



HAL
open science

Analyse de quelques inégalités fonctionnelles et équations aux dérivées partielles liées à de grands systèmes quantiques

Julien Sabin

► **To cite this version:**

Julien Sabin. Analyse de quelques inégalités fonctionnelles et équations aux dérivées partielles liées à de grands systèmes quantiques. Mathematical Physics [math-ph]. Université Paris-Saclay, 2022. tel-03920898

HAL Id: tel-03920898

<https://hal.science/tel-03920898v1>

Submitted on 3 Jan 2023

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Analyse de quelques inégalités fonctionnelles et équations aux dérivées partielles liées à de grands systèmes quantiques

**Habilitation à diriger des recherches
de l'Université Paris-Saclay**

présentée et soutenue à Orsay, le 25/10/22, par

Julien Sabin

Composition du jury

Maria Esteban Directrice de recherche au CNRS, Université Paris-Dauphine	Rapportrice
Patrick Gérard Professeur, Université Paris-Saclay	Rapporteur
Benjamin Schlein Professeur, Universität Zürich	Rapporteur
Volker Bach Professeur, Technische Universität Braunschweig	Examineur
Rémi Carles Directeur de recherche au CNRS, Université de Rennes I	Examineur
François Golse Professeur, Ecole polytechnique	Examineur
Luis Vega Professeur, Universidad del País Vasco & BCAM	Examineur

Résumé: Nous étudions plusieurs questions d'analyse provenant de la mécanique quantique à plusieurs corps. Plus précisément, nous considérons des équations aux dérivées partielles non-linéaires ainsi que des inégalités fonctionnelles décrivant un grand nombre de fermions dans une approximation de champ moyen. Cela mène à l'analyse mathématique des matrices densités à un corps dont la trace est grande ou infinie, avec un intérêt particulier pour la distribution spatiale de particules associée. Nous adaptons des outils d'EDPs dispersives, d'analyse de Fourier et semi-classique à ce cadre. Dans une première partie, nous établissons un lien entre plusieurs équations d'évolution qui décrivent de grands systèmes quantiques (relativistes et non-relativistes) dans certains régimes asymptotiques, en utilisant des méthodes de compacité ou de convergence forte. Dans une deuxième partie, nous développons des méthodes d'analyse harmonique qui impliquent des inégalités fonctionnelles sur des matrices densités que nous appelons *fermioniques*. Dans une troisième et dernière partie (sans lien avec la mécanique quantique à plusieurs corps), nous étudions l'existence de fonctions optimales pour des inégalités d'extension de la transformée de Fourier, à l'aide de méthodes de compacité.

Abstract: We study several topics in analysis which come from many-body quantum mechanics. More specifically, we consider nonlinear partial differential equations and functional inequalities that describe a large number of fermions in a mean-field approximation. This leads to the mathematical analysis of one-body density matrices with large or infinite trace, with a special emphasis on the associated spatial distribution of particles. We adapt tools from dispersive PDEs, Fourier and semiclassical analysis to this setting. In a first part, we relate several evolution equations describing large quantum systems (relativistic and non-relativistic) in various asymptotic regimes, employing both compactness methods or strong estimates. In a second part, we develop methods in harmonic analysis that imply functional inequalities on density matrices that we call *fermionic*. In a third and last part (unrelated to many-body quantum mechanics), we study the existence of optimizers for Fourier extension inequalities by compactness methods.

Liste de publications

ARTICLES PUBLIÉS

- [1] J. Sabin. Static Electron-Positron Pair Creation in Strong Fields for a Nonlinear Dirac Model. *Ann. Henri Poincaré* **14** (2013), no 5, pp. 1349–1382.
- [2] J. Sabin. Charge Renormalization and Static Electron-Positron Pair Production for a Nonlinear Dirac model with Weak Interactions. *SIAM J. Math. Anal.* **45** (2013), no 4, pp 2099–2147.
- [3] M. Lewin et J. Sabin. The Hartree equation for infinitely many particles. I. Well-posedness theory. *Comm. Math. Phys.* **334** (2015), no 1, pp 117–170.
- [4] M. Lewin et J. Sabin. The Hartree equation for infinitely many particles. II. Dispersion and scattering in 2D. *Analysis & PDE*, **7** (2014), no 6, pp 1339–1363.
- [5] M. Lewin et J. Sabin. A family of monotone quantum relative entropies. *Lett. Math. Phys.* **104** (2014), no 6, pp 691–705.
- [6] J. Sabin. Global well-posedness for a nonlinear wave equation coupled to the Dirac sea. *Appl. Math. Res. Express*, (2014), no 2, pp 312–331.
- [7] R. L. Frank et J. Sabin. Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates. *Amer. J. Math.* (2017), 139, no 6, 1649–1691.
- [8] J. Sabin. Littlewood-Paley decomposition of operator densities and application to a new proof of the Lieb-Thirring inequality. *Math. Phys. Anal. Geom.* (2016), no 2, 1–11.
- [9] R. L. Frank, E. H. Lieb, et J. Sabin. Maximizers for the Stein-Tomas inequality. *J. Geom. Funct. Anal.* (2016), no 26, 1095–1134.
- [10] R. L. Frank et J. Sabin. Spectral cluster bounds for orthonormal systems and oscillatory integral operators in Schatten spaces. *Advances in Mathematics* (2017), no 317, 157–192.
- [11] R. L. Frank et J. Sabin. Extremizers for the Airy-Strichartz inequality. *Math. Ann.* (2018), no 372, 1121–1166
- [12] M. Lewin et J. Sabin. The Hartree and Vlasov equations at positive density. *Comm. Partial Differential Equations*. (2020), no 45, 1702–1754.

PREPRINTS

- [13] R. L. Frank et J. Sabin. Sharp Weyl laws with singular potentials. arXiv:2007.04284.
- [14] J. Lampart, L. Le Treust, S. Rota Nodari et J. Sabin. The Dirac-Klein-Gordon system in the strong coupling limit. arXiv:2110.09087.

PROCEEDINGS

- [a] J. Sabin. The Hartree equation for infinite quantum systems. *Journées équations aux dérivées partielles* (2014), Exp. No. 8.
- [b] R.L. Frank et J. Sabin. The Stein-Tomas inequality in trace ideals. *Séminaire Laurent Schwartz* (2016), Exp. No. 15.

REVIEWS

- [a] J. Sabin. Compactness methods in Lieb’s work. arXiv:2203.09779.

Contents

Introduction (FR)	7
Introduction (ENG)	9
Remerciements	11
Chapter 1. Around some mean-field dynamics	13
1. From the Hartree equation to the Vlasov equation at positive density	14
2. From the Dirac-Klein-Gordon equation to the nonlinear Dirac equation	28
Appendix: The density of a non-negative operator	36
Chapter 2. Fermionic functional inequalities	39
1. Fourier restriction and Strichartz inequalities	42
2. Sogge's L^p spectral cluster bounds	50
3. Weyl laws for singular potentials	52
4. Littlewood-Paley decomposition for operator densities	58
Chapter 3. Optimizers for restriction inequalities	65
1. The Stein-Tomas inequality	66
2. The Airy-Strichartz inequality	71
Bibliography	75

Introduction (FR)

Cette thèse est consacrée à la présentation de quelques résultats d'équations aux dérivées partielles et d'inégalités fonctionnelles, motivés par la mécanique quantique à plusieurs corps. La mécanique quantique est une théorie physique décrivant la matière à l'échelle microscopique. Sa formulation repose sur la théorie spectrale et les espaces de Hilbert. Lorsque l'on étudie les propriétés statistiques des systèmes quantiques, on doit considérer un grand nombre d'objets quantiques individuels interagissant entre eux. Dans ce régime d'un grand nombre d'objets, d'autres modèles émergent comme limite rigoureuse ou comme approximation. Ces modèles peuvent impliquer d'autres objets mathématiques comme des fonctionnelles/équations aux dérivées partielles non-linéaires. On présente certains de ces modèles non-linéaires et on étudie leurs propriétés. On insistera en particulier sur certains outils développés pour comprendre ces grands systèmes quantiques dans quelques approximations non-linéaires. Cette famille d'outils comporte les inégalités fonctionnelles, auxquelles on consacre une bonne partie de ce manuscrit.

Dans le Chapitre 1, nous présentons quelques EDPs non-linéaires modélisant l'évolution en temps d'un grand nombre de particules en interaction. Elles partagent la propriété d'appartenir aux modèles dits de *champ moyen*, dans lesquels l'interaction entre les particules est simplifiée d'une telle manière que chaque particule n'interagit pas avec chaque autre particule individuellement, mais plutôt avec (le potentiel engendré par) la distribution moyenne des particules. Nous considérons deux modèles de champ moyen, le premier décrivant des particules non-relativistes dont la spécificité est qu'elles sont réparties de manière uniforme dans l'espace \mathbb{R}^d (ou sont une perturbation locale d'une telle distribution) et est appelé *équation de Hartree à densité positive*. Ce modèle a été introduit dans les articles [95, 94] et la contribution présentée ici est sa relation avec un modèle similaire décrivant des particules classiques, l'équation de Vlasov non-linéaire, dans un limite semi-classique. De manière intéressante, cette limite semi-classique permet de mieux comprendre l'équation classique. Le second modèle que nous présentons concerne la dynamique de particules relativistes et provient de la physique du noyau, dans laquelle il décrit l'interaction entre nucléons et mésons. Nous relierons une fois de plus deux équations utilisées dans ce contexte dans un régime asymptotique, les équations de Dirac-Klein-Gordon et de Dirac non-linéaire. Ces deux parties sont complémentaires dans le sens où elles illustrent deux approches déterminant l'asymptotique d'EDPs d'évolution: la première utilise des techniques de compacité et de convergence faible, alors que la seconde utilise des estimées forte et mène à la convergence en norme.

Dans le Chapitre 2, nous nous tournons vers le sujet de certaines inégalités fonctionnelles apparaissant lorsqu'on étudie des problèmes de champ moyen. Inspirés par les travaux

fondateurs de Lieb et Thirring, nous étudions plusieurs versions d'inégalités fonctionnelles connues (Strichartz, Stein-Tomas, Sogge) adaptées au contexte des systèmes quantiques de champ moyen. De manière informelle, les versions standard de ces inégalités correspondent au cas d'une seule particule, et on cherche à généraliser ces inégalités à plusieurs particules. Un point important est de quantifier précisément la dépendance de ces nouvelles estimées en le nombre de particules, idéalement de manière optimale. C'est particulièrement important lorsque l'on considère de grands systèmes, où cette dépendance optimale permet de traiter plus de cas. Nous insistons sur une méthode développée en collaboration avec Rupert Frank et basée sur de l'interpolation complexe dans certains espaces d'opérateurs, qui permet de déduire ces inégalités fonctionnelles 'fermioniques' à partir d'hypothèses assez générales. Nous appliquons cette méthode aux inégalités de Stein-Tomas, de Strichartz, de Sobolev uniforme, et de Sogge. Nous présentons également une application à la descriptions des valeurs propres non-réelles d'opérateurs de Schrödinger non auto-adjoints. Même si nous ne l'explicitons pas dans ce manuscrit, les inégalités de Strichartz fermioniques jouent également un rôle fondamental dans le caractère bien posé des EDPs étudiées dans la première partie. Ce chapitre contient également une section consacrée aux lois de Weyl ponctuelles et optimales, qui ne sont pas des inégalités fermioniques à proprement parler mais y sont fortement reliées et donnent une description de la distribution spatiale d'un nombre suffisamment grand de fermions libres.

Le dernier Chapitre 3 est tourné vers une question reliée à quelques estimées présentes dans la partie précédente, les estimées d'extension de la transformée de Fourier. Nous considérons la version standard de ces inégalités (pas la version fermionique), et nous présentons quelques contributions à la détermination de leurs constantes optimales. Notre point de vue est celui de la compacité, afin de comprendre le comportement des suites de fonctions approchant le cas d'égalité dans l'inégalité. Nous expliquons comment adapter les outils développés par Lieb et Lions dans les années 80 pour donner des conditions optimales pour que ces suites convergent, et par conséquent, pour qu'il existe des fonctions réalisant l'égalité dans l'inégalité.

Introduction (ENG)

This manuscript is devoted to presenting some results in the fields of partial differential equations and functional inequalities, with a motivation coming from many-body quantum mechanics. Quantum mechanics is a physical theory describing matter at the microscopic scale. Its formulation relies on Hilbert spaces and spectral theory. When studying the statistical properties of quantum systems, one has to consider a large number of interacting individual quantum objects. In this regime of a large number of objects, other models arise either a rigorous limit or as an approximation. These models may involve other mathematical theories like nonlinear functionals/partial differential equations. We present some of these nonlinear models and study their properties. A focus will also be made on some tools developed to understand these large quantum systems in some nonlinear approximations. A family of such tools is the one of functional inequalities, on which we devote a big part of this manuscript.

In Chapter 1, we present some nonlinear PDEs which model the time evolution of a large number of interacting particles. They share the common property of belonging to *mean-field* models, where the interaction between particles is simplified in a way that each particle does not interact with the individual other particles but rather with (the potential generated by) their average distribution. We consider two mean-field models, the first one describing non-relativistic particles with the specificity that they are uniformly distributed in the full space \mathbb{R}^d (or are a local perturbation of such a distribution) and usually referred to as the *Hartree equation at positive density*. This model was introduced in [95, 94] and the contribution presented in this thesis is its relation to a similar model describing classical particles, the nonlinear Vlasov equation, in a semi-classical limit. Interestingly, this semi-classical limit helps to understand the limiting model. The second model we present is concerned with the dynamics of relativistic particles and comes from nuclear physics where it describes the interaction between nucleons and mesons. We also relate two equations used in this field in a specific asymptotic regime, the Dirac-Klein-Gordon and nonlinear Dirac equations. These two parts are complementary in the sense that they illustrate two approaches to study asymptotics of evolutions PDEs: in the first one, we use compactness methods leading to weak convergence of solutions while in the second one we use strong estimates which lead to norm convergence.

In Chapter 2, we turn towards the topic of some functional inequalities which appear when studying mean-field problems. Following the fundamental work of Lieb and Thirring, we investigate several versions of known functional inequalities (Strichartz, Stein-Tomas, Sogge) adapted to the context of mean-field quantum systems. Loosely speaking, the standard version of these inequalities can be seen as describing a single particle, and one asks how to

generalize these inequalities to several particles. An important point is to quantify precisely how these new estimates depend on the number of particles involved, ideally in an optimal way. This is particularly important when considering large systems, where this optimal dependence allow to treat a wider class of infinite systems. We will emphasize a method that we devised with Rupert Frank, based on complex interpolation in some operator spaces, which allows to deduce these 'fermionic' functional inequalities from a general framework. We applied this method to the setting of Stein-Tomas, Strichartz, uniform Sobolev, and Sogge inequalities. We also present some applications such as understanding the behaviour of eigenvalues of non-self-adjoint Schrödinger operators. Even if we do not make it explicit in this manuscript, the fermionic Strichartz estimates also play a key role in the well-posedness theory of the PDEs studied in the first chapter. This chapter also contains a section on sharp pointwise Weyl laws, which are not fermionic inequalities but are both closely related to them and give a description on the spatial distribution of sufficiently large systems of non-interacting fermions.

The final Chapter 3 is devoted to a question related to some estimates mentioned in the previous chapter, the Fourier extension estimates. We consider the standard version of these inequalities, unrelated to many-body quantum mechanics, and present some contributions to the study of their optimal constants. Our point of view of is the one of compactness, understanding the behaviour of sequences of functions approaching the case of equality in the considered inequality. We explain how to adapt the tools developed by Lieb and Lions in the 80s to give sharp conditions for the convergence of these sequences which, as a byproduct, implies the existence of functions realizing the equality in the inequality.

Remerciements

Je voudrais tout d'abord remercier les rapporteurs de ce manuscrit pour leur travail de lecture et d'évaluation, ainsi que les membres du jury pour leur présence lors de la soutenance. Leur participation est un plaisir et un honneur pour moi.

J'apprends toujours énormément aux côtés de Mathieu et de Rupert, et je voudrais les remercier pour ces années d'échanges et de découvertes mathématiques communes. Merci aux amis et collègues de l'ANR, Simona, Jonas, et Loïc pour les aventures relativistes.

J'ai eu la chance de bénéficier de conditions de travail exceptionnelles et d'une ambiance scientifique stimulante que ce soit à Orsay, à l'Ecole polytechnique, ou maintenant à Rennes. Merci à tous les collègues qui y participent, et merci pour leur accueil bienveillant.

Enfin, merci à mes amis, à ma famille, à Mirza et à Marie pour leur support et leur présence.

Articles presented in this thesis

[1] J. SABIN, *Littlewood-Paley decomposition of operator densities and application to a new proof of the Lieb-Thirring inequality*, Math. Phys. Anal. Geom., 19 (2016), pp. Art. 11, 11.

[2] J. SABIN, *The Hartree equation for infinite quantum systems*, Journées Équations aux dérivées partielles, (2014), pp. 1–18.

[3] R. L. FRANK AND J. SABIN, *Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates*, Amer. J. Math., 139 (2017), pp. 1649–1691.

[4] R. L. FRANK AND J. SABIN, *The Stein-Tomas inequality in trace ideals*, Séminaire Laurent Schwartz—EDP et applications, (2016), pp. 1–12.

[5] R. L. FRANK AND J. SABIN, *Spectral cluster bounds for orthonormal systems and oscillatory integral operators in Schatten spaces*, Adv. Math., 317 (2017), pp. 157–192.

[6] R. L. FRANK, E. H. LIEB, AND J. SABIN, *Maximizers for the Stein-Tomas inequality*, Geom. Funct. Anal., 26 (2016), pp. 1095–1134.

[7] R. L. FRANK AND J. SABIN, *Extremizers for the Airy–Strichartz inequality*, Math. Ann., (2018), pp. 1–46.

[8] M. LEWIN AND J. SABIN, *The Hartree and Vlasov equations at positive density*, Comm. Partial Differential Equations, 45 (2020), pp. 1702–1754.

[9] R. L. FRANK AND J. SABIN, *Sharp Weyl laws with singular potentials*, arXiv preprint arXiv:2007.04284, (2020).

[10] J. LAMPART, L. L. TREUST, S. R. NODARI, AND J. SABIN, *The Dirac-Klein-Gordon system in the strong coupling limit*, arXiv preprint arXiv:2110.09087, (2021).

CHAPTER 1

Around some mean-field dynamics

This chapter is devoted to the study of some nonlinear evolution PDEs arising from mean-field many-body quantum mechanics, and in particular to their relation to other equations in some asymptotic regimes. In a first part, we explain the results of [96] concerning the relation between the Hartree and Vlasov equations in the semi-classical limit, around specific homogeneous backgrounds. In a second part, we turn to a problem from nuclear physics in which a convergence of solutions to a Dirac-Klein-Gordon equation to solutions to a nonlinear Dirac equation is obtained, in a strong coupling regime [90].

Mean-field models have an old history in situations involving interacting quantum particles. Basically, they amount to replace the full (or first principle) many-body interaction by a specific one-body interaction where each particle interacts with the classical field (the mean-field) generated by the density of the other (or more simply, all the) particles. Such models have a huge success from a computational/simulation perspective, since the full many-body problem, which is posed in very high dimension, is replaced by a problem posed in a fixed dimension. The price to pay is that the new Hamiltonian to diagonalize is non-linear (because the force field now depends on the position of the particles submitted to this force field). Still, many efficient methods to compute numerically the resulting spectrum or states are widely used [31, 108].

It is interesting to ask for a mathematical justification of this replacement of the true many-body interaction by the mean-field one. Historically, one of the first instances where it appeared was Hartree-Fock theory [55, 138], where it is shown that a mean-field model arises when the many-body wavefunction is assumed to be a Slater determinant (uniquely determined by a finite-rank orthogonal projection on the one-body Hilbert space). The Hartree-Fock theory has then been generalized to quantum states that are fully determined not by a projection, but by any self-adjoint operator between 0 and 1 called the one-body density matrix of the state (which is relevant for positive temperature systems) [100, 8]. A full presentation of Hartree-Fock theory can be found in [10, 141], including quantum states (called quasi-free states) which are characterized by a density matrix and also a pairing matrix (relevant in BCS theory). In all these theories, the complexity of the full many-body problem is reduced by assuming that the quantum state belongs to a subclass of states that are fully characterized by a less complex object (either a Slater determinant, a general density matrix, or a density matrix together with a pairing matrix).

The reduction to this particular subclass of states has been rigorously justified in some cases. For instance, it is known that the ground state energy of a large, molecular (quasi-neutral) Hamiltonian is given to leading order by its Hartree-Fock reduction [104, 98, 8]. A similar result holds in the theory of stellar collapse [107]. These results have been generalized

to include any reasonable many-body Hamiltonian, in a combined semi-classical and mean-field regime [58]. Notice that in this last work, it is also shown that the mean-field reduction is correct at the level of the energy but also at the level of states (in a weak sense). Similar results have been obtained concerning the dynamics of many-body systems: there, the question is a bit different since one cannot reasonably expect that any quantum trajectory can be approximated by a 'reduced' one (in the same way that one cannot expect that the energy of arbitrary excited states of the many-body Hamiltonian is related to a 'reduced' energy, the above works only show it for the low part of the spectrum). Hence, a more natural question is to assume that the quantum state belongs (or is close) to the 'mean-field' subclass at the initial time, and to ask whether this property is preserved for later times (similarly to the propagation of chaos in kinetic theory). There are many works showing this kind of result, in various asymptotic regimes [78, 117, 142, 113, 115, 11, 53, 4, 70, 18, 9, 123, 17, 74, 75, 51]. There, we only mentioned works related to fermionic systems since these are relevant to what we detail below. The bosonic case has been also extensively treated, but we will not try to list the corresponding results.

In the following chapters, quantum states of several fermionic particles will thus be described by a *one-body density matrix*, which is a self-adjoint operator γ on the underlying one-body Hilbert space (typically $L^2(\mathbb{R}^d)$ for a particle moving in \mathbb{R}^d) satisfying $0 \leq \gamma \leq 1$. The constraint $\gamma \leq 1$ is typical to fermions and reflects Pauli's exclusion principle. The way to associate to γ a unique quantum (quasi-free) state is recalled in [126, Chap. 1, App. C]. For instance, if γ is a projection on the space generated by normalized vectors u_1, \dots, u_N , the quantum state associated to γ is just the Slater determinant $u_1 \wedge \dots \wedge u_N$. For such an operator γ , the number $\text{Tr } \gamma \in [0, \infty]$ is interpreted as the number of particles of the system. As we will see below, the density matrix formulation is particularly useful to treat states with infinitely many particles, since these states are merely described by density matrices γ such that $\text{Tr } \gamma = +\infty$ (natural examples including translation-invariant operators, see below). At the full quantum level, such states are not as easily described (for instance, one would need to define a Slater determinant with infinitely many factors). An important quantity associated to a one-body density matrix γ is its *density* $\rho_\gamma : \mathbb{R}^d \rightarrow \mathbb{R}$ which represents its spatial distribution of particles and is formally defined by $\rho_\gamma(x) = \gamma(x, x)$ for all $x \in \mathbb{R}^d$, where $\gamma(\cdot, \cdot)$ is the integral kernel of γ . We recall its precise definition in an appendix.

1. From the Hartree equation to the Vlasov equation at positive density

We begin with presenting a result about the Hartree equation in a semi-classical regime

$$\begin{cases} i\hbar\partial_t\gamma = [-\hbar^2\Delta + \hbar^d w * \rho_\gamma, \gamma], \\ \gamma|_{t=0} = \gamma_0, \end{cases} \quad (1.1)$$

for which the unknown $\gamma = \gamma(t)$ at time $t \in \mathbb{R}$ is a non-negative, bounded operator on $L^2(\mathbb{R}^d)$, $w : \mathbb{R}^d \rightarrow \mathbb{R}$ is the interaction potential, and $\hbar > 0$ is a parameter which plays the role of Planck's constant but we will later study what happens to solutions as $\hbar \rightarrow 0$. The small coefficient \hbar^d in front of the interaction is typical of the mean-field regime [53, 18, 17]. Recall that ρ_γ denotes the density of γ as defined in Proposition 1.5. The equation (1.1) is

non-linear since the Hamiltonian $-\hbar^2\Delta + \hbar^d w * \rho_\gamma$ depends on the solution itself. In this formulation, the Hilbert space $L^2(\mathbb{R}^d)$ is independent of the number of particles $N = \text{Tr } \gamma$, which is not the case in full many-body quantum mechanics where the Hilbert space $L^2(\mathbb{R}^{dN})$ depends on the number of particles.

We study the dynamics of (1.1) in a neighborhood of a specific family of stationary states. Namely, for any bounded and integrable $g : \mathbb{R}^d \rightarrow \mathbb{R}_+$, the density matrix

$$\gamma_{\text{ref}} = g(-i\hbar\nabla) \tag{1.2}$$

is a time-independent solution to 1.1 with a constant density

$$\rho_{\gamma_{\text{ref}}} \equiv \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(v) dv.$$

Indeed, it obviously commutes with $-\hbar^2\Delta$ since they are both Fourier multipliers while the non-linear potential $w * \rho_{\gamma_{\text{ref}}}$ is constant (since $\rho_{\gamma_{\text{ref}}}$ is constant), if $w \in L^1(\mathbb{R}^d)$, so that it commutes with any operator. Such solutions describe fluid/gaseous phases of matter since the spatial distribution of particles $\rho_{\gamma_{\text{ref}}}$ is homogeneous. Physical examples of such states include

- the Fermi gas at zero temperature and chemical potential $\mu > 0$,

$$g(v) = \mathbf{1}(|v|^2 \leq \mu),$$

- the Fermi gas at temperature $T > 0$ and chemical potential $\mu \in \mathbb{R}$,

$$g(v) = \frac{1}{e^{(|v|^2 - \mu)/T} + 1},$$

- the Bose gas at temperature $T > 0$ and chemical potential $\mu < 0$,

$$g(v) = \frac{1}{e^{(|v|^2 - \mu)/T} - 1},$$

- the Boltzmann gas at temperature $T > 0$ and chemical potential $\mu \in \mathbb{R}$,

$$g(v) = e^{-(|v|^2 - \mu)/T}.$$

All these physical examples have a variational interpretation which will be very important for our purposes. For instance, the Fermi gas at zero temperature is a formal minimizer of the (kinetic energy) functional

$$\gamma \mapsto \text{Tr}(-\hbar^2\Delta - \mu)\gamma$$

when γ varies among all the self-adjoint operators with spectrum in $[0, 1]$. Notice that in our context, it makes sense to assume that $0 \leq \gamma \leq 1$ since this condition is preserved along the flow of (1.1) for reasonable solutions [95, 154]. Similarly, the Fermi, Bose, and Boltzmann gases at positive temperature are formal minimizers of the (free energy) functional

$$\gamma \mapsto \text{Tr}(-\hbar^2\Delta - \mu)\gamma - T \text{Tr } S(\gamma),$$

where S is an entropy functional adapted to each case, more precisely

$$S(x) = \begin{cases} -x \log x - (1-x) \log(1-x) & \text{(Fermi gas),} \\ -x \log x + (1+x) \log(1+x) & \text{(Bose gas),} \\ -x \log + x & \text{(Boltzmann gas).} \end{cases}$$

We only talked about 'formal' minimizers because their energy is actually $-\infty$. Below, we will detail the correct interpretation of the energy in order to ensure that this variational interpretation is well-defined. This variational property will be key to study the limit $\hbar \rightarrow 0$ of solutions to (1.1). We will also explain why these entropy functionals S are special and how more general S may be considered.

The particular solutions γ_{ref} all satisfy $\text{Tr } \gamma_{\text{ref}} = +\infty$ (as well as having infinite energy, as we mentioned above) and hence describe an infinite number of particles. From a mathematical point of view, this greatly complicates the analysis of the local and global well-posedness of the equation, as we will detail below.

We also consider another mean-field model for classical interacting particles, called the Vlasov equation

$$\begin{cases} \partial_t m + 2v \cdot \nabla_x m - \nabla_x (w * \rho_m) \cdot \nabla_v m = 0, \\ m|_{t=0} = m_0. \end{cases} \quad (1.3)$$

There, the unknown $m = m(t)$ at time $t \in \mathbb{R}$ is a bounded, non-negative function on the phase-space $\mathbb{R}_x^d \times \mathbb{R}_v^d$ which describes where the classical particles are distributed in position and velocity. We also denoted by ρ_m the marginal giving only the position distribution of the particles,

$$\rho_m(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} m(x, v) dv, \quad x \in \mathbb{R}^d$$

which is the analogue of the density ρ_γ in the quantum case. Similarly to the Hartree equation, the particular choices

$$m_{\text{ref}}(x, v) = g(v), \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,$$

for the same class of g as above, are all stationary solutions to (1.3) describing a spatially homogeneous distribution of particles. The physical examples of g above are all of interest, and they have the same variational interpretation for the classical versions of the energies listed above; that is they are formal minimizers of the functionals

$$m \mapsto \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[(|v|^2 - \mu)m(x, v) - TS(m(x, v)) \right] dx dv.$$

Again, they are only 'formal' minimizers since their energy is actually $-\infty$. We will see below how to set this property on rigorous grounds. Notice that they also satisfy

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_{\text{ref}}(x, v) dx dv = +\infty,$$

meaning again that they describe an infinite number of particles.

The Vlasov equation (1.3) is the classical analogue of the Hartree (1.1) in the sense that they both describe a system of particles interacting via a mean-field potential. The only

difference is that in the Vlasov equation, these particles are classical while in the Hartree equation, the particles are quantum. A relation between the two equations can be made more precise in the semi-classical limit $\hbar \rightarrow 0$. The fact that the Vlasov equation is in some sense the 'limit' of the Hartree equation as $\hbar \rightarrow 0$ has been understood for a long time (basically, from the origins of quantum mechanics as illustrated by the work of Wigner [153] for instance). In this chapter, we explain the results of [96] where we show that this semi-classical limit still holds in the neighborhood of the special stationary solutions described above. This can be viewed as an extension of the results of [113], which treats the case of a finite number of particles. We also explain how to use this semi-classical limit to prove a global well-posedness result for the Vlasov equation around these stationary states, a result that seemed to be unknown. In this regard, one can thus interpret the Hartree equation as a form of regularization of the Vlasov equation.

1.1. Local well-posedness. Let us first comment quickly on the local well-posedness properties of the Hartree equation around translation-invariant stationary states. Since this question is independent of the semi-classical limit (in other words, $\hbar > 0$ is fixed for this question), we set $\hbar = 1$ for simplicity here. To discuss it, it is first useful to consider the question without background (i.e. $\gamma_{\text{ref}} = 0$). In this context, the well-posedness theory for the Hartree equation has been developed in [26, 27, 43, 155] where the natural regularity of operators considered is the trace-class \mathfrak{S}^1 . This class has nice properties with respect to the equation: i) it is invariant by the linear part $i\partial_t\gamma = [-\Delta, \gamma]$, ii) the non-linear part also stabilizes this class under mild assumption on the mean-field potential (because $[V, \gamma] \in \mathfrak{S}^1$ if $\gamma \in \mathfrak{S}^1$ and $V \in L^\infty$ for instance), and iii) it allows to define and control the density easily (because $\rho_\gamma \in L^1$ if $\gamma \in \mathfrak{S}^1$). From all these properties, it is not hard to show that the Hartree equation is locally well-posed in \mathfrak{S}^1 if $w \in L^\infty$. Interestingly, one can also show that such solutions are global since the \mathfrak{S}^1 -norm is preserved along the flow. Another natural quantity which is conserved along the flow is the energy (if the interaction potential w is even),

$$\mathcal{E}(\gamma) := \text{Tr}(-\Delta)\gamma + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_\gamma(x)w(x-y)\rho_\gamma(y) dy.$$

One can also show that the Hartree equation is well-posed in a space where the kinetic energy is well-defined, that is $(1 - \Delta)\gamma \in \mathfrak{S}^1$, and that the energy is indeed conserved for such solutions. Notice that these solutions are also trace-class, so we already know that they are global by the preceding argument. It would be interesting to study the equation in energy space for which $(-\Delta)\gamma \in \mathfrak{S}^1$ (notice that such an operator is not necessarily trace-class; it can even be non-compact due to the spectrum of $-\Delta$ close to zero). To our knowledge, this question has not been treated anywhere. Notice that in this case, the density can be controlled due to the Lieb-Thirring inequality [105, 106]

$$\int_{\mathbb{R}^d} \rho_\gamma(x)^{1+2/d} dx \leq C \text{Tr}(-\Delta)\gamma,$$

valid for any operator γ such that $0 \leq \gamma \leq 1$. Local well-posedness in spaces unrelated to conserved quantities is also an interesting question; a natural scale of spaces which satisfy

the properties i) ii) iii) listed above are those such that $(1 - \Delta)^{s/2}\gamma \in \mathfrak{S}^\alpha$. While these spaces are clearly invariant by the free evolution, they are also nice with respect to the control of the density if $s > d/\alpha'$ (indeed, one can show that $\rho_\gamma \in L^\alpha$ in this case; see [95, Lemma 1]). Then, it can be shown that these spaces are also invariant by the nonlinear evolution under suitable assumptions on w (which ensure that $w * \rho_\gamma$ sends H^s to H^s ; since $\rho_\gamma \in L^\alpha$ this amounts to have $w \in W^{s, \alpha'}$). Interestingly, all these arguments can be adapted to the Vlasov equation, the analogue condition being that $(1 + |v|^2)^s m \in L^\alpha(\mathbb{R}^d \times \mathbb{R}^d)$ (which is still invariant by the free evolution and controls the classical density). The conditions on w are different however, due to the fact that one wants to define the Newton dynamics associated to the mean-field potential $w * \rho_m$.

