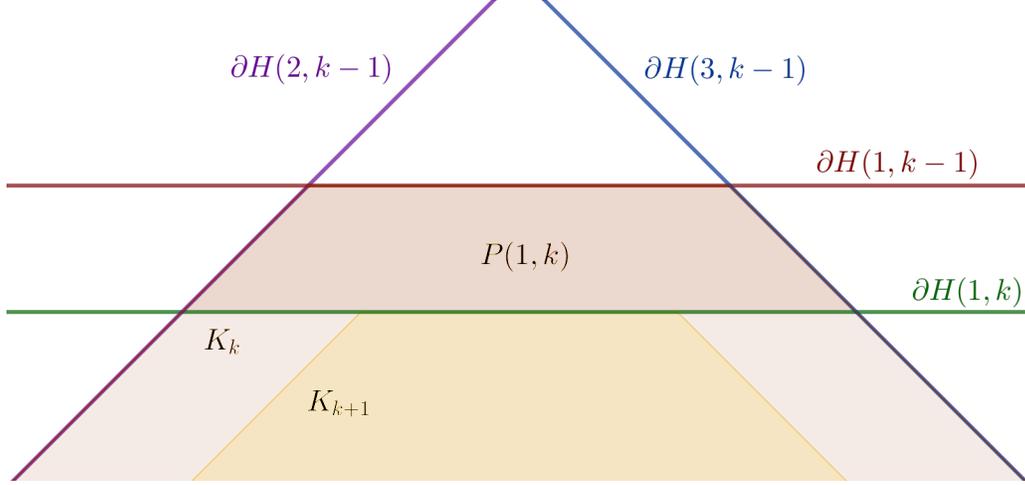


Figure 2.3 – Picture of  $P(i, k)$  and the hyperplanes defining its facets.

In particular we have

$$K_k = \bigcap_{i=1}^N H(i, k)^c.$$

Our next goal is to give an upper bound for the maximum layer number of a point outside of  $K_{m^2}$ . The following observation is essential to the whole reasoning. Let us fix  $k \geq 1$  and  $1 \leq i \leq N$ . We denote by  $J$  the set of  $j$  such that  $H(j, k-1)$  supports a facet of  $P(i, k)$ . We have

$$H(i, k) \subset P(i, k) \cup \bigcup_{j \in J} H(j, k-1). \quad (2.4)$$

This can be seen on Figure 2.3. Let  $d_{\max} := \max_{\mathcal{P}_\lambda \cap (\cup_{j \in J} H(j, k-1))} \ell_\lambda$ . If we remove the  $d_{\max}$  first layers, in particular we remove every point in  $\bigcup_{j \in J} H(j, k-1)$  and  $\max_{\mathcal{P}_\lambda \cap P(i, k)} \ell_\lambda$  becomes smaller than  $\text{card}(P(i, k) \cap \mathcal{P}_\lambda)$ . We deduce that

$$\max_{\mathcal{P}_\lambda \cap P(i, k)} \ell_\lambda \leq \text{card}(\mathcal{P}_\lambda \cap P(i, k)) + \max_{\mathcal{P}_\lambda \cap (\cup_{j \in J} H(j, k-1))} \ell_\lambda$$

and it follows from (2.4) that

$$\max_{\mathcal{P}_\lambda \cap H(i, k)} \ell_\lambda \leq \text{card}(\mathcal{P}_\lambda \cap P(i, k)) + \max_{j \in J} \max_{\mathcal{P}_\lambda \cap H(j, k-1)} \ell_\lambda. \quad (2.5)$$

We now iterate the inequality (2.5). If the max in the right hands side of (2.5) is reached for some  $i_1 \in J$ , we denote by  $J_1$  the  $J$ -set associated with  $i_1$ , i.e. the set of  $j$  such that  $H(j, k-2)$  supports a facet of  $P(i_1, k-1)$ . Applying twice (2.5) to  $i$  then to  $i_1$ , we deduce that

$$\max_{\mathcal{P}_\lambda \cap H(i, k)} \ell_\lambda \leq \text{card}(P(i, k) \cap \mathcal{P}_\lambda) + \text{card}(P(i_1, k-1) \cap \mathcal{P}_\lambda) + \max_{j \in J_1} \max_{\mathcal{P}_\lambda \cap H(j, k-2)} \ell_\lambda.$$

It is in fact possible to use this formula recursively. Let  $G$  be the directed acyclic graph whose vertices are the sets  $P(i, k)$  for  $i = 1, \dots, N$  and  $k = 1, \dots, m^2$  and such that there is an edge going

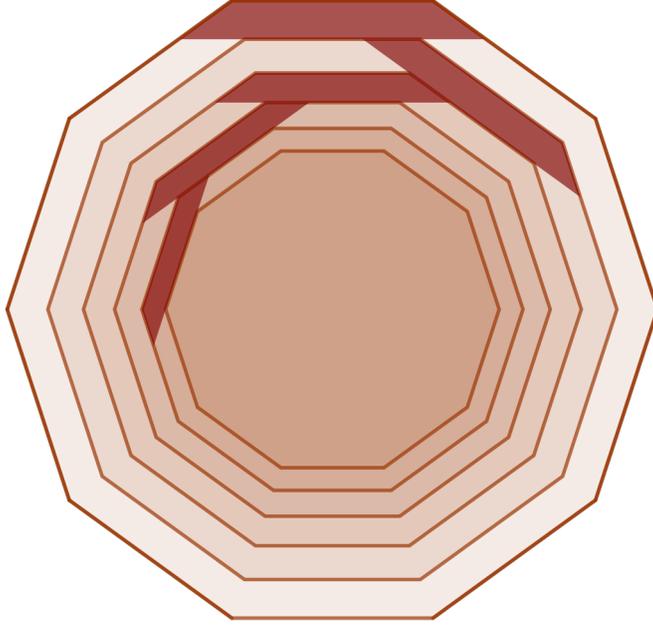


Figure 2.4 – The sequence of polygons in dark red from the innermost one to the outermost one is a path in  $G$ .

from  $P(i, k)$  to  $P(j, k - 1)$  if and only if a facet of  $P(i, k)$  and a facet of  $P(j, k - 1)$  have a non empty intersection, see Figure 2.4 for an example of path in  $G$ .

Let us write  $p_{\text{ext}} := \max_{\mathcal{P}_\lambda \cap (R\mathbb{B}^d \setminus K_{m^2})} \ell_\lambda$ . We have  $p_{\text{ext}} = \max_{i=1, \dots, N} \max_{\mathcal{P}_\lambda \cap H(i, m^2)} \ell_\lambda$ . Using (2.5) inductively, we get that  $p_{\text{ext}}$  is bounded from above by the maximum number of points contained in the vertices of  $G$  along any path starting from a vertex of type  $P(i, m^2)$  to a vertex of type  $P(j, 1)$ .

We now aim to make explicit the previous upper bound of  $p_{\text{ext}}$ . This requires in particular to estimate the maximal number of Poisson points along a path in the graph  $G$ . To do so, we need four geometric claims. Claim 1 provides an estimate for the packing number introduced at (2.3). In Claim 2, we sandwich  $K$  in an annulus between  $R\mathbb{B}^d$  and  $\frac{R}{\cos(2\varphi)}\mathbb{B}^d$  while Claims 3 and 4 deal with the geometry of the thin polytopes  $P_i$ . The proofs of these 4 claims are omitted.

**Claim 1.** There exists a constant  $c_a$  independent from  $\lambda$  such that the number  $N$  of facets of  $K$  is at most  $c_a m^{d-1}$

**Claim 2.**  $K \subset \frac{R}{\cos(2\varphi)}\mathbb{B}^d$ .

**Claim 3.** There exists a constant  $c_{\text{vol}}$  that does not depend on  $\lambda$  such that for each  $i$

$$\text{Vol}_d(P_i) \leq c_{\text{vol}}/\lambda.$$

**Claim 4.** There exists a constant  $c_{\text{fac}}$  that does not depend on  $\lambda$  such that for each  $i$  the number of facets of  $P_i$  is at most  $c_{\text{fac}}$ .

We now show that the probability that there exists a path in  $G$  with more than a constant times  $m^2$  points is exponentially small. This will allow us to get a bound on the  $p^{\text{th}}$  moment of the maximum layer number outside of  $K_{m^2}$  denoted by  $p_{\text{ext}}$ .

We will call a vertex of  $G$  of the form  $P(i, m^2)$  a root and a vertex of the form  $P(j, 1)$  a leaf. In what follows we will just say path for a path from a root to a leaf.

The number of points in a path is Poisson distributed with mean the sum of the volume along the path times  $\lambda$ . From Claim 3, we get that the sum of the volumes of the vertices along a path is at most  $c_{\text{vol}}m^2/n$ . Consequently, we get that for any  $\delta > 0$  and any path, the probability that this path has more than  $(1 + \delta)c_{\text{vol}}m^2$  points is at most  $\mathbb{P}(X \geq (1 + \delta)c_{\text{vol}}m^2)$  for  $X \sim \mathcal{P}(c_{\text{vol}}m^2)$ . Using the concentration inequality for the Poisson distribution given in Proposition 1.1.12, we obtain

$$\mathbb{P}(X \geq (1 + \delta)c_{\text{vol}}m^2) \leq \exp\left(-\frac{\delta^2(c_{\text{vol}})^2}{2(1 + \delta)}m^2\right). \quad (2.6)$$

Moreover, the number of paths in the directed acyclic graph  $G$  is at most the number of roots times the maximum degree in  $G$  at the power  $m^2$  since our paths are of length  $m^2$ . The number of roots is bounded by Claim 1 and the maximum degree is at most  $c_{\text{fac}}$  by Claim 4. Thus, there is a positive constant  $c > 0$  such that the number of paths in  $G$  is at most

$$Nc_{\text{fac}}^{m^2} \leq \exp(cm^2). \quad (2.7)$$

Let us denote by  $A$  the event ‘there exists a path with more than  $(1 + \delta)c_{\text{vol}}m^2$  points’. Using (2.6) and (2.7), we get that

$$\mathbb{P}(A) \leq \exp\left(cm^2 - \frac{\delta^2(c_{\text{vol}})^2}{2(1 + \delta)}m^2\right). \quad (2.8)$$

Using Minkowski’s inequality, we get

$$(\mathbb{E}[p_{\text{ext}}^p])^{1/p} = (\mathbb{E}[(p_{\text{ext}}\mathbf{1}_A + p_{\text{ext}}\mathbf{1}_{A^c})^p])^{1/p} \leq (\mathbb{E}[(p_{\text{ext}}\mathbf{1}_A)^p])^{1/p} + (\mathbb{E}[(p_{\text{ext}}\mathbf{1}_{A^c})^p])^{1/p} \quad (2.9)$$

On the event  $A$ , we bound  $p_{\text{ext}}$  by the number of points of the process, which is Poisson distributed with mean  $\lambda$ . Its moment of order  $2\lceil p \rceil$  is a polynomial  $Q_p$  in  $\lambda$ . Using the Cauchy-Schwartz inequality and recalling that  $m = \lambda^{1/(d+1)}$ , we get

$$(\mathbb{E}[(p_{\text{ext}}\mathbf{1}_A)^p])^{1/p} \leq (\mathbb{E}[\text{card}(\mathcal{P}_\lambda)^p \mathbf{1}_A])^{1/p} \leq \left(\mathbb{E}[\text{card}(\mathcal{P}_\lambda)^{\lceil p \rceil} \mathbf{1}_A]\right)^{1/p} \leq (Q_p(\lambda)\mathbb{P}(A))^{1/(2p)}$$

Since (2.8) is satisfied for any  $\delta$ , we choose a suitable large enough  $\delta$  so that for any  $\lambda$  large enough,  $Q_p(\lambda)\mathbb{P}(A) \leq 1$ . We then have

$$(\mathbb{E}[(p_{\text{ext}}\mathbf{1}_A)^p])^{1/p} \leq 1. \quad (2.10)$$

On the event  $A^c$ , since  $p_{\text{ext}}$  is upper bounded by the maximum number of points in a path of  $G$ , we have

$$p_{\text{ext}} \leq (1 + \delta)c_{\text{vol}}m^2.$$

Recalling that  $m = \lambda^{1/(d+1)}$ , we deduce from the previous inequality that

$$(\mathbb{E}[(p_{\text{ext}}\mathbf{1}_{A^c})^p])^{1/p} \leq (1 + \delta)c_{\text{vol}}\lambda^{2/(d+1)}. \quad (2.11)$$

Inserting (2.10) and (2.11) into (2.9), we get the existence of a positive constant  $c$  such that for every  $\lambda$

$$\mathbb{E}[p_{ext}^p] \leq c\lambda^{2p/(d+1)}. \quad (2.12)$$

We now have an upper bound for the  $p^{\text{th}}$  moment of the maximum layer number outside of  $K_{m^2}$ . It remains to show a similar result for the whole ball  $R\mathbb{B}^d$ . We do so by applying the same reasoning recursively. Using Claim 2 and recalling  $\varphi = \frac{1}{m} = \lambda^{-1/(d+1)}$ , we obtain that for  $\lambda$  large enough

$$K_{m^2} \subset \left(1 - \frac{1}{m^2}\right)^{m^2} \frac{R}{\cos(2\varphi)} \mathbb{B}^d \subset \frac{2R}{e} \mathbb{B}^d.$$

Removing the points outside of  $\frac{2R}{e}\mathbb{B}^d$  cancels at most  $p_{ext}$  layers. This means that

$$\max_{R\mathbb{B}^d} \ell_\lambda \leq p_{ext} + \max_{\frac{2R}{e}\mathbb{B}^d} \ell_{\mathcal{P}_\lambda \cap \frac{2R}{e}\mathbb{B}^d}. \quad (2.13)$$

We now rescale  $\mathcal{P}_\lambda \cap 2R/e\mathbb{B}^d$  to get a Poisson point process on  $R\mathbb{B}^d$  with intensity only a fraction of the intensity of  $\mathcal{P}_\lambda$ . Indeed,  $\frac{e}{2}\mathcal{P}_\lambda \cap \frac{2R}{e}\mathbb{B}^d$  is a Poisson point process with intensity measure  $(2/e)^d$  times the Lebesgue measure in  $\frac{2R}{e}\mathbb{B}^d$ . Furthermore, by scaling invariance, see Proposition 2.1.2, the number of layers in the convex hull peeling of  $\mathcal{P}_\lambda \cap \frac{2R}{e}\mathbb{B}^d$  is the same as the number of layers in the convex hull peeling of  $\frac{e}{2}\mathcal{P}_\lambda \cap \frac{2R}{e}\mathbb{B}^d$ . Consequently, we get the stochastic domination

$$\max_{\frac{2R}{e}\mathbb{B}^d} \ell_{\mathcal{P}_\lambda \cap \frac{2R}{e}\mathbb{B}^d} \leq \ell_{\lceil(2/e)^d\rceil\lambda}. \quad (2.14)$$

For any  $\lambda$ , we write

$$D_p(\lambda) := \mathbb{E}[\max_{R\mathbb{B}^d} \ell_\lambda^p]^{1/p}. \quad (2.15)$$

Using successively (2.13), the Minkowski inequality, (2.14) and (2.12), we obtain

$$D_p(\lambda) \leq c\lambda^{2/(d+1)} + D_p(\lceil(2/e)^d\rceil\lambda). \quad (2.16)$$

By induction, we obtain the existence of a constant  $c > 0$  such that for any  $\lambda \in \mathbb{N}^*$

$$D_p(\lambda) \leq c\lambda^{2/(d+1)}.$$

Taking the power  $p$  yields the result.  $\square$

**Theorem 2.2.4.** *Let  $A$  be a compact subset of  $\mathbb{R}^d$  and  $\mathcal{P}_\lambda$  be a Poisson point process with intensity  $\lambda\mathbb{1}_A dx$ . There exists a constant  $c > 0$  such that for all  $\lambda > 0$  :*

$$\mathbb{E}[(\max \ell_\lambda)^p] \leq c\lambda^{2p/(d+1)}.$$

*Proof.* Let  $R > 0$  such that  $A \subset B(R)$ . For all  $\lambda$  let  $\tilde{\mathcal{P}}_\lambda$  be a Poisson point process with intensity  $\lambda\mathbb{1}_{B(R)} dx$ . Let  $\tilde{\ell}_\lambda := \ell_{\tilde{\mathcal{P}}_\lambda}$  and  $\tilde{\ell}_{\lambda,A} := \ell_{\tilde{\mathcal{P}}_\lambda \cap A}$

$\tilde{\mathcal{P}}_\lambda \cap A$  and  $\mathcal{P}_\lambda$  have the same distribution so

$$\mathbb{E}[(\max \ell_\lambda)^p] = \mathbb{E}[(\max \tilde{\ell}_{\lambda,A})^p]. \quad (2.17)$$

Since  $\tilde{\mathcal{P}}_\lambda \cap A \subset \tilde{\mathcal{P}}_\lambda$  Proposition 2.1.5 gives

$$\mathbb{E}[(\max \tilde{\ell}_{\lambda,A})^p] \leq \mathbb{E}[(\max \tilde{\ell}_\lambda)^p]. \quad (2.18)$$

Using jointly (2.17), (2.18) and applying Proposition 2.2.3 to  $\tilde{\mathcal{P}}_\lambda$  we get for all  $\lambda > 0$  :

$$\mathbb{E}[(\max \ell_\lambda)^p] = \mathbb{E}[(\max \tilde{\ell}_{\lambda,A})^p] \leq \mathbb{E}[(\max \tilde{\ell}_\lambda)^p] \leq c\lambda^{2p/(d+1)}.$$

□

The previous upper bound can actually be extended to point processes distributed as  $\mathcal{P}_f$  where  $f$  is continuous compact set of  $\mathbb{R}^d$ . A similar proof would work for a lower bound under the hypothesis that  $f$  is lower bounded by a constant  $c > 0$ .

**Theorem 2.2.5.** *Let  $f \geq 0$  be a continuous function on a compact set  $K$  of  $\mathbb{R}^d$ . For all  $\lambda$ , let  $\mathcal{P}_{\lambda f}$  be a Poisson point process with intensity  $\lambda f(x)dx$  in  $K$ , and  $\ell_\lambda := \ell_{\mathcal{P}_{\lambda f}}$ . There exists a constant  $c > 0$  such that for all  $\lambda > 0$  :*

$$\mathbb{E}[(\max \ell_\lambda)^p] \leq c\lambda^{2p/(d+1)}.$$

*Proof.* Since  $f$  is continuous on  $K$  and  $K$  is a compact set, there exists  $R > 0$  such that  $f \leq R$ . For all  $\lambda$ , we can assume that the Poisson point process  $\mathcal{P}_{\lambda R}$  with intensity  $\lambda R dx$  and  $\mathcal{P}_{\lambda f}$  are defined on the same probability space with  $\mathcal{P}_{\lambda f} \subset \mathcal{P}_{\lambda R}$  as in Proposition 1.1.11. We denote by  $\tilde{\ell}_\lambda$  the analogue of  $\ell_\lambda$  for the peeling of  $\mathcal{P}_{\lambda R}$ . Since  $\mathcal{P}_{\lambda f} \subset \mathcal{P}_{\lambda R}$ , thanks to Proposition 2.1.5 we have for all  $n$

$$\mathbb{E}[(\max \ell_\lambda)^p] \leq \mathbb{E}[(\max \tilde{\ell}_\lambda)^p].$$

Applying Theorem 2.2.4 to  $\tilde{\ell}_\lambda$ , there exists  $C > 0$  such that for any  $\lambda > 0$

$$\mathbb{E}[(\max \ell_\lambda)^p] \leq \mathbb{E}[(\max \tilde{\ell}_\lambda)^p] \leq CR^{2p/(d+1)}\lambda^{2p/(d+1)}.$$

□

## 2.3 Convex height

This section is devoted to the recent work of Calder and Smart [22], definitely a breakthrough in the study of the layers of the convex hull peeling of a Poisson point process in some convex body  $K$ . Their main theorem is a major improvement of the results of Dalal [33] in many ways. Indeed, the originality of their approach lies in particular in their ability of dealing with a functional point of view and making a bridge with the domains of game theory and partial differential equations. More precisely, they show that the convex hull peeling induces a random field on  $K$ , called the *convex height function* which is proved to converge after renormalization to a deterministic function. By interpreting the peeling procedure as a game, they get the intuition that the limit function satisfies a particular explicit non-linear PDE and this is the basis for their main result.

In the whole subsection,  $f : K \rightarrow \mathbb{R}$  denotes a continuous positive function on  $K$ . We consider a Poisson point process  $\mathcal{P}_{\lambda f}$  in  $K$  with intensity measure  $\lambda f(x)\mathbb{1}_K(x)dx$ . First, we define their main object of study, i.e. the convex height function.

**Definition 2.3.1** (Convex height function). Let  $X$  be a finite point set included in a convex body  $K$ . We define for each  $x \in K$  the convex height function of  $X$  at  $x$  by

$$h_X(x) := \sum_{i=1}^{\infty} \mathbb{1}_{\text{int}(\text{conv}_i(X))}(x).$$

The function  $h_{\mathcal{P}_{\lambda f}}$  is a random field over  $K$  which is simply denoted by  $h_\lambda$ .

For any  $x \in K$ , we notice that  $h_X(x) = \ell_X(x) + 1$ . In particular, the two functionals  $h_X$  and  $\ell_X$  share obviously the same asymptotic behavior.

### 2.3.1 Main results

We rewrite below the main theorem from [22] which makes explicit both the correct normalization of  $\lambda^{2/(d+1)}$  for  $h_\lambda$  and the limit function.

**Theorem 2.3.2** ([22], Theorem 1.2). *Let  $K$  be a convex body of  $\mathbb{R}^d$ . For any  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$\mathbb{P}(\sup_K |\lambda^{-2/(d+1)} h_\lambda - \alpha h| > \varepsilon) \leq \exp(-\delta(\log^{-2}(\lambda))\lambda^{3/(d+1)})$$

where  $\alpha > 0$  is constant that only depends on the dimension and  $h$  is the unique viscosity solution of the partial differential equation

$$\begin{cases} \langle Dh, \text{cof}(-D^2 h) Dh \rangle = f^2 \text{ in } \text{int}(K) \\ h = 0 \text{ on } \partial K \end{cases}, \quad (2.19)$$

$\text{cof}(\cdot)$  being the cofactor matrix.

A consequence of this result is the almost sure uniform convergence of the rescaled convex height function to  $\alpha h$ . Additionally the rescaled total number of layers, which differs from  $\max_K h_\lambda$  by at most 1, also converges to  $\alpha \max_K h$ . These two results are discussed in [22] though not precisely proved therein. They are summarized in Corollary 2.3.3 below

**Corollary 2.3.3.** *We have almost surely*

$$\lambda^{-2/(d+1)} h_\lambda \xrightarrow[\lambda \rightarrow \infty]{\text{unif.}} \alpha h \quad \text{in } K \quad (2.20)$$

and

$$\lambda^{-2/(d+1)} \max_K h_\lambda \xrightarrow[\lambda \rightarrow \infty]{} \alpha \max_K h. \quad (2.21)$$

*Proof.* We start with a proof of (2.20). It is enough to prove that

$$\lambda_n^{-2/(d+1)} h_{\lambda_n} \xrightarrow[n \rightarrow \infty]{\text{unif.}} \alpha h$$

for any sequence  $(\lambda_n)_n$  that goes to infinity as  $n$  goes to infinity.

Let  $\varepsilon > 0$ . For all  $n \in \mathbb{N}^*$ , let  $A_n := \{\sup_K |\lambda_n^{-2/(d+1)} h_{\lambda_n} - \alpha h| > \varepsilon\}$ . Theorem 2.3.2 implies that

$$\sum_n \mathbb{P}(A_n) < \infty$$

so the Borel-Cantelli lemma gives

$$\mathbb{P}(\overline{\lim} A_n) = 0.$$

Thus for every  $\varepsilon > 0$  we have

$$\mathbb{P}(\forall N \in \mathbb{N}^*, \exists n \geq N, \sup_{\bar{U}} |\lambda_n^{-2/(d+1)} h_{\lambda_n} - \alpha h| > \varepsilon) = 0.$$

It implies the desired almost sure convergence.

The result on the maximum of  $h_\lambda$  is a direct consequence of the following result: if a sequence of bounded functions  $(f_n)_n$  converges uniformly to a limit function  $f$ , then  $\sup f_n$  converges to  $\sup f$ .  $\square$

Theorem 2.3.2 is an improvement over the results of Dalal, see Subsection 2.2, in three ways. First, this induces a limit for the total number of layers. Secondly, the convergence is strengthened to a functional version for the underlying random field. Finally, the result holds for a whole class of Poisson point processes whose intensity measure has a continuous and positive density. The only additional assumption with respect to Proposition 2.2.2 and Theorem 2.2.5 is the convexity of the underlying set  $K$  which is not needed in Dalal's work.

Calder and Smart state that their almost sure uniform convergence result also implies the convergence in expectation of the total number of layers but they do not provide a proof. In Corollary 2.3.4 below, we propose to use our slightly improved rewriting of Dalal's upper bound, i.e. Theorem 2.2.5, to deduce the required convergence in mean and in  $L^p$ .

**Corollary 2.3.4.** *We have*

$$\lambda^{-2/(d+1)} \mathbb{E}[\max_K h_\lambda] \xrightarrow{\lambda \rightarrow \infty} \alpha \mathbb{E}[\max_K h]$$

and for any  $p \geq 1$ ,

$$\lambda^{-2/(d+1)} \max_K h_\lambda \xrightarrow{\lambda \rightarrow \infty} \alpha \max_K h.$$

*Proof.* This is a direct consequence of the almost sure convergence (2.21) combined with the Vitali convergence theorem and the uniform integrability of the sequence  $(h_\lambda)_\lambda$  deduced from Theorem 2.2.5.  $\square$

**Heuristics for the PDE.** We propose below an account of the heuristic ingredients which are presented in [22] as the main inspiration for getting Theorem 2.3.2: a game interpretation leading to a dynamic programming principle which is then used to establish heuristically the PDE.

Let  $X \subset K$  be finite. The convex hull peeling game can be described as follows. It starts at a point  $x_0 \in \text{int}(K)$ . Then Player 1 chooses  $p_0 \in \mathbb{R}^d \setminus \{0\}$  that should be seen as the ingoing normal vector of a half-space going through  $x_0$ . Inductively, when Player 2 chooses a point  $x_k$ , Player 1 chooses  $p_k \in \mathbb{R}^d \setminus \{0\}$  and once Player 1 chooses  $p_k$ , Player 2 chooses in turn a point  $x_{k+1}$  such that  $\langle p_k, x_{k+1} - x_k \rangle > 0$ , i.e. in the open half-space with  $x_k$  on its boundary and with  $p_k$  as an ingoing normal vector. The game ends after the choice of a half-space that contains no point of  $X$ . The goal of Player 1 is to minimize the final score  $\sum_{k \geq 1} \mathbb{1}_X(x_k)$  while Player 2 aims at maximizing this quantity. An optimal choice of  $p_k$  for Player 1 is such that every point  $x$  such that  $\langle p_k, x - x_k \rangle > 0$  verifies  $h_X(x) \leq h_X(x_k) - 1$ . The best possible choice of  $x_{k+1}$  for Player 2 is then a point of  $X$  such that  $h_X(x_{k+1}) = h_X(x_k) - 1$ . Each of these choices is possible by definition of the peeling. Thus, assuming that both players play optimally, each time they make their move, the game shifts to the previous layer. As a consequence the final score under the assumption of optimal play from both players is  $h_X(x_0)$ .

This game can be summarized in the so-called *dynamic programming principle* that is verified by  $h_X$ . For any  $x \in \text{int}(K)$

$$h_X(x) = \inf_{p \in \mathbb{R}^d \setminus \{0\}} \sup_{y: \langle p, y-x \rangle > 0} (\mathbb{1}_X(y) + h_X(y)).$$

Incidentally, we notice that this equality is closely related to our criterion in Lemma 2.1.4.

This connection between a game and a stochastic dynamic principle can already be found in Kohn and Serfaty [51] and in the same spirit as in this article it is possible to derive a PDE from the dynamic programming principle. First, Calder and Smart assume for their heuristics that

$\tilde{h} = \lambda^{-2/(d+1)}h_{\lambda f}$  is  $C^\infty$ , even if it is in fact discontinuous. They also add the assumptions that it has uniformly convex level sets and a non-zero gradient. When a point  $x = x_k$  has been chosen, the best possible choice of  $p = p_k$  for Player 1 is  $p = -Dh(x)$  up to a multiplicative constant. Indeed, because of the assumptions above, it ensures that  $\{y : \langle p_k, y - x \rangle > 0\}$  only contains points  $y$  such that  $h_{\lambda f}(y) \leq h_{\lambda f}(x) - 1$ . As a consequence, the dynamic programming principle can be rewritten as

$$\sup_{y: -\langle Dh(x), y-x \rangle > 0} (\lambda^{-2/(d+1)}\mathbf{1}_{\mathcal{P}_{\lambda f}}(y) + \tilde{h}(y) - \tilde{h}(x)) = 0.$$

In particular it implies that the set

$$\{y \in K : -\langle Dh(x), y-x \rangle > 0 \text{ and } \tilde{h}(y) \geq \tilde{h}(x) - \lambda^{-2/(d+1)}\}$$

contains at least one point of  $\mathcal{P}_{\lambda f}$ , otherwise the indicator function above would always be equal to 0. Then a computation involving the Taylor expansion of the function  $\tilde{h}$  yields

$$\langle D\tilde{h}(x), \text{cof}(-D^2h(x))D\tilde{h}(x) \rangle \approx Cf(x)^2.$$

In other words,  $\tilde{h} = \lambda^{-2/(d+1)}h_{\lambda f}(x)$  verifies equation (2.19) up to the constant  $C$  that this heuristic does not allow to determine precisely.

**Sketch of proof of Theorem 2.3.2.** We propose here a tentative rough description of the proof of Theorem 2.3.2.

A first ingredient is of geometric nature. The authors consider the (full) paraboloid  $P$  defined as the set of  $(x_1, \dots, x_d) \in \mathbb{R}^d$  such that  $\varphi(x) \geq 0$  where  $\varphi(x) = x_d - \frac{1}{2} \sum_{i=1}^{d-1} x_i^2$ . They then exhibit a map  $\pi$  which sends  $P$  to a half-space and transforms the convex hull peeling of a point set inside the paraboloid into what they call the *semiconvex peeling* of the image of the point set inside the half-space. In fact, the semiconvex peeling is the direct analogue of the convex hull peeling where the role of the half-spaces in the classical convex hull peeling is now played by paraboloids.

The second step consists in observing locally the convex hull peeling in  $K$  and sending it after rescaling to a semiconvex peeling. For a fixed point  $z$  in  $K$ , they consider the scaling transform

$$\tau_z(x) = (\lambda^{\frac{1}{d+1}}(x_1 - z_1), \dots, \lambda^{\frac{1}{d+1}}(x_{d-1} - z_{d-1}), \lambda^{\frac{2}{d+1}}(x_d - z_d - \sum_{i=1}^{d-1} z_i(x_i - z_i))). \quad (2.22)$$

They show that  $\tau_z$  sends a well-chosen neighborhood of  $z$  to a part of the paraboloid  $P$  and composing with the map  $\pi$ , they finally send the convex hull peeling in that neighborhood to a semiconvex hull peeling.

The crucial ingredient is then probabilistic. They study the semiconvex peeling and obtain the asymptotics for the convex height function in this context by using martingale and subadditivity-type arguments. First they derive estimates on the distribution of the layer number of a point at height  $r$  in the parabolic model. They deduce from them a result reminiscent of stabilization results i.e. they show that the layer number of a point at height  $r$  only depends on points in a cylinder of width  $cr$  with  $c > 1$  with probability going to 1 exponentially fast with  $r$ . This lets them reduce the domain to a cylinder, that they periodize. On this periodized domain they study the martingale  $(X_n)_n$  where each  $X_n$  is defined as the conditional expectation of the layer number of  $re_d$  given the  $\sigma$ -algebra generated by the  $n$ -th parabolic hull and the knowledge of every point on the first  $n$  layers. Through the use of Azuma's inequality, they get a fluctuation bound for the layer number

of  $re_d$  where  $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$ . Combining this with sub-additivity arguments yields a law of large number for the layer number of  $re_d$  that evolves as  $\alpha r$ . This is then extended to a uniform fluctuation result for every point at height  $r$  in a cylinder.

Finally, the connection between the PDE given at (2.19) and the equation  $\varphi$  of the paraboloid  $P$  is the following:  $\varphi$  is a particular solution of the PDE when  $f^2$  is equal to 1. Moreover, any solution in the sense of viscosity can be approximated by a supersolution and subsolution which are constructed piecewise as  $\mathcal{C}^\infty$ -deformations of  $\varphi$ .

**Comparison with [24].** We would like to emphasize here two connections between Calder and Smart's approach and the techniques developed in [24] and described at the end of Subsection 1.3.2.

First, we observe that the two rescalings given at (1.7) and (2.22) are of similar nature. Indeed, in both cases, the first  $(d-1)$  coordinates (spherical coordinates in (1.7)) are rescaled by  $\lambda^{1/(d+1)}$  while the last coordinate (radial coordinate in (1.7)) is rescaled by  $\lambda^{2/(d+1)}$ . The main difference is that the rescaling from [22] is applied around any point of  $K$  while the rescaling from [24] is only relevant in the vicinity of the boundary of the unit ball.

Secondly, in both papers, the parabolic convexity in the upper half-space plays a pivotal role and it will prove to be as crucial in Chapter 3 when studying the first layers of the convex hull peeling in the unit ball. Incidentally, the study of the asymptotics for the parabolic hull process (semiconvex peeling in [22]) involves the use of a stabilization property in both papers, see e.g. [22, Lemma 2.5].

### 2.3.2 Conjectures and possible extensions

Calder and Smart's work includes a specific conjecture which is a law of large numbers for the number of points on a layer of the convex hull peeling with label proportional to  $\lambda^{2/(d+1)}$ . Their heuristic reasoning is based on a geometric interpretation of the PDE that we describe below.

**Geometric interpretation of the PDE** A general differential geometry formula [43, 42] states that, as long as  $h \in C^2$  and  $Dh \neq 0$ , the Gaussian curvature of the level sets of  $h$  is given by

$$\kappa_G = \frac{\langle Dh, \text{cof}(-D^2h)Dh \rangle}{\|Dh\|^{d+1}}.$$

As a consequence of this formula, the PDE (2.19) can be formally rewritten as

$$\|Dh\| \kappa_G^{1/(d+1)} = f^{2/(d+1)}. \quad (2.23)$$

According to [22], the geometric interpretation of the PDE then goes as follows. Heuristically, we consider two nearby level sets  $\{h = t\}$  and  $\{h = t + \Delta t\}$  and write  $\Delta x$  for the normal distance between these two level sets at some point  $x \in \mathbb{R}^d$  such that  $h(x) = t$ . Thus, we have  $\|Dh(x)\| \approx \Delta t / \Delta x$  and by combining this with (2.23), we obtain in turn

$$\Delta x \approx \kappa_G^{1/(d+1)} f^{-2/(d+1)} \Delta t.$$

The equality above means roughly that  $h(x)$  can be interpreted as the arrival time of the boundary  $\partial K$  as it evolves with normal velocity  $\kappa_G^{1/(d+1)} f^{-2/(d+1)}$ . Incidentally, we notice that this velocity involves the power  $1/(d+1)$  of the Gaussian curvature exactly as in the integrand which appears

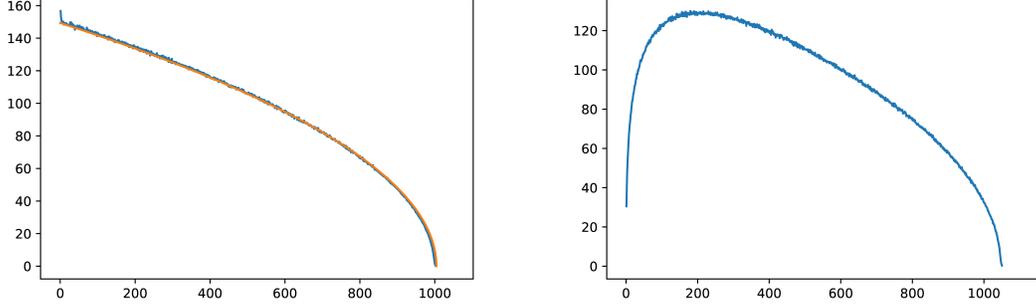


Figure 2.5 – In blue: the number of points on each layer as a function of the layer number, averaged over 100 peelings of  $10^5$  i.i.d. points. The figure on the left is in the unit disk in  $\mathbb{R}^2$  while the figure on the right is in a square. We plotted the function giving the number of points on each layer given by the conjecture of Calder and Smart in orange in the case of the unit disk.

in the definition of the affine surface area at (1.2).

**Conjecture for the number of points on the layers in the regime  $\lambda^{2/(d+1)}$ .** The heuristic argument of Calder and Smart goes as follows. They fix  $0 < a < b$  and assert that the number of Poisson points which fall in a region located between the two level sets  $\{h_\lambda = a\lambda^{2/(d+1)}\}$  and  $\{h_\lambda = b\lambda^{2/(d+1)}\}$  is Poisson distributed with mean  $\lambda \int_{a \leq \lambda^{-2/(d+1)}h_\lambda \leq b} f(x)dx$ . Since  $\lambda^{-2/(d+1)}h_\lambda \rightarrow \alpha h$  a.s. when  $\lambda$  goes to infinity, they formally replace  $\lambda^{-2/(d+1)}h_\lambda$  by its limit  $\alpha h$  and get the approximation for  $\lambda$  large

$$\frac{1}{\lambda} \sum_{n=\lfloor a\lambda^{2/(d+1)} \rfloor}^{\lceil b\lambda^{2/(d+1)} \rceil} f_0(\text{conv}_n(\mathcal{P}_{\lambda f})) \approx \int_{a \leq \alpha h \leq b} f(x)dx. \quad (2.24)$$

Using the co-area formula and (2.23), they get

$$\int_{a \leq \alpha h \leq b} f(x)dx = \frac{1}{\alpha} \int_a^b \int_{\{\alpha h=r\}} \frac{f}{\|Dh\|} d\sigma dr = \int_a^b \int_{\{\alpha h=r\}} f^{(d-1)/(d+1)} \kappa_G^{1/(d+1)} d\sigma dr \quad (2.25)$$

where  $\sigma$  is the surface measure on the level set  $\{\alpha h = r\}$ .

With the choice  $a = t\lambda^{2/(d+1)}$  and  $b = a + \lambda^{-2/(d+1)}$ , which, by the way, is not rigorous as  $a$  and  $b$  were initially assumed to be constant, they combine (2.24) and (2.25) to obtain for any  $t > 0$

$$\lambda^{-\frac{d-1}{d+1}} f_0(\text{conv}_{\lfloor t\lambda^{2/(d+1)} \rfloor}(\mathcal{P}_{\lambda f})) \approx \frac{1}{\alpha} \int_{\{\alpha h=r\}} f^{(d-1)/(d+1)} \kappa_G^{1/(d+1)} d\sigma. \quad (2.26)$$

**Perspectives.** Calder and Smart’s impressive work yields a series of relevant open problems.

- *Constant  $\alpha$ .* There is of course the question of the constant  $\alpha$  in Theorem 2.3.2 which is unknown, as emphasized by the authors [22, Problem 1.4]. They observe by simulation that

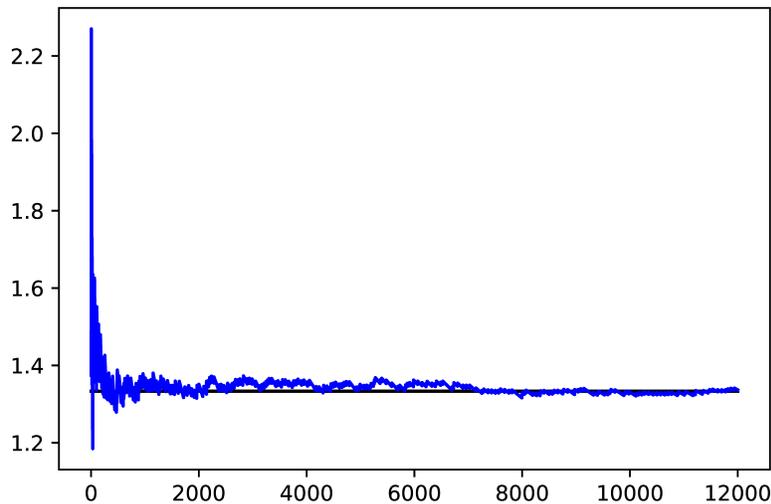


Figure 2.6 – Evolution of  $n^{-2/(d+1)} \max h_n / \max h$  with  $n$  for *i.i.d.* realizations in the unit disk. A depoissonized version of Corollary 2.3.3 implies that it should converge to  $\alpha$ , which is the height of the black horizontal line.

$\alpha$  is close to  $4/3$  in dimension two. Our own simulations in the case of the unit disk with the explicit calculation of  $h$  provided in [22, display (1.10)] are in agreement with this conjecture, see Figure 2.6.

- *Number of points on each layer.* We recall the conjecture (2.26) described above, which asserts that the layers with label proportional to  $\lambda^{2/(d+1)}$  should contain asymptotically  $\lambda^{(d-1)/(d+1)}$  points. This extends to a similar question for the layers with label  $g(\lambda)$  for any function  $g(\lambda)$  smaller than  $\lambda^{2/(d+1)}$ . In Chapters 3 and 4, we answer that question when  $g$  is constant in the cases of the unit ball and a simple polytope respectively. In the smooth case, we think that the growth rate  $\lambda^{\frac{d-1}{d+1}}$  should stay the same for any  $g$ . Even if this fact were justified, the evolution of the constant in front of the power should prove to be a very delicate matter. In the polytope case, we expect a phase transition for the growth rate. Indeed, it is proved in Chapter 4 that the first layers contain  $\log^{d-1}(\lambda)$  up to a multiplicative constant and in addition to the conjecture (2.26), Theorem 2.2.5 derived from Dalal’s work implies that there are at most  $\lambda^{2/(d+1)}$  layers up to a multiplicative constant and consequently, that the growth rate should reach at least  $\lambda^{\frac{d-1}{d+1}}$  for some layers. Finally, a completely unexplored topic concerns the study of the stochastic process constituted with the number of Poisson points on the layer with label  $t\lambda^{2/(d+1)}$  as a function of  $t$ . In particular, each layer is a random surface which could be studied in the spirit of other random growth processes, like for instance a percolation front.
- *Unbounded  $K$ .* The result of Calder and Smart requires the set  $K$  to be bounded. However, as shown by their simulations, it seems that their result should be true for more general

Poisson point processes. For instance, the case of a Poisson point process with a Gaussian intensity is shown in their simulations but Theorem 2.3.2 does not apply. This case could be investigated in the future.

- *Non-convex  $K$* . As mentioned earlier, the only drawback of Calder and Smart's work with respect to Dalal's result lies in their assumption of convexity of the mother body  $K$  which is not required in [33]. When the Poisson points fall into a bounded and possibly non-convex region, Dalal proves that the total number of layers still grows like  $\lambda^{2/(d+1)}$ . The asymptotics for the corresponding convex height function and in particular its connection with a certain PDE are unknown.

## Chapter 3

# Limit theory for the first layers in the ball

This chapter is a co-authored article with Pierre Calka that has been submitted for publication [23]. We study the number of  $k$ -dimensional faces and the outer defect intrinsic volumes of the first layers of the convex hull peeling of a homogeneous Poisson point process in the unit ball when its intensity goes to infinity. More precisely we provide asymptotic limits for their expectation and variance as well as a central limit theorem. In particular, we prove that the growth rates do not depend on the layer.

The work relies heavily on techniques introduced in [24]. However, the proof of stabilization estimates for the layers of the peeling requires a new approach that constitutes the most novel part of our contribution.

### 3.1 Introduction

#### 3.1.1 Context

Random polytopes as convex hulls of random points have been extensively studied in stochastic geometry. An overview of the subject can be found in [63] and [71, Chapter 8.2] for instance. Let  $\mathcal{P}_\lambda$  be a Poisson point process with intensity measure  $\lambda dx$  in a convex body  $K$  of  $\mathbb{R}^d$ . The study of the asymptotic behaviour as  $\lambda \rightarrow \infty$  of  $\text{conv}(\mathcal{P}_\lambda)$  started with the work of Rényi and Sulanke in [64, 65], in a binomial setting. They obtain in particular a different growth rate for the mean number of extreme points when  $K$  is a smooth convex body with a  $\mathcal{C}^2$ -regular boundary and when  $K$  is a polytope, namely polynomial for the former and logarithmic for the latter. Since then diverse results on the number of  $k$ -dimensional faces and on the defect intrinsic volumes of  $\text{conv}(\mathcal{P}_\lambda)$  have been proved for both choices of  $K$ . We only consider the smooth case in this paper. Asymptotic expectations are shown notably in [68, 62, 3]. In particular, it is known that the mean number of

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*American Mathematical Society 2010 subject classifications.* Primary 60D05; Secondary 52A22. 60G55e  
*Key words and phrases.* Poisson point process, random convex hulls, convex hull peeling, paraboloid hull process, stabilization

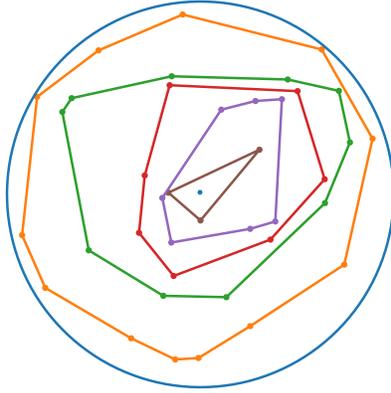


Figure 3.1 – Example of a convex hull peeling in  $\mathbb{B}^2$  with six layers (here the last layer has only one point).

extreme points grows like  $\lambda^{\frac{d-1}{d+1}}$  up to a multiplicative constant. First-order results have then been complemented by central limit theorems in [61, 78, 24]. Results on the variance of these quantities go from general bounds in [11, 8] to explicit limits in [24, 25]. More recently, concentration inequalities have been derived in [44]. The explicit formulas obtained by [21, 49, 50] are worth noting among the very few non-asymptotic results available.

The subject of this paper is a generalization of the study of the convex hull of random points to the so-called *convex hull peeling*. We start by taking the convex hull of the whole process and then repeatedly take the convex hull of the points that were not extreme at the previous step until no point remains. Let us write  $\text{conv}_1(\mathcal{P}_\lambda) := \text{conv}(\mathcal{P}_\lambda)$  and by induction for any  $n \geq 1$ ,  $\text{conv}_{n+1}(\mathcal{P}_\lambda) := \text{conv}(\text{int}(\text{conv}_n(\mathcal{P}_\lambda)) \cap \mathcal{P}_\lambda)$ . The boundary of the  $n$ -th convex hull  $\partial \text{conv}_n(\mathcal{P}_\lambda)$  will be called the  $n$ -th *layer* of the convex hull peeling of  $\mathcal{P}_\lambda$ . The words *peeling* and *layers* were chosen by analogy with the peeling of an onion, see Figure 3.1.

The convex hull peeling was first introduced by Barnett in [13] as a way to order multivariate data and give a meaning to how central a point is with respect to a dataset. Indeed, the layer number of a point can be interpreted as the *depth* of that point with respect to the input and we expect it to be all the larger the more central the point is. The convex hull peeling has then been used in robust statistics and outlier detection, see [38, 47, 66]. It fits into a list of classical techniques for ordering multivariate data, including half-space depth, simplicial depth or zonoid depth, see e.g. [30] for an overview on these techniques.

However it seems that very few theoretical results exist on the convex hull peeling of a random sample. For instance, the survey [63] devotes a small section on convex hull peeling but does not provide any reference and states that in continuation with the available asymptotic results for the convex hull, *investigations concerning expectations and deviation inequalities for [the subsequent layers of the convex hull peeling] are unknown*. To the best of our knowledge, there are mostly two papers which deal with the asymptotic properties of the convex hull peeling of random points. The first one due to Dalal [33] states that the mean total number of layers of the convex hull peeling of  $n$  i.i.d. uniform points in any bounded region of  $\mathbb{R}^d$  is lower and upper bounded by multiples of  $n^{\frac{2}{d+1}}$ . Very recently, a breakthrough work by Calder and Smart [22] greatly improves Dalal's estimate.

Let  $\mathcal{P}_{\lambda f}$  be a Poisson point process of intensity measure  $\lambda f(x)dx$  in  $K$  with  $f$  a continuous and positive function on  $K$ . They consider the convex height function  $h_\lambda(x)$  of any point  $x$  in  $\mathbb{R}^d$  as the largest  $n$  such that  $x \in \text{int}(\text{conv}_n(\mathcal{P}_{\lambda f}))$ . Note that Dalal's work covers the particular problem of estimating the expectation of  $\max h_\lambda$  when  $f = 1$ . Calder and Smart show that  $\lambda^{-\frac{2}{d+1}} h_\lambda$  converges uniformly in probability with an explicit exponential bound and almost surely to a multiple of a function  $h$  which is the unique viscosity solution of an explicit PDE. In other words, they obtain in particular that when  $\lambda \rightarrow \infty$ , almost surely

$$\lambda^{-\frac{2}{d+1}} h_\lambda \xrightarrow{\text{unif}} \alpha h \quad (3.1)$$

where  $\alpha$  is a positive constant which depends only on dimension  $d$  and  $h$  is the unique viscosity solution of

$$\begin{cases} \langle Dh, \text{cof}(-D^2h)Dh \rangle = f^2 \text{ in } \text{int}(K) \\ h = 0 \text{ on } \partial K \end{cases},$$

$\text{cof}(\cdot)$  being the cofactor matrix. Denoting by  $\mathbb{B}^d$  the unit ball of  $\mathbb{R}^d$ , we observe that (3.1) implies in the particular case when  $K = \mathbb{B}^d$  and  $f = 1$  that the rescaled total number of layers of the peeling satisfies almost surely

$$\lambda^{-\frac{2}{d+1}} \max h_\lambda \longrightarrow \beta_d := \frac{(d+1)\alpha}{2d^{\frac{d-1}{d+1}} \text{Vol}_{d-1}^{\frac{2}{d+1}}(\mathbb{S}^{d-1})} \quad (3.2)$$

where  $\text{Vol}_{d-1}(\mathbb{S}^{d-1})$  is the surface area of the  $(d-1)$ -dimensional unit sphere  $\mathbb{S}^{d-1}$ .

Because of the normalization of  $h_\lambda$  in (3.1), this uniform convergence result can only provide information on the regime of the peeling limited to layers numbered  $\lambda^{\frac{2}{d+1}}$  up to a multiplicative constant. The authors do not investigate any combinatorial or geometric functional of these layers. Nonetheless, they conjecture with a short heuristic argument that the number  $N_{n(\lambda,t),0,\lambda}$  of Poisson points on a layer numbered  $n(\lambda,t) := \lfloor t\lambda^{\frac{2}{d+1}} \rfloor$  should satisfy a law of large numbers when  $\lambda \rightarrow \infty$ , i.e. almost surely

$$\lambda^{-\frac{d-1}{d+1}} N_{n(\lambda,t),0,\lambda} \longrightarrow \int_{\{\alpha h=t\}} f^{\frac{d-1}{d+1}} \kappa^{\frac{1}{d+1}} dS$$

where  $\kappa$  is the Gauss curvature of the level set  $\{\alpha h = t\}$  and  $dS$  is the Hausdorff measure of that set. In particular, when  $K$  is the unit ball and  $f = 1$ , the conjectured result should read, see [22, display (1.18)],

$$\lambda^{-\frac{d-1}{d+1}} N_{n(\lambda,t),0,\lambda} \longrightarrow \frac{d+1}{2\beta_d} (1 - \beta_d^{-1}t)_+^{\frac{d-1}{2}} \quad (3.3)$$

where the constant  $\beta_d$  is introduced at (3.2).

