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Mattia Giaccagli

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## **Stabilité incrémentale et applications pour systèmes de contrôle non linéaires**

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**Mots clés:** Systèmes non linéaires, stabilité incrémentale, contraction, métrique Riemannienne, forwarding, inégalité des matrices linéaires, régulation de sortie, action intégrale, régulation harmonique, systèmes multi-agents, synchronisation, réseau neuronal profond, contrôleur de flux de puissance, micro-réseaux maillés, systèmes bilinéaires, ventilation mécanique, application médicale, critère circulaire, systèmes Lur'e.



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# ABSTRACT

## Context

This thesis deals with the notion of incremental stability and its application in the context of control design for nonlinear systems. The manuscript is divided into four main chapters, each of them dealing with different topics but strictly related among them. In the first chapter, we study the notion of incremental stability for nonlinear control systems. In short, a system is said to be incrementally stable if trajectories starting from different initial conditions asymptotically converge towards each other. Such a notion is of interest due to several properties that incrementally stable systems share, such as periodicity of trajectories, robustness with respect to external perturbations, and many others. Among the different tools to study such a notion, we focus on the so-called (Riemanniann) ‘metric-based’ approach. Despite the theory of incrementally stable systems is receiving a lot of interest from the worldwide control community, several open questions need to be answered yet, concerning the analysis of incremental properties and the feedback control design achieving incremental stability.

In the second chapter, we focus on the output regulation problem. The goal is to design a (dynamic) control law such that the output of a nonlinear system can asymptotically track a reference and, at the same time, reject perturbations. In particular, we aim to achieve “global” output regulation, meaning that the regulation task must be achieved independently of the initial conditions and on the amplitude of the external signals. The challenge is to guarantee the existence of a steady-state solution on which the regulation error is zero for every value of the external signals, and the convergence of trajectories towards such a solution for every initial condition. While tools achieving regulation for minimum phase systems in normal form are well developed, much less is known for more general classes of systems, especially when global regulation is the goal. Therefore, new tools need to be developed. In our approach, in particular, we cast the regulation task into the incremental framework and we tackle the problem with tools derived from the first chapter of the manuscript.

In the third chapter, we focus on the multiagent synchronization problem. Here, we consider a group of single identical entities which communicate among them through a communication protocol. The objective is the design of a distributed coupling control law such that these entities reach an agreement on their state evaluation. While the theory for linear systems is well developed, many questions remain open for nonlinear ones. In our approach, we cast the synchronization problem into the incremental framework. Such a

choice is motivated by the fact that, if agents are described by the same dynamical model, then the synchronization problem corresponds to the design of a distributed control law such that different trajectories of the same differential equation asymptotically converge towards each other.

In the fourth and last chapter, we focus on two practical applications. In particular, we consider two separate problems. The first problem is a robust output set-point tracking problem for a power flow controller. A power flow controller is an electric circuit whose role is to regulate the power of the lines to which it is attached, despite the uncertainties of the plant parameters and of the references to be tracked. The second problem is a periodic trajectory tracking for a ventilation machine. A ventilation machine is a piece of medical equipment used to support patients breathing. The objective here is to design a control law such that the machine can track a periodic pressure signal representing the breathing phase, despite the uncertainties in the plant.

At the end of the last chapter, a summary of the thesis written in French is present.

## Main contributions

In Chapter 1, we study incremental properties of nonlinear systems with a metric-based approach. Some literature results are recalled. Then, we study incremental ISS properties. To conclude, we propose some designs of feedback control laws for the closed-loop system to possess incremental properties. The main contribution of this chapter can be found in the author's publications Giaccagli et al. (2022a,b,c,d).

**Contribution 1:** Through the notion of ‘Killing vector field’, we derive metric-based sufficient conditions for a system to be incremental ISS.

**Contribution 2:** We provide a set of sufficient conditions for the design of feedback control laws achieving incremental properties for the closed-loop system. In particular: i) we derive a metric-based incremental small gain theorem; ii) we develop an incremental version of forwarding control techniques; iii) we provide a set of LMI conditions for systems with sector-bound nonlinearities.

In Chapter 2, we study the global output regulation problem. We cast the problem into the contraction framework. We study two problems separately: the problem of tracking a constant reference and the problem of harmonic regulation. The main contributions can be found in the author's publications Giaccagli et al. (2020, 2021b, 2022d).

**Contribution 3:** We provide a set of sufficient conditions to achieve global constant output set-point tracking and disturbance rejection for nonlinear systems which do not necessarily admit a globally defined diffeomorphism that allows rewriting them in normal form.

**Contribution 4:** We provide a set of sufficient conditions to achieve global harmonic regulation of a periodic trajectory of a nonlinear system which does not necessarily admit a globally defined diffeomorphism that allows to rewrite it in normal form.

In Chapter 3, we study the multiagent synchronization problem. We tackle the problem with tools derived from incremental stability theory. We study the synchronization of connected directed networks in the Euclidean framework and leader-synchronization of

undirected networks in the Riemaniann framework. Then, in this second case, we provide a Deep Neural Network-based algorithm for the practical implementation of the distributed control law. The results in this chapter have been obtained together with Samuele Zoboli, LAGEPP, University of Lyon 1 (Villeurbanne, France) who took care of the machine learning experimental part. The main contributions can be found in the author's publications Giaccagli et al. (2021a, 2022e).

**Contribution 5:** We derive a set of sufficient conditions based on a Euclidean metric analysis for the synchronization of connected directed networks of nonlinear systems.

**Contribution 6:** We derive a set of sufficient conditions based on a Riemannian metric analysis for the leader-synchronization of connected undirected networks of nonlinear systems. A general deep neural network-based algorithm for the implementation of the control law is presented.

In Chapter 4, we focus on two practical applications. In particular, we consider a robust output set-point tracking problem for a power flow controller and a harmonic regulation problem for a mechanical ventilation machine. The results in this chapter have been obtained together with Tanguy Simon and collaborators at Laboratory Ampere, INSA Lyon (Villeurbanne, France) for the power flow controller and with Joey Reinders and collaborators at Department of Mechanical Engineering, Eindhoven University of Technology (Eindhoven, the Netherlands) for the ventilation machine. In both cases, the other teamwork took care of the modelling and of the experimental part. The main contributions can be found in the author's publications Simon et al. (2021a, 2022); Reinders et al. (2022).

**Contribution 7:** We design and implement a forwarding-based dynamic robust control law for the power flow controller achieving robust output set-point tracking. The control law is validated through experimental results.

**Contribution 8:** We design and implement an incremental circle criterion-based dynamic control law for the mechanical ventilation machine achieving robust harmonic regulation. The control law is validated through experimental results.

## Publications

During this thesis, the total account of published/submitted/in preparation articles is of 8 journal articles and 4 conference papers. The publications are listed here below.

### Journal articles:

- Giaccagli, M., Astolfi, D., Andrieu, V., and Marconi, L. (2021). Sufficient conditions for global integral action via incremental forwarding for input-affine nonlinear systems. *IEEE Transactions on Automatic Control*.
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- Giaccagli, M., Andrieu, V., Tarbouriech, S., and Astolfi, D. (2022). LMI conditions for contraction, integral action and output feedback stabilization for a class of nonlinear systems. Submitted to *Automatica*.
- Giaccagli, M., Astolfi, D., Andrieu, V., and Marconi, L. (2022). Incremental stabilization of cascade nonlinear systems and harmonic regulation: a forwarding-based design. Submitted to *IEEE Transactions on Automatic Control*.
- Giaccagli, M., Astolfi, D., and Andrieu, V. (2022). A metric approach to incremental stability. In preparation.
- Simon, T., Giaccagli, M., Tréguët, J.F., Astolfi, D., Andrieu, V., Lin-Shi, X., and Morel, H. (2022). Robust regulation of a power flow controller via nonlinear integral action, *Prov.* accepted in *IEEE Transaction on Control Systems Technology*.
- Reinders, J., Giaccagli, M., Hunnekens, B., Astolfi, D., Oomen, T., and van de Wouw, N. (2022). Repetitive control for Lur'e-type systems: application to mechanical ventilation. Submitted to *IEEE Transaction on Control Systems Technology*.

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- Giaccagli, M., Astolfi, D., Andrieu, V., and Marconi, L. (2020). Sufficient conditions for output reference tracking for nonlinear systems: a contractive approach. In *59th IEEE Conference on Decision and Control (CDC)*, pp. 4580-4585, IEEE.
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- Giaccagli, M., Andrieu, V., Tarbouriech, S., and Astolfi, D. (2022). Infinite gain margin, contraction and optimality: a LMI-based design. In *20th European Control Conference (ECC)*.
- Simon, T., Giaccagli, M., Tréguët, J. F., Astolfi, D., Andrieu, V., Lin-Shi, X., and Morel, H. (2021). Robust output set-point tracking for a power flow controller via forwarding design. In *60th IEEE Conference on Decision and Control (CDC)*, pp. 6385-6390, IEEE.

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# NOTATION

Unless where explicitly specified, we will use the following notations.

## General notation

- $\dot{x} := \frac{dx}{dt}(t)$  ;
- $\frac{\partial f}{\partial x} :=$  partial derivative of  $f$  with respect to  $x$ ;
- $\frac{\partial^k f}{\partial x^k} := k$ -th partial derivative of  $f$  with respect to  $x$ ;
- $A^\top :=$  transpose of a vector/matrix  $A$ ;
- $\forall :=$  for all;
- Given a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ ,  $A = A^\top$ , we say that  $A$  is positive definite (respectively, semi-positive definite) and we write  $A \succ 0$  ( $\succeq 0$ ) if  $x^\top A x > 0$  ( $\geq 0$ ) for all  $x \in \mathbb{R}^n$ ;
- Given a symmetric matrix  $A \in \mathbb{R}^{n \times n}$ ,  $A = A^\top$ , we say that  $A$  is negative definite (respectively, semi-negative definite) and we write  $A \prec 0$  ( $\preceq 0$ ) if  $-A$  is positive definite (respectively, semi-positive definite);

## Vectors

- $\mathbf{1} := (1 \ 1 \ \dots \ 1)^\top$  is the column vector full of ones;
- Given a vector  $a \in \mathbb{R}^n$ , the notation  $a_k$  refers to the  $k$ -th element of  $a$ , with 1 being the index of the first element;

## Sets

- $\mathbb{R} := (-\infty, +\infty)$ , real numbers;
- $\mathbb{R}^n := \underbrace{(-\infty, +\infty) \times \dots \times (-\infty, +\infty)}_{n\text{-times}}$ ,  $n$ -dimensional real numbers;
- $\mathbb{R}_{\geq 0} := [0, +\infty)$ , non-negative real numbers;

- $\mathbb{R}_{>0} := (0, +\infty)$ , positive real numbers;
- $\mathbb{N} := \{0, 1, 2, \dots\}$ , natural numbers;
- $\mathbb{C} := \{a + jb \mid a \in \mathbb{R}, b \in \mathbb{R}\}$ , complex numbers where  $j$  is the imaginary term satisfying  $j^2 = -1$ ;
- Given a set  $\mathcal{A} \subseteq \mathbb{R}^n$  and  $1 \leq p \leq +\infty$ , we indicate with  $\mathcal{L}^p(\mathcal{A})$  the set of locally  $p$ -integrable functions defined on  $\mathcal{A}$ , that is the set of functions  $f : \mathcal{A} \mapsto \mathbb{R}^m$  satisfying

$$\int_{\mathcal{A}} |f|^p < +\infty;$$

- Given a set  $\mathcal{S} \subset \mathbb{R}^n$ , we indicate with  $\text{cl}\{\mathcal{S}\}$  the closure of  $\mathcal{S}$ , that is, the set containing all the elements of  $\mathcal{S}$  together with its boundary elements;
- Given a set  $\mathcal{S} \subseteq \mathbb{R}^n$ , we indicate with  $\mathcal{S}^m$  the set  $\underbrace{\mathcal{S} \times \dots \times \mathcal{S}}_{m\text{-times}} \subseteq \mathbb{R}^{nm}$ ;
- Given a set  $\mathcal{S}$ ,  $\text{card}\{\mathcal{S}\}$  denotes its cardinality, that is, the number of the elements in  $\mathcal{S}$ ;
- Given a set  $\mathcal{S}$ ,  $\text{int}\{\mathcal{S}\}$  denotes its interior, that is, the elements of  $\mathcal{S}$  that do not belong to its closure.

## Functions

- A function  $f : \mathbb{R}^n \mapsto \mathbb{R}^m$  is  $C^k$  (for some  $k \in \mathbb{N}$ ) if it is differentiable up to  $k$ -times and all the  $k$  partial derivatives are continuous functions
- A function  $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  is *positive definite* if it verifies the identity  $\{x \in \mathbb{R} \mid \alpha(x) = 0\} = \{0\}$ ;
- A function  $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  is of *class- $\mathcal{K}$*  if it is strictly increasing and  $\alpha(0) = 0$ ;
- A function  $\alpha : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  is of *class- $\mathcal{K}_\infty$*  if it is of class- $\mathcal{K}$  and  $\lim_{s \rightarrow \infty} \alpha(s) = +\infty$ .
- A function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \mapsto \mathbb{R}_{\geq 0}$  is of *class- $\mathcal{KL}$*  if, for each fixed  $s$ ,  $\beta(r, s)$  is of class- $\mathcal{K}$  with respect to  $r$  and, for each fixed  $r$ , the mapping  $\beta(r, s)$  is decreasing with respect to  $s$  and  $\lim_{s \rightarrow \infty} \beta(r, s) = 0$ ;
- Define the following asymmetric scalar saturation function  $\text{sat}_{\underline{s}}^{\bar{s}} : \mathbb{R} \rightarrow \mathbb{R}$  as

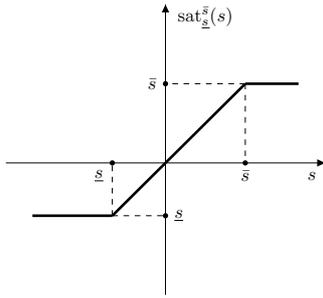
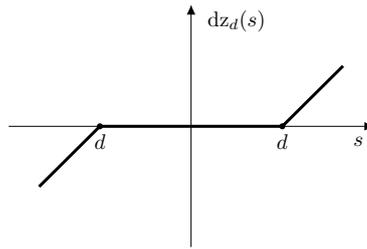
$$\text{sat}_{\underline{s}}^{\bar{s}}(s) = \begin{cases} \underline{s} & \text{if } s < \underline{s} \\ s & \text{if } \underline{s} \leq s \leq \bar{s} \\ \bar{s} & \text{if } s > \bar{s} \end{cases}$$

for some constants  $\underline{s} \leq \bar{s} \in \mathbb{R}$  (see Figure 1);

- Define the following scalar dead-zone function  $\text{dz}_d : \mathbb{R} \rightarrow \mathbb{R}$  as

$$\text{dz}_d(s) = s - \text{sat}_{-d}^d(s)$$

for some constant  $d > 0$  (see Figure 2);

Figure 1: Function  $\text{sat}_{\bar{s}}(s)$ Figure 2: Function  $\text{dz}_d(s)$ 

- Given any two vectors  $\bar{s} = [\bar{s}_1 \ \dots \ \bar{s}_m]^\top \in \mathbb{R}^m$ ,  $\underline{s} = [\underline{s}_1 \ \dots \ \underline{s}_m]^\top \in \mathbb{R}^m$ , define the following (vector) saturation function  $\text{Sat}_{\bar{s}}: \mathbb{R}^m \rightarrow \mathbb{R}^m$  as

$$\text{Sat}_{\bar{s}}(s) = \begin{bmatrix} \text{sat}_{\bar{s}_1}(s_1) \\ \vdots \\ \text{sat}_{\bar{s}_m}(s_m) \end{bmatrix}$$

for some vector  $s = [s_1 \ \dots \ s_m]^\top \in \mathbb{R}^m$ , i.e. as the functions that implement element-wise the sat function;

- Given any two vectors  $\bar{s} = [\bar{s}_1 \ \dots \ \bar{s}_m]^\top \in \mathbb{R}^m$ ,  $\underline{s} = [\underline{s}_1 \ \dots \ \underline{s}_m]^\top \in \mathbb{R}^m$ , define the following (vector) dead-zone function  $\text{Dz}_d: \mathbb{R}^m \rightarrow \mathbb{R}^m$  as

$$\text{Dz}_d(s) = \begin{bmatrix} \text{dz}_d(s_1) \\ \vdots \\ \text{dz}_d(s_m) \end{bmatrix}, \quad (1)$$

for some vector  $s = [s_1 \ \dots \ s_m]^\top \in \mathbb{R}^m$ , i.e. as the functions that implement element-wise the dz function;

## Operators

- Given a complex number  $c \in \mathbb{C}$ ,  $c = a + jb$  we indicate its real part with  $\mathcal{R}\{c\} = a$ ;
- Given a complex number  $c \in \mathbb{C}$ ,  $c = a + jb$  we indicate its imaginary part with  $\mathcal{I}\{c\} = b$ ;
- Given a matrix  $A \in \mathbb{R}^{n \times m}$ , we indicate with  $\text{Im}\{A\}$  its image, namely  $\text{Im}\{A\} := \{\lambda \in \mathbb{R}^n \mid Av = \lambda, \text{ for some } v \in \mathbb{R}^m\}$ ;
- Given a matrix  $A \in \mathbb{R}^{n \times m}$ , we indicate with  $\text{rank}\{A\} \in \mathbb{N}$  the rank of  $A$ ;
- Given a square matrix  $A \in \mathbb{R}^{n \times n}$ , we indicate with  $\text{spec}\{A\}$  the spectrum of  $A$ , i.e. the set whose elements are the eigenvalues of  $A$ ;
- Given a partially ordered set  $\mathcal{S}$ , we indicate with  $\text{inf}_{\mathcal{S}}\{\cdot\}$  the infimum over the set  $\mathcal{S}$ , namely the greatest element in  $\mathcal{S}$  that is less than or equal to each element of  $\mathcal{S}$ ;
- Given a partially ordered set  $\mathcal{S}$ , we indicate with  $\text{sup}_{\mathcal{S}}\{\cdot\}$  the supremum over the set  $\mathcal{S}$ , namely the least element in  $\mathcal{S}$  that is greater or equal to each element of  $\mathcal{S}$ ;

- Given a square matrix  $A \in \mathbb{R}^{n \times n}$  we indicate the Hermitian operator as  $\text{He}\{A\} = A + A^\top$ ;
- Given  $N$  square matrices (possibly with different dimensions)  $A_1, \dots, A_N$ , we indicate with  $\text{blkdiag}\{A_1, \dots, A_N\}$  the square matrix with  $A_1, \dots, A_N$  in the main diagonal and zeros everywhere else

$$\text{blkdiag}\{A_1, \dots, A_N\} := \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & A_N \end{pmatrix};$$

- Given  $N$  column vectors  $x_i \in \mathbb{R}^{n_i}$  for  $i \in \{1, \dots, N\}$ , we indicate with  $\text{col}\{x_1, \dots, x_N\}$  the column vector of dimension  $n = \sum_{i=1}^N n_i$  whose elements are the ordered set of vectors  $x_i$ ,

$$\text{col}\{x_1, \dots, x_N\} := \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix};$$

- Given a  $n \times m$  matrix  $B$ , we indicate with  $\text{vec}(B)$  the  $nm$  column vector where the elements are the ordered elements of matrix  $B$ .
- $|x| := \sqrt{x^\top x}$ , Euclidean norm of  $x \in \mathbb{R}^n$ ;
- $|x|_P := \sqrt{x^\top P x}$ , norm of  $x \in \mathbb{R}^n$  in the matrix operator  $P \in \mathbb{R}^{n \times n}$ ;
- $\|A\|_P := \sup_{x \neq 0} \frac{|Ax|_P}{|x|_P}$ , induced matrix norm of the matrix  $A \in \mathbb{R}^{n \times n}$  in the matrix operator  $P \in \mathbb{R}^{n \times n}$ ;
- Given a set  $\mathcal{S} \subset \mathbb{R}^n$  and a vector  $x \in \mathbb{R}^n$ , we indicate with  $|x|_{\mathcal{S}} := \inf_{z \in \mathcal{S}} |x - z|$ ;
- $A \otimes B :=$  Kronecker product between  $A$  and  $B$ ;
- Given a vector field  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and a  $C^1$  mapping  $h : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , we denote the Lie derivative of  $h$  along  $f$  at  $x$  as  $L_f h(x) = \frac{\partial h}{\partial x}(x) f(x)$ ;
- Given a vector field  $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$  and a 2-tensor  $P : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^{n \times n}$  both  $C^1$ , we indicate with  $L_f P(x, t)$  the Lie derivative of the tensor  $P$  along  $f$  defined as

$$L_f P(x, t) = \lim_{h \rightarrow 0} \frac{(I + h \frac{\partial f}{\partial x}(x, t))^\top P(x + h f(x, t), t + h) (I + h \frac{\partial f}{\partial x}(x, t)) - P(x, t)}{h}$$

with coordinates

$$(L_f P(x, t))_{i,j} = \sum_k \left[ 2P_{ik}(x, t) \frac{\partial f_k}{\partial x_j}(x, t) + \frac{\partial P_{ij}}{\partial x_k}(x, t) f_k(x, t) + \frac{\partial P_{ij}}{\partial t}(x, t) \right].$$

Equivalently,

$$L_f P(x, t) := \mathfrak{d}_f P(x, t) + P(x, t) \frac{\partial f}{\partial x}(x, t) + \frac{\partial f^\top}{\partial x}(x, t) P(x, t)$$

where

$$\mathfrak{d}_f P(x, t) := \lim_{h \rightarrow 0} \frac{P(\mathcal{X}(x, t + h, t), t) - P(x, t)}{h} + \frac{\partial P}{\partial t}(x, t),$$

and where  $\mathcal{X}(x_0, t, t_0)$  is the solution of

$$\frac{\partial}{\partial t} \mathcal{X}(x_0, t, t_0) = f(\mathcal{X}(x_0, t, t_0), t), \quad \mathcal{X}(x_0, t_0, t_0) = x_0, \quad \forall t \geq 0;$$

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## Acronyms

- SISO := Single-Input Single-Output;
- MIMO := Multi-Input Multi-Output;
- ODE:= Ordinary Differential Equation;
- PDE := Partial Differential Equation;
- LMI := Linear Matrix Inequality;
- BMI := Bilinear Matrix Inequality;
- ARE := Algebraic Riccati Equation;
- LQR:= Linear Quadratic Regulator;
- HJB := Hamilton-Jacobi-Bellman
- RC := Repetitive Control;
- PFC := Power Flow Controller;
- VM := Ventilation Machine;
- DC := Direct Current;
- AW := Anti Windup;
- DNN := Deep Neural Network.



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# CHAPTER 1

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## INCREMENTAL STABILITY

The only thing that one really knows about human nature is that it changes. Change is the one quality we can predicate of it. The systems that fail are those that rely on the permanency of human nature, and not on its growth and development.

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O. Wilde

### 1.1 Introduction

Stability is a wide concept. Depending on the kind of problem that we face or on the kind of analysis that we aim to pursue, one can exploit different types of stability notions. The most common is the stability of an equilibrium point of a vector field in the sense of Lyapunov with the  $(\varepsilon - \delta)$  formalism (see (Khalil, 2002, Section 4.1) for an overview). Roughly speaking, an equilibrium point is said to be stable if solutions of an Ordinary Differential Equation (ODE) that are initialized ‘close’ to it, remain ‘close’ for all positive times. If moreover, such an equilibrium is attractive, that is, the solutions asymptotically converge towards it, then we recover the classical notion of asymptotic stability of an equilibrium point. Such a concept is generally exploited when dealing with stabilization problems of nonlinear systems. But the stability of an equilibrium point is not the only notion of stability. One can for instance look at the stability of a manifold. In this sense, similar concepts regarding the boundness of solutions and asymptotic convergence can be derived. This is of interest for instance, when dealing with engineering problems such as output regulation or multiagent synchronization. Similarly, if we deal with a trajectory tracking problem, one can look at the stability properties of a particular solution of the system.

It is clear that, depending on the kind of problem that one aims to solve, it is of interest to exploit one or another stability notion. All of the above, however, have a common point: they’re all stability notions of a particular *attractor*. In other words,

the interest is to study how the solutions of a dynamical system behave with respect to a well-defined mathematical object, be it an equilibrium point, a manifold, or a particular solution. It is straightforward that the knowledge of such a mathematical object is a central point. This is evident if we look at a stabilization problem of an equilibrium point. Most of the existing results in the literature that rely on a Lyapunov-based analysis, consider (without loss of generality) the origin as the equilibrium point that has to be stabilized using a control law. However, especially in practical problems, this is never the case. What is commonly done, is to first apply a constant control law to shift the equilibrium to the origin, and then rely on stabilizing control actions. This, however, requires the knowledge of the equilibrium point, and also of the constant preliminary control action that has to be applied. The problem is that these two pieces of information might not be perfectly known since they are strongly dependent on the plant. Moreover, they are very sensitive in case of external perturbations, up to the point that, in limited cases, the existence itself of an equilibrium cannot be guaranteed.

From this limitation, it becomes of interest the study of stability notions that are not dependent on a ‘fixed’ element. Indeed, instead of linking any solution of the system with respect to a specific object, we may be interested in studying the relations between any solution of the system with respect any other solution. This led to the definition of the notion of *incremental stability*, also known as *contraction*, which will be the main subject of this first section. Of course, a trajectory initialized in an equilibrium point (if it exists), is itself a valid solution to a dynamic system. For this reason, the kind of analysis and the conditions that we will obtain will be generally stronger than the ones for the stability of an equilibrium point. On the other hand, the properties that will be derived will be stronger as well.

This chapter is structured as follows. In Section 1.2 we study incremental stability properties for systems with no control action. We introduce the main definition, provide a quick overview of existing results on incremental stability theory and recall some of the main properties of such systems. In particular, we focus on the so-called *metric-based* approach, which will be the main tool for analysis and design of incremental stability properties that we will exploit throughout the whole manuscript. Then, in Section 1.3, we focus on incremental properties for nonlinear systems coupled with an external input. To conclude, in Section 1.4, we focus on the design of feedback control laws achieving incremental stability for the closed-loop system. Conclusions and future perspectives are in Section 1.5. The novel results in this section can be found in the author’s publications Giaccagli et al. (2022a,b,c,d).

## 1.2 Autonomous systems

### 1.2.1 Main definition

We consider a dynamic system that can be described using a differential equation of the form

$$\dot{x} = f(x, t) \tag{1.1}$$

where  $x \in \mathbb{R}^{n_x}$  is the state of the system and  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$  is a vector field. We will assume  $f$  to be a  $C^2$  vector field in the first argument and piecewise continuous in the second. We indicate with  $\mathcal{X}(x_0, t, t_0)$  the solution of (1.1)

initialized at  $x_0 \in \mathbb{R}^{n_x}$  and evaluated at time  $t \geq t_0 \in \mathbb{R}$ , that is, the solution of the initial value problem

$$\frac{\partial}{\partial t} \mathcal{X}(x_0, t, t_0) = f(\mathcal{X}(x_0, t, t_0), t), \quad \mathcal{X}(x_0, t_0, t_0) = x_0. \quad (1.2)$$

In the following, we define the notion of incremental stability.

**Definition 1.2.1** (Incremental stability). *Let  $\mathcal{D} \subseteq \mathbb{R}^{n_x}$ . We say that system (1.1) is*

- ( $\delta US$ ) *Incrementally Uniformly Stable in  $\mathcal{D}$  if there exists a class- $\mathcal{K}$  function  $\alpha$  such that*

$$|\mathcal{X}(x_1, t, t_0) - \mathcal{X}(x_2, t, t_0)| \leq \alpha(|x_1 - x_2|); \quad (1.3)$$

- ( $\delta UES$ ) *Incrementally Uniformly Exponentially Stable in  $\mathcal{D}$  if there exist two strictly positive real numbers  $k, \lambda > 0$  such that*

$$|\mathcal{X}(x_1, t, t_0) - \mathcal{X}(x_2, t, t_0)| \leq k |x_1 - x_2| \exp(-\lambda(t - t_0)); \quad (1.4)$$

for any couple of initial conditions  $(x_1, x_2) \in \mathcal{D} \times \mathcal{D}$  and for all  $t \geq t_0$ . In case  $\mathcal{D} = \mathbb{R}^{n_x}$ , we say that the system is, respectively, *Incrementally Globally Uniformly Stable ( $\delta GUS$ )* and *Incrementally Globally Uniformly Exponentially Stable ( $\delta GUES$ )*.

## 1.2.2 Existing results

The study of incremental stability properties has a long history. A nice bibliography summary can be found in the recent review Giesl et al. (2022). The earliest results on contractive systems can be tracked back down 100 years ago, when Banach showed that in a complete metric space,  $(\mathcal{S}, |\cdot|)$ , a mapping  $T : \mathcal{S} \mapsto \mathcal{S}$  satisfying  $|T(x) - T(y)| \leq \rho |x - y|$  for some  $\rho \in (0, 1)$  would lead to the existence of a unique fixed point  $x^*$  for the map  $T$  satisfying  $x^* = T(x^*)$ , see Banach (1922). Later on, in the late '40, Lewis showed that under the solution of a differential equation, the geodesic (i.e. the shortest curve) between two points in a Finsler space is exponentially decreasing in time and this would imply asymptotic convergence of two different solutions of an ODE Lewis (1949, 1951). Then, in Hartman (1961) the author showed analogous results, by proving that the distance between two solutions is decreasing in time and that, for time-invariant systems, this would lead to the existence of an asymptotically stable equilibrium point. Similar results have been found independently by Krasovskii when considering a Euclidean space in Krasovskii (1959) and by Demidovich (see Pavlov et al. (2004) for a historical review). In particular, Demidovich studied the convergent properties of solutions of ODEs. Under the solution of a differential equation, he claimed that solutions of a nonlinear time-varying vector field have a distance associated with a Euclidean metric that is exponentially decreasing in time, extending Krasovskii's result in the case of non-existence of equilibrium points. Demidovich's results led to the definition of *convergent* systems (see Section 1.2.3.2).

The interest in incremental stability and contractive properties in control theory started to become popular after the publication of Lohmiller and Slotine (1998) (and intensively extended in the numerous publications by Slotine and co-workers). The authors studied an infinitesimal displacement (i.e. an infinitesimal variation of the initial conditions) between two different solutions of an ODE. They showed that

different solutions approach each other, provided that they remain inside a *contraction region*, a subset of the state space where the Jacobian of the vector field satisfies a differential equation. In this sense, the authors recovered the results previously obtained by Lewis, Hartman, Krasovskii, and Demidovich. From the '2000 to the present days, incremental stability in control theory has been intensively studied in numerous publications. Among them, we recall for instance the Lyapunov-based analysis in Forni and Sepulchre (2013a) and Angeli (2002, 2009) (where the concept of *incremental ISS* was first introduced), the analysis based on matrix measures (also called logarithmic norms) in Sontag (2010), Aminzare and Sontag (2014) and the related applications for infinite-dimensional systems (see, for instance Russo et al. (2010), Aminzare et al. (2014) and references therein). It is worth also recalling the work Simpson-Porco and Bullo (2014) where a geometrical interpretation of contractive systems was studied in Riemannian manifolds, and Andrieu et al. (2016) where the authors studied *transverse* exponential stability, a generalized notion of incremental exponential stability. Incremental stability has been studied also in case only a 'subset' of the state-space is contractive. This has led to the (similar) notions of *partial contraction* Wang and Slotine (2005), *horizontal contraction* Forni and Sepulchre (2013a) and, more recently, of *k-contraction* Wu et al. (2022). Despite this manuscript focuses on continuous-time smooth nonlinear systems, for the sake of completeness it is worth recalling also some results on incremental stability for discrete-time nonlinear systems Tran et al. (2018), Tsukamoto et al. (2021), for hybrid systems Biemond et al. (2018), stochastic incremental stability Kawano and Hosoe (2021) and with non-Euclidean norms Davydov et al. (2021a,b). Incremental stability properties have been used also in several practical applications. For instance, in Fromion et al. (1999), a PI control for a missile was designed through contractive tools implemented via LMIs. In Reinders et al. (2022), a repetitive control scheme was implemented for a ventilation machine, where stability properties are guaranteed using incrementally stable and convergent arguments. In Jayawardhana et al. (2007), incremental passivity (see Forni and Sepulchre (2013b)) was exploited for the control of nonlinear RCL circuits.

Giving an exhaustive and detailed list of references for incremental stability is out of the scope of this section. For this, we have just recalled some of the most important works existing in the literature and we won't go further into details. The interested readers can find additional results in the references of the various articles that have been cited. Here, the main objective is to point out the following three aspects:

1. incremental stability and contractive systems have a long history and therefore some of the results that will be presented in this first chapter are not new;
2. nevertheless, there are still several aspects and open questions, and this makes contraction theory still a 'hot topic' for control engineers and mathematicians;
3. the interest in incremental stability is motivated by its employment in practical applications, since it provides stability properties that, generally, cannot be obtained by solving an equilibrium stabilization problem.

With this in mind, the objective of what will follow will be:

- to provide an overview of the main aspects of incremental stability, by characterizing the main properties and the various shades with a detailed mathematical formalism. In particular, we will focus on the 'metric-based' approach, which will be the leading analysis and design tool of the whole manuscript;

- to present some additional new results that have been developed, to provide an answer to some open problems;
- to do all this with ‘an eye towards control design’, that is, to present all of these aspects with the objective to use them as tools to solve control engineering problems. In particular, this last aspect will be of interest as we will rely on incremental stability theory as the main tool to provide a possible solution to the output regulation and the multiagent synchronization problem, respectively, in Chapter 2 and Chapter 3 of the manuscript, together with some practical applications in Chapter 4.

### 1.2.3 Metric-based approach to incremental stability

#### 1.2.3.1 Tools to study incremental stability

Among the different methodologies to study incremental stability properties of nonlinear systems, we recognize 3(+1) different approaches. The first approach is the one proposed for instance in Sontag (2010) and Aminzare and Sontag (2014) for the case of incremental exponential stability. It is based on the study of system (1.1) by means of *matrix measures* (also called *logarithmic norms*) on normed spaces, see Söderlind (2006). We recall in the following the main aspects of Euclidean spaces. Given a symmetric positive definite matrix  $P \in \mathbb{R}^{n_x \times n_x}$ ,  $P = P^\top \succ 0$  and the associated induced matrix norm  $\|\cdot\|_P$ , we define the matrix measure  $\mu_P : \mathbb{R}^{n_x \times n_x} \mapsto \mathbb{R}$  associated to the matrix function  $P$  as

$$\mu_P(A) := \lim_{h \rightarrow 0^+} \frac{\|I + hA\|_P - 1}{h} \quad (1.5)$$

**Remark 1.2.1.** *In case  $\|\cdot\|$  is the Euclidean 2-norm, then  $P = I$  and  $\mu_I(A)$  is the maximum eigenvalue of the symmetric part of  $A$ , that is,  $\frac{A+A^\top}{2}$ . Note moreover that the logarithmic norm can be extended to more general norms, such as the infinity norm. See Davydov et al. (2021a,b) for a contraction theory analysis in such a case.*

Incremental exponential stability is claimed under the existence of a symmetric positive definite matrix  $P$  and associated matrix norm such that the matrix measure of the Jacobian of the vector field  $f$  in (1.1) is uniformly negative definite, see (Sontag, 2010, Theorem 1). In particular, if there exists a strictly positive real number  $\lambda > 0$  such that

$$\mu_P \left( \frac{\partial f}{\partial x}(x, t) \right) \leq -\lambda, \quad (1.6)$$

then (1.4) holds with such a  $\lambda$  and with  $k = 1$ .

The second approach follows a Lyapunov-based analysis. Within this context, we can recognize two sub-cases. The first one is the one proposed in Forni and Sepulchre (2013a) (see also Wu and Duan (2021)). In this work, the authors study incremental stability properties of a system of the form (1.1) on a forward invariant manifold  $\mathcal{D}$  with Finsler-Lyapunov functions. In other words, they lift the usual concept of Lyapunov function to the tangent bundle<sup>1</sup>. This point-wise tangent characterization allows associating to the local convergence of trajectories, a Lyapunov function by path-integrating along the solutions of the system. It is shown in (Forni and Sepulchre, 2013a, Theorem 1) that it is possible to characterize the behaviour of

<sup>1</sup>Given a manifold  $\mathcal{D}$ , the *tangent bundle* of  $\mathcal{D}$  is defined as the union of all the  $x \in \mathbb{R}^{n_x}$  belonging to the *tangent space* of  $\mathcal{D}$ .

distances between different trajectories of (1.1) under the existence of a  $C^1$  Finsler-Lyapunov function  $V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  satisfying

$$\frac{\partial V}{\partial x}(x, \delta_x)f(x, t) + \frac{\partial V}{\partial \delta_x}(x, \delta_x)\frac{\partial f}{\partial x}(x, t) \leq -\alpha(V(x, \delta_x)), \quad (1.7)$$

together with the usual ‘sandwich-like’ bounds on  $V$  guaranteeing positivity, homogeneity, and strict convexity, which are generally required to show that if the Lyapunov function decreases along the trajectories, so does the distance between two solutions. The function  $\alpha(\cdot)$  characterizes the behavior of the convergence, depending if it is the zero-function  $\alpha(s) = 0$  (non-increasing distances), a class- $\mathcal{K}$  function (stable and asymptotically convergent to zero distances) or if it is linear in its argument (stable and exponential convergency to zero of distances). A converse theorem with Finsler-Lyapunov functions has been provided in Wu and Duan (2021) in case the vector field  $f$  has bounded first derivatives.

The second sub-case regarding a Lyapunov-based approach to study incremental stability is the one proposed in Angeli (2002, 2009). Here, the author considers time-invariant systems affected by an external signal  $d$  taking values in a compact set, and studies the notion of incremental global asymptotic stability. The key intuition is to study a system and a copy of itself, that is, to study

$$\begin{aligned} \dot{x}_1 &= f(x_1, d) \\ \dot{x}_2 &= f(x_2, d) \end{aligned} \quad (1.8)$$

and to analyze the stability properties of the manifold where  $x_1 = x_2$ . Sufficient and necessary conditions guaranteeing incremental global asymptotic stability are obtained through the existence of a continuous function  $V : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  satisfying

$$V(\mathcal{X}(x_1, t, d), \mathcal{X}(x_2, t, d)) - V(x_1, x_2) \leq -\int_0^t \alpha(|\mathcal{X}(x_1, \tau, d) - \mathcal{X}(x_2, \tau, d)|) d\tau. \quad (1.9)$$

In such a case,  $\alpha$  is any positive function and  $V$  still presents the ‘sandwich-like’ bounds guaranteeing radial unboundedness, positivity, and homogeneity, where such bounds are imposed to hold on the difference  $|x_1 - x_2|$ . It is important to remark that in Angeli (2002) (with further extensions in Angeli (2009)), the author first introduces the concept of *incremental input-to-state stability*, that will be discussed in this chapter later on, in Section 1.3.

The last of the approaches to study incremental stability properties is the one based on the study of Riemannian metrics. Its application in control system design has been characterized in several works. Among the most well-known and complete, we recall for instance Lohmiller and Slotine (1998) (and the several works by Slotine and co-workers) and Andrieu et al. (2016) for the  $\delta$ GUES case. In this case, incremental exponential stability is obtained if there exists a Riemannian metric along which the flow of the vector field (1.1) generates trajectories for which the distance associated with such Riemannian metric is monotonically decreasing in forward time (i.e. for each  $t \geq t_0$ , we have that the mapping  $x_0 \mapsto \mathcal{X}(x_0, t, t_0)$  is a contraction). Such a condition can be verified under the solution of a differential equation involving the Jacobian of the vector field  $f$  of the form<sup>2</sup>

$$L_f P(x, t) \preceq -qI, \quad (1.10)$$

<sup>2</sup>see the Notation for the definition of the Lie derivative of a vector field along a 2-tensor

see for instance Andrieu et al. (2020) for the time-invariant case. In such a case,  $q$  is a strictly positive real number,  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  is a  $C^1$  symmetric and positive definite matrix function representing the Riemannian metric (and coupled with the norm  $|\cdot|_{P(x,t)}$ ), which is uniformly upper and lower bounded by strictly positive definite constant matrices, to guarantee completeness of the metric space and to provide uniformity in the convergence of different solutions of the system (i.e. to guarantee equivalence of norms between the Euclidean one and the Riemannian one).

In this manuscript, we focus on this fourth and last method, that is, on the analysis of incremental stability properties using a *metric-based* approach. All of these approaches provide similar results. However, at the current time, the author of this manuscript is not aware of works that provide a comparative study among all of these techniques, by providing for instance a constructive design to pass from one of the four conditions (1.6), (1.7), (1.9), (1.10) to any of the others. The main interest in employing a metric-based analysis can be addressed for instance, in the fact that we will be able to provide tractable conditions for the analysis of incrementally stable systems (and the design of control laws) for some specific classes of systems.

### 1.2.3.2 A few words on convergent systems

The notion of incremental stability is strictly related to the notion of *convergent* systems. Such a notion has been first introduced by B.P. Demidovich (see Demidovich (1959) and Pavlov et al. (2004)) and has been intensively studied in the manuscript Pavlov et al. (2006) and in various articles by the same authors. For the sake of completeness, we recall such a definition (see (Pavlov et al., 2005, Definition 1)).

**Definition 1.2.2** (Convergent). *System (1.1) is said to be convergent if:*

- *there exists a solution  $\bar{X}(x_0, t, t_0)$  defined and bounded for all  $t \in \mathbb{R}$ ;*
- *the solution  $\bar{X}(x_0, t, t_0)$  is globally asymptotically stable.*

*If the solution  $\bar{X}(x_0, t, t_0)$  is globally exponentially stable, then the system is said to be exponentially convergent.*

Despite Definition 1.2.1 and Definition 1.2.2 may seem similar, they're actually different notions and many counterexamples can be found in this sense, see the comparative study Ruffer et al. (2013). The main difference in this sense is that convergent systems assume the existence of a 'steady-state' bounded solution, while incremental systems do not. This difference becomes evident and relevant when considering time-varying vector fields.

### 1.2.3.3 Metric characterization of incremental stability: sufficient conditions

In this section, we provide a characterization of the different shades of incremental stability in Definition 1.2.1 using an analysis based on Riemannian metrics. Sufficient conditions to claim global uniform incremental stability for a system of the form (1.1) are presented in the following (see e.g. Andrieu et al. (2016) for an equivalent proof).

**Theorem 1.2.2** (Sufficient conditions for incremental stability). *Consider system (1.1). Assume there exists a  $C^1$  matrix function taking symmetric positive real values*

$P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$ , two strictly positive real numbers  $\bar{p} \geq \underline{p} > 0$  and a non-negative continuous function  $q : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  such that the following conditions hold:

$$\underline{p}I \preceq P(x, t) \preceq \bar{p}I, \quad (1.11a)$$

$$L_f P(x, t) \preceq -q(x)P(x, t) \quad (1.11b)$$

for all  $x \in \mathbb{R}^{n_x}$ , for all  $t \geq t_0$ . Then:

1. if  $q(x) \geq 0$  for all  $x \in \mathbb{R}^{n_x}$ , then the system is  $\delta GUS$ ;
2. if there exists a strictly positive real number  $\lambda > 0$  such that  $q(x) \geq \lambda$  for all  $x \in \mathbb{R}^{n_x}$ , then the system is  $\delta GUES$ .

**Proof.** Given any vector  $v \in \mathbb{R}^{n_x}$  define the norm operator associated to the metric  $P(x, t)$  as

$$|v|_{P(x,t)}^2 := v^\top P(x, t)v. \quad (1.12)$$

Let  $[s_1, s_2] \subset \mathbb{R}$  and fix an instant of time  $t \geq t_0$ . Define  $\gamma_t : [s_1, s_2] \mapsto \mathbb{R}^{n_x}$  as any  $C^2$  curve (parametrized by  $t$ ) such that, at time  $t$ , we have that  $\gamma_t(s_1) = x_1$  and  $\gamma_t(s_2) = x_2$  for any two points  $(x_1, x_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ . Define the length of the curve  $\gamma_t$  in the operator norm  $P(x, t)$  as

$$\ell_P(\gamma_t) := \int_{s_1}^{s_2} |\gamma_t(s)|_{P(\gamma(s), t)} ds \quad (1.13)$$

where the pedix  $t$  is included to specify the fact that the length of the curve depends on the considered time instant. Then, define the energy of the curve  $\gamma_t$  as

$$E_P(\gamma_t) := \frac{1}{2} \int_{s_1}^{s_2} |\gamma_t(s)|_{P(\gamma(s), t)}^2 ds. \quad (1.14)$$

Then, for any fixed  $t \geq t_0$ , define the distance between the two points  $(x_1, x_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$  as the infimum of the length in the operator norm  $P$  in all the possible curves  $\gamma$  at time  $t$ , i.e.

$$\text{dist}_P(x_1, x_2) := \inf_{\gamma_t} \{\ell_P(\gamma_t)\}. \quad (1.15)$$

Note that the  $\inf\{\cdot\}$  operator is in  $\gamma_t$  as, for fixed  $(x_1, x_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ , the curve with minimum length might change depending on the considered time instant. Now, fix  $t_0 \in \mathbb{R}$ . Let  $\Gamma(s, t, t_0)$  be the solution of the system (1.1) parametrized along the curve  $\gamma_{t_0}(s)$ , namely,  $\Gamma(s, t, t_0) = \mathcal{X}(\gamma_{t_0}(s), t, t_0)$  and define the function

$$\xi(s, t, t_0) := \left| \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right|_{P(\Gamma(s, t, t_0), t)}^2. \quad (1.16)$$

Taking its time derivative and by using (1.11b) it yields

$$\begin{aligned}
\frac{\partial}{\partial t} \xi(s, t, t_0) &= \frac{\partial}{\partial t} \left| \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right|_{P(\Gamma(s, t, t_0), t)}^2 \\
&= \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) L_f P(\Gamma(s, t, t_0), t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \\
&\leq -q(\Gamma(s, t, t_0)) \left| \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right|_{P(\Gamma(s, t, t_0), t)}^2 \\
&\leq -q(\Gamma(s, t, t_0)) \xi(s, t, t_0).
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial}{\partial t} E_P(\Gamma(s, t, t_0)) &= \frac{\partial}{\partial t} \int_{s_1}^{s_2} \xi(s, t, t_0) ds \\
&\leq - \int_{s_1}^{s_2} q(\Gamma(s, t, t_0)) \xi(s, t, t_0) ds.
\end{aligned}$$

We now show all the items separately.

*Item 1:* If  $q(\Gamma(s, t, t_0)) \geq 0$ , then  $\frac{\partial}{\partial t} E_P(\Gamma(s, t, t_0)) \leq 0$ . This implies  $E_P(\Gamma(s, t, t_0)) \leq E_P(\gamma_{t_0}(s))$ . Let  $E_P^*(\Gamma(s, t, t_0))$  be the minimum-energy path, that is

$$E_P^*(\Gamma(s, t, t_0)) := \inf_{\gamma_{t_0}} E_P(\Gamma(s, t, t_0)) \quad (1.17)$$

Using the lower bound in (1.11a), we get that the energy of a curve in the norm  $P$  is lower-bounded by the energy in a Euclidean metric, namely

$$\frac{p}{2} \int_{s_1}^{s_2} \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \frac{\partial \Gamma}{\partial s}(s, t, t_0) ds \leq E_P(\Gamma(s, t, t_0)).$$

Recall now that the minimum energy path for an Euclidean metric is the straight line  $s \mapsto x_1 + \frac{s_1-s}{s_1-s_2}(x_2 - x_1)$ . To check this, consider the path  $s \mapsto \delta(s)$  such that  $\delta(s_1) = \delta(s_2) = 0$ . Consider now the function

$$\chi(h) := \int_{s_1}^{s_2} \left( \frac{\partial \mathcal{X}}{\partial s}(\gamma_{t_0}^*(s), t, t_0) + h \frac{\partial \delta}{\partial s}(s) \right)^\top \left( \frac{\partial \mathcal{X}}{\partial s}(\gamma_{t_0}^*(s), t, t_0) + h \frac{\partial \delta}{\partial s}(s) \right) ds,$$

where  $\gamma_{t_0}^*(s)$  is the minimum energy path. In this sense,  $\delta(\cdot)$  represents a perturbation to the (Euclidean) minimum energy of a curve. This implies that the function  $\chi$  reaches its minimum at  $h = 0$ . By integrating per part and recalling the boundary conditions of  $\delta$ , we get

$$0 = \frac{\partial \chi}{\partial h}(0) = 2 \int_{s_1}^{s_2} \frac{\partial \delta^\top}{\partial s}(s) \frac{\partial \mathcal{X}}{\partial s}(\gamma_{t_0}^*(s), t, t_0) ds = -2 \int_{s_1}^{s_2} \delta^\top(s) \frac{\partial^2 \mathcal{X}}{\partial s^2}(\gamma_{t_0}^*(s), t, t_0) ds.$$

Since this doesn't depends on  $\delta$ , it follows that

$$\frac{\partial^2 \mathcal{X}}{\partial s^2}(\gamma_{t_0}^*(s), t, t_0) = 0,$$

namely, for the Euclidean case, the minimum energy path is the straight (linear) path. Hence

$$\frac{p}{2} \int_{s_1}^{s_2} \frac{\partial \mathcal{X}^\top}{\partial s}(\gamma_{t_0}^*(s), t, t_0) \frac{\partial \mathcal{X}}{\partial s}(\gamma_{t_0}^*(s), t, t_0) ds \geq \underline{c} |x_1 - x_2|^2$$

for some  $\underline{c} > 0$ . Note now that, by definition of infimum, for all  $n \in \mathbb{N}$ , there exists a curve  $\gamma_{t_0}^n$  such that

$$E_P(\mathcal{X}(\gamma_{t_0}^n(s), t, t_0)) - E_P^*(\mathcal{X}(\gamma_{t_0}(s), t, t_0)) \leq \frac{1}{n}. \quad (1.18)$$

By taking the limit for  $n \mapsto +\infty$ , we get

$$\underline{c} |x_1 - x_2|^2 \leq E_P^*(\mathcal{X}(\gamma_{t_0}(s), t, t_0)). \quad (1.19)$$

On the other hand, by definition, any other curve than the geodesic has a greater energy. Taking for instance the energy of the straight line, it follows that there exists  $\bar{c} > 0$  such that

$$E_P^*(\mathcal{X}(\gamma_{t_0}(s), t, t_0)) \leq \bar{c} |x_1 - x_2|^2. \quad (1.20)$$

Since

$$\begin{aligned} E_P^*(\mathcal{X}(\gamma_{t_0}(s), t, t_0)) &\leq E_P(\mathcal{X}(\gamma_{t_0}(s), t, t_0)) \\ &\leq E_P(\gamma_{t_0}(s)) \\ &\leq \left( E_P^*(\gamma_{t_0}(s)) + \frac{1}{n} \right), \end{aligned}$$

by employing (1.19) and (1.20) and by taking the limit  $n \mapsto +\infty$ , inequality (1.3) follows.

*Item 2:* If there exists  $\lambda > 0$  such that  $q(x) \geq \lambda$  for any  $x \in \mathbb{R}^{n_x}$ , then

$$\frac{\partial}{\partial t} E_P(\Gamma(s, t, t_0)) \leq -\lambda E_P(\Gamma(s, t, t_0)).$$

Hence by Gronwall's lemma

$$E_P(\Gamma(s, t, t_0)) \leq E_P(\gamma_{t_0}(s)) \exp(-\lambda(t - t_0)).$$

Let  $E_P^*(\Gamma(s, t, t_0))$  be the minimum-energy path defined as in (1.17). By definition, for all  $n \in \mathbb{N}$  there exists a curve  $\gamma_{t_0}^n$  such that (1.18) holds. Therefore it follows that

$$\begin{aligned} E_P^*(\mathcal{X}(\gamma_{t_0}(s), t, t_0)) &\leq E_P(\mathcal{X}(\gamma_{t_0}(s), t, t_0)) \\ &\leq E_P(\gamma_{t_0}(s)) \exp(-\lambda(t - t_0)) \\ &\leq \left( E_P^*(\gamma_{t_0}(s)) + \frac{1}{n} \right) \exp(-\lambda(t - t_0)). \end{aligned}$$

By taking the limit for  $n \mapsto +\infty$  and following the same exact steps as in Item 1, inequality (1.4) follows. □

**Remark 1.2.3.** *The lower bound in (1.11a) is required to make sure that the whole  $\mathbb{R}^{n_x}$  space is endowed with the Riemannian metric  $P$  is complete. Such a condition guarantees that every geodesic can be maximally extended to  $\mathbb{R}$  (see Sanfelice and*

*Praly (2011)*). By Hopf-Rinow's Theorem (see (Sakai, 1996, Theorem 1.1)) this implies that the metric is complete and hence that the minimum of the length of any curve  $\gamma$  connecting two points  $(x_1, x_2)$  is given by the length of the geodesic at any time instant. Similarly, it guarantees that the Lyapunov function defined as the distance associated with the norm operator in the metric  $P$  is radially unbounded, and therefore incremental properties are obtained globally in the state space. Moreover, the upper bound in (1.11a) is introduced for solutions to be uniformly decreasing. As shown in the proof, by guaranteeing that both the upper and the lower bound are provided with respect to two strictly positive real numbers  $\underline{p}, \bar{p}$ , we can link the decay of trajectories in the distance operator  $\text{dist}_P(\cdot, \cdot)$  to the Euclidean norm, i.e. we can go from the Riemannian distance to the Euclidean one and its associated norm.

**Remark 1.2.4.** In case the metric  $P$  is a constant symmetric positive definite matrix (i.e. we consider a Euclidean metric), the function  $f$  can be only  $C^1$  in its first argument. Indeed, in this case, an equivalent proof of Theorem 1.2.2 can be given by considering the Lyapunov function  $V(e(t)) = e^\top(t)Pe(t)$  with  $e := x_1 - x_2$  and by using the Mean Value Theorem.

**Remark 1.2.5.** Theorem 1.2.2 provides only sufficient conditions to claim uniform incremental stability. In literature, there exist some converse theorems. In particular, in Andrieu et al. (2016) a converse metric-based result for the  $\delta$ GUES property is given in case the vector field  $f$  is time-invariant and has a bounded first derivative. Recently, a similar result has been given in Wu and Duan (2021) with Finsler-Lyapunov functions.

The same result of Theorem 1.2.2 holds when trajectories remain inside a compact invariant set. The following result shows such a case for the  $\delta$ GUES property. Similar considerations can be done for the  $\delta$ GUS one.

**Corollary 1.2.5.1.** Suppose that there exists a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  taking positive definite symmetric values and two positive real numbers  $\bar{p}, \underline{p}$ , such that (1.11a) holds for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ . Let  $\mathcal{D}$  be compact subset of  $\mathbb{R}^{n_x}$  that is invariant for system (1.1) and assume that there exist a positive function  $q : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{>0}$  and a strictly positive real number  $\lambda > 0$  such that  $q(x) \geq \lambda$  for all  $x \in \mathbb{R}^{n_x}$  and such that inequality (1.11b) holds for all  $x \in \mathcal{O}$ , where

$$\mathcal{O} := \left\{ x \in \mathbb{R}^{n_x}, \max_{s \in \mathcal{D}} |x - s| < r \right\}, \quad (1.21)$$

where  $r > 0$  is any positive real number satisfying

$$r > \frac{\sqrt{\bar{p}}}{2\sqrt{\underline{p}}} \max_{(x_1, x_2) \in \mathcal{D}^2} |x_1 - x_2|. \quad (1.22)$$

Then, system (1.1) is  $\delta$ UES in  $\mathcal{D}$ .

**Proof.** Let  $[s_1, s_2] \subset \mathbb{R}$  and fix an instant of time  $t \geq t_0$ . Define  $\gamma_t : [s_1, s_2] \mapsto \mathbb{R}^{n_x}$  as any curve (parametrized by  $t$ ) such that, at time  $t$ , we have that  $\gamma_t(s_1) = x_1$  and  $\gamma_t(s_2) = x_2$  for any two points  $(x_1, x_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ . Define then, the length of the curve  $\gamma_t$  in the operator norm  $P(x, t)$  as (1.13). Then, for any fixed  $t \geq t_0$ , define the distance between the two points  $(x_1, x_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$  as (1.15). Note that for all  $(x_1, x_2)$  both in  $\mathbb{R}^{n_x}$ , inequalities (1.11a) imply

$$\underline{p}|x_1 - x_2|^2 \leq \text{dist}_P^2(x_1, x_2) \leq \bar{p}|x_1 - x_2|^2. \quad (1.23)$$

Let  $r_1$  be such that

$$r > r_1 > \frac{\sqrt{\underline{p}}}{2\sqrt{\underline{p}}} \max_{(x_1, x_2) \in \mathcal{D}^2} |x_1 - x_2|, \quad (1.24)$$

and let

$$\mathcal{O}_1 = \left\{ x \in \mathbb{R}^{n_x}, \max_{s \in \mathcal{D}} |x - s| < r_1 \right\}. \quad (1.25)$$

Note that  $\mathcal{D} \subset \mathcal{O}_1$  and  $\text{cl}\{\mathcal{O}_1\} \subset \mathcal{O}$ . The proof is divided into two steps. In the first step, it is shown that the (Riemannian) convex hull of  $\mathcal{D}$  is included in  $\mathcal{O}_1$  defined in (1.25). In a second step, it is shown that the Riemannian distance between any pair of points in  $\mathcal{D}$  converges exponentially to zero.

**Step 1 :** Let  $(x_1, x_2)$  be in  $\mathcal{D}$ . From (1.24), we can find  $\epsilon > 0$  such that

$$\frac{\epsilon}{\sqrt{\underline{p}}} < 2r_1 \sqrt{\frac{\underline{p}}{\underline{p}}} - \max_{(x_1, x_2) \in \mathcal{D}^2} |x_1 - x_2|. \quad (1.26)$$

Fix  $t_0 \in \mathbb{R}$ . We aim at showing that for all  $C^2$  time-parametrized path  $\gamma_{t_0} : [s_1, s_2] \rightarrow \mathbb{R}^{n_x}$  between  $x_1 = \gamma_{t_0}(s_1)$  and  $x_2 = \gamma_{t_0}(s_2)$  such that

$$\ell_P(\gamma_{t_0}) \leq \text{dist}_P(x_1, x_2) + \epsilon, \quad (1.27)$$

this implies that  $\gamma_{t_0}(s) \in \mathcal{O}_1$  for all  $s \in [s_1, s_2]$ . Assume the opposite. In other words, assume that there exists  $s^*$  in  $[s_1, s_2]$  such that  $\gamma_{t_0}(s^*) \notin \mathcal{O}_1$ . With (1.25), this implies that  $|\gamma_{t_0}(s^*) - x_1| \geq r_1$  and  $|\gamma_{t_0}(s^*) - x_2| \geq r_1$ . Consequently, in combination with the left-hand side of (1.23), it implies

$$\begin{aligned} \ell_P(\gamma_{t_0}) &\geq \text{dist}_P(x_1, \gamma_{t_0}(s^*)) + \text{dist}_P(x_2, \gamma_{t_0}(s^*)) \\ &\geq \sqrt{\underline{p}} (|x_1 - \gamma_{t_0}(s^*)| + |x_2 - \gamma_{t_0}(s^*)|) \\ &\geq 2r_1 \sqrt{\underline{p}}. \end{aligned} \quad (1.28)$$

On the other hand, with (1.27), the right-hand side of (1.23) and (1.26), it yields

$$\ell_P(\gamma_{t_0}) \leq \sqrt{\underline{p}} |x_1 - x_2| + \epsilon < 2r_1 \sqrt{\underline{p}}. \quad (1.29)$$

This yields a contradiction. Consequently, for all  $C^2$  paths such that (1.27) holds,  $\gamma_{t_0}(\cdot)$  takes values in  $\mathcal{O}_1$ .

**Step 2:** Let now  $T > 0$  be a real number such that

$$\mathcal{X}(x, t, t_0) \in \mathcal{O}, \quad \forall (x, t) \in \mathcal{O}_1 \times [t_0, T]. \quad (1.30)$$

Such a  $T$  exists since there exists a minimal distance between the boundary of the two sets, and  $\mathcal{O}$  is bounded. Consider a  $C^2$  path between  $(x_1, x_2) \in \mathcal{D} \times \mathcal{D}$  satisfying (1.27). With Step 1, we know that  $\gamma_{t_0} : [s_1, s_2] \mapsto \mathcal{O}_1$ . For all  $(s, t, t_0) \in [s_1, s_2] \times [t_0, T] \times \mathbb{R}$  we can define

$$\Gamma(s, t, t_0) := \mathcal{X}(\gamma_{t_0}(s), t; t_0), \quad \xi(s, t, t_0) := \left| \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right|_{P(\Gamma(s, t, t_0))}^2.$$

Due to (1.30),  $\Gamma(s, t, t_0) \in \mathcal{O}$  for all  $(s, t, t_0) \in [s_1, s_2] \times [t_0, T] \times \mathbb{R}$ . Then  $\xi$  is  $C^1$  and equation (1.11b) yields

$$\begin{aligned} \frac{\partial \xi}{\partial t}(s, t, t_0) &= \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) L_f P(\Gamma(s, t, t_0), t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \\ &\leq -\lambda \left| \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right|_{P(\Gamma(s, t, t_0))}^2 \\ &= -\lambda \xi(s, t, t_0) \end{aligned}$$

which yields

$$\begin{aligned} \frac{\partial \ell_P(\Gamma(\cdot, t, \cdot))}{\partial t} &= \frac{\partial}{\partial t} \int_{s_1}^{s_2} \sqrt{\xi(s, t, t_0)} ds \\ &= \int_{s_1}^{s_2} \frac{1}{2\sqrt{\xi(s, t, t_0)}} \frac{\partial \xi}{\partial t}(s, t, t_0) ds \\ &\leq -\frac{\lambda}{2} \ell_P(\Gamma(\cdot, t, \cdot)). \end{aligned}$$

Hence,  $\ell_P(\Gamma(\gamma_{t_0}, T, t_0)) \leq \ell_P(\gamma_{t_0}) \exp(-\frac{\lambda}{2}(T-t_0))$ . Choosing a sequence of paths  $(\gamma_{t_0}^n)_{n \in \mathbb{N}}$  such that  $\ell_P(\gamma_{t_0}^n) \mapsto \text{dist}_P(x_1, x_2)$  and satisfying (1.27) and passing to the limit, we obtain

$$\text{dist}_P(\mathcal{X}(x_1, T, t_0), \mathcal{X}(x_2, T, t_0)) < \kappa \text{dist}_P(x_a, x_b). \quad (1.31)$$

for some  $0 < \kappa < 1$ . This property is true for all  $(x_1, x_2)$  in  $\mathcal{D} \times \mathcal{D}$ . Recalling the positive invariance of  $\mathcal{D}$ , it implies that (1.4) holds.  $\square$

## 1.2.4 Main properties

### 1.2.4.1 Periodicity of trajectories and equilibrium point

In this section, we aim to review the main properties of incremental systems. In particular, we will focus on the following two aspects: periodicity of trajectories (and the existence of equilibrium points) and invariance with respect to diffeomorphisms. A property of interest of incremental systems is that if the vector field  $f$  is periodic with respect to time, then the trajectories will asymptotically converge to a periodic solution with the same period. This is recalled in the following. See for instance Angeli (2002); Sontag (2010) for a proof.

**Lemma 1.2.6** (Periodic trajectories). *Consider system (1.1) and assume that is  $\delta GUES$ . Assume that  $f$  is periodic, namely, there exists a positive real number  $T > 0$  such that  $f(x, t) - f(x, t+T) = 0$  for all  $t \geq t_0$ . Then the trajectories of the system asymptotically converge towards a periodic solution with the same period.*

An incremental system doesn't necessarily admit the existence of an equilibrium point. The following examples show that.

**Example 1.2.7.** *We consider the  $\delta GUS$  and  $\delta GUES$  property separately.*

- Consider the system  $\dot{x} = 1$ . The system is  $\delta GUS$ . This can be claimed through Theorem 1.2.2 with the metric  $P = 1$ , or by explicitly calculating the trajectory of the system, that is

$$\mathcal{X}(x, t) = x + t$$

and by showing that

$$|\mathcal{X}(x_1, t) - \mathcal{X}(x_2, t)| = |x_1 - x_2|.$$

The system does not have an equilibrium point.

- Consider the system  $\dot{x} = -x + t$ . The system is  $\delta GUES$ . This can be claimed through Theorem 1.2.2 with the metric  $P = 1$ , or by explicitly calculating the trajectory of the system, that is

$$\mathcal{X}(x, t, t_0) = \exp(-(t - t_0))x + \int_{t_0}^t \exp(s - t)s \, ds$$

and by showing that

$$|\mathcal{X}(x_1, t) - \mathcal{X}(x_2, t)| = |x_1 - x_2| \exp(-(t - t_0)).$$

Again, the system does not have an equilibrium point.

However, if an equilibrium point exists then it is necessarily unique and stable.

**Corollary 1.2.7.1** (Equilibrium implies stability). *Consider system (1.1) and assume that there exists  $x^* \in \mathbb{R}^{n_x}$  such that  $f(x^*, t) = 0$  for all  $t \geq t_0$ . Then:*

- if the system is  $\delta GUS$ , then  $x^*$  is Globally Stable (GS);
- if the system is  $\delta GUES$ , then  $x^*$  is Globally Exponentially Stable (GES).

**Proof.** Since  $x^*$  is an equilibrium, then  $x^* = \mathcal{X}(x^*, t, t_0)$  for all  $t \geq t_0$ . As (1.3) (respectively, (1.4)) hold for all  $(x_1, x_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$  and for all  $t \geq t_0$ , the result follows.  $\square$

If Lemma 1.2.6 holds for any  $T > 0$ , then system (1.1) is time-invariant, i.e. it can be described by a differential equation of the form

$$\dot{x} = f(x). \quad (1.32)$$

If a time-invariant system (1.32) presents some incrementally uniformly globally stability properties on a closed forward invariant set, then it must admit a stable equilibrium point. This is recalled in the following.

**Corollary 1.2.7.2** (Time-invariant implies equilibrium). *Consider system (1.32). If it is  $\delta UES$  in a closed forward invariant set  $\mathcal{D}$ , then there exists a unique equilibrium  $x^*$  which attracts all solutions initiated in  $\mathcal{D}$ .*

**Proof.** Let  $\tau$  be such that  $k \exp(-\lambda\tau) = \rho < 1$  and define  $\mathcal{T}_\tau$  as the mapping that associates to any initial condition  $x_0$  of system  $\dot{x} = f(x)$ , its corresponding solution  $\mathcal{X}(x_0, \tau)$  at time  $\tau$ , that is  $\mathcal{T}_\tau(x_0) := \mathcal{X}(x_0, \tau)$ . Since  $\mathcal{D}$  is forward invariant, the function so defined maps points in  $\mathcal{D}$  into  $\mathcal{D}$ . Furthermore, by using the inequality (1.4) guaranteed by the  $\delta UES$  property of system (1.1) on  $\mathcal{D}$ , for any two given points  $x_1, x_2 \in \mathcal{D}$ , we have

$$|\mathcal{T}_\tau(x_1) - \mathcal{T}_\tau(x_2)| \leq \rho |x_1 - x_2|$$

for some  $\rho < 1$ . Recall that the Euclidean space endowed with standard Euclidean norm is a complete metric. Hence, the map  $\mathcal{T}_\tau$  is a contraction and by Banach fixed point theorem (see, e.g., Almezal et al. (2014) or (Khalil, 2002, Appendix B)), there exists a unique fixed point  $x^*$  in  $\mathcal{D}$  such that  $\mathcal{T}_\tau(x^*) = x^*$ .

Hence, for any  $x$  in  $\mathcal{D}$ , inequality (1.4), implies,

$$\begin{aligned} |\mathcal{X}(x, t) - x^*| &= |\mathcal{X}(x, t) - \mathcal{X}(x^*, t)| \\ &\leq k \exp(-\lambda t) |x - x^*|. \end{aligned}$$

Hence,  $x^*$  attracts all trajectories initiated from  $\mathcal{D}$ . An equivalent proof of this Corollary has been developed independently in (Heemels et al., 2020, Section IV-B).  $\square$

**Remark 1.2.8.** *In case  $\mathcal{D}$  in Corollary 1.2.7.2 coincides with  $\mathbb{R}^{n_x}$ , then the equilibrium point is Globally Exponentially Stable.*

A time-invariant  $\delta$ GUES system always admits a globally exponentially stable equilibrium point. The same however does not hold for the  $\delta$ GUS property. Example 1.2.7 confirms so. In this case, indeed, we have to assume the existence of the equilibrium. In some cases, such an equilibrium is not only stable but also attractive. The following result claims so.

**Proposition 1.2.9** (Non-uniform  $\delta$ GUES). *Suppose there exist a  $C^1$  function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking positive definite symmetric values, a continuous strictly positive function  $q : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{>0}$  and positive real numbers  $\bar{p}, \underline{p} > 0$ , such that (1.11a), (1.11b) hold for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ . If there exists an equilibrium point  $x^* \in \mathbb{R}^{n_x}$ , then  $x^*$  is unique and it is globally attractive for system (1.1).*

**Proof.** Since system (1.1) satisfies (1.11b), (1.11a), following the same steps that in Step 2 of the proof of Corollary 1.2.5.1 it can be shown that picking any two points  $(x_1, x_2)$  in  $\mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ , it yields  $\text{dist}_P(\mathcal{X}(x_1, t, t_0), \mathcal{X}(x_2, t, t_0)) \leq \text{dist}_P(x_1, x_2)$ . Recalling that  $\underline{p}|x_1 - x_2|^2 \leq \text{dist}^2(x_1, x_2) \leq \bar{p}|x_1 - x_2|^2$ , it follows that  $|\mathcal{X}(x_1, t, t_0) - \mathcal{X}(x_2, t, t_0)| \leq k|x_1 - x_2|$  for some  $k > 0$ . In the particular case in which there exists an equilibrium  $x^*$ , then for all  $\bar{x}$  in  $\mathbb{R}^{n_x}$ , it follows that  $\text{dist}_P(\mathcal{X}(\bar{x}, t, t_0), x^*) \leq \text{dist}_P(\bar{x}, x^*)$ . Hence, for all  $\bar{d} \geq 0$ , the set  $\mathcal{C} := \text{cl}\{\bar{x}, \text{dist}_P(\bar{x}, x^*) \leq \bar{d}\}$  is a compact invariant subset of  $\mathbb{R}^{n_x}$ . Let  $q_{\mathcal{O}} := \min_{\{x \in \mathcal{O}\}} q(x) > 0$ , where  $\mathcal{O} := \{x \in \mathbb{R}^{n_x}, \max_{s \in \mathcal{C}} |x - s| < r, \}$ , for some positive real number  $r$  satisfying  $r > \frac{\sqrt{\bar{p}}}{2\sqrt{\underline{p}}} \max_{(x_1, x_2) \in \mathcal{C}^2} |x_1 - x_2|$ . Hence, in view of Corollary 1.2.5.1, this implies that the system (1.1) is  $\delta$ UES on  $\mathcal{C}$ , and application of Corollary 1.2.7.2 implies the existence of a unique equilibrium which is attractive from  $\mathcal{C}$ . Since these arguments hold for any  $\bar{d} \geq 0$ , we conclude that  $x^*$  is the unique equilibrium globally attractive.  $\square$

#### 1.2.4.2 Preservation via diffeomorphism

Incremental stability is a property of trajectories and hence it is coordinate-free. Therefore, it is preserved via diffeomorphism. This has been shown in (Angeli, 2002, Proposition 4.6) in case convergence between trajectories is only asymptotic. Following the same lines, we provide such a result.

**Proposition 1.2.10** (Invariance via diffeomorphism). *Consider system (1.1) and assume that it is  $\delta$ GUS (respectively,  $\delta$ GUES). Let  $\varphi : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x}$  be a global diffeomorphism such that there exist two strictly positive real numbers  $L_1, L_2 > 0$*

such that

$$|\varphi(x) - \varphi(y)| \leq L_1 |x - y|, \quad (1.33a)$$

$$|\varphi^{-1}(x) - \varphi^{-1}(y)| \leq L_2 |x - y| \quad (1.33b)$$

for all  $x, y \in \mathbb{R}^{n_x}$ , where  $\varphi(\varphi^{-1}(z)) = z$ . Consider the change of coordinates  $x \mapsto z := \varphi(x)$  so that system (1.1) can be rewritten as

$$\dot{z} = \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) f(\varphi^{-1}(z), t) := f_z(z, t). \quad (1.34)$$

Then also (1.33) is  $\delta GUS$  (respectively,  $\delta GUES$ ).

**Proof.** Let  $\mathcal{Z}(z, t, t_0)$  indicates the trajectory of (1.34). We prove the two cases separately.

*Case 1: ( $\delta GUS$ ).* We have that

$$\begin{aligned} |\mathcal{Z}(z_1, t, t_0) - \mathcal{Z}(z_2, t, t_0)| &= |\varphi(\mathcal{X}(\varphi^{-1}(z_1), t, t_0)) - \varphi(\mathcal{X}(\varphi^{-1}(z_2), t, t_0))| \\ &\leq L_1 |\mathcal{X}(\varphi^{-1}(z_1), t, t_0) - \mathcal{X}(\varphi^{-1}(z_2), t, t_0)| \\ &\leq L_1 \alpha (|\varphi^{-1}(z_1) - \varphi^{-1}(z_2)|) \\ &\leq L_1 \alpha (L_2 |z_1 - z_2|) \\ &:= \alpha_z (|z_1 - z_2|), \end{aligned}$$

where  $\alpha_z$  is again a class- $\mathcal{K}$  function.