Let us now consider a non-trivial stationary state $\gamma_{\text{ref}} = g(-i\nabla)$. One wants to know if the dynamics of (1.1) is well-posed in a neighborhood of γ_{ref} , at least locally in time. At initial time ($t = 0$, say), we thus set $\gamma_{|t=0} = \gamma_{\text{ref}} + Q_0$ for some perturbation operator Q_0 , and it is natural to study the dynamics of the perturbation $Q(t) := \gamma(t) - \gamma_{\text{ref}}$ which satisfies the equation

$$i\partial_t Q = [-\Delta + w * \rho_Q, Q] + [w * \rho_Q, \gamma_{\text{ref}}]. \quad (1.4)$$

This equation shares a similar structure with the Hartree equation, the only difference being the (linear) driving term $[w * \rho_Q, \gamma_{\text{ref}}]$. Compared to the above discussion of local well-posedness, we already see that this term introduces some new features. For instance, if one wants to study the well-posedness of (1.4) in the trace-class, a natural question is to ask under which assumptions on w, g does the term $[w * \rho_Q, \gamma_{\text{ref}}]$ is also trace-class. An answer can be provided by the Birman-Solomjak inequality [133, Theorem 4.5], which states that $f(x)g(-i\nabla)$ belongs to the trace-class if $f, g \in \ell^1 L^2$ (that is, $\|f\|_{L^2(z+[0,1]^d)} \in \ell^1(\mathbb{Z}^d)$). This can be shown to hold under suitable assumptions on w . Our point here is that one needs more assumptions on w than in the case $\gamma_{\text{ref}} = 0$ (in which $w * \rho_Q \in L^\infty$ was sufficient, while here we need $w * \rho_Q \in \ell^1 L^2$ as well). In this respect, the regularity $(1 - \Delta)^{s/2}Q \in \mathfrak{S}^\alpha$ is a bit nicer when $\alpha \geq 2$, due to the Kato-Seiler-Simon inequality [133, Theorem 4.1] which states that $f(x)g(-i\nabla)$ belongs to \mathfrak{S}^α if $f, g \in L^\alpha$. The setting with $\alpha \geq 2$ thus involves L^α -norms, contrary to $\ell^\alpha L^2$ -norms for $1 \leq \alpha < 2$. With these kind of ideas, one can prove the local well-posedness of the (1.4) under suitable assumptions on w, g , for Q satisfying $(1 - \Delta)^{s/2}Q \in \mathfrak{S}^\alpha$ with $s > d/\alpha'$ [95, Theorem 3].

The main difference between the cases $\gamma_{\text{ref}} = 0$ and $\gamma_{\text{ref}} \neq 0$ are related to conserved quantities; indeed, the conserved quantities mentioned in the case $\gamma_{\text{ref}} = 0$ (namely, the trace norm and the energy) are both infinite for $\gamma(t) = \gamma_{\text{ref}} + Q(t)$ with $Q(t)$ in the classes of well-posedness. There are no general analogue of these quantities for the equation (1.4) on the perturbation, so that it looks impossible to globalize solutions by using conserved coercive quantities in this case. However, we explain below that for a special class on backgrounds γ_{ref} (namely, the ones which has the variational interpretation mentioned above), one can still build a notion of *relative* energy which can be used as a conserved coercive quantity to globalize solutions. The main difficulty associated to these notions of relative energy are the fact that they live at a very low regularity scale (essentially, $(1 - \Delta)^{1/2}Q \in \mathfrak{S}^2$) which are too weak to control the density (since the condition $s > d/\alpha'$ is not satisfied, except for

$d = 1$) and fall outside of the local well-posedness theory mentioned above. In $d \geq 2$, one thus has to develop a local well-posedness theory at low regularity for which the dispersive properties of the equation (in the form of Strichartz estimates) are key. Such low regularity well-posedness results were obtained for $d = 2, 3$ in [95, Theorem 5 & 6]. The restriction $d \leq 3$ comes from the fact that the control in the relative energy space of $(1 - \Delta)^{1/2}Q \in \mathfrak{S}^2$ becomes too weak in large dimensions (the number of “derivatives” of Q that we control being fixed).

1.2. Relative energies and global well-posedness. Let us now explain the concept of relative energy and why it can be used to globalize solutions to (1.4), as introduced in [95]. As we mentioned, the main problem is that the standard conserved quantities like the energy $\mathcal{E}(\gamma)$ are infinite for the class of considered solutions. There are two ideas behind the concept of relative energies: i) use only relative quantities, that is measure everything with respect to the background γ_{ref} and ii) use a suitable combination of conserved quantities to obtain something coercive. A simple illustration of point i) is for instance to replace the trace $\text{Tr } \gamma$ (which is a conserved quantity for (1.1) but infinite for our class of solutions) by the relative object “ $\text{Tr } \gamma - \text{Tr } \gamma_{\text{ref}}$ ” which, written in this way, does not make sense since it is a difference of two $+\infty$. However, the rigorous version of this object is clear; namely it is $\text{Tr}(\gamma - \gamma_{\text{ref}}) = \text{Tr } Q$ which makes perfect sense for trace-class solutions of (1.4) and is also conserved along the flow since it is the difference between the conserved quantity $\text{Tr } \gamma$ and the “constant” (thus also conserved) $\text{Tr } \gamma_{\text{ref}}$. One can apply the same strategy to the energy, to consider the relative energy “ $\mathcal{E}(\gamma) - \mathcal{E}(\gamma_{\text{ref}})$ ” which is also formally conserved. It is also not hard to see that what should be the rigorous version of this quantity, because one formally has

$$\begin{aligned} \mathcal{E}(\gamma) - \mathcal{E}(\gamma_{\text{ref}}) &= \text{Tr}(-\Delta)(\gamma - \gamma_{\text{ref}}) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_{\gamma - \gamma_{\text{ref}}}(x) w(x - y) \rho_{\gamma - \gamma_{\text{ref}}}(y) dy \\ &\quad + \rho_{\gamma_{\text{ref}}} \left(\int w \right) \text{Tr } \gamma, \end{aligned}$$

and the sum of the first two term (the last term being formally conserved) is a good candidate for the relative energy since only the relative operator $\gamma - \gamma_{\text{ref}}$ appears in this expression. It turns out that this construction (shifted again by a suitable multiple of $\text{Tr } \gamma$) is enough to obtain a coercive quantity in the special case of the Fermi gas at zero temperature. However, we will not detail it here since this case is not relevant for the semi-classical limit. In the positive temperature case, this functional is in general not coercive, essentially due to the fact that the kinetic energy term $\text{Tr}(-\Delta)(\gamma - \gamma_{\text{ref}})$ has no sign. The key idea is to replace the kinetic energy $\text{Tr}(-\Delta)\gamma$ by the *free energy* $\text{Tr}(-\Delta)\gamma - \text{Tr } S(\gamma)$ for some suitable function $S : [0, 1] \rightarrow \mathbb{R}$. Notice that the quantity $\text{Tr } S(\gamma)$ is also formally conserved along the flow of (1.1) since the spectrum of γ is invariant. Applying the philosophy of i), we obtain the *relative entropy* functional

$$\begin{aligned} \mathcal{H}_S(\gamma, \gamma_{\text{ref}}) &= \text{Tr}(-\Delta)(\gamma - \gamma_{\text{ref}}) - \text{Tr } S(\gamma) + \text{Tr } S(\gamma_{\text{ref}}) \\ &= - \text{Tr} [S(\gamma) - S(\gamma_{\text{ref}}) - (-\Delta)(\gamma - \gamma_{\text{ref}})] \end{aligned}$$

and an easy condition to obtain the non-negativity of $\mathcal{H}_S(\gamma, \gamma_{\text{ref}})$ is the *concavity* of the function S as well as the relation

$$-\Delta = S'(\gamma_{\text{ref}}) \iff \gamma_{\text{ref}} = (S')^{-1}(-\Delta).$$

Defining properly this notion of relative entropy is not an obvious task since the operator under the trace has no sign in general (even if the function $(x, y) \mapsto S(x) - S(y) - S'(y)(x - y)$ is non-negative). In [93], we proved that one could define $\mathcal{H}_S(\gamma, \gamma_{\text{ref}})$ for general $0 \leq \gamma \leq 1$ if the function $-S$ is operator monotone, a condition which ensures (and actually is equivalent to) the monotonicity of relative entropy (meaning that $\mathcal{H}_S(A, B) \geq \mathcal{H}_S(PAP, PBP)$ for all $0 \leq P \leq 1$). This condition is natural if one wants to reduce to finite dimensions where the relative entropy is much easier to define. The condition that $-S$ is operator monotone is quite stringent (there is a representation theorem of all such functions), but all the physical examples (Fermi-Dirac, Bose-Einstein, Boltzmann) satisfy this property. Once the relative entropy is defined, one can show using the strict concavity of S that it indeed controls some norm on $Q = \gamma - \gamma_{\text{ref}}$, namely

$$\mathcal{H}_S(\gamma, \gamma_{\text{ref}}) \geq C \text{Tr}(1 - \Delta)Q^2, \quad (1.5)$$

as we mentioned above. It also leads to the notion of *relative free energy*

$$\mathcal{F}(\gamma, \gamma_{\text{ref}}) := \mathcal{H}_S(\gamma, \gamma_{\text{ref}}) + \frac{1}{2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \rho_{\gamma - \gamma_{\text{ref}}}(x) w(x - y) \rho_{\gamma - \gamma_{\text{ref}}}(y) dy,$$

which is formally conserved along the flow and coercive (if for instance $\widehat{w} \geq 0$, so that the second term is non-negative). Let us mention that the relative entropy also controls the density in the spirit of the Lieb-Thirring inequality stated above: following the zero-temperature work [61], we proved a Lieb-Thirring inequality for the relative entropy [95, Theorem 7] which states that $\mathcal{H}_S(\gamma, \gamma_{\text{ref}})$ controls the $(L^2 + L^{\min(1+2/d, 2)})$ -norm of $\rho_{\gamma - \gamma_{\text{ref}}}$ (under some integrability conditions on S' needed to ensure that $\rho_{\gamma_{\text{ref}}}$ is well-defined). To sum up, the relative free energy is well-defined in the space

$$\mathcal{K}_S := \{0 \leq \gamma \leq 1 : \mathcal{H}_S(\gamma, \gamma_{\text{ref}}) < +\infty\}, \quad \gamma_{\text{ref}} = (S')^{-1}(-\Delta),$$

which is the natural space in which to study the global well-posedness of (1.1). In [95, Theorem 9], we indeed proved that, for $d = 1, 2, 3$, the Hartree equation is globally well-posed in \mathcal{K}_S with a conserved relative free energy, for a general class of entropy functionals S .

1.3. Semi-classical limit. The semi-classical limit $\hbar \rightarrow 0$ of the Hartree equation (1.1) around the stationary states γ_{ref} is based on the global well-posedness results of the preceding section. Indeed, a first advantage of global solutions is that their maximal interval of existence is independent of \hbar (while for general solutions it may depend badly on \hbar). Another nice property of global solutions is the conservation of the coercive relative free energy which provides bounds on the solution which are uniform in \hbar .

Before going into the details of the semi-classical limit, let us recall the mechanism behind it. First, let us explain how to relate solutions of the Hartree and Vlasov equations (after all, they are objects of a different nature). From a one-body density matrix γ , one can define its

Wigner transform W_γ^h for any $\hbar \rightarrow 0$. This is a function defined on the phase-space $\mathbb{R}^d \times \mathbb{R}^d$, by the formula

$$\forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d, W_\gamma^h(x, v) = \int_{\mathbb{R}^d} \gamma(x + y/2, x - y/2) e^{-iv \cdot y/\hbar} dy.$$

In many references, the Wigner transform is defined with an additional factor $(2\pi\hbar)^{-d}$. Our choice of normalization is made so that we have $W_{\gamma_{\text{ref}}}^h(x, v) = g(v)$. Another way to view the Wigner transform is via duality, because it satisfies for all $\varphi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$

$$\langle W_\gamma^h, \varphi \rangle = (2\pi\hbar)^d \text{Tr } \gamma \text{Op}^h(\varphi), \quad (1.6)$$

where $\text{Op}^h(\varphi)$ denotes the Weyl quantization of φ [157]. As its name suggests, this transform was invented by Wigner [153] exactly for the purpose of linking quantum and classical mechanics. Indeed, the relation (1.6) means that the classical 'state' W_γ^h and the quantum state γ have the same expectation (up to the factor $(2\pi\hbar)^d$) against observables modeled by the classical symbol φ (recall that $\text{Op}^h(\varphi(x)) = \varphi(x)$ and $\text{Op}^h(\varphi(v)) = \varphi(-i\hbar\nabla)$). To capture all these expectations into a classical state is a powerful property, which is counter-balanced by the fact that W_γ^h is not exactly a classical state simply because it is not non-negative in general. However, as we will recall below, the limit of W_γ^h as $\hbar \rightarrow 0$ is non-negative (formally, this can be seen from the expression of $W_\gamma^h(x, v) \sim_{\hbar \rightarrow 0} (2\pi\hbar)^d \gamma(x, x) \delta_{v=0} \geq 0$). Another important property of the Wigner transform which is particularly relevant in our context is the fact that if γ is a solution to the free Schrödinger equation $i\hbar\partial_t\gamma = [-\hbar^2\Delta, \gamma]$, then its Wigner transform is a solution to the free transport equation $\partial_t W_\gamma^h + 2v \cdot \nabla_x W_\gamma^h = 0$ for each $\hbar > 0$. From this point of view, quantum and classical dynamics can be related at *every* $\hbar > 0$! The usefulness of the semi-classical limit $\hbar \rightarrow 0$ is apparent when looking at more complicated dynamics, for instance adding a potential $V : \mathbb{R}^d \rightarrow \mathbb{R}$. Indeed, if γ is a solution to the equation $i\hbar\partial_t\gamma = [-\hbar^2\Delta + V, \gamma]$ and if $W_\gamma^h \rightarrow W$ when $\hbar \rightarrow 0$ in a suitable sense (which will be made more explicit below), it can be shown that W solves the (linear) Vlasov equation $\partial_t W + 2v \cdot \nabla_x W - \nabla_x V \cdot \nabla_v W = 0$. This result has been put on solid mathematical ground for instance by Lions and Paul [113], or it can be seen as a consequence of Egorov's theorem [157, Theorem 11.1] in some cases.

In their article, Lions and Paul also treat the non-linear case where the potential V depends on the solution in a mean-field way like $V = w * \rho_\gamma$ (this case has simultaneously been treated in [115]). Since their argument is the basis of our analysis in [96], we provide some details on the ideas behind it. The first step is to obtain a limit for the Wigner transform $W_{\gamma(t)}^h$ as $\hbar \rightarrow 0$, for any $t \in \mathbb{R}$. A simple way to find a limit is through compactness and uniform bounds on $W_{\gamma(t)}^h$ in \hbar . Such bounds can follow for instance from the identity (which follows from Plancherel's theorem)

$$\|W_\gamma^h\|_{L_{x,v}^2}^2 = (2\pi\hbar)^d \|\gamma\|_{\mathfrak{S}^2}^2$$

which relate in a simple way a norm on the Wigner transform and a norm of γ . For the Hartree equation (1.1), the norm $\|\gamma\|_{\mathfrak{S}^2}$ is invariant so that if $(2\pi\hbar)^d \|\gamma\|_{\mathfrak{S}^2}^2$ is bounded in \hbar at initial time, it is bounded for all times. By compactness, we obtain in this way a sequence (h_n) converging to 0 and $W \in L_{x,v}^2$ such that $W_{\gamma}^{h_n} \rightarrow W$ weakly in $L_{x,v}^2$ as $n \rightarrow \infty$

(technically, the sequence (h_n) obtained in this could depend on t ; a way to obtain a single sequence is to use compactness in $L_t^\infty L_{x,v}^2$ instead). The weak convergence of $(W_\gamma^{h_n})$ is a priori not enough to take the limit in the equation satisfied by W_γ^h (this equation is called the *Wigner equation*), due to the nonlinear term involving the density ρ_γ . One has to first extract a weak limit on ρ_γ itself, which can be done for instance using the Lieb-Thirring inequality which in its \hbar -dependent version reads

$$\int_{\mathbb{R}^d} (\hbar^d \rho_\gamma(x))^{1+2/d} dx \leq C \hbar^d \text{Tr}(-\hbar^2 \Delta) \gamma.$$

Notice that the right side can be shown to be uniformly bounded in t and \hbar because it is controlled by (the \hbar -dependent version of) the energy $\mathcal{E}(\gamma)$, so that if the energy is bounded in \hbar at initial time, one gets a uniform bound on $\rho_{W_\gamma^h} = \hbar^d \rho_\gamma$ in $L_t^\infty L_x^{1+2/d}$. From this uniform bound, one obtains a weak-* limit ν of $\rho_{W_\gamma^h}$ in $L_t^\infty L_x^{1+2/d}$, up to a subsequence of (h_n) that we still denote by (h_n) . It seems that a weak limit on the couple $(W_\gamma^h, \rho_{W_\gamma^h})$ is not enough to take a limit in the nonlinearity (which is a bilinear expression in (γ, ρ_γ)) since weak limits behave badly under products. However, the mean-field potential $V_h := \hbar^d w * \rho_\gamma = w * \rho_{W_\gamma^h}$ is better behaved if w is nice enough (here, it means $\nabla w \in L^{1+d/2}$), so that $V_{h_n} \rightarrow w * \nu$ *strongly* in $L_{t,x}^\infty$. This strong convergence is enough to take the limit of the Wigner equation, to find that (W, ν) satisfy the 'decoupled' Vlasov equation

$$\partial_t W + 2v \cdot \nabla_x W - \nabla_x(w * \nu) \cdot \nabla_v W = 0. \quad (1.7)$$

Here, we strongly used the regularity of w and let us mention that [113, 115] go beyond this simple assumption to treat less regular, more realistic potentials (like the Coulomb one for instance). Let us also mention that the assumptions on the initial data ($\|\gamma\|_{\mathfrak{S}^2}^2 + \text{Tr}(-\hbar^2 \Delta) \gamma \leq C \hbar^{-d}$) are not void and are satisfied for instance for $\gamma = \text{Op}^h(\varphi)$ for any $\varphi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. For this last example, we have $W_\gamma^h = \varphi$.

From these elementary arguments, it remains to understand why we have $\nu = \rho_W$ in (1.7), where we recall that ν is the weak limit of $\rho_{W_\gamma^{h_n}}$ while ρ_W is the density of W , the weak limit of $W_\gamma^{h_n}$. Hence, we have to prove that the weak limit of the density is the density of the weak limit. This property is not automatic, since ν is characterized by the limit

$$\begin{aligned} \int_{\mathbb{R}^d} \nu(x) \psi(x) dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} \rho_{W_\gamma^{h_n}}(x) \psi(x) dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W_\gamma^{h_n}(x, v) dv \psi(x) dx, \end{aligned}$$

for any test function $\psi \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^d)$. There, the weak limit of $W_\gamma^{h_n}$ cannot be used because the function $(x, v) \mapsto \psi(x)$ is not a test function for $W_\gamma^{h_n}$: it has no decay when $|v| \rightarrow \infty$. Notice that the computation above is a bit formal since there is no reason that $W_\gamma^{h_n}$ should be integrable in v to begin with. The way to make it more rigorous is to use the *Husimi transform* of γ , defined by

$$m_\gamma^h(x, v) := \langle \chi_{x,v}^h, \gamma \chi_{x,v}^h \rangle, \quad (x, v) \in \mathbb{R}^d \times \mathbb{R}^d,$$

where $\chi^{\hbar}x, v(z) := \chi^{\hbar}(z - x)e^{iv \cdot z/\hbar}$, $\chi^{\hbar}z := \hbar^{-d/4}\chi(\hbar^{-1/2}z)$ for all $z \in \mathbb{R}^d$ (for some fixed $\chi \in L^2(\mathbb{R}^d, \mathbb{R})$ with $\int \chi^2 = 1$). The states $(\chi_{x,v}^{\hbar})_{(x,v) \in \mathbb{R}^d \times \mathbb{R}^d}$ are called *coherent states*. This is another way to create a classical object from the quantum state γ , the main advantage compared to the Wigner transform is that $m_{\gamma}^{\hbar} \geq 0$ for all $\hbar > 0$ (since $\gamma \geq 0$), meaning that they are 'true' classical states. Now it can be shown that we also have $m_{\gamma}^{\hbar_n} \rightarrow W$ and $\rho_{m_{\gamma}^{\hbar_n}} \rightarrow \nu$ as $n \rightarrow \infty$ (in the sense of distributions for instance). This proves in particular that $W \geq 0$. For the Husimi transform, the above computation is rigorous for $\psi \in C_c^{\infty}(\mathbb{R}^d)$ with $\psi \geq 0$ since it only involves non-negative functions. Hence, we indeed have

$$\int_{\mathbb{R}^d} \nu(x)\psi(x) dx = \lim_{n \rightarrow \infty} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_{\gamma}^{\hbar_n}(x, v)\psi(x) dx dv.$$

A sufficient condition to take the limit of the right side is then the following *tightness* property of the sequence $(m_{\gamma}^{\hbar_n})$:

$$\lim_{R \rightarrow +\infty} \limsup_{n \rightarrow \infty} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}(|v| \geq R) m_{\gamma}^{\hbar_n}(x, v) dx dv = 0. \quad (1.8)$$

Indeed, assuming this property, if we introduce $f \in C_c^{\infty}(\mathbb{R}^d)$ such that f is radially decreasing, $f \geq 0$, $f \equiv 1$ on $B(0, 1)$, then one can estimate

$$\mathcal{R}_f(n, R) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (1 - f(v/R)) m_{\gamma}^{\hbar_n}(x, v) dx dv \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbf{1}(|v| \geq R) m_{\gamma}^{\hbar_n}(x, v) dx dv.$$

On the other hand, for any $R > 0$ we have

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_{\gamma}^{\hbar_n}(x, v)\psi(x) dx dv - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} m_{\gamma}^{\hbar_n}(x, v)\psi(x)f(v/R) dx dv \right| \leq \mathcal{R}_f(n, R)\|\psi\|_{L^{\infty}},$$

so that in the limit $n \rightarrow \infty$ we get

$$\left| (2\pi)^d \int_{\mathbb{R}^d} \nu(x)\psi(x) dx - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x, v)\psi(x)f(v/R) dx dv \right| \leq \limsup_{n \rightarrow \infty} \mathcal{R}_f(n, R)\|\psi\|_{L^{\infty}},$$

due to the fact that $(x, v) \mapsto \psi(x)f(v/R) \in C_c^{\infty}(\mathbb{R}^d \times \mathbb{R}^d)$. In the limit $R \rightarrow +\infty$, using the tightness property and monotone convergence, we indeed deduce that

$$(2\pi)^d \int_{\mathbb{R}^d} \nu(x)\psi(x) dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} W(x, v)\psi(x) dx dv,$$

showing that $\rho_W = \nu$. It remains to explain why the tightness property (1.8) holds. First, from the conservation of energy we deduce that $\text{Tr}(-\hbar^2\Delta)\gamma(t) \leq C\hbar^{-d}$ for a constant $C > 0$ independent of \hbar and t . Now this quantum kinetic energy can be related to the classical kinetic energy of $m_{\gamma(t)}^{\hbar}$ by the identity

$$\frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^2 m_{\gamma}^{\hbar}(x, v) dx dv = \text{Tr}(-\hbar^2\Delta)\gamma + \hbar\|\nabla\chi\|_{L^2}^2 \text{Tr} \gamma. \quad (1.9)$$

If $\text{Tr} \gamma$ (which is also a conserved quantity) is also of the order \hbar^{-d} (which is the case if $\gamma = \text{Op}^{\hbar}(\varphi)$), then we obtain

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |v|^2 m_{\gamma(t)}^{\hbar}(x, v) dx dv \leq C$$

for $C > 0$ independent of \hbar and t , from which the tightness property follows easily. This finishes the proof of the semi-classical limit via the Lions-Paul strategy (in a simplified setting).

The same method can be applied to solutions to the Hartree equation around stationary states γ_{ref} (or rather the \hbar -dependent version of (1.4)). Indeed, the compactness step to obtain weak limits for W_Q^\hbar and $\hbar^d \rho_Q$ can be done in the same way once uniform bounds are obtained for these objects. Contrary to the case $\gamma_{\text{ref}} = 0$, the norm $\|Q\|_{\mathfrak{S}^2}$ is not invariant under the dynamics of (1.4). However, it can be controlled via the only conserved quantity: the relative free energy. The conservation of the relative free energy implies a global bound on the relative entropy $\hbar^d \mathcal{H}_S(\gamma, \gamma_{\text{ref}})$ which itself controls $\hbar^d \|Q\|_{\mathfrak{S}^2}^2$ by (1.5). We may find a weak limit W on $W_\gamma^\hbar = W_Q^\hbar + g(v)$ in this way. Similarly, a global bound on $\hbar^d \rho_Q$ can be obtained using the control of the relative entropy and the Lieb-Thirring inequality for the relative entropy that we mentioned above. We deduce a weak limit ν for the density $\hbar^d \rho_\gamma = \hbar^d \rho_{\gamma_{\text{ref}}} + \hbar^d \rho_Q$. In this way, the same argument as above leads to an equation of the type (1.7) where the unknown W and the density ν are decoupled. As in the case $\gamma_{\text{ref}} = 0$, all boils down to showing that $\rho_W = \nu$, which again would follow from the tightness property (1.8) (which itself follows from a control on the quantum kinetic energy and (1.9)). In the case $\gamma_{\text{ref}} \neq 0$, we argued that the natural replacement of the kinetic energy was the relative entropy, which is indeed controlled uniformly. However, no formula as explicit as (1.9) exists to relate classical and quantum relative entropies. Instead, we rely on *Berezin-Lieb inequalities*.

1.4. Berezin-Lieb inequalities and tightness. The idea of Berezin-Lieb inequalities is to give bounds between classical and quantum quantities. In our context, one has a quantum object (the one-body density matrix γ) and one tries to obtain bounds on a classical transform of it (its Husimi transform m_γ^\hbar). We already saw one instance of such a relation in the form of the identity (1.9). Berezin-Lieb inequalities [20, 99, 134] have the same flavour, in the form

$$\text{Tr } F(\gamma) \geq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} F(m_\gamma^\hbar(x, v)) \frac{dx dv}{(2\pi\hbar)^d},$$

where on the left side is the expectation of the quantum observable $F(\gamma)$ (with $F : [0, 1] \rightarrow \mathbb{R}_+$ a *convex* function) and on the right side its classical counterpart. Such an inequality follows from the elementary remark that

$$F(m_\gamma^\hbar(x, v)) = F(\langle \chi_{x,v}^\hbar, \gamma \chi_{x,v}^\hbar \rangle) \leq \langle \chi_{x,v}^\hbar, F(\gamma) \chi_{x,v}^\hbar \rangle,$$

which follows from the spectral decomposition of γ and the convexity of F . Integrating this relation in (x, v) together with the resolution of the identity

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\langle \chi_{x,v}^\hbar, f \rangle|^2 \frac{dx dv}{(2\pi\hbar)^d} = \|f\|_{L^2}^2, \quad f \in L^2(\mathbb{R}^d)$$

leads to the claimed lower bound on $\text{Tr } F(\gamma)$. The situation is less clear when dealing with *relative* quantities where two quantum objects (and their two classical counterparts) appear. In [96, Proposition 4.3], we provide conditions on the entropy functional S so that

an inequality

$$\mathcal{H}_S(\gamma, \gamma_{\text{ref}}) \geq C \mathcal{H}_S^{\text{cl}}(m_\gamma^{\hbar}, m_{\gamma_{\text{ref}}}^{\hbar}) \quad (1.10)$$

holds, where $\mathcal{H}_S^{\text{cl}}$ is the *classical relative entropy* defined by

$$\mathcal{H}_S^{\text{cl}}(m, m_0) = - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \left[S(m(x, v)) - S(m_0(x, v)) - S'(m_0(x, v))(m(x, v) - m_0(x, v)) \right] \frac{dx dv}{(2\pi\hbar)^d}.$$

This property holds for the von Neumann entropy $S(x) = -x \log x + x$, using the crucial property that it is monotone under completely positive, trace-preserving maps [97]. This was already remarked in [92, Lemma 7.2], and using elementary manipulations it is not hard to prove it also for the Fermi-Dirac and Bose-Einstein entropy functionals. It is an open question to determine the set of all entropy functionals S for which a relative Berezin-Lieb inequality (1.10) holds. Let us just emphasize that in our context, while the global well-posedness results of [94] hold on fairly general assumptions on S (essentially, $-S$ being operator monotone), this Berezin-Lieb property introduces additional restrictions on S so that we are only able to treat the Boltzmann, Fermi-Dirac, and Bose-Einstein entropy functionals.

Once a uniform in \hbar control on $\mathcal{H}_S^{\text{cl}}(m_\gamma^{\hbar}, m_{\gamma_{\text{ref}}}^{\hbar})$ is obtained, one has to understand why it controls large velocities to deduce the tightness property (1.8). Of course, due to the complicated expression for the classical relative entropy, it is less obvious than for the kinetic energy $\int |v|^2 m dx dv$. We proved in [96, Theorem 5.1] that for all $R > 0$ we have

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \mathbb{1}(|v| \geq R) m_\gamma^{\hbar}(x, v) |\varphi(x)| dx dv &\leq C \|\varphi\|_{L^\infty} \frac{\mathcal{H}_S^{\text{cl}}(m_\gamma^{\hbar}, m_{\gamma_{\text{ref}}}^{\hbar})}{R} \\ &\quad + C \|\varphi\|_{L^1} \int_{|v|^2 \geq R/128} (S')^{-1}(|v|^2) dv, \end{aligned}$$

which provides the desired tightness. Interestingly, this result has a quantum analogue [94, Theorem 8] which is a key technical component of our proof of global well-posedness around stationary states (to show that finite energy solutions can be approximated by regular solutions).

1.5. Uniqueness. Let us sum up the results of the arguments presented above. We fix an entropy functional S which is either the Boltzmann, Fermi-Dirac, or Bose-Einstein one (to ensure that the relative entropy associated with S satisfies a Berezin-Lieb inequality). This defines a quantum background $\gamma_{\text{ref}} = (S')^{-1}(-\hbar^2 \Delta)$ for any $\hbar > 0$ as well as a classical background $m_{\text{ref}}(x, v) = (S')^{-1}(|v|^2)$. We also assume $d = 1, 2, 3$, which is the condition under which the global well-posedness of the Hartree equation was obtained in [95]. For any initial data $0 \leq \gamma_0 \leq 1$ such that $\hbar^d \mathcal{H}_S(\gamma_0, \gamma_{\text{ref}})$ is bounded as $\hbar \rightarrow 0$, we have found a sequence (h_n) converging to zero and $W : \mathbb{R}_t \times \mathbb{R}_x^d \times \mathbb{R}_v^d \rightarrow [0, 1]$ a (distributional) solution to the nonlinear Vlasov equation (1.3) such that $W_{\gamma(t)}^{h_n} \rightarrow W(t)$, weakly-* in $L_t^\infty(m_{\text{ref}} + L_{x,v}^2)$, where $\gamma(t)$ is the unique global solution to the Hartree equation (1.1). Additional properties of this solution W can be obtained from the proof. First, its time regularity can be improved to $C_t^0(L_{x,v}^\infty, *)$ due to the bounds $0 \leq W \leq 1$ and the Vlasov equation which controls $\partial_t W$. Secondly, we have informations about the density of the solution ρ_W because we showed that

it is the weak limit of the quantum density, which satisfies some bounds coming from the Lieb-Thirring inequality. This implies that $\rho_W \in \rho_{m_{\text{ref}}} + L_t^\infty(L_x^2 + L_x^{\min(1+2/d, 2)})$.

The goal of this step is to show that such solutions to the nonlinear Vlasov equation are unique. This has two motivations: i) to obtain the convergence of W_γ^{\hbar} to W not only up to a sequence (\hbar_n) but as $\hbar \rightarrow 0$, and ii) to give a first step towards the well-posedness of the Vlasov equation in this setting. Indeed, the existing results on uniqueness/well-posedness of the Vlasov equation all concern (to our knowledge) solutions m which are at least $L_{x,v}^1$ (or finite measures), which is not the case here. Notice also that in order to have uniqueness, the initial data for the Vlasov equation must be unique so it is natural to furthermore assume that at initial time, $W_{\gamma_0}^{\hbar} \rightarrow W(0)$ as $\hbar \rightarrow 0$, in the sense of distributions for instance. The uniqueness is proved in [96, Theorem 6.3], where we follow the method of Loeper [114] based on optimal transportation techniques. We won't go too much into details, but let us just mention that Loeper's method is based on considering two solutions W_1 and W_2 with the above regularity (we also denote by ρ_1 and ρ_2 their respective densities) and the goal is to show that $W_1 = W_2$. To do it, the key is to consider the Wasserstein distance between ρ_1 and ρ_2 . The interesting point here is that $\int \rho_1 = \int \rho_2 = +\infty$, so that ρ_1 and ρ_2 are infinite-mass measures. Our point is that the classical results of optimal transportation of Brenier and McCann (which are stated for *finite* measures) extend to the case of infinite measures, as long as their Wasserstein distance is finite. For instance, one has the following result:

LEMMA 1.1 (Lemma B.1 in [96]). *Let $\rho_1, \rho_2 \in L_{\text{loc},+}^1(\mathbb{R}^d)$. Define $\Gamma(\rho_1, \rho_2)$ as the set of all Radon measures on $\mathbb{R}^d \times \mathbb{R}^d$ having marginals ρ_1 and ρ_2 . Define the optimal transport problem*

$$C(\rho_1, \rho_2) = \inf_{\Pi \in \Gamma(\rho_1, \rho_2)} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 d\Pi(x, y).$$

If $C(\rho_1, \rho_2) < +\infty$, then the infimum defining $C(\rho_1, \rho_2)$ is attained.

In the finite-mass setting, the proof of this result is standard [152, Theorem 4.1] and exploits the fact that the minimizing set $\Gamma(\rho_1, \rho_2)$ is *compact*, so that any sequence has a convergent subsequence. Interestingly, in the infinite-mass setting one has to use that a sequence is *minimizing* to obtain a convergent subsequence. To show that ρ_1 and ρ_2 are at a finite Wasserstein distance in our setting, one exploits the fact that $\rho_1 - \rho_2$ has some decay (even if ρ_1 and ρ_2 do not decay), due to the fact that they both have the same background $\rho_{m_{\text{ref}}}$.

Adding up all the arguments presented above, we deduce the following result on the semi-classical limit of the Hartree equation around translation-invariant stationary states.