In comparison to [22], our approach is different, i.e. we consider the case  $K = \mathbb{B}^d$  and  $f = 1$ , we choose to fix a layer numbered  $n$  that does not depend on  $\lambda$  and study the geometric properties of  $\partial \text{conv}_n(\mathcal{P}_\lambda)$  as  $\lambda \rightarrow \infty$ . In other words, we investigate a different regime, namely the regime of the first layers in the context of uniform points in the ball. There are several reasons to do so: when applying the convex hull peeling to outlier detection, we expect the outliers to be located on the first layers of the peeling, which provides some motivation for understanding the cardinality of these particular layers. Moreover, we intend to use a global scaling transformation on the ball which has been introduced in [24] for the study of the convex hull and which is expected to bring exhaustive information on the visible layers after rescaling, namely the first layers.

### 3.1.2 Model

Let  $\mathcal{P}_\lambda$  be a Poisson point process of intensity measure  $\lambda dx$  in the unit ball  $\mathbb{B}^d$  of  $\mathbb{R}^d$ . We construct the consecutive hulls  $\text{conv}_n(\mathcal{P}_\lambda)$ ,  $n \geq 1$ , of the peeling of  $\mathcal{P}_\lambda$ . For  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$ , we denote by  $N_{n,k,\lambda}$  the number of  $k$ -dimensional faces of the  $n$ -th layer  $\partial \text{conv}_n(\mathcal{P}_\lambda)$  and for  $k \in \{1, \dots, d\}$ , by  $V_{n,k,\lambda}$  the defect  $k$ -dimensional intrinsic volume of  $\text{conv}_n(\mathcal{P}_\lambda)$ , i.e.

$$V_{n,k,\lambda} = V_k(\mathbb{B}^d) - V_k(\text{conv}_n(\mathcal{P}_\lambda)) \quad (3.4)$$

where  $V_k$  stands for the  $k$ -th intrinsic volume, see for example [71, p. 600] for a definition and some properties of the intrinsic volumes.

We focus on these two families of random variables and aim at studying their first and second-order properties.

### 3.1.3 Main results

For two non-negative functions  $f$  and  $g$ , we write  $f = O(g)$  if there exist a constant  $C > 0$  and  $\lambda_0 > 0$  such that for any  $\lambda \geq \lambda_0$  we have  $f(\lambda) \leq Cg(\lambda)$ . Theorem 3.1.1 below provides expectation and variance asymptotics as well as a central limit theorem for the variables  $N_{n,k,\lambda}$ .

**Theorem 3.1.1.** *For any  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$  there exist  $C_{n,k,d}, C'_{n,k,d} \in (0, \infty)$  such that*

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-\frac{d-1}{d+1}} \mathbb{E}[N_{n,k,\lambda}] = C_{n,k,d} \text{ and } \lim_{\lambda \rightarrow +\infty} \lambda^{-\frac{d-1}{d+1}} \text{Var}[N_{n,k,\lambda}] = C'_{n,k,d}.$$

Moreover, when  $\lambda \rightarrow \infty$ , we have

$$\sup_t \left| \mathbb{P} \left( \frac{N_{n,k,\lambda} - \mathbb{E}[N_{n,k,\lambda}]}{\sqrt{\text{Var}[N_{n,k,\lambda}]}} \leq t \right) - \mathbb{P}(\mathcal{N}(0, 1) \leq t) \right| = O \left( \lambda^{-\frac{d-1}{2(d+1)}} (\log \lambda)^{3d+1} \right).$$

In Theorem 3.1.2, we derive similar results for the variables  $V_{n,k,\lambda}$ .

**Theorem 3.1.2.** *For any  $n \geq 1$  and  $k \in \{1, \dots, d\}$  there exist  $C_{V,n,k,d}, C'_{V,n,k,d} \in (0, \infty)$  such that*

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\frac{2}{d+1}} \mathbb{E}[V_{n,k,\lambda}] = C_{V,n,k,d} \text{ and } \lim_{\lambda \rightarrow +\infty} \lambda^{\frac{d+3}{d+1}} \text{Var}[V_{n,k,\lambda}] = C'_{V,n,k,d}.$$

Moreover, when  $\lambda \rightarrow \infty$ , we have

$$\sup_t \left| \mathbb{P} \left( \frac{V_{n,k,\lambda} - \mathbb{E}[V_{n,k,\lambda}]}{\sqrt{\text{Var}[V_{n,k,\lambda}]}} \leq t \right) - \mathbb{P}(\mathcal{N}(0, 1) \leq t) \right| = O \left( \lambda^{-\frac{d-1}{2(d+1)}} (\log \lambda)^{3d+1} \right).$$

The rates in Theorems 3.1.1 and 3.1.2 are identical to those obtained for the first layer, i.e. the convex hull of  $\mathcal{P}_\lambda$ , as described in [24]. In particular, the underlying limiting expectations and variances are proved to be different from zero. They have an explicit formulation in terms of a random process derived from a homogeneous Poisson point process in the product space  $\mathbb{R}^{d-1} \times \mathbb{R}_+$ , see Theorems 3.2.8 – 3.2.11. This solves the conjecture discussed in [22] and along the lines above (3.3) in the particular regime when the layer number does not depend on the size of the input.

Our tools are those of stabilization theory that were used in [24] to prove precise variance asymptotics for the first layer. The key idea consists in writing  $N_{n,k,\lambda}$  and  $V_{n,k,\lambda}$  as a sum  $\sum_{x \in \mathcal{P}_\lambda} \xi(x, \mathcal{P}_\lambda)$

for some functional  $\xi$  and proving that for a given point  $x$  this functional only depends on the process in a neighbourhood of  $x$ . That is what we call stabilization. The importance of the stabilization can already be seen in the study of the variance of  $N_{n,k,\lambda}$  as it implies that  $\xi(x, \mathcal{P}_\lambda)$  and  $\xi(y, \mathcal{P}_\lambda)$  are independent when  $x$  and  $y$  are far enough from each other, which simplifies the calculation of the variance. The stabilization of  $\xi$  is in fact used much more extensively for all six results stated in Theorems 3.1.1 and 3.1.2 and constitutes the main difficulty of this paper. Indeed, the formation of each layer of the peeling requires a global knowledge of the point set and also of the history of the previously constructed layers. In particular, for a given point  $x$ , there is no easy local criterion for checking that  $x$  is on the  $n$ -th layer of the peeling. In this regard, the problem is significantly different from the study of the convex hull as done in [24]. The only characterization that we can use is incremental, see Lemma 3.2.5, and this explains why the proof of stabilization is done by induction on the layer number. Incidentally, this also requires to estimate the position of each layer, see e.g. Lemma 3.3.2.

The strategy of proof of the expectation and variance asymptotics in Theorem 3.1.1 is the following.

- Using the decomposition of each variable as a sum over  $x \in \mathcal{P}_\lambda$  of a functional  $\xi(x, \mathcal{P}_\lambda)$ , we rewrite the expectation and variance of  $N_{n,k,\lambda}$  as an integral thanks to Mecke’s formula for Poisson point processes.
- Dealing with this multiple integral, we intend to use Lebesgue’s dominated convergence theorem after applying a suitable change of variables inside the integral. To do so, we need to rescale the model. This leads us to introducing the notions of parabolic hull peeling in the upper half-space, see Section 3.2.
- It then remains to show the convergence and domination of the integrands, rewritten as either an expectation or a covariance of a local functional of the parabolic hull peeling. This requires to show the so-called stabilization of the functionals, see Section 3.3. The stabilization implies in turn general moment bounds and the convergence of the integrands, see Section 3.4.

The proof of the central limit theorem also relies on the stabilization results from Section 3.3 as well as a Gauss approximation result in the particular setting of dependency graphs.

Finally, showing the positivity of the limiting expectations and variances represents another challenge. It requires to introduce a particular configuration where the determination of the layers and the calculation of the considered variables are natural and then to randomize this idealized configuration. This general principle has been used previously for proving the positivity of the limiting variances of  $N_{0,k,\lambda}$  and  $V_{0,k,\lambda}$ , see e.g. [61] and [8]. The construction that we do in the context of the  $n$ -th layer is partly inspired by [33].

We have chosen to concentrate mainly on the variables  $N_{n,k,\lambda}$  throughout the paper and to discuss briefly the adaptations that are needed in the case of the variables  $V_{n,k,\lambda}$  at the end of the paper, see Section 3.5.2.

### 3.1.4 Outline

The paper is structured as follows.

- In Section 3.2 we introduce the scaling transformation and study its effect on the point process and on the subsequent convex hulls. Incidentally we state a few basic properties on the peeling. We then define the scores as functionals of a point  $x$  and of the point process such that the variables  $N_{n,k,\lambda}$  and  $V_{n,k,\lambda}$  can be decomposed as sums of such scores. We conclude

with statements of more refined versions of the expectation and variance asymptotics of Theorem 3.1.1, with precise limiting constants.

- Section 3.3 is devoted to proving the stabilization of the rescaled scores, i.e. that with probability exponentially close to 1 they only depend on the process in the neighbourhood of the point considered.
- In Section 3.4 we use stabilization properties shown to prove  $L^p$  bounds and a convergence of in expectation of the rescaled scores.
- Section 3.5 contains the proofs of our main results.
- Finally, Section 3.6 collects several concluding remarks about possible extensions of our work and open problems.

## 3.2 Rescaling and scores

In this section, we introduce an *ad hoc* scaling procedure originated in [73] and [24]. We then study the image by that scaling transformation of the Poisson point process and of the layers of the underlying convex hull peeling. Next we prove general properties on the construction of the rescaled layers which are analogues of similar properties of the initial convex hull peeling. Finally we introduce functionals that we call scores and we decompose  $N_{n,k,\lambda}$  as the sum over every point of the process of these scores. This leads us to write explicit formulas for the constants in Theorem 3.1.1 and 3.1.2, where scores are involved.

### 3.2.1 Scaling transformation

To describe the scaling transformation that we will use on the point process, we first recall a few definitions. We write  $T_{e_d}$  for the tangent space of  $\mathbb{S}^{d-1}$  at point  $e_d = (0, 0, \dots, 0, 1)$ . The exponential map  $\exp_{e_d} : T_{e_d} \cong \mathbb{R}^{d-1} \rightarrow \mathbb{S}^{d-1}$  maps a vector  $v$  of  $T_{e_d}$  to the vector  $u$  that lies at the end of the geodesic of  $\mathbb{S}^{d-1}$  of length  $\|v\|$  that starts at  $e_d$  with direction  $v$ . The function  $\exp_{e_d}$  induces a one-to-one map between  $B_{d-1}(\pi)$  and  $\mathbb{S}^{d-1} \setminus \{-e_d\}$  of inverse  $\exp^{-1}$  where  $B_l(r)$  denotes the open ball centered at 0 of radius  $r$  in  $\mathbb{R}^l$ . This lets us define a one-to-one map  $T^{(\lambda)}$  between  $\mathbb{B}^d \setminus [0, -e_d]$  and  $W_\lambda := \lambda^{\frac{1}{d+1}} B_{d-1}(\pi) \times [0, \lambda^{\frac{2}{d+1}}]$  by

$$T^{(\lambda)}(x) := \left( \lambda^{\frac{1}{d+1}} \exp^{-1} \left( \frac{x}{\|x\|} \right), \lambda^{\frac{2}{d+1}} (1 - \|x\|) \right)$$

for all  $x \in \mathbb{B}^d \setminus [0, -e_d]$ . In general we will denote by  $w = (v, h)$  with  $v \in \mathbb{R}^{d-1}$  and  $h \in \mathbb{R}_+$  a generic point in  $W_\lambda$ . The transformation  $T^{(\lambda)}$  was already used in [24] to obtain variance asymptotics of functionals of the convex hull of  $\mathcal{P}_\lambda$  that include the number of  $k$ -faces (short for *k-dimensional faces*) and the  $k$ -th intrinsic volume. This transformation enjoys two important properties. First, unit volume subsets of  $W_\lambda$  near the hyperplane  $\mathbb{R}^{d-1} \times \{0\}$  contain  $\Theta(1)$  rescaled points and actually, we can show that the limit point process is Poisson and has intensity 1, see Lemma 3.2.1. Secondly, the transformation preserves the parabolic shape of both the defect radius-vector function and the defect support function of the random polytope  $\text{conv}(\mathcal{P}_\lambda)$ , as described in [24, p. 53–54]. These properties have been crucial in the proofs of the results contained in [24] on the convex hull  $\text{conv}(\mathcal{P}_\lambda)$ . It turns out that  $T^{(\lambda)}$  plays a similar role for the first  $n$  layers as long as we take a fixed  $n$  that does not vary with  $\lambda$ . Indeed, at the limit, the convex hull is mapped by  $T^{(\lambda)}$  to what we call the parabolic hull of the limit rescaled process. In the same way it maps the convex hull peeling to the

analogue of the peeling procedure in the parabolic picture that we name *parabolic hull peeling* of the limit rescaled process. We give more details below, after describing the effect of  $T^{(\lambda)}$  on  $\mathcal{P}_\lambda$ .

Our scaling transformation maps the Poisson point process  $\mathcal{P}_\lambda$  to a Poisson point process on  $W_\lambda$  that we denote by  $\mathcal{P}^{(\lambda)}$ . Its intensity has a density with respect to the Lebesgue measure given by

$$(v, h) \mapsto \frac{\sin^{d-2} \left( \lambda^{-\frac{1}{d+1}} \|v\| \right)}{\|\lambda^{-\frac{1}{d+1}} v\|^{d-2}} \left( 1 - \lambda^{-\frac{2}{d+1}} h \right), \quad (3.5)$$

see [24, p. 57] for the computation. As proved in [24, p. 71], this point process converges in distribution to a homogeneous Poisson point process of intensity one on  $\mathbb{R}^{d-1} \times \mathbb{R}_+$ , that we denote by  $\mathcal{P}$  or  $\mathcal{P}^{(\infty)}$ .

**Lemma 3.2.1.** *We have  $\lim_{\lambda \rightarrow +\infty} \mathcal{P}^{(\lambda)} = \mathcal{P}$  in distribution.*

Next we recall from [24] the effect of the rescaling on the spherical caps in the ball. This will then allow us to deduce the images of the consecutive layers by the rescaling. Any spherical cap in the unit ball of  $\mathbb{R}^d$  can be written

$$\text{cap}(x_0) := \{x \in \mathbb{B}^d : \langle x, \frac{x_0}{\|x_0\|} \rangle > \|x_0\|\}, \quad x_0 \in \mathbb{B}^d. \quad (3.6)$$

One can see that  $\text{cap}(x_0)$  is the cap orthogonal to  $x_0$  at distance  $\|x_0\|$  of the origin. Let us write  $(v_0, h_0) := T^{(\lambda)}(x_0)$ . The cap  $\text{cap}(x_0)$  is sent by  $T^{(\lambda)}$  to a so-called *downward quasi-paraboloid*  $[\Pi^\downarrow]^{(\lambda)}(v_0, h_0)$ . Furthermore, the quasi-paraboloids  $[\Pi^\downarrow]^{(\lambda)}(v_0, h_0)$  converge to a paraboloid

$$[\Pi^\downarrow(v_0, h_0)]^{(\infty)}(v_0, h_0) = \Pi^\downarrow(v_0, h_0) := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h < h_0 - \frac{\|v - v_0\|^2}{2}\}.$$

These results are made precise in Lemma 3.2.2 below, whose proof can be found in [24, p. 72-73] or in [26, Lemma 3.1] up to a small adaptation. Note that in this convergence result, with a slight abuse, we use the notation  $\partial[\Pi^\downarrow]^{(\lambda)}$  (resp.  $\partial\Pi^\downarrow$ ) for the function from  $\mathbb{R}^{d-1}$  to  $\mathbb{R}_+$  whose graph is the boundary of the set  $[\Pi^\downarrow]^{(\lambda)}$  (resp.  $\Pi^\downarrow$ ).

Beforehand, we need to introduce useful notation for several types of cylinders that are used in the rest of the paper. For any  $v \in \mathbb{R}^{d-1}$  and  $r > 0$ ,  $C_v(r)$  denotes the vertical cylinder  $B_{d-1}(v, r) \times [0, \infty)$  with the convention  $C(r) = C_0(r)$ . We also define the truncated cylinders  $C_v^{\geq t}(r) := C_v(r) \cap \{(v', h') \in \mathbb{R}^d : h' \geq t\}$ ,  $C_v^{\leq t}(r) := C_v(r) \cap \{(v', h') \in \mathbb{R}^d : h' \leq t\}$  and  $C_v^I(t) := C_v(r) \cap \{(v', h') \in \mathbb{R}^d : h' \in I\}$  for any  $t > 0$  and any interval  $I \subset \mathbb{R}_+$ .

**Lemma 3.2.2.** *Let  $\lambda > 0$  and  $x_0 \in \mathbb{B}^d$ . We write  $(v_0, h_0) = T^{(\lambda)}(x_0)$ . Then we have*

$$\begin{aligned} T^{(\lambda)}(\text{cap}(x_0)) &= [\Pi^\downarrow]^{(\lambda)}(v_0, h_0) \\ &:= \{(v, h) \in W_\lambda : h < \lambda^{\frac{2}{d+1}} \left( 1 - \frac{1 - \lambda^{-\frac{2}{d+1}} h_0}{\cos(e_\lambda(v, v_0))} \right)\} \end{aligned} \quad (3.7)$$

where  $e_\lambda(v, v_0) := d_{\mathbb{S}^{d-1}} \left( \exp_{d-1}(\lambda^{-\frac{1}{d+1}} v), \exp_{d-1}(\lambda^{-\frac{1}{d+1}} v_0) \right)$  for  $v, v_0 \in \mathbb{R}^{d-1}$ .

Additionally, for any  $L \geq 1$ , we have the following convergence result.

$$\lim_{\lambda \rightarrow \infty} \partial[\Pi^\downarrow]^{(\lambda)}(v_0, h_0) \cap C_{(v_0, h_0)}(L) = \partial\Pi^\downarrow(v_0, h_0) \cap C_{(v_0, h_0)}(L). \quad (3.8)$$

**Remark 3.2.3.** As in [24], when  $\lambda \in (0, \infty)$ , we can introduce a dual set

$$\begin{aligned} [\Pi^\uparrow]^{(\lambda)}(v_0, h_0) &:= T^{(\lambda)}\left(\partial B\left(\frac{x_0}{2}, \frac{\|x_0\|}{2}\right)\right) \\ &= \{(v, h) \in W_\lambda : h > \lambda^{\frac{2}{d+1}}(1 - (1 - \lambda^{-\frac{2}{d+1}} h_0) \cos(e_\lambda(v, v_0)))\} \end{aligned} \quad (3.9)$$

that we call the upward quasi-paraboloid with apex  $(v_0, h_0)$ . Similarly, we define

$$[\Pi^\uparrow]^{(\infty)}(v_0, h_0) = \Pi^\uparrow(v_0, h_0) := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+ : h > h_0 + \frac{\|v - v_0\|^2}{2}\}. \quad (3.10)$$

In particular, for any  $\lambda \in (0, \infty]$  and any  $(v_0, h_0), (v_1, h_1) \in W_\lambda$ ,

$$((v_1, h_1) \in [\Pi^\uparrow]^{(\lambda)}(v_0, h_0)) \iff ((v_0, h_0) \in [\Pi^\downarrow]^{(\lambda)}(v_1, h_1)).$$

This fact will be used on many occasions in the forthcoming proofs.

Recalling that for a locally finite point set  $X$ ,

$$\text{conv}(X) = \bigcup_{\substack{H^+ \text{ half-space} \\ X \cap H^+ = \emptyset}} (H^+)^c,$$

we are led by Lemma 3.2.2 to the following definition, that corresponds to the analogue of the convex hull in the (quasi-)parabolic setting, where the role of the half-spaces is played by downward quasi-paraboloids or paraboloids. For  $\lambda \in (0, \infty]$  and a locally finite point set  $Y$  in  $W_\lambda$ , we write

$$\Phi^{(\lambda)}(Y) := \bigcup_{\substack{w \in W_\lambda \\ Y \cap [\Pi^\downarrow]^{(\lambda)}(w) = \emptyset}} [\Pi^\downarrow]^{(\lambda)}(w)^c$$

that we call the quasi-parabolic hull of  $Y$ , or parabolic hull when  $\lambda = \infty$ . We will generally write  $\Phi$  instead of  $\Phi^{(\infty)}$  for sake of simplicity. Thanks to Lemma 3.2.2, we obtain that  $T^{(\lambda)}$  maps the convex hull of a point set to the quasi-parabolic hull of the image of this point set, i.e.

$$T^{(\lambda)}(\text{conv}(X)) = \Phi^{(\lambda)}(T^{(\lambda)}(X)),$$

provided that  $\text{conv}(X)$  contains the origin in its interior. In particular, when  $X = \mathcal{P}_\lambda$ , Wendel's formula [81] shows that the event  $\{0 \in \text{int}(\text{conv}(\mathcal{P}_\lambda))\}$  has a probability going to 1 exponentially fast when  $\lambda \rightarrow \infty$  and we implicitly condition on that event when working with  $T^{(\lambda)}$ .

We call *extreme points* of  $Y$  the points of  $Y \cap \partial\Phi_1^{(\lambda)}(Y)$ . They are naturally images by  $T^{(\lambda)}$  of the extreme points of the convex hull of  $[T^{(\lambda)}]^{-1}(Y)$ .

When  $\lambda \rightarrow \infty$ , the intersection  $\partial\Phi^{(\lambda)}(\mathcal{P}^{(\lambda)}) \cap C(R)$ ,  $R > 0$ , converges in distribution to  $\partial\Phi(\mathcal{P}) \cap C(R)$  where each quasi-parabolic hull is seen as a continuous function over  $C(R)$  and the set of continuous functions on  $C(R)$  is endowed with the topology of the uniform convergence, see [24, Theorem 4.1].

Let us now investigate the action of the transform  $T^{(\lambda)}$  on the convex hull peeling procedure. Quite naturally, it maps the convex hull peeling to a quasi-parabolic hull peeling that will converge in some sense to a parabolic hull peeling. We define the hulls of the quasi-parabolic and parabolic

hull peeling recursively with the following formula: for all  $n \geq 2$ ,  $\lambda \in (0, \infty]$  and a locally finite point set  $Y \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$ , we set

$$\Phi_n^{(\lambda)}(Y) = \Phi^{(\lambda)}(Y \cap \text{int}(\Phi_{n-1}^{(\lambda)}(Y))).$$

When  $\lambda = \infty$ , we speak of parabolic hull peeling and we write  $\Phi_n$  instead of  $\Phi_n^{(\infty)}$ . In the same way as for the the first layer of the convex hull peeling, the subsequent ones are mapped by  $T^{(\lambda)}$  to the corresponding parabolic hulls, i.e.

$$T^{(\lambda)}(\text{conv}_n(X)) = \Phi_n^{(\lambda)}(T^{(\lambda)}(X)), \quad (3.11)$$

provided that the origin lies in the interior of  $\text{conv}_n(X)$ . It is a direct consequence of [22, Theorem 1.2], see also (3.1), and [22, Equation (1.10) for  $x = 0$ ] that the event  $\{0 \in \text{int}(\text{conv}_n(\mathcal{P}_\lambda))\}$  has a probability going to 1 exponentially fast when  $\lambda \rightarrow \infty$ . Henceforth, when dealing with  $\text{conv}_n$ , we implicitly condition on that particular event.

We call the sets  $\partial\Phi_n^{(\lambda)}(Y)$  the *layers* of the quasi-parabolic or parabolic hull peeling of  $Y$ .

For any  $n \geq 1$ , the set  $\Phi_n^{(\lambda)}(Y)$  (resp.  $\Phi_n(Y)$ ) is the complement of a union of down quasi-paraboloids (resp. paraboloids). As the quasi-paraboloids converge to paraboloids, see Lemma 3.2.2, and  $W_\lambda$  goes to  $\mathbb{R}^{d-1} \times \mathbb{R}_+$  as  $\lambda$  goes to infinity, we can extend [24, Theorem 4.1] to the convergence in distribution of the subsequent rescaled layers of the original convex peeling to the corresponding layers of the parabolic peeling associated with the limit Poisson point process. As a side result, we also obtain the convergence of the point process of points of  $\mathcal{P}^{(\lambda)}$  on the  $n$ -th layer. This is summarized in Proposition 3.2.4 below.

**Proposition 3.2.4.** *Let  $R > 0$  and  $n \geq 1$ . When  $\lambda \rightarrow \infty$ , we have that*

$$\partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)}) \xrightarrow{\mathcal{L}} \partial\Phi_n(\mathcal{P})$$

where the set of continuous functions over  $\mathbb{R}^{d-1}$  is endowed with the topology of the uniform convergence on every compact set. Moreover,

$$\mathcal{P}^{(\lambda)} \cap \partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)}) \xrightarrow{\mathcal{L}} \mathcal{P} \cap \partial\Phi_n(\mathcal{P}).$$

In other words, Proposition 3.2.4 explains to what extent the parabolic hull peeling is the rescaled limiting model of the convex hull peeling in the ball. Since Proposition 3.2.4 is a natural analogue of the results stated and proved in [24, Theorem 4.1] and [26, Theorem 1.1] for the (quasi)-parabolic hull process, its proof is omitted. Let us note that when  $\lambda = \infty$ , the parabolic hull peeling of  $\mathcal{P}$ , as seen in Figure 3.2, is also a crucial tool of [22] under the name of *semiconvex peeling*.

### 3.2.2 Properties of the rescaled layers

For any  $w \in \mathbb{R}^{d-1} \times \mathbb{R}_+$  and  $Y \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$ , we introduce the number

$$\ell^{(\lambda)}(w, Y) = n \text{ such that } w \in \partial\Phi_n^{(\lambda)}(Y \cup \{w\}). \quad (3.12)$$

In particular, for any  $x \in \mathbb{B}^d$ ,  $\ell^{(\lambda)}(T^{(\lambda)}(x), T^{(\lambda)}(X))$  is the number of the layer of  $x$  in the initial convex hull peeling of  $X \cup \{x\}$ . We now aim at proving an explicit criterion for determining  $\ell^{(\lambda)}(w, Y)$ , see Lemma 3.2.5, and the monotonicity of  $\ell^{(\lambda)}(w, Y)$  with respect to  $Y$ , see Lemma

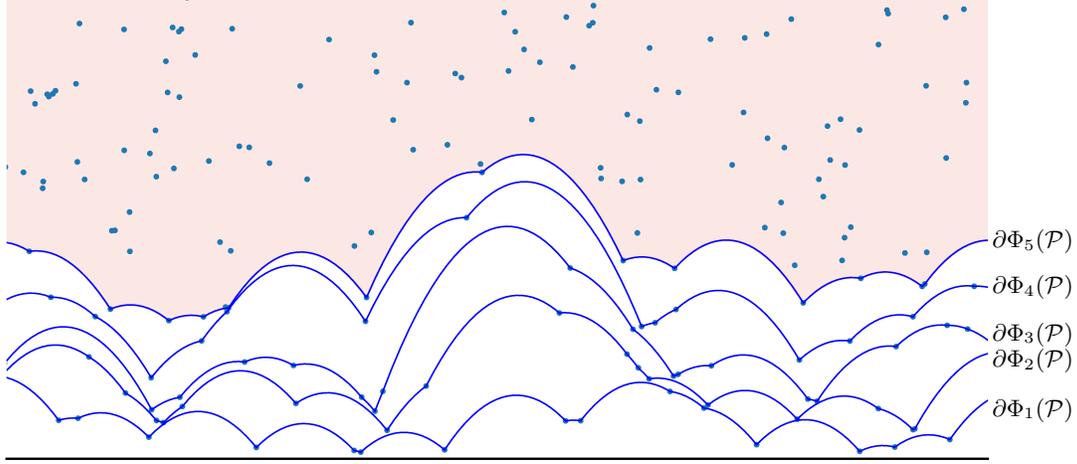


Figure 3.2 – First layers of a parabolic hull peeling, with the interior of  $\Phi_5(\mathcal{P})$  in pink.

**3.2.6.** Both of these properties could be stated for the initial convex hull peeling but we will only use the rescaled versions below.

Let us recall, see e.g. [24, pages 66-67], that a point  $w$  in  $\mathbb{R}^{d-1} \times \mathbb{R}_+$  is extreme if and only if there exists  $(v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w)$  such that  $[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap Y = \emptyset$ .

Lemma 3.2.5 provides a geometric interpretation of the function  $\ell^{(\lambda)}$  that extends the result above and that we will use frequently – and sometimes implicitly – in the rest of the paper, see Figure 3.3 for an illustration of this lemma.

**Lemma 3.2.5.** *Let  $Y$  be a locally finite subset of  $\mathbb{R}^{d-1} \times \mathbb{R}_+$ ,  $w \in Y$ ,  $\lambda \in (0, \infty]$  and  $n \geq 1$ . Then we have the two following equivalences.*

$$(i) \quad (\ell^{(\lambda)}(w, Y) \geq n) \iff (\forall (v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w) : Y \cap [\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \not\subset \cup_{i=1}^{n-2} \partial[\Phi_i]^{(\lambda)}(Y)).$$

$$(ii) \quad (\ell^{(\lambda)}(w, Y) \leq n) \iff (\exists (v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w) : Y \cap [\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \subset \cup_{i=1}^{n-1} \partial[\Phi_i]^{(\lambda)}(Y)).$$

*Proof.* (i) Let us assume that  $\ell^{(\lambda)}(w, Y) \geq n$  and take  $(v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w)$ . If we had  $[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap Y \subset \cup_{i=1}^{n-2} \partial[\Phi_i]^{(\lambda)}(Y)$ , this quasi-paraboloid would not intersect  $Y$  after removing the first  $(n-2)$  layers, meaning that  $w$  would be at most of layer  $(n-1)$  so the first implication holds.

Conversely, let us assume that  $[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap Y \not\subset \cup_{i=1}^{n-2} \partial[\Phi_i]^{(\lambda)}(Y)$ . Then after removing the first  $(n-2)$  layers, any down quasi-paraboloid whose boundary contains  $w$  has to meet  $Y$ . This implies that  $w$  is not extreme after removing the first  $(n-2)$  layers and thus  $\ell^{(\lambda)}(w, Y) \geq n$ .

(ii) If  $\ell^{(\lambda)}(w, Y) = m \leq n$ ,  $w$  is extreme when we remove the first  $(m-1)$  layers. Thus let  $(v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w)$  be such that  $[\Pi^\downarrow]^{(\lambda)}(v_1, h_1)$  does not contain any point of  $Y$  after removing the first  $(m-1)$  layers. This implies that

$$[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap Y \subset \cup_{i=1}^{m-1} \partial[\Phi_i]^{(\lambda)}(Y) \subset \cup_{i=1}^{n-1} \partial[\Phi_i]^{(\lambda)}(Y).$$

Conversely, if we assume that

$$\exists (v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w) : [\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap Y \subset \cup_{i=1}^{n-1} \partial[\Phi_i]^{(\lambda)}(Y),$$

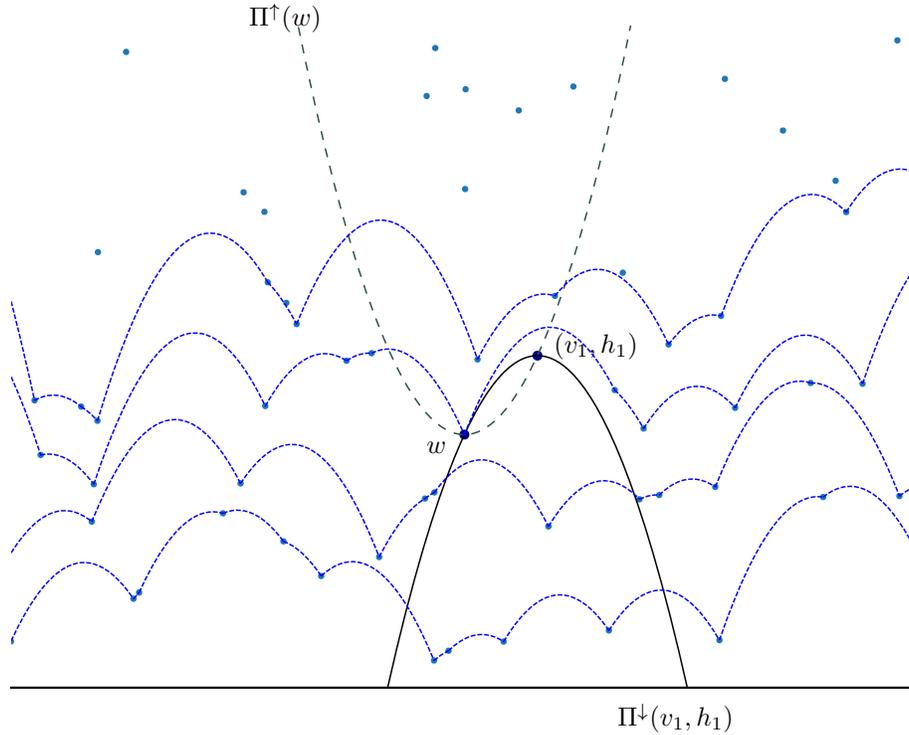


Figure 3.3 – Illustration of the criteria (i) and (ii) of Lemma 3.2.5 in the case  $\lambda = \infty$  and  $l^{(\infty)}(w, Y) = 3$ . As in (ii), the paraboloid  $\Pi^\downarrow(v_1, h_1)$  contains  $w$  on its boundary and only contains points on layers at most 2. As in (i), any translate of  $\Pi^\downarrow(v_1, h_1)$  containing  $w$  in its boundary (i.e. with  $(v_1, h_1) \in \partial\Pi^\uparrow(w)$ ) contains at least one point of layer at least 2.

either  $w$  belongs to the first  $(n - 1)$  layers or it is extreme once the first  $(n - 1)$  layers are removed. Thus  $\ell^{(\lambda)}(w, Y) \leq n$ .  $\square$

Lemma 3.2.6 below, which is of frequent use in our proofs, shows that the variables  $\ell^{(\lambda)}(w, X)$  are increasing with respect to the set  $X$ . It slightly rephrases [33, Lemma 3.1] in the context of the parabolic hull peeling and [22, Lemma 2.1]. For sake of completeness, we include a short proof below.

**Lemma 3.2.6.** *For  $\lambda \in (0, \infty]$ , if  $X \subset Y \subset W_\lambda$ , we have for every  $w \in W_\lambda$ ,  $\ell^{(\lambda)}(w, X) \leq \ell^{(\lambda)}(w, Y)$ .*

*Proof.* We prove the result by induction on  $n = \ell^{(\lambda)}(w, Y)$ . When  $n = 1$ ,  $w$  is extreme for the point set  $Y \cup \{w\}$  so is also extreme for the smaller point set  $X \cup \{w\}$ . When  $n > 1$ , by Lemma 3.2.5 (ii),  $w$  lies on the boundary of a down quasi-paraboloid such that each point  $w'$  of  $Y$  in its interior satisfies  $\ell^{(\lambda)}(w', Y \cup \{w\}) \leq (n - 1)$ . When  $w' \in X$ , the induction hypothesis applied to  $w'$  and the point sets  $X \cup \{w\}$  and  $Y \cup \{w\}$  shows that  $\ell^{(\lambda)}(w', X \cup \{w\}) \leq \ell^{(\lambda)}(w', Y \cup \{w\}) \leq (n - 1)$ . Consequently, using again Lemma 3.2.5 (ii), we obtain that  $\ell^{(\lambda)}(w, X) \leq n$ . This completes the proof.  $\square$

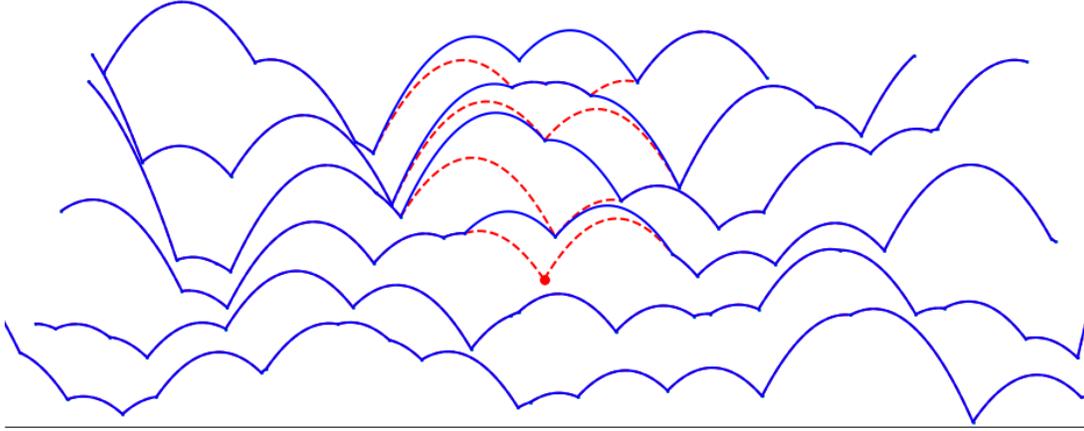


Figure 3.4 – The effect of adding a (red) point to the parabolic hull peeling. The red dashed lines are the changes applied to the layers.

**Remark 3.2.7.** In fact, when  $Y \setminus X$  is finite, we can show by arguments similar to the proof of [33, Lemma 3.1] that  $\ell^{(\lambda)}(w, Y) \leq \ell^{(\lambda)}(w, X) + \#(Y \setminus X)$  where  $\#(\cdot)$  denotes the cardinality, see Figure 3.4.

### 3.2.3 Scores and correlation functions

In this subsection we associate to each point of  $\mathbb{B}^d$  (resp.  $W_\lambda$ ) a random variable depending on that point and on the Poisson point process which we call a *score*. We start by defining the score of a point in the initial convex hull peeling before rescaling, i.e. for every  $x \in \mathbb{B}^d$ ,  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$ , we introduce the r.v.

$$\xi_{n,k}(x, X) := \begin{cases} \frac{1}{k+1} \#\mathcal{F}_{n,k}(x, X) & \text{if } x \in \partial \text{conv}_n(X \cup \{x\}) \\ 0 & \text{otherwise} \end{cases}$$

where  $\mathcal{F}_{n,k}(x, X)$  is the set of all  $k$ -faces containing  $x$  of  $\partial \text{conv}_n(X \cup \{x\})$ . The factor  $\frac{1}{k+1}$  is needed to take into account the fact that the faces are counted multiple times since a  $k$ -face contains a.s.  $(k+1)$  points of  $\mathcal{P}_\lambda$ . In particular, we get the identity

$$N_{n,k,\lambda} = \sum_{x \in \mathcal{P}_\lambda} \xi_{n,k}(x, \mathcal{P}_\lambda). \quad (3.13)$$

We now extend this notion of score to the rescaled model. Let  $\lambda \in (0, \infty]$ ,  $Y$  be a locally finite subset of  $W_\lambda$ ,  $w \in W_\lambda$ ,  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$ . We denote by  $\mathcal{F}_{n,k}^{(\lambda)}(w, Y)$  the set of  $k$ -faces of  $\partial \Phi_n^{(\lambda)}(Y \cup \{w\})$  containing  $w$ , i.e. the image by  $T^{(\lambda)}$  of the set of  $k$ -faces of  $\text{conv}_n([T^{(\lambda)}]^{-1}(Y \cup \{w\}))$  containing  $[T^{(\lambda)}]^{-1}(w)$  when  $\lambda < \infty$ . When  $\lambda = \infty$ ,  $\mathcal{F}_{n,k}^{(\lambda)}(w, Y)$  is the set of  $k$ -dimensional parabolic faces of  $\Phi_{n,k}^{(\lambda)}(Y \cup \{w\})$ , as defined in [24, p. 65–66], containing  $w$ . For any fixed  $\lambda \in (0, \infty]$ , we

define the score

$$\xi_{n,k}^{(\lambda)}(w, Y) := \begin{cases} \frac{1}{k+1} \#\mathcal{F}_{n,k}^{(\lambda)}(w, Y) & \text{if } w \in \partial\Phi_n^{(\lambda)}(Y \cup \{w\}) \\ 0 & \text{otherwise} \end{cases}. \quad (3.14)$$

We deduce from (3.14) and (3.11) that for every  $w \in W_\lambda$ ,

$$\xi_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi_{n,k} \left( [T^{(\lambda)}]^{-1}(w), \mathcal{P}_\lambda \right). \quad (3.15)$$

Note that the r.v.  $\xi_{n,k}^{(\lambda)}$  are calibrated such that  $\sum_{w \in \mathcal{P}^{(\lambda)}} \xi_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$  is a.s. the total number of  $k$ -faces of  $\partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)})$ .

We then introduce the two-point correlation function which is crucial for deriving the limiting variance. For any  $\lambda \in (0, \infty]$  let us write

$$c_{n,k}^{(\lambda)}((0, h_0), (v_1, h_1)) := \mathbb{E}[\xi_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \cup \{(v_1, h_1)\}] \xi_{n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(0, h_0)\}) \\ - \mathbb{E}[\xi_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)})] \mathbb{E}[\xi_{n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)})]. \quad (3.16)$$

We conclude by giving a more precise statement of Theorem 3.1.1, as we have now introduced every notation involved in the limiting constants.

**Theorem 3.2.8.** *For any  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$  we have*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{d-1}{d+1}} \mathbb{E}[N_{n,k,\lambda}] = \text{Vol}_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h), \mathcal{P})] dh \in (0, \infty).$$

**Theorem 3.2.9.** *For any  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$  we have*

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{d-1}{d+1}} \text{Var}[N_{n,k,\lambda}] = \text{Vol}_{d-1}(\mathbb{S}^{d-1}) (I_1 + I_2) \in (0, \infty)$$

where

$$I_1 := \int_0^\infty \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h), \mathcal{P})^2] dh \quad (3.17)$$

and

$$I_2 := \int_0^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} c_{n,k}^{(\infty)}((0, h_0), (v_1, h_1)) dv_1 dh_0 dh_1. \quad (3.18)$$

We restate in a similar way Theorem 3.1.2 for the intrinsic volumes, using the definitions of  $\xi_{V,n,k}^{(\infty)}$  and  $c_{V,n,k}^{(\infty)}$  introduced at (3.94) and (3.95) respectively.

**Theorem 3.2.10.** *For any  $n \geq 1$  and  $k \in \{1, \dots, d\}$  we have*

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\frac{2}{d+1}} \mathbb{E}[V_{n,k,\lambda}] = \text{Vol}_{d-1}(\mathbb{S}^{d-1}) \int_0^{+\infty} \mathbb{E}[\xi_{V,n,k}^{(\infty)}((0, h), \mathcal{P})] dh \in (0, \infty).$$

**Theorem 3.2.11.** *For any  $n \geq 1$  and  $k \in \{1, \dots, d\}$  we have*

$$\lim_{\lambda \rightarrow +\infty} \lambda^{\frac{d+3}{d+1}} \text{Var}[V_{n,k,\lambda}] = \text{Vol}_{d-1}(\mathbb{S}^{d-1}) (I_1 + I_2) \in (0, \infty)$$

where

$$I_1 := \int_0^\infty \mathbb{E}[\xi_{V,n,k}^{(\infty)}((0, h), \mathcal{P})^2] dh$$

and

$$I_2 := \int_0^{+\infty} \int_0^{+\infty} \int_{\mathbb{R}^{d-1}} c_{V,n,k}^{(\infty)}((0, h_0), (v_1, h_1)) dv_1 dh_0 dh_1.$$

### 3.3 Stabilization

The aim of this section is to show stabilization results for the considered scores. This means roughly that the score calculated at one particular fixed point requires the knowledge of the Poisson points outside of a lateral neighborhood of that fixed point with an exponentially decreasing probability. This tool is essential to get moment bounds in Lemma 3.4.1, then the convergence of the mean of one score and of the covariance of the scores and ultimately our main results, i.e Theorems 3.1.1 and 3.1.2.

#### 3.3.1 Local scores and stabilization radius

First we extend the notion of score to a local score in an angular sector around a point in the following way. For  $x \in \mathbb{B}^d$  and  $r > 0$ , we introduce  $S(x, r) = \{y \in \mathbb{R}^d : d_{\mathbb{S}^{d-1}}(x/\|x\|, y/\|y\|) \leq r\}$  where  $d_{\mathbb{S}^{d-1}}$  is the geodesic distance along  $\mathbb{S}^{d-1}$  and

$$\xi_{n,k,[r]}(x, \mathcal{P}_\lambda) := \xi_{n,k}(x, \mathcal{P}_\lambda \cap S(x, r)).$$

We define the stabilization radius in the initial model as

$$R_{n,k}(x, \mathcal{P}_\lambda) := \inf\{R > 0 : \xi_{n,k}(x, \mathcal{P}_\lambda) = \xi_{n,k,[r]}(x, \mathcal{P}_\lambda) \forall r \geq R\}.$$

In particular, thanks to the rotation invariance of  $\mathcal{P}_\lambda$ , we get the identity in law

$$R_{n,k}(x, \mathcal{P}_\lambda) \stackrel{(d)}{=} R_{n,k}(\|x\|e_d, \mathcal{P}_\lambda). \quad (3.19)$$

We formally introduce the stabilization radius in the rescaled model as

$$R_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := R_{n,k}([T^{(\lambda)}]^{-1}(w), \mathcal{P}_\lambda). \quad (3.20)$$

Combining (3.20) with (3.19), we obtain the invariance under horizontal translation of  $R_{n,k}^{(\lambda)}$ , namely for any  $(v, h) \in W_\lambda$ ,

$$R_{n,k}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)}) \stackrel{(d)}{=} R_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}). \quad (3.21)$$

Since the image by  $T^{(\lambda)}$  of  $S(e_d, r)$  for  $r \in (0, \pi)$  is a cylinder, we also extend the notion of score in the rescaled picture to a local score in a cylinder of radius  $r$  around a point in the following way. For any  $r > 0$ ,  $\lambda \in (0, \infty]$  and  $w = (v, h) \in W_\lambda$  we write

$$\xi_{n,k,[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) := \xi_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cap C_v(r)). \quad (3.22)$$

When  $w = (0, h)$ , the following equality provides a convenient expression of the stabilization radius, which is the one we use most of the time:

$$R_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}) = \inf\{R > 0 : \xi_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}) = \xi_{n,k,[r]}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}) \forall r \geq R\}.$$

We provide estimates for the distribution tail of  $R_{n,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$  in Section 3.3.2 and of  $R_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$  in Section 3.3.3.

### 3.3.2 Stabilization for points

A prerequisite for Sections 3.3.2 and 3.3.3 is the following geometric lemma that we will use extensively when showing the stabilization property. Though standard, we include its proof below for the reader's convenience.

**Lemma 3.3.1.** *Let  $\lambda \in (0, \infty]$  and  $w_0, w_1 \in H$  such that  $\partial[\Pi^\downarrow]^{(\lambda)}(w_1)$  goes through  $w_0$ . Then there exists a half-space  $P^+$  delimited by a hyperplane  $P$  going through  $w_0$  with direction containing  $(0, 0, \dots, 0, 1)$  such that  $[\Pi^\downarrow]^{(\lambda)}(w_0) \cap P^+ \subset [\Pi^\downarrow]^{(\lambda)}(w_1)$ .*

*Proof.* For finite  $\lambda$ , this is a direct consequence of the following fact in the non-rescaled model: for any point  $x_0 = re_d$ ,  $r \in (0, 1)$  and any point  $x_1 \in \partial B(\frac{x_0}{2}, \frac{\|x_0\|}{2})$ , the set  $\text{cap}(x_1)$  as defined in (3.6) contains at least half of  $\text{cap}(x_0)$ . Using (3.7) and (3.9), we deduce the required result in the rescaled model.

For  $\lambda = \infty$ , we proceed along the following lines. An orthogonal transformation allows us to assume that  $w_0 = (0, 0, \dots, 0, h_0)$  for some  $h_0 > 0$  and  $w_1 = (a, 0, \dots, 0, h_1)$  with  $a, h_1 > 0$ . Since  $\partial\Pi^\downarrow(w_1)$  goes through  $w_0$ , we must have  $a = \sqrt{2(h_1 - h_0)}$ . Let us show that  $\Pi^\downarrow(w_0) \cap \{(v, h) : (v)_1 > 0\} \subset \Pi^\downarrow(w_1)$  where  $(v)_1, \dots, (v)_{d-1}$  denote the consecutive coordinates of  $v$ . The equations of both paraboloids are

$$\Pi^\downarrow(w_0) : \left( h < h_0 - \frac{1}{2} \sum_{i=1}^{d-1} (v)_i^2 \right) \quad \text{and} \quad \Pi^\downarrow(w_1) : \left( h < h_1 - \frac{1}{2} \left( ((v)_1 - a)^2 + \sum_{i=2}^{d-1} (v)_i^2 \right) \right).$$

For  $(v, h) \in \Pi^\downarrow(w_0) \cap \{(v, h) : (v)_1 > 0\}$ , using  $a = \sqrt{2(h_1 - h)}$ , we get

$$\begin{aligned} h &< h_0 - \frac{1}{2} \left( ((v)_1 - a)^2 + \sum_{i=2}^{d-1} (v)_i^2 \right) + \frac{1}{2} \sqrt{2(h_1 - h_0)^2} - a(v)_1 \\ &< h_1 - \frac{1}{2} \left( ((v)_1 - a)^2 + \sum_{i=2}^{d-1} (v)_i^2 \right). \end{aligned}$$

This completes the proof.  $\square$

In the next lemma we show that the maximal height of the Poisson points on the  $n$ -th layer  $\partial\Phi_n^{(\lambda)}$  of the quasi-parabolic hull peeling inside a cylinder is bounded with a probability going to 1 exponentially fast with respect to the bound. This will be essential for proving the stabilization result in Proposition 3.3.3 and will also be useful when proving Lemma 3.3.5 which provides a stabilization in height.

Here and in the sequel we denote by  $c, c_1, c_2, \dots$  generic positive constants that only depend on  $n, k$  and  $d$  and which may change from line to line.

**Lemma 3.3.2.** *For all  $n \geq 1$ , there exist  $\lambda_0, c_1, c_2 > 0$  such that for all  $t \geq 0$ ,  $\lambda \in [\lambda_0, \infty]$  and  $1 \leq r < \pi\lambda^{\frac{1}{d+1}}$ , we have*

$$\mathbb{P} \left( \exists (v, h) \in \mathcal{P}^{(\lambda)} \cap \partial\Phi_n^{(\lambda)} \left( \mathcal{P}^{(\lambda)} \cap C(r) \right) \cap C(r/2^n) \text{ with } h \geq t \right) \leq c_1 r^{n(d-1)} e^{-c_2 t (r \wedge \sqrt{t})^{d-1}}$$

and

$$\mathbb{P}\left(\exists(v, h) \in \mathcal{P}^{(\lambda)} \cap \partial\Phi_n^{(\lambda)}\left(\mathcal{P}^{(\lambda)}\right) \cap C(r/2^n) \text{ with } h \geq t\right) \leq c_1 r^{n(d-1)} e^{-c_2 t(r \wedge \sqrt{t})^{d-1}}.$$

*Proof.* We only prove the first inequality as the method for getting the second one is very similar. We begin with the proof for  $\lambda = \infty$  as the case  $\lambda < \infty$  is a bit more technical. We are going to show it by induction on  $n$ . We first prove the induction step as it contains the main ideas. Then we describe what needs to be changed to prove the induction step for  $\lambda < \infty$  and finally we explain the slight modifications that are needed to prove the base case.

*Proof of the induction step for  $\lambda = \infty$ .* We assume that the result holds for all  $l < n$  with a fixed  $n > 1$  and we show that it holds for  $n$ . Let  $w = (v, h) \in C(r/2^n)$  with  $h \geq t$ . Our first step is to show that the event

$$\{w \in \partial\Phi_n((\mathcal{P} \cup \{w\}) \cap C(r))\} \quad (3.23)$$

occurs with probability smaller than  $c_1 \exp(-c_2 h(r \wedge \sqrt{h})^{d-1})$ . Here we add  $w$  to the point process because we plan to use Mecke's formula later to deal with a union over all  $w$ .