*Case 2: ( $\delta GUES$ ).* We have that

$$\begin{aligned} |\mathcal{Z}(z_1, t, t_0) - \mathcal{Z}(z_2, t, t_0)| &= |\varphi(\mathcal{X}(\varphi^{-1}(z_1), t, t_0)) - \varphi(\mathcal{X}(\varphi^{-1}(z_2), t, t_0))| \\ &\leq L_1 (|\mathcal{X}(\varphi^{-1}(z_1), t, t_0) - \mathcal{X}(\varphi^{-1}(z_2), t, t_0)|) \\ &\leq L_1 k |\varphi^{-1}(z_1) - \varphi^{-1}(z_2)| \exp(-\lambda(t - t_0)) \\ &\leq L_1 k L_2 |z_1 - z_2| \exp(-\lambda(t - t_0)). \end{aligned}$$

□

**Remark 1.2.11.** Condition (1.33a) implies that the considered diffeomorphism is uniformly continuous, while (1.33b) implies that it is uniformly injective.

If incremental properties are claimed using Theorem 1.2.2, then the construction of the metric in the new coordinates  $z = \varphi(x)$  can be easily obtained using the following.

**Proposition 1.2.12** (Metric in different coordinates). *Consider system (1.1) and assume to know a  $C^1$  symmetric and positive definite matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$ , two strictly positive real numbers  $\bar{p} \geq \underline{p} > 0$  and a continuous function  $q : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  such that (1.11) holds. Let  $\varphi : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x}$  be a global diffeomorphism such that (1.33) holds. Then, the system (1.34) satisfies*

$$0 \prec \underline{p}_z I \preceq P_z(z, t) \preceq \bar{p}_z I, \quad (1.35a)$$

$$L_{f_z} P_z(z, t) \preceq -q_z(z) P_z(z, t) \quad (1.35b)$$

for all  $z \in \mathbb{R}^{n_x}$ , for all  $t \geq t_0$ , for some strictly positive real numbers  $\bar{p}_z, \underline{p}_z > 0$  and some function  $q_z : \mathbb{R}^{n_x} \mapsto \mathbb{R}$  where

$$P_z(z, t) = \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-\top} P(\varphi^{-1}(z), t) \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-1}. \quad (1.36)$$

Moreover:

- if  $q(x) \geq 0$  for all  $x \in \mathbb{R}^{n_x}$ , then (1.34) is  $\delta GUS$ ;
- if there exists a positive real number  $\lambda > 0$  such that  $q(x) \geq \lambda$  for all  $x \in \mathbb{R}^{n_x}$ , then (1.34) is  $\delta GUES$ .

**Proof.** The proof is divided into three steps. First, we show that (1.34), (1.36) satisfy (1.35b). Then, we show that the bound (1.35a) are guaranteed. Finally, we show that the incremental properties are preserved.

*Part 1.* We have that

$$\begin{aligned} L_{f_z} \bar{P}_z(z, t) &= \mathfrak{d}_{f_z} \bar{P}_z(z, t) + \text{He} \left\{ P_z(z, t) \left[ \frac{\partial}{\partial z} \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right) f(\varphi^{-1}(z), t) \right] \right\} \\ &\quad + \text{He} \left\{ P_z(z, t) \left[ \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \frac{\partial f}{\partial x}(\varphi^{-1}(z), t) \frac{\partial}{\partial z}(\varphi^{-1}(z)) \right] \right\}. \end{aligned}$$

Given any invertible matrix function  $A$ , we have that

$$\frac{\partial}{\partial x}(A^{-1}(x)) = -A^{-1}(x) \frac{\partial}{\partial x}(A(x)) A^{-1}(x).$$

Therefore we have that

$$\begin{aligned} \mathfrak{d}_{f_z} P_z(z, t) &= \left[ \text{He} \left\{ - \left( \left( \frac{\partial \varphi}{\partial x} \right)^{-1} \frac{\partial}{\partial z} \left( \frac{\partial \varphi}{\partial x} \right) \left( \frac{\partial \varphi}{\partial x} \right)^{-1} \right)^\top P \left( \frac{\partial \varphi}{\partial x} \right)^{-1} \right\} \right. \\ &\quad \left. + \left( \frac{\partial \varphi}{\partial x} \right)^{-\top} \frac{\partial}{\partial z}(P) \left( \frac{\partial \varphi}{\partial x} \right)^{-1} \right] \frac{\partial \varphi}{\partial x} f \end{aligned}$$

where every function is evaluated in  $\varphi^{-1}(z)$  (besides  $f$  and  $P$  which are in  $(\varphi^{-1}(z), t)$ ) and where the notation has been omitted for space reasons. Finally, by using the definition (1.36) we get

$$\begin{aligned} L_{f_z} P_z(z, t) &= \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-\top} [L_f P(\varphi^{-1}(z), t)] \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-1} \\ &\preceq -q(\varphi^{-1}(z)) P_z(z, t) \end{aligned}$$

By defining  $q_z(z) := q(\varphi^{-1}(z))$ , the relation (1.35b) follows.

*Part 2.* Since the inverse of the Jacobian of the diffeomorphism is always full rank and therefore invertible, we have that

$$\varphi(\varphi^{-1}(z)) = z \implies \frac{\partial}{\partial z}(\varphi(\varphi^{-1}(z))) = 1 \quad (1.37a)$$

$$\implies \frac{\partial \varphi^{-1}}{\partial z}(z) = \left[ \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right]^{-1}. \quad (1.37b)$$

Therefore,

$$\begin{aligned} P_z(z, t) &\succeq \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-\top} \underline{p} \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-1} \\ &= \left( \frac{\partial \varphi^{-1}}{\partial z}(z) \right)^{-\top} \underline{p} \left( \frac{\partial \varphi^{-1}}{\partial z}(z) \right)^{-1} \\ &\succeq \frac{1}{L_2^2} \underline{p} I \end{aligned}$$

By doing similar computations, a uniform constant upper-bound can be obtained. *Part 3.* The third item is satisfied since for the  $\delta$ GUS and the  $\delta$ GUES case it's possible to take, respectively,  $q_z = q = 0$  and  $q_z = q = \lambda > 0$ . Hence the result follows from direct application of Theorem 1.2.2.  $\square$

## 1.3 Nonautonomous systems

### 1.3.1 Incremental Input-to-State Stability

#### 1.3.1.1 Main definition

In this section, we study nonlinear systems whose dynamics are affected by an external input  $u$ . Such an input may represent a control action and/or a disturbance. In particular, we consider systems defined by

$$\dot{x} = f(x, t) + g(x, t)\rho(u). \quad (1.38)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathcal{U}$  is an exogenous signal where  $\mathcal{U} \subset \mathbb{R}^{n_u}$  is a compact set,  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$  and  $g : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_\rho}$  are  $C^2$  in their first argument and piecewise continuous in the second and  $\rho : \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_\rho}$  is any function for which there exists a class- $\mathcal{K}_\infty$  function  $\delta_\rho$  such that

$$|\rho(u_1) - \rho(u_2)| \leq \delta_\rho(|u_1 - u_2|). \quad (1.39)$$

We denote by  $\mathcal{X}(x, u, t, t_0)$  the solution of system (1.38) starting at initial condition  $x$  at time  $t$  with input  $u = u(t)$  and satisfying the initial value problem

$$\begin{aligned} \mathcal{X}(x, u, t, t_0) &= x, \\ \frac{\partial \mathcal{X}}{\partial t}(x, u, t, t_0) &= f(\mathcal{X}(x, u, t, t_0), t) + g(\mathcal{X}(x, u, t, t_0), t)\rho(u). \end{aligned} \quad (1.40)$$

**Definition 1.3.1** (Incremental ISS). *We say that system (1.1) is*

- ( $\delta$ GUISS) *Incrementally Globally Uniformly Input-to-State Stable if there exists a class- $\mathcal{K}$  function  $\alpha$  such that*

$$|\mathcal{X}(x_1, u_1, t, t_0) - \mathcal{X}(x_2, u_2, t, t_0)| \leq \alpha(|x_1 - x_2|) + \sup_{s \in [t_0, t]} [\alpha_d(|u_1(s) - u_2(s)|)]; \quad (1.41)$$

- ( $\delta$ GUEISS) *Incrementally Globally Uniformly Exponentially Input-to-State Stable* if there exist positive real numbers  $k, \lambda$  such that

$$\begin{aligned} & |\mathcal{X}(x_1, u_1, t, t_0) - \mathcal{X}(x_2, u_2, t, t_0)| \\ & \leq k |x_1 - x_2| \exp(-\lambda(t - t_0)) + \sup_{s \in [t_0, t]} [\alpha_d (|u_1(s) - u_2(s)|)]; \end{aligned} \quad (1.42)$$

for all initial conditions  $x_1, x_2 \in \mathbb{R}^{n_x}$  and for all inputs  $u_1, u_2 \in \mathcal{U}$ , for all  $t \geq t_0$  and for some class- $\mathcal{K}$  function  $\alpha_d$ .

### 1.3.1.2 Sufficient conditions for incremental-ISS: the Killing vector notion

In this section, we look for sufficient conditions for a system of the form (1.38) to possess an incremental ISS property. To the best of the knowledge of the author of this manuscript, such a notion has been studied in a general framework only using a Lyapunov-based analysis in Angeli (2002, 2009). In this section, we aim to look for some metric-based sufficient conditions to have an incremental ISS property. For this, we introduce the notion of Killing Vector field.

**Definition 1.3.2** (Killing vector field). *Given a  $C^1$  2-tensor  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  and a  $C^1$  vector field  $g : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x}$ , we say that  $g$  is a Killing vector field with respect to  $P$  if*

$$L_g P(x, t) = 0 \quad (1.43)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ .

**Remark 1.3.1.** *In case  $g$  is a  $C^1$  matrix function i.e.  $g : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_\rho}$ , we say that  $g$  is a Killing vector field (or it possesses the Killing vector field property), meaning that  $L_{g_i} P(x, t) = 0$  for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ , and for all  $i = 1, \dots, n_\rho$  with  $g_i$  being the  $i$ -th column of  $g$ .*

The notion of Killing vector field takes the name from Wilhelm Killing, a German mathematician (see, for instance, Nomizu (1960); Yano (1952)). In a few words, such a Killing Vector property implies that distances between different trajectories generated by the vector field  $g(x, t)$  in the norm  $|\cdot|_{P(x, t)}$  are invariant. Basically, the signals that enter in the directions of the vector field  $g$  do not affect the distances, in the sense that different trajectories of the differential equation  $\dot{x} = g(x, t)$  have a distance (associated with the norm provided by  $P$ ) among them which is constant for any  $t \geq t_0$ .

**Remark 1.3.2.** *Note that:*

- if  $P$  is constant and  $g(x, t) = G$  where  $G \in \mathbb{R}^{n_x \times n_\rho}$  is any constant matrix, then  $G$  has always the Killing vector property concerning such a  $P$ ;
- for any  $C^1$  scalar vector field  $g : \mathbb{R} \mapsto \mathbb{R}$  such that there exists  $\bar{g} \in \mathbb{R}_{>0}$  such that  $|g(x)| \leq \bar{g}$  for all  $x \in \mathbb{R}$ , then  $P(x) = g^{-2}(x)$  always satisfies the Killing

vector property. Indeed

$$\begin{aligned}
L_g P(x) &= \frac{\partial P}{\partial x}(x)g(x) + 2\frac{\partial g}{\partial x}(x)P(x) \\
&\quad \frac{\partial g^{-2}}{\partial x}(x)g(x) + 2\frac{\partial g}{\partial x}(x)g^{-2}(x) \\
&= -2\frac{g'(x)}{g^3(x)}g(x) + 2\frac{g'(x)}{g^2(x)} \\
&= 0
\end{aligned}$$

The requirement on  $\bar{g}$  is due to the bounds that  $P$  should have.

The Killing Vector property can be used to derive sufficient conditions for a system of the form (1.38) to be  $\delta$ GUISS or  $\delta$ GUEISS, as stated in the following.

**Theorem 1.3.3** (Incremental + Killing vector implies  $\delta$ ISS). *Consider system (1.38) and assume there exists a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values, two strictly positive real numbers  $\bar{p} \geq \underline{p} > 0$  and a continuous function  $q : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  satisfying (1.11). Suppose moreover that there exists a strictly positive real number  $\bar{g} > 0$  such that  $\|g(x, t)\| \leq \bar{g}$  for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ . Furthermore, assume that  $g$  is a Killing vector field for  $P$ , namely  $L_g P(x, t) = 0$  for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ . Then*

1. if  $q(x) \geq 0$  for all  $x \in \mathbb{R}^{n_x}$ , then the system is  $\delta$ GUISS;
2. if there exists a strictly positive real number  $\lambda > 0$  such that  $q(x) \geq \lambda$  for all  $x \in \mathbb{R}^{n_x}$ , then the system is  $\delta$ GUEISS.

**Proof.** We prove the theorem only for the  $\delta$ GUEISS case. The proof for the other item is analogous. First, we indicate with  $\nu := \rho(u)$ , so that system (1.38) can be rewritten as

$$\dot{x} = f(x, t) + g(x, t)\nu.$$

Associated to the matrix function  $P$  we can define a Riemannian metric and its associated norm (1.12) on  $\mathbb{R}^{n_x}$ . The (Riemannian) distance  $d_P$  on  $\mathbb{R}^{n_x}$  between a pair of points of  $\mathbb{R}^{n_x}$  can be defined similarly as in the proof of Theorem 1.2.2. Indeed, define  $\gamma_t : [s_1, s_2] \mapsto \mathbb{R}^{n_x}$  as any  $C^2$   $t$ -parametrized curve such that, at time  $t$ , we have that  $\gamma_t(s_1) = x_1$  and  $\gamma_t(s_2) = x_2$ . Define the length of the curve  $\gamma_t$  in the operator  $P(x, t)$  as (1.13) and the distance between  $x_1$  and  $x_2$  as (1.15). Consider a  $C^2$  path between  $(x_1, x_2) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x}$ . For all  $(s, t, t_0) \in [s_1, s_2] \times \mathbb{R} \times \mathbb{R}$ , let  $\Gamma(s, t, t_0) = \mathcal{X}(\gamma_{t_0}(s), v(s), t, t_0)$  and define the function  $\xi$  as in (1.16) where

$$v(s) = \frac{(s_1 - s)\nu_2 + (s - s_2)\nu_1}{s_1 - s_2}.$$

The function  $\xi$  is  $C^1$  and, by using the relations (1.11a) and (1.43), we obtain

$$\begin{aligned} \frac{\partial \xi}{\partial t}(s, t, t_0) &= \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) L_f P(\Gamma(s, t, t_0), t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \\ &\quad + 2 \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) P(\Gamma(s, t, t_0), t) g(\Gamma(s, t, t_0), t) \frac{\nu_1 - \nu_2}{s_1 - s_2} \\ &\leq -\lambda \xi(s, t, t_0) + 2\sqrt{\xi(s, t)} \sqrt{\bar{p}\bar{g}} \frac{|\nu_1 - \nu_2|}{s_2 - s_1}. \end{aligned}$$

Hence, it yields

$$\begin{aligned} \frac{\partial \ell_P}{\partial t}(\Gamma(s, t, t_0)) &= \frac{\partial}{\partial t} \int_{s_1}^{s_2} \sqrt{\xi(s, t, t_0)} ds \\ &= \int_{s_1}^{s_2} \frac{1}{2\sqrt{\xi(s, t, t_0)}} \frac{\partial \xi}{\partial t}(s, t, t_0) ds \\ &\leq -\frac{\lambda}{2} \ell_P(\Gamma(s, t, t_0)) + \sqrt{\bar{p}\bar{g}} |\nu_1 - \nu_2|. \end{aligned}$$

Since  $\nu = \rho(u)$ , by using (1.39) we get

$$\frac{\partial \ell_P}{\partial t}(\Gamma(s, t, t_0)) \leq -\frac{\lambda}{2} \ell(\Gamma(s, t, t_0)) + \sqrt{\bar{p}\bar{g}} \delta_\rho(|u_1 - u_2|).$$

Hence for each  $t$

$$\begin{aligned} \text{dist}_P(\mathcal{X}(x_1, u_1, t, t_0), \mathcal{X}(x_2, u_2, t, t_0)) &\leq \ell_P(\Gamma(s, t, t_0)) \\ &\leq \ell_P(\gamma_{t_0}) e^{-\frac{\lambda}{2}(t-t_0)} + \sqrt{\bar{p}\bar{g}} \int_{t_0}^t e^{-\frac{\lambda}{2}(t-s)} \delta_\rho(|u_1(s) - u_2(s)|) ds \\ &\leq \ell(\gamma_{t_0}) e^{-\frac{\lambda}{2}(t-t_0)} + \sqrt{\bar{p}\bar{g}} \sup_{s \in [t_0, t]} [\delta_\rho(|u_1(s) - u_2(s)|)] \end{aligned}$$

Following similar arguments as in the proof of Theorem 1.2.2, the result follows with  $\sqrt{\bar{p}\bar{g}} \delta_\rho(s) =: \alpha_d(s)$ .  $\square$

### 1.3.2 Interconnected systems

In this section, we present some results about incrementally stable systems that are interconnected with each other. Most of these results are well known in the literature, see e.g. (Lohmiller and Slotine, 1998, Section 4).

1. Consider two systems in *parallel* described by

$$\dot{x}_1 = f_1(x_1, t) + g_1(x_1, t) \rho_1(u_1) \quad (1.44a)$$

$$\dot{x}_2 = f_2(x_2, t) + g_2(x_2, t) \rho_2(u_2). \quad (1.44b)$$

Of course, if both sub-systems are  $\delta$ GUEISS, the parallel interconnection  $(x_1, x_2)$  is so. In particular, if there exist two metrics  $P_1(x_1, t)$ ,  $P_2(x_2, t)$  such that Theorem 1.3.3 holds for the  $x_1$  subsystem with  $P_1$  and, respectively, for the  $x_2$  subsystem with  $P_2$ , then  $\text{blkdiag}\{P_1(x_1, t), P_2(x_2, t)\}$  is a metric for the parallel interconnection.

2. Consider now two sub-systems in a cascade interconnection

$$\dot{x}_1 = f_1(x_1, t) + g_1(x_1, t)\rho_1(u_1) \quad (1.45a)$$

$$\dot{x}_2 = f_2(x_2, t) + g_2(x_2, t)\rho_2(x_1). \quad (1.45b)$$

where  $x_1 \in \mathbb{R}^{n_{x_1}}, u_1 \in \mathbb{R}^{n_{u_1}}, x_2 \in \mathbb{R}^{n_{x_2}}, u_2 \in \mathbb{R}^{n_{u_2}}$  and all the functions are sufficiently smooth. If (1.45a) is  $\delta$ GUES and (1.45b) is  $\delta$ GUEISS, then the interconnection is  $\delta$ GUES. This has been shown in (Angeli, 2002, Proposition 4.7). A suitable metric can be defined as the following.

**Proposition 1.3.4** (Metric for cascade interconnection). *Consider system (1.45). Assume that there exist two  $C^1$  matrix functions  $P_1 : \mathbb{R}^{n_{x_1}} \times \mathbb{R} \mapsto \mathbb{R}^{n_{x_1} \times n_{x_1}}, P_2 : \mathbb{R}^{n_{x_2}} \times \mathbb{R} \mapsto \mathbb{R}^{n_{x_2} \times n_{x_2}}$  taking symmetric and positive definite values, nine strictly positive real numbers  $\lambda_1, \lambda_2, \bar{p}_1, \underline{p}_1, \bar{p}_2, \underline{p}_2, \bar{g}_1, \bar{g}_2, \bar{\rho}$  such that  $\|g_1(x_1, t)\| \leq \bar{g}_1, \|g_2(x_2, t)\| \leq \bar{g}_2, \left\| \frac{\partial \rho_2}{\partial x_1}(x_1) \right\| \leq \bar{\rho}$ . Then if*

$$\begin{aligned} L_{f_1}P_1(x_1, t) &\preceq -\lambda_1 I, & \underline{p}_1 I &\preceq P_1(x_1, t) \preceq \bar{p}_1 I, & L_{g_1}P_1(x_1, t) &= 0 \\ L_{f_2}P_2(x_2, t) &\preceq -\lambda_2 I, & \underline{p}_2 I &\preceq P_2(x_2, t) \preceq \bar{p}_2 I, & L_{g_2}P_2(x_2, t) &= 0 \end{aligned} \quad (1.46)$$

for all  $(x_1, x_2, t)$ , then Theorem 1.3.3 holds for the cascade system (1.45) with  $P(x_1, x_2, t) = \text{blkdiag}\{P_1(x_1, t), \epsilon P_2(x_2, t)\}$  for some  $\epsilon > 0$  sufficiently small and therefore the system is  $\delta$ GUEISS with respect to  $u_1$ .

**Proof.** Follows from the Lipschitz condition on  $\rho_2$ . A similar proof has been obtained in (Sontag, 2010, Theorem 3) via matrix measures.  $\square$

## 1.4 Feedback design

### 1.4.1 Existing solutions

In this section, we try to provide an answer to the question: ‘how to design a feedback control to achieve incremental properties for the closed-loop system?’. The question is not trivial and several results already exist in the literature. We can divide such results into three categories.

- The first category considers control laws that are specific for particular classes of systems. Among them, we recall for instance a backstepping approach in Zamani and Tabuada (2011), impulsive control in Liu et al. (2021), a piecewise approximation design for Lur’e systems Waitman et al. (2017); Fromion et al. (2003) and a small-gain design in Angeli (2002); Sontag (2010). In these cases, the proposed control laws generally provide satisfactory and tractable conditions. On the other hand, the main drawback of the proposed designs is that they apply only to the considered class of systems.
- The second category considers control laws that provide a design for notions that are similar to the one of incremental stability. We recall for instance the control laws based on the notion of convergent systems (see the manuscript Pavlov et al. (2006) and references therein), or the one of quadratic stability D’Alto and Corless (2013). It’s important to highlight however the following aspect. These notions are similar, but not equivalent to incremental stability. Many counterexamples can be constructed. In this sense, we already recalled

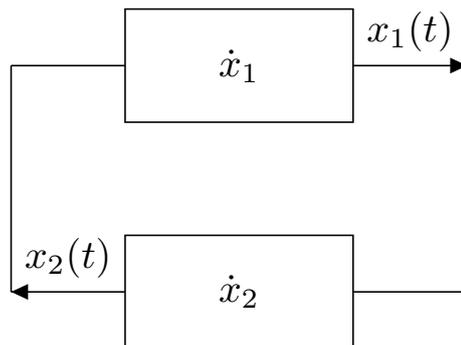


Figure 1.1: Feedback interconnection

the strict link between incremental stability and convergent systems, see Section 1.2.3.2. For this reason, we can generally apply these results to have incremental properties, but each case has to be carefully addressed.

- The last category includes control designs that make use of machine learning tools and/or optimization techniques. See for instance Manchester and Slotine (2017); Tsukamoto et al. (2021) and references therein. This last category provides more general control designs that can be applied to many different classes of systems. See for instance, the control contraction metric approach in Manchester and Slotine (2017). On the other hand, these results are difficult to be practically implemented since they usually require solving an online optimization problem along the geodesic, whose knowledge is usually unknown. For this reason, the implementation usually relies on machine learning tools. If the use of artificial intelligence simplifies the practical aspects, it reflects in a loss of guarantees that an analytic solution could provide.

In this section, we claim some results about the design of feedback control laws guaranteeing incremental properties for the closed-loop system. The results that we will give, will take place in the first category of the one mentioned above. In other words, we will consider specific classes of systems.

#### 1.4.2 A metric-based incremental small-gain theorem

The first result that we aim to show is an incremental version of the small gain theorem (Jiang et al. (1994)). A small gain result for incremental stability is not new in the literature. It has been already stated in (Angeli, 2002, Proposition 4.8), but it can practically be difficult to verify since it requires the knowledge of the function(s)  $\alpha_d$  in (1.42). We want to obtain the same, but easier-to-check, kind of result through a metric analysis. Consider therefore a system of the form (1.38) where  $x = (x_1, x_2)$  and

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, t) + g_1(x_1, t)\rho_1(x_2) \\ \dot{x}_2 &= f_2(x_2, t) + g_2(x_2, t)\rho_2(x_1) \end{aligned} \quad (1.47)$$

connected as in Figure 1.1. The following holds

**Theorem 1.4.1** (Incremental Small Gain). *Consider system (1.47). Assume that there exist two  $C^1$  matrix functions  $P_1 : \mathbb{R}^{n_{x_1}} \times \mathbb{R} \mapsto \mathbb{R}^{n_{x_1} \times n_{x_1}}$ ,  $P_2 : \mathbb{R}^{n_{x_2}} \times \mathbb{R} \mapsto \mathbb{R}^{n_{x_2} \times n_{x_2}}$  taking symmetric and positive definite values, ten strictly positive real numbers  $\lambda_1, \lambda_2, \bar{p}_1, \underline{p}_1, \bar{p}_2, \underline{p}_2, \bar{g}_1, \bar{g}_2, \bar{\rho}_1, \bar{\rho}_2$  such that (1.46) holds and such that*

$\|g_1(x_1, t)\| \leq \bar{g}_1$ ,  $\|g_2(x_2, t)\| \leq \bar{g}_2$ ,  $\left\| \frac{\partial \rho_1}{\partial x_2}(x_2) \right\| \leq \bar{\rho}_1$  and  $\left\| \frac{\partial \rho_2}{\partial x_1}(x_1) \right\| \leq \bar{\rho}_2$ . If moreover

$$\lambda_1 \lambda_2 > 8\bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2 \bar{\rho}_1 \bar{\rho}_2, \quad (1.48)$$

then system (1.47) is  $\delta$ GUES.

**Proof.** Consider the metric

$$P(x_1, x_2, t) = \begin{bmatrix} \epsilon P_1(x_1, t) & 0 \\ 0 & P_2(x_2, t) \end{bmatrix}$$

where  $P_1, P_2$  are the metrics satisfying (1.46) and  $\epsilon > 0$  is a strictly positive real number to be defined. Let  $x = (x_1, x_2)$  and

$$f(x, t) = \begin{bmatrix} f_1(x_1, t) + g_1(x_1, t)\rho_1(x_2) \\ f_2(x_2, t) + g_2(x_2, t)\rho_2(x_1) \end{bmatrix}.$$

Let  $\mathcal{L}$  be defined as

$$\mathcal{L}(x, t) := L_f P(x, t) + \lambda I = \begin{bmatrix} \ell_1(x, t) & \ell_2(x, t) \\ \ell_2^\top(x, t) & \ell_3(x, t) \end{bmatrix}$$

where

$$\ell_1(x, t) = \epsilon L_{f_1} P_1(x_1, t) + \lambda I$$

$$\ell_2(x, t) = \epsilon P_1(x_1, t) g_1(x_1, t) \frac{\partial \rho_1}{\partial x_2}(x_2) + \left( P_2(x_2, t) g_2(x_2, t) \frac{\partial \rho_2}{\partial x_1}(x_1) \right)^\top$$

$$\ell_3(x, t) = L_{f_2} P_2(x_2, t) + \lambda I.$$

We have to show that there exists  $\lambda > 0$  such that  $\mathcal{L} \preceq 0$ . First note that, for any  $\lambda \leq \frac{\lambda_2}{2}$ , it follows from (1.46) that  $\ell_3(x, t) \preceq 0$ . Therefore, we have just to check that the Schur complement of  $L_f P$  is negative definite, namely if  $\ell_1 - \ell_2 \ell_3^{-1} \ell_2^\top \prec 0$ . From the bounds on  $\rho_i, g_i$  and  $P_i$ , we have

$$\begin{aligned} \ell_1 - \ell_2 \ell_3^{-1} \ell_2^\top &= \epsilon L_{f_1} P_1(x_1, t) + \lambda I \\ &- \left( \epsilon P_1(x_1, t) g_1(x_1, t) \frac{\partial \rho_1}{\partial x_2}(x_2) + \left( P_2(x_2, t) g_2(x_2, t) \frac{\partial \rho_2}{\partial x_1}(x_1) \right)^\top \right) (L_{f_2} P_2(x_2, t) + \lambda I)^{-1} \\ &\quad \times \left( \epsilon P_1(x_1, t) g_1(x_1, t) \frac{\partial \rho_1}{\partial x_2}(x_2) + \left( P_2(x_2, t) g_2(x_2, t) \frac{\partial \rho_2}{\partial x_1}(x_1) \right)^\top \right)^\top \\ &\preceq \left[ -\epsilon \lambda_1 + \lambda + \frac{2}{\lambda_2} (\epsilon \bar{p}_1 \bar{g}_1 \bar{\rho}_1 + \bar{p}_2 \bar{g}_2 \bar{\rho}_2)^2 \right] I. \end{aligned}$$

Therefore,  $\mathcal{L} \preceq 0$  if

$$-\epsilon \lambda_1 + \lambda + \frac{2}{\lambda_2} (\epsilon \bar{p}_1 \bar{g}_1 \bar{\rho}_1 + \bar{p}_2 \bar{g}_2 \bar{\rho}_2)^2 \leq 0. \quad (1.49)$$

We look at (1.49) as a quadratic inequality in the unknown  $\epsilon$  and we look for the positive value of  $\epsilon$  such that (1.49) holds. Hence, we derive the left-hand

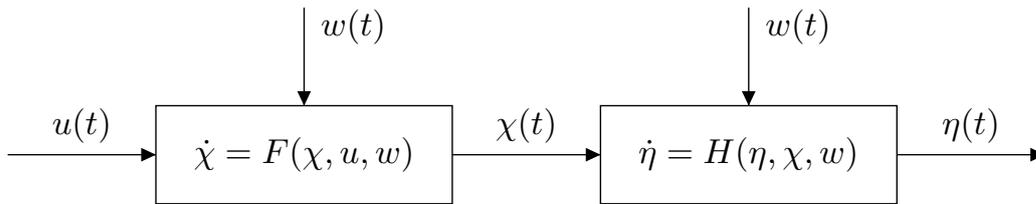


Figure 1.2: Feed-forward interconnection

side of (1.49) with respect to  $\epsilon$  and set it equal to 0 and we get

$$\epsilon = \frac{\lambda_1 \lambda_2 - 4\bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2 \bar{\rho}_2 \bar{\rho}_1}{4\bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2}. \quad (1.50)$$

From (1.48), such  $\epsilon$  is positive definite. To conclude, we have to show that, with such a choice of  $\epsilon$ , we have that (1.49) holds. For this, we plug (1.50) into (1.49) and get

$$\begin{aligned} & -\epsilon \lambda_1 + \lambda + \frac{2}{\lambda_2} (\epsilon \bar{p}_1 \bar{g}_1 \bar{\rho}_1 + \bar{p}_2 \bar{g}_2 \bar{\rho}_2)^2 \\ &= -\lambda_1 \left( \frac{\lambda_1 \lambda_2 - 4\bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2 \bar{\rho}_2 \bar{\rho}_1}{4\bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2} \right) + \lambda + \frac{2}{\lambda_2} \left[ \left( \frac{\lambda_1 \lambda_2 - 4\bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2 \bar{\rho}_2 \bar{\rho}_1}{4\bar{p}_1 \bar{g}_1 \bar{\rho}_2} \right) + \bar{p}_2 \bar{g}_2 \bar{\rho}_2 \right]^2 \\ &= -\lambda_1 \left( \frac{\lambda_1 \lambda_2 - 4\bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2 \bar{\rho}_2 \bar{\rho}_1}{4\bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2} \right) + \lambda + \frac{2}{\lambda_2} \left[ \frac{\lambda_1^2 \lambda_2^2}{16\bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2} \right] \\ &= \frac{-4\lambda_1 \lambda_2 (\lambda_1 \lambda_2 - 4\bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2 \bar{\rho}_2 \bar{\rho}_1) + 2\lambda_1^2 \lambda_2^2 + 16\lambda \lambda_2 \bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2}{16\lambda_2 \bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2} \\ &= \frac{-2\lambda_1 \lambda_2 (\lambda_1 \lambda_2 - 8\bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2 \bar{\rho}_2 \bar{\rho}_1) + 16\lambda \lambda_2 \bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2}{16\lambda_2 \bar{p}_1^2 \bar{g}_1^2 \bar{\rho}_2^2} < 0 \end{aligned}$$

where the last equality follows from (1.48) and by taking  $\lambda > 0$  sufficiently small. And this concludes the proof.  $\square$

### 1.4.3 Incremental forwarding design for cascade systems

#### 1.4.3.1 Introduction and context

In this section, we focus on the control design for the class of systems in *cascade*. With the term ‘cascade’, we mean that the dynamics of (1.38) can be decomposed into two terms. A first component, denoted with  $\chi$  and whose time derivative is driven by the control action  $u$ , and a second component, denoted with  $\eta$ , whose input is a (possibly nonlinear) function of  $\chi$ . Both the two dynamics are affected by an external uncontrolled disturbance  $w(t)$ . A block scheme representing such a control structure can be found in Figure 1.2. This class of systems (also called *feed-forward* form) has been intensively studied for stabilization purposed, especially in the ‘90, see for instance Arca et al. (2001); Mazenc and Praly (1996); Sepulchre et al. (2012); Teel (1992); Sepulchre et al. (1997); Jankovic et al. (1996) and references therein. In particular, we will consider a system of the form (1.38) which can be described

by the following ODE

$$\dot{\chi} = f_{\chi}(\chi) + g_{\chi}(\chi)(u + w(t)), \quad (1.51a)$$

$$\dot{\eta} = \Phi\eta + v(\chi) + R w(t). \quad (1.51b)$$

In this case, the state is  $x = (\chi, \eta)$  where  $\chi \in \mathbb{R}^{n_{\chi}}$ ,  $\eta \in \mathbb{R}^{n_{\eta}}$ . The plant is excited by a control input  $u \in \mathbb{R}^{n_u}$  and by an external perturbation  $w : \mathbb{R} \mapsto \mathcal{W} \subset \mathbb{R}^{n_w}$ . The functions  $f_{\chi} : \mathbb{R}^{n_{\chi}} \mapsto \mathbb{R}^{n_{\chi}}$ ,  $g_{\chi} : \mathbb{R}^{n_{\chi}} \mapsto \mathbb{R}^{n_{\chi} \times n_u}$ ,  $v : \mathbb{R}^{n_{\chi}} \mapsto \mathbb{R}^{n_{\eta}}$  are sufficiently smooth and  $f(0) = 0$  and  $v(0) = 0$ . Moreover  $\Phi, R$  are constant matrices.

**Remark 1.4.2.** *The perturbation  $w$  acting on the  $\chi$ -dynamics is assumed to satisfy a ‘matching condition’, i.e. the disturbance enters in the same directions as the controller. The matching condition assumption is quite common in the nonlinear control literature and verified by many practical applications. See for instance the works Qu (1993); Praly and Wang (1996) and references therein. From a practical point of view, matching disturbances are always verified when considering perturbations acting on the actuators of the plant.*

The objective will be to design the control action  $u$  for the closed-loop system to be incrementally ISS with respect to  $w$ .

**Remark 1.4.3.** *The interest in the design of control laws for such a class of systems is relevant since this system’s structure arises in several case studies. This is the case for instance when dealing with dynamic control laws architectures, where the  $\chi$ -dynamics is the plant and the  $\eta$ -dynamics is part of the control design. For example, the results presented in this section will play a fundamental role in providing solutions to the output regulation problem, which will be considered in Chapter 2 of this manuscript.*

In particular, we will focus on extending the *forwarding*-based results in Mazenc and Praly (1996); Praly et al. (2001) to the incremental framework, presenting a set of sufficient conditions for the closed-loop system to possess some incremental properties. This will lead to a control design that we will call *incremental forwarding*. The results that will be presented in this section can be found in the author’s article Giaccagli et al. (2022d) (see also Giaccagli et al. (2020, 2021b) for a simplified version).

#### 1.4.3.2 Forwarding design for stabilization

Before presenting the main results of this section, we recall the main aspects and peculiarities of forwarding-based control design (in the sense of Mazenc and Praly (1996)) for stabilization purposes. Forwarding-based control laws for stabilization purposes have been studied in several works, see Mazenc and Praly (1996); Praly et al. (2001); Astolfi and Praly (2017); Kaliora and Astolfi (2004); Poulain and Praly (2010); Benachour et al. (2013); Zitte et al. (2020).

Therefore, consider system (1.51) with  $w = 0$ , that is, system

$$\begin{aligned} \dot{\chi} &= f_{\chi}(\chi) + g_{\chi}(\chi)u \\ \dot{\eta} &= \Phi\eta + v(\chi). \end{aligned} \quad (1.52)$$

The forwarding design starts with the assumption that the  $\chi$ -dynamics of (1.52) is stabilizable using a feedback control law.

**Assumption 1.4.1** (Open-loop stabilizability). *We know a  $C^1$  functions  $\phi_0 : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  such that the origin of*

$$\dot{\chi} = f_0(\chi), \quad f_0(\chi) := f_\chi(\chi) + g_\chi(\chi)\phi_0(\chi), \quad (1.53)$$

*is Locally Exponentially Stable and Globally Asymptotically Stable.*

The second assumption is on the  $\eta$ -dynamics. In particular, it is assumed that it is stable, as in the following.

**Assumption 1.4.2** (Stability of  $\Phi$ ). *There exists a symmetric positive definite matrix  $H = H^\top \succ 0$  such that*

$$H\Phi + \Phi^\top H \preceq 0. \quad (1.54)$$

In a nutshell, the main intuition behind forwarding is the following one. The  $\eta$ -dynamics can be seen as a linear system (having a stable origin) that is forced by the external signal  $v(\chi(t))$ . Since the origin of  $\chi$  is GAS and LES and  $v(0) = 0$ , such an external signal will asymptotically vanish along the trajectories of the  $\chi$ -dynamics, i.e.  $\lim_{t \rightarrow +\infty} v(\chi(t)) = 0$ . As a consequence, the  $\eta$ -dynamics will converge on a (stable) manifold  $M(\chi)$ , which depends of the vector field  $f_0$  and on  $v, \Phi$ . The existence of such a manifold is recalled in the following result.

**Lemma 1.4.4** (Existence of the  $M$ ). *Consider system (1.52) and let Assumption 1.4.1 and Assumption 1.4.2 hold. Then, there exists a  $C^1$  function  $M : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_\eta}$  such that  $M(0) = 0$  and  $M$  is solution of*

$$L_{f_0}M(\chi) = \Phi M(\chi) + v(\chi). \quad (1.55)$$

**Remark 1.4.5.** *The main intuition behind such a result is the existence of a stable manifold*

$$\Omega := \left\{ (\chi, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \mid \eta = M(\chi), \quad M \text{ solution to (1.55)} \right\} \quad (1.56)$$

*that contains the origin and that it is forward invariant for the system's dynamics.*

The forwarding approach relies on a state-feedback control law of the form

$$u = \phi_0(\chi) + \ell(\chi)L_{g_\chi}M^\top(\chi)(\eta - M(\chi)), \quad (1.57)$$

where  $\ell : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{>0}$  is a sufficiently small function. We have then the following result.

**Proposition 1.4.6** (Forwarding for stabilization). *Consider system (1.53) and let Assumption 1.4.1 and Assumption 1.4.2 hold. Let  $M : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_\eta}$  be defined as in Lemma 1.4.4 and assume that*

$$\text{rank} \begin{pmatrix} A - \sigma I & B \\ C & 0 \end{pmatrix} = n_\eta \quad (1.58)$$

where

$$A := \frac{\partial}{\partial \chi} (f_\chi(\chi) + g_\chi(\chi)\phi_0(x))_{\chi=0}, \quad B := g_\chi(0) \quad C := \frac{\partial v}{\partial \chi}(0) \quad (1.59)$$

*for any  $\sigma \in \text{spec}\{\Phi\}$ . Then, there exists  $\ell : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{>0}$  such that the origin of system (1.51) in closed-loop with (1.57) is Globally Asymptotically Stable and Locally Exponentially Stable.*

**Proof.** The main idea is to take the Lyapunov function  $W(\chi, \eta) = V(\chi) + (\eta - M(\chi))^\top (\eta - M(\chi))$  where  $V$  is a Lyapunov function for the  $\chi$ -dynamics. The proof is concluded with standard LaSalle's invariance principle. See (Astolfi and Praly, 2017, Section III) or Poulain and Praly (2010) and references therein for a complete proof.  $\square$

**Remark 1.4.7.** *The condition (1.58) is to make sure that the system (1.51) is controllable in the origin. Indeed, by considering (1.59), note first that the condition (1.55) reduces to the solution of the Sylvester equation*

$$M(0)A = \sigma M(0) + C.$$

*In such a case, for every  $C$  the solution always exists and it is unique since  $A$  and  $-\Phi$  have disjoint spectra. Indeed  $A$  is Hurwitz by local exponential stability in Assumption 1.4.1, and  $-\Phi$  has no negative eigenvalue by Assumption 1.4.2. Such a property is also known in regulation theory as non-resonance condition. This aspect will be recalled later on in Chapter 2.*

Although the existence of  $M$  is always guaranteed under the conditions of Lemma 1.4.4, its explicit expression may not be always easy to find, since it involves the solution of the differential equation (1.55). As a consequence, in order to provide additional degrees of freedom for the design of the control law (1.57), several extensions have been developed. See, for instance, (Astolfi and Praly, 2017, Section III.B) for an overview.

A particular design of interest is the one proposed in Praly et al. (2001), called 'forwarding mod $\{L_g V\}$ '. We recall such a result in the following proposition. Note that the forwarding mod $\{L_g V\}$  has been formulated only in the single-input case with unitary dimension of the  $\eta$ -dynamics ( $n_u = n_\eta = 1$ ) and for the case in which  $\Phi = 0$ .

**Proposition 1.4.8** (Forwarding mod $\{L_g V\}$ ). *Consider system (1.51) with  $\Phi = 0$  and  $n_\eta = n_u = 1$ , and suppose to know a positive definite Lyapunov function  $V$  satisfying  $L_{f_0} V(\chi) < 0$  for all  $\chi \neq 0$ . Assume, moreover, to know a  $C^1$  function  $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}$  with  $\bar{M}(0) = 0$  and a  $C^0$  function  $k(\chi)$  such that the following holds.*

1.  $L_{f_0} \bar{M}(\chi) = v(\chi) + k(\chi) L_{g_x} V(\chi)$ .
2.  $L_{g_x} \bar{M}(0) \neq 0$ .
3. For all  $\{\chi \in \mathbb{R}^{n_x} : L_{g_x} \bar{M}(\chi) \neq 0, \chi \neq 0\}$ , then

$$L_{f_0} \bar{M}(\chi) - \frac{k(\chi) (L_{g_x} V)^2(\chi)}{L_{g_x} \bar{M}(\chi)} < 0. \quad (1.60)$$

*Then, the origin of (1.51) in closed-loop with*

$$u = \phi_0(\chi) - \frac{k(\chi) L_{g_x} V(\chi)}{L_{g_x} \bar{M}(\chi)} - [L_{g_x} V(\chi) - L_{g_x} \bar{M}(\chi) (\eta - \bar{M}(\chi))], \quad (1.61)$$

*is Globally Asymptotically Stable and Locally Exponentially Stable.*

| **Proof.** See (Praly et al., 2001, Proposition 1).  $\square$

The interest of the condition 1 in Proposition 1.4.8 concerns the computation of  $\bar{M}$ , which can be interpreted as an approximated version of the solution of (1.55). Now, the  $\bar{M}$  can be ‘chosen’ to simplify the design of the control action. However, this choice add some extra constraints, which are represented by the *mismatching* term  $k(\chi)$ . The open-loop system must be ‘sufficiently stable’ to stand the presence of such a mismatching. This is represented by the condition (1.60) In such a case indeed, the mismatch needs to satisfy the extra condition on item 3 that is crucial to conclude the global asymptotic stability of the closed-loop system. On the other hand, item 2 is the counterpart of (1.58) when considering  $\sigma = 0$ .

**Remark 1.4.9.** *Note that in case one selects  $k(\chi) = 0$ , the condition (1.60) is automatically satisfied by assumption and item 1 recovers (1.55) with  $\bar{M}(\chi) = M(\chi)$ .*

### 1.4.3.3 Uniform incremental forwarding mod $\{L_gV\}$

In this section, we aim to extend the result on forwarding design to the incremental framework. In other words, the objective is to develop an *incremental forwarding* design for the class of systems (1.38) of the form

$$\dot{\chi} = f_\chi(\chi) + g_\chi(\chi)(u + w(t)), \quad (1.62a)$$

$$\dot{\eta} = \Phi\eta + v(\chi) + R w(t). \quad (1.62b)$$

where  $x = (\chi, \eta)$  and where we assume that there exists  $\bar{g} > 0$  such that  $\|g_\chi(\chi)\| \leq \bar{g}$  for all  $\chi \in \mathbb{R}^{n_\chi}$ . Motivated by Remark 1.4.9, we focus directly on the forwarding mod $\{L_gV\}$  design. We start by assuming the following.

**Assumption 1.4.3** (Pre-contractive feedback + Killing vector). *Consider system (1.62). We know a  $C^2$  function  $\phi_0 : \mathbb{R}^{n_\chi} \mapsto \mathbb{R}^{n_u}$ , a  $C^1$  matrix function  $P_\chi : \mathbb{R}^{n_\chi} \mapsto \mathbb{R}^{n_\chi \times n_\chi}$  taking symmetric and positive values  $P_\chi = P_\chi^\top \succ 0$  and three positive real numbers  $\underline{p}_\chi, \bar{p}_\chi, p_\chi$  such that the function  $f_0(\chi) = f_\chi(\chi) + g_\chi(\chi)\phi_0(\chi)$  satisfies*

$$L_{f_0}P_\chi(\chi) \preceq -p_\chi I, \quad \underline{p}_\chi I \preceq P_\chi(\chi) \preceq \bar{p}_\chi I \quad (1.63)$$

$$L_{g_\chi}P_\chi(\chi) = 0 \quad (1.64)$$

for all  $\chi \in \mathbb{R}^{n_\chi}$ .

Assumption 1.4.3 asks for the knowledge of a pre-stabilizing feedback control action such that the  $\chi$ -dynamics is incrementally uniformly exponentially stable in the sense of Definition 1.2.1, with respect to a Riemannian metric induced by the matrix function  $P_\chi$ . Also, (1.64) implies that  $g_\chi$  is a Killing Vector field for this metric, see Definition 1.3.2. In the linear framework, this corresponds to a stabilizability assumption, where (1.64) is always satisfied as  $P_\chi$  is taken as a constant positive definite matrix solution of a Lyapunov equation. The design of  $\phi_0$  and  $P_\chi$  can be obtained following, for instance, the techniques recalled in Section 1.4.1

**Remark 1.4.10.** *Note moreover that, in light of Theorem 1.3.3, the Killing vector property (1.64) guarantees that the  $\chi$ -dynamics is incrementally globally uniformly exponentially ISS with respect to any perturbation  $w$ .*

From now on, we consider  $f_\chi(\chi) = f_0(\chi)$  without loss of generality (that is, Assumption 1.4.3 holds for  $\phi_0(\chi) = 0$  for all  $\chi$ , for some metric  $P_\chi$ ).

**Assumption 1.4.4** (Stability of  $\Phi$ ). *There exists a symmetric positive definite matrix  $H = H^\top \succ 0$  such that*

$$H\Phi + \Phi^\top H \preceq 0 \quad (1.65)$$

As we aim to develop an incremental version of forwarding  $\text{mod}\{L_g V\}$  as in Proposition 1.4.8, we assume the following.

**Assumption 1.4.5** (Incremental forwarding  $\text{mod}\{L_g V\}$ ). *We know three  $C^1$  functions  $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$ ,  $\Delta : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$  and  $\varrho : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  such that, for all  $\chi \in \mathbb{R}^{n_x}$ , the following hold:*

1. *the functions  $\bar{M}$  and  $\Delta$  are solution of*

$$L_{f_x} \bar{M}(\chi) = \Phi \bar{M}(\chi) + v(\chi) + \Delta(\chi); \quad (1.66)$$

2. *there exists a matrix  $\Lambda$  such that*

$$L_{g_x} \bar{M}(\chi) = \Lambda \quad (1.67)$$

*and such that the couple  $(\Phi, (H\Lambda)^\top)$  is detectable;*

3. *the function  $\varrho$  satisfies*

$$\Lambda \frac{\partial \varrho}{\partial \chi}(\chi) = -\frac{\partial \Delta}{\partial \chi}(\chi); \quad (1.68)$$

4. *the following inequality holds*

$$L_{f_x} P_\chi(\chi) + \text{He} \left\{ P_\chi(\chi) g_x(\chi) \frac{\partial \varrho}{\partial \chi}(\chi) \right\} \preceq -\lambda I \quad (1.69)$$

*for some  $\lambda > 0$ .*

Assumption 1.4.5 corresponds to a MIMO version of the assumptions in Proposition 1.4.8. In particular:

- item 1 corresponds to a more general version of (1.66). In particular, a solution  $\bar{M}(\chi) = M(\chi)$  of (1.66) is known to exist for  $\Delta(\chi) = 0$ . This follows from Assumption 1.4.3, Assumption 1.4.2 and from the fact that a time-invariant system that is incrementally uniformly globally exponentially stable, admits a globally exponentially stable equilibrium point, see Corollary 1.2.7.2;
- about item 2, the term  $L_{g_x} \bar{M}(\chi)$  can be seen as a controllability assumption on the control  $u$  to act on the dynamics of  $\eta$  of (1.62) in any point of the state space  $\chi$ . This is a counterpart of item 2 of Proposition 1.4.8;
- about item 3, as the dynamics of  $\eta$  in the most general case, can have a higher dimension than the one of  $u$ , we ask for the mismatch term  $\Delta$  to be mapped in a (possibly lower) space of the dimension of the input and to be integrable. This additional item is required to extend the results of Proposition 1.4.8 to the MIMO case;
- item 4 asks for a robustness-like property for the autonomous system. Indeed, to rely on a free-to-choose differential equation solution  $\bar{M}$  rather than the exact one  $M$ , the open-loop system must be *sufficiently contractive* to merge the mismatch represented by  $\Delta$ . This last item is the counterpart of (1.60).

We're now ready to state the main result of this section.

**Theorem 1.4.11** (Incremental stability of feedforward systems). *Consider system (1.62) and let Assumption 1.4.3, 1.4.4 and 1.4.5 hold. Moreover assume that there exists  $L_M \geq 0$  such that  $\left\| \frac{\partial \bar{M}}{\partial \chi}(\chi) \right\| \leq L_M$  for all  $\chi \in \mathbb{R}^{n_x}$ . Then, for any gain  $\kappa > 0$ , the system (1.62) in closed-loop with the control law*

$$u = \phi_0(\chi) + \kappa(H\Lambda)^\top(\eta - \bar{M}(\chi)) + \varrho(\chi) \quad (1.70)$$

is incrementally globally uniformly ISS with respect to  $w$ .

**Proof.** The proof is divided into two steps. First, we consider the unforced system (1.62) with  $w = 0$  and we show that the closed-loop system is  $\delta$ GUES by constructing a suitable metric. Then, we show that a Killing vector property holds, and therefore, by Theorem 1.3.3, the system is  $\delta$ GUEISS. For this, we first consider the change of coordinates  $\eta \mapsto z := \eta - \bar{M}(\chi)$  with  $\bar{M}$  solving (1.66). By making use of (1.67), the closed-loop system can be written in the form

$$\dot{\mathbf{x}} = F(\mathbf{x}), \quad F(\mathbf{x}) := \begin{bmatrix} f_\chi(\chi) + g_\chi(\chi) [\kappa(H\Lambda)^\top z + \varrho(\chi)] \\ (\Phi - \kappa\Lambda(H\Lambda)^\top)z - \Lambda\varrho(\chi) - \Delta(\chi) \end{bmatrix} \quad (1.71)$$

with  $\mathbf{x} = (\chi^\top, z^\top)^\top$ . Following Theorem 1.2.2, in order to claim incremental uniform global exponential stability, we look for a  $C^1$  matrix function  $\mathcal{P} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \mapsto \mathbb{R}^{(n_x+n_\eta) \times (n_x+n_\eta)}$  of the form

$$\mathcal{P}(\mathbf{x}) := \begin{pmatrix} P_\chi(\chi) & 0 \\ 0 & \mu(H + b\bar{S}) \end{pmatrix} \quad (1.72)$$

with  $b, \mu$  being strictly positive real numbers to be defined,  $P_\chi$  taken as in Assumption 1.4.3,  $H$  as in Assumption 1.4.4 and  $\bar{S}$  being a strictly positive definite matrix to be defined. The main intuition behind this choice is that given Assumption 1.4.4 the matrix  $H$  by itself doesn't provide negativity in all the components of  $z$ . In order to 'strictify' the metric, we rely on a design inspired by Praly (2019) (also used in Astolfi et al. (2022b)) by means of an observer. Indeed, by item 2 of Assumption 1.4.5, the couple  $(\Phi, (H\Lambda)^\top)$  is detectable, and therefore there exist two matrices  $S = S^\top \succ 0$  and  $K$  solving

$$\text{He} \{S(\Phi - K(H\Lambda)^\top)\} \preceq -2I. \quad (1.73)$$

Hence let  $\bar{S} = S$  and consider the matrix function  $\mathcal{L} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \mapsto \mathbb{R}^{(n_x+n_\eta) \times (n_x+n_\eta)}$

$$\mathcal{L}(\mathbf{x}) := L_F \mathcal{P}(\mathbf{x}) + \begin{pmatrix} \mathbf{p}_1 I & 0 \\ 0 & \mathbf{p}_2 I \end{pmatrix} \quad (1.74)$$

for some  $\mathbf{p}_1, \mathbf{p}_2$  strictly positive real numbers to be chosen. If  $\mathcal{L}(\mathbf{x}) \preceq 0$  for all  $\mathbf{x}$  and for some  $\mathbf{p}_1, \mathbf{p}_2$ , then the conditions of Theorem 1.2.2 hold with  $\mathbf{p} = \min\{\mathbf{p}_1, \mathbf{p}_2\}$  (conditionally to a constant, which always exists thanks to the constant upper bound of  $\mathcal{P}$ ). Thanks to (1.68) and to the Killing Vector property  $L_{g_\chi} P_\chi(\chi) = 0$  in Assumption 1.4.3, we have that

$$\mathcal{L}(\mathbf{x}) = \begin{pmatrix} \ell_1(\mathbf{x}) & \ell_2(\mathbf{x}) \\ \ell_2^\top(\mathbf{x}) & \ell_3(\mathbf{x}) \end{pmatrix} \quad (1.75)$$

where

$$\begin{aligned}\ell_1(\mathbf{x}) &= L_{f_x} P_x(\chi) + \text{He} \left\{ P_x(\chi) g_x(\chi) \frac{\partial g}{\partial x}(\chi) \right\} + \mathbf{p}_1 I \\ \ell_2(\mathbf{x}) &= \kappa P_x(\chi) g_x(\chi) (H\Lambda)^\top \\ \ell_3(\mathbf{x}) &= \mu \text{He} \left\{ (H + bS)(\Phi - \kappa\Lambda(H\Lambda)^\top) \right\} + \mathbf{p}_2 I.\end{aligned}$$

By adding and subtracting the term  $\mu b \text{He}\{SK(H\Lambda)^\top\}$  and by Assumption 1.4.4 we get

$$\begin{aligned}\ell_3(\mathbf{x}) &\preceq \mathbf{p}_2 I - \mu \left[ 2\kappa H\Lambda(H\Lambda)^\top + 2bI - b \text{He}\{S(K - \kappa\Lambda)(H\Lambda)^\top\} \right] \\ &\preceq \mathbf{p}_2 I - \mu \left[ 2\kappa H\Lambda(H\Lambda)^\top + 2bI - \frac{b^2}{2\beta} S(K - \kappa\Lambda)(K - \kappa\Lambda)^\top S - \frac{\beta}{2} H\Lambda(H\Lambda)^\top \right]\end{aligned}$$

for any real number  $\beta > 0$ , where we used Young's inequality. Therefore, we select  $\beta = 4\kappa$ ,  $b < \frac{2\beta}{|S(K - \kappa\Lambda)|^2}$  and  $\mathbf{p}_2 < \frac{\mu b}{2}$  and we get  $\ell_3(\mathbf{x}) \prec 0$  for all  $\mathbf{x}$ , for any  $\kappa, \mu > 0$ . To have  $\mathcal{L} \prec 0$ , it remains to check the sign of its Schur complement  $\mathcal{S}_{\mathcal{L}}(\mathbf{x})$ . In particular, we need  $\mathcal{S}_{\mathcal{L}}(\mathbf{x}) := \ell_1(\mathbf{x}) + \ell_2(\mathbf{x})\ell_3^{-1}(\mathbf{x})\ell_2^\top(\mathbf{x}) \prec 0$ . Keeping in mind item 4 of Assumption 1.4.5 and the upper bounds of  $P$  and  $g$ , we get

$$\mathcal{S}_{\mathcal{L}}(\mathbf{x}) \preceq -(\lambda - \mathbf{p}_1)I + \frac{4\kappa^2 \bar{p}_x^2 \bar{g}^2}{\mu b} (H\Lambda)^\top (H\Lambda).$$

Hence, for any  $\kappa > 0$ , set

$$\mathbf{p}_1 \leq \frac{\lambda}{4}, \quad \mu \geq \frac{16\kappa^2 \bar{p}_x^2 \bar{g}^2 |H\Lambda|^2}{\lambda b}$$

so that  $\mathcal{S}_{\mathcal{L}}(\mathbf{x}) \prec 0$  and therefore  $\mathcal{L}(\mathbf{x}) \prec 0$  for all  $\mathbf{x}$ . Note that the metric  $\mathcal{P}$  has been obtained in the  $\mathbf{x} = (\chi, z)$ -coordinates. To complete the proof, we need to come back into the original coordinates  $x = (\chi, \eta)$ . This can be done with a globally Lipschitz diffeomorphism, see Proposition 1.2.10. In particular, the metric  $P$  in the original coordinates is defined as

$$P(\chi, \eta) := E^\top(\chi) \mathcal{P}(\mathbf{x}) E(\chi), \quad E(\chi) := \begin{pmatrix} I & 0 \\ -\frac{\partial \bar{M}}{\partial \chi}(\chi) & I \end{pmatrix}, \quad (1.76)$$

namely

$$P(x) = \begin{pmatrix} P_x(\chi) + \mu(H + bS) \frac{\partial \bar{M}^\top}{\partial \chi}(\chi) \frac{\partial \bar{M}}{\partial \chi}(\chi) & \star^\top \\ -\mu(H + bS) \frac{\partial \bar{M}}{\partial \chi}(\chi) & \mu(H + bS) \end{pmatrix}.$$

Note that

$$E^{-1}(\chi) = \begin{pmatrix} I & 0 \\ \frac{\partial \bar{M}}{\partial \chi}(\chi) & I \end{pmatrix},$$

and, since  $\bar{M}$  is Lipschitz,

$$|E(\chi)| \leq 1 + L_M, \quad |E^{-1}(\chi)| \leq 1 + L_M, \quad \forall \chi \in \mathbb{R}^{n_x}.$$

Hence, for all vectors  $v$ ,

$$\begin{aligned} v^\top P(\chi, \eta)v &\geq \min\{\underline{p}_\chi, \mu |H + bS|\} |E(\chi)v|^2 \\ &\geq \min\{\underline{p}_\chi, \mu |H + bS|\} \frac{|v|^2}{|E^{-1}(\chi)|^2} \\ &\geq \frac{\min\{\underline{p}_\chi, \mu |H + bS|\}}{(1 + L_M)^2} |v|^2. \end{aligned}$$

On another hand,

$$\begin{aligned} v^\top P(\chi, \eta)v &\leq \max\{\bar{p}_\chi, \mu |H + bS|\} |E(\chi)v|^2 \\ &\leq \max\{\bar{p}_\chi, \mu |H + bS|\} (1 + L_M)^2 |v|^2. \end{aligned}$$

Hence, the closed-loop system is incrementally globally exponentially stable with respect to the contraction metric  $P(\chi, \eta)$  satisfying  $\underline{p}I \preceq P(x) \preceq \bar{p}I$  and  $L_f P(x) \preceq -pI$  with  $p = \min\{\mathfrak{p}_1, \mathfrak{p}_2\}$  and

$$\begin{aligned} \bar{p} &:= \max\{\bar{p}_\chi, \mu |H + bS|\} (1 + L_M)^2, \\ \underline{p} &:= \frac{\min\{\underline{p}_\chi, \mu |H + bS|\}}{(1 + L_M)^2}. \end{aligned} \tag{1.77}$$

Note now that the closed-loop system can be written in the form

$$\dot{x} = f(x) + \beta(x)w(t)$$

where

$$f(x) = \begin{pmatrix} f_\chi(\chi) + \kappa g_\chi(\chi)[(H\Lambda)^\top(\eta - \bar{M}(\chi)) + \varrho(\chi)] \\ \Phi\eta + v(\chi) \end{pmatrix}, \quad \beta(x) = \begin{pmatrix} g_\chi(\chi) \\ R \end{pmatrix}$$

It is not hard to verify that the matrix function  $\beta$  is a Killing vector field with respect to the metric  $P(x)$  (since  $L_{g_\chi}\bar{M}(\chi)$  is constant for all  $\chi$ ). Therefore, in virtue of Theorem 1.3.3, the closed-loop system is incrementally globally uniformly exponentially ISS with respect to  $w$ , completing the proof.  $\square$

**Example 1.4.12.** We consider the example of a nonlinear liquid level control resonant circuit system described in Li et al. (2021). Such a system can be described by a nonlinear model of the form (1.62), where

$$f_\chi(\chi) = \begin{pmatrix} -\chi_1 \\ \chi_1 - \sin(\chi_2) \end{pmatrix}, \quad g_\chi(\chi) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \Phi = 0, \quad v(\chi) = -\chi_1 - \chi_2 + \sin(\chi_2).$$

First thing, we apply a preliminary feedback in order to satisfy Assumption 1.4.3. We look for a linear feedback of the form

$$\phi_0(\chi) = K_1\chi_1 + K_2\chi_2 + K_3\sin(\chi_2).$$

The choice of the nonlinear term in the preliminary feedback will be clear in a second moment. In particular, we select  $K_1 = -2.8$ ,  $K_2 = -4.4$  and  $K_3 = 3.8$ . The closed-loop system satisfies Assumption 1.4.3 with respect to the constant metric

$$P_\chi(\chi) = \begin{pmatrix} 0.7 & 1 \\ 1 & 6 \end{pmatrix}.$$

Note moreover that, since the metric is Euclidean and the vector field  $g_\chi$  is constant, the Killing vector property is automatically satisfied. Then, note that since  $\Phi = 0$ , it follows that Assumption 1.65 is automatically satisfied for any  $H \succ 0$ . We take  $H = 1$ . About Assumption 1.4.5 we pick the following choice. Since  $g_\chi$  is constant, we look for a function  $\bar{M}(\chi)$  that is linear in its argument in order to satisfy the constraint on  $L_{g_\chi}\bar{M}$  to be constant. Namely, we take

$$\bar{M}(\chi) = M_1\chi_1 + M_2\chi_2.$$

Note that, since  $L_{g_\chi}\bar{M} = M_1$  and  $n_\eta = 1$ , it's sufficient to have  $M_1 \neq 0$  to satisfy the detectability assumption on  $(\Phi, (H\Lambda)^\top) = (0, M_1)$ . Now, equation (1.66) reads

$$(M_1 \quad M_2) \begin{pmatrix} -3.8\chi_1 - 4.4\chi_2 + 3.8\sin(\chi_2) \\ \chi_1 - \sin(\chi_2) \end{pmatrix} = -\chi_1 - \chi_2 + \sin(\chi_2) + \Delta(\chi).$$

We set  $\Delta(\chi) = 0$ . With this choice, we have that (1.68) and (1.69) are automatically satisfied with  $\varrho(\chi) = 0$ . With this choice, since equation (1.66) has to be satisfied for any  $(\chi_1, \chi_2)$ , we can rewrite it in the unknowns  $M_1, M_2$  as

$$\begin{pmatrix} -3.8 & 1 \\ 1 & 0 \\ 3.8 & -1 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Note that the first and third rows are a linear combination. This implies that the solution is

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \begin{pmatrix} -3.8 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4.8 \end{pmatrix}$$

which leads to the choice

$$\bar{M}(\chi) = -\chi_1 - 4.8\chi_2.$$

Therefore, the system in closed-loop with the feedback

$$\begin{aligned} u &= \phi_0(\chi) + \kappa(H\Lambda)^\top(\eta - \bar{M}(\chi)) + \varrho(\chi) \\ &= -2.8\chi_1 - 4.4\chi_2 + 3.8\sin(\chi_2) - \kappa(\eta + \chi_1 + 4.8\chi_2) \end{aligned}$$

is incrementally globally uniformly exponentially stable for any  $\kappa > 0$ . Note that the nonlinear term in the preliminary feedback  $\phi_0$  played an important role. Indeed, thanks to such a term, Assumption 1.4.5 was satisfied with very simple choices (linear  $\bar{M}$  and  $\varrho = \Delta = 0$ ).

## 1.4.4 LMI-based design

### 1.4.4.1 Introduction and context

In this section, we focus on the analysis and on the control design for a class of nonlinear systems that can be described by a combination of a linear term plus a nonlinearity which we will assume to satisfy a monotonic or a sector-bound condition. Such a class of system has been intensively studied in the literature, see for instance Arcak et al. (2003); Zemouche and Boutayeb (2013); Fan and Arcak (2003) for the design of observers, Zames and Falb (1968); Castelan et al. (2008); Tarbouriech et al. (2011); Turner and Kerr (2014); Waitman et al. (2017) for stabilization purposes and Zhang et al. (2014, 2015); Zhao et al. (2013); Tang et al. (2018); Andrieu and Tarbouriech (2019) for multiagent synchronization and all the references therein. The interest in considering such a class of systems can be seen from the following two main aspects.

- First of all, there exist several real-life engineering systems that can be modeled as a linear part plus a nonlinearity satisfying a monotonic/sector-bound condition. Some examples can be found for instance in Reinders et al. (2022) for a mechanical ventilation machine for patient respiration support, in Tang et al. (2018) for Chua’s circuit, in Andersson et al. (2014) for a surge’s systems, in Pavlov et al. (2006) for the TORA systems, in Praly et al. (2001) for the disk inertia pendulum and in (Isidori, 1985, Section 4.10) for a link robot arm.
- The second reason is that through the monotonic/sector-bound assumption on the nonlinearity, the analysis and the control design for this class of systems simplify and tractable conditions can be derived.

In particular, for such a class of systems, it is common in the literature to rely on Linear Matrix Inequalities (LMIs). LMIs are an efficient and practical easy-to-use tools of interest in control design as they provide constructive ways to provide possible solutions to many problems. Examples of such problems are for instance stabilization Henrion and Tarbouriech (1999); Tarbouriech and Gomes da Silva Jr. (2000); Šiljak and Stipanovic (2000), observer design Arcak et al. (2003); Zemouche and Boutayeb (2013); Zemouche et al. (2008) stability region estimation Gomes da Silva Jr and Tarbouriech (2005); Pittet et al. (1997) and multiagent synchronization Zhang et al. (2015); Dal Col et al. (2018); Nguyen et al. (2017); Andrieu and Tarbouriech (2019).

In conclusion, in this section, we will study the analysis and the design of control action for systems composed of a linear term plus a nonlinearity satisfying a monotonic or sector-bound condition. The focus will be made with respect to the *uniform global incremental exponential stability* (and the uniform global incremental exponential ISS) property described in Definition 1.2.1. The objective will be to derive LMI conditions for the analysis and the control design for the considered class of systems. Along the section, several examples taken from the articles cited above will be presented to check the validity (and the feasibility) of the proposed conditions. The results that will be presented can be found in the author’s work Giaccagli et al. (2022a,b).

#### 1.4.4.2 Sufficient LMI conditions for uniform global exponential stability

In this section, we consider a nonlinear non-autonomous system described by the following differential equation

$$\begin{aligned}\dot{x} &= Ax + G\varphi(z) + Rw(t), \\ z &= Hx\end{aligned}\tag{1.78}$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $z \in \mathbb{R}^{n_z}$  is a linear combination of the state,  $w : \mathbb{R} \mapsto \mathcal{W}$  is an external signal taking values on a compact set  $\mathcal{W} \subset \mathbb{R}^{n_w}$  that includes the origin,  $A, G, R$  and  $H$  are constant matrices of appropriate dimensions and  $\varphi : \mathbb{R}^{n_z} \mapsto \mathbb{R}^{n_\varphi}$  is a  $C^1$  function. In particular we assume that the nonlinearity  $\varphi$  satisfies a *sector-bound* or a *monotonic* condition. Such an assumption is stated in the following.

**Assumption 1.4.6.** *Consider system (1.78). The function  $\varphi$  satisfies one of the following conditions.*

(A<sub>1</sub>) (**Sector-bound**): *there exist a symmetric positive definite matrix  $S \in \mathbb{R}^{n_\varphi \times n_\varphi}$ ,*

$S = S^\top \succ 0$  and a matrix  $\Omega \in \mathbb{R}^{n_\varphi \times n_z}$  such that

$$\text{He} \left\{ \frac{\partial \varphi}{\partial z}(z)^\top S \left[ \frac{\partial \varphi}{\partial z}(z) + \Omega \right] \right\} \preceq 0; \quad (1.79)$$

(A<sub>2</sub>) (**Monotonic**): there exists a symmetric positive definite matrix  $\Gamma = \Gamma^\top \succ 0$  such that

$$\frac{\partial \varphi}{\partial z}(z) = \frac{\partial \varphi^\top}{\partial z}(z), \quad 0 \preceq \text{He} \left\{ \frac{\partial \varphi}{\partial z}(z) \right\} \preceq \Gamma \quad (1.80)$$

for all  $z \in \mathbb{R}^{n_z}$ .

Figures 1.3 and 1.4 illustrate an example of a nonlinearity and its Jacobian when  $\varphi$  is a function from  $\mathbb{R}$  to  $\mathbb{R}$ . Note that, in the scalar case, the monotonic and the sector-bound conditions are equivalent.

**Remark 1.4.13.** Note that both the sector bound and the monotonic conditions imply that  $\varphi$  is a globally Lipschitz nonlinearity.

**Remark 1.4.14.** Note that in the monotonic case, we require the Jacobian of  $\varphi$  to be symmetric. This requires  $\varphi$  to be a mapping from  $\mathbb{R}^{n_\varphi}$  to  $\mathbb{R}^{n_\varphi}$ .

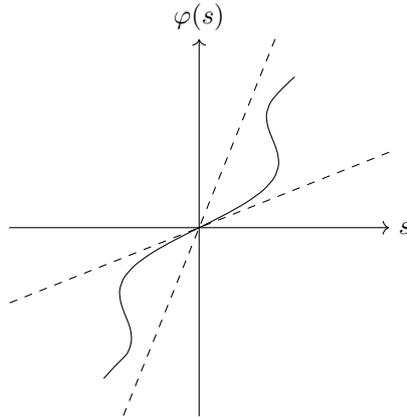


Figure 1.3: Non-linearity

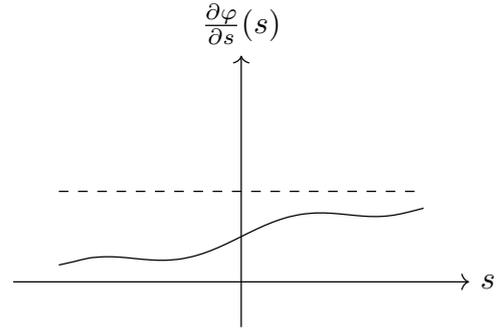


Figure 1.4: Derivative term

As previously said, in this section we aim to study the notion of incremental uniform global exponential stability of (1.78) and incremental uniform global exponential ISS with respect to the external signal  $w$ . In this sense, Theorem 1.2.2 and Theorem 1.3.3 provide respectively some sufficient conditions to claim such properties. Indeed system (1.78) is uniformly incrementally globally exponentially ISS with respect to the input  $w$ , if there exists a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_x}$  and three strictly positive real numbers  $\underline{p}, \bar{p}, \lambda > 0$  such that

$$\begin{aligned} L_f P(x) &\preceq -\lambda I, & \underline{p} I &\preceq P(x) \preceq \bar{p} I \\ L_g P(x) &= 0 \end{aligned} \quad (1.81)$$

for all  $x \in \mathbb{R}^{n_x}$ , where, for shortness of notation, we indicated with  $f(x) = Ax + G\varphi(Hx)$  and  $g(x) = B$ .

**Remark 1.4.15.** Note that, in light of Remark 1.4.13, the condition for the incremental uniform exponential stability (i.e. for  $w = 0$ ) in (1.81) is also necessary by Andrieu et al. (2016).

Equations (1.81) require the solution of a first-order differential equation in the unknown  $P(x)$ . In such a case, an explicit solution may be difficult to be computed, especially for high-dimensional systems. For this reason, we restrict ourselves to the case in which the metric is Euclidean, that is,  $P$  is a strictly positive constant matrix  $P = P^\top \succ 0$ . From such a restriction, the condition (1.81) simplifies and we have the following.

**Lemma 1.4.16.** *Consider system (1.78). If there exists a symmetric positive definite matrix  $P = P^\top \succ 0$  and a strictly positive real number  $\lambda > 0$  such that*

$$\text{He} \left( P \left[ A + G \frac{\partial \varphi}{\partial z}(z) H \right] \right) \preceq -\lambda I, \quad (1.82)$$

then the system is uniformly incrementally globally exponentially ISS with respect to  $w$ .