THEOREM 1.1 (Theorem 2.24 in [96]). *Let $d \in \{1, 2, 3\}$, $\hbar > 0$, and assume that either*

$$S(x) = \begin{cases} -x \log x - (1-x) \log(1-x), \\ -x \log x + (1+x) \log(1+x), \\ -x \log +x. \end{cases}$$

Define $\gamma_{\text{ref}} = (S')^{-1}(-\hbar^2 \Delta)$, $m_{\text{ref}}(x, v) := (S')^{-1}(|v|^2)$. Let $w \in (W^{2,1} \cap W^{2,\infty})(\mathbb{R}^d)$ be an even, real-valued function. Let $\gamma \in \gamma_{\text{ref}} + C_t^0(\mathbb{R}, \mathfrak{S}^2)$ be the unique solution to the Hartree

equation (1.1) associated with an initial condition γ_0 satisfying

$$\liminf_{\hbar \rightarrow 0} \hbar^d \mathcal{H}_S(\gamma_0, \gamma_{\text{ref}}) < +\infty$$

and $W_{\gamma_0}^{\hbar} \rightarrow W_0$ as $\hbar \rightarrow 0$ in the sense of distributions on the phase space $\mathbb{R}^d \times \mathbb{R}^d$. Then, there exists $C_S > 0$ and

$$W \in \left\{ m_{\text{ref}} + L_t^\infty(\mathbb{R}, L_{x,v}^2(\mathbb{R}^d \times \mathbb{R}^d)) \right\} \cap C_t^0(\mathbb{R}, \mathcal{D}'_{x,v}(\mathbb{R}^d \times \mathbb{R}^d))$$

such that:

- (1) $0 \leq W(t) \leq 1$ for all $t \in \mathbb{R}$;
- (2) $\mathcal{H}_S^{\text{cl}}(W(t), m_{\text{ref}}) \leq C_S \liminf_{\hbar \rightarrow 0} \hbar^d \mathcal{H}_S(\gamma(t), \gamma_{\text{ref}})$ for all $t \in \mathbb{R}$;
- (3) $W_{\gamma(t)}^{\hbar} \rightarrow W(t)$ as $\hbar \rightarrow 0$ in the sense of distributions on \mathbb{R}^{2d} , uniformly on compact sets in t ;
- (4) $\hbar^d \rho_\gamma - \hbar^d \rho_{\gamma_{\text{ref}}} \rightarrow \rho_W - \rho_{m_{\text{ref}}}$ as $\hbar \rightarrow 0$, weakly-* in $L_t^\infty(L^2 + L^{\min(1+2/d, 2)})$;
- (5) W is the unique solution to the nonlinear Vlasov equation

$$\begin{cases} \partial_t W + 2v \cdot \nabla_x W - \nabla_x(w * \rho_W) \cdot \nabla_v W = 0, \\ W|_{t=0} = W_0 \end{cases}$$

in $\left\{ m_{\text{ref}} + L_t^\infty(\mathbb{R}, L_{x,v}^2(\mathbb{R}^d \times \mathbb{R}^d)) \right\} \cap C_t^0(\mathbb{R}, \mathcal{D}'_{x,v}(\mathbb{R}^d \times \mathbb{R}^d))$ such that $\mathcal{H}_S^{\text{cl}}(W(t), m_{\text{ref}}) \in L_t^\infty$.

Let us comment on the elements of the theorem which did not appear in our previous arguments. First, the convergence of $W_{\gamma(t)}^{\hbar}$ to $W(t)$ as $\hbar \rightarrow 0$ locally uniform in t follows from Ascoli's theorem together with bounds on $\partial_t W_{\gamma(t)}^{\hbar}$ coming from the equation. More importantly, the control on the classical relative entropy $\mathcal{H}_S^{\text{cl}}(W(t), m_{\text{ref}})$ follows from the Berezin-Lieb inequality (1.10) and the fact that the classical relative entropy decreases under weak limits (perhaps up to a multiplicative constant). This last fact is not automatic and can be proved easily if $(x, y) \mapsto S(x) - S(y) - S'(y)(x - y)$ is *convex* (essentially because in this case, weak limits can be replaced by a.e. limits for which the decrease follows from Fatou's lemma) [96, Lemma 5.3]. This introduces a priori another constraint on the entropy functional S , which is satisfied by our three options for S . We insist on this control on the classical relative entropy because if one could achieve a bound of the type $\mathcal{H}_S^{\text{cl}}(W(t), m_{\text{ref}}) \leq \liminf_{\hbar \rightarrow 0} \hbar^d \mathcal{H}_S(\gamma(t), \gamma_{\text{ref}})$, it would be a first step to show that the classical free energy is preserved along the flow of the nonlinear Vlasov equation (by deducing it from the quantum case).

1.6. Applications. We can use Theorem 1.1 to deduce a global well-posedness result about the nonlinear Vlasov equation itself. The idea is to build solutions to the Vlasov equation as limits of (Wigner transforms of) solutions to the Hartree equation. In this sense, the quantum equation acts as a regularization of the Vlasov equation. Looking at Theorem 1.1, if one starts with an initial datum $W(0)$ for the Vlasov equation satisfying $0 \leq W(0) \leq 1$ and $\mathcal{H}_S^{\text{cl}}(W(0), m_{\text{ref}}) < +\infty$, one would like to construct an initial datum γ_0 for the Hartree equation such that $W_{\gamma_0}^{\hbar} \rightarrow W(0)$ as $\hbar \rightarrow 0$ in the distributional sense, and such that $\hbar^d \mathcal{H}_S(\gamma_0, m_{\text{ref}})$ is bounded as $\hbar \rightarrow 0$. The first property is easy to obtain (just choose

$\gamma_0 = \text{Op}^{\hbar}(W(0))$), but the second property is much harder to obtain due to the complicated definition of the quantum relative entropy. We show the second property for a particular class of $W(0)$, namely those who can be written as $W(0)(x, v) = (S')^{-1}(|v|^2 + a(x, v))$ for some $a \in C_c^\infty(\mathbb{R}^d \times \mathbb{R}^d)$. For those, it turns that the choice $\gamma_0 = (S')^{-1}(-\hbar^2\Delta + \text{Op}^{\hbar}(a))$ can be shown to satisfy the two desired properties. Then, one can show that this class of $W(0)$ is dense in the full classical relative energy space to obtain the following result.

THEOREM 1.2 (Theorem 2.28 in [96]). *Let $d \in \{1, 2, 3\}$ and assume that either*

$$S(x) = \begin{cases} -x \log x - (1-x) \log(1-x), \\ -x \log x + (1+x) \log(1+x), \\ -x \log + x. \end{cases}$$

Define $m_{\text{ref}}(x, v) := (S')^{-1}(|v|^2)$. Let $w \in (W^{2,1} \cap W^{2,\infty})(\mathbb{R}^d)$ be an even, real-valued function. Let $W_0 \in m_{\text{ref}} + L_{x,v}^2$ be such that $0 \leq W_0 \leq 1$ and such that $\mathcal{H}_S^{\text{cl}}(W_0, m_{\text{ref}}) < +\infty$. Then, there exists a unique

$$W \in (m_{\text{ref}} + L_t^\infty(\mathbb{R}, L_{x,v}^2)) \cap C_t^0(\mathbb{R}, \mathcal{D}'_{x,v})$$

such that $0 \leq W(t) \leq 1$ and $\mathcal{H}_S^{\text{cl}}(W(t), m_{\text{ref}}) \leq C$ for all $t \in \mathbb{R}$, which solves the nonlinear Vlasov equation

$$\begin{cases} \partial_t W + 2v \cdot \nabla_x W - \nabla_x(w * \rho_W) \cdot \nabla_v W = 0, \\ W|_{t=0} = W_0. \end{cases}$$

To our knowledge, Theorem 1.2 is the first result about the existence and uniqueness of non-perturbative solutions to the nonlinear Vlasov equation around a non-trivial homogeneous state m_{ref} in \mathbb{R}^d , in the energy space. Perturbative solutions with high regularity have been obtained in [13] in the context of Landau damping, leaving the case of global-in-time non-perturbative solutions open (see Remark 2.2 in [13]). Of course, our result has two drawbacks: (i) we consider only nice interaction potentials w and (ii) our assumptions on the reference state m_{ref} coming from the entropies are quite stringent. Relaxing these assumptions on w and m_{ref} is a very interesting challenge. Notice also that it seems hard to adapt the proof strategy of [95] concerning the global well-posedness of the Hartree equation to obtain a direct proof of Theorem 1.2, since the quantum result heavily relies on dispersive tools such as Strichartz estimates. These tools also exist in the classical setting, but it is not obvious to us how to apply them to obtain local well-posedness results at low regularity for the Vlasov equation, for instance. Let us finally notice that Theorem 1.2 does not imply Landau damping (that is, weak convergence of $W(t)$ to m_{ref} as $t \rightarrow +\infty$, in other words the asymptotic stability of m_{ref}). However, it implies the orbital (or rather Lyapounov, since the orbit is reduced to a single point) stability of m_{ref} in the sense of the relative entropy: given $\varepsilon > 0$, there exists $\eta > 0$ such that for all initial W_0 with $\mathcal{H}_S^{\text{cl}}(W_0, m_{\text{ref}}) < \eta$, then $\mathcal{H}_S^{\text{cl}}(W(t), m_{\text{ref}}) < \varepsilon$ for all $t \in \mathbb{R}$.

2. From the Dirac-Klein-Gordon equation to the nonlinear Dirac equation

We next present the work of [90] which is in some sense complementary to the one presented above, because we also relate two PDEs in some asymptotic regime. While the

preceding result relied on compactness methods and weak limits, the result below will provide quantitative estimates and strong limits. This model also has a many-body component, but we begin with its one-body version which is simpler and which motivates the many-body strategy.

2.1. The one-body problem. We start from the Dirac-Klein-Gordon equation used in the relativistic mean-field theory of nuclei [125] and models the interaction between nucleons (described by the Dirac equation) and mesons (modeled by the Klein-Gordon part). It reads

$$\begin{cases} i\partial_t\Psi = D\Psi + \beta S\Psi, \\ (\partial_t^2 - \Delta + M^2)S = -g^2\langle\beta\Psi, \Psi\rangle_{\mathbb{C}^4}, \end{cases} \quad (1.11)$$

where $\Psi : \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{C}^4$ is the Dirac spinor (with mass $m \geq 0$) describing the nucleons and $S : \mathbb{R}_t \times \mathbb{R}_x^3 \rightarrow \mathbb{R}$ describes (part of) the meson field (with mass $M > 0$). Here, D denotes the Dirac operator $D := -i\boldsymbol{\alpha} \cdot \nabla + \beta m$ where $(\beta, \boldsymbol{\alpha})$ are the complex 4×4 Dirac matrices defined by

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \text{ with } \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix}, \quad (1.12)$$

for $k = 1, 2, 3$, and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (1.13)$$

the Pauli matrices. We chose a simplified formulation of (1.11) where the meson field S is scalar-valued; a more realistic model would include a field with 5 (real) components (see [90] for details). We don't include the full model here to highlight the mathematical features of the equations, and because the full model is not significantly harder to treat.

We are interested in the strong coupling regime where g and M are large and of the same order. We set $g = M$ for simplicity, and we expect formally that in the limit $M \rightarrow +\infty$, the second equation of (1.11) implies that $S = -\langle\beta\Psi, \Psi\rangle_{\mathbb{C}^4}$, which reinserted into the first equation of (1.11) leads to the nonlinear Dirac equation

$$i\partial_t\Psi = D\Psi - \langle\beta\Psi, \Psi\rangle_{\mathbb{C}^4}\beta\Psi. \quad (1.14)$$

Of course, this argument is purely formal since it relies on the smallness of $(\partial_t^2 - \Delta)S$ which will typically be wrong due to fast oscillations of S as $M \rightarrow +\infty$.

A potentially more problematic question concerning the convergence of solutions as $M \rightarrow +\infty$ is the question of the time of existence. Indeed, if the maximal time of existence $T_M^* > 0$ of solutions to (1.11) vanishes as $M \rightarrow +\infty$, it becomes hard to make sense of the convergence of Ψ as $M \rightarrow +\infty$. This is particularly relevant for such equations which are based on the Dirac operator, which is neither bounded from above or below (contrary to its non-relativistic counterpart $-\Delta$). This implies that one cannot hope to control the time of existence by using a conserved quantity like the energy, based on the unsigned relativistic kinetic energy $\langle\Psi, D\Psi\rangle$. Another natural conserved quantity which one could hope to use to obtain global-in-time solutions is the L^2 -norm $\|\Psi\|_{L^2}$, which has a definite sign and is coercive. The new problem that arises for this conservation law is the low regularity at which it lives, and for which well-posedness results are harder to obtain. This being said, global well-posedness

for the equation (1.11) for Ψ in L^2 was obtained in this way in [25] in one space dimension (see [42] for a related earlier result), and in [77] in two space dimensions. A corresponding result for the nonlinear Dirac equation in one space dimension is in [49]. To our knowledge, no such result exists either for the Dirac-Klein-Gordon equation in three space dimensions or for the nonlinear Dirac equation in two or three space dimensions. This is due to the fact that the former is L^2 -critical while the latter is $H^{1/2}$ -critical in $2d$ and H^1 -critical in $3d$. Meanwhile, some global well-posedness results exist for these equations for *small* solutions, see for instance [16, 15, 14].

Here, we take the opposite point of view of high regularity solutions. At such regularity, local-in-time well-posedness is very easy to obtain and their time of existence is controlled by a blow-up criterion, as summed up in the following result.

LEMMA 1.2. *Let $s > 3/2$ and $(\Psi_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}}) \in H^s(\mathbb{R}^3, \mathbb{C}^4) \times H^s(\mathbb{R}^3, \mathbb{R}) \times H^{s-1}(\mathbb{R}^3, \mathbb{R})$.*

(i) *For all $g, M > 0$ there exist $T_{\text{min}}, T_{\text{max}} \in (0, +\infty]$ and a unique maximal solution*

$$(\Psi, S) \in C^0((-T_{\text{min}}, T_{\text{max}}), H^s(\mathbb{R}^3, \mathbb{C}^4) \times H^s(\mathbb{R}^3, \mathbb{R})),$$

to the Dirac-Klein-Gordon equation (1.11) such that $(\Psi, S, \partial_t S)|_{t=0} = (\Psi_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}})$.

If $T_{\text{max}/\text{min}} < +\infty$ then

$$\limsup_{t \rightarrow T_{\text{max}/\text{min}}} \|(\Psi, S)(t)\|_{L^\infty} = +\infty. \quad (1.15)$$

(ii) *There exist $T_{\text{min}}^{\text{nl}}, T_{\text{max}}^{\text{nl}} \in (0, +\infty]$ and a unique maximal solution*

$$\Psi_{\text{nl}} \in C^0((-T_{\text{min}}^{\text{nl}}, T_{\text{max}}^{\text{nl}}), H^s(\mathbb{R}^3, \mathbb{C}^4)),$$

to the nonlinear Dirac equation (1.14) such that $(\Psi_{\text{nl}})|_{t=0} = \Psi_{\text{in}}$. If $T_{\text{max}/\text{min}}^{\text{nl}} < +\infty$ then

$$\limsup_{t \rightarrow T_{\text{max}/\text{min}}^{\text{nl}}} \|\Psi_{\text{nl}}(t)\|_{L^\infty} = +\infty.$$

Lemma 1.2 can be proved by a fixed point argument on the Duhamel formulations of both equations. Working in high regularity simplifies things because H^s is an algebra for $s > 3/2$, so that the nonlinearities stabilize H^s . An important remark is the blow-up criteria which are stated for the L^∞ -norm (and not the H^s -norm), and which follow from the Kato-Ponce inequality [85, Lemma X.4]

$$\|uv\|_{H^s} \leq C(\|u\|_{L^\infty}\|v\|_{H^s} + \|v\|_{L^\infty}\|u\|_{H^s}). \quad (1.16)$$

which holds for any $u, v \in H^s$ with $s > 3/2$. This standard fact about the blow-up criterion will be important to deal with the limit $M \rightarrow +\infty$.

Solutions of (1.11) are related to the solutions to (1.14) in the limit $M \rightarrow +\infty$, as shown by the following result.

THEOREM 1.3 (Theorem 1 in [90]). *Let $s > 5/2$ and $(\Psi_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}}) \in H^s(\mathbb{R}^3, \mathbb{C}^4) \times H^s(\mathbb{R}^3, \mathbb{R}) \times H^{s-1}(\mathbb{R}^3, \mathbb{R})$. Let Ψ_{nl} be the maximal solution to (1.14) with initial condition $\Psi_{\text{nl}}|_{t=0} = \Psi_{\text{in}}$ given by Lemma 1.2. Let $M > 0$ and let (Ψ, S, ω) be the maximal solution to*

(1.11) with $g = M$ and initial conditions $(\Psi, S, \partial_t S)|_{t=0} = (\Psi_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}})$ given by Lemma 1.2. Then, we have

$$\liminf_{M \rightarrow +\infty} T_{\text{min/max}} \geq T_{\text{min/max}}^{\text{nl}} \quad (1.17)$$

and, for all $0 < T_1 < T_{\text{min}}^{\text{nl}}$, $0 < T_2 < T_{\text{max}}^{\text{nl}}$, and all $0 \leq s' < s$,

$$\lim_{M \rightarrow +\infty} \|\Psi - \Psi_{\text{nl}}\|_{L^\infty([-T_1, T_2], H^{s'}(\mathbb{R}^3, \mathbb{C}^4))} = 0. \quad (1.18)$$

Let us make a few comments on the statement of Theorem 1.3. First, the assumption that $s > 5/2$ is stronger than the one of Lemma 1.2 ($s > 3/2$) due to the method of proof that we will describe below (essentially, we use a different equation where derivatives appear in the nonlinearity, so that we need that H^{s-1} is also an algebra). Of course, we don't think that this assumption is sharp in any way since we already mentioned that these equations are well-posed for lower values of s , so we expect that the convergence result of Theorem 1.3 can be extended to the values of s so that the nonlinear Dirac equation is well posed (hence, $s > 1$ or even the critical exponent $s = 1$). An interesting related question is what happens for $0 < s < 1$, where the Dirac-Klein-Gordon system is well-posed but not the nonlinear Dirac equation. Another comment we can make is about the convergence (1.18). Notice that it holds for the $H^{s'}$ -topology (which is weaker than the H^s topology of the solution). This is related to the fact that in this weaker topology, we actually obtain a quantitative rate of convergence (essentially $M^{s'-s}$), which is important in the argument to control the blow-up time as in (1.17). This being said, we expect that the convergence (1.18) also holds in the maximal topology $s' = s$, probably without an explicit rate. To illustrate this phenomenon, if $s - 1 < s' \leq s$ and $u \in H^{s-1}$, we have

$$\|(-\Delta + M^2)^{-1/2} u\|_{H^{s'}} \leq M^{s'-s} \|u\|_{H^{s-1}}$$

which converges to 0 as $M \rightarrow +\infty$ if $s' < s$ with a rate $M^{s'-s}$, while for $s' = s$ convergence to 0 also holds by dominated convergence (with no rate). Last but not least, we want to emphasize that Theorem 1.3 holds without any assumption on the initial datum for the meson field $(S_{\text{in}}, \dot{S}_{\text{in}})$, in particular since it would look natural to assume that S_{in} is close to $-\langle \beta \Psi_{\text{in}}, \Psi_{\text{in}} \rangle$. That we do not need such an assumption is related to the fact that the meson field S does not need to be close to $-\langle \beta \Psi, \Psi \rangle$ for the limit (1.18) to hold. Our proof even shows that $S + \langle \beta \Psi, \Psi \rangle$ has a (fast) oscillating component which may not vanish in norm, but which regardless does not influence the dynamics of the Dirac spinor. This fact has an interesting consequence when Ψ_{in} is small enough in some norm to ensure that $T_{\text{min/max}}^{\text{nl}} = +\infty$. Then, (1.17) shows that solutions the Dirac-Klein-Gordon equation exist for arbitrarily large time if M is chosen large enough, regardless of the size of initial meson field. This kind of result seems new to us, since global solutions to the Dirac-Klein-Gordon equation seem to be known to exist only when all the initial data (spinor *and* field) are small (see for instance [16]).

Let us now comment on how the proof is made, and let us begin by motivating the strategy. To simplify notations, assume that $S_{\text{in}} = -\langle \beta \Psi_{\text{in}}, \Psi_{\text{in}} \rangle$ and that $\dot{S}_{\text{in}} = 0$. Then, the Duhamel formulation for the equation on S implies that for all t in the interval of existence

we have

$$S(t) = \cos(t\sqrt{-\Delta + M^2})S_{\text{in}} - M^2 \int_0^t \frac{\sin((t-s)\sqrt{-\Delta + M^2})}{\sqrt{-\Delta + M^2}} \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4} ds.$$

Using integration by parts, we find that

$$\begin{aligned} S(t) + \langle \beta\Psi(t), \Psi(t) \rangle_{\mathbb{C}^4} &= \frac{-\Delta}{-\Delta + M^2} \left(\langle \beta\Psi(t), \Psi(t) \rangle_{\mathbb{C}^4} - \cos(t\sqrt{-\Delta + M^2}) \langle \beta\Psi_{\text{in}}, \Psi_{\text{in}} \rangle_{\mathbb{C}^4} \right) \\ &\quad + \frac{M^2}{-\Delta + M^2} \int_0^t \cos((t-s)\sqrt{-\Delta + M^2}) \partial_s \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4} ds, \end{aligned}$$

with a first term that vanishes as $M \rightarrow +\infty$ if we have uniform bounds on $\Psi(t)$ in H^s for instance. To see that the integral term vanishes as well, it is natural to do another integration by parts to obtain

$$\begin{aligned} \int_0^t \cos((t-s)\sqrt{-\Delta + M^2}) \partial_s \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4} ds &= \frac{\sin(t\sqrt{-\Delta + M^2})}{\sqrt{-\Delta + M^2}} (\partial_s)_{|s=0} \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4} \\ &\quad + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta + M^2})}{\sqrt{-\Delta + M^2}} \partial_s^2 \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4} ds \end{aligned}$$

To show that this last term vanishes in H^s as $M \rightarrow +\infty$, it thus seems natural to try to control $\partial_s^2 \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4}$ in H^{s-1} uniformly in M . Using the equation satisfied by $\Psi(t)$, one finds that

$$\partial_s^2 \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4} = 2 \operatorname{Re} \left[\langle \beta D\Psi, D\Psi \rangle_{\mathbb{C}^4} - \langle \beta\Psi, D^2\Psi \rangle_{\mathbb{C}^4} - 2 \langle \Psi, S(i\alpha \cdot \nabla\Psi) \rangle_{\mathbb{C}^4} \right],$$

so that a control on $\partial_s^2 \langle \beta\Psi(s), \Psi(s) \rangle_{\mathbb{C}^4}$ in H^{s-1} would follow from a control on Ψ in H^{s+1} (due to the term with two derivatives on Ψ). Using now the Duhamel formulation of the equation on Ψ ,

$$\Psi(t) = e^{-itD}\Psi_{\text{in}} - i \int_0^t e^{-i(t-s)D} \beta S(s)\Psi(s) ds,$$

we see that a control on $\Psi(t)$ in H^{s+1} would follow from a control of S in H^{s+1} . We therefore cannot close the argument in such a way (because we began by trying to control S in H^s). Let us mention also that this strategy appeared in previous works where such loss of derivatives does not happen (due to a regularizing effect in the nonlinearity) [129, 1, 48, 76, 12].

The solution we proposed to bypass this issue was to rather work on the reduced variable

$$\bar{S}(t) = S(t) + \langle \beta\Psi(t), \Psi(t) \rangle_{\mathbb{C}^4},$$

and the new system of equations is then

$$\begin{cases} i\partial_t \Psi = D\Psi - \langle \beta\Psi, \Psi \rangle_{\mathbb{C}^4} \beta\Psi + \beta\bar{S}\Psi, \\ (\partial_t^2 - \Delta + M^2)\bar{S} = (\partial_t^2 - \Delta)\langle \beta\Psi, \Psi \rangle_{\mathbb{C}^4}. \end{cases} \quad (1.19)$$

The key point is that the new nonlinearity $(\partial_t^2 - \Delta)\langle \beta\Psi, \Psi \rangle_{\mathbb{C}^4}$ has a hidden regularizing effect, in the sense that it actually does not lose two derivatives on Ψ : using the equation on Ψ , one can show that $(\partial_t^2 - \Delta)\langle \beta\Psi, \Psi \rangle_{\mathbb{C}^4}$ is actually a polynomial in Ψ , $\nabla\Psi$, and $\nabla\bar{S}$ that we denote by $P(\Psi, \nabla\Psi, \nabla\bar{S})$. We will see that this gain of one derivative is enough to close the argument. Notice that this cancellation of higher order derivatives is not related to

the particular form of the nonlinearity but rather on the fact that Ψ satisfies a Dirac type equation, so that to leading order we have $(\partial_t^2 - \Delta)\Psi = 0$.

Using equation (1.19), we infer the following technical key result.

LEMMA 1.3 (Lemma 2.4, 2.6, and Prop. 2.7 in [90]). *Let $s > 5/2$ and $(\Psi_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}}) \in H^s(\mathbb{R}^3, \mathbb{C}^4) \times H^s(\mathbb{R}^3, \mathbb{R}) \times H^{s-1}(\mathbb{R}^3, \mathbb{R})$. Let $T > 0$, $M \geq 1$, and $(\Psi, S) \in C_t^0([0, T], H^s(\mathbb{R}^3, \mathbb{C}^4) \times H^s(\mathbb{R}^3, \mathbb{R}))$ be a solution to (1.11) with $g = M$ and initial data $(\Psi_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}})$. Let $\Psi_{\text{nl}} \in C_t^0([0, T], H^s(\mathbb{R}^3, \mathbb{C}^4))$ be a solution to (1.14) with initial data Ψ_{in} . Define $\bar{S} := S + \langle \beta \Psi, \Psi \rangle_{\mathbb{C}^4}$. Let $s' \in [s - 1, s]$. Then, for any $R > 0$ there exists $C(R) > 0$ independent of M such that*

(1)

$$\|(\Psi, \bar{S})\|_{L_t^\infty([0, T], W^{1, \infty})} \leq R \implies \forall t \in [0, T], \|(\Psi, \bar{S})(t)\|_{H^s} \leq \|(\Psi, \bar{S})(0)\|_{H^s} e^{C(R)t},$$

(2)

$$\|(\Psi, \bar{S})\|_{L_t^\infty([0, T], H^s)} \leq R \implies \|\bar{S}(t) - \cos(t\sqrt{-\Delta + M^2})\bar{S}(0)\|_{L_t^\infty([0, T], H^{s'})} \leq C(R)M^{s'-s},$$

(3)

$$\|(\Psi, \bar{S})\|_{L_t^\infty([0, T], H^s)} \leq R \implies \|\Psi - \Psi_{\text{nl}}\|_{L^\infty([0, T], H^{s'})} \leq C(R)M^{s'-s}.$$

Point (1) is a kind of quantitative blowup criterion which is uniform in M : (Ψ, \bar{S}) does not blow up in H^s (uniformly in $M \geq 1$) as long as the $W^{1, \infty}$ -norm of (Ψ, \bar{S}) does not blow up. It follows from the Duhamel formulation of the reduced equation (1.19), the Kato-Ponce inequality (we need a control in $W^{1, \infty}$ since the nonlinearity now involves first order derivatives of (Ψ, \bar{S})), and the Gronwall lemma. Point (2) gives the leading order as $M \rightarrow +\infty$ of $\bar{S}(t)$ as long as one has a control on the H^s -norm of (Ψ, \bar{S}) . It can be proved using the Duhamel formulation of the equation on $\bar{S}(t)$ together with the bound (that we mentioned above) $\|(-\Delta + M^2)^{-1/2}\|_{H^{s-1} \rightarrow H^{s'}} \leq M^{s'-s}$. Finally, point (3) shows that the desired convergence from Dirac-Klein-Gordon to the nonlinear Dirac equation holds as long as we have a control on the H^s -norm of (Ψ, \bar{S}) which is uniform in M . It can be proved by computing the difference $\Psi - \Psi_{\text{nl}}$ using the Duhamel formulation of the equations on Ψ and Ψ_{nl} together with point (2) and Gronwall's lemma. Interestingly, one uses here that the oscillation in $\bar{S}(t)$ does not influence the dynamics of $\Psi(t)$: indeed, this leads to a term of the type

$$\int_0^t e^{-i(t-s)D} \cos(s\sqrt{-\Delta + M^2})\bar{S}(0)\beta\Psi(s) ds,$$

which can be shown to vanish in $H^{s'}$ as $M \rightarrow +\infty$ by integration by parts.

To deduce Theorem 1.3 from Lemma 1.3, we let $0 < T < T_{\text{max}}^{\text{nl}}$, $5/2 < s' < s$, and we prove simultaneously that for M large enough we have $T_{\text{max}} > T$ and that $\|\Psi - \Psi_{\text{nl}}\|_{L_t^\infty([0, T], H^{s'})} \rightarrow 0$ as $M \rightarrow +\infty$. To do so, we let $R > \|(\Psi, \bar{S})(0)\|_{W^{1, \infty}}$ and define

$$T' = \sup\{0 < t < \min(T, T_{\text{max}}), \|(\Psi, \bar{S})\|_{L^\infty([0, t], W^{1, \infty})} \leq R\}.$$

Notice that by the blow-up criterion for (Ψ, S) , we have $0 < T' < T_{\text{max}}$. To prove that $T_{\text{max}} > T$, we prove that $T' = T$ for M large enough and for an appropriate choice of R (independent of M). Notice that once we know that $T = T' < T_{\text{max}}$, we deduce that $\|(\Psi, \bar{S})\|_{L_t^\infty([0, T], W^{1, \infty})} \leq R$ by definition of T' so that by Lemma 1.3 points (1) and (3), we

have $\|\Psi - \Psi_{\text{nl}}\|_{L_t^\infty([0, T], H^{s'})} \rightarrow 0$ as $M \rightarrow +\infty$ (with even a rate of convergence). We prove that $T' = T$ by contradiction, assuming that $T' < T$ (along a subsequence of $M \rightarrow +\infty$, say). Then, using the maximality of T' we deduce that $\|(\Psi, \bar{S})\|_{L^\infty([0, T'], W^{1, \infty})} = R$. From Lemma 1.3 point (1), we deduce that $\|(\Psi, \bar{S})\|_{L^\infty([0, T'], H^s)} \leq R'$ for some $R' > 0$ independent of M and from point (3) we have $\|\Psi - \Psi_{\text{nl}}\|_{L^\infty([0, T'], H^{s'})} \leq C(R')M^{s'-s}$. Using the Duhamel formulation of the equation on \bar{S} ,

$$\bar{S}(t) = \cos(t\sqrt{-\Delta + M^2})\bar{S}(0) + \int_0^t \frac{\sin((t-s)\sqrt{-\Delta + M^2})}{\sqrt{-\Delta + M^2}} P(\Psi(s), \nabla\Psi(s), \nabla\bar{S}(s)) ds,$$

as well as $\|(-\Delta + M^2)^{-1/2}\|_{H^{s-1} \rightarrow H^{s'}} \leq M^{s'-s}$, we deduce that

$$\|\bar{S}\|_{L_t^\infty([0, T'], H^{s'})} \leq \|\bar{S}(0)\|_{H^{s'}} + CM^{s'-s}$$

due to the uniform H^s -bound on (Ψ, \bar{S}) . We also have

$$\|\Psi\|_{L_t^\infty([0, T'], H^{s'})} \leq \|\Psi_{\text{nl}}\|_{L_t^\infty([0, T'], H^{s'})} + \|\Psi - \Psi_{\text{nl}}\|_{L_t^\infty([0, T'], H^{s'})} \leq \|\Psi_{\text{nl}}\|_{L_t^\infty([0, T'], H^{s'})} + CM^{s'-s}$$

and hence since $s' > 5/2$,

$$\|(\Psi, \bar{S})\|_{L_t^\infty([0, T'], W^{1, \infty})} \leq C\|(\Psi, \bar{S})\|_{H^{s'}} \leq C(\|\bar{S}(0)\|_{H^{s'}} + \|\Psi_{\text{nl}}\|_{L_t^\infty([0, T'], H^{s'})}) + CM^{s'-s}.$$

Choosing $R = 2C(\|\bar{S}(0)\|_{H^{s'}} + \|\Psi_{\text{nl}}\|_{L_t^\infty([0, T'], H^{s'})})$, we deduce that $\|(\Psi, \bar{S})\|_{L_t^\infty([0, T'], W^{1, \infty})} < R$ for M large enough, the desired contradiction.

2.2. The many-body problem. In [90], we also considered a many-body version of (1.11) where several nucleons are considered in a mean-field way. The corresponding equation is

$$\begin{cases} i\partial_t \gamma = [D + \beta S, \gamma], \\ (\partial_t^2 - \Delta + M^2)S = -g^2 \rho_{\beta\gamma}, \end{cases} \quad (1.20)$$

where γ is the one-body density matrix describing the state of the nucleons (which, in this setting, is a non-negative bounded operator on $L^2(\mathbb{R}^3, \mathbb{C}^4)$) and where $\rho_{\beta\gamma}(x) := \text{Tr}_{\mathbb{C}^4}(\beta\gamma(x, x))$. In the limit $M \rightarrow +\infty$ with $g = M$, one expects to recover the following nonlinear many-body Dirac equation

$$i\partial_t \gamma = [D - \beta \rho_{\beta\gamma}, \gamma]. \quad (1.21)$$

As in the one-body case (which corresponds to the special case where γ is rank-one), it is first relevant to discuss the well-posedness theory of these equations. A natural extension of Sobolev spaces to density matrices (that we already encountered in the previous chapter) are the spaces of γ such that $(1 - \Delta)^{s/2} \gamma (1 - \Delta)^{s/2} \in \mathfrak{S}^\alpha$ for some $\alpha \geq 1$ (indeed, for γ of rank-one, it relates to standard Sobolev spaces H^s for any choice of α). For reasons that we detail below, we can treat the case $\alpha = 2$, therefore we introduce the notation

$$\mathfrak{H}^s := \{\gamma \in \mathfrak{S}^2, (1 - \Delta)^{s/2} \gamma (1 - \Delta)^{s/2} \in \mathfrak{S}^2\}$$

endowed with its natural norm. In the one-body case, the well-posedness theory is easy because the space H^s is an algebra (implying that the nonlinearity stabilizes H^s). In the

many-body case, the situation is a bit different due to the more general structure of the nonlinearity. However, it is still well-behaved due to the estimates valid for all $s > 3/2$

$$\begin{aligned} \|f\gamma\|_{\mathfrak{H}^s} + \|\gamma f\|_{\mathfrak{H}^s} &\leq C\|f\|_{H^s}\|\gamma\|_{\mathfrak{H}^s}, \\ \|\rho_\gamma\|_{H^s} &\leq C\|\gamma\|_{\mathfrak{H}^s}. \end{aligned}$$

The first one follows from the fact that the multiplication by f is a bounded operator on H^s if $f \in H^s$ since H^s is an algebra, and the second one follows for instance from the fact that

$$\|\gamma\|_{\mathfrak{H}^s}^2 = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |p|^2)^s (1 + |q|^2)^s |\widehat{\gamma}(p, q)|^2 dp dq,$$

where $\widehat{\gamma}(p, q)$ denotes the integral kernel of $\mathcal{F}\gamma\mathcal{F}^*$ (\mathcal{F} being the Fourier transform on L^2). While the first bound clearly holds for all $\alpha \geq 1$ (if we replace \mathfrak{S}^2 by \mathfrak{S}^α in the definition of \mathfrak{H}^s), we do not know if the second holds for some $\alpha > 2$. This is a first reason to consider $\alpha = 2$ (in the sense that it is the maximal α for which we know the second estimate). Using these two estimates, it is easy to show that the equations (1.20) and (1.21) are locally well-posed in \mathfrak{H}^s (with $S \in H^s$ as well).