Let  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$  such that  $w \in \partial\Pi^\downarrow(v_1, h_1)$  and  $\Pi^\downarrow(v_1, h_1)$  only contains points of layer at most  $(n-1)$  for  $C(r)$ . Lemma 3.2.5 (ii) guarantees that such a  $(v_1, h_1)$  exists. By Lemma 3.3.1, this downward paraboloid contains at least half of  $\Pi^\downarrow(w)$ . Consequently, denoting by  $A_1, \dots, A_{2^{d-1}}$  the intersections of  $\Pi^\downarrow(w)$  with the product of an orthant of  $\mathbb{R}^{d-1}$  translated by  $v$  with  $\mathbb{R}_+$ , we have  $\text{Vol}_d(A_i \cap C^{\geq h/2}(r/2^{n-1})) = \frac{1}{2^{d-1}} \text{Vol}_d(\Pi^\downarrow(w) \cap C^{\geq h/2}(r/2^{n-1}))$  and  $\Pi^\downarrow(v_1, h_1)$  contains at least one of the  $A_i$ .

Let us write

$$B_{i,n}(h, r) := A_i \cap \{(v', h') : h' \geq h/2\} \cap C(r/2^{n-1}). \quad (3.24)$$

From the preceding reasoning we deduce that

$$\{w \in \partial\Phi_n((\mathcal{P} \cup \{w\}) \cap C(r))\} \subset \{\exists i : \mathcal{P} \cap B_{i,n}(h, r) \subset [\Phi_n((\mathcal{P} \cup \{w\}) \cap C(r))]^c\}.$$

For fixed  $i$ ,  $B_{i,n}(h, r)$  is either empty, which happens with probability smaller than  $c_1 \exp(-c_2 h(r \wedge \sqrt{h})^{d-1})$  or it contains a point at height larger than  $h/2$  on a layer at most  $(n-1)$ , which happens with probability smaller than  $c_1 r^{(n-1)(d-1)} \exp(-c_2 h(r \wedge \sqrt{h})^{d-1})$  by the induction hypothesis. Consequently we have

$$\mathbb{P}(\{w \in \partial\Phi_n((\mathcal{P} \cup \{w\}) \cap C(r))\}) \leq c_1 r^{(n-1)(d-1)} \exp(-c_2 h(r \wedge \sqrt{h})^{d-1}). \quad (3.25)$$

We can write

$$\begin{aligned} & \mathbb{P}(\exists(v, h) \in \mathcal{P} \cap \partial\Phi_n(\mathcal{P} \cap C(r)) \cap C(r/2^n) \text{ with } h \geq t) \\ & \leq \mathbb{E} \left[ \sum_{w \in \mathcal{P} \cap C^{\geq t}(r/2^n)} \mathbf{1}_{w \in \partial\Phi_n(\mathcal{P} \cap C(r))} \right]. \end{aligned}$$

We combine this with Mecke's formula and (3.25) to get

$$\begin{aligned}
& \mathbb{P}(\exists(v, h) \in \mathcal{P} \cap \partial\Phi_n(\mathcal{P} \cap C(r)) \cap C(r/2^n) \text{ with } h \geq t) \\
& \leq \int_{\|v\| \leq r/2^n} \int_{h \in ]t, +\infty[} \mathbb{P}((v, h) \in \partial\Phi_n((\mathcal{P} \cup \{(v, h)\}) \cap C(r))) \, dh dv \\
& \leq \int_{\|v\| \leq r/2^n} \int_{h \in ]t, +\infty[} c_1 r^{(n-1)(d-1)} \exp(-c_2 h(r \wedge \sqrt{h})^{d-1}) \, dh dv \\
& \leq c_1 r^{n(d-1)} \exp(-c_2 t(r \wedge \sqrt{t})^{d-1}).
\end{aligned}$$

This proves the induction step.

*Proof of the induction step for  $\lambda < \infty$ .* Let us check that the proof above still holds. The only difference here is that the intensity of the process is no longer constant. However let us recall that this intensity has a density given by (3.5) so it is uniformly bounded from below for any  $\|v\| \leq \frac{3}{4}\lambda^{\frac{1}{d+1}}\pi$  and  $h \leq \frac{3}{4}\lambda^{\frac{2}{d+1}}$  by a constant that does not depend on  $\lambda$  and is upper bounded by 1. The same proof as in the case  $\lambda = \infty$  shows that

$$\{w \in \partial\Phi_n^{(\lambda)}((\mathcal{P}^{(\lambda)} \cup \{w\}) \cap C(r))\} \subset \{\exists i : \mathcal{P}^{(\lambda)} \cap B_{i,n}(h, r) \subset [\Phi_n^{(\lambda)}((\mathcal{P}^{(\lambda)} \cup \{w\}) \cap C(r))]^c\}.$$

with  $B_{i,n}(h, r)$  introduced at (3.24). If for a fixed  $i$ ,  $B_{i,n}(h, r)$  is empty then in particular the set  $B_{i,n}(h, r) \cap \{(v', h') : \frac{1}{2}h \leq h' \leq \frac{3}{4}h\} \cap C(r/2^{n-1})$  is also empty and included in the region  $\{(v', h') : \|v'\| \leq \frac{3}{4}\lambda^{\frac{1}{d+1}}\pi \text{ and } h' \leq \frac{3}{4}\lambda^{\frac{2}{d+1}}\}$  on which the density at (3.5) is bounded from below by a constant. Consequently,

$$\mathbb{P}(\mathcal{P}^{(\lambda)} \cap B_{i,n}(h, r) = \emptyset) \leq e^{-c_1 \text{Vol}_d(B_{i,n}(h, r))} \leq e^{-c_2 h(r \wedge \sqrt{h})^{d-1}}.$$

The use of the induction hypothesis remains unchanged so we still have (3.25) in the case  $\lambda < \infty$ . To get the result we then follow the same steps as before except that we upper-bound the intensity density by 1 after the use of Mecke's formula.

*Proof of the base case  $n = 1$  for both  $\lambda = \infty$  and  $\lambda < \infty$ .* We define for every  $1 \leq i \leq 2^{d-1}$

$$\tilde{B}_{i,1}(h, r) = A_i \cap \{(v', h') : \frac{1}{2}h \leq h' \leq \frac{3}{4}h\} \cap C(3r/4).$$

which guarantees that the intensity measure of  $\tilde{B}_{i,1}(h, r)$  is lower bounded by its Lebesgue measure up to a multiplicative constant. Using the inclusion

$$\{w \in \partial\Phi_1((\mathcal{P}^{(\lambda)} \cup \{w\}) \cap C(r))\} \subset \{\exists i : \mathcal{P}^{(\lambda)} \cap \tilde{B}_{i,1}(h, r) = \emptyset\},$$

we get an analogue of (3.25) which, combined with Mecke's formula, proves the base case.

This completes the proof of Lemma 3.3.2.  $\square$

We are now ready to prove a stabilization result for the 0-faces, i.e. the extreme points of the  $n$ -th layer. It is a crucial step towards a general stabilization for  $k$ -faces.

**Proposition 3.3.3.** *For all  $n \geq 1$ , there exist  $\lambda_0, c_1, c_2 > 0$  such that for any  $h_0 > 0$ ,  $\lambda \in [\lambda_0, +\infty]$  and  $1 \leq r < \pi\lambda^{\frac{1}{d+1}}$  we have*

$$\mathbb{P}\left(R_{n,0}^{(\lambda)}(0, h_0) \geq r\right) \leq c_1 \exp(-c_2 r^{d+1}). \quad (3.26)$$

*Proof.* We give a detailed proof for the case  $\lambda = \infty$  and briefly describe at the end how to adapt the proof to make it work for finite  $\lambda$ .

We show this result by induction. The case  $n = 1$  corresponds to [24, Lemma 6.1]. We now fix  $n \geq 2$  and assume that (3.26) is verified for all  $m < n$ . Let us show (3.26) for  $n$ .

We first notice that

$$\{R_{n,0}^{(\lambda)}(0, h_0) \geq r\} = \bigcup_{s \geq r} \{\xi_{n,0}^{(\infty)}((0, h_0), \mathcal{P} \cap C(s)) \neq \xi_{n,0}^{(\infty)}((0, h_0), \mathcal{P})\}$$

Let us introduce

$$E_1 := \bigcup_{s \geq r} \bigcup_{l < n} \{(0, h_0) \in \partial\Phi_l(\mathcal{P} \cap C(s)) \cap \partial\Phi_n(\mathcal{P})\},$$

$$E_2 := \bigcup_{s \geq r} \bigcup_{l > n} \{(0, h_0) \in \partial\Phi_n(\mathcal{P} \cap C(s)) \cap \partial\Phi_l(\mathcal{P})\}.$$

Since  $\{R_{n,0}^{(\lambda)}(0, h_0) \geq r\} = E_1 \cup E_2$ , it is enough to prove that for any  $r \geq 1$ ,

$$\mathbb{P}(E_1) \leq c_1 \exp(-c_2 r^{d+1}) \quad (3.27)$$

and

$$\mathbb{P}(E_2) \leq c_1 \exp(-c_2 r^{d+1}). \quad (3.28)$$

*Decomposition of  $E_1$ .* The strategy is the following: we plan to select a down-paraboloid which contains  $(0, h_0)$  on its boundary and a point  $w$  in its interior to which we can apply the induction hypothesis, recalling (3.21).

To do so, we introduce the two events

$$F_1 := \{ \exists (v_1, h_1) \in \partial\Pi^\uparrow(0, h_0), h_1 \leq r^2/32 : \\ \mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_{n-1}(\mathcal{P} \cap C(r))]^c \} \cap \{(0, h_0) \in \partial\Phi_n(\mathcal{P})\},$$

$$F_2 := \{ \exists (v_1, h_1) \in \partial\Pi^\uparrow(0, h_0), h_1 \geq r^2/32 : \\ \mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_{n-1}(\mathcal{P} \cap C(r))]^c \} \cap \{(0, h_0) \in \partial\Phi_n(\mathcal{P})\},$$

see Figure 3.5.

When  $n = 2$ , we replace the inclusion  $\mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_{n-1}(\mathcal{P} \cap C(r))]^c$  in the definition of  $F_1$  and  $F_2$  above by  $\mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) = \emptyset$ .

In particular,  $E_1 \subset F_1 \cup F_2$ . Indeed if one of the events of the union in the definition of  $E_1$  occurs for fixed  $l < n$  and  $s \geq r$ , there exists  $(v_1, h_1) \in \partial\Pi^\uparrow(0, h_0)$  such that for every  $w \in \mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1)$ ,  $\ell^{(\infty)}(w, \mathcal{P} \cap C(s)) \leq (l-1)$ . Lemma 3.2.6 then implies that  $\ell^{(\infty)}(w, \mathcal{P} \cap C(r)) \leq (l-1) \leq (n-2)$  for any such  $w$ .

Consequently, it suffices to upper bound  $\mathbb{P}(F_1)$  and  $\mathbb{P}(F_2)$  which we do with two different strategies. In the case of  $F_1$ , there is a downward paraboloid  $\Pi^\downarrow(v_1, h_1)$  which is low enough to be

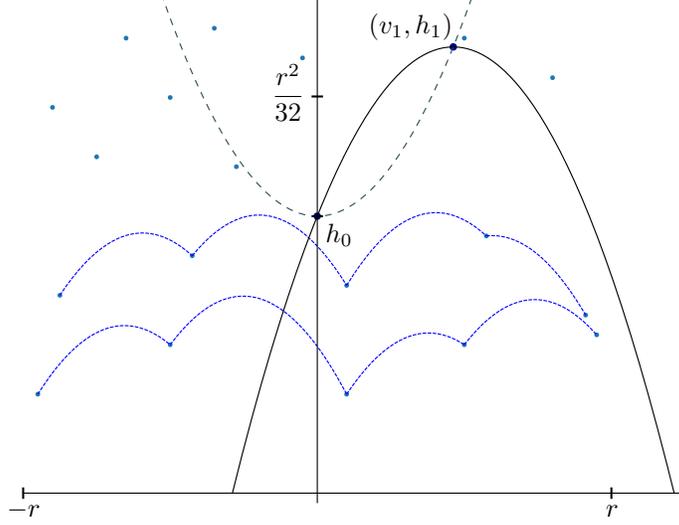


Figure 3.5 – Example of a situation where event  $F_2$  from Proposition 3.3.3 occurs when  $(0, h_0)$  is on layer 3 for the peeling of  $\mathcal{P} \cap C(r)$ .

contained in a cylinder smaller than  $C(r)$ . This implies that we can apply the induction hypothesis to a well chosen point in  $\Pi^\downarrow(v_1, h_1)$ . When on  $F_2$ , the downward paraboloid is high enough so that there is a high point  $w$  with  $\ell^{(\infty)}(w, \mathcal{P} \cap C(r)) \leq n$  and we deduce from Lemma 3.3.2 that it happens with exponentially small probability.

*Upper bound for  $\mathbb{P}(F_1)$ .* Let us fix  $(v_1, h_1)$  with  $h_1 \leq r^2/32$  as in the event  $F_1$ . Using that  $(v_1, h_1) \in \partial\Pi^\uparrow(0, h_0)$ , we get

$$\|v_1\| = \sqrt{2(h_1 - h_0)} \leq \sqrt{2h_1} = r/4. \quad (3.29)$$

Furthermore if we take  $(v_2, h_2) \in \Pi^\downarrow(v_1, h_1)$ , the norm of  $v_2$  is smaller than the norm of  $v_1$  plus half of the width of the paraboloid  $\Pi^\downarrow(v_1, h_1)$ , so

$$\|v_2\| \leq \|v_1\| + \|v_2 - v_1\| \leq \frac{r}{4} + \sqrt{2h_1} \leq r/2. \quad (3.30)$$

This implies that  $\Pi^\downarrow(v_1, h_1) \subset C(r/2)$ . In particular, when  $n = 2$ , we get that  $(0, h_0)$  is an extremal point of  $(\mathcal{P} \cup \{(0, h_0)\})$  and subsequently that  $F_1 = \emptyset$ . In the case  $n > 2$ , we proceed in the following way. Since  $\ell^{(\infty)}((0, h_0), \mathcal{P}) = n$ , we can choose a point  $w \in \Pi^\downarrow(v_1, h_1)$  such that  $\ell^{(\infty)}(w, \mathcal{P}) \geq (n-1)$ . Then because we are on the event  $F_1$ , we also have  $\ell^{(\infty)}(w, \mathcal{P} \cap C(r)) \leq (n-2)$ . By Lemma 3.2.6, this implies that  $\ell^{(\infty)}(w, \mathcal{P} \cap C_w(r/2)) \leq (n-2)$ , which means that

$$\exists m \leq (n-2) : R_{m,0}^{(\infty)}(w) \geq r/2. \quad (3.31)$$

Using the induction hypothesis

$$\mathbb{P}(R_{m,0}^{(\infty)}(w) \geq r/2) \leq c_1 \exp(-c_2 r^{d+1}).$$

Since  $w$  belongs to  $\mathcal{P} \cap C \leq \frac{r^2}{32}(r/2)$ , we rewrite

$$\mathbb{P}(F_1) \leq \mathbb{P}\left(\bigcup_{m=1}^{n-2} \bigcup_{w \in \mathcal{P} \cap C \leq \frac{r^2}{32}\left(\frac{r}{2}\right)} \{R_{m,0}^{(\infty)}(w, \mathcal{P}) \geq \frac{r}{2}\}\right) \leq \mathbb{E}\left[\sum_{m=1}^{n-2} \sum_{w \in \mathcal{P} \cap C \leq \frac{r^2}{32}\left(\frac{r}{2}\right)} \mathbb{1}_{\{R_{m,0}^{(\infty)}(w, \mathcal{P}) \geq \frac{r}{2}\}}\right].$$

Now we use Mecke's formula to obtain

$$\begin{aligned} \mathbb{P}(F_1) &\leq \sum_{m \leq n-2} \int_{C \leq r^2/32(r/2)} \mathbb{P}\left(R_{m,0}^{(\infty)}(w, \mathcal{P}) \geq r/2\right) dw \\ &\leq c_1 \exp(-c_2 r^{d+1}). \end{aligned} \tag{3.32}$$

*Decomposition of  $F_2$ .* We rewrite  $F_2 = G_1 \cup G_2$  where

$$\begin{aligned} G_1 &:= \{\exists(v_1, h_1), h_1 \geq r^2/32, \|v_1\| \leq r/6 : \\ &\quad \mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_{n-1}(\mathcal{P} \cap C(r))]^c \cap \{(0, h_0) \in \partial\Phi_n(\mathcal{P})\}\}. \\ G_2 &:= \{\exists(v_1, h_1), h_1 \geq r^2/32, \|v_1\| \geq r/6 : \\ &\quad \mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_{n-1}(\mathcal{P} \cap C(r))]^c \cap \{(0, h_0) \in \partial\Phi_n(\mathcal{P})\}\}. \end{aligned}$$

Again, when  $n = 2$ , we replace the inclusion  $\mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_{n-1}(\mathcal{P} \cap C(r))]^c$  in the definition of  $F_1$  and  $F_2$  above by  $\mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) = \emptyset$ .

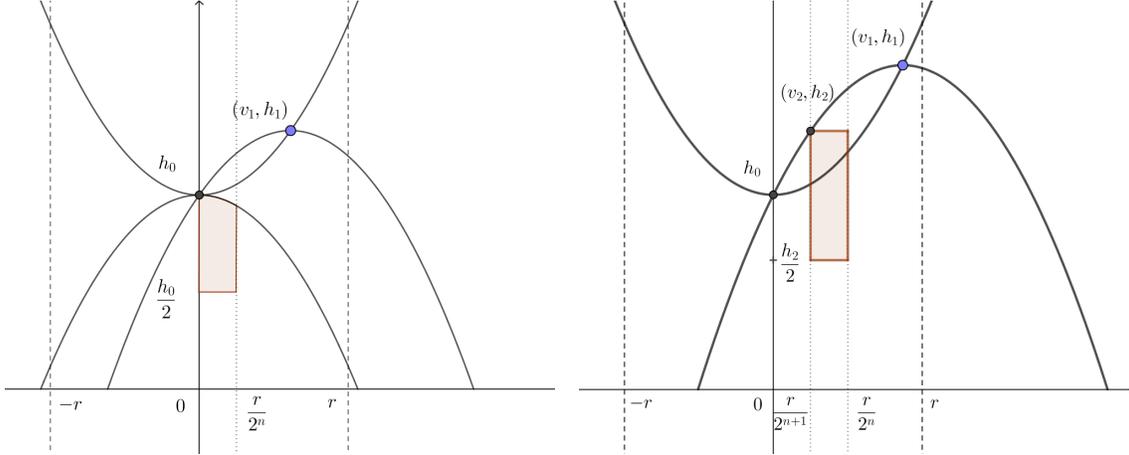


Figure 3.6 – Illustration of the geometric constructions leading to the upper-bounds of  $\mathbb{P}(G_1)$  (left) and  $\mathbb{P}(G_2)$  (right)

*Upper bound for  $\mathbb{P}(G_1)$ .* We fix  $(v_1, h_1)$  with  $\|v_1\| \leq r/6$  and  $h_1 \geq r^2/32$  as in the event  $G_1$ . In particular, we get

$$h_0 = h_1 - \frac{\|v_1\|^2}{2} \geq \frac{5}{288} r^2. \tag{3.33}$$

By Lemma 3.3.1, the down-paraboloid  $\Pi^\downarrow(v_1, h_1)$  contains half of  $\Pi^\downarrow(0, h_0) \cap C \geq \frac{h_0}{2}\left(\frac{r}{2^n}\right)$ , see Figure 3.6 (left). As in the proof of Lemma 3.3.2, we can find a deterministic subset  $A_i$  of  $\Pi^\downarrow(0, h_0)$  for

some  $1 \leq i \leq 2^{d-1}$  such that  $A_i \subset \Pi^\downarrow(v_1, h_1)$  and

$$\text{Vol}_d\left(A_i \cap C^{\geq \frac{h_0}{2}}\left(\frac{r}{2^n}\right)\right) = \frac{1}{2^{d-1}} \text{Vol}_d\left(\Pi^\downarrow(0, h_0) \cap C^{\geq \frac{h_0}{2}}\left(\frac{r}{2^n}\right)\right). \quad (3.34)$$

The set  $A_i \cap C^{\geq \frac{h_0}{2}}\left(\frac{r}{2^n}\right)$  is empty with probability bounded by  $\exp(-cr^{d+1})$  thanks to (3.33) and (3.34). If  $A_i \cap C^{\geq \frac{h_0}{2}}\left(\frac{r}{2^n}\right)$  is not empty, which happens only when  $n \geq 3$ , it contains a point  $w$  at height at least  $\frac{h_0}{2} \geq cr^2$  with  $\ell^{(\infty)}(w, \mathcal{P} \cap C(r)) \leq (n-2)$ . Using Lemma 3.3.2 this happens with probability smaller than  $c_1 \exp(-c_2 r^{d+1})$ .

A union bound on the finite number of sets  $A_1, \dots, A_{2^{d-1}}$  yields

$$\mathbb{P}(G_1) \leq c_1 \exp(-c_2 r^{d+1}). \quad (3.35)$$

*Upper bound for  $\mathbb{P}(G_2)$ .* We fix  $(v_1, h_1)$  with  $\|v_1\| \geq r/6$  and  $h_1 \geq r^2/32$  as in the event  $G_2$ . Let  $P$  be the vertical plane containing the origin and  $(v_1, h_1)$  and let  $(v_2, h_2)$  be the highest intersection point between  $\Pi^\downarrow(v_1, h_1)$  and  $\partial C(r/2^{n+1})$  in  $P$ . Using  $\|v_1\| = \sqrt{2(h_1 - h_0)}$ , we get

$$h_2 = h_1 - \frac{1}{2}(\|v_1\| - \frac{r}{2^{n+1}})^2 = h_0 + \frac{r}{2^{n+1}} \sqrt{2(h_1 - h_0)} - \frac{r^2}{2(2^{n+1})^2} \geq cr^2. \quad (3.36)$$

We claim that  $\Pi^\downarrow(v_1, h_1) \cap C^{(\frac{h_2}{2}, h_2)}\left(\frac{r}{2^n}\right)$  contains a deterministic cylinder  $C_0$  with width proportional to  $r$  and height proportional to  $h_2$ . Indeed, for  $n \geq 3$ , the cylinder inscribed in  $C\left(\frac{r}{2^n}\right) \setminus C\left(\frac{r}{2^{n+1}}\right)$  of radius  $\frac{r}{2^{n+2}}$  between heights  $\frac{h_2}{2}$  and  $h_2$  and with axis included in  $P$  as in Figure 3.6 (right) is fully included in  $\Pi^\downarrow(v_1, h_1)$ , thanks to the inequality  $\|v_1\| \geq \frac{r}{6} \geq \frac{r}{2^n}$ . For  $n = 2$ , the inequality is not satisfied and that is why we take a thinner cylinder inscribed in  $C\left(\frac{r}{6}\right) \setminus C\left(\frac{r}{8}\right)$  instead.

The cylinder  $C_0$  constructed above is empty with probability smaller than  $\exp(-ch_2 r^{d-1})$ . If  $C_0$  is not empty, which happens only when  $n \geq 3$ , it contains a point  $w$  of height at least  $\frac{h_2}{2}$  with  $\ell^{(\infty)}(w, \mathcal{P} \cap C(r)) \leq (n-2)$ . By Lemma 3.3.2 combined with (3.36), this happens with probability smaller than  $c_1 r^{d-1} \exp(-c_2 h_2 r^{d-1})$ .

Finally, discretizing  $\partial \Pi^\uparrow(0, h_0)$  for

$$h_1 \geq h_0 + \frac{1}{2} \left(\frac{r}{6}\right)^2, \quad (3.37)$$

we get

$$\begin{aligned} \mathbb{P}(G_2) &\leq c_1 \int_{h_0+r^2/72}^{\infty} (h_1 - h_0)^{(d-2)/2} r^{d-1} e^{-c_2(h_0 + \frac{r}{2^{n+1}} \sqrt{2(h_1 - h_0)} - \frac{r^2}{2(2^{n+1})^2}) r^{d-1}} dh_1 \\ &\leq c_1 \int_{h_0+r^2/72}^{\infty} (h_1 - h_0)^{(d-2)/2} r^{d-1} e^{-c_2 \frac{r}{2^{n+2}} \sqrt{2(h_1 - h_0)} r^{d-1}} dh_1 \\ &\leq c_1 \exp(-c_2 r^{d+1}). \end{aligned} \quad (3.38)$$

*Conclusion for  $\lambda = \infty$ .* Using the inclusion  $E_1 \subset F_1 \cup G_1 \cup G_2$  and the estimates for  $\mathbb{P}(F_1)$ ,  $\mathbb{P}(G_1)$  and  $\mathbb{P}(G_2)$  obtained above, we deduce (3.27). The estimate (3.28) is obtained in a very similar

fashion, where  $n$  plays the role of  $l$ , by considering the decomposition  $E_2 \subset F'_1 \cup F'_2$  where

$$\begin{aligned} F'_1 &= \{\exists(v_1, h_1) \in \partial\Pi^\uparrow(0, h_0), h_1 \leq r^2/32 : \\ &\quad \mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_n(\mathcal{P} \cap C(r))]^c\} \cap \{\exists l > n, (0, h_0) \in \partial\Phi_l(\mathcal{P})\}, \\ F'_2 &= \{\exists(v_1, h_1) \in \partial\Pi^\uparrow(0, h_0), h_1 \geq r^2/32 : \\ &\quad \mathcal{P} \cap C(r) \cap \Pi^\downarrow(v_1, h_1) \subset [\Phi_n(\mathcal{P} \cap C(r))]^c\} \cap \{\exists l > n, (0, h_0) \in \partial\Phi_l(\mathcal{P})\}. \end{aligned}$$

For the sake of brevity, the proof of (3.28) is omitted. Combining (3.27) and (3.28), we complete the proof of Proposition 3.3.3 when  $\lambda = \infty$ .

*Case  $\lambda < \infty$ .* We recall the two necessary updates for finite  $\lambda$ .

1. *Density.* The density of the intensity measure of  $\mathcal{P}^{(\lambda)}$  is lower bounded by a positive constant in a compact subset of  $W_\lambda$  only.
2. *Quasi-paraboloids.* The calculations that have been done with paraboloids are valid for quasi-paraboloids up to a small error.

The second update above implies that for  $\lambda$  large enough, (3.29) and (3.30) can be replaced by  $\|v_1\| \leq \frac{r}{3}$  and  $\|v_2\| \leq \frac{2r}{3}$  respectively. The assertion (3.31) is in turn replaced by  $R_{m,0}^{(\lambda)}(w) \geq r/3$  for some  $m \leq (n-2)$  and we proceed with the same reasoning as before to get (3.32).

Regarding the upper bounds of  $\mathbb{P}(G_1)$  and  $\mathbb{P}(G_2)$ , we make the following modifications.

- Because of the second update, the inequalities (3.33), (3.36) and (3.37) deduced from the actual equation of a paraboloid become  $h_0 \geq cr^2$ ,  $h_2 \geq cr^2$  and  $h_1 \geq h_0 + cr^2$  for  $\lambda$  large enough thanks to (3.8). Moreover, the second equality at (3.36) stays the same up to a multiplicative constant.
- In view of the first update, we replace  $A_i \cap C^{\geq \frac{h_0}{2}}(\frac{r}{2^n})$  by  $A_i \cap C^{(\frac{h_0}{2}, \frac{3h_0}{4})}(\frac{r}{2^n})$  in (3.34) and in the rest of the proof of the upper bound for  $\mathbb{P}(G_1)$ .

Moreover, the deterministic cylinder  $C_0$  constructed in the proof of the upper bound for  $\mathbb{P}(G_2)$  on page 66 is replaced by a cylinder included in  $C^{(\frac{h_2}{2}, \frac{3h_2}{4})}(\frac{r}{2^n})$  and with radius  $\frac{r}{2^{n+3}}$ , say.

We then obtain (3.35) and (3.38) and conclude as in the case  $\lambda = \infty$ . □

Proposition 3.3.3 is a general stabilization result for the score at one fixed point and this stabilization is lateral, meaning that the point process is intersected with a cylinder. Lemma 3.3.4 below is a complementary stabilization result in a cylinder and the stabilization there is in height, meaning that the point process is intersected with a horizontal strip. Combining Lemma 3.3.4 with Proposition 3.3.3, we can deduce a general stabilization result both in width and height. The stabilization in height is required to restrict the peeling to a cylinder bounded in height later on and use the continuous mapping theorem, see Lemma 3.4.2. This will ultimately imply in particular a convergence result for the mean of the functional  $\xi_{n,k}^{(\lambda)}$ , see Proposition 3.4.3. An extra refinement contained in Lemma 3.3.4 is that the stabilization in height is proved to be uniform for all the points inside a small cylinder. This will be needed for getting the stabilization in height of the  $k$ -face score, see Lemma 3.3.7.

**Lemma 3.3.4.** *For all  $n \geq 1$ , there exist  $\lambda_0, c_1, c_2 > 0$  such that for all  $\lambda \in [\lambda_0, \infty]$  and  $1 \leq r < \pi\lambda^{\frac{1}{d+1}}$ , we have*

$$\begin{aligned} \mathbb{P}\left(\exists w \in \mathcal{P}^{(\lambda)} \cap C^{\leq l_n r^2}\left(\frac{r}{2^{n+1}}\right) : \xi_{n,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cap C(r)) \neq \xi_{n,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cap C^{\leq l_n r^2}(r))\right) \\ \leq c_1 \exp(-c_2 r^{d+1}) \end{aligned}$$

with  $l_n = \frac{1}{2^{2n+7}}$ .

*Proof.* As in the previous proofs, we proceed in the case  $\lambda = \infty$  and explain at the end how to adapt the arguments in the case  $\lambda < \infty$ . For fixed  $n$ , we prove by induction on  $m$  that for all  $m \leq n$  there exists  $c_1, c_2 > 0$  such that for all  $r \geq 1$  we have

$$\begin{aligned} \mathbb{P}\left(\exists w \in \mathcal{P} \cap C^{\leq l_n r^2}\left(\frac{r}{2^{m+1}}\right) : \xi_{m,0}^{(\infty)}(w, \mathcal{P} \cap C(r)) \neq \xi_{m,0}^{(\infty)}(w, \mathcal{P} \cap C^{\leq l_n r^2}(r))\right) \\ \leq c_1 \exp(-c_2 r^{d+1}). \end{aligned} \quad (3.39)$$

Lemma 3.3.4 is then derived from (3.39) by taking  $m = n$ .

*Proof of the base case  $m = 1$  for  $\lambda = \infty$ .* Let  $w \in C^{\leq l_n r^2}\left(\frac{r}{4}\right)$  and assume that  $\xi_{1,0}^{(\infty)}(w, \mathcal{P} \cap C(r)) \neq \xi_{1,0}^{(\infty)}(w, \mathcal{P} \cap C^{\leq l_n r^2}(r))$ . Then there exists a downward paraboloid  $\Pi^\downarrow(v_1, h_1)$ , whose boundary contains  $w$ , that contains no point of  $\mathcal{P} \cap C^{\leq l_n r^2}(r)$  and contains at least one point of  $\mathcal{P} \cap C(r)$ .

If  $h_1 \leq l_n r^2$ , we observe that thanks to  $l_n = \frac{1}{2^{2n+7}}$  and the fact that  $w \in C\left(\frac{r}{4}\right)$ , we get for any  $(v', h') \in \Pi^\downarrow(v_1, h_1)$ ,

$$\begin{aligned} \|v'\| &\leq \|v' - v_1\| + \|v_1\| \leq \sqrt{2h_1} + \|v - v_1\| + \|v\| \leq 2\sqrt{2h_1} + \frac{r}{4} \\ &\leq 2\sqrt{2l_n r^2} + \frac{r}{4} \leq r. \end{aligned} \quad (3.40)$$

The last inequality in (3.40) implies that  $\Pi^\downarrow(v_1, h_1)$  is contained in  $C^{\leq l_n r^2}(r)$  which is excluded.

If  $h_1 > l_n r^2$ , we claim that the intersection between  $\Pi^\uparrow(w)$  and the vertical plane containing  $w$  and  $(v_1, h_1)$  contains exactly two points at height equal to  $l_n r^2$  and we call  $(v_2, h_2)$  the one which is closer to  $(v_1, h_1)$ , see Figure 3.7. Thanks to Lemma 3.3.1, the paraboloid  $\Pi^\downarrow(v_1, h_1)$  contains half of the down paraboloid with apex at the vertical projection of  $(v_2, h_2)$  onto  $\partial\Pi^\downarrow(v_1, h_1)$ . Consequently,  $\Pi^\downarrow(v_1, h_1)$  also contains  $\Pi_+^\downarrow(v_2, h_2) \cap C^{\leq l_n r^2}(r)$  where  $\Pi_+^\downarrow(v_2, h_2)$  denotes half of  $\Pi^\downarrow(v_2, h_2)$ . This latter set has volume  $cr^{d+1}$  so we can show by using deterministic orthants as in the proof of Lemma 3.3.2 that

$$\mathbb{P}(\mathcal{P} \cap \Pi_+^\downarrow(v_2, h_2) \cap C^{\leq l_n r^2}(r) = \emptyset) \leq \exp(-cr^{d+1}). \quad (3.41)$$

Denoting by  $D$  the set of points of  $\partial\Pi^\uparrow(w)$  at height  $l_n r^2$  and discretizing  $D$ , we obtain

$$\mathbb{P}(\exists (v_2, h_2) \in D : \mathcal{P} \cap \Pi_+^\downarrow(v_2, h_2) \cap C^{\leq l_n r^2}(r) = \emptyset) \leq c_1 \exp(-c_2 r^{d+1}).$$

Using Mecke's formula we get

$$\begin{aligned} \mathbb{P}\left(\exists w \in \mathcal{P}^{(\lambda)} \cap C^{\leq l_n r^2}\left(\frac{r}{4}\right) : \xi_{1,0}^{(\infty)}(w, \mathcal{P}^{(\lambda)} \cap C(r)) \neq \xi_{1,0}^{(\infty)}(w, \mathcal{P}^{(\lambda)} \cap C^{\leq l_n r^2}(r))\right) \\ \leq c_1 \text{Vol}_d\left(C^{\leq l_n r^2}\left(\frac{r}{4}\right)\right) \exp(-c_2 r^{d+1}) \leq c_1 \exp(-c_2 r^{d+1}). \end{aligned}$$

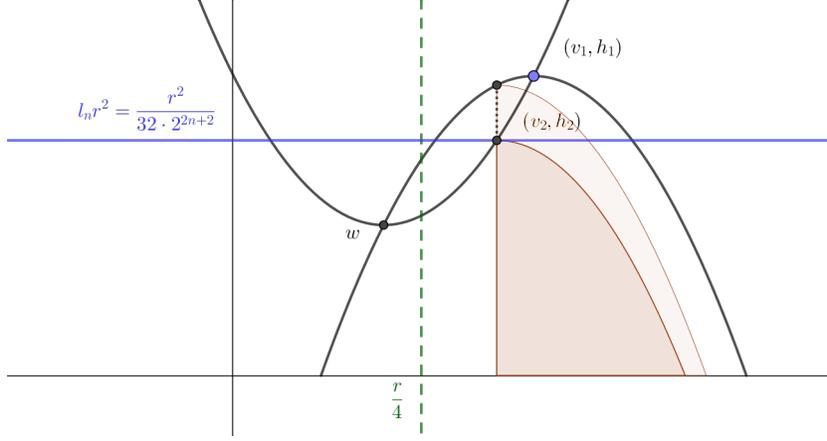


Figure 3.7 – Case  $m = 1$  of Lemma 3.3.4.

This proves the base case for  $\lambda = \infty$ .

*Proof of the induction step for  $\lambda = \infty$ .* Now for fixed  $2 \leq m \leq n$  we assume that the result holds for any  $p < m$  and we show that it remains true for  $m$ . Let  $w \in C^{\leq l_n r^2}(\frac{r}{2^{m+1}})$ . We denote by  $E_w$  the event

$$E_w := \{\xi_{m,0}^{(\infty)}(w, \mathcal{P} \cap C(r)) \neq \xi_{m,0}^{(\infty)}(w, \mathcal{P} \cap C^{\leq l_n r^2}(r))\}.$$

When on  $E_w$ , we also assume that  $\xi_{m,0}^{(\infty)}(w, \mathcal{P} \cap C^{\leq l_n r^2}(r)) = 1$ , i.e.  $\ell^{(\infty)}(w, \mathcal{P} \cap C^{\leq l_n r^2}(r)) = m$ . Indeed, the case  $\xi_{m,0}^{(\infty)}(w, \mathcal{P} \cap C(r)) = 1$  can be treated in a similar way, see what we did when dealing with events  $E_1$  and  $E_2$  in the proof of Proposition 3.3.3. In particular, when  $\ell^{(\infty)}(w, \mathcal{P} \cap C^{\leq l_n r^2}(r)) = m$ , the depth  $\ell^{(\infty)}(w, \mathcal{P} \cap C(r))$  is larger than  $(m + 1)$  thanks to Lemma 3.2.6.

In other words, there exists a downward paraboloid  $\Pi^\downarrow(v_1, h_1)$  whose boundary contains  $w$  and that only contains points on a layer of order at most  $(m - 1)$  for the peeling in  $C^{\leq l_n r^2}(r)$  and at least one point denoted by  $(v_3, h_3)$  such that

$$\ell^{(\infty)}((v_3, h_3), \mathcal{P} \cap C(r)) \geq m \text{ and } \ell^{(\infty)}((v_3, h_3), \mathcal{P} \cap C^{\leq l_n r^2}(r)) \leq m - 1. \quad (3.42)$$

If  $h_1 \leq l_n r^2$ , then for  $(v', h') \in \Pi^\downarrow(v_1, h_1)$ , we obtain by the same method as in (3.40) that

$$\|v'\| \leq 2\sqrt{2h_1} + \frac{r}{2^{m+1}} \leq 2\sqrt{2l_n r^2} + \frac{r}{2^{m+1}} \leq \frac{r}{2^m}. \quad (3.43)$$

We notice that the last inequality in (3.43) still holds when  $l_n$  is replaced by  $\frac{1}{2^{2n+5}}$ , which means that our current calibration takes into account the case  $\lambda < \infty$  which is discussed at the end of the proof. The estimate (3.43) shows that the paraboloid  $\Pi^\downarrow(v_1, h_1)$  stays in  $C^{\leq l_n r^2}(\frac{r}{2^m})$ . Consequently the point  $(v_3, h_3)$  introduced above belongs to  $C^{\leq l_n r^2}(\frac{r}{2^m})$  and by (3.42), it satisfies  $\xi_{p,0}^{(\infty)}((v_3, h_3), \mathcal{P} \cap C^{\leq l_n r^2}(r)) \neq \xi_{p,0}^{(\infty)}((v_3, h_3), \mathcal{P} \cap C(r))$  for  $p = \ell^{(\infty)}((v_3, h_3), \mathcal{P} \cap C^{\leq l_n r^2}(r)) < m$ . Using the induction hypothesis, this happens with probability smaller than  $c_1 \exp(-c_2 r^{d+1})$ .

If  $h_1 > l_n r^2$  we proceed as in the case  $m = 1$  by using the construction described in Figure 3.7. Namely, we show that there exists half of a paraboloid whose apex is on  $\partial\Pi^\uparrow(w)$  at height  $l_n r^2$

and that only contains points on a layer of order at most  $(m-1)$  for the peeling of  $\mathcal{P} \cap C^{\leq l_n r^2}(r)$ . Let us denote by  $A$  this half-paraboloid. Thanks to (3.43), the set  $A$  is included in  $C\left(\frac{r}{2^m}\right)$  which implies that

$$\mathbb{P}\left(\mathcal{P} \cap A \cap C^{\geq l_n r^2/2}\left(\frac{r}{2^m}\right) = \emptyset\right) \leq \exp(-cr^{d+1}). \quad (3.44)$$

If  $\mathcal{P} \cap A \cap C^{\geq l_n r^2/2}\left(\frac{r}{2^m}\right) \neq \emptyset$ , we choose  $(v_4, h_4)$  in that set and two cases arise.

If  $\ell^{(\infty)}((v_4, h_4), \mathcal{P} \cap C(r)) = \ell^{(\infty)}((v_4, h_4), \mathcal{P} \cap C^{\leq l_n r^2}(r))$ , then  $\ell^{(\infty)}((v_4, h_4), \mathcal{P} \cap C(r)) \leq (m-1)$  and  $h_4 \geq l_n r^2/2 = cr^2$ . Using Lemma 3.3.2, this happens with probability smaller than  $c_1 \exp(-c_2 r^{d+1})$ .

If  $\ell^{(\infty)}((v_4, h_4), \mathcal{P} \cap C(r)) \neq \ell^{(\infty)}((v_4, h_4), \mathcal{P} \cap C^{\leq l_n r^2}(r))$ , we can use the induction hypothesis and a union bound for  $p < m$  to prove that this happens with probability smaller than  $c_1 \exp(-c_2 r^{d+1})$ .

To sum up, we have shown that for any  $w \in C^{\leq l_n r^2}\left(\frac{r}{2^n}\right)$

$$\mathbb{P}(E_w) \leq c_1 r^{(m-1)(d+1)} \exp(-c_2 r^{d+1}).$$

Using Mecke's formula, we finally obtain

$$\begin{aligned} \mathbb{P}\left(\exists w \in C^{\leq l_n r^2}\left(\frac{r}{2^{m+1}}\right) : \xi_{m,0}^{(\infty)}(w, \mathcal{P}^{(\lambda)} \cap C(r)) \neq \xi_{m,0}^{(\infty)}(w, \mathcal{P}^{(\lambda)} \cap C^{\leq l_n r^2}(r))\right) \\ \leq c_1 r^{m(d+1)} \exp(-c_2 r^{d+1}). \end{aligned}$$

*Case  $\lambda < \infty$ .* We have to adapt the arguments where either the actual equation of a paraboloid or a lower bound of the intensity measure of  $\mathcal{P}$  is used, namely (3.40), (3.41), (3.43) and (3.44). Thanks to (3.8), for  $\lambda$  large enough, the series of estimates leading to (3.40) can be replaced by

$$\|v'\| \leq \dots \leq 4\sqrt{2h_1} + \frac{r}{4} \leq 4\sqrt{2l_n r^2} + \frac{r}{4} \leq r. \quad (3.45)$$

The adaptation of (3.43) is identical.

In order to show (3.41) for  $\lambda < \infty$ , we use (3.45) to show that  $[\Pi^\downarrow]^{(\lambda)}(v_2, h_2)$  is included in  $C\left(\frac{r}{2}\right)$  for  $\lambda$  large enough. In particular, the set  $[\Pi^\downarrow]^{(\lambda)}(v_2, h_2) \cap C^{\leq l_n r^2}(r)$  is contained in  $C^{\leq \lambda^{\frac{2}{d+1}}/2}\left(\frac{1}{2}\lambda^{\frac{1}{d+1}}\pi\right)$  which is a domain where the intensity measure of  $\mathcal{P}^{(\lambda)}$  is bounded from below. Consequently, (3.41) occurs for  $\lambda < \infty$  as well. In the same way, the estimate (3.44) holds for  $\lambda < \infty$  as well because  $A \cap C^{\geq l_n r^2/2}\left(\frac{r}{2^m}\right)$  is included in  $C^{\leq \lambda^{\frac{2}{d+1}}/2}\left(\frac{1}{2}\lambda^{\frac{1}{d+1}}\pi\right)$ .  $\square$

### 3.3.3 Stabilization for k-faces

Henceforth, we fix  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$ . We aim at proving Proposition 3.3.6 which states a stabilization result for the quantities  $\xi_{n,k}^{(\lambda)}$  introduced at (3.14). To do so, we start with a intermediary lemma on the distribution tail of the height of the parabolic facets containing a fixed point from the  $n$ -th layer.

Let us recall the definition of the set  $\mathcal{F}_{n,d-1}^{(\lambda)}(w, Y)$  given on page 57. For any locally finite set  $Y \subset \mathbb{R}^{d-1} \times \mathbb{R}_+$  and  $w = (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ , we introduce the maximal height of the facets from  $\mathcal{F}_{n,d-1}^{(\lambda)}(w, Y)$ , i.e.  $H_n^{(\lambda)}(w, Y) = 0$  if  $w \notin \partial\Phi_n(Y)$  and otherwise,

$$H_n^{(\lambda)}(w, Y) = \sup\{h_1 > 0 : \exists v_1 \in \mathbb{R}^{d-1}, F \in \mathcal{F}_{n,d-1}^{(\lambda)}(w, Y) \text{ s.t. } F \subset \partial[\Pi^\downarrow]^{(\lambda)}(v_1, h_1)\}. \quad (3.46)$$

**Lemma 3.3.5.** For all  $n \geq 1$  there exist  $\lambda_0, c_1, c_2 > 0$  such that for all  $\lambda \in [\lambda_0, +\infty]$ ,  $h_0 \in (0, \lambda^{\frac{2}{d+1}})$ ,  $t > 0$  and  $1 \leq r < \pi \lambda^{\frac{1}{d+1}}$ , we have

$$\mathbb{P}(\exists s \geq r : H_n^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)} \cap C(s)) \geq t) \leq c_1 \exp(-c_2 \sqrt{t}(r \wedge \sqrt{t})^d) \quad (3.47)$$

and

$$\mathbb{P}(H_n^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \geq t) \leq c_1 \exp(-c_2 t^{\frac{d+1}{2}}). \quad (3.48)$$

*Proof.* We only prove (3.47) as the proof of (3.48) is almost identical.

Case  $\lambda = \infty$ . For  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ , we introduce the events

$$E_{(v_1, h_1)} = \{\exists s \geq r, F \in \mathcal{F}_{n, d-1}((0, h_0), \mathcal{P} \cap C(s)), F \subset \partial \Pi^\downarrow(v_1, h_1)\}$$

and

$$E = E(t) := \bigcup_{(v_1, h_1) \in \Pi^\uparrow(0, h_0), h_1 \geq t} E_{(v_1, h_1)}.$$

It is enough to prove that

$$\mathbb{P}(E) \leq c_1 \exp(-c_2 \sqrt{t}(r \wedge \sqrt{t})^d). \quad (3.49)$$

In the same spirit as in the proof of Proposition 3.3.3, we consider two cases depending on the value of  $h_0$ . The left- (resp. right-) hand side of Figure 3.6 reflects the first case  $h_0 \geq t/2$  (resp.  $h_0 \leq t/2$ ).

Case  $h_0 \geq t/2$ . As in the proof of Lemma 3.3.2, we start by decomposing  $\Pi^\downarrow(0, h_0)$  into  $2^{d-1}$  deterministic subparts  $A_1, \dots, A_{2^{d-1}}$  such that

$$\begin{aligned} \text{Vol}_d(A_i \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n)) &= \frac{1}{2^{d-1}} \text{Vol}_d(\Pi^\downarrow(0, h_0) \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n)) \\ &\geq ct(r \wedge \sqrt{t})^{d-1}. \end{aligned} \quad (3.50)$$

Let  $(v_1, h_1) \in \Pi^\uparrow(0, h_0)$  such that  $h_1 \geq t$  and  $\partial \Pi^\downarrow(v_1, h_1)$  contains a facet of  $\partial \Phi_n(\mathcal{P} \cap C(s))$  going through  $(0, h_0)$  for some  $s \geq r$ . By Lemma 3.3.1,  $\Pi^\downarrow(v_1, h_1)$  contains half of  $\Pi^\downarrow(0, h_0)$ , which implies that it contains a set  $A_i \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n)$  for some  $i$ . Consequently,

$$\begin{aligned} \mathbb{P}(E) &= \mathbb{P}\left(\bigcup_{(v_1, h_1) \in \Pi^\uparrow(0, h_0), h_1 > t} E_{(v_1, h_1)}\right) \\ &\leq \mathbb{P}\left(\bigcup_{s \geq r} \bigcup_{i=1}^{2^{d-1}} \{\mathcal{P} \cap A_i \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n) \subset [\Phi_n(\mathcal{P} \cap C(s))]^c\}\right) \\ &\leq \mathbb{P}\left(\bigcup_{i=1}^{2^{d-1}} \{\mathcal{P} \cap A_i \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n) \subset [\Phi_n(\mathcal{P} \cap C(r))]^c\}\right) \\ &\leq 2^{d-1} \mathbb{P}(\mathcal{P} \cap A_1 \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n) \subset [\Phi_n(\mathcal{P} \cap C(r))]^c). \end{aligned} \quad (3.51)$$

where the inclusion  $\Phi_n(\mathcal{P} \cap C(r)) \subset \Phi_n(\mathcal{P} \cap C(s))$  is due to Lemma 3.2.6. By (3.50),  $\mathcal{P}$  does not meet  $A_1 \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n)$  with probability smaller than  $\exp(-ct(r \wedge \sqrt{t})^{d-1})$ . Otherwise it

contains a point  $(v_3, h_3)$  with  $h_3 \geq h_0/2 \geq t/4$  and such that  $\ell^{(\infty)}((v_3, h_3), \mathcal{P} \cap C(r)) \leq (n-1)$ . Using Lemma 3.3.2, we obtain that

$$\mathbb{P}(\mathcal{P} \cap A_1 \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n) \subset \Phi_n^c(\mathcal{P}_\lambda \cap C(r))) \leq c_1 \exp(-c_2 t (r \wedge \sqrt{t})^{d-1}),$$

which implies (3.49) thanks to (3.51) and the inequality  $t(r \wedge \sqrt{t})^{d-1} \geq \sqrt{t}(r \wedge \sqrt{t})^d$ .

*Case  $h_0 \leq t/2$ .* Let  $(v_1, h_1) \in \partial\Pi^\uparrow(0, h_0)$  such that  $h_1 \geq t$  and  $\partial\Pi^\downarrow(v_1, h_1)$  contains a facet of  $\partial\Phi_n(\mathcal{P} \cap C(s))$  going through  $(0, h_0)$  for some  $s \geq r$ . For  $u > 0$ , the height of the highest point of intersection between  $\Pi^\downarrow(v_1, h_1)$  and  $C(u/2^{n+1})$  in the vertical plane containing  $(v_1, h_1)$  and the origin is

$$h_2 := h_0 + \sqrt{2(h_1 - h_0)} \frac{u}{2^{n+1}} - \frac{u^2}{2^{2n+3}}. \quad (3.52)$$

We take  $u = \sqrt{2(h_1 - h_0)} \wedge r$  and fit a cylinder  $C_0$  of radius  $\frac{u}{2^{n+2}}$  between height  $h_2/2$  and  $h_2$  in  $C(\frac{u}{2^n}) \cap \Pi^\downarrow(v_1, h_1)$ , see the right-hand side of Figure 3.6 with  $r$  replaced by  $u$ . Either the point process  $\mathcal{P}$  does not meet the cylinder  $C_0$  which happens with probability  $\exp(-ch_2 u^{d-1})$  or it contains a point  $(v_3, h_3)$  with  $h_3 \geq h_2/2$  and such that

$$\ell^{(\infty)}((v_3, h_3), \mathcal{P} \cap C(r)) \leq \ell^{(\infty)}((v_3, h_3), \mathcal{P} \cap C(s)) \leq (n-1).$$

Using Lemma 3.3.2, this happens with probability smaller than  $c_1 \exp(-c_2 h_2 (u \wedge \sqrt{h_2})^{d-1})$ . Consequently, we get

$$\mathbb{P}(E_{(v_1, h_1)}) \leq c_1 u^{d-1} \exp\left(-c_2 h_2 (u \wedge \sqrt{h_2})^{d-1}\right). \quad (3.53)$$

It remains to make explicit the right-hand side of (3.53) in the two cases  $u = \sqrt{2(h_1 - h_0)}$  and  $u = r$ .