**Proof.** The result follows from Theorem 1.2.2 and Theorem 1.3.3 since, by taking the metric  $P$  as a constant, the Killing vector assumption  $L_g P = 0$  is automatically satisfied.  $\square$

By considering only Euclidean metrics, we restrict the set of solutions of (1.81). Because of this choice, we will neglect the possibility of obtaining necessary conditions. However, on the other hand, this will allow us to obtain tractable easy-to-use sufficient conditions, that we will express in form of LMIs. The condition (1.82) is still an infinite-dimensional LMI due to the presence of the nonlinearity  $\varphi$ . It is now where the assumption on the sector-bound/monotonic condition of  $\varphi$  will play a role.

In the following, we show a set of sufficient conditions to claim uniformly incrementally exponentially ISS with respect to  $w$  for systems of the form (1.78). First, we show some LMI-based conditions in case  $\varphi$  satisfies the sector-bound condition.

**Proposition 1.4.17** (Sufficient conditions for sector bound). *Consider system (1.78) and suppose that  $\varphi$  satisfies the sector bound assumption ( $A_1$ ) in Assumption 1.4.6. If there exist a symmetric positive definite matrix  $P \in \mathbb{R}^{n_x \times n_x}$ ,  $P = P^\top \succ 0$  and a positive real number  $\lambda > 0$  such that the following inequality holds*

$$\begin{bmatrix} A^\top P + PA + \lambda I & PG - H^\top \Omega^\top S \\ (PG - H^\top \Omega^\top S)^\top & -2S \end{bmatrix} \preceq 0, \quad (1.83)$$

then system (1.78) is incrementally uniformly globally exponentially ISS with respect to  $w$ .

**Proof.** Let us define

$$\mathcal{L} := \text{He} \left\{ P \left[ A + G \frac{\partial \varphi}{\partial z}(z) H \right] \right\} \quad (1.84)$$

$$= PA + A^\top P + PG \frac{\partial \varphi}{\partial z}(z) H + H^\top \frac{\partial \varphi^\top}{\partial z}(z) G^\top P \quad (1.85)$$

for all  $z \in \mathbb{R}^{n_z}$ . In view of Assumption 1.4.6, it yields, by pre-multiplying (respectively, post-multiplying) inequality (1.79) by  $H^\top$  (respectively,  $H$ ):

$$\mathcal{L} \preceq \mathcal{L} - H^\top \frac{\partial \varphi}{\partial z}(z)^\top S \left[ \frac{\partial \varphi}{\partial z}(z) + \Omega \right] H - H^\top \left[ \frac{\partial \varphi}{\partial z}(z) + \Omega \right]^\top S \frac{\partial \varphi}{\partial z}(z) H. \quad (1.86)$$

By defining

$$\mathcal{Q} = \begin{bmatrix} A^\top P + PA + \eta I & PG - H^\top \Omega^\top S \\ (PG - H^\top \Omega^\top S)^\top & -2S \end{bmatrix},$$

and using the definition of  $\mathcal{L}$ , inequality (1.86) gives

$$\mathcal{L} \preceq -\lambda I + \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial z}(z)H \end{bmatrix}^\top \mathcal{Q} \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial z}(z)H \end{bmatrix}.$$

Hence, if relation (1.83) is satisfied,  $\mathcal{Q} \preceq 0$  and one can conclude from previous inequality that

$$\mathcal{L} \preceq -\lambda I.$$

In other words, relation (1.82) holds and then, according to Lemma 1.4.16, system (1.78) is uniformly incrementally globally exponentially ISS with respect to input  $w$ . This concludes the proof.  $\square$

Similar considerations can be done in case  $\varphi$  satisfies the monotonic condition. This is recalled in the following.

**Proposition 1.4.18** (Sufficient conditions for monotonic). *Consider system (1.78) and assume that  $\varphi$  satisfies the monotonic condition  $(A_2)$  in Assumption 1.4.6. If there exist a symmetric positive definite matrix  $P \in \mathbb{R}^{n_x \times n_x}$ ,  $P = P^\top \succ 0$  and a strictly positive real number  $\lambda > 0$  such that*

$$\begin{bmatrix} A^\top P + PA + \lambda I & PG + H^\top \\ (PG + H^\top)^\top & -4\Gamma^{-1} \end{bmatrix} \preceq 0 \quad (1.87)$$

then system (1.78) is incrementally uniformly globally exponentially ISS with respect to  $w$ .

**Proof.** Similarly as in the proof of Proposition 1.4.17, let us define

$$\mathcal{L} := \text{He} \left\{ P \left[ A + G \frac{\partial \varphi}{\partial z}(z)H \right] \right\} \quad (1.88)$$

$$= PA + A^\top P + PG \frac{\partial \varphi}{\partial z}(z)H + H^\top \frac{\partial \varphi^\top}{\partial z}(z)G^\top P \quad (1.89)$$

Such a function  $\mathcal{L}$  defined in (1.88) can be rewritten as:

$$\mathcal{L} := PA + A^\top P + w^\top \frac{\partial \varphi}{\partial z}(z)v + v^\top \frac{\partial \varphi}{\partial z}(z)^\top w,$$

with the notation  $v = H$  and  $w = G^\top P$ . The latter gives

$$\begin{aligned} \mathcal{L} = PA + A^\top P - \frac{1}{4}(v - w)^\top \left( \frac{\partial \varphi}{\partial z}(z) + \frac{\partial \varphi^\top}{\partial z}(z) \right) (v - w) \\ + \frac{1}{4}(v + w)^\top \left( \frac{\partial \varphi}{\partial z}(z) + \frac{\partial \varphi^\top}{\partial z}(z) \right) (v + w). \end{aligned}$$

With the monotonic condition in Assumption 1.4.6, this implies

$$\mathcal{L} \preceq PA + A^\top P + \frac{1}{4}(v + w)^\top \Gamma (v + w),$$

or, equivalently,

$$\mathcal{L} \preceq -\lambda I + PA + A^\top P + pI + \frac{1}{4}(v+w)^\top \Gamma(v+w). \quad (1.90)$$

If relation (1.87) holds then one also gets by using the Schur's complement of the matrix in the left-hand term of (1.87) that

$$PA + A^\top P + \lambda I + \frac{1}{4}(H^\top + PG)\Gamma(H + G^\top P) \preceq 0$$

corresponding to the right-hand term of (1.90). In other words, we obtain

$$\mathcal{L} \preceq -\lambda I.$$

Then, one can conclude that if relation (1.87) holds then the system (1.78) is exponentially contractive in view of Lemma 1.4.16.  $\square$

**Remark 1.4.19.** *An interesting aspect of the approach developed from the monotonic condition  $(A_2)$  in Assumption 1.4.6 is that the condition (1.80) can be relaxed to*

$$0 \preceq \frac{\partial \varphi}{\partial z}(z) + \frac{\partial \varphi}{\partial z}(z)^\top, \quad \forall z \in \mathbb{R}^{n_z}. \quad (1.91)$$

*In this case, no global Lipschitz assumption is imposed on the mapping  $\varphi$ . In this case, the matrix inequality (1.87) reads*

$$\begin{bmatrix} A^\top P + PA + \lambda I & PG + H^\top \\ (PG + H^\top)^\top & 0 \end{bmatrix} \preceq 0. \quad (1.92)$$

*This implies that  $P = P^\top \succ 0$  has to satisfy the following constraints*

$$A^\top P + PA + \lambda I \prec 0, \quad H^\top = -PG.$$

*Hence, no restriction on the slope of the nonlinearity has to be imposed provided that a part of the LMI is replaced by an equality constraint.*

**Remark 1.4.20.** *Note that all the result that have been provided in Section 1.4.4.2 still hold in case  $\varphi$  is an explicit function of time  $\varphi = \varphi(z, t)$ .*

**Remark 1.4.21.** *In case one aims to obtain these properties on a specific compact invariant set rather than on the whole state space, it is sufficient to restrict the analysis on the considered set, see Corollary 1.2.5.1.*

#### 1.4.4.3 LMI conditions for state-feedback control design

The LMI-based conditions proposed in Section 1.4.4.2 focus on the analysis of systems of the form (1.78) under a monotonic/sector-bound assumption as in Assumption 1.4.6. In this section, we aim to keep a similar framework, but we want to focus on a control design problem. In particular, we focus on a nonlinear non-autonomous system of the form

$$\begin{aligned} \dot{x} &= Ax + G\varphi(z) + Bu + Rw, \\ z &= Hx \end{aligned} \quad (1.93)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $z \in \mathbb{R}^{n_z}$  is a linear combination of the state,  $u \in \mathbb{R}^{n_u}$  is a control action,  $w : \mathbb{R} \mapsto \mathcal{W}$  is an external signal taking values on a compact set  $\mathcal{W} \subset \mathbb{R}^{n_w}$  that includes the origin,  $A, G, R, B$  and  $H$  are constant matrices of appropriate dimensions and  $\varphi : \mathbb{R}^{n_z} \mapsto \mathbb{R}^{n_\varphi}$  is a  $C^1$  function that will satisfy, again, a monotonic or sector-bound assumption. The problem that we aim to solve is the following one.

**Problem 1.4.22** (State-feedback design). *Consider system (1.93). Find a  $C^1$  function  $\alpha : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  such that system in closed-loop with the control action*

$$u = \alpha(x) \quad (1.94)$$

*is incrementally uniformly globally exponentially ISS with respect to  $w$ .*

In the proposed framework, we restrict ourselves to a specific feedback control of the form

$$\alpha(x) = Kx + N\varphi(Hx) \quad (1.95)$$

where  $K$  and  $N$  are constant matrices of suitable dimensions. The main reason behind this choice is that the closed-loop system dynamics reads

$$\dot{x} = (A + BK)x + (G + BN)\varphi(Hx) + Rw. \quad (1.96)$$

With this choice, the closed-loop structure reassembles the one of (1.78). Therefore, similar considerations as in Lemma 1.4.16 can be stated, which are expressed in the following.

**Lemma 1.4.23.** *Consider system (1.93). If there exists a symmetric positive definite matrix  $P = P^\top \succ 0$ , two matrices  $K, N$  and a strictly positive real number  $\lambda > 0$  such that*

$$\text{He} \left( P \left[ A + BK + (G + BN) \frac{\partial \varphi}{\partial z}(z) H \right] \right) \preceq -\lambda I, \quad (1.97)$$

*then the system is uniformly incrementally globally exponentially ISS with respect to  $w$ .*

| **Proof.** Straightforward from Lemma 1.4.16. □

The first idea that one may have, is to re-use the conditions that have been obtained in Proposition 1.4.17 and Proposition 1.4.18 by properly substituting in the related LMIs (1.83) and (1.87) the particular choice of the matrices  $K$  and  $N$  of control action 1.94. This however would lead to two major problems:

- the control designer should select beforehand the matrices  $K$  and  $N$ , evaluate the closed-loop matrices, plug everything into the previous LMIs, and check if they admit a solution. In this case, the conditions would be just a ‘try-and-check’ result, that wouldn’t lead to a constructive design of the control;
- in case the control designer aims to keep  $K$  and  $N$  as free design parameters that would appear as an unknown in the LMIs, the final form of (1.83) and (1.87) would become Bilinear Matrix Inequalities (BMIs).

It is clear that to obtain a constructive design, the LMI conditions previously presented should be somehow revisited so that the presence of the unknowns  $K$  and  $N$  of the control action are included. However, such a modification can be done

by exploiting the properties of the Schur's complement. In case the nonlinearity  $\varphi$  in (1.93) satisfies a sector-bound property, sufficient LMI-based conditions for the design of a state-feedback control law to have incremental uniform global exponential stability with respect to  $w$  can be obtained. Such a result is shown in the following.

**Proposition 1.4.24** (Feedback design for sector bound). *Consider system (1.93). Assume that the nonlinearity  $\varphi$  satisfies the sector bound condition  $(A_1)$  in Assumption 1.4.6. If there exist a symmetric positive definite matrix  $W \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $Z \in \mathbb{R}^{n_u \times n_x}$ ,  $N \in \mathbb{R}^{n_u \times n_\varphi}$  and a positive real number  $\nu$  such that the following LMI holds*

$$\text{He} \left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ G^\top + N^\top B^\top - S\Omega HW & -S & 0 \\ W & 0 & -\frac{\nu}{2}I \end{bmatrix} \right\} \leq 0, \quad (1.98)$$

then the system (1.93) in closed-loop with the control action (1.94) with  $K = ZW^{-1}$  and such a  $N$  is incrementally globally uniformly exponentially ISS with respect to  $w$ .

**Proof.** By applying the Schur complement to relation (1.98), and by pre- and post-multiplying the resulting one by  $\text{blkdiag}\{P, I\}$  where  $P = W^{-1}$  and  $K = ZW^{-1}$ , we obtain

$$\begin{bmatrix} \text{He}\{P(A + BK)\} + \nu^{-1}I & P(G + BN) - H^\top \Omega^\top S \\ (P(G + BN) - H^\top \Omega^\top S)^\top & -2S \end{bmatrix} \prec 0.$$

By denoting  $\nu^{-1} = \lambda$ , one retrieves relation (1.83) of Proposition 1.4.17. Hence, by applying Proposition 1.4.17 to the closed-loop system (1.93), (1.94), we conclude that the closed-loop system is incrementally globally uniformly exponentially ISS with respect to the input  $w$ .  $\square$

Similar considerations can be done in case  $\varphi$  satisfies the monotonic condition. This is shown in the following.

**Proposition 1.4.25** (Feedback design for monotonic). *Consider system (1.93). Assume that the nonlinearity  $\varphi$  satisfies the monotonic condition in  $(A_2)$  of Assumption 1.4.6. If there exist a symmetric positive definite matrix  $W \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $Z \in \mathbb{R}^{n_u \times n_x}$ ,  $N \in \mathbb{R}^{n_u \times n_\varphi}$  and a positive real number  $\nu > 0$  such that the following LMI holds*

$$\text{He} \left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ HW + (G + BN)^\top & -2\Gamma^{-1} & 0 \\ W & 0 & -\frac{\nu}{2}I \end{bmatrix} \right\} \preceq 0, \quad (1.99)$$

then the closed-loop system (1.93), (1.94) with  $K = ZW^{-1}$  and such a  $N$  is incrementally globally uniformly exponentially ISS with respect to  $w$ .

**Proof.** By applying the Schur complement to relation (1.99), and by pre- and post-multiplying the resulting one by  $\text{blkdiag}\{P, I\}$  with  $P = W^{-1}$  and  $K = ZW^{-1}$ , we obtain

$$\begin{bmatrix} \text{He}\{P(A + BK)\} + \nu^{-1}I & P(G + BN) + H^\top \\ (P(G + BN) + H^\top)^\top & -4\Gamma^{-1} \end{bmatrix} \preceq 0.$$

By denoting  $\nu^{-1} = \lambda$ , one retrieves relation (1.87) of Proposition 1.4.18. Hence, by Proposition 1.4.18 to the closed-loop system (1.93), (1.94), we conclude that the closed-loop system is incrementally globally uniformly exponentially ISS with respect to the input  $w$ .  $\square$

**Remark 1.4.26.** *Similarly considerations as the one of Remark 1.4.19 and Remark 1.4.20 can be done in this case. Indeed if the function  $\varphi$  satisfies the inequality (1.91), the LMI condition (1.99) is transformed into an equality constraint of the form*

$$\begin{bmatrix} \text{He}\{AW + BZ\} & W \\ W^\top & -\nu I \end{bmatrix} \preceq 0, \quad W \succ 0, \quad (1.100)$$

$$WH^\top = -(G + BN). \quad (1.101)$$

*Similarly, the same conditions hold if  $\varphi$  is explicitly dependent on time. Furthermore, similar results can be obtained by restricting the analysis to desired forward invariant compact sets.*

**Example 1.4.27.** *Consider the surge subsystem of an axial compressor (see for instance (Andersson et al., 2014, Eq. (2))), with system dynamics described by*

$$\begin{cases} \dot{x}_1 &= -x_2 - \frac{3}{2}x_1^2 - \frac{1}{2}x_1^3 \\ \dot{x}_2 &= x_1 - u, \end{cases} \quad (1.102)$$

*with  $x = (x_1, x_2) \in \mathbb{R}^2$ . Following Arcak et al. (2003), we can rewrite this system as*

$$\begin{cases} \dot{x}_1 &= -x_2 + \ell x_1 - \varphi_\ell(x_1) \\ \dot{x}_2 &= v \end{cases}, \quad (1.103)$$

*where*

$$\varphi_\ell(z) = \ell z + \frac{3}{2}z^2 + \frac{1}{2}z^3 \quad (1.104)$$

*and where  $\ell \in \mathbb{R}$  is a free parameter,  $u = x_1 - v$  and  $v$  is an additional input. It is shown in Arcak et al. (2003) that  $\varphi_\ell$  satisfies the sector condition  $\varphi_\ell(z)z \geq 0$  if  $\ell \geq \frac{9}{8}$ . If  $\ell \geq 6$ , it yields*

$$\frac{\partial \varphi_\ell}{\partial z}(z) = \ell + 3z + \frac{3}{2}z^2 \geq \ell - 6. \quad (1.105)$$

*Hence, equation (1.91) is satisfied and we may try to apply the former approach. In particular, for all  $\ell \geq 6$ , there exist  $K$  in  $\mathbb{R}^{1 \times 2}$  and  $N$  in  $\mathbb{R}$ , such that the feedback law*

$$v = Kx + N\varphi_\ell(x_1), \quad (1.106)$$

*makes the system (1.103) incrementally uniformly globally exponentially stable. Note that it can be shown that the conditions (1.100)-(1.101) are always feasible. Indeed, the system (1.102) is in the form (1.93) with*

$$A = \begin{bmatrix} \ell & -1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \quad H = [1 \quad 0], \quad R = 0. \quad (1.107)$$

*Then, letting*

$$W = \begin{bmatrix} w_{11} & w_{12} \\ w_{12} & w_{22} \end{bmatrix}, \quad Z = [z_1 \quad z_2],$$

the equation (1.101) is solved with  $w_{11} = 1$  and  $w_{12} = -N$ . Now, take  $w_{22}$  sufficiently large ( $w_{22} - N^2 > 0$ ) such that  $W$  is positive definite. Then, by ignoring the term  $\nu$  and selecting  $w_{11} = 1$ , inequality (1.100) becomes

$$\text{He}\{AW + BZ\} = \text{He} \left\{ \begin{bmatrix} \ell + N & -N\ell - w_{22} \\ z_1 & z_2 \end{bmatrix} \right\}. \quad (1.108)$$

By taking  $N < -\ell < 0$ , this matrix can be always made negative definite provided  $z_2$  is taken sufficiently negative. For example, one can choose  $z_1 = w_{22}$  and  $z_2 \geq \frac{\ell^2 N^2}{4|l+N|}$ . Applying Proposition 1.4.25, the result follows. The introduction of the nonlinearity explicitly appearing in the controller, in this case, plays an important role. Note indeed that in case  $N = 0$ , the LMIs have no solution since the top-left term of (1.108) reduces to  $2\ell$  and therefore is always positive definite.

#### 1.4.4.4 Incremental infinite gain margin control design

The results that we have presented so far in this section, provide a set of tools for the analysis and design that can be used in practical applications. For this, we focused on a class of systems with several real-life examples and we derived tractable and easy-to-use conditions that we expressed in LMI form. This has been remarked in Section 1.4.4.1. However, most of the control designs that exist in literature and that look towards a practical implementation, focus on the design of *robust* controllers. In the most general case, with ‘robust’ we mean that the stabilization problem that the control action aims to solve (that in our case will be an *incremental stabilization* problem) has to be achieved in case the plant to be stabilized is not perfectly known.

In this section, we aim to extend the control action proposed in Section 1.4.4.3 to guarantee some form of robustness to our design. To do this, we focus on the notion of *infinite gain margin* control laws. Such a concept is not new in control theory for stabilization purposes. A detailed analysis of such a property can be found for instance in the manuscript (Sepulchre et al., 2012, Section 3). Roughly speaking, a stabilizing control action  $u = \alpha(\cdot)$  is of infinite gain margin if the system in closed-loop with a control of the form  $u = \kappa\alpha(\cdot)$  has an equilibrium point that is stable for any gain  $\kappa \geq 1$ . Such a property is of interest as closed-loop stability is preserved in presence of static and/or unmodeled fast dynamics plant uncertainties. This robustness property is guaranteed because infinite gain margin feedbacks stabilize a given plant always in the ‘right direction’ and therefore can be re-scaled by any positive (large enough) parameter. For linear systems, this is translated to the fact that the Nyquist plot of the transfer function between the control and the state never crosses the negative part of the real axis.

In this section, we aim to extend the concept of infinite gain margin to the design of incrementally uniform exponential stabilizing control actions. To the author’s knowledge, such a concept has only been exploited in Sanfelice and Praly (2011) for the design of observers, but never for the design of incremental control actions. We have therefore the following definition.

**Definition 1.4.1** (Incremental infinite gain margin). *Consider a system of the form*

$$\dot{x} = f(x, t) + g(x, t)u$$

where  $x \in \mathbb{R}^{n_x}$ ,  $u \in \mathbb{R}^{n_u}$  and  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$ ,  $g : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_u}$  are both  $C^2$  in their first argument and piecewise continuous in the second. We say that the  $C^2$

function  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_u}$  is an incremental infinite gain margin control law if there exist a  $C^1$  symmetric and positive definite matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  and three strictly positive real numbers  $\underline{p}, \bar{p}, \lambda > 0$  such that, by letting

$$f_\kappa(x, t) := f(x, t) + \kappa g(x, t) \alpha(x, t),$$

we have that

$$L_{f_\kappa} P(x, t) \preceq -\lambda I, \quad \underline{p}I \preceq P(x, t) \preceq \bar{p}I$$

for all  $\kappa \geq 1$  and for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ .

**Remark 1.4.28.** Note that Definition 1.4.1 is uniform with respect to  $P$ , that is, the metric does not depend on the gain  $\kappa$ .

We focus again on the class of systems defined by (1.93) with  $\varphi$  satisfying a monotonic or a sector-bound assumption. By keeping the same control structure (1.94), we aim to derive sufficient conditions based on LMIs to have an incremental infinite gain margin state-feedback control law. If we consider Euclidean metrics  $P = P^\top \succ 0$ , by means of Lemma 1.4.23 the incrementally uniformly globally exponential ISS property with respect to  $w$  is automatically guaranteed if the system with  $w = 0$  is incrementally uniformly globally exponentially stable. Therefore, the problem that we aim to solve is the following.

**Problem 1.4.29** (Infinite gain margin state-feedback design). Consider system (1.93). Find matrices  $K, N$  such that the state-feedback control (1.94) is an incremental infinite gain margin control law for system (1.93), namely, there exist  $P = P^\top \succ 0$  and  $\lambda > 0$  such that, the following holds

$$\text{He} \left\{ P \left[ (A + \kappa BK) + (G + \kappa BK) \frac{\partial \varphi}{\partial y} (Hx) H \right] \right\} \preceq -\lambda I \quad (1.109)$$

for all  $x \in \mathbb{R}^{n_x}$  and for all  $\kappa \geq 1$ .

Of course, the problem is trivially satisfied if the system with no control action, i.e.  $u = 0$  is already incrementally uniformly globally exponentially stable. However, following the same lines as in Section 1.4.4.3, some LMI conditions for the control law design can be derived. If the nonlinearity  $\varphi$  satisfies a sector-bound condition, this is expressed in the following.

**Proposition 1.4.30** (Infinite gain margin state-feedback design for sector-bound). Consider system (1.93) and assume that  $\varphi$  satisfies the incremental sector-bound condition ( $A_1$ ) in Assumption (1.4.6) for some matrices  $\Omega$  and  $S = S^\top \succ 0$ . If there exist a matrix  $W \in \mathbb{R}^{n_x \times n_x}$ ,  $W = W^\top \succ 0$ , two matrices  $Z \in \mathbb{R}^{n_u \times n_x}$ ,  $N \in \mathbb{R}^{n_u \times n_\varphi}$  and a real number  $\nu > 0$  such that the following LMIs hold

$$\text{He} \left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ G^\top + N^\top B^\top - S\Omega HW & -S & 0 \\ W & 0 & -\frac{\nu}{2}I \end{bmatrix} \right\} \preceq 0, \quad (1.110)$$

$$\text{He} \left\{ \begin{bmatrix} BZ & 0 \\ N^\top B^\top - S\Omega HW & -S \end{bmatrix} \right\} \preceq 0, \quad (1.111)$$

then Problem 1.4.29 is solved with (1.94) where  $K = ZW^{-1}$  and such a  $N$ .

**Proof.** The closed-loop system can be written in the form

$$\begin{aligned}\dot{x} &= (A + \kappa BK)x + (G + \kappa BN)\varphi(Hx) \\ &= (A + BK)x + (G + BN)\varphi(Hx) \\ &\quad + (\kappa - 1)[BKx + BN\varphi(Hx)]\end{aligned}\tag{1.112}$$

Let

$$\begin{aligned}\xi(x) &= (A + BK)x + (G + BN)\varphi(Hx) \\ \zeta(x, \kappa) &= (\kappa - 1)[BKx + BN\varphi(Hx)].\end{aligned}\tag{1.113}$$

By Proposition 1.4.24, if (1.110) holds, then  $\xi(x)$  is incrementally uniformly globally exponentially stable with respect to the metric  $P = W^{-1}$  for  $K = ZW^{-1}$  and such  $N$ . To conclude the proof, by the linearity of the Lie derivative, we have to show that  $\zeta$  satisfies  $L_\zeta P(x, \kappa) \preceq 0$  for all  $\kappa \geq 1$  with respect to the same  $K$  and  $P$ . Then, by employing the sector bound condition (1.79), we have that, for  $\kappa \geq 1$  and for any vector  $b \in \mathbb{R}^{n_x}$ ,

$$b^\top L_\zeta P(x, \kappa)b \leq b^\top L_\zeta P(x, \kappa)b + (\kappa - 1)b^\top (\Psi(x) + \Psi^\top(x))b$$

with

$$\Psi(x) = H^\top \frac{\partial \varphi}{\partial z}(z) S \left[ \frac{\partial \varphi}{\partial y}(y) + \Omega \right] H.$$

From the definition of  $\zeta$  in (1.113), we get

$$b^\top L_\zeta P(x, \kappa)b \leq (\kappa - 1)b^\top \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial z}(z) \end{bmatrix}^\top \Lambda \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial z}(z) \end{bmatrix} b$$

with

$$\Lambda := \text{He} \left\{ \begin{bmatrix} PBK & 0 \\ N^\top B^\top - PS\Omega H & -S \end{bmatrix} \right\}.$$

By pre and post-multiplying (1.110) by  $\text{blkdiag}\{W, I\}$ , we retrieve  $\Lambda$ . Hence  $\Lambda \succeq 0$  and so  $b^\top L_\zeta P(x, \kappa)b \leq 0$  for all  $b, x$  and  $\kappa \geq 1$ , concluding the proof.  $\square$

Similar considerations can be done in case  $\varphi$  satisfies a monotonic condition. In such a case, we have the following proposition.

**Proposition 1.4.31** (Infinite gain margin state-feedback design for monotonic). *Consider system (1.93) and assume that  $\varphi$  satisfies the monotonic condition  $(A_2)$  in Assumption 1.4.6 for some matrix  $\Gamma = \Gamma^\top \succ 0$ . If there exist a matrix  $W \in \mathbb{R}^{n_x \times n_x}$ ,  $W = W^\top \succ 0$ , two matrices  $Z \in \mathbb{R}^{n_u \times n_x}$ ,  $N \in \mathbb{R}^{n_u \times n_\varphi}$  and a real number  $\nu > 0$  such that the following LMIs hold*

$$\text{He} \left\{ \begin{bmatrix} AW + BZ & 0 & 0 \\ HW + (G + BN)^\top & -2\Gamma^{-1} & 0 \\ W & 0 & -\frac{\nu}{2}I \end{bmatrix} \right\} \preceq 0,\tag{1.114}$$

$$\text{He} \left\{ \begin{bmatrix} BZ & 0 \\ HW + (BN)^\top & -2\Gamma^{-1} \end{bmatrix} \right\} \preceq 0,\tag{1.115}$$

then Problem 1.4.29 is solved with (1.94) where  $K = ZW^{-1}$  and such a  $N$ .

**Proof.** The closed-loop system can be written in the form

$$\begin{aligned}\dot{x} &= (A + \kappa BK)x + (G + \kappa BN)\varphi(Hx) \\ &= (A + BK)x + (G + BN)\varphi(Hx) \\ &\quad + (\kappa - 1)[BKx + BN\varphi(Hx)]\end{aligned}\tag{1.116}$$

Let

$$\begin{aligned}\xi(x) &= (A + BK)x + (G + BN)\varphi(Hx) \\ \zeta(x, \kappa) &= (\kappa - 1)[BKx + BN\varphi(Hx)]\end{aligned}\tag{1.117}$$

By Proposition 1.4.25, if (1.114) admits a solution, then  $\xi(x)$  defines a contraction with respect to the metric  $P = W^{-1}$  for  $K = ZW^{-1}$  and  $N$ . To conclude the proof, by the linearity of the Lie derivative, we have to show that  $\zeta$  satisfies  $L_\zeta P(x, \kappa) \preceq 0$  for all  $\kappa \geq 1$  with respect to the same  $K$  and  $P$ . As a consequence, compute, for any vector  $b \in \mathbb{R}^{n_x}$ ,

$$\begin{aligned}b^\top L_\zeta P(x, \kappa)b &= (\kappa - 1)b^\top \text{He} \left\{ PBK + PBN \frac{\partial \varphi}{\partial z}(z)H \right\} b \\ &= (\kappa - 1)b^\top \left( PBK + (BK)^\top P - \frac{1}{4}(PBN - H^\top) \text{He} \left\{ \frac{\partial \varphi}{\partial z}(z) \right\} ((BN)^\top P - H) \right. \\ &\quad \left. + \frac{1}{4}(PBN + H^\top) \text{He} \left\{ \frac{\partial \varphi}{\partial z}(z) \right\} ((BN)^\top P + H) \right) b \\ &\leq (\kappa - 1)b^\top (\text{He}\{PBK\})b + \frac{1}{4}b^\top (PBN + H^\top) \Gamma ((BN)^\top P + H)b \\ &= (\kappa - 1)b^\top \begin{bmatrix} PBK + (BK)^\top P & (PBN + H^\top) \\ (H + (BN)^\top P) & 4\Gamma^{-1} \end{bmatrix} b.\end{aligned}$$

By left and right-multiplying by  $\text{blkdiag}\{W, I\}$ , we finally obtain

$$\begin{aligned}b^\top L_\zeta P(x, \kappa)b &\leq b^\top (\kappa - 1) \text{He} \left\{ \begin{bmatrix} BZ & 0 \\ HW + (BN)^\top & -2\Gamma^{-1} \end{bmatrix} \right\} b \\ &\leq 0\end{aligned}$$

for all  $b$  and all  $\kappa \geq 1$  by (1.115), concluding the proof.  $\square$

#### 1.4.4.5 Optimality of infinite-gain margin control laws

For linear systems of the form

$$\dot{x} = Ax + Bu\tag{1.118}$$

with  $A, B$  being constant matrices, the infinite gain margin property is linked to the concept of passivity (see (Sepulchre et al., 2012, Section 3.5.3)). Indeed in such a case, the construction of an infinite gain margin stabilizing control law boils down to the solution of an Algebraic Riccati Equation (ARE)

$$PA + A^\top P - PBR^{-1}B^\top P = -Q\tag{1.119}$$

where  $P = P^\top \succ 0$  and  $R = R^\top \succ 0$  are strictly positive matrices and  $Q = Q^\top \succeq 0$  is a semi-positive definite matrix. In such a case, the control action (1.94) is

$$u = Kx, \quad K = R^{-1}B^\top P\tag{1.120}$$

On the other hand, the solution of the ARE (1.119) is linked to a (infinite horizon) Linear Quadratic Regulator (LQR) problem. Given a strictly positive matrix  $R = R^\top \succ 0$  and a semi-positive definite matrix  $Q = Q^\top \succeq 0$ , the objective is to find a control action  $u$  such that the cost function  $\mathcal{J} : \mathbb{R}^{n_x} \times \mathcal{L}^p(\mathbb{R}_{\geq 0}) \mapsto \mathbb{R}_{\geq 0}$  defined as

$$\mathcal{J}(x, u) = \int_0^\infty \mathcal{X}^\top(x, u, t) Q \mathcal{X}(x, u, t) + u^\top(t) R u(t)$$

subject to (1.118) is minimized. Again, the *optimal* control action is given by (1.120), with the matrix  $P$  solution of the ARE (1.119).

For nonlinear systems, some results have been recently developed in Faulwasser and Kellett (2021), where the relations between infinite time-horizon optimal control, dissipativity, and transversality are studied. To the knowledge of the author of this manuscript, however, the link between infinite gain margin and optimality in an incremental framework has not been studied. In the last part of this section, we aim to study such a relation for the class of systems (1.93), where we take  $w = 0$  for the sake of simplicity. Even in an incremental framework, we can show that a strong link between infinite gain margin control action and optimality exists. For this, define the cost function  $\mathcal{J} : \mathbb{R}^{n_x} \times \mathcal{L}^p(\mathbb{R}_{\geq 0}) \mapsto \mathbb{R}_{\geq 0}$  of the form

$$\mathcal{J}(x, u) := \int_0^\infty \mathcal{Q}(\mathcal{X}(x, u, t)) + u(t)^\top \mathcal{R}(\mathcal{X}(x, u, t))^{-1} u(t) dt \quad (1.121)$$

where  $\mathcal{X}(x, u, t)$  is the solution of (1.93) for some chosen control action  $u$  and for  $w(t) = 0$  for all  $t \geq t_0$ ,  $\mathcal{Q} : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  is a semi-positive definite function and  $\mathcal{R} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u \times n_u}$  is a matrix-valued function such that  $\mathcal{R}(x) > 0$  for all  $x$  in  $\mathbb{R}^{n_x}$ . In particular, we can show that, under a monotonic assumption with no right-hand upper bound as in (1.91) of the nonlinearity  $\varphi$ , we have that:

- an incremental infinite gain margin control law can be obtained under the solution of some LMIs;
- it is possible to construct a cost function of the form (1.121) such that, if these LMIs admit a solution, the incremental infinite gain margin control law minimizes (1.121).

Such a result is formalized in the following.

**Theorem 1.4.32** (Incremental infinite gain margin and optimality). *Consider system (1.93) where  $w = 0$  and with  $\varphi$  satisfying the monotonic condition (1.91). Assume that there exist two symmetric positive definite matrices  $P \in \mathbb{R}^{n_x \times n_x}$  and  $R \in \mathbb{R}^{n_u \times n_u}$ , two matrices  $M \in \mathbb{R}^{n_x \times n_\varphi}$ ,  $N \in \mathbb{R}^{n_u \times n_\varphi}$ , and three real numbers  $(\mu_1, \mu_2)$  and  $\lambda > 0$  such that*

$$\begin{aligned} \text{He} \left\{ P \left( A + M \frac{\partial \varphi}{\partial z}(z) H \right) \right\} - P B R B^\top P &\preceq -\lambda I, \\ P(G - M) = \mu_1 H^\top, \quad P B N = \mu_2 H^\top. \end{aligned} \quad (1.122)$$

Then there exists a positive real number  $\gamma$  such that the control law

$$\alpha(x) = -\gamma (R B^\top P x + \mu_2 N \varphi(Hx)), \quad (1.123)$$

is a solution to Problem 1.4.29, namely the control action (1.94), (1.123) is an incrementally infinite gain margin control law. Moreover if  $\varphi(0) = 0$ , then for all  $\kappa \geq 2$  there exist  $\mathcal{Q} : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  and  $\mathcal{R} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u \times n_u}$  both depending on  $\kappa$  such that  $u = \kappa \alpha(x)$  minimizes the cost function (1.121).

**Proof.** Noting that the control law  $\alpha(x)$  defined in (1.123) depends on the parameter  $\gamma$ , we define

$$\begin{aligned} f(x) &:= Ax + G\varphi(Hx), & \mathfrak{L}(\gamma, x) &:= P \frac{\partial f_\gamma}{\partial x}(x) + \frac{\partial f_\gamma^\top}{\partial x}(x)P, \\ f_\gamma(x) &:= Ax + G\varphi(Hx) + B\alpha(x). \end{aligned} \quad (1.124)$$

**About infinite gain margin.** Let  $\gamma$  be defined as

$$\gamma := \max \left\{ \frac{\mu_1}{\mu_2^2}, \frac{1}{2} \right\}. \quad (1.125)$$

Using the definition of  $\mathfrak{L}$  in (1.124), we obtain

$$\mathfrak{L}(\gamma, x) := \text{He} \left\{ P \left[ A + G \frac{\partial \varphi}{\partial z}(z) H \right] \right\} - 2\gamma PBRB^\top P - \gamma \mu_2 \text{He} \left\{ PBN \frac{\partial \varphi}{\partial z}(z) H \right\}.$$

Adding and subtracting the term  $M \frac{\partial \varphi}{\partial z}(z) H$  to the right-hand side of the above definition of  $\mathfrak{L}$  yields

$$\begin{aligned} \mathfrak{L}(\gamma, x) &= \text{He} \left\{ P \left[ A + M \frac{\partial \varphi}{\partial z}(z) H \right] \right\} + \text{He} \left\{ P(G - M) \frac{\partial \varphi}{\partial z}(z) H \right\} - 2\gamma PBRB^\top P \\ &\quad - \gamma \mu_2 \text{He} \left\{ PBN \frac{\partial \varphi}{\partial z}(z) H \right\}. \end{aligned}$$

Furthermore, using the second line in (1.122), the latter gives

$$\begin{aligned} \mathfrak{L}(\gamma, x) &= \text{He} \left\{ P \left( A + M \frac{\partial \varphi}{\partial z}(z) H \right) \right\} - 2\gamma PBRB^\top P \\ &\quad + (\mu_1 - \gamma \mu_2^2) H^\top \text{He} \left\{ \frac{\partial \varphi}{\partial z}(z) \right\} H. \end{aligned} \quad (1.126)$$

Using the first line of (1.122), it implies for all  $(\gamma, x)$

$$\mathfrak{L}(\gamma, x) \preceq -\lambda I - (2\gamma - 1)PBRB^\top P + (\mu_1 - \gamma \mu_2^2) H^\top \text{He} \left\{ \frac{\partial \varphi}{\partial z}(z) \right\} H. \quad (1.127)$$

Hence, with  $\gamma$  defined in (1.125), and using inequality (1.91), it follows that the right-hand term of (1.127) is negative definite for all  $x$ . Consequently, the control action (1.94) with (1.123) makes the closed-loop system incrementally uniformly globally exponentially stable. Note moreover that for all  $\kappa \geq 1$ ,  $u = \kappa \alpha(x)$  makes also the closed-loop system incrementally uniformly globally exponentially stable. Hence, (1.123) is a control action with an infinite gain margin.

**About optimality.** The idea of the proof is first to show that the Lyapunov function  $V(x) = x^\top P x$  is the value function associated with an optimal control problem and satisfies a Hamilton-Jacobi-Bellman (HJB) equation. Second, one

proves that the control law  $u = \kappa\alpha(x)$  is the optimal one. To this end, note that since  $\varphi(0) = 0$ , by the Mean Value Theorem, we have that

$$\varphi(Hx) = \int_0^1 \frac{\partial \varphi}{\partial z}(sz) ds Hx \quad (1.128)$$

for all  $x \in \mathbb{R}^{n_x}$ . Hence, the function  $\alpha$  given in (1.123) reads

$$\alpha(x) = -\gamma \left( RB^\top Px + \mu_2 N \int_0^1 \frac{\partial \varphi}{\partial z}(sz) ds Hx \right). \quad (1.129)$$

Consequently, employing  $PBN = \mu_2 H^\top$  from (1.122), the following equation may be obtained.

$$\begin{aligned} \kappa\alpha(x) &= -\kappa\gamma RB^\top Px - \kappa\gamma\mu_2 N \int_0^1 \frac{\partial \varphi}{\partial z}(sz) ds Hx \\ &= -\kappa\gamma \left[ R + \mu_2 N \int_0^1 \frac{\partial \varphi}{\partial z}(sz) ds N^\top \right] B^\top Px \\ &= -\frac{1}{2} \mathcal{R}(x) (2B^\top Px), \end{aligned}$$

where  $\mathcal{R} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u \times n_u}$  depends on  $\kappa$  and is defined as

$$\mathcal{R}(x) := \kappa\gamma \left( R + \mu_2 N \int_0^1 \frac{\partial \varphi}{\partial z}(sz) ds N^\top \right).$$

Note that, since  $\varphi$  has a symmetric Jacobian and satisfies (1.91), we have that  $\mathcal{R}$  takes symmetric positive definite values. Then, recalling the definition of  $f$  given in (1.124), let the function  $\mathcal{Q} : \mathbb{R}^{n_x} \mapsto \mathbb{R}$  be defined as

$$\mathcal{Q}(x) := -L_f V(x) + \frac{1}{4} (2x^\top PB) \mathcal{R}(x) (2B^\top Px), \quad (1.130)$$

The former equation (1.130) is a HJB equation with respect to the cost function (1.121) if  $\mathcal{Q}$  takes positive semi-definite values. Recalling the definitions of  $V$  and  $f$ , the term  $L_f V$  is computed as

$$L_f V(x) = x^\top (PA + A^\top P)x + \text{He} \left\{ x^\top PG\varphi(Hx) \right\},$$

which gives by using (1.128)

$$L_f V(x) = x^\top \left[ \int_0^1 \text{He} \left\{ P \left[ A + G \frac{\partial \varphi}{\partial z}(sz) H \right] \right\} ds \right] x.$$

Moreover,

$$x^\top PBR\mathcal{R}(x)B^\top Px = \kappa\gamma x^\top \left[ \int_0^1 \left( PBRB^\top P + \frac{\mu_2}{2} \text{He} \left\{ PBN \frac{\partial \varphi}{\partial z}(sz) H \right\} \right) ds \right] x.$$

Hence, using the definition of  $\mathfrak{L}$  given in (1.124) and equation (1.126), we obtain

$$\mathcal{Q}(x) = -x^\top \left( \int_0^1 \mathfrak{L}(\kappa\gamma, sx) ds \right) x - x^\top PBR\mathcal{R}(x)B^\top Px .$$

With (1.127), it implies

$$-\mathfrak{L}(\kappa\gamma, x) \succeq \lambda I + (2\kappa\gamma - 1)PBRB^\top P - (\mu_1 - \kappa\gamma\mu_2^2)H^\top \text{He} \left\{ \frac{\partial\varphi}{\partial z}(z) \right\} H .$$

Hence,

$$\mathcal{Q}(x) \succeq \lambda|x|^2 + (\kappa\gamma - 1)x^\top PBRB^\top Px - \left( \mu_1 - \kappa\gamma\frac{\mu_2^2}{2} \right) H^\top \text{He} \left\{ \frac{\partial\varphi}{\partial z}(z) \right\} H . \quad (1.131)$$

With the definition of  $\gamma$  in (1.125), previous inequality implies

$$\mathcal{Q}(x) \succeq \lambda|x|^2 + \left( \frac{\kappa}{2} - 1 \right) x^\top PBRB^\top Px + \mu_1 \left( \frac{\kappa}{2} - 1 \right) H^\top \text{He} \left\{ \frac{\partial\varphi}{\partial z}(z) \right\} H . \quad (1.132)$$

Consequently  $\mathcal{Q}$  is a positive definite function if  $\kappa \geq 2$ . Hence,  $u = \kappa\alpha(x)$  minimizes the cost function (1.121).  $\square$

**Remark 1.4.33.** Using the monotonic condition (1.91), the conditions in (1.122) can be also rewritten as

$$\begin{aligned} WA^\top + AW - BRB^\top &< 0, \\ G = WH^\top, \quad BN = \mu_2WH^\top, \end{aligned}$$

with unknown  $W = W^\top \succ 0$ ,  $\mu_2$  and  $N$  so that inequality (1.122) is satisfied with  $P = W^{-1}$ ,  $M = G$ ,  $\mu_1 = 0$ .

**Remark 1.4.34.** If one includes the upper-bound in the monotonic condition, that is, consider (1.80), the condition (1.122) can be written as

$$\begin{aligned} \begin{bmatrix} A^\top P + PA - PBRB^\top P + \lambda I & H^\top + PM \\ H + M^\top P & -4\Gamma^{-1} \end{bmatrix} &\preceq 0, \\ P(G - M) = \mu_1 H^\top, \quad PBN = \mu_2 H^\top. \end{aligned}$$

Again, with the choice  $P = W^{-1}$ ,  $M = G$ ,  $\mu_1 = 0$ , by matrix manipulation one can recover the conditions given in Proposition 1.4.31, namely the LMIs (1.114), (1.115).

**Remark 1.4.35.** If  $\varphi$  is a function satisfying (1.91) and the sector-bound condition (1.79) rather than the monotonic condition (1.80), similar LMIs conditions can be derived.

**Example 1.4.36.** To show that the proposed design for incremental infinite gain margin control law can be applied to practical case studies, in this example we consider a single-link robot arm system (see (Isidori, 1985, Section 4.10)). Its model is

given in the form (1.93) with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k+mgdl}{J_2} & -\frac{F_2}{J_2} & \frac{k}{J_2b} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{J_1b} & 0 & -\frac{k}{J_2b} & -\frac{F_1}{J_1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ -\frac{mgd}{J_2} \\ 0 \\ 0 \end{bmatrix},$$

$$H = [1 \ 0 \ 0 \ 0] \quad \varphi(s) = \cos(s) + \ell s, \quad R = 0$$

where  $k = 0.4, m = 0.8, g = 9.81, d = 0.6, J_2 = 0.2, F_2 = 0.15, b = 2, J_1 = 0.15, F_1 = 0.1$  are plant parameters and  $\ell$  is a degree of freedom. Note that if  $\ell > 1$ , then the monotonic condition (1.80) holds with  $\Gamma = \frac{\ell+1}{2}$ . With  $\ell = 1.01$ , it's possible to check that the LMIs (1.114), (1.115) are satisfied with  $N = 6.15 \cdot 10^3$  and

$$K = [-1.03 \cdot 10^6 \quad -0.35 \cdot 10^6 \quad -0.5 \cdot 10^8 \quad -2.22 \cdot 10^3].$$

We obtain that the closed-loop system with the control law  $u = \kappa(Kx + \varphi(Hx))$  for any  $\kappa \geq 1$  is a contraction. Figure 1.5 shows the dynamics of the closed-loop system for different values of  $\kappa$ . As the closed-loop system is a contraction, the trajectories converge to a globally exponentially stable equilibrium point, that is the origin.

#### 1.4.4.6 Uniformly incrementally stable observers and a nonlinear separation principle

In this section, we consider the case in which the dynamical system (1.93) is complemented with a measured output of the form

$$y = Cx \tag{1.133}$$

where  $y \in \mathbb{R}^{n_y}$  and  $C$  is a matrix of appropriate dimensions. Incremental stability properties have been deeply investigated for the design of observers. See for instance the recent survey Bernard et al. (2022) and more in particular Sanfelice and Praly (2011, 2015, 2021); Andrieu et al. (2020). In our context, following Arcak and Kokotović (2001), we consider an observer of the form

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(y - C\hat{x}) + G\varphi(\hat{z}), \\ \hat{z} &= H\hat{x} - E(y - C\hat{x}), \end{aligned} \tag{1.134}$$

where  $E, L$  are matrices in  $\mathbb{R}^{n_z \times n_y}$  and  $\mathbb{R}^{n_x \times n_y}$  respectively. In particular, differently from many other standard designs (such as Luenberger observers, Kalman-like observers, high-gain observers and so on), the observer (1.134) is characterized by the presence of a correction term inside the nonlinear term. In case  $w = 0$ , a sufficient condition to guarantee the convergence of the observer, namely, that the observer trajectories  $\hat{\mathcal{X}}(\hat{x}, x, t)$  converge to the plant's trajectory  $\mathcal{X}(x, t)$ , is to select the matrices  $E$  and  $L$  to ensure that the observer defines is uniformly (with respect to  $y$ ) incrementally exponentially stable. For systems of the form (1.93), (1.133) with an observer selected as (1.134), this is achieved for instance if there exists a positive definite matrix  $Q$  in  $\mathbb{R}^{n_x \times n_x}$  such that

$$\text{He} \left\{ Q \left[ A - LC + G \frac{\partial \varphi}{\partial z}(z)(H - EC) \right] \right\} \preceq -qI \tag{1.135}$$

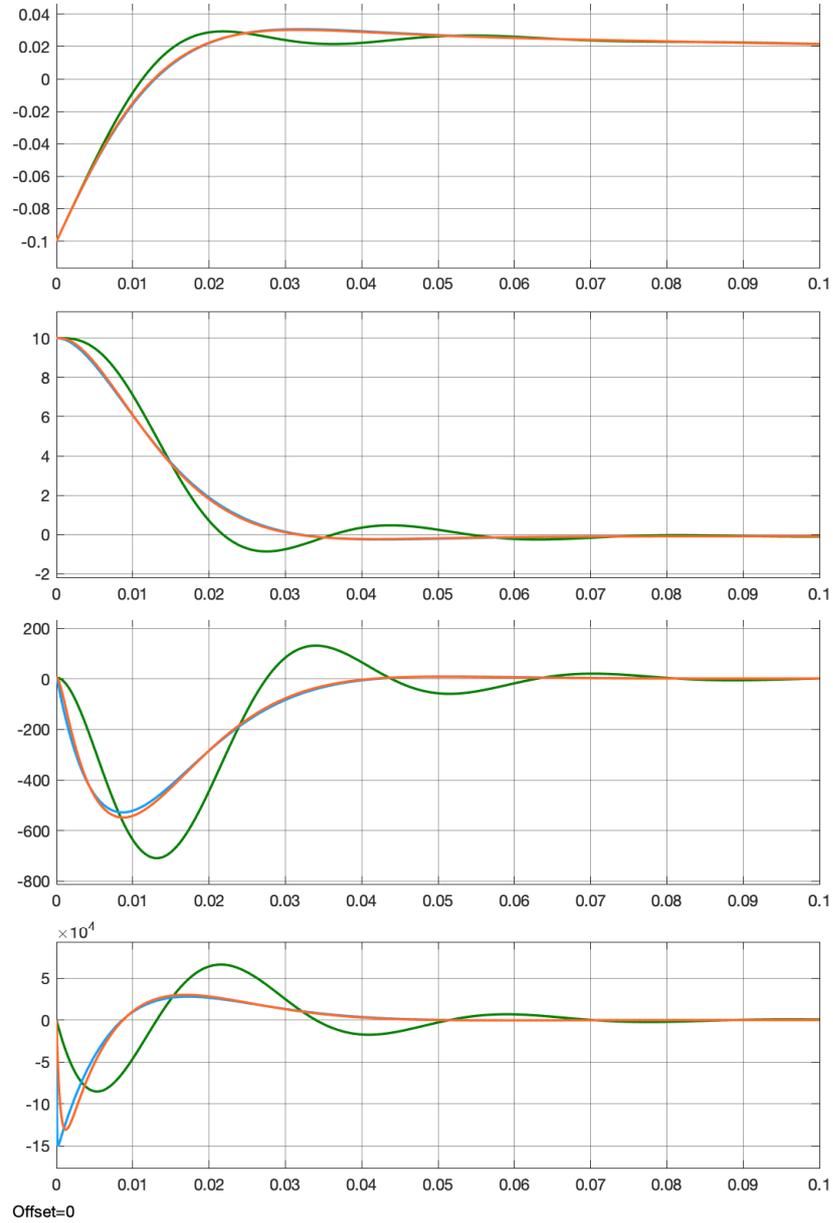


Figure 1.5:  $x$ -dynamics for  $\kappa = 1$  (green),  $\kappa = 10$  (red) and  $\kappa = 100$  (blue)

for all  $z$ , for some strictly positive real number  $q > 0$ . Indeed, under the condition (1.135), it is easy to show<sup>3</sup> the existence of strictly positive real numbers  $(\tilde{k}, \tilde{\lambda})$  such that for all  $(x, \hat{x})$  in  $\mathbb{R}^{2n_x}$ ,

$$\left| \mathcal{X}(x, t) - \hat{\mathcal{X}}(\hat{x}, x, t) \right| \leq \tilde{k} \exp(-\tilde{\lambda}t) |x - \hat{x}| ,$$

for all  $t \geq 0$  in the time domain of existence of solutions.

Similar to the results presented in the former sections, it is possible to give sufficient conditions in the form of LMI to obtain constructive conditions for the design of  $L$  and  $E$ . For instance, based on the sector-bound condition in item  $(A_1)$  of Assumption 1.4.6, we obtain the following result.

**Corollary 1.4.36.1** (Observer for sector-bound). *Consider system (1.93) with  $w = 0$  coupled with a measurement output (1.133). Assume that  $\varphi$  satisfies the sector bound condition  $(A_1)$  in Assumption 1.4.6. If there exist a symmetric positive definite matrix  $Q \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $R$  and  $E$  of appropriate dimensions, and a strictly positive real number  $q > 0$  such that*

$$\text{He} \left\{ \begin{bmatrix} QA - RC + \frac{q}{2}I & 0 \\ G^\top Q - S\Omega(H - EC) & -S \end{bmatrix} \right\} \prec 0, \quad (1.136)$$

then (1.134) is a incrementally uniformly globally exponentially stable observer with  $L = Q^{-1}R$  and such a  $E$ .

| **Proof.** Straightforward from Proposition 1.4.17. □

Based on monotonic nonlinearities, the same result has been obtained in Arcak and Kokotović (2001). Equivalently, under a monotonic assumption on  $\varphi$ , system (1.134) is an exponential observer for system (1.93), (1.133). This is recalled in the following.

**Corollary 1.4.36.2** (Observer for monotonic). *Consider system (1.93) where  $w = 0$  coupled with a measurement output (1.133). Assume that  $\varphi$  satisfies the monotonic condition  $(A_2)$  in Assumption 1.4.6. Then if there exist a symmetric positive definite matrix  $Q \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $R$  and  $E$  of appropriate dimensions and a strictly positive real number  $q > 0$  such that*

$$\text{He} \left\{ \begin{bmatrix} QA - RC + \frac{q}{2}I & 0 \\ H - EC + G^\top Q & -2\Gamma^{-1} \end{bmatrix} \right\} \prec 0, \quad (1.137)$$

then (1.134) is a incrementally uniformly globally exponentially stable observer with  $L = Q^{-1}R$  and such a  $E$ .

| **Proof.** Straightforward from Proposition 1.4.18. □

In the following, we specialize such a result for a nonlinearity  $\varphi$  satisfying (1.91).

**Corollary 1.4.36.3.** *Consider system (1.93) where  $w = 0$  coupled with a measurement output (1.133). Assume that  $\varphi$  satisfies 1.91. If there exist a symmetric positive definite matrix  $Q \in \mathbb{R}^{n_x \times n_x}$ , two matrices  $R$  and  $E$  of appropriate dimensions and a strictly positive real number  $q > 0$  such that*

$$\text{He} \{ QA - RC + \frac{q}{2}I \} \preceq 0 \quad (H - EC)^\top = -QG. \quad (1.138)$$

---

<sup>3</sup>This can be done, for instance, by applying Lemma 1.4.16 to the error-coordinates dynamics  $e := x - \hat{x}$ .

then (1.134) is an incrementally uniformly globally exponentially stable observer with  $L = Q^{-1}R$  and such a  $E$ .

| **Proof.** Straightforward from Proposition 1.4.18 and Remark 1.4.19.  $\square$

**Example 1.4.37.** Consider again the surge subsystem studied in Example 1.4.27. Recall that the nonlinearity satisfies relation (1.91). The output matrix  $C$  is given as  $C = [c_1, c_2]$ . Depending on the values of  $c_1$  and  $c_2$  different cases can be considered. In the following, we consider the case in which  $c_1 = 0$  and  $c_2 = 1$ . It can be shown that for all  $\ell \geq 6$ , there exists  $(L_1, L_2, E)$  in  $\mathbb{R}^3$  such that the system

$$\begin{cases} \dot{\hat{x}}_1 &= -\hat{x}_2 + \ell\hat{x}_1 - \varphi_\ell(\hat{x}_1 + E(y - \hat{x}_2)) + L_1(y - \hat{x}_2) \\ \dot{\hat{x}}_2 &= \hat{x}_1 - u + L_2(y - \hat{x}_2) \end{cases}$$

with  $\varphi_\ell$  defined in (1.104) is a incrementally uniformly globally exponentially stable observer. As shown in (1.105), the function  $\varphi_\ell$  satisfies inequality (1.91) if  $\ell \geq 6$ . Moreover, the system (1.102) is in the form of (1.93) with matrices  $A, B, G, H, R$  as in (1.107) and  $C = [0 \ 1]$ . Let

$$Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{bmatrix}, \quad R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}.$$

Then, the equality constraint in (1.138) reads  $q_{11} = 1$  and  $q_{12} = -E$ . To satisfy  $Q > 0$  one has to satisfy  $q_{22} - E^2 > 0$ . From inequality (1.138), one also obtain

$$\text{He}\{QA - RC\} = \text{He}\left\{ \begin{bmatrix} \ell - E & -1 - R_1 \\ -E\ell + q_{22} & E - R_2 \end{bmatrix} \right\},$$

which can be made definite negative selecting

$$R_1 = -E\ell + q_{22} - 1, \quad E - \ell > 0, \quad R_2 - E > 0.$$

Hence, (1.138) is satisfied for sufficiently small  $q$ .

One of the motivations of observer design is surely the design of output feedback control laws. In this section, we establish a separation principle by showing that a globally stabilizing output feedback law can be obtained by first designing an incrementally uniformly exponentially stable state-feedback control law, and then replacing the state with an estimate provided by a contractive observer. Recall in this sense, that an incrementally uniformly exponentially stable system admits an exponentially stable equilibrium point, see Corollary 1.2.7.2. In contrast with most of nonlinear separation principle, with the proposed conditions one may recover standard results of linear systems in which the design of gains of the state-feedback law and the observer output injection are independent. As a matter of fact, most of the existing results about the separation principle for nonlinear systems rely on time-separation scale conditions, see, e.g. Atassi and Khalil (1999); Teel and Praly (1994); Andrieu and Praly (2009). In other words, the typical nonlinear approach consists in first designing a state-feedback law, and then replacing the state with an estimate which converges to the true state trajectory sufficiently fast. Such a goal is typically achieved using (tunable) high-gain observers Atassi and Khalil (1999); Teel and Praly (1994); Andrieu and Praly (2009). In our case, such a time-separation scale condition is no anymore needed thanks to the uniform incremental stability

properties. As a result, the observer's convergence may be tuned independently of the feedback design (for instance, it can be selected slower than the convergence of the state-feedback law). We recover in this sense, the standard linear separation principle results (Khalil, 2002, Section 12).

To state the main result of this section, consider system (1.93) with  $w = 0$  coupled with a measurement output (1.133) together with an output feedback control law of the form

$$u = K\hat{x} + N\varphi(H\hat{x}) \quad (1.139)$$

in which the estimate  $\hat{x}$  is provided by an observer of the form (1.134). In this section we consider only the case of functions  $\varphi$  satisfying the monotonic condition ( $A_2$ ) in Assumption 1.4.6, but similar results can be extended to the case of sector-bound condition ( $A_1$ ) of Assumption 1.4.6. Now, following the framework in Andrieu and Praly (2009) based on the notion of input-to-state stability (ISS), we recall that two different paths may be followed for the design of an output feedback law:

**Direct approach:** An ISS property is imposed on measurement error in the state feedback law to cope with the mismatch between  $x$  and  $\hat{x}$ .

**Indirect approach:** An ISS property is imposed to cope with the observer correction term.

In our context, both cases may be pursued under an extra assumption, as shown in the following.

**Theorem 1.4.38** (Output-feedback stabilization for monotonic). *Consider the system (1.93) with  $w = 0$  coupled with a measurement output (1.133). Suppose that the nonlinearity  $\varphi$  satisfies the monotonic condition ( $A_2$ ) in Assumption 1.4.6. Consider the feedback output law given by the observer (1.134) and control law (1.139). Suppose that the LMIs (1.99) and (1.137) are feasible for some symmetric strictly positive matrices  $Q = Q^\top \succ 0$ ,  $W = W^\top \succ 0$  and some matrices  $R, E, Z, N$  of appropriate dimensions and some strictly positive scalar  $\lambda, q > 0$ . Assume moreover that one of the following properties holds:*

- *The nonlinear term in the controller (1.139) is zero ( $N = 0$ );*
- *The injection term inside the nonlinearity in (1.134) is zero ( $E = 0$ ).*

*Then the origin of the closed-loop system (1.93), (1.133), (1.134), (1.139) is globally exponentially stable with  $K = ZW^{-1}$ ,  $L = Q^{-1}R$ , and  $N = 0$  or  $E = 0$ .*

**Proof. Direct approach ( $N = 0$ ):** With the change of coordinates  $\hat{x} \mapsto e := x - \hat{x}$ , the closed-loop dynamics  $(x, e)$  reads

$$\begin{aligned} \dot{x} &= (A + BK)x - BKe + G\varphi(Hx) \\ \dot{e} &= (A - LC)e + G[\varphi(Hx) - \varphi(Hx - (H + EC)e)]. \end{aligned}$$

Now, consider the Lyapunov function  $V(x) = x^\top Px$  where  $P = W^{-1}$  with  $W$

defined as solution to (1.99). By the Mean Value Theorem, we have that

$$\begin{aligned}\dot{V} &= 2x^\top P [(A + BK)x + G\varphi(Hx) - BK e] \\ &\leq 2x^\top P \left[ A + BK + G \int_0^1 \frac{\partial \varphi}{\partial z}(sz) ds H \right] x + 2x^\top PBK e \\ &\leq -2\nu|x|^2 + 2x^\top PBK e\end{aligned}$$

for some  $\nu > 0$ , where the last inequality comes from Proposition 1.4.25. From Sontag and Wang (1995), the  $x$ -dynamics is ISS with respect to the input  $e$ . Since (1.137) holds, then by Arcak et al. (2003) we have that

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

Hence the closed-loop can be seen as a cascade of a system having an equilibrium that is globally asymptotically stable and an ISS system. Hence,  $(x, e) = (0, 0)$  is globally asymptotically stable for the closed loop system.

**Indirect approach ( $E = 0$ ):** With the change of coordinates  $x \mapsto e := x - \hat{x}$ , the closed-loop dynamics  $(\hat{x}, e)$  reads

$$\begin{aligned}\dot{\hat{x}} &= (A + BK)\hat{x} + (BN + G)\varphi(H\hat{x}) + LCe \\ \dot{e} &= (A - LC)e + G\varphi(\hat{x} + e) - G\varphi(\hat{x})\end{aligned}$$

Consider the Lyapunov function  $V(\hat{x}) = \hat{x}^\top P \hat{x}$  where  $P = W^{-1}$  with  $W$  defined as solution to (1.99). We have that

$$\begin{aligned}\dot{V} &= 2\hat{x}^\top P [(A + BK)\hat{x} + LCe + (BN + G)\varphi(H\hat{x})] \\ &\leq 2\hat{x}^\top P \left[ A + BK + (BN + G) \int_0^1 \frac{\partial \varphi}{\partial s}(sH\hat{x}) ds H \right] \hat{x} \\ &\quad + 2\hat{x}^\top PLCe \\ &\leq -\nu|\hat{x}|^2 + 2\hat{x}^\top PLCe\end{aligned}$$

for some  $\nu > 0$ . Again, we have obtained a cascade of a contractive system and an ISS system. Therefore, the origin of the closed loop is globally exponentially stable.  $\square$

**Remark 1.4.39.** *In the general case in which  $E \neq 0$  and  $N \neq 0$ , no stability results can be stated a priori. However, one might still be able to claim stability of the closed loop by addressing the problem through a small-gain analysis (see Jiang et al. (1994)). In this case, the designs of the feedback and the observer cannot be made disjointed and typically a time-scale separation between these dynamics is needed.*

**Remark 1.4.40.** *The output feedback law (1.134), (1.139) doesn't ensure in general any contractivity property for the closed-loop system (1.93), (1.133), (1.134), (1.139). As a consequence, if one aims at obtaining a contractive output feedback law, a more general dynamic output feedback of the form*

$$\begin{aligned}u &= K_1 x_c + K_2 y + \sum_{j=1}^m N_j \varphi(H J_j x_c + E_j y), \\ \dot{x}_c &= A_c x_c + \sum_{j=1}^m M_j \varphi(H J_j x_c + E_j y)\end{aligned}\tag{1.140}$$

*needs to be considered. Indeed in the proposed design, we considered a dynamical feedback control action based on an observer where, in particular, we focused on the*

observer proposed in Arcač and Kokotović (2001). This is however a restriction on the choice of the observer structure. More general dynamical designs can be constructed, following for instance the ideas in Scherer et al. (1997). To maintain an LMI-based analysis, the objective would just be to look for a dynamic control law such that the closed-loop system can be written as a linear term plus a nonlinearity satisfying either a monotonic or a sector-bound condition. By rewriting the closed-loop system in the form (1.93), LMI conditions similar to those derived in Section 1.4.4.3 can be established to show contractivity of the closed-loop system in the sense of Lemma 1.4.23.

**Example 1.4.41.** Consider the example of a flexible link manipulator presented in Wu et al. (2015) where system (1.93) defined by matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k}{I_m} & -\frac{C_{vf}}{I_m} & \frac{k}{I_m} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k}{I_l} & 0 & -\frac{k}{I_l} - b\frac{mgh}{I_m} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{k_T}{I_m} \\ 0 \\ 0 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{mgh}{I_m} \end{bmatrix},$$

$$H = [0 \ 0 \ 1 \ 0], \quad \varphi(z) = \sin(z) + bz, \quad R = 0.$$

The nonlinearity  $\varphi$  satisfies the monotonic condition ( $A_2$ ) in Assumption 1.4.6 with  $\Gamma = 2(b-1)$  where  $b > 1$  is a degree of freedom parameter. The system is coupled with a linear output  $C = [1 \ 0 \ 0 \ 0]$ . For such a system with plant parameters selected as  $k = 1.8, I_m = 3.7; I_l = 9.3, C_{vf} = 4.6, k_T = 8, m = 2.1, g = 9.81, h = 1$  and  $b = 1.1$ , the assumptions of Theorem 1.4.38 hold with a zero nonlinear injection term  $E = 0$  and the system is output-feedback stabilizable by means of the dynamical controller (1.134), (1.139) with gains selected as

$$K = [-43.2 \ -5.4 \ 203.8 \ -599.9 \ ], \quad N = 16.9, \\ L = [-9.4 \ -87.1 \ -858.7 \ -1177.5 \ ],$$

and with positive scalars  $p = 1$  and  $q = 0.5$ . In this case, the convergence of the observer is slower than the guaranteed convergence of the state-feedback law (differently from most of output feedback results for nonlinear systems). Figure 1.6 shows the trajectory of the system with initial conditions  $x(0) = (-10, 5, -3, 8)$  and observer initial conditions  $\hat{x}(0) = (7, 6, 0, -4)$ .

## 1.5 Conclusions and perspectives

In this chapter, we studied incremental properties for nonlinear systems. First, we analyzed autonomous time-varying systems. We provided a set of sufficient conditions with a metric-based approach guaranteeing incremental properties for a nonlinear time-invariant system. Then, we recalled the main properties of incremental systems. Later, we considered systems coupled with an input. Through the notion of ‘Killing vector’ field, we provided a set of novel sufficient conditions for a system to be incremental ISS. Then, we focused on the design of control laws guaranteeing incremental properties for the closed-loop. We considered three different sets of control laws for three different classes of systems: a small-gain analysis for systems in feedback, an incremental forwarding control law for systems in cascade, and a

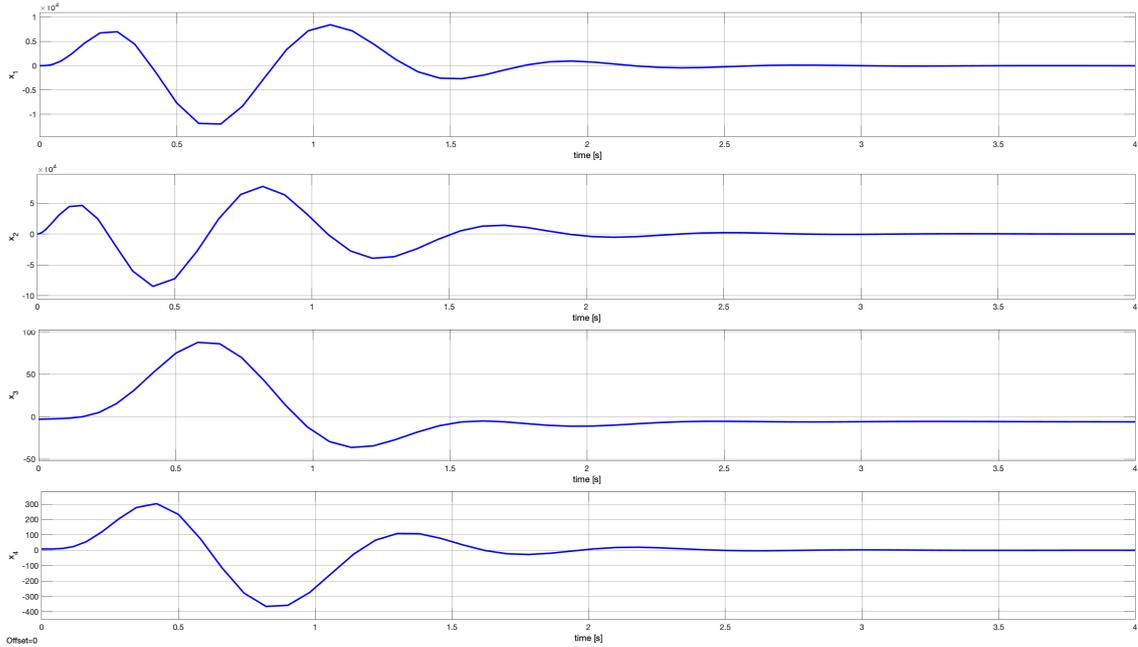


Figure 1.6: Output feedback stabilization for the flexible link manipulator

set of LMI conditions for linear systems coupled with a monotonic or sector-bound nonlinearity.

Future studies will focus on the following aspects:

- Theorem 1.2.2 provides sufficient conditions for a system to have incremental properties with respect to the Euclidean metric. An interesting point could be to see whenever this result can be generalized concerning any Riemannian metric (and the associated distance operator), similarly to Forni and Sepulchre (2013a).
- Another interesting analysis could focus on the notion of incremental asymptotic stability (not exponential), i.e. when the left-hand side of (1.3) is a class- $\mathcal{KL}$  function. To the best of the author's knowledge, such notions have not been studied with a metric-based analysis.
- The study that we pursued considers vector fields that have to be sufficiently smooth. On the other hand, incremental stability is a property of trajectories and therefore is not linked with the smoothness of the vector field. Therefore, a future study could involve the analysis of incremental properties for non-smooth systems, for instance by means of a study based on differential inclusions.
- An interesting point could be to study more general sufficient conditions to have incremental ISS properties concerning the one provided with the Killing vector field notion. On one hand, it could be of interest to look for a feedback control design guaranteeing that the Killing vector field property is satisfied. On the other hand, another analysis can focus on the robustness of the Killing vector field property and how model-uncertainties can cause a loss of  $\delta$ ISS properties
- A very interesting field is related to the results that have been provided con-

cerning the infinite gain margin control law and optimality. A future perspective could be to study in deeper detail the relations between incremental stability and optimality for more general classes of systems and to see how this is related, in a discrete-time context, to existing results on Model Predictive Control.

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## CHAPTER 2

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# GLOBAL OUTPUT REGULATION

When it is obvious that the goals cannot be reached, don't adjust the goals, adjust the action steps.

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Confucius

### 2.1 Introduction and problem statement

#### 2.1.1 Introduction

One of the most important problems in control theory is the so-called *output regulation problem* (also called the *servomechanism problem*). Given a system, the goal of the output regulation problem is to regulate some desired outputs of the system to some references while rejecting some external perturbations. In case such an objective has to be achieved without the perfect knowledge of the plant's dynamics, it is called the *robust output regulation problem*. The robust output regulation problem in the linear framework for Multi-Input Multi-Output (MIMO) system has been completely solved in the '70 by B.A. Francis and W.M. Wonham in their works Francis and Wonham (1975, 1976) and by E. Davison in Davison (1976). In their contributions, Francis and Wonham provided the notion of what nowadays is commonly known as the *Internal Model Principle*. Such a principle says that '*a regulator synthesis is structurally stable only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process.*'. If such a problem has been completely solved for linear systems, for nonlinear systems there are still several questions and open problems. The objective of this chapter is to try to provide an answer to some of these questions.

In detail, in this Chapter, we focus on the global output regulation problem for nonlinear systems. Roughly speaking, with the word 'global' we mean that we want to achieve the regulation task independently on the initial conditions of the system

and independently on the amplitude of the external references and disturbances. The peculiarity of the proposed design is that the main tool that we will use for the control design is incremental stability, which has been presented in Chapter 1. In other words, we will cast the output regulation problem into an incremental framework, by using the tools for the analysis and control synthesis previously developed. The main intuition behind this choice is that, if the system in closed-loop has some incremental global uniform ISS properties with respect to the external signals, then, independently of the initial conditions and on the amplitude of these signals, the system reaches a bounded steady-state solution. In this case, the role of the control action will be to provide such an incremental ISS property, as well as to force the steady-state solution to be the one in which the regulation task is achieved.