As we saw in the one-body case, obtaining well-posedness in \mathfrak{H}^s in this way is however not enough: an important part of the argument was that we have a blow-up criterion in L^∞ (so that it is controlled by lower $H^{s'}$ -norms, which are the one for which we have quantitative convergence). This blow-up criterion in L^∞ followed in the one-body case from the Kato-Ponce inequality. In [90, Lemma 3.3], we proved the following replacement of the Kato-Ponce inequality for density matrices: for all $s > s' > 3/2$, $f \in H^s$, and all non-negative $\gamma \in \mathfrak{H}^s$, we have

$$\begin{aligned} \|f\gamma\|_{\mathfrak{H}^s} + \|\gamma f\|_{\mathfrak{H}^s} &\leq C(\|f\|_{L^\infty}\|\gamma\|_{\mathfrak{H}^s} + \|f\|_{H^s}\|\gamma\|_{\mathfrak{H}^s}^{1/2}\|\gamma\|_{\mathfrak{H}^{s'}}^{1/2}), \\ \|\rho_\gamma\|_{H^s} &\leq C\|\gamma\|_{\mathfrak{H}^s}^{1/2}\|\gamma\|_{\mathfrak{H}^{s'}}^{1/2}, \end{aligned}$$

In this inequality, we use a $\mathfrak{H}^{s'}$ for the operator γ since it is not clear what should replace the L^∞ norm which appears in the usual Kato-Ponce inequality. Notice also that we proved this inequality for *non-negative* γ , which is actually necessary since we can show that the second inequality fails for γ which are not non-negative. An important remark is that non-negativity is preserved along the flow. To show these inequalities, we also used the specificity of the \mathfrak{S}^2 -norm which can be computed using the L^2 -norm of the integral kernel. Hence, we do not know whether these inequalities hold for other Schatten exponents α . Using these tools, we can prove the following result.

PROPOSITION 1.4 (Prop. 3.4 and Lemma 3.5 in [90]). *Let $s > 3/2$ and $(\gamma_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}}) \in \mathfrak{H}^s \times H^s(\mathbb{R}^3, \mathbb{R}) \times H^{s-1}(\mathbb{R}^3, \mathbb{R})$ be such that $\gamma_{\text{in}} \geq 0$.*

(i) *For all $g, M > 0$ there exist $T_{\min}, T_{\max} \in (0, +\infty]$ and a unique maximal solution*

$$(\gamma, S) \in C^0((-T_{\min}, T_{\max}), \mathfrak{H}^s \times H^s(\mathbb{R}^3, \mathbb{R})),$$

to the Dirac-Klein-Gordon equation (1.11) such that $(\gamma, S, \partial_t S)|_{t=0} = (\gamma_{\text{in}}, S_{\text{in}}, \dot{S}_{\text{in}})$. If $T_{\max/\min} < +\infty$ then for all $3/2 < s' < s$,

$$\limsup_{t \rightarrow T_{\max/\min}} \|(\gamma, S)(t)\|_{\mathfrak{H}^{s'} \times H^{s'}} = +\infty. \quad (1.22)$$

(ii) There exist $T_{\min}^{\text{nl}}, T_{\max}^{\text{nl}} \in (0, +\infty]$ and a unique maximal solution

$$\gamma_{\text{nl}} \in C^0((-T_{\min}^{\text{nl}}, T_{\max}^{\text{nl}}), \mathfrak{H}^s),$$

to the nonlinear Dirac equation (1.14) such that $(\gamma_{\text{nl}})|_{t=0} = \gamma_{\text{in}}$. If $T_{\max/\min}^{\text{nl}} < +\infty$ then for all $3/2 < s' < s$,

$$\limsup_{t \rightarrow T_{\max/\min}^{\text{nl}}} \|\gamma_{\text{nl}}(t)\|_{\mathfrak{H}^{s'}} = +\infty.$$

While this result is interesting in its own, it is still not enough to deal with the limit $M \rightarrow +\infty$ as in the one-body case. Again, it is useful to use the equation on the reduced field $\bar{S} := S + \rho_{\beta\gamma}$ and it turns out that we have the same regularizing phenomenon for the equation on \bar{S} as we have in the one-body case. From this reduced equation and the estimates listed above, it is not too hard to adapt the one-body strategy to obtain the following convergence theorem.

THEOREM 1.4 (Theorem 2 in [90]). *Let $s > 5/2$, $\gamma_{\text{in}} \in \mathfrak{H}^s$ be a non-negative operator, and $(S_{\text{in}}, \dot{S}_{\text{in}}) \in H^s(\mathbb{R}^3, \mathbb{R}) \times H^{s-1}(\mathbb{R}^3, \mathbb{R})$. Let*

$$\gamma_{\text{nl}} \in C^0((-T_{\min}^{\text{nl}}, T_{\max}^{\text{nl}}), \mathfrak{H}^s)$$

be the maximal solution to (1.21) with initial condition $\gamma_{\text{nl}}|_{t=0} = \gamma_{\text{in}}$. Let $M > 0$ and let

$$(\gamma, S) \in C^0((-T_{\min}, T_{\max}), \mathfrak{H}^s \times H^s(\mathbb{R}^3, \mathbb{R})),$$

be the maximal solution to (1.20) with $g = M$, and initial conditions

$$\gamma|_{t=0} = \gamma_{\text{in}}, (S, \partial_t S)|_{t=0} = (S_{\text{in}}, \dot{S}_{\text{in}}).$$

Then, we have

$$\liminf_{M \rightarrow +\infty} T_{\min/\max} \geq T_{\min/\max}^{\text{nl}}$$

and, for all $0 < T_1 < T_{\min}^{\text{nl}}$, $0 < T_2 < T_{\max}^{\text{nl}}$, and all $0 \leq s' < s$,

$$\lim_{M \rightarrow +\infty} \|\gamma - \gamma_{\text{nl}}\|_{L^\infty([-T_1, T_2], \mathfrak{H}^{s'})} = 0.$$

This result may be a first step towards the “true” many-body problem associated to relativistic particles, where one has to take into account the *Dirac sea* filling all negative energy states. Since this state describes an infinite number of particles with a uniform spatial distribution, this would extend the theory of the first part to relativistic systems.

Appendix: The density of a non-negative operator

For a density matrix γ , it is a well-known fact that one can define the density ρ_γ if γ is trace-class (in which case ρ_γ is integrable) or if γ is locally trace-class (in which case ρ_γ is locally trace-class). The following result shows that the density is always well-defined for a non-negative operator.

PROPOSITION 1.5 (Density of an operator).

- (1) Let γ be a non-negative bounded operator on $L^2(\mathbb{R}^d)$. Then, there exists a unique measurable $\rho_\gamma : \mathbb{R}^d \rightarrow [0, +\infty]$ called its density such that for any Borel set $B \subset \mathbb{R}^d$, we have $\int_B \rho_\gamma = \text{Tr} \sqrt{\gamma} \mathbb{1}_B \sqrt{\gamma} = \text{Tr} \mathbb{1}_B \gamma \mathbb{1}_B$.
- (2) In particular, when γ is trace-class and non-negative, $\rho_\gamma \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} \rho_\gamma = \text{Tr} \gamma$. In this case, the map $\gamma \in \mathfrak{S}^1 \mapsto \rho_\gamma \in L^1(\mathbb{R}^d)$ can be extended to the whole trace-class (not assuming non-negativity anymore) in a unique way such that for all Borel sets $B \subset \mathbb{R}^d$, $\int_B \rho_\gamma = \text{Tr} \mathbb{1}_B \gamma \mathbb{1}_B = \text{Tr} \mathbb{1}_B \gamma = \text{Tr} \gamma \mathbb{1}_B$.
- (3) Finally, the map $\gamma \in \mathfrak{S}^1 \mapsto \rho_\gamma \in L^1(\mathbb{R}^d)$ can be extended to a map $\gamma \in \mathfrak{S}_{\text{loc}}^1 \mapsto \rho_\gamma \in L_{\text{loc}}^1(\mathbb{R}^d)$ (where we recall that $\mathfrak{S}_{\text{loc}}^1$ is the set of all bounded operators γ such that $\mathbb{1}_K \gamma \mathbb{1}_K$ is trace-class for any compact set $K \subset \mathbb{R}^d$) in a unique way such that for all bounded Borel sets $B \subset \mathbb{R}^d$, $\int_B \rho_\gamma = \text{Tr} \mathbb{1}_B \gamma \mathbb{1}_B$.

PROOF. Let γ be a non-negative bounded operator and $B \subset \mathbb{R}^d$ be a Borel set. Then, the operator $\sqrt{\gamma} \mathbb{1}_B \sqrt{\gamma}$ is also bounded and non-negative so that its trace is always well-defined in $[0, +\infty]$ [137, Thm. 3.6.2]. Furthermore, the map $B \mapsto \text{Tr} \sqrt{\gamma} \mathbb{1}_B \sqrt{\gamma}$ is clearly a Borel measure (it vanishes on the empty set and it is σ -additive by exchanging sums of positive terms) which is absolutely continuous with respect to the Lebesgue measure (since $|B| = 0$ implies that $\mathbb{1}_B$ is the zero operator), so that the existence and uniqueness of ρ_γ in (1) follows from the Radon-Nikodym theorem [47, Thm. 4.1.6] (notice that in standard statements of the Radon-Nikodym theorem, the measures are required to be σ -finite. This is not necessarily the case here; however for the theorem to hold only the σ -finiteness of the Lebesgue measure is required, see [47, Exercise 4.1.(3)]). The identity $\text{Tr} \sqrt{\gamma} \mathbb{1}_B \sqrt{\gamma} = \text{Tr} \mathbb{1}_B \gamma \mathbb{1}_B$ holds because for any bounded operator A , we have $\text{Tr} AA^* = \text{Tr} A^*A$ [137, Thm. 3.6.1]. When γ is trace-class, then $\text{Tr} \mathbb{1}_B \gamma \mathbb{1}_B = \text{Tr} \mathbb{1}_B \gamma = \text{Tr} \gamma \mathbb{1}_B$ [137, Thm. 3.6.7], and the existence of the map $\gamma \in \mathfrak{S}^1 \mapsto \rho_\gamma \in L^1(\mathbb{R}^d)$ follows from the Radon-Nikodym theorem applied to the complex measure $B \mapsto \text{Tr} \mathbb{1}_B \gamma$ which is absolutely continuous with respect to the Lebesgue measure [47, Thm. 4.2.4]. By the uniqueness part of the Radon-Nikodym theorem, the two notions of densities from (1) and (2) coincide when γ is non-negative and trace-class. Finally, when γ is only assumed to be locally trace-class, one can apply point (2) to construct $\rho_{\mathbb{1}_K \gamma \mathbb{1}_K} \in L^1(\mathbb{R}^d)$ for any compact $K \subset \mathbb{R}^d$. Clearly, $\rho_{\mathbb{1}_K \gamma \mathbb{1}_K}$ is supported on K and if $K \subset K'$, then $\rho_{\mathbb{1}_K \gamma \mathbb{1}_K} = \rho_{\mathbb{1}_{K'} \gamma \mathbb{1}_{K'}}$ a.e. on K . Hence, $\rho_\gamma \in L_{\text{loc}}^1(\mathbb{R}^d)$ can be uniquely defined by $\rho_\gamma = \rho_{\mathbb{1}_K \gamma \mathbb{1}_K}$ a.e. on K , for any compact $K \subset \mathbb{R}^d$. For any bounded Borel set B , letting K any compact such that $B \subset K$, we then have $\int_B \rho_\gamma = \int_B \rho_{\mathbb{1}_K \gamma \mathbb{1}_K} = \text{Tr}(\mathbb{1}_K \gamma \mathbb{1}_K \mathbb{1}_B) = \text{Tr} \mathbb{1}_B \gamma \mathbb{1}_B$. If γ is trace-class, then this notion of density reduces to the one defined in (2) because in this case $\rho_{\mathbb{1}_K \gamma \mathbb{1}_K} = \mathbb{1}_K \rho_\gamma$. If γ is non-negative, then the two notions of density defined in (1) and (3) are also the same because we again have $\rho_{\mathbb{1}_K \gamma \mathbb{1}_K} = \mathbb{1}_K \rho_\gamma$. \square

REMARK 1.6. Point (1) is useful in situations where we consider operators which are not necessarily locally trace-class (typically, if it belongs to a Schatten class \mathfrak{S}^α with $\alpha > 1$). For such a non-negative compact operator γ , one can find a sequence (γ_n) of finite-rank operators such that $0 \leq \gamma_n \leq \gamma$ for all n , converging weakly to γ in the sense of operators, and such that for any $f \in L^2(\mathbb{R}^d)$, $\langle f, \gamma_n f \rangle \rightarrow \langle f, \gamma f \rangle$ as $n \rightarrow \infty$ increasingly. This implies

that $\int_B \rho_{\gamma_n} \rightarrow \int_B \rho_\gamma$ increasingly for all Borel sets B , so that (ρ_{γ_n}) converges increasingly to ρ_γ , almost everywhere. This is useful for density arguments, where one can prove a property for finite-rank operators and then extend it to a more general class using this approximation.

REMARK 1.7. In a similar way, we can define $\rho_{A\gamma A^*}$ where γ is a non-negative operator on a separable Hilbert space \mathcal{H} and $A : \mathcal{H} \rightarrow L^p(\mathbb{R}^d)$ for some $p \geq 2$. Indeed, for any finite measure Borel set $B \subset \mathbb{R}^d$, the operator $\mathbf{1}_B A \sqrt{\gamma} : \mathcal{H} \rightarrow L^2(\mathbb{R}^d)$ is bounded so the trace $\text{Tr}_{L^2(\mathbb{R}^d)} \mathbf{1}_B A \gamma A^* \mathbf{1}_B = \text{Tr}_{\mathcal{H}} \sqrt{\gamma} A^* \mathbf{1}_B A \sqrt{\gamma}$ is well-defined.

Fermionic functional inequalities

In their seminal works [105, 106], Lieb and Thirring proved the following inequality: for any $d \geq 1$ and any $(f_j)_{j=1}^N \subset H^1(\mathbb{R}^d)$ which are orthonormal in $L^2(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} \left(\sum_{j=1}^N |f_j|^2 \right)^{1+2/d} dx \leq C \sum_{j=1}^N \int_{\mathbb{R}^d} |\nabla f_j|^2 dx, \quad (2.1)$$

for some $C > 0$ independent of (f_j) and of N . Their motivation to introduce such a result was coming from the stability of matter in many-body quantum mechanics. Indeed, (2.1) gives a lower bound on the kinetic energy of N fermions occupying the states f_1, \dots, f_N . Notice that for $N = 1$, (2.1) reduces to the Gagliardo-Nirenberg-Sobolev interpolation inequality,

$$\|f\|_{L^{2+4/d}(\mathbb{R}^d)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^d)}^{d/(d+2)}, \quad (2.2)$$

for any $f \in H^1(\mathbb{R}^d)$ such that $\|f\|_{L^2} = 1$. Similarly, one can deduce a version of (2.1) combining the one-function inequality (2.2) together with the triangle inequality:

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\sum_{j=1}^N |f_j|^2 \right)^{1+2/d} dx &= \left\| \sum_{j=1}^N |f_j|^2 \right\|_{L^{1+2/d}}^{1+2/d} \\ &\leq \left(\sum_{j=1}^N \|f_j\|_{L^{2+4/d}}^2 \right)^{(d+2)/d} \\ &\leq C \left(\sum_{j=1}^N \|\nabla f_j\|_2^{2d/(d+2)} \right)^{(d+2)/d} \\ &\leq C \left(N^{2/(d+2)} \left(\sum_{j=1}^N \|\nabla f_j\|_{L^2}^2 \right)^{d/(d+2)} \right)^{(d+2)/d} \\ &= CN^{2/d} \sum_{j=1}^N \int_{\mathbb{R}^d} |\nabla f_j|^2 dx. \end{aligned}$$

Compared to (2.1), this argument leads to an additional $N^{2/d}$ on the right side of the inequality, which of course behaves badly as $N \rightarrow +\infty$. One can thus see (2.1) as an improvement over (2.2), and this improvement is due to the orthogonality condition on the (f_j) (indeed, if this condition was dropped, one could take all the f_j 's equal to the same function, in which case the prefactor $N^{2/d}$ becomes sharp). Many different proofs of (2.1) have been found since the original work of Lieb and Thirring, and we will give one proof in Section 4 below, where

the usefulness of the orthogonality condition will become apparent. After the works of Lieb and Thirring, the same kind of result was obtained by Lieb [101] for the Sobolev inequality, and more recently by Frank, Lewin, Lieb and Seiringer [62] for the Strichartz inequality.

In this chapter, we discuss various extensions of the previous phenomenon which we call *fermionic* functional inequalities; in the sense that one generalizes a “one-function” inequality to systems of orthonormal functions, in such a way that the estimate depends non-trivially on the number of functions involved. We call these estimates “fermionic” because of the key orthogonality condition, which is a reflection of Pauli’s exclusion principle stating that fermions cannot occupy the same quantum state (hence their wavefunctions must be orthogonal). The general setting (which is not quite applicable to the Lieb-Thirring inequality above) can be summed up in the following way. Let \mathcal{H} be a Hilbert space, X a measure space and $p \geq 2$. Let $A : \mathcal{H} \rightarrow L^p(X)$ be a bounded linear operator. The question is then to find the best $C(N) > 0$ such that for any orthonormal system $(f_j)_{j=1}^N \subset \mathcal{H}$ one has the bound

$$\left\| \sum_{j=1}^N |Af_j|^2 \right\|_{L^{p/2}(X)} \leq C(N). \quad (2.3)$$

Notice that the triangle inequality implies that (2.3) holds with $C(N) = N\|A\|_{\mathcal{H} \rightarrow L^p}^2$, and one may ask whether the N -dependence of $C(N)$ as $N \rightarrow +\infty$ can be reduced (to N^θ for some $\theta \in (0, 1)$, typically). The mechanism behind such an answer was discovered by Lieb [101], who realized that by introducing the operator γ which is the orthogonal projection on the space spanned by the (f_j) , i.e.

$$\gamma = \sum_{j=1}^N |f_j\rangle\langle f_j|,$$

the estimate (2.3) is equivalent by duality to

$$C(N)\|V\|_{L^{(p/2)'}(X)} \geq \int_X V(x) \sum_{j=1}^N |Af_j|^2 dx = \sum_{j=1}^N \langle Af_j, VAf_j \rangle_{L^2(X)} = \text{Tr}_{\mathcal{H}}(A^*VA\gamma)$$

for all non-negative V . By this argument, (2.3) would follow from the *Schatten space* estimate

$$\|A^*VA\|_{\mathfrak{S}^\alpha(\mathcal{H})} \leq C\|V\|_{L^{(p/2)'}(X)} \quad (2.4)$$

for some $\alpha \geq 1$, where we recall that $\|B\|_{\mathfrak{S}^\alpha(\mathcal{H})} = (\text{Tr}_{\mathcal{H}} |B|^\alpha)^{1/\alpha}$ for any Hilbert space \mathcal{H} and any operator B on \mathcal{H} . Indeed, Schatten spaces satisfy the Hölder inequality

$$\text{Tr}_{\mathcal{H}}(A^*VA\gamma) \leq \|A^*VA\|_{\mathfrak{S}^\alpha} \|\gamma\|_{\mathfrak{S}^{\alpha'}},$$

and we also have $\|\gamma\|_{\mathfrak{S}^{\alpha'}} = N^{1/\alpha'}$ (this is the step where the orthogonality of the (f_j) is used!). Notice also that

$$\|A^*VA\|_{\mathfrak{S}^\alpha(\mathcal{H})} = \|\sqrt{V}AA^*\sqrt{V}\|_{\mathfrak{S}^\alpha(L^2(X))} = \|\sqrt{V}A\|_{\mathfrak{S}^{2\alpha}(\mathcal{H} \rightarrow L^2(X))}^2,$$

which provides another interpretation for fermionic inequality: the “one-function” estimate states that A is bounded from \mathcal{H} to $L^p(X)$, which by the Hölder inequality is equivalent to

$$\|\sqrt{V}A\|_{\mathcal{H} \rightarrow L^2(X)} \leq C\|V\|_{L^{(p/2)'}(X)}.$$

The fermionic version of the “one-function” inequality thus states that one can improve this $\mathcal{H} \rightarrow L^2(X)$ operator bound into a Schatten space bound; meaning that the operator $\sqrt{V}A$ is better than a bounded operator, it is a compact operator with some explicit rate of decay of its singular values.

We will see below how to obtain such Schatten bounds for some specific operator A , but let us already give an example where such a principle can be applied. Consider the operator $A = (1 - \Delta)^{-s/2}$ defined on $\mathcal{H} = L^2(\mathbb{R}^d)$. Then, Sobolev embeddings imply that A is a bounded operator from $L^2(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$, for any $p \geq 2$ such that $1/p \geq 1/2 - s/d$. This “one-function” property can be improved to a Schatten bound using the Kato-Seiler-Simon inequality [133, Theorem 4.1],

$$\|f(x)g(-i\nabla)\|_{\mathfrak{S}^\alpha(L^2(\mathbb{R}^d))} \leq C\|f\|_{L^\alpha(\mathbb{R}^d)}\|g\|_{L^\alpha(\mathbb{R}^d)},$$

valid for all $\alpha \in [2, +\infty]$. One deduces that

$$\|\sqrt{V}(1 - \Delta)^{-s/2}\|_{\mathfrak{S}^{2(p/2)'}(L^2(\mathbb{R}^d))} \leq C\|V\|_{L^{(p/2)'(\mathbb{R}^d)}}$$

for all $p \geq 2$ such that $2s(p/2)' > d \iff 1/p > 1/2 - s/d$. The borderline case $1/p = 1/2 - s/d$ can also be treated in this way (replacing the Kato-Seiler-Simon bound by the Cwikel-Lieb-Rozenblum bound, as was actually done by Lieb in [101]), but we will not detail it here.

Finally, let us notice that the Schatten bound (2.4) is not equivalent to (2.3). To find an equivalent formulation in terms of orthonormal functions, one needs a stronger version of (2.3), namely

$$\left\| \sum_j \lambda_j |Af_j|^2 \right\|_{L^{p/2}(X)} \leq C \left(\sum_j |\lambda_j|^{\alpha'} \right)^{1/\alpha'},$$

for any orthonormal system (f_j) and any coefficients $(\lambda_j) \subset \mathbb{C}$. One can encode the (λ_j) and the (f_j) into the density matrix $\gamma = \sum_j \lambda_j |f_j\rangle\langle f_j|$ so that the inequality becomes

$$\|\rho_{A\gamma A^*}\|_{L^{p/2}(X)} \leq C\|\gamma\|_{\mathfrak{S}^{\alpha'}}, \quad (2.5)$$

which is equivalent to (2.4). This formulation will be useful later on.

The plan of this chapter is the following. In Section 1, we explain the results of [65] where we identify a general strategy to prove estimates of the type (2.4), and which we apply to various cases such as Fourier restriction estimates, Strichartz estimates, and uniform Sobolev estimates. In Section 2, we apply the same strategy to Sogge’s L^p spectral cluster estimates and interpret the results in terms of measuring the concentration of orthonormal systems of quasimodes, investigating their optimality as well. In Section 3, we leave the topic of fermionic estimates and go to the related question of asymptotics of large fermionic systems as described pointwise Weyl laws. Finally, we finish by Section 4 where we give a fermionic version of the Littlewood-Paley theorem and apply it to prove the Lieb-Thirring inequality (2.1).

1. Fourier restriction and Strichartz inequalities

We begin with presenting the work of [65] where, motivated by Strichartz and Fourier restriction inequalities, we found a general framework in which to prove (2.4). We first give this framework before applying it to several contexts.

1.1. Complex interpolation in Schatten spaces. Many inequalities implying that a linear operator $A : \mathcal{H} \rightarrow L^p(X)$ is bounded can be proved via *complex interpolation*. Using that \mathcal{H} is a Hilbert space, it is equivalent to the fact that $AA^* : L^{p'}(X) \rightarrow L^p(X)$ is bounded, and if one can find an analytic family $(T_z)_{z \in S}$ of operators defined on the strip $S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$ such that

- (1) $\exists \theta \in [0, 1], T_\theta = AA^*$;
- (2) $\exists a_0, C_0 > 0, \forall t \in \mathbb{R}, \|T_{it}\|_{L^2 \rightarrow L^2} \leq C_0 e^{a_0|t|}$,
- (3) $\exists a_1, C_1 > 0, \forall t \in \mathbb{R}, \|T_{1+it}\|_{L^1 \rightarrow L^\infty} \leq C_1 e^{a_1|t|}$,

then one can apply a theorem of Stein [143] to infer that $T_\theta = AA^*$ is bounded from $L^{p'}$ to L^p with $p = 2/(1 - \theta)$. The idea behind these assumptions is that one can deform the operator AA^* using a complex parameter z , in such a way that it becomes bounded from L^2 to L^2 for $\operatorname{Re} z = 0$ and from L^1 to L^∞ for $\operatorname{Re} z = 1$. We will see how to perform such deformation in some explicit cases below. Notice that in the case where $z \mapsto T_z$ is constant (that is, AA^* is itself bounded from L^2 to L^2 and from L^1 to L^∞), one deduces that AA^* is bounded from $L^{p'}$ to L^p for all $p \in [2, +\infty]$, and this special case is called the Riesz-Thorin theorem [136]. It is interesting to note that both the Stein and Riesz-Thorin follow from Hadamard's three line theorem, but noticing that one could consider an analytic dependence on z in the family of operators T_z as Stein did leads to far more reaching applications.

Our main input is that under the *same* assumptions on the operator A , one gets automatically a Schatten improvement in the form (2.4) as given by the following result.

PROPOSITION 2.1 (Proposition 1 in [65]). *Let $d \geq 1$. Let (T_z) be an analytic family of operators from $(L^1 \cap L^\infty)(\mathbb{R}^d)$ to $(L^1 + L^\infty)(\mathbb{R}^d)$ defined on the strip $S = \{z \in \mathbb{C} : 0 \leq \operatorname{Re} z \leq 1\}$. Assume that we have the bounds*

$$\|T_{it}\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \leq C_0 e^{a_0|t|}, \quad \|T_{1+it}\|_{L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)} \leq C_1 e^{a_1|t|}, \quad \forall t \in \mathbb{R}, \quad (2.6)$$

for some $a_0, a_1 \geq 0$ and for some $C_0, C_1 \geq 0$. Then, for any $\theta \in [0, 1]$ and for all $W_1, W_2 \in L^{2/\theta}(\mathbb{R}^d, \mathbb{C})$, the operator $W_1 T_\theta W_2$ belongs to $\mathfrak{S}^{2/\theta}(L^2(\mathbb{R}^d))$ and we have the estimate

$$\|W_1 T_\theta W_2\|_{\mathfrak{S}^{2/\theta}(L^2(\mathbb{R}^d))} \leq C_0^{1-\theta} C_1^\theta \|W_1\|_{L^{2/\theta}(\mathbb{R}^d)} \|W_2\|_{L^{2/\theta}(\mathbb{R}^d)}. \quad (2.7)$$

Proposition 2.1 follows from the simple remark that for $\operatorname{Re} z = 1$, knowing that the operator T_z is bounded from L^1 to L^∞ actually implies more than the operator $W_1 T_z W_2$ is bounded from L^2 to L^2 if $W_1, W_2 \in L^2$. Indeed, the Dunford-Pettis theorem [52, Theorem 2.2.5] implies that T_z has an integral kernel $T_z(\cdot, \cdot)$ which satisfies

$$\|T_z(\cdot, \cdot)\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} = \|T_z\|_{L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)}.$$

This implies that for any $W_1, W_2 \in L^2(\mathbb{R}^d)$ we have

$$\|W_1 T_z W_2\|_{\mathfrak{S}^2}^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |W_1(x)|^2 |T_z(x, y)|^2 |W_2(y)|^2 dx dy \leq \|T_z\|_{L^1 \rightarrow L^\infty}^2 \|W_1\|_{L^2}^2 \|W_2\|_{L^2}^2.$$

Here, we used the characterization [133, Theorem 2.11] of Hilbert-Schmidt operators K by their integral kernel $K(\cdot, \cdot)$, namely

$$\|K\|_{\mathfrak{S}^2}^2 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |K(x, y)|^2 dx dy.$$

Hence, the operator $W_1 T_z W_2$ is better than bounded from L^2 to L^2 , it is actually Hilbert-Schmidt. Interpolating this result with the boundedness from L^2 to L^2 for $\operatorname{Re} z = 0$, one obtains Proposition 2.1.

1.2. Application to Fourier restriction inequalities. We first apply the principle of Proposition 2.1 to the Fourier restriction inequalities due to Stein [145], Tomas [151], and Strichartz [147]. Their setting is the following. Consider an hypersurface $S \subset \mathbb{R}^N$ (with $N \geq 2$), endowed with a Borel measure σ (in all cases considered, σ is a –possibly singular– factor $m(\xi)$ of the Hausdorff measure on S). For any $f \in L^1(S, d\sigma)$, consider its *Fourier extension* $\mathcal{E}_S(f) : \mathbb{R}^N \rightarrow \mathbb{C}$ defined by

$$\mathcal{E}_S(f)(x) = \int_S f(\omega) e^{ix \cdot \omega} d\sigma(\omega), \quad x \in \mathbb{R}^N.$$

For the same reason as the standard Fourier transform, $\mathcal{E}_S(f)$ is a bounded continuous function on \mathbb{R}^N for any $f \in L^1(S, d\sigma)$, with $\|\mathcal{E}_S(f)\|_{L^\infty(\mathbb{R}^N)} \leq \|f\|_{L^1(S)}$. The general question asked by Stein is to find other exponents (p, q) for which an inequality of the type

$$\|\mathcal{E}_S(f)\|_{L^q(\mathbb{R}^N)} \leq C \|f\|_{L^p(S)} \tag{2.8}$$

holds, for all $f \in (L^1 \cap L^p)(S, d\sigma)$. Clearly, if $\sigma(S)$ is finite, then this property holds for $(q, p) = (\infty, p)$ for any $p \geq 1$. Finding some $q < +\infty$ and $p \geq 1$ for which this holds is not obvious, and wrong in full generality (if for instance σ is a Dirac measure). Having $q < +\infty$ means that $\mathcal{E}_S(f)$ has some decay at infinity, which should be understood from the fast oscillations of $e^{ix \cdot \omega}$ as $|x| \rightarrow +\infty$. Another interpretation of this question comes from its dual version, which states that $\|\widehat{F}|_S\|_{L^{p'}(S)} \leq C \|F\|_{L^{q'}(\mathbb{R}^N)}$ for any $F \in (L^1 \cap L^{q'})(\mathbb{R}^N)$. This dual version means that the Fourier transform of a function $F \in L^{q'}(\mathbb{R}^N)$ can be restricted to S in an $L^{p'}(S)$ sense. This property fails for $q = 2$ (if the measure σ is the Hausdorff measure on S for instance) since \widehat{F} can be any function in $L^2(\mathbb{R}^N)$ and thus cannot be restricted to the zero-measure set S in any sense.

The striking result due to Stein-Tomas [145, 151] is the fact that, in the case where S is compact with non-vanishing Gauss curvature and σ is the Hausdorff measure on S , this Fourier extension property holds for $p = 2$ and $q = 2(N + 1)/(N - 1)$. Furthermore, this q is the smallest possible exponent for $p = 2$ (this can be understood from the Knapp counterexample, which amounts to test the inequality against a function which concentrates around a point on S). This result has been extended by Strichartz [147] (still with $p = 2$ but with some different values of q 's) to some non-compact hypersurfaces S , namely zero sets of non-degenerate degree two polynomials (for instance, paraboloids, cones, or hyperboloids).

The case $p = 2$ is particular because $L^2(S)$ is a Hilbert space, so that (2.8) is equivalent to the boundedness of $T = \mathcal{E}_S(\mathcal{E}_S)^*$ from $L^q(\mathbb{R}^N)$ to $L^q(\mathbb{R}^N)$. There are several ways to show this boundedness, but we focus on the approach of Stein relying on complex interpolation to make the connection with Proposition 2.1. For this argument, let us assume that S is the graph

$$\{\xi = (\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \xi_N = \varphi(\xi')\}$$

of some smooth function $\varphi : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ (for a compact S , this can be assumed locally by reducing to local patches and summing up the estimates, while some quadratic surfaces like the paraboloid $\xi_N = |\xi'|^2$ also have this form. We will not mention how to treat other quadratic surfaces like the cone or the hyperboloid for brevity). In this case, we have for any $f : S \rightarrow \mathbb{C}$,

$$\int_S f(\xi) d\sigma(\xi) = \int_{\mathbb{R}^{N-1}} f(\xi', \varphi(\xi')) \chi(\xi') d\xi'$$

for some $\chi \in C^\infty(\mathbb{R}^{N-1})$ (for compact surfaces, χ can be assumed to be compactly supported and for quadratic surfaces, σ is such that $\chi \equiv 1$. In all cases, χ is a bounded function). Then, one can deform the operator T by defining T_z the Fourier multiplier by the function

$$M_z(\xi) = \Gamma(z+1)^{-1} (\xi_N - \varphi(\xi'))_+^z \chi(\xi').$$

Using the fact that $t_+^z \sim_{z \rightarrow -1} \delta_0 / (z+1)$ in $\mathcal{D}'_t(\mathbb{R})$, we recover our operator T when evaluating T_z at $z = -1$. The factor $\Gamma(z+1)^{-1}$ compensates the singularities of the distribution t_+^z at negative integers to ensure that $(T_z)_{z \in \mathbb{C}}$ is analytic. For $\operatorname{Re} z = 0$, the function $\xi \mapsto (\xi_N - \varphi(\xi'))_+^z \chi(\xi')$ is bounded so T_z is bounded from L^2 to L^2 if $\operatorname{Re} z = 0$. By complex interpolation, it thus remains to bound T_z from L^1 to L^∞ , for $\operatorname{Re} z = \lambda$, for some $\lambda < -1$. Since T_z is a Fourier multiplier, such a bound would follow from estimating \widehat{M}_z in $L^\infty(\mathbb{R}^N)$. For any $x \in \mathbb{R}^N$, $\widehat{M}_z(x)$ is proportional to

$$\int_{\mathbb{R}^N} (\xi_N - \varphi(\xi'))_+^z \chi(\xi') e^{-ix \cdot \xi} d\xi = \int_{\mathbb{R}} (\xi_N)_+^z e^{-ix_N \xi_N} \int_{\mathbb{R}^{N-1}} \chi(\xi') e^{-ix_N \varphi(\xi') - ix' \cdot \xi'} d\xi'.$$

The first factor is the Fourier transform of the distribution t_+^z , which is known to be proportional to $(x_N + i0)^{-z-1}$. If $\lambda = \operatorname{Re} z < -1$, we have $\operatorname{Re}(-z-1) > 0$ and hence $(x_N + i0)^{-z-1} \in L^1_{\text{loc}}(\mathbb{R})$ with $|(x_N + i0)^{-z-1}| \leq C|x_N|^{-\lambda-1}$. The second factor can be estimated using stationary phase estimates if $\chi \in C_c^\infty(\mathbb{R}^{N-1})$, or explicit Fourier transform in the quadratic case (for instance, if $\varphi(\xi') = |\xi'|^2$ and $\chi \equiv 1$, this factor is just a complex Gaussian). In all cases, since the Hessian of φ is non-degenerate (due to the curvature condition for compact surfaces, and always the case for quadratic surfaces) the second factor can be estimated by

$$\left| \int_{\mathbb{R}^{N-1}} \chi(\xi') e^{-ix_N \varphi(\xi') - ix' \cdot \xi'} d\xi' \right| \leq C(1 + |x_N|)^{-(N-1)/2}.$$

We deduce that \widehat{M}_z is bounded if $\operatorname{Re} z = \lambda = -(N+1)/2$ (the z -dependence of the constants can be also explicitly estimated, they are all at most exponential). By complex interpolation (Proposition 2.1), one deduces a Schatten bound on the operator T (or, $L^q \rightarrow L^q$ bounds as in the Stein-Tomas and Strichartz works).