When  $u = \sqrt{2(h_1 - h_0)} \leq r$ , we deduce from (3.52) and the fact that  $h_1 \geq t$  that

$$h_2 = h_0 + \frac{h_1 - h_0}{2^n} - \frac{h_1 - h_0}{2^{2(n+1)}} \geq c(h_1 - h_0) \geq c't. \quad (3.54)$$

Combining (3.53) with (3.54), we obtain

$$\mathbb{P}(E_{(v_1, h_1)}) \leq c_1 \exp(-c_2 (h_1 - h_0) \sqrt{t}^{d-1}). \quad (3.55)$$

When  $u = r \leq \sqrt{2(h_1 - h_0)}$ , we obtain in the same way

$$\mathbb{P}(E_{(v_1, h_1)}) \leq c_1 \exp(-c_2 (h_1 - h_0) r^d). \quad (3.56)$$

Discretizing and integrating the right-hand sides of (3.55) and (3.56) over the set  $\{(v_1, h_1) \in \Pi^\uparrow(0, h_0) : h_1 \geq t\}$ , we deduce (3.49) when  $h_0 \leq t/2$ .

This completes the proof of (3.47) in the case  $\lambda = \infty$ .

*Case  $\lambda < \infty$ .* In the case  $h_0 \geq t/2$ , we need to replace  $A_i \cap C^{\geq h_0/2}((r \wedge \sqrt{t})/2^n)$  with  $A_i \cap C^{(h_0/2, \frac{3h_0}{4})}((r \wedge \sqrt{t})/2^n)$  which is included in  $C^{\leq \frac{3}{4}\lambda^{\frac{2}{d+1}}}(\frac{\pi}{2}\lambda^{\frac{1}{d+1}})$  so that we can lower bound by a constant the density of the intensity measure of  $\mathcal{P}^{(\lambda)}$  when on that particular set. We adapt the case  $h_0 \leq t/2$  in the exact same way as in the proof of Proposition 3.3.3 by replacing the equality (3.52) by an inequality up to a multiplicative constant and reducing the cylinder  $C_0$  so that it is included in  $C^{(h_2/2, \frac{3}{4}h_2)}(\frac{u}{2^n})$ , which makes it possible to lower bound by a constant the density of the intensity measure of  $\mathcal{P}^{(\lambda)}$  on  $C_0$ .  $\square$

**Proposition 3.3.6.** *For any  $n \geq 1$  there exists  $\lambda_0, c_1, c_2 > 0$  such that for any  $h_0 > 0, \lambda \in [\lambda_0, +\infty]$  and  $1 \leq r < \pi\lambda^{\frac{1}{d+1}}$ , we have*

$$\mathbb{P}\left(R_{n,k}^{(\lambda)}(0, h_0) \geq r\right) \leq c_1 \exp(-c_2 r^{d+1}).$$

*Proof.* Let us assume that  $R_{n,k}^{(\lambda)}(0, h_0) > r$ .

First, we can also assume that  $R_{n,0}^{(\lambda)}(0, h_0) < r$  since the complement occurs with probability smaller than  $c_1 \exp(-c_2 r^{d+1})$  by Proposition 3.3.3. In particular we have  $\xi_{n,0}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) = \xi_{n,0}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)} \cap C(s))$  for every  $s \geq r$ . We can also assume that both are different from 0, or equivalently that  $(0, h_0)$  is on the  $n$ -th layer for both peelings because otherwise we would have  $R_{n,k}^{(\lambda)}(0, h_0) \leq r$ .

Thanks to Lemma 3.3.5, we have  $\min(H_n((0, h_0), \mathcal{P}^{(\lambda)}), \max_{s \geq r} H_n((0, h_0), \mathcal{P}^{(\lambda)} \cap C(s))) > r^2/32$  with probability smaller than  $c_1 \exp(-c_2 r^{d+1})$ . Consequently, we can assume that for any  $s \in [r, \infty]$ ,  $H_n((0, h_0), \mathcal{P}^{(\lambda)} \cap C(s)) \leq r^2/32$ .

Let

$$\mathcal{U} = \mathcal{U}(h_0, r^2/32) := \bigcup_{(v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(0, h_0), h_1 \leq r^2/32} \Pi^\downarrow(v_1, h_1). \quad (3.57)$$

When  $\lambda = \infty$ , a point  $(v, h) \in \mathcal{U}$  verifies  $\|v\| \leq \sqrt{2(r^2/32 - h_0)} + \sqrt{2r^2/32} \leq 2\sqrt{2r^2/32} \leq r/2$ . By (3.8), this implies that for  $\lambda$  large enough, a point  $(v, h) \in \mathcal{U}$  satisfies  $\|v\| \leq \frac{3}{4}r$ .

The set  $\mathcal{U}$  is designed to include all points of  $\mathcal{P}^{(\lambda)}$  which lie on a common  $k$ -face of  $\partial\Phi_n(\mathcal{P}^{(\lambda)} \cap C(s))$  with  $(0, h_0)$  for every  $s \in [r, \infty]$ . We assert that

$$\{R_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \geq r\} \subset \{\exists w \in \mathcal{P}^{(\lambda)} \cap \mathcal{U} : R_{n,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \geq r/4\}. \quad (3.58)$$

Indeed, if every point  $w \in \mathcal{P}^{(\lambda)} \cap \mathcal{U}$  verifies  $R_{n,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \leq r/4$ , then the status of these points with respect to the  $n$ -th layer is the same for both  $\mathcal{P}^{(\lambda)}$  and  $\mathcal{P}^{(\lambda)} \cap C(s)$ , for any  $s \geq r$ . Consequently, the  $k$ -faces of the  $n$ -th layer containing  $(0, h_0)$  are the same for the peeling of any  $\mathcal{P}^{(\lambda)} \cap C(s)$ ,  $s \geq r$ , which implies that  $R_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \leq r$ . Using consecutively (3.58), Mecke's formula and the fact that the density of the intensity measure of  $\mathcal{P}^{(\lambda)}$  is upper bounded by 1, we have

$$\begin{aligned} \mathbb{P}(R_{n,k}^{(\lambda)}(0, h_0) \geq r) &\leq \mathbb{E} \left[ \sum_{x \in \mathcal{U} \cap \mathcal{P}^{(\lambda)}} \mathbf{1}_{R_{n,0}^{(\lambda)}(x) \geq r/4} \right] \\ &\leq \int_{\mathcal{U}} \mathbb{P}(R_{n,0}^{(\lambda)}(x) \geq r/4) dx \\ &\leq \text{Vol}_d(\mathcal{U}) c_1 \exp(-c_2 r^{d+1}) \\ &\leq c_1 \exp(-c_2 r^{d+1}). \end{aligned}$$

This yields the result.  $\square$

The final result of this section is a slight analogue of Proposition 3.3.6 for the stabilization in height, i.e. we prove that with high probability the calculation of the score of  $(0, h_0)$  inside the cylinder  $C(r)$  does not depend on the points of the point process which are higher than  $r^2$

up to a multiplicative constant. The statement of that result uses the following notation: for  $w = (v, h) \in W_\lambda$ ,  $r, h > 0$ ,  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$ , we denote by  $\xi_{n,k,[r,h]}^{(\lambda)}(w)$  the quantity

$$\xi_{n,k,[r,h]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)} \cap C_v^{\leq h}(r)).$$

Let us also recall the notation  $\xi_{n,k,[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$  introduced at (3.22).

**Lemma 3.3.7.** *For all  $n \geq 1$  and there exists  $\lambda_0, c > 0$  such that for all  $h_0 > 0$ ,  $\lambda \in [\lambda_0, +\infty]$  and  $1 \leq r < \pi\lambda^{\frac{1}{d+1}}$ , we have*

$$\mathbb{P}\left(\xi_{n,k,[r]}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \neq \xi_{n,k,[r,l_n r^2]}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)})\right) \leq c_1 \exp(-c_2 r^{d+1})$$

with  $l_n = \frac{1}{2^{2n+\tau}}$ .

*Proof.* We denote by  $A$  the event  $\{H_n((0, h_0), \mathcal{P}^{(\lambda)} \cap C(r)) > l_n r^2\}$ . Using Lemma 3.3.5 with the choice  $r$  for  $t$  and  $l_n r^2$  for  $r$ , we get

$$\mathbb{P}(A) \leq c_1 \exp(-c_2 r^{d+1}). \quad (3.59)$$

On the event  $A^c$ , every paraboloid containing a facet going through  $(0, h_0)$  of  $\partial\Phi_n(\mathcal{P}^{(\lambda)} \cap C(r))$  has height smaller than  $l_n r^2$ . This implies that every point that shares a facet of  $\partial\Phi_n(\mathcal{P}^{(\lambda)} \cap C(r))$  with  $(0, h_0)$  is on the boundary of such a paraboloid and is consequently included in  $C^{\leq l_n r^2}(\frac{r}{2^{n+\tau}})$  by the same method as in (3.41). Using Lemma 3.3.4, we obtain that with probability larger than  $1 - c_1 \exp(-c_2 r^{d+1})$ , any  $w \in C^{\leq l_n r^2}(\frac{r}{2^{n+\tau}})$  verifies  $\xi_{n,0,[r]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = \xi_{n,0,[r,l_n r^2]}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$ . Thus we deduce that with probability larger than  $1 - c_1 \exp(-c_2 r^{d+1})$ , the  $k$ -faces containing  $w$  of  $\partial\Phi_n(\mathcal{P}^{(\lambda)} \cap C(r))$  coincide with those of  $\partial\Phi_n(\mathcal{P}^{(\lambda)} \cap C^{\leq l_n r^2}(r))$ . In other terms we have proved that

$$\mathbb{P}\left(\{\xi_{n,k,[r]}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \neq \xi_{n,k,[r,l_n]}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)})\} \cap A^c\right) \leq c_1 \exp(-c_2 r^{d+1}). \quad (3.60)$$

Combining (3.59) and (3.60), we obtain the result.  $\square$

## 3.4 $L^p$ bounds and pointwise convergences

In this section, we prove intermediary results on the functionals  $\xi_{n,k}^{(\lambda)}$  and their variations which depend on the stabilization properties of Section 3.3 and pave the way to the proofs of Theorems 3.2.8 and 3.2.9. More precisely, Lemma 3.4.1 states some moment bounds, Lemma 3.4.2 and Proposition 3.4.3 show in two steps the convergence of the expectation of  $\xi_{n,k}^{(\lambda)}$  to the expectation of  $\xi_{n,k}^{(\infty)}$ . We then deduce similar results for covariances of scores in Proposition 3.4.4 and Lemma 3.4.5.

**Lemma 3.4.1.** *For any  $p \in [1, +\infty)$ , there exist constants  $\lambda_0, c > 0$  such that for any  $(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ ,  $\lambda \in [\lambda_0, \infty]$  and  $1 \leq r < \pi\lambda^{\frac{1}{d+1}}$*

$$\mathbb{E}\left[\xi_{n,k}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})^p\right] \leq c_1 h^{pk} \exp(-c_2 h^{\frac{d+1}{2}}), \quad (3.61)$$

$$\mathbb{E} \left[ \xi_{n,k,[r]}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})^p \right] \leq cr^{pk(d-1)+1} (h \vee r^2)^{pk}, \quad (3.62)$$

$$\mathbb{E} \left[ \xi_{n,k,[r,l_n r^2]}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})^p \right] \leq cr^{pk(d+1)+1}. \quad (3.63)$$

with  $l_n = \frac{1}{2^{2n+7}}$ .

*Proof.* The method is almost identical to [26, Lemma 4.4]. For the sake of completeness and because the variants at (3.62) and (3.63) are new in comparison to [26], we provide as an example the proof of (3.62).

We assert that  $\xi_{n,k,[r]}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)}) \stackrel{d}{=} \xi_{n,k,[r]}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})$  because of the rotation-invariance of the initial model in the unit ball.

We introduce the variables  $H := H_n^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)} \cap C(r))$  as defined at (3.46) and  $R$  as the smallest  $s > 0$  such that  $C(s)$  contains  $\bigcup_{(v_1, h_1) \in \mathcal{P}^{(\lambda)} \cap \mathcal{U}(h, H)} C_{v_1}(R_{n,k}^{(\lambda)}(v_1, h_1))$  where  $\mathcal{U}(\cdot, \cdot)$  has been introduced at (3.57). We assert that the proof of Proposition 3.3.6 implies that

$$\mathbb{P}(R \geq r) \leq c_1 e^{-c_2 r^{d+1}}. \quad (3.64)$$

In particular, only the points in  $C^{\leq H}(R)$  can be part of a potential facet containing  $(0, h)$  on  $\partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)})$ . This implies that

$$\xi_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}) \leq \frac{1}{k+1} \binom{N}{k}$$

where  $N = \text{card}(C^{\leq H}(R) \cap \mathcal{P}^{(\lambda)})$ . Consequently, it is enough to show that there exist  $c > 0$  such that

$$\mathbb{E}[N^{pk}] \leq cr^{pk(d-1)+1} (h \vee r^2)^{pk}. \quad (3.65)$$

We decompose the expectation in the following way:

$$\begin{aligned} \mathbb{E}[N^{pk}] &= \sum_{i=1}^{\lceil r \rceil} \sum_{j=\lfloor h \rfloor+1}^{\infty} \mathbb{E}[N^{pk} \mathbf{1}_{(i-1) \leq R < i} \mathbf{1}_{(j-1) \leq H < j}] \\ &\leq \sum_{i=1}^{\lceil r \rceil} \sum_{j=\lfloor h \rfloor+1}^{\infty} \mathbb{E}[\text{card}(C^{\leq j}(i) \cap \mathcal{P}^{(\lambda)})^{pk} \mathbf{1}_{(i-1) \leq R < i} \mathbf{1}_{(j-1) \leq H < j}] \\ &\leq \sum_{i=1}^{\lceil r \rceil} \sum_{j=\lfloor h \rfloor+1}^{\infty} \mathbb{E}[\text{card}(C^{\leq j}(i) \cap \mathcal{P}^{(\lambda)})^{3pk}]^{1/3} \mathbb{P}(R \geq i-1)^{1/3} \mathbb{P}(H \geq j-1)^{1/3} \end{aligned} \quad (3.66)$$

where the last line is a consequence of Hölder's inequality. For any  $i, j \geq 0$ , we observe that  $C^{\leq j}(i)$  has volume  $ci^{d-1}j$  and thanks to (3.5), the  $d\mathcal{P}^{(\lambda)}$ -measure of  $C^{\leq j}(i)$  is bounded by its volume. Consequently, the variable  $\text{card}(\mathcal{P} \cap C^{\leq j}(i))$  is stochastically dominated by a Poisson variable  $\text{Po}(ci^{d-1}j)$ . Combining this fact, the moment bound  $\mathbb{E}[Po(\lambda)^r] \leq c\lambda^r$  for any  $r \geq 1$  and (3.66), we obtain

$$\mathbb{E}[N^{pk}] \leq \sum_{i=1}^{\lceil r \rceil} \sum_{j=\lfloor h \rfloor+1}^{\infty} ci^{pk(d-1)} j^{pk} \mathbb{P}(R \geq (i-1))^{1/3} \mathbb{P}(H \geq (j-1))^{1/3}.$$

Let us assume that  $h \leq r^2$ . We can now use Lemma 3.3.5 and (3.64) to get

$$\begin{aligned} \mathbb{E}[N^{pk}] &\leq \sum_{i=1}^{\lfloor r \rfloor} c_1 i^{pk(d-1)} e^{-c_2 i^{d+1}} \left( \sum_{j=\lfloor h \rfloor + 1}^{\lfloor r^2 \rfloor} j^{pk} e^{-c_3 j^{\frac{d+1}{2}}} + \sum_{j=\lfloor r^2 \rfloor + 1}^{\infty} j^{pk} e^{-c_4 \sqrt{j} r^d} \right) \\ &\leq c_1 r^{pk(d-1)+1} r^{2pk} \left( e^{-c_2 h^{\frac{d+1}{2}}} + e^{-c_3 r^{d+1}} \right). \end{aligned}$$

This shows (3.65) and consequently (3.62). We proceed in the same way when  $h \geq r^2$ .  $\square$

The next lemma is a first step towards the convergence of the expectation of  $\xi_{n,k}$  stated in Proposition 3.4.3. It relies on the application of the continuous mapping theorem to the intersection of the point process  $\mathcal{P}^{(\lambda)}$  with the compact set  $C^{\leq l_n r^2}(r)$ .

**Lemma 3.4.2.** *There exists  $c > 0$  such that for all  $h_0 \geq 0$  and  $r \geq 1$  we have*

$$\overline{\lim}_{\lambda \rightarrow +\infty} |\mathbb{E}[\xi_{n,k,[r]}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)})] - \mathbb{E}[\xi_{n,k,[r]}^{(\infty)}((0, h_0), \mathcal{P})]| \leq c_1 r^{k(d-1)+1/2} (h_0 \vee r^2)^k e^{-c_2 r^{d+1}}.$$

*Proof.* For sake of simplicity, we use the following abbreviations:

$$\xi_{n,k,[r]}^{(\lambda)} := \xi_{n,k,[r]}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \quad \text{and} \quad \xi_{n,k,[r]}^{(\infty)} := \xi_{n,k,[r]}^{(\infty)}((0, h_0), \mathcal{P}).$$

Recalling  $l_n = \frac{1}{2^{2n+7}}$ , we use similar notations  $\xi_{n,k,[r,l_n r^2]}^{(\lambda)}$  and  $\xi_{n,k,[r,l_n r^2]}^{(\infty)}$  and obtain

$$\begin{aligned} &|\mathbb{E}[\xi_{n,k,[r]}^{(\lambda)}] - \mathbb{E}[\xi_{n,k,[r]}^{(\infty)}]| \\ &\leq |\mathbb{E}[\xi_{n,k,[r]}^{(\lambda)} - \xi_{n,k,[r,l_n r^2]}^{(\lambda)}]| + |\mathbb{E}[\xi_{n,k,[r,l_n r^2]}^{(\lambda)}] - \mathbb{E}[\xi_{n,k,[r,l_n r^2]}^{(\infty)}]| + |\mathbb{E}[\xi_{n,k,[r,l_n r^2]}^{(\infty)}] - \mathbb{E}[\xi_{n,k,[r]}^{(\infty)}]|. \end{aligned} \quad (3.67)$$

We start by bounding the first term in the rhs of (3.67). Using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} |\mathbb{E}[\xi_{n,k,[r]}^{(\lambda)} - \xi_{n,k,[r,l_n r^2]}^{(\lambda)}]| &= |\mathbb{E}[(\xi_{n,k,[r]}^{(\lambda)} - \xi_{n,k,[r,l_n r^2]}^{(\lambda)}) \mathbb{1}_{\{\xi_{n,k,[r]}^{(\lambda)} \neq \xi_{n,k,[r,l_n r^2]}^{(\lambda)}\}}]| \\ &\leq \mathbb{E}[(\xi_{n,k,[r]}^{(\lambda)} - \xi_{n,k,[r,l_n r^2]}^{(\lambda)})^2]^{1/2} \mathbb{P}(\xi_{n,k,[r]}^{(\lambda)} \neq \xi_{n,k,[r,l_n r^2]}^{(\lambda)})^{1/2}. \end{aligned}$$

Combining Lemma 3.3.7 and Lemma 3.4.1, we obtain for  $\lambda$  large enough

$$|\mathbb{E}[\xi_{n,k,[r]}^{(\lambda)} - \xi_{n,k,[r,l_n r^2]}^{(\lambda)}]| \leq c_1 r^{k(d-1)+1/2} (h_0 \vee r^2)^k \exp(-c_2 r^{d+1}). \quad (3.68)$$

In the same way, we bound the third term in the rhs of (3.67) to get

$$|\mathbb{E}[\xi_{n,k,[r]}^{(\infty)} - \xi_{n,k,[r,l_n r^2]}^{(\infty)}]| \leq c_1 r^{k(d-1)+1/2} (h_0 \vee r^2)^k \exp(-c_2 r^{d+1}). \quad (3.69)$$

Let us now prove that

$$\lim_{\lambda \rightarrow +\infty} \mathbb{E}[\xi_{n,k,[r,l_n r^2]}^{(\lambda)}] = \mathbb{E}[\xi_{n,k,[r,l_n r^2]}^{(\infty)}]. \quad (3.70)$$

To do so, we first prove the convergence in distribution of  $\xi_{n,k,[r,l_n r^2]}^{(\lambda)}$  to  $\xi_{n,k,[r,l_n r^2]}^{(\infty)}$ . We denote by  $\mathcal{X}(r, l_n r^2)$  the set of finite points sets in  $C^{\leq l_n r^2}(r)$  and we endow it with the discrete topology. A

sequence of point sets  $(\eta_i)_i$  of  $\mathcal{X}(r, l_n r^2)$  converges to a point set  $\eta$  if  $\eta_i = \eta$  for all  $i \geq i_0$  for some  $i_0 \in \mathbb{N}$ .

We define for all  $w \in W_\lambda$  and  $\eta \in \mathcal{X}(r, l_n r^2)$

$$g_{n,\lambda}(w, \eta) := \xi_{n,k,[r,l_n r^2]}^{(\lambda)}(w, \eta).$$

Considering that  $\mathcal{P}^{(\lambda)} \rightarrow \mathcal{P}$  in distribution in  $C^{\leq l_n r^2}(r)$  by Lemma 3.2.1, we intend to show the convergence in distribution  $g_{n,\lambda}(w_0, \mathcal{P}^{(\lambda)}) \rightarrow g_{n,\infty}(w_0, \mathcal{P})$  by using [17, Theorem 5.5, p. 34]. To do so, we observe that since  $\mathcal{P}$  is in general position with probability 1, it is enough to prove that for any  $\eta \in \mathcal{X}(r, l)$  in general position,  $g_{n,\lambda}(w_0, \eta) = g_{n,\infty}(w_0, \eta)$  for all  $\lambda$  large enough.

Let  $\eta \in \mathcal{X}(r, l_n r^2)$  be in general position. We explain here why for each point  $w_1$  of  $\eta$ , the number of the layer containing  $w_1$  and the local structure of that layer around  $w_1$  are fixed for  $\lambda$  large enough. We start by considering an extreme point  $w_1$  for the parabolic hull peeling of  $\eta$ . We choose a  $(d-1)$ -dimensional parabolic face containing  $w_1$  which is generated by the extreme points  $w_1, w_2, \dots, w_d$  and we also denote by  $\Pi^\downarrow(v, h)$  the downward paraboloid containing that face on its boundary. For  $\varepsilon > 0$  small enough, the intersection of  $\Pi^\downarrow(v, h + \varepsilon)$  with  $\eta$  is  $\{w_1, \dots, w_d\}$ . Since the quasi-paraboloids converge to the real paraboloids as  $\lambda$  goes to infinity, for  $\lambda$  large enough the quasi-paraboloid with  $w_1, \dots, w_d$  on its boundary is contained in  $\Pi^\downarrow(v, h + \varepsilon)$  and does not contain any point of  $\eta$  in its interior. This means that for  $\lambda$  large enough,  $w_1$  is extreme and the facets containing it are generated by the same points. Applying this to every extreme point, we deduce that the first layer is stable for  $\lambda$  large enough. By induction on the number of the layer, we prove similarly that the subsequent layers of the quasi-parabolic hull peeling of  $\eta$  are stable for  $\lambda$  large enough. This means that  $g_{n,\lambda}(w_0, \eta) = g_{n,\infty}(w_0, \eta)$  for all  $\lambda$  large enough and as a consequence completes the proof of the convergence in distribution of  $\xi_{n,k,[r,l_n r^2]}^{(\lambda)}(w_0, \mathcal{P}^{(\lambda)})$  to  $\xi_{n,k,[r,l_n r^2]}^{(\infty)}(w_0, \mathcal{P})$ . This extends to (3.70) by [17, Theorem 5.4] since the considered sequence is bounded in  $L^p$  with  $p > 1$  thanks to Lemma 3.4.1.

Inserting the results (3.68), (3.69) and (3.70) into (3.67), we deduce Lemma 3.4.2.  $\square$

We are now ready to state the required convergence in expectation in Proposition 3.4.3 below.

**Proposition 3.4.3.** *For any  $n \geq 1$  and all  $h_0 > 0$  we have*

$$\lim_{\lambda \rightarrow +\infty} \mathbb{E} \left[ \xi_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \right] = \mathbb{E} \left[ \xi_{n,k}^{(\infty)}((0, h_0), \mathcal{P}) \right].$$

*Proof.* With the same notation as in the proof of Lemma 3.4.2, we get from the triangle inequality that

$$|\mathbb{E}[\xi_{n,k}^{(\lambda)}] - \mathbb{E}[\xi_{n,k}^{(\infty)}]| \leq |\mathbb{E}[\xi_{n,k}^{(\lambda)}] - \mathbb{E}[\xi_{n,k,[r]}^{(\lambda)}]| + |\mathbb{E}[\xi_{n,k,[r]}^{(\lambda)}] - \mathbb{E}[\xi_{n,k,[r]}^{(\infty)}]| + |\mathbb{E}[\xi_{n,k,[r]}^{(\infty)}] - \mathbb{E}[\xi_{n,k}^{(\infty)}]|. \quad (3.71)$$

Using the same method as in the proof of (3.68), we obtain thanks to Proposition 3.3.6 and Lemma 3.4.1 that for  $r$  large enough,

$$\max(|\mathbb{E}[\xi_{n,k}^{(\lambda)}] - \mathbb{E}[\xi_{n,k,[r]}^{(\lambda)}]|, |\mathbb{E}[\xi_{n,k,[r]}^{(\infty)}] - \mathbb{E}[\xi_{n,k}^{(\infty)}]|) \leq r^{k(d-1) + \frac{1}{2} + 2k} e^{-cr^{\frac{d+1}{2}}}. \quad (3.72)$$

Inserting (3.72) and the result of Lemma 3.4.2 into (3.71), we obtain that for any  $\varepsilon > 0$ , there exists  $r$  large enough such that  $\overline{\lim}_{\lambda \rightarrow \infty} |\mathbb{E}[\xi_{n,k}^{(\lambda)}] - \mathbb{E}[\xi_{n,k}^{(\infty)}]| \leq \varepsilon$ . This completes the proof of Proposition 3.4.3.  $\square$

In view of the required variance estimates, we now aim at extending Proposition 3.4.3 when the expectation of a score is replaced by the covariance of two scores  $\xi_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)})$  and  $\xi_{n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)})$  at two points  $(0, h_0)$  and  $(v_1, h_1)$  belonging to  $W_\lambda$ . To do so, we study the two-point correlation function defined at (3.16).

**Proposition 3.4.4.** *For all  $h_0 \geq 0$  and  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$  we have*

$$\lim_{\lambda \rightarrow \infty} c_{n,k}^{(\lambda)}((0, h_0), (v_1, h_1)) = c_{n,k}^{(\infty)}((0, h_0), (v_1, h_1)).$$

*Proof.* For sake of simplicity, let us use the following reduced notation for  $\lambda \in [\lambda_0, \infty]$ :

$$\xi^{(\lambda)}(0) := \xi_{n,k}^{(\lambda)}\left((0, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}\right), \xi^{(\lambda)}(1) := \xi_{n,k}^{(\lambda)}\left((v_1, h_1), \mathcal{P} \cup \{(0, h_0)\}\right). \quad (3.73)$$

By Proposition 3.4.3, we get

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[ \xi_{n,k}^{(\lambda)}\left((0, h_0), \mathcal{P}^{(\lambda)}\right) \right] \mathbb{E} \left[ \xi_{n,k}^{(\lambda)}\left((v_1, h_1), \mathcal{P}^{(\lambda)}\right) \right] = \mathbb{E} \left[ \xi_{n,k}^{(\infty)}\left((0, h_0), \mathcal{P}\right) \right] \mathbb{E} \left[ \xi_{n,k}^{(\infty)}\left((v_1, h_1), \mathcal{P}\right) \right]$$

so we only need to prove

$$\lim_{\lambda \rightarrow \infty} \mathbb{E} \left[ \xi^{(\lambda)}(0) \xi^{(\lambda)}(1) \right] = \mathbb{E} \left[ \xi^{(\infty)}(0) \xi^{(\infty)}(1) \right]. \quad (3.74)$$

Let us fix  $\varepsilon > 0$ . We can show that for  $r$  large enough

$$\overline{\lim}_{\lambda \rightarrow \infty} \left| \mathbb{E} \left[ \xi^{(\lambda)}(0) \xi^{(\lambda)}(1) \right] - \mathbb{E} \left[ \xi_{[r]}^{(\lambda)}(0) \xi_{[r]}^{(\lambda)}(1) \right] \right| \leq \varepsilon, \quad (3.75)$$

$$\overline{\lim}_{\lambda \rightarrow \infty} \left| \mathbb{E} \left[ \xi_{[r]}^{(\lambda)}(0) \xi_{[r]}^{(\lambda)}(1) \right] - \mathbb{E} \left[ \xi_{[r, l_n r^2]}^{(\lambda)}(0) \xi_{[r, l_n r^2]}^{(\lambda)}(1) \right] \right| \leq \varepsilon, \quad (3.76)$$

$$\lim_{\lambda \rightarrow \infty} \left| \mathbb{E} \left[ \xi_{[r, l_n r^2]}^{(\lambda)}(0) \xi_{[r, l_n r^2]}^{(\lambda)}(1) \right] - \mathbb{E} \left[ \xi_{[r, l_n r^2]}^{(\infty)}(0) \xi_{[r, l_n r^2]}^{(\infty)}(1) \right] \right| = 0, \quad (3.77)$$

$$\left| \mathbb{E} \left[ \xi_{[r]}^{(\infty)}(0) \xi_{[r]}^{(\infty)}(1) \right] - \mathbb{E} \left[ \xi_{[r, l_n r^2]}^{(\infty)}(0) \xi_{[r, l_n r^2]}^{(\infty)}(1) \right] \right| \leq \varepsilon, \quad (3.78)$$

$$\left| \mathbb{E} \left[ \xi^{(\infty)}(0) \xi^{(\infty)}(1) \right] - \mathbb{E} \left[ \xi_{[r]}^{(\infty)}(0) \xi_{[r]}^{(\infty)}(1) \right] \right| \leq \varepsilon \quad (3.79)$$

where we have used the reduced notation for  $\xi_{n,k,[r]}^{(\lambda)}$  and  $\xi_{n,k,[r, l_n r^2]}^{(\lambda)}$  and their counterparts in the limit model similar to the one introduced at (3.73). These five assertions imply (3.74) so it is enough to show each of them. We claim that (3.75), (3.76), (3.78) and (3.79) are obtained by similar methods. Consequently we omit the proofs of (3.76), (3.78) and (3.79) and concentrate on getting (3.75).

$$\begin{aligned} & \left| \mathbb{E} \left[ \xi^{(\lambda)}(0) \xi^{(\lambda)}(1) \right] - \mathbb{E} \left[ \xi_{[r]}^{(\lambda)}(0) \xi_{[r]}^{(\lambda)}(1) \right] \right| \\ & \leq \left| \mathbb{E} \left[ (\xi^{(\lambda)}(0) - \xi_{[r]}^{(\lambda)}(0)) \xi^{(\lambda)}(1) \right] \right| + \left| \mathbb{E} \left[ \xi_{[r]}^{(\lambda)}(0) (\xi^{(\lambda)}(1) - \xi_{[r]}^{(\lambda)}(1)) \right] \right|. \end{aligned} \quad (3.80)$$

We derive an upper bound for the first term in the rhs of (3.80), the second term being treated identically. Using Hölder's inequality, we obtain

$$\begin{aligned} & \left| \mathbb{E} \left[ (\xi^{(\lambda)}(0) - \xi_{[r]}^{(\lambda)}(0)) \xi^{(\lambda)}(1) \right] \right| \\ & = \left| \mathbb{E} \left[ (\xi^{(\lambda)}(0) - \xi_{[r]}^{(\lambda)}(0)) \xi^{(\lambda)}(1) \mathbf{1}_{\{\xi^{(\lambda)}(0) \neq \xi_{[r]}^{(\lambda)}(0)\}} \right] \right| \\ & \leq \mathbb{E} \left[ |\xi^{(\lambda)}(0) - \xi_{[r]}^{(\lambda)}(0)|^3 \right]^{1/3} \mathbb{E} \left[ \xi^{(\lambda)}(1)^3 \right]^{1/3} \mathbb{P}(\xi^{(\lambda)}(0) \neq \xi_{[r]}^{(\lambda)}(0))^{1/3}. \end{aligned}$$

The estimate (3.75) is then a consequence of Proposition 3.3.6 and Lemma 3.4.1. The convergence (3.77) is derived analogously to (3.70) by using [17, Theorem 5.4] and Lemma 3.4.1. This proves Proposition 3.4.4.  $\square$

Lemma 3.4.5 below asserts that the correlation function  $c_{n,k}^{(\lambda)}$  decays exponentially fast as a function of the heights of the two points and of the distance between them. This means in particular that the scores get closer to being independent as the distance between the points increases. The proof of Lemma 3.4.5 relies on Proposition 3.3.6 and Lemma 3.4.1 and is identical to the proof of [26, Lemma 4.8]. For that reason, it is omitted.

**Lemma 3.4.5.** *For all  $n \geq 1$ , there exist  $c_1, \dots, c_4 > 0$  such that for any  $h > 0$ ,  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$ ,  $p \geq 1$  and  $\lambda \in [1, \infty]$  we have*

$$|c_{n,k}^{(\lambda)}((0, h_0), (v_1, h_1))| \leq c_1 h_0^{c_2} h_1^{c_3} \exp\left(-c_4 \left(\|v_1\|^{d+1} + h_0^{\frac{d+1}{2}} + h_1^{\frac{d+1}{2}}\right)\right).$$

## 3.5 Proofs of the main results

In this section, Theorems 3.2.8–3.2.11 are proved. We recall that they include the statements on the limiting expectations and variances of Theorems 3.1.1 and 3.1.2. We have chosen to omit the proof of the central limit theorems as it is the exact replicate line by line of [24, pages 93–98] when the convex hull is replaced by the  $n$ -th layer of the peeling.

### 3.5.1 Results on $k$ -dimensional faces

*Proof of Theorem 3.2.8.*

*Proof of the convergence of the normalized expectation.* The first step consists in taking the expectation in (3.13) and using the Mecke formula to get

$$\mathbb{E}[N_{n,k,\lambda}] = \lambda \int_{\mathbb{B}^d} \mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda)] dx.$$

Let us introduce  $e_d = (0, \dots, 0, 1)$ . By rotation-invariance of  $\mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda)]$ , we obtain

$$\mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda)] = \mathbb{E}[\xi_{n,k}(|x|e_d, \mathcal{P}_\lambda)].$$

Then we apply a change of coordinates in spherical coordinates to get

$$\mathbb{E}[N_{n,k,\lambda}] = \lambda \int_{\mathbb{S}^{d-1}} \int_0^1 \mathbb{E}[\xi_{n,k}(re_d, \mathcal{P}_\lambda)] r^{d-1} dr d\sigma_{d-1}(u), \quad (3.81)$$

where  $\sigma_{d-1}$  is the unnormalized area measure on  $\mathbb{S}^{d-1}$ .

Recall that for every  $h > 0$ ,  $\xi_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}) = \xi_{n,k}([T^{(\lambda)}]^{-1}((0, h)), \mathcal{P}_\lambda)$ , see (3.15). Consequently, an application of the change of variables  $h = \lambda^{\frac{2}{d+1}}(1-r)$  in (3.81) leads to

$$\begin{aligned} \mathbb{E}[N_{n,k,\lambda}] &= \lambda \int_{\mathbb{S}^{d-1}} \int_0^{\lambda^{\frac{2}{d+1}}} \mathbb{E}[\xi_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})] \lambda^{-\frac{2}{d+1}} (1 - \lambda^{-\frac{2}{d+1}} h)^{d-1} dh d\sigma_{d-1}(u) \\ &= \lambda^{\frac{d-1}{d+1}} \int_{\mathbb{S}^{d-1}} \int_0^{\lambda^{\frac{2}{d+1}}} \mathbb{E}[\xi_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})] (1 - \lambda^{-\frac{2}{d+1}} h)^{d-1} dh d\sigma_{d-1}(u). \end{aligned} \quad (3.82)$$

We now wish to use the dominated convergence theorem. Lemma 3.4.3 implies that for any  $h > 0$

$$\mathbb{E}[\xi_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})](1 - \lambda^{-\frac{2}{d+1}} h)^{d-1} \mathbf{1}_{0 \leq h \leq \lambda^{\frac{2}{d+1}}} \xrightarrow{\lambda \rightarrow +\infty} \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h), \mathcal{P})].$$

Thanks to Lemma 3.4.1, the integrand in the rhs of (3.82) is bounded from above by an integrable function of  $h$ , i.e.

$$\mathbb{E}[\xi_{n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})](1 - \lambda^{-\frac{2}{d+1}} h)^{d-1} \mathbf{1}_{0 \leq h \leq \lambda^{\frac{2}{d+1}}} \leq c_1 \exp(-c_2 h^{\frac{d+1}{2}}).$$

Applying the dominated convergence theorem in (3.82) shows that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{d-1}{d+1}} \mathbb{E}[N_{n,k,\lambda}] = \text{Vol}_{d-1}(\mathbb{S}^{d-1}) \int_0^\infty \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h), \mathcal{P})] dh < \infty. \quad (3.83)$$

*Proof of the positivity of the limiting expectation.* Noticing that

$$\mathbb{E}[\xi_{n,k}^{(\infty)}((0, h), \mathcal{P})] \geq \frac{1}{k+1} \mathbb{P}\left((0, h_0) \in \partial\Phi_n^{(\infty)}(\mathcal{P})\right),$$

we deduce that the positivity of the rhs of (3.83) is a consequence of Lemma 3.5.1 below.  $\square$

**Lemma 3.5.1.** *For any  $(0, h_0)$  with  $h_0 > 0$  and any  $n \geq 1$  :*

$$\mathbb{P}\left((0, h_0) \in \partial\Phi_n^{(\infty)}(\mathcal{P})\right) \neq 0.$$

*Proof.* Let us write  $w_0 := (0, h_0)$  and put  $\mathcal{T}_0 := \{w_0\}$ . Our first step is purely deterministic and consists in constructing for  $n = 2$  an idealized point configuration which puts  $w_0$  on the second layer. We then extend the procedure for every  $n > 2$  to get a point set which puts  $w_0$  on the  $n$ -th layer. In a second step, we introduce randomness and show that this property is stable with respect to a small random perturbation of the configuration.

For each  $i \in \{1, \dots, d-1\}$ , we write  $w_{(i,+)} = (\frac{\sqrt{2h_0}}{2} e_i, \frac{h_0}{8})$  and  $w_{(i,-)} = (-\frac{\sqrt{2h_0}}{2} e_i, \frac{h_0}{8})$ . A direct calculation shows that for any  $i$ ,  $\Pi^\downarrow(w_{(i,+)}) \subset \Pi^\downarrow(0, h_0)$ ,  $\Pi^\downarrow(w_{(i,-)}) \subset \Pi^\downarrow(w_0)$  and  $\Pi^\downarrow(w_{(i,s_1)}) \cap \Pi^\downarrow(w_{(j,s_2)}) = \emptyset$  for any  $i, j \in \{1, \dots, d-1\}$ ,  $s_1, s_2 \in \{+, -\}$  such that  $(i, s_1) \neq (j, s_2)$ .

Let us consider the deterministic point set  $\mathcal{T}_1 := \{w_0\} \cup \{w_{(i,s)} : i \in \{1, \dots, d-1\}, s \in \{+, -\}\}$  and describe the peeling of  $\mathcal{T}_1$ . The points  $w_{(i,s)}$  are on the first layer because each  $\Pi^\downarrow(w_{(i,s)})$  is empty. Any downward paraboloid whose boundary goes through  $w_0$  contains at least half of  $\Pi^\downarrow(w_0)$  by Lemma 3.3.1 so it contains at least one of the  $w_{(i,s)}$ . This implies in turn that  $w_0$  is not on the first layer. Furthermore  $\Pi^\downarrow(w_0)$  only contains points on the first layer so  $w_0$  is on the second layer.

We are going to iterate this construction by induction to obtain for every  $n > 2$  an extended deterministic point configuration  $\mathcal{T}_{n-1} = \{w_a : a \in \cup_{l=0}^{n-1} (\{1, \dots, d-1\} \times \{+, -\})^l\}$ , with the convention  $(\cdot)^0 = \{0\}$ , which guarantees that  $w_0$  is on the  $n$ -th layer of the peeling of  $\mathcal{T}_{n-1}$  and that for any  $1 \leq l \leq (n-1)$ ,  $\mathcal{T}_l \setminus \mathcal{T}_{l-1}$  is included on its  $(n-l)$ -th layer. To do so, let us assume that we have constructed  $\mathcal{T}_{n-2}$  with that property. Let us fix  $a \in (\{1, \dots, d-1\} \times \{+, -\})^{n-2}$  and write  $w_a = (v, h)$ . We define  $w_{a,(i,+)} := (v + \frac{1}{2} \sqrt{2 \frac{h_0}{8^{n-2}}} e_i, \frac{h}{8^{n-1}})$  and  $w_{a,(i,-)} := (v - \frac{1}{2} \sqrt{2 \frac{h_0}{8^{n-2}}} e_i, \frac{h}{8^{n-1}})$ .

The construction is just a rescaling of the situation described for the case  $n = 2$  with  $w_a$  playing the role of  $w_0$  and  $w_{a,(i,s)}$  that of  $w_{(i,s)}$ . Consequently, we get the same properties, i.e. for any  $(i, s)$

$$\Pi^\downarrow(w_{a,(i,s)}) \subset \Pi^\downarrow(w_a) \quad (3.84)$$

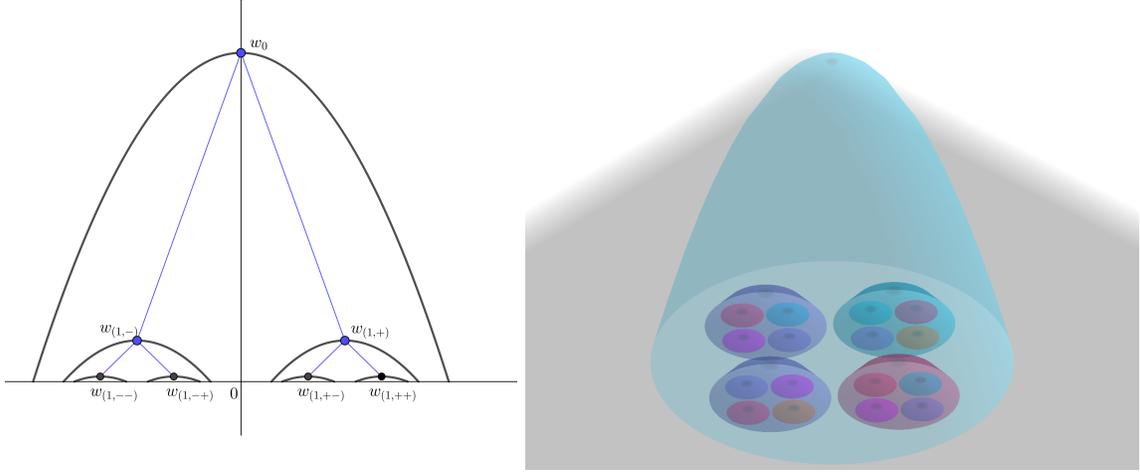


Figure 3.8 – The tree of Lemma 3.5.1 for  $n = 3$  and  $d = 2, 3$ .

and for  $(i, s_1) \neq (j, s_2)$

$$\Pi^\downarrow(w_{a,(i,s_1)}) \cap \Pi^\downarrow(w_{a,(j,s_2)}) = \emptyset. \quad (3.85)$$

If we put an edge from  $w_a$  to each  $w_{a,(i,s)}$  then we endow  $\mathcal{T}_{n-1}$  with a structure of tree with root  $w_0$ , see Figure 3.8 in the case  $n = 3$  in dimension 2 and 3. We also take the convention  $\mathcal{T}_{-1} = \emptyset$ .

The construction has been done such that for every  $l \in \{0, \dots, (n-1)\}$ , each point  $w_a \in (\mathcal{T}_l \setminus \mathcal{T}_{l-1})$  is on the  $(n-l)$ -th layer of the peeling of  $\mathcal{T}_{n-1}$ .

We prove now by downward induction the stability of the property above when the idealized point set is subject to a small random perturbation. Let us fix  $\varepsilon > 0$  and consider the event

$$A_n = \cap_{w_a \in \mathcal{T}_{n-1} \setminus \{w_0\}} \{\#(\mathcal{P} \cap B(w_a, \varepsilon)) = 1\} \cap \{\mathcal{P} \cap (\Pi^\downarrow(w_0) \setminus \cup_{w_a \in \mathcal{T}_{n-1}} B(w_a, \varepsilon)) = \emptyset\}.$$

On the event  $A_n$ , for every  $w_a \in \mathcal{T}_{n-1}$  we denote by  $\tilde{w}_a$  the unique point belonging to  $\mathcal{P} \cap B(w_a, \varepsilon)$  and call it a perturbed point from the tree (with same depth as the original point  $w_a$ ). We then assert and prove below that the stability occurs, i.e. that the perturbed point  $\tilde{w}_a$  is on the  $(n-l)$ -th layer of the peeling of  $\mathcal{P} \cup \{w_0\}$  as soon as the original point  $w_a$  is at depth  $l$  in  $\mathcal{T}_{n-1}$ .

We make the following preliminary observation: the calibrations in the construction of the tree  $\mathcal{T}_{n-1}$  allow us to strengthen (3.84) and (3.85) by claiming that for  $\varepsilon$  small enough, the  $\varepsilon$ -neighborhood of  $\Pi^\downarrow(w_a)$  is included in  $\Pi^\downarrow(w_b)$  for any  $w_a \in \mathcal{T}_{n-1}$  with parent  $w_b$  and that the  $\varepsilon$ -neighborhoods of the downward parabolas associated with two distinct perturbed points at same depth do not meet.

We start with our base case, which is depth  $(n-1)$ . For any  $a \in (\{1, \dots, d-1\} \times \{+, -\})^{n-1}$ , thanks to the observation above, the perturbed point  $\tilde{w}_a$  satisfies that  $\Pi^\downarrow(\tilde{w}_a)$  does not intersect  $\mathcal{P}$  when  $\varepsilon$  is small enough. This shows that all perturbed points at depth  $(n-1)$  are extreme.

Now we assume that for some  $l \in \{1, \dots, n-1\}$ , each perturbed point at depth  $k$  is on the  $(n-k)$ -th layer for every  $l \leq k \leq (n-1)$ . We consider a perturbed point  $\tilde{w}_a$  at depth  $(l-1)$ , which means  $a \in (\{1, \dots, d-1\} \times \{+, -\})^{(l-1)}$ . Because of the preliminary observation, for  $\varepsilon$  small enough, the downward paraboloid  $\Pi^\downarrow(\tilde{w}_a)$  only contains points that can be written  $\tilde{w}_{a,b}$  for some  $b \in (\{1, \dots, d-1\} \times \{+, -\})^p$  and  $1 \leq p \leq (n-l)$ . Thus  $\tilde{w}_a$  is at most on the  $(n-l+1)$ -th layer.

Moreover, noticing that any downward paraboloid whose boundary goes through  $\tilde{w}_a$  contains at least one of the  $\tilde{w}_{a,(i,s)}$ , which are on the  $(n-l)$ -th layer. This shows that  $\tilde{w}_a$  is on the  $(n-l+1)$ -st layer. A finite induction on  $l$  thus gives us that on the event  $A_n$ ,  $w_0$  is on the  $n$ -th layer.

It remains to show that  $\mathbb{P}(A_n) > 0$ . Using the Poisson property and (3.84)-(3.85), we get

$$\mathbb{P}(A_n) = \mathbb{P}(\mathcal{P} \cap (\Pi^\downarrow(w_0) \setminus \cup_{w_a \in \mathcal{T}_{n-1}} B(w_a, \varepsilon)) = \emptyset) \prod_{w_a \in \mathcal{T}_{n-1} \setminus \{w_0\}} \mathbb{P}(\#\mathcal{P} \cap B(w_a, \varepsilon) = 1) > 0.$$

□

*Proof of Theorem 3.2.9.*

*Proof of the convergence of the normalized variance.* In the same way as for the expectation asymptotics, the first idea consists in using Mecke's formula. Combining it with Fubini's theorem and writing  $\xi'_{n,k,\lambda}(x, \mathcal{P}_\lambda)$  for the same functional as  $\xi_{n,k,\lambda}(x, \mathcal{P}_\lambda)$  save for the fact that  $x$  is not added to the process, we get

$$\begin{aligned} & \text{Var}(N_{n,k,\lambda}) \\ &= \mathbb{E} \left[ \left( \sum_{x \in \mathcal{P}_\lambda} \xi'_{n,k,\lambda}(x, \mathcal{P}_\lambda) \right)^2 \right] - \mathbb{E} \left[ \sum_{x \in \mathcal{P}_\lambda} \xi'_{n,k,\lambda}(x, \mathcal{P}_\lambda) \right]^2 \\ &= \mathbb{E} \left[ \sum_{x \in \mathcal{P}_\lambda} \xi'_{n,k,\lambda}(x, \mathcal{P}_\lambda)^2 \right] + \mathbb{E} \left[ \sum_{\substack{x, y \in \mathcal{P}_\lambda \\ x \neq y}} \xi'_{n,k,\lambda}(x, \mathcal{P}_\lambda) \xi'_{n,k,\lambda}(y, \mathcal{P}_\lambda) \right] - \mathbb{E} \left[ \sum_{x \in \mathcal{P}_\lambda} \xi'_{n,k,\lambda}(x, \mathcal{P}_\lambda) \right]^2 \\ &= I_1(\lambda) + I_2(\lambda) \end{aligned}$$

where

$$I_1(\lambda) := \lambda \int_{\mathbb{B}^d} \mathbb{E} [\xi_{n,k,\lambda}(x, \mathcal{P}_\lambda)^2] dx \quad \text{and} \quad I_2(\lambda) := \lambda^2 \iint_{(\mathbb{B}^d)^2} c_{n,k,\lambda}(x, y) dx dy.$$

with

$$c_{n,k,\lambda}(x, y) = \mathbb{E} [\xi_{n,k,\lambda}(x, \mathcal{P}_\lambda \cup \{y\}) \xi_{n,k,\lambda}(y, \mathcal{P}_\lambda \cup \{x\})] - \mathbb{E} [\xi_{n,k,\lambda}(x, \mathcal{P}_\lambda)] \mathbb{E} [\xi_{n,k,\lambda}(y, \mathcal{P}_\lambda)].$$

We treat  $I_1(\lambda)$  as in the proof of Theorem 3.2.8 to get

$$\lim_{\lambda \rightarrow +\infty} \lambda^{-\frac{d-1}{d+1}} I_1(\lambda) = I_1$$

where we recall the definition of  $I_1$  at (3.17). We now prove the convergence of  $\lambda^{-\frac{d-1}{d+1}} I_2(\lambda)$ . We follow line by line the method used in [24, calculation of II on pages 92-93], i.e. rewriting in spherical coordinates, then use of a change of variables provided by the scaling transformation  $T^{(\lambda)}$ , to obtain that

$$I_2(\lambda) = \lambda^{\frac{d-1}{d+1}} \int_{\mathbb{S}^{d-1}} \int_0^{\lambda^{\frac{2}{d+1}}} \int_{T_u^{(\lambda)}(\mathbb{S}^{d-1})} f^{(\lambda)}(u_0, h_0, v_1, h_1) dv_1 dh_1 dh_0 d\sigma_{d-1}(u_0)$$

where

$$f^{(\lambda)}(u_0, h_0, v_1, h_1) := (1 - \lambda^{-\frac{2}{d+1}} h_0)^{d-1} \frac{\sin^{d-2} \left( \lambda^{-\frac{1}{d+1}} |v_1| \right)}{|\lambda^{-\frac{1}{d+1}} v_1|^{d-2}} (1 - \lambda^{-\frac{2}{d+1}} h_1)^{d-1} c_{n,k}^{(\lambda)}((0, h_0), (v_1, h_1)),$$

with  $c_{n,k}^{(\lambda)}$  defined at (3.16).