The Chapter is structured as follows. In Section 2.1.2, we provide an overview of existing results and open problems on linear and nonlinear output regulation. In Section 2.2, we study the problem of global set-point tracking and constant disturbance rejection problem. In particular in Section 2.2.1, we show why incremental stability tools can be used to solve such a problem. Then, in Section 2.2.2 and Section 2.2.3, we provide, respectively, a state-feedback and output-feedback control design for the solution to such a problem. To conclude, in Section 2.2.4, we show that the proposed design applies to some cases of interest. After this, in Section 2.3, we approach the global harmonic regulation problem. In details, in Section 2.3.1, we show how incremental properties can be useful to solve such a problem. Then, in Section 2.3.2, we propose some state-feedback conditions. The proposed design is characterized in Section 2.3.3 for some case studies. The results that will be presented, can be found in the author's publications Giaccagli et al. (2020, 2021b, 2022d).

## 2.1.2 The global output regulation problem

### 2.1.2.1 Linear output regulation

In this section, we recall some aspects of linear output regulation theory. The objective here is not to provide a detailed analysis of existing literature results, but rather to recall the main aspects and to put in evidence the main differences with respect to the nonlinear case. For the sake of brevity, we focus only on state-feedback control designs. The results that will be presented can be found for instance in Francis and Wonham (1975, 1976); Davison (1976) or in several books of control of linear and nonlinear systems theory, such as Isidori et al. (2003); Byrnes et al. (2012); Huang (2004).

Let's consider a linear system defined by the following dynamical equations

$$\dot{w} = Sw \tag{2.1a}$$

$$\dot{x} = Ax + Bu + Pw \tag{2.1b}$$

$$e = Cx + Qw \tag{2.1c}$$

where  $x \in \mathbb{R}^{n_x}$  is the state of the plant,  $u \in \mathbb{R}^{n_u}$  is the control action and  $w \in \mathbb{R}^{n_w}$  is an external signal generated by the autonomous system (2.1a) called *exosystem*. Such an autonomous signal  $w$  may physically represent some disturbances acting on the system and/or some signals that have to be tracked. We denote with  $e \in \mathbb{R}^{n_e}$  the error signal between an output  $y = Cx$  and the external signal  $Qw(t)$ . We have that  $A, B, P, S, C, Q$  are constant matrices of appropriate dimensions. The linear output regulation framework is based on the following assumption.

**Assumption 2.1.1** (Neutral stability). *The exosystem (2.1a) is neutrally stable, that is, the matrix  $S$  can be expressed as a skew-symmetric matrix, i.e.*

$$S = -S^\top.$$

In other words,  $S$  has eigenvalues only on the imaginary axis. This guarantees that the trajectories of the autonomous exosystem (2.1a) are well-defined and bounded for all positive and negative times. To have a well-posed problem, we assume that the system is not underactuated, that is  $n_u \geq n_e$ . As we will recall, such a constraint is indeed necessary for the solution of the problem.

The problem that we aim to solve is to find a dynamic feedback control action that asymptotically regulates the error  $e$  to zero while keeping the trajectories of the closed-loop bounded. This can be done with or without the full knowledge of the plant's dynamics (2.1). Such an objective is formulated in the following.

**Problem 2.1.1** (Linear output regulation). *Consider system (2.1). Find a dynamical control law of the form*

$$\dot{\eta} = \Phi\eta + \Gamma e \tag{2.2a}$$

$$u = Kx + N\eta \tag{2.2b}$$

with  $\eta \in \mathbb{R}^{n_\eta}$ , for some matrices  $\Phi, \Gamma, K, N$  such that the following hold.

( $P_1$ ) (**Output regulation**) *We have that:*

- the origin  $(x, \eta) = (0, 0)$  of the unforced ( $w(t) = 0$ ) closed-loop system (2.1), (2.2) is globally asymptotically stable;
- the trajectories of the forced ( $w(t) \neq 0$ ) closed-loop system (2.1), (2.2) are bounded and

$$\lim_{t \rightarrow \infty} e(t) = 0$$

for every initial condition  $(x(0), \eta(0), w(0)) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \times \mathbb{R}^{n_w}$ .

( $P_2$ ) (**Robust output regulation**) *For the same control action (2.2) with fixed matrices  $\Phi, \Gamma, K, N$  and the same exosystem (2.1a) with fixed matrix  $S$ , we have that:*

- the output regulation problem ( $P_1$ ) is solved for nominal matrices  $A, B, P, C, Q$ ;
- the output regulation problem ( $P_1$ ) is solved for perturbed matrices  $\tilde{A}, \tilde{B}, \tilde{P}, \tilde{C}, \tilde{Q}$  belonging to a neighborhood of the nominal matrices in the space of parameters  $\mathbb{P} = \mathbb{R}^{n_x \times n_x} \times \mathbb{R}^{n_x \times n_u} \times \mathbb{R}^{n_x \times n_w} \times \mathbb{R}^{n_e \times n_x} \times \mathbb{R}^{n_e \times n_w}$ .

We consider two different problems. In the (linear) output regulation problem ( $P_1$ ), we aim to find a dynamical control action of the form (2.2) such that:

- if we disconnect the exosystem, i.e.  $w(t) = 0$  for all  $t \geq 0$ , then the origin of the closed-loop system is globally asymptotically stable;
- If the plant is excited by the exosystem (2.1a) satisfying Assumption 2.1.1, then, for any initial condition, the trajectories of the closed-loop system are bounded and, asymptotically, the regulation error  $e$  goes to zero.

On the other hand, the (linear) robust output regulation problem ( $P_2$ ) requires that:

- the linear output regulation problem ( $P_1$ ) is solved;
- for the same control law (2.2), even if we do not have a perfect knowledge of the plant's matrices (under certain limits), the asymptotic regulation task on the error is still achieved.

Since the origin of the closed-loop system (2.1), (2.2) has to be asymptotically stable if  $w = 0$ , a trivial necessary condition is that the following must hold.

**Assumption 2.1.2** (Closed-loop stabilizability). *There exist two symmetric positive definite matrices  $\mathcal{P} = \mathcal{P}^\top \succ 0$  and  $\mathcal{Q} = \mathcal{Q}^\top \succ 0$  such that*

$$\mathcal{P}A + A^\top \mathcal{P} \preceq -\mathcal{Q}, \quad A := \begin{pmatrix} A + BK & BN \\ \Gamma C & \Phi \end{pmatrix}$$

By Assumption 2.1.2, the unforced plant has the origin which is globally asymptotically stable. If now we excite the plant with the signal  $w(t)$ , since the matrix  $S$  is assumed to have poles only on the imaginary axis, the closed-loop system (2.1), (2.2) will have bounded trajectories.

An apparently simple yet incredibly powerful result is the following one, which provides a sufficient and necessary condition for the solution of the output regulation problem.

**Theorem 2.1.2** (Linear regulator equations). *Consider the system (2.1) with the controller (2.2). There exists a controller solving the output regulation problem if and only if there exist matrices  $\Pi, \Psi$  solution of*

$$\Pi S = A\Pi + B\Psi + P \tag{2.3a}$$

$$0 = C\Psi + Q. \tag{2.3b}$$

The conditions (2.3) are generally called *regulator equations* (also known as Francis-Byrnes-Isidori equations). It can be shown that, if the regulation problem is solved, then the steady-state solution of the  $x$ -dynamics on which the regulation error  $e$  is zero is given by  $x = \Pi w$ . In a similar way, the steady-state input (known as the *friend*) is given by  $u = \Psi w$ . We have then the following assumption.

**Assumption 2.1.3** (Non-resonance condition). *Consider the system (2.1) for some matrices  $(A, B, C, S)$ . The matrix*

$$\begin{pmatrix} A - \lambda I & B \\ C & 0 \end{pmatrix}$$

*has rank equal to the number of its rows, for any  $\lambda$  in the spectrum of  $S$ .*

Such an assumption is generally known as *non-resonance* condition. It can be understood as the assumption that the transfer function  $C(sI - A)^{-1}B$  between input  $u$  and the output  $y = Cx$  (where here  $s$  denotes the Laplace's variable) does not have zeros at the same frequency at the eigenvalues of  $S$ . Note that such a condition can be verified only if the number of columns of  $B$  is higher than the number of rows of  $C$ , which justifies the fact that we assumed  $n_u \geq n_e$ . Note moreover that, in virtue of (Byrnes et al., 2012, Proposition 1.6), such an assumption is necessary to solve the robust linear output regulation problem.

The design of the matrices of the control action (2.2) follows a two step procedure:

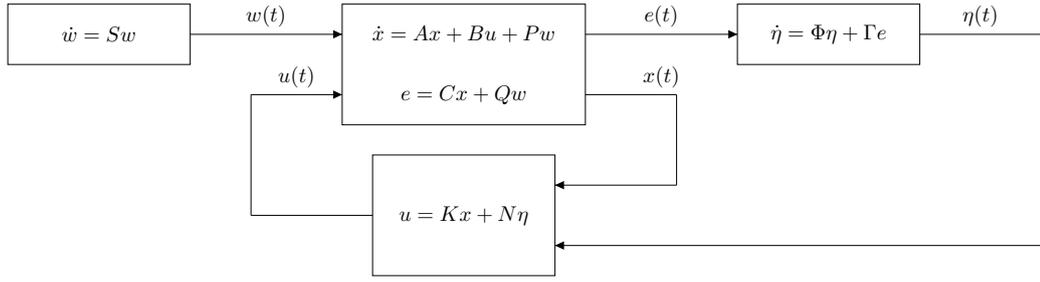


Figure 2.1: Block-scheme solving the output regulation problem for linear systems

1. design the internal model unit (2.2a) which has to process the regulation error  $e$ ;
2. design the stabilizing action (2.2b) for the  $(x, \eta)$ -dynamics.

The internal model principle states that the control action must encode in its structure a suitable copy of the exosystem. For this, the solution to the problem consists in the design of the dynamics of  $\eta$  in (2.2a) with a state-space matrix that possesses the same characteristic polynomial of  $S$  and that processes as input the error  $e$ . For the MIMO case, a possible choice for the design of the internal model is to pick the matrix

$$\Phi = \text{blkdiag}\{\Phi_1, \dots, \Phi_{n_e}\} = \begin{pmatrix} \Phi_1 & & \\ & \ddots & \\ & & \Phi_{n_e} \end{pmatrix}$$

where each  $\Phi_i$  possesses the same characteristic polynomial of  $S$  (the copy of the exosystem) and any matrix  $\Gamma = \text{blkdiag}\{\Gamma_1, \dots, \Gamma_{n_e}\}$  such that any pair  $(\Phi_i, \Gamma_i)$  is controllable (to ensure that the  $\eta$ -dynamics correctly processes  $e(t)$ ). Then, the stabilizer (2.2b) can be selected as any couple of matrices  $K, N$  such that Assumption 2.1.2 is satisfied. A block scheme of the resulting closed-loop system is shown in Figure 2.1. This result is summed up in the following.

**Theorem 2.1.3** (Linear output regulation). *Consider system (2.1) in closed-loop with a dynamic controller of the form 2.2. Let Assumption 2.1.1 hold. Select the matrices  $\Phi, \Gamma$  as  $\Phi = \text{blkdiag}\{\Phi_1, \dots, \Phi_{n_e}\}$  and  $\Gamma = \text{blkdiag}\{\Gamma_1, \dots, \Gamma_{n_e}\}$  where  $\Phi_i$  and  $S$  have the same eigenvalues and  $(\Phi_i, \Gamma_i)$  is controllable pair for every  $i$ . Let the non-resonance condition in Assumption 2.1.3 hold and select any  $K, N$  such that Assumption 2.1.2 holds. Then the linear output regulation problem  $(P_1)$  in Problem 2.1.1 is solved.*

**Remark 2.1.4.** *Two important aspects are that:*

- *the problem is solved independently on the stabilizer  $u$ , as long as the closed-loop system is stable for  $w = 0$ . This implies that different control designs can be applied (passivity,  $\mathcal{H}_\infty$ , LQR, ...) and the asymptotic behavior of the error  $e(t)$  is independent of the stabilizer (as long as it is linear). This is achieved thanks to the Internal Model Principle;*
- *the problem is solved for any initial condition of the plant  $x$ , of the internal model unit  $\eta$ , and of the exosystem  $w$ . This means that independently of the amplitude of the perturbations and the signals to be tracked, the control action (which depends only on the frequencies of  $w$  and not on  $P, Q$ ) asymptotically regulates the error to zero.*

An even more surprising result is that the same control action and the same conditions are sufficient for the solution of the robust output regulation problem ( $P_2$ ) in Problem 2.1.1. In particular, if the matrices  $A, B, C, P, Q$  are not well known, as long as the non-resonance condition in Assumption 2.1.3 is satisfied and the stabilizer (2.2b) makes the unforced closed-loop system stable, output regulation is still achieved. This is recalled in the following.

**Theorem 2.1.5** (Robust linear output regulation). *Consider system (2.1) in closed-loop with (2.2). Assume that Theorem 2.1.3 holds, that is, the linear output regulation problem ( $P_1$ ) in Problem 2.1.1 is solved for some nominal plant matrices  $(A, B, P, C, Q)$ . Then for the same control action (2.2), the linear output regulation problem is solved for any matrices  $(\tilde{A}, \tilde{B}, \tilde{P}, \tilde{C}, \tilde{Q})$ , as long as Assumption 2.1.2 and Assumption 2.1.3 are satisfied for  $(\tilde{A}, \tilde{B}, \tilde{P}, \tilde{C}, \tilde{Q})$  instead of  $(A, B, P, C, Q)$ , i.e. the robust linear output regulation problem ( $P_2$ ) in Problem 2.1.1 is solved.*

**Remark 2.1.6.** *In the results that we presented, we considered a state-feedback design for the stabilizer. Similar results can be obtained with an output-feedback design, through an additional detectability assumption and the design of a Luenberger observer for such an output.*

### 2.1.2.2 Nonlinear output regulation

In this section, we review the main aspects of nonlinear output regulation theory and we illustrate some open problems. In a nutshell, for nonlinear systems, there are two main differences with respect to the linear case.

1. A nonlinear version of the Internal Model Principle for general MIMO input-affine systems does not exist (yet?). This implies that it is generally not possible to follow the same two-step procedure as in the linear case, by first designing the internal model unit and then (any!) suitable stabilizer. This is because the internal model might not just depend on the exosystem, but it can depend also on the plant and the stabilizer. At the same time, the stabilizing unit must act on both the plant and the internal model unit. This creates a *chicken-egg* dilemma (Bin and Marconi (2018)), in which it is not generally possible to design the internal model and the stabilizer independently with respect to the other. This dependence is of particular importance when the goal is to achieve robust regulation (with particular attention to what ‘robust’ now means, see Bin et al. (2018, 2022)).
2. In linear output regulation, the problem is solved independently on the initial conditions and independently on the amplitude of the exosystem (i.e. for every fixed  $S$ ). For nonlinear systems, this is not always the case. Depending on the domain of attraction on which convergence to the steady state solution must be achieved and on the amplitude of the external signals  $w$ , different stability properties can be exploited. In other words, weaker or stronger properties might be required, depending if regulation has to be achieved locally, regionally, semi-globally, or globally.

Consider for this a general nonlinear system of the form

$$\dot{w} = s(w) \tag{2.4a}$$

$$\dot{x} = f(x) + g(x)u + p(x)w \tag{2.4b}$$

$$e = h(x) + q(w) \tag{2.4c}$$

where  $x \in \mathbb{R}^{n_x}$  is the plant's state,  $u \in \mathbb{R}^m$  is a control action,  $w \in \mathbb{R}^{n_w}$  is an external signal generated by the autonomous exosystem (2.4a) and  $e \in \mathbb{R}^{n_e}$  is the error that has to be asymptotically regulated to zero. We assume  $f : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x}$ ,  $g : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_u}$ ,  $p : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_w}$ ,  $h : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$ ,  $q : \mathbb{R}^{n_w} \mapsto \mathbb{R}^{n_e}$  and  $s : \mathbb{R}^{n_w} \mapsto \mathbb{R}^{n_w}$  to be sufficiently smooth and  $f(0) = 0$ ,  $h(0) = 0$  and  $s(0) = 0$ . Similarly to the linear case, we assume that  $n_u \geq n_e$  and that the exosystem (2.4a) is Poisson stable<sup>1</sup> in a neighborhood of the origin. The problem to be solved is the following.

**Problem 2.1.7** (Nonlinear output regulation). *Consider system (2.4). Find a dynamic control action of the form*

$$\dot{\eta} = \phi(\eta, e) \quad (2.5a)$$

$$u = \alpha(x, \eta) \quad (2.5b)$$

such that

1. The origin of

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x, \eta) \\ \dot{\eta} &= \phi(\eta, h(x)) \end{aligned}$$

is asymptotically stable and locally exponentially stable;

2. there exists a neighborhood  $\mathcal{N}_x \times \mathcal{N}_\eta \times \mathcal{N}_w$  of the origin such that, for any initial condition  $(x(0), \eta(0), w(0)) \in \mathcal{N}_x \times \mathcal{N}_\eta \times \mathcal{N}_w$ , the solutions of the closed-loop system (2.4), (2.5) are bounded and satisfy

$$\lim_{t \rightarrow \infty} e(t) = 0.$$

Similarly to Theorem 2.1.2, for the nonlinear case, a sufficient and necessary condition for the solution of the problem can be found. This is recalled in the following, whose proof can be found in (Isidori and Byrnes, 1990, Theorem 1).

**Theorem 2.1.8** (Nonlinear regulator equations). *Consider system (2.4) and assume that  $(A, B)$  is stabilizable, where  $A := \frac{\partial f}{\partial x}(0)$ ,  $B := \frac{\partial g}{\partial x}(0)$ . Then, the nonlinear output regulation problem in Problem 2.1.7 is solvable if and only if there exist two sufficiently smooth mapping  $\pi : \mathbb{R}^{n_w} \mapsto \mathbb{R}^{n_x}$  and  $\psi : \mathbb{R}^{n_w} \mapsto \mathbb{R}^{n_u}$  satisfying  $\pi(0) = 0$  and  $\psi(0) = 0$  that are well-defined in a neighborhood of  $w = 0$  such that*

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w)) + g(\pi(w))\psi(w) + p(\pi(w))w \quad (2.6a)$$

$$0 = h(\pi(w)) + q(w). \quad (2.6b)$$

The conditions (2.6) are a nonlinear version of the regulator equations in (2.3). The mapping  $\pi$  is the steady-state solution to which the plant has to asymptotically converge  $x = \pi(w)$  and on which the regulation error is zero. Similarly,  $\psi$  is the 'friend', that is, the steady-state input  $u = \psi(w)$  that is required to guarantee

<sup>1</sup>The system (2.4a) is said to be Poisson stable if, for any initial condition  $w(0)$ , the solution  $\mathcal{W}(w(0), t)$  of the system is well-defined for all  $t \in \mathbb{R}$  and for each neighborhood  $\mathcal{N}$  of  $w(0)$ . Moreover, for each real number  $T > 0$ , there exist  $\tau_1 > T$  and  $\tau_2 < -T$  such that  $\mathcal{W}(w(0), \tau_1) \in \mathcal{N}$  and  $\mathcal{W}(w(0), \tau_2) \in \mathcal{N}$ . See Isidori and Byrnes (1990). The concept of Poisson stability is a generalization of a system having a periodic solution (e.g. an oscillator). A linear system (2.1a) is Poisson stable if and only if  $S = -S^\top$ .

that the manifold on which  $x = \pi(w)$  is forward-invariant for the dynamics of the closed-loop system.

The question now is: how to design the control action? As we said, a nonlinear version of the Internal Model Principle does not exist. For this, most of the results that can be found in the literature, approach the problem with techniques to overcome such a missing point. An important class of control designs is the one in which the internal model unit behaves as an observer for the friend (see for instance Marconi et al. (2007); Byrnes and Isidori (2004); Priscoli et al. (2006)). In such a case, the key observation is that the existence of an attractive and invariant manifold on which the regulated variable vanishes can be related to the problem of designing an observer. This kind of approach has however the drawback that the knowledge of the plant is generally required. This is not the case for control designs that make use of adaptive techniques (Serrani et al. (2001); Priscoli et al. (2006); Bin et al. (2020); Bernard et al. (2020)). In this case, the internal model unit is capable to adapt and self-tune itself to the dynamics that generate the steady-state control that is necessary to maintain the regulation error zero. Other more recent approaches make use of a pre-processing internal model design (Bin and Marconi (2020); Wang et al. (2020)), where the input of the internal model unit is the combination of the error dynamics and of the stabilizing unit or some infinite-dimensional internal models (Astolfi et al. (2021)), where the internal model unit is modeled as a transport equation (a delay).

Other kinds of designs tackle the problem with the objective to achieve *practical and/or harmonic* regulation rather than the asymptotic one. With practical regulation, we mean that the regulation error  $e$  does not vanish asymptotically, but can be upper bounded by an arbitrarily small constant, i.e.

$$\lim_{t \rightarrow \infty} |e(t)| \leq \varepsilon, \quad \text{for some } \varepsilon > 0.$$

Such a notion is generally satisfying in practical cases, especially because more tractable conditions can be derived for the design of the controller, see Marconi and Praly (2008); Astolfi et al. (2022b,a). In some cases, the internal model unit can be designed independently of the plant's model. Another design is to have a control action that aims to achieve harmonic regulation, see e.g. Blin et al. (2021); Riedinger and Daafouz (2022). With harmonic regulation, we mean that the control design is realized to cancel the harmonic content of the steady-state periodic solution to which the regulation error asymptotically converges. Despite no formal proof exists that shows that harmonic cancellation implies that the regulation error decreases in norm (besides for some particular cases, see Astolfi et al. (2022b)), such a design has been successfully implemented in some practical applications (see for instance the results in Section 4.3 and references therein), due to its simple synthesizing.

**Remark 2.1.9.** *Practical and harmonic regulation designs have a strong link with Repetitive Control (RC) theory. In RC's control design, the idea is to include in the transfer function of the controller a (finite) number of complex conjugates poles at the reference's frequency and its multiplies (see Ghosh and Paden (2000)). Note that such a technique is a finite-dimensional approximation of an infinite-dimensional internal model. This because a generator of periodic signals can be modeled through a delay  $T$ , which has the effect to place an infinite number of complex conjugates poles at the frequency  $\frac{2\pi}{T}$  and all its multiplies (see Astolfi et al. (2021)). This is particularly effective for discrete-time systems, as a delay can be easily modeled with*

a *buffer*.

The literature on nonlinear output regulation is wide, and the above references represent just a small portion of existing results. Nevertheless, it is worth pointing out an important aspect. The vast majority of the results on nonlinear output regulation, consider systems with a globally defined normal form that possesses a minimum phase property (see (Isidori, 1995, Section IV)). For linear systems, the design is ‘coordinate-free’, i.e. it can be done in the original given set of coordinates. On the contrary, for nonlinear systems, this condition simplifies the analysis and, usually, it allows to obtain more tractable conditions. This however poses three main problems.

- The first one is that finding the change of coordinates that allows rewriting the system in normal form is generally not an easy task, especially for MIMO systems, see for instance Wang et al. (2016).
- The second problem is that uncertainties in the ‘original-coordinates’ plant would be manipulated as well, once the change of coordinates is applied. This implies that ‘easy-to-handle’ uncertainties (such as parametric and globally Lipschitz uncertainties) in the original coordinate plant might pose some problems in normal form coordinates.
- The third problem is the domain of existence of the change of coordinates. Indeed, the diffeomorphism needed to write the system in normal form is always well-defined locally around the origin. However, we are not guaranteed that such a diffeomorphism is well defined globally, i.e. for all  $x$ .

### 2.1.2.3 Global output regulation: existing solutions

In this chapter, we focus on the *global output regulation* problem. With the word ‘global’, we mean that we aim to solve the problem for every initial condition of the system (global domain of attraction) and any amplitude of the external signal. Concerning the structural properties of the system, we can classify the existing results into three main categories.

1. In the first group of works, it is commonly supposed the existence of a change of coordinates that can put the system into a normal form, (Isidori, 1995, Chapter 4). Depending on the properties of the zero-dynamics, different control design have been proposed: see, among others, Khalil (2000); Behtash (1990); Jiang and Marcelis (2001) for single-input single-output (SISO) minimum-phase systems; Wang et al. (2020); Bin and Marconi (2020) for MIMO minimum-phase systems; Huang et al. (2018) for SISO non-minimum phase systems. In such case, output regulation has been achieved (semi)globally in the initial conditions and in the size of the references, using a feedback control law composed of a term depending on the internal model and a high-gain feedback (see for instance Serrani et al. (2001); Astolfi et al. (2022b) and references therein). In these settings, output regulation can be ensured for arbitrarily large references and/or disturbances with a semi-global (or global) domain of attraction, provided the zero-dynamics possess good uniform attractivity properties. However, as said in the previous section, such approaches cannot be followed when a global normal form is not globally defined or its explicit expression is hard to be computed.

2. The second group of works approached the problem in a ‘coordinate-free’ framework, trying to develop a feedback design in the original coordinates, see Astolfi and Praly (2017); Astolfi et al. (2022a). Such an approach provides results that are semi-global in the domain of attraction, but only local in the size of the exosystem. In this case, the analysis is generally divided into two parts. First, the exosystem is disconnected from the plant, and the stability of the origin is studied. Then, solutions are studied in presence of external signals. For this, stability is guaranteed only in case of sufficiently small external signals (locally around the equilibrium point).
3. The third group of results makes use of the notion of *convergent* systems. Such technique has been developed in the manuscript Pavlov et al. (2006) and in several articles by the same authors. Such an approach works in the original coordinates and provides control action which allows obtaining output regulation globally in the domain of attraction and globally in the size of external signals. The main drawback of such results is that the control design generally requires the exact knowledge of the regulator equations (2.6), i.e. the knowledge of a solution to a PDE. This is generally a difficult task and strongly model-dependent.

In our design, we will try to provide a solution for the global output regulation problem. The main idea is to combine the advantages of the three approaches proposed above. Indeed we will aim to provide a solution working in the original coordinates without explicitly relying on the existence of a globally defined diffeomorphism to write the system in normal form. We will look for conditions to solve the problem globally in the domain of attraction and globally in the amplitude of the external signals. For this, we will cast the problem by using incremental stability tools derived in Chapter 1. We already recalled the strong link between incrementally stable and convergent systems. In the proposed design, however, we will look for control designs that do not explicitly require the solution of the regulator equations. We will split the problem into two parts. First, in Section 2.2, we will consider the global integral action problem. In this case, we can show that asymptotic regulation can be achieved and a ‘perfect’ internal model can be designed. Then, in Section 2.3 we will focus on the global harmonic regulation problem. By ‘adding oscillators’, we present some sufficient conditions to achieve harmonic regulation independently on the initial conditions and on the amplitude of external signals.

## 2.2 Global integral action

### 2.2.1 From incremental stability to global integral action

In this chapter, we focus on a nonlinear system of the form

$$\dot{x} = f(x) + g(x)(u + d) \quad (2.7a)$$

$$e = h(x) - r. \quad (2.7b)$$

where  $x \in \mathbb{R}^{n_x}$  is the system’s state,  $u \in \mathbb{R}^{n_u}$  is the control input,  $e \in \mathbb{R}^{n_e}$  is the error between the output  $y = h(x)$  to be regulated and a constant (possibly unknown) reference  $r \in \mathbb{R}^{n_e}$ , and  $d \in \mathbb{R}^{n_u}$  is a constant unknown disturbance. We suppose that the functions  $f : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x}$ ,  $g : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_u}$ ,  $h : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$  are  $C^2$ . To have a well-posed problem, we assume that the system is not underactuated, that is, we assume  $n_u \geq n_e$ . Furthermore, we assume that  $f(0) = 0$  and  $h(0) = 0$ .

**Remark 2.2.1.** *Note that:*

- *system (2.7) is in input-affine form, satisfying a matching-conditions for the disturbances  $d$ . The matching condition assumption is quite common in the nonlinear control literature and verified by many practical applications. See for instance the works Qu (1993); Praly and Wang (1996) and references therein. From a practical point of view, matching disturbances are always verified when considering perturbations acting on the actuators of the plant,*
- *The results that will be presented may be generalized to systems having a more complex structure such as (2.4). In such a case, however, the knowledge of functions  $p, q$  is required as well as an additional assumption on a Killing vector property (see Assumption 2.2.1).*

The problem that we consider in this section is the design of a state feedback control law to achieve output set-point tracking and disturbance rejection, that is,  $\lim_{t \rightarrow \infty} e(t) = 0$ . In other words, we focus on the output regulation problem in case  $w = (d, r)$  is a constant signal. Following Poulain and Praly (2010); Astolfi and Praly (2017) and in the spirit of Francis and Wonham (1976); Davison (1976), we focus on the implementation of a dynamic controller with the internal model designed as an integral action post-processing the regulation error. Specifically, we extend the system with a bunch of integrators of the form

$$\dot{\eta} = e \tag{2.8a}$$

with  $\eta \in \mathbb{R}^{n_e}$ . After this, we look for a feedback stabilizer for the closed-loop system  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \mapsto \mathbb{R}^{n_u}$  of the form

$$u = \alpha(x, \eta) \tag{2.8b}$$

with  $\alpha(0, 0) = 0$ . We formalize the problem in the following.

**Problem 2.2.2** (Constant nonlinear output regulation). *Consider system (2.7). Find a dynamic control law of the form (2.8a), (2.8b) such that there exist two sets  $\mathcal{S} \subseteq \mathbb{R}^{n_x+n_e}$  and  $\mathcal{W} \subseteq \mathbb{R}^{n_u+n_e}$ , containing their respective origins, such that the following properties are fulfilled:*

1. *if  $(d, r) = (0, 0)$ , the origin of the closed-loop system (2.7),(2.8a), (2.8b) is globally asymptotically stable;*
2. *for all constant  $(d, r) \in \mathcal{W}$  and all initial conditions  $(x_0, \eta_0) \in \mathcal{S}$ , the closed-loop system (2.7),(2.8a), (2.8b) has bounded trajectories and*

$$\lim_{t \rightarrow +\infty} e(t) = 0.$$

*Then, we have the following problem definitions.*

- *If  $\mathcal{S}$  and  $\mathcal{W}$  are non-empty, the regional constant output regulation problem is solved.*
- *If  $\mathcal{S} = \mathbb{R}^{n_x+n_e}$  and  $\mathcal{W} = \mathbb{R}^{n_u+n_e}$ , the global constant output regulation problem is solved.*
- *If  $\mathcal{S}$  is non-empty and  $\mathcal{W}$  is of the form  $\mathcal{W} = \emptyset \times \mathcal{R}$  for some non-empty set  $\mathcal{R} \subseteq \mathbb{R}^{n_e}$ , then the constant output set-point tracking problem is solved.*

Without the use of normal forms the construction of the control law  $\alpha$  is not straightforward, especially whenever the size of the disturbances is large in amplitude. The crucial point in the synthesis of an integral controller is to ensure the existence of an equilibrium for every value of  $(d, r)$ . This amounts to asking that, for every external signal, the flow of the closed-loop system admits a fixed point which can be obtained via Banach fixed point Theorem (see for instance (Khalil, 2002, Appendix B)) if the flow itself defines a contraction.

System (2.7), (2.8a), (2.8b) leads to a closed-loop system of the form

$$\dot{\chi} = \varphi(\chi) + \Gamma(\chi)w, \quad (2.9)$$

where  $\chi = (x^\top, \eta^\top)^\top \in \mathbb{R}^{n_x+n_e}$ ,  $w = (d^\top, r^\top)^\top \in \mathbb{R}^{n_u+n_e}$ , and the functions  $\varphi$  and  $\Gamma$  are defined as

$$\varphi(\chi) := \begin{pmatrix} f(x) + g(x)\alpha(x, \eta) \\ h(x) \end{pmatrix}, \quad (2.10)$$

$$\Gamma(\chi) := (\Gamma_d(\chi) \quad \Gamma_r), \quad \Gamma_d(\chi) := \begin{pmatrix} g(x) \\ 0 \end{pmatrix}, \quad \Gamma_r := \begin{pmatrix} 0 \\ -I \end{pmatrix}. \quad (2.11)$$

Given any initial condition  $\chi_0$  in  $\mathbb{R}^{n_x+n_e}$  and  $w \in \mathbb{R}^{n_u+n_e}$ , the corresponding solution of system (2.9) is denoted with  $\mathcal{X}(\chi_0, w, t)$ .

The control law (2.8b) solves the constant global output regulation problem if, for any value of  $w \in \mathcal{W}$ , there exists a unique equilibrium  $\chi^* = (x^*, \eta^*)$  that is globally asymptotically stable. Indeed, thanks to the integral action, at the equilibrium we have

$$0 = \dot{\eta} = h(x^*) - r \quad \implies \quad h(x^*) = r$$

and consequently the error  $e$  converges asymptotically to zero along any solution of the closed-loop system. Contraction theory can be used to solve the problem at hand, as highlighted in the following theorem.

**Theorem 2.2.3** (Contraction implies constant regulation). *Consider system (2.9). Suppose that there exist a  $C^2$  function  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \mapsto \mathbb{R}^{n_u}$ , a  $C^1$  matrix function  $\mathcal{P} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \rightarrow \mathbb{R}^{(n_x+n_e) \times (n_x+n_e)}$  taking symmetric positive definite values, a  $C^0$  function  $\mathbf{q} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}_{>0}$  taking positive values and two strictly positive real numbers  $\bar{\mathbf{p}}, \underline{\mathbf{p}} > 0$  such that the vector fields  $\varphi, \Gamma$  defined in (2.10), (2.11) satisfy the following set of properties*

$$\underline{\mathbf{p}}I \preceq \mathcal{P}(\chi) \preceq \bar{\mathbf{p}}I \quad (2.12a)$$

$$L_\varphi \mathcal{P}(\chi) \preceq -\mathbf{q}(\chi)I \quad (2.12b)$$

$$L_{\Gamma_r} \mathcal{P}(\chi) = 0, \quad (2.12c)$$

$$L_{\Gamma_d} \mathcal{P}(\chi) = 0, \quad (2.12d)$$

for all  $\chi \in \mathbb{R}^{n_x+n_e}$ . Then, the following holds.

1. If there exists a positive real number  $\underline{\mathbf{q}} > 0$  such that  $\mathbf{q}(\chi) \geq \underline{\mathbf{q}}$  for all  $\chi \in \mathbb{R}^{n_x+n_e}$ , then the global constant output regulation problem is solved.
2. Otherwise, we have the following properties.
  - (a) Let  $\mathcal{W}$  be a non-empty subset of  $\mathbb{R}^{n_u+n_e}$  such that, for all  $w$  in  $\mathcal{W}$ , there exists an initial condition  $\chi_0 \in \mathbb{R}^{n_x+n_e}$  such that the corresponding closed-loop system trajectory  $\mathcal{X}(\chi_0, w, t)$  is bounded. Then, the regional constant output regulation problem is solved for  $\mathcal{S} = \mathbb{R}^{n_x+n_e}$  and such a  $\mathcal{W}$ .

- (b) *There exists  $\bar{w} > 0$  such that, the regional constant output regulation problem is solved for  $\mathcal{S} = \mathbb{R}^{n_x+n_e}$  and  $\mathcal{W} = \{w \in \mathbb{R}^{n_d+n_e}, |w| \leq \bar{w}\}$ .*

*Furthermore, if the condition (2.12d) is not verified, then the above results 1) and 2) hold for the constant output set-point tracking problem, in particular with  $\mathcal{W}$  of the form  $\mathcal{W} = \emptyset \times \mathcal{R}$ , for some  $\mathcal{R} \subseteq \mathbb{R}^{n_e}$ .*

The statement of Theorem 2.2.3 involves many different results based on a contraction analysis of the vector field  $\varphi$  of the closed-loop system (2.9) and how perturbations and references  $w$  affect the system through the vector  $\Gamma$ .

First, conditions (2.12a) and (2.12b), require the unforced closed-loop system (2.9), i.e, for  $w = 0$ , to be incrementally globally stable (see Definition 1.2.1), but possibly non-uniformly with respect to the full-state space  $\chi$ . In other words, since  $\mathbf{q}$  can be non-uniformly lower-bounded, we cannot conclude that the unforced closed-loop system is uniformly  $\delta$ GUES. However, since  $\varphi(0) = 0$ , the origin is globally asymptotically stable (see Proposition 1.2.9). If the contraction is uniform (see item 1), that is  $\mathbf{q}(x) \geq \underline{\mathbf{q}} > 0$ , and the Killing vector conditions (2.12c), (2.12d), with respect to  $\Gamma$  are verified, then by Theorem 1.3.3 the system is incrementally globally uniformly ISS with respect to  $w$ , for any  $\chi \in \mathbb{R}^{n_x+n_e}$  and any  $w \in \mathbb{R}^{n_u+n_e}$ . In this case, the key role of the Killing vector property is to ensure that the distance among different trajectories is invariant with respect to the amplitude of external signals. In turn, such a geometric condition ensures that the contractivity property of system (2.9) is invariant with respect to  $w$ , and that the convergence to an equilibrium is guaranteed. Note that such a Killing property is always verified when  $\Gamma$  is a linear mapping and  $\mathcal{P}$  is a constant matrix. This is the case, for instance, of linear systems.

As we shall see in the sequel, the existence of a feedback law (2.8b) ensuring a uniform contraction is not always possible. However, under milder assumptions, we shall see that it is in general possible to obtain a non-uniform contraction, according to the conditions of Item 2. In such a case, output regulation is obtained globally in the initial conditions, but only regionally with respect to the exosignal  $w$ , that is, only for  $w \in \mathcal{W}$  for some compact set  $\mathcal{W} \subset \mathbb{R}^{n_u+n_e}$ . Item 2 (a) and (b) provide two different conditions for the estimation of such a set  $\mathcal{W}$ . It is of particular interest the first set of conditions (a), in which it suffices to verify whether the resulting trajectory is bounded forward in time to conclude asymptotic regulation. Nonetheless, condition (b) ensures that the set  $\mathcal{W}$  is non-empty. In other words, asymptotic regulation is always guaranteed globally in the state  $\chi$  and at least locally in the size of the exosignal  $w$ . Note that such a result is also new with respect to Astolfi and Praly (2017), where output regulation is obtained only semi-globally in  $\chi$  and locally in  $w$ .

Finally, since the existence of a metric  $\mathcal{P}$  which is of Killing for both vectors  $\Gamma_r$  and  $\Gamma_d$  is not always possible, the last part of the theorem states that whenever only  $\Gamma_r$  is a Killing vector for  $\mathcal{P}$ , then, all previous arguments hold by considering only the case of tracking problem, i.e. with  $d = 0$ . As a matter of fact, as we shall see in the subsequent sections, it is easier to design a feedback law satisfying the condition (2.12c) than (2.12d), because  $\Gamma_r$  is constant while  $\Gamma_d$  is not.

**Proof.** [Proof of Theorem 2.2.3] First, note that the origin of (2.9) for  $w = 0$  is an equilibrium. Hence, direct application of Proposition 1.2.9 implies that the

equilibrium is globally attractive when  $w = 0$ . Furthermore, it is also locally exponentially stable. This can be proved by using the first order approximation of system (2.9) at the origin and the Lyapunov function  $V(\chi) = \chi^\top \mathcal{P}(0)\chi$ , and inequality (2.12b), which is  $L_\varphi \mathcal{P}(0) \leq -\mathbf{q}(0)I < 0$  at the origin. This establishes the first property 1). Concerning the second property, we will now prove the two items separately.

*Item 1.* Since  $\mathbf{q}(\chi) \geq \underline{\mathbf{q}} > 0$ , the inequality (2.12b) is strict for all  $\chi \in \mathbb{R}^{n_x+n_e}$ . Furthermore, by using the the Killing matrix properties (2.12c), (2.12d), by linearity of the Lie derivative we have

$$\begin{aligned} L_{\varphi+\Gamma w} \mathcal{P}(\chi) &= L_\varphi \mathcal{P}(\chi) + L_\Gamma \mathcal{P}(\chi)w \\ &= L_\varphi \mathcal{P}(\chi) \\ &\preceq -\underline{\mathbf{q}}I, \end{aligned} \tag{2.13}$$

for all  $\chi \in \mathbb{R}^{n_x+n_e}$  and all  $w \in \mathbb{R}^{n_u+n_e}$ . As a consequence, the system (2.9) is an autonomous system satisfying the conditions of Theorem 1.2.2 in Chapter 1, Section 1.2. We deduce that system (2.9) is  $\delta$ GUES. Direct application of Corollary 1.2.7.2 in the Chapter 1, Section 1.2 ensures that, for any  $w$ , there exists a unique equilibrium point  $\chi^* = (x^*, \eta^*)$  which exponentially attracts all solutions. On such equilibrium, we obtain  $\dot{\eta} = 0$  and hence  $e = 0$ , concluding the proof of the statement.

*Item 2 (a).* Let  $w$  be in  $\mathcal{W}$  and let  $\chi_0 \in \mathbb{R}^{n_x+n_e}$  such that  $\mathcal{X}(\chi_0, w, t)$  is bounded. Let  $\mathcal{C} \subset \mathbb{R}^{n_x+n_e}$  be defined as

$$\mathcal{C} := \text{cl} \left\{ \bigcup_{t \geq 0} \mathcal{X}(\chi_0, w, t) \right\}$$

and note that  $\mathcal{C}$  is a forward invariant compact set, in view of (Teschl, 2012, Lemma 6.4). Let

$$\underline{\mathbf{q}}_\mathcal{O} := \min_{\chi \in \mathcal{O}} \mathbf{q}(\chi) > 0$$

where  $\mathcal{O}$  is the set defined as

$$\mathcal{O} := \left\{ \chi \in \mathbb{R}^{n+p} \mid \max_{s \in \mathcal{C}} |\chi - s| < \mathbf{r} \right\},$$

for some positive real number  $\mathbf{r}$  satisfying

$$\mathbf{r} > \frac{\sqrt{\mathbf{p}}}{2\sqrt{\underline{\mathbf{p}}}} \max_{(\chi_1, \chi_2) \in \mathcal{C}^2} |\chi_1 - \chi_2|.$$

Keeping in mind that  $\Gamma_d, \Gamma_r$  satisfy the Killing matrix properties (2.12c), (2.12d) for  $\mathcal{P}$ , by using computations similar to (2.13), we obtain

$$L_{\varphi+\Gamma w} \mathcal{P}(\chi) \preceq -\underline{\mathbf{q}}_\mathcal{O} I$$

for all  $(\chi, w) \in \mathcal{O} \times \mathcal{W}$ . Hence Corollary 1.2.5.1 applies and therefore system (2.9) is  $\delta$ UES in  $\mathcal{C}$ . Then, Corollary 1.2.7.2 implies the existence of an equilibrium  $\chi^* \in \mathcal{C}$ . From Proposition 1.2.9 it yields that this equilibrium is unique and it

is globally attractive. Again, in such an equilibrium, we have  $\dot{\eta} = 0$ , and hence  $e = 0$ .

*Item 2 (b).* We already proved that the equilibrium of (2.9) is locally exponentially stable for  $w = 0$ . Hence, by (Astolfi and Praly, 2017, Lemma 5), there exists  $\bar{w} > 0$  such that, for all  $w \in \mathcal{W}$ , where

$$\mathcal{W} := \{w \in \mathbb{R}^{n_u+n_e} : |w| \leq \bar{w}\},$$

system (2.9) admits an equilibrium  $\chi^* \in \mathbb{R}^{n_x+n_e}$  which is locally exponentially stable. Hence, employing Item 2 (a) with this set  $\mathcal{W}$ , the result follows.  $\square$

Motivated by the conditions of Theorem 2.2.3, in the rest of the section we will design a control law able to guarantee both the contraction of  $\varphi$  for some metric  $\mathcal{P}$  and the Killing vector property with respect to  $\Gamma$ .

## 2.2.2 State feedback design

### 2.2.2.1 Assumptions

In this section, we state the main assumptions for system (2.7) which are needed in order to design a feedback-law of the form (2.8) able to satisfy the contractive conditions of Theorem 2.2.3. Motivated by the closed-loop structure and by Remark 1.4.3, the idea is to make use of *incremental forwarding* control techniques presented in Section 1.4.3.3 to design the stabilizing unit (2.8b). Indeed, the cascade (2.7), (2.8a) for  $(d, r) = (0, 0)$  reassemble the structure in (1.52) with  $\Phi = 0$  and  $v(x) = h(x)$ . For this, we first assume the existence of a pre-stabilizing feedback ensuring incremental uniform global exponential stability of the plant (2.7) according to a metric satisfying the Killing vector property with respect to the function  $g$ .

**Assumption 2.2.1** (Open-loop  $\delta$ GUES and Killing vector). *Consider system (2.7) with the matrix function  $g$  fulfilling  $\|g(x)\| \leq \bar{g}$  for some positive  $\bar{g} > 0$  for all  $x \in \mathbb{R}^{n_x}$ . There exist a known  $C^2$  function  $\alpha_0 : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$ ,  $\alpha_0(0) = 0$ , a  $C^1$  function  $P : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values  $P(x) = P(x)^\top \succ 0$  for all  $x \in \mathbb{R}^{n_x}$ , and three strictly positive real numbers  $\underline{p}, \bar{p}, \lambda_0 > 0$  such that, by denoting*

$$\dot{x} = f_0(x), \quad f_0(x) := f(x) + g(x)\alpha_0(x), \quad (2.14)$$

*the following properties hold for all  $x \in \mathbb{R}^{n_x}$*

$$\underline{p}I \preceq P(x) \preceq \bar{p}I, \quad L_{f_0}P(x) \preceq -2\lambda_0I, \quad (2.15)$$

$$L_gP(x) = 0. \quad (2.16)$$

According to Theorem 1.2.2 (see Section 1.2), the feedback law  $u = \alpha_0(x)$  ensures the vector field  $f_0$  to be incrementally uniformly globally exponentially stable ( $\delta$ UGES) with respect to the metric  $P$ . The computation of  $\alpha_0$  can be obtained, for instance, by the design techniques illustrated in Section 1.4 (and references therein). Furthermore, in (2.16), we ask that the (Riemannian) metric induced by  $P$  is invariant along  $g$ . This means that the system preserves its contractive properties in the directions provided by  $g$ , namely, in the directions in which the control law and the perturbations  $d$  act. In other words, by Theorem 1.3.3, Assumption 2.2.1 implies that system

$$\dot{x} = f_0(x) + g(x)d$$

is  $\delta$ GUEISS (see Definition 1.3.1) with respect to  $d$ .

Since system (2.14) is  $\delta$ GUES, there exists an equilibrium point (the origin, without loss of generality) that is globally exponentially stable. As a consequence, by applying Lemma 1.4.4, we know the existence of a function  $M : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$  satisfying

$$L_{f_0}M(x) = h(x). \quad (2.17)$$

This is a particular case of (1.55), due to the system's structure (2.7), (2.8a). Similarly to Assumption 2.1.3, we suppose then that  $L_gM(x)$  satisfies a controllability-like condition, which, in our incremental framework, is stated as follows.

**Assumption 2.2.2** (Controllability along trajectories). *There exists a positive real number  $b > 0$  satisfying*

$$L_gM(x)L_gM^\top(x) \succeq bI \quad \forall x \in \mathbb{R}^{n_x} \quad (2.18)$$

where  $M$  is the solution of (2.17).

**Remark 2.2.4.** *Note that for a linear system of the form*

$$\dot{x} = Ax + Bu, \quad \dot{\eta} = Cx$$

Assumptions 2.2.1 and 2.2.2 boil down to ask, respectively, for  $(A, B)$  to be stabilizable (in this case the metric  $\mathcal{P}$  is constant and (2.16) is automatically satisfied) and the non-resonance condition in Assumption 2.1.3 to hold. In particular:

- since  $(A, B)$  is stabilizable, then there exists  $K$  such that  $A_0 := A + BK$  is Hurwitz. In such a case,  $f_0(x) = A_0x$ . Therefore, (2.17) reduces to the existence of a function  $M$  solution to

$$\frac{\partial M}{\partial x}A_0x = Cx.$$

Since  $A_0$  is Hurwitz and therefore invertible, a solution is given by  $M(x) = CA_0^{-1}x$ ;

- with such a choice of  $M$  (and remembering that  $n_u \geq n_e$ ), we have that  $L_gM(x) = CA_0^{-1}B$ . Therefore, the controllability-like condition (2.18) is equivalent to ask for  $\text{rank}\{CA_0^{-1}B\} = n_e$ . This is the counterpart of the non-resonance condition for the linear systems' case. Indeed since  $w$  is constant, the dynamics of the exosystem is given by  $\dot{w} = Sw = 0$ , namely  $S = 0$ . The non-resonance condition in Assumption 2.1.3 reduces to ask for the matrix

$$\begin{pmatrix} A_0 & B \\ C & 0 \end{pmatrix}$$

to have full rank. By left-multiplying by the non-null vector  $(CA_0^{-1} \quad -I)$  we obtain

$$(CA_0^{-1} \quad -I) \begin{pmatrix} A_0 & B \\ C & 0 \end{pmatrix} = (0 \quad CA_0^{-1}B).$$

Since Assumption 2.1.3 is known to be necessary for the linear systems case, necessarily  $CA_0^{-1}B$  must be full-rank;

- in light of the previous item, note that the term  $L_gM(x)$  can be understood as a non-linear version of the DC-gain of the transfer function between the control input  $u$  and the output  $y = h(x)$ .

As remarked in Section 2.1.2.1, in linear output regulation theory both these two conditions are necessary and sufficient to solve the linear global output regulation Problem.

### 2.2.2.2 Design for constant output set-point tracking

In this Section, we aim to propose a solution for the constant output set-point tracking problem proposed in Problem 2.2.2. In other words, along all this section, we will consider system (2.7) when  $d = 0$ . Under the assumptions stated in Section 2.2.2.1, we are now in the position to state the first result, concerning the design of a forwarding-based control law for system (2.7) satisfying the hypothesis (2.12a) (2.12b) and (2.12c) of Theorem 2.2.3. Taking inspiration from (1.57), we focus on a control law of the form

$$\alpha(x, \eta) = \alpha_0(x) + \kappa \Psi(x) \beta(\eta - M(x)), \quad (2.19)$$

where the function  $\Psi : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_u \times n_e}$  is defined as

$$\Psi(x) := L_g M(x)^\top (L_g M(x) L_g M(x)^\top)^{-1}, \quad (2.20)$$

$\kappa \in \mathbb{R}$  is a control gain parameter and  $\beta : \mathbb{R}^{n_e} \rightarrow \mathbb{R}^{n_e}$  is a  $C^1$  function to be specified. We have the following result.

**Proposition 2.2.5.** *Consider system (2.7) and suppose Assumptions 2.2.1 and 2.2.2 hold. Suppose, moreover, that there exist two positive real numbers  $L_M > 0$  and  $k_1 \geq 0$  such that the following inequalities are verified*

$$\left\| \frac{\partial M}{\partial x}(x) \right\| \leq L_M, \quad (2.21)$$

$$\left\| \frac{\partial \Psi v}{\partial x}(x) \right\| \leq k_1 |v|, \quad \forall v \in \mathbb{R}^{n_e} \quad (2.22)$$

for all  $x \in \mathbb{R}^{n_x}$ , with  $\Psi$  defined as in (2.20) and with  $M$  defined as in (2.17). Finally, select  $\beta : \mathbb{R}^{n_e} \rightarrow \mathbb{R}^{n_e}$  as any  $C^1$  function satisfying

$$|\beta(s)| \leq \frac{1}{k_1}, \quad \frac{\partial \beta}{\partial s}(s) = \frac{\partial \beta^\top}{\partial s}(s), \quad 0 \prec \frac{\partial \beta}{\partial s}(s) \preceq I, \quad (2.23)$$

for all  $s \in \mathbb{R}^{n_e}$ . Then, there exists a positive real number  $\bar{\kappa} > 0$  such that, for any  $\kappa \in (0, \bar{\kappa}]$ , the closed-loop system (2.7),(2.8), with  $\alpha$  selected as in (2.19), satisfies the properties (2.12a), (2.12b), and (2.12c) and the conditions of Item 2 of Theorem 2.2.3.

**Remark 2.2.6.** For instance, when  $p = 1$ , a simple choice of  $\beta$  satisfying the conditions in (2.23) is  $\beta(s) = \mu \operatorname{atan}(s)$  with  $\mu > 0$  to be selected sufficiently small.

A direct consequence of Proposition 2.2.5 and Theorem 2.2.3, is the following corollary.

**Corollary 2.2.6.1** (Regional constant output set-point tracking). *Under the assumptions of Proposition 2.2.5, the control law (2.8), with  $\alpha$  selected as in (2.19), satisfies the regional constant output set-point tracking problem for system (2.7) for some non-empty  $\mathcal{R} \subset \mathbb{R}^{n_e}$ , according to the statement of Item 2 of Theorem 2.2.3.*

**Proof.** [Proof of Proposition 2.2.5] Consider then the following change of coordinates

$$\chi = (x, \eta) \mapsto \mathbf{x} := (x, z), \quad z := \eta - M(x), \quad (2.24)$$

and recall that in view of Assumption 2.2.2 and the definition of  $\Psi$  in (2.20), we have  $L_g M(x)\Psi(x) = I$  for all  $x \in \mathbb{R}^{n_x}$ . As a consequence, in the  $\mathbf{x}$ -coordinates, the closed-loop system (2.7), (2.8), (2.19), with  $d = 0$  reads

$$\dot{\mathbf{x}} = F(\mathbf{x}) + \Gamma_r r$$

with

$$F(\mathbf{x}) := \begin{pmatrix} f_0(x) + \kappa g(x)\Psi(x)\beta(z) \\ -\kappa\beta(z) \end{pmatrix}, \quad \Gamma_r := \begin{pmatrix} 0 \\ -I \end{pmatrix}. \quad (2.25)$$

Now, from the triangular structure of  $F$ , the Jacobian  $J_F$  of  $F$  is given by

$$J_F(\mathbf{x}) := \frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}) = \begin{pmatrix} J_{11}(\mathbf{x}) & J_{12}(\mathbf{x}) \\ 0 & J_{22}(\mathbf{x}) \end{pmatrix} \quad (2.26)$$

where the components  $J_{11}$ ,  $J_{12}$  and  $J_{22}$  are defined as

$$J_{11}(\mathbf{x}) := \frac{\partial f_0}{\partial x}(x) + \kappa \frac{\partial}{\partial x} \left( g(x)\Psi(x) \right) \beta(z)$$

$$J_{12}(\mathbf{x}) := \kappa g(x)\Psi(x) \frac{\partial \beta}{\partial z}(z)$$

$$J_{22}(\mathbf{x}) := -\kappa \frac{\partial \beta}{\partial z}(z).$$

Consider a matrix-valued function  $\bar{\mathcal{P}} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}^{(n_x+n_e) \times (n_x+n_e)}$  of the form

$$\bar{\mathcal{P}}(\mathbf{x}) := \begin{pmatrix} P(x) & 0 \\ 0 & \mathbf{b}I \end{pmatrix}, \quad (2.27)$$

where  $P$  is given by Assumption 2.2.1 and  $\mathbf{b} > 0$  is some positive real number to be defined yet. By construction, the function  $\bar{\mathcal{P}}$  takes symmetric positive values. Then, let also the function  $\mathcal{R} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}^{(n_x+n_e) \times (n_x+n_e)}$  be defined as

$$\mathcal{R}(\mathbf{x}) := L_F \bar{\mathcal{P}}(\mathbf{x}) + \begin{pmatrix} \lambda_0 I & 0 \\ 0 & \kappa \frac{\partial \beta}{\partial z}(z) \end{pmatrix} \quad (2.28)$$

with  $\lambda_0$  given by Assumption 2.2.1. Observe that  $\mathcal{R}$  takes symmetric values. We want to show now that  $\mathcal{R}$  takes only negative definite values if  $\kappa$  is selected small enough. To this end, let us decompose  $\mathcal{R}$  as follows

$$\mathcal{R}(\mathbf{x}) := \begin{pmatrix} \mathcal{R}_{11}(\mathbf{x}) & \mathcal{R}_{12}(\mathbf{x}) \\ \mathcal{R}_{12}^\top(\mathbf{x}) & \mathcal{R}_{22}(\mathbf{x}) \end{pmatrix}. \quad (2.29)$$

By inspecting its components, by using the definition of  $J_F$  and the Killing vector property  $L_g P(x) = 0$ , see (2.16) in Assumption 2.2.1, we have

$$\mathcal{R}_{11}(\mathbf{x}) := L_{f_0} P(x) + \lambda_0 I + \kappa P(x) g(x) \frac{\partial}{\partial x} \Psi(x) \beta(z) + \kappa \left( P(x) g(x) \frac{\partial}{\partial x} \Psi(x) \beta(z) \right)^\top,$$

$$\mathcal{R}_{12}(\mathbf{x}) := \kappa P(x) g(x) \Psi(x) \frac{\partial \beta}{\partial z}(z),$$

$$\mathcal{R}_{22}(\mathbf{x}) := -\kappa(2\mathbf{b} - 1) \frac{\partial \beta}{\partial z}(z).$$

By using the bound in the Jacobian of  $\beta$  in (2.23), for any  $\mathbf{b} > \frac{1}{2}$ , we get

$$\mathcal{R}_{22}(\mathbf{x}) \prec 0$$

for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . Furthermore, this also guarantees that  $\mathcal{R}_{22}(\mathbf{x})$  is invertible for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . By the Schur's complement, it follows that  $\mathcal{R}(\mathbf{x})$  is negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$  if its Schur complement, denoted as  $\mathcal{S}_{\mathcal{R}}(\mathbf{x})$ , is also negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ , i.e. if

$$\mathcal{S}_{\mathcal{R}}(\mathbf{x}) = \mathcal{R}_{11} - \mathcal{R}_{12} \mathcal{R}_{22}^{-1} \mathcal{R}_{12}^\top \prec 0.$$

Now, by using the fact that the open-loop system is already incrementally uniformly globally exponentially stable, i.e.  $L_{f_0} P$  satisfies inequality (2.15), and that the functions  $P$ ,  $g$ ,  $\beta$  and the derivative of  $\Psi$  are bounded by assumption, see (2.15), (2.16), (2.21), and (2.23), it yields

$$\begin{aligned} \mathcal{R}_{11}(\mathbf{x}) &:= L_{f_0} P(x) + \lambda_0 I + \kappa P(x) g(x) \frac{\partial}{\partial x} \Psi(x) \beta(z) + \kappa \left( P(x) g(x) \frac{\partial}{\partial x} \Psi(x) \beta(z) \right)^\top \\ &\preceq -2\lambda_0 I + \lambda_0 I + 2\kappa \bar{p} \|g(x)\| \left\| \frac{\partial}{\partial x} \Psi(x) \right\| |\beta(z)| I \\ &\preceq -(\lambda_0 - 2\kappa \bar{p} \bar{g}) I \end{aligned} \tag{2.30}$$

for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . Moreover, by combining the bounds on  $M$  and  $g$  in (2.18), the controllability along trajectories assumption in (2.21) and by recalling the definition of  $\Psi$  in (2.20), we also obtain

$$\begin{aligned} \Psi(x) &\preceq \|\Psi(x)\| I \\ &= \|L_g M(x) (L_g M^\top L_g M(x))^{-1}\| I \\ &\preceq \frac{\left\| \frac{\partial M}{\partial x}(x) \right\| \|g(x)\|}{\|(L_g M^\top L_g M(x))^{-1}\|} I \\ &\preceq \frac{L_M \bar{g}}{b} I \end{aligned}$$

By combining all previous bounds together, we obtain

$$\begin{aligned} \mathcal{S}_{\mathcal{R}}(\mathbf{x}) &:= \mathcal{R}_{11}(\mathbf{x}) - \mathcal{R}_{12}(\mathbf{x})\mathcal{R}_{22}(\mathbf{x})^{-1}\mathcal{R}_{12}(\mathbf{x})^\top \\ &\preceq -(\lambda_0 - 2\kappa\bar{p}\bar{g})I + \frac{\kappa}{2\mathbf{b} - 1}P(x)g(x)\Psi(x)\frac{\partial\beta}{\partial z}(z)\Psi^\top(x)g^\top(x)P(x) \\ &\preceq -\left(\lambda_0 - \kappa\bar{p}\bar{g}\left(2 + \frac{\bar{p}L_M^2\bar{g}^3}{2(2\mathbf{b} - 1)b^2}\right)\right)I. \end{aligned}$$

Hence, with  $\mathbf{b} > \frac{1}{2}$ , selecting

$$\bar{\kappa} = \frac{\lambda_0}{2}\left(\bar{p}\bar{g}\left(2 + \frac{\bar{p}\bar{g}^3L_M^2}{2(2\mathbf{b} - 1)b^2}\right)\right)^{-1},$$

it implies that  $\mathcal{S}_{\mathcal{R}}(\mathbf{x}) \preceq 0$  for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$  and therefore  $\mathcal{R}(\mathbf{x}) \preceq 0$  for all  $\kappa \in (0, \bar{\kappa}]$  and for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . Consequently, recalling the definition of  $\mathcal{R}$  in (2.28), it yields

$$L_F\bar{\mathcal{P}}(\mathbf{x}) \preceq -\min\left\{\lambda_0, \kappa\frac{\partial\beta}{\partial z}(z)\right\}I. \quad (2.31)$$

Note that the metric  $\bar{\mathcal{P}}$  has been obtained in the  $\mathbf{x}$ -coordinates. To complete the proof, we need to come back into the original coordinates  $\chi = (x, \eta)$ . In particular by Proposition 1.2.12, the metric  $\mathcal{P}$  is defined as

$$\mathcal{P}(\chi) := E(x)^\top\bar{\mathcal{P}}(\chi)E(x), \quad E(x) := \begin{pmatrix} I & 0 \\ -\frac{\partial M}{\partial x}(x) & I \end{pmatrix}, \quad (2.32)$$

giving

$$\mathcal{P}(\chi) = \begin{pmatrix} P(x) + \mathbf{b}\frac{\partial M^\top}{\partial x}(x)\frac{\partial M}{\partial x}(x) & -\mathbf{b}\frac{\partial M^\top}{\partial x}(x) \\ -\mathbf{b}\frac{\partial M}{\partial x}(x) & \mathbf{b}I \end{pmatrix}.$$

Note that

$$E(x)^{-1} = \begin{pmatrix} I & 0 \\ \frac{\partial M}{\partial x}(x) & I \end{pmatrix},$$

and, with the Lipschitz condition on the norm of Jacobian of  $M$  in (2.21), we get

$$\|E(x)\| \leq 1 + L_M, \quad \|E^{-1}(x)\| \leq 1 + L_M,$$

for all  $x \in \mathbb{R}^{n_x}$ . Hence, for all  $v$  in  $\mathbb{R}^{n_x+n_e}$  and  $x \in \mathbb{R}^{n_x}$ ,

$$\begin{aligned} v^\top\mathcal{P}(\chi)v &\geq \min\{\underline{p}, \mathbf{b}\}|E(x)v|^2 \\ &\geq \min\{\underline{p}, \mathbf{b}\}\frac{|v|^2}{\|E^{-1}(x)\|^2} \\ &\geq \frac{\min\{\underline{p}, \mathbf{b}\}}{(1 + L_M)^2}|v|^2. \end{aligned}$$

On another hand,

$$\begin{aligned} v^\top \mathcal{P}(\chi)v &\leq \max\{\bar{p}, \mathfrak{b}\} |E(x)v|^2 \\ &\leq \max\{\bar{p}, \mathfrak{b}\} (1 + L_M)^2 |v|^2. \end{aligned}$$

Hence, the metric  $\mathcal{P}$  is uniformly upper and lower bounded by two constants, that is, inequality (2.12a) holds with

$$\bar{\mathfrak{p}} := \max\{\bar{p}, \mathfrak{b}\} (1 + L_M)^2, \quad \underline{\mathfrak{p}} := \frac{\min\{\underline{p}, \mathfrak{b}\}}{(1 + L_M)^2}. \quad (2.33)$$

Finally, by combining inequality (2.31) with the definition of  $\mathcal{P}$  in (2.32) and previous bounds, we have that the Lie derivative of the closed-loop vector field with respect to  $\mathcal{P}$  is non-uniformly negative definite, that is, inequality (2.12b) holds with the function  $\mathfrak{q} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}_{>0}$  defined as

$$\mathfrak{q}(\chi) = \frac{1}{(1 + L_M)^2} \min \left\{ \lambda_0, \kappa \frac{\partial \beta}{\partial x} (\eta - M(x)) \right\}, \quad (2.34)$$

which is always positive in view of the property of  $\beta$  in (2.23). Finally, since the metric  $\mathcal{P}$  is constant in the  $z$ -coordinates (indeed, by its definition, it does not depend on  $z$ ), and  $\Gamma_r$  is constant, it follows that  $L_{\Gamma_r} \mathcal{P}(\chi) = 0$  for all  $\chi \in \mathbb{R}^{n_x+n_e}$ , that is (2.12c) is satisfied.  $\square$

**Remark 2.2.7.** *Note that, according to Definition 1.2.1 in Chapter 1, Section 1.2, the closed-loop system is not  $\delta$ GUES. This implies that it is not possible to guarantee the existence of an equilibrium point for every value of  $r \in \mathbb{R}^{n_e}$ , but an equilibrium point exists if trajectories remain bounded. The main reason why the closed-loop system loses its incremental uniformity property, can be attributed to the term  $L_g M(x)$  being non-constant.*

### 2.2.2.3 Design for global constant output regulation

In the previous section, we have seen that under Assumptions 2.2.1 and 2.2.2, it is always possible to solve the regional constant output set-point tracking Problem. Three main drawbacks of the previous procedure can be highlighted.

- The first concerns the fact that the proposed design relies on the solution of (2.17) that may be hard to compute in practice.
- The second is related to the fact that the condition (2.12d) is not satisfied as  $\Gamma_d$  depends on  $x$  and therefore only the second part of Theorem 2.2.3 can be applied. In other words, global disturbance rejection is not generically ensured.
- The third is that the regulation task is achieved regionally and not globally.

To handle these two points, the idea now is to rely on a control structure that follows the lines of the *forwarding mod* $\{L_g V\}$  presented in Section 1.4.3.3. With such a choice:

- the proposed conditions, although more conservative, will allow for a design that is easier to apply since it doesn't rely on the exact solution of (2.17), but only on an approximation version of it. This generally simplifies the design of the control action, adding more degrees of freedom;

- this will allow us to include the presence of disturbances  $d \neq 0$  in (2.7) and to obtain a design that is global in the domain of attraction and in the size of the external signals  $(d, r)$ .

In particular, as highlighted in Remark 2.2.7, the idea is to use this freedom in the choice of the solution of (2.17) to force the term  $L_g M(x)$  to be constant. This particular condition will allow us to have a result that is global in the size of the external signals and to take into account the presence of disturbances  $d$ .

To make this Chapter more self-contained, we will highlight again the main assumptions, theorems, proofs, remarks, and comments that have been done in Section 1.4.3.3, with a specific viewpoint concerning the output regulation problem. To this end, we state a new assumption. It is a particular case of Assumption 1.4.5, due to the system's structure.

**Assumption 2.2.3** (Incremental forwarding mod $\{L_g V\}$ ). *We know a  $C^2$  function  $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$ , a  $C^2$  function  $\Delta : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$  a constant matrix  $\Lambda \in \mathbb{R}^{n_e \times n_u}$  and a positive real number  $\lambda_1 > 0$  satisfying the following properties:*

1.  $L_{f_0} \bar{M}(x) = h(x) + \Delta(x)$  for all  $x \in \mathbb{R}^{n_x}$ .
2.  $L_g \bar{M}(x) = \Lambda$  for all  $x \in \mathbb{R}^{n_x}$ .
3.  $\text{rank}(\Lambda) = n_e$ .
4. The following inequality holds

$$L_{f_0} P(x) - P(x)g(x)\Psi \frac{\partial \Delta}{\partial x}(x) - \frac{\partial \Delta^\top}{\partial x}(x)\Psi^\top g(x)^\top P(x) \leq -2\lambda_1 I, \quad (2.35)$$

for all  $x \in \mathbb{R}^{n_x}$ , with  $P$  given by Assumption 2.2.1, and  $\Psi$  defined as

$$\Psi := L_g \bar{M}(x)^\top [L_g \bar{M}(x)L_g \bar{M}(x)^\top]^{-1} = \Lambda^\top (\Lambda \Lambda^\top)^{-1}. \quad (2.36)$$

**Remark 2.2.8.** *Note that:*

- in Assumption 2.2.3 we ask  $L_g \bar{M}(x)$  to be constant for all  $x \in \mathbb{R}^{n_x}$ . Although such an assumption is in general much more stringent than asking for a constant rank as in Assumption 2.2.2, it is worth however noting that the definition of  $\bar{M}$  is now different, see item 1. In particular, the function  $\Delta$  represents the mismatch between the definition of  $M$  in (2.17) and  $\bar{M}$  in Assumption 2.2.3. Such a term  $\Delta$  can be therefore used as an extra degree of freedom, to achieve the desired regularity on  $L_g \bar{M}$  or to provide an ‘easy-to-compute’ approximate solution of (2.17);
- in the contraction analysis, the remaining term  $\Delta$  is managed via the robustness properties of the  $x$ -dynamics, which is assumed to be ‘sufficiently contractive’. This is expressed in the inequality (2.35). Evidently, in case  $\Delta(x) = 0$  for all  $x \in \mathbb{R}^{n_x}$ , the designs of  $\bar{M}$  and  $M$  coincide and items 2 and 3 read as a strongest version of Assumption 2.2.2, while item 4 is automatically satisfied by Assumption 2.2.1 with  $\lambda_1 := \lambda_0$ .

**Remark 2.2.9.** *Item 2 of Assumption 2.2.3 can be interpreted as the existence of a DC-gain approximation  $\Lambda$  of the system (2.14), which is constant for any linearization around any point of the state space.*

Based on the previous assumption, we focus on a control law of the form

$$\alpha(x, \eta) = \alpha_0(x) + \kappa \Psi (\eta - \bar{M}(x)) - \Psi \Delta(x), \quad (2.37)$$

where the matrix  $\Psi$  is defined as in (2.36),  $\Delta$  is the function of item 1 satisfying (2.35), and  $\kappa \in \mathbb{R}$  is a control gain parameter. We have then the following result.

**Proposition 2.2.10.** *Consider system (2.7) and suppose Assumptions 2.2.1 and 2.2.3 hold. Suppose, moreover, that there exists a positive real number  $L_M > 0$  such that*

$$\left\| \frac{\partial \bar{M}}{\partial x}(x) \right\| \leq L_M \quad (2.38)$$

holds for all  $x \in \mathbb{R}^{n_x}$ . Then, for any  $\kappa > 0$ , the closed-loop system (2.7),(2.8) with  $\alpha$  selected as in (2.37), satisfies the properties (2.12a), (2.12b), (2.12c), (2.12d) and the conditions of Item 1 of Theorem 2.2.3, namely, there exists  $\underline{q} > 0$  such that the function  $\mathbf{q}$  in (2.12b) satisfies  $\mathbf{q}(\chi) \geq \underline{q}$  for all  $\chi \in \mathbb{R}^{n_x+n_e}$ .

A direct consequence of Proposition 2.2.5 and Theorem 2.2.3 is the following result.