THEOREM 2.1 (Theorem 2 in [65]). *Let $N \geq 2$ and S be a compact hypersurface of \mathbb{R}^N with non-zero Gauss curvature. Let $1 \leq q \leq (N+1)/2$. Then, there exists $C > 0$ such that for all $W_1, W_2 \in L^{2q}(\mathbb{R}^N)$, one has*

$$\|W_1 \mathcal{E}_S(\mathcal{E}_S)^* W_2\|_{\mathfrak{S}^{(N-1)q/(N-q)}(L^2(\mathbb{R}^N))} \leq C \|W_1\|_{L^{2q}} \|W_2\|_{L^{2q}}.$$

The case $q = (N+1)/2$ follows from the complex interpolation argument sketched above, while the other extreme case $q = 1$ is specific to compact surfaces. In this case, the estimate states that $W_1 \mathcal{E}_S(\mathcal{E}_S)^* W_2$ is actually trace-class, which may look like a strong property. However, it just follows from the fact that the operator $W_1 \mathcal{E}_S$ is Hilbert-Schmidt as an operator from $L^2(S)$ to $L^2(\mathbb{R}^N)$ when $W_1 \in L^2(\mathbb{R}^N)$, since its integral kernel is $\mathbb{R}^N \times S \ni (x, \xi) \mapsto W_1(x) e^{ix \cdot \xi}$ and belongs to $L^2(\mathbb{R}^N \times S)$ since $\sigma(S)$ is finite. Let us insist once again that Theorem 2.1 is a strenghtening of the Stein-Tomas result (which states that the operator $W_1 \mathcal{E}_S(\mathcal{E}_S)^* W_2$ is merely bounded).

In [65, Theorem 6], we also proved that the Schatten exponent $(N-1)q/(N-q)$ of Theorem 2.1 is optimal; that is the inequality cannot hold with a lower Schatten exponent. To obtain this optimality, we employ the equivalent formulation (2.5) on the test density matrix on $L^2(S)$,

$$\gamma_h := \int_{\mathbb{R}^N} \mathbf{1}(h|x| \leq 1) |e_x\rangle \langle e_x| dx, \quad h > 0,$$

where $e_x(\xi) = e^{-ix \cdot \xi}$ for any $\xi \in S$. To estimate the Schatten norm of γ_h , we first use its non-negativity to infer that $\|\gamma_h\|_{\mathfrak{S}^1} = \text{Tr } \gamma_h = Ch^{-N}$. Secondly, we use that for any $f \in L^2(S)$,

$$\langle f, \gamma_h f \rangle = \int_{|x| \leq 1/h} \left| \int_S f(\xi) e^{ix \cdot \xi} d\sigma(\xi) \right|^2 dx \leq Ch^{-1} \int_S |f(\xi)|^2 d\sigma(\xi)$$

due to the Agmon-Hörmander bound [2] (which follows from writing S as a graph $\xi_N = \varphi(\xi')$, estimating $\mathbf{1}(h|x| \leq 1) \leq \mathbf{1}(h|x_N| \leq 1)$, and applying Plancherel's identity in the x' -integration). We deduce that $\|\gamma_h\|_{L^2 \rightarrow L^2} \leq Ch^{-1}$, so by interpolation we have $\|\gamma_h\|_{\mathfrak{S}^r} \leq C(h^{-1})^{(N-1)/r+1}$. Meanwhile, we also have for any $y \in \mathbb{R}^N$,

$$\rho_{\mathcal{E}_S \gamma_h (\mathcal{E}_S)^*}(y) = \int_{h|x| \leq 1} |\widehat{d\sigma}(y-x)|^2 dx, \quad \widehat{d\sigma}(y) = \int_S e^{iy \cdot \xi} d\sigma(\xi).$$

By stationary phase estimates, there is a lower bound $|\widehat{d\sigma}(y)| \geq c|y|^{-(N-1)/2}$ for large $|y|$, so that $\rho_{\mathcal{E}_S \gamma_h (\mathcal{E}_S)^*}(y) \gtrsim h^{-1}$ for $|y| \lesssim h^{-1}$. Hence, $\|\rho_{\mathcal{E}_S \gamma_h (\mathcal{E}_S)^*}\|_{L^{q'}}$ $\gtrsim h^{-1-N/q'}$, so that

$$\frac{\|\rho_{\mathcal{E}_S \gamma_h (\mathcal{E}_S)^*}\|_{L^{q'}}}{\|\gamma_h\|_{\mathfrak{S}^r}} \gtrsim (h^{-1})^{N/q' - (N-1)/r}.$$

The right side is bounded as $h \rightarrow 0$ only if $r \leq (N-1)q'/N$, that is $\alpha = r' \geq (N-1)q/(N-q)$, which is exactly the exponent found in Theorem 2.1.

1.3. Application to Strichartz estimates. We mentioned above that the complex interpolation argument applies to quadratic surfaces as well, with both Schatten and Lebesgue exponents adapted to the surface (see [65] for details). In the case of the paraboloid $\xi_N = -|\xi'|^2$, we found that

$$\|W_1 \mathcal{E}_S(\mathcal{E}_S)^* W_2\|_{\mathfrak{S}^{N+1}(L^2(\mathbb{R}^N))} \leq C \|W_1\|_{L^{N+1}(\mathbb{R}^N)} \|W_2\|_{L^{N+1}(\mathbb{R}^N)} \quad (2.9)$$

First, let us argue why our argument can be refined to lead to a wider range of estimates. In the complex interpolation, we estimated our deformed operator T_z for $\operatorname{Re} z = -(N+1)/2$ from L^1 to L^∞ using that the function \widehat{M}_z is bounded for these z 's. Recall also that this uniform bound on \widehat{M}_z implies a Hilbert-Schmidt bound on the operator $W_1 T_z W_2$, which integral kernel is proportional to $(x, y) \mapsto W_1(x) \widehat{M}_z(x-y) W_2(y)$ and hence belongs to $L^2(\mathbb{R}^N \times \mathbb{R}^N)$ if $W_1, W_2 \in L^2$ and \widehat{M}_z is bounded. We now argue that this estimate can be made differently, using the bound $|\widehat{M}_z(x_N)| \leq |x_N|^{-(\operatorname{Re} z + (N+1)/2)}$ that we found. Indeed, the Hardy-Littlewood-Sobolev inequality implies that

$$\begin{aligned} \|W_1 T_z W_2\|_{\mathfrak{S}^2}^2 &= c \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |W_1(x)|^2 |\widehat{M}_z(x-y)|^2 |W_2(y)|^2 dx dy \\ &\leq C \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{\|W_1(\cdot, x_N)\|_{L_{x'}^2}^2 \|W_2(\cdot, y_N)\|_{L_{y'}^2}^2}{|x_N|^{2\operatorname{Re} z + N + 1}} dx_N dy_N \\ &\leq C \|W_1\|_{L_{x_N}^{\frac{4}{1-N-2\operatorname{Re} z}} L_{x'}^2}^2 \|W_2\|_{L_{x_N}^{\frac{4}{1-N-2\operatorname{Re} z}} L_{x'}^2}^2 \end{aligned}$$

when $-(N+1)/2 \leq \operatorname{Re} z < -N/2$. By complex interpolation, this leads to the following result.

THEOREM 2.2 (Theorem 9 in [65]). *Let $N \geq 2$ and S be the paraboloid*

$$S = \{(\xi', \xi_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : \xi_N = -|\xi'|^2\}.$$

Let $p, q \geq 2$ be two exponents satisfying

$$\frac{2}{p} + \frac{N-1}{q} = 2, \quad q > \frac{N}{2}.$$

Then, there exists $C > 0$ such that for any $W_1, W_2 \in L_{x_N}^{2p} L_{x'}^{2q}$ we have

$$\|W_1 \mathcal{E}_S(\mathcal{E}_S)^* W_2\|_{\mathfrak{S}^{2q}(L^2(\mathbb{R}^N))} \leq C \|W_1\|_{L_{x_N}^{2p} L_{x'}^{2q}} \|W_2\|_{L_{x_N}^{2p} L_{x'}^{2q}}. \quad (2.10)$$

When $q = (N+1)/2$, we recover the estimate of the previous section. Of course, one can obtain similar estimates for compact surfaces but in this case, the mixed Lebesgue spaces $L_{x_N}^{2p} L_{x'}^{2q}$ are less relevant because the x_N -direction is not intrinsic to the surface (contrary to the case of the paraboloid) and depends on the patch in which one is working.

These mixed norms estimates have a very important role which is related to the relation between the operator \mathcal{E}_S when S is a paraboloid and solutions to the linear Schrödinger equation. Indeed, if $u \in L^2(\mathbb{R}^{N-1})$, then one can define $f : S \rightarrow \mathbb{C}$ by $f(\xi', -|\xi'|^2) = \widehat{u}(\xi')$ for any $\xi' \in \mathbb{R}^{N-1}$. In this case, we have for all $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$,

$$\mathcal{E}_S(f)(x', x_N) = \int_{\mathbb{R}^{N-1}} e^{ix' \cdot \xi' - ix_N |\xi'|^2} \widehat{u}(\xi) d\xi = (2\pi)^{(N-1)/2} (e^{ix_N \Delta_{x'}} u)(x').$$

This means that $U := (2\pi)^{-(N-1)/2} \mathcal{E}_S(f)$ is the solution to the Schrödinger equation

$$\begin{cases} i\partial_{x_N} U = -\Delta_{x'} U, \\ U|_{x_N=0} = u. \end{cases}$$

With this point of view, knowing that $\mathcal{E}_S : L^2(S) \rightarrow L^q(\mathbb{R}^N)$ is a bounded operator is the same as having the *Strichartz estimate*

$$\|e^{ix_N \Delta} u\|_{L^q(\mathbb{R}^N)} \leq C \|u\|_{L^2(\mathbb{R}^{N-1})},$$

which is a way to quantify how small $e^{ix_N \Delta} u$ is as the “time” x_N becomes large, if the initial condition u is merely in $L^2(\mathbb{R}^{N-1})$. In this respect, the mixed norms estimates of Theorem 2.2 imply that

$$\|e^{ix_N \Delta} u\|_{L_{x_N}^{2p'} L_{x'}^{2q'}(\mathbb{R}^N)} \leq C \|u\|_{L^2(\mathbb{R}^{N-1})},$$

for all $p', q' \geq 2$ such that

$$\frac{2}{p'} + \frac{N-1}{q'} = N-1, \quad q' < \frac{N}{N-2}.$$

Such mixed norms estimates are very useful to study nonlinear perturbations of the Schrödinger equation [41, 149]. They are known at least since the works of Ginibre and Velo [73], and hold for a larger range of exponents (namely, $q' < (N-1)/(N-3)$ for $N \geq 4$) than the many-body version of Theorem 2.2. In the many-body case, Theorem 2.2 was proved for the first time by Frank, Lewin, Lieb and Seiringer [62] in the restricted range $q \geq (N+1)/2$, by a different method than the one we presented. They furthermore proved that the Schatten exponent $2q$ in (2.10) is sharp for any value of q , and that (2.10) *fails* for $q = N/2$ (at this endpoint, it is conjectured that (2.10) holds with the Schatten space \mathfrak{S}^N replaced by the weak Schatten space $\mathfrak{S}^{N,\infty}$). The complex interpolation argument of [65] that we presented thus provided the missing range of exponents $N/2 < q < (N+1)/2$. As we already mentioned, the ‘one-body’ version holds for a larger range of exponents, namely $q \geq (N-1)/2$ for $N \geq 4$. This implies that (2.10) holds for $q = (N-1)/2$ with the Schatten norm replaced by the operator norm. In [64, Lemma 2], we proved that at this ‘Keel-Tao’ endpoint $q = (N-1)/2$, one cannot replace the operator norm by a Schatten norm: the operator is not even compact for some adequate choice of W ! Finally, interpolating this operator norm bound at $q = (N-1)/2$ with the would-be weak bound at $q = N/2$, one would deduce a version of (2.10) in the range $(N-1)/2 < q < N/2$ with a Schatten exponent $2q/(2q-N+1)$. In [64, Proposition 3], we proved that this exponent is the best possible.

The dual version of (2.10) as stated in (2.5) has also a dynamical interpretation; namely that for any density matrix γ on $L^2(\mathbb{R}^{N-1})$ one has

$$\|\rho_{e^{ix_N \Delta_{x'}} \gamma e^{-ix_N \Delta_{x'}}}\|_{L_{x_N}^{p'} L_{x'}^{q'}} \leq C \|\gamma\|_{(2q)'}. \quad (2.11)$$

Note here that the operator $\Gamma = e^{ix_N \Delta_{x'}} \gamma e^{-ix_N \Delta_{x'}}$ is a solution to the equation

$$\begin{cases} i\partial_{x_N} \Gamma = [-\Delta_{x'}, \Gamma], \\ \Gamma|_{x_N=0} = \gamma \end{cases}$$

which is the many-body analogue of the linear Schrödinger equation. The many-body Strichartz estimates (2.11) can be used in the same fashion as their one-body version to study some nonlinear mean-field dynamics as the ones of the first chapter, see for instance [127] for some immediate applications.

Finally, let us mention that from the estimates (2.11) one can recover the Strichartz estimates of Castella and Perthame [40] for the linear transport equation (see [127, Lemma 9]) with the same exponents, motivating the form of the estimates (2.11).

1.4. Application to uniform Sobolev estimates. The same complex interpolation strategy can be applied to the 'uniform' Sobolev estimates of Kenig, Ruiz, and Sogge [87], to prove the following result.

THEOREM 2.3 (Theorem 12 in [65]). *Let $N \geq 2$ and assume that*

$$\begin{cases} 4/3 \leq q \leq 3/2 & \text{if } N = 2; \\ N/2 \leq q \leq (N+1)/2 & \text{if } N \geq 3. \end{cases}$$

Then, there exists $C > 0$ such that for all $z \in \mathbb{C} \setminus [0, \infty)$ and all $W_1, W_2 \in L^{2q}(\mathbb{R}^N)$ we have

$$\|W_1(-\Delta - z)^{-1}W_2\|_{\mathfrak{S}^{(N-1)q/(N-q)}(L^2(\mathbb{R}^N))} \leq C|z|^{-1+N/(2q)}\|W_1\|_{L^{2q}}\|W_2\|_{L^{2q}}.$$

If the Schatten norm above is replaced by an operator norm, one recovers the estimate of Kenig, Ruiz, and Sogge. Interestingly, it shows that the resolvent of the laplacian on \mathbb{R}^N does not blow up as the spectral parameter approaches $(0, +\infty)$ if one looks at it as an operator from L^p to $L^{p'}$ (with $p' = 2q'$ with the notations above). In [87], the estimate is proved by complex interpolating an $L^2 \rightarrow L^2$ bound with a $L^1 \rightarrow L^\infty$ bound so that the Schatten bound of Theorem 2.3 follows from applying Proposition 2.1. The Schatten exponent found is actually sharp (notice that it is the same as in the Stein-Tomas bound of Theorem 2.1) since one can recover the Stein-Tomas operator from the resolvent as the spectral parameter approaches $(0, +\infty)$ (as can be understood from the fact that $(t - i\eta)^{-1} - (t + i\eta)^{-1} \rightarrow_{\eta \rightarrow 0_+} 2\pi i\delta_0$ in $\mathcal{D}'_t(\mathbb{R})$).

In [65, Theorem 13], we gave an application of Theorem 2.3 to a limiting absorption principle in Schatten spaces, but let us give another application here to estimating the eigenvalues of Schrödinger operators with complex potentials. For simplicity, let $N \geq 3$ (although the results can be extended $N = 1, 2$ by appropriately modifying the various exponents), $N/2 \leq q < +\infty$ and $V \in L^q(\mathbb{R}^N, \mathbb{C})$. Since the operator $\sqrt{|V|}(1 - \Delta)^{-1/2}$ is compact, one can define the Schrödinger operator $-\Delta + V$ as a m -sectorial operator (see for instance [60, Section 4]) with essential spectrum $[0, +\infty)$ and such that its spectrum outside of $[0, +\infty)$ is discrete and consists of eigenvalues of finite algebraic multiplicities. Notice that since the potential V may be complex-valued, non-real eigenvalues may appear and since we are not dealing with a self-adjoint operator, techniques to estimate these eigenvalues such as the variational principle are not available. In particular, it is possible in principle that some eigenvalues accumulate at a point in $[0, +\infty)$ or even at infinity. In [59], Frank noticed that one could apply the uniform Sobolev estimates of Kenig, Ruiz, and Sogge to estimate these

eigenvalues: if $\lambda \in \sigma(-\Delta + V) \setminus [0, +\infty)$, one has

$$|\lambda|^{q-N/2} \leq C \int_{\mathbb{R}^N} |V(x)|^q dx,$$

if $N/2 < q \leq (N+1)/2$, which excludes for instance accumulation of eigenvalues at infinity. For $q = N/2$, one can only say that for $\|V\|_{L^{N/2}}$ small enough, $-\Delta + V$ has no eigenvalue outside $[0, +\infty)$. In this case, one can still exclude accumulation at infinity but the argument is different. Related to this question, let us mention the article [23] where examples of V (with $q > N$) are constructed such that eigenvalues of $-\Delta + V$ outside $[0, +\infty)$ accumulate at *any* point on $[0, \infty)$. From the Schatten bound of Theorem 2.3 (which is a many-body version of the Kenig, Ruiz, Sogge estimate), we can estimate several eigenvalues as the following result shows.

THEOREM 2.4 (Theorem 16 in [65]). *Let $N \geq 3$, $N/2 \leq q \leq (N+1)/2$, and $V \in L^q(\mathbb{R}^N, \mathbb{C})$. Denote by $(\lambda_j)_{j \in \mathbb{N}}$ the discrete set of eigenvalues outside of $[0, +\infty)$ of $-\Delta + V$, repeated according to their algebraic multiplicity. Then, we have*

- if $q > N/2$, then

$$\sum_{j \geq 0} \frac{d(\lambda_j, [0, +\infty))}{|\lambda_j|^{1-\sigma}} \leq C_\sigma \|V\|_{L^q}^{2\sigma q/(2q-N)},$$

with

$$\begin{cases} \sigma \geq 1/2 & \text{if } q < N^2/(2N-1), \\ \sigma > (N-1)(2q-N)/(2(N-q)) & \text{if } q \geq N^2/(2N-1). \end{cases}$$

- if $q = N/2$, then

$$\sum_{j \geq 0} \frac{\operatorname{Im} \sqrt{\lambda_j}}{1 + |\lambda_j|} < +\infty$$

where the branch of the square root is chosen to have positive imaginary part.

The estimates of Theorem 2.4 control the possible rate of accumulation of eigenvalues on $[0, \infty)$. In particular, it shows that eigenvalues accumulating at a given point on $(0, +\infty)$ must have imaginary parts that converge sufficiently fast to zero such that they are summable. This improves earlier results on the question where it was shown that such imaginary parts must be ℓ^p for larger p .

The connection between Theorem 2.4 and Theorem 2.3 comes from a method of proof developed in [24, 50], which states that estimating the eigenvalues of $-\Delta + V$ amounts to estimating the 'zeroes' of the analytic function

$$z \mapsto A(z) := 1 + \sqrt{V}(-\Delta - z)^{-1} \sqrt{|V|},$$

where we denoted $\sqrt{V} := V/\sqrt{|V|}$. That eigenvalues of $-\Delta + V$ are related to zero eigenvalues of $A(z)$ is well-known and is called the Birman-Schwinger principle (see [60, Prop. 4.1] for the version adapted to this context). Encoding the z 's for which $A(z)$ has a zero eigenvalue can be done via *regularized determinants* [60, Lemma 3.2], denoted by $\det_n A(z)$, which are well-defined when $A(z) - 1 \in \mathfrak{S}^n$ (similarly to the fact that the standard determinant

$\det A$ is well-defined when $A - 1$ is trace-class). Using this object, estimating the eigenvalues of $-\Delta + V$ amounts to estimating the zeroes of the analytic function $z \mapsto \det_n A(z)$ for an adequate value of n (actually, the multiplicities also coincide). The other ingredient is a family of results from complex analysis that relate the accumulation of zeroes of analytic functions at the boundary of their domains to their possible growth at the boundary; the oldest of such results is perhaps Jensen's theorem which states that zeroes (z_n) of a bounded analytic function on the unit disk centered at the origin satisfy $\sum_n (1 - |z_n|) < +\infty$. Here, the 'growth' at the boundary (here, the boundedness up to the boundary) has a consequence on the possible accumulation of $|z_n|$ at 1: it must be fast enough so that $(1 - |z_n|)$ is summable. Such a result was then vastly generalized to account for various behaviours at the boundary beside boundedness (like blow-up at several points or uniform blow-up at the boundary) in [24]. In our setting, the relevant blow-up at the boundary can be estimated using the bound

$$|\det_n A(z)| \leq C \|A(z) - 1\|_{\mathfrak{S}^n}^n,$$

which combined with the Schatten bound of Theorem 2.3 explains the idea behind the proof of Theorem 2.4. Notice that the estimate of Theorem 2.3 blows up only at $z = 0$.

2. Sogge's L^p spectral cluster bounds

We give a final application of the complex interpolation in Schatten spaces [66], to the L^p spectral cluster bounds following Sogge [139, 140]. In this context, the relevant operator is the spectral projection

$$\Pi_\lambda := \mathbf{1}(-\Delta_g \in [\lambda^2, (\lambda + 1)^2]), \quad \lambda \geq 1,$$

associated to the (non-negative) Laplace-Beltrami operator $-\Delta_g$ on a smooth, compact, boundaryless Riemannian manifold (M, g) of dimension $N \geq 2$. Sogge proved that for any $p \geq 2$, one has

$$\|\Pi_\lambda\|_{L^2(M) \rightarrow L^p(M)} \leq C \lambda^{s(p)} \tag{2.12}$$

for all $\lambda \geq 1$ with $C > 0$ independent of λ , where

$$s(p) = \begin{cases} N(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2} & \text{if } \frac{2(N+1)}{N-1} \leq p \leq +\infty, \\ \frac{N-1}{2}(\frac{1}{2} - \frac{1}{p}) & \text{if } 2 \leq p \leq \frac{2(N+1)}{N-1}. \end{cases}$$

The interpretation of (2.12) is that it gives an estimate on the possible *concentration* of functions $f \in L^2(M)$ satisfying $\Pi_\lambda f = f$ and $\|f\|_{L^2(M)} = 1$, as $\lambda \rightarrow +\infty$. Indeed, since $s(p) > 0$ the right side of (2.12) blows up as $\lambda \rightarrow +\infty$, and a growth of an L^p -norm for $p > 2$ while the L^2 -norm remains fixed is a manifestation of concentration on small sets. Functions satisfying $\Pi_\lambda f = f$, i.e. functions in the *spectral cluster* $E_\lambda := \Pi_\lambda L^2(M)$ are linear combinations of eigenfunctions of $-\Delta_g$ associated to eigenvalues in $[\lambda^2, (\lambda + 1)^2]$. As $\lambda \rightarrow +\infty$, these eigenfunctions display a fast oscillating behaviour and linear combinations of them may concentrate in small regions of $-\Delta_g$. Furthermore, the different behaviour of the concentration exponent $s(p)$ according to whether p is larger or lower than $2(N + 1)/(N - 1)$ is a manifestation that different concentration scenarii may happen according to the value of p . Sogge actually proved that the exponent $s(p)$ is sharp (that is, it cannot be lowered),

constructing explicit $f_\lambda \in E_\lambda$ such that $\|f_\lambda\|_{L^2} = 1$ and $\|f_\lambda\|_{L^p} \geq c\lambda^{s(p)}$. In the case of the sphere \mathbb{S}^2 , such functions are called *Gaussian beams* for $p \leq 6$ (they concentrate around an equator), and *zonal spherical harmonics* for $p > 6$ (they concentrate around a point). On the sphere, it thus turns out that one can find *eigenfunctions* saturating the bounds while on any manifold, Sogge showed that such examples can only be found in E_λ .

The link with the themes presented before is that since Π_λ is a projection, (2.12) is equivalent to $\|\Pi_\lambda\|_{L^{p'} \rightarrow L^p} \leq C\lambda^{2s(p)}$. Sogge proved such a bound by reduction to an oscillatory integral operator, which $L^{p'} \rightarrow L^p$ properties are given by theorems of Carleson-Sjölin [33] and Stein [145, Theorem 10]. As it turns out, Stein's proof of these bounds proceed by using complex interpolation between a $L^2 \rightarrow L^2$ and a $L^1 \rightarrow L^\infty$ estimate, so that Proposition 2.1 automatically leads to a many-body version of them:

THEOREM 2.5 (Theorem 2 in [66]). *Let $2 \leq p \leq +\infty$. Then, there exists $C > 0$ such that for all $\lambda \geq 1$ and all density matrix γ we have*

$$\|\rho_{\Pi_\lambda \gamma \Pi_\lambda}\|_{L^{p/2}(M)} \leq C\lambda^{2s(p)} \|\gamma\|_{\mathfrak{S}^{\alpha(p)}}, \quad (2.13)$$

where

$$\alpha(p) = \begin{cases} \frac{p(N-1)}{2N} & \text{if } \frac{2(N+1)}{N-1} \leq p \leq +\infty, \\ \frac{2p}{p+2} & \text{if } 2 \leq p \leq \frac{2(N+1)}{N-1}. \end{cases}$$

Of course, Theorem 2.5 reduces to Sogge's estimate in the case where γ is rank-one. In particular, this shows that the exponent $s(p)$ is optimal in (2.13). Furthermore, the exponent $\alpha(p)$ is also sharp, as can be seen by taking $\gamma = \Pi_\lambda$ itself. Such a fact follows by the pointwise Weyl law, which states that

$$\mathbf{1}(-\Delta_g \leq \lambda^2)(x, x) = (2\pi)^{-N} (|\mathbb{S}^{N-1}|/N)\lambda^N + O_{\lambda \rightarrow +\infty}(\lambda^{N-1}), \quad (2.14)$$

where the remainder is uniform in $x \in M$. Such a strong version of Weyl law was proved by Avakumovic [7], Levitan [91], and Hörmander [79], and has two important consequences related to our question:

- (1) it implies that $\|\rho_{\Pi_\lambda}\|_{L^\infty(M)} \leq C\lambda^{N-1}$, essentially by taking the difference between (2.14) for $\lambda + 1$ and (2.14) for λ ;
- (2) it also implies that

$$\limsup_{\lambda \rightarrow +\infty} \lambda^{-(N-1)} \|\rho_{\Pi_\lambda}\|_{L^1(M)} = \limsup_{\lambda \rightarrow +\infty} \lambda^{-(N-1)} \dim E_\lambda > 0.$$

These two facts imply that all the $L^{p/2}$ -norms of ρ_{Π_λ} are of the same order λ^{N-1} , a manifestation of *delocalization* instead of concentration. Using that $\|\rho_{\Pi_\lambda}\|_{L^{p/2}} \geq c\lambda^{N-1}$ and that $\|\Pi_\lambda\|_{\mathfrak{S}^\alpha} \leq C\lambda^{(N-1)/\alpha}$, one verifies easily that $\alpha(p)$ in (2.13) is sharp.

The above discussion allows to interpret (2.13) as measuring the transition between concentration (which happens for 'one-particle', rank $\gamma = 1$) and delocalization (which happens for the maximal number of particles, rank $\gamma = \dim E_\lambda \sim \lambda^{N-1}$). In [66], we gave a partial answer to a stronger version of optimality of the exponents $s(p)$, $\alpha(p)$. Namely, if one fixes an intermediate number of particles N_λ such that $1 \ll N_\lambda \ll \dim E_\lambda$, can one construct a projection γ_λ with $\gamma_\lambda = \Pi_\lambda \gamma_\lambda \Pi_\lambda$ such that rank $\gamma_\lambda \sim N_\lambda$ and for which $\|\rho_{\gamma_\lambda}\|_{L^{p/2}} \geq c\lambda^{2s(p)} N_\lambda^{1/\alpha(p)}$?

We constructed such optimal 'intermediate' concentrating objects on \mathbb{S}^2 , using precise asymptotics of spherical harmonics [66, Theorem 4].

3. Weyl laws for singular potentials

Let us now switch to a related topic where fermionic inequalities do not appear explicitly. We mentioned in the previous section the importance of Weyl laws when interpreted as a manifestation of the delocalization of a sufficient number of particles. The goal of this section is to present the results of [68] where we study more general version of the Weyl law, adding a (singular) potential to the Laplacian.

It is a natural question to ask what becomes of (2.14) when $-\Delta_g$ is replaced by $-\Delta_g + V$, when $V : M \rightarrow \mathbb{R}$ is a potential. The first easy answer is that (2.14) is still valid when V is smooth, since the result of Hörmander [79] actually covers the case of any elliptic pseudo-differential operator. From this perspective, one can ask whether this property persists when V is singular. Of course, V must not be too singular since we still want $-\Delta_g + V$ to be well-defined as a self-adjoint operator (with compact resolvent say, to ensure that it still has discrete spectrum). For this property, a natural class to consider is $V \in L^{N/2}(M)$ (where we recall that N is the dimension of the manifold) due to Sobolev embeddings. This class is sharp in the L^p scale and includes all point singularities of the type $d(\cdot, x_0)^{-\alpha}$ for all $\alpha < 2$ (where $x_0 \in M$ is the point of singularity and d_g is the Riemannian distance on M), which is again natural since $\alpha = 2$ is the known threshold to have a relatively bounded perturbation of $-\Delta_g$. Now we remark that (2.14) *cannot* hold for all $V \in L^{N/2}(M)$, for the simple reason that it may happen that $-\Delta_g + V$ has unbounded eigenfunctions (while (2.14) clearly implies that all eigenfunctions are $L^\infty(M)$). An example given in [135, Sec. A.3] is $u(x) = \log|x|$ (in the flat case, say) for $|x| \leq \varepsilon$, which satisfies $u \notin L^\infty$ and $V := \Delta u/u \sim (|x|^2 \log|x|)^{-1}$ as $|x| \rightarrow 0$, so that $V \in L^{N/2}$ for $N \geq 3$.

The correct replacement of $L^{N/2}$ which is still 'critical' with respect to the Laplacian and such that eigenfunctions of $-\Delta_g + V$ are bounded is the *Kato class*, as emphasized in [3]. For $N \geq 3$ (for simplicity), a potential $V : M \rightarrow \mathbb{R}$ belongs to the Kato class if $V \in L^1(M)$ and if

$$\limsup_{r \rightarrow 0} \int_{x \in M} \int_{d_g(x,y) < r} \frac{|V(y)|}{d_g(x,y)^{N-2}} dv_g(y) = 0.$$

Similarly to $L^{N/2}$, the Kato class contains all L^q spaces for $q > N/2$ and in particular all point singularities of the type $d_g(\cdot, x_0)^{-\alpha}$ with $\alpha < 2$. However, the Kato class is neither contained nor contains $L^{N/2}$. This can be seen already for radial potentials, where being $L^{N/2}$ means that $\int_0^1 |V(r)|^{N/2} r^{N-1} dr < +\infty$ while being Kato class means that $\int_0^1 |V(r)| r dr < +\infty$. Hence, $V(r) = (r^2 \log r)^{-1}$ belongs to $L^{N/2}$ but not to the Kato class, while $V(r) = (1-r)^{-\alpha}$ belongs to the Kato class but not to $L^{N/2}$ for $2/N < \alpha < 1$.

In [68], we investigated whether the Weyl law (2.14) was still valid when a Kato class potential is added to $-\Delta_g$. At a similar time, Huang and Sogge [80] answered the weaker question of the *integrated* Weyl law and proved that for any Kato class potential V one indeed has

$$\mathrm{Tr} \mathbf{1}(-\Delta_g \leq \lambda^2) = (2\pi)^{-N} (|\mathbb{S}^{N-1}|/N) \mathrm{Vol}_g(M) \lambda^N + O_{\lambda \rightarrow +\infty}(\lambda^{N-1}), \quad (2.15)$$

so that adding a Kato class potential does not influence the sharp Weyl law. Let us notice that they are also able to treat specific manifolds where the remainder $\mathcal{O}(\lambda^{N-1})$ can be improved (like the torus or negatively curved manifolds). In [68, Theorem 1.3], we noticed that (2.15) holds when $V \in L^{N/2}(M)$ by a simple perturbative argument using the Cwikel-Lieb-Rosenblum inequality.

The situation of the pointwise Weyl law is less clear than the integrated one. Indeed, one of the main findings of [68] is that (2.14) is actually *wrong* for some singular enough V in the Kato class, and that in order to recover (2.14) one needs stronger assumptions on the potential. Let us summarize our results.