It remains to apply the dominated convergence theorem. Using Lemma 3.4.5, we obtain

$$|f^{(\lambda)}(u_0, h_0, v_1, h_1)| \leq c_1 h_0^{c_2} h_1^{c_3} \exp\left(-c_4 \left(\|v_1\|^{d+1} + h_0^{\frac{d+1}{2}} + h_1^{\frac{d+1}{2}}\right)\right)$$

which is integrable on  $\mathbb{S}^{d-1} \times \mathbb{R}_+ \times \mathbb{R}^{d-1} \times \mathbb{R}_+$ . We deduce from Proposition 3.4.4 that

$$\lim_{\lambda \rightarrow \infty} f^{(\lambda)}(u_0, h_0, v_1, h_1) = c_{n,k}^{(\infty)}((0, h_0), (v_1, h_1)).$$

This implies that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{d-1}{d+1}} I_2(\lambda) = I_2,$$

where  $I_2$  has been defined at (3.18).

*Proof of the positivity of the limiting variance.* This proof is essentially adapted from [28, Section 4.5]. The main difference is that we make sure that our points are on the  $n$ -th layer instead of being extremal.

Similarly to what is done in [24, Lemma 7.6], we can use the same arguments as in the proof of Theorem 3.2.9 to show that

$$\lim_{\lambda \rightarrow \infty} \lambda^{-\frac{d-1}{d+1}} \text{Var}[N_{n,k,\lambda}] = \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{d-1}{d+1}} \text{Var}[\tilde{N}_{n,k,\lambda}] \quad (3.86)$$

where we write  $\tilde{N}_{n,k,\lambda} := \sum_{w \in \mathcal{P} \cap W_\lambda} \xi_{n,k}^{(\infty)}(w, \mathcal{P})$ .

Our strategy to prove that the limit in the rhs of (3.86) is positive is the following: we start by discretizing  $W_\lambda$  and construct in each parallelepiped of that discretization two different configurations, said to be *good*, which have a positive probability to occur and which give birth to two different values for the local number of  $k$ -faces of the  $n$ -th layer. Then we check that this counting is not affected by the external configuration and finally, we find a lower bound for the total variance conditional on the intersection of the point process with the outside of the parallelepipeds which contain one of the two *good* configurations.

*Step 1. Construction of a good configuration in a thin parallelepiped.* First, we consider a cube  $Q \subset \mathbb{R}^{d-1}$  and we take  $\rho \in (0, \infty)$  smaller than the diameter of  $Q$ . We take  $\delta > 0$  sufficiently small such that the paraboloids  $\Pi^\downarrow(w)$  are pairwise disjoint for  $w$  belonging to the grid  $(\rho\mathbb{Z}^{d-1} \cap Q) \times \{\delta\}$ . For each  $w \in (\rho\mathbb{Z}^{d-1} \cap Q) \times \{\delta\}$ , we make the same tree construction as in the proof of Lemma 3.5.1 inside  $\Pi^\downarrow(w)$ . We obtain a forest that we call  $\mathcal{T}_{n,\rho}$ . In particular, the construction ensures that all points  $w \in (\rho\mathbb{Z}^{d-1} \cap Q) \times \{\delta\}$  belong to the  $n$ -th layer of  $\mathcal{T}_{n,\rho}$ . Let us write  $F_{n,k}(Q, \rho, \delta)$  for the number of  $k$ -faces of the  $n$ -th layer of  $(\mathcal{P} \setminus (Q \times [0, \infty))) \cup \mathcal{T}_{n,\rho}$  going through any  $w \in (\rho\mathbb{Z}^{d-1} \cap Q) \times \{\delta\}$ . If we ignore the points of  $\mathcal{P} \setminus (Q \times [0, \infty))$ , as the diameter of  $Q$  gets large compared to  $\rho$ , boundary effects become negligible and we get

$$F_{n,k}(Q, \rho/2, \delta) \sim 2^{d-1} F_{n,k}(Q, \rho, \delta).$$

Then we consider  $\varepsilon > 0$  such that

$$\delta + \varepsilon \leq \frac{\rho^2}{8}. \quad (3.87)$$

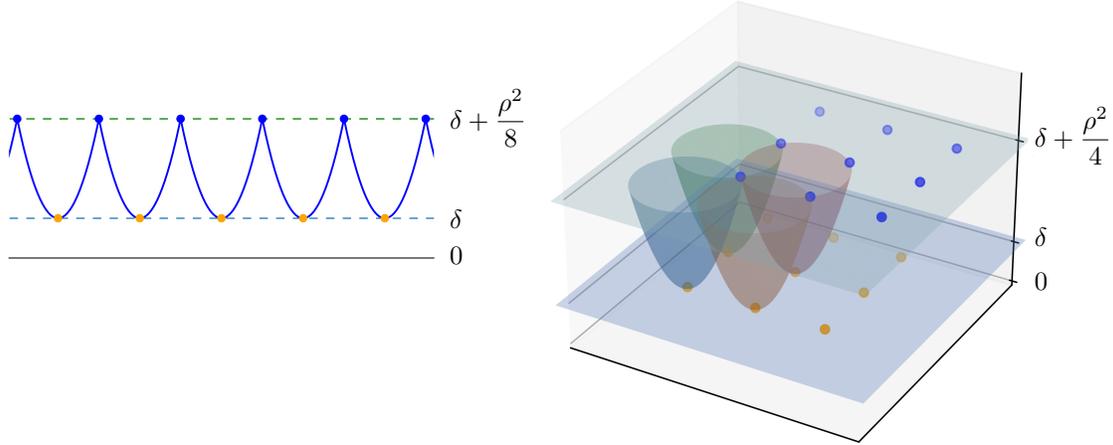


Figure 3.9 – A part of the dual grid for non-perturbed points in dimensions 2 and 3.

This guarantees in particular that for any  $w = (v, \delta + \varepsilon)$  and  $w' = (v', \delta + \varepsilon)$  with  $|v - v'| \geq \rho$ , we have  $\Pi^\downarrow(w) \cap \Pi^\downarrow(w') = \emptyset$ . As in the proof of the positivity of the expectation, we consider a random perturbation of the points of  $\mathcal{T}_{n,\rho}$  by at most  $\varepsilon$  for the Euclidean distance, i.e. we assume that  $\mathcal{P} \cap B(w, \varepsilon)$  consists of one point for each  $w \in \mathcal{T}_{n,\rho}$ . We notice that the obtained perturbed points have a height at most equal to  $\delta + \varepsilon$  and are distant by at most  $2\rho$  when  $\varepsilon < \rho/2$ . Consequently, for  $\varepsilon$  small enough, the maximal height of  $(Q \times [0, \infty)) \setminus \bigcup_{w' \in \mathcal{P} \cap \bigcup_{w \in \mathcal{T}_{n,\rho}} B(w, \varepsilon)} \Pi^\uparrow(w')$  is smaller than  $\alpha$  where  $\alpha$  is the maximal height of a point in  $(Q \times [0, \infty)) \setminus \bigcup_{w \in (Q \cap 2\rho\mathbb{Z}^{d-1}) \times \{\delta + \varepsilon\}} \Pi^\uparrow(w)$ . In particular, we claim that there is a constant  $c_\alpha > 1$  depending only on dimension  $d$  such that

$$\alpha \leq c_\alpha \rho^2. \quad (3.88)$$

Indeed, up to boundary effects, the set of apices of down paraboloids which contain  $2^{d-1}$  points of  $Q \cap 2\rho\mathbb{Z}^{d-1} \times \{\delta + \varepsilon\}$  is located on a translated dual grid at height  $((d-1)\frac{\rho^2}{2} + \delta + \varepsilon)$ , see Figure 3.9.

Consequently, let us consider the event

$$A_{n,\rho} = \bigcap_{w \in \mathcal{T}_{n,\rho}} \{ \#(\mathcal{P} \cap B(w, \varepsilon)) = 1 \} \cap \{ (\mathcal{P} \cap [(Q \times [0, \alpha]) \setminus \bigcup_{w \in \mathcal{T}_{n,\rho}} B(w, \varepsilon)]) = \emptyset \}.$$

Let us write  $F_{n,k}(Q, \rho, \delta, \varepsilon)$  for the total number of  $k$ -faces of the  $n$ -th layer of  $\mathcal{P}$  going through at least one point in  $\mathcal{P} \cap \bigcup_{w \in (\rho\mathbb{Z}^{d-1} \cap Q) \times \{\delta\}} B(w, \varepsilon)$ . Conditional on  $A_{n,\rho}$  and when the points of  $\mathcal{P} \setminus (Q \times [0, \alpha])$  are ignored, for  $\varepsilon$  small enough, this quantity is in fact equal to  $F_{n,k}(Q, \rho, \delta)$  and we keep the relation

$$F_{n,k}(Q, \rho/2, \delta, \varepsilon) \sim 2^{d-1} F_{n,k}(Q, \rho, \delta, \varepsilon). \quad (3.89)$$

*Step 2. Influence of the points outside of the thin parallelepiped  $Q \times [0, \alpha]$ .* For any closed set  $C \subset \mathbb{R}^{d-1}$ , we define for any  $\gamma > 0$   $C^{(\gamma)} := \{x \in C : d(x, \partial C) > \gamma\}$ . We claim that on  $A_{n,\rho}$ , for any  $w \in \mathcal{P} \cap (Q^{(\rho)} \times [0, \alpha])$ , the status of  $w$  does not depend on points outside  $Q \times [0, \alpha]$ , i.e.

$$\ell^{(\infty)}(w, \mathcal{P} \cap (Q \times [0, \alpha])) = \ell^{(\infty)}(w, \mathcal{P}).$$

This is due to the fact that the condition (3.87) guarantees that the down paraboloids with apices at points from  $\mathcal{P} \cap (Q^{(\rho)} \times [0, \alpha])$  are included in  $Q \times [0, \alpha]$ . Moreover, for  $\mathbf{c} = \sqrt{2c_\alpha} + 2$ , we assert that the facial structure around any point inside  $\mathcal{P} \cap (Q^{(\mathbf{c}\rho)} \times [0, \alpha])$  which belongs to the  $n$ -th layer of the peeling of  $\mathcal{P}$  does not depend on points outside  $Q \times [0, \alpha]$ . Indeed, let us consider  $w \in \mathcal{P} \cap (Q^{(\mathbf{c}\rho)} \times [0, \alpha])$ . We choose  $(d-1)$  points from  $\mathcal{P} \cap (Q^{(\sqrt{2c_\alpha}\rho)} \times [0, \alpha])$  which share a common facet of the  $n$ -th layer of the peeling of  $\mathcal{P} \cap (Q \times [0, \alpha])$ . The down paraboloid which contains this facet has an apex in  $Q^{(\sqrt{2c_\alpha}\rho)} \times [0, \alpha]$ . Consequently, thanks to (3.88), that down paraboloid is included in  $Q \times [0, \alpha]$ , which implies that the facet containing  $w$  and the  $(d-1)$  other points is a facet of the  $n$ -th layer of the peeling of  $\mathcal{P}$ .

*Step 3. Discretization of  $W_\lambda$  and lower bound for the variance.* We are now ready to discretize  $W_\lambda$  and isolate the *good* parallelepipeds from the discretization, according to the two previous steps. We choose  $\delta$  and  $\varepsilon$  which satisfy (3.87) with the choice  $\rho = 1$ . We take a large positive number  $M$  and we partition  $\left[-\frac{\lambda^{\frac{1}{d+1}}}{2}, \frac{\lambda^{\frac{1}{d+1}}}{2}\right]^{d-1}$  into  $L := \left[\frac{\lambda^{\frac{1}{d+1}}}{M}\right]^{d-1}$  cubes  $Q_1, \dots, Q_L$ . We consider the cubes  $Q_i$  satisfying the following properties:

(a) For each  $z \in (\mathbb{Z}^{d-1} \cap (Q_i \setminus Q_i^{(\mathbf{c})})) \times \{\delta\}$ ,  $\mathcal{P} \cap B(z, \varepsilon)$  is a singleton and is put on the  $n$ -th layer using the tree construction associated with  $\mathcal{T}_{n,1}$  and the perturbation of each point by at most  $\varepsilon$  as in Step 1.

(b) One of these two conditions holds:

1. For each  $z \in (\mathbb{Z}^{d-1} \cap Q_i^{(\mathbf{c})}) \times \{\delta\}$ ,  $\mathcal{P} \cap B(z, \varepsilon)$  is a singleton and this point is put on the  $n$ -th layer as in property (a).
2. For each  $z \in (\frac{1}{2}\mathbb{Z}^{d-1} \cap Q_i^{(\mathbf{c})}) \times \{\delta\}$ ,  $\mathcal{P} \cap B(z, \varepsilon)$  is a singleton and this point is put on the  $n$ -th layer as before.

(c) Aside from the points described above,  $\mathcal{P}$  has no point in  $Q_i \times [0, \alpha]$ .

After a possible relabeling, let  $I := \{1, \dots, K\}$  be the indices of cubes partitioning  $\left[-\frac{\lambda^{\frac{1}{d+1}}}{2}, \frac{\lambda^{\frac{1}{d+1}}}{2}\right]^{d-1}$  that verify properties (a) to (c). Since any cube has a positive probability to verify these properties, we have

$$\mathbb{E}[K] \geq c\lambda^{\frac{d-1}{d+1}}. \quad (3.90)$$

Let  $\mathcal{F}_\lambda$  be the  $\sigma$ -algebra generated by  $I$ , the positions of points in  $W_\lambda \setminus (\bigcup_{i \in I} Q_i^{(\mathbf{c})} \times [0, \alpha])$  and the scores  $\xi_{n,k}^{(\infty)}(x, \mathcal{P})$  at these points. For each  $i \in I$  we claim that

$$\text{Var} \left[ \sum_{x \in \mathcal{P} \cap (Q_i \times [0, \alpha])} \xi_{n,k}^{(\infty)}(x, \mathcal{P}) \middle| \mathcal{F}_\lambda \right] = \text{Var} \left[ \sum_{x \in \mathcal{P} \cap (Q_i^{(3)} \times [0, \alpha])} \xi_{n,k}^{(\infty)}(x, \mathcal{P}) \middle| \mathcal{F}_\lambda \right] \geq c_0. \quad (3.91)$$

Indeed, either condition (b1) or condition (b2) occurs in  $Q_i^{(\mathbf{c})} \times [0, \alpha]$ , each with positive probability. Moreover, we observe that  $\sum_{x \in \mathcal{P} \cap (Q_i \times [0, \alpha])} \xi_{n,k}^{(\infty)}(x, \mathcal{P})$  is larger when (b2) is satisfied. This is due to the scaling result (3.89) which implies that the contribution of points inside  $Q_i^{(\mathbf{c})} \times [0, \alpha]$  provides a quantity almost  $2^{d-1}$  times larger when (b2) is satisfied.

Since only scores in  $\cup_{i \in I} Q_i \times [0, \alpha]$  have any variability conditional on  $\mathcal{F}_\lambda$  we get

$$\begin{aligned} \text{Var}[\tilde{N}_{n,k,\lambda}] &= \text{Var}[\mathbb{E}[\tilde{N}_{n,k,\lambda} | \mathcal{F}_\lambda] + \mathbb{E}[\text{Var}[\tilde{N}_{n,k,\lambda} | \mathcal{F}_\lambda]]] \\ &\geq \mathbb{E}[\text{Var}[\tilde{N}_{n,k,\lambda} | \mathcal{F}_\lambda]] \\ &= \mathbb{E}\left[\text{Var}\left[\sum_{i \in I} \sum_{x \in \mathcal{P} \cap (Q_i^{(3)} \times [0, \alpha])} \xi_{n,k}^{(\infty)}(x, \mathcal{P}) \middle| \mathcal{F}_\lambda\right]\right]. \end{aligned}$$

Then the sums of scores in  $Q_i^{(c)} \times [0, \alpha]$  and  $Q_j^{(c)} \times [0, \alpha]$  for  $i \neq j$  are independent conditional on  $\mathcal{F}_\lambda$  since the scores in  $Q_i^{(c)} \times [0, \alpha]$  only depend on points in  $Q_i \times [0, \alpha]$ . Thus we can write

$$\begin{aligned} \text{Var}[\tilde{N}_{n,k,\lambda}] &\geq \mathbb{E}\left[\sum_{i \in I} \text{Var}\left[\sum_{x \in \mathcal{P} \cap (Q_i \times [0, \alpha])} \xi_{n,k}^{(\infty)}(x, \mathcal{P}) \middle| \mathcal{F}_\lambda\right]\right] \\ &\geq c_0 \mathbb{E}[K] \\ &\geq c\lambda^{\frac{d-1}{d+1}}. \end{aligned}$$

where the second inequality comes from (3.91) applied to each  $Q_i$  and the last inequality comes from (3.90). Thanks to (3.86), this proves the positivity of the limiting variance.  $\square$

### 3.5.2 Results on intrinsic volumes

We define the score and two-point correlation function in the case of intrinsic volumes by following closely the method of [24, pp. 54–55], which relies on Kubota's formula, see [71, equation (6.11)]. For convenience, let us write  $\text{conv}_{n,\lambda} := \text{conv}_n(\mathcal{P}_\lambda)$  and let us denote by  $\kappa_m$  the volume of the  $m$ -dimensional unit ball. By Kubota's formula applied to  $\text{conv}_{n,\lambda}$ , we get

$$V_k(\text{conv}_{n,\lambda}) = \frac{d! \kappa_d}{k! \kappa_k (d-k)! \kappa_{d-k}} \int_{G(d,k)} \text{Vol}_k(\text{conv}_{n,\lambda} | L) d\nu_k(L) \quad (3.92)$$

where  $\nu_k$  is the normalized Haar measure on the  $k$ -th Grassmanian  $G(d, k)$  of  $\mathbb{R}^d$  and  $\text{conv}_{n,\lambda} | L$  is the orthogonal projection of  $\text{conv}_{n,\lambda}$  onto  $L$ . For every  $x \in \mathbb{R}^d \setminus \{0\}$  we define  $\vartheta_L(x, \text{conv}_{n,\lambda}) := \mathbb{1}_{\{x \notin \text{conv}_{n,\lambda} | L\}}$  and the projection avoidance functionals

$$\vartheta_k(x, \text{conv}_{n,\lambda}) := \int_{G(\text{lin}[x], k)} \vartheta_L(x, \text{conv}_{n,\lambda}) d\nu_k^{\text{lin}[x]}(L)$$

where  $\text{lin}[x]$  is the linear space spanned by  $x$ ,  $G(\text{lin}[x], k)$  is the set of all  $k$ -dimensional linear subspaces of  $\mathbb{R}^d$  containing  $\text{lin}[x]$  and  $\nu_k^{\text{lin}[x]}$  is the normalized Haar measure on  $G(\text{lin}[x], k)$ . Combining (3.92) and Fubini's theorem we can rewrite the defect  $k$ -th intrinsic volume of  $\text{conv}_{n,\lambda}$  as

$$V_{n,k,\lambda} = \frac{\binom{d-1}{k-1}}{\kappa_{d-k}} \int_{\mathbb{R}^d} \vartheta_k(x, \text{conv}_{n,\lambda}) \frac{dx}{|x|^{d-k}}.$$

The functional  $V_{n,k,\lambda}$  can then be written as a sum of scores, i.e.

$$V_{n,k,\lambda} = \sum_{x \in \mathcal{P}_\lambda} \xi_{V,n,k}(x, \mathcal{P}_\lambda)$$

with

$$\xi_{V,n,k}(x, \mathcal{P}_\lambda) := \begin{cases} \frac{\binom{d-1}{k-1}}{d\kappa_{d-k}} \int_{\text{cone}_n(x, \mathcal{P}_\lambda)} \vartheta_k(y, \text{conv}_n(\mathcal{P}_\lambda \cup \{x\})) \frac{dy}{|y|^{d-k}} & \text{if } x \in \partial \text{conv}_{n,\lambda} \\ 0 & \text{otherwise} \end{cases}$$

where  $\text{cone}_n(x, \mathcal{P}_\lambda) := \{ry : r > 0 \text{ and } y \in \mathcal{F}_{n,d-1}(x, \mathcal{P}_\lambda)\}$ . We denote by

$$\xi_{V,n,k}^{(\lambda)}(x, \mathcal{P}^{(\lambda)}) := \lambda \xi_{V,n,k,\lambda}([T^{(\lambda)}]^{-1}(x), \mathcal{P}_\lambda) \quad (3.93)$$

their rescaled counterparts, see [24, page 84] for an explanation on the factor  $\lambda$ . Let us explain how to define the limit versions of the rescaled scores. For a point  $w = (v, h) \in \mathbb{R}^{d-1} \times \mathbb{R}_+$  we write  $w^\uparrow$  for the set  $\{v\} \times \mathbb{R}_+$ . We denote by  $A(w^\uparrow, k)$  the set of all  $k$ -dimensional affine spaces in  $\mathbb{R}^{d-1} \times \mathbb{R}_+$  containing  $w^\uparrow$ . For any  $k$  and any affine space  $L \in A(w^\uparrow, k)$  we define the orthogonal parabolic volume

$$\Pi^\perp(w, L) := (w \oplus L^\perp) \cap \Pi^\perp(w).$$

We put  $\vartheta_{n,L}^{(\infty)}(w) = 1$  if  $\Pi^\perp(w, L) \cap \Phi_n(\mathcal{P}) = \emptyset$  and 0 otherwise. Then we define

$$\vartheta_{n,k}^{(\infty)}(w) := \int_{A(w^\uparrow, k)} \vartheta_{n,L}^{(\infty)}(w) d\mu_k^{w^\uparrow}(L)$$

where  $\mu_k^{w^\uparrow}$  is the normalized Haar measure on  $A(w^\uparrow, k)$ . We are finally able to define the limit score

$$\xi_{V,n,k}^{(\infty)}(w, \mathcal{P}) := \frac{\binom{d-1}{k-1}}{d\kappa_{d-k}} \int_{\text{v-cone}(\mathcal{F}_{n,d-1}^{(\infty)}(w, \mathcal{P}))} \vartheta_{n,k}^{(\infty)}(w') dw' \quad (3.94)$$

where  $\text{v-cone}(\mathcal{F}_{n,d-1}^{(\infty)}(w, \mathcal{P})) := \{(v', h'), \exists h'' : (v', h'') \in \mathcal{F}_{n,d-1}^{(\infty)}(w)\}$ .

For any  $\lambda \in (0, \infty]$ , the corresponding two-point correlation function is then defined by the identity

$$\begin{aligned} & c_{V,n,k}^{(\lambda)}((0, h_0), (v_1, h_1)) \\ & := \mathbb{E}[\xi_{V,n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}) \xi_{V,n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(0, h_0)\})] \\ & \quad - \mathbb{E}[\xi_{V,n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)})] \mathbb{E}[\xi_{V,n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)})]. \end{aligned} \quad (3.95)$$

*Proof of Theorems 3.2.10 and 3.2.11.*

*Proof of the convergence of the normalized expectation and variance.* The proofs of Theorems 3.2.10 and 3.2.11 go along the same lines as the proofs of Theorems 3.2.8 and 3.2.9 : after application of Mecke's formula and a suitable change of variables in the integral, we need to apply Lebesgue's dominated convergence theorem which requires the convergence of  $\mathbb{E}[\xi_{V,n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})]$ ,  $\mathbb{E}[(\xi_{V,n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}))^2]$  and  $c_{V,n,k}^{(\lambda)}(w, w')$  in the same spirit as in Propositions 3.4.3 and 3.4.4 as well as moment bounds similar to those in Lemma 3.4.1. All these results rely on stabilization results identical to the tail estimates in Propositions 3.3.6 and 3.3.7.

As an example, we explain how to adapt Proposition 3.3.6, Lemma 3.4.1 and Proposition 3.4.3 to get the convergence of the expectation of  $\xi_{V,n,k}^{(\lambda)}$ .

We claim that as soon as  $H_n^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \leq \frac{r^2}{32}$ , the calculation of  $\xi_{V,n,k}^{(\lambda)}$  only depends on the set  $\mathcal{U}$  defined at (3.57). Consequently, the radius of stabilization for  $\xi_{V,n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})$  is the same as the one considered in Proposition 3.3.6.

To prove the moment bounds for  $\xi_{V,n,k}^{(\lambda)}$ , we notice that  $\xi_{V,n,k}^{(\lambda)}(0, h)$  is smaller than the volume of  $C^{\leq H}(R)$  where  $R$  is the stabilization radius of  $\xi_{V,n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})$  and  $H = H_n^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)})$ . Using Lemmas 3.3.5 and 3.3.6 as in the proof of Lemma 3.4.1, we deduce that for some  $c_1, c_2 > 0$ ,

$$\mathbb{E} \left[ (\xi_{V,n,k}^{(\lambda)}((0, h), \mathcal{P}^{(\lambda)}))^p \right] \leq c_1 h^p \exp(-c_2 h^{\frac{d+1}{2}}).$$

These two ingredients allow us to prove the convergence of  $\mathbb{E}[\xi_{V,n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)})]$  with the help of the continuous mapping theorem as in Lemmas 3.4.2 and 3.4.3.

*Proof of the positivity of the limiting constants for the intrinsic volumes.* The defect intrinsic volumes are increasing with respect to set inclusion. As the limiting constant is positive for  $\text{conv}_1(\mathcal{P}^{(\lambda)})$  [1, Theorem 3], it remains true for  $\text{conv}_n(\mathcal{P}_\lambda)$ .

It remains to prove that the limiting constant for the variance is positive. The strategy is the same as the one that we used in the case of the  $k$ -faces. Because of the renormalization by  $\lambda$  in the definition of  $\xi_{V,n,k}^{(\lambda)}$ , see (3.93), the equality (3.86) in the case of the volume scores becomes

$$\lim_{\lambda \rightarrow \infty} \lambda^{\frac{d+3}{d+1}} \text{Var}[V_{n,k,\lambda}] = \lim_{\lambda \rightarrow \infty} \lambda^{-\frac{d-1}{d+1}} \text{Var}[\tilde{V}_{n,k,\lambda}],$$

where  $\tilde{V}_{n,k,\lambda} := \sum_{w \in \mathcal{P} \cap W_\lambda} \xi_{V,n,k}^{(\infty)}(w, \mathcal{P})$ . The paragraph right after (3.86) details the rest of our strategy. We repeat word for word the first two steps of the proof of the positivity of the limiting variance on page 83, save for all the statements regarding  $F_{n,k}$ . In the sequel we use the notations of the aforementioned proof and go directly to Step 3, i.e. the choice of the *good* configurations.

We discretize  $W_\lambda$  by decomposing it into parallelepipeds and isolate the *good* parallelepipeds from the discretization as we have done for the  $k$ -faces. We choose  $\delta$  and  $\varepsilon$  which satisfy (3.87) with the choice  $\rho = 1$ . We take a large positive number  $M$  and we partition  $\left[ -\frac{\lambda^{\frac{1}{d+1}}}{2}, \frac{\lambda^{\frac{1}{d+1}}}{2} \right]^{d-1}$  into

$L := \left[ \frac{\lambda^{\frac{1}{d+1}}}{M} \right]^{d-1}$  cubes  $Q_1, \dots, Q_L$ . We consider the cubes  $Q_i$  satisfying the following properties, note that the main difference is the content of conditions b)1) and b)2):

(a) For each  $z \in (\mathbb{Z}^{d-1} \cap (Q_i \setminus Q_i^{(c)})) \times \{\delta\}$ ,  $\mathcal{P} \cap B(z, \varepsilon)$  is a singleton and is put on the  $n$ -th layer using the tree construction associated with  $\mathcal{T}_{n,1}$  and the perturbation of each point by at most  $\varepsilon$  as in Step 1.

(b) One of these two conditions holds:

1. For each  $z \in (\mathbb{Z}^{d-1} \cap Q_i^{(c)}) \times \{\delta\}$ ,  $\mathcal{P} \cap B(z, \varepsilon)$  is a singleton and this point is put on the  $n$ -th layer as in property (a).
2. For each  $z \in (\mathbb{Z}^{d-1} \cap Q_i^{(c)}) \times \{\delta/2\}$ ,  $\mathcal{P} \cap B(z, \varepsilon)$  is a singleton and this point is put on the  $n$ -th layer as before.

(c) Aside from the points described above,  $\mathcal{P}$  has no point in  $Q_i \times [0, \alpha]$ .

After a possible relabeling, let  $I := \{1, \dots, K\}$  be the indices of cubes partitioning  $\left[-\frac{\lambda^{\frac{1}{d+1}}}{2}, \frac{\lambda^{\frac{1}{d+1}}}{2}\right]^{d-1}$  that verify properties (a) to (c) and it remains true that

$$\mathbb{E}[K] \geq c\lambda^{\frac{d-1}{d+1}}. \quad (3.96)$$

Let  $\mathcal{F}_\lambda$  be the  $\sigma$ -algebra generated by  $I$ , the positions of points in  $W_\lambda \setminus (\bigcup_{i \in I} Q_i^{(c)} \times [0, \alpha])$  and the scores  $\xi_{V,n,k}^{(\infty)}(x, \mathcal{P})$  at these points. For each  $i \in I$  we claim that

$$\text{Var} \left[ \sum_{x \in \mathcal{P} \cap (Q_i \times [0, \alpha])} \xi_{V,n,k}^{(\infty)}(x, \mathcal{P}) \middle| \mathcal{F}_\lambda \right] = \text{Var} \left[ \sum_{x \in \mathcal{P} \cap (Q_i^{(3)} \times [0, \alpha])} \xi_{V,n,k}^{(\infty)}(x, \mathcal{P}) \middle| \mathcal{F}_\lambda \right] \geq c_0. \quad (3.97)$$

Again, either condition (b1) or condition (b2) occurs in  $Q_i^{(c)} \times [0, \alpha]$ , each with positive probability. To ensure that the remaining part of the proof for the  $k$ -faces works here as well, it remains to prove that  $\sum_{x \in \mathcal{P} \cap (Q_i \times [0, \alpha])} \xi_{V,n,k}^{(\infty)}(x, \mathcal{P})$  is larger when (b1) is satisfied. We do it when  $k = d$ , i.e. in the case of the volume, as it contains all the ingredients needed to prove it for any intrinsic volume but with slightly less technicality that would only obfuscate the ideas.

If each point  $z$  were deterministic in  $(\mathbb{Z}^{d-1} \cap Q_i^{(c)}) \times \{\delta\}$  or  $(\mathbb{Z}^{d-1} \cap Q_i^{(c)}) \times \{\delta/2\}$  then the  $n$ -th layer in case (b2) would be a copy of the  $n$ -th layer in case (b1) at a lower height. The difference of volume between the two cases (b1) and (b2) would thus be a constant times the difference of height, i.e.  $c\delta$ . In the general case when the points are random, since we can choose  $\varepsilon > 0$  as small as we want, we can make the difference of volume as close as needed to this deterministic situation. Thus we can make sure that there exists a constant  $c > 0$  such that  $\sum_{x \in \mathcal{P} \cap (Q_i \times [0, \alpha])} \xi_{n,k}^{(\infty)}(x, \mathcal{P})$  in situation (b2) is smaller than  $c$  times  $\sum_{x \in \mathcal{P} \cap (Q_i \times [0, \alpha])} \xi_{n,k}^{(\infty)}(x, \mathcal{P})$  in situation (b1). Thus we have proved (3.97).

The end of the proof follows along the same lines as in the case of the  $k$ -faces.  $\square$

### 3.6 Concluding remarks

Let us conclude with a list of possible extensions of our results and related open problems.

— *Other intensity measures.* In [26], the convex hull of a Poisson point process with a Gaussian intensity measure is studied. While the intensity of the process and therefore the rescaling are different, the techniques that are used are very similar so these results should extend to the  $n$ -th layer as we did in this paper for the uniform measure in the unit ball. We also expect that our results extend to a stationary Poisson point process in any smooth convex body as in [25] where the case of the first layer is treated. The  $n$ -th layer should have the same behaviour as the first one in this case as well. However, [25] uses a sandwiching result stating that with high probability the first layer lies between two floating bodies, see [61]. Such a result has not been proved for the  $n$ -th layer and would be interesting on its own. The same kind of problem arises in [27] about the convex hull of the peeling of a uniform point set in a polytope. Again, an argument on the sandwiching of the first layer is needed, see [12]. An extension to the  $n$ -th layer inside a polytope will be the subject of a future work.

Finally, in [9] asymptotic results on the expected number of vertices of the convex hull of an i.i.d. sample of an arbitrary non atomic probability distribution on  $\mathbb{R}^d$  are derived from an

approximation of the convex hull with floating bodies. We may hope that an approximation of the same kind should be possible for the  $n$ -th layer.

- *Invariance principles.* Let us consider the processes defined as the integrated versions of the defect support function and radius support function of each of the first layers. It is proved in [24, Theorem 8.3] that in the case of the first layer, such processes when properly rescaled and centered converge to a Brownian sheet process. We expect to get analogous functional central limit theorems for subsequent layers as we think that the arguments, for both the convergence of the finite-dimensional distributions and the tightness, should translate well to the case of the  $n$ -th layer. This would certainly be more challenging to do so when the number of the layer depends on  $\lambda$ , see below.
- *Depoissonization.* We left aside the case where we have a fixed number of i.i.d. random points uniformly distributed in the unit ball, i.e. a binomial point process instead of a Poisson point process. We expect a result of depoissonization in the spirit of [25, Theorem 1.2] and [28, Theorem 1.1] to occur.
- *Optimal Berry-Esseen bounds.* Recently, a method to derive central limit theorems for stabilizing functionals has been derived in [52]. The case of the number of  $k$ -faces of a convex hull of Poisson point processes in a smooth convex body is given as an application with an improved rate of convergence, i.e.  $O\left(\lambda^{-\frac{d-1}{2(d+1)}}\right)$  instead of  $O\left(\lambda^{-\frac{d-1}{2(d+1)}}(\log \lambda)^{3d+1}\right)$  as in Theorem 3.1.1. We conjecture that the same rate of convergence should hold for the  $n$ -th layer but extending the method from [52, Theorem 3.5] seems somehow challenging.
- *Monotonicity.* A monotonicity problem arises naturally from Theorem 3.1.1. Denoting by  $C_{n,k,d}$  the limit obtained for the expectation in Theorem 3.1.1, we might wonder how the constants  $C_{n,k,d}$  evolve with  $n$ . For  $k = 0$ , we expect that the sequence  $(C_{n,0,d})_n$  decreases with  $n$ . This is what our simulations suggest. In general, monotonicity problems in random polytopes are difficult, some insightful results on the monotonicity of the number of  $k$ -faces of the convex hull when the number of points increases can be found in [37], [15] and [18].
- *Other regimes.* Until now we have only considered a fixed layer number  $n$  that does not depend  $\lambda$  which means that we have only studied the first layers. It would be interesting to study different regimes, where  $n$  would vary with  $\lambda$ . Thanks to [33] and [22] we know that the expected number of layers is equivalent to  $\lambda^{\frac{2}{d+1}}$  up to a constant. Thus a natural regime to study would be the case where  $n = c\lambda^{\frac{2}{d+1}}$ . Calder and Smart conjecture in [22] that the mean number of points in this regime is still equivalent to  $\lambda^{\frac{d-1}{d+1}}$  up to an explicit constant. They provide in particular a heuristic argument and simulations that we were able to reproduce.

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## Chapter 4

# Limit theory for the first layers in a simple polytope

This chapter contains the current state of an ongoing work with Pierre Calka, where we study the first layers of the convex hull peeling of a homogeneous Poisson point process in a simple polytope as its intensity goes to infinity. We obtain a limit for the expectation and the variance of the number of  $k$ -faces of the  $n$ -th layer for any fixed  $n$  that does not vary with  $\lambda$ .

We draw our inspiration from the methods of [27] and [12]. The originality of our contribution lies notably in Sections 4.2 and 4.4. In the former, we generalize the sandwiching result from [12] to all of the first layers, i.e. we show that the first  $n$ -layers can be sandwiched in a thin region delimited by two floating bodies. In the latter, we prove a stabilization result that requires new arguments, of the same flavor as those used in Chapter 3 in the case of the unit ball.

We expect that our techniques would provide additional results such as a central limit theorem, analogous asymptotics for the defect volume and the positivity of the underlying constants. This should take place in a future version.

### 4.1 Introduction

Random polytopes constructed as convex hulls of a random point set have attracted attention for sixty years since the seminal work due to Rényi and Sulanke [64, 65]. For a general overview, we refer to the classical surveys [6, 69, 63, 48]. The basic question is: given  $n$  random points which are independent and uniformly distributed in some fixed convex body  $K$  in  $\mathbb{R}^d$ ,  $d \geq 2$ , what can be said about the convex hull of these points? Even easier to deal with is the Poissonized counterpart of that question, when the random input is a Poisson point process with intensity measure equal to  $\lambda$  times the Lebesgue measure in  $K$ . With a few notable exceptions [81, 41] including a very recent one [50], most of the existing work tackles asymptotic issues, including the behavior for large  $n$  or large  $\lambda$  of functionals of the random convex hull which are either combinatorial, namely the number of  $k$ -dimensional faces, or geometric, namely the volume and intrinsic volumes. Rényi and Sulanke's results in dimension two have revealed a fundamental dichotomy, depending on whether the convex body  $K$  has a smooth boundary with a  $\mathcal{C}^2$  regularity or is a polytope itself. In the first case, for any  $d \geq 2$ , the expectations of the number of extreme points and subsequently of any

number of  $k$ -dimensional faces are proved to grow polynomially fast, with a rate proportional to  $n^{\frac{d-1}{d+1}}$  or  $\lambda^{\frac{d-1}{d+1}}$  [2, 74, 62]. In the second case, the obtained growth rate is logarithmic [7, 62]. More precisely, for  $0 \leq k \leq (d-1)$ , let us denote by  $f_k(\cdot)$  the number of  $k$ -dimensional faces (or  $k$ -faces in short) of a polytope and by  $\text{conv}(\cdot)$  the convex hull of a point set. When  $K$  is a convex polytope and  $\mathcal{P}_\lambda$  is a Poisson point process with intensity measure  $\lambda$  times the Lebesgue measure in  $K$ , a Poissonized version of a result due to Reitzner [62, Theorem] says that

$$\lim_{\lambda \rightarrow \infty} \log^{-(d-1)}(\lambda) \mathbb{E}(f_k(\text{conv}(\mathcal{P}_\lambda))) = c_d T(K) \quad (4.1)$$

where  $c_d$  is an explicit positive constant depending only on dimension  $d$  and  $T(K)$  is the number of towers of  $K$ , a tower being an increasing chain  $F_0 \subset F_1 \dots \subset F_{d-1}$  of  $k$ -faces  $F_k$  of  $K$ . A similar asymptotic estimate has been obtained for the variance of  $f_k(\text{conv}(\mathcal{P}_\lambda))$  when the convex polytope  $K$  is assumed to be simple, in which case the number  $T(K)$  is proportional to the number of vertices of  $K$  [27, Theorem 1.3]. In other words, when  $K$  is a simple polytope, we get

$$\lim_{\lambda \rightarrow \infty} \log^{-(d-1)}(\lambda) \text{Var}(f_k(\text{conv}(\mathcal{P}_\lambda))) = c'_d f_0(K) \quad (4.2)$$

where  $c'_d$  is a positive constant depending only on dimension  $d$ .

For a general polytope  $K$ , Bárány and Reitzner have obtained lower and upper bounds for the variance which match with the growth rate  $\log^{d-1}(\lambda)$  up to multiplicative constants as well as a central limit theorem satisfied by  $f_k(\text{conv}(\mathcal{P}_\lambda))$ , see [12, 11].

In this work, we concentrate on the case when  $K$  is a  $d$ -dimensional simple polytope and we investigate the so-called convex hull peeling of a Poisson input  $\mathcal{P}_\lambda$  in  $K$ .

The procedure of *convex hull peeling* consists in constructing a decreasing sequence  $\text{conv}_n(\mathcal{P}_\lambda)$ ,  $n \geq 1$ , of convex hulls as follows: we initialize the process by taking  $\text{conv}_1(\mathcal{P}_\lambda) := \text{conv}(\mathcal{P}_\lambda)$ . We then remove the set of extreme points  $\partial \text{conv}_1(\mathcal{P}_\lambda) \cap \mathcal{P}_\lambda$  and define  $\text{conv}_2(\mathcal{P}_\lambda) := \text{conv}(\mathcal{P}_\lambda \setminus \partial \text{conv}_1(\mathcal{P}_\lambda))$ . We extend recursively the process so that for every  $n \geq 1$ ,

$$\text{conv}_n(\mathcal{P}_\lambda) = \text{conv}(\mathcal{P}_\lambda \setminus (\cup_{i=1}^{n-1} \partial \text{conv}_i(\mathcal{P}_\lambda))). \quad (4.3)$$

We call  $n$ -th layer the boundary  $\partial \text{conv}_n(\mathcal{P}_\lambda)$  and for any  $x \in K$ , we denote by  $\ell_{\mathcal{P}_\lambda}(x)$  the label of the layer which contains  $x$  in the peeling of the point set  $\mathcal{P}_\lambda \cup \{x\}$ .

To the best of our knowledge, the literature on the theoretical study of random convex hull peeling remains sparse, as emphasized in [63, Section 2.2.7]. Introduced by Barnett in [13] in 1976, the procedure of peeling finds its roots in spatial statistics, in particular it is used for defining a notion of depth inside multivariate data [47, 66]. We draw attention to two particular references which investigate the asymptotics of the depth related to the peeling. In [33], Dalal shows that the total number of layers, which differs from  $\max_{x \in K} \ell_{\mathcal{P}_\lambda}(x)$  by at most 1, is upper and lower bounded in expectation by  $\lambda^{\frac{2}{d+1}}$  up to a multiplicative constant. Actually, his statement is given for a binomial input, i.e. with deterministic cardinality, but it can be Poissonized straightforwardly. In any case, the result is remarkable as it does not depend on the nature of the convex body  $K$ , which means that it includes both cases of a polytope and of a smooth convex body. In fact, Dalal does not even assume that the underlying mother body is convex.

More recently, in a breakthrough paper [22], Calder and Smart have strengthened Dalal's result by deriving a limit expectation and a law of large numbers for the total number of layers when the input is a Poisson point process in a convex body  $K$ , such that its intensity measure has a continuous density  $f$ . Above all, they produce a functional version of the convergence by showing

that the rescaled convex height function, equal to  $\lambda^{-\frac{2}{d+1}}(\ell_{\mathcal{P}_\lambda}(x) + 1)$  according to their convention therein, converges uniformly and almost surely to a limit function which is proved to be the viscosity solution of an explicit non-linear partial differential equation, see [22, Theorem 1.2]. Naturally, their renormalization involves layers of the peeling which have a label proportional to  $\lambda^{\frac{2}{d+1}}$ . In particular, let us fix  $n(\lambda, t) = \lfloor t\lambda^{\frac{2}{d+1}} \rfloor$  and denote by  $N_{n(\lambda, t), 0, \lambda}$  the number of Poisson points lying on the  $n(\lambda, t)$ -th layer of the peeling. Calder and Smart then conjecture [22, display (1.18)] with a short heuristic argument that when  $\lambda \rightarrow \infty$ , almost surely

$$\lambda^{-\frac{d-1}{d+1}} N_{n(\lambda, t), 0, \lambda} \longrightarrow \frac{1}{\alpha} \int_{\{\alpha h = t\}} f^{\frac{d-1}{d+1}} \kappa^{\frac{1}{d+1}} dS \quad (4.4)$$

where  $\alpha$  is a positive constant depending only on dimension  $d$ ,  $\alpha h$  is the limit function of  $\lambda^{-\frac{2}{d+1}} \ell_{\mathcal{P}_\lambda}(x)$ ,  $\kappa$  is the Gaussian curvature of the level set  $\{\alpha h = t\}$  and  $dS$  is the Hausdorff measure of that level set.

Again, they believe that (4.4) should not depend on the nature of  $K$ , i.e. whether  $K$  is a polytope or is smooth. This would suggest in particular a polynomial growth for the number of vertices of the  $n(\lambda, t)$ -th layer, i.e. asymptotically equal to  $\lambda^{\frac{d-1}{d+1}}$  up to a multiplicative constant. This is especially noticeable in view of (4.1) which claims that when  $K$  is a polytope, the first layer for a uniform input contains in mean a logarithmic number of Poisson points. Incidentally, Dalal's result on the growth rate for the total number of layers confirms that the number  $N_{n, 0, \lambda}$  of Poisson points on the  $n$ -th layer could not be logarithmic for all  $n$  as there are, in mean and up to a multiplicative constant, at most  $\lambda^{\frac{2}{d+1}}$  layers and a total of  $\lambda$  Poisson points in  $K$ .

Inspired by Calder and Smart's conjecture (4.4) and by its patent discrepancy with the case of the first layer when  $K$  is a polytope, we devote this paper to the study of the combinatorial functionals, i.e. the number of  $k$ -faces for  $0 \leq k \leq (d-1)$ , of the consecutive layers of the convex hull peeling of a Poisson point process  $\mathcal{P}_\lambda$  in a simple polytope  $K$ . We focus on the first layers of the peeling, i.e. with a label independent of  $\lambda$ . The regime that we investigate is different from the regime which is covered by Calder and Smart's study and conjecture (4.4) but in our opinion, this choice is meaningful for two reasons. First, when applying the peeling procedure to outlier detection in a point set, we expect the outliers to appear in the first layers rather than in the last ones. Secondly, we anticipate the first layers to be located near the boundary of  $K$  so that the methods developed for deriving (4.1) and (4.2) in the case of the first layer could extend to subsequent layers.

For a fixed  $d$ -dimensional simple polytope  $K$  and a Poisson point process  $\mathcal{P}_\lambda$  in  $K$  with intensity measure equal to  $\lambda$  times the Lebesgue measure in  $K$ , we denote by  $N_{n, k, \lambda}$  the number of Poisson points on the  $n$ -th layer  $\text{conv}_n(\mathcal{P}_\lambda)$ , i.e.  $N_{n, k, \lambda} = f_0(\text{conv}_n(\mathcal{P}_\lambda))$ . Our main result below provides expectation and variance asymptotics for  $N_{n, k, \lambda}$  when  $\lambda \rightarrow \infty$ .

**Theorem 4.1.1.** *For any  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$  there exist constants  $C_{n, k, d}, C'_{n, k, d} \in [0, \infty)$  which only depend on  $n, k$  and  $d$  such that*

$$\lim_{\lambda \rightarrow \infty} \log(\lambda)^{-(d-1)} \mathbb{E}[N_{n, k, \lambda}] = C_{n, k, d} f_0(K) \text{ and } \lim_{\lambda \rightarrow \infty} \log(\lambda)^{-(d-1)} \text{Var}[N_{n, k, \lambda}] = C'_{n, k, d} f_0(K).$$

Theorem 4.1.1 shows that the growth rate for both the expectation and variance of the number of  $k$ -faces of the  $n$ -th layer does not depend on  $n$  up to a multiplicative constant, i.e. that for any  $n \geq 1$ , the  $n$ -th layer behaves like the first one. In view of (4.4), this suggests in particular that we should expect a phase transition when  $n$  gets close to  $\lambda^{\frac{2}{d+1}}$ . Only, the actual evolution of  $N_{n, k, \lambda}$

as a function of  $n$  which would depend on  $\lambda$  is unfortunately not covered by our methods. In any case, the transition from a logarithmic regime to a polynomial regime remains a pivotal question that would deserve further attention in the future.

Moreover, Theorem 4.1.1 is intended as an obvious companion to [23, Theorem 1.1]. Indeed, in [23], we investigate the first layers of the convex hull peeling of a Poisson input in the  $d$ -dimensional unit ball and show that, in the same way, the expectation and variance of the number of  $k$ -faces of the  $n$ -th layer behave asymptotically like the first layer and have a growth rate proportional to  $\lambda^{\frac{d-1}{d+1}}$ .

In fact, the connection with [23] is even more visible when we address the strategy of proof. Indeed, the main results in [23] rely on the use of a global scaling transformation in the unit ball which was introduced in the context of the first layer [24]. This transformation sends the consecutive layers of the convex hull peeling to the layers of a different peeling with a so-called parabolic convexity. In particular, only the layers with label independent of  $\lambda$  are visible in this new picture and this explains a posteriori the choice for this particular regime. In the case of Theorem 4.1.1, we take our inspiration from [27] which contains in particular the proof of (4.2) and we show that near each vertex of  $K$ , we can construct a scaling transformation which sends the consecutive layers of the initial convex hull peeling to layers of a new peeling. The role played by the paraboloids in [23] is now played by so-called cone-like grains. Again, the rescaling makes visible the first layers with label independent of  $\lambda$  and only them.

The reason why the scaling transformation is needed is the following: in both [23] and the present paper, we can show properties of stabilization for the rescaled process. In other words, the status of one point, meaning the number of the layer containing that particular point, should only depend on the intersection of the Poisson point process with a vicinity of that point. This guarantees in turn mixing properties which imply, thanks to Mecke's formula for Poisson point processes, the calculation of the limit expectation and variance of the number of  $k$ -faces in the neighborhood of a vertex of  $K$ .

Still, the analysis of the peeling in the neighborhood of each vertex of  $K$  is not enough for deriving Theorem 4.1.1, namely global asymptotics for the number of  $k$ -faces of each layer. We need to show that on the one hand the contribution of the flat parts, i.e. the regions far from the vertices of  $K$ , is negligible and on the other hand that the contributions of all the neighborhoods of the vertices of  $K$  can be added. In the case of the variance, this means that these contributions must decorrelate asymptotically. Both of these issues are treated with the same method as in [27] which is in turn inspired by geometric techniques developed in [12]. One of the key ingredients in [12] is a so-called sandwiching result, which states that with high probability, the boundary of the convex hull of  $\mathcal{P}_\lambda$  lies between two so-called floating bodies related to  $K$ . The shape of such a sandwich is very thin along the facets of  $K$  and it is wider in the neighborhood of the vertices. As such, it plays the role of a deterministic approximation for the location of the boundary of the random convex hull of  $\mathcal{P}_\lambda$ . Subsequently, it is partitioned into smaller regions with the help of an economic cap covering and this paves the way for the construction of a dependency graph. This, in turn, induces both the decorrelation of the contributions of the neighborhoods of the vertices and the negligibility of the contribution of the flat parts. Our goal in this paper consists in adapting the technique to the first layers of the peeling, starting with the sandwiching property.

In comparison with the previous works on the convex hull of a Poisson input in a polytope [12, 27] and on the peeling in the unit ball [23], the most challenging and novel parts in the proof of Theorem 4.1.1 are the following.

— The proof of the stabilization result after rescaling, see notably Proposition 4.4.4, differs

significantly from its counterpart [23, Proposition 3.6] in the case of the parabolic peeling for two reasons: first, the geometry of the cone-like grains which play the role held by the paraboloids in [23] prevents us from making explicit analytical calculations. This induces a specific study of the shape of a cone-like grain, see Lemma 4.3.8 and a careful use of a spherical cone included in a cone-like grain. Secondly, the scaling transformation sends the initial Poisson point process  $\mathcal{P}_\lambda$  to a Poisson point process in the whole space  $\mathbb{R}^{d-1} \times \mathbb{R}$  which has a limit density equal to  $\sqrt{d}e^{dh}$ . In comparison, the limit Poisson point process in [23] is homogeneous with intensity 1 in the product space  $\mathbb{R}^{d-1} \times \mathbb{R}_+$ . This new exponential density induces several adaptations in the proofs of the main results in Section 4.4.

- The extension of the sandwiching result to the layers of the convex hull peeling requires new ideas: we study the peeling construction in each subset given by the economic cap covering of the sandwich, then use a general estimate for the probability of having points in convex position combined with the monotonicity of the layer numbers of the peeling.
- The proof in Section 4.5 for the negligibility of the flat parts and decorrelation of the parts in the neighborhoods of the vertices of  $K$  contains a specific twist with respect to its counterpart in [27, Section 3]. Indeed, the technique in [27] relies on the knowledge of an actual upper bound for the variance of the number of  $k$ -faces of the first layer provided by [11]. This falls down in the case of the subsequent layers of the convex hull peeling. Our proof then requires a careful study of each term appearing in our decomposition without any prior prediction on the growth rate of either the expectation or the variance.