**Corollary 2.2.10.1** (Global constant output regulation). *Under the assumptions of Proposition 2.2.10, the control law (2.8b), with  $\alpha$  selected as in (2.37), solves the global constant output regulation problem in Problem 2.2.2 for system (2.7).*

**Proof.** [Proof of Proposition 2.2.10] Consider the change of coordinates

$$\chi := (x, \eta) \mapsto \mathbf{x} := (x, z), \quad z := \eta - \bar{M}(x)$$

with  $\bar{M}$  defined as in Assumption 2.2.3. By item 2 of Assumption 2.2.3, we have that  $L_g \bar{M}(x)$  is a constant matrix for all  $x \in \mathbb{R}^{n_x}$ . Moreover, by definition of  $\Psi$  in (2.36), we have that

$$L_g \bar{M}(x) \Psi = I.$$

Hence, the  $z$ -dynamics reads

$$\begin{aligned} \dot{z} &= h(x) - L_{f_0} \bar{M}(x) - L_g \bar{M}(x) \alpha(x, \eta) \\ &= h(x) - L_{f_0} \bar{M}(x) - \kappa L_g \bar{M}(x) \Psi z + L_g \bar{M}(x) \Psi \Delta(x) \\ &= -\kappa z, \end{aligned}$$

where we used the relation

$$L_{f_0} \bar{M}(x) = h(x) + \Delta(x)$$

given in Item 1 of Assumption 2.2.3. The closed-loop system (2.7), (2.8), (2.37) can be then compactly written as

$$\dot{\mathbf{x}} = F(\mathbf{x}) + \tilde{\Gamma}(\mathbf{x})w$$

with  $F$  and  $\tilde{\Gamma}$  defined as

$$F(\mathbf{x}) := \begin{pmatrix} f_0(x) + \kappa g(x)\Psi z - g(x)\Psi\Delta(x) \\ -\kappa z \end{pmatrix}, \quad \tilde{\Gamma}(\mathbf{x}) := \begin{pmatrix} g(x) & 0 \\ -\Lambda & -I \end{pmatrix}.$$

The Jacobian  $J_F$  of  $F$  is given by

$$J_F(\mathbf{x}) := \frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}) = \begin{pmatrix} J_{11}(\mathbf{x}) & J_{12}(\mathbf{x}) \\ 0 & J_{22}(\mathbf{x}) \end{pmatrix} \quad (2.39)$$

where the components  $J_{11}$ ,  $J_{12}$  and  $J_{22}$  are defined as

$$\begin{aligned} J_{11}(\mathbf{x}) &:= \frac{\partial f_0}{\partial x}(x) + \kappa \frac{\partial g}{\partial x}(x)\Psi z - g(x)\Psi \frac{\partial \Delta}{\partial x}(x), \\ J_{12}(\mathbf{x}) &:= \kappa g(x)\Psi, \\ J_{22}(\mathbf{x}) &:= -\kappa I. \end{aligned}$$

Consider a matrix-valued function  $\bar{\mathcal{P}} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}^{(n_x+n_e) \times (n_x+n_e)}$  of the form

$$\bar{\mathcal{P}}(\mathbf{x}) := \begin{pmatrix} P(x) & 0 \\ 0 & \mathbf{b}I \end{pmatrix}, \quad (2.40)$$

where  $P$  is the metric associated with the incremental stability property of the open-loop  $x$ -dynamics and it is given by Assumption 2.2.1 and  $\mathbf{b} > 0$  is some positive real number to be defined yet. Let the matrix function  $\mathcal{R} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}^{(n_x+n_e) \times (n_x+n_e)}$  be defined as

$$\mathcal{R}(\mathbf{x}) := L_F \bar{\mathcal{P}}(\mathbf{x}) + \begin{pmatrix} \lambda_1 I & 0 \\ 0 & \kappa I \end{pmatrix} \quad (2.41)$$

with  $\lambda_1$  given by (2.35). By construction,  $\mathcal{R}$  takes symmetric values. We want to show that for any value of  $\kappa$ , the function  $\mathcal{R}$  so defined is negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . To this end, let us decompose  $\mathcal{L}$  as follows

$$\mathcal{R}(\mathbf{x}) := \begin{pmatrix} \mathcal{R}_{11}(\mathbf{x}) & \mathcal{R}_{12}(\mathbf{x}) \\ \mathcal{R}_{12}^\top(\mathbf{x}) & \mathcal{R}_{22}(\mathbf{x}) \end{pmatrix}. \quad (2.42)$$

By using the Killing vector property  $L_g P(x) = 0$  in (2.16), we obtain

$$\begin{aligned} \mathcal{R}_{11}(\mathbf{x}) &:= L_{f_0} P(x) + \lambda_1 I - P(x)g(x)\Psi \frac{\partial \Delta}{\partial x}(x) - \frac{\partial \Delta^\top}{\partial x}(x)\Psi^\top g^\top(x)P(x), \\ \mathcal{R}_{12}(\mathbf{x}) &:= \kappa P(x)g(x)\Psi, \\ \mathcal{R}_{22}(\mathbf{x}) &:= -\kappa(2\mathbf{b} - 1)I. \end{aligned}$$

Trivially, for any  $\mathbf{b} > \frac{1}{2}$  we have that

$$\mathcal{R}_{22}(\mathbf{x}) \prec 0$$

for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . As a consequence,  $\mathcal{R}(\mathbf{x})$  is negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$  if its Schur complement, denoted as

$$\mathcal{S}_{\mathcal{R}}(\mathbf{x}) = \mathcal{R}_{11} - \mathcal{R}_{12}\mathcal{R}_{22}^{-1}\mathcal{R}_{12}^\top$$

is also negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . By using (2.35), we compute

$$\begin{aligned} \mathcal{S}_{\mathcal{R}}(\mathbf{x}) &:= \mathcal{R}_{11}(\mathbf{x}) - \mathcal{R}_{12}(\mathbf{x})\mathcal{R}_{22}(\mathbf{x})^{-1}\mathcal{R}_{12}(\mathbf{x})^\top \\ &\preceq L_{f_0}P(x) + \lambda_1 I - P(x)g(x)\Psi \frac{\partial \Delta}{\partial x}(x) - \frac{\partial \Delta^\top}{\partial x}(x)\Psi^\top g^\top(x)P(x) \\ &\quad - \kappa(P(x)g(x)\Psi)(-\kappa(2\mathbf{b}-1)I)^{-1}(\kappa P(x)g(x)\Psi)^\top \\ &\preceq -\lambda_1 I + \frac{\kappa}{2\mathbf{b}-1}P(x)g(x)\Psi\Psi^\top P(x)g(x)P(x) \\ &\preceq -\left(\lambda_1 - \frac{\kappa(\bar{p}\bar{g}k_1)^2}{2\mathbf{b}-1}\right)I, \end{aligned}$$

where  $k_1 = \sqrt{\|\Psi\Psi^\top\|}$ . Hence, by selecting

$$\mathbf{b} = \frac{1}{2} \left( 1 + \frac{\kappa(\bar{p}\bar{g}k_1)^2}{\lambda_1} \right). \quad (2.43)$$

we obtain  $\mathcal{S}_{\mathcal{R}}(\mathbf{x}) \preceq 0$  and consequently  $\mathcal{R}(\mathbf{x}) \prec 0$  for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . Consequently, recalling the definition of  $\mathcal{R}$  in (2.41), it yields

$$L_F\bar{\mathcal{P}}(\mathbf{x}) \preceq -\min\{\lambda_1, \kappa\}I. \quad (2.44)$$

Moreover, in light of the structure of  $\tilde{\Gamma}$  and the fact that  $\bar{\mathcal{P}}$  is independent of  $z$ , we deduce that  $L_{\tilde{\Gamma}}\bar{\mathcal{P}}(\mathbf{x}) = 0$  for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . In order to complete the proof, we need to come back into the original coordinates  $\chi = (x, \eta)$ . This can be done following the same steps as in the proof of Proposition 2.2.5 and get

$$\mathcal{P}(\chi) = \begin{pmatrix} P(x) + \mathbf{b} \frac{\partial \bar{M}^\top}{\partial x}(x) \frac{\partial \bar{M}}{\partial x}(x) & -\mathbf{b} \frac{\partial \bar{M}^\top}{\partial x}(x) \\ -\mathbf{b} \frac{\partial \bar{M}}{\partial x}(x) & \mathbf{b}I \end{pmatrix}.$$

Finally, by combining inequality (2.44) with the definition of  $\mathcal{P}$  and previous bounds, we have that the Lie derivative of the closed-loop with respect to the metric  $\mathcal{P}$  is uniformly negative definite, that is, inequality (2.12b) holds with the function  $\mathbf{q} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}_{>0}$  defined as

$$\mathbf{q} := \frac{1}{(1+L_M)^2} \min\{\lambda_1, \kappa\}, \quad (2.45)$$

which is always positive. Finally, since the metric  $\mathcal{P}$  is constant in the  $z$ -coordinates (indeed, by its definition, it does not depend on  $z$ ), it is not hard to see that the Killing vector property is satisfied, namely  $L_{\Gamma}\mathcal{P}(\chi) = 0$  for all  $\chi \in \mathbb{R}^{n_x+n_e}$ , that is, both (2.12c) and (2.12d) are satisfied.  $\square$

**Remark 2.2.11.** *The fact that the closed loop defines a uniform global contraction with respect to a metric  $\mathcal{P}(\chi)$  and that  $\Gamma$  is of Killing with respect to such  $\mathcal{P}$ , is a sufficient condition to show that the closed loop is incremental uniformly globally ISS (see Theorem 1.3.3) with respect to the input  $w$ . Therefore, in case  $w = (d, r)$  is a time-varying bounded and integrable signal, the trajectory of the closed loop is guaranteed to be bounded for all positive times. Moreover, if  $\lim_{t \rightarrow \infty} w(t) = w^*$  for some constant  $w^*$ , then the closed loop trajectory asymptotically tends to an equilibrium point where constant output regulation is achieved.*

**Remark 2.2.12.** *Note that Proposition 2.2.10 holds for any positive gain  $\kappa > 0$ . Therefore, the stabilizer (2.37) (besides  $\alpha_0$ ) is a control action with infinite gain margin.*

### 2.2.3 Output feedback design

In this section, we explore the case in which the control law used to stabilize the cascade system is not dependent on  $x$  but only on the state of the integrator itself. Such a problem has been studied for instance in Desoer and Lin (1985) or, more recently, in Simpson-Porco (2020); Lorenzetti and Weiss (2022)). Solutions in which the stabilization of the cascade system is achieved by a control law only dependent on the local state are not new in literature. For linear systems, this corresponds to the use of a pure  $I$ -regulator with a small gain, instead of a classical  $PI$ -regulator. In this section, we show how, by slightly strengthening the condition (2.35), the feedback law (2.37) can be modified into

$$\alpha(x, \eta) = \alpha_0(x) + \kappa \Psi \eta \quad (2.46)$$

where  $\alpha_0$  is the pre-stabilizing feedback needed to achieve contractivity of the  $x$ -dynamics (recall that for an open-loop contractive system,  $\alpha_0$  can be zero), and  $\kappa > 0$  is a positive gain to be selected, which will be taken *sufficiently small* (we lose the infinite gain margin property). With respect to control law (2.37), the term  $\kappa \Psi \bar{M}(x) + \Delta(x)$  is not needed. To this end, we state the following assumption.

**Assumption 2.2.4** (Robustness for output-feedback). *There exist positive real numbers  $\mathbf{a}, \lambda_2 > 0$  such that all the items of Assumption 2.2.3 are verified and moreover the following inequality*

$$\left( \mathbf{a}P(x)g(x)\Psi - \frac{1}{\mathbf{a}} \frac{\partial \Delta^\top}{\partial x}(x) \right) \left( \mathbf{a}P(x)g(x)\Psi - \frac{1}{\mathbf{a}} \frac{\partial \Delta^\top}{\partial x}(x) \right)^\top + L_{f_0}P(x) \preceq -2\lambda_2 I \quad (2.47)$$

holds for all  $x \in \mathbb{R}^{n_x}$ .

**Remark 2.2.13.** *Simple computation show that (2.47) implies (2.35) for some  $\lambda_1 \geq \lambda_2$ . Moreover, if  $\Delta(x) = 0$  for all  $x \in \mathbb{R}^{n_x}$ , the inequality (2.47) is always verified under open-loop incremental stability in Assumption 2.2.1. In particular, it can be directly derived from inequality (2.15) by selecting*

$$\mathbf{a} = \frac{1}{\bar{p}\bar{g}} \sqrt{\frac{\lambda_0}{2\|\Psi\Psi^\top\|}}, \quad \lambda_2 = \frac{\lambda_0}{2},$$

and  $\lambda_0, \bar{p}, \bar{g}$  as in (2.15), (2.16), and (2.21).

Under the above assumption, we can design a pure output-feedback control design to solve the global constant output regulation problem. This is shown in the following.

**Proposition 2.2.14.** *Consider system (2.7) and suppose Assumptions 2.2.1 and 2.2.4 hold and that there exists a strictly positive real number  $L_M > 0$  such that the inequality (2.38) holds for all  $x \in \mathbb{R}^{n_x}$ . Then, there exists a positive real number  $\bar{\kappa} > 0$ , such that, for any  $\kappa \in (0, \bar{\kappa}]$ , the closed-loop system (2.7),(2.8), with  $\alpha$  selected as in (2.46), satisfies the properties (2.12a), (2.12b), (2.12c), (2.12d) and the conditions of Item 1 of Theorem 2.2.3, namely, there exists  $\underline{q} > 0$  such that the function  $\mathbf{q}$  in (2.12b) satisfies  $\mathbf{q}(\chi) \geq \underline{q}$  for all  $\chi \in \mathbb{R}^{n_x+n_e}$ .*

Combining Proposition 2.2.14 with Theorem 2.2.3, we directly obtain the following result.

**Corollary 2.2.14.1** (Output-feedback global constant output regulation). *Under the assumptions of Proposition 2.2.14, the control law (2.8b), with  $\alpha$  selected as in (2.46), satisfies the global constant output regulation problem for system (2.7) for  $\kappa > 0$  small enough.*

**Proof.** [Proof of Proposition 2.2.14] Consider the change of coordinates

$$\chi := (x, \eta) \mapsto \mathbf{x} := (x, z), \quad z := \eta - \bar{M}(x)$$

with  $\bar{M}$  defined as in Assumption 2.2.3. The closed-loop system (2.7), (2.8), (2.46) reads as

$$\dot{\mathbf{x}} = F(\mathbf{x}) + \tilde{\Gamma}(\mathbf{x})w$$

with  $F$  and  $\tilde{\Gamma}$  defined as

$$F(\mathbf{x}) := \begin{pmatrix} f_0(x) + \kappa g(x)\Psi(z + \bar{M}(x)) \\ -\kappa z - \Delta(x) - \kappa \bar{M}(x) \end{pmatrix}, \quad \tilde{\Gamma}(\mathbf{x}) := \begin{pmatrix} g(x) & 0 \\ -\Lambda & -I \end{pmatrix}.$$

The Jacobian  $J_F$  of  $F$  is now of the form

$$J_F(\mathbf{x}) := \frac{\partial F}{\partial \mathbf{x}}(\mathbf{x}) = \begin{pmatrix} J_{11}(\mathbf{x}) & J_{12}(\mathbf{x}) \\ J_{21}(\mathbf{x}) & J_{22}(\mathbf{x}) \end{pmatrix},$$

where now the components  $J_{11}$ ,  $J_{12}$ ,  $J_{21}$  and  $J_{22}$  are defined as

$$J_{11}(\mathbf{x}) := \frac{\partial f_0}{\partial x}(x) + \kappa \frac{\partial g}{\partial x}(x)\Psi(z + \bar{M}(x)) + \kappa g(x)\Psi \frac{\partial \bar{M}}{\partial x}(x),$$

$$J_{12}(\mathbf{x}) := \kappa g(x)\Psi,$$

$$J_{21}(\mathbf{x}) := -\kappa \frac{\partial \bar{M}}{\partial x}(x) - \frac{\partial \Delta}{\partial x}(x),$$

$$J_{22}(\mathbf{x}) := -\kappa I.$$

Consider a matrix-valued function  $\bar{\mathcal{P}} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}^{(n_x+n_e) \times (n_x+n_e)}$  of the form

$$\bar{\mathcal{P}}(\mathbf{x}) := \begin{pmatrix} P(x) & 0 \\ 0 & \mathfrak{b}I \end{pmatrix}, \quad (2.48)$$

where  $P$  is given by Assumption 2.2.1 and  $\mathfrak{b} > 0$  is some positive real number to be defined yet. Let the function  $\mathcal{R} : \mathbb{R}^{n_x+n_e} \rightarrow \mathbb{R}^{(n_x+n_e) \times (n_x+n_e)}$  be defined as

$$\mathcal{R}(\mathbf{x}) := L_F \bar{\mathcal{P}}(\mathbf{x}) + \begin{pmatrix} \lambda_2 I & 0 \\ 0 & \frac{\kappa \mathfrak{b}}{2} I \end{pmatrix} \quad (2.49)$$

with  $\lambda_2$  given by (2.47). By construction,  $\mathcal{R}$  takes symmetric values. We want to show that, for  $\kappa$  sufficiently small, the function  $\mathcal{R}$  so defined is negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . To this end, let us decompose  $\mathcal{L}$  as follows

$$\mathcal{R}(\mathbf{x}) := \begin{pmatrix} \mathcal{R}_{11}(\mathbf{x}) & \mathcal{R}_{12}(\mathbf{x}) \\ \mathcal{R}_{12}^\top(\mathbf{x}) & \mathcal{R}_{22}(\mathbf{x}) \end{pmatrix}. \quad (2.50)$$

By using the Killing vector property  $L_g P(x) = 0$  in (2.16), we obtain

$$\mathcal{R}_{11}(\mathbf{x}) := L_{f_0} P(x) + \lambda_2 I + \kappa P(x) g(x) \Psi \frac{\partial \bar{M}}{\partial x}(x) + \kappa \frac{\partial \bar{M}^\top}{\partial x}(x) \Psi^\top g^\top(x) P(x),$$

$$\mathcal{R}_{12}(\mathbf{x}) := \kappa P(x) g(x) \Psi - \kappa \mathfrak{b} \frac{\partial \bar{M}^\top}{\partial x}(x) - \mathfrak{b} \frac{\partial \Delta^\top}{\partial x}(x),$$

$$\mathcal{R}_{22}(\mathbf{x}) := -\frac{3}{2} \kappa \mathfrak{b} I.$$

Trivially,

$$\mathcal{R}_{22}(\mathbf{x}) \prec 0$$

for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$  and for any  $\mathfrak{b} > 0$ . As a consequence,  $\mathcal{R}(\mathbf{x})$  is negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$  if its Schur complement, denoted as  $\mathcal{S}_{\mathcal{R}}(\mathbf{x}) = \mathcal{R}_{11} - \mathcal{R}_{12} \mathcal{R}_{22}^{-1} \mathcal{R}_{12}^\top$ ,

is also negative definite for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . By using Young's inequality we obtain

$$\begin{aligned}
\mathfrak{S}_{\mathcal{R}}(\mathbf{x}) &= \mathcal{R}_{11}(\mathbf{x}) - \mathcal{R}_{12}(\mathbf{x})\mathcal{R}_{22}(\mathbf{x})^{-1}\mathcal{R}_{12}(\mathbf{x})^\top \\
&= L_{f_0}P(x) + \lambda_2 I + \kappa P(x)g(x)\Psi \frac{\partial \bar{M}}{\partial x}(x) + \kappa \frac{\partial \bar{M}^\top}{\partial x}(x)\Psi^\top g^\top(x)P(x) \\
&\quad + \frac{2}{3\kappa \mathfrak{b}} \left[ \kappa P(x)g(x)\Psi - \kappa \mathfrak{b} \frac{\partial \bar{M}^\top}{\partial x}(x) - \mathfrak{b} \frac{\partial \Delta^\top}{\partial x}(x) \right] \\
&\quad \times \left[ \kappa P(x)g(x)\Psi - \kappa \mathfrak{b} \frac{\partial \bar{M}^\top}{\partial x}(x) - \mathfrak{b} \frac{\partial \Delta^\top}{\partial x}(x) \right]^\top \\
&\preceq L_{f_0}P(x) + \lambda_2 I + \kappa P(x)g(x)\Psi \frac{\partial \bar{M}}{\partial x}(x) + \kappa \frac{\partial \bar{M}^\top}{\partial x}(x)\Psi^\top g^\top(x)P(x) \\
&\quad + \left( \sqrt{\frac{\kappa}{\mathfrak{b}}} P(x)g(x)\Psi - \sqrt{\frac{\mathfrak{b}}{\kappa}} \frac{\partial \Delta^\top}{\partial x}(x) \right) \left( \sqrt{\frac{\kappa}{\mathfrak{b}}} P(x)g(x)\Psi - \sqrt{\frac{\mathfrak{b}}{\kappa}} \frac{\partial \Delta^\top}{\partial x}(x) \right)^\top \\
&\quad + 2\kappa \mathfrak{b} \frac{\partial \bar{M}^\top}{\partial x}(x) \frac{\partial \bar{M}}{\partial x}(x).
\end{aligned}$$

Hence, select  $\mathfrak{b} = \kappa/\mathfrak{a}^2$  with  $\mathfrak{a}$  given by (2.47). It gives

$$\begin{aligned}
\mathfrak{S}_{\mathcal{R}}(\mathbf{x}) &\preceq L_{f_0}P(x) + \lambda_2 I + \kappa P(x)g(x)\Psi \frac{\partial \bar{M}}{\partial x}(x) + \kappa \frac{\partial \bar{M}^\top}{\partial x}(x)\Psi^\top g^\top(x)P(x) \\
&\quad + 2\kappa \mathfrak{b} \frac{\partial \bar{M}^\top}{\partial x}(x) \frac{\partial \bar{M}}{\partial x}(x) + \left( \mathfrak{a} P g(x) \Psi - \frac{1}{\mathfrak{a}} \frac{\partial \Delta^\top}{\partial x}(x) \right) \left( \mathfrak{a} P g(x) \Psi - \frac{1}{\mathfrak{a}} \frac{\partial \Delta^\top}{\partial x}(x) \right)^\top
\end{aligned}$$

and therefore, by using (2.47), and the bounds (2.15), (2.16) and (2.21), we obtain

$$\mathfrak{S}_{\mathcal{R}}(\mathbf{x}) \preceq - \left( \lambda_2 - 2\kappa \bar{p} \bar{g} \|\Psi\| L_M - 2\kappa^2 \frac{L_M^2}{\mathfrak{a}^2} \right) I,$$

for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$ . Hence by selecting  $\kappa > 0$  small enough so that

$$\lambda_2 - 2\kappa \bar{p} \bar{g} \|\Psi\| L_M - 2\kappa^2 \frac{L_M^2}{\mathfrak{a}^2} \geq 0$$

and by recalling the definition of  $\mathcal{R}$  in (2.49) and the structure of  $\tilde{\Gamma}$ , we conclude that

$$L_F \bar{\mathcal{P}}(\mathbf{x}) \preceq - \min \left\{ \lambda_2, \frac{\kappa^2}{2\mathfrak{a}^2} \right\} I, \quad L_{\tilde{\Gamma}} \mathcal{P}(\mathbf{x}) = 0 \quad (2.51)$$

for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_e}$  and for all  $\kappa \in (0, \bar{\kappa}]$ . The proof concludes by coming back into the original coordinates  $\chi = (x, \eta)$  and by showing that the Killing vector property is satisfied. This can be done as in the proof of Proposition 2.2.10,

where inequality (2.12b) holds with

$$\underline{q} := \frac{1}{(1 + L_M)^2} \min \left\{ \lambda_2, \frac{\kappa^2}{2\mathfrak{a}^2} \right\}, \quad (2.52)$$

which is always positive. □

## 2.2.4 Examples

### 2.2.4.1 A simple motivating example

Consider a nonlinear system of the form (2.7) with  $n_x = 3$ ,  $n_u = 2$ ,  $n_e = 2$  and functions  $f, g, h$  selected as

$$f(x) = \begin{pmatrix} -x_1 - x_1^3 \\ -x_2 - \frac{1}{2}x_2 \cos(x_2) \\ -2x_3 \end{pmatrix}, \quad g(x) = \begin{pmatrix} 1 & 1 \\ 2 + \cos(x_2) & 0 \\ 0 & 1 \end{pmatrix},$$

$$h(x) = \begin{pmatrix} x_1^3 - 2x_2 - \sin(x_2) \\ x_3 \end{pmatrix}$$

Following (Isidori, 1995, Chapter 5), we compute

$$L_g h(x) = \begin{pmatrix} 3x_1^2 - (2 + \cos(x_2))^2 & 3x_1^2 \\ 0 & 1 \end{pmatrix}$$

which is not invertible for all  $x$ , as it contains some singularities, for instance in  $(x_1, x_2) = (\pm\sqrt{3}, 1)$ . Therefore, it does not exist a global diffeomorphism transforming this system into a *normal form* (Isidori, 1995, Chapter 5). Yet, a solution to the global constant output regulation Problem does exist. Indeed, it's possible to check that Assumptions 2.2.1 and 2.2.3 hold. First, we can check that inequality (2.15) is verified with  $\alpha_0(x) = 0$ ,

$$P(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2 + \cos(x_2))^{-2} & 0 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}$$

and  $\lambda_0 = 1$ . Indeed,

$$L_f P(x) = \begin{pmatrix} -2 - 6x_1^2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \preceq -I.$$

Furthermore, it can be verified that the Killing vector property (2.16) is verified. Hence Assumption 2.2.1 holds. Then, selecting

$$\begin{aligned}\bar{M}(x) &= \begin{pmatrix} -x_1 \\ 1 \\ -\frac{1}{2}x_3 \end{pmatrix}, & \Lambda &= \begin{pmatrix} -1 & -1 \\ 0 & -\frac{1}{2} \end{pmatrix}, \\ \Delta(x) &= \begin{pmatrix} x_1 + 2x_2 + \sin(x_2) \\ 0 \end{pmatrix}, & \Psi &= \begin{pmatrix} -1 & 2 \\ 0 & -2 \end{pmatrix}.\end{aligned}$$

verifies items 1-3 of Assumption 2.2.3. Finally, item 4 and in particular inequality (2.35) is satisfied for  $\lambda_1 = \frac{1}{2}$ . Hence, Corollary 2.2.10.1 holds and the control law (2.37), reading

$$\begin{aligned}\dot{\eta} = h(x) - r &= \begin{pmatrix} x_1^3 - 2x_2 - \sin(x_2) - r_1 \\ x_3 - r_2 \end{pmatrix} \\ u = \alpha(x, \eta) &= \begin{pmatrix} -\kappa\eta_1 + 2\kappa\eta_2 + (1 - \kappa)x_1 + (2 + \kappa)x_2 + \sin(x_2) \\ -\kappa(\eta_2 + x_3) \end{pmatrix},\end{aligned}$$

solves the global constant output regulation Problem for any  $\kappa > 0$ .

#### 2.2.4.2 Lipschitz systems

In this section, we specialize our previous results for the class of linear systems coupled with a Lipschitz nonlinearity. In particular, consider a system of the form

$$\begin{aligned}\dot{x} &= Ax + B(u + d) + G\vartheta(\nu), & \nu &= Hx \\ e &= Cx + D\vartheta(\nu) - r\end{aligned}\tag{2.53}$$

where  $A, B, C, G, D, H$  are constant matrices of suitable dimension and  $\vartheta : \mathbb{R}^{n_\nu} \mapsto \mathbb{R}^{n_\vartheta}$  is a  $C^1$  Lipschitz function with Lipschitz constant  $\vartheta_L$ . In order to apply the results proposed in Section 2.2.2, we specialize the assumptions for such a class of systems. Therefore, we suppose that the following assumption holds.

**Assumption 2.2.1'.** *Consider system (2.53). There exist a constant symmetric positive definite matrix  $P$  and a positive real number  $\lambda_0 > 0$  satisfying the inequality*

$$PA + A^\top P + PG \frac{\partial \vartheta}{\partial \nu}(\nu)H + H^\top \frac{\partial \vartheta^\top}{\partial \nu}(\nu)G^\top P \preceq -2\lambda_0 I$$

for all  $x \in \mathbb{R}^{n_x}$ .

Such an assumption is a particular case of Assumption 2.2.1 in which we considered a constant Euclidean metric  $P$ . Assumption 2.2.1' can be satisfied after a preliminary state-feedback, by following for instance the results in Section 1.4.4. Following an incremental forwarding  $\text{mod}\{L_g V\}$  design, instead of looking for the exact solution  $M$  of (2.17), we look for an approximation  $\bar{M}$  that is obtained by considering only the linear terms of (2.53). Let us define the following functions and matrices

$$\begin{aligned}\bar{M}(x) &:= CA^{-1}x, & \Lambda &:= CA^{-1}B, \\ \Delta(x) &:= (CA^{-1}G - D)\vartheta(Hx), & \Psi &= \Lambda^\top (\Lambda \Lambda^\top)^{-1}.\end{aligned}$$

With these definitions and by using the Lipschitz constant of  $\vartheta$ , it follows that inequality (2.35) in Assumption 2.2.3 is satisfied if

$$PB\Psi(CA^{-1}G - D)H + [PB\Psi(CA^{-1}G - D)H]^\top \preceq \frac{2(\lambda_0 - \lambda_1)}{\vartheta_L} I, \quad (2.54)$$

for some  $\lambda_1 > 0$ . Note that inequality (2.54) is verified with  $\lambda_1 = \lambda_0$  when  $CA^{-1}D = G$  or for some  $\lambda_1 < \lambda_0$  when  $\vartheta_L$  is sufficiently small compared to  $\lambda_0$ . Furthermore, inequality (2.47) in Assumption 2.2.4 reads

$$\left( \mathbf{a}PB\Psi - \frac{\vartheta_L}{\mathbf{a}}(CA^{-1}G - D)H \right) \left( \mathbf{a}PB\Psi - \frac{\vartheta_L}{\mathbf{a}}(CA^{-1}G - D)H \right)^\top \preceq 2(\lambda_0 - \lambda_2)I \quad (2.55)$$

for some  $\lambda_2, \mathbf{a} > 0$ . Hence, the following can be stated.

**Corollary 2.2.14.2** (Global constant output regulation for Lipschitz systems). *Consider system (2.53) and suppose Assumption 2.2.1' holds. Then, the following holds.*

- If  $\Lambda$  is full rank and the inequality (2.54) is satisfied for some  $\lambda_1 > 0$ , the dynamic control law

$$\begin{aligned} \dot{\eta} &= e, \\ u &= \kappa\Psi(\eta - CA^{-1}x) - \Psi(CA^{-1}G - D)\vartheta(Hx) \end{aligned}$$

solves the global constant output regulation problem for any  $\kappa > 0$ .

- Moreover if the inequality (2.55) is satisfied for some  $\lambda_2, \mathbf{a} > 0$ , then the output-feedback control law

$$\begin{aligned} \dot{\eta} &= e \\ u &= \kappa\Psi\eta \end{aligned}$$

solves the global constant output regulation problem for  $\kappa$  sufficiently small.

### 2.2.4.3 Minimum phase systems

As stated in Section 2.1.2, most of the results in output regulation literature focus on systems possessing a well defined relative degree ((Isidori, 1995, Chapter IV)). See, for instance, Khalil (2000); Behtash (1990); Jiang and Marcelis (2001). In this section, we aim to show that the proposed conditions for the incremental forwarding  $\text{mod}\{L_g V\}$  control action, are satisfied for the class of systems with a globally well-defined normal form possessing a minimum phase property. We focus on SISO systems that admit a globally defined normal form with a unitary relative degree. The extension to a higher relative degree can be dealt with with canonical tools, see, e.g., (Serrani et al., 2001, Section V)). In such a case, system (2.7) can be rewritten as

$$\begin{aligned} \dot{z} &= \psi(z, y) \\ \dot{y} &= q(z, y) + b(z, y)(u + d), \\ e &= y - r \end{aligned} \quad (2.56)$$

where  $z \in \mathbb{R}^{n_x-1}$  is the so-called zero-dynamics,  $y \in \mathbb{R}$  is the output to be regulated,  $u \in \mathbb{R}$  is the control,  $r \in \mathbb{R}$  is the reference and  $d \in \mathbb{R}$  represents some constant perturbation.

By using the change of coordinates  $y \mapsto e := y - r$ , system (2.56) can be alternatively rewritten as

$$\begin{aligned}\dot{z} &= \bar{\psi}(w, z, e), \\ \dot{e} &= \bar{q}(w, z, e) + \bar{b}(w, z, e)u,\end{aligned}$$

with

$$\bar{\psi}(w, z, e) := \psi(z, e+r), \quad \bar{q}(w, z, e) := q(z, e+r) + b(z, e+r)d, \quad \bar{b}(w, z, e) := b(z, e+r).$$

Concerning the zero-dynamics, it is worth recalling that very few works addressed the case of non-minimum-phase, that is when the dynamics

$$\dot{z} = \bar{\psi}(w, z, 0) \tag{2.57}$$

is possibly unstable. Most of the works, indeed, focused on the case in which such a zero-dynamics possess a unique steady-state trajectory (possibly that depends on  $w$ ) which is attractive with a given domain of attraction. See, for instance, (Serrani et al., 2001, Assumption V.1), (Khalil, 2000, Assumption 4), (Mahmoud and Khalil, 1996, Assumption 4), (Seshagiri and Khalil, 2005, Assumption 4), (Marino and Tomei, 2007, Assumption H2), and many others. This, in turn, corresponds to asking that the subsystem (2.57) possesses some incremental stability property on a given set of interest, since the attractivity properties are uniform with respect to  $w$ . In the global framework considered in this work, this corresponds to our Assumption 2.2.1.

We focus here on minimum-phase systems (2.56) possessing a constant, and therefore, without loss of generality, unitary high frequency gain, that is  $b(z, y) = 1$ . System (2.56) reads

$$\begin{aligned}\dot{z} &= \psi(z, y) \\ \dot{y} &= q(z, y) + u + d \\ e &= y - r.\end{aligned} \tag{2.58}$$

In our framework, the minimum-phase assumption is stated as follows.

**Assumption 2.2.5** (Incremental minimum-phase). *Consider system (2.58). There exist positive real numbers  $\bar{q}$ ,  $\bar{\psi}_y$ ,  $\bar{p}_z$ ,  $\underline{p}_z$ ,  $\lambda_z > 0$  and a function  $P_z : \mathbb{R}^{n_x-1} \mapsto \mathbb{R}^{(n_x-1) \times (n_x-1)}$  taking symmetric positive values so that the following inequalities<sup>2</sup> hold*

$$\left| \frac{\partial q}{\partial x}(z, y) \right| \leq \bar{q}, \quad \left| \frac{\partial \psi}{\partial y}(z, y) \right| \leq \bar{\psi}_y, \tag{2.59}$$

$$\underline{p}_z I \preceq P_z(z) \preceq \bar{p}_z I, \quad L_\psi P_z(z, y) \preceq -2\lambda_z I, \tag{2.60}$$

for all  $(z, y) \in \mathbb{R}^{n_x}$ .

We have the following result.

**Proposition 2.2.15.** *Consider system (2.58) and suppose Assumption 2.2.5 is satisfied. Then, there exists a positive real number  $\underline{\sigma} \geq 1$  such that, for any fixed  $\sigma \geq \underline{\sigma}$ ,*

<sup>2</sup>The notation  $L_\psi P_z(z, y)$  has to be understood as the Lie derivative of  $P_z$  along the vector field  $z \mapsto \psi(z, y)$  where  $y$  is fixed.

Assumptions 2.2.1, 2.2.3 and 2.2.4 are satisfied with

$$\begin{aligned} P(z, y) &= \begin{pmatrix} P_z(z) & 0 \\ 0 & \sigma^{-\frac{2}{3}} \end{pmatrix}, & \underline{p} &= \min \left\{ \underline{p}_z, \sigma^{-\frac{2}{3}} \right\}, & \bar{p} &= \max \{ \bar{p}_z, 1 \}, \\ \alpha_0(z, y) &= -\sigma y, & \lambda_0 &= \frac{\lambda_z}{2}, & \lambda_1 &= \lambda_2 = \frac{\lambda_z}{4}, & \mathbf{a} &= \sigma^{-\frac{2}{3}}, \\ \bar{M}(z, y) &= -\frac{y}{\sigma}, & \Delta(z, y) &= -\frac{q(z, y)}{\sigma}, & \Lambda &= \Psi^{-1} = -\frac{1}{\sigma}, \end{aligned}$$

with  $P_z$ ,  $\bar{p}_z$ ,  $\underline{p}_z$  and  $\lambda_z$  given by Assumption 2.2.5.

As a consequence of the previous proposition, we have the following corollary that can be derived from Proposition 2.2.15 and Corollary 2.2.14.1.

**Corollary 2.2.15.1** (Global constant output regulation for minimum-phase systems). *Under the assumptions of Proposition 2.2.15, the control law*

$$\begin{aligned} \dot{\eta} &= e, \\ u &= -\sigma(y + \kappa\eta), \end{aligned} \tag{2.61}$$

solves the global constant output regulation Problem for system (2.58) for two strictly positive real numbers  $\kappa > 0$  small enough and  $\sigma > 0$  large enough.

**Proof.** [Proof of Proposition 2.2.15] Let  $\underline{\sigma}$  be defined as

$$\underline{\sigma}^{\frac{1}{3}} := \max \left\{ \bar{q} + \frac{\lambda_z}{2} + \frac{(\bar{p}_z \bar{\psi}_y + \bar{q})^2}{2\lambda_z}, 2\sqrt{\frac{\bar{q}}{\lambda_z}}, \sqrt{\frac{2}{\lambda_z}}(1 + 2\bar{q}), 1 \right\}. \tag{2.62}$$

Now, with the control  $u = \alpha_0(x, z)$  system (2.58) is in the form (2.7) and (2.14) with  $x = (z, y)$  and

$$f_0(x) = \begin{pmatrix} \psi(z, y) \\ q(z, y) - \sigma y \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h(x) = y.$$

With the function,  $P$  defined as in the statement of the proposition, let us define

$$\mathcal{J}(x) := L_{f_0}P(x) + \lambda_z I = \begin{pmatrix} \mathcal{J}_{11}(x) & \mathcal{J}_{12}(x) \\ \mathcal{J}_{12}(x)^\top & \mathcal{J}_{22}(x) \end{pmatrix}, \tag{2.63}$$

where

$$\begin{aligned} \mathcal{J}_{11}(x) &:= L_\psi P_z(z) + \lambda_z I, \\ \mathcal{J}_{12}(x) &:= P_z(z) \frac{\partial \psi}{\partial y}(z, y) + \sigma^{-\frac{2}{3}} \frac{\partial q}{\partial z}(z, y), \\ \mathcal{J}_{22}(x) &:= -2\sigma^{\frac{1}{3}} + 2\sigma^{-\frac{2}{3}} \frac{\partial q}{\partial y}(z, y) + \lambda_z. \end{aligned}$$

By using the bound on the Jacobian of  $q$  given in Assumption 2.2.5 and the definition of  $\underline{\sigma}$  in (2.62), we have

$$\mathcal{J}_{11}(x) \prec 0$$

and

$$\mathcal{J}_{22}(x) \prec 0$$

for all  $x \in \mathbb{R}^{n_x}$  and all  $\sigma \geq \underline{\sigma}$ . Then, by denoting with  $\mathcal{S}_{\mathcal{J}} = \mathcal{J}_{11} - \mathcal{J}_{12}\mathcal{J}_{22}^{-1}\mathcal{J}_{12}^{\top}$  the Schur complement of  $\mathcal{J}$ , by using again the bounds on the Jacobian of  $q$  and  $\psi$  and  $\sigma^{-\frac{2}{3}} \leq 1$ , we compute

$$\mathcal{S}_{\mathcal{J}}(x) \preceq - \left( \lambda_z - \frac{(\bar{p}_z \bar{\psi}_z + \bar{q})^2}{2\sigma^{\frac{1}{3}} - 2\bar{q} - \lambda_z} \right) I \preceq 0 \quad \forall x \in \mathbb{R}^{n_x},$$

for all  $\sigma \geq \underline{\sigma}$ . Therefore  $\mathcal{J}(x) \preceq 0$  for all  $x \in \mathbb{R}^{n_x}$  which shows that the system with the pre-stabilizing control action  $\alpha_0(x)$  is incrementally uniformly globally exponentially stable with respect to the metric  $P$ , namely, inequality (2.15) with  $\lambda_0$  selected as in the statement of the theorem holds. Finally, since the function,  $P$  is defined in the statement of the proposition and the function  $g$  are constant in the  $y$ -coordinates (and zero otherwise), the Killing vector property  $L_g P(x) = 0$  is trivially satisfied. This shows that Assumption 2.2.1 holds.

**Remark 2.2.16.** *Note that an alternative proof can be done, in the case of constant metric  $P_z$ , by following (Pavlov et al., 2009, Theorem 2).*

Now, in order to show Assumption 2.2.3, first note that with the definition of  $\bar{M}, \Delta, \Lambda, \Psi$  given in the statement, Items 1, 2, and 3 are trivially satisfied with simple computations. Then, to show inequality (2.35), we define

$$Q_1(x) := -P(x)g(x)\Psi \frac{\partial \Delta}{\partial x}(x) - \frac{\partial \Delta^{\top}}{\partial x}(x)\Psi^{\top}g(x)^{\top}P(x).$$

By omitting computations and using the bounds on  $q$  in (2.59), we have

$$Q_1(x) = \sigma^{-\frac{2}{3}} \begin{pmatrix} 0 & \frac{\partial q}{\partial z}q(z, y) \\ \frac{\partial q^{\top}}{\partial z}q(z, y) & \frac{\partial q}{\partial y}q(z, y) \end{pmatrix} \preceq 2\sigma^{-\frac{2}{3}}\bar{q}I,$$

that is,  $Q_1(x) \preceq \frac{\lambda_z}{2}I$  for all  $x \in \mathbb{R}^{n_x}$  and all  $\sigma \geq \underline{\sigma}$ . Therefore, by recalling the definition of  $\mathcal{J}$  given in (2.63) and recalling that  $\mathcal{J}(x) \preceq 0$  for all  $x \in \mathbb{R}^{n_x}$ , inequality (2.35) reads

$$L_{f_0}P(x) + Q_1(x) \preceq \mathcal{J}(x) - \frac{\lambda_z}{2}I \preceq -\frac{\lambda_z}{2}I$$

for all  $x \in \mathbb{R}^{n_x}$  and for all  $\sigma \geq \underline{\sigma}$ . Hence, inequality (2.35) holds with  $\lambda_1$  given as in the statement of the proposition showing Assumption 2.2.3.

Finally, in order to show Assumption 2.2.4, we define

$$Q_2(x) := T(x)T^{\top}(x), \quad T(x) := \mathbf{a}P(x)g(x)\Psi - \frac{1}{\mathbf{a}} \frac{\partial \Delta^{\top}}{\partial x}(x),$$

which gives

$$T^{\top}(x) = \sigma^{-\frac{1}{3}} \begin{pmatrix} \frac{\partial q}{\partial z}(z, y) & \frac{\partial q}{\partial y}(z, y) - 1 \end{pmatrix}$$

with  $\mathbf{a}$  selected as in the statement of the Proposition. By using the bounds in (2.59), we obtain

$$Q_2(x) \preceq \|T(x)\|^2 I \preceq \sigma^{-\frac{2}{3}}(2\bar{q} + 1)^2 I \preceq \frac{\lambda_z}{2} I$$

for all  $x \in \mathbb{R}^{n_x}$  and all  $\sigma \geq \underline{\sigma}$ . Hence, by following the previous steps, we obtain

$$L_{f_0}P(x) + Q_2(x) \preceq \mathcal{J}(x) - \frac{\lambda_z}{2} I \preceq -\frac{\lambda_z}{2} I$$

for all  $x \in \mathbb{R}^{n_x}$  and for all  $\sigma \geq \underline{\sigma}$ . This shows inequality (2.47) with  $\lambda_2$  given as in the statement of the proposition and concludes the proof.  $\square$

**Example 2.2.17.** We consider the example of a flexible joint robot manipulator presented in Zemouche and Boutayeb (2013). The system can be described with a dynamical model of the form (2.53) where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 + 3.33\ell & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3.33 \end{pmatrix},$$

$$H = (0 \ 0 \ 1 \ 0), \quad C = (1 \ 0 \ 0 \ 0), \quad D = 0,$$

where  $\vartheta(\nu) = \sin(\nu) + \ell\nu$  and where  $\ell$  is an additional degree of freedom. We aim to achieve global constant output regulation. For this, first we look for a feedback design so that the system satisfies Assumption 2.2.1. To this end, as the open-loop system is composed by a linear part and a monotonic nonlinearity, we rely on the results presented in Section 1.4.4. In particular, the system satisfies the monotonic Assumption 1.4.6 for any  $\ell > 1$  with  $\Gamma = \ell - 1$ . Therefore, for the design of the preliminary feedback we rely on Proposition 1.4.25. The LMI is solvable for  $\ell = 1.1$  and the system in closed-loop with the preliminary feedback with

$$u = Kx + N\vartheta(Hx), \quad K = [-0.0040 \quad -0.0002 \quad -0.0009 \quad -0.0011], \quad N = -5.3490$$

is incrementally globally uniformly exponentially stable with respect to a constant metric  $P$ . Moreover, the Killing vector assumption is automatically satisfied. We extend now the system with an integral action processing the regulation error

$$\dot{\eta} = Cx - r$$

where  $r$  is the constant reference that has to be tracked. Now, we aim to satisfy Assumption 2.2.3. For this, we follow the suggestion in Section 2.2.4.2. We take  $\bar{M}(x) = CA^{-1}x$  and  $\Delta(x) = CA^{-1}G\vartheta(\nu)$ . This choice is admissible since  $L_g\bar{M}(x) = CA^{-1}B \neq 0$ . A simple computation moreover shows that (2.35) is satisfied. Therefore, the system in closed-loop with the dynamic control law

$$\dot{\eta} = Cx - r$$

$$u = Kx + N\vartheta(Hx) + \frac{\kappa}{CA^{-1}B}(\eta - CA^{-1}x) - \frac{CA^{-1}G}{CA^{-1}B}\vartheta(Hx)$$

for any  $\kappa > 0$  solves the global constant output regulation problem, for any  $r \in \mathbb{R}$ .

## 2.3 Global harmonic regulation

### 2.3.1 From incremental stability to global harmonic regulation

#### 2.3.1.1 Introduction

In Section 2.2 we showed that it is possible to solve the global constant output regulation Problem by casting the control design problem into the incremental framework. If the closed-loop system possesses some incremental uniform ISS properties with respect to the (constant) external signal, then its trajectories asymptotically converge to an equilibrium point where, thanks to the presence of the integral action, the regulation error is zero. By taking advantage of the closed-loop structure, we proposed a design of the stabilizing unit based on incremental forwarding tools developed in Section 1.4.3.3.

Similarly to the previous section, we consider now a system of the form (2.7). In this case, however, we do not assume the external signals to be constant, but  $T$ -periodic. In other words, we assume the following.

**Assumption 2.3.1** ( $T$ -periodic exosystem). *Consider system (2.7). The signals  $r : \mathbb{R} \mapsto \mathbb{R}^{n_e}$ ,  $d : \mathbb{R} \mapsto \mathbb{R}^{n_u}$  are smooth time-varying external references with finite  $\mathcal{L}^\infty$ -norm which are  $T$ -periodic, i.e. there exists  $T > 0$  such that*

$$r(t) = r(t + T), \quad \text{and} \quad d(t) = d(t + T). \quad (2.64)$$

**Remark 2.3.1.** *Note that the case in which  $(r, d)$  is constant signals, is a sub-case of Assumption 2.3.1 (take any  $T > 0$ ).*

As we said in Section 2.1.2 when considering non-constant exosignals the design of the control action becomes much more complicated. It is clear indeed that the integral action is a *perfect* internal model unit for constant exosignals. In other words, the design of the dynamical part of the closed-loop control action can be designed without the knowledge of the plant's dynamics. Nevertheless, *asymptotic* output regulation can be achieved. This is not the case in presence of non-constant external signals. In such a case, the internal model unit strongly depends on the plant's dynamic and on the stabilizing unit.

In this section, we focus on a control design achieving *harmonic regulation*. With 'harmonic regulation' we mean that we aim to cancel the Fourier coefficients of the steady state solution of the regulation error. Despite no formal proof exists guaranteeing that such a condition implies that the norm of the regulation error reduces (except for the special case of unitary relative degree minimum phase systems in normal form, see Astolfi et al. (2022b)), our interest is motivated by the following aspects.

- Harmonic regulation is a useful tool in practical applications, see for instance the results in Section 4.3.
- There's a strong link with Repetitive Control design structures, see Remark 2.1.9.
- 'Perfects' internal model units can be designed using an infinite dimensional internal model modeled as a delay of time  $T$ , see Astolfi et al. (2021). A delay has the effect of placing an infinite number of poles at  $T$  and its multiples,

canceling all the harmonic contents (i.e. the Fourier coefficients) at frequencies of the delay and all its multiplies.

As we will see in the following, the proposed approach relies on the technique of *adding oscillators and feedback stabilizers*. In other words, we will design the internal model unit as a bunch of linear oscillators at frequencies that should vanish. Then, we will design the stabilizing unit to guarantee an incremental uniform global ISS property of the closed-loop with respect to the exosystem. Indeed:

- if the plant is incrementally globally uniformly ISS with respect to the exosignal  $w = (d, r)$ , the trajectories of the closed-loop system will converge asymptotically to a periodic solution with the same period  $T$  of the exosystem (see Lemma 1.2.6). However, because of the nonlinearities, they will present a harmonic content involving the exosystem's frequencies and also *higher-order terms*, see Khalil (1993); Huang and Lin (1993); Priscoli (1993).
- if the closed-loop system possesses some incremental stability properties, because of the choice of the internal model unit as a bunch of the linear oscillators, it is possible to show that the Fourier coefficients associated with the frequencies of such oscillators will be zero.

We stress again that we focus on the *global* harmonic regulation problem, that is, we aim to achieve harmonic regulation independently on the domain of attraction and on the amplitude of the external signals. In other words, we allow the external signals  $(r, d)$  to span the whole  $\mathbb{R}^{n_e} \times \mathbb{R}^{n_u}$  and we look for a global result in the domain of attraction of the state-space. The existing results have been developed only for the class of nonlinear systems having a globally defined normal form and possessing a minimum phase property Astolfi et al. (2022b, 2021). On the other hand, for systems working in the 'original' coordinates the existing results are only local in the amplitude of the external signals (Astolfi et al. (2022a); Ghosh and Paden (2000)), that is, with the external signal's amplitude being sufficiently small. As we do not want to make use of normal forms but still provide a global result, we cast again the problem into the incremental framework.

**Remark 2.3.2.** *For the sake of completeness, we recall a different approach to achieve harmonic regulation, that is for instance the one proposed in Riedinger and Daafouz (2022); Blin et al. (2021). The authors study the problem in the frequency domain through a Fourier decomposition and design the control action in different coordinates. As such an approach presents several differences with respect to the proposed one, we won't enter in details.*

### 2.3.1.2 Problem statement and internal model unit design

The problem that we address in this section is the design of a state-feedback control law that can achieve *harmonic regulation*. With 'harmonic regulation' we mean that the trajectories of system (2.7) in closed-loop with a dynamical control action of the form

$$\dot{\eta} = \phi(\eta, e), \quad (2.65a)$$

$$u = \alpha(x, \eta) \quad (2.65b)$$

are bounded in forward time and the error  $e$  does not present a frequency content at particular selected frequencies. We formalize our problem in the following.

**Problem 2.3.3** (Global harmonic regulation). *Consider system (2.7). Find a dynamical control law of the form (2.65) such that:*

1. *if  $(d, r) = (0, 0)$ , the origin of the closed-loop (2.7), (2.65) is globally asymptotically stable;*
2. *For any  $(d, r)$  satisfying Assumption 2.3.1 the closed-loop system (2.7), (2.65) has bounded trajectories and there exists a trajectory  $e^*(t)$  that is globally attractive for the error  $e$  and such that it does not present an harmonic content at the frequencies  $\omega_\ell = \ell \frac{2\pi}{T}$  for  $\ell = 0, 1, \dots, L$  for some  $L \geq 0$ . Namely, the first  $L$ -Fourier coefficients of  $e^*$  are zero, i.e.*

$$c_\ell := \frac{1}{T} \int_0^T e^*(t) \exp\left(2i\ell\pi \frac{t}{T}\right) dt = 0, \quad \forall \ell = 0, \dots, L. \quad (2.66)$$

A common approach to solve such a problem is to *add oscillators and feedback stabilizers*. In other words, the design makes use of an internal-model-based control design of the form (2.65) processing the regulation error and containing linear oscillators at the desired frequencies, together with a stabilizer for the closed-loop system. In particular, we consider a dynamical controller (2.65) of the form

$$\dot{\eta} = \Phi\eta + \Gamma e \quad (2.67)$$

with  $\eta \in \mathbb{R}^{n_\eta}$  and where the matrices  $\Phi, \Gamma$  are selected as

$$\Phi = \text{blkdiag}\left(0 \quad \omega_1\Phi_1 \quad \dots \quad \omega_L\Phi_1\right) = \begin{pmatrix} 0 & & & \\ & \frac{2\pi}{T}\Phi_1 & & \\ & & \ddots & \\ & & & L\frac{2\pi}{T}\Phi_1 \end{pmatrix}, \quad (2.68)$$

$$\Gamma = \text{blkdiag}\left(\Gamma_0 \quad \Gamma_1 \quad \dots \quad \Gamma_L\right) = \begin{pmatrix} \Gamma_0 & & & \\ & \Gamma_1 & & \\ & & \ddots & \\ & & & \Gamma_L \end{pmatrix}$$

where  $\Phi_1 \in \mathbb{R}^{2n_e \times 2n_e}$  and  $\Gamma_\ell \in \mathbb{R}^{2n_e}$  are selected as

$$\Phi_1 = \text{blkdiag}\left(\phi, \dots, \phi\right) = \begin{pmatrix} \phi & & & \\ & \phi & & \\ & & \ddots & \\ & & & \phi \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad (2.69)$$

and  $\Gamma_\ell = (\gamma, \dots, \gamma)$  such that each couple  $(\omega_\ell\phi, \gamma)$  is controllable, together with a stabilizer (2.65b) for the closed-loop.

## 2.3.2 State feedback design

### 2.3.2.1 Assumptions

To provide a solution to problem 2.3.3, the idea is to rely on the results developed in Section 1.4.3.3 to make the closed-loop system incrementally uniformly globally exponentially ISS with respect to the external signals. To make the Section more self-contained, we will highlight again the main assumptions, theorems, proofs, remarks,

and comments that have been done in Section 1.4.3.3, with a specific viewpoint concerning the output regulation problem. Therefore, we start by assuming the following.

**Assumption 2.3.2** (Open-loop  $\delta$ GUES and Killing vector). *Consider system (2.7). There exist a  $C^2$  function  $\alpha_0 : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$ , a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_x}$  taking symmetric and positive values and three positive real numbers  $\underline{p}, \bar{p}, p$  such that the function  $f_0(x) = f(x) + g(x)\alpha_0(x)$  satisfies*

$$L_{f_0}P(x) \preceq -pI, \quad \underline{p}I \preceq P(x) \preceq \bar{p}I \quad (2.70)$$

$$L_gP(x) = 0 \quad (2.71)$$

for all  $x \in \mathbb{R}^{n_x}$ .

Assumption 2.3.2 asks for the knowledge of a pre-stabilizing feedback control action such that the  $x$ -dynamics generates a contraction with respect to a Riemannian metric induced by the matrix function  $P$ . Also, (2.71) implies that  $g$  is a Killing Vector field for this metric. Moreover, by, the Killing vector property (1.64) guarantees that the  $x$ -dynamics is incrementally uniformly exponentially ISS with respect to any input  $u$ .

**Remark 2.3.4.** *In the linear framework, this corresponds to a stabilizability assumption, where (2.71) is always satisfied as  $P$  is taken as a constant positive definite matrix solution of a Lyapunov equation.*

The design of  $\alpha_0$  and  $P$  can be obtained following, for instance, the techniques in Section 1.4 and references therein. From now on, we consider  $f(x) = f_0(x)$  without loss of generality (that is, Assumption 2.3.2 holds for  $\alpha_0(x) = 0$  for all  $x$ , for some metric  $P$ ).

The open-loop system (2.7) together with the internal model (2.65) is in *feedforward form*. The structure of the stabilizing unit (2.65b) that we aim to use is based on the incremental forwarding  $\text{mod}\{L_gV\}$  developed in Section 1.4.3.3. For this, we recall the following assumption.

**Assumption 2.3.3** (Incremental Forwarding  $\text{mod}\{L_gV\}$ ). *We know three  $C^1$  functions  $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_n}$ ,  $\Delta : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_n}$  and  $\varrho : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  such that, for all  $x \in \mathbb{R}^{n_x}$ , the following hold:*

1. the functions  $\bar{M}$  and  $\Delta$  are solution of

$$L_f\bar{M}(x) = \Phi\bar{M}(x) + \Gamma h(x) + \Delta(x); \quad (2.72)$$

2. there exists a matrix  $\Lambda$  such that

$$L_g\bar{M}(x) = \Lambda \quad (2.73)$$

and such that the couple  $(\Phi, \Lambda^\top)$  is detectable;

3. the function  $\varrho$  satisfies

$$\Lambda \frac{\partial \varrho}{\partial x}(x) = -\frac{\partial \Delta}{\partial x}(x); \quad (2.74)$$

4. the following inequality holds

$$L_fP(x) + \text{He} \left\{ P(x)g(x) \frac{\partial \varrho}{\partial x}(x) \right\} \leq -\lambda I \quad (2.75)$$

for some  $\lambda > 0$ .

Assumption 2.3.3 corresponds to a MIMO version of the assumptions in Proposition 1.4.8. In particular:

- item 1 corresponds to a generalization of Praly et al. (2001). In particular, a solution  $\bar{M}(x) = M(x)$  of (2.72) is known to exist for  $\Delta(x) = 0$ . This follows from Assumption 2.3.2 and from the fact that  $\Phi$  has only poles on the imaginary axis. Indeed, by Corollary 1.2.7.2, the  $x$ -dynamics possesses an equilibrium point (the origin, without loss of generality), that is globally exponentially stable;
- about item 2, the term  $L_g \bar{M}(x)$  can be seen as a controllability assumption on the control  $u$  to act on the dynamics of  $\eta$  of (2.67) in any point of the state space  $x$ ;
- as the dynamics of  $\eta$  in the most general case can have a higher dimension than the one of  $u$  (because the internal model unit contains the bunch of oscillators at period  $T$  and its multiples), in item 3 we ask for the mismatch term  $\Delta$  to be mapped in a (possibly lower) space of the dimension of the input and to be integrable;
- finally, item 4 asks for a robustness-like property for the autonomous system. Indeed, in order to rely on a free-to-choose solution  $\bar{M}$  of (2.72) rather than the exact one  $M$ , the open-loop system must be *sufficiently contractive* to merge the mismatch represented by  $\Delta$ .

### 2.3.2.2 State-feedback design for global harmonic regulation

We're ready to state the main result of this section

**Proposition 2.3.5** (Global harmonic regulation). *Consider system (2.7) extended with the dynamical system (2.65), (2.68), (2.69) and let Assumptions 2.3.2 and 2.3.3 hold. Suppose, moreover, that there exists a positive real number  $L_M > 0$  such that*

$$\left\| \frac{\partial \bar{M}}{\partial x}(x) \right\| \leq L_M \quad (2.76)$$

*holds for all  $x \in \mathbb{R}^{n_x}$ . Then for any  $(r, d)$  satisfying Assumption 2.3.1 and any initial conditions  $(x_0, \eta_0) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta}$ , system (2.7) in closed-loop with the dynamic control law*

$$\begin{aligned} \dot{\eta} &= \Phi \eta + \Gamma e \\ u &= \alpha_0(x) + \kappa \Lambda^\top (\eta - \bar{M}(x)) + \varrho(x) \end{aligned} \quad (2.77)$$

*with any gain  $\kappa > 0$  solves the global harmonic regulation Problem in Problem 2.3.3.*

**Proof.** We consider the change of coordinates

$$\chi := (x, \eta) \mapsto \mathbf{x} := (x, z), \quad z := \eta - \bar{M}(x)$$

with  $\bar{M}$  solving (2.72). Since  $L_g \bar{M}(x)$  is a constant matrix by (2.73), the closed-loop system can be then written in form

$$\dot{\mathbf{x}} = F(\mathbf{x}) + \Omega(\mathbf{x})w, \quad (2.78)$$

$$\Omega(\mathbf{x}) = \begin{pmatrix} g(x) & 0 \\ \Lambda & I \end{pmatrix}, \quad F(\mathbf{x}) := \begin{bmatrix} f(x) + g(x) [\kappa \Lambda^\top z + \varrho(x)] \\ (\Phi - \kappa \Lambda \Lambda^\top)z - \Lambda \varrho(x) - \Delta(x) \end{bmatrix}$$

where  $w := (d, r)$ . The proof is divided into three steps:

1. we show that the unforced closed-loop system (i.e. for  $w(t) = 0$  for all  $t \geq 0$ ) is incrementally uniformly globally exponentially stable. This will be done by showing that it is possible to find a metric for which the closed-loop vector field  $F$  has a uniformly negative definite Lie derivative;
2. we show that, in case  $w(t) \neq 0$  satisfies Assumption 2.3.1, then the forced closed-loop system is incrementally uniformly globally exponentially ISS with respect to  $w(t)$  and that the trajectories are bounded and asymptotically converge to a solution that is  $T$ -periodic. This will be done by showing that the matrix  $\Omega$  possesses the Killing vector field property with respect to the metric found in item 1;
3. we show that, thanks to the presence of a bunch of oscillators in the internal model unit, the Fourier coefficients associated with the frequencies  $\omega_\ell = \ell \frac{2\pi}{T}$  are zero for  $\ell = 0, \dots, L$ , i.e. harmonic regulation is achieved.

Hence, we first look for a  $C^1$  matrix function  $\bar{\mathcal{P}} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \mapsto \mathbb{R}^{(n_x+n_\eta) \times (n_x+n_\eta)}$  taking symmetric positive definite values of the form

$$\bar{\mathcal{P}}(\mathbf{x}) := \begin{pmatrix} P(x) & 0 \\ 0 & \mu(I + b\bar{S}) \end{pmatrix} \quad (2.79)$$

with  $b, \mu$  being strictly positive real numbers to be defined,  $P$  taken as in Assumption 2.3.2,  $I$  is the Identity matrix and  $\bar{S}$  being a strictly positive definite matrix to be defined. The main intuition behind this choice is that, since  $\Phi$  is a skew-symmetric matrix (with eigenvalues only on the imaginary axis), the identity matrix by itself doesn't provide negativity in all the components of  $z$ . In order to "strictify" the metric, we rely on a design inspired by Praly (2019) (also used in Astolfi et al. (2022b)) by means of an observer. Indeed, by item 2 of Assumption 2.3.3, the couple  $(\Phi, \Lambda^\top)$  is detectable, and therefore there exist two matrices  $S = S^\top \succ 0$  and  $K$  solving

$$S(\Phi - K\Lambda^\top) + (\Phi - K\Lambda^\top)^\top S \preceq -2I. \quad (2.80)$$

Hence let  $\bar{S} = S$  and consider the matrix function  $\mathcal{R} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \mapsto \mathbb{R}^{(n_x+n_\eta) \times (n_x+n_\eta)}$

$$\mathcal{R}(\mathbf{x}) := L_F \bar{\mathcal{P}}(\mathbf{x}) + \begin{pmatrix} \bar{\mathfrak{p}}_1 I & 0 \\ 0 & \bar{\mathfrak{p}}_2 I \end{pmatrix} \quad (2.81)$$

for some  $\bar{\mathfrak{p}}_1, \bar{\mathfrak{p}}_2$  strictly positive real numbers to be chosen. If  $\mathcal{R}(\mathbf{x}) \preceq 0$  for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_\eta}$  and for some  $\bar{\mathfrak{p}}_1, \bar{\mathfrak{p}}_2$ , then by Theorem 1.2.2 the closed-loop

system (2.78) with  $w(t) = 0$  for all  $t \geq 0$  is incrementally uniformly globally exponentially stable. We can express the matrix function  $\mathcal{R}$  as

$$\mathcal{R}(\mathbf{x}) = \begin{pmatrix} \mathcal{R}_{11}(\mathbf{x}) & \mathcal{R}_{12}(\mathbf{x}) \\ \mathcal{R}_{12}^\top(\mathbf{x}) & \mathcal{R}_{22}(\mathbf{x}) \end{pmatrix}. \quad (2.82)$$

Thanks to (2.74) and to the Killing Vector property  $L_g P(x) = 0$  in Assumption 2.3.2, we have that

$$\mathcal{R}_{11}(\mathbf{x}) = L_f P(x) + \text{He} \left\{ P(x)g(x) \frac{\partial g}{\partial x}(x) \right\} + \bar{\mathbf{p}}_1 I$$

$$\mathcal{R}_{12}(\mathbf{x}) = \kappa P(x)g(x)\Lambda^\top$$

$$\mathcal{R}_{22}(\mathbf{x}) = \mu \text{He} \left\{ (I + bS)(\Phi - \kappa\Lambda\Lambda^\top) \right\} + \bar{\mathbf{p}}_2 I.$$

Remember now that, since  $\Phi$  is skew-symmetric,

$$\Phi + \Phi^\top = 0.$$

By adding and subtracting the term  $\mu b \text{He}\{SK\Lambda^\top\}$  and by making use of (2.80) we get

$$\begin{aligned} \mathcal{R}_{22}(\mathbf{x}) &= \mu \text{He} \left\{ (I + bS)(\Phi - \kappa\Lambda\Lambda^\top) \right\} + \bar{\mathbf{p}}_2 I \\ &= \mu \text{He} \left\{ \Phi - \kappa\Lambda\Lambda^\top + bS\Phi - b\kappa S\Lambda\Lambda^\top \right\} + \bar{\mathbf{p}}_2 I \\ &= \mu \left[ -2\kappa\Lambda\Lambda^\top + bS\Omega + b\Omega^\top S - b\kappa S\Lambda\Lambda^\top - b\kappa\Lambda\Lambda^\top S \right] + \bar{\mathbf{p}}_2 I \\ &\quad \pm \mu b (SK\Lambda^\top + \Lambda K^\top S) \\ &= \mu \left[ -2\kappa\Lambda\Lambda^\top + b \text{He} \left\{ S(\Omega - K\Lambda^\top) \right\} - b \text{He} \left\{ S(K - \kappa\Lambda\Lambda^\top) \right\} \right] + \bar{\mathbf{p}}_2 I \\ &\preceq -\mu \left[ 2\kappa\Lambda\Lambda^\top + 2bI - b \text{He} \left\{ S(K - \kappa\Lambda\Lambda^\top) \right\} \right] + \bar{\mathbf{p}}_2 I \\ &\preceq -\mu \left[ 2\kappa\Lambda\Lambda^\top + 2bI - \frac{b^2}{2\beta} S(K - \kappa\Lambda)(K - \kappa\Lambda)^\top S - \frac{\beta}{2}\Lambda\Lambda^\top \right] + \bar{\mathbf{p}}_2 I \end{aligned}$$

for any real number  $\beta > 0$ , where we used Young's inequality. Therefore, we select

$$\beta = 4\kappa, \quad b < \sqrt{\frac{2\beta}{\|S(K - \kappa\Lambda)\|^2}}, \quad \bar{\mathbf{p}}_2 < \frac{\mu b}{2}$$

and we get

$$\mathcal{R}_{22}(\mathbf{x}) \prec -\frac{\mu b}{2} I$$

for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_\eta}$ , for any  $\kappa, \mu > 0$ . In order to have  $\mathcal{R} \prec 0$ , it remains to check the negative definiteness of its Schur complement  $\mathcal{S}_{\mathcal{R}}(\mathbf{x}) = \mathcal{R}_{11} - \mathcal{R}_{12}\mathcal{R}_{22}^{-1}\mathcal{R}_{12}^\top \prec 0$ . Keeping in mind Item 4 of Assumption 2.3.3 and the upper bounds of  $P$  and  $g$ ,

we get

$$\begin{aligned}\mathfrak{S}_{\mathfrak{R}}(\mathbf{x}) &= L_f P(x) + \text{He} \left\{ P(x) g(x) \frac{\partial g}{\partial x}(x) \right\} + \bar{\mathfrak{p}}_1 I \\ &\quad - \left( \kappa P(x) g(x) \Lambda^\top \right) \mathfrak{R}_{22}^{-1}(\mathbf{x}) \left( \kappa P(x) g(x) \Lambda^\top \right)^\top \\ &\preceq -(\lambda - \bar{\mathfrak{p}}_1) I + \frac{4\kappa^2 \bar{p}^2 \bar{g}^2}{\mu b} \Lambda^\top \Lambda.\end{aligned}$$

Hence, for any  $\kappa > 0$ , set

$$\bar{\mathfrak{p}}_1 \leq \frac{\lambda}{4}, \quad \mu \geq \frac{16\kappa^2 \bar{p}^2 \bar{g}^2 |\Lambda|^2}{\lambda b}$$

so that  $\mathfrak{S}_{\mathfrak{R}}(\mathbf{x}) \prec 0$  and therefore  $\mathfrak{R}(\mathbf{x}) \prec 0$  for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_\eta}$ . Note that the metric  $\mathcal{P}$  has been obtained in the  $\mathbf{x} = (x, z)$ -coordinates. In order to complete the proof, we need to come back to the original coordinates  $\chi = (x, \eta)$ . This can be done with a globally Lipschitz diffeomorphism similarly to the proof of Proposition 2.2.10, where we get

$$\mathcal{P}(\chi) = \begin{pmatrix} P(x) + \mu(I + bS) \frac{\partial \bar{M}^\top}{\partial x}(x) \frac{\partial \bar{M}}{\partial x}(x) & \star^\top \\ -\mu(I + bS) \frac{\partial \bar{M}}{\partial x}(x) & \mu(I + bS) \end{pmatrix}.$$

Hence, the closed-loop system is incrementally uniformly globally exponentially stable with respect to the contraction metric  $\mathcal{P}(x, \eta)$  satisfying

$$\underline{\mathfrak{p}} I \preceq \mathcal{P} \preceq \bar{\mathfrak{p}} I \quad L_{\bar{F}} \mathcal{P}(x, \eta) \preceq -\mathfrak{p} I$$

with  $\bar{F}(\chi)$  being the closed-loop (2.78) in the original  $(x, \eta)$ -coordinates, and

$$\begin{aligned}\mathfrak{p} &= \min\{\bar{\mathfrak{p}}_1, \bar{\mathfrak{p}}_2\} \\ \bar{\mathfrak{p}} &:= \max\{\bar{p}, \mu \|I + bS\|\} (1 + L_M)^2, \\ \underline{\mathfrak{p}} &:= \frac{\min\{p, \mu \|I + bS\|\}}{(1 + L_M)^2}.\end{aligned}\tag{2.83}$$

This concludes the first part of the proof.

Note now that, since  $\mathcal{P}(\chi)$  is independent on  $\eta$  and thanks to the Killing vector property in Assumption 2.3.3, then  $\Omega(\chi)$  possesses the Killing vector property with respect to such  $\mathcal{P}(\mathbf{x})$ , i.e.  $L_\Omega \mathcal{P}(\mathbf{x}) = 0$  for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_\eta}$ . Since  $\Omega(\mathbf{x})$  is bounded for all  $\mathbf{x} \in \mathbb{R}^{n_x+n_\eta}$ , by Theorem 1.3.3 the closed-loop system is incrementally uniformly globally exponentially ISS with respect to  $w(t)$ . Since  $(r, d)$  satisfy Assumption 2.3.1 with period  $T$ , then by Corollary 1.2.6 the trajectories of the system and of the error  $e$  are bounded and asymptotically converge toward a periodic trajectory  $e^*(t)$  with same period  $T$ . The proof concludes since, by construction of the matrices  $\Phi, \Gamma$  as in (2.68), (2.69) and by (Astolfi et al., 2022a, Proposition 1), the first  $L$ -Fourier coefficients of the error  $e^*$  are zero, i.e. (2.66) holds.  $\square$

### 2.3.3 Examples

#### 2.3.3.1 A test design for a class of nonlinear systems

In this section, we propose a possible design to apply the results in Proposition 2.3.5. We consider a nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)(u + d) \\ e &= h(x) - r, \end{aligned}$$

where

$$\begin{aligned} f(x) &= Ax + G\vartheta(\nu), & g(x) &= B, \\ h(x) &= Cx + D\vartheta(\nu), & \nu &= Hx \end{aligned} \tag{2.84}$$

where  $A, G, H, B, C, D$  are constant matrices of suitable dimension and  $\vartheta : \mathbb{R}^{n_\nu} \mapsto \mathbb{R}$  is a scalar  $C^1$  nonlinearity with  $\vartheta(0) = 0$  without loss of generality. For such class of systems we first assume that Assumption 2.3.2 is satisfied with respect to some constant metric  $P = P^\top \succ 0$ . In other words, we assume that there exists a symmetric and positive definite matrix  $P = P^\top \succ 0$  such that

$$\text{He} \left\{ P \left( A + G \frac{\partial \vartheta}{\partial \nu}(\nu) H \right) \right\} \preceq -pI \tag{2.85}$$

for some strictly positive real number  $p > 0$ .

**Remark 2.3.6.** *In case such a condition is not verified, a preliminary feedback design may be applied. To have a preliminary design that can maintain the same system's structure, it's possible to use for instance the techniques in Section 1.4.4.*

In order to apply the control law as in Proposition 2.3.5, we propose the following design. We want to stress that such a design is not unique, as the solution of (2.72) is a choice. Let  $\mathcal{Q}$  be defined as the following matrix parametrized by two scalars  $a$  and  $\omega$

$$\mathcal{Q}(a, \omega) := \begin{pmatrix} A^\top \otimes I - I \otimes \Phi(\omega) & -C^\top \otimes I \\ H^\top (aB^\top - G^\top) \otimes I & -H^\top D^\top \otimes I \end{pmatrix} \tag{2.86}$$

where  $\Phi$  is as in (2.68) where we explicitly expressed the dependency on the parameter  $\omega = (0, \omega_1, \dots, \omega_L)$  for some  $L \geq 0$  and let  $\mathcal{A} \subset \mathbb{R}$  be the set defined as

$$\mathcal{A}(\lambda) := \left\{ a \in \mathbb{R} \mid \text{He} \left\{ P(A + (G + Ba) \frac{\partial \vartheta}{\partial \nu}(Hx)H) \right\} \preceq -\lambda I, \quad \forall x \in \mathbb{R}^{n_x} \right\}. \tag{2.87}$$

Then the following holds.