THEOREM 2.6 (Theorems 1.1, 1.2, 1.3 in [68]). *Let (M, g) be a three dimensional compact, smooth, Riemannian manifold without boundary. Denote by $-\Delta_g$ the Laplace-Beltrami operator on M and let $V : M \rightarrow \mathbb{R}$ be a Kato class potential. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exists $r_\varepsilon^V : (0, +\infty) \times M \rightarrow \mathbb{R}$ such that*

- (1) (Sharp pointwise Weyl law with an additional term) *As $\lambda \rightarrow +\infty$, we have uniformly in $x \in M$*

$$\mathbb{1}(-\Delta_g + V \leq \lambda^2)(x, x) = \frac{\lambda^3}{6\pi^2} + \lambda^3 r_\varepsilon^V(\lambda, x) + \mathcal{O}^{(\varepsilon)}(\lambda^2).$$

- (2) (Smallness of the additional term) *There exists $C > 0$ such that for all $\lambda > 0$ we have*

$$\sup_{x \in M} |r_\varepsilon^V(\lambda, x)| \leq C \|V\|_{\mathcal{K}(\varepsilon)}$$

where $\|V\|_{\mathcal{K}(\varepsilon)} = \sup_{x \in M} \int_{d_g(x, y) < \varepsilon} |V(y)| d_g(x, y)^{-1} dv_g(y)$. In particular, we always have as $\lambda \rightarrow +\infty$ and uniformly in $x \in M$,

$$\mathbb{1}(-\Delta_g + V \leq \lambda^2)(x, x) = \frac{\lambda^3}{6\pi^2} + o(\lambda^3).$$

- (3) (Integrated Weyl law always persists) *As $\lambda \rightarrow +\infty$ we have*

$$\int_M r_\varepsilon^V(\lambda, x) dv_g(x) = \mathcal{O}^{(\varepsilon)}(\lambda^{-1}).$$

In particular, as $\lambda \rightarrow +\infty$ we have

$$\mathrm{Tr} \mathbb{1}(-\Delta_g + V \leq \lambda^2) = \frac{\mathrm{Vol}_g(M)}{6\pi^2} \lambda^3 + \mathcal{O}(\lambda^2).$$

- (4) (When does the additional term disappear) *If V satisfies furthermore for some $r > 0$ that $\sup_{x \in M} \int_{d_g(x, y) < r} |V(y)| d_g(x, y)^{-2} dv_g(y)$, one has*

$$\sup_{x \in M} |r_\varepsilon^V(\lambda, x)| = \mathcal{O}_{\lambda \rightarrow +\infty}(\lambda^{-1}).$$

- (5) (Failure of the sharp Weyl law for specific potentials) *Let $x_0 \in M$, $\eta \in (0, 1)$, $\gamma \in \mathbb{R}$ and assume that for all $x \in M$,*

$$V(x) = \gamma \frac{\chi(d_g(x, x_0))}{d_g(x, x_0)^{2-\eta}},$$

for some smooth $\chi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\chi(0) = 1$ and with support close to the origin. Then, there exists $a_\eta > 0$ such that

$$r_\varepsilon^V(\lambda, x_0) \sim_{\lambda \rightarrow +\infty} -\gamma a_\eta \lambda^{-\eta}.$$

Point (1) means that adding a potential adds a new term $\lambda^3 r_\varepsilon^V(\lambda, x)$ to the Weyl law, which can be put into the remainder $\mathcal{O}(\lambda^2)$ by Point (4) if the potential V is less singular than Kato class. In general, this term cannot be put into the remainder $\mathcal{O}(\lambda^2)$ as shown by Point (5) where we can compute this additional term for a specific potential with a point singularity: in this case, it leads to a correction $\lambda^{3-\eta}$ to the Weyl law, showing that it may be arbitrarily close to the main order term λ^3 as $\eta \rightarrow 0_+$ (which means that the singularity gets stronger). Meanwhile, this additional term never contributes to the main order as shown by Point (2). Finally, Point (3) shows that while the additional term may contribute non-trivially to the *pointwise* asymptotics, it never contributes to the *integrated* asymptotics. Notice that our result holds only in three dimensions, which is due to our method of proof that we describe below. Some extensions of our result to other dimensions have been since proved [81, 82]. Let us mention that our proof provides an explicit formula for the additional term $r_\varepsilon^V(\lambda, x)$ in terms of a series involving the potential V , and the properties listed in the theorem above follow from estimating this series under various assumptions on V .

Let us now comment on our method of proof. As we already mentioned, the now widely popular method of Hörmander to obtain such spectral asymptotics seems hard to adapt to this case of general singular potentials (in particular since for a general potential in the Kato class, one does not know what kind of singularities it exhibits, where they are located, etc.). To treat this problem, we rather used the older method of Avakumovic [7] based on resolvent estimates and Tauberian theorems. His method was obtained in three dimensions and this is where our dimensional restriction comes from (although one could reasonably expect that it extends to other dimensions as well, but we do not know how to do it yet). From a naive perspective, his method seems more suited to include singular potentials since it relies on resolvent or heat kernels which are regularizing in nature and hence which may deal better with the singularity of the potential, compared to the wave propagator method of Levitan/Hörmander which is less regularizing.

The basic idea of Avakumovic's approach to study the asymptotics of $\mathbf{1}(-\Delta_g \leq t)$ as $t \rightarrow +\infty$ is to deduce them from the asymptotics of a more regular object, like the heat kernel $e^{t\Delta_g}$ as $t \rightarrow 0_+$ or powers of the resolvent $(-\Delta_g + \lambda)^{-k}$ as $\lambda \rightarrow +\infty$. These objects are easier to study because they are solutions to PDEs, and asymptotics can be found using appropriate approximate solutions. Relating the asymptotics of these transforms (going from the counting functions to the heat kernel is called the *Laplace transform* while going from the counting function to powers of the resolvent is called the *Stieltjes transform*) to the asymptotics of the original object is the content of *Tauberian theorems*. This approach was originally applied to the prime number theorem, and it was Carleman [32] who had the idea to apply it to spectral counting. The main contribution of Avakumovic to this story is to apply this strategy to obtain *sharp* asymptotics. Indeed, Tauberian theorems typically only give the main term in the asymptotics. For instance, the famous theorem of Hardy and

Littlewood states that if a non-negative measure μ on $[0, +\infty)$ satisfies

$$\int_{[0, +\infty)} e^{-tx} d\mu(x) \sim_{t \rightarrow 0_+} ct^{-a}$$

for some $c > 0$ and $a \geq 0$, then one has

$$\mu([0, \lambda]) \sim_{\lambda \rightarrow +\infty} \frac{c}{\Gamma(a+1)} \lambda^a.$$

This result can actually be applied to obtain the main term in the Weyl law (see for instance [156, Theorem XIII.4.1] for a proof). A natural question is to ask under which conditions do we have a version of this result with stronger remainder term. The following (which does not hold in full generality but we state it anyway to give an idea of the philosophy behind this kind of results) is a result of Freud [69]: if for some remainder $r(t) = o_{t \rightarrow 0_+}(1)$ one has

$$\int_{[0, +\infty)} e^{-tx} d\mu(x) = (c + r(t))t^{-a},$$

then one deduces that

$$\mu([0, \lambda]) = \left(\frac{c}{\Gamma(a+1)} + \mathcal{O}_{\lambda \rightarrow +\infty} \left(\frac{1}{|\log r(1/\lambda)|} \right) \right) \lambda^a.$$

This means that one has a logarithmic loss in the remainders of the Tauberian theorem. Surprisingly, this result is sharp! Applied to our problem where the main term of the Weyl law is λ^N and the remainder term is λ^{N-1} , this means that we would need an *exponentially small* remainder in the asymptotics of the Laplace transform in order to derive such a Weyl law with remainder. This strategy can be applied to the flat case of the Dirichlet Laplacian on an open bounded set in the Euclidean space [6], but in the general case of a Riemannian manifold exponentially small remainder terms in the asymptotics of the heat kernel are wrong (the correction to the main term is polynomial, see for instance [21, Sec. III.E]). Despite this fact, Avakumovic managed to apply Tauberian theorems to prove the Weyl law with sharp remainder term by noticing that exponentially small remainder terms in the Laplace transform can be obtained up to a structured term which is itself a Laplace transform; namely he could find a measure μ_0 such that

$$\int_{[0, +\infty)} e^{-tx} d\mu(x) = ct^{-a} + \int_{[0, +\infty)} e^{-tx} d\mu_0(x) + \mathcal{O}_{t \rightarrow 0_+}(e^{-b/t}),$$

for some $b > 0$, so that a Freud like theorem could be applied to the measure $\mu - \mu_0$, which leads to the desired asymptotics if $\mu_0([0, \lambda])$ does not grow too fast as $\lambda \rightarrow +\infty$. Of course, one has to be precise because such Tauberian theorems typically hold under some assumption like the non-negativity of the measure μ which would break down when applied to $\mu - \mu_0$ instead. Thus, Avakumovic's contributions was twofold: i) provide Tauberian theorems with weak enough 'monotonicity' assumptions so that they can be applied to measures of the type $\mu - \mu_0$ and ii) explain why an exponentially small remainder term can be obtained up to a structured term for the heat kernel asymptotics on a Riemannian manifold.

In [68], we explained how to adapt Avakumovic's argument to include Kato class potentials, and in particular the two points mentioned above. We first had to prove a Tauberian theorem with remainders, which is a bit stronger than the one Avakumovic used.

THEOREM 2.7 (Theorem 3.1 in [68]). *Let $A : [0, +\infty) \rightarrow \mathbb{R}$ be such that $A(t) = B_0(t)t^{3/2} + B_1(t) + B_2(t)$ where*

- (1) $\exists C > 0, \forall t \geq 0, |B_2(t)| \leq C(1+t)$;
- (2) B_1 is non-decreasing;
- (3) B_0 is bounded and such that there exists $c > 0$ such that for all $0 < u \leq v \leq u+1$ we have $B_0(v^2) - B_0(u^2) \geq -c/u$.
- (4) There exists $\varepsilon > 0$ such that as $\lambda \rightarrow +\infty$, we have

$$\int_0^\infty \frac{A(t)}{(t+\lambda)^3} dt = \mathcal{O}(e^{-\varepsilon\sqrt{\lambda}}).$$

Then, $A(t) = \mathcal{O}(t)$ as $t \rightarrow +\infty$.

This Tauberian theorem is stated for the Stieltjes transform rather than for the Laplace transform because the method relies on resolvent estimates rather than on heat kernel estimates. The idea behind it is that the function B_1 is the one we want to estimate (like the counting function $\text{Tr } \mathbf{1}(-\Delta_g + V \leq t)$), and for which we don't know anything more than it is non-decreasing. In Avakumovic's argument, the function B_0 is constant so that $B_0(t)t^{3/2}$ corresponds to the main term in the asymptotics and its Stieltjes transform is proportional to $\lambda^{-1/2}$. Hence, the way to interpret Theorem 2.7 compared to our previous discussion is that (still in the case B_0 constant) under the assumption

$$\int_0^\infty \frac{B_1(t)}{(t+\lambda)^3} dt = c\lambda^{-1/2} + \int_0^\infty \frac{B_2(t)}{(t+\lambda)^3} dt + \mathcal{O}_{\lambda \rightarrow +\infty}(e^{-\varepsilon\sqrt{\lambda}}),$$

we deduce

$$B_1(t) = -B_0(t)t^{3/2} - B_2(t) + \mathcal{O}_{t \rightarrow +\infty}(t),$$

so that together with the assumption $B_2(t) = \mathcal{O}_{t \rightarrow +\infty}(t)$ this implies $B_1(t) = -B_0(t)t^{3/2} + \mathcal{O}_{t \rightarrow +\infty}(t)$ as desired. When including a Kato class potential, B_0 is not constant anymore (due to the additional term r_ε^V in Theorem 2.6) and one has to use assumption (3) to deal with it.

The assumptions of Theorem 2.7 in the case $B_1(t) = \mathbf{1}(-\Delta_g + V \leq t)(x, x)$ for some fixed $x \in M$ follow from the following result.

PROPOSITION 2.2 (Proposition 5.1 in [68]). *Let $V : M \rightarrow \mathbb{R}$ be a potential in the Kato class. Then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ there exist functions $r_\varepsilon^V, s_\varepsilon^V : (0, +\infty) \times M \rightarrow \mathbb{R}$ such that*

- (1) $\exists C > 0, \forall t \geq 0, \forall x \in M, |s_\varepsilon^V(t, x)| \leq C(1+t)$;
- (2) $\exists C > 0, \forall t \geq 0, \forall x \in M, |r_\varepsilon^V(t, x)| \leq C\|V\|_{\mathcal{K}(\varepsilon)}$;
- (3) $\exists C > 0, \forall t' \geq t > 0, \forall x \in M, |r_\varepsilon^V(t, x) - r_\varepsilon^V(t', x)| \leq C(\sqrt{t'} - \sqrt{t})/\sqrt{t}$;

(4) As $\lambda \rightarrow +\infty$ and uniformly in $x \in M$ we have

$$(-\Delta_g + V + \lambda)^{-2}(x, x) = \frac{1}{8\pi\sqrt{\lambda}} + \int_0^\infty \frac{t^{3/2}r_\varepsilon^V(t, x) + s_\varepsilon^V(t, x)}{(t + \lambda)^3} dt + \mathcal{O}(e^{-\varepsilon\sqrt{\lambda}/4}).$$

The combination of Proposition 2.2 and Theorem 2.7 lead to Theorem 2.6 by taking $B_0(t) = 1/(6\pi^2) + r_\varepsilon^V(t, x)$ and $B_2(t) = s_\varepsilon^V(t, x)$ (for fixed $x \in M$). To ensure the uniformity in x , we actually prove a more quantitative statement of the Tauberian theorem 2.7 to track the dependence of the final result on the constant appearing in its assumptions.

Let us explain briefly the ideas behind the resolvent asymptotics of Proposition 2.2, which we illustrate on the simpler case of the first power of the resolvent

$$G_\lambda^V := (-\Delta_g + V + \lambda)^{-1}.$$

Avakumovic's idea is to use Hadamard's parametrix to approximate G_λ^0 , that is

$$T_\lambda^0(x, y) = \frac{e^{-\sqrt{\lambda}d_g(x, y)}}{4\pi d_g(x, y)} U(x, y),$$

where the first factor is just the resolvent of the Laplacian on \mathbb{R}^3 with the Euclidean distance replaced by the Riemannian distance, and the function U is supported close to the diagonal $x = y$ (actually, supported on $d_g(x, y) \leq \varepsilon$: this is where the ε -dependence comes from). The idea behind Hadamard's parametrix is that for a certain choice of U , defining

$$R_\lambda^0(x, y) := (-\Delta_g + \lambda)_x(T_\lambda^0(x, y) - G_\lambda^0(x, y))$$

we have

$$|R_\lambda^0(x, y)| \leq C \frac{e^{-\sqrt{\lambda}d_g(x, y)/2}}{d_g(x, y)}, \quad R_\lambda^0(x, y) = \frac{e^{-\sqrt{\lambda}d_g(x, y)}}{4\pi d_g(x, y)} \tilde{U}(x, y),$$

where the first relation holds for all x, y while the second relation only holds for $d_g(x, y) \leq \varepsilon/2$, for some λ -independent function \tilde{U} . This means that despite taking two derivatives of the singular object T_λ^0 , R_λ^0 is as singular as T_λ^0 due to the choice of the function U . Now the relation defining R_λ^0 is equivalent to the fact that, as operators,

$$G_\lambda^0 = T_\lambda^0 - G_\lambda^0 R_\lambda^0$$

and the bounds on R_λ^0 imply that one can iterate this relation to obtain the series representation

$$G_\lambda^0 = \sum_{n \geq 0} (-1)^n T_\lambda^0 (R_\lambda^0)^n.$$

Now each of the terms $T_\lambda^0 (R_\lambda^0)^n$ for $n \geq 1$ may be split as a sum of an exponentially small term (as $\lambda \rightarrow +\infty$) and a structured term which is a Stieltjes transform. Indeed, the integral kernel of $T_\lambda^0 (R_\lambda^0)^n$ can be written as an n -fold integral,

$$T_\lambda^0 (R_\lambda^0)^n(x, y) = \int_M dv_g(z_1) \cdots \int_M dv_g(z_n) T_\lambda^0(x, z_1) R_\lambda^0(z_1, z_2) \cdots R_\lambda^0(z_n, y).$$

Now if all integrals are restricted to the region $d_g(z_k, z_{k+1}) \leq \varepsilon/2$, we are in the region where the function R_λ^0 is known explicitly and in particular, the λ -dependence of this term is just

$$e^{-\sqrt{\lambda}(d_g(x, z_1) + d_g(z_1, z_2) + \cdots + d_g(z_n, y))},$$

which is a Stieltjes transform in λ (after all, by definition the resolvent is a Stieltjes transform and we saw that its λ -dependence is also of the form $e^{-a\sqrt{\lambda}}$ for some $a > 0$). Hence, this term where all integrations are taken close to the diagonal is the structured term which is a Stieltjes transform. For the other term, at least one of the integration domain is $d_g(z_k, z_{k+1}) \geq \varepsilon/2$ and since the function R_λ^0 is exponentially decaying away from the diagonal, this term is exponentially small in λ .

All we just discussed did not include the potential V ; this argument is the one given by Avakumovic in the case $V = 0$ to obtain the asymptotics of G_λ^0 when $\lambda \rightarrow +\infty$. To include a potential, we use the *same* parametrix T_λ^0 , which now satisfies

$$R_\lambda^V := (-\Delta_g + V + \lambda)_x(T_\lambda^0(x, y) - G_\lambda^0(x, y)) = R_\lambda^0 + VT_\lambda^0,$$

which can be rewritten as operators as

$$G_\lambda^V = T_\lambda^0 - G_\lambda^V R_\lambda^V.$$

The Kato class property of V can be used to infer that one has the convergent series representation

$$G_\lambda^V = \sum_{n \geq 0} (-1)^n T_\lambda^0 (R_\lambda^V)^n = \sum_{n \geq 0} (-1)^n T_\lambda^0 (R_\lambda^0 + VT_\lambda^0)^n.$$

Now one can use the same idea as above to split each term $T_\lambda^0 (R_\lambda^0 + VT_\lambda^0)^n$ for $n \geq 1$ as a sum of a structured term and an exponentially small term, just by splitting the integration domain in two parts. While in Avakumovic's argument, the structured term could be shown to be the Stieltjes transform of a function which is $\mathcal{O}_{t \rightarrow +\infty}(t)$, it is not the case when $V \neq 0$ due to the term $T_\lambda^0 (VT_\lambda^0)^n$ which is the most singular and leads to the additional term in the Weyl asymptotics when summed over n .

In this sketch of proof, one sees where the restriction of three dimensions comes from: in this special case, the λ -dependence of the resolvent of the Euclidean Laplacian is particularly simple (essentially $e^{-\sqrt{\lambda}|x-y|}$), so that when taking products of it, it remains of the same form and in particular is still a Stieltjes transform. In other dimensions, products of resolvents are perhaps still Stieltjes transforms but it is far from obvious. Of course, it would be very important to understand how to extend this argument to any dimension.

As a concluding remark, let us mention that this method is robust enough to obtain off-diagonal asymptotics $\mathbf{1}(-\Delta_g + V \leq t)(x, y)$ as well [68, Corollary 8.2] and also treat the case of Kato class perturbations of the Dirichlet Laplacian on a bounded domain in \mathbb{R}^3 [68, Prop. 7.6]. This last result is weaker than the one on manifolds due to the boundary of the domain, at which estimates degenerate. Understanding what happens at the boundary with a Kato class potential is an interesting challenge, which could also lead to more complicated questions associated to the semi-classical regime or to confining potentials.

4. Littlewood-Paley decomposition for operator densities

The last topic we want to discuss in the theme of fermionic inequalities is the one presented in [128] related to the Littlewood-Paley decomposition, that we first recall in the

one-body setting. Let $d \geq 1$ and $\psi \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ be a non-negative function such that

$$\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \psi_j(\xi) = 1, \quad \psi_j(\xi) := \psi(2^{-j}\xi).$$

The function ψ_j should be thought as localizing at the dyadic frequency $|\xi| \sim 2^j$, through the Fourier multiplier P_j defined by

$$\forall u \in L^2(\mathbb{R}^d), \quad \widehat{P_j u} := \psi_j \widehat{u}.$$

The classical Littlewood-Paley theorem (see for instance [144, Theorem 8.3]) states that for any $1 < p < +\infty$ there exists $C > 0$ such that for any $u \in L^p(\mathbb{R}^d)$ one has

$$C \|u\|_{L^p} \leq \left\| \left(\sum_{j \in \mathbb{Z}} |P_j u|^2 \right)^{1/2} \right\|_{L^p} \leq (1/C) \|u\|_{L^p},$$

meaning that one can deduce L^p -properties of u from knowing the L^p -properties of the frequency localized $(P_j u)_{j \in \mathbb{Z}}$. This statement has many applications in PDEs for instance, and let us illustrate its usefulness by deducing from it the Sobolev inequality

$$\|u\|_{L^q} \leq C \|(-\Delta)^{s/2} u\|_{L^2} \tag{2.16}$$

where $s \in [0, d/2)$ and $q = 2d/(d - 2s)$. We first claim that (2.16) is true for frequency localized functions. Indeed, if $j \in \mathbb{Z}$ and $u \in \dot{H}^s(\mathbb{R}^d)$, we have by the Hausdorff-Young, Hölder inequalities and by Plancherel's identity,

$$\|P_j u\|_{L^q} \lesssim \|\widehat{P_j u}\|_{L^{2d/(d+2s)}} \lesssim 2^{sj} \|\widehat{P_j u}\|_{L^2} \lesssim \|(-\Delta)^{s/2} P_j u\|_{L^2}.$$

Now using the Littlewood-Paley theorem, we have

$$\|u\|_{L^q} \lesssim \left(\sum_{j \in \mathbb{Z}} \|P_j u\|_{L^q}^2 \right)^{1/2} \lesssim \left(\sum_{j \in \mathbb{Z}} \|(-\Delta)^{s/2} P_j u\|_{L^2}^2 \right)^{1/2} \lesssim \|(-\Delta)^{s/2} u\|_{L^2}^2,$$

which is exactly (2.16).

In [128], we provided a version of the Littlewood-Paley theorem for density matrices.

LEMMA 2.3 (Lemma 1 in [128]). *Let $d \geq 1$ and $1 < p < +\infty$. Then, there exists $C > 0$ such that for any non-negative finite rank operator γ on $L^2(\mathbb{R}^d)$ we have*

$$C \|\rho_\gamma\|_{L^{p/2}} \leq \left\| \sum_{j \in \mathbb{Z}} \rho_{P_j \gamma P_j} \right\|_{L^{p/2}} \leq (1/C) \|\rho_\gamma\|_{L^{p/2}}.$$

Lemma 2.3 reduces to the Littlewood-Paley theorem when γ is rank-one. Before talking about the proof of Lemma 2.3, let us discuss an application to the analogue of the Sobolev embeddings for density matrices that we already discussed at the beginning of this chapter: the Lieb-Thirring inequality. When rewritten in terms of density matrices, it reads

$$\int_{\mathbb{R}^d} \rho_\gamma(x)^{1+2/d} \lesssim \text{Tr} \sqrt{\gamma} (-\Delta) \sqrt{\gamma},$$

for any density matrix $0 \leq \gamma \leq 1$. In the one body setting, it amounts to the subcritical Sobolev embedding $H^1(\mathbb{R}^d) \hookrightarrow L^{2+4/d}(\mathbb{R}^d)$. If we try to adapt the one-body proof above

using the Littlewood-Paley theorem to density matrices, we first notice that a frequency-localized version of the Lieb-Thirring inequality holds; indeed $0 \leq \gamma \leq 1$ implies that $0 \leq \rho_{P_j \gamma P_j} \leq \rho_{P_j^2} \lesssim 2^{jd}$, so that

$$\begin{aligned} \|\rho_{P_j \gamma P_j}\|_{L^{1+2/d}} &\leq \|\rho_{P_j \gamma P_j}\|_{L^1}^{d/(d+2)} \|\rho_{P_j \gamma P_j}\|_{L^\infty}^{2/(d+2)} \\ &\lesssim 2^{2jd/(d+2)} (\text{Tr } \sqrt{\gamma} P_j^2 \sqrt{\gamma})^{d/(d+2)} \\ &= (\text{Tr } \sqrt{\gamma} 2^{2j} P_j^2 \sqrt{\gamma})^{d/(d+2)} \\ &\lesssim (\text{Tr } \sqrt{\gamma} P_j (-\Delta) P_j \sqrt{\gamma})^{d/(d+2)}, \end{aligned}$$

which is indeed the Lieb-Thirring inequality for the operator $P_j \gamma P_j$. Now contrary to the one-body case, one encounters an issue when trying to sum this inequality over j with the Littlewood-Paley theorem. Indeed, Lemma 2.3 implies that

$$\|\rho_\gamma\|_{L^{1+2/d}} \lesssim \sum_{j \in \mathbb{Z}} \|\rho_{P_j \gamma P_j}\|_{L^{1+2/d}} \lesssim \sum_{j \in \mathbb{Z}} (\text{Tr } \sqrt{\gamma} P_j (-\Delta) P_j \sqrt{\gamma})^{d/(d+2)},$$

and the right side cannot be compared to $\text{Tr } \sqrt{\gamma} (-\Delta) \sqrt{\gamma} \sim \sum_j \text{Tr } \sqrt{\gamma} P_j (-\Delta) P_j \sqrt{\gamma}$ since $d/(d+2) < 1$. Of course, the issue comes from using the crude consequence of Lemma 2.3 and the triangle inequality which is

$$\|\rho_\gamma\|_{L^{1+2/d}} \lesssim \sum_{j \in \mathbb{Z}} \|\rho_{P_j \gamma P_j}\|_{L^{1+2/d}},$$

which was enough in the one-body case but not in the many-body case. The way to correct this argument is to rather estimate from below

$$\text{Tr } \sqrt{\gamma} (-\Delta) \sqrt{\gamma} \gtrsim \sum_j 2^{2j} \text{Tr } \sqrt{\gamma} P_j^2 \sqrt{\gamma} = \int_{\mathbb{R}^d} \sum_j 2^{2j} \rho_{P_j \gamma P_j}(x) dx, \quad (2.17)$$

and remark that for fixed $x \in \mathbb{R}^d$, $\alpha_j := \rho_{P_j \gamma P_j}(x)$ satisfies $0 \leq \alpha_j \lesssim 2^{jd}$ as we already said, so that

$$\sum_{j \in \mathbb{Z}} \alpha_j = \sum_{j \leq J} \alpha_j + \sum_{j > J} \alpha_j \lesssim 2^{Jd} + 2^{-2J} \sum_j 2^{2j} \alpha_j,$$

which when optimizing over J leads to

$$\sum_{j \in \mathbb{Z}} \alpha_j \lesssim \left(\sum_j 2^{2j} \alpha_j \right)^{d/(d+2)}.$$

Inserted into (2.17) and Lemma 2.3, this leads to

$$\text{Tr } \sqrt{\gamma} (-\Delta) \sqrt{\gamma} \gtrsim \int_{\mathbb{R}^d} \left(\sum_{j \in \mathbb{Z}} \rho_{P_j \gamma P_j}(x) \right)^{1+2/d} dx \gtrsim \int_{\mathbb{R}^d} \rho_\gamma(x)^{1+2/d} dx.$$

Let us now comment on the proof of Lemma 2.3. In the one-body case, it relies on Khintchine's inequality which states that if $0 < p < +\infty$ and if (r_j) is a sequence of independent

random variables taking values in ± 1 with equal probability $1/2$, then there exists $C > 0$ such that for any complex numbers $(a_j) \subset \mathbb{C}$ one has

$$C(\sum_j |a_j|^2)^{p/2} \leq \mathbb{E} \left| \sum_j a_j r_j \right|^p \leq (1/C)(\sum_j |a_j|^2)^{p/2},$$

which is an equality for $p = 2$ due to independence. With this result, one can estimate for $u \in L^p(\mathbb{R}^d)$,

$$\|(\sum_{j \in \mathbb{Z}} |P_j u|^2)^{1/2}\|_{L^p}^p \lesssim \mathbb{E} \int_{\mathbb{R}^d} \left| \sum_j r_j (P_j u)(x) \right|^p dx$$

and then use that the Fourier multiplier $\sum_j r_j P_j$ is bounded on $L^p(\mathbb{R}^d)$ uniformly in the values of (r_j) by the Mikhlin multiplier theorem since $\Psi := \xi \mapsto \sum_j r_j \psi_j(\xi)$ satisfies that $|\xi^\alpha| \partial^\alpha \Psi$ is uniformly bounded in ξ and (r_j) , for all $\alpha \in \mathbb{N}^d$.

The proof of the many-body case Lemma 2.3 can be made along the same lines, but let us give another way to see the proof proposed in [128]. The idea is to use the operator $\sqrt{\gamma}$ and the relation

$$\rho_\gamma(x) = \int_{\mathbb{R}^d} |\sqrt{\gamma}(x, y)|^2 dy = \|\sqrt{\gamma}(x, \cdot)\|_{L_y^2}^2,$$

so that

$$\|\rho_\gamma\|_{L^{p/2}} = \|\sqrt{\gamma}(\cdot, \cdot)\|_{L_x^p L_y^2}, \quad \|\sum_{j \in \mathbb{Z}} \rho_{P_j \gamma P_j}\|_{L^{p/2}} = \left\| \left(\sum_{j \in \mathbb{Z}} \|P_j \sqrt{\gamma}(\cdot, \cdot)\|_{L_y^2}^2 \right)^{1/2} \right\|_{L_x^p}.$$

Hence, Lemma 2.3 may be seen as a one-body Littlewood-Paley estimate but applied to *Hilbert space valued* functions. That the Littlewood-Paley theorem extends to this case seems to be well-known in the harmonic analysis community, and it can be proved quite straightforwardly from this slight extension of Khintchine's inequality which can be found for instance in [144, Appendix D]:

$$\left(\sum_{jk} |a_{jk}|^2 \right)^{p/2} \lesssim \mathbb{E}_{jk} \left| \sum_{jk} a_{jk} r_j \otimes r_k \right|^p.$$

Then, for any separable Hilbert space \mathcal{H} with Hilbert basis (e_k) and for any strongly measurable $f : \mathbb{R}^d \rightarrow \mathcal{H}$, one has

$$\begin{aligned}
\left\| \left(\sum_{j \in \mathbb{Z}} \|P_j f\|_{\mathcal{H}}^2 \right)^{1/2} \right\|_{L_x^p}^p &= \int_{\mathbb{R}^d} \left(\sum_{jk} |\langle (P_j f)(x), e_k \rangle_{\mathcal{H}}|^2 \right)^{p/2} dx \\
&\lesssim \mathbb{E}_{jk} \int_{\mathbb{R}^d} \left| \sum_{jk} \langle (P_j f)(x), e_k \rangle_{\mathcal{H}} r_j \otimes r_k \right|^p dx \\
&\lesssim \mathbb{E}_k \int_{\mathbb{R}^d} \left| \sum_k \langle f(x), e_k \rangle_{\mathcal{H}} r_k \right|^p dx \\
&\lesssim \int_{\mathbb{R}^d} \left(\sum_k |\langle f(x), e_k \rangle_{\mathcal{H}}|^2 \right)^{p/2} dx \\
&\lesssim \int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}}^p dx,
\end{aligned}$$

where in the third line we used again that the multiplier $\sum_j r_j P_j$ is bounded on L^p , and in the fourth line we used the standard Khintchine inequality. The reverse inequality can be proved in a standard way by duality.

As a final remark, let us notice that the Lieb-Thirring inequality can be proved directly using this idea of stating it for the operator $\sqrt{\gamma}$; indeed in this representation it amounts to

$$\|\sqrt{\gamma}(\cdot, \cdot)\|_{L_x^{2+4/d} L_y^2} \lesssim \|\nabla_x \sqrt{\gamma}(\cdot, \cdot)\|_{L_{x,y}^2},$$

so that the Lieb-Thirring inequality may be seen as a Sobolev inequality applied to Hilbert space valued functions, with the following subtlety: the above inequality holds in the one-body case under the L^2 -constraint $\|\sqrt{\gamma}(\cdot, \cdot)\|_{L_{x,y}^2} \leq 1$ which at the level of operators would mean that $\text{Tr } \gamma \leq 1$, which is the *wrong* constraint: it should be $\gamma \leq 1$ instead! The constraint $\gamma \leq 1$ means at the level of the integral kernel of $\sqrt{\gamma}$ that for all $u \in L^2(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \sqrt{\gamma}(x, y) u(y) dy \right|^2 dx \leq \|u\|_{L^2}^2,$$

which is of course a weaker assumption than $\|\sqrt{\gamma}(\cdot, \cdot)\|_{L_{x,y}^2} \leq 1$. Hence, the Lieb-Thirring inequality is rather the following statement.

PROPOSITION 2.4 (Hilbert space valued improved Sobolev inequality). *Let $d \geq 1$ and \mathcal{H} be a separable Hilbert space. Then, there exists $C > 0$ such that for all strongly measurable $f : \mathbb{R}^d \rightarrow \mathcal{H}$ we have*

$$\int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}}^{2+4/d} dx \leq C \left(\sup_{\|u\|_{\mathcal{H}}=1} \int_{\mathbb{R}^d} |\langle f(x), u \rangle_{\mathcal{H}}|^2 dx \right)^{2/d} \int_{\mathbb{R}^d} \|\nabla_x f(x)\|_{\mathcal{H}}^2 dx.$$

As we explained, this inequality is called 'improved' because its standard statement would be the weaker

$$\int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}}^{2+4/d} dx \leq C \|f\|_{L^2}^{4/d} \int_{\mathbb{R}^d} \|\nabla_x f(x)\|_{\mathcal{H}}^2 dx.$$

Interestingly, this proposition can be proved using the Chemin-Xu method [44].