The paper is structured as follows. In Section 4.2, we introduce a few geometric objects including the floating bodies and show the required sandwiching property for the first layers of the convex hull peeling. This represents our starting point for the proof of Theorem 4.1.1. Section 4.3 is devoted to the introduction of a scaling transformation in the neighborhood of a vertex of  $K$ , which is the same as in [27], and to the description of its effect on the layers of the convex hull peeling. We show in Section 4.4 the required stabilization properties for the functionals of interest in the rescaled picture. This implies the asymptotics for the expectation and variance of the number of  $k$ -faces in the neighborhood of a vertex of  $K$  given in Proposition 4.4.9. Section 4.5 deals with the negligibility of the contribution of the remaining parts of the sandwich and the asymptotic decorrelation of the contributions of all the vertices of  $K$ . We then prove Proposition 4.4.9 and Theorem 4.1.1 in Section 4.6. The final Section 4.7 includes a list of selected comments and prospects suggested by our work.

Throughout the paper, unless noted otherwise,  $c, c', c_1, c_2 \dots$  denote positive constants which only depend on dimension  $d$  and possibly  $n$  and  $k$  and whose value may change at each occurrence.

## 4.2 Floating bodies, Macbeath regions and sandwiching

Floating bodies and Macbeath regions are classical objects from convex geometry that have proved to play a significant role in the study of random polytopes [10, 12]. This section aims at introducing them and using them to prove a probabilistic result of asymptotic nature, which goes as follows: with high probability, the first layers of the convex hull peeling are located in a small vicinity of the boundary of  $K$  which is precisely described as the region between two well-calibrated deterministic floating bodies. This extends a similar result due to Bárány and Reitzner in the context of the convex hull [12, Section 5] and which is called by them *sandwiching property* in a welcome figurative way. That property is indeed crucial for the study of the second-order asymptotics of the convex hull inside a polytope [12, 27]. Unsurprisingly, we show in the rest of the paper that this is also the case for the subsequent layers of the peeling.

### 4.2.1 Floating bodies and Macbeath regions

In this subsection, we recall the definitions of floating bodies and Macbeath regions and state the classical economic cap covering due to Bárány and Larman [10] and Bárány [1].

Floating bodies are convex bodies included in  $K$  which provide notably a deterministic approximation of the consecutive convex hulls of the random input in  $K$ . As in [12] and [27], we define  $v : K \rightarrow \mathbb{R}$  as

$$v(z) := \min\{\text{Vol}_d(K \cap H) : H \text{ is a half-space of } \mathbb{R}^d \text{ and } z \in H\},$$

where  $\text{Vol}_d$  is the  $d$ -dimensional Lebesgue measure. For  $t \in [0, \infty)$  we call floating body of  $K$  with parameter  $t$  the set

$$K(v \geq t) := \{z \in K : v(z) \geq t\}$$

and define similarly the sets  $K(v \leq t)$  and  $K(v = t)$ .

When  $K$  is locally a cube around one of its vertices, it is possible to make explicit the equation of  $K(v = t)$  and show that its shape is pseudo hyperbolic. We recall without proof the following result.

**Lemma 4.2.1** ([27], Lemma 7.1). *There exists  $\Delta_d \in [1, \infty)$  depending only on  $d$  such that when  $K$  contains  $[0, \Delta_d]$  and is contained in some multiple of that cube, then for any  $t \in (0, \infty)$  :*

$$K(v = t) \cap [0, \frac{1}{2}]^d = \{(z_1, \dots, z_d) \in [0, \frac{1}{2}]^d : \prod_{i=1}^d z_i = \frac{d!}{d^d} t\}$$

Bárány and Larman were the first to show a connection between floating bodies and random polytopes, namely that the mean defect volume of the random polytope generated as the convex hull of  $n$  i.i.d. uniform points in any convex body is well approximated up to a multiplicative constant by the volume of the floating body with parameter  $\frac{1}{n}$  [10, Theorem 1]. Such a property relies on the construction of a specific deterministic covering of the set  $K(v \leq t)$  which is called the *economic cap covering* and which has been introduced in [10] and [1]. Subsequently, the economic cap covering has been instrumental in Bárány and Reitzner's proof of the sandwiching property of the first layer of the peeling [12] and will remain so for proving the analogue for the next layers. Its statement, given in Theorem 4.2.2 below, relies on the notion of Macbeath regions defined as follows: the Macbeath region, or M-region for short, with center  $z$  and factor  $\lambda > 0$  is the set  $M(z, \lambda) = M_K(z, \lambda) := z + \lambda[(K - z) \cap (z - K)]$ .

Let  $s_0 := (2d)^{-2d}$ . For any  $s \in [0, s_0]$ , we choose a maximal set of points  $z_1(s), \dots, z_{m(s)}(s)$  on  $K(v = s)$  having pairwise disjoint M-regions  $M(z_i, \frac{1}{2})$ . We call such a system saturated. We assert that it exists for each  $s$  although it is not necessarily unique. For any  $z$  in the interior of  $K$ , we denote by  $C(z)$  the minimal cap of  $z$ , i.e. the intersection of  $K$  with a half-space  $H$  containing  $z$  which has minimal volume equal to  $v(z)$ . The hyperplane bounding  $H$  is at some distance  $t > 0$  from the support hyperplane in same direction  $u \in \mathbb{S}^{d-1}$  and we denote by  $C^\gamma(z)$  the intersection of  $K$  with the translate of  $H$  by  $-(\gamma - 1)tu$ . We then write  $K'_i(s) := M(z_i, \frac{1}{2}) \cap C(z_i)$  and  $K_i(s) := C^\gamma(z_i)$  for  $1 \leq i \leq m(s)$  and state the economic cap covering result as follows.

**Theorem 4.2.2** ([10], Theorem 6). *Assume that  $\text{Vol}_d(K) = 1$ . For all  $s \in [0, s_0]$ , we have :*

1.  $\cup_{i=1}^{m(s)} K'_i(s) \subset K(v \leq s) \subset \cup_{i=1}^{m(s)} K_i(s)$ ,
2.  $s \leq \text{Vol}_d(K_i(s)) \leq 6^d s$  for all  $i = 1, \dots, m(s)$ ,
3.  $(6d)^{-d} s \leq \text{Vol}_d(K'_i(s)) \leq 2^{-d} s$  for all  $i = 1, \dots, m(s)$ .

## 4.2.2 Sandwiching

In this section, we aim at proving Theorem 4.2.3 below, i.e. constructing two floating bodies with small parameters such that the first  $n$  layers of the peeling are located between those floating bodies with high probability. This result extends Bárány and Reitzner's work [12, Claims 5.1 and 5.2] on the convex hull, i.e. the first layer, and for that reason, we keep their original expression of *sandwiching result*.

Following their notation, we write

$$s := \frac{1}{\lambda(\log \lambda)^{4d^2+d-1}}, \quad T := \alpha \frac{\log \log(\lambda)}{\lambda} \text{ and } T^* = d6^d T \quad (4.5)$$

where  $\alpha = 16 \cdot 2^{-d} \cdot (6d)^{2d}(4d^2 + d - 1)$ . We make the observation here that the definitions of  $s$ ,  $T$  and  $T^*$  are the same as in [12, Section 5] but the constant  $\alpha$  is larger. The reason why  $\alpha$  has been recalibrated for our purpose is visible in the proof of Lemma 4.2.5.

We introduce the sandwich set

$$\mathcal{A}(s, T^*, K) := K(v \geq s) \setminus K(v \geq T^*)$$

and the sandwiching event, for fixed  $n \geq 1$ ,

$$\tilde{A}_\lambda := \{\cup_{l=1}^n \partial \text{conv}_l(\mathcal{P}_\lambda) \subset \mathcal{A}(s, T^*, K)\}. \quad (4.6)$$

Theorem 4.2.3 below shows that the event  $\tilde{A}_\lambda$  above occurs with high probability.

**Theorem 4.2.3.** *Assume that  $\text{Vol}_d(K) = 1$ . There exists a positive constant  $c > 0$  such that for all  $\lambda$  large enough,*

$$\mathbb{P}(\tilde{A}_\lambda) \geq 1 - c \log^{-4d^2}(\lambda).$$

This result will prove to be of utmost importance further on. First, it is instrumental when studying the convex hull peeling near a vertex of  $K$ : indeed, in Section 4.3, the neighborhood of a vertex is transformed after rescaling into a product space in which the convex peeling becomes a peeling of another sort. Proposition 4.3.2 therein is proved with the help of Theorem 4.2.3 and shows that the faces of the first layers of the initial convex peeling are sent to the faces of the new peeling. Later on, Theorem 4.2.3 is again essential all the way through Section 4.5 when proving Proposition 4.5.15 which says in particular that the contribution of the Poisson points far from a vertex of  $K$  is negligible in the asymptotic estimate of both the expectation and the variance of the variables  $N_{n,k,\lambda}$ . Intuitively, this is due to the fact that the sandwich between the two floating bodies is asymptotically much thinner in the region far away from the vertices of  $K$  and consequently contains less Poisson points in that particular region.

The proof of Theorem 4.2.3 relies first on the use of Theorem 4.2.2 which guarantees the existence of sets  $K_i$  and  $K'_i$ ,  $1 \leq i \leq m(T)$  for the coverage of  $K(v \leq T)$  and then on the essential fact that with high probability, the peeling of the Poisson points in each set  $K'_i$  gives birth to at least  $n$  layers, see Lemma 4.2.5 below.

A prerequisite to the proof of Lemma 4.2.5 is a general estimate on the probability denoted by  $p(n, L)$  that  $n$  i.i.d. points which are uniformly distributed in a fixed convex body  $L$  in  $\mathbb{R}^d$  are in convex position, i.e. belong to the boundary of their common convex hull. That particular question is a classical extension to any finite point set and any dimension of the famous Sylvester's four-point problem which asks for the probability that 4 *random points* in the plane are the vertices

of a convex quadrilateral. In dimension 2, the precise asymptotic behavior of the probability  $p(n, L)$  has been obtained by Bárány [4], see also [46] for further discussion. In particular, he shows that for any planar convex body  $L$ , when  $n \rightarrow +\infty$ ,

$$\log(p(n, L)) = -2n \log n + n \log(e^2 A^3(L)/(4\text{Vol}_2(L))) + o(n), \quad (4.7)$$

where  $A(L)$  means the maximal affine perimeter of any convex body included in  $L$ . The case of higher dimension is considered in [4] and for any  $d \geq 3$ , an expansion similar to (4.7) is conjectured, see display (3.1) therein. To the best of our knowledge, the existence of such a result is still open. Fortunately, this is not required for our purpose. We state below a weaker estimate which has been obtained by Bárány in 2001.

**Lemma 4.2.4** ([5], Theorem). *There exist two positive constants  $c_1 < c_2$  and  $n_0 \geq 1$  such that for any convex body  $L$  in  $\mathbb{R}^d$  and  $n \geq n_0$ ,*

$$c_1^n n^{-\frac{2}{d-1}n} \leq p(n, L) \leq c_2^n n^{-\frac{2}{d-1}n}. \quad (4.8)$$

As emphasized above, Lemma 4.2.4 is the key ingredient for proving Lemma 4.2.5 below which investigates the peeling in each set  $K'_i$  from the economic cap covering applied to the convex body  $K$  and the parameter  $T$ . For any  $1 \leq i \leq m(T)$ , we denote by  $L_{i,\lambda}$  the number of layers in the peeling of  $\mathcal{P}_\lambda \cap K'_i$ , i.e.

$$L_{i,\lambda} := \max_{x \in \mathcal{P}_\lambda \cap K'_i} \ell_{\mathcal{P}_\lambda \cap K'_i}(x). \quad (4.9)$$

**Lemma 4.2.5.** *Let  $K'_1, \dots, K'_{m(T)}$  be the  $m(T)$  sets provided by the economic cap covering applied to  $K$  with the parameter  $T$  given at (4.5). For every  $n \geq 1$ , there exists  $c > 0$  such that*

$$\mathbb{P}(\exists i \in \{1, \dots, m(T)\} : L_{i,\lambda} \leq n) \leq c \log^{-4d^2}(\lambda).$$

*Proof.* Let us introduce  $m_{i,\lambda} := \mathbb{E}[\text{card}(\mathcal{P}_\lambda \cap K'_i)]$  for  $1 \leq i \leq m(T)$ . Thanks to part 3 of Theorem 4.2.2 and (4.5), we obtain

$$(6d)^{-d} \alpha \log \log(\lambda) \leq m_{i,\lambda} \leq 2^{-d} \alpha \log \log(\lambda). \quad (4.10)$$

Consequently, we get for every  $1 \leq i \leq m(T)$

$$\mathbb{P}(L_{i,\lambda} \leq n) \leq T_1 + T_2 \quad (4.11)$$

where

$$T_1 := \mathbb{P}\left(|\text{card}(\mathcal{P}_\lambda \cap K'_i) - m_{i,\lambda}| \geq \frac{1}{2}(6d)^{-d} \alpha \log \log(\lambda)\right)$$

and

$$T_2 := \mathbb{P}(L_{i,\lambda} \leq n, c_1 \log \log(\lambda) \leq \text{card}(\mathcal{P}_\lambda \cap K'_i) \leq c_2 \log \log(\lambda)),$$

with  $c_1 = \frac{1}{2}(6d)^{-d} \alpha$  and  $c_2 = (\frac{1}{2}(6d)^{-d} + 2^{-d}) \alpha$ .

Using the fact that  $\text{card}(\mathcal{P}_\lambda \cap K'_i)$  is Poisson distributed with mean  $m_{i,\lambda}$ , we observe that the term  $T_1$  can be estimated through a classical concentration inequality for the Poisson distribution,

see e.g. [20, Section 2.2]. Indeed, using that any Poisson variable  $\text{Pois}(\mu)$  with mean  $\mu$  satisfies for every  $x > 0$

$$\mathbb{P}(|\text{Pois}(\mu) - \mu| \geq x) \leq 2 \exp\left(-\frac{x^2}{2(\mu + x)}\right),$$

combining it with (4.10) and recalling the value of  $\alpha$ , we obtain that

$$T_1 \leq 2 \log(\lambda)^{-(4d^2+d-1)}. \quad (4.12)$$

We now estimate  $T_2$ . By the law of total probability,

$$T_2 \leq \sum_{j=\lceil c_1 \log \log(\lambda) \rceil}^{\lceil c_2 \log \log(\lambda) \rceil} \mathbb{P}(L_{i,\lambda} \leq n \mid \text{card}(\mathcal{P}_\lambda \cap K'_i) = j). \quad (4.13)$$

Given that  $\text{card}(\mathcal{P}_\lambda \cap K'_i) = j$ ,  $\mathcal{P}_\lambda \cap K'_i$  consists of  $j$  i.i.d. variables which are uniformly distributed in  $K'_i$ . Moreover, whenever  $L_{i,\lambda} \leq n$ , at least one layer with label smaller than  $n$  must contain at least  $j/n$  points. In particular at least  $j/n$  points in  $\mathcal{P}_\lambda \cap K'_i$  need to be in convex position. This implies that

$$\mathbb{P}(L_{i,\lambda} \leq n \mid \text{card}(\mathcal{P}_\lambda \cap K'_i) = j) \leq \binom{j}{\lceil j/n \rceil} p(\lceil j/n \rceil, K'_i)$$

where we recall the notation  $p(n, L)$  introduced before (4.7). Using Lemma 4.2.4, the fact that  $j$  is proportional to  $\log \log(\lambda)$  and the inequality  $\binom{j}{\lceil j/n \rceil} \leq 2^j$ , we deduce from the previous inequality that for  $\lambda$  large enough,

$$\mathbb{P}(L_{i,\lambda} \leq n \mid \text{card}(\mathcal{P}_\lambda \cap K'_i) = j) \leq c^{\log \log(\lambda)} \log \log(\lambda)^{-c' \log \log(\lambda)} \quad (4.14)$$

where  $c$  and  $c'$  are two positive constants which depend only on dimension  $d$ . Summing (4.14) over  $j$  between  $\lceil c_1 \log \log(\lambda) \rceil$  and  $\lceil c_2 \log \log(\lambda) \rceil$  and using (4.13), we obtain the existence of  $c > 0$  such that for  $\lambda$  large enough

$$T_2 \leq \log \log(\lambda)^{-c \log \log(\lambda)}. \quad (4.15)$$

Combining (4.11), (4.12) and (4.15), we get for every  $1 \leq i \leq m(T)$

$$\mathbb{P}(L_{i,\lambda} \leq n) \leq c \log(\lambda)^{-(4d^2+d-1)}.$$

It remains to use the estimate  $m(T) \leq c \log^{d-1}(\lambda)$  from [12, Theorem 2.7] to complete the proof of Lemma 4.2.5. □

We are finally ready to prove Theorem 4.2.3.

*Proof of Theorem 4.2.3.* We start by recalling the sandwiching result for the convex hull of  $\mathcal{P}_\lambda$  [12, Claims 5.1 and 5.2], which implies in particular that

$$\mathbb{P}(\text{conv}_1(\mathcal{P}_\lambda) \subset K(v \geq s)) \geq 1 - c \log^{-4d^2}(\lambda).$$

Since the consecutive convex hulls  $\text{conv}_n(\mathcal{P}_\lambda)$ ,  $n \geq 1$  produced by the peeling are decreasing, this also implies that for every  $n \geq 1$ ,

$$\mathbb{P}(\text{conv}_n(\mathcal{P}_\lambda) \subset K(v \geq s)) \geq 1 - c \log^{-4d^2}(\lambda).$$

Consequently, only the bound

$$\mathbb{P}(K(v \geq T^*) \subset \text{conv}_n(\mathcal{P}_\lambda)) \geq 1 - c \log^{-4d^2}(\lambda) \quad (4.16)$$

requires an explanation. We consider again the sets  $K_i = K_i(T)$  and  $K'_i = K'_i(T)$ ,  $1 \leq i \leq m(T)$ , provided by Theorem 4.2.2 applied to the convex body  $K$  and the parameter  $T$ . Thanks to Lemma 4.2.5, each  $K'_i$  contains at least  $n$  layers for the peeling of  $\mathcal{P}_\lambda \cap K'_i$  with probability greater than  $1 - c \log^{-4d^2}(\lambda)$ . On this event, we pick a point  $x_i \in \mathcal{P}_\lambda \cap K'_i$  for each  $1 \leq i \leq m(T)$  such that  $\ell_{\mathcal{P}_\lambda \cap K'_i}(x_i) = n$ . In particular, the monotonicity of the function  $\ell$  with respect to the input set [33, Lemma 3.1] implies that  $\ell_{\mathcal{P}_\lambda}(x_i) \geq n$  for every  $i$  so  $\text{conv}(x_1, \dots, x_{m(T)}) \subset \text{conv}_n(\mathcal{P}_\lambda)$ . Moreover, [12, Claim 4.5] provides the inclusion  $K(v \geq T^*) \subset \text{conv}(x_1, \dots, x_{m(T)})$ . Consequently, we have proved that

$$\{\forall i \in \{1, \dots, m(T)\}, L_{i,\lambda} \leq n\} \subset \{K(v \geq T^*) \subset \text{conv}_n(\mathcal{P}_\lambda)\}$$

where we recall the definition of  $L_{i,\lambda}$  at (4.9). Lemma 4.2.5 now justifies (4.16) and completes the proof of Theorem 4.2.3. □

### 4.3 Scaling transform and scores

In this section, we introduce a scaling transform that lets us study the layers of the peeling in a more efficient way. This transform was already used in [27] in the case of the first layer and we expect it to bring the same benefits in our setting. As in [23], the scaling transform leads to a new type of peeling where the hyperplanes are replaced by a different geometric shape. In [23], that shape was a paraboloid while it is here a shape close to a cone. An important difference with the convex hull peeling in the ball [23] is the following: in the case of the simple polytope, the rescaling procedure is tailored for the study of the layers in the neighborhood of a vertex of  $K$  but we are unable to build a suitable global scaling transform, see Remark ii) after Theorem 4 in [27] for an explanation of this fact. A large part of this paper, see Section 4.5, is then dedicated to proving that the contribution of the Poisson points far from the vertices is negligible and that the contributions of the Poisson points near different vertices of  $K$  are independent. With that in mind, we devote Sections 4.3 and 4.4 to the estimation through the scaling transform of the *local* numbers of  $k$ -faces of the layers in the vicinity of a fixed vertex of  $K$ .

The outline of this section is the following. We first define the scaling transform and describe its effect on the Poisson point process, see Lemma 4.3.1. We then study its effect on the layers. In order to do this, we observe that only the so-called cone-extreme faces find their counterparts in the rescaled picture but fortunately, on the sandwiching event defined in Section 4.2.2, all regular faces are proved to be cone-extreme faces, see Lemma 4.3.2. We are then led to study a new type of rescaled peeling procedure where so-called cone-like grains defined with a specific function  $G$ , see (4.18), play the role of the half-spaces in the classical convex hull peeling and where the same basic properties as for the regular peeling are proved to occur, see Lemmas 4.3.4 and 4.3.5. Next, we introduce a sequence of random variables called scores and its equivalent in the rescaled

picture. Incidentally, we give a more precise version of Theorem 4.1.1 where the constants are rewritten explicitly through integral formulas involving the scores. Finally, we state a few analytical properties for the function  $G$  that are required in Section 4.4, see notably Lemma 4.3.8.

### 4.3.1 Rescaling

First of all, we introduce some useful notation related to the simple polytope  $K$ . We denote by  $\mathcal{V}_K := \{\mathcal{V}_i\}$  the set of vertices of  $K$ . Rescaling  $K$  if necessary, for each vertex  $\mathcal{V}_i \in \mathcal{V}_K$ , we introduce an associated volume preserving affine transformation  $a_i : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , with  $a_i(\mathcal{V}_i) = 0$ , and such that the facets of  $a_i(K)$  containing 0 also contain the facets of  $K' := [0, \Delta_d]^d$  belonging to the coordinate hyperplanes, with  $\Delta_d$  fixed by Lemma 4.2.1. This is possible because  $K$  is a simple polytope. For any  $\delta \in (0, \Delta_d)$ , we define the parallelepiped  $p_d(\mathcal{V}_i, \delta) := a_i^{-1}([0, \delta]^d)$ .

In practice, it means that the behavior of the convex hull peeling in any of the sets  $p_d(\mathcal{V}_i, \delta)$  can be deduced from the behavior of the convex hull peeling in  $[0, \delta]^d$  through the use of  $a_i$ . For this reason, we focus in the sequel on the convex hull peeling in a neighborhood of the vertex 0. We begin with a definition of a rescaling in the corner  $(0, \infty)^d$  that is suitable for our purpose.

We extend the definition of the logarithm and exponential functions to  $d$ -dimensional vectors by writing for any  $z = (z_1, \dots, z_d) \in (0, \infty)^d$  and  $v = (v_1, \dots, v_d) \in \mathbb{R}^d$ ,

$$\log(z) = (\log(z_1), \dots, \log(z_d)) \text{ and } e^v = (e^{v_1}, \dots, e^{v_d}).$$

We write  $p_V : \mathbb{R}^d \rightarrow V$  for the orthogonal projection onto  $V$  where

$$V = \{z \in \mathbb{R}^d : \sum_{i=1}^d z_i = 0\}.$$

The  $(d-1)$  dimensional vector space  $V$  is frequently identified with  $\mathbb{R}^{d-1}$  in the sequel. For any  $\lambda \in [1, \infty)$  we define the scaling transform

$$T^{(\lambda)} : \begin{cases} (0, \infty)^d & \rightarrow V \times \mathbb{R} \\ (z_1, \dots, z_d) & \mapsto (p_V(\log(z)), \frac{1}{d}(\log(\lambda) + \sum_{i=1}^d \log(z_i))) \end{cases} \quad (4.17)$$

The transform  $T^{(\lambda)}$  comes from [27, Equation (1.5)] and [14]. It is a one-to-one map between  $(0, \infty)^d$  and  $V \times \mathbb{R}$  of inverse

$$[T^{(\lambda)}]^{-1} : \begin{cases} V \times \mathbb{R} & \rightarrow (0, \infty)^d \\ (v, h) & \mapsto \lambda^{-1/d} e^h e^{l(v)} \end{cases}$$

where for any  $v \in V$ ,  $l(v) = (l_1(v), \dots, l_d(v))$  and  $l_i(v)$  is the  $i$ -th coordinate of  $v$  in the standard basis of  $\mathbb{R}^d$ .

Let us consider the cube  $Q_0 = [0, \delta_0]^d$  where  $\delta_0 = \exp(-\log(\lambda)^{1/d})$ . Its image by  $T^\lambda$  is

$$W_\lambda := T^{(\lambda)}(Q_0) = \{(v, h) \in V \times \mathbb{R} : h \leq -l_i(v) + \log(\lambda)^{1/d} \delta_0\}.$$

We also define the image in  $W_\lambda$  of the initial Poisson point process, i.e.

$$\mathcal{P}^{(\lambda)} := T^{(\lambda)}(\mathcal{P}_\lambda \cap Q_0).$$

We recall [27, Lemma 4.2] that gives the distribution and the limit in distribution of  $\mathcal{P}^{(\lambda)}$ .

**Lemma 4.3.1** ([27], Lemma 4.2). *Let  $\mathcal{P}$  be the Poisson point process in  $\mathbb{R}^{d-1} \times \mathbb{R}$  with intensity measure  $\sqrt{d}e^{dh}dv dh$ . The Poisson point process  $\mathcal{P}^{(\lambda)}$  satisfies*

$$\mathcal{P}^{(\lambda)} \stackrel{(d)}{=} \mathcal{P} \cap W_\lambda \quad \text{and} \quad \mathcal{P}^{(\lambda)} \xrightarrow{\mathcal{L}} \mathcal{P} \quad \text{when } \lambda \rightarrow \infty$$

where  $\stackrel{(d)}{=}$  means the equality in distribution and  $\xrightarrow{\mathcal{L}}$  means the convergence in distribution.

For sake of simplicity, when  $\lambda = \infty$  we identify  $W_\lambda$  with  $V \times \mathbb{R}$  and  $\mathcal{P}^{(\lambda)}$  with  $\mathcal{P}$ .

### 4.3.2 Effect on the layers

We now focus on the effect of the scaling transform  $T^{(\lambda)}$  on the layers of the peeling procedure. In agreement with [27, Section 3], we anticipate that  $T^{(\lambda)}$  makes visible only a fraction of the faces of a layer, namely the so-called *cone-extreme* faces.

Adapting [27, Definition 3.1] to the  $n$ -th layer, we call a face  $F$  of  $\text{conv}_n(\mathcal{P}_\lambda)$  a ‘cone-extreme’ face if the collection  $C_F(\text{conv}_n(\mathcal{P}_\lambda) \cap Q_0)$  of outward normals to  $F$  belongs to the normal cone  $C_0(K) := (-\infty, 0)^d$ .

Proposition 4.3.2 below states that on the event  $\tilde{A}_\lambda$  defined at (4.6), every face of  $\text{conv}_n(\mathcal{P}_\lambda)$  is indeed cone-extreme. It extends to the subsequent layers the analogous result [27, Proposition 3.1] proved in the case of the first layer.

**Proposition 4.3.2.** *On the event  $\tilde{A}_\lambda$  we have for  $\lambda$  large enough  $C_F(\text{conv}_n(\mathcal{P}_\lambda \cap Q_0)) \subset C_0(K)$  for any face  $F$  of  $\text{conv}_n(\mathcal{P}_\lambda \cap Q_0)$ .*

The proof of [27, Proposition 3.1] relies on two ingredients: the sandwiching estimate [12, Claims 5.1 and 5.2] and a construction of explicit dyadic Macbeath regions in a region containing  $Q_0$ . Unsurprisingly, the proof of Proposition 4.3.2 goes along very similar lines, i.e. it builds upon Theorem 4.2.3 and the explicit Macbeath regions given in Section 4.5. Consequently, for sake of brevity, we omit its proof.

We say that a hyperplane (resp. a half-space) is cone-extreme, if it has a unit normal vector (resp. an outward unit normal vector) in  $C_0(K)$ . We describe in the next lines how to encode in a suitable way such a hyperplane.

For any  $t > 0$ , we call pseudo-hyperboloid the surface

$$\mathcal{H}_t := \{(z_1, \dots, z_d) \in (0, \infty)^d : \prod_{i=1}^d z_i = t\}.$$

For every  $z^{(0)} \in (0, \infty)^d$ , we denote by  $H(z^{(0)})$  the hyperplane tangent to the unique pseudo-hyperboloid  $\mathcal{H}_t$  containing  $z^{(0)}$ . The reasoning before [27, (4.2)] shows that the equation of this hyperplane is given by

$$H(z^{(0)}) := \{(z_1, \dots, z_d) \in \mathbb{R}^d : \sum_{i=1}^d \frac{z_i}{z^{(0)}_i} = d\}.$$

In particular, any cone-extreme hyperplane can be written as  $H(z^{(0)})$  for some  $z^{(0)} \in (0, \infty)^d$ .

Let us define for any  $v \in V$  the function

$$G(v) := \log\left(\frac{1}{d} \sum_{i=1}^d e^{l_i(v)}\right). \tag{4.18}$$

Next, we define the downward cone-like grain as

$$\Pi^\downarrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h < -G(v)\}.$$

For any  $w \in \mathbb{R}^d$ , we then denote by  $\Pi^\downarrow(w) := w + \Pi^\downarrow$  the translate of  $\Pi^\downarrow$  by  $w$ , often called cone-like grain with apex at  $w$ . In a similar way, we introduce the upward cone-like grain

$$\Pi^\uparrow := \{(v, h) \in \mathbb{R}^{d-1} \times \mathbb{R} : h > -G(v)\}$$

and its translate  $\Pi^\uparrow(w) := w + \Pi^\uparrow$ . The duality between the upward and downward cone-like grains is given by the equivalence

$$w \in \overline{\Pi^\uparrow(w')} \iff w' \in \overline{\Pi^\downarrow(w)}.$$

Lemma 4.3.3 below is taken from [27, Lemma 4.3]. It shows that any cone-extreme half-space is mapped to a downward cone-like grain. In other words, the cone-like grains play the role in the rescaled model of the cone-extreme half-spaces in the original model.

**Lemma 4.3.3** ([27], Lemma 4.3). *(i) For every  $c \in (0, \infty)$ , we have*

$$T^{(\lambda)}(\mathcal{H}_{c/\lambda}) = V \times \left\{ \frac{1}{d} \log(c) \right\}.$$

*(ii) For every cone-extreme half-space  $H^+(z^{(0)})$ ,  $z^{(0)} \in (0, \infty)^d$ , we have*

$$T^{(\lambda)}(H^+(z^{(0)})) = \Pi^\downarrow(T^{(\lambda)}(z^{(0)})).$$

A consequence of Lemmas 4.3.2 and 4.3.3 is that on the event  $\tilde{A}_\lambda$ , the convex hull peeling of  $\mathcal{P}_\lambda$  is mapped to a new peeling procedure with cone-like grains instead of half-spaces. We now describe properly that peeling.

For any  $\lambda \in [1, \infty]$  and any locally finite point set  $Y$  in  $W_\lambda$ , we introduce  $[\Pi^\downarrow]^{(\lambda)} = \Pi^\downarrow \cap W_\lambda$  and the cone-like hull of  $Y$  as

$$\Phi^{(\lambda)}(Y) := \bigcup_{\substack{w \in W_\lambda \\ Y \cap [\Pi^\downarrow]^{(\lambda)}(w) = \emptyset}} [\Pi^\downarrow]^{(\lambda)}(w)^c. \quad (4.19)$$

We then define recursively the consecutive hulls  $\Phi_n^{(\lambda)}(Y)$ ,  $n \geq 2$ , of the cone-like peeling with the formula

$$\Phi_n^{(\lambda)}(Y) = \Phi^{(\lambda)}(Y \cap \text{int}(\Phi_{n-1}^{(\lambda)}(Y))). \quad (4.20)$$

When  $\lambda = \infty$ , we generally write  $\Phi_n$  instead of  $\Phi_n^{(\infty)}$ . As a result of Lemma 4.3.3, we obtain that on the event  $\tilde{A}_\lambda$ , the transformation  $T^{(\lambda)}$  maps the layers of the convex hull peeling of  $\mathcal{P}_\lambda \cap Q_0$  to the layers of the cone-like hull peeling of its image, i.e. for any  $n \geq 1$

$$T^{(\lambda)}(\text{conv}_n(\mathcal{P}_\lambda \cap Q_0)) = \Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)}).$$

### 4.3.3 Properties of the rescaled layers

This section gathers two relevant results on the rescaled layers that will be used frequently in Section 4.4. We omit the proofs as they are direct transpositions to the context of the cone-like peeling of [23, Lemma 2.5 and 2.6.] written for a parabolic peeling.

For any  $w \in \mathbb{R}^{d-1} \times \mathbb{R}$ , we introduce its layer number as

$$\ell^{(\lambda)}(w, Y) = n \text{ such that } w \in \partial\Phi_n^{(\lambda)}(Y \cup \{w\}). \quad (4.21)$$

The number  $\ell^{(\lambda)}(w, Y)$  is the counterpart for the cone-like peeling of  $Y$  of the layer number  $\ell_X(x)$  of a point  $x$  in the classical convex hull peeling of a point set  $X$ . According to the construction of the cone-like hull given at (4.19), we recall that a point  $w$  in  $\mathbb{R}^{d-1} \times \mathbb{R}$  is extreme if and only if there exists  $(v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w)$  such that  $[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap Y = \emptyset$ . The following result is a generalization of this criterion to the subsequent layers.

**Lemma 4.3.4.** *Let  $Y$  be a locally finite subset of  $\mathbb{R}^{d-1} \times \mathbb{R}$ ,  $w \in Y$ ,  $\lambda \in [1, \infty]$  and  $n \geq 1$ . Then we have the two following equivalences.*

$$(i) \ (\ell^{(\lambda)}(w, Y) \geq n) \iff (\forall (v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w) : Y \cap [\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \not\subset \cup_{i=1}^{n-2} \partial[\Phi_i]^{(\lambda)}(Y)).$$

$$(ii) \ (\ell^{(\lambda)}(w, Y) \leq n) \iff (\exists (v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(w) : Y \cap [\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \subset \cup_{i=1}^{n-1} \partial[\Phi_i]^{(\lambda)}(Y)).$$

Another important property of the peeling is that the layer number of any point increases with the point set. This is stated in Lemma 4.3.5 below.

**Lemma 4.3.5.** *For  $\lambda \in [1, \infty]$ , if  $X \subset Y \subset W_\lambda$ , we have for every  $w \in W_\lambda$ ,  $\ell^{(\lambda)}(w, X) \leq \ell^{(\lambda)}(w, Y)$ .*

### 4.3.4 Scores

To each point in  $K$ , we associate a random variable that depends on the point and the point process and that we call *score*. It represents the contribution of that point to the number of  $k$ -faces of the  $n$ -th layer of the convex hull peeling of the point process. For any  $x \in K$ ,  $n \geq 1$ ,  $k \in \{0, \dots, d-1\}$  and any point set  $X$  of  $\mathbb{R}^d$ , we introduce the function

$$\xi_{n,k}(x, X) := \begin{cases} \frac{1}{k+1} \text{card}(\mathcal{F}_{n,k}(x, X)) & \text{if } x \in \partial\text{conv}_n(X \cup \{x\}) \\ 0 & \text{otherwise} \end{cases}$$

where  $\mathcal{F}_{n,k}(x, X)$  is the set of all  $k$ -faces of  $\partial\text{conv}_n(X \cup \{x\})$  containing  $x$ . The factor  $\frac{1}{k+1}$  is needed to take into account the fact that the faces are counted multiple times since a  $k$ -face contains a.s.  $(k+1)$  points of  $\mathcal{P}_\lambda$ . In particular, we get the identity

$$N_{n,k,\lambda} = \sum_{x \in \mathcal{P}_\lambda} \xi_{n,k}(x, \mathcal{P}_\lambda). \quad (4.22)$$

We now extend this notion of score to the rescaled model. Let  $\lambda \in [1, \infty]$  and  $Y$  be a locally finite subset of  $W_\lambda$ . For any  $w \in W_\lambda$ , we define the score of  $w$  in the rescaled model as

$$\xi_{n,k}^{(\lambda)}(w, Y) = \xi_{n,k} \left( [T^{(\lambda)}]^{-1}(w), [T^{(\lambda)}]^{-1}(Y) \right). \quad (4.23)$$

When  $w \in \partial\Phi_n^{(\lambda)}(Y)$ , we put

$$\hat{\xi}_{n,k}^{(\lambda)}(w, Y) := \begin{cases} \frac{1}{k+1} [\text{number of } k\text{-faces of } \partial\Phi_n^{(\lambda)}(Y) \text{ containing } w] & \text{if } w \in \partial\Phi_n^{(\lambda)}(Y) \\ 0 & \text{otherwise} \end{cases} \quad (4.24)$$

where as in [27, Section 4], we call  $k$ -face of  $\partial\Phi_n^{(\lambda)}(Y)$  the image by  $T^{(\lambda)}$  of a cone-extreme  $k$ -face of  $[T^{(\lambda)}]^{-1}(\partial\Phi_n^{(\lambda)}(Y))$ .

A consequence of Proposition 4.3.2 and Lemma 4.3.3 is that on the event  $\tilde{A}_\lambda$ , the score of a point  $x \in \mathcal{P}_\lambda \cap Q_0$  can be rewritten as  $\hat{\xi}_{n,k}^{(\lambda)}(T^{(\lambda)}(x), \mathcal{P}^{(\lambda)})$ . More precisely, putting  $w = T^{(\lambda)}(x)$ , we have for  $x \in Q_0$

$$\xi_{n,k}(x, \mathcal{P}_\lambda) \mathbb{1}_{\tilde{A}_\lambda} = \hat{\xi}_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbb{1}_{\tilde{A}_\lambda} = \xi_{n,k}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \mathbb{1}_{\tilde{A}_\lambda}. \quad (4.25)$$

In fact, because of boundary effects, the scores of the points close to the boundary of  $Q_0$  may be different when the peeling is restricted to  $Q_0$ , making the equation above possibly false for these points. A solution to that issue would consist in applying the scaling transform in a larger cube  $Q_1$ . Because of Lemma 4.5.5, the scores of the points in  $Q_0$  would only depend on the points in  $Q_1$  for a suitable choice of  $Q_1$ . The rescaled model would then be constructed inside a window slightly larger than  $W_\lambda$ . This would not lead to any significant change and the identity (4.25) would then be true for any point in  $Q_0$ . For sake of simplicity, we ignore these considerations in the rest of the paper and assume that (4.25) holds for any point  $x \in Q_0$ .

For all  $\lambda \in [1, \infty]$  and  $(v_0, h_0), (v_1, h_1) \in W_\lambda$ , we introduce the two-point correlation function

$$\begin{aligned} c_{n,k}^{(\lambda)}((v_0, h_0), (v_1, h_1)) &:= \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \cup \{(v_1, h_1)\}] \hat{\xi}_{n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)}) \cup \{(v_0, h_0)\}] \\ &\quad - \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})] \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)})]. \end{aligned}$$

This function plays a crucial role in the proof of the convergence of the variance of  $N_{n,k,\lambda}$ .

Let us denote by  $S(d)$  the regular  $d$ -dimensional simplex of edge length  $\sqrt{2}d$  given by

$$S(d) := \{(x_1, \dots, x_d) \in (-\infty, 1] : \sum_{i=1}^d x_i = 0\}. \quad (4.26)$$

Now that we have introduced every notation involved in the limiting constants of Theorem 4.1.1, we can state a more precise result which clearly implies Theorem 4.1.1 and which is proved in Section 4.6.

**Theorem 4.3.6.** *For any  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$ , we get*

$$\lim_{\lambda \rightarrow \infty} \log(\lambda)^{-(d-1)} \mathbb{E}[N_{n,k,\lambda}] = f_0(K) d^{-d+3/2} \text{Vol}_d(S(d)) \int_{-\infty}^{\infty} \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h_0), \mathcal{P})] e^{dh_0} dh_0$$

and

$$\lim_{\lambda \rightarrow +\infty} \log(\lambda)^{-(d-1)} \text{Var}[N_{n,k,\lambda}] = f_0(K) d^{-d+1} \text{Vol}_d(S(d)) (I_1(\infty) + I_2(\infty))$$

where

$$I_1(\infty) := \sqrt{d} \int_{-\infty}^{\infty} \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h_0), \mathcal{P})^2] e^{dh_0} dh_0$$

and

$$I_2(\infty) := d \int_{-\infty}^{\infty} \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{\infty} c^{(\infty)}((0, h_0), (v_1, h_1)) e^{d(h_0+h_1)} dh_1 dv_1 dh_0.$$

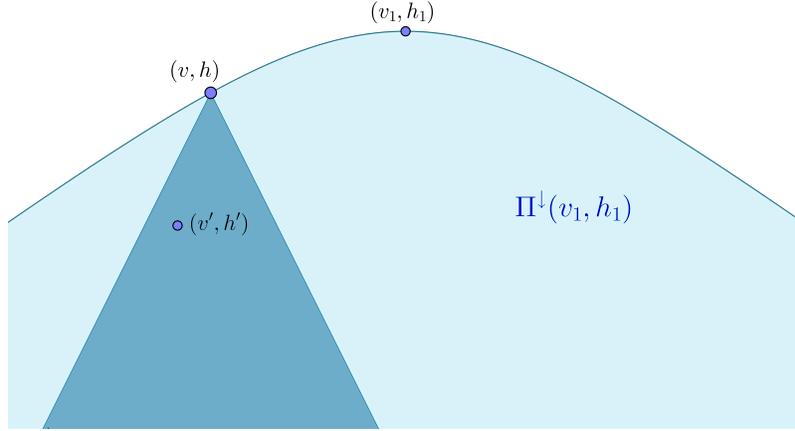


Figure 4.1 – The cone inclusion of Lemma 4.3.7 (2) .

We finally introduce some abbreviated notation used for practical reasons in Sections 4.4 and 4.5. For  $n \geq 1$ ,  $k \in \{1, \dots, d-1\}$ ,  $\delta \in (0, 1/2)$  and  $1 \leq i \leq f_0(K)$ , we put

$$Z := \sum_{x \in \mathcal{P}_\lambda} \xi_{n,k}(x, \mathcal{P}_\lambda) = N_{n,k,\lambda} \quad \text{and} \quad Z_i(\delta) := \sum_{x \in \mathcal{P}_\lambda \cap \mathcal{P}_d(\mathcal{V}_i, \delta)} \xi_{n,k}(x, \mathcal{P}_\lambda). \quad (4.27)$$

### 4.3.5 Properties of the function $G$

In preparation for Section 4.4, we state here a few properties satisfied by the function  $G$  introduced at (4.18).

Lemma 4.3.7 below comes from [27, Lemma 4.4 and 4.5]. It shows that the graph of the function  $G$  is convex and sandwiched between two circular cones.

**Lemma 4.3.7** ([27], Lemmas 4.4 and 4.5). *The function  $G$  is convex and positive. Moreover, there exists two constants  $\underline{c}$  and  $\bar{c} \in (0, \infty)$  such that for every  $v \in V$ ,*

$$\underline{c}\|v\| - \log(d) \leq G(v) \leq \bar{c}\|v\|.$$

Lemma 4.3.8 provides a new property of the function  $G$  that is heavily required to prove the stabilization of the scores in Section 4.4. In particular, it can be geometrically reinterpreted by stating that there exists a downward circular cone  $\mathcal{C}$  such that for any point  $w$  on the boundary of a cone-like grain  $\Pi^\downarrow(v_1, h_1)$ , we have  $\mathcal{C} + w \subset \Pi^\downarrow(v_1, h_1)$ , see Figure 4.1.

**Lemma 4.3.8.**

1. *There exists a constant  $c > 0$  such that for any  $v \in V$ ,*

$$\|\nabla G(v)\| \leq c\|v\|.$$

2. For any  $(v_1, h_1)$ ,  $(v, h)$  and  $(v', h') \in V \times \mathbb{R}$  such that  $h = h_1 - G(v - v_1)$  and  $h' \leq h - c\|v' - v\|$ , we have

$$h' \leq h_1 - G(v' - v_1).$$

*Proof.* We fix a basis  $\mathcal{B} = (\varepsilon_1, \dots, \varepsilon_{d-1})$  of  $V$  and a vector  $v \in V$  which will be identified with its coordinates  $(v_1, \dots, v_{d-1})$  in  $\mathcal{B}$ .

1. For any  $i = 1, \dots, (d-1)$ , we introduce  $\alpha_{i,1}, \dots, \alpha_{i,d-1}$  such that for any  $v \in V$  we can write  $l_i(v) = \alpha_{i,1}v_1 + \dots + \alpha_{i,d-1}v_{d-1}$ . It is enough to prove that for any  $1 \leq j \leq (d-1)$  and  $v \in V$ , there exists  $c_j$  such that  $|\frac{\partial G}{\partial v_j}(v)| \leq c_j$ . Since for any  $i, j$ , we have  $\frac{\partial l_i}{\partial v_j}(v) = \alpha_{i,j}$ , we deduce from (4.18) that

$$\frac{\partial G}{\partial v_j}(v) = \frac{\sum_{i=1}^d \frac{\partial l_i}{\partial v_j}(v) e^{l_i(v)}}{\sum_{i=1}^d e^{l_i(v)}} = \frac{\sum_{i=1}^d \alpha_{i,j} e^{l_i(v)}}{\sum_{i=1}^d e^{l_i(v)}}.$$

Thus we get the bound

$$\left| \frac{\partial G}{\partial v_j}(v) \right| \leq \max_{k,l} |\alpha_{k,l}|$$

and taking  $c_j = \max_{k,l} |\alpha_{k,l}|$  yields the result.

2. Using part 1 and the mean value theorem applied to the function  $G$ , we obtain that

$$G(v' - v_1) \leq G(v - v_1) + c\|v' - v\|. \quad (4.28)$$

Using consecutively the two assumptions on  $(v_1, h_1)$ ,  $(v, h)$  and  $(v', h')$  then (4.28), we then get

$$\begin{aligned} h' &\leq h - c\|v' - v\| \\ &\leq h_1 - G(v - v_1) - c\|v' - v\| \\ &\leq h_1 - G(v' - v_1), \end{aligned}$$

which completes the proof of Lemma 4.3.8.  $\square$

## 4.4 Stabilization and limit theory near a vertex

The final goal of this section is the convergence of the expectation and the variance of the variables  $Z_i(\delta_0)$ ,  $1 \leq i \leq f_0(K)$ , introduced at (4.27), i.e. the sum of the scores of Poisson points in a neighborhood of a vertex of  $K$ . Starting from (4.27), we apply Mecke's theorem then a rescaling to rewrite both the expectation and the variance as an integral of a functional which involves either an expected score or the correlation function introduced in Section 4.3. The proof of the convergence then relies on the use of the dominated convergence theorem, in the spirit of [27]. In order to get the required convergence and domination of the integrand, we follow a series of papers including [27] and [23], i.e. we appeal to a crucial ingredient, which consists in proving stabilization results for rescaled scores. In other words, we show that with probability exponentially close to one, the score of a point  $w$  only depends on the points located in a neighborhood of  $w$ . An additional difficulty compared to the case of the first layer studied by [27] comes from the lack of an easy criterion for determining the layer number of a point without the knowledge of all the preceding layers. To tackle this problem, we prove a lemma on the localisation in height of the  $n$ -th layer, as in the case of the unit ball [23].

Beforehand, we need to introduce useful notation for several types of cylinders that are used in the rest of the paper. For any  $v \in \mathbb{R}^{d-1}$  and  $r > 0$ ,  $C_v(r)$  denotes the vertical cylinder  $B_{d-1}(v, r) \times \mathbb{R}$  with the convention  $C(r) = C_0(r)$ . We also define the truncated cylinders  $C_v^{\geq t}(r) := C_v(r) \cap \{(v', h') \in \mathbb{R}^d : h' \geq t\}$ ,  $C_v^{\leq t}(r) := C_v(r) \cap \{(v', h') \in \mathbb{R}^d : h' \leq t\}$  and  $C_v^I(t) := C_v(r) \cap \{(v', h') \in \mathbb{R}^d : h' \in I\}$  for any  $t > 0$  and any interval  $I \subset \mathbb{R}$ .

In this first lemma, we show that the maximal height of the Poisson points on the  $n$ -th layer of the cone-like hull peeling inside a cylinder is bounded with a probability going to 1 exponentially fast with respect to the bound. This represents an essential ingredient of the proof of the stabilization result in height of Lemma 4.4.2.

**Lemma 4.4.1.** *For all  $n \geq 1$  and  $\varepsilon > 0$ , there exist  $\lambda_0 \geq 1$  and  $c > 0$  such that for all  $t > 0$ ,  $r \geq \varepsilon$ ,  $v_0 \in \mathbb{R}^{d-1}$  and  $\lambda \in [\lambda_0, \infty]$  we have*

$$\begin{aligned} \mathbb{P}\left(\exists(v, h) \in \partial\Phi_n^{(\lambda)}\left(\mathcal{P}^{(\lambda)} \cap C_{v_0}(r)\right) \cap \mathcal{P}^{(\lambda)} \cap C_{v_0}(r/2^n) \text{ with } h \geq t\right) \\ \leq cr^{(n-1)(d-1)} \exp\left(-e^{t/c}\right) \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\left(\exists(v, h) \in \partial\Phi_n^{(\lambda)}\left(\mathcal{P}^{(\lambda)}\right) \cap \mathcal{P}^{(\lambda)} \cap C_{v_0}(r/2^n) \text{ with } h \geq t\right) \\ \leq cr^{(n-1)(d-1)} \exp\left(-e^{t/c}\right). \end{aligned}$$

*Proof.* We only prove the first statement as the proof of the second one is identical. We start with the case  $\lambda = \infty$  and explain at the end why it still works for  $\lambda < \infty$ .

We show the result by induction, we start with the induction step and assume that  $n \geq 2$  and that the result is verified for any  $p < n$ .

Our first step is to show that for any fixed  $w = (v, h) \in C_{v_0}(r/2^n)$  with  $h \geq t$  the event

$$\{w \in \partial\Phi_n((\mathcal{P} \cup \{w\}) \cap C_{v_0}(r))\} \quad (4.29)$$

occurs with probability smaller than  $c \exp(-e^{h/c})$ . We consider a fixed  $w$  that verifies these conditions. Then there exists  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}$  such that  $w \in \partial\Pi^\downarrow(v_1, h_1)$  and  $\Pi^\downarrow(v_1, h_1)$  only contains points of layer number at most  $(n-1)$  for the peeling in  $C_{v_0}(r)$ . This downward cone-like grain contains a fixed circular cone  $\mathcal{C}_w$  with apex  $w$  as given by Lemma 4.3.8.

From the preceding reasoning and because  $\mathcal{C}_w$  does not depend on  $(v_1, h_1)$ , we deduce that

$$\{w \in \partial\Phi_n((\mathcal{P} \cup \{w\}) \cap C_{v_0}(r))\} \subset \{\mathcal{P} \cap \tilde{\mathcal{C}}_w \subset \cup_{i=1}^{n-1} \partial\Phi_i((\mathcal{P} \cup \{w\}) \cap C_{v_0}(r))\}$$

where  $\tilde{\mathcal{C}}_w := \mathcal{C}_w \cap \{(v', h') : h' \geq h/2\} \cap C_{v_0}(r/2^{n-1})$ . The set  $\mathcal{P} \cap \tilde{\mathcal{C}}_w$  is empty with probability smaller than  $c \exp(-e^{h/c})$ . If  $\mathcal{P} \cap \tilde{\mathcal{C}}_w$  is not empty, it contains a point at height larger than  $h/2$  on a layer at most  $(n-1)$ , which happens with probability smaller than  $cr^{(n-2)(d-1)} \exp(-e^{h/c})$  thanks to the induction hypothesis. Consequently we have

$$\mathbb{P}(\{w \in \partial\Phi_n((\mathcal{P} \cup \{w\}) \cap C_{v_0}(r))\}) \leq cr^{(n-2)(d-1)} \exp(-e^{h/c}). \quad (4.30)$$

We can write

$$\begin{aligned} \mathbb{P}(\exists(v, h) \in \partial\Phi_n(\mathcal{P} \cap C_{v_0}(r)) \cap \mathcal{P} \cap C_{v_0}(r/2^n) \text{ with } h \geq t) \\ \leq \mathbb{E} \left[ \sum_{w \in \mathcal{P} \cap C_{v_0}^{\geq t}(r/2^n)} \mathbf{1}_{w \in \partial\Phi_n(\mathcal{P} \cap C_{v_0}(r))} \right]. \end{aligned}$$

We combine this with the Mecke formula and (4.30) to get

$$\begin{aligned}
& \mathbb{P}(\exists(v, h) \in \partial\Phi_n(\mathcal{P} \cap C_{v_0}(r)) \cap C_{v_0}(r/2^n) \text{ with } h \geq t) \\
& \leq \int_{\|v-v_0\| \leq r/2^n} \int_{h \in ]t, \infty[} \mathbb{P}((v, h) \in \partial\Phi_n((\mathcal{P} \cup \{(v, h)\}) \cap C_{v_0}(r))) \sqrt{d} e^{dh} dh dv \\
& \leq \int_{\|v-v_0\| \leq r/2^n} \int_{h \in ]t, \infty[} c r^{(n-2)(d-1)} \exp(-e^{h/c}) \sqrt{d} e^{dh} dh dv \\
& \leq c r^{(n-1)(d-1)} \exp(-e^{t/c}).
\end{aligned}$$

This proves the induction step.