**Corollary 2.3.6.1.** *Consider system (2.84) extended with the dynamical system (2.65), (2.68), (2.69) and assume that (2.85) holds for some constant matrix  $P = P^\top \succ 0$  and some  $p > 0$ . Given  $\omega$ , let  $\lambda > 0$  and suppose there exists  $\bar{a} \in \mathcal{A}(\lambda)$  such that  $\det(Q(\bar{a}, \omega)) = 0$ . Let  $M, \Gamma$  be any solution to*

$$\mathcal{Q}(\bar{a}, \omega) \begin{pmatrix} \text{vec}(M) \\ \text{vec}(\Gamma) \end{pmatrix} = 0.$$

If  $(\Phi, B^\top M^\top)$  is detectable and  $(\Phi, \Gamma)$  is controllable, then Assumption 2.3.3 holds with  $\bar{M}, \Delta, \varrho, \Lambda$  given as

$$\begin{aligned}\bar{M}(x) &= Mx, & \Delta(x) &= (MG - \Gamma D)\vartheta(Hx), \\ \varrho(x) &= \bar{a}\vartheta(Hx), & \Lambda &= MB.\end{aligned}\tag{2.88}$$

**Proof.** First, note that by Assumption 2.3.2 and by continuity, the set  $\mathcal{A}$  is non-empty. Then, let  $\bar{M}, \Delta, \varrho, \Lambda$  be defined as in (2.88). For the considered class of systems (2.84), the main idea behind this choice is to pick the function  $\bar{M}(x)$  as a linear function which satisfies the linear part of (2.72) (and therefore also (2.73)), and to stick all the nonlinearities in the term  $\Delta(x)$ , which will be handled by the robustness of the open-loop system with (2.75) through the existence of a mapping  $\varrho(x)$  satisfying (2.74). With this choice, the conditions (2.72), (2.74) reduce to the existence of constant matrices  $M, \Gamma$  and a real number  $a$  solution of the matrix equalities

$$MA = \Phi(\omega)M + \Gamma C \tag{2.89a}$$

$$aMBH = (MG - \Gamma D)H \tag{2.89b}$$

where (2.89a) comes from the linear terms of (2.72) and (2.89b) from the definition of  $\Delta$  in (2.88) and (2.74). Recalling the definition of the Sylvester equation, the conditions (2.89a), (2.89b) can be rewritten with the Kronecker operator  $\otimes$  as a linear problem of the form

$$\mathcal{Q}(a, \omega)\mathcal{Y} = 0 \tag{2.90}$$

where  $\mathcal{Y}$  is a  $2L \times n_x + 2L$  column vector of unknowns defined as

$$\mathcal{Y} = (\text{vec}(M), \text{vec}(\Gamma))^\top$$

and  $\mathcal{Q}(a, \omega)$  is the matrix defined as in (2.86). For fixed  $\omega$ , if there exists  $\bar{a} \in \mathcal{A}$  such that the matrix  $\mathcal{Q}(\bar{a}, \omega)$  has a non-null kernel, then there exists at least one non-null vector  $\bar{\mathcal{Y}}$  such that  $\mathcal{Q}(\bar{a}, \omega)\bar{\mathcal{Y}} = 0$ . In such case, the matrices  $M, \Gamma$  can be constructed from the vector  $\bar{\mathcal{Y}}$  and the function  $\varrho$  is selected as  $\varrho(x) = \bar{a}\vartheta(Hx)$ . From such choice, (2.72), (2.74) are satisfied. Moreover, since  $\bar{a} \in \mathcal{A}$ , then also (2.75) holds. Since  $(\Phi, B^\top M^\top)$  is detectable and  $(\Phi, \Gamma)$  is controllable by assumption, then Assumption 2.3.3 holds.  $\square$

We have then the following result.

**Corollary 2.3.6.2** (Global harmonic regulation for Lipschitz systems). *Consider system (2.84) (2.65), (2.68), (2.69) and assume that Corollary 2.2.14.2 holds. Then system (2.84) in closed-loop with the dynamical control law*

$$\dot{\eta} = \Phi\eta + \Gamma e \tag{2.91}$$

$$u = \kappa(MB)^\top(\eta - Mx) + \bar{a}\vartheta(Hx) \tag{2.92}$$

*solves the global harmonic regulation Problem in Problem 2.3.3.*

**Remark 2.3.7.** *Note that the results of Corollary 2.3.6.1 can be seen also from a different perspective. In particular, for a given system and a fixed set  $\mathcal{A}$ , such*

a result shows the frequencies  $w$  that can be naturally regulated with the proposed design. In this case, the set of  $w$  that can be regulated is a subset of the  $w$  satisfying the non-resonance condition

$$\text{rank} \begin{pmatrix} A - \sigma I & B \\ C & 0 \end{pmatrix} = n_x + n_e$$

with  $\sigma$  being any eigenvalue of the matrix  $\Phi$ , see Astolfi et al. (2022a).

**Remark 2.3.8.** Note that the design proposed in Corollary 2.3.6.1 provides a test that can be performed on the system: for a given set of frequencies  $\omega = (\omega_1, \dots, \omega_L)$ .

1. Compute the matrix  $\mathcal{Q}(a, \omega)$ .
2. Compute its determinant which is a polynomial in the variable  $a$ .
3. Find the (finite) values of  $a$  that nullify the determinant.
4. Check if such values are in the set  $\mathcal{A}$ .
5. If this is the case, then it's sufficient to check the detectability of  $(\Phi, B^\top M^\top)$  and the controllability of  $(\Phi, \Gamma)$ .

**Example 2.3.9.** Consider a simple academic example where we take a system of the form (2.7) with

$$f(x) = \begin{pmatrix} -5x_1 + x_2 + \sin(x_1) \\ x_1 - 2x_2 \end{pmatrix}, \quad g(x) = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad h(x) = \sin(x_1) - x_2.$$

We aim to cancel the harmonic content at the frequency  $\omega = \frac{2\pi}{T} = 1$ . Note that such a system does not admit a globally defined normal form. Indeed  $L_g h(x) = \cos(x_1) + 1$ , which is non constant for  $x_1 = (2j+1)\pi$  for any integer  $j$ . Yet, a solution does exist. First, Assumption 2.3.2 is satisfied with  $P = I$  where the Killing Vector property holds as both  $P$  and  $g$  are constant. Then, following Proposition 2.3.5, we extend the plant with a dynamical system (2.67) where we choose

$$\Phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 0.478 \\ 0.433 \end{pmatrix}.$$

We select  $\bar{M}, \Delta, \varrho, \Gamma, \Lambda$  solution of (2.72), (2.73), (2.74) according to Corollary 2.3.6.1 where

$$A = \begin{pmatrix} -5 & 1 \\ 1 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ C = (0 \quad -1), \quad D = 1, \quad H = (1 \quad 0),$$

and  $\vartheta(s) = \sin(s)$ . In such case, we can select  $\bar{a} = 0.4962 \in \mathcal{A}(1)$  and check the detectability of  $(\Phi, \Lambda^\top)$  and the controllability of  $(\Phi, \Gamma)$ . With such choice, Assumption 2.3.3 holds and global harmonic regulation is achieved with the dynamic controller (2.67), (1.70).

### 2.3.3.2 The case of minimum phase systems

The main drawback of the design provided by Proposition 2.3.3 when applied to the case of harmonic regulation, is that in the most general case a constructive design

that is uniform on the number of oscillators  $L$  and independent of the considered frequency  $\omega_\ell$  to check the validity of Assumption 2.3.3 might not be always possible, but instead depends on the considered system. In this section, we show that this is not the case for instance if we consider the special case of nonlinear systems which admit a globally defined normal form with a zero-dynamics that is incrementally stable. We want to stress that the result in Proposition 2.3.5 does not assume a-priori the existence of a globally defined normal form. We consider, for the sake of simplicity, a single-input single-output (SISO) system with unitary relative degree (the extension to a higher relative degree can be dealt with canonical tools, see, e.g., (Serrani et al., 2001, Section V)) of the form

$$\begin{aligned}\dot{z} &= \psi(z, y) \\ \dot{y} &= q(z, y) + u - r(t)\end{aligned}\tag{2.93}$$

where  $x = (z^\top, y)^\top \in \mathbb{R}^{n_x-1} \times \mathbb{R}$  is the state, and the vector fields  $\psi : \mathbb{R}^{n_x-1} \times \mathbb{R} \mapsto \mathbb{R}^{n_x-1}$  and  $q : \mathbb{R}^{n_x-1} \times \mathbb{R} \mapsto \mathbb{R}$  are sufficiently smooth. Most of the existing results on output regulation for nonlinear systems focus on systems having a minimum phase zero-dynamics. As our approach is to cast the problem in the contraction framework, we will assume that system (2.93) possesses a contractive zero-dynamics. This is assumed in the following.

**Assumption 2.3.4** (Incremental minimum phase). *Consider system (2.93). There exist positive real numbers  $\bar{q}, \bar{\psi}_y, \bar{p}_z, \underline{p}_z, \lambda_z > 0$  and a  $C^1$  matrix function  $P_z : \mathbb{R}^{n_x-1} \mapsto \mathbb{R}^{(n_x-1) \times (n_x-1)}$  taking symmetric positive values such that the following inequalities<sup>3</sup> hold*

$$\left| \frac{\partial q}{\partial x}(z, y) \right| \leq \bar{q}, \quad \left| \frac{\partial \psi}{\partial y}(z, y) \right| \leq \bar{\psi}_y,\tag{2.94}$$

$$\underline{p}_z I \preceq P_z(z) \preceq \bar{p}_z I, \quad L_\psi P_z(z) \preceq -2\lambda_z I,\tag{2.95}$$

for all  $(z, y) \in \mathbb{R}^{n_x-1} \times \mathbb{R}$ .

Under such an assumption we have the following result.

**Proposition 2.3.10.** *Consider system (2.93) extended with the dynamics (2.67), (2.68), (2.69) and let Assumption 2.3.4 hold. Then for any number of oscillators  $L \geq 0$ , Assumption 2.3.2 and 2.3.3 are satisfied with  $h(x) = y$  and*

$$\begin{aligned}P(x) &= \begin{pmatrix} P_z(z) & 0 \\ 0 & \varepsilon \end{pmatrix}, \quad \alpha_0(x) = -ky, \quad \varrho(x) = -q(z, y) + (k-1)y, \\ \bar{M}^\top(x) &= [y \quad \bar{M}_1^\top(x) \quad \dots \quad \bar{M}_L^\top(x)], \quad \bar{M}_\ell^\top(x) = (y \quad y), \\ \Lambda^\top &= [1 \quad \Lambda_1^\top \quad \dots \quad \Lambda_L^\top], \quad \Lambda_\ell^\top = (1 \quad 1), \\ \Gamma^\top &= [-1 \quad \Gamma_1^\top \quad \dots \quad \Gamma_L^\top], \quad \Gamma_\ell^\top = [-\omega_\ell - 1 \quad \omega_\ell - 1], \\ \Delta^\top(x) &= [\Delta_0(x) \quad \Delta_1^\top(x) \quad \dots \quad \Delta_{2L}^\top(x)], \quad \Delta_0(x) = \Delta_\ell(x) = -\varrho(x),\end{aligned}\tag{2.96}$$

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<sup>3</sup>The notation  $L_\psi P_z(z, y)$  has to be understood as the Lie derivative of  $P_z$  along the vector field  $z \mapsto \psi(z, y)$  where  $y$  is fixed.

with  $\ell = 1, \dots, L$  for some gain  $k > 0$  sufficiently large and  $\varepsilon > 0$  sufficiently small, both independent on  $L$ .

**Proof.** We first show that Assumption 2.3.2 holds. After the pre-stabilizing action  $\alpha_0$ , the open-loop system (2.93) is of the form (2.7), (2.65) where  $x = (z, y)$  and

$$f(x) = \begin{pmatrix} \psi(z, y) \\ q(x, y) - ky \end{pmatrix}, \quad g(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Consider the metric  $P(x)$  defined as in (2.96). First, note that the Killing Vector property is satisfied, i.e.  $L_g P(x) = 0$  for all  $x$ . We look for  $p > 0$  such that  $\mathcal{T}(x) \prec 0$  where

$$\begin{aligned} \mathcal{T}(x) &:= L_f P(x) + pI = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{12}^\top & \mathcal{T}_{22} \end{pmatrix} \\ &= \begin{pmatrix} L_{\psi_z} P_z(z) & P_z(z) \frac{\partial \psi}{\partial y}(x) + \left( \varepsilon \frac{\partial q}{\partial z}(x) \right)^\top \\ \left[ P_z(z) \frac{\partial \psi}{\partial y}(x) + \left( \varepsilon \frac{\partial q}{\partial z}(x) \right)^\top \right]^\top & 2\varepsilon \left( \frac{\partial q}{\partial y}(x) - k \right) \end{pmatrix} + pI \end{aligned}$$

Fix any

$$\varepsilon > \frac{\lambda_z}{2(\bar{p}_z^2 \bar{\psi}_y^2)}$$

and set

$$k \geq \bar{k} := \bar{q} + \frac{\lambda_k}{2\varepsilon\lambda_z} + \frac{(\bar{p}_z \bar{\psi}_y + \varepsilon \bar{q})^2}{2\varepsilon\lambda_z}$$

for some  $\lambda_k > 0$ . From the bounds on  $q$  in Assumption 2.3.4, the bottom-right term of  $L_f P(x)$  is negative definite for any  $\varepsilon, \lambda_k > 0$ , namely  $\mathcal{T}_{22} \prec 0$  for any  $p \leq \varepsilon(k - \bar{q})$ . Hence to show the existence of a metric for the open-loop system, namely, (2.70), we check the sign of the Schur's complement  $\mathcal{S}_{\mathcal{T}}(x) = \mathcal{T}_{11} - \mathcal{T}_{12} \mathcal{T}_{22}^{-1} \mathcal{T}_{12}^\top$  of  $\mathcal{T}(x)$ . By making use of the bounds on  $\psi, q$  and since there exists  $\lambda_z$  such that  $L_{\psi_z} P_z(z) \preceq -2\lambda_z I$  from Assumption 2.3.4, we get

$$\begin{aligned} \mathcal{S}_{\mathcal{T}}(x) &= L_{\psi_z} P_z(z) + pI \\ &\quad - \left[ P_z(z) \frac{\partial \psi}{\partial y}(x) + \left( \varepsilon \frac{\partial q}{\partial z}(x) \right)^\top \right] \mathcal{T}_{22}^{-1} \left[ P_z(z) \frac{\partial \psi}{\partial y}(x) + \left( \varepsilon \frac{\partial q}{\partial z}(x) \right)^\top \right]^\top \\ &\preceq \left( -2\lambda_z + p + \frac{(\bar{p}_z \bar{\psi}_y + \varepsilon \bar{q})^2}{\frac{1}{\lambda_z} (\lambda_k + (\bar{p}_z \bar{\psi}_y + \varepsilon \bar{q})^2)} \right) I \\ &\preceq -2\lambda_k I \end{aligned}$$

Therefore, Assumption 2.3.2 holds with  $P$  defined in (2.96) and

$$\underline{p} = \min\{\underline{p}_z, \varepsilon\}, \quad \bar{p} = \max\{\bar{p}_z, \varepsilon\}, \quad p = \min\{2\lambda_k, 2\lambda_z, \frac{2\lambda_k}{\lambda_z}, \varepsilon(k - \bar{q})\}.$$

We then check each point of Assumption 2.3.3 separately. For Item 1 note that (2.72) holds. Indeed for  $\ell = 0$ , we get

$$\begin{aligned} \frac{\partial M_0}{\partial x}(x)f(x) &= h(x) + \Delta_0(x) \\ q(x) - ky &= -y + \Delta_0 \end{aligned}$$

and we recover the definition of  $\Delta_0(x)$  in (2.96). Similarly, for  $\ell = 1, \dots, L$  from the definitions of  $\Phi, \Gamma$  as in (2.68), (2.69), (2.96) we have

$$\frac{\partial M_\ell}{\partial x}(x)f(x) = \begin{pmatrix} 0 & \omega_\ell \\ -\omega_\ell & 0 \end{pmatrix} M_\ell(x) + \Gamma_\ell h(x) + \Delta_\ell(x)$$

which is a set of  $2L$  identities of the form

$$\begin{cases} (q(x) - ky) &= \omega_\ell y + (-\omega_\ell - 1)y + \Delta_\ell(x) \\ (q(x) - ky) &= -\omega_\ell y + (\omega_\ell - 1)y + \Delta_\ell(x) \end{cases}$$

from the definition of  $\Delta$  as in (2.96). About Item 2, note that  $L_g M(x) = \Lambda$ . Moreover, since  $M$  is linear and  $\Phi$  is block diagonal, the observability matrix  $\mathcal{O}$  of  $(\Phi, \Lambda^\top)$  is

$$\mathcal{O} = \text{blkdiag}\{1, \mathcal{O}_1, \dots, \mathcal{O}_L\}, \quad \mathcal{O}_\ell = \begin{pmatrix} 1 & \omega_\ell \\ 1 & -\omega_\ell \end{pmatrix}$$

which has non-zero determinant for all  $\omega_\ell \neq 0$ . This shows that  $(\Phi, \Lambda^\top)$  is detectable. About Item 3, again, we recover a set of  $2L + 1$  identities. Indeed for all  $\ell = 0, \dots, 2L$  we defined  $\varrho(x) = -\Delta_\ell(x)$ . About Item 4, equation (2.75) reads

$$\begin{pmatrix} L_{\psi_z} P_z(z) & P_z(z) \frac{\partial \psi}{\partial e}(x) + (\varepsilon \frac{\partial q}{\partial z}(x))^\top \\ \star^\top & 2\varepsilon \left( \frac{\partial q}{\partial e}(x) - k \right) \end{pmatrix} - \begin{pmatrix} 0 & (\varepsilon \frac{\partial q}{\partial z}(x))^\top \\ \star^\top & 2\varepsilon \left( \frac{\partial q}{\partial y}(x) - k + 1 \right) \end{pmatrix} \preceq -\lambda_z I$$

where the right-hand-side follows a similar analysis as previously, from the definition of  $\varepsilon$  and Assumption 2.3.4. To conclude the proof, note that the controllability matrix  $\mathcal{C}$  of  $(\Phi, \Gamma)$  has a block-diagonal structure of the form

$$\mathcal{C} = \text{blkdiag}\{1, \mathcal{C}_1, \dots, \mathcal{C}_L\}, \quad \mathcal{C}_\ell = \begin{pmatrix} -\omega_\ell - 1 & \omega_\ell(\omega_\ell - 1) \\ \omega_\ell - 1 & \omega_\ell(\omega_\ell + 1) \end{pmatrix}$$

where each  $\mathcal{C}_\ell$  is full rank and hence  $\mathcal{C}$  is so.  $\square$

From Proposition 2.3.5, the following then holds.

**Corollary 2.3.10.1** (Global harmonic regulation for minimum-phase systems). *Consider system (2.93) with any  $r \in \mathbb{R}$  satisfying (2.64) and let Assumption 2.3.4 hold. Then for any  $L \geq 0$  and any initial condition  $(z_0, y_0, \eta_0)$ , the system in closed loop with the dynamical control law*

$$\begin{aligned} \dot{\eta} &= \Phi\eta + \Gamma e \\ u &= \alpha_0(x) + \kappa \Lambda^\top (\eta - \bar{M}(x)) + \varrho(x) \end{aligned}$$

with (2.68), (2.69), (2.96) for any gain  $\kappa > 0$  solves the global harmonic regulation Problem in Problem 2.3.3.

**Remark 2.3.11.** *The control (2.77) recovers more classical designs used in nonlinear output regulation theory for minimum phase systems (see Serrani et al. (2001); Astolfi et al. (2022a)). The term  $\alpha_0 = -ky$  is a high-gain that acts as a pre-stabilizer for the  $y$ -dynamics where  $k$  is sufficiently large to handle the Lipschitz constant of  $q$  while the term  $\Lambda^\top(\eta - \bar{M}(x))$  provides negativity in the directions of the internal model. In the design (2.77) a particular feature is the term  $\rho$  which might be seen as a feedback linearization term in the input-output mapping between  $u$  and  $\eta$ . Such a term allows obtaining an upper-triangular structure in the Jacobian of the closed-loop system under the change of coordinates  $\eta \mapsto z = \eta - \bar{M}(x)$ .*

## 2.4 Conclusions and perspectives

In this chapter, we studied the global output regulation problem for input-affine nonlinear systems which do not necessarily admit the existence of a globally defined normal form. To do so, we framed the regulation task with tools from incremental stability theory. We considered, separately, the global integral action problem and the global harmonic regulation one. In the first case, the objective is to design a feedback control law such that the output of a nonlinear system asymptotically tracks a constant reference while rejecting external (constant) disturbances. The second case focuses on the design of a feedback control law such that the output of a nonlinear system is bounded and asymptotically converges towards a periodic solution that does not contain a harmonic content at certain frequencies. In both cases, the objective was to provide a solution to these problems independently of the initial conditions and independently of the amplitude of the external signals (references and disturbances). In both cases, we proposed a set of sufficient conditions. For the global integral action problem, the control law consists of an integral action processing the regulation error and a feedback design (state or output feedback) based on incremental forwarding techniques so that the closed-loop system is incrementally globally uniformly input-to-state stable with respect to the external signals. For the global harmonic regulation problem, the control law consists of a dynamical system composed of a bunch of linear oscillators processing the regulation error and a state-feedback design based on incremental forwarding techniques so that the closed-loop system is incrementally globally uniformly input-to-state stable with respect to the external signals. The proposed designs have been characterized for some classes of systems such as minimum phase systems and linear systems possessing a monotonic nonlinearity.

Future studies will focus on the following aspects.

- Despite incremental properties are preserved under sufficiently small model uncertainties, it could be of interest to try to quantify the maximum uncertainties and how they affect the control design and the closed-loop stability.
- For the harmonic regulation problem, the proposed design is generally not easily extendable in case the number of oscillators changes (except for the case of minimum phase systems in normal form). A possible extension could be to try to provide a design that does not have such a limitation.
- It is known that, for the class of minimum phase systems in normal form with a

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unitary relative degree, canceling harmonics with the proposed design implies that the norm of the regulation error decreases with respect to the number of oscillators in the internal model unit (see Astolfi et al. (2022b)). An interesting extension could be to see if such a condition is true for more general classes of system. In such a way, the proposed harmonic regulation design could achieve also practical regulation.

- A future perspective could be also to look for a pre-processing design for the internal model unit that provides incremental stability properties for the closed-loop, see Wang et al. (2020).

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# CHAPTER 3

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## MULTIAGENT SYNCHRONIZATION

Concord is a union of wills, not of  
opinions.

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T. D'aquino

### 3.1 Introduction

#### 3.1.1 The multiagent synchronization problem

The problem in which a group of systems has to exchange information among them in order to cooperate and achieve an agreement is generally called the *synchronization* or *consensus* problem. In this framework, we generally consider a group of single separated entities, each of them described by their dynamical equation and with their inputs and outputs. Each of these systems is called an *agent*. These entities however are not isolated but can exchange information among them through a communication protocol. We consider therefore a *network* of agents. Each agent can send and/or receive information from/to some other agents in the network. Each agent is not in general allowed to communicate with every other agent of the network, but only to a subset of them. The agents with which a single agent  $i$  can communicate are called *neighbors* of agent  $i$ . In particular, if agent  $j$  can send an information to agent  $i$ , then  $j$  is an *in-neighbor* of agent  $i$ , while  $i$  is an *out-neighbor* of  $j$ . In this sense, the synchronization problem is the problem of designing a distributed diffusing coupling control action, so that all the agents in the network reach consensus, that is, they reach some sort of agreement in their state evaluation.

Over the last years, synchronization and consensus problems have become very popular in the control community due to the several applications in which such problems arise. Power networks Dörfler et al. (2013), heat networks Scholten et al. (2016), robot swarms Bullo et al. (2009); Olfati-Saber (2006); Chiddarwar and Babu (2011), urban traffic Abdoos et al. (2013); Balaji and Srinivasan (2010), fault detection Hu

et al. (2018), microgrids Simpson-Porco et al. (2013); Shen et al. (2018) and biological systems Roche et al. (2008) are just a few of possible real-life cases in which the study and the control of multi-agent networks play a fundamental role.

In a synchronization problem, the design of the control action must take into consideration two fundamental aspects that do not appear in other control problems. The first one is the model of the network. In other words, the first aspect is how to model the interaction and exchange of information among agents. This is done with tools from graph theory (see, for instance, Godsil and Royle (2001)). Depending on the properties of the graph describing the network (directed/undirected, weighted/unweighted, ...), different control strategies may be exploited. The second aspect is the model of each agent. In this case, it is not only important the form of the equation describing the agent (linear/nonlinear, continuous-time/discrete-time/hybrid, ...), but also if the network is *homogeneous* or *heterogeneous*. In the first case, all the agents are described by the same model, while in the second case each agent may be modeled differently.

### 3.1.2 Existing solutions and proposed contributions

Due to the various form of characterization that such a problem can have, the literature on multi-agent synchronization is very large. In this manuscript, we will consider only networks whose agents are modeled by a *continuous-time* ODE. In such a framework, systems modeled by a linear ODE were first considered. In this case, fundamental results were obtained in Scardovi and Sepulchre (2008) for homogeneous networks and Wieland et al. (2011) for heterogeneous ones.

Nowadays, the control community is focusing on the study of control actions designed for nonlinear systems. In this manuscript, we will consider only *homogeneous* networks. The majority of results that have been obtained for homogeneous networks, approach the problem using control tools specifically adapted to deal with a distributed framework. Among them, we recall for instance a passivity approach in Arcak (2007), a dissipativity one in Stan and Sepulchre (2007) and an ISS one in Casadei et al. (2019a,b). High-gain techniques, inherited from high-gain observers theory or high-gain domination approaches (see, e.g., Chopra and Spong (2008); Isidori et al. (2014); Casadei and Astolfi (2017); Panteley and Loría (2017)), form another notable class of solutions. Such controllers share pros and cons with their observers counterparts. This led researchers to investigate alternative tools such as nonlinear integral control, see Pavlov et al. (2022).

In the proposed approach, we attack the synchronization problem using tools derived from incremental stability. The reason for this choice is very intuitive. In a homogeneous network, every agent is modeled by the same dynamical equation, and they are different from each other only by their initial conditions. In other words, we have copies of the same ODE. In this sense, the synchronization task is to design a distributed control action so that each agent loses its dependence on the initial conditions. Since the difference between two solutions of the same ODE (the trajectories of two different agents) must asymptotically converge to zero, we recover the basic concept of incremental asymptotic stability described in Chapter 1. The explicit use of incremental stability tools to solve a synchronization task is not new in the literature. Some results can be found for instance in Andrieu et al. (2016); Andrieu and Tarbouriech (2019); Yin et al. (2021); Zhang et al. (2015, 2014); Pavlov et al. (2022). The existing results, however, either focus on specific classes

of systems such as monotonic systems Andrieu and Tarbouriech (2019) or Lur'e and sector bound Zhang et al. (2014, 2015); Zhao et al. (2013), or solve the synchronization task only locally i.e. if the initial conditions of each agent are close one to each other Andrieu et al. (2016), or propose control strategies that are strongly dependent on the topology of the network Pavlov et al. (2018, 2022).

In the tackled problem:

- we will consider homogeneous networks where agents are described by a non-linear MIMO time-varying ODE;
- we aim to achieve global synchronization of the network, that is, the consensus problem will be solved independently on the initial conditions of the agents.

The contributions are various. First, we approach the problems with an Euclidean-based contractive analysis. We consider a directed and connected network of input-affine nonlinear systems with constant input vector fields. We show that synchronization can be achieved under the design of an infinite gain margin control law that involves the solution of a Riccati-like inequality. This extends more classical results on synchronization of linear systems, see (Isidori, 2017, Section 5). In this context, the proposed design provides a unifying framework with respect to many literature results (see e.g. the high-gain observer approach in Casadei et al. (2014), the incremental passive approach in Pavlov et al. (2022) and the results in Zhang et al. (2014); Andrieu and Tarbouriech (2019) when considering particular classes of systems). Moreover, we show how the proposed design extends the results in Andrieu et al. (2018) in terms of control gain (that is different on every agent) and domain of attraction (local synchronization), as well as the results in Pavlov et al. (2022) in terms of network topology (which in our case is only assumed to be connected). As a second contribution, we show that in leader-synchronization scenarios with undirected networks the proposed control laws can be generalized further using Riemannian metrics. This allows considering more general classes of systems with significant nonlinearities. However, in the Riemannian case, the proposed control design requires the solution of a nonlinear differential equation. This turns out to be a difficult task in practice. To deal constructively with such complexity, we provide the following additional results. i) First, we show that synchronization can be achieved in a regional context under less stringent assumptions. ii) Thanks to this relaxation, we provide a formulation of a practical Deep Neural Network (DNN)-based algorithm to check the solvability of such differential equation. The use of DNNs for solving partial differential equations is not new to the Machine Learning community, see e.g., Raissi et al. (2019); Cai et al. (2022). In this context, we recall recent results on the use of Deep Learning tools for the estimation of a Riemannian metric (see for instance Tsukamoto et al. (2021); Wei et al. (2021) and references therein).

The results that will be presented can be found in the articles of the manuscript's author Giaccagli et al. (2021a, 2022e). The chapter is structured as follows. In Section 3.2, we provide some preliminaries. In particular, in Section 3.2.1, we provide basic results on graph theory and introduce the notation while in Section 3.2.3 we recall the main aspects of synchronization of linear systems. In Section 3.3, we provide some results for the design of a distributed feedback control law (state and output feedback) to achieve synchronization of connected networks of nonlinear systems. Such an analysis is based on Euclidean incremental stability tools. In Section 3.4 we

provide a state-feedback and an output-feedback distributed control law based on a Riemannian metric analysis for the synchronization of a leader-connected network of nonlinear systems. In Section 3.5 we show that our design applies to some classes of systems. To conclude, in Section 3.6, using Deep Learning tools we provide a Deep Neural Network-based algorithm for the practical implementation of the proposed design. The procedure is then successfully implemented to achieve synchronization for a leader-connected network of Lorentz oscillators.

## 3.2 Preliminaries

### 3.2.1 Basics on graph theory

Before entering in details of the results in this section, it is useful to introduce some preliminaries. In particular, we will cover the following aspects.

- We will remind basic aspects of graph theory and introduce the notation that will be used, together with some useful results that will play a role in the upcoming analysis.
- We will recall the main aspects of synchronization for networks of linear systems.

In a general framework, a communication graph is described by a triplet  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  in which  $\mathcal{V} = \{v_1, v_2, \dots, v_N\}$  is a set of  $N$  *vertexes* (or nodes),  $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$  is the set of *edges*  $e_{jk}$  that models the interconnection between the vertexes with the flow of information from vertex  $j$  to vertex  $k$  *weighted* by the  $(k, j)$ -th entry  $a_{kj} \geq 0$  of the *adjacency matrix*  $\mathcal{A} \in \mathbb{R}^{N \times N}$ . We denote by  $\mathcal{L} \in \mathbb{R}^{N \times N}$  the *Laplacian matrix* of the graph, defined as

$$\ell_{kj} = -a_{kj} \quad \text{for } k \neq j, \quad \ell_{kj} = \sum_{i=1}^N a_{ki} \quad \text{for } k = j,$$

where  $\ell_{j,k}$  is the  $(j, k)$ -th entry of  $\mathcal{L}$ . We denote with  $\mathcal{N}_i$  the set of in-neighbors of node  $i$ , i.e. the set  $\mathcal{N}_i := \{j \in \{1, \dots, N\} \mid e_{ji} \in \mathcal{E}\}$ .

**Definition 3.2.1** (Connected graph). *A time-invariant communication graph is said to be connected if, for any two nodes  $(i, j) \in \mathcal{V} \times \mathcal{V}$ , there exists a (directed) path connecting  $v_i$  to  $v_j$ , i.e. there exists a sequence of nodes  $v_q, v_{q+1}, \dots, v_{q_n}$  such that  $(e_{i,q}, e_{q,q+1}, \dots, e_{q_n,j}) \in \mathcal{E} \times \dots \times \mathcal{E}$ .*

**Proposition 3.2.1.** *A time-invariant graph is connected if and only if  $\mathcal{L}$  has only one trivial eigenvalue  $\lambda_1(\mathcal{L}) = 0$  and all other eigenvalues  $\lambda_2(\mathcal{L}), \dots, \lambda_n(\mathcal{L})$  have strictly positive real parts*

| **Proof.** See (Isidori, 2017, Theorem 5.1). □

In this chapter, two different types of graphs will be considered: directed and undirected. In particular, let us identify the Laplacian of the network as

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}_{11} & \mathcal{L}_{12} \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix} \quad (3.1)$$

where  $\mathcal{L}_{11}$  is a scalar,  $\mathcal{L}_{12}$  is a  $N - 1$  row vector,  $\mathcal{L}_{21}$  is a  $N - 1$  column vector and  $\mathcal{L}_{22}$  is a  $(N - 1) \times (N - 1)$  matrix. The following then holds.

**Lemma 3.2.2.** *Let the graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  be directed and connected. Then, there exists a strictly positive real number  $\mu > 0$  and a symmetric positive definite matrix  $Q = Q^\top \succ 0$  such that the Laplacian  $\mathcal{L}$  satisfies*

$$\text{He}\{Q(\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12})\} \succeq \mu I. \quad (3.2)$$

**Proof.** Since the graph is connected, the Laplacian  $\mathcal{L}$  has one zero eigenvalue and  $N - 1$  strictly positive eigenvalues by Proposition 3.2.1. Consider the matrix

$$\mathcal{R} = \mathcal{R}^{-1} = \begin{pmatrix} 1 & 0 \\ \mathbf{1} & -I \end{pmatrix}.$$

and let  $\nu^\top = (1 \ 0 \ \dots \ 0)$  and note that  $\mathcal{R}\nu = \mathbf{1}$ . Consider the change of coordinates on the Laplacian defined by

$$\tilde{\mathcal{L}} = \mathcal{R}\mathcal{L}\mathcal{R}^{-1} = \begin{pmatrix} 0 & -\mathcal{L}_{12} \\ 0 & \mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12} \end{pmatrix}.$$

Since  $\mathcal{R}$  is full rank, the new matrix  $\tilde{\mathcal{L}}$  has still one eigenvector associated with the eigenvalue zero, and all the others are strictly positive definite. In particular, the zero eigenvalue is associated with the eigenvector  $\nu$ , since  $\mathcal{R}\nu = \mathbf{1}$  and  $\mathcal{L}\mathbf{1} = 0$ . Because of the block-diagonal structure of  $\tilde{\mathcal{L}}$ , all the eigenvalues of  $\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12}$  are strictly positive. Hence (3.2) holds.  $\square$

**Definition 3.2.2** (Leader-connected graph). *A time-invariant communication graph is said to be leader-connected if there exists a node, called ‘leader’, for which the graph contains at least one spanning tree with the leader as a root.*

For an undirected graph, instead, we have the following property proved in Godsil and Royle (2001).

**Lemma 3.2.3.** *Consider the graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  be undirected and leader-connected. Then the Laplacian  $\mathcal{L}$  can be partitioned as*

$$\mathcal{L} = \begin{pmatrix} 0 & 0 \\ \mathcal{L}_{21} & \mathcal{L}_{22} \end{pmatrix}. \quad (3.3)$$

Moreover, there exists a strictly positive real number  $\mu > 0$  such that

$$\mathcal{L}_{22} \succeq \mu I. \quad (3.4)$$

### 3.2.2 Synchronization with nonlinear diffusive coupling

The dynamics of each agent of the network are described by a general time-varying nonlinear ODE

$$\dot{x}_i = f(t, x_i, u_i), \quad y = h(t, x_i), \quad i = 1, \dots, N, \quad (3.5)$$

where  $x_i \in \mathbb{R}^{n_x}$  is the state of node  $i$ ,  $u_i \in \mathbb{R}^{n_u}$  is the control action on node  $i$ , and  $y_i \in \mathbb{R}^{n_y}$  is the output which is exchanged to the neighborhoods of the network. We denote the state of the entire network as

$$\mathbf{x} := \text{col}\{x_1^\top, \dots, x_N^\top\}^\top \in \mathbb{R}^{Nn_x}. \quad (3.6)$$

Furthermore, we denote with  $\mathcal{X}_i(x_i^\circ, t, t_0)$  the trajectory of agent  $i$  starting from the initial condition  $x_i^\circ$  and initial time  $t_0 \in \mathbb{R}$  evaluated at time  $t \geq t_0$ , and with  $\mathcal{X}(\mathbf{x}^\circ, t, t_0)$  the trajectory of the entire network (3.6) evaluated at initial condition  $\mathbf{x}^\circ$ , initial time  $t_0$  at time  $t \geq t_0$ . Our synchronization objective is to design a nonlinear diffusive coupling, namely a distributed feedback control law of the form

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} \left[ \varphi(x_i, y_j, t) - \varphi(x_i, y_i, t) \right] = - \sum_{j=1}^N \ell_{ij} \varphi(x_i, y_j, t) \quad (3.7)$$

for all  $i = 1, \dots, N$ , for some sufficiently smooth function  $\varphi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_y} \times \mathbb{R} \rightarrow \mathbb{R}^{n_u}$ , that stabilizes the dynamics (3.16) on the so-called *synchronization manifold*  $\mathcal{D}$  defined as

$$\mathcal{D} := \{ \mathbf{x} \in \mathbb{R}^{Nn_x} \mid x_i = x_j, \text{ for all } i, j \in \{1, \dots, N\} \}, \quad (3.8)$$

where the states of all the agents of the network agree with each other. By construction, the  $i$ -th agent uses only the output information  $y_j$  of its neighborhoods  $j \in \mathcal{N}_i$  and its own local information  $y_i, x_i$ . Furthermore, the control action  $u_i$  is equal to zero on the synchronization manifold. In other words, when consensus is achieved, no correction term is needed for each agent. As a consequence, stabilizing all the agents on the desired equilibrium point is generally not a valid solution in such a framework. We formalize our synchronization problem as follows.

**Problem 3.2.4** (Network synchronization). *Let the function  $\varphi$  be such that the manifold  $\mathcal{D}$  defined in (3.8) is globally uniformly exponentially stable for the closed-loop system*

$$\dot{x}_i = f(t, x_i, - \sum_{j \in \mathcal{N}_i} \ell_{ij} \varphi(x_i, y_j, t)), \quad i = 1, \dots, N,$$

namely, there exist positive constants  $k$  and  $\lambda > 0$  such that for all  $(\mathbf{x}^\circ, t_0)$  in  $\mathbb{R}^{Nn_x} \times \mathbb{R}$  and for all  $t \geq t_0$  in the time domain of existence of solutions  $\mathcal{T} \subseteq \mathbb{R}$  we have

$$|\mathcal{X}(\mathbf{x}^\circ, t, t_0)|_{\mathcal{D}} \leq k \exp(-\lambda(t - t_0)) |\mathbf{x}^\circ|_{\mathcal{D}}. \quad (3.9)$$

Then, we say that the distributed feedback control law (3.7) solves the global exponential synchronization problem for the network (3.6).

### 3.2.3 Network synchronization for linear systems

To better contextualize and frame the results in this chapter, it is useful to recall some important aspects of synchronization of a homogeneous network of linear systems, see, e.g. (Isidori, 2017, Chapter 5). Consider a network where each agent is described by

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i \quad (3.10)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control action,  $y \in \mathbb{R}^{n_y}$  is a linear combination of the state and  $A, B, C$  are matrices of appropriate dimension. The following result holds, see (Isidori, 2017, Proposition 5.2).

**Proposition 3.2.5** (Synchronization of linear systems). *Consider a connected network where each agent is described by (3.10). Assume there exists a matrix  $K$  such*

that the matrix  $A - \lambda_{\mathcal{L}} BKC$  is Hurwitz for all  $\lambda_{\mathcal{L}} \in \text{spec}\{\mathcal{L}\} \setminus \{0\}$ . Then the network in closed-loop with the distributed control law

$$u_i = K \sum_{j \in \mathcal{N}_i} a_{ij}(y_j - y_i) = -K \sum_{j=1}^n \ell_{ij} y_j \quad (3.11)$$

achieves consensus, i.e.  $\lim_{t \rightarrow +\infty} |x_i(t) - x_j(t)| = 0$  for all  $(i, j) \in \mathcal{V} \times \mathcal{V}$ .

In other words, for a connected network of linear systems, the synchronization problem can be seen as a *robust* (or simultaneous) stabilization problem. With ‘robust’ we mean that the stabilization problem must be achieved for any strictly positive eigenvalue of the Laplacian  $\lambda_{\mathcal{L}}$ , which can be seen as a gain acting on the control term. To fulfill this requirement, a solution is given by employing an *infinite gain margin* feedback (see e.g. (Sepulchre et al., 2012, Section 3)), that is a feedback law that achieves stability in the presence of an uncertain factor in front of the gain matrix  $B$ .

For linear systems, a general sufficient condition can be stated as follows.

**Lemma 3.2.6.** *Suppose there exists a symmetric positive definite matrix  $P = P^\top \succ 0$ , matrices  $D, E$  and positive real numbers  $\varepsilon, \rho > 0$  such that the following holds*

$$\begin{aligned} PA + A^\top P - \rho C^\top E^\top EC &\preceq -\varepsilon P, \\ PBD &= C^\top E^\top. \end{aligned} \quad (3.12)$$

Then, the feedback gain  $K = DE$  is an infinite-gain margin static output feedback law for the triplet  $(A, B, C)$ , namely  $A - \kappa BKC$  is Hurwitz for any  $\kappa \in [\frac{\rho}{2}, \infty)$ .

**Proof.** Let  $P$  given by (3.12) and compute

$$\begin{aligned} &P(A - \kappa BKC) + (A - \kappa BKC)^\top P \\ &= PA + A^\top P - \kappa PBB^\top C^\top E^\top EC - \kappa C^\top E^\top D^\top B^\top P \\ &= PA + A^\top P - 2\rho C^\top E^\top EC - (2\kappa - \rho)C^\top E^\top EC \\ &\preceq -\varepsilon P \end{aligned}$$

for all  $\kappa \geq \frac{\rho}{2}$  concluding the proof.  $\square$

We remark that condition (3.12) is slightly different from the one established for instance in Kučera and De Souza (1995), where sufficient and necessary conditions for the existence of a static output feedback stabilizing control law are given. The conditions in Kučera and De Souza (1995) takes the form

$$\begin{aligned} PA + A^\top P - PBB^\top P + C^\top C + E^\top E &= 0, \\ KC - B^\top P &= -E, \end{aligned}$$

but don’t necessarily have the infinite gain margin property (except for the passivity-like case  $E = 0$ ). Therefore, (3.12) is more restrictive. An exhaustive discussion about the necessity and feasibility of (3.12) is out of the scope of this section. Some specific cases are discussed at the end of this section.

Coming back to our synchronization problem, we may state now the following result.

**Lemma 3.2.7.** *Consider the network described by (3.10) and suppose there exists a solution to (3.12). Then, the distributed control law (3.11) solves the synchronization Problem 3.2.4 with  $K = \kappa DE$  with  $\kappa \geq \frac{\rho}{2\mu}$  and  $\mu$  given by Lemma 3.2.2.*

The proof of such a result is shown, for instance, in (Li et al., 2009, Section II.B). See also (Isidori, 2017, Chapter 5) and references therein for more details. We may observe that the aforementioned condition (3.12) generalizes the following classical results.

- For a *state-feedback* synchronization problem

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = x_i,$$

condition (3.12) recovers the feedback design

$$PA + A^\top P - PBB^\top P \preceq -\varepsilon P, \quad K = B^\top P \quad (3.13)$$

with  $D = I$ ,  $E = B^\top P$ ,  $\rho = 1$ . Note that the Algebraic Riccati Equation (ARE) (3.13) always admits a solution if  $(A, B)$  is stabilizable.

- For the *observer-form output-feedback* synchronization problem

$$\dot{x}_i = Ax_i + u_i, \quad y_i = Cx_i,$$

the condition (3.12) recovers the feedback design

$$PA + A^\top P - C^\top C \preceq -\varepsilon P, \quad K = P^{-1}C^\top, \quad (3.14)$$

with  $D = P^{-1}C^\top$ ,  $E = I$ ,  $\rho = 1$ . The ARE (3.14) always admits a solution if  $(A, C)$  is detectable.

- For the square  $n_y = n_u$  MIMO output-feedback synchronization problem

$$\dot{x}_i = Ax_i + Bu_i, \quad y_i = Cx_i,$$

the condition (3.12) recovers the *passivity condition*

$$PA + A^\top P - C^\top C \preceq -\varepsilon P, \quad PB = C^\top, \quad (3.15)$$

with  $D = I_{n_u}$ ,  $E = I_{n_y}$ ,  $\rho = 1$  and  $K = I_{n_y}$ .

Therefore, the take-away message we aim to highlight in this section is that, if we aim at developing a theory for general nonlinear dynamics and generic connected networks, we need to be able to solve a robust stabilization problem as in Proposition 3.2.5. As a consequence, the key property is given by the extension of the aforementioned infinite-gain margin law in the contraction framework, see, e.g. Section 1.4.4.4. This symmetry will be further developed in the next sections.

### 3.3 Network synchronization with Euclidean contraction

#### 3.3.1 Feedback design for network synchronization

In this first section, we will consider a network of  $N$  homogeneous nonlinear multi-agent systems, i.e., described by identical dynamics. The objective is to derive sufficient conditions for the synchronization of the full network. Motivated by the linear systems case, we aim to provide a design that requires no additional assumptions on the network, except for its connectivity. This is recalled in the following assumption.

**Assumption 3.3.1** (Graph network synchronization). *The graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$  is directed and connected.*

Concerning the dynamics of the agents of the network, in this section we focus on time-varying nonlinear dynamics described by the following nonlinear ODE

$$\dot{x}_i = f(x_i, t) + Bu_i, \quad i = 1, \dots, N, \quad (3.16)$$

where  $x_i \in \mathbb{R}^{n_x}$  is the state of node  $i$ ,  $u_i \in \mathbb{R}^{n_u}$  is the control action on node  $i$ ,  $B$  is a constant matrix and  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$  is a vector field which is  $C^1$  in the first argument and piece-wise continuous in the second. Motivated by the analysis in Section 3.2.3, we consider the case in which agent (3.16) can exchange with their neighborhoods only an output of the form

$$y_i = Cx_i, \quad i = 1, \dots, N, \quad (3.17)$$

with  $y_i \in \mathbb{R}^{n_y}$  and where  $C$  is a constant matrix. In particular, our objective is to design a feedback distributed control law stabilizing the dynamics (3.16) on the so-called *synchronization manifold*  $\mathcal{D}$  defined in (3.8), where the states of all the agents of the network agree with each other. Following Section 3.2.2, we look for a diffusive coupling law of the form (3.7) for all  $i = 1, \dots, N$ . Along the lines of the linear case, we focus on a control design which involves the solution of a Riccati-like inequality in order to get a control law possessing an infinite gain margin property. We have the following result.

**Theorem 3.3.1** (Feedback network synchronization). *Consider system (3.16), (3.17) and let Assumption 3.3.1 hold. Assume there exists a symmetric positive definite matrix  $P = P^\top \succ 0$ , two matrices  $D, E$  and two real numbers  $\varepsilon, \rho > 0$  such that*

$$\begin{aligned} L_f P(x, t) - \rho C^\top E^\top E C &\leq -\varepsilon P, \\ PBD &= C^\top E^\top. \end{aligned} \quad (3.18)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ . Then, for any  $\kappa \geq \frac{\rho}{2\mu}$ , with  $\mu$  given by Lemma 3.2.2, the distributed feedback law (3.7) with

$$\varphi(x, t) = \kappa Ky, \quad K = DE, \quad (3.19)$$

solves the synchronization Problem 3.2.4 for the network (3.16).

As for the linear case, condition (3.18) includes many published results. For instance, in the case of output-feedback form in which  $B = I$ , we recover the high-gain observer approach proposed in (Casadei et al., 2014, Proposition 1). In the state-feedback case with  $C = I$ , we recover the results in (Andrieu et al., 2018, Theorem 3) which are generalized concerning the assumption on the graph and in terms of gain of the control law, which in our case is the same for each agent. When considering system dynamics expressed in Lur'e form, we recover the results of Zhang et al. (2014). When considering the passivity condition as in (3.15), we recover the context of incrementally passivity with respect to a constant Euclidean metric  $P$ , see for instance Forni and Sepulchre (2013b), (Pavlov and Marconi, 2008, Section 5) and we generalize the result in (Pavlov et al., 2022, Theorem 4) concerning the assumption on the graph.

**Proof.** The idea behind the proof is to set a *virtual leader* and show that the dynamics of the error between any other agent and such a leader exponentially goes to zero. Therefore, let  $\tilde{x} = (\tilde{x}_2^\top, \dots, \tilde{x}_N^\top)^\top$  where  $\tilde{x}_i = x_i - x_1$  represents the error between agent  $x_i$  and  $z = x_1$ . Since  $\ell_{ij} = 0$  if  $j \notin \mathcal{N}_i$ , the  $\tilde{x}_i$ -dynamics can be written as

$$\begin{aligned} \dot{\tilde{x}}_i &= f(z + \tilde{x}_i, t) - \kappa B \sum_{j=1}^N \ell_{ij} DEC(z + \tilde{x}_j) \\ &\quad - f(z, t) + \kappa B \sum_{j=1}^N \ell_{1j} DEC(z + \tilde{x}_j). \end{aligned}$$

Since  $\sum_{j=1}^N \ell_{ij} = 0$  for all  $i = 1, \dots, N$ , we can add the terms  $-\kappa B \left( \sum_{j=1}^N \ell_{1j} \right) DECz = 0$  and  $\kappa B \left( \sum_{j=1}^N \ell_{ij} \right) DECz = 0$ . This leads to

$$\begin{aligned} \dot{\tilde{x}}_i &= f(z + \tilde{x}_i, t) - f(z, t) - \kappa B \sum_{j=1}^N \ell_{ij} DEC \tilde{x}_j \\ &\quad + \kappa B \sum_{j=1}^N \ell_{1j} DEC \tilde{x}_j \\ &= f(z + \tilde{x}_i, t) - f(z, t) - \kappa B \sum_{j=1}^N (\ell_{ij} - \ell_{1j}) DEC \tilde{x}_j. \end{aligned}$$

Note that in these new coordinates, the synchronization manifold defined in (3.8) corresponds to the origin of the  $\tilde{x}$ -dynamics. Let  $t_0$  be in  $\mathbb{R}$  and consider a solution  $\mathcal{Z}(z^\circ, t, t_0)$ ,  $\tilde{\mathcal{X}}(t, t_0) = (\tilde{\mathcal{X}}_2(t, t_0), \dots, \tilde{\mathcal{X}}_N(t, t_0))$  of the closed-loop system which is defined for all  $t$  in the time domain of definition  $\mathcal{T} \subseteq \mathbb{R}$ . Consider the following Lyapunov function  $V$

$$V(t) := \tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) \tilde{\mathcal{X}}(t, t_0) \quad (3.20)$$

defined for all  $t$  in  $\mathcal{T}$ , where  $P$  solves (3.18) and  $Q$  is chosen as in Lemma 3.2.2. Taking the time-derivative  $V$  and recalling that  $(Q \otimes P)^\top = (Q^\top \otimes P^\top) = (Q \otimes P)$ , it yields

$$\begin{aligned} \dot{V}(t) &= 2\tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) \text{blkdiag} \left\{ f(\mathcal{Z} + \tilde{\mathcal{X}}_i, t) - f(\mathcal{Z}, t) \right\}_{i=2, \dots, N} \\ &\quad - 2\kappa \tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) ((\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12}) \otimes BDEC) \tilde{\mathcal{X}}(t, t_0). \end{aligned}$$

Since  $Q \otimes P = (Q \otimes I)(I \otimes P)$ , by Lemma 3.2.2 and by using the Mean Value

Theorem, it follows that

$$\begin{aligned} \dot{V}(t) &= 2\tilde{\mathcal{X}}(t, t_0)^\top \left[ (Q \otimes P) \text{blkdiag} \left\{ \int_0^1 \frac{\partial f}{\partial x}(z - s\tilde{\mathcal{X}}_i, t) ds \right\} \right. \\ &\quad \left. - 2\kappa (Q(\mathcal{L}_{22} - \mathbf{1}\mathcal{L}_{12}) \otimes PBDEC) \right] \tilde{\mathcal{X}}(t, t_0) \\ &\leq 2\tilde{\mathcal{X}}(t, t_0)^\top \left[ (Q \otimes I) \text{blkdiag} \left\{ \int_0^1 L_f P(z - s\tilde{\mathcal{X}}_i, t) ds \right\} \right. \\ &\quad \left. - 2\kappa (\mu I \otimes C^\top E^\top EC) \right] \tilde{\mathcal{X}}(t, t_0). \end{aligned}$$

By using (3.18) and by choosing any  $\kappa \geq \frac{\rho}{2\mu}$  with  $\mu$  given by Lemma 3.2.2, we get

$$\dot{V}(t) \leq -\varepsilon \tilde{\mathcal{X}}(t, t_0)^\top (Q \otimes P) \tilde{\mathcal{X}}(t, t_0) = -\varepsilon V(t). \quad (3.21)$$

From Gronwall's Lemma this implies

$$V(t) \leq \exp(-\varepsilon(t - t_0)) V(t_0)$$

for all  $t$  in  $\mathcal{T}$ . Using the definition of  $V$  in (3.20), the latter inequality gives

$$|\tilde{\mathcal{X}}(t, t_0)|^2 \leq \frac{\lambda_M(Q \otimes P)}{\lambda_m(Q \otimes P)} \exp(-\varepsilon(t - t_0)) |\tilde{\mathcal{X}}(t_0, t_0)|^2, \quad (3.22)$$

for all  $t$  in  $\mathcal{T}$ , where  $\lambda_m(Q \otimes P)$ ,  $\lambda_M(Q \otimes P)$  are respectively the minimum and the maximum (strictly positive) eigenvalues of  $Q \otimes P$ . Remembering that by definition  $\tilde{x}_i = x_i - z$ , by equivalence of norms in finite dimensional spaces it follows that there exist two strictly positive real numbers  $\underline{c}, \bar{c} > 0$  such that

$$\underline{c} |\mathcal{X}(\mathbf{x}^\circ, t, t_0)|_{\mathcal{D}} \leq |\tilde{\mathcal{X}}(t, t_0)| \leq \bar{c} |\mathcal{X}(\mathbf{x}^\circ, t, t_0)|_{\mathcal{D}}, \quad (3.23)$$

for all  $t$  in  $\mathcal{T}$ , which implies (3.9) and concludes the proof.  $\square$

We highlight again the symmetry between the results proposed in Theorem 3.3.1 and the linear case in Section 3.2.3. Both results involve the solution of a Riccati-like inequality to guarantee an infinite-gain margin property for the control action. This aspect allows relying on the sole assumption of the network being simply connected.

As a last remark of this section, we stress that the Riccati-like condition (3.18) is in general only sufficient and not necessary. Indeed, in the context of contraction analysis and incremental stability, the existence of a Riemannian metric is proved to be equivalent (under a globally Lipschitz property of the first derivative of the vector field) to the desired contraction properties, see Andrieu et al. (2016). Similar considerations obviously, can be done in a multi-agent context, see Andrieu et al. (2018). However, when we restrict to Euclidean metrics, this equivalence is lost. As a consequence, a set of Euclidean metric-based conditions is only sufficient.

## 3.4 Leader synchronization with Riemannian contraction

### 3.4.1 Network structure and problem statement

In Section 3.3 the synchronization problem has been solved with an analysis based on incremental stability tools concerning a Euclidean metric. To extend such a result to more general classes of systems, in this section, we aim to pursue a similar analysis with a study based on Riemannian metrics. Consider a network of  $N$  identical agents described by

$$\dot{x}_i = f(x_i, t) + g(x_i, t)u_i, \quad i = 1, \dots, N, \quad (3.24)$$

where  $x \in \mathbb{R}^{n_x}$  is the state,  $u_i \in \mathbb{R}^{n_u}$  is the control action and  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$ ,  $g : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_u}$  are  $C^2$  functions in the first argument and piecewise continuous in the second.

In what follows, we focus on the leader-synchronization problem. In other words, we suppose the presence of a *leader*, that is, an agent of the network (3.24) for which the control action is zero, as specified in the following assumption. Without loss of generality, we label the leader as the node 1.

**Assumption 3.4.1.** *The graph  $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$  is undirected and leader-connected.*

As a consequence, the network (3.25) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= f(x_1, t) \\ \dot{x}_i &= f(x_i, t) + g(x_i, t)u_i, \quad i = 2, \dots, N. \end{aligned} \quad (3.25)$$

The control objective is to design a feedback distributed control law of the form

$$u_i = \sum_{j \in \mathcal{N}_i} a_{ij} [\varphi(x_j, t) - \varphi(x_i, t)] = - \sum_{j=1}^N \ell_{ij} \varphi(x_j, t) \quad (3.26)$$

for all  $i = 2, \dots, N$ , stabilizing the dynamics of (3.25) to the synchronization manifold  $\mathcal{D}$  (3.8). For the moment, we focus on the case in which the  $i$ -th agent uses only the state information of its neighborhoods and of itself. To this end, we look again for a feedback design that is derived from a Riccati-like inequality. In this case, however, we follow the contraction analysis based on Riemannian metrics.

### 3.4.2 Feedback design for leader synchronization

#### 3.4.2.1 State-feedback leader synchronization

In this section, we consider the problem of designing a state-feedback control law such that the global exponential multiagent leader-synchronization problem is solved. To consider a more general class of systems as in (3.25), we approach the problem with a contraction analysis based on Riemannian metrics. The following result holds.

**Theorem 3.4.1** (State-feedback leader-synchronization). *Consider system (3.25) and assume Assumption 3.4.1 holds. Assume moreover that there exists a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values such that the following conditions hold:*

- the State Control Matrix Function (SCMF) condition holds:

$$\begin{aligned} L_f P(x, t) - \rho P(x, t) g(x, t) g^\top(x, t) P(x, t) &\preceq -\varepsilon P(x, t) \\ \underline{p}I &\preceq P(x, t) \preceq \bar{p}I, \end{aligned} \quad (3.27)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$  and for some positive real numbers,  $\underline{p}, \bar{p}, \varepsilon, \rho > 0$ ;

- the matrix function  $g$  has the Killing vector field property with respect to  $P$ , i.e.,

$$L_g P(x, t) = 0 \quad (3.28)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ ;

- there exists a  $C^2$  function  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_u}$  such that

$$\frac{\partial \alpha^\top}{\partial x}(x, t) = P(x, t) g(x, t) \quad (3.29)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ .

Then, for any gain  $\kappa \geq \frac{\rho}{2\mu}$  for some  $\mu > 0$  given by Lemma 3.2.3, the distributed state-feedback control law (3.26) with

$$\varphi_j(x_j, t) = -\kappa \ell_{ij} \alpha(x_j, t), \quad (3.30)$$

solves the global exponential leader-synchronization problem for (3.25).

**Proof.** Similarly to the proof of Theorem 3.3.1, the main goal is to show that the dynamics of the error between any agent and the leader exponentially decreases to zero. Therefore, let us denote  $\tilde{x} = (\tilde{x}_2^\top \dots \tilde{x}_N^\top)^\top$  where  $\tilde{x}_i = x_i - x_1$  represents the error between agent  $i$  and  $z = x_1$ . Since  $\ell_{ij} = 0$  for all  $j \notin \mathcal{N}_i$ , the dynamics of the error  $\tilde{x}_i$  for all  $i = 2, \dots, N$  with the control law (3.30) can be rewritten as

$$\dot{\tilde{x}}_i = f(z + \tilde{x}_i, t) - f(z, t) - \kappa g(z + \tilde{x}_i, t) \sum_{j=1}^N \ell_{ij} \alpha(z + \tilde{x}_j, t).$$

Note that there's no term on  $g(z, t)$  since no control action is acting on the leader. Since  $\sum_{j=1}^N \ell_{ij} = 0$  for all  $i = 1, \dots, N$ , we can add the term  $\kappa g(z + \tilde{x}_i, t) \left( \sum_{j=1}^N \ell_{ij} \right) \alpha(z, t) = 0$  and get

$$\begin{aligned} \dot{\tilde{x}}_i &= f(z + \tilde{x}_i, t) - f(z, t) \\ &\quad - \kappa g(z + \tilde{x}_i, t) \sum_{j=1}^N \ell_{ij} [\alpha(z + \tilde{x}_j, t) - \alpha(z, t)]. \end{aligned} \quad (3.31)$$

Note that in this new coordinates, the synchronization manifold defined in (3.8) corresponds to the origin of the  $\tilde{x}$ -dynamics. Let  $t_0$  be in  $\mathbb{R}$  and consider a solution  $\mathcal{Z}(z^\circ, t, t_0)$  and  $\tilde{\mathcal{X}}(t, t_0) = (\tilde{\mathcal{X}}_2(t, t_0), \dots, \tilde{\mathcal{X}}_N(t, t_0))$  of the closed-loop system which is defined for all  $t$  in the time domain of definition  $\mathcal{T}_1 \subset \mathbb{R}$ . Consider the function  $\Gamma : [0, 1] \times \mathcal{T}_2 \times \mathbb{R} \mapsto \mathbb{R}^{Nn_x}$  which satisfies

$$\Gamma(s, t_0, t_0) = s \tilde{\mathcal{X}}(t_0, t_0)$$

and where its  $i$ -th component  $\Gamma_i$  is the solution of the following ordinary differential equation

$$\begin{aligned} \frac{\partial \Gamma_i}{\partial t}(s, t, t_0) &= f(\zeta_i, t) - f(\mathcal{Z}(z^\circ, t, t_0), t) \\ &\quad - \kappa g(\zeta_i, t) \sum_{j=1}^N \ell_{ij} (\alpha(\zeta_j, t) - \alpha(\mathcal{Z}(z^\circ, t, t_0), t)) \end{aligned}$$

where we indicated with  $\zeta_i = \mathcal{Z}(z^\circ, t, t_0) + \Gamma_i(s, t, t_0)$  for shortness of notation and where  $\mathcal{Z}(z^\circ, t, t_0)$  is the solution of the virtual leader  $z$  initialized at  $(z^\circ, t_0) \in \mathbb{R}^{n_x} \times \mathbb{R}$  and evaluated at time  $t \geq t_0$ . Finally  $\mathcal{T}_2 \subset \mathcal{T}_1$  is the time domain of definition of  $\Gamma$ . We will show in the following that  $\mathcal{T}_2 = \mathcal{T}_1$ . Consider the function  $V_i$  for  $i = 2, \dots, N$  defined for  $t$  in  $\mathcal{T}_2$  by

$$V_i(t) = \int_0^1 \frac{\partial \Gamma_i}{\partial s}(s, t, t_0)^\top P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) ds. \quad (3.32)$$

Note that we have for all  $(k, l)$  in  $\{1, \dots, n_x\}^2$

$$\frac{d}{dt} [P(\zeta_i, t)_{kl}] = \frac{\partial P_{kl}}{\partial x}(\zeta_i, t) [f(\mathcal{Z}(z^\circ, t, t_0) + \frac{\partial \Gamma_i}{\partial t}(s, t, t_0))] + \frac{\partial P_{kl}}{\partial t}(\zeta_i, t).$$

This implies that for all vector  $\nu$  in  $\mathbb{R}^{n_x}$ ,

$$\begin{aligned} \frac{d}{dt} [\nu^\top P(\zeta_i, t) \nu] &= \nu^\top \mathfrak{d}_f P(\zeta_i, t) \nu \\ &\quad - 2\kappa \sum_{j=1}^N \left[ \ell_{ij} \sum_{\iota=1}^{n_u} \nu^\top \mathfrak{d}_{g_\iota} P(\zeta_i, t) \nu (\alpha_\iota(\zeta_j, t) - \alpha_\iota(\mathcal{Z}(z^\circ, t, t_0), t)) \right]. \end{aligned}$$

By using the Killing vector assumption (3.28) and the integrability one on the function  $\alpha$  in (3.29), the time derivative of  $V_i$  becomes

$$\begin{aligned} \dot{V}_i(t) &= \int_0^1 \left[ \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) L_f P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) \right. \\ &\quad \left. - 2\kappa \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) \sum_{j=1}^N \ell_{ij} P(\zeta_i, t) g(\zeta_i, t) g(\zeta_j, t)^\top P(\zeta_j, t) \frac{\partial \Gamma_j}{\partial s}(s, t, t_0) \right] ds. \end{aligned}$$

Let

$$D_f P(\zeta, t) := \text{blkdiag}\{L_f P(\zeta_2, t), \dots, L_f P(\zeta_N, t)\}$$

and

$$\Psi(\zeta, t) := \text{col}\{P(\zeta_1, t)g(\zeta_1, t), \dots, P(\zeta_N, t)g(\zeta_N, t)\}.$$

By considering all the  $V_i$  we have that

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) = \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_f P(\zeta, t) \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \right. \\ \left. - 2\kappa \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \Psi(\zeta, t) \mathcal{L}_{22} \Psi^\top(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right] ds. \end{aligned} \quad (3.33)$$

Hence by Lemma 3.2.3,

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) \leq \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_f P(\zeta, t) \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \right. \\ \left. - 2\kappa \mu \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \Psi(\zeta, t) \Psi^\top(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right] ds. \end{aligned} \quad (3.34)$$

Therefore by selecting  $\kappa \geq \frac{\rho}{2\mu}$  with  $\rho$  satisfying the SCMF defined in (3.27) and  $\mu > 0$  given by Lemma 3.2.3 we get

$$\dot{V}(t) = \sum_{i=2}^N \dot{V}_i(t) \leq -\varepsilon \sum_{i=2}^N V_i(t) = -\varepsilon V(t). \quad (3.35)$$

From Gronwall's Lemma this implies for all  $t$  in  $\mathcal{T}_2$

$$V(t) \leq \exp(-\varepsilon(t - t_0)) V(t_0). \quad (3.36)$$

From this inequality, we first deduce that  $\mathcal{T}_2 = \mathcal{T}_1$  since the path  $\Gamma$  has finite (Riemanniann) energy and has boundary defined in  $\mathcal{T}_1$ . Moreover,

$$\begin{aligned} V_i(t) &\geq \underline{p} \int_0^1 \frac{\partial \Gamma_i}{\partial s}(s, t, t_0)^\top \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) ds \\ &\geq \underline{p} \tilde{\mathcal{X}}_i(t, t_0)^\top \tilde{\mathcal{X}}_i(t, t_0). \end{aligned}$$

Since moreover,

$$V(t_0) \leq \bar{p} \tilde{\mathcal{X}}(t_0, t_0)^\top \tilde{\mathcal{X}}(t_0, t_0)$$

it yields for all  $t$  in  $\mathcal{T}_1$

$$|\tilde{\mathcal{X}}(t, t_0)|^2 \leq \exp(-\varepsilon(t - t_0)) \frac{\bar{p}}{\underline{p}} |\tilde{\mathcal{X}}(t_0, t_0)|^2.$$

With (3.23), inequality (3.9) follows.  $\square$

### 3.4.2.2 Incrementally passive output feedback leader synchronization

In this section, we tackle the problem of designing a static output-feedback distributed control law of the form (3.26) achieving leader-synchronization for the network. In other words, we consider a network defined by

$$\begin{aligned} \dot{x}_1 &= f(x_1, t), & y_1 &= h(x_1, t), \\ \dot{x}_i &= f(x_i, t) + g(x_i, t)u_i, & y_i &= h(x_i, t), \end{aligned} \quad (3.37)$$

where  $y_i \in \mathbb{R}^{n_y}$  and  $h : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_y}$  is  $C^2$  in the first argument and piecewise continuous in the second. We assume that the number of inputs is equal to the number of outputs, i.e.  $n_u = n_y$ . Similar to the previous section, the following result establishes a set of sufficient conditions for the existence of a distributed output feedback nonlinear diffuse coupling.

**Corollary 3.4.1.1** (Output-feedback leader-synchronization). *Consider the network (3.37). Let Assumption 3.4.1 hold and suppose that there exists a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values such that the following conditions hold.*

1. *The Output Control Matrix Function (OCMF) condition holds:*

$$\begin{aligned} L_f P(x, t) - \rho \frac{\partial h^\top}{\partial x}(x, t) \frac{\partial h}{\partial x}(x, t) &\preceq -\varepsilon P(x, t) , \\ \underline{p}I &\preceq P(x, t) \preceq \bar{p}I , \end{aligned} \quad (3.38)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$  for some positive constants  $\underline{p}, \bar{p}, \varepsilon, \rho > 0$ ;

2. *the following holds*

$$P(x, t)g(x, t) = \frac{\partial h^\top}{\partial x}(x, t) \quad (3.39)$$

and  $g$  has the Killing vector property for  $P$ , i.e.

$$L_g P(x, t) = 0 \quad (3.40)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ .

Then, for any  $\kappa \geq \frac{\rho}{2\mu}$  with  $\mu > 0$  given by Lemma 3.2.3, the output-feedback distributed control law (3.26) with

$$\varphi_j(x_j, t) = -\kappa \ell_{ij} h(x_j, t) \quad (3.41)$$

solves the global exponential leader synchronization problem.

**Proof.** The proof is similar as in Theorem 3.4.1. We recall only the main aspects. Therefore, let us denote  $\tilde{x} = (\tilde{x}_1^\top \dots \tilde{x}_N^\top)^\top$  where  $\tilde{x} = x_i - x_1$  represents the error between agent  $i$  and  $z = x_1$ . The dynamics of the error  $\tilde{x}$  for all  $i = 2, \dots, N$  with the control law (3.30) can be rewritten as

$$\dot{\tilde{x}}_i = f(z + \tilde{x}_i, t) - f(z, t) - \kappa g(z + \tilde{x}_i, t) \sum_{j=1}^N \ell_{ij} y_j$$

where there's no term in  $g(z, t)$  since the control action on the leader is zero.

Since  $\sum_{j=1}^N \ell_{ij} = 0$  for all  $i = 1, \dots, N$ , we can add the term  $\kappa g(z + \tilde{x}_i, t) \left( \sum_{j=1}^N \ell_{ij} \right) y_1$  and get

$$\dot{\tilde{x}}_i = f(z + \tilde{x}_i, t) - f(z, t) - \kappa g(z + \tilde{x}_i, t) \sum_{j=1}^N \ell_{ij} [y_j - y_1] .$$

Let  $t_0$  be in  $\mathbb{R}$  and consider a solution  $\mathcal{Z}(z^\circ, t, t_0)$  and  $\tilde{\mathcal{X}}(t, t_0) = (\tilde{\mathcal{X}}_2(t, t_0), \dots, \tilde{\mathcal{X}}_N(t, t_0))$  of the closed-loop system which is defined for all  $t$  in the time domain of defini-

tion  $\mathcal{T}_1 \subset \mathbb{R}$ . Consider the function  $\Gamma : [0, 1] \times \mathcal{T}_2 \times \mathbb{R} \mapsto \mathbb{R}^{Nn_x}$  which satisfies

$$\Gamma(s, t_0, t_0) = s\tilde{\mathcal{X}}(t_0, t_0)$$

and where its  $i$ -th component  $\Gamma_i$  is the solution of the following ordinary differential equation

$$\begin{aligned} \frac{\partial \Gamma_i}{\partial t}(s, t, t_0) &= f(\zeta_i, t) - f(\mathcal{Z}(z^\circ, t, t_0), t) \\ &\quad - \kappa g(\zeta_i, t) \sum_{j=1}^N \ell_{ij} (\alpha(\zeta_j, t) - \alpha(\mathcal{Z}(z^\circ, t, t_0), t)) \end{aligned}$$

where we indicated with  $\zeta_i = \mathcal{Z}(z^\circ, t, t_0) + \Gamma_i(s, t, t_0)$  for shortness of notation and where  $\mathcal{Z}(z^\circ, t, t_0)$  is the solution of the virtual leader  $z$  initialized at  $(z^\circ, t_0) \in \mathbb{R}^{n_x} \times \mathbb{R}$  and evaluated at time  $t \geq t_0$ . Finally  $\mathcal{T}_2 \subset \mathcal{T}_1$  is the time domain of definition of  $\Gamma$ . We will show in the following that  $\mathcal{T}_2 = \mathcal{T}_1$ . Consider the function  $V_i$  for  $i = 2, \dots, N$  defined by (3.32). By using the Killing vector assumption (3.40) and the integrability one (3.39), the time derivative of  $V_i$  becomes

$$\begin{aligned} \dot{V}_i(t) &= \int_0^1 \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) \left[ L_f P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) \right. \\ &\quad \left. - 2\kappa \sum_{j=1}^N \ell_{ij} \frac{\partial h^\top}{\partial x}(\zeta_i, t) \frac{\partial h}{\partial x}(\zeta_j, t) \frac{\partial \Gamma_j}{\partial s}(s, t, t_0) \right] ds. \end{aligned}$$

Let

$$D_f P(\zeta, t) := \text{blkdiag}\{L_f P(\zeta_2, t), \dots, L_f P(\zeta_N, t)\}$$

and

$$H(\zeta, t) := \text{col} \left\{ \frac{\partial h^\top}{\partial x}(\zeta_2, t), \dots, \frac{\partial h^\top}{\partial x}(\zeta_N, t) \right\}.$$

By Lemma 3.2.3, we have that

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) &= \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_f P(\zeta, t) \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \right. \\ &\quad \left. - 2\kappa \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) H(\zeta, t) \mathcal{L}_{22} H^\top(\zeta, t) \Psi(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right] ds \\ &\leq \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_f P(\zeta, t) \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \right. \\ &\quad \left. - 2\kappa \mu \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) H(\zeta, t) H^\top(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right] ds. \end{aligned}$$

Therefore by selecting  $\kappa \geq \frac{\rho}{2\mu}$  with  $\rho$  satisfying the OCMF condition in (3.38) and  $\mu > 0$  given by Lemma 3.2.3 we have that (3.35) holds. From Gronwall's Lemma we have that (3.36) is satisfied. From this inequality, we first deduce that  $\mathcal{T}_2 = \mathcal{T}_1$  since the path  $\Gamma$  has finite (Riemanniann) energy and has boundary defined in  $\mathcal{T}_1$ . The proof then concludes as in Theorem 3.4.1.  $\square$

**Remark 3.4.2.** Note that the control law (3.41) is linear in the output  $y$ . A similar result can be claimed by considering a controller  $\xi(y)$  which is nonlinear in  $y$  and such that  $\xi(0) = 0$ . In such a case, the relations (3.38) and (3.39) become

$$\begin{aligned} L_f P(x, t) - \rho \left( \frac{\partial \xi}{\partial x}(h(x, t)) \right)^\top \left( \frac{\partial \xi}{\partial x}(h(x, t)) \right) &\preceq -\varepsilon P(x, t) \\ P(x, t)g(x, t) &= \left( \frac{\partial \xi}{\partial x}(h(x, t)) \right)^\top \end{aligned}$$

and synchronization is achieved with the output-feedback control law

$$u_i = -\kappa \sum_{j \in \mathcal{N}_i} \ell_{ij} \xi(y_j).$$

### 3.4.2.3 Full-input-feedback leader synchronization

The output-feedback control design for the Riemannian case in Section 3.4.2.2 makes use of incremental passive tools. This requires the dimension of the output to be the same as the dimension of the input. In this section, we aim to show that such a limitation can be avoided in some cases. We suppose in this case that each agent is described by (3.25) where  $g(x, t) = I$  and that it is coupled with a scalar nonlinear output. In other words, we consider a system of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, t), & y_1 &= h(x_1, t), \\ \dot{x}_i &= f(x_i, t) + u_i, & y_i &= h(x_i, t), \end{aligned} \quad (3.42)$$

for all  $i = 2, \dots, N$ , where  $x \in \mathbb{R}^{n_x}$ ,  $u \in \mathbb{R}^{n_x}$  and  $y \in \mathbb{R}$ . For such a class of systems, the following result holds.