PROOF OF PROPOSITION 2.4. We first extend the properties of the Fourier transform to functions taking values in Hilbert spaces. For any $f : \mathbb{R}^d \rightarrow \mathcal{H}$ such that

$$\int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}} < +\infty,$$

we define its Fourier transform by

$$\forall \xi \in \mathbb{R}^d, \widehat{f}(\xi) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx \in \mathcal{H}.$$

We clearly have the bound for all $\xi \in \mathbb{R}^d$,

$$\begin{aligned} \|\widehat{f}(\xi)\|_{\mathcal{H}} &= \sup_{\|u\|=1} |\langle \widehat{f}(\xi), u \rangle_{\mathcal{H}}| \\ &= \sup_{\|u\|=1} \frac{1}{(2\pi)^{d/2}} \left| \int_{\mathbb{R}^d} \langle f(x), u \rangle_{\mathcal{H}} e^{-ix \cdot \xi} dx \right| \\ &\leq \sup_{\|u\|=1} \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |\langle f(x), u \rangle_{\mathcal{H}}| dx. \end{aligned}$$

Here, we see that we gain compared to the stronger inequality

$$\|\widehat{f}\|_{L^\infty \mathcal{H}} \leq (2\pi)^{-d/2} \int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}} dx,$$

and this small gain is exactly what we need in the Chemin-Xu proof to go from the Sobolev inequality to the stronger Lieb-Thirring inequality. Note that we also have the Plancherel identities

$$\int_{\mathbb{R}^d} \|\widehat{f}(\xi)\|_{\mathcal{H}}^2 d\xi = (2\pi)^{-d} \int_{\mathbb{R}^{3d}} \langle f(x), f(x') \rangle_{\mathcal{H}} e^{-i\xi \cdot (x-x')} dx dx' d\xi = \int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}}^2 dx,$$

$$\int_{\mathbb{R}^d} |\langle \widehat{f}(\xi), u \rangle_{\mathcal{H}}|^2 d\xi = (2\pi)^{-d} \int_{\mathbb{R}^{3d}} \langle f(x), u \rangle_{\mathcal{H}} \langle u, f(x') \rangle_{\mathcal{H}} e^{-i\xi \cdot (x-x')} dx dx' d\xi = \int_{\mathbb{R}^d} |\langle f(x), u \rangle_{\mathcal{H}}|^2 dx,$$

for all $u \in \mathcal{H}$, which is the second main ingredient in Chemin-Xu's proof. To do it, let $q = 2 + 4/d$ and write

$$\int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}}^q dx = q \int_0^\infty |\{x : \|f(x)\|_{\mathcal{H}} \geq t\}| t^{q-1} dt.$$

For any $t > 0$, we split $f = f_{>} + f_{<}$, where $\widehat{f}_{<}(\xi) := \mathbf{1}(|\xi| \leq R_t) \widehat{f}(\xi)$, where R_t is chosen such that

$$\begin{aligned} \|f_{<}\|_{L^\infty \mathcal{H}} &\leq (2\pi)^{-d/2} \sup_{\|u\|_{\mathcal{H}}=1} \int_{|\xi| \leq R_t} |\langle \widehat{f}(\xi), u \rangle| d\xi \\ &\leq \frac{R_t^{d/2} |\mathbb{S}^{d-1}|^{1/2}}{d^{1/2} (2\pi)^{d/2}} \left(\sup_{\|u\|_{\mathcal{H}}=1} \int_{\mathbb{R}^d} |\langle \widehat{f}(\xi), u \rangle|^2 d\xi \right)^{1/2} \\ &\leq \frac{R_t^{d/2} |\mathbb{S}^{d-1}|^{1/2}}{d^{1/2} (2\pi)^{d/2}} \left(\sup_{\|u\|_{\mathcal{H}}=1} \int_{\mathbb{R}^d} |\langle f(x), u \rangle|^2 dx \right)^{1/2} \\ &\leq t/2, \end{aligned}$$

that is

$$R_t^d := \frac{t^2 d (2\pi)^d}{4 |\mathbb{S}^{d-1}|} \left(\sup_{\|u\|_{\mathcal{H}}=1} \int_{\mathbb{R}^d} |\langle f(x), u \rangle|^2 dx \right)^{-1}.$$

With this choice of R_t , we deduce that for all $t > 0$,

$$\begin{aligned} |\{x : \|f(x)\|_{\mathcal{H}} \geq t\}| &= |\{x : \|f_{>}(x)\|_{\mathcal{H}} \geq t/2\}| \\ &\leq \frac{4}{t^2} \int_{\mathbb{R}^d} \|f_{>}(x)\|_{\mathcal{H}}^2 dx \\ &= \frac{4}{t^2} \int_{|\xi| > R_t} \|\widehat{f}(\xi)\|_{\mathcal{H}}^2 d\xi \end{aligned}$$

and integrating in t we get

$$\begin{aligned} \int_{\mathbb{R}^d} \|f(x)\|_{\mathcal{H}}^q dx &\leq 4q \int_{\mathbb{R}^d} \|\widehat{f}(\xi)\|_{\mathcal{H}}^2 \int_0^\infty t^{q-3} \mathbf{1}(R_t \leq |\xi|) dt d\xi \\ &= (2d+4) \left(\frac{4 |\mathbb{S}^{d-1}|}{d (2\pi)^d} \right)^{2/d} \left(\sup_{\|u\|_{\mathcal{H}}=1} \int_{\mathbb{R}^d} |\langle f(x), u \rangle|^2 dx \right)^{2/d} \int_{\mathbb{R}^d} |\xi|^2 \|\widehat{f}(\xi)\|_{\mathcal{H}}^2 d\xi, \end{aligned}$$

which ends the proof. \square

It would be very interesting to investigate if this point of view on fermionic inequalities seen as Hilbert space valued “standard” inequalities can lead to more one-body techniques being adapted to the many-body setting, like compactness tools that we present in the next chapter.

CHAPTER 3

Optimizers for restriction inequalities

This last chapter is devoted to the study of the best constants in the restriction inequalities

$$\|\mathcal{E}_S(f)\|_{L^q(\mathbb{R}^N)} \leq C \|f\|_{L^p(S, d\sigma)}$$

that we already discussed above. We will focus on the case $p = 2$ and S being either the sphere \mathbb{S}^{N-1} or the cubic curve $y = x^3$ in \mathbb{R}^2 . In these cases, we investigate conditions under which there exist functions realizing the equality in the inequality above, using *compactness* methods. Indeed, the best constant can be defined as

$$\mathcal{S}_{S, \sigma, q} := \sup \left\{ \|\mathcal{E}_S(f)\|_{L^q(\mathbb{R}^N)} : \|f\|_{L^2(S, d\sigma)} = 1 \right\},$$

and we wish to find functions realizing this supremum by studying the convergence properties of *maximizing sequences*, i.e. sequences $(f_n) \subset L^2(S, d\sigma)$ with $\|f_n\|_{L^2(S)} = 1$ for all n and $\|\mathcal{E}_S(f_n)\|_{L^q(\mathbb{R}^N)} \rightarrow \mathcal{S}_{S, \sigma, q}$ as $n \rightarrow +\infty$. Ideally, one would try to show that such sequence converges (perhaps up to a subsequence) in $L^2(S)$ so that the limit would be a maximizer for $\mathcal{S}_{S, \sigma, q}$. We will see below that there are major obstacles to this property and the main goal will be to understand precisely the origin of this *loss of compactness*. Beside the question of existence of maximizers, we will also try to describe all maximizing sequences which has an independent interest. This type of questions has been vastly developed in the 80s for instance by Lieb [103, 28, 102] and Lions [109, 110, 111, 112] in his works around concentration-compactness. We extend these ideas to the case of Fourier extension inequalities, with specific features that we emphasize below.

The question of best constants for Fourier extension inequalities has been treated in several directions before. Kunze [89] first showed existence of maximizers when S is the paraboloid in \mathbb{R}^2 (i.e. we are in the context of the Strichartz inequality, with $q = 6$) by concentration-compactness methods and profile decompositions. This result was later generalized to all dimensions by Shao [131]. Their methods also allow to determine the behaviour of all maximizing sequences. In parallel to these results, Foschi [56] and Hundertmark-Zharnitsky [84] found the set of maximizers for the paraboloids in \mathbb{R}^2 and \mathbb{R}^3 : they are all obtained from Gaussian functions using the symmetries of the inequality (see also [19]). Then, existence of optimizers for $S = \mathbb{S}^2$ and for the endpoint $q = 4$ was proved by Christ and Shao [45], a result that was later extended to $S = \mathbb{S}^1$ (for the endpoint $q = 6$) by Shao [132] (see also [118] for the same result on convex arcs). Notice that the existence of maximizers in the subcritical cases (i.e. for q which is not the endpoint) in all dimensions is much easier to obtain [54], due to the fact that the sphere has finite measure. In [57], Foschi determined the set of maximizers for $S = \mathbb{S}^2$: they are all obtained from constant functions, using the symmetries of the inequality. The same result for $S = \mathbb{S}^1$ is still open, even

though some progress has been made in this direction [34, 122]. Maximizers have also been found for some related inequalities (subcritical, restricted set of maximization, weighted) [36, 38, 121, 35]. Let us also mention very interesting works related to the determination of best constants and the *non-existence* of optimizers, typically on hyperboloids or related surfaces [124, 119, 120, 37, 30, 39]. These techniques were also recently extended to the harder case $p \neq 2$ [146, 22].

The works cited above may be split into two categories: i) those related to the question of existence of maximizers using compactness techniques (these typically do not say anything about the exact form of the maximizer, but work in a larger range of cases) and ii) those who determine precisely the set of maximizers (which typically use a specific structure of the problem only happening for some exponents or in low dimensions, and are hard to generalize to other cases). Our approach belongs to the first category, and was greatly inspired by the work of Christ and Shao [45] on \mathbb{S}^2 .

1. The Stein-Tomas inequality

We begin with our work [63] on the existence of maximizers for the Stein-Tomas inequality on spheres in any dimension. For $N \geq 2$, let us thus define

$$\mathcal{R}_N := \sup_{f \in L^2(\mathbb{S}^{N-1}) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\check{f}(x)|^q dx}{\|f\|_{L^2(\mathbb{S}^{N-1})}^q},$$

where \check{f} is the Fourier transform of any function $f \in L^1(\mathbb{S}^{N-1})$ defined by

$$\forall x \in \mathbb{R}^N, \check{f}(x) = (2\pi)^{-(N-1)/2} \int_{\mathbb{S}^{N-1}} e^{ix \cdot \omega} f(\omega) d\omega,$$

where $d\omega$ is the standard Hausdorff measure on \mathbb{S}^{N-1} . Above, we chose

$$q = \frac{2(N+1)}{N-1}$$

to be the endpoint exponent, because as we mentioned above the non-endpoint case is much easier to treat [54]. We are interested in the question of the existence of maximizers for \mathcal{R}_N and in the description of maximizing sequences. Let $(f_n) \subset L^2(\mathbb{S}^{N-1})$ be such a sequence, that is $\|f_n\|_{L^2} = 1$ for all n and $\int_{\mathbb{R}^N} |\check{f}_n|^q \rightarrow \mathcal{R}_N$ as $n \rightarrow \infty$. As we mentioned in the introduction, we would ideally like to extract of subsequence of (f_n) which converges strongly in $L^2(\mathbb{S}^{N-1})$, so that its limit is a maximizer for \mathcal{R}_N . A first reason why such a hope is vain is the presence of the *non-compact symmetry* $f(\omega) \rightarrow e^{ia \cdot \omega} f(\omega)$ for any $a \in \mathbb{R}^N$, which clearly preserve the L^2 -norm of f and the L^q -norm of \check{f} . Hence, if $(a_n) \subset \mathbb{R}^N$ is any sequence, then $(e^{ia_n \cdot \omega} f_n)$ is also a maximizing sequence for \mathcal{R}_N , which may have no strongly convergent subsequence (for instance, if (f_n) is independent of n , equal to a maximizer of \mathcal{R}_N — if it exists — and $|a_n| \rightarrow +\infty$, then $(e^{ia_n \cdot \omega} f_n)$ converges weakly to zero in $L^2(\mathbb{S}^{N-1})$ and has fixed non-zero L^2 -norm, so that it does not converge strongly). This motivates the following definition.

DEFINITION 3.1. A sequence $(f_n) \subset L^2(\mathbb{S}^{N-1})$ is precompact up to modulations if there exists $(a_n) \subset \mathbb{R}^N$ such that $(e^{-ia_n \cdot \omega} f_n)$ has a strongly convergent subsequence in $L^2(\mathbb{S}^{N-1})$.

As we explain below, the problem of precompactness (up to modulations) of maximizing sequences to \mathcal{R}_N is closely related to the value of the best constant of the Fourier extension problem on the paraboloid (the Strichartz estimate), which for any $d \geq 1$ is

$$\mathcal{S}_d := \sup_{\psi \in L^2(\mathbb{R}^d) \setminus \{0\}} \frac{\int_{\mathbb{R}} \int_{\mathbb{R}^d} |e^{it\Delta/2}\psi(x)|^{2+4/d} dx dt}{\|\psi\|_{L^2(\mathbb{R}^d)}^{2+4/d}}.$$

Our main result is the following.

THEOREM 3.1 (Theorem 1.1 in [63]). *Let $N \geq 2$. Then, all L^2 -normalized maximizing sequences for \mathcal{R}_N are precompact up to modulations if and only if*

$$\mathcal{R}_N > a\mathcal{S}_{N-1}, \quad a := \frac{2^{q/2} \Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{q+2}{2}\right)}. \quad (3.1)$$

In this case, maximizers for \mathcal{R}_N exist.

The necessary and sufficient condition (3.1) is reminiscent of Lions' *binding inequality* in this theory of concentration-compactness. Indeed, we will see that the right side of (3.1) can be interpreted as the maximal 'energy' (here, energy refers to the quantity we are maximizing) of a sequence that 'escapes to infinity' along a certain non-compact transformation (we will see below that this transformation corresponds to concentration around two antipodal points on the sphere). From this point of view, Theorem 3.1 states that this is the only possible source of loss of compactness for maximizing sequences.

A natural question is to ask when (3.1) can be proved. We have the following result in this direction.

PROPOSITION 3.2 (Proposition 1.3 in [63]). *Let $N \geq 2$. If $\psi(x) = e^{-|x|^2/2}$ ($x \in \mathbb{R}^{N-1}$) is a maximizer for \mathcal{S}_{N-1} , then (3.1) holds.*

It is conjectured that Gaussians indeed maximize \mathcal{S}_d for all $d \geq 1$, so that a positive answer to this conjecture would imply precompactness up to modulations of maximizing sequences for \mathcal{R}_N by Theorem 3.1. Foschi [56] proved this fact for $d = 1, 2$, which was used by Christ-Shao [46] and Shao [132] to obtain this precompactness and thus existence of maximizers for \mathcal{R}_2 and \mathcal{R}_3 .

To understand why Proposition 3.2 holds, one first has to understand how the condition (3.1) arises in this problem. As we said, our goal amounts to understand the potential ways that a maximizing sequence fails to be precompact up to modulations. One natural such loss of compactness could arise from concentration around a point on the sphere. Indeed, a sequence which concentrates around a point such that its L^2 -mass remains constant typically converges weakly to zero and hence cannot converge strongly. Furthermore, modulating such a sequence does not alter this property: it still cannot converge strongly. One way to rule out the possibility that a maximizing sequence behaves like that is to compute its energy: if its energy converges to a number strictly less than \mathcal{R}_N , it cannot be a maximizing sequence and hence this source of loss of compactness is ruled out. It is thus instructive to compute the energy of a sequence of functions concentrating at a point. Since the inequality is invariant

by rotations, we may assume that this point is the north pole $(0, \dots, 0, 1)$. For $\psi \in \mathcal{S}(\mathbb{R}^{N-1})$ such that $\widehat{\psi} \in C_0^\infty(\mathbb{R}^{N-1})$ and $\varepsilon > 0$, we may thus define $f_\varepsilon : \mathbb{S}^{N-1} \rightarrow \mathbb{C}$ by

$$\begin{cases} f_\varepsilon(\eta', \sqrt{1 - |\eta'|^2}) = \varepsilon^{-(N-1)/2} \widehat{\psi}(\eta'/\varepsilon), \\ f_\varepsilon(\eta', -\sqrt{1 - |\eta'|^2}) = 0, \end{cases}$$

for all $\eta' \in \mathbb{R}^{N-1}$ with $|\eta'| < 1$. The function f_ε concentrates around the north pole at scale ε . Then, one has for ε small enough,

$$\|f\|_{L^2(\mathbb{S}^{N-1})}^2 = \varepsilon^{-(N-1)} \int_{\mathbb{R}^{N-1}} |\widehat{\psi}(\eta'/\varepsilon)|^2 \frac{d\eta'}{\sqrt{1 - |\eta'|^2}} \sim_{\varepsilon \rightarrow 0} \|\psi\|_{L^2(\mathbb{R}^{N-1})}^2,$$

and also for all $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$,

$$\begin{aligned} \check{f}_\varepsilon(x) &= \frac{\varepsilon^{-(N-1)/2}}{(2\pi)^{(N-1)/2}} \int_{\mathbb{R}^{N-1}} e^{ix' \cdot \eta' + ix_N \sqrt{1 - |\eta'|^2}} \widehat{\psi}(\eta'/\varepsilon) \frac{d\eta'}{\sqrt{1 - |\eta'|^2}} \\ &= \frac{\varepsilon^{(N-1)/2}}{(2\pi)^{(N-1)/2}} \int_{\mathbb{R}^{N-1}} e^{i\varepsilon x' \cdot \eta' + ix_N \sqrt{1 - \varepsilon^2 |\eta'|^2}} \widehat{\psi}(\eta') \frac{d\eta'}{\sqrt{1 - \varepsilon^2 |\eta'|^2}} \\ &= \frac{\varepsilon^{(N-1)/2} e^{ix_N}}{(2\pi)^{(N-1)/2}} \int_{\mathbb{R}^{N-1}} e^{i\varepsilon x' \cdot \eta' + ix_N (\sqrt{1 - \varepsilon^2 |\eta'|^2} - 1)} \widehat{\psi}(\eta') \frac{d\eta'}{\sqrt{1 - \varepsilon^2 |\eta'|^2}} \\ &\sim_{\varepsilon \rightarrow 0} \varepsilon^{(N-1)/2} e^{ix_N} (e^{i\varepsilon^2 x_N \Delta} \psi)(\varepsilon x'), \end{aligned}$$

so that

$$\int_{\mathbb{R}^N} |\check{f}_\varepsilon(x)|^q dx \sim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |(e^{ix_N \Delta} \psi)(x')|^q dx' dx_N.$$

As a conclusion, the 'energy' of the concentrating sequence (f_ε) is

$$\frac{\int_{\mathbb{R}^N} |\check{f}_\varepsilon(x)|^q dx}{\|f_\varepsilon\|_{L^2(\mathbb{S}^{N-1})}^q} \sim_{\varepsilon \rightarrow 0} \frac{\int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |(e^{ix_N \Delta} \psi)(x')|^q dx' dx_N}{\|\psi\|_{L^2(\mathbb{R}^{N-1})}^q},$$

whose maximal value is \mathcal{S}_{N-1} . The geometric interpretation of this phenomenon is that since the function concentrates at a point, it only sees the local geometry of the sphere which is the one of the paraboloid; hence one only sees in the limit the Fourier extension of the paraboloid which is exactly controlled by Strichartz estimates and the constant \mathcal{S}_{N-1} . If we have $\mathcal{R}_N > \mathcal{S}_{N-1}$, this means that a maximizing sequence for \mathcal{R}_N cannot concentrate at a point and thus this kind of loss of compactness is ruled out. For subcritical q , we have $\int_{\mathbb{R}^N} |\check{f}_\varepsilon(x)|^q dx \rightarrow 0$ as $\varepsilon \rightarrow 0$ so this phenomenon does not happen for maximizing sequences. Let us note that ruling out concentration at a point is the mechanism employed to obtain solutions for the Brézis-Nirenberg [29] and Yamabe problems [5].

In Theorem 3.1, our assumption is stronger than $\mathcal{R}_N > \mathcal{S}_{N-1}$: it is $\mathcal{R}_N > a\mathcal{S}_{N-1}$ where one can check that $a > 1$. The reason for that comes from a remark which was first made by Christ and Shao, namely that there is a bigger enemy than concentration at a point: concentration at two (antipodal) points! Indeed, if ones defines in a similar way as above

$$F_\varepsilon = f_\varepsilon + \widetilde{f}_\varepsilon$$

where \check{f}_ε is the reflection of f_ε through the equatorial hyperplane (that is, $\check{f}_\varepsilon(\omega', \omega_N) = f_\varepsilon(\omega', -\omega_N)$ for all $\omega = (\omega', \omega_N) \in \mathbb{S}^{N-1}$), then F_ε has half its mass that concentrates at the north pole and half its mass that concentrates at the south pole. A computation then shows its energy satisfies

$$\frac{\int_{\mathbb{R}^N} |\check{F}_\varepsilon(x)|^q dx}{\|F_\varepsilon\|_{L^2(\mathbb{S}^{N-1})}^q} \sim_{\varepsilon \rightarrow 0} a \frac{\int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |(e^{ix_N \Delta} \psi)(x')|^q dx' dx_N}{\|\psi\|_{L^2(\mathbb{R}^{N-1})}^q},$$

whose maximal value is $a\mathcal{S}_{N-1}$. Hence, Theorem 3.1 states that concentration at two antipodal points is the only enemy to precompactness of maximizing sequences. This construction is also the reason behind Proposition 3.2. Indeed, to prove it one considers the sequence F_ε with ψ which is a Gaussian. Since by assumption Gaussians are maximizers for \mathcal{S}_{N-1} , the leading order of the energy of F_ε is exactly $a\mathcal{S}_{N-1}$, and the idea is to compute the next order term in the energy. Luckily, it turns out that this term is strictly positive so that we indeed have $\mathcal{R}_N > a\mathcal{S}_{N-1}$. Notice that the same strategy was applied by Aubin [5] in the Yamabe problem, where he glued a Euclidean Sobolev maximizer on a manifold by concentrating it around a point, and computed the next order term in the energy which has the good sign.

It is instructive to understand why it is energetically better to concentrate at *antipodal* points. Our computation of \check{f}_ε above showed that it lives at scale ε^{-1} in the direction x' and at scale ε^{-2} in the direction x_N . Here, x_N is exactly the direction of the north pole which is the point of concentration of f_ε . If f_ε concentrates rather at $\omega \in \mathbb{S}^{N-1}$, then \check{f}_ε lives at scale ε^{-2} in the direction ω and at scale ε^{-1} in the direction orthogonal to ω . Hence, if F_ε concentrates at two points $\omega \in \mathbb{S}^{N-1}$ and $\omega' \in \mathbb{S}^{N-1}$ ($F_\varepsilon = f_\varepsilon^{(\omega)} + f_\varepsilon^{(\omega')}$ with $f_\varepsilon^{(\omega)}$ concentrating around ω and $f_\varepsilon^{(\omega')}$ concentrating around ω') and if ω and ω' are not colinear then $\check{f}_\varepsilon^{(\omega)}$ and $\check{f}_\varepsilon^{(\omega')}$ live at different scales so that

$$\int_{\mathbb{R}^N} |\check{F}_\varepsilon|^q \sim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^N} |\check{f}_\varepsilon^{(\omega)}|^q + \int_{\mathbb{R}^N} |\check{f}_\varepsilon^{(\omega')}|^q \sim_{\varepsilon \rightarrow 0} 2 \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |(e^{ix_N \Delta} \psi)(x')|^q dx' dx_N,$$

while $\int_{\mathbb{S}^{N-1}} |F_\varepsilon|^2 \sim_{\varepsilon \rightarrow 0} 2 \int_{\mathbb{R}^{N-1}} |\psi|^2$, so that

$$\frac{\int_{\mathbb{R}^N} |\check{F}_\varepsilon|^q}{\|F_\varepsilon\|_{L^2}^q} \sim_{\varepsilon \rightarrow 0} 2^{1-q/2} \frac{\int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |(e^{ix_N \Delta} \psi)(x')|^q dx' dx_N}{\|\psi\|_{L^2(\mathbb{R}^{N-1})}^q}$$

which is even worse than concentration around one point since $2^{1-q/2} < 1!$ Notice that this reasoning fails when ω and ω' are colinear (thus, equal or antipodal) since then $\check{f}_\varepsilon^{(\omega)}$ and $\check{f}_\varepsilon^{(\omega')}$ live at the same scale and one cannot infer that $\int_{\mathbb{R}^N} |\check{F}_\varepsilon|^q$ splits. An explicit computation then shows that in the antipodal case, some 'constructive interference' happens which leads to the best case energetically. Notice that this argument also shows why it is energetically unfavorable to concentrate at a higher number of points.

Now that we have explained the reasoning behind the statement of Theorem 3.1, let us comment a bit on its proof. We first used a method invented by Lieb [103] which states that a maximizing sequence converges in $L^2(\mathbb{S}^{N-1})$ if and only if it converges weakly to a non-zero function. Indeed, if $(f_n) \subset L^2$ is such that $\|f_n\|_{L^2} = 1$ for all n and such that $f_n \rightharpoonup f \neq 0$ in L^2 , $\int_{\mathbb{R}^N} |\check{f}_n|^q \rightarrow \mathcal{R}_N$ as $n \rightarrow +\infty$, we deduce by weak convergence that

$\|f_n - f\|_{L^2}^2 \rightarrow 1 - \|f\|_{L^2}^2$ as $n \rightarrow +\infty$ and that $\check{f}_n(x) \rightarrow \check{f}(x)$ as $n \rightarrow +\infty$ for all $x \in \mathbb{R}^N$ (since $\omega \mapsto e^{-ix \cdot \omega} \in L^2(\mathbb{S}^{N-1})$). By this pointwise convergence and the Brézis-Lieb lemma [28], we deduce

$$\begin{aligned} \mathcal{R}_N &= \int_{\mathbb{R}^N} |\check{f}_n|^q + o_{n \rightarrow +\infty}(1) \\ &= \int_{\mathbb{R}^N} |\check{f}|^q + \int_{\mathbb{R}^N} |\check{f}_n - \check{f}|^q + o_{n \rightarrow +\infty}(1) \\ &\leq \mathcal{R}_N(\|f\|_{L^2}^q + \|f_n - f\|_{L^2}^q) + o_{n \rightarrow +\infty}(1) \end{aligned}$$

which in the limit $n \rightarrow +\infty$ leads to

$$\mathcal{R}_N \leq \mathcal{R}_N(\|f\|_{L^2}^q + (1 - \|f\|_{L^2}^2)^{q/2}).$$

Now since $\mathcal{R}_N > 0$ and since $a^{q/2} + b^{q/2} \leq (a+b)^{q/2}$ for all $a, b \geq 0$ with equality if and only if $a = 0$ or $b = 0$ due to $q > 2$, we deduce that $\|f\|_{L^2} = 0$ or $\|f\|_{L^2} = 1$. Since $f \neq 0$, we deduce $\|f\|_{L^2} = 1$ and hence $\|f_n - f\|_{L^2} \rightarrow 0$ as $n \rightarrow +\infty$, proving the claim. Hence, if we define

$$\mathcal{R}_N^* := \sup \left\{ \limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N} |\check{f}_n|^q : \|f_n\|_{L^2} = 1, f_n \rightharpoonup_{\text{mod}} 0 \right\},$$

where $f_n \rightharpoonup_{\text{mod}} 0$ means that $e^{ia_n \cdot \omega} f_n \rightarrow 0$ in L^2 for all (a_n) , then $\mathcal{R}_N > \mathcal{R}_N^*$ implies that all normalized maximizing sequences for \mathcal{R}_N are precompact up to symmetries (because $\mathcal{R}_N > \mathcal{R}_N^*$ implies that maximizing sequences for \mathcal{R}_N have a non-zero weak limit up to modulation). The proof of Theorem 3.1 is thus reduced to proving $\mathcal{R}_N^* = a\mathcal{S}_{N-1}$, and more precisely $\mathcal{R}_N^* \leq a\mathcal{S}_{N-1}$ since the reverse inequality is true due to our example of functions concentrating at two antipodal points.

To prove that $\mathcal{R}_N^* \leq a\mathcal{S}_{N-1}$, we fix a sequence $(f_n) \subset L^2(\mathbb{S}^{N-1})$ with $\|f_n\|_{L^2} = 1$ and $f_n \rightharpoonup_{\text{mod}} 0$. We may also assume $\limsup_{n \rightarrow +\infty} \int_{\mathbb{R}^N} |\check{f}_n|^q \geq (1/2)\mathcal{R}_N^* \geq (a/2)\mathcal{S}_{N-1} > 0$ so that \check{f}_n does not converge to 0 in $L^q(\mathbb{R}^N)$. The goal is to detect some mass of (f_n) that concentrates around a point. To do so, we consider a family \mathcal{D} of dyadic cubes Q covering the sphere (technically, one starts with cubes on \mathbb{R}^{N-1} centered on $(2^j\mathbb{Z})^{N-1}$ and of side length 2^j for all $j \in \mathbb{Z}$, and lift them on the sphere by using a finite number of small enough patches). The main technical tool is the following refined Stein-Tomas inequality.

PROPOSITION 3.3 (Proposition 5.1 in [63]). *There exist $C > 0$ and $\theta \in (0, 1)$ such that for all $f \in L^2(\mathbb{S}^{N-1})$ we have*

$$\|\check{f}\|_{L^q(\mathbb{R}^N)} \leq C \left(\sup_{Q \in \mathcal{D}} |Q|^{-1/2} \|(\mathbf{1}_Q f)^\vee\|_{L^\infty(\mathbb{R}^N)} \right)^\theta \|f\|_{L^2}^{1-\theta}.$$

Such an inequality is called refined since

$$\|(\mathbf{1}_Q f)^\vee\|_{L^\infty(\mathbb{R}^N)} \leq C \|\mathbf{1}_Q f\|_{L^1(\mathbb{S}^{N-1})} \leq C |Q|^{1/2} \|f\|_{L^2(\mathbb{S}^{N-1})}$$

and hence it implies the standard Stein-Tomas estimate. Furthermore it has the important consequence that if $(f_n) \subset L^2$ is a normalized sequence such that $\|\check{f}\|_{L^q(\mathbb{R}^N)}$ does not converge

to zero as $n \rightarrow +\infty$ (as the previous test sequence (f_n) for \mathcal{R}_N^*), then there exist $c > 0$, $(x_n) \subset \mathbb{R}^N$ and $(Q_n) \subset \mathcal{D}$ such that

$$= \left| |Q_n|^{-1/2} \int_{Q_n} e^{ix_n \cdot \omega} f_n(\omega) d\omega \right| = (2\pi)^{(N-1)/2} |(1_{Q_n} f_n)^\vee(x_n)| \geq c.$$

This implies that the sequence (f_n) 'rescaled around (Q_n) ' has a non-zero weak limit (in $L^2(\mathbb{R}^{N-1})$, because we use patches to rescale the sequence). With this method, we found some non-zero mass concentrating along the sequence of cubes (Q_n) . Since $f_n \rightharpoonup_{\text{mod}} 0$, we have $|Q_n| \rightarrow 0$ meaning that (f_n) indeed has some mass concentrating around a point (the center of the cube). The end of argument revolves around the idea of proving that *all* the mass concentrates in (Q_n) and its antipodal version if (f_n) is maximizing for \mathcal{R}_N^* . We do so by adapting Lieb's argument above to this setting of concentrating objects, which requires several technical tools like a Brézis-Lieb lemma for sequences. In the end, this proves that $\mathcal{R}_N^* \leq a\mathcal{S}_{N-1}$.

The proof of Proposition 3.3 is very similar to the one of Tao [150] and Killip-Visan [88] in the case of the paraboloid, which themselves rely heavily on deep bilinear estimates due to Tao [148]. Other refinements of the Stein-Tomas exist in the terms of the so-called X_p -spaces [116, 118, 132], but as emphasized by the above works of Tao and Killip-Visan, a version with an L^∞ -norm on the right side is particularly convenient for compactness purposes. In the context of homogeneous Sobolev spaces, a similar compactness argument has been given by Gérard [71], where the corresponding refined inequality with an L^∞ on the right side is taken from [72].

2. The Airy-Strichartz inequality

In [67], we applied the same strategy to the Airy-Strichartz inequality [86]:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| |D_x|^\gamma (e^{-t\partial_x^3} u)(x) \right|^q \right)^{p/q} dt \leq C \|u\|_{L^2(\mathbb{R})}^p, \quad (3.2)$$

measuring the space-time decay of $v(t, x) := (e^{-t\partial_x^3} u)(x)$ which satisfies the Airy equation $\partial_t v + \partial_x^3 v = 0$ with initial condition $u \in L^2(\mathbb{R})$. The inequality (3.2) is valid for all exponents satisfying

$$2 \leq p, q < +\infty, \quad -\gamma + \frac{3}{p} + \frac{1}{q} = \frac{1}{2}, \quad -\frac{1}{2} < \gamma \leq \frac{1}{p}.$$

The inequality (3.2) may be seen as a kind of Fourier extension inequality, since the space-time Fourier transform of v is supported on the cubic curve $\omega = \xi^3$. We are also interested in the best constants

$$\mathcal{A}_p := \sup_{u \in L^2(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| |D_x|^{1/p} (e^{-t\partial_x^3} u)(x) \right|^q dx \right)^{p/q} dt}{\|u\|_{L^2(\mathbb{R})}^p}$$

in the critical case $\gamma = 1/p$, for any $p > 4$ and q such that $2/p + 1/q = 1/2$. Again, an important role is played by the symmetries of the inequality. For any $(t_0, x_0, \lambda_0) \in$

$\mathbb{R} \times \mathbb{R} \times (0, +\infty)$, if we define for any $u \in L^2(\mathbb{R})$,

$$\forall x \in \mathbb{R}, (g_{t_0, x_0, \lambda_0} u)(x) = \lambda_0^{1/2} (e^{-t_0 \partial_x^3} u)(\lambda_0 x + x_0),$$

and

$$G = \{g_{t_0, x_0, \lambda_0} : (t_0, x_0, \lambda_0) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty)\},$$

we have

$$\|gu\|_{L^2} = \|u\|_{L^2}, \quad \|\Psi_p[gu]\|_{L_t^p L_x^q} = \|\Psi_p[u]\|_{L_t^p L_x^q},$$

for all $g \in G$ and $u \in L^2(\mathbb{R})$, where we used the notation

$$\Psi_p[u](t, x) := |D_x|^{1/p} (e^{-t \partial_x^3} u)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\xi|^{1/p} e^{ix\xi + it\xi^3} \widehat{u}(\xi) d\xi.$$

The (t_0, x_0) -part of the symmetry corresponds to the modulation symmetry of the previous part, while the λ_0 -part of the symmetry is the scaling symmetry which is not a symmetry on the sphere (but is rather the transformation responsible for the arising of the paraboloid by concentrating functions around points). Since G is not compact, it may also be responsible for a loss of compactness of maximizing sequences, which motivates the following definition.

DEFINITION 3.4. A sequence $(u_n) \subset L^2(\mathbb{R})$ is *precompact up to symmetries* if there exists $(g_n) \subset G$ such that $(g_n u_n)$ has a subsequence which converges strongly in $L^2(\mathbb{R})$.

As in the case of the Stein-Tomas inequality on the sphere, the Strichartz inequality for the paraboloid will also play a role so we define

$$\mathcal{S}_p := \sup_{u \in L^2(\mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \left| (e^{-3it \partial_x^2} u)(x) \right|^q dx \right)^{p/q} dt}{\|u\|_{L^2(\mathbb{R})}^p}.$$

Our result in this case is the following.

THEOREM 3.2 (Theorem 1 in [67]). *Let $4 < p < \infty$ and q such that $2/p + 1/q = 1/2$. Then, all normalized maximizing sequences for \mathcal{A}_p are precompact up to symmetries if and only if*

$$\mathcal{A}_p > a_p \mathcal{S}_p, \quad a_p := \frac{2^{p/2}}{\pi^{p/(2q)}} \left(\frac{\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{q+2}{2}\right)} \right)^{p/q}. \quad (3.3)$$

In particular, if (3.3) holds, then there is a maximizer for \mathcal{A}_p .

Having seen Theorem 3.1, the statement of Theorem (3.2) is not surprising if we understand how the paraboloid arises from the cubic curve. It does so when looking at functions u such that \widehat{u} concentrates around a non-zero frequency (for instance, $\xi = 1$). This is not surprising since the cubic curve is indeed curved (!) away from the origin so it looks like a paraboloid locally around any of those points. The constant $a_p > 1$ appears when looking at functions which Fourier transform concentrate at two opposite frequencies. This concentration phenomenon appears in the case of the cubic curve due to the non-invariance of the inequality by translation in Fourier variables, so that it makes sense to look at what happens to the inequality asymptotically in such a transformation. This problem is in some sense complementary to what happens on the sphere, where ξ -translations correspond to

rotations on the sphere which are actual symmetries. On the contrary, the Airy-Strichartz inequality is scaling invariant while the Stein-Tomas inequality on the sphere is not; and it is the non-invariance by scaling which leads to the concentration phenomenon.

The proof of Theorem 3.2 follows the same lines as the proof of Theorem 3.1, except that one has to adapt the tools to include mixed Lebesgue space, which creates some technical challenges. Interestingly, the refined inequality (which is the main tool to follow the mass) can be proved in a self-contained manner since we are in one space dimension. While the Stein-Tomas refined inequality relied on the bilinear estimates of Tao, the bilinear estimates in the case of the cubic curve have a simple proof using the Hausdorff-Young inequality (see [67, Lemma 3.7]). This leads to the following refined Airy-Strichartz inequality.

THEOREM 3.3 (Theorem 3 in [67]). *There exist $C > 0$ and $\theta \in (0, 1)$ such that for any $u \in L^2(\mathbb{R})$ we have*

$$\| |D_x|^{1/6} e^{-t\partial_x^3} u \|_{L_{t,x}^6} \leq C \left(\sup_{I \in \mathcal{D}} |c(I)|^{-1/6} |I|^{-1/2} \| |D_x|^{1/6} e^{-t\partial_x^3} u_I \|_{L_{t,x}^\infty} \right)^\theta \|u\|_{L^2(\mathbb{R})}^{1-\theta},$$

where \mathcal{D} denotes the family of all intervals of the type $I = [k, k+1)2^\ell$ with $k, \ell \in \mathbb{Z}$, $|I|$ denotes the length of I , $c(I)$ denotes the center of I , and $\widehat{u}_I := \mathbf{1}_I \widehat{u}$.

Notice the appearance of the *center* of the dyadic interval explicitly in the estimate, related to the non-invariance of the estimate by ξ -translations.

Our theorem in the case $p = 6$ has been stated before in [130], but there was a missing point in the proof (concentration at antipodal points was not noticed). Since the cubic curve has infinite measure, one cannot use the result of [54] to infer that subcritical inequalities always have maximizers. We proved in [67, Theorem 4] that all subcritical inequalities $\gamma < 1/p$ have maximizers. The subcritical result corresponding to $p = q = 8$ was proved before in [83].

An interesting point is that we have no equivalent of Proposition 3.2 in this case, even if we *know* that Gaussians are maximizers of the one-dimensional Strichartz inequality for $p = 6$. This is due to the fact that the next order term in the concentration parameter when using Gaussians is *negative*, contrary to the Stein-Tomas case, preventing us to show that (3.3) holds for $p = 6$. Let us also notice that, to our knowledge, there is no conjecture about what the maximizers for the Airy-Strichartz look like.

As a conclusion, the topic of optimal constants for extension inequalities remains vastly open compared to the similar one for Sobolev inequalities, for instance. Even if these questions have an interest for themselves, it is reasonable to hope that the tools developed to study them, like the compactness tools of this chapter, could be applicable to other non-local and/or oscillatory problems.

Bibliography

- [1] H. ADDED AND S. ADDED, *Equations of Langmuir turbulence and nonlinear Schrödinger equation: smoothness and approximation*, J. Funct. Anal., 79 (1988), pp. 183–210.
- [2] S. AGMON AND L. HÖRMANDER, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Analyse Math., 30 (1976), pp. 1–38.
- [3] M. AIZENMAN AND B. SIMON, *Brownian motion and Harnack inequality for Schrödinger operators*, Comm. Pure Appl. Math., 35 (1982), pp. 209–273.
- [4] G. L. AKI, P. A. MARKOWICH, AND C. SPARBER, *Classical limit for semirelativistic Hartree systems*, J. Math. Phys., 49 (2008), pp. 102110, 10.
- [5] T. AUBIN, *Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire*, J. Math. Pures Appl. (9), 55 (1976), pp. 269–296.
- [6] V. G. AVAKUMOVIĆ, *Über die Eigenfunktionen der Schwingungsgleichung*, Acad. Serbe Sci. Publ. Inst. Math., 4 (1952), pp. 95–96.
- [7] ———, *Über die Eigenfunktionen auf geschlossenen Riemannschen Mannigfaltigkeiten*, Math. Z., 65 (1956), pp. 327–344.
- [8] V. BACH, *Error bound for the Hartree-Fock energy of atoms and molecules*, Commun. Math. Phys., 147 (1992), pp. 527–548.
- [9] V. BACH, S. BRETEAUX, S. PETRAT, P. PICKL, AND T. TZANETEAS, *Kinetic energy estimates for the accuracy of the time-dependent Hartree-Fock approximation with Coulomb interaction*, J. Math. Pures Appl. (9), 105 (2016), pp. 1–30.
- [10] V. BACH, E. LIEB, AND J.-P. SOLOVEJ, *Generalized Hartree-Fock theory and the Hubbard model*, J. Statist. Phys., 76 (1994), pp. 3–89.
- [11] C. BARDOS, F. GOLSE, A. GOTTLIEB, AND N. MAUSER, *Mean field dynamics of fermions and the time-dependent Hartree-Fock equation*, J. Math. Pures Appl. (9), 82 (2003), pp. 665–683.
- [12] S. BAUMSTARK, G. SCHNEIDER, K. SCHRATZ, AND D. ZIMMERMANN, *Effective slow dynamics models for a class of dispersive systems*, J. Dyn. Diff. Eq., 32 (2020), pp. 1867–1899.
- [13] J. BEDROSSIAN, N. MASMOUDI, AND C. MOUHOT, *Landau damping in finite regularity for unconfined systems with screened interactions*, Comm. Pure Appl. Math., 71 (2018), pp. 537–576.
- [14] I. BEJENARU AND S. HERR, *The cubic Dirac equation: small initial data in $H^1(\mathbb{R}^3)$* , Comm. Math. Phys., 335 (2015), pp. 43–82.
- [15] ———, *The cubic Dirac equation: small initial data in $H^{\frac{1}{2}}(\mathbb{R}^2)$* , Comm. Math. Phys., 343 (2016), pp. 515–562.
- [16] ———, *On global well-posedness and scattering for the massive Dirac-Klein-Gordon system*, J. Eur. Math. Soc. (JEMS), 19 (2017), pp. 2445–2467.
- [17] N. BENEDIKTER, M. PORTA, C. SAFFIRIO, AND B. SCHLEIN, *From the Hartree dynamics to the Vlasov equation*, Arch. Ration. Mech. Anal., 221 (2016), pp. 273–334.
- [18] N. BENEDIKTER, M. PORTA, AND B. SCHLEIN, *Mean-field evolution of fermionic systems*, Comm. Math. Phys., 331 (2014), pp. 1087–1131.
- [19] J. BENNETT, N. BEZ, A. CARBERY, AND D. HUNDERTMARK, *Heat-flow monotonicity of Strichartz norms*, Anal. PDE, 2 (2009), pp. 147–158.
- [20] F. A. BEREZIN, *Covariant and contravariant symbols of operators*, Izv. Akad. Nauk SSSR Ser. Mat., 36 (1972), pp. 1134–1167.

- [21] M. BERGER, P. GAUDUCHON, AND E. MAZET, *Le spectre d'une variété riemannienne*, Lecture Notes in Mathematics, Vol. 194, Springer-Verlag, Berlin-New York, 1971.
- [22] C. BISWAS AND B. STOVALL, *Existence of extremizers for fourier restriction to the moment curve*, arXiv preprint arXiv:2012.01528, (2020).
- [23] S. BÖGLI, *Schrödinger operator with non-zero accumulation points of complex eigenvalues*, Comm. Math. Phys., 352 (2017), pp. 629–639.
- [24] A. BORICHEV, L. GOLINSKII, AND S. KUPIN, *A Blaschke-type condition and its application to complex Jacobi matrices*, Bulletin of the London Mathematical Society, 41 (2009), pp. 117–123.
- [25] N. BOURNAVEAS, *A new proof of global existence for the Dirac Klein-Gordon equations in one space dimension*, J. Funct. Anal., 173 (2000), pp. 203–213.
- [26] A. BOVE, G. DA PRATO, AND G. FANO, *An existence proof for the Hartree-Fock time-dependent problem with bounded two-body interaction*, Commun. Math. Phys., 37 (1974), pp. 183–191.
- [27] ———, *On the Hartree-Fock time-dependent problem*, Commun. Math. Phys., 49 (1976), pp. 25–33.
- [28] H. BRÉZIS AND E. H. LIEB, *A relation between pointwise convergence of functions and convergence of functionals*, Proceedings of the American Mathematical Society, 88 (1983), pp. 486–490.
- [29] H. BRÉZIS AND L. NIRENBERG, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math., 36 (1983), pp. 437–477.
- [30] G. BROCCHI, D. OLIVEIRA E SILVA, AND R. QUILODRÁN, *Sharp Strichartz inequalities for fractional and higher-order Schrödinger equations*, Anal. PDE, 13 (2020), pp. 477–526.
- [31] E. CANCÈS, M. DEFRANCESCHI, W. KUTZELNIGG, C. LE BRIS, AND Y. MADAY, *Computational quantum chemistry: a primer*, in Handbook of numerical analysis, Vol. X, Handb. Numer. Anal., X, North-Holland, Amsterdam, 2003, pp. 3–270.
- [32] T. CARLEMAN, *Über die asymptotische Verteilung der Eigenwerte partielle Differentialgleichungen*, Berichten der mathematisch-physisch Klasse der Sächsischen Akad. der Wissenschaften zu Leipzig, LXXXVIII Band, Sitzung, 15 (1936).
- [33] L. CARLESON AND P. SJÖLIN, *Oscillatory integrals and a multiplier problem for the disc*, Studia Math., 44 (1972), pp. 287–299. (errata insert). Collection of articles honoring the completion by Antoni Zygmund of 50 years of scientific activity, III.
- [34] E. CARNEIRO, D. FOSCHI, D. OLIVEIRA E SILVA, AND C. THIELE, *A sharp trilinear inequality related to Fourier restriction on the circle*, Rev. Mat. Iberoam., 33 (2017), pp. 1463–1486.
- [35] E. CARNEIRO, G. NEGRO, AND D. OLIVEIRA E SILVA, *Stability of sharp fourier restriction to spheres*, arXiv preprint arXiv:2108.03412, (2021).
- [36] E. CARNEIRO AND D. OLIVEIRA E SILVA, *Some sharp restriction inequalities on the sphere*, Int. Math. Res. Not. IMRN, (2015), pp. 8233–8267.
- [37] E. CARNEIRO, D. OLIVEIRA E SILVA, AND M. SOUSA, *Extremizers for Fourier restriction on hyperboloids*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 36 (2019), pp. 389–415.
- [38] ———, *Sharp mixed norm spherical restriction*, Adv. Math., 341 (2019), pp. 583–608.
- [39] E. CARNEIRO, D. OLIVEIRA E SILVA, M. SOUSA, AND B. STOVALL, *Extremizers for adjoint Fourier restriction on hyperboloids: the higher-dimensional case*, Indiana Univ. Math. J., 70 (2021), pp. 535–559.
- [40] F. CASTELLA AND B. PERTHAME, *Estimations de Strichartz pour les équations de transport cinétique*, CR Acad. Sci. Paris Sér. I Math, 322 (1996), pp. 535–540.
- [41] T. CAZENAVE, *Semilinear Schrödinger Equations*, Courant Lecture Notes in Mathematics, American Mathematical Society, 2003.
- [42] J. M. CHADAM, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac equations in one space dimension*, J. Functional Analysis, 13 (1973), pp. 173–184.
- [43] ———, *The time-dependent Hartree-Fock equations with Coulomb two-body interaction*, Commun. Math. Phys., 46 (1976), pp. 99–104.

- [44] J.-Y. CHEMIN AND C.-J. XU, *Inclusions de Sobolev en calcul de Weyl-Hörmander et champs de vecteurs sous-elliptiques*, Ann. Sci. École Norm. Sup. (4), 30 (1997), pp. 719–751.
- [45] M. CHRIST AND S. SHAO, *Existence of extremals for a Fourier restriction inequality*, Anal. PDE, 5 (2012), pp. 261–312.
- [46] ———, *On the extremizers of an adjoint Fourier restriction inequality*, Adv. Math., 230 (2012), pp. 957–977.
- [47] J. B. CONWAY, *A course in abstract analysis*, vol. 141 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2012.
- [48] M. DAUB, G. SCHNEIDER, AND K. SCHRATZ, *From the Klein-Gordon-Zakharov system to the Klein-Gordon equation*, Math. Meth. Appl. Sc., 39 (2016), pp. 5371–5380.
- [49] V. DELGADO, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac and other nonlinear Dirac equations in one space dimension*, Proc. Amer. Math. Soc., 69 (1978), pp. 289–296.
- [50] M. DEMUTH, M. HANSMANN, AND G. KATRIEL, *Eigenvalues of non-selfadjoint operators: A comparison of two approaches*, in Mathematical Physics, Spectral Theory and Stochastic Analysis, Springer, 2013, pp. 107–163.
- [51] E. DIETLER, S. RADEMACHER, AND B. SCHLEIN, *From Hartree dynamics to the relativistic Vlasov equation*, J. Stat. Phys., 172 (2018), pp. 398–433.
- [52] N. DUNFORD AND B. J. PETTIS, *Linear operations on summable functions*, Trans. Amer. Math. Soc., 47 (1940), pp. 323–392.
- [53] A. ELGART, L. ERDŐS, B. SCHLEIN, AND H.-T. YAU, *Nonlinear Hartree equation as the mean field limit of weakly coupled fermions*, J. Math. Pures Appl., 83 (2004), pp. 1241–1273.
- [54] L. FANELLI, L. VEGA, AND N. VISCIGLIA, *On the existence of maximizers for a family of restriction theorems*, Bull. Lond. Math. Soc., 43 (2011), pp. 811–817.
- [55] V. FOCK, *Näherungsmethode zur Lösung des quantenmechanischen Mehrkörperproblems*, Zeitschrift für Physik, 61 (1930), pp. 126–148.
- [56] D. FOSCHI, *Maximizers for the Strichartz inequality*, J. Eur. Math. Soc. (JEMS), 9 (2007), pp. 739–774.
- [57] ———, *Global maximizers for the sphere adjoint Fourier restriction inequality*, J. Funct. Anal., 268 (2015), pp. 690–702.
- [58] S. R. FOURNAIS, M. LEWIN, AND J. P. SOLOVEJ, *The semi-classical limit of large fermionic systems*, Calc. Var. Partial Differential Equations, 57 (2018), pp. Paper No. 105, 42.
- [59] R. L. FRANK, *Eigenvalue bounds for Schrödinger operators with complex potentials*, Bull. Lond. Math. Soc., 43 (2011), pp. 745–750.
- [60] ———, *Eigenvalue bounds for Schrödinger operators with complex potentials. III*, Trans. Amer. Math. Soc., 370 (2018), pp. 219–240.
- [61] R. L. FRANK, M. LEWIN, E. H. LIEB, AND R. SEIRINGER, *A positive density analogue of the Lieb-Thirring inequality*, Duke Math. J., 162 (2012), pp. 435–495.
- [62] ———, *Strichartz inequality for orthonormal functions*, J. Eur. Math. Soc. (JEMS), 16 (2014), pp. 1507–1526.
- [63] R. L. FRANK, E. H. LIEB, AND J. SABIN, *Maximizers for the Stein-Tomas inequality*, Geom. Funct. Anal., 26 (2016), pp. 1095–1134.
- [64] R. L. FRANK AND J. SABIN, *The Stein-Tomas inequality in trace ideals*, Séminaire Laurent Schwartz—EDP et applications, (2016), pp. 1–12.
- [65] ———, *Restriction theorems for orthonormal functions, Strichartz inequalities, and uniform Sobolev estimates*, Amer. J. Math., 139 (2017), pp. 1649–1691.
- [66] ———, *Spectral cluster bounds for orthonormal systems and oscillatory integral operators in Schatten spaces*, Adv. Math., 317 (2017), pp. 157–192.
- [67] ———, *Extremizers for the Airy-Strichartz inequality*, Math. Ann., (2018), pp. 1–46.
- [68] ———, *Sharp Weyl laws with singular potentials*, arXiv preprint arXiv:2007.04284, (2020).

- [69] G. FREUD, *Restglied eines Tauberschen Satzes. I*, Acta Math. Acad. Sci. Hungar., 2 (1951), pp. 299–308.
- [70] J. FRÖHLICH AND A. KNOWLES, *A microscopic derivation of the time-dependent Hartree-Fock equation with Coulomb two-body interaction*, J. Stat. Phys., 145 (2011), pp. 23–50.
- [71] P. GÉRARD, *Description du défaut de compacité de l'injection de Sobolev*, ESAIM Control Optim. Calc. Var., 3 (1998), pp. 213–233.
- [72] P. GERARD, Y. MEYER, AND F. ORU, *Inégalités de Sobolev précisées*, in Séminaire sur les Équations aux Dérivées Partielles, 1996–1997, École Polytech., Palaiseau, 1997, pp. Exp. No. IV, 11.
- [73] J. GINIBRE AND G. VELO, *The global Cauchy problem for the nonlinear Schrödinger equation revisited*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 2 (1985), pp. 309–327.
- [74] F. GOLSE, C. MOUHOT, AND T. PAUL, *On the mean field and classical limits of quantum mechanics*, Comm. Math. Phys., 343 (2016), pp. 165–205.
- [75] F. GOLSE AND T. PAUL, *The Schrödinger equation in the mean-field and semiclassical regime*, Arch. Ration. Mech. Anal., 223 (2017), pp. 57–94.
- [76] M. GRIESEMER, J. SCHMID, AND G. SCHNEIDER, *On the dynamics of the mean-field polaron in the high-frequency limit*, Lett. Math. Phys., 107 (2017), pp. 1809–1821.
- [77] A. GRÜNROCK AND H. PECHER, *Global solutions for the Dirac-Klein-Gordon system in two space dimensions*, Comm. Partial Differential Equations, 35 (2010), pp. 89–112.
- [78] K. HEPP, *The classical limit for quantum mechanical correlation functions*, Commun. Math. Phys., 35 (1974), pp. 265–277.
- [79] L. HÖRMANDER, *The spectral function of an elliptic operator*, Acta Math., 121 (1968), pp. 193–218.
- [80] X. HUANG AND C. D. SOGGE, *Weyl formulae for Schrödinger operators with critically singular potentials*, Comm. Partial Differential Equations, 46 (2021), pp. 2088–2133.
- [81] X. HUANG AND C. ZHANG, *Pointwise weyl laws for Schrödinger operators with singular potentials*, arXiv preprint arXiv:2103.05531, (2021).
- [82] ———, *Sharp pointwise weyl laws for Schrödinger operators with singular potentials on flat tori*, arXiv preprint arXiv:2109.13370, (2021).
- [83] D. HUNDERTMARK AND S. SHAO, *Analyticity of extremizers to the Airy-Strichartz inequality*, Bull. Lond. Math. Soc., 44 (2012), pp. 336–352.
- [84] D. HUNDERTMARK AND V. ZHARNITSKY, *On sharp Strichartz inequalities in low dimensions*, Int. Math. Res. Not., (2006), pp. Art. ID 34080, 18.
- [85] T. KATO AND G. PONCE, *Commutator estimates and the Euler and Navier-Stokes equations*, Commun. Pure Applied Math., 41 (1988), pp. 891–907.
- [86] C. E. KENIG, G. PONCE, AND L. VEGA, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J., 40 (1991), pp. 33–69.
- [87] C. E. KENIG, A. RUIZ, AND C. D. SOGGE, *Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators*, Duke Math. J., 55 (1987), pp. 329–347.
- [88] R. KILLIP AND M. VISAN, *Nonlinear Schrödinger equations at critical regularity*, Evolution equations, 17 (2013), pp. 325–437.
- [89] M. KUNZE, *On the existence of a maximizer for the Strichartz inequality*, Commun. Math. Phys., 243 (2003), pp. 137–162.
- [90] J. LAMPART, L. L. TREUST, S. R. NODARI, AND J. SABIN, *The Dirac-Klein-Gordon system in the strong coupling limit*, arXiv preprint arXiv:2110.09087, (2021).
- [91] B. M. LEVITAN, *On the asymptotic behavior of the spectral function of a self-adjoint differential equation of the second order*, Izvestiya Akad. Nauk SSSR. Ser. Mat., 16 (1952), pp. 325–352.
- [92] M. LEWIN, P. T. NAM, AND N. ROUGERIE, *Derivation of nonlinear gibbs measures from many-body quantum mechanics*, Journal de l'École polytechnique - Mathématiques, 2 (2015), pp. 65–115.
- [93] M. LEWIN AND J. SABIN, *A family of monotone quantum relative entropies*, Lett. Math. Phys., 104 (2014), pp. 691–705.

- [94] ———, *The Hartree equation for infinitely many particles. II. Dispersion and scattering in 2D*, Analysis and PDE, 7 (2014), pp. 1339–1363.
- [95] ———, *The Hartree equation for infinitely many particles I. Well-posedness theory*, Comm. Math. Phys., 334 (2015), pp. 117–170.
- [96] ———, *The Hartree and Vlasov equations at positive density*, Comm. Partial Differential Equations, 45 (2020), pp. 1702–1754.
- [97] E. LIEB AND M.-B. RUSKAI, *Proof of the strong subadditivity of quantum-mechanical entropy*, J. Math. Phys., 14 (1973), pp. 1938–1941. With an appendix by B. Simon.
- [98] E. LIEB AND B. SIMON, *The Thomas-Fermi theory of atoms, molecules and solids*, Advances in Math., 23 (1977), pp. 22–116.
- [99] E. H. LIEB, *The classical limit of quantum spin systems*, Comm. Math. Phys., 31 (1973), pp. 327–340.
- [100] ———, *Variational principle for many-fermion systems*, Phys. Rev. Lett., 46 (1981), pp. 457–459.
- [101] ———, *An L^p bound for the Riesz and Bessel potentials of orthonormal functions*, J. Funct. Anal., 51 (1983), pp. 159–165.
- [102] ———, *On the lowest eigenvalue of the Laplacian for the intersection of two domains*, Invent. Math., 74 (1983), pp. 441–448.
- [103] ———, *Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities*, Ann. Math., 118 (1983), pp. 349–374.
- [104] E. H. LIEB AND B. SIMON, *The Hartree-Fock theory for Coulomb systems*, Commun. Math. Phys., 53 (1977), pp. 185–194.
- [105] E. H. LIEB AND W. E. THIRRING, *Bound on kinetic energy of fermions which proves stability of matter*, Phys. Rev. Lett., 35 (1975), pp. 687–689.
- [106] ———, *Inequalities for the moments of the eigenvalues of the Schrödinger hamiltonian and their relation to Sobolev inequalities*, Studies in Mathematical Physics, Princeton University Press, 1976, pp. 269–303.
- [107] E. H. LIEB AND H.-T. YAU, *The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics*, Commun. Math. Phys., 112 (1987), pp. 147–174.
- [108] L. LIN AND J. LU, *A mathematical introduction to electronic structure theory*, vol. 4 of SIAM Spotlights, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2019.
- [109] P.-L. LIONS, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part I*, Ann. Inst. Henri Poincaré, 1 (1984), pp. 109–149.
- [110] ———, *The concentration-compactness principle in the calculus of variations. The locally compact case, Part II*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), pp. 223–283.
- [111] ———, *The concentration-compactness principle in the calculus of variations. The limit case. I*, Rev. Mat. Iberoamericana, 1 (1985), pp. 145–201.
- [112] ———, *The concentration-compactness principle in the calculus of variations. The limit case. II*, Rev. Mat. Iberoamericana, 1 (1985), pp. 45–121.
- [113] P.-L. LIONS AND T. PAUL, *Sur les mesures de Wigner*, Rev. Mat. Iberoamericana, 9 (1993), pp. 553–618.
- [114] G. LOEPER, *Uniqueness of the solution to the Vlasov-Poisson system with bounded density*, J. Math. Pures Appl. (9), 86 (2006), pp. 68–79.
- [115] P. A. MARKOWICH AND N. J. MAUSER, *The classical limit of a self-consistent quantum-Vlasov equation in 3D*, Math. Models Methods Appl. Sci., 3 (1993), pp. 109–124.
- [116] A. MOYUA, A. VARGAS, AND L. VEGA, *Restriction theorems and maximal operators related to oscillatory integrals in \mathbb{R}^3* , Duke Math. J., 96 (1999), pp. 547–574.
- [117] H. NARNHOFER AND G. L. SEWELL, *Vlasov hydrodynamics of a quantum mechanical model*, Comm. Math. Phys., 79 (1981), pp. 9–24.
- [118] D. OLIVEIRA E SILVA, *Extremizers for Fourier restriction inequalities: convex arcs*, J. Anal. Math., 124 (2014), pp. 337–385.
- [119] ———, *Nonexistence of extremizers for certain convex curves*, Math. Res. Lett., 25 (2018), pp. 973–987.

- [120] D. OLIVEIRA E SILVA AND R. QUILODRÁN, *On extremizers for Strichartz estimates for higher order Schrödinger equations*, Trans. Amer. Math. Soc., 370 (2018), pp. 6871–6907.
- [121] ———, *Global maximizers for adjoint Fourier restriction inequalities on low dimensional spheres*, J. Funct. Anal., 280 (2021), pp. Paper No. 108825, 73.
- [122] D. OLIVEIRA E SILVA, C. THIELE, AND P. ZORIN-KRANICH, *Band-Limited Maximizers for a Fourier Extension Inequality on the Circle*, Exp. Math., 31 (2022), pp. 192–198.
- [123] S. PETRAT AND P. PICKL, *A new method and a new scaling for deriving fermionic mean-field dynamics*, Math. Phys. Anal. Geom., 19 (2016).
- [124] R. QUILODRÁN, *Nonexistence of extremals for the adjoint restriction inequality on the hyperboloid*, J. Anal. Math., 125 (2015), pp. 37–70.
- [125] P. RING, *Relativistic mean field theory in finite nuclei*, Prog. Part. Nucl. Phys., 37 (1996), pp. 193–263.
- [126] J. SABIN, *Stability, dispersion, and pair production for some infinite quantum systems*, PhD thesis, Université de Cergy-Pontoise, 2013. tel-00924084.
- [127] ———, *The Hartree equation for infinite quantum systems*, Journées Équations aux dérivées partielles, (2014), pp. 1–18.
- [128] ———, *Littlewood-Paley decomposition of operator densities and application to a new proof of the Lieb-Thirring inequality*, Math. Phys. Anal. Geom., 19 (2016), pp. Art. 11, 11.
- [129] S. H. SCHOCHET AND M. I. WEINSTEIN, *The nonlinear Schrödinger limit of the Zakharov equations governing Langmuir turbulence*, Comm. Math. Phys., 106 (1986), pp. 569–580.
- [130] S. SHAO, *The linear profile decomposition for the Airy equation and the existence of maximizers for the Airy Strichartz inequality*, Anal. PDE, 2 (2009), pp. 83–117.
- [131] ———, *Maximizers for the Strichartz inequalities and the Sobolev-Strichartz inequalities for the Schrödinger equation*, Electronic J. of Differential Equations, (2009), pp. 1–13.
- [132] ———, *On existence of extremizers for the Tomas-Stein inequality for S^1* , J. Funct. Anal., 270 (2016), pp. 3996–4038.
- [133] B. SIMON, *Trace ideals and their applications*, vol. 35 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1979.
- [134] ———, *The classical limit of quantum partition functions*, Comm. Math. Phys., 71 (1980), pp. 247–276.
- [135] ———, *Schrödinger semigroups*, Bull. Amer. Math. Soc. (N.S.), 7 (1982), pp. 447–526.
- [136] ———, *Basic complex analysis*, A Comprehensive Course in Analysis, Part 2A, American Mathematical Society, Providence, RI, 2015.
- [137] ———, *Operator theory*, A Comprehensive Course in Analysis, Part 4, American Mathematical Society, Providence, RI, 2015.
- [138] J. C. SLATER, *Note on Hartree’s method*, Physical Review, 35 (1930), p. 210.
- [139] C. D. SOGGE, *Oscillatory integrals and spherical harmonics*, Duke Math. J., 53 (1986), pp. 43–65.
- [140] ———, *Concerning the L^p norm of spectral clusters for second-order elliptic operators on compact manifolds*, J. Funct. Anal., 77 (1988), pp. 123–138.
- [141] J.-P. SOLOVEJ, *Many body quantum mechanics*. Lectures notes at LMU München, 2007. Available at <http://web.math.ku.dk/~solovej/MANYBODY/mbnotes-ptn-5-3-14.pdf>.
- [142] H. SPOHN, *On the Vlasov hierarchy*, Math. Methods Appl. Sci., 3 (1981), pp. 445–455.
- [143] E. M. STEIN, *Interpolation of linear operators*, Trans. Amer. Math. Soc., 83 (1956), pp. 482–492.
- [144] ———, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, 30, Princeton University Press, 1970.
- [145] ———, *Oscillatory integrals in Fourier analysis*, in Beijing lectures in harmonic analysis (Beijing, 1984), vol. 112 of Ann. of Math. Stud., Princeton Univ. Press, Princeton, NJ, 1986, pp. 307–355.
- [146] B. STOVALL, *Extremizability of Fourier restriction to the paraboloid*, Adv. Math., 360 (2020), pp. 106898, 18.
- [147] R. STRICHARTZ, *Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J., 44 (1977), pp. 705–714.

- [148] T. TAO, *A sharp bilinear restriction estimate for paraboloids*, *Geom. Funct. Anal.*, 13 (2003), pp. 1359–1384.
- [149] ———, *Nonlinear Dispersive Equations: Local And Global Analysis*, no. 106 in CBMS Regional Conference Ser. in Mathematics Series, American Mathematical Society, 2006.
- [150] ———, *A pseudoconformal compactification of the nonlinear Schrödinger equation and applications*, *New York J. Math.*, 15 (2009), pp. 265–282.
- [151] P. A. TOMAS, *A restriction theorem for the Fourier transform*, *Bull. Amer. Math. Soc.*, 81 (1975), pp. 477–478.
- [152] C. VILLANI, *Optimal transport*, vol. 338 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 2009. Old and new.
- [153] E. WIGNER, *On the quantum correction for thermodynamic equilibrium*, *Phys. Rev.*, 40 (1932), pp. 749–759.
- [154] K. YAJIMA, *Existence of solutions for Schrödinger evolution equations*, *Comm. Math. Phys.*, 110 (1987), pp. 415–426.
- [155] S. ZAGATTI, *The Cauchy problem for Hartree-Fock time-dependent equations*, *Ann. Inst. H. Poincaré Phys. Théor.*, 56 (1992), pp. 357–374.
- [156] C. ZUILY, *Éléments de distributions et d'équations aux dérivées partielles: cours et problèmes résolus*, vol. 130, Dunod, 2002.
- [157] M. ZWORSKI, *Semiclassical analysis*, vol. 138 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2012.

Titre : Analyse de quelques inégalités fonctionnelles et équations aux dérivées partielles liées à de grands systèmes quantiques

Résumé : Nous étudions plusieurs questions d'analyse provenant de la mécanique quantique à plusieurs corps. Plus précisément, nous considérons des équations aux dérivées partielles non-linéaires ainsi que des inégalités fonctionnelles décrivant un grand nombre de fermions dans une approximation de champ moyen. Cela mène à l'analyse mathématique des matrices densités à un corps dont la trace est grande ou infinie, avec un intérêt particulier pour la distribution spatiale de particules associée. Nous adaptons des outils d'EDPs dispersives, d'analyse de Fourier et semi-classique à ce cadre.

Dans une première partie, nous établissons un lien entre plusieurs équations d'évolution qui décrivent de grands systèmes quantiques (relativistes et non-relativistes) dans certains régimes asymptotiques, en utilisant des méthodes de compacité ou de convergence forte. Dans une deuxième partie, nous développons des méthodes d'analyse harmonique qui impliquent des inégalités fonctionnelles sur des matrices densités que nous appelons fermioniques. Dans une troisième et dernière partie (sans lien avec la mécanique quantique à plusieurs corps), nous étudions l'existence de fonctions optimales pour des inégalités d'extension de la transformée de Fourier, à l'aide de méthodes de compacité.

Title : Analysis of some functional inequalities and partial differential equations related to large quantum systems

Abstract : We study several topics in analysis which come from many-body quantum mechanics. More specifically, we consider nonlinear partial differential equations and functional inequalities that describe a large number of fermions in a mean-field approximation. This leads to the mathematical analysis of one-body density matrices with large or infinite trace, with a special emphasis on the associated spatial distribution of particles. We adapt tools from dispersive PDEs, Fourier and semiclassical analysis to this setting.

In a first part, we relate several evolution equations describing large quantum systems (relativistic and non-relativistic) in various asymptotic regimes, employing both compactness methods or strong estimates. In a second part, we develop methods in harmonic analysis that imply functional inequalities on density matrices that we call fermionic. In a third and last part (unrelated to many-body quantum mechanics), we study the existence of optimizers for Fourier extension inequalities by compactness methods.