The base case is easier as the same reasoning holds except that we only have to consider the case where the circular cone  $\mathcal{C}_w$  is empty.

*Case  $\lambda < \infty$ .* We claim that as long as the circular cone  $\mathcal{C}_w$  is included in  $W_\lambda$  the proof remains valid because the intensity of the point process  $\mathcal{P}^{(\lambda)}$  is the same as the intensity of  $\mathcal{P}$  except that the process is restricted to  $W_\lambda$ . Let us describe how we can ensure that  $\mathcal{C}_w$  is included in  $W_\lambda$ . Let  $C_m = C_m(\lambda)$  be the largest cone contained in  $W_\lambda$  with the same apex as  $W_\lambda$ . As for  $\lambda \geq \lambda'$ ,  $W_\lambda$  is only a vertical translation of  $W_{\lambda'}$ , the aperture of  $C_m(\lambda)$  stays the same. Thus, for  $\lambda \geq \lambda_0$  and any  $(v', h') \in W_\lambda$  the cone  $C_m(\lambda_0)$  is contained in  $W_\lambda$ . If we take for  $\mathcal{C}_w$  the smallest cone between the translation of  $C_m(\lambda_0)$  with apex  $(v, h)$  and the choice of  $\mathcal{C}_w$  in the case  $\lambda = \infty$  we obtain a circular cone that is contained in  $W_\lambda$  and verifies the volume estimates in the proof of the case  $\lambda = \infty$ . This makes the proof valid in the case  $\lambda < \infty$ .  $\square$

When  $w \in \partial\Phi_n(\mathcal{P}^{(\lambda)})$  we define  $H_n^{(\lambda)}(w, \mathcal{P}^{(\lambda)})$  as the maximal height of an apex of a down cone-like grain belonging to the  $n$ -th layer and containing  $w$ . Otherwise, we put  $H_n^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) = 0$ . The next result has the flavor of a stabilization in height. It provides an upper bound on the distribution tail of the height of the rescaled cone-like facets containing a fixed point from the  $n$ -th layer.

**Lemma 4.4.2.** *There exist a constant  $c > 0$  and  $\lambda_0 \geq 1$  such that for all  $\lambda \in [\lambda_0, \infty]$ ,  $(v_0, h_0) \in W_\lambda$  and  $t \geq h_0 \vee 0$ ,*

$$\mathbb{P}(H_n^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \geq t) \leq c \exp(-e^{t/c}).$$

*Furthermore, for any  $\varepsilon > 0$ , there exist a constant  $c > 0$  and  $\lambda_0 \geq 1$  such that for all  $\lambda \in [\lambda_0, \infty]$ ,  $(v_0, h_0) \in W_\lambda$ ,  $t \geq h_0 \vee 0$  and  $r > \varepsilon$ ,*

$$\mathbb{P}(\exists s \geq r : H_n^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(s)) \geq t) \leq c \exp(-e^{t/c}).$$

*Proof.* Let us concentrate on the first statement. We can assume that  $n \geq 2$  as the case  $n = 1$  is proved in [27, Lemma 5.1]. Let us assume that  $H_n^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \geq t$ . It implies that there exists a downward cone-like grain with apex  $(v_1, h_1) \in [\Pi^\dagger]^{(\lambda)}(v_0, h_0)$  and  $h_1 \geq t$  that only contains points on layer at most  $(n - 1)$ .

Let  $c_{min}$  be the minimum between the aperture of the circular cone contained in  $\Pi^\downarrow(v_1, h_1)$  given by Lemma 4.3.7 and the aperture of the largest circular cone contained in  $W_\lambda$  with the same apex as  $W_\lambda$ . We can fit a circular cone  $\mathcal{C}_{(v_1, h_1)}$  with apex  $(v_1, h_1)$  and aperture  $c_{min}$  in  $\Pi^\downarrow(v_1, h_1)$  that is entirely contained in  $W_\lambda$ .

Let  $\tilde{\mathcal{C}} := \mathcal{C}_{(v_1, h_1)} \cap \{(v', h') : h_1/4 \leq h' \leq h_1/2\} \cap C_{v_1}(t/2^n)$ . Then either  $\mathcal{P}^{(\lambda)} \cap \tilde{\mathcal{C}} = \emptyset$ , which happens with probability smaller than  $c \exp(-e^{h_1/c})$  or it contains at least one point which is thus on layer at most  $(n-1)$ . This last event occurs with probability smaller than  $c \exp(-e^{h_1/c})$  because of Lemma 4.4.1.

Discretizing and integrating over  $(v_1, h_1) \in [\Pi^\uparrow]^{(\lambda)}(v_0, h_0)$ ,  $h_1 \in [t, \infty)$  yields the desired result.

The second statement follows along similar lines with a proper updating of the cylinder  $C_{v_1}(t/2^n)$  in the intersection defining the set  $\tilde{\mathcal{C}}$ . □

We are now ready to prove in Proposition 4.4.3 below a result of stabilization in width for the score  $\hat{\xi}_{n,0}^{(\lambda)}$  corresponding to the number of vertices of the  $n$ -th layer. It is a necessary step towards the proof of the stabilization in width for the general score  $\hat{\xi}_{n,k}^{(\lambda)}$  which is stated in Proposition 4.4.4 that follows directly.

For any  $h_0$  we write

$$\tilde{h}_0 := \left(\frac{6}{c} \log(d)\right) \vee \left(-\frac{6}{c} h_0\right) \mathbf{1}_{\{h_0 < 0\}}.$$

For any  $\lambda \geq 1$  and  $(v_0, h_0) \in W_\lambda$ , we define the radius of stabilization as

$$R_{n,k}^{(\lambda)} := \inf\{r > 0 : \hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) = \hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cap C(s)) \text{ for all } s \geq r\}.$$

**Proposition 4.4.3.** *For all  $n \geq 1$ , there exist  $\lambda_0 \geq 1$  and  $c > 0$  such that for any  $(v_0, h_0) \in W_\lambda$ ,  $r \geq \tilde{h}_0$  and  $\lambda \in [\lambda_0, +\infty]$  we have*

$$\mathbb{P}\left(R_{n,0}^{(\lambda)}(v_0, h_0) \geq r\right) \leq c \exp(-r/c).$$

*Proof.* For sake of simplicity, we only treat  $v_0 = 0$ , the proof for general  $v_0$  is analogous. Let us write  $R_{n,0}$  for  $R_{n,0}^{(\lambda)}(0, h_0)$ . We use an induction reasoning on  $n$ . The base case was already proved in [27]. We assume that the result holds for any  $p < n$  with  $n \geq 2$ .

Let us suppose that  $R_{n,0} \geq r$ . We can further assume that  $(0, h_0) \in \partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)})$  and  $(0, h_0) \notin \partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)} \cap C(r))$  as the other case, which is  $(0, h_0) \notin \partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)})$  and  $(0, h_0) \in \partial\Phi_n^{(\lambda)}(\mathcal{P}^{(\lambda)} \cap C(r))$ , can be proved almost identically. Let us write  $c_1 := \frac{c}{12}$  with  $c$  the constant from Lemma 4.3.7 and  $c_2 = \inf(c_1, \frac{1}{6dc_{stab}})$  with  $c_{stab}$  the minimum (that makes the exponential the smallest) of all the constants  $c$  in the exponential given by the induction hypothesis for  $p < n$ .

Let  $l = \ell^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)} \cap C(r))$ . In particular, due to Lemma 4.3.5,  $\ell^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \geq l$ . We choose a pseudo-cone  $[\Pi^\downarrow]^{(\lambda)}(v_1, h_1)$ , with  $(v_1, h_1) \in [\Pi^\uparrow]^{(\lambda)}(0, h_0)$ , that only contains points of layer at most  $(l-1) \leq (n-2)$  for the peeling of  $\mathcal{P}^{(\lambda)} \cap C(r)$  and at least one point of layer at least  $(n-1)$  for the whole peeling. As Lemma 4.4.2 implies that

$$\mathbb{P}(H_l^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)} \cap C(r)) \geq c_2 r) \leq c \exp(-e^{r/c}),$$

we can assume that  $h_1 \leq c_2 r$ . We write  $E$  for the corresponding event. We assert that

$$[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap \{(v', h') : h' \geq -c_1 r\} \subset C(2r/3). \quad (4.31)$$

Indeed let us take  $(v, h) \in [\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap \{(v', h') : h' \geq -c_1 r\}$ . The first step is to prove that  $\|v - v_1\| \leq r/3$ . The equation of the cone-like grains and Lemma 4.3.7 imply that

$$h \leq h_1 - G(v - v_1) \leq h_1 - \underline{c}\|v - v_1\| + \log(d).$$

In turn

$$\|v - v_1\| \leq \frac{1}{\underline{c}}[h_1 - h + \log(d)].$$

As  $h_1$  and  $h$  are both bounded by  $c_1 r = \frac{c}{12}r$  and  $\log(d) \leq \frac{c}{6}r$  we obtain

$$\|v - v_1\| \leq r/3. \quad (4.32)$$

Additionally, a similar reasoning yields

$$h_1 = h_0 + G(-v_1) \geq h_0 + \underline{c}\|v_1\| - \log(d)$$

and then  $\|v_1\| \leq r/3$ . Combining this, (4.32) and the triangle inequality gives (4.31).

Now we write

$$E_1 := E \cap \{[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap \{h \leq -c_1 r\} \cap \mathcal{P}^{(\lambda)} \cap \partial\Phi_{n-1}^{(\lambda)}(\mathcal{P}^{(\lambda)}) = \emptyset\}$$

and

$$E_2 = E \cap \{[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap \{h \leq -c_1 r\} \cap \mathcal{P}^{(\lambda)} \cap \partial\Phi_{n-1}^{(\lambda)}(\mathcal{P}^{(\lambda)}) \neq \emptyset\}.$$

a) If  $E_1$  occurs, we have a point  $(v, h)$  of layer  $(n-1)$  for the peeling of  $\mathcal{P}^{(\lambda)}$  with  $h \geq -c_1 r$ . It is in particular included in  $C(2r/3)$ , see (4.31). This point must have a stabilization radius greater than  $r/3$  as its layer number can not be  $(n-1)$  for the peeling of  $\mathcal{P}^{(\lambda)} \cap C(r)$ .

$$\begin{aligned} \mathbb{P}(E_1) &\leq \mathbb{P}\left(\bigcup_{m \leq n-1} \bigcup_{w \in \mathcal{P}^{(\lambda)} \cap C^{[-c_1 r, c_2 r]}(2r/3)} \{R_{m,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \geq r/3\}\right) \\ &\leq \mathbb{E}\left[\sum_{m \leq n-1} \sum_{w \in \mathcal{P}^{(\lambda)} \cap C^{[-c_1 r, c_2 r]}(2r/3)} \mathbf{1}_{R_{m,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \geq r/3}\right]. \end{aligned}$$

Now we use the Mecke formula, the induction hypothesis and the calibration of  $c_2$  to obtain

$$\mathbb{P}(E_1) \leq \sqrt{d} \sum_{m \leq n-1} \int_{C^{[-c_1 r, c_2 r]}(2r/3)} \mathbb{P}\left(R_{m,0}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)}) \geq r/3\right) e^{dh} dv dh \leq c \exp(-r/c).$$

b) If  $E_2$  occurs, it means that  $[\Pi^\downarrow]^{(\lambda)}(v_1, h_1) \cap \{h \leq -c_1 r\} \cap \mathcal{P}^{(\lambda)} \neq \emptyset$ . This happens with probability smaller than  $c \exp(-r/c)$ , see [27, p.32].

Combining a) and b) we get  $\mathbb{P}(E) \leq c \exp(-r/c)$ . □

**Proposition 4.4.4.** *For all  $n \geq 1$ , there exist  $\lambda_0 \geq 1$  and  $c > 0$  such that for any  $(v_0, h_0) \in W_\lambda$ ,  $r \geq h_0$  and  $\lambda \in [\lambda_0, \infty]$  we have*

$$\mathbb{P}\left(R_{n,k}^{(\lambda)}(v_0, h_0) \geq r\right) \leq c \exp\left(-e^{r/c}\right).$$

*Proof.* Let us consider  $(v_0, h_0)$  such that  $R_{n,k}^{(\lambda)}(v_0, h_0) \geq r$ . We also write  $c_1 = \frac{c}{12}$  as in Proposition 4.4.3 and  $c_2 = \inf(c_1, \frac{1}{6dc_{stab}})$  with  $c_{stab}$  the constant from Proposition 4.4.3.

Guided by the previous arguments used in Proposition 4.4.3, we can assume several facts without loss of generality.

- $v_0 = 0$ .
- $R_{n,0}^{(\lambda)}(0, h_0) < r$ . Indeed, the complement event occurs with probability smaller than  $c \exp(-r/c)$  thanks to Proposition 4.4.3. This implies that  $(0, h_0)$  is on the  $n$ -th layer of the peeling of  $\mathcal{P}^{(\lambda)}$  if and only if it is on the  $n$ -th layer of the peeling of  $\mathcal{P}^{(\lambda)} \cap C(s)$  for any  $s \geq r$ .
- $(0, h_0)$  is on the  $n$ -th layer for the peelings of both  $\mathcal{P}^{(\lambda)}$  and  $\mathcal{P}^{(\lambda)} \cap C(s)$  for any  $s \geq r$ . Otherwise, the score  $\xi_{n,k}^{(\lambda)}(0, h_0)$  would be equal to  $\xi_{n,0}^{(\lambda)}(0, h_0) = 0$  and would share the same stabilization radius.
- $\mathcal{P}^{(\lambda)}$  does not meet  $C(r) \cap \{(v', h') : h' \leq -c_1 r\}$  as the complement set occurs with probability smaller than  $c \exp(-r/c)$ .
- $\min(H_n((0, h_0), \mathcal{P}^{(\lambda)}), \sup_{s \geq r} H_n((0, h_0), \mathcal{P}^{(\lambda)} \cap C(s))) \leq c_2 r$ . Indeed, Lemma 4.4.2 implies that the complement event occurs with probability smaller than  $c \exp(-e^{r/c})$ .

We denote by  $E$  the event corresponding to all of the assumptions above. We consider the set

$$\mathcal{U} := \bigcup_{(v_1, h_1) \in \partial[\Pi^\uparrow]^{(\lambda)}(0, h_0), h_1 \leq c_2 r} [\Pi^\uparrow]^{(\lambda)}(v_1, h_1) \cap \{(v', h') : h' \geq -c_1 r\}.$$

On the event  $E$ , we obtain thanks to (4.31) that  $\mathcal{U} \subset C^{[-c_1 r, c_2 r]}(2r/3)$ . Moreover, because of the assumption on the variables  $H_n((0, h_0), \mathcal{P}^{(\lambda)} \cap C(s))$ ,  $\mathcal{U}$  contains every point that shares a common face of the  $n$ -layer with  $(0, h_0)$  for the peeling of  $\mathcal{P}^{(\lambda)} \cap C(s)$  for any  $s \in [r, \infty]$ . We can now proceed as in the proof of [23, Lemma 3.6]. We assert that

$$\{R_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \geq r\} \cap E \subset \{\exists w \in \mathcal{P}^{(\lambda)} \cap \mathcal{U} : R_{n,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \geq r/3\} \cap E. \quad (4.33)$$

Indeed, if every point  $w \in \mathcal{P}^{(\lambda)} \cap \mathcal{U}$  verifies  $R_{n,0}^{(\lambda)}(w, \mathcal{P}^{(\lambda)}) \leq r/3$ , then the status of these points with respect to the  $n$ -th layer is the same for both  $\mathcal{P}^{(\lambda)}$  and  $\mathcal{P}^{(\lambda)} \cap C(s)$ , for any  $s \geq r$ . Consequently, the  $k$ -faces of the  $n$ -th layer containing  $(0, h_0)$  are the same for the peeling of any  $\mathcal{P}^{(\lambda)} \cap C(s)$ ,  $s \geq r$ , which implies that  $R_{n,k}^{(\lambda)}((0, h_0), \mathcal{P}^{(\lambda)}) \leq r$ . Using consecutively (4.33) and Mecke's formula we have

$$\begin{aligned} \mathbb{P}(\{R_{n,k}^{(\lambda)}(0, h_0) \geq r\} \cap E) &\leq \mathbb{E} \left[ \sum_{(v,h) \in \mathcal{U} \cap \mathcal{P}^{(\lambda)}, h \geq -c_1 r} \mathbb{1}_{\{R_{n,0}^{(\lambda)}(v,h) \geq r/3\} \cap E} \right] \\ &\leq \sqrt{d} \int_{C^{[-c_1 r, c_2 r]}(2r/3)} \mathbb{P}(R_{n,0}^{(\lambda)}((v, h)) \geq r/3) e^{dh} dv dh \\ &\leq c \exp(-r/c). \end{aligned}$$

□

As said before, our strategy to derive the limit of the sum of the scores in the vicinity of the origin relies on the dominated convergence theorem. This requires to dominate the expectation of the score or one of its powers. This is done in Lemma 4.4.5 below. Its proof which relies on Lemmas 4.4.4 and 4.4.2 is omitted, as it is almost identical to the proof given in the case of the first layer, see [27, Lemma 5.3] and also very close to the proof of [23, Lemma 4.1].

**Lemma 4.4.5.** *For all  $p \in [1, \infty)$ , there exists a constant  $c > 0$  and  $\lambda_0 \geq 1$  such that for all  $(v_0, h_0) \in W_\lambda$  and  $\lambda \geq \lambda_0$*

$$\mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}(v_0, h_0)^p] \leq c_1(|h_0| + 1) \exp(-c_2 e^{h_0 v_0}).$$

In the next lemma, we prove the convergence of  $\mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}(v_0, h_0)]$  to  $\mathbb{E}[\hat{\xi}_{n,k}^{(\infty)}(v_0, h_0)]$ .

**Lemma 4.4.6.** *For all  $(v_0, h_0) \in \mathbb{R}^{d-1} \times \mathbb{R}$ , we have*

$$\lim_{\lambda \rightarrow \infty} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})] = \mathbb{E}[\hat{\xi}_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P})].$$

*Proof.* First, we prove the a.s. convergence of  $\hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)})$  to  $\hat{\xi}_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P})$ . Let us write

$$B(v_0, h_0) := C_{v_0}^{\leq H_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P})} (R_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P})).$$

By definition of  $R_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P})$  and  $H_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P})$ , we have

$$\hat{\xi}_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P} \cap B(v_0, h_0)) = \hat{\xi}_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P}).$$

Almost surely, we have for  $\lambda$  large enough  $B(v_0, h_0) \subset W_\lambda$  and subsequently

$$\hat{\xi}_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P}) = \hat{\xi}_{n,k}^{(\infty)}((v_0, h_0), \mathcal{P} \cap W_\lambda) = \hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}).$$

It implies in turn the almost sure convergence. To extend it to the convergence of the expectation, we use the uniform integrability of  $(\hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}))_\lambda$  that comes from Lemma 4.4.5.  $\square$

The next two lemmas imply analogous results of domination and convergence for the two-point correlation function. They are used in the proof of the convergence of the variance of the number of  $k$ -faces in a neighborhood of a vertex of  $K$ . Proofs of these results are again omitted, as they are almost identical to the proofs of [27, Lemma 5.4], [27, Lemma 5.5] and [27, Lemma 5.6].

**Lemma 4.4.7.** *For all  $n \geq 1$  and  $k \in \{0, \dots, d-1\}$ , there exists a constant  $c > 0$  such that for all  $\lambda \geq 1$  and  $(v_0, h_0), (v_1, h_1) \in W_\lambda$  satisfying*

$$\|v_1 - v_0\| \geq 2 \max \left( \frac{6}{\underline{c}} \log(d), -\frac{6}{\underline{c}} \mathbf{1}_{\{h_0 < 0\}}, -\frac{6}{\underline{c}} \mathbf{1}_{\{h_1 < 0\}} \right)$$

with  $\underline{c}$  as in Lemma 4.3.7, we have

$$|c_{n,k}^{(\lambda)}((v_0, h_0), (v_1, h_1))| \leq c(\|h_0\| + 1)^c (\|h_1\| + 1)^c \exp \left( -\frac{1}{c} (\|v_1 - v_0\| + e^{h_0 v_0} + e^{h_1 v_0}) \right).$$

Additionally, there is an integrable function  $g : \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R} \rightarrow \mathbb{R}_+$  such that for all  $\lambda \in [1, \infty]$  we have

$$|c_{n,k}^{(\lambda)}((0, h_0), (v_1, h_1))| e^{dh_0} e^{dh_1} \leq g_{n,k}(h_0, v_1, h_1).$$

**Lemma 4.4.8.** *For all  $h_0 \in \mathbb{R}$ ,  $(v_1, h_1) \in \mathbb{R}^{d-1} \times \mathbb{R}$ , we have*

$$\lim_{\lambda \rightarrow \infty} c_{n,k}^{(\lambda)}((0, h_0), (v_1, h_1)) = c_{n,k}^{(\infty)}((0, h_0), (v_1, h_1)).$$

The combination of the previous results allows us to obtain Proposition 4.4.9 which provides the asymptotics for the expectation and variance of the contribution of Poisson points in a neighborhood of a vertex of  $K$ . We recall the definition of  $Z_i(\delta_0)$  given at (4.27). The event  $A_\lambda$  therein is a slight modification of the event  $\tilde{A}_\lambda$  which is introduced later on, at (4.35). Proposition 4.4.9 is used in Section 4.5 where the event  $A_\lambda$  is required for technical reasons. Its proof is postponed to Section 4.6, dedicated to the proof of the main results. It is an adaptation of the core of the proof of [27, Theorem 2.1].

**Proposition 4.4.9.** *For each  $1 \leq i \leq f_0(K)$  we have*

$$\lim_{\lambda \rightarrow \infty} \frac{\mathbb{E}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}]}{\log^{d-1}(\lambda)} = d^{-d+3/2} \text{Vol}_d(S(d)) \int_{\mathbb{R}} \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h_0), \mathcal{P})] e^{dh_0} dh_0$$

and

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}]}{\log^{d-1}(\lambda)} = I_1(\infty) + I_2(\infty)$$

where

$$I_1(\infty) = d^{-d+3/2} \text{Vol}_d(S(d)) \int_{\mathbb{R}} \mathbb{E}[\xi_{n,k}^{(\infty)}((0, h_0), \mathcal{P})^2] e^{dh_0} dh_0$$

and

$$I_2(\infty) = d^{-d+2} \text{Vol}_d(S(d)) \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \int_{\mathbb{R}} c_{n,k}^{(\infty)}((0, h_0), (v', h_1)) e^{d(h_0+h_1)} dh_0 d(v', h_1).$$

## 4.5 Decomposition of the expectation and variance

The aim of this section is to prove Proposition 4.5.15 which combines the two following facts.

- The variance is additive over the vertices of  $K$ , i.e. for a suitable  $\delta$  that depends on  $\lambda$ , the variables  $Z_i(\delta)$ ,  $1 \leq i \leq f_0(K)$ , decorrelate asymptotically where we recall that  $Z_i(\delta)$  is the sum of the scores of points in sets  $p_d(\mathcal{V}_i, \delta)$ .
- Asymptotically, the contribution of the Poisson points far from the vertices to the expectation and variance of  $Z$  is negligible compared to the contribution of the points near the vertices. In other words, the sum of scores in the complement of  $\cup_{i=1}^{f_0(K)} p_d(\mathcal{V}_i, \delta)$  in the sandwich  $\mathcal{A}(s, T^*, K)$  is negligible in the asymptotic calculation of both the expectation and the variance of  $Z$ .

In order to do so, we use an explicit construction of Macbeath regions appearing near the vertices in the economic cap covering theorem, see Theorem 4.2.2. This construction allows us to partition the annulus  $\mathcal{A}(s, T^*, K)$  in a way that makes it easier to control the spatial dependence of the scores, conditional on the sandwiching event. This dependence is described through a dependency graph. The strategy described above is heavily inspired by [27, Section 3].

The results of this section justify a posteriori the work of Sections 4.3 and 4.4. Indeed it shows that it is enough to study the asymptotics of the sums of the scores near a vertex, which is precisely the goal of Sections 4.3 and 4.4.

In the whole section, unless stated otherwise, we consider  $\delta \in (0, 1/2)$ .

### 4.5.1 Dyadic M-regions and supersets

This part is dedicated to the construction of the so-called supersets. It is done via an explicit description of the Macbeath regions of the economic cap covering near a corner. This construction comes from [27, Section 3].

Let us write, for all  $z \in K$ ,  $M_K(z) := M_K(z, 1/2)$ . Given  $\delta \in (0, 1/2)$  and integers  $k_i \in \mathbb{Z}$  with  $3^{k_i} \in (0, 1/(3\delta))$  for  $1 \leq i \leq d$  (so that  $3^{k_i+1}\delta/2 \leq 1/2$ ), the dyadic rectangular solids  $\prod_{i=1}^d [\frac{3^{k_i}\delta}{2}, \frac{3^{k_i+1}\delta}{2}]$  coincide with the M-regions  $M_K((3^{k_1}\delta, \dots, 3^{k_d}\delta))$ . When  $\log_3(\frac{dT}{d^d\delta^d}) \in \mathbb{Z}$ , an M-region  $M_K(z)$  has center  $z$  belonging to  $K(v = t)$  as soon as  $\sum_{i=1}^d k_i = \log_3(\frac{dT}{d^d\delta^d})$ , thanks to Lemma 4.2.1 and a direct computation.

For any  $\delta \in (0, 1/2)$  with  $\log_3(\frac{dT}{d^d\delta^d}) \in \mathbb{Z}$ , we denote by  $\mathcal{M}(0, \delta)$  the set of all M-regions that can be written as

$$M_K((3^{k_1}\delta, \dots, 3^{k_d}\delta)) = \prod_{i=1}^d \left[ \frac{3^{k_i}\delta}{2}, \frac{3^{k_i+1}\delta}{2} \right]$$

with  $k_i \in \mathbb{Z}$ ,  $3^{k_i} \in (0, 1/(3\delta))$  for  $1 \leq i \leq d$  and  $\sum_{i=1}^d k_i = \log_3(\frac{dT}{d^d\delta^d})$ . The set  $\mathcal{M}(0, \delta)$  is maximal in the sense that it can not be enlarged to include another M-region with center on  $K(v = T) \cap [0, 1/2]^d$ , see [27, Lemme 3.1]. Furthermore, the economic cap covering that we recalled in Theorem 4.2.2 can be constructed such that the set of all the Macbeath regions of the saturated system covering  $K(v = T)$  with center in  $[0, 1/2]^d$  is exactly  $\mathcal{M}(0, \delta)$ .

We can now introduce the supersets induced by the net of the M-regions in  $\mathcal{M}_K(0, \delta)$ . The goal is to partition  $\mathcal{A}(s, T^*, K)$  with these supersets. This construction is the same as in [27, Section 3.3]. We start with the supersets in  $[0, 1/2]^d$ .

We describe the construction of the cone sets and cylinder sets associated to an M-region in  $\mathcal{M}(0, \delta)$  as they intervene in the construction of the supersets. For any M-region  $M_j$  that meets  $[0, (T^*)^{1/d}]^d$ , we define its associated cone set  $\text{Co}_j$  as the intersection of  $K(v \leq T^*)$  with the smallest cone with apex at  $((T^*)^{1/d}, \dots, (T^*)^{1/d})$  that contains  $M_j$ . For any M-region  $M_j$  with center at  $(3^{k_1}\delta, \dots, 3^{k_d}\delta)$  that meets  $([0, (T^*)^{1/d}]^d)^c$ , we define its associated cylinder set as follows. Here and in the sequel, we denote by  $H_l$ ,  $1 \leq l \leq d$ , the coordinate hyperplanes. We first define for every  $1 \leq l \leq d$  the cylinder

$$C_l(k_1, \dots, k_d) := \prod_{i=1}^{l-1} \left[ \frac{3^{k_i}\delta}{2}, \frac{3^{k_i+1}\delta}{2} \right] \times \mathbb{R} \times \prod_{i=l+1}^d \left[ \frac{3^{k_i}\delta}{2}, \frac{3^{k_i+1}\delta}{2} \right] \cap ([0, (T^*)^{1/d}]^d)^c.$$

It is the smallest cylinder containing  $M_j$  and oriented in direction  $n_{H_l}$  where  $n_{H_l}$  is a unit normal vector of the hyperplane  $H_l$ . We define the regions

$$\tilde{S}_j := \tilde{S}_j(k_1, \dots, k_d) := \bigcup_{l: k_l = \min(k_1, \dots, k_d)} C_l(k_1, \dots, k_d) \cap K.$$

When  $k_l$  is the unique minimum,  $\tilde{S}_j$  is a single cylinder  $C_l$  and it simply extends  $M_j \cap ([0, (T^*)^{1/d}]^d)^c$  in direction  $n_{H_l}$ . We wish the cylinder sets to cover  $K(v \leq T^*) \cap [0, 1/2]^d \setminus [0, (T^*)^{1/d}]^d$  but the union of all the regions  $\tilde{S}_j$  does not cover all of it. The uncovered parts are rectangular regions produced by exactly one M-region having a cubical face. For this reason, we define the cylinder set  $\text{Cyl}_j$  of  $M_j$  as the union of  $\tilde{S}_j$  and the rectangular regions produced by the ties in the minimum of  $k_1, \dots, k_d$ .

Let  $M_j$  be an M-region in  $\mathcal{M}(0, \delta)$ . The superset  $S'_j$  associated with  $M_j$  is defined as follows.

- If  $M_j$  is entirely included in  $[0, (T^*)^{1/d}]^d$ , the cone set associated with  $M_j$  is also included in  $[0, (T^*)^{1/d}]^d$  and the superset associated with  $M_j$  is defined as  $S'_j = \text{Co}_j \cap \mathcal{A}(s, T^*, K)$ . We say that  $S'_j$  is a cone set.
- If  $M_j$  meets  $[0, (T^*)^{1/d}]^d$ , but is not entirely included in  $[0, (T^*)^{1/d}]^d$ , the superset associated with  $M_j$  is  $S'_j = (\text{Co}_j \cup \text{Cyl}_j) \cap \mathcal{A}(s, T^*, K)$ . In this case we say that  $S'_j$  is a cone-cylinder set.
- If  $M_j$  is included in  $([0, (T^*)^{1/d}]^d)^c$  we simply put  $S'_j = \text{Cyl}_j \cap \mathcal{A}(s, T^*, K)$  and call  $S'_j$  a cylinder set.

The supersets associated with an M-region in  $\mathcal{M}_K(0, \delta)$  are now well defined. For any vertex  $\mathcal{V}_i \in \mathcal{V}_K \setminus \{0\}$ , we may likewise define a collection  $\mathcal{M}_K(\mathcal{V}_i, \delta)$  of dyadic M-regions. We can then generate an associated collection of supersets in a similar way. We embed the union of all the  $\mathcal{M}_K(\mathcal{V}_i, \delta)$  into a (not necessarily unique) larger collection of M-regions  $\mathcal{M}_K(m(T, \delta))$  with cardinality  $m(T, \delta)$ , that is maximal for the entirety of  $K(v = T)$ . This is always possible because among all the collections of M-regions containing the union of the  $\mathcal{M}_K(\mathcal{V}_i, \delta)$ , there is always at least one that is maximal. To each additional M-region that is not in any of the  $\mathcal{M}_K(\mathcal{V}_i, \delta)$ , we associate a superset as in [12], i.e. we take them such that  $M_j \subset S'_j$ , the sets  $S'_j$  are pairwise internally disjoint and such that their union cover the part of the sandwich that is not already covered by the supersets associated with the M-regions in one of the  $\mathcal{M}_K(\mathcal{V}_i, \delta)$ . We also ask for an additional technical condition on these  $S'_j$  which is that  $S'_j \subset K_j^{(\gamma)}$  where  $\gamma = 3d^36^d$ . This is possible because  $K(v \leq T^*)$  is covered by  $\cup_i K_i^{(\gamma)}$ , see [12, Claim 2.6] where  $T$  and  $d6^d$  play the role of  $s$  and  $\lambda$  therein respectively. We claim in Lemma 4.5.1 below that all of these properties are in fact satisfied by every superset.

**Lemma 4.5.1.** *The supersets  $(S'_j)_j$  are pairwise interior disjoint,  $\cup_j S'_j = \mathcal{A}(s, T^*, K)$  and for every  $1 \leq j \leq m(T, \delta)$ , we have  $M_j \subset S'_j$  and  $S'_j \subset K_j^{(\gamma)}$ .*

Note that the M-regions  $M_j$  also verify the properties of Theorem 4.2.2. In particular, combining them with the previous lemma, we deduce the following volume estimates for the supersets. For all  $1 \leq j \leq m(T, \delta)$ , we have

$$\frac{(6d)^{-d} \alpha \log \log(\lambda)}{\lambda} \leq \text{Vol}_d(M_j) \leq \text{Vol}_d(S'_j) \leq \text{Vol}_d(K_j^{(\gamma)}) \leq \frac{(6\gamma)^d \alpha \log \log(\lambda)}{\lambda}. \quad (4.34)$$

In the sequel, we denote by  $\mathcal{S}(\delta) := \{S'_j\}_{j=1}^{m(T, \delta)}$  the set of all the supersets generated by M-regions in  $\mathcal{M}_K(m(T, \delta))$ . Now that we have defined the supersets, we are able to define the event  $A_\lambda$ , which is a refinement of  $\tilde{A}_\lambda$  where we ask additionally for each superset to contain less than  $c \log \log(\lambda)$  Poisson points.

$$A_\lambda = A_\lambda(\delta) := \{\cup_{l=1}^n \partial \Phi_l(\mathcal{P}_\lambda) \subset \mathcal{A}(s, T^*, K), \forall j \text{ card}(S'_j \cap \mathcal{P}_\lambda) \leq 3(6\gamma)^d \alpha \log \log(\lambda)\}. \quad (4.35)$$

The event  $A_\lambda$  depends on  $\delta$  through the explicit construction of the M-regions in each  $\mathcal{M}(\mathcal{V}_i, \delta)$ . However, the next lemma shows that the probability of  $A_\lambda$  can be estimated independently from  $\delta$ . As a result, we omit the dependency on  $\delta$  in the sequel. A particular choice of  $\delta$  will be made in the next subsection.

The proof of the following estimation of  $\mathbb{P}(A_\lambda)$  follows from [12, Claim 5.2].

**Lemma 4.5.2.** *There exists a constant  $c > 0$  such that for all  $\lambda \in [1, \infty)$*

$$\mathbb{P}(A_\lambda) \geq 1 - c \log^{-4d^2}(\lambda).$$

*Proof.* From Theorem 4.2.3, we deduce that it is enough to prove that

$$\mathbb{P}(\exists j : \text{card}(S'_j \cap \mathcal{P}_\lambda) \geq 3(6\gamma)^d \alpha \log \log(\lambda)) \leq \log^{-4d^2}(\lambda).$$

We recall that for any Poisson random variable  $\text{Pois}(\mu)$  with mean  $\mu$ ,

$$\mathbb{P}(\text{Pois}(\mu) \geq 3\mu) \leq \frac{3}{3-e} e^{-\mu}.$$

Setting  $\mu = \lambda \text{Vol}_d(S'_j)$  and using the volume estimate at (4.34), we get that

$$\mathbb{P}(\text{card}(\mathcal{P}_\lambda \cap S'_j) \geq 3(6\gamma)^d \alpha \log \log(\lambda)) \leq \frac{3}{3-e} \exp(-(6d)^{-d} \alpha \log \log(\lambda)) \leq c \log^{-(4d^2+d-1)}(\lambda).$$

As the number of M-regions and thus of supersets in  $m(T, \delta)$  is bounded by  $c \log^{d-1}(\lambda)$ , thanks to the estimate on the number of Macbeath regions in the economic cap covering provided by [12, Theorem 2.7], we obtain the result with a union bound.  $\square$

Using the estimate [12, Theorem 2.7] and the definition of  $A_\lambda$  yields the following lemma, that provides an estimate of the number of points of  $\mathcal{P}_\lambda$  in the sandwich on the event  $A_\lambda$ .

**Lemma 4.5.3.** *There exists a constant  $c > 0$  such that on the event  $A_\lambda$ ,*

$$\text{card}(\mathcal{P}_\lambda \cap \mathcal{A}(s, T^*, K)) \leq c \log^{d-1}(\lambda) \log \log(\lambda).$$

## 4.5.2 Dependency graph

In this subsection, we use the previous construction of the supersets to build a dependency graph as in [27, Section 3.4].

We define a graph  $\mathcal{G} := (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ , where  $\mathcal{V}_\mathcal{G}$  is the set  $\{1, \dots, m(T, \delta)\}$ . With a slight abuse of notation, we may sometimes identify the vertex  $i$  with the corresponding superset  $S'_i$  and say for instance that there is an edge between  $S'_i$  and  $S'_j$  instead of  $i$  and  $j$ . We define the edges of the graph as follows. For any  $1 \leq j \leq m(T, \delta)$ , we define  $L_j$  to be the union of all the supersets  $S'_k \in \mathcal{S}(\delta)$  such that there exist  $a \in S'_j$  and  $b \in S'_k$  with the segment  $[a, b]$  disjoint from  $K(v \geq T^*)$ . In particular  $S'_j \subset L_j$  and  $S'_j \subset L_k$  if and only if  $S'_k \subset L_j$ . We join the vertices  $i$  and  $j$  in  $\{1, \dots, m(T, \delta)\}$  by an edge if and only if  $L_i$  and  $L_j$  contain at least one superset  $S'_k$  in common. The remaining part of this subsection consists in stating some spatial conditions for having no edge between two vertices in the graph and deducing from them that conditional on  $A_\lambda$ , it is indeed a dependency graph.

We first prove a technical geometric lemma.

**Lemma 4.5.4.** *For  $\lambda$  large enough and any choice of  $\delta = \delta(\lambda)$  such that  $T = o(\delta)$  as  $\lambda$  goes to infinity,*

$$\bigcup_{j: S'_j \subset [0, \frac{3}{2}\delta]^d} S'_j = [0, \frac{3}{2}\delta]^d \cap \mathcal{A}(s, T^*, K).$$

*Proof.* For any  $i$ , we consider the M-regions

$$M_{i,p} := \prod_{l=1}^{i-1} \left[ \frac{3^{k_{l,p}}}{2} \delta, \frac{3^{k_{l,p}+1}}{2} \delta \right] \times \left[ \frac{1}{2} \delta, \frac{3}{2} \delta \right] \times \prod_{l=i+1}^d \left[ \frac{3^{k_{l,p}}}{2} \delta, \frac{3^{k_{l,p}+1}}{2} \delta \right]$$

such that  $M_{i,p} \in \mathcal{M}(0, \delta)$  and  $k_{i,p} \leq 0$  for all  $i$  and  $p$ . As  $T = o(\delta)$  and  $\sum_{i=1}^d k_i = \log_3(d!T/(d\delta)^d)$ , for  $\lambda$  large enough, these regions are included in  $[0, \frac{3}{2}\delta]^d$ . We claim that the cylinder sets  $S'_j$  corresponding to these regions cover the part of  $\partial[0, \frac{3}{2}\delta]^d$  that is contained in  $\mathcal{A}(s, T^*, K)$ . Since the supersets  $S'_j$  are pairwise interior disjoint and cover all of  $\mathcal{A}(s, T^*, K)$ , we deduce that  $[0, \frac{3}{2}\delta] \setminus \cup_{i,p} M_{i,p}$  is covered by supersets  $S'_j$  that can only be contained in  $[0, \frac{3}{2}\delta]^d$ . The result follows.  $\square$

An important consequence of Lemma 4.5.4 is the following decomposition of  $Z_1(\frac{3}{2}\delta)$ , corresponding to the vertex 0 of  $K$ :

$$Z_1(\frac{3}{2}\delta) = \sum_{j: S'_j \subset [0, \frac{3}{2}\delta]^d} \sum_{x \in S'_j} \xi_{n,k}(x, \mathcal{P}_\lambda). \quad (4.36)$$

It justifies our focus on the supersets that are included in  $[0, \frac{3}{2}\delta]^d$  in the next lemmas.

For  $0 < a < b < \infty$  and  $1 \leq i \leq d$ , we denote by  $H_i[a, b]$  the ‘parallel slab’ between hyperplanes  $H_i + an_{H_i}$  and  $H_i + bn_{H_i}$  where  $n_{H_i} \in [0, \infty)^d$  is a normal unit vector of  $H_i$ . We define for any bounded  $A \subset \mathbb{R}^d$ , the diameter  $\text{diam}_i(A)$  in direction  $n_{H_i}$  as the width of the maximal parallel slab containing  $A$ . Let  $\cup_{i=1}^{J'} S'_j$  be connected and assume that it meets  $H_i[0, \delta]$ . We recall the following bound from [27],

$$\text{diam}_i(\cup_{j=1}^{J'} S'_j) \leq c' \delta 3^{J'}. \quad (4.37)$$

Since (4.37) relies on the construction of the sets  $S'_j$ , which themselves only depends on  $T$  and  $T^*$  and since  $T$  and  $T^*$  in this work only differ from the  $T$  and  $T^*$  in [27] by multiplicative constants, we claim that this bound still holds in our context.

From the bound (4.37) and the inequality  $\text{card}(\{k; S'_k \subset L_j\}) \leq c \log \log(\lambda)^{3(d-1)}$  provided by [12, Lemma 6.1], we deduce as in [27] that as long as  $L_j$  has a non-empty intersection with  $H_i[0, \delta_1]$ , there exist constants  $c^*, c_{\text{diam}} \in (0, \infty)$  such that its diameter in the direction  $n_{H_i}$  satisfies

$$\text{diam}_i(L_j) \leq c_{\text{diam}} \delta 3^{c^* L(\lambda)} \quad (4.38)$$

where  $L(\lambda) := T(K)^3 (\log \log(\lambda))^{3(d-1)}$  and  $T(K)$  is the number of towers of  $K$ . This is the essential ingredient of the proof of the next lemma, which is an adaptation of [27, Lemma 3.3]. It is a first step towards proving that the sums of scores in supersets sufficiently far apart are independent.

Beforehand, we specify our choice of  $\delta \in (0, 1/2)$ . Recalling that  $\delta_0 = \exp(-\log(\lambda)^{1/d})$ , we define  $\delta_1 := r(\lambda, d)\delta_0$  where  $r(\lambda, d) \in [1, 3^{1/d}]$  is chosen so that  $\log_3(d!T/(d\delta)^d) \in \mathbb{Z}$ . In particular it verifies the hypotheses of Lemma 4.5.4.

**Lemma 4.5.5.** *For  $\lambda$  large enough, if  $S'_j \subset [0, \frac{3}{2}\delta_1]^d$  and if there exist an  $i$  and  $1 \leq l \leq d$  such that  $S'_i \cap H_l[4c_{\text{diam}}\delta_1 3^{c^* L(\lambda)}, \text{diam}(K)] \neq \emptyset$ , then there is no edge in  $\mathcal{E}_G$  between  $j$  and  $i$ .*

*Proof.* Let us assume that such an  $S'_i$  exists. We distinguish two different cases.

Case i). Let us assume that  $S'_i$  is generated by an M-region in  $\mathcal{M}_K(0, \delta_1)$ . We prove that in this case, it is enough to have  $S'_i \cap H_l[c_{\text{diam}}\delta_1 3^{c^* L(\lambda)+1}, \text{diam}(K)] \neq \emptyset$  to conclude that there is no edge between  $S'_i$  and  $S'_j$ . Indeed, from the diameter bound (4.38), we deduce that  $L_i \subset H_l[2c_{\text{diam}}\delta_1 3^{c^* L(\lambda)}, \text{diam}(K)]$  while  $L_j \subset H_i[0, c_{\text{diam}}\delta_1 3^{c^* L(\lambda)}]$ . In particular, we deduce that  $L_i \cap L_j = \emptyset$  and thus, there is no edge between  $j$  and  $i$  in this case.

Case ii). We now assume that  $S'_i$  is generated by an M-region in  $\mathcal{M}_K(m(T, \delta_1)) \setminus \mathcal{M}_K(0, \delta_1)$ . We

proceed by contradiction and assume that there is an edge between  $i$  and  $j$ . As  $c_{\text{diam}}\delta_1 3^{c^*L(\lambda)} \rightarrow 0$ , for  $\lambda$  large enough, a path from  $S'_j$  to  $S'_i$  has to cross a superset  $S'_m$  that verifies  $S'_m \cap H_l[c_{\text{diam}}\delta_1 3^{c^*L(\lambda)+1}, \text{diam}(K)] \neq \emptyset$  before leaving  $[0, 1/2]^d$ . Therefore, we can find a superset  $S'_m$  generated by an M-region in  $[0, 1/2]^d$  that verifies  $S'_m \cap H_l[c_{\text{diam}}\delta_1 3^{c^*L(\lambda)+1}, \text{diam}(K)] \neq \emptyset$  such that there is an edge between  $S'_j$  and  $S'_m$ . This contradicts the first case.  $\square$

The next lemma expands on the result of Lemma 4.5.5. Its goal is to prove that two regions at a long distance from each other have no edge between them. This is a slight rephrasing of [27, Lemma 3.5].

**Lemma 4.5.6.** *There exists a constant  $c' \in (0, \infty)$  such that if  $S'_0 \in \mathcal{S}(\delta_1)$  is a subset of  $[0, \frac{3}{2}\delta_1]$  and  $S' \in \mathcal{S}(\delta_1)$  is at distance at least  $c'\delta_1 3^{c^*L(\lambda)}$  from  $[0, \frac{3}{2}\delta_1]^d$ , then conditional on  $A_\lambda$  and for  $\lambda$  large enough, there is no edge between  $S'_0$  and  $S'$ .*

*Proof.* We can choose  $c' > 0$  such that anytime the distance between  $S'_0$  and  $S'$  is larger than  $c'\delta_1 3^{c^*L(\lambda)}$ , the distance between them in some direction  $n_{H_l}$  is larger than  $4\delta_1 3^{c^*L(\lambda)}$ . In particular,  $S'_0$  and  $S'$  verify the hypotheses of Lemma 4.5.5 for  $l$ . Thus there is no edge between them.  $\square$

An important corollary of Lemma 4.5.6 is the fact that the sums of scores around different vertices of  $K$  are independent for  $\lambda$  large enough. Beforehand, we make a connection between the fact of having no connection between two supersets and the fact of having the independence of the two sums of the scores inside each of them. This result stated in Lemma 4.5.7 below implies that conditional on  $A_\lambda$ , the constructed graph is a dependency graph.

**Lemma 4.5.7.** *Let  $m \geq 2$  be an integer  $\mathcal{W}_1, \dots, \mathcal{W}_m$  be disjoint subsets of  $\mathcal{V}_G$  having no edge between them. Then conditional on  $A_\lambda$ , the random variables*

$$\left( \sum_{x \in \mathcal{P}_\lambda \cap (\cup_{j \in \mathcal{W}_l} S'_j)} \xi_{n,k}(x, \mathcal{P}_\lambda) \right)_{l=1, \dots, m}$$

are independent.

*Proof.* Let  $j \in \{1, \dots, m(T, \delta_1)\}$ . At least one facet of each layer with number  $\leq n$  must intersect  $S'_j$ . The vertices of these facets can only be points in  $L_j$  because they are at one end of a segment crossing  $S'_j$ . We deduce from this observation that for any  $p \leq n$ , the construction of  $\text{conv}_p(\mathcal{P}_\lambda)$  in  $S'_j$  only depends on the points in  $L_j$ .

Let  $L^{(l)} = \cup_{i \in \mathcal{W}_l} L_i$  for  $l = 1, \dots, m$ . The sets  $L^{(l)}$  are unions of sets  $S'_j$  and have disjoint interior because there is no edge between the  $\mathcal{W}_l$ . Since  $\sum_{x \in \mathcal{P}_\lambda \cap (\cup_{j \in \mathcal{W}_l} S'_j)} \xi_{n,k}(x, \mathcal{P}_\lambda)$  only depends on  $L^{(l)} \cap \mathcal{P}_\lambda$  and the point sets  $(L^{(l)} \cap \mathcal{P}_\lambda)_{l=1, \dots, m}$  are independent conditional of  $A_\lambda$ , the result follows.  $\square$

### 4.5.3 Decomposition

This section contains the proof of Proposition 4.5.15. Using the dependency graph described in Section 4.5.2, we obtain the asymptotic independence of the variables  $Z_i(\frac{3}{2}\delta_1)$  conditional on the high probability event  $A_\lambda$ , see Lemma 4.5.8. In Lemma 4.5.11, we prove that conditional on

$A_\lambda$ , the contribution of  $Z_0(\frac{3}{2}\delta_1)$  is negligible. The conditioning is then removed thanks to Lemmas 4.5.13 and 4.5.14.

Our first result is the required asymptotic independence of the variables  $Z_i(\frac{3}{2}\delta_1)$ , up to conditioning on the high probability event  $A_\lambda$ . This result is a direct consequence of Lemmas 4.5.6 and 4.5.7.

**Lemma 4.5.8.** *Conditional on  $A_\lambda$ , for  $\lambda$  large enough the variables  $Z_i(\frac{3}{2}\delta_1)$ , for  $1 \leq i \leq f_0(K)$ , are independent.*

*Proof.* We first recall that thanks to Lemma 4.5.4, the variable  $Z_i(\frac{3}{2}\delta_1)$ ,  $1 \leq i \leq f_0(K)$ , is the sum of the scores of all points in the supersets included in  $p_d(\mathcal{V}_i, \frac{3}{2}\delta_1)$ . Consequently, thanks to the dependency graph property proved in Lemma 4.5.7, we obtain the required conditional independence of the variables  $Z_i(\frac{3}{2}\delta_1)$  as soon as we can show that there is no edge in the graph between any superset included in  $p_d(\mathcal{V}_i, \frac{3}{2}\delta_1)$  and any superset included in  $p_d(\mathcal{V}_j, \frac{3}{2}\delta_1)$ ,  $j \neq i$ . Let us then consider two such supersets  $S^{(i)}$  and  $S^{(j)}$ . We have

$$d(S^{(i)}, S^{(j)}) \geq d(\mathcal{V}_i, \mathcal{V}_j) - c\delta_1.$$

which implies in particular that for  $\lambda$  large enough, the Euclidean distance between  $S^{(i)}$  and  $S^{(j)}$  is larger than  $c'\delta_1 3^{c^*L(\lambda)}$ . Thus, applying Lemma 4.5.6 after application of the respective affine transformations  $a_i$  and  $a_j$ , we obtain that there is no edge between  $S^{(i)}$  and  $S^{(j)}$ . This completes the proof of Lemma 4.5.8.  $\square$

Lemma 4.5.9 below is a purely geometrical one and basically states that most of the volume of the sandwich  $\mathcal{A}(s, T^*, K)$  is located near the vertices. This will be an important ingredient for showing the negligibility of the flat parts of the sandwich in the calculation of the expectation and variance of  $Z$  as it implies that most of the Poisson points and subsequently facets of the layers of the peeling should lie near the vertices of  $K$ .