**Proposition 3.4.3** (Full-input output feedback leader synchronization). *Consider a network of systems of the form (3.42) and let Assumption 3.4.1 hold. Suppose that there exists a  $C^1$  function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values such that the following conditions hold.*

- The Output Control Matrix Function (OCMF) condition in (3.38) holds;
- The vector field  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \rightarrow \mathbb{R}^{n_x}$  defined as

$$\alpha(x, t) = P^{-1}(x, t) \frac{\partial h^\top}{\partial x}(x, t) \quad (3.43)$$

is a Killing vector for  $P$ , i.e.

$$L_\alpha P(x, t) = 0 \quad (3.44)$$

for all  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ .

Then, for any  $\kappa \geq \frac{\rho}{2\mu}$ , with  $\rho$  satisfying (3.38) and  $\mu$  given by Lemma 3.2.2, the distributed control law (3.26) with

$$\varphi_j(x_j, t) = -\kappa \ell_{ij} \alpha(x_i, t) y_j \quad (3.45)$$

solves the global exponential leader synchronization problem for system (3.42).

**Proof.** First of all note that, in virtue of Lemma 3.2.3, the control action on the leader is identically zero since  $\ell_{1j} = 0$  for all  $j$ . Similarly to the proof of Theorem 3.3.1 then, the idea behind the proof is to show that the dynamics of the error between any agent and the leader exponentially decrease to zero. Consider, without loss of generality, that  $x_1$  is the leader node in the network. Define  $N - 1$  error coordinates  $\tilde{x} := (\tilde{x}_2, \dots, \tilde{x}_N)$  with  $\tilde{x}_i := x_i - x_1$  and  $z := x_1$ .

The dynamics of these errors reads

$$\dot{\tilde{x}}_i = f(\tilde{x}_i + z, t) - f(z, t) - \kappa \sum_{j=1}^N \ell_{ij} \alpha(\tilde{x}_i + z, t) [h(\tilde{x}_j + z, t) - h(z, t)]$$

for all  $i = 2, \dots, N$ . Let  $t_0$  be in  $\mathbb{R}$  and consider a solution  $\mathcal{Z}(z^\circ, t, t_0)$  and  $\tilde{\mathcal{X}}(t, t_0) = (\tilde{\mathcal{X}}_2(t, t_0), \dots, \tilde{\mathcal{X}}_N(t, t_0))$  of the closed-loop system which is defined for all  $t$  in the time domain of definition  $\mathcal{T}_1 \subset \mathbb{R}$ . Consider the function  $\Gamma : [0, 1] \times \mathcal{T}_2 \times \mathbb{R} \mapsto \mathbb{R}^{Nn_x}$  which satisfies

$$\Gamma(s, t_0, t_0) = s\tilde{\mathcal{X}}(t_0, t_0)$$

and where its  $i$ -th component  $\Gamma_i$  is the solution of the following ordinary differential equation

$$\begin{aligned} \frac{\partial \Gamma_i}{\partial t}(s, t, t_0) &= f(\zeta_i, t) - f(\mathcal{Z}(z^\circ, t, t_0), t) \\ &\quad - \kappa g(\zeta_i, t) \sum_{j=1}^N \ell_{ij} (\alpha(\zeta_j, t) - \alpha(\mathcal{Z}(z^\circ, t, t_0), t)) \end{aligned}$$

where we indicated with  $\zeta_i = \mathcal{Z}(z^\circ, t, t_0) + \Gamma_i(s, t, t_0)$  for shortness of notation and where  $\mathcal{Z}(z^\circ, t, t_0)$  is the solution of the virtual leader  $z$  initialized at  $(z^\circ, t_0) \in \mathbb{R}^{n_x} \times \mathbb{R}$  and evaluated at time  $t \geq t_0$ . Finally  $\mathcal{T}_2 \subset \mathcal{T}_1$  is the time domain of definition of  $\Gamma$ . We will show in the following that  $\mathcal{T}_2 = \mathcal{T}_1$ . Consider the function  $V_i$ ,  $i = 2, \dots, N$ , defined by (3.32). Employing the Killing vector field assumption (3.44) and the definition of  $\alpha$  in (3.43), we compute the time-derivative of  $V_i$  as

$$\begin{aligned} \dot{V}_i(t) &= \int_0^1 \frac{\partial \Gamma_i^\top}{\partial s}(s, t) \left[ L_f P(Z(t) + \Gamma_i(s, t)) \right. \\ &\quad \left. - 2\kappa \sum_{j=1}^N \ell_{ij} \frac{\partial h^\top}{\partial z}(Z(t)) \frac{\partial h}{\partial z}(Z(t)) \right] \frac{\partial \Gamma_i^\top}{\partial s}(s, t) ds \end{aligned}$$

Hence

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) &= \int_0^1 \sum_{i=2}^N \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) \left[ L_f P(Z(t, t_0) + \Gamma_i(s, t, t_0), t) \right. \\ &\quad \left. - 2\kappa \sum_{j=1}^N \ell_{ij} \frac{\partial h^\top}{\partial z}(Z(t, t_0), t) \frac{\partial h}{\partial z}(Z(t, t_0), t) \right] \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) ds \\ &= \int_0^1 \sum_{i=2}^N \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) L_f P(Z(t, t_0) + \Gamma_i(s, t, t_0), t) \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) \\ &\quad - 2\kappa \mu v^\top(s, t, t_0) v(s, t, t_0) ds \end{aligned}$$

where we defined

$$v(s, t, t_0) = \begin{bmatrix} \frac{\partial h}{\partial z}(Z(t, t_0), t) \frac{\partial \Gamma_2}{\partial s}(s, t, t_0) \\ \vdots \\ \frac{\partial h}{\partial z}(Z(t, t_0), t) \frac{\partial \Gamma_N}{\partial s}(s, t, t_0) \end{bmatrix}.$$

Therefore by selecting  $\kappa \geq \frac{\rho}{2\mu}$  with  $\rho$  satisfying the OCMF condition in (3.38) and  $\mu > 0$  given by Lemma 3.2.3 we have that (3.35) holds. From Gronwall's Lemma we have that (3.36) is satisfied. The proof then concludes as in Theorem 3.4.1.  $\square$

**Remark 3.4.4.** *Concerning the Euclidean case in Section 3.3, the following critical aspects should be highlighted.*

- *In this section we considered a network with the presence of a leader. The Riemannian extension to network synchronization (without a leader) is non-trivial. The main challenge is the dependence of the control vector field  $g$  on the state  $x_i$ . Indeed, without an agent that acts as a leader, we are left with an additional term in (3.31) which depends on  $g(z, t)$ . As a consequence, the sum of  $V_i(t)$  is not (in general) a good candidate to be a Lyapunov function. This problem does not arise when the matrix function multiplying the control action  $u$  is state independent, such as in Section 3.3. In this case, to satisfy the Killing vector condition the metric is Euclidean, that is,  $P$  is a constant positive definite matrix.*
- *In this section we considered an undirected network while in Section 3.3 we allowed the communication links between agents to be unidirectional. The main limitation in the Riemannian case is, again, the choice of the Lyapunov function. Indeed in the Euclidean case, the metric is equivalent for each agent since it is a constant matrix  $P$ . In the Riemannian framework, this is not the case. As a consequence, the Kronecker product  $Q \otimes P$  in the Riemannian framework cannot be defined since  $P$  depends on  $x$ . In such a case the Lyapunov function for the entire network takes the form of a block-diagonal structure of all the  $P(x_i, t)$  (or, equivalently, the sum of the Lyapunov functions for each agent). This requires the matrix  $\mathcal{L}_{22}$  to be symmetric, with the consequence that (besides the leader) the graph must be undirected.*
- *We conclude this section by noting that, differently from the Euclidean case developed in Section 3.3, we don't have a complete extension to the Riemannian case of the general ARE condition (3.12): we only extended the three cases (3.13), (3.14), (3.15). The main reason is that, again, in a Riemannian framework, the dependency of the state in the vector field  $g(x, t)$  does not allow to easily derive an equivalent condition to  $PBD = C^T E^T$  as in the nonlinear framework.*

## 3.5 Examples

### 3.5.1 Minimum-phase systems

In this section, we specialize the previous conditions to specific classes of nonlinear systems. In particular, we will consider minimum-phase systems and nonlinear systems that are described in Lur'e form with an incremental monotonic nonlinearity.

A significant amount of results in synchronization considers systems either in normal form or for which there exists a globally defined diffeomorphism that allows rewriting the dynamics in normal form, see, e.g. Isidori et al. (2014); Panteley and Loría (2017); Chopra and Spong (2008). For such a class of systems, the zero-dynamics is generally assumed to possess a unique steady-state trajectory which is attractive with a given domain of attraction (minimum-phase). We show below that the conditions provided in Section 3.4 recover these results. In particular, consider a network where each agent is modeled as a SISO system whose dynamics can be described by

$$\begin{aligned}\dot{z}_i &= F(z_i, y_i) \\ \dot{y}_i &= q(z_i, y_i) + u_i\end{aligned}\tag{3.46}$$

where  $x_i = (z_i, y_i) \in \mathbb{R}^{n_x}$  is the state, with  $i = 1, \dots, N$ . In particular, we consider systems with unitary relative degree, that is,  $z_i \in \mathbb{R}^{n_x-1}$  and  $y \in \mathbb{R}$ . The functions  $F, q$  are  $C^2$  in their arguments. The high-frequency gain (i.e. the coefficient in front of  $u$ ) is selected as 1 without loss of generality. With respect to the representation (3.37), we have

$$f(x, t) = \begin{pmatrix} F(z, y) \\ q(z, y) \end{pmatrix}, \quad g(x, t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h(x, t) = (0 \quad 1).$$

We cast the synchronization problem to the incremental framework with the following (incremental) minimum-phase assumption.

**Assumption 3.5.1** (Incremental minimum-phase). *There exists a  $C^1$  symmetric and positive definite matrix function  $S : \mathbb{R}^{n_x-1} \mapsto \mathbb{R}^{(n_x-1) \times (n_x-1)}$  and positive real numbers  $\underline{s}, \bar{s}, \epsilon, \chi > 0$  such that the following inequalities<sup>1</sup> hold*

$$\underline{s}I \preceq S(z) \preceq \bar{s}I, \quad L_F S(z, y) \preceq -\epsilon S(z), \tag{3.47}$$

$$\left| \frac{\partial F}{\partial y}(z, y) \right| \leq \chi, \quad \left| \frac{\partial q}{\partial z}(z, y) \right| \preceq \chi, \quad \left| \frac{\partial q}{\partial y}(z, y) \right| \leq \chi, \tag{3.48}$$

for all  $(z, y) \in \mathbb{R}^{n_x}$ .

Under the previous assumption, the problem of synchronization is solved, as established in the following result. It recovers the results of Chopra and Spong (2008).

**Proposition 3.5.1** (Synchronization of minimum phase systems). *Consider system (3.46) and let Assumptions 3.4.1 and 3.5.1 hold. Then, there exists  $\kappa^* > 0$  such that, for any  $\kappa \geq \kappa^*$ , the distributed state-feedback control law (3.7) with  $\varphi(x, y, t) = \kappa y$  solves the synchronization Problem 3.2.4 for (3.46). Furthermore, if the metric  $S$  is Euclidean (i.e. constant), then Assumption 3.4.1 can be relaxed with Assumption 3.3.1.*

**Proof.** The proof follows by noticing that condition (3.39) and (3.40) are satisfied with the metric  $P(x) = \text{blkdiag}(S(z), 1)$ . Then, it is completed with standard high-gain arguments in order to find a sufficiently large  $\rho$  satisfying the condition (3.38). As a consequence, the value of  $\kappa^*$  depends on the parameter  $\mu$  of Lemma 3.2.2 and the Lipschitz constant  $\chi$  of Assumption 3.5.1. As the proof follows the same steps as in the proof Proposition 2.2.15, the details are omitted

<sup>1</sup>The notation  $L_F S(z, y)$  has to be understood as the Lie derivative of  $S$  along the vector field  $z \mapsto F(z, y)$  where  $y$  is fixed.

for space reasons.  $\square$

### 3.5.2 Monotonic systems

In this Section, we specify our design for systems of the form

$$\dot{x}_i = Ax_i + G\theta(v_i, t) + Bu_i + \omega(t), \quad v_i = Hx_i, \quad (3.49a)$$

$$y_i = Cx_i \quad (3.49b)$$

where  $A, G, B, H, C$  are constant matrices of suitable dimension,  $v_i \in \mathbb{R}^{n_v}$  is a linear combination of the state,  $\omega : \mathbb{R} \mapsto \mathcal{W}$  is a time-varying piecewise continuous signal taking values in a compact set  $\mathcal{W} \subset \mathbb{R}^{n_x}$  and  $\theta : \mathbb{R}^{n_v} \times \mathbb{R} \mapsto \mathbb{R}^{n_\theta}$  is a  $C^1$  nonlinearity satisfying the following *monotonic* condition

**Assumption 3.5.2** (Monotonic). *There exists a symmetric positive matrix  $Q = Q^\top \succ 0$  such that*

$$0 \preceq \frac{\partial \theta}{\partial v}(v, t) + \frac{\partial \theta^\top}{\partial v}(v, t) \preceq Q \quad (3.50)$$

for all  $(v, t) \in \mathbb{R}^{n_v} \times \mathbb{R}$ .

In the following, we want to show that it is possible to rewrite the design proposed in Theorem 3.4.1 (and Theorem 3.3.1) in form of a Linear Matrix Inequality (LMI).

**Proposition 3.5.2.** *Consider system (3.49) and let Assumption 3.5.2 hold. Assume that there exists a symmetric positive definite matrix  $W = W^\top \succ 0$ , and two strictly positive real numbers  $q, \rho > 0$  such that the LMI*

$$\begin{pmatrix} AW + W^\top A^\top - \rho BB^\top & G + W^\top H^\top & W \\ G^\top + HW & -4Q^{-1} & 0 \\ W & 0 & -qI \end{pmatrix} \preceq 0 \quad (3.51)$$

holds. Then the assumptions of Theorem 3.4.1 hold with  $P(x) = W^{-1}$ ,  $\varepsilon = \frac{1}{q\rho}$ ,  $\alpha(x_j, t) = PBx_j$  and such a  $\rho$ . Moreover, if  $B = WC^\top$ , then the Assumptions of Theorem 3.3.1 are satisfied with  $D = I$  and  $E = I$  and synchronization is achieved with  $\varphi(x, t) = \kappa y$ .

**Proof.** The proof follows the same lines as the ones in Proposition 1.4.25 and hence it's omitted for space reasons.  $\square$

Since the chosen metric is Euclidean, the following trivially holds.

**Corollary 3.5.2.1** (Synchronization of monotonic systems). *Consider a network where each agent is described by (3.50) and let Assumption 3.5.2 hold. Assume that the LMI (3.51) is satisfied and consider the distributed control law*

$$u_i = -\kappa \sum_{j=1}^N \ell_{ij} B^\top P x_j.$$

Then:

1. if Assumption 3.3.1 holds, the global exponential synchronization problem is solved;

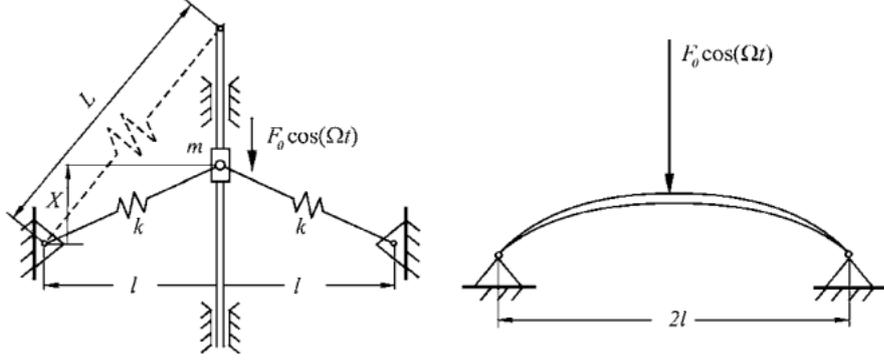


Figure 3.1: Mechanical structure of the elastic bar

2. if Assumption 3.4.1 holds, the global exponential leader-synchronization problem is solved.

**Remark 3.5.3.** Note the similarity between the LMI in Proposition 3.5.2 and the ones proposed in Zhang et al. (2014).

**Example 3.5.4** (Synchronization of archetypal oscillators of an elastic arc). In this section, we apply the proposed design for the synchronization of a network of forced archetypal oscillators described in Cao et al. (2006). The model of each agent is inspired by the model of a mass-spring elastic arc. It is composed of a mass  $m$  linked by a pair of inclined elastic springs which are capable of resisting both tension and compression. Each spring of stiffness  $k$  is pinned to a rigid support. A picture describing each agent taken from (Cao et al., 2006, Figure 1) can be found in Figure 3.1. The space-state model describing each agent is given by

$$\begin{aligned} \dot{x}_{i,1} &= x_{i,2} \\ \dot{x}_{i,2} &= -\frac{1}{m} \left[ 2kx_{i,1} \left( 1 - \frac{L}{\sqrt{(x_{i,1})^2 + l}} \right) + F_0 \cos(\Omega t) + u_i \right] \end{aligned}$$

where  $(x_{i,1}, x_{i,2}) \in \mathbb{R}^2$  is the state of agent  $i$ . Here,  $L$  is the equilibrium length,  $x_i^1$  is the mass displacement (in Figure 3.1 indicated with the symbol  $X$ ) and  $l$  is half the distance between the rigid supports. Each agent is excited by the same external co-sinusoidal signal  $F_0 \cos(\Omega t)$  and by the control action  $u_i$ . Without the control action, each agent represents a nonlinear oscillator where the dynamics of the two states have a different amplitude and phase (which depends on the plant parameters). The system can be written in form (3.49) where

$$A = \begin{pmatrix} 0 & 1 \\ -\frac{2k}{m} & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad H = (1 \ 0), \quad \theta(\nu, t) = \frac{2kL\nu}{m\sqrt{\nu^2 + l}}.$$

The nonlinearity is monotonic according to Assumption 3.5.2 with

$$Q = \frac{2kL}{m\sqrt{l}},$$

and therefore we aim to apply Proposition 3.5.2. We picked as parameters  $m = 1, k = 0.5, L = 0.5, l = 1$ . The system is excited by a high-frequency external signal

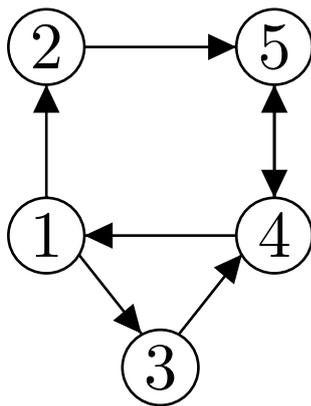


Figure 3.2: Considered network of archetypal oscillators of elastic arcs

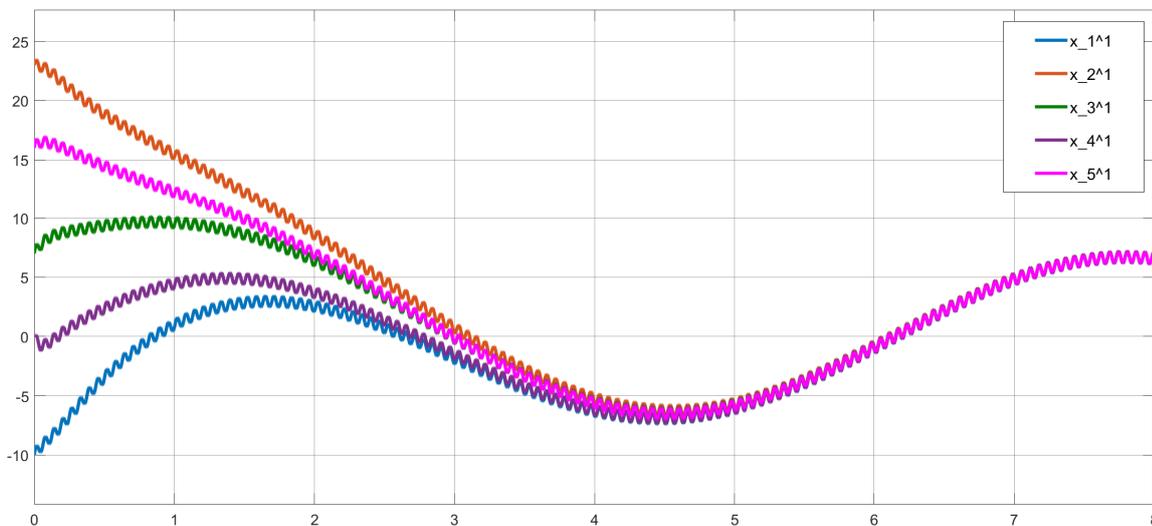


Figure 3.3: First state of each agent

$F_0 \cos(\Omega t)$  with  $F_0 = 50$  and  $\Omega = 100$ . With the aid of a MATLAB LMI solver, we compute the LMI (3.51) which is feasible with

$$\rho = 3 \quad \text{and} \quad W = \begin{pmatrix} 0.3441 & -0.2558 \\ -0.2558 & 0.6499 \end{pmatrix}.$$

Note that the solution of the LMI does not depend on the external force  $F_0 \cos(\Omega t)$ . We consider a network of 5 agents connected by a directed graph as in Figure 3.2. The initial conditions of the network are taken as

$$\mathbf{x}^\circ = (-10 \quad 21 \quad 23 \quad -0.5 \quad 7 \quad 19 \quad 0 \quad -34 \quad 16 \quad 22)^\top.$$

Figure 3.3 and Figure 3.4 show respectively the first and second state of each agent of the network.

## 3.6 Deep Learning for metric estimation

### 3.6.1 Killing-less leader synchronization

The main limitation of the approach presented in Section 3.4 is the complexity of finding a metric  $P$  solving (3.27) and, at the same time, satisfying the Killing

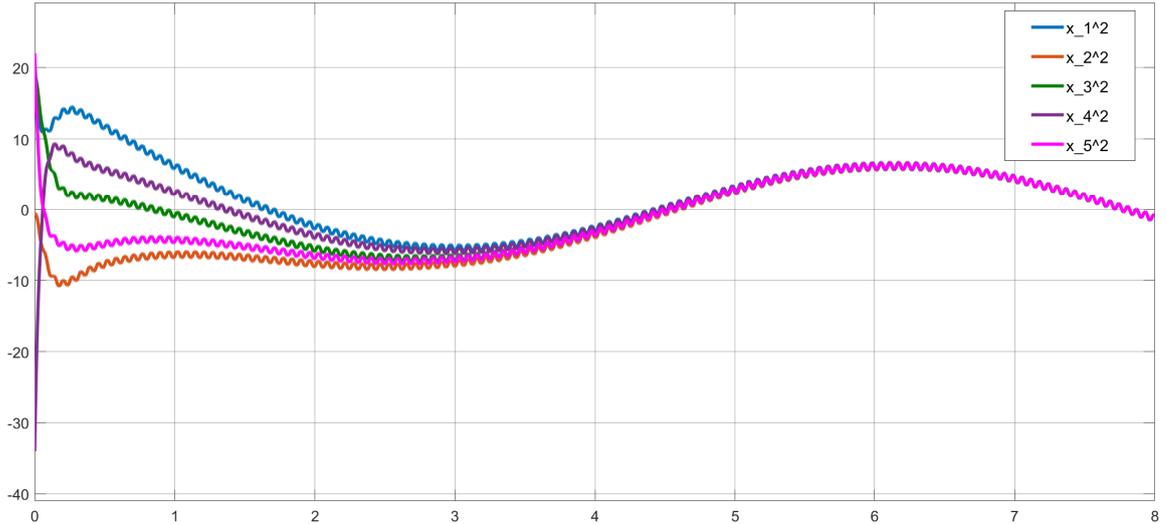


Figure 3.4: Second state of each agent

vector field property in (3.28). For several classes of systems (see Section 3.5) these conditions can be easily verified. However, this is not an easy task in general, especially when considering high-dimensional systems with significant nonlinearities. In this section, we aim to provide a practical solution to such limitations. First, we relax the Killing vector assumption. Hence, we show that synchronization can still be achieved (in compact sets) when the  $L_g P(x, t)$ -term is non-zero, yet sufficiently small. Second, we circumvent the need of computing a suitable metric by relying on Machine Learning tools. We provide a general optimization-based algorithm allowing the approximation of both the metric and the integrability condition in (3.29) via Deep Neural Networks (DNNs). The proposed design is then validated on a synchronization problem for a network of Lorentz oscillators. We present the results for the state-feedback control design. Nevertheless, the same tools can be used to derive conditions for the output-feedback design. For the sake of notation, we consider the single-input case. Yet, similar results can be applied to the multi-input one. We now present the theoretical result relaxing the Killing vector field property in (3.28).

**Proposition 3.6.1** (Killing-less synchronization). *Consider system (3.25). Let Assumption 3.4.1 hold and assume that there exists a  $C^1$  matrix function  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  taking symmetric positive definite values, some strictly positive constants  $\underline{p}, \bar{p}, \varepsilon, \rho$  and a  $C^2$  function  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_u}$  such that (3.27), (3.29) hold. Let  $X \subset \mathbb{R}^{n_x}$  be a compact set and fix  $\kappa \geq \frac{\rho}{2\mu}$ , for some  $\mu > 0$  given by Lemma 3.2.3. Then, there exists a strictly positive real number  $\epsilon_X > 0$  such that, if  $\|L_g P(x, t)\| \leq \epsilon_X$  for all  $(x, t) \in X \times \mathbb{R}$ , the trajectory of network in closed-loop with the state-feedback distributed control law (3.26) with (3.30) exponentially converges towards the synchronization manifold (3.8) for all time existence of solutions in  $X$ , i.e. for all  $t_f \geq t_0$  such that  $\mathcal{X}_i(x_i^\circ, t, t_0) \in X$  for all  $t \in [t_0, t_f)$  and all  $i = 1, \dots, N$ .*

**Proof.** The proof is identical to the proof of Theorem 3.4.1 up to equation (3.32). Therefore, the first part is omitted. Then, by using the integrability

assumption on the function  $\alpha$  in (3.29), the time derivative of  $V_i$  becomes

$$\begin{aligned} \dot{V}_i(t) = & \int_0^1 \left[ \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) L_f P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) \right. \\ & - 2\kappa \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) \sum_{j=1}^N \ell_{ij} P(\zeta_i, t) g(\zeta_i, t) g^\top(\zeta_j, t) P(\zeta_j, t) \frac{\partial \Gamma_j}{\partial s}(s, t, t_0) \\ & \left. - 2\kappa \frac{\partial \Gamma_i^\top}{\partial s}(s, t, t_0) L_g P(\zeta_i, t) \frac{\partial \Gamma_i}{\partial s}(s, t, t_0) \sum_{j=1}^N \ell_{ij} \alpha(\zeta_j, t) \right] ds. \end{aligned}$$

Consequently, by selecting  $\kappa \geq \frac{\rho}{2\mu}$  with  $\rho$  satisfying the SCMF condition in (3.27) and by following similar computations as in the proof of Theorem 3.4.1, we get

$$\begin{aligned} \sum_{i=2}^N \dot{V}_i(t) \leq & -\varepsilon \int_0^1 \left[ \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) P(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) \right. \\ & \left. - 2\kappa \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_g P(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) L_{22} \Lambda(\zeta, t) \right] ds \quad (3.52) \end{aligned}$$

where we indicated with

$$\begin{aligned} D_g P(\zeta, t) &:= \text{blkdiag}\{L_g P(\zeta_2, t), \dots, L_g P(\zeta_N, t)\} \\ \Lambda(\zeta, t) &:= [\alpha^\top(\zeta_2, t), \dots, \alpha^\top(\zeta_N, t)]^\top. \end{aligned}$$

Now, for any  $X \subset \mathbb{R}^{n_x}$ , let

$$\epsilon_X := \frac{\varepsilon p}{4\kappa N^{\frac{3}{2}} \bar{\mu} \bar{\alpha}}$$

with  $\bar{\mu}$  being the largest eigenvalue of  $\mathcal{L}_{22}$  and

$$\bar{\alpha} := \sup_{x \in X, t \geq t_0} \{\alpha(x, t)\}.$$

Then if  $\|L_g P(x, t)\| \leq \epsilon_X$ , it follows that

$$\begin{aligned} 2\kappa \int_0^1 \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) D_g P(\zeta, t) \frac{\partial \Gamma}{\partial s}(s, t, t_0) L_{22} \Lambda(\zeta, t) ds \\ \leq \frac{\varepsilon p}{2} \int_0^1 \frac{\partial \Gamma^\top}{\partial s}(s, t, t_0) \frac{\partial \Gamma}{\partial s}(s, t, t_0) ds \end{aligned}$$

which by (3.52) it implies

$$\dot{V}(t) = \sum_{i=2}^N \dot{V}_i(t) \leq -\frac{\varepsilon}{2} \sum_{i=2}^N V_i(t) = -\frac{\varepsilon}{2} V(t).$$

The proof concludes by following the same lines of Theorem 3.4.1.  $\square$

**Remark 3.6.2.** Note that a bound on  $\epsilon_X$  can be given also by linearizing the er-

ror dynamics around the equilibrium  $e = 0$ . This recovers known results on local exponential synchronization, see Andrieu et al. (2018).

### 3.6.2 A DNN-based algorithm for metric estimation

As mentioned in the previous section, a drawback of the proposed approach lies in the fact that metrics may not be easy to find in the Riemannian scenario. Moreover, even when a metric has been found, it may not be straightforward to design a control law satisfying the integrability condition (3.29). A way of overcoming such difficulties is to combine the proposed control design with Machine Learning tools. In the last years, DNNs turned out to be effective tools for solving complex differential equations, see e.g., Raissi et al. (2019); Cai et al. (2022). As a matter of fact, recent works began mixing learning tools and control. Such a combined framework tackles the complexity of computing control theoretic exact solutions by exploiting deep approximators, e.g., Zoboli et al. (2021); Janny et al. (2021); Sanchez-Escalonilla et al. (2021). Hence, the idea is to set up an optimization problem for approximating the solution. In this section, we circumvent the need of computing a suitable metric by approximating it via a DNN. Once a suitable metric has been found, we train a second DNN to satisfy the integrability condition.

The idea of relying on Machine Learning tools to estimate a Riemannian metric is not new in literature and similar approaches have been used for instance in Tsukamoto et al. (2021); Wei et al. (2021) (see also references therein). In Tsukamoto et al. (2021), the authors propose a convex optimization problem to compute a suitable metric. Yet, they successively suggest approximating the solution via a DNN. Hence, the convex optimization is solved on a finite number of samples and the DNN provides a continuous interpolation through those points. This overcomes the need of solving such an optimization in each point of the state space. Our approach, however, is inspired by Wei et al. (2021). The authors propose a Siamese DNN structure Sheng and Fazekas (2019). The goal is to minimize a loss function defined by the matrix conditions required for contraction. Once such a function reaches 0, the DNN provides the entries of a suitable metric for each point in the training/test datasets. Even if similar, our solution differs in three main points.

1. First, we rely on the continuous time framework. Hence, we avoid the need of a Siamese network and compute the DNN's Jacobian. Note that, usually, it can be easily obtained thanks to the automatic differentiation tools provided by common libraries such as PyTorch.
2. Second, we add a separate estimator which looks for the best parameters in the cost function. It works jointly with the DNN during the optimization process.
3. Finally, we rely on (3.29), instead of computing the control law via approximate integration over the geodesic. This greatly simplifies the algorithm, since geodesics are not easy to find in general.

We now describe the proposed algorithm. Let us consider the problem of finding a suitable approximation of the metric first. The neural metric is constructed as

$$P(x_i, \vartheta') = \begin{pmatrix} p_1(x_i, \vartheta') & p_2(x_i, \vartheta') & \cdots & p_n(x_i, \vartheta') \\ p_2(x_i, \vartheta') & p_{n+1}(x_i, \vartheta') & \cdots & p_{2n}(x_i, \vartheta') \\ \vdots & \vdots & \ddots & \vdots \\ p_n(x_i, \vartheta') & p_{2n}(x_i, \vartheta') & \cdots & p_M(x_i, \vartheta') \end{pmatrix},$$

where  $M = \frac{n(n+1)}{2}$  is the total number of entries to be learned, the vector

$$p = (p_0(x_i, \vartheta'), \dots, p_M(x_i, \vartheta'))^\top$$

is the output of the neural network  $\text{DNN}_P : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\vartheta'}} \mapsto \mathbb{R}^M$  and  $\vartheta' \in \mathbb{R}^{n_{\vartheta'}}$  is the vector of  $\text{DNN}_P$  parameters. To train the  $\text{DNN}_P$  parameters, we relax the Killing-vector assumption and rely on Proposition 3.6.1. We set up an optimization problem asking for the minimization of the following cost function

$$J_P(x, \vartheta') = \sum_{i=1}^4 w_i J_i(x, \vartheta'), \quad (3.53)$$

being  $w = (w_1, \dots, w_4)$  a vector of scalar weights and

$$\begin{aligned} M_1 &= L_f P(x, \vartheta') - \rho P(x, \vartheta') g(x) g^\top(x) P(x, \vartheta') + \varepsilon I \\ M_2 &= L_g P(x, \vartheta') - \epsilon I \\ M_3 &= -L_g P(x, \vartheta') - \epsilon I \\ M_4 &= -P(x, \vartheta') + \underline{p} I \\ J_i(x, \vartheta') &= \ln \left( \max \left( \Re \left\{ \lambda_M(M_i) \right\}, 0 \right) + 1 \right), \quad i = 1, \dots, 4 \end{aligned}$$

where  $\rho, \epsilon, \underline{p}$  are positive scalars,  $\varepsilon > \epsilon$ . Note that each cost  $J_i$  serves the purpose of satisfying a particular condition for the neural metric. While  $J_1$  provides a positive cost if the contraction condition is not satisfied,  $J_2$  and  $J_3$  encourage the boundedness of  $L_g P$  and  $J_4$  steers the solution towards positive definite matrices. The natural logarithm is used as a regularization term between costs  $J_i$ . It allows the rescaling of widely different costs to similar values and a more precise selection of their importance through the weight vector  $w$ . In parallel to the  $\text{DNN}_P$ , we train a parameter estimator outputting the values of  $\rho, \varepsilon, \epsilon, \underline{p}$ . The estimator and  $\text{DNN}_P$  work together, trying to minimize (3.53). Note that if the cost reaches 0, all the contraction conditions are satisfied for the dataset and the learned estimator outputs, hence learning can be stopped.

The second step is to find a suitable law satisfying the integrability condition (3.29). We train the parameters  $\vartheta'' \in \mathbb{R}^{n_{\vartheta''}}$  of the second network  $\text{DNN}_\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\vartheta''}} \mapsto \mathbb{R}^{n_u}$  such that

$$J_\alpha(x, \vartheta'') = \left\| \frac{\partial \text{DNN}_\alpha}{\partial x}(x, \vartheta'') - g(x)^\top P(x, \vartheta') \right\|^2 \quad (3.54)$$

is minimized.

Finally, the controller is synthesized as in (3.30) with the approximation  $\alpha(x) \approx \text{DNN}_\alpha(x, \vartheta'')$ . We rely on the robustness properties of contractive systems (see Sontag (2010)) to compensate for the DNNs' approximation errors. The full learning procedure is summarized by Algorithm 1.

**Remark 3.6.3.** *Note that Algorithm 1 refers to time-invariant agents. However, things can be generalized in case the dynamics of each node are modeled by a time-varying equation.*

**Algorithm 1** DNN-based controller learning

- 
- 1: Input: Dataset of  $(x, f(x), g(x), \frac{\partial f}{\partial x}(x), \frac{\partial g}{\partial x}(x))$ ,  
DNN<sub>P</sub>, DNN<sub>α</sub>;
  - 2: **while**  $J_P(x, \vartheta') \neq 0$  **do**
  - 3:   Train DNN<sub>P</sub> and the estimator with (3.53);
  - 4: **end while**
  - 5: Train the DNN<sub>α</sub> with (3.54);
  - 6: Set the distributed law
- 

$$u_i(x_1, \dots, x_N) = -\kappa \sum_{j \in \mathcal{N}_i} \ell_{ij} \text{DNN}_\alpha(x_j, \vartheta'');$$


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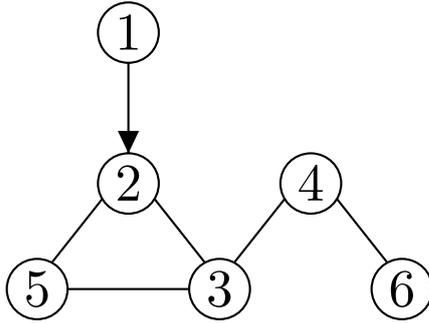


Figure 3.5: Considered network of Lorentz’s oscillators

**3.6.3 Leader-synchronization of a network of Lorentz oscillators with DNNs**

In the following, we apply the proposed algorithm to a leader-synchronization problem. The code for reproducing the experiments proposed in this section can be found at <https://github.com/SamueleZoboli/Control-learning-multiagent-lorenz.git>. We consider a network of  $N = 6$  identical Lorenz attractors. Such systems are particularly interesting since they can present a chaotic behavior. Each agent  $i = 1, \dots, N$  is described by the following dynamics

$$\begin{cases} \dot{x}_i^1 = a(x_i^2 - x_i^1) + u_i \\ \dot{x}_i^2 = x_i^1(b - x_i^3) - x_i^2 + (2 + \sin(x_1))u_i \\ \dot{x}_i^3 = x_i^1 x_i^2 - c x_i^3 \end{cases}$$

where  $a, b, c$  are positive scalars. Similarly to (Casadei et al., 2018, Section 5), we pick  $a = 10, b = \frac{8}{3}, c = 28$ , guaranteeing the chaotic behavior. We consider the control matrix  $g(x) = (1 \ 2 + \sin(x_1) \ 0)^\top$  to exclude the possibility of feedback linearizing solutions. The agents communicate with each other following the leader-connected graph represented in Figure 3.5.

We code and train our fully-connected DNNs and estimator using PyTorch Paszke et al. (2019). For the metric network, we select an architecture composed of 4 hidden layers, with sizes 30, 20, 20, 10 respectively and tanh activation functions. The output layer passes through a saturation function as a final activation, limiting the single elements of the metric. The second network presents 3 hidden layers, with size 30, 20, 10 respectively and tanh activation functions. We select the identity function as the output layer activation function.

We select a weight vector  $w = (1, 10, 10, 20)$ , directing the learning towards positive matrices first and successively satisfying the Killing-less assumptions and the contraction condition. We train both the networks and the estimator using Adam optimizer (see Kingma and Ba (2014)). The learning rate for the metric network and the estimator is set as  $3 \times 10^{-3}$ , while  $\text{DNN}_\alpha$  uses a learning rate of  $5 \times 10^{-3}$ . The DNNs' learning rates are scheduled according to a cosine annealing policy (see Loshchilov and Hutter (2016)), while the estimator one remains constant. We train the neural metric and the estimator over 100 epochs (yet stopped after 15 epochs due to the cost reaching 0) and the second DNN over 200 epochs. For both of the learning phases (the metric learning and the integrability learning), the dataset is composed of  $2 \times 10^5$  samples coming from a Gaussian distribution  $\mathcal{N}(0, 10)$ . We use 80% of the dataset as the training set, with a batch size of 512. The remaining 20% is used as a test set.

We select a  $\kappa = 5$  and we apply the controller in a noisy-measurements scenario, i.e.,  $u_i = \varphi(x_i + \omega_i)$  where  $\omega_i \sim \mathcal{N}(0, 0.2)$ . This allows for testing the robustness properties of the proposed neural control law. Each agents' initial condition is randomly sampled from a Gaussian distribution  $\mathcal{N}(0, 20)$ . Figure 3.6 and Figure 3.7 show the controller performances once the DNNs have been trained. Figure 3.6 presents the mean and standard deviation between agents of the norm of the error concerning the leader trajectory. Figure 3.7 directly shows the state trajectories of each agent. As synchronization is achieved, we can see that the DNN optimized with (3.53) provides a suitable approximation of the metric, while the one trained with (3.54) effectively learns a primitive of  $g^\top(x)P(x, \vartheta')$ . The parameter estimator provided a decay rate  $\varepsilon \approx 4.7$  and  $\rho \approx 36.3$ . From Figure 3.7 it's possible to see that the agents quickly synchronize, despite having significantly different initial conditions. It has to be noticed that such an approach depends on the generalization capabilities of the DNN. Since it is trained on a finite number of data, we cannot guarantee perfect behavior for all the state space. Nevertheless, due to the robustness properties of contracting systems, training a neural metric on a sufficiently big dataset is a valuable tool to tackle the complexity of the proposed solution. This is proved by the experimental results.

### 3.7 Conclusions and perspectives

In this chapter, we studied the multiagent synchronization problem for nonlinear MIMO time-varying systems. To do so, we tackled the problem with tools taken from the contraction framework. To this end, we derived sufficient conditions for the design of a distributed nonlinear diffusive coupling control law. First, through Euclidean metric-based incremental stability tools, we achieved network consensus between the agents, globally in the domain of attraction. Then, through a Riemannian metric-based analysis, we achieved synchronization of undirected graphs in the presence of a leader. Static state and output feedback laws have been investigated. Then we showed that our design applies to classical case studies, such as minimum phase systems and linear systems coupled with a monotonic nonlinearity. To conclude, we showed that the proposed conditions for the Riemannian case can be further relaxed by asking for synchronization in compact sets. We provided a constructive algorithm based on Deep Learning tools to estimate the metric. We applied such a design for the synchronization of a network of Lorentz oscillators.

Future perspectives will involve the following aspects:

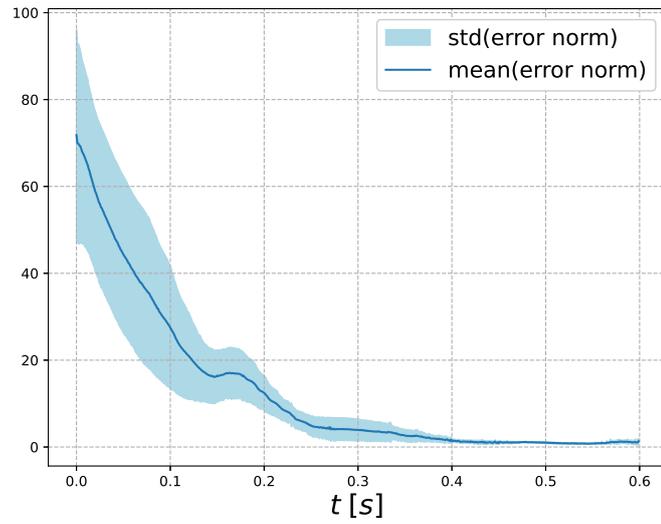


Figure 3.6: Evolution of the mean error norm between agents with respect to the leader

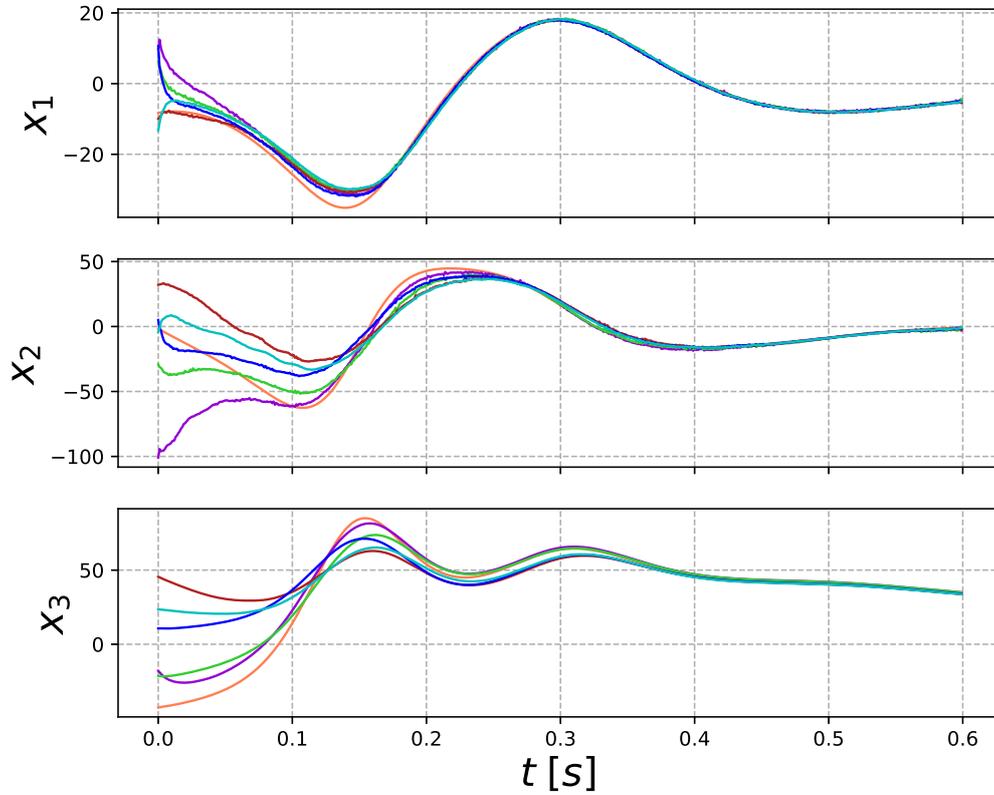


Figure 3.7: States' evolution of agents  $a_i$ ,  $i = 1, \dots, N$ , orange is leader

- 
- A first future perspective will involve the generalization of the results obtained in the Euclidean framework to the Riemannian one. As recalled in Remark 3.4.4, this is due to the choice of the Lyapunov function. We believe that this generalization could be achieved by considering the Lyapunov function based on a potential energy, similar to what has been done in Andrieu et al. (2018). The main reason is that, for instance, in the proofs of the Euclidean case we look for the error between any node of the network with respect to a virtual leader, that is taken as node 1. However, in practice, a ‘true’ leader does not exist.
  - Another development could be to look for dynamic output feedback distributed control laws. This could be done, for instance, by coupling each agent with an internal model unit, similarly to Isidori et al. (2014), and then looking for a feedback controller.
  - The last important future perspective could involve the analysis of the robustness of the proposed control law. It is known that the infinite gain margin property already provides robustness concerning fast dynamics and specific plant uncertainties. The interest in this case would be to try to see if these properties persist also in a distributed framework.

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# CHAPTER 4

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## APPLICATIONS

Experience without theory is blind,  
but theory without experience is mere  
intellectual play.

---

I. Kant

### 4.1 Introduction

In this chapter, we consider two output regulation tasks for two real-life engineering applications. In particular, we will consider two separate problems.

- First, we will consider a constant output regulation problem for a Power Flow Controller (PFC). PFCs are power electronic devices used to control the power flow at a node of a meshed grid and may act as DC circuit breakers. The structure of the PFC can be modeled as a bilinear system whose ODE is strongly dependent on a set of parameters. The system is then coupled with a polynomial output. The objective is to design a control action that can regulate the power to the lines of the terminals attached to the PFC to a constant set-point.
- Then, we will consider a harmonic regulation problem for a mechanical Ventilation Machine (VM). Such machines are essential equipment in Intensive Care Units (ICUs) to assist patients who cannot breathe on their own or need support to breathe sufficiently. Such a machine can be modeled as a Lur'e system. As the model takes also into consideration the patient, it is strongly parameter-dependent. In this case, the task is to track a desired pressure reference signal. As such a signal represents the pressure in and out of the patient's lungs, it is a periodic signal.

The PFC's problem is addressed with tools presented in Section 2.2. The system is extended with an integral action processing the regulation error and a forwarding-based control action has been implemented. To keep in consideration the physics of the actuators, a saturated controller has been implemented, together with an

anti-windup strategy to cope with possible instability behaviors due to such a saturation. A tuning procedure for the control gains is given. Stability and set-point tracking have been achieved, despite parametric uncertainties of the plant and of the references. The VM's problem is approached with tools from Section 2.3. The system is extended with a bunch of linear oscillators at the reference's trajectories and their multiplies. Through an incremental-like version of the Circle Criterion, an output-feedback control law is implemented and harmonic regulation is achieved. The proposed solutions for both problems have been validated theoretically and through real-life experiments.

Both of these applications have been realized in collaboration with another research team, which took care of the modelling and the implementation. In particular, the regulation problem on the PFC has been done together with Tanguy Simon and collaborators at Laboratory Ampere, INSA Lyon (Villeurbanne, France) and the results are presented in Simon et al. (2021a, 2022). The regulation problem on the VM has been made together with Joey Reinders and collaborators at the Department of Mechanical Engineering, Eindhoven University of Technology (Eindhoven, the Netherlands) and the results are presented in Reinders et al. (2022).

## 4.2 Robust output set-point tracking of a Power Flow Controller

### 4.2.1 About Power Flow Controllers

In the light of the current climate breakdown, it appears of paramount importance to reduce our impact on the environment. The access to electrical energy holds an important place in these discussions, and direct current (DC) micro-grids respond favorably to those issues. This type of electrical power network enhances the penetration of small renewable energy generators and lowers the energy losses while helping the clean access to electrical energy and the transition to a more energy-frugal lifestyle (see, e.g., Sen and Kumar (2018); Simon et al. (2021b) and references therein for a broader discussion on the subject). In this sense, meshed structures play an important role. With *meshed*, we mean that there might be multiple paths between two nodes in the electrical network. The meshed structure of micro-grids improves this result by reducing the amount of copper needed. This because there can be multiple paths between two points and the average power in each line in a building is low, as well as improving the reliability, modularity, and efficiency of the system, see Mackay et al. (2015).

To control a meshed DC grid, a DC Power Flow Controller (PFC) is required. It is a multi-terminal DC-DC converter located at a node in the mesh, sometimes called a smart-node Zafeiratou et al. (2018). Its objective is the regulation of the power in each line of the node (see Fig. 4.1), despite the high intermittency of renewable generators.

Although PFCs for high voltage DC applications (HVDC) have received a strong academic interest (see, e.g., the recent survey Balasubramaniam et al. (2019)) very little has been done for low-voltage applications (LVDC). The lower voltage rating leads to a completely different converter topology, and therefore to different control schemes. Among them, recall a multi-terminal PFC with a compensation node in Takahashi et al. (2015) improved by removing the compensation node in Natori et al.

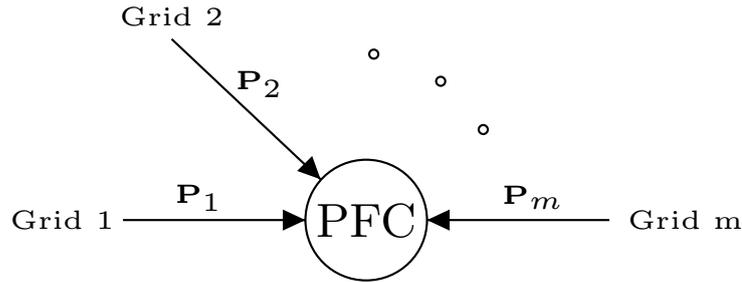


Figure 4.1:  $m$ -terminal PFC at a node in the grid

(2018), a PFC made of two separate Split-PI converters in Barara et al. (2017), and a three-terminal PFC in Simon et al. (2021b,a). The main shortcomings of Takahashi et al. (2015); Natori et al. (2018); Barara et al. (2017) are the absence of a dynamic model and their control strategies, which fail to give any proof of stability or robustness. No direct control of the power is achieved, and the control laws are applied to two-terminal devices, which do not constitute a node. In Takahashi et al. (2015), the control law is a constant ratio determined by a look-up table, based on the knowledge of the voltage at the end of the line, an uncertain parameter in practice. In Natori et al. (2018), a PI controller is used to regulate the current instead of the power, whose reference is again computed using the voltage at the end of the line. Moreover, the reservoir voltage, which is the voltage on a capacitor inside the converter, is not controlled and can drift outside the physical boundaries. Finally, in Barara et al. (2017), the authors propose a current-limited voltage controller using the RST technique with hysteresis switching. The reservoir voltage is properly controlled but power flow control is not achieved.

In this section, an  $n_T$ -terminal power flow controller is considered. Assume a synchronous PWM switching scheme on each branch, and suppose that the grid's dynamics are partially unknown to take into account its high variability. In this context, the PFC in the grid has been recently modeled using a state-space approach Simon et al. (2021b). The resulting continuous-time and finite-dimensional model is uncertain and bilinear. Moreover, the output to be regulated is a second-order polynomial. Indeed, this signal corresponds to the electrical power in all the lines, and this quantity is the product of couples of state variables, i.e. the voltage and current in each line.

While many papers dealing with stabilization problems for bilinear systems can be found in the literature (see, for instance, Koditschek and Narendra (1983); Gutman (1981); Quinn (1980); Longchamp (1980); Banks (1986); Tarbouriech et al. (2009); Andrieu and Tarbouriech (2012)), very few addressed the more general problem of output regulation, e.g., Grasselli et al. (1979), and more recently Cisneros et al. (2015); Tang et al. (2006), and all of them are focused on control designs for systems having a pure linear output and therefore not directly applicable to the problem explained above. Despite this lack of literature on output regulation, bilinear systems are a class of systems commonly employed to model physical systems, such as a heat exchanger Zitte et al. (2020), hydraulic systems Guo and Schwarz (1989), power factor compensators, and HVDC converters Cisneros et al. (2015), microbial cell growth Williamson (1977) and many others Pardalos and Yatsenko (2010). Direct power flow control is achieved in Simon et al. (2021b) on a three-terminal PFC, using a state feedback on the linearised dynamics after adding integrators. Yet, such a result only gives local stability results in the state-space, and although its

robustness has been tested, no proofs are given for local stability in the parametric space. Furthermore, the saturation of the duty cycles is not taken into account when designing the controller.

About the proposed control design, the contributions concerning the literature are therefore the following.

- The proposed controller achieves semi-global asymptotic stability in the state-space and provides robustness with respect to small parameter variations. To this end, model uncertainties and non-linearities are directly taken into account at the design step. In a nutshell, the PFC model is first extended with an integral action processing the regulation error. Then, a stabilizer for the extended system is derived with a forwarding-based control law, which has been presented in Section 1.4.3.3.
- An arbitrary number of terminals is considered, i.e.  $n_T \geq 2$  can be any integer. This guarantees that the proposed model and control design is independent of the number of terminals.
- As the control inputs are physically represented by the duty cycles of each terminal, a saturation is applied to the control action to meet such constraints. Compared to the unconstrained case, the closed-loop basin of attraction is preserved through the implementation of an anti-windup correction term to deal with (possible) unstable behaviors of the integrator dynamics due to such an input saturation.
- A tuning procedure for the controller gains is provided.
- All these achievements are successfully assessed on a real tenth-scale test bench for  $n_T = 3$ , and via simulations for a larger number of terminals.

#### 4.2.2 PFC's model

The Power Flow Controller (PFC) is an electrical device whose objective is to control the electric power in the lines it is connected to. We consider a general  $n_T$ -terminal version as in Fig. 4.1. The chosen electrical circuit to achieve this function is made of  $n_T$  identical buck-boost converters whose high-side are connected in parallel to a unique reservoir capacitor noted  $C_R$ . Each branch (buck-boost) is depicted on the left of Fig. 4.2. The grid connected to terminal  $k$ , as seen by the PFC, is modeled by a Thevenin equivalent circuit, as drawn on the right-hand side of the same figure.

This circuit cannot operate properly if the reservoir voltage is not controlled. Indeed, if  $\mathbf{v}_R < \mathbf{v}_k$  for some  $k \in \{1, \dots, n_T\}$ , the current flows freely through the diode in the upper transistor of the  $k^{\text{th}}$  branch, which cannot be controlled by the PWM switching (see Fig. 4.2). This implies that, for a constant power reference, there exists an infinite set of equilibrium points on which the power tracking task is achieved. From a practical viewpoint, this may be problematic, as the control action may overcome the bounds imposed by the saturation. This adds another control objective which can be solved knowing that such a voltage is constant if and only if the sum of average powers is equal to zero. Consequently, if the reservoir voltage is regulated to a constant value  $\mathbf{v}_R^r$  and all the lines but one are regulated to a constant power reference  $\mathbf{P}_k^r$  for  $k \in \{1, \dots, n_T - 1\}$ , the power in the last line naturally converges to the overall power balance, i.e.  $\mathbf{P}_{n_T}^r$  tends to  $-\sum_{k=1}^{n_T-1} \mathbf{P}_k^r$ .

Table 4.1: Symbols

Electrical			
$C_f$	Filter capacitance	$C_R$	Reservoir capacitance
$\mathbf{i}_k, i_k$	Current in $L_f$	$\mathbf{i}_{Gk}, i_{Gk}$	Grid current
$L_f$	Filter inductance	$L_{Gk}$	Grid inductance
$\mathbf{P}_k$	Power in line $k$	$R_{Gk}$	Grid resistance
$\mathbf{v}_k, v_k$	Line voltage (on $C_f$ )	$V_{Gk}$	Grid voltage
$\mathbf{v}_R, v_R$	Reserv. volt. (on $C_R$ )	$n_T$	number of terminals
State-space			
$\mathbf{A}, A$	Space state matrix	$B$	Input matrix
$\mathbf{C}, C$	Output matrix	$\mathbf{H}, H$	Polynomial out. mat.
$\mathbf{J}$	"Inertia" matrix	$\mathbf{N}, N$	Bilinear input mat.
$\mathbf{q}$	Constant vector	$\mathbf{r}$	Reference vector
$\mathbf{u}, u$	Input vector	$\boldsymbol{\chi}, \chi$	Aug. state vector
$\mathbf{x}, x$	State vector	$\mathbf{y}, y$	Output vector
$\boldsymbol{\eta}, \eta$	Integrator vector	$\delta$	Set-point vector
$\boldsymbol{\theta}$	Parameter vector	$\varphi(\cdot)$	aug. cl.-loop dynamics
$n_x$	number of states	$n_u$	number of inputs
$n_y$	number of outputs	$n_\eta$	number of integrators
$\phi$	integral action	$\alpha, \psi$	control laws
$\zeta$	anti-windup	$P, \Omega, M$	control actions
$d, \underline{d}$	AW tuning		
Sets			
$\mathcal{D}$	Possible set-points	$\mathcal{E}(\delta)$	Equilibrium points
$\mathcal{R}$	Possible references	$\mathcal{S}$	Admissible set-points
$\mathcal{U}$	Admissible inputs	$\Theta$	Possible parameters
Accents, indexes and exponents			
$\sim$	see Section 4.2.3.5	$r$	Reference
*	Equilibrium	$nom$	On nominal set-point
<b>bold</b>	absolute coordinates	$slim$	error coordinates

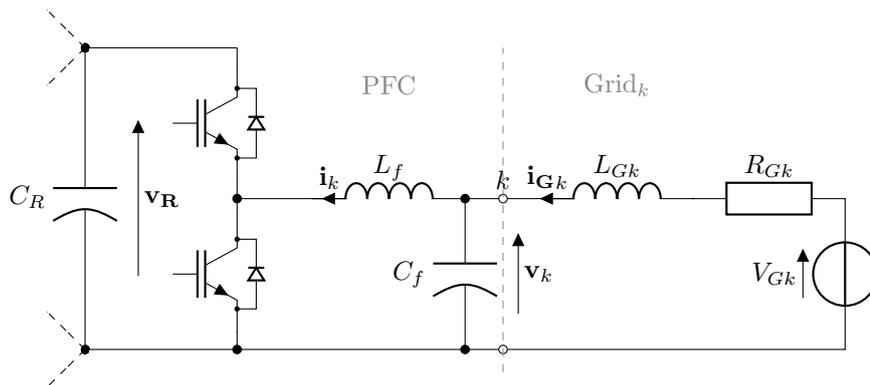


Figure 4.2: Detail of the  $k$ -th branch of the PFC (left) and the proposed Thevenin grid model as seen by this terminal (right)

The problem is the following: design a state-feedback controller delivering the duty ratios for the pulse-width modulation (PWM) switching of the transistors to achieve power control in each line of the node, while maintaining the reservoir voltage to a fixed given constant value, despite the uncertainty of the parameters characterizing the grid. Moreover, to ensure flexibility and modularity, assume limited knowledge of the grid model at the end of each line.

In Fig. 4.2, the currents are denoted by  $\mathbf{i}$  and measured in Amps, and the dynamic voltages by  $\mathbf{v}$  and measured in Volts. Note that the line voltage should be positive in a grid (more specifically within a precise tolerance of the nominal voltage), therefore every steady-state voltage  $\mathbf{v}_k^*$  will be considered non-negative. The component parameters are denoted by  $L_f/L_{Gk}$ ,  $C_f/C_R$ ,  $R_{Gk}$  and  $V_{Gk}$  for, respectively, inductors (Henry), capacitors (Farad), resistors (Ohm) and constant voltages (Volts). The dynamic model of the system can be derived by using Kirchhoff's and Ohm's electrical laws, along with the dynamic electrical laws for inductors and capacitors while assuming ideal components, see, e.g. Simon et al. (2021b) for a more detailed construction of the PFC's model.

A synchronous PWM switching is implemented, and the dynamics are averaged over a switching period. The input vector  $\mathbf{u} = [\mathbf{u}_1, \dots, \mathbf{u}_{n_u}]^\top \in \mathbb{R}^{n_u}$  is made of the duty ratio of each terminal and hence each of the  $\mathbf{u}_i(t)$  must be included in the set  $[0, 1] \subset \mathbb{R}$  for all  $t \geq 0$ . Therefore, the set of admissible inputs is defined as

$$\mathcal{U} := [0, 1]^{n_u}. \quad (4.1)$$

Note that, since the control  $u_i$  is the duty ratio at terminal  $i$ , it follows that  $n_u = n_T$ . The dynamic variables are all gathered in the state vector  $\mathbf{x} \in \mathbb{R}^{n_x}$  with  $n_x = 3n_u + 1$ , and the uncertain model parameters are collected in the vector  $\theta \in \mathbb{R}^{3n_u}$ , as shown in Table 4.2. The output vector corresponds to the control objectives  $\mathbf{y} = [\mathbf{P}_1, \dots, \mathbf{P}_{n_y-1}, \mathbf{v}_R]^\top \in \mathbb{R}^{n_y}$ , with  $\mathbf{P}_k = \mathbf{i}_{Gk} \mathbf{v}_k$ . Note that, since the output vector is made by the power in each terminal (but one) and the reservoir voltage, it follows that  $n_y = n_T = n_u$ . To avoid confusion, we will use a common notation and indicate with

$$m = n_y = n_T = n_u.$$

A state-space model can then be obtained for the PFC, following the results in Simon et al. (2021b). The system can be represented as a bilinear system. The states are the currents  $\mathbf{i}_k, \mathbf{i}_{Gk}$  and the voltages  $\mathbf{v}_k$  for  $k \in \{1, \dots, m\}$  and  $\mathbf{v}_R$ . Additionally,

the PFC is coupled with a polynomial output on which the regulation task has to be achieved. Such an output is composed of the powers  $\mathbf{P}_k$  for  $k \in \{1, \dots, m\}$  and the reservoir voltage  $\mathbf{v}_R$ . In detail, it can be described by a set of dynamical equations of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(\theta)\mathbf{x} + \mathbf{N}(\mathbf{x})\mathbf{u} + \mathbf{q}(\theta), \\ \mathbf{y} &= \mathbf{C}\mathbf{x} + \mathbf{H}(\mathbf{x})\mathbf{x}, \\ \mathbf{u} &\in \mathcal{U},\end{aligned}\tag{4.2}$$

with

$$\mathbf{A}(\theta) = \mathbf{J}(\theta)^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & I \\ 0 & 0 & -I & -\text{blkdiag}\{R_G\} \end{pmatrix},\tag{4.3a}$$

$$\mathbf{N}(\mathbf{x})\mathbf{u} = \sum_{j=1}^m (\mathbf{N}_j \mathbf{u}_j) \mathbf{x} = \mathbf{J}(\theta)^{-1} \begin{pmatrix} 0 & \mathbf{u}^\top & 0 \\ -\mathbf{u} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x},\tag{4.3b}$$

$$\mathbf{q}(\theta) = \mathbf{J}(\theta)^{-1} [0, V_G^\top]^\top,\tag{4.3c}$$

$$\mathbf{J}(\theta) = \text{blkdiag}\{C_R, \mathbf{1}^\top L_f, \mathbf{1}^\top C_f, L_G^\top\}^\top,\tag{4.3d}$$

$$\mathbf{C} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\tag{4.3e}$$

$$\mathbf{H}(\mathbf{x}) = \begin{pmatrix} 0 & \frac{1}{2} \text{blkdiag}\{\mathbf{i}_{G1}, \dots, \mathbf{i}_{Gm-1}\} & 0 & \frac{1}{2} \text{blkdiag}\{\mathbf{v}_1, \dots, \mathbf{v}_{m-1}\} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\tag{4.3f}$$

where, for each  $j = 1, \dots, m$ , the  $\mathbf{N}_j$  are constant  $n_x \times n_x$  matrices defined as  $\mathbf{N}_j = \mathbf{J}(\theta)^{-1} N_j$  where  $N_j$  are  $m$  different matrices full of zeros with a 1 in the first row and  $j + 1$  column and a  $-1$  in the first column and  $j + 1$  row, and with

$$L_G = [L_{G1}, \dots, L_{Gm}], \quad R_G = [R_{G1}, \dots, R_{Gm}], \quad V_G = [V_{G1}, \dots, V_{Gm}].$$

Let  $\Theta \subset \mathbb{R}^{3m}$  be the non-empty set of possible system parameters that are compatible with the physics of the system. It is defined as

$$\Theta := \{\theta \in \mathbb{R}^{3m} \mid L_{Gk} > 0, R_{Gk} > 0, V_{Gk} \geq 0, k \in \{1, \dots, m\}\}.\tag{4.4}$$

There is no loss of generality in these constraints since  $L_G$ , and  $R_G$  represent physical properties (inductance and resistance) that are always strictly positive. As shown later,

$$\mathbf{P}_k^r = 0 \quad \implies \quad \lim_{t \rightarrow +\infty} \mathbf{v}_k(t) = \mathbf{v}_k^* = V_{Gk},$$

i.e. when the power reference is null, the line voltage tends to  $V_{Gk}$ . As stated before, this line voltage should always be non-negative, then so should  $V_{Gk}$ .

The vector of references corresponds to the control objectives, i.e.

$$\mathbf{r} = [\mathbf{P}_1^r, \dots, \mathbf{P}_{m-1}^r, \mathbf{v}_R^r]^\top,$$

and the non-empty set of possible references  $\mathcal{R} \subseteq \mathbb{R}^m$  is defined as

$$\mathcal{R} := \left\{ \mathbf{r} = (\mathbf{P}_1^r, \dots, \mathbf{P}_{m-1}^r, \mathbf{v}_R^r) \in \mathbb{R}^m \mid (\mathbf{P}_1^r, \dots, \mathbf{P}_{m-1}^r) \neq (0, \dots, 0), \mathbf{v}_R^r > 0 \right\}.\tag{4.5}$$

Table 4.2: State vector and uncertain parameter vector.

$\mathbf{x}$	$\mathbf{x}_1$	$\mathbf{x}_2$	$\dots$	$\mathbf{x}_{m+1}$	$\mathbf{x}_{m+2}$	$\dots$	$\mathbf{x}_{2m+1}$	$\mathbf{x}_{2m+2}$	$\dots$	$\mathbf{x}_{3m+1}$
	$\mathbf{v}_R$	$\mathbf{i}_1$	$\dots$	$\mathbf{i}_m$	$\mathbf{v}_1$	$\dots$	$\mathbf{v}_m$	$\mathbf{i}_{G1}$	$\dots$	$\mathbf{i}_{Gm}$
$\theta$	$\theta_1$	$\dots$	$\theta_m$	$\theta_{m+1}$	$\dots$	$\theta_{2m}$	$\theta_{2m+1}$	$\dots$	$\theta_{3m}$	
	$L_{G1}$	$\dots$	$L_{Gm}$	$R_{G1}$	$\dots$	$R_{Gm}$	$V_{G1}$	$\dots$	$V_{Gm}$	

where, as stated before,  $\mathbf{v}_R^r$  should be sufficiently high for the device to operate properly, and hence only strictly positive values have been taken into consideration. The case in which all the reference powers  $\mathbf{P}_j^r$  are null has not been taken into consideration, as it makes the control objective structurally impossible, as shown later on.

### 4.2.3 Robust output set-point tracking for the PFC

#### 4.2.3.1 Control problem

Define beforehand the compact notation  $\delta := (\theta, \mathbf{r})$  and  $\mathcal{D} := \Theta \times \mathcal{R}$ . The tackled problem is stated as follows.

**Problem 4.2.1** (Robust output set-point tracking for the PFC). *Given a nominal set of parameters  $\mathcal{D}$  and  $\delta_{nom} \in \mathcal{D}$ , find functions  $\phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \rightarrow \mathbb{R}^{n_\eta}$  and  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \rightarrow \mathcal{U}$  such that for any arbitrarily large compact set of initial conditions  $\mathcal{X} \times \mathcal{Z} \subseteq \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta}$  there exists  $\bar{\delta} > 0$  such that, for any  $\delta$  satisfying  $|\delta - \delta_{nom}| \leq \bar{\delta}$ , the resulting trajectories of system (4.2) in closed-loop with the regulator*

$$\begin{aligned}\dot{\boldsymbol{\eta}} &= \phi(\mathbf{x}, \boldsymbol{\eta}), \\ \mathbf{u} &= \alpha(\mathbf{x}, \boldsymbol{\eta})\end{aligned}$$

are bounded forward in time and

$$\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{r}.$$

Following an *internal model* based design, the system (4.2) is first extended with an integral action processing the regulation error as follows

$$\dot{\boldsymbol{\eta}} = \mathbf{y} - \mathbf{r}. \quad (4.6)$$

Furthermore, the control law is saturated since  $\mathbf{u} \in \mathcal{U}$ , and this physical constraint may lead to unstable behaviors in the dynamics of the integrator. To deal with this issue, an anti-windup design is implemented (see Tarbouriech and Turner (2009); Zheng et al. (1994); Galeani et al. (2009)) to mitigate the effects of uncontrollable integral action. This leads to the following integral dynamics

$$\dot{\boldsymbol{\eta}} = \phi(\mathbf{x}, \boldsymbol{\eta}) := \mathbf{y} - \mathbf{r} + \zeta(\mathbf{x}, \boldsymbol{\eta}) \quad (4.7)$$

where the function  $\zeta$  represents such an anti-windup. The overall proposed control scheme is depicted in Fig. 4.3.

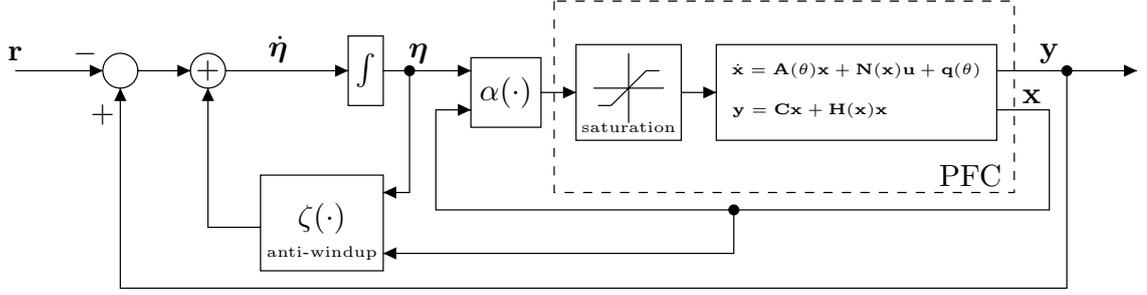


Figure 4.3: Proposed control structure for the PFC

#### 4.2.3.2 Set of solutions

The considered problem may not be solvable for all  $\delta = (\theta, \mathbf{r}) \in \mathcal{D}$  since for some values of  $\delta$  there may not exist an equilibrium pair  $(\mathbf{x}^*, \mathbf{u}^*)$  satisfying  $\mathbf{y} = \mathbf{r}$  within the input constraints. Therefore, for a given  $\delta \in \mathcal{D}$ , let  $\mathcal{E}(\delta)$  be the set of admissible equilibrium points, namely the set of steady-state solutions on which output regulation is achieved, i.e.

$$\mathcal{E}(\delta) := \left\{ (\mathbf{x}^*, \mathbf{u}^*) \in \mathbb{R}^{n_x} \times \mathcal{U} \mid \mathbf{A}(\theta)\mathbf{x}^* + \mathbf{N}(\mathbf{x}^*)\mathbf{u}^* + \mathbf{q}(\theta) = \mathbf{0}, \mathbf{C}\mathbf{x}^* + \mathbf{H}(\mathbf{x}^*)\mathbf{x}^* = \mathbf{r} \right\}. \quad (4.8)$$

The set  $\mathcal{S}$  is then defined as the set of admissible parameters and references  $\delta$  for which there exist such equilibrium points:

$$\mathcal{S} := \{ \delta \in \mathcal{D} \mid \text{card}\{\mathcal{E}(\delta)\} > 0 \}. \quad (4.9)$$

A characterization of  $\mathcal{S}$  can then be provided through model inversion as shown below.

**Proposition 4.2.2** (Equilibrium points for the PFC). *Consider system (4.2). Then*

$$\mathcal{S} = \left\{ \delta \in \mathcal{D} \mid \Delta_k(\delta) \geq 0, 0 \leq \frac{1}{2\mathbf{v}_{\mathbf{R}^r}} \left( V_{Gk} \pm \sqrt{\Delta_k(\delta)} \right) \leq 1, k \in \{1, \dots, m\} \right\}, \quad (4.10)$$

where

$$\Delta_k(\delta) := V_{Gk}^2 - 4R_{Gk}\mathbf{P}_k^r \quad (4.11)$$

with  $\mathbf{P}_m^r := -\sum_{k=1}^{m-1} \mathbf{P}_k^r$ . Moreover, for a given  $\delta \in \mathcal{S}$ , there exist from one to  $2^m$  pairs  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{E}(\delta)$ .

**Proof.** By considering (4.2) and posing  $\dot{\mathbf{x}} = \mathbf{0}$ , the following set of equations is found

$$\begin{cases} \sum_{k=1}^m \mathbf{i}_k^* \mathbf{u}_k^* = 0, & (4.12a) \end{cases}$$

$$\begin{cases} \mathbf{v}_k^* - \mathbf{v}_{\mathbf{R}^r}^* \mathbf{u}_k^* = 0, & (4.12b) \end{cases}$$

$$\begin{cases} -\mathbf{i}_k^* + \mathbf{i}_{\mathbf{G}k}^* = 0, & (4.12c) \end{cases}$$

$$\begin{cases} -\mathbf{v}_k^* - R_{Gk} \mathbf{i}_{\mathbf{G}k}^* + V_{Gk} = 0, & (4.12d) \end{cases}$$

where  $k \in \{1, \dots, m\}$ , and

$$\begin{cases} \mathbf{v}_k^* \mathbf{i}_{Gk}^* = \mathbf{P}_k^r, & k \in \{1, \dots, m-1\} \\ \mathbf{v}_{\mathbf{R}^*} = \mathbf{v}_{\mathbf{R}^r}. \end{cases} \quad (4.13a)$$

$$(4.13b)$$

Solving (4.12b) and (4.12c) results in

$$\mathbf{u}_k^* = \frac{\mathbf{v}_k^*}{\mathbf{v}_{\mathbf{R}^*}} \quad \text{and} \quad \mathbf{i}_k^* = \mathbf{i}_{Gk}^* \quad (4.14)$$

where  $\mathbf{v}_{\mathbf{R}^*} = \mathbf{v}_{\mathbf{R}^r} > 0$ . Hence the first condition for the existence of solutions is

$$0 \leq \frac{\mathbf{v}_k^*}{\mathbf{v}_{\mathbf{R}^*}} \leq 1.$$

Feeding (4.12c) into (4.12a) leads to

$$\frac{1}{\mathbf{v}_{\mathbf{R}^*}} \sum_{k=1}^m \mathbf{i}_{Gk}^* \mathbf{v}_k^* = 0. \quad (4.15)$$

Taking (4.13) in (4.15) results in

$$\frac{1}{\mathbf{v}_{\mathbf{R}^r}} \left( \sum_{k=1}^{m-1} \mathbf{P}_k^r + \mathbf{i}_{Gm}^* \mathbf{v}_{\mathbf{m}^*} \right) = 0, \quad (4.16)$$

from which  $\mathbf{P}_m^r := -\sum_{k=1}^{m-1} \mathbf{P}_k^r = \mathbf{i}_{Gm}^* \mathbf{v}_{\mathbf{m}^*}$  is defined. Then, multiplying (4.12d) by  $\mathbf{v}_k^*$  and using (4.13a), it follows that

$$-(\mathbf{v}_k^*)^2 + V_{Gk} \mathbf{v}_k^* - R_{Gk} \mathbf{P}_k^r = 0 \quad (4.17)$$

hence the second condition for the existence of real solutions is  $\Delta_k(\delta) = V_{Gk}^2 - 4R_{Gk} \mathbf{P}_k^r \geq 0$ . If  $\Delta_k(\delta) \geq 0$ , the characterisation of  $\mathcal{E}(\delta)$  can be given using (4.12) and (4.13) as

$$\mathcal{E}(\delta) = \begin{cases} \mathbf{v}_k^* = \frac{1}{2} \left( V_{Gk} \pm \sqrt{\Delta_k(\delta)} \right) & (4.18a) \\ \mathbf{i}_k^* = \mathbf{i}_{Gk}^* = \frac{1}{R_{Gk}} (V_{Gk} - \mathbf{v}_k^*) & (4.18b) \\ \mathbf{v}_{\mathbf{R}^*} = \mathbf{v}_{\mathbf{R}^r} & (4.18c) \\ \mathbf{u}_k^* = \frac{\mathbf{v}_k^*}{\mathbf{v}_{\mathbf{R}^r}} & (4.18d) \end{cases}$$

with  $k \in \{1, \dots, m\}$  and  $\mathbf{v}_{\mathbf{R}^r} > 0$  by definition of  $\mathcal{R}$ .

□

**Remark 4.2.3.** *Two important aspects should be highlighted.*

- *Note that the control problem for the PFC does not fit in the incremental framework proposed in Section 2.2. The main reason is that the system is bilinear. Indeed, satisfying the Killing vector property for such a class of systems is not an easy task, especially because the vector field multiplying the control input is*

not bounded.

- Nevertheless, this is not a problem. As explained, the references to be tracked do not belong to the whole state-space, but only to the set  $\mathcal{R}$ . In other words, the interest is not in achieving global output set-point tracking. For this reason, the control action that we will implement will not make use of incremental forwarding tools, but of forwarding tools for equilibrium point stability (see Section 1.4.3.2).

#### 4.2.3.3 Forwarding for bilinear systems with polynomial output

In light of Remark 4.2.3, in this section we aim to specialize the forwarding-based Lyapunov stability tool for the class of systems which are bilinear in the  $x$ -dynamics, feeding a polynomial  $\eta$ -dynamics. For this, consider a system composed of a cascade of a bilinear system having a second-order polynomial output feeding an integrator of the form

$$\begin{aligned}\dot{x} &= Ax + (N(x) + B)u, \\ \dot{\eta} &= Cx + H(x)x,\end{aligned}\tag{4.19}$$

where  $(x, \eta) \in \mathbb{R}^{n_x+n_\eta}$  is the state,  $u \in \mathbb{R}^{n_u}$  is the control input,  $A, B, C$  are matrices of suitable dimensions, and functions  $H : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_\eta \times n_x}$  and  $N : \mathbb{R}^{n_x} \rightarrow \mathbb{R}^{n_x \times n_u}$  defined as  $H(x) = [H_1^\top x \ \dots \ H_{n_\eta}^\top x]^\top$  and  $N(x) = [N_1 x \ \dots \ N_{n_u} x]$  for some  $n_x \times n_x$  matrices  $H_1, \dots, H_{n_\eta}, N_1, \dots, N_{n_u}$ . Similarly to Section 1.4.3.2, we assume the following.

**Assumption 4.2.1** (Over-actuated system). *The number of inputs is not smaller than the number of outputs, i.e.  $n_u \geq n_\eta$ .*

Stabilizing forwarding design for systems of the form (4.19) is presented in the following.

**Proposition 4.2.4** (Forwarding for polynomial systems). *Consider system (4.19). Suppose that  $A$  is Hurwitz and that the matrix  $CA^{-1}B$  is full rank. Select  $P = P^\top \succ 0$  and  $M_i = M_i^\top$  such that*

$$PA + A^\top P \prec 0,\tag{4.20}$$

$$M_i A + A^\top M_i = \frac{1}{2}(H_i + H_i^\top), \quad \forall i \in \{1, \dots, n_\eta\},\tag{4.21}$$

and let  $M_0 = CA^{-1}$ . Then, for any matrix  $\Omega = \Omega^\top \succ 0$ , the origin of system (4.19) in closed-loop with  $u = \psi(x, \eta)$ , with the function  $\psi : \mathbb{R}^{n_x+n_\eta} \rightarrow \mathbb{R}^{n_u}$  defined as

$$\psi(x, z) = -\left(x^\top P(N(x) + B) - (\eta - M(x))^\top \Omega (M_0 + 2R(x)) (N(x) + B)\right)^\top,\tag{4.22}$$

with

$$R(x) := [M_1 x \ \dots \ M_{n_\eta} x]^\top\tag{4.23}$$

$$M(x) := M_0 x + R(x)x,\tag{4.24}$$

is globally asymptotically stable and locally exponentially stable.

**Proof.** The proof is based on a Lyapunov function construction which follows the results presented in Section 1.4.3.2. In particular, let  $W : \mathbb{R}^{n_x+n_\eta} \rightarrow \mathbb{R}$  be defined as

$$W(x, \eta) = \frac{1}{2}x^\top Px + \frac{1}{2}(\eta - M(x))^\top \Omega(\eta - M(x)), \quad (4.25)$$

with  $P$  and  $M$  defined respectively by (4.20) and (4.24) as in the statement of the proposition. Note that  $W$  is proper and positive definite. By construction, the function  $M$  satisfies

$$\begin{aligned} \frac{\partial M}{\partial x}(x)Ax &= M_0Ax + \begin{bmatrix} x^\top(M_1A + A^\top M_1)x \\ \vdots \\ x^\top(M_pA + A^\top M_p)x \end{bmatrix} = Cx + \frac{1}{2} \begin{bmatrix} x^\top(H_1 + H_1^\top)x \\ \vdots \\ x^\top(H_p + H_p^\top)x \end{bmatrix} \\ &= Cx + H(x)x. \end{aligned}$$

Hence, the time derivative of  $W$  along the solutions of system (4.19) satisfies

$$\begin{aligned} \dot{W}(x, z) &= x^\top P [Ax + (N(x) + B)u] + (\eta - M(x))^\top \Omega \left[ -\frac{\partial M}{\partial x}(x)(N(x) + B)u \right] \\ &= \frac{1}{2}x^\top (PA + A^\top P)x \\ &\quad + [x^\top P(N(x) + B) - (\eta - M(x))^\top \Omega(M_0 + 2R(x))(N(x) + B)] u. \end{aligned}$$

Using again the definition of  $M$  in (4.24) and the definition of  $\psi$  in (4.22) yields

$$\dot{W}(x, \eta) = \frac{1}{2}x^\top (PA + A^\top P)x - \psi^\top(x, \eta)\psi(x, \eta) \leq 0.$$

Furthermore, note that

$$\{(x, \eta) \mid \dot{W}(x, \eta) = 0\} = \{(x, z) \mid x = 0, \psi(x, \eta) = 0\}. \quad (4.26)$$

Moreover, using the definition of  $M_0$ , one obtains  $\psi^\top(0, \eta) = \eta^\top \Omega C A^{-1} B$ . Since by assumption,  $C A^{-1} B$  and  $\Omega$  are full rank, the set  $\{(x, \eta) \mid \dot{W}(x, \eta) = 0\}$  coincides with the origin and therefore  $(x, \eta) \mapsto \dot{W}(x, \eta)$  is negative definite. Consequently,  $W$  is a Lyapunov function of the closed-loop system and the origin is globally asymptotically stable. Finally, employing the same method, note that the quadratic function

$$W_0(x, \eta) = \frac{1}{2}x^\top Px + \frac{1}{2}(\eta - M_0x)^\top \Omega(\eta - M_0x),$$

is a Lyapunov function for the first-order approximation

$$\dot{x} = Ax + B\psi_0(x, \eta), \quad \dot{\eta} = Cx,$$

with  $\psi_0(x, \eta)$  being the first-order approximation of  $\psi$ . Hence, local exponential stability of the equilibrium is obtained.  $\square$

#### 4.2.3.4 Controller design

To solve the regulation problem in Problem 4.2.1, we aim to follow the control structure presented in Fig. 4.3. The controller is designed for some known nominal parameters and references  $\delta_{\text{nom}} := (\theta_{\text{nom}}, \mathbf{r}_{\text{nom}})$ . Furthermore, it is assumed that this pair belongs to  $\text{int}\{\mathcal{S}\}$ . This allows to prove that output set-point tracking is still achieved for  $\delta$  distinct from  $\delta_{\text{nom}}$  but sufficiently close to it, as specified later on. The main result of the section is stated in the following.

**Theorem 4.2.5** (Robust output set-point tracking for the PFC). *Consider the set  $\mathcal{S}$  defined in (4.9). Select any  $\delta_{\text{nom}} = (\theta_{\text{nom}}, \mathbf{r}_{\text{nom}}) \in \text{int}\{\mathcal{S}\}$  and any corresponding  $(\mathbf{x}_{\text{nom}}^*, \mathbf{u}_{\text{nom}}^*) \in \mathcal{E}(\delta_{\text{nom}})$ . Then, for  $d > 0$  sufficiently high, the robust output set-point tracking Problem for system (4.2) is solved by the dynamic control law*

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \phi(\mathbf{x}, \boldsymbol{\eta}) := \mathbf{y} - \mathbf{r} - \text{Dz}_d(\boldsymbol{\eta} - M(\mathbf{x} - \mathbf{x}_{\text{nom}}^*)) \\ \mathbf{u} &= \alpha(\mathbf{x}, \boldsymbol{\eta}) := \mathbf{u}_{\text{nom}}^* + \text{Sat}_{-\mathbf{u}_{\text{nom}}^*}^{1-\mathbf{u}_{\text{nom}}^*}(\psi(\mathbf{x} - \mathbf{x}_{\text{nom}}^*, \boldsymbol{\eta})) \end{aligned} \quad (4.27)$$

where the functions  $\text{Sat}$  and  $\text{Dz}$  are defined in (1), the function  $\psi$  is chosen as (4.22) with the matrices  $A, B, C$  and functions  $N, H$  defined as

$$\begin{aligned} A &:= \mathbf{A}(\theta_{\text{nom}}) + \sum_{j=1}^m \mathbf{N}_j \mathbf{u}_{\text{nom},j}^*, & B &:= \mathbf{N}(\mathbf{x}_{\text{nom}}^*) \\ C &:= \mathbf{C} + 2\mathbf{H}(\mathbf{x}_{\text{nom}}^*), & N(x) &:= \mathbf{N}(x), \end{aligned}$$

$H(x) := \mathbf{H}(x)$ ,  $P, M$  defined as in (4.20), (4.24) and any  $\Omega = \Omega^\top \succ 0$ .

**Remark 4.2.6.** *Note that the integral action dynamics is designed in the form of (4.7) with the anti-windup term that takes the form of a dead-zone function. As it will be shown in the following, if  $\delta = \delta_{\text{nom}}$  one could avoid the anti-windup design, as a simple feed-forward controller is sufficient to stir the system's dynamics to its (nominal) target equilibrium point and solve the regulation problem. However, because of the plant's uncertainties, the dead-zone effect must be present not only in the nominal equilibrium, but also in a neighborhood of it, and hence one requires  $d > 0$ . On the other hand, the function  $\alpha$  is selected as a first (nominal) feed-forward action  $\mathbf{u}_{\text{nom}}^*$  plus a second term made by saturating the function  $\psi$  derived from the forwarding approach in Proposition 4.2.4. Thanks to the saturation, the stabilizer satisfies the input constraints as  $\alpha$  takes only values in  $\mathcal{U}$ , ensuring the validity of the control law with respect to the model (4.2). Indeed the following holds*

$$0 \leq \mathbf{u} = \mathbf{u}_{\text{nom}}^* + \text{Sat}_{-\mathbf{u}_{\text{nom}}^*}^{1-\mathbf{u}_{\text{nom}}^*}(\psi(x, \boldsymbol{\eta})) \leq \mathbf{1}$$

In this sense, note that the control law  $\psi$  in Proposition 4.2.4 is of infinite gain margin, i.e.  $\kappa\psi(\cdot)$  is still a stabiliser for (4.19) for all gains  $\kappa > 0$ , and hence the saturation does not restrict the set of solutions. To this end, recall the well-known link between forwarding and small-input control (Kaliora and Astolfi (2004)).

**Proof.** [Proof of Theorem 4.2.5] The proof is divided into three parts. It is first shown that the origin of the closed-loop system (4.2), (4.27) is globally asymptotically stable for  $\delta = \delta_{\text{nom}}$ . Then, it is proven that for  $d$  sufficiently large, the anti-windup effect disappears in the target equilibrium (and so regulation is achieved). Finally, it is shown that the proposed design is robust to sufficiently small model parameter variations and semi-global asymptotic stability on an equilibrium is guaranteed (on which the regulation objective is satisfied).

**Part 1: Global output set-point tracking.** Consider  $\delta = \delta_{\text{nom}} \in \text{int}\{\mathcal{S}\}$  and  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{E}(\delta)$ . Define the following change of coordinates:

$$\begin{pmatrix} \mathbf{u} \\ \mathbf{x} \\ \boldsymbol{\eta} \end{pmatrix} \mapsto \begin{pmatrix} u \\ x \\ \eta \end{pmatrix} := \begin{pmatrix} \mathbf{u} - \mathbf{u}_{\text{nom}}^* \\ \mathbf{x} - \mathbf{x}_{\text{nom}}^* \\ \boldsymbol{\eta} - \boldsymbol{\eta}_{\text{nom}}^* \end{pmatrix}$$

in which  $\boldsymbol{\eta}_{\text{nom}}^* = 0$ . In these coordinates, the system (4.2) reads

$$\dot{x} = \mathbf{A}(\theta)(x + \mathbf{x}_{\text{nom}}^*) + \mathbf{N}(x + \mathbf{x}_{\text{nom}}^*)(u + \mathbf{u}_{\text{nom}}^*) + \mathbf{q}(\theta), \quad (4.28)$$

$$= \left( \mathbf{A}(\theta) + \sum_{j=1}^m \mathbf{N}_j \mathbf{u}_{\text{nom},j}^* \right) x + \mathbf{N}(x + \mathbf{x}_{\text{nom}}^*) u \quad (4.29)$$

while the  $\eta$ -dynamics reads

$$\dot{\eta} = (\mathbf{C} + 2\mathbf{H}(\mathbf{x}_{\text{nom}}^*))x + \mathbf{H}(x)x,$$

for which the relations  $\mathbf{C}\mathbf{x}_{\text{nom}}^* + \mathbf{H}(\mathbf{x}_{\text{nom}}^*)\mathbf{x}_{\text{nom}}^* = \mathbf{r}$  and  $\mathbf{H}(x)\mathbf{x}_{\text{nom}}^* = \mathbf{H}(\mathbf{x}_{\text{nom}}^*)x$  have been used. By selecting the matrices  $A, B, C$  and the functions  $N, H$  as in the statement of the theorem, one obtains a system in the form of (4.19). The following two technical lemmas show that  $A$  is Hurwitz and that the non-resonance condition  $CA^{-1}B$  full rank holds. The proofs are post-poned in Section 4.2.5.

**Lemma 4.2.7.** *Pick any  $\delta \in \text{int}\{\mathcal{S}\}$ . Then for all  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{E}(\delta)$ , the matrix  $A = \mathbf{A}(\theta) + \sum_{j=1}^m \mathbf{N}_j \mathbf{u}_j^*$  is Hurwitz.*

**Lemma 4.2.8.** *Pick any  $\delta \in \text{int}\{\mathcal{S}\}$ . Then for all  $(\mathbf{x}^*, \mathbf{u}^*) \in \mathcal{E}(\delta)$  the matrix  $CA^{-1}B$  is full rank.*

**Remark 4.2.9.** *Lemma 4.2.7 can be understood as the natural stability of the system: in practice, for any constant duty ratio, the system stabilizes to a steady-state equilibrium point. Concerning Lemma 4.2.8, it has been shown in Astolfi and Praly (2017) that such a condition is necessary to achieve output set-point tracking in case of (sufficiently small) parametric uncertainties. Since the linearized model around the equilibrium point is stabilizable, this condition implies controllability of the extended (plant and integral action) linearised system and represents the non-resonance condition (see Section 2.1.2.1). It follows from the proof of Lemma 4.2.8 that the points  $\delta \in \partial\mathcal{S} := \mathcal{S} \setminus \text{int}\{\mathcal{S}\}$  do not satisfy such a condition.*

Following the proof of Proposition 4.2.4, consider the Lyapunov function

$$W(x, \eta) = \frac{1}{2}x^\top Px + \frac{1}{2}(\eta - M(x))^\top \Omega(\eta - M(x))$$

where  $P = P^\top \succ 0$  verifies  $PA + A^\top P \prec 0$ , the function  $M$  is defined as in (4.24) and  $\Omega$  is any symmetric positive definite matrix. Note that  $P$  always exists in view of Lemma 4.2.7. Computing the time derivative and following the same steps of the proof of Proposition 4.2.4, one obtains

$$\dot{W}(x, \eta) = -x^\top (PA + A^\top P)x - \Psi(x, \eta) - (\eta - M(x))^\top \text{Dz}_d(\eta - M(x)) \quad (4.30)$$

with

$$\Psi(x, \eta) := \psi(x, \eta)^\top \text{Sat}_{-\mathbf{u}_{\text{nom}}^*}^{1-\mathbf{u}_{\text{nom}}^*}(\psi(x, \eta))$$

with  $\psi$  defined as in the statement of the theorem, i.e. in (4.27). The function  $\Psi$  verifies

$$\Psi(x, \eta) > 0 \quad \forall (x, \eta) \neq (0, 0). \quad (4.31)$$

To show the previous inequality, let  $\psi_i(x, \eta)$  denote the  $i$ -th element of  $\psi(x, \eta)$ . By definition of the saturation function in (1),

$$\Psi(x, \eta) = \sum_{i=1}^m \Psi_i(x, \eta) = \sum_{i=1}^m \psi_i(x, \eta) \text{sat}_{-\mathbf{u}_{\text{nom},i}^*}^{1-\mathbf{u}_{\text{nom},i}^*}(\psi_i(x, \eta))$$

and, for any  $i \in \{1, \dots, m\}$ , the following conditions hold :

1. if  $\psi_i(x, \eta) < -\mathbf{u}_{\text{nom},i}^*$ , then  
 $\Psi_i(x, \eta) = -\psi_i(x, \eta)\mathbf{u}_{\text{nom},i}^* > 0$ ;
2. if  $-\mathbf{u}_{\text{nom},i}^* \leq \psi_i(x, \eta) \leq 1 - \mathbf{u}_{\text{nom},i}^*$ , then  
 $\Psi_i(x, \eta) = \psi_i(x, \eta)^2 > 0$ ;
3. if  $\psi_i(x, \eta) > 1 - \mathbf{u}_{\text{nom},i}^*$ , then  
 $\Psi_i(x, \eta) = \psi_i(x, \eta)(1 - \mathbf{u}_{\text{nom},i}^*) > 0$ ,

proving (4.31) since each element of the sum is positive. Moreover, since  $s \text{Dz}_d(s) \geq 0$  for any  $s, d$ , it follows from (4.30) that  $\dot{W}(x, \eta) < 0$ . Since this derivative is a sum of non-negative terms,  $\dot{W}(x, z) = 0$  if and only if each term is null. Moreover,  $x^\top (PA + A^\top P)x = 0$  if and only if  $x = 0$ . It follows that

$$\begin{aligned} \Psi(0, z) = 0 &\iff \psi_i(0, z) \text{sat}_{-\mathbf{u}_{\text{nom},i}^*}^{1-\mathbf{u}_{\text{nom},i}^*}(\psi_i(0, z)) \quad \forall i \in \{1, \dots, m\} \\ &\iff \psi(0, z) = 0 \iff z = M(0) = 0, \end{aligned}$$

hence that  $\dot{W}(x, z) = 0 \iff (x, z) = (0, 0)$  and that the origin of the closed-loop system is globally asymptotically stable. Furthermore, following the proof of Proposition 4.2.4, it is possible to show that the origin of the closed-loop system is globally asymptotically stable and locally exponentially stable.

**Part 2: Anti-windup function design.** The dead-zone function  $\text{Dz}_d$  has been introduced to implement an anti-windup in the integral action. It follows from the Lyapunov analysis that such a function does not compromise the stability of the nominal closed-loop. However, the constant  $d$  must be chosen sufficiently large so that when trajectories are close to the equilibrium point for which  $\mathbf{y} = \mathbf{r}$ , the anti-windup has no effect (i.e. the dead-zone is equal to zero), and set-point tracking is achieved. The dead-zone constant  $d$  must therefore satisfy

$$d \geq \underline{d}, \quad \underline{d} := \sup_{(x, \eta) \in \mathcal{D}_z \times \mathcal{D}_M} \{|\eta - M(x)|\}$$

where  $\mathcal{D}_z, \mathcal{D}_M$  are defined as the sets containing all possible equilibrium of  $(x, \eta)$ . On one hand, when  $\delta = \delta_{\text{nom}}$ , i.e. the system converges to the nominal equilibrium,  $\underline{d}$  can be taken equal to 0 as the feed-forward action  $\mathbf{u} = \mathbf{u}_{\text{nom}}^*$  is sufficient to bring the system to the equilibrium where regulation is achieved. On the other hand, when  $\delta \neq \delta_{\text{nom}}$ ,  $\underline{d}$  must be sufficiently high: the dead-zone constant

must be chosen concerning the plant's uncertainties so that the anti-windup effect vanishes on the (new) equilibrium point. Giving a detailed formulation of  $\underline{d}$  is quite complicated and out of the scope of the practical implementation, as its value affects the largest admissible bounds on the set-points  $\bar{\delta}$  (see Part 3 of the proof of this Theorem) and vice versa. Its computation has therefore been omitted. For practical implementations, once a bound on the possible uncertainties for the parameters and references is known (and hence a neighborhood of the nominal equilibrium), taking  $\underline{d}$  sufficiently high will guarantee that the anti-windup disappears on the new target equilibrium. Such aspects have been remarked also in the experimental part in Section 4.2.4. Taking  $\underline{d}$  too high will not compromise the stability or the regulation, but will simply result in a delay of the effect of the anti-windup.

**Part 3: Robustness analysis.** To conclude the proof of the theorem, we aim to show that for every compact set of initial conditions, there exists a bound of the parameters' uncertainty  $\bar{\delta}$  such that, if  $|\delta - \delta_{nom}| \leq \bar{\delta}$ , then the following can still be guaranteed: i) that an equilibrium point for the closed-loop system exists; ii) that such an equilibrium is asymptotically and locally exponentially stable; iii) that the regulation task is still achieved. To this end, let  $w := (\mathbf{x}, \boldsymbol{\eta})$  and let the nominal closed-loop (4.2), (4.27) be defined by

$$\dot{w} = \varphi(w, \delta_{nom}). \quad (4.32)$$

From the Lyapunov analysis in *Part 1*, there exists a radially unbounded Lyapunov function  $W$  and a positive definite function  $V$  such that

$$\frac{\partial W}{\partial \chi}(\chi) \varphi(\chi, \delta_{nom}) \leq -V(\chi) < 0$$

where  $\chi = (\mathbf{x} - \mathbf{x}_{nom}^*, \boldsymbol{\eta} - \boldsymbol{\eta}_{nom}^*)$ , for which the origin of the closed-loop is globally asymptotically stable and locally exponentially stable. Therefore, for each compact set  $\mathcal{D}$ , there exists two compact sets of initial conditions containing the origin and denoted  $\underline{\mathcal{C}}$  and  $\bar{\mathcal{C}}$ , both in  $\mathcal{X} \times \mathcal{Z}$ , such that  $\bar{\mathcal{C}}$  is forward invariant<sup>a</sup> for the closed-loop system (4.32). Therefore, by (Astolfi and Praly, 2017, Lemma 5) there exists  $\rho > 0$  such that, for each  $C^1$  vector field  $\varphi_p$  satisfying

$$|\varphi_p(\chi, \delta) - \varphi(\chi, \delta_{nom})| \leq \rho, \quad \forall \chi \in \bar{\mathcal{C}}, \quad (4.33)$$

$$\left\| \frac{\partial \varphi_p}{\partial x}(\chi, \delta) - \frac{\partial \varphi}{\partial x}(\chi, \delta_{nom}) \right\| \leq \rho, \quad \forall \chi \in \underline{\mathcal{C}}, \quad (4.34)$$

there exists an exponentially stable equilibrium of

$$\dot{\chi} = \varphi_p(\chi, \delta),$$

whose basin of attraction contains the compact set  $\bar{\mathcal{C}}$ . Let us define now the function  $\bar{\rho} : \mathcal{D} \rightarrow \mathbb{R}_{>0}$  as

$$\bar{\rho}(\delta) := \max_{\chi \in \bar{\mathcal{C}}} \left\{ |\varphi(\chi, \delta) - \varphi(\chi, \delta_{nom})|, \left\| \frac{\partial \varphi}{\partial x}(\chi, \delta) - \frac{\partial \varphi}{\partial x}(\chi, \delta_{nom}) \right\| \right\}.$$

Such a function is continuous and satisfies  $\bar{\rho}(\delta_{nom}) = 0$ . Any positive real number  $\bar{\delta} > 0$  can now be selected such that,  $|\delta - \delta_{nom}| \leq \bar{\delta}$  implies  $\bar{\rho}(\delta) \leq \rho$ .

This parameter  $\bar{\delta} > 0$  is a solution to the third part of the proof. Indeed, for each  $\delta$  such that  $|\delta - \delta_{nom}| \leq \bar{\delta}$ , then the closed-loop system

$$\dot{\chi} = \varphi(\chi, \delta),$$

satisfies (4.33) and (4.34) and consequently admits an exponentially stable equilibrium with a basin of attraction containing  $\mathcal{D} \times \{0\}$ . Thanks to the integral action, the output set-point tracking is achieved on this equilibrium, which concludes the proof.  $\square$

<sup>a</sup>Simply pick  $\bar{\mathcal{C}} = \{(x, z) : W(x, z) \leq c_0\}$  for sufficiently large  $c_0$ .

**Remark 4.2.10.** *A more precise characterization of the robustness bound  $\bar{\delta}$  can be given by explicitly computing its value. Indeed the closed-loop is a polynomial system, hence computation solvers or polynomial optimization tools (see Henrion and Lasserre (2003); Henrion et al. (2009)) could be used.*

#### 4.2.3.5 Tuning of the control gains

The proposed control law  $\alpha(\mathbf{x}, \boldsymbol{\eta})$  in (4.27) admits some free-to-choose parameters. In particular, any matrices  $P = P^\top$  satisfying (4.20),  $\Omega = \Omega^\top \succ 0$ . This section aims to give guidelines for choosing the matrices  $P$  and  $\Omega$  with respect to some cost function around the nominal equilibrium point.

Consider the nominal system: (4.2), (4.27) with  $\delta = \delta_{nom}$ . Since only local performances are sought around the nominal equilibrium, the tuning procedure is designed without taking into account the saturation function and the dead-zone anti-windup, as their effect disappears sufficiently close to the equilibrium point. Taking the nominal error coordinates  $(\mathbf{x}, \boldsymbol{\eta}) \mapsto \chi = (x, \eta) = (\mathbf{x} - \mathbf{x}_{nom}^*, \boldsymbol{\eta} - \boldsymbol{\eta}_{nom}^*)$ , the closed-loop system writes

$$\begin{aligned} \dot{x} &= Ax + (N(x) + B)u \\ \dot{\eta} &= Cx + H(x)x \\ u &= \psi(x, \eta) \end{aligned} \tag{4.35}$$

where  $u = \mathbf{u} - \mathbf{u}_{nom}^*$ ,  $A, B, N, C$  and  $H$  are taken as in Theorem 4.2.5 and  $\psi(\cdot)$  is taken accordingly to Proposition 4.2.4.

Consider the cost function  $\mathcal{J} : \mathbb{R}^{n_x} \times \mathbb{R}^m \rightarrow \mathbb{R}_{>0}$  defined as

$$\mathcal{J} := \int_0^\infty \left[ \chi^\top(t) \tilde{Q} \chi(t) + u^\top(t) \tilde{R} u(t) \right] dt \tag{4.36}$$

for some given matrices  $\tilde{Q} = \tilde{Q}^\top \succeq 0$  and  $\tilde{R} = \tilde{R} \succ 0$ , where  $\chi(t) = (x(t), \eta(t))$  is the trajectory of the closed-loop at time  $t$  for some initial conditions. Consider the linearisation of the closed-loop system (4.35) around  $\chi = (x, \eta) = (0, 0)$ :

$$\begin{aligned} \dot{\tilde{\chi}} &= \tilde{A} \tilde{\chi} + \tilde{B} \tilde{u} \\ \tilde{u} &:= \tilde{K} \tilde{\chi} \end{aligned} \tag{4.37}$$

where  $\tilde{\chi} \in \mathbb{R}^{n_x+m}$ ,  $\tilde{u} \in \mathbb{R}^m$  and

$$\begin{aligned}\tilde{A} &:= \begin{bmatrix} A & 0 \\ C & 0 \end{bmatrix}, & \tilde{B} &:= \begin{bmatrix} B \\ 0 \end{bmatrix} \\ \tilde{K} &:= -[B^\top P + B^\top M_0^\top \Omega M_0 & -B^\top M_0^\top \Omega]\end{aligned}\quad (4.38)$$

Note first that the  $M_i$  ( $i = 1, \dots, m$ ) do not play a role in this local optimization design, as they are associated with a quadratic term and hence disappear during the linearization. For a linear system of the form (4.37), a linear state-feedback optimal control law (see (Kirk, 2004, Section 3.12)) is given by

$$\tilde{u} = \tilde{K}_{opt} \tilde{\chi} = -\tilde{R}^{-1} \tilde{B}^\top \tilde{S} \tilde{\chi} \quad (4.39)$$

where  $\tilde{S} = \tilde{S}^\top \succ 0$  is the solution of the algebraic Riccati equation

$$\tilde{S} \tilde{A} + \tilde{A}^\top \tilde{S} - \tilde{S} \tilde{B} \tilde{R}^{-1} \tilde{B}^\top \tilde{S} = -\tilde{Q} \quad (4.40)$$

The idea is to choose the control law degrees of freedom such that its linearisation  $\tilde{K}$  is as close as possible to the optimal control law for linear systems  $\tilde{K}_{opt}$ . In this sense, it can be rewritten as an optimization problem of the form

$$\begin{aligned}\min_{P, \Omega, \varepsilon} & \varepsilon \\ \text{s.t.} & (\tilde{K} - \tilde{K}_{opt})(\tilde{K} - \tilde{K}_{opt})^\top - \varepsilon I \preceq 0 \\ & P \succ 0, \Omega \succ 0, \varepsilon \geq 0 \\ & A^\top P + PA \prec 0\end{aligned}\quad (4.41)$$

where  $\tilde{K}$  depends on  $P$  and  $\Omega$ , as seen in (4.38). This problem can be expressed in LMI form using the Schur complement:

$$\begin{aligned}\min_{P, \Omega, \varepsilon} & \varepsilon \\ \text{s.t.} & \begin{bmatrix} -\varepsilon I & (\tilde{K} - \tilde{K}_{opt}) \\ (\tilde{K} - \tilde{K}_{opt})^\top & -I \end{bmatrix} \preceq 0 \\ & P \succ 0, \Omega \succ 0, \varepsilon \geq 0 \\ & A^\top P + PA \prec 0\end{aligned}$$

which is a semi-definite program for which efficient solvers exist, see for instance VanAntwerp and Braatz (2000); Henrion and Lasserre (2003); Henrion et al. (2009).

**Remark 4.2.11.** *As the system's dynamics involve the plant and the integrator, in most applications there is no interest in choosing the cost function to be dependent on the full state space, i.e.  $\tilde{Q}$  is generally positive semi-definite. Hence one can select  $\tilde{Q} := \tilde{C}^\top \tilde{C}$  for some matrix  $\tilde{C}$  of appropriate dimensions, where  $(\tilde{A}, \tilde{C})$  is assumed to be detectable to still provide convergence of the closed-loop system towards an equilibrium point. Moreover, note that the plant's uncertainties play a role in this tuning and hence more advanced techniques could be used, such as robust optimal control or stochastic optimal control. As this focus is out of the main scope of this section, the details will not be explained, and the interested readers may refer for instance to Van Handel (2007); Savkin and Petersen (1995); Chen (1993) and the references therein.*

#### 4.2.4 Experimentations

This section presents the tenth-scale experimental measurements of the PFC in closed-loop with the proposed controller. The experimental setup is presented in Fig. 4.4 and 4.5. A dSPACE MicroLabBox rapid prototyping system ( $\mu$ LB) is used to control the PFC. A voltage-controlled electronic load in series with a resistor (EL+R) is connected to the first terminal of the PFC through thirty meters of standard U1000 RV2 cable ( $L_G$ ). The two other terminals are connected to a resistor (R) and a power supply (PS), also through thirty meters of the same cable.

The proposed controller is tested through different scenarios. First, it is verified that the control objectives can be properly reached, despite perturbations of the uncertain parameters. Second, a test is performed to show the efficiency of the anti-windup function. Third, an experiment illustrates the effect of the proposed tuning of the parameters. Finally, the generalization to  $m$ -terminal is illustrated by a simulation on MATLAB Simulink, for  $m = 5$ .

Unless otherwise stated, the parameters of control law (4.27) are selected as follows: the matrices  $P$  and  $M_i$ ,  $i \in \{1, \dots, m\}$  are computed using CVX, a package for specifying and solving convex programs Grant and Boyd (2014), Grant and Boyd (2008), such that

$$A(\theta_{\text{nom}})^\top P + PA(\theta_{\text{nom}}) \preceq -I,$$

$$M_i A(\theta_{\text{nom}}) + A^\top(\theta_{\text{nom}}) M_i = \frac{1}{2}(H_i + H_i^\top),$$

the function  $\psi$  in (4.27) is multiplied by a scalar positive gain  $\kappa = 0.01$  inside the saturation (see Remark 4.2.6) and the tuning matrix  $\Omega$  is chosen equal to  $\text{blkdiag}\{1, 1, 5\}$ . These values have been found by experimentally tuning the parameters to reach satisfactory dynamics.

**Robust regulation.** The system is initialised on the nominal set-point  $\delta_{\text{nom}} = (\theta_{\text{nom}}, r_{\text{nom}})$ , whose numerical values are displayed in Table 4.3. At  $t = 21$  ms, the reference is changed to  $r_a$ , and at  $t = 61.5$  ms, the system is perturbed with a voltage step on the electronic load, from 2 V to 10 V, so that  $\theta = \theta_a$  (see Table 4.3). From Fig. 4.6, it can be concluded that the objectives are properly reached in 20 ms so that modeling and measurement errors are properly compensated for. Moreover, observe that although no reference is tracked on the third line, a constant power value is asymptotically reached. This asymptotic value is determined by the total power balance so that it equals  $-\mathbf{P}_1^r - \mathbf{P}_2^r$  plus the power losses in the converter.

**Saturation and anti-windup.** Bearing in mind that the power references are given by a higher-layer control algorithm, observe that if a constant perturbation occurring on a line makes the given reference unreachable, it will induce a saturation of the duty cycle and lead to an integral windup, until the higher-layer controller can deliver a new attainable reference. This scenario has been tested and the most relevant signals are displayed in Fig. 4.7.  $\mathbf{P}_1^r$  is set to  $-85$  W, which is attainable when  $V_{G1} = 2$  V, but not when  $V_{G1} = 10$  V. Such a step on  $V_{G1}$  is implemented via the electronic load at  $t = 0.1$  s. This prevents the power reference to be reached due to duty cycle constraints. Assume that the higher-layer controller delivers a new *reachable* reference  $\mathbf{P}_1^r = -75$  W at  $t = 0.5$  s. Without anti-windup action, i.e.  $\zeta = 0$  on (4.7), the value of  $\boldsymbol{\eta}_1$  increases continuously, and the integral wind-up

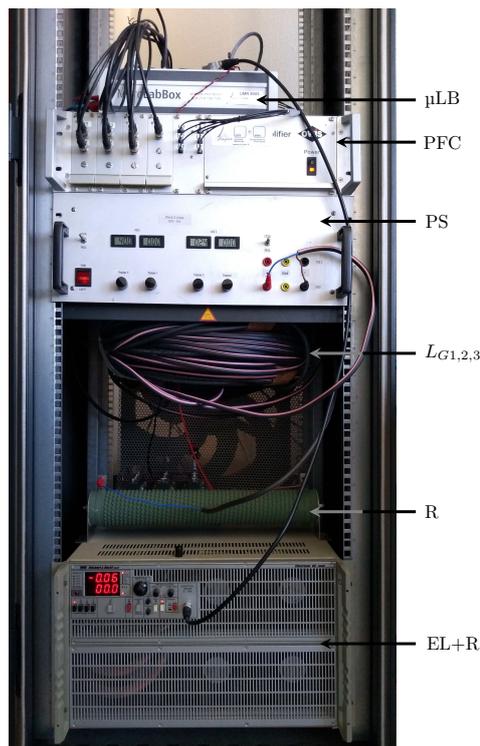


Figure 4.4: Picture of the experimental setup

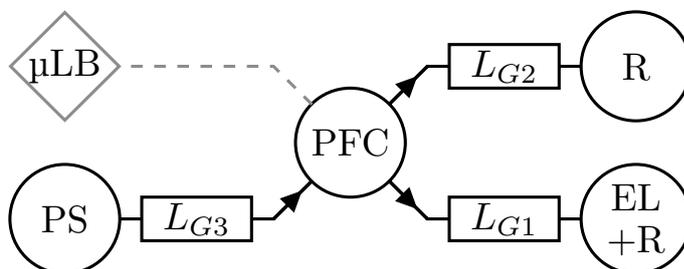


Figure 4.5: Drawing of the experimental setup

Table 4.3: Set-point numerical values

Parameter	Test value			Unit	Parameter				
	$k = 1$	$k = 2$	$k = 3$		Parameter	Test value	Unit		
$\theta_{\text{nom}}$	$L_{Gk}$	18	18	18	$\mu\text{H}$	$\mathbf{r}_{\text{nom}}$	$\mathbf{P}_1^r$	-50	W
	$R_{Gk}$	21.7	24.5	1.2	$\Omega$		$\mathbf{P}_2^r$	-50	W
	$V_{Gk}$	2	0	40	V		$\mathbf{v}_{\mathbf{R}}^r$	50	V
$\theta_a$	$L_{Gk}$	18	18	18	$\mu\text{H}$	$\mathbf{r}_a$	$\mathbf{P}_1^r$	<b>-60</b>	W
	$R_{Gk}$	21.7	24.5	1.2	$\Omega$		$\mathbf{P}_2^r$	<b>-60</b>	W
	$V_{Gk}$	<b>10</b>	0	40	V		$\mathbf{v}_{\mathbf{R}}^r$	50	V

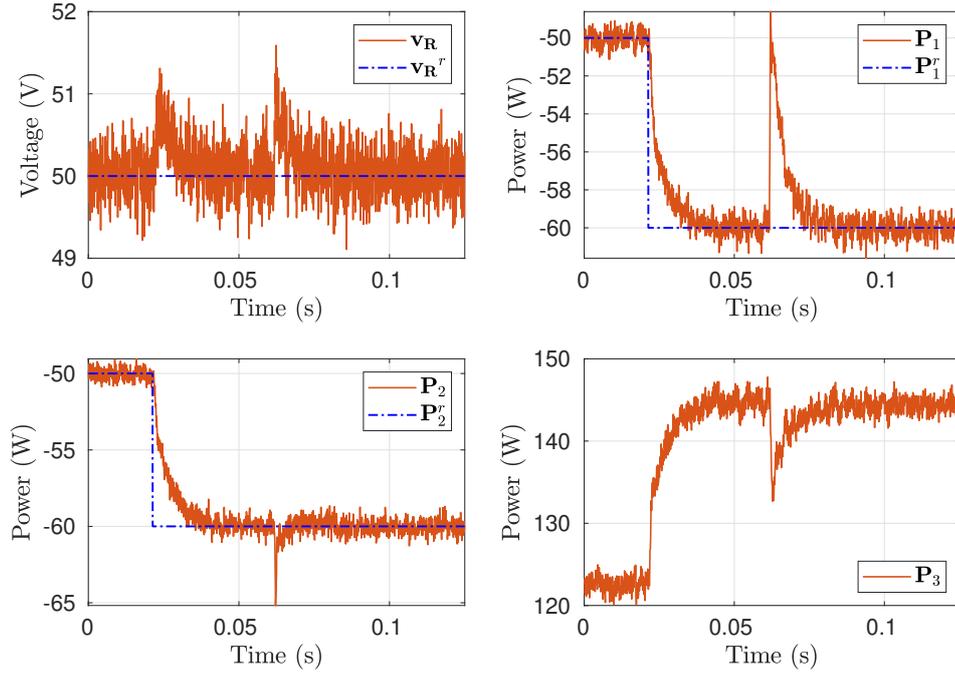


Figure 4.6: Experimental measurements of the PFC in closed-loop with control (4.27). Reservoir voltage and power in each line with their references

problem is clearly illustrated as the tracking capability is only recovered after 600 ms (in blue). This is in contrast with  $\eta_{1,AW}$  which saturates around 0.2 J, leading to a much faster recovery of the tracking capability (in red).

**Tuning of the converter parameters.** In Section 4.2.3.5, a procedure is given to assist the tuning of the control parameters: define a cost function in (4.36), solve the Riccati equation in (4.40) and deduce the control gain using (4.39). Then, chose  $P$  and  $\Omega$  along the optimisation problem (4.41). This procedure has been followed: for simplicity, the cost function is chosen using the matrices  $\tilde{Q} = I$  and  $\tilde{R} = 0.01 \times I$ . Indeed, since the inputs are saturated duty cycles, there is no need to minimize their magnitude, so the associated cost is small. The Riccati equation has been solved using the `icare`( $\cdot$ ) function in MATLAB. To evaluate the impact of  $\Omega$ , matrix  $P$  is kept unchanged while different values for  $\Omega$  are tested with the optimization problem. One then notices that for  $\Omega = \epsilon I$ , smaller values of  $\epsilon$  lead to a smaller  $\epsilon$ , i.e.  $\tilde{K}$  is closer to  $\tilde{K}_{opt}$ . This is verified during a startup test on the nominal system. Considering  $(\theta_{nom}, \mathbf{r}_{nom})$  given by Table 4.3 at  $t = 2$  ms, the reservoir voltage transients are displayed in Fig. 4.8 for different  $\epsilon$ . It can be noticed that decreasing the value of  $\Omega$  improves the dynamics on the nominal set-point. However, note that this matrix multiplies the integral part  $\boldsymbol{\eta}$  of the current law (see (4.22)). Therefore, the selection of  $\Omega$  appears to be a trade-off between nominal performance, suggesting  $\Omega = 0$ , and accuracy of the steady-state, requiring  $\Omega \succ 0$ .

**Simulation of a 5-terminal PFC.** A MATLAB-Simulink simulation validates this extension for  $m = 5$ . The two new branches have the same parameters and references as branches one and two and have been inserted between the second and the third ones, i.e. they are numbered 3 and 4. Thus, the numerical values are  $L_{G1,2,3,4,5} = 18 \mu\text{H}$ ,  $R_{G1,3} = 21.7 \Omega$ ,  $R_{G2,4} = 24.5 \Omega$ ,  $R_{G5} = 1.2 \Omega$ ,  $V_{G2,4} = 0 \text{ V}$

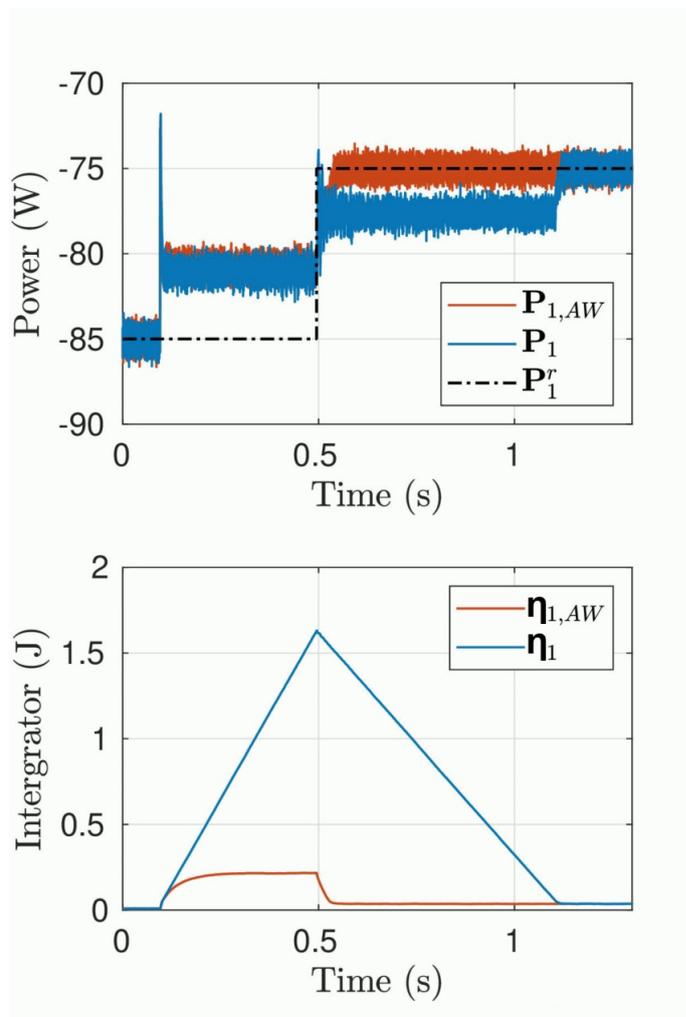


Figure 4.7: Experimental measurements illustrating the efficiency of the dead-zone function method to prevent an integral wind-up when the set-point is unreachable. Power on the first line with its reference (top) and corresponding integrator (bottom)

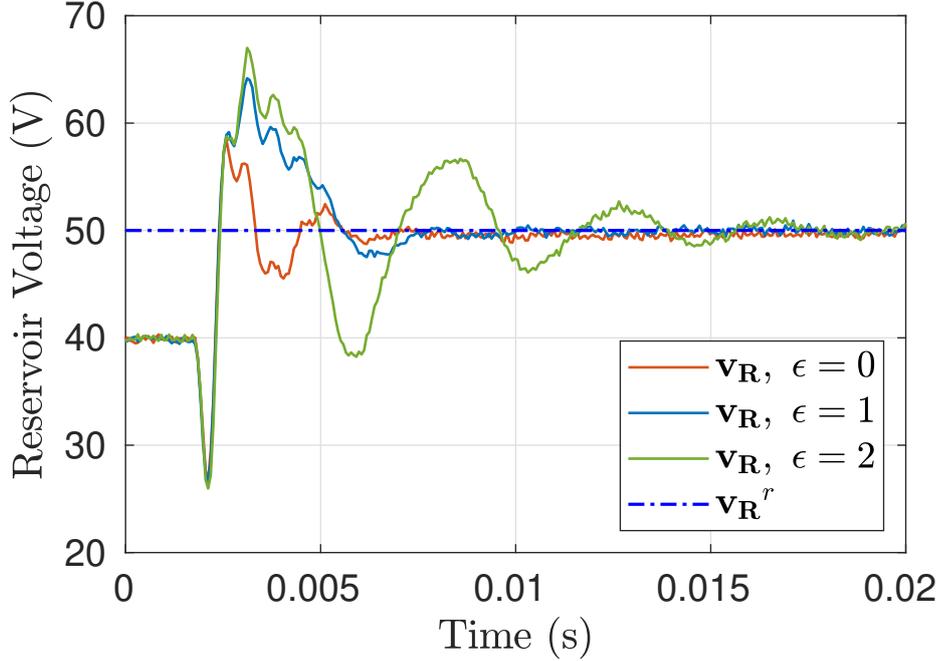


Figure 4.8: Experimental measurements of the startup of the converter on the nominal set-point for different values of  $\Omega = \epsilon I$ . Only  $\mathbf{v}_R$  is shown, but the same behavior is observed on the other outputs

and  $V_{G5} = 40$  V.  $V_{G1,3}$  is initially set to 2 V and changed to 10 V at  $t = 61.5$  ms. The power references are all initially at  $\mathbf{P}_{1,2,3,4}^r = -50$  W, and stepped to  $-60$  W at  $t = 21$  ms. The reservoir voltage reference  $\mathbf{v}_R^r$  remains at 50 V. The resulting reservoir voltage and power signals are drawn in Fig. 4.9. Their comparison to those of Fig. 4.6, shows that the proposed model can accurately capture the dynamics of the system. Furthermore, this simulation suggests that the generalization of the control law to more than three terminals maintains both the stability and the performance of the system.

#### 4.2.5 Proof of technical lemmas

In the following, we provide the proofs of Lemma 4.2.7 and of Lemma 4.2.8.

**Proof.** [of Lemma 4.2.7] To show that  $A$  is Hurwitz, it is sufficient to show that the origin of the system  $\dot{x} = Ax$  is globally asymptotically stable. Taking the quadratic Lyapunov function  $V_J(x) = x^\top \mathbf{J}(\theta)x$  yields

$$\begin{aligned} \dot{V}_J(x) &= x^\top (A^\top \mathbf{J}(\theta) + \mathbf{J}(\theta)A)x \\ &= -x^\top \text{blkdiag}\{0, R_{G1}, \dots, R_{Gm}\}x \leq 0 \end{aligned}$$

by definition of  $\Theta$ . By applying LaSalle's invariance principle, the system converges to the following set  $\{x \mid V_J(x) = 0\} = \text{Im}\{[I, 0]^\top\}$ . Pick any element  $x$  so that  $i_{Gk} = 0$ , for all  $k \in \{1, \dots, m\}$ . Plugging it in the  $i_{Gk}$ -dynamic, see (4.3a) results in

$$0 = \frac{d}{dt} i_{Gk} = -\frac{1}{L_{Gk}} v_k,$$

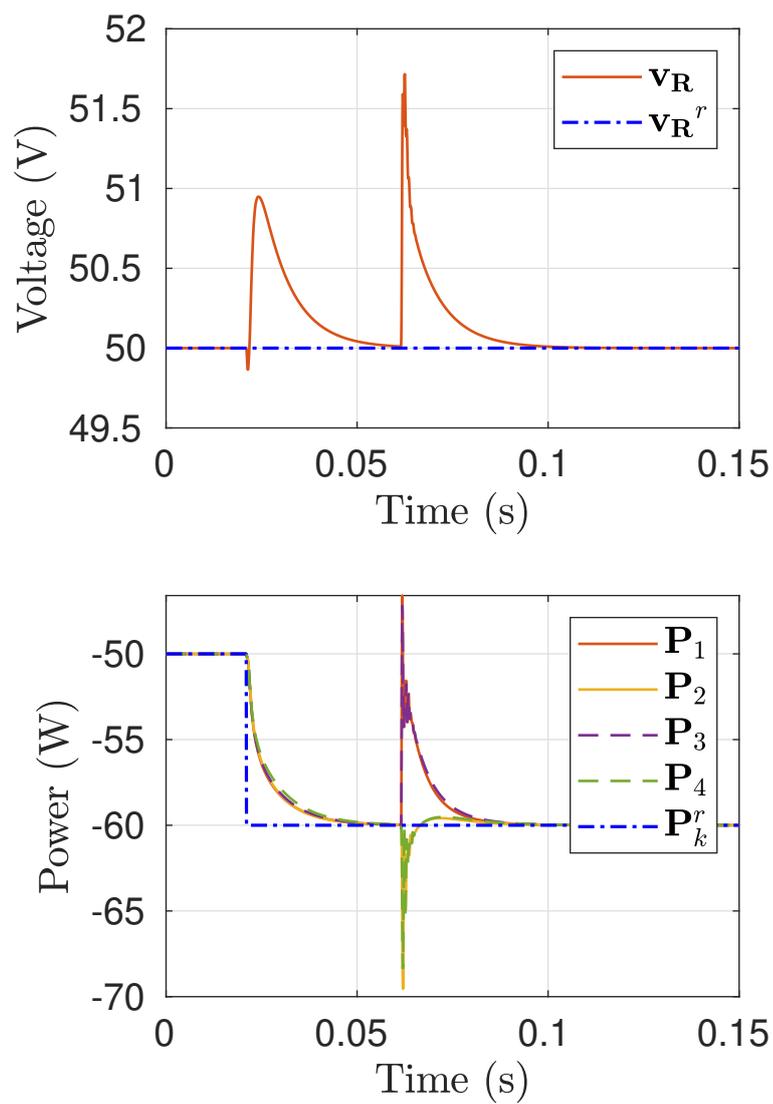


Figure 4.9: Simulation of the PFC model (4.2) for  $m = 5$ , in closed-loop with control (4.27). Reservoir voltage and power in the  $m - 1$  first lines with their references

which leads to  $v_k = 0$ . From the  $v_k$ -dynamic this means

$$0 = \frac{d}{dt} v_k = -\frac{1}{C} i_k,$$

and so  $i_k = 0$ . Finally, by looking at the  $i_k$ -dynamic, the following holds

$$0 = \frac{d}{dt} i_k = -\frac{1}{L} \mathbf{u}_k^* v_R.$$

Before concluding that  $v_R = 0$ , it is necessary to prove that  $(\mathbf{u}_1^*, \dots, \mathbf{u}_m^*) \neq (0, \dots, 0)$ . From (4.18d) and the definition of  $\mathcal{R}$ , a necessary and sufficient condition is  $(\mathbf{v}_1^*, \dots, \mathbf{v}_m^*) \neq (0, \dots, 0)$ . From (4.18a), and the definition of  $\Theta$ ,  $\mathbf{v}_k^*$  equals zero on two conditions: the first one is  $V_{Gk} = 0$  and  $\Delta_k = 0$ , which is excluded by taking  $\delta \in \text{int}\{\mathcal{S}\}$ . The second one is  $V_{Gk} - \sqrt{V_{Gk}^2 - 4R_{Gk}\mathbf{P}_k^r} = 0$ , which occurs if and only if  $\mathbf{P}_k^r = 0$ , by definition of  $\Theta$ . Yet, the definition of  $\mathcal{R}$  ensures that  $(\mathbf{P}_1^r, \dots, \mathbf{P}_{m-1}^r) \neq (0, \dots, 0)$ , meaning there is at least one  $\mathbf{u}_k^* \neq 0$  and so  $v_R \equiv 0$  for all  $(\theta, r) \in \text{int}\{\mathcal{S}\}$ , which concludes the proof of the lemma.  $\square$

**Proof.** [of Lemma 4.2.8] As the number of inputs equals the number of outputs,  $CA^{-1}B$  is full rank if and only if the following matrix is injective:

$$T = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

Let us show that  $T(x^\top, u^\top)^\top = 0$  if and only if  $(x^\top, u^\top)^\top = 0$ . We have that

$$Ax + Bu = 0$$

if and only if

$$\frac{1}{C_R} \sum_{j=1}^m \mathbf{u}_j^* i_j + \mathbf{i}_k^* u_k = 0 \quad (4.42)$$

$$\frac{1}{L} (-\mathbf{u}_k^* v_R + v_k - \mathbf{v}_R^r u_k) = 0 \quad (4.43)$$

$$\frac{1}{C} (-i_k + i_{Gk}) = 0 \quad (4.44)$$

$$\frac{1}{L_{Gk}} (-v_k - R_{Gk} i_{Gk}) = 0 \quad (4.45)$$

and

$$Cx = 0$$

if and only if

$$2 \left( \frac{\mathbf{i}_{Gk}^*}{2} v_k + \frac{\mathbf{v}_k^*}{2} i_{Gk} \right) = 0 \quad (4.46)$$

$$v_R = 0. \quad (4.47)$$

Using (4.43),(4.44),(4.45) and (4.47), one gets

$$v_k = \mathbf{v}_R^r u_k, \quad i_k = i_{Gk} \quad \text{and} \quad v_k = -R_{Gk} i_{Gk}. \quad (4.48)$$

Placing the latter in (4.46) yields

$$(-R_{Gk} \mathbf{i}_{Gk}^* + \mathbf{v}_k^*) i_{Gk} = 0.$$

Using (4.12d), we get  $-R_{Gk} \mathbf{i}_{Gk}^* = -V_{Gk} + \mathbf{v}_k^*$  which then gives

$$(-V_{Gk} + 2\mathbf{v}_k^*) i_{Gk} = 0$$

and, from (4.18a) we get  $\mathbf{v}_k^* = \frac{1}{2} \left( V_{Gk} \pm \sqrt{\Delta_k(\delta)} \right)$ . Therefore, one finds

$$\pm i_{Gk} \sqrt{\Delta_k(\theta, r)} = 0$$

for  $k \in \{1, \dots, m\}$ . In  $\text{int}\{\mathcal{S}\}$ ,  $\Delta_j(\theta, \mathbf{r}) > 0$ , hence  $i_{Gk} = 0$ . Remembering that by definition of  $\Theta$  and  $\mathcal{R}$ ,  $R_{Gk} > 0$  for  $k \in \{1, \dots, m\}$  and  $\mathbf{v}_R^r > 0$  and using (4.48), it can be concluded that  $(x^\top, u^\top)^\top = 0$  and therefore that  $T$  is full rank for all  $(\theta, \mathbf{r}) \in \text{int}\{\mathcal{S}\}$ .  $\square$

## 4.3 Harmonic regulation of a Ventilation Machine

### 4.3.1 About the Ventilation Machine

In this section, a mechanical Ventilation Machine (VM) is considered. Mechanical ventilators are essential equipment in Intensive Care Units (ICUs) to assist patients who cannot breathe on their own or need support to breathe sufficiently. A large number of patients require mechanical ventilation. According to Needham et al. (2005), 19 186 people required mechanical ventilation in ON, Canada, in 2000. Therefore, improvements in ventilation benefit a large population worldwide. The goal of mechanical ventilation is to ensure adequate oxygenation and carbon dioxide elimination, see Warner and Patel (2013), thereby sustaining the patient's life.

In this section, a Pressure Controlled Ventilation (PVC) is considered. Due to its critical importance, modeling and control of PVCs devices have been intensively studied. An overview of techniques for modeling and control of mechanical ventilation machines can be found in Borrello (2005). About the control strategies, basic techniques make use of linear time-invariant control laws. This generally results in sub-optimal behaviors, especially in terms of tracking performance. For this reason, several solutions were proposed in the last two decades. In Van De Wouw et al. (2018); Hunnekens et al. (2018) a variable control gain is proposed, but the control law makes use of measurements that are usually non-available in practice and resulted in a tracking performance with significant overshoots. In Borrello (2001) an adaptive feedback control is presented. The drawback, in this case, is that the control action is based on strong assumptions about the model's knowledge, a task generally difficult. In Pomprapa et al. (2015) a funnel-based control is implemented, achieving however limited performances in terms of tracking. In Scheel et al. (2017) and Li and Haddad (2012) a model-based control and, respectively, a model predictive control are provided. Both such control strategies require however the knowledge of a set of parameters representing the patient's dynamics, a piece of information

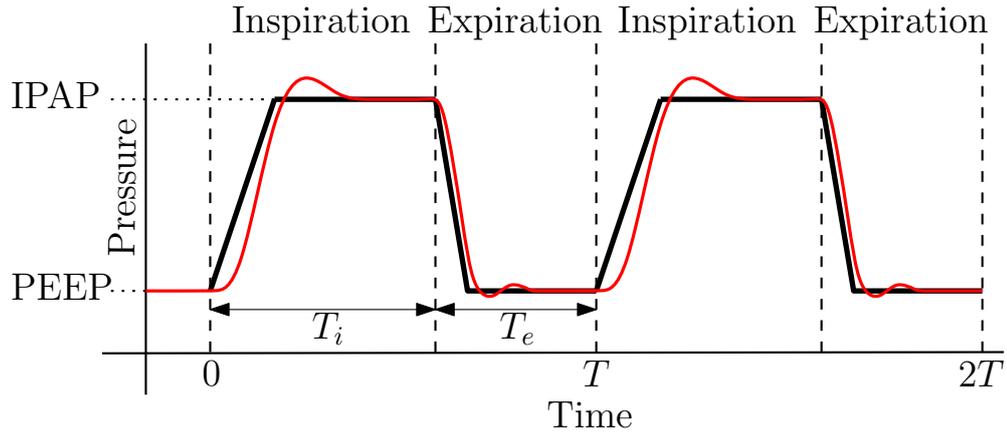


Figure 4.10: Typical airway pressure for two breathing cycles of pressure-controlled ventilation, showing the set-point (—) and the typical response (—).

that is generally not perfectly known due to the different typologies of patients. An iterative control strategy is presented in Scheel et al. (2015).

Giving a detailed analysis of existing results and strategies to model and control PVCs devices is out of the scope of the chapter, and hence it's omitted. The interested reader may refer to Borrello (2005); Reinders et al. (2020) and references therein. What is important to highlight, is how the control of the VM assuring PVC is of interest in the scope of this manuscript. From a structural point of view, the PVC system is driven by a blower. Such a blower compresses the air to achieve the desired pressure reference profile in the patient's lungs. The airway pressure is increased during inspiration, in order to achieve the Inspiratory Positive Airway Pressure (IPAP), filling the patient's lungs with air. Then, the blower decreases the pressure to the Positive End-Expiratory Pressure (PEEP), so that the air in the lungs comes out. A figure representing the respiratory phase can be seen in Figure 4.10. From a control viewpoint, the objective of a VM is to achieve tracking of a signal. Such a signal represents the pressure in the patient's lungs during the breathing phase. Therefore, it is a periodic signal.

The considered ventilation machine system can be modeled as a nonlinear dynamical system in Lur'e form. Lur'e-type systems consist of the interconnection of linear time-invariant dynamics with a static nonlinearity in the feedback loop. The main components of this system are the blower, the hose-filter system, and the patient. A centrifugal blower compresses ambient air to achieve the desired blower outlet pressure  $p_{out}$ . The difference between  $p_{out}$  and the airway pressure  $p_{aw}$  results in the outlet flow  $Q_{out}$  through the hose. This hose is modeled using a nonlinear hose model. The flow through the hose, i.e., the outlet flow  $Q_{out}$ , is divided into a patient flow  $Q_{pat}$  and a leak flow  $Q_{leak}$ . The intended leak near the patient is used to flush  $\text{CO}_2$ -rich air from the system. Finally, the patient's lungs are inflated and deflated by the patient's flow. A block-scheme structure can be seen in Figure 4.11.

In the PCV system the pressure near the patient's mouth, the airway pressure  $p_{aw}$ , should track a desired pressure target reference  $p_{target}$ , i.e.,  $r := p_{target}$ . On a preset periodic interval, of length  $T$ , the pressure level is increased to the IPAP and consequently lowered to the PEEP. These varying pressure levels ensure the desired airflow in and out of the patient's lungs. The total breath length  $T$  consists of the inspiration time  $T_i$  and expiration time  $T_e$ , i.e.,  $T = T_i + T_e$ . The control goal for the

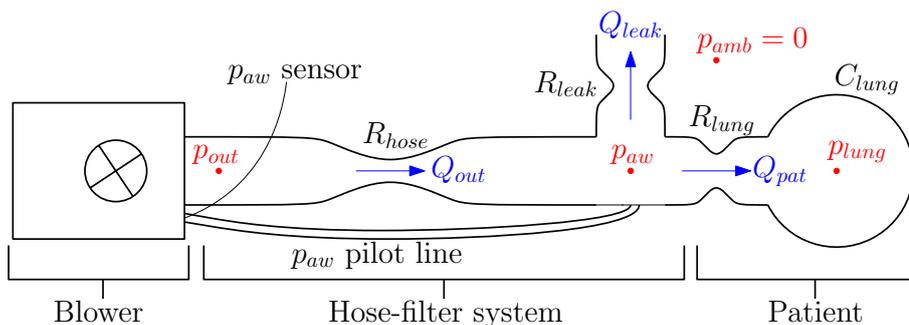


Figure 4.11: Schematic representation of the blower-hose-patient system, with the corresponding resistances, lung compliance, pressures, and flows.

PCV system is to achieve a small tracking error  $e := r - p_{aw}$ , where the reference  $r(t)$  is a time-varying signal that is perfectly periodic with an interval length  $T$ , i.e.,  $r(t) = r(t + T)$  for a known  $T > 0$  and all  $t \geq 0$ , see Figure 4.10.

Because of this periodicity property of the reference to be tracked and the nonlinear nature of the hose model, the harmonic regulation tool presented in Section 2.3 is particularly effective for the application. The control strategy would be therefore to extend the plant's dynamics with a finite number  $L$  of linear oscillators in the control loop. This results in  $L$  poles on the imaginary axes at the frequency of the periodic reference and its multiples. Therewith, if the resulting closed-loop trajectories converge to a periodic solution, *harmonic regulation* of the tracking error is guaranteed. More precisely, the Fourier coefficients of the steady-state trajectory of the error signal corresponding to the frequencies embedded in the linear oscillators are zero. As experiments will show, in this case, also the  $\mathcal{L}^2$ -norm of the error signal will be significantly reduced for a value of  $L$  large enough. To guarantee the existence of globally asymptotically stable periodic solutions, incremental properties are exploited. To this end, we suppose that the static nonlinearity in the Lur'e-type system satisfies an incremental sector-bound condition. Then, using an incremental version of the Circle Criterion, sufficient conditions for the design of an output-feedback control law achieving incremental uniform global exponential stability are established. From a practical point of view, such an approach is interesting because the conditions can be checked by visual inspection of the Nyquist plot and linear analysis tools (potentially using measured data only).

Summarizing, the main contributions of this section are:

- a harmonic regulation strategy for nonlinear Lur'e-type systems including a formal circle criterion-based incremental stability analysis, and
- the implementation and analysis of this RC scheme on the practical use-case of a nonlinear mechanical ventilator, including experimental validation.

### 4.3.2 Model of the Ventilation Machine

For controller design and stability analysis, a mathematical Lur'e-type system model is derived. The ventilation model is based on Reinders et al. (2020). First, the separate models for the plant components are derived, i.e., blower model  $G_b$ , hose model  $R_{hose}$ , and patient-leak model  $G_p$ . Thereafter, these models are combined to obtain the open-loop Lur'e-type ventilation system model for the controller design and associated stability analysis. The complete plant and the considered control

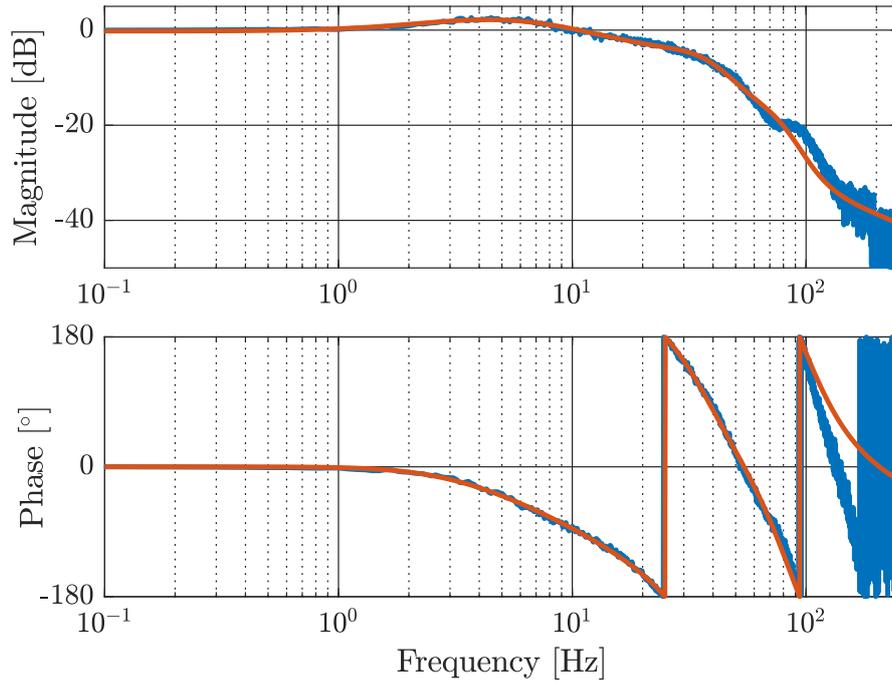


Figure 4.12: Frequency response measurement (—) and 6<sup>th</sup> order identified parametric model (—) of the blower, i.e., from  $p_c$  to  $p_{out}$ .

strategy are visualized in the block diagram in Fig. 4.13.

The blower model  $G_b$  is obtained using a sixth-order fit of a Frequency Response Measurement (FRF) of the actual blower dynamics, see Pintelon and Schoukens (2012). This state-space model accurately describes the input-output relation of the blower, i.e., from the control signal  $p_c$  to the blower output  $p_{out}$ . The measured FRF and the blower model  $G_b$  are depicted in Fig. 4.12, showing that  $G_b$  is an accurate representation of the FRF measurement of the actual blower. The blower  $G_b$  is modeled as the following state-space system:

$$\begin{aligned} \dot{x}_b &= A_b x_b + B_b p_c \\ p_{out} &= C_b x_b, \end{aligned} \quad (4.49)$$

with  $x_b \in \mathbb{R}^6$ ,  $p_c \in \mathbb{R}$ ,  $p_{out} \in \mathbb{R}$ , and system matrices  $A_b, B_b, C_b$  of appropriate dimensions.

The hose is modeled by the nonlinear hose resistance  $R_{hose}$ , which