The statement of Lemma 4.5.9 is in fact a slight variation around [27, display (7.13)] which itself relies on [7, display (4.1)]. We omit the proof as the only change compared to these two references comes from the fact that our constants  $T$  and  $T^*$  differ by a multiplicative constant from their respective definitions in [27] and [12]. This easily leaves the growth rate of the volume unchanged.

**Lemma 4.5.9.** *For  $\delta \in (0, 1/2)$ , let*

$$\mathcal{A}(s, T^*, K, \delta) := \mathcal{A}(s, T^*, K) \setminus \bigcup_{i=1}^{f_0(K)} p_d(\mathcal{V}_i, \delta).$$

When  $\lambda \rightarrow \infty$ , we get

$$\text{Vol}_d(\mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)) = O(\log \log(\lambda)(\log(\lambda))^{d-2+1/d}/\lambda).$$

In the sequel we write for each  $\delta \in (0, 1/2)$

$$Z_0(\delta) := \sum_{x \in \mathcal{P}_\lambda \cap \mathcal{A}(s, T^*, K, \delta)} \xi_{n,k}(x, \mathcal{P}_\lambda).$$

The next lemma is again deterministic and provides information on the maximal degree of the underlying dependency graph. It will be another ingredient needed to show Lemma 4.5.11. Its proof is omitted as well as it is an exact rewriting of [12, Theorem 6.2].

**Lemma 4.5.10.** *The maximal degree  $D(\lambda)$  of the dependency graph  $(\mathcal{V}_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}})$  satisfies*

$$D(\lambda) = O(\log \log(\lambda)^{6(d-1)}).$$

Next, we derive the equivalent of [27, Lemma 3.7]. Note that the statement is slightly different, i.e.  $o(\text{Var}(Z))$  is replaced by  $o(\log^{d-1}(\lambda))$  since in [27] the growth rate of  $\text{Var}[Z]$  was already known and the challenge was to obtain a precise limit. Here, in the context of the subsequent layers of the convex hull peeling, we do not even know an order of magnitude for the variance of the number of  $k$ -faces. The proof does not differ much from [27, Lemma 3.7] but still, we write it in detail in order to justify that the issue mentioned above is in fact harmless. It also brings to light the importance of having the same  $T$  and  $T^*$  as in [27] and [12] up to a constant factor.

**Lemma 4.5.11.** *We have  $\mathbb{E}[Z_0(\frac{3}{2}\delta_1)|A_\lambda] = o(\log^{d-1}(\lambda))$  and  $\text{Var}[Z_0(\frac{3}{2}\delta_1)|A_\lambda] = o(\log^{d-1}(\lambda))$ .*

*Proof.* We prove the result on the conditional variance as it is the most challenging estimate and we mention at the end how to adapt the method for the conditional expectation. For sake of simplicity, we write  $Z_0$  for  $Z_0(\frac{3}{2}\delta_1)$  in this proof.

(1) First, we prove that  $\text{Var}[Z_0 \mathbb{1}_{A_\lambda}] = o(\log^{d-1}(\lambda))$ . Let us write  $\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda) := \xi_{n,k}(x, \mathcal{P}_\lambda) \mathbb{1}_{A_\lambda}$  and  $\mathcal{P}_\lambda^x := \mathcal{P}_\lambda \cup \{x\}$  for any  $x \in K$ . Using the Mecke formula we obtain the following decomposition

$$\text{Var}[Z_0 \mathbb{1}_{A_\lambda}] = \text{Var}\left[\sum_{x \in \mathcal{P}_\lambda(s, T^*, K, \frac{3}{2}\delta_1)} \tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda)\right] = V_1 + V_2$$

where

$$V_1 := \lambda \int_{\mathcal{A}_\lambda(s, T^*, K, \frac{3}{2}\delta_1)} \tilde{\xi}(x, \mathcal{P}_\lambda)^2 dx$$

and

$$V_2 := \lambda^2 \int_{\mathcal{A}(s, T, K, \frac{3}{2}\delta_1)^2} (\mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x)] - \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda)] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda)]) dx dy.$$

(a) We start by finding a bound for  $V_1$ . Each  $x \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)$  is in some  $S'_i \in (S'_j)_{j=1}^{m(T, \frac{3}{2}\delta_1)}$ . Let  $\mathcal{S}_x$  be the union of all the  $S'_j$  connected to  $S'_i$  in the dependency graph. As the cardinal of  $\mathcal{S}_x$  is bounded by the maximal degree in the graph, Lemma 4.5.10 implies

$$\text{card}(\{j : S'_j \subset \mathcal{S}_x\}) = O(\log \log(\lambda)^{6(d-1)}).$$

Since only the points in  $\mathcal{S}_x$  can be in a  $k$ -face containing  $x$ , see the definition of the edges in the graph, and since on  $A_\lambda$ , each  $S'_j$  contains at most  $c \log \log(\lambda)$  points of the process, only  $O(\log \log(\lambda)^{6(d-1)+1})$  points of  $\mathcal{P}_\lambda$  can contribute to a  $k$ -face containing  $x$ . Using McMullen's bound, see [54], the number of  $k$ -faces on a set of  $l$  points contains a number of  $k$ -faces of at most  $cl^{d/2}$ . Thus, we obtain the following bound

$$\sup_{x, y \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)} |\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y)| = O(\log \log(\lambda)^{(6(d-1)+1)/2}). \quad (4.39)$$

Combining (4.39) and Lemma 4.5.9, we obtain

$$\begin{aligned} V_1 &= \lambda \int_{\mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)} \mathbb{E}[\tilde{\xi}(x, \mathcal{P}_\lambda)^2] dx \\ &= O(\lambda \text{Vol}_d(\mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)) (\log \log(\lambda))^{6(d-1)+1}) \\ &= o(\log^{d-1}(\lambda)). \end{aligned}$$

(b) We now bound  $V_2$ . It goes in two parts, i.e. we first treat the sum over  $x \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)$ ,  $y \notin \mathcal{S}_x$  and then the sum over  $x \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)$ ,  $y \in \mathcal{S}_x$  separately.

(b1) We start with the sum over  $x \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)$ ,  $y \notin \mathcal{S}_x$ . In this case, as the scores of  $x$  and  $y$  only depend on points in  $\mathcal{S}_x$  and  $\mathcal{S}_y$  respectively, see the beginning of the proof of Lemma 4.5.7 for an explanation, and since  $y \notin \mathcal{S}_x$ , conditional on  $A_\lambda$ , the scores  $\xi_{n,k}(x, \mathcal{P}_\lambda^x)$  and  $\xi_{n,k}(y, \mathcal{P}_\lambda^y)$  are independent. Therefore

$$\mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda^x) \xi_{n,k}(y, \mathcal{P}_\lambda^y) | A_\lambda] - \mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda^x) | A_\lambda] \mathbb{E}[\xi_{n,k}(y, \mathcal{P}_\lambda^y) | A_\lambda] = 0.$$

Consequently, we get

$$\begin{aligned} &\mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x)] - \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda)] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda)] \\ &= \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x) | A_\lambda] \mathbb{P}(A_\lambda) - \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda) | A_\lambda] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda) | A_\lambda] \mathbb{P}(A_\lambda)^2 \\ &= \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x) | A_\lambda] \mathbb{P}(A_\lambda) - \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda) | A_\lambda] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda) | A_\lambda] \mathbb{P}(A_\lambda) (1 - \mathbb{P}(A_\lambda^c)) \\ &= \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda) | A_\lambda] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda) | A_\lambda] \mathbb{P}(A_\lambda) \mathbb{P}(A_\lambda^c). \end{aligned}$$

Combining this last equality with (4.39), slightly modified for  $\mathcal{P}_\lambda$  instead of  $\mathcal{P}_\lambda^y$ , Lemma 4.5.2 and Lemma 4.5.9, we obtain

$$\lambda^2 \int_{\mathcal{A}(s, T, K, \frac{3}{2}\delta_1)^2} (\mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x)] - \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda)] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda)]) \mathbb{1}_{y \notin \mathcal{S}_x} dx dy = o(\log^{d-1}(\lambda)). \quad (4.40)$$

(b2) We now deal with the case  $y \in \mathcal{S}_x$ . Equation (4.39) implies that

$$\sup_{x, y} |\mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x)] - \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda)] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda)]| = O((\log \log(\lambda))^{(6(d-1)+1)d}) \quad (4.41)$$

Moreover, using [12, equation (5.4)] yields

$$\sup_{x \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)} \text{Vol}_d(\mathcal{S}_x) \leq \sup_{x \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)} \text{card}(\{j : S'_j \subset \mathcal{S}_x\}) \sup_j \text{Vol}_d(S'_j) = O\left(\frac{(\log \log(\lambda))^{(6(d-1)+1)d}}{\lambda}\right). \quad (4.42)$$

Combining (4.41), (4.42) and Lemma 4.5.9, we obtain

$$\begin{aligned} &\lambda^2 \int_{\mathcal{A}(s, T, K, \frac{3}{2}\delta_1)^2} (\mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x)] - \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda)] \mathbb{E}[\tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda)]) \mathbb{1}_{y \in \mathcal{S}_x} dx dy \\ &\leq \lambda^2 \text{Vol}_d(\mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)) \sup_{x \in \mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)} \text{Vol}_d(\mathcal{S}_x) (\log \log(\lambda))^{(6(d-1)+1)d} \\ &= o(\log^{d-1}(\lambda)). \end{aligned} \quad (4.43)$$

To summarize, we have proved that  $V_2 = o(\log^{d-1}(\lambda))$ . We have shown the same bound in (a) for  $V_1$  and we can thus conclude that  $\text{Var}[Z_0 \mathbf{1}_{A_\lambda}] = o(\log^{d-1}(\lambda))$ .

(2) Our last step is to show that  $\text{Var}[Z_0 | A_\lambda] = o(\log^{d-1}(\lambda))$ . First, we notice that

$$\text{Var}[Z_0 | A_\lambda] = \mathbb{P}(A_\lambda)^{-2} (\text{Var}[Z_0 \mathbf{1}_{A_\lambda}] - \mathbb{E}[Z_0^2 \mathbf{1}_{A_\lambda}] (1 - \mathbb{P}(A_\lambda))).$$

As  $\mathbb{P}(A_\lambda) \rightarrow 1$ , see Lemma 4.5.2, and we already know that  $\text{Var}[Z_0 \mathbf{1}_{A_\lambda}] = o(\log^{d-1}(\lambda))$ , we only need to prove that  $\mathbb{E}[Z_0^2 \mathbf{1}_{A_\lambda}] (1 - \mathbb{P}(A_\lambda)) = o(\log^{d-1}(\lambda))$ . To achieve this, we use the estimates already obtained in Lemma 4.5.9 and (4.39). We then get

$$\begin{aligned} \mathbb{E}[Z_0^2 \mathbf{1}_{A_\lambda}] &\leq \lambda^2 \int_{\mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1)^2} \mathbb{E}[\tilde{\xi}_{n,k}(x, \mathcal{P}_\lambda^y) \tilde{\xi}_{n,k}(y, \mathcal{P}_\lambda^x)] dx dy \\ &= O(\lambda^2 \text{Vol}_d(\mathcal{A}(s, T^*, K, \frac{3}{2}\delta_1))^2 (\log \log(\lambda))^{(6(d-1)+1)d}) \\ &= O((\log(\lambda))^{2(d-2)+2/d} (\log \log(\lambda))^{(6(d-1)+1)d+2}). \end{aligned}$$

Using Lemma 4.5.2 again gives the desired result and concludes the proof.

To prove the result on the conditional expectation, one can adapt the proof of the upper bound for  $V_1$  where we integrate  $\tilde{\xi}(x, \mathcal{P}_\lambda)$  instead of its square.  $\square$

For now, on the one hand, we have shown the additivity of  $\text{Var}[\sum_i Z_i(\frac{3}{2}\delta_1) | A_\lambda]$  and the negligibility of  $\mathbb{E}[Z_0(\frac{3}{2}\delta_1) | A_\lambda]$  and  $\text{Var}[Z_0(\frac{3}{2}\delta_1) | A_\lambda]$ . On the other hand, we found the asymptotics of  $\mathbb{E}[Z_i(\delta_0) \mathbf{1}_{A_\lambda}]$  and  $\text{Var}[Z_i(\delta_0) \mathbf{1}_{A_\lambda}]$ .

The goal in the sequel is to estimate the error made when the indicator function is replaced by the conditioning given  $A_\lambda$  and the error made when we remove the conditioning.

For the next step, we are going to need an additional technical result. We put  $U = \frac{\log(\lambda)}{\lambda}$  and  $U^* = d6^d U$ . Then we call  $B_\lambda$  the event that  $K(v \geq U^*) \subset \Phi_n(\mathcal{P}_\lambda)$  and  $K(v \leq U^*)$  contains at most  $c \log^d(\lambda)$  points of  $\mathcal{P}_\lambda$ . The same method as in the previous sections yields the following estimate, which is an adaptation of [12, Lemma 5.3].

**Lemma 4.5.12.**

$$\mathbb{P}(B_\lambda^c) = O(\lambda^{-3d}).$$

The following lemma proves that the conditional expectation and the conditional variance of  $Z$  are close to the expectation and the variance of  $Z$  respectively. The proof follows from [12, Section 8].

**Lemma 4.5.13.** *For  $\lambda$  large enough we have*

$$\max\{|\mathbb{E}[Z] - \mathbb{E}[Z | A_\lambda]|, |\text{Var}[Z] - \text{Var}[Z | A_\lambda]|\} = o(\log^{d-1}(\lambda))$$

*Proof.* We first estimate  $\mathbb{E}[Z^k | A_\lambda]$  and  $\mathbb{E}[Z^k | A_\lambda^c]$  for  $k = 1, 2$ .

1) *Estimation of  $\mathbb{E}[Z^k | A_\lambda]$ .* On the event  $A_\lambda$ , the sandwich  $\mathcal{A}(s, T^*, K)$  contains at most  $c \log^{d-1}(\lambda) \log \log(\lambda)$  points of  $\mathcal{P}_\lambda$ , see Lemma 4.5.3. Using MacMullen's bound [54], the fact that  $\mathbb{E}[Z^k | A_\lambda] = \frac{1}{\mathbb{P}(A_\lambda)} \mathbb{E}[Z^k \mathbf{1}_{A_\lambda}]$  and Lemma 4.5.2, we deduce that for  $\lambda$  large enough

$$\mathbb{E}[Z^k | A_\lambda] \leq c \log^{d(d-1)}(\lambda) \log \log(\lambda)^d. \quad (4.44)$$

2) *Estimation of  $\mathbb{E}[Z^k|A_\lambda^c]$ .* We use the event  $B_\lambda$  here, we have

$$\mathbb{E}[Z^k|A_\lambda^c] = \mathbb{E}[Z\mathbf{1}_{B_\lambda}|A_\lambda^c] + \mathbb{E}[Z\mathbf{1}_{B_\lambda^c}|A_\lambda^c]. \quad (4.45)$$

On the event  $B_\lambda$ , there is at most  $c \log^d(\lambda)$  points in  $B_\lambda$  and from MacMullen's bound [54], we deduce that

$$\mathbb{E}[Z^k\mathbf{1}_{B_\lambda}|A_\lambda^c] \leq c \log^{d^2}(\lambda). \quad (4.46)$$

The estimate of  $\mathbb{E}[Z^k\mathbf{1}_{B_\lambda^c}|A_\lambda^c]$  is more difficult as on the event  $B_\lambda^c$ , the variable  $Z$  is not necessarily bounded. Using the conditional total probability formula, writing  $E_m$  for the event  $\text{card}(\mathcal{P}_\lambda) = m$  and MacMullen's bound we get

$$\begin{aligned} \mathbb{E}[Z^k\mathbf{1}_{B_\lambda^c}|A_\lambda^c] &= \sum_{m=0}^{\infty} \mathbb{E}[Z^k\mathbf{1}_{B_\lambda^c}|A_\lambda^c \cap E_m] \mathbb{P}(E_m|A_\lambda^c) \\ &\leq \sum_{m=0}^{3\lambda} (3\lambda)^{\frac{kd}{2}} \mathbb{E}[\mathbf{1}_{B_\lambda^c}|A_\lambda^c \cap E_m] \mathbb{P}(E_m|A_\lambda^c) + \sum_{m \geq 3\lambda} m^{\frac{kd}{2}} \mathbb{E}[\mathbf{1}_{B_\lambda^c}|A_\lambda^c \cap E_m] \mathbb{P}(E_m|A_\lambda^c) \\ &\leq c \left( \sum_{m=0}^{3\lambda} (3\lambda)^{\frac{kd}{2}} \mathbb{P}(B_\lambda^c|A_\lambda^c \cap E_m) \mathbb{P}(E_m|A_\lambda^c) + \sum_{m \geq 3\lambda} m^{\frac{kd}{2}} \mathbb{P}(E_m|A_\lambda^c) \right) \end{aligned}$$

Then we use Lemma 4.5.12 and Lemma 4.5.2 to get

$$\mathbb{P}(B_\lambda^c|A_\lambda^c \cap E_m) \mathbb{P}(E_m|A_\lambda^c) = \frac{\mathbb{P}(B_\lambda^c \cap A_\lambda^c \cap E_m)}{\mathbb{P}(A_\lambda^c \cap E_m)} \frac{\mathbb{P}(A_\lambda^c \cap E_m)}{\mathbb{P}(A_\lambda^c)} \leq \frac{\mathbb{P}(B_\lambda^c)}{\mathbb{P}(A_\lambda^c)} \leq c\lambda^{-3d+1}.$$

Thus we have

$$\mathbb{E}[Z^k\mathbf{1}_{B_\lambda^c}|A_\lambda^c] \leq c \left( \lambda^{-2d+1} + \sum_{m \geq 3\lambda} m^{\frac{kd}{2}} \frac{\mathbb{P}(E_m)}{\mathbb{P}(A_\lambda^c)} \right)$$

As  $\mathbb{P}(A_\lambda^c)$  goes to 1 as  $\lambda$  goes to infinity, we deduce, after an estimation of  $\sum_{m \geq 3\lambda} m^{\frac{kd}{2}} \mathbb{P}(E_m)$  that we leave to the reader, that for  $\lambda$  large enough

$$\mathbb{E}[Z^k\mathbf{1}_{B_\lambda^c}|A_\lambda^c] \leq c\lambda^{-2d+1}. \quad (4.47)$$

Combining (4.45), (4.46) and (4.47), we obtain

$$\mathbb{E}[Z^k|A_\lambda^c] \leq c \log^{d^2}(\lambda). \quad (4.48)$$

We are now able to prove the result on the expectation. For any positive random variable  $\zeta$  and event  $A$ , we appeal to the following result,

$$|\mathbb{E}[\zeta] - \mathbb{E}[\zeta|A]| \leq (\mathbb{E}[\zeta|A] + \mathbb{E}[\zeta|A^c]) \mathbb{P}(A^c) \quad (4.49)$$

which is proved in [12, Claim 8.3]. Applying it to our context, we get

$$|\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]| \leq (\mathbb{E}[Z|A_\lambda] + \mathbb{E}[Z|A_\lambda^c]) \mathbb{P}(A_\lambda^c).$$

From the upper bound  $\mathbb{P}(A_\lambda^c) \leq c \log^{-4d^2}(\lambda)$  given by Lemma 4.5.2 and equations (4.44) and (4.48), we deduce that

$$|\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]| \leq c \log^{-3d^2}(\lambda) = o(\log^{d-1}(\lambda)).$$

Let us prove the result about the variance. We have

$$\begin{aligned} |\text{Var}[Z] - \text{Var}[Z|A_\lambda]| &= |\mathbb{E}[Z^2] - \mathbb{E}[Z]^2 - (\mathbb{E}[Z^2|A_\lambda] - \mathbb{E}[Z|A_\lambda]^2)| \\ &\leq |\mathbb{E}[Z^2] - \mathbb{E}[Z^2|A_\lambda]| + |\mathbb{E}[Z]^2 - \mathbb{E}[Z|A_\lambda]^2| \\ &= |\mathbb{E}[Z^2] - \mathbb{E}[Z^2|A_\lambda]| + |\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]| |\mathbb{E}[Z] + \mathbb{E}[Z|A_\lambda]| \\ &\leq |\mathbb{E}[Z^2] - \mathbb{E}[Z^2|A_\lambda]| + |\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]| (|\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]| + \mathbb{E}[Z|A_\lambda]). \end{aligned}$$

The first term of this last line is bounded by  $\log^{-3d^2}(\lambda)$ , thanks to (4.49) applied to  $Z^2$ , Lemma 4.5.2 and the previous estimates for  $k = 2$ . As we know that  $|\mathbb{E}[Z] - \mathbb{E}[Z|A_\lambda]| \leq c \log^{-3d^2}(\lambda)$  and  $\mathbb{E}[Z|A_\lambda] \leq c \log^{d^2}(\lambda)$ , the second term is also a  $o(\log^{d-1}(\lambda))$  and the result follows.  $\square$

Next we estimate the difference between the expectation (resp. variance) of the variable  $Z_i(\delta_0)$  conditional on  $A_\lambda$  and the expectation (resp. variance) of  $Z_i(\delta_0)\mathbb{1}_{A_\lambda}$ . Notice that this concerns  $Z_i(\delta_0)$  and not the variable  $Z_i(\frac{3}{2}\delta_1)$  as opposed to most of the previous lemmas of this subsection.

**Lemma 4.5.14.** *For every  $1 \leq i \leq f_0(K)$  we have*

$$|\mathbb{E}[Z_i(\delta_0)|A_\lambda] - \mathbb{E}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}]| = o(\log^{d-1}(\lambda))$$

and

$$|\text{Var}[Z_i(\delta_0)|A_\lambda] - \text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}]| = o(\log^{d-1}(\lambda)).$$

*Proof.* We only prove the result on the variance. Let  $1 \leq i \leq f_0(K)$ .

The key is to use the formula

$$\text{Var}[Z_i(\delta_0)|A_\lambda] - \text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}] = (\mathbb{P}(A_\lambda)^{-2} - 1) \text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}] - \mathbb{P}(A_\lambda)^{-2} \mathbb{P}(A_\lambda^c) \mathbb{E}[Z_i(\delta_0)^2 \mathbb{1}_{A_\lambda}],$$

which comes from

$$\text{Var}[Z_i(\delta_0)|A_\lambda] = \mathbb{P}(A_\lambda)^{-2} (\text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}] - \mathbb{E}[Z_i(\delta_0)^2 \mathbb{1}_{A_\lambda}] (1 - \mathbb{P}(A_\lambda))).$$

Theorem 4.2.3 and Proposition 4.4.9 imply that  $(\mathbb{P}(A_\lambda)^{-2} - 1) \text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}] = o(\log^{d-1}(\lambda))$  so we only need to study  $\mathbb{P}(A_\lambda)^{-2} \mathbb{P}(A_\lambda^c) \mathbb{E}[Z_i(\delta_0)^2 \mathbb{1}_{A_\lambda}]$ . On the event  $A_\lambda$ ,  $\text{card}(\mathcal{P}_\lambda \cap \mathcal{A}(s, T^*, K)) \leq c \log^{d-1}(\lambda) \log \log(\lambda)$ , see Lemma 4.5.3, and only these points can contribute to a  $k$ -face of  $\partial \text{conv}_n(\mathcal{P}_\lambda)$ . Using MacMullen's bound [54] we deduce that

$$\mathbb{E}[Z_i(\delta_0)^2 \mathbb{1}_{A_\lambda}] \leq c \log \log(\lambda)^d \log^{d(d-1)}(\lambda).$$

As  $\mathbb{P}(A_\lambda^c) \leq c \log^{-4d^2}(\lambda)$ , see Lemma 4.5.2, we get  $\mathbb{P}(A_\lambda)^{-2} \mathbb{P}(A_\lambda^c) \mathbb{E}[Z_i(\delta_0)^2 \mathbb{1}_{A_\lambda}] = o(1)$  and the result follows.  $\square$

We are now finally able to prove the main result of this section, which yields the decomposition of  $\mathbb{E}[Z]$  and  $\text{Var}[Z]$  into the sum of the variables  $\mathbb{E}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}]$  and the sum of the variables  $\text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}]$  respectively, plus a negligible term. There is a small technical issue that needs to be addressed. Some of the results of this section concern the variables  $Z_i(\frac{3}{2}\delta_1)$ . As the transformation in the corner is applied to  $Q_0 = [0, \delta_0]^d$ , we need to replace  $Z_i(\frac{3}{2}\delta_1)$  by  $Z_i(\delta_0)$  and estimate the error.

**Proposition 4.5.15.** *When  $\lambda \rightarrow \infty$ , we get*

$$\mathbb{E}[Z] = \sum_{\mathcal{V}_i \in \mathcal{V}_K} \mathbb{E}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}] + o(\log^{d-1}(\lambda)).$$

and

$$\text{Var}[Z] = \sum_{\mathcal{V}_i \in \mathcal{V}_K} \text{Var}[Z_i(\delta_0)\mathbb{1}_{A_\lambda}] + o(\log^{d-1}(\lambda)).$$

*Proof.* We focus on the variance, the proof for the expectation is similar and actually easier. Our first step is to prove that the variances conditional on  $A_\lambda$  of  $Z_i(\frac{3}{2}\delta_1)$  and  $Z_i(\delta_0)$  are close to each other. More precisely, we show that for every  $i$

$$\text{Var}[Z_i(\frac{3}{2}\delta_1)|A_\lambda] = \text{Var}[Z_i(\delta_0)|A_\lambda] + o(\log^{d-1}(\lambda)). \quad (4.50)$$

The idea is to write the difference  $Z_i(\frac{3}{2}\delta_1) - Z_i(\delta_0)$  as a sum of scores in the flat part for a  $\delta'_1$  that is slightly smaller than  $\delta_0$  and allows the construction of dyadic Macbeath region in the same spirit as what we have done for  $\frac{3}{2}\delta_1$ . As for  $\frac{3}{2}\delta_1$ , the scores in this flat part are negligible. We introduce  $\delta'_1 = r'(\lambda, d)\delta_0$  where  $r'(\lambda, d) \in (3^{-1/d}, 1]$  is chosen so that  $\log_3(T/\delta_1^d) \in \mathbb{Z}$ . We claim that such a  $\delta'_1$  exists and it can be checked by looking for a condition on  $r'(\lambda, d)$  to have  $\log_3(T/\delta_1^d) = \lfloor \log_3(T/\delta_0^d) \rfloor + 1$ . Methods similar to the proof of Lemma 4.5.11 show that

$$\text{Var}[Z_0(\frac{1}{2}\delta'_1)|A_\lambda] = o(\log^{d-1}(\lambda))$$

and more generally for any subset  $\mathcal{B}$  of  $\mathcal{P}_\lambda(s, T^*, K, \frac{1}{2}\delta'_1)$ , we have

$$\text{Var}[\sum_{x \in \mathcal{B}} \xi_{n,k}(x, \mathcal{P}_\lambda)|A_\lambda] = o(\log^{d-1}(\lambda)). \quad (4.51)$$

As for any  $i$  the set  $\mathcal{B}_i := (p_d(\mathcal{V}_i, \frac{3}{2}\delta_1) \setminus p_d(\mathcal{V}_i, \delta_0)) \cap \mathcal{A}(s, T^*, K)$  is a subset of  $\mathcal{P}_\lambda(s, T^*, K, \frac{1}{2}\delta'_1)$ , we deduce that

$$\text{Var}[Z_i(\frac{3}{2}\delta_1) - Z_i(\delta_0)|A_\lambda] = \text{Var}[\sum_{x \in \mathcal{B}_i} \xi_{n,k}(x, \mathcal{P}_\lambda)|A_\lambda] = o(\log^{d-1}(\lambda)). \quad (4.52)$$

Moreover

$$\text{Cov}(Z_i(\frac{3}{2}\delta_1) - Z_i(\delta_0), Z_i(\delta_0)|A_\lambda) \leq \sqrt{\text{Var}[Z_i(\frac{3}{2}\delta_1) - Z_i(\delta_0)|A_\lambda]} \sqrt{\text{Var}[Z_i(\delta_0)|A_\lambda]}. \quad (4.53)$$

From Lemma 4.5.14 and Proposition 4.4.9 we know that  $\text{Var}[Z_i(\delta_0)|A_\lambda] = O(\log^{d-1}(\lambda))$ . Therefore, combining this, (4.52) and (4.53) yields (4.50).

We are now able to prove the lemma. First, we use Lemma 4.5.13 to reduce to the variance of  $Z$  conditional on  $A_\lambda$ . We get

$$\text{Var}[Z] = \text{Var}[Z|A_\lambda] + o(\log^{d-1}(\lambda)) = \text{Var}[Z_0(\frac{3}{2}\delta_1) + \sum_{i=1}^{f_0(K)} Z_i(\frac{3}{2}\delta_1)|A_\lambda] + o(\log^{d-1}(\lambda)).$$

Next, we want to remove  $Z_0$  from the variance. We can write

$$\text{Var}[Z_0(\frac{3}{2}\delta_1) + \sum_{i=1}^{f_0(K)} Z_i(\frac{3}{2}\delta_1)|A_\lambda] = \text{Var}[Z_0(\frac{3}{2}\delta_1)|A_\lambda] + \text{Var}[\sum_{i=1}^{f_0(K)} Z_i(\frac{3}{2}\delta_1)|A_\lambda] + 2\text{Cov}(\sum_{i=1}^{f_0(K)} Z_i(\delta_1), Z_0(\delta_1)|A_\lambda). \quad (4.54)$$

Lemma 4.5.11 implies that  $\text{Var}[Z_0(\frac{3}{2}\delta_1)|A_\lambda] = o(\log^{d-1}(\lambda))$ . Additionally, we use the Cauchy-Schwarz inequality, Lemma 4.5.8, (4.50), Lemma 4.5.14 and Proposition 4.4.9 to obtain

$$\begin{aligned} \text{Cov}(\sum_{i=1}^{f_0(K)} Z_i(\frac{3}{2}\delta_1), Z_0(\delta_1)) &\leq \left( \text{Var}[\sum_{i=1}^{f_0(K)} Z_i(\frac{3}{2}\delta_1)|A_\lambda] \right)^{1/2} \left( \text{Var}[Z_0(\frac{3}{2}\delta_1)|A_\lambda] \right)^{1/2} \\ &= \left( \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\frac{3}{2}\delta_1)|A_\lambda] \right)^{1/2} \left( o(\log^{d-1}(\lambda)) \right)^{1/2} \\ &= \left( \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\delta_0)|A_\lambda] + o(\log^{d-1}(\lambda)) \right)^{1/2} \left( o(\log^{d-1}(\lambda)) \right)^{1/2} \\ &= \left( \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\delta_0)\mathbf{1}_{A_\lambda}] + o(\log^{d-1}(\lambda)) \right)^{1/2} \left( o(\log^{d-1}(\lambda)) \right)^{1/2} \\ &= \left( O(\log^{d-1}(\lambda)) \right)^{1/2} \left( o(\log^{d-1}(\lambda)) \right)^{1/2} \\ &= o(\log^{d-1}(\lambda)) \end{aligned} \quad (4.55)$$

Inserting  $\text{Var}[Z_0(\delta_1)|A_\lambda] = o(\log^{d-1}(\lambda))$  and (4.55) into (4.54) gives

$$\text{Var}[Z] = \text{Var}[Z_0(\frac{3}{2}\delta_1) + \sum_{i=1}^{f_0(K)} Z_i(\frac{3}{2}\delta_1)|A_\lambda] = \text{Var}[\sum_{i=1}^{f_0(K)} Z_i(\frac{3}{2}\delta_1)|A_\lambda] + o(\log^{d-1}(\lambda)).$$

Finally, we combine this equation with the additivity of the variance for  $Z_i(\delta_1)$  implied by Lemma 4.5.8, (4.50) and Lemma 4.5.14 to obtain

$$\begin{aligned} \text{Var}[Z] &= \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\frac{3}{2}\delta_1)|A_\lambda] + o(\log^{d-1}(\lambda)) = \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\delta_0)|A_\lambda] + o(\log^{d-1}(\lambda)) \\ &= \sum_{i=1}^{f_0(K)} \text{Var}[Z_i(\delta_0)\mathbf{1}_{A_\lambda}] + o(\log^{d-1}(\lambda)) \end{aligned}$$

which is the desired result.  $\square$

## 4.6 Proof of the main results

*Proof of Proposition 4.4.9.* This proof is an adaptation of the proof of [27, Theorem 2.1]. As we have already proved the same kind of required results of stabilization and domination adapted to the  $n$ -th layer in Section 4.4 and an analogous sandwiching result in Section 4.2, we merely have to check that no new problem arises.

Appealing to the volume-preserving affine transformations  $a_i$  introduced at the beginning of Section 4.3 are volume-preserving, we claim that we only need to prove the result for a vertex artificially placed at 0 as the proof works in the exact same way for the other vertices. In the lines below, the label  $i$  refers to the actual label of the vertex 0.

1) Let us first prove the result on the expectation. Applying the transformation  $T^{(\lambda)}$ , Mecke's formula, the fact that  $\xi_{n,k}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})\mathbb{1}_{A_\lambda} = \hat{\xi}_{n,k}^{(\lambda)}((v, h), \mathcal{P}^{(\lambda)})\mathbb{1}_{A_\lambda}$ , see (4.25), and the translation invariance of  $\hat{\xi}_{n,k}^{(\lambda)}$  we have

$$\begin{aligned}\mathbb{E}[Z_i\mathbb{1}_{A_\lambda}] &= \int_{Q_0} \mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda)\mathbb{1}_{A_\lambda}]\lambda dx = \sqrt{d} \int_{(v,h) \in W_\lambda} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v, h), W_\lambda)\mathbb{1}_{A_\lambda}]e^{dh} dh dv \\ &= \sqrt{d} \int_{(v,h) \in W_\lambda} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W_\lambda - v))\mathbb{1}_{A_\lambda}]e^{dh} dh dv\end{aligned}$$

Then we apply the change of variable  $u = (\frac{1}{d} \log(\lambda))^{-1}v$ ,  $dv = d^{-(d-1)} \log^{d-1}(\lambda)du$  to get

$$\frac{\mathbb{E}[Z_i\mathbb{1}_{A_\lambda}]}{\log^{d-1}(\lambda)} = d^{-d+3/2} \int_{(u,v) \in W'_\lambda} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda))\mathbb{1}_{A_\lambda}]e^{dh} dh du \quad (4.56)$$

where  $W'_\lambda = \{((\frac{1}{d} \log(\lambda))^{-1}v, h) : (v, h) \in W_\lambda\}$  and writing for any  $s \in \mathbb{R}$  and  $B \subset \mathbb{R}^{d-1} \times \mathbb{R}$   $sB := \{(sv, h) : (v, h) \in B\}$ .

Now our goal is to use Lebesgue's dominated convergence theorem. The equation of  $W'_\lambda$  is  $l_i(u) \leq \left(1 + \frac{\log(\delta_0) - h}{\log^{1/d}(\lambda)}\right)$  for all  $1 \leq i \leq d$ . Therefore the limit of  $\mathbb{1}_{W'_\lambda}$  is the indicator function of the cylinder  $V \cap \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_i \leq 1 \text{ for all } i\} \times \mathbb{R}$ . The base of this cylinder is exactly  $S(d)$  defined at (4.26).

As  $(W'_\lambda - u) \log^{1/d}(\lambda)$  goes to  $\mathbb{R}^d$  as  $\lambda \rightarrow \infty$ , we can show that

$$\lim_{\lambda \rightarrow \infty} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda))] = \mathbb{E}[\hat{\xi}_{n,k}^{(\infty)}((0, h), \mathcal{P})]$$

with a small modification of the proof of Lemma 4.4.6. Lemma 4.4.5 shows that  $\mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda))e^{dh}]$  is dominated by an integrable function on  $\mathbb{R}^{d-1} \times \mathbb{R}$ . Using Lebesgue's dominated convergence theorem, we get

$$\begin{aligned}\lim_{\lambda \rightarrow \infty} \int_{(u,v) \in W'_\lambda} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda))]e^{dh} dh du \\ = \text{Vol}_d(S(d)) \int_{\mathbb{R}} \mathbb{E}[\hat{\xi}_{n,k}^{(\infty)}((0, h), \mathcal{P})]e^{dh} dh.\end{aligned} \quad (4.57)$$

In view of (4.56), we only need to get rid of the indicator function of  $A_\lambda$ . By the Cauchy-Schwarz inequality and a moment bound similar to the one in Lemma 4.4.5, we have uniformly in  $u$

$$\begin{aligned} & \int_{\mathbb{R}} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda)) \mathbb{1}_{A_\lambda^c}] e^{dh} dh \\ & \leq \int_{\mathbb{R}} (\mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda))^2])^{1/2} \mathbb{P}(A_\lambda^c)^{1/2} e^{dh} dh \\ & \leq c \log^{-2d^2}(\lambda). \end{aligned}$$

Thus combining this with (4.56), we obtain

$$\frac{\mathbb{E}[Z_i \mathbb{1}_{A_\lambda}]}{\log^{d-1}(\lambda)} = \lim_{\lambda \rightarrow \infty} d^{-d+3/2} \int_{(u,v) \in W'_\lambda} \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((0, h), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda))] e^{dh} dh du + o(1).$$

Using (4.57) we obtain the result.

2) Next, let us prove the limit of the variance. As for the expectation, we begin with an application of Mecke's formula. We get

$$\text{Var}[Z_i \mathbb{1}_{A_\lambda}] = I_1(\lambda) + I_2(\lambda).$$

where

$$I_1(\lambda) = \int_{Q_0} \mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda)^2 \mathbb{1}_{A_\lambda}] \lambda dx$$

and

$$\begin{aligned} I_2(\lambda) := & \int_{Q_0^2} (\mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda \cup \{y\}) \mathbb{E}[\xi_{n,k}(y, \mathcal{P}_\lambda \cup \{x\}) \mathbb{1}_{A_\lambda}]] \\ & - \mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda) \mathbb{1}_{A_\lambda}] \mathbb{E}[\xi_{n,k}(y, \mathcal{P}_\lambda) \mathbb{1}_{A_\lambda}]) \lambda^2 dx dy. \end{aligned}$$

A slight adaptation of the proof of the convergence of the expectation shows that

$$\lim_{\lambda \rightarrow \infty} \frac{I_1(\lambda)}{\log^{d-1}(\lambda)} = d^{-d+3/2} \text{Vol}_d(S(d)) \int_{\mathbb{R}} \mathbb{E}[\xi_{n,k}(x, \mathcal{P}_\lambda)^2] e^{dh} dh.$$

It remains to find a limit for  $I_2(\lambda)$ , rescaled by  $\log^{d-1}(\lambda)$ . Let us write

$$\begin{aligned} c^{\hat{\xi}_{n,k}^{(\lambda)} \mathbb{1}_{A_\lambda}}((v_0, h_0), (v_1, h_1)) := & \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)} \cup \{(v_1, h_1)\}) \hat{\xi}_{n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)} \cup \{(v_0, h_0)\}) \mathbb{1}_{A_\lambda}] \\ & - \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v_0, h_0), \mathcal{P}^{(\lambda)}) \mathbb{1}_{A_\lambda}] \mathbb{E}[\hat{\xi}_{n,k}^{(\lambda)}((v_1, h_1), \mathcal{P}^{(\lambda)}) \mathbb{1}_{A_\lambda}]. \end{aligned}$$

Applying the transformation  $T^{(\lambda)}$ , we can rewrite  $I_2(\lambda)$  as

$$I_2(\lambda) = d \int_{(v_0, h_0) \in W_\lambda} \int_{(v_1, h_1) \in W_\lambda} c^{\hat{\xi}_{n,k}^{(\lambda)} \mathbb{1}_{A_\lambda}}((v_0, h_0), (v_1, h_1)) e^{dh_0} e^{dh_1} dh_0 dh_1 dv_0 dv_1.$$

The translation invariance of  $\hat{\xi}$  yields

$$I_2(\lambda) = d \int_{(v_0, h_0) \in W_\lambda} \int_{(v_1, h_1) \in W_\lambda} c^{\hat{\xi}_{n,k}^{(\lambda)} \mathbb{1}_{A_\lambda}}((v_0, h_0), (v_1 - v_0, h_1), \mathcal{P} \cap (W_\lambda - v_0)) e^{dh_0} e^{dh_1} dh_0 dh_1 dv_0 dv_1.$$

As for the expectation, we make the change of variable  $u = (\frac{1}{d} \log(\lambda))^{-1} v_0$ ,  $dv_0 = d^{-(d-1)} \log^{d-1}(\lambda) du$ . We obtain

$$\begin{aligned} & \frac{I_2(\lambda)}{\log^{d-1}(\lambda)} \\ &= d^{-d+2} \int_{(u, h_0) \in W'_\lambda} \int_{(v_1, h_1) \in W_\lambda} c_{n,k}^{\xi(\lambda)} \mathbb{1}_{A_\lambda}((v_0, h_0), (v_1 - \frac{1}{d} \log(\lambda)u, h_1), \mathcal{P} \cap (W_\lambda - \frac{1}{d} \log(\lambda)u)) \\ & \quad e^{dh_0} e^{dh_1} dh_0 dh_1 dv_0 dv_1. \end{aligned} \tag{4.58}$$

We determine the limit of the integral above in four steps (i)–(iv). First, we take the limit of the integration domain, then we remove the indicator of  $A_\lambda$ , then we find the limit of the integrand and finally, we bound it to finally apply Lebesgue's dominated convergence theorem.

(i) As in part 1), we have

$$\lim_{\lambda \rightarrow \infty} \mathbb{1}_{(u, h_0) \in W'_\lambda} \mathbb{1}_{(v_1, h_1) \in W_\lambda} = \mathbb{1}_{u \in S(d)}.$$

(ii) We now justify that we can remove the indicator of  $A_\lambda$  in  $c_{n,k}^{\xi(\lambda)} \mathbb{1}_{A_\lambda}$  with an error that goes to 0 as  $\lambda$  goes to infinity. We assert that with a small modification of the proof of Lemma 4.4.5, we can show that

$$\begin{aligned} & |c_{n,k}^{\xi(\lambda)} \mathbb{1}_{A_\lambda} \left( (0, h_0), (v_1, -\frac{1}{d} \log(\lambda)u, h_1), \mathcal{P} \cap (W_\lambda - \frac{1}{d} \log(\lambda)u) \right) \\ & - c_{n,k}^{(\lambda)} \left( (0, h_0), (v_1, -\frac{1}{d} \log(\lambda)u, h_1), \mathcal{P} \cap (W_\lambda - \frac{1}{d} \log(\lambda)u) \right)| e^{d(h_0+h_1)} \end{aligned}$$

is bounded by  $\mathbb{P}(A_\lambda^c) G(h_0, v_1, h_1)$  where  $G$  is an integrable function with respect to  $(h_0, v_1, h_1) \in \mathbb{R} \times \mathbb{R}^{d-1} \times \mathbb{R}$  that does not depend on  $u$ . As a result, using Lemma 4.5.2, we get that the integrated error is bounded uniformly in  $u$  by

$$\int_{h_0 \in \mathbb{R}} \int_{(v_1, h_1) \in W_\lambda} \mathbb{P}(A_\lambda^c)^{1/4} G(h_0, v_1, h_1) dh_0 dv_1 dh_1 \leq c \log^{-d^2}(\lambda) = o(\log^{d-1}(\lambda)).$$

Thus we obtain that removing the indicator function of  $A_\lambda$  in (4.58) does not change the asymptotics.

(iii) For any fixed  $u$  and for all  $(v_1, h_1) \in W_\lambda$ , we make the change of variable  $v' = v_1 - \frac{1}{d} \log(\lambda)u$ ,  $dv' = dv_1$ . This changes

$$c_{n,k}^{(\lambda)} \left( (0, h_0), (v_1 - \frac{1}{d} \log(\lambda)u, h_1), \mathcal{P} \cap (W'_\lambda - \frac{1}{d} \log(\lambda)u) \right)$$

into

$$c_{n,k}^{(\lambda)} \left( (0, h_0), (v', h_1), \mathcal{P} \cap (W'_\lambda - u) \log^{1/d}(\lambda) \right) e^{d(h_0+h_1)}.$$

By small modifications of Lemma 4.4.8 and Lemma 4.4.7, we can show that it is bounded uniformly in  $\lambda$  by an integrable function of  $h_0, v'$  and  $h_1$  and it converges to  $c_{n,k}^{(\infty)}((0, h_0), (v_1, h_1), \mathcal{P})$  as  $\lambda$  goes

to infinity.

(iv) The change of variable  $v' = v_1 - \frac{1}{d} \log(\lambda)u$  transforms the integration domain  $W_\lambda$  into  $\{(v', h_0) \in (W'_\lambda - u) \log^{1/d}(\lambda)\}$ , which increases up to  $\mathbb{R}^{d-1} \times \mathbb{R}$ . Combining this with points (i) to (iii) and the dominated convergence theorem, we finally obtain

$$\lim_{\lambda \rightarrow \infty} \frac{I_2(\lambda)}{\log^{d-1}(\lambda)} = d^{-d+2} \text{Vol}_d(S(d)) \int_{\mathbb{R}^{d-1} \times \mathbb{R}} \int_{\mathbb{R}} c_{n,k}^{(\infty)}((0, h_0), (v', h_1)) e^{d(h_0+h_1)} dh_0 dv', h_1.$$

□

*Proof of Theorem 4.3.6.* We only prove the result on the variance as the proof of the expectation is analogous. From Proposition 4.5.15, we deduce that

$$\text{Var}[Z] = \sum_{\mathcal{V}_i \in \mathcal{V}_K} \text{Var}[Z_i(\delta_0) \mathbf{1}_{A_\lambda}] + o(\log^{d-1}(\lambda)).$$

Combining this with the rewriting of the limit

$$\lim_{\lambda \rightarrow \infty} \frac{\text{Var}[Z_i(\delta_0) \mathbf{1}_{A_\lambda}]}{\log^{d-1}(\lambda)} = I_1(\infty) + I_2(\infty)$$

that is obtained in Proposition 4.4.9, we get

$$\lim_{\lambda \rightarrow \infty} \text{Var}[Z] = f_0(K)(I_1(\infty) + I_2(\infty)).$$

□

## 4.7 Concluding remarks

- To conclude, we give some possible extensions of our results and related open problems.
- *Positivity of the limiting constants.* While we have proved that the expectation and variance of  $\frac{N_{n,k,\lambda}}{\log^{d-1}(\lambda)}$  converge to constants, we did not show that these constants do not vanish. The renormalization  $\log^{d-1}(\lambda)$  could theoretically be too large. However, we believe it is the correct renormalization as we think that the order of magnitude is the same as for the first layer and this is corroborated by our simulations. An adaptation of the methods of [23, Section 5] might be possible.
  - *Central limit theorem* Some of the methods of Section 4.5 are direct adaptations of those developed in [12] in order to obtain a central limit theorem for the volume and the number of  $k$ -faces of the convex hull. In particular, we use the same dependency graph. As a consequence, we expect to be able to find a central limit theorem for the number of  $k$ -faces of the  $n$ -th layer in a near future by adapting the remaining part of their methods.
  - *Intrinsic volumes.* The functionals under investigation are currently limited to the number of  $k$ -faces of the  $n$ -th layer. In the case of the first layer in a simple polytope, limiting expectation and variance for the defect volume of  $\text{conv}(\mathcal{P}_\lambda)$  have been derived as well in [7] and [27] respectively. In the case of the unit ball, we obtained in [23] the asymptotics for the defect intrinsic volume of  $\text{conv}_n(\mathcal{P}_\lambda)$  for any fixed  $n$ . An interesting perspective would

then be to do the same in the case of the convex hull peeling in a (simple) polytope. We expect the defect volume corresponding to the  $n$ -th layer to be decomposed as a sum of scores in a similar way as in [27] for the convex hull so we think the method developed in this paper may extend to the defect volume. The case of the defect intrinsic volumes of order  $k \in \{1, \dots, (d-1)\}$  seems more intricate, especially because in the case of the convex hull, the problem remains open for both the expectation and the variance. To the best of our knowledge, only the expectation for the defect mean width, i.e. for  $k = 1$ , has been derived in [67].

- *Monotonicity.* Theorem 4.1.1 induces a natural monotonicity problem, i.e. how do the constants  $C_{n,k,d}$  evolve with  $n$ ? We may start with  $C_{n,0,d}$ , i.e. the expected renormalized number of extreme points on the  $n$ -th layer. We recall that this particular issue was already raised in [23, Section 6] in the context of the convex hull peeling in the unit ball. It turns out that our simulations show a significant difference between these two models regarding the behaviour of  $C_{n,0,d}$ . While  $C_{n,0,d}$  seems to be decreasing with  $n$  in the case of the unit ball, it looks like it is increasing up to a certain point and then decreasing in the case of a polytope. We assert that the two behaviors are in fact consistent with our current knowledge of the convex hull peeling. Indeed, the expected total number of layers estimated by Dalal in [33] and then more precisely by Calder and Smart in [22] is proved to behave like  $\Theta(\lambda^{2/(d+1)})$  in any case, i.e. whether  $K$  is smooth or is a polytope. In the particular case of the polytope, we know that the first layers are occupied by (at most)  $\Theta(\log^{d-1}(\lambda))$  Poisson points while the expected total number of Poisson points is  $\lambda \text{Vol}_d(K)$ . Consequently, that regime of  $\Theta(\log^{d-1}(\lambda))$  could not be shared by all the  $\Theta(\lambda^{2/(d+1)})$  layers up to the last one. We know that  $N_{n,k,\lambda}$  has to increase and reach, for some  $n$  depending on  $\lambda$ , a phase transition where it starts to grow polynomially fast. That phase transition should probably occur when  $n$  is proportional to  $\lambda^{2/(d+1)}$ , i.e. in the regime investigated in [22]. This heuristic reasoning would imply the increase of the constant  $C_{n,0,d}$  up to the phase transition. After the point when  $N_{n,0,\lambda}$  reaches the polynomial regime, we think that the Markov property of the peeling construction implies that the subsequent layers should behave in the same way as for the convex hull peeling in the unit ball, i.e. we expect a decrease of  $N_{n,0,\lambda}$ . This is indeed what we observe in our simulations.
- *Other regimes.* In view of the previous point, it appears that a change in the growth rate of  $N_{n,0,\lambda}$  should occur. We expect the mean of  $N_{n,0,\lambda}$  to grow at most like  $\Theta(\log^{d-1}(\lambda))$  when  $n$  is fixed from the start and does not vary with  $\lambda$ , because of Theorem 4.1.1, then to grow like  $\Theta(\lambda^{\frac{d-1}{d+1}})$  for the last layers, i.e. when  $n$  is proportional to  $\lambda^{2/(d+1)}$ . This last prediction comes from both the conjecture (4.4) stated by Calder and Smart [22] and our own simulations. We would then wish to determine, if possible, the specific ‘time’  $n$  when the phase transition occurs and the behavior of the expectation of  $N_{n,0,\lambda}$  when  $n = f(\lambda)$  reaches all the intermediate regimes between  $f$  being constant and  $f(\lambda) = c\lambda^{2/(d+1)}$ . In fact, this would mean being able to describe  $N_{n,0,\lambda}$  as a random process depending on time  $n = f(\lambda)$ .
- *Depoissonization.* In this paper, the input set has been assumed to be a Poisson point process, i.e. the total number of i.i.d. uniform points in  $K$  is Poisson distributed, which involves notably nice independence properties and Mecke’s formula. When the input set is a binomial point process, i.e. the total number of points is deterministic, the formula for the limiting expectation of the number of  $k$ -faces of the convex hull is known to depoissonize [7]. We expect that Theorem 4.1.1 should depoissonize as well, even though there is no mention

in [27] of a binomial version of the limiting variance in the case of the convex hull. Methods of depossionization in the spirit of [25, Theorem 1.2] and [28, Theorem 1.1] could be relevant here.

- *Position of the subsequent layers.* Lemma 3.3.2 provides a rough estimate of the distribution tail of the maximal height of the  $n$ -th layer in the rescaled picture. We did not try to discuss how this height should depend on  $n$  even though we expect it to be asymptotically proportional to  $n$ , in the spirit of the results of Calder and Smart for the parabolic hull peeling [22, Section 2]. This would first require to improve Lemma 3.3.2 by taking  $t$  and the constant  $c$  therein depending on  $n$ .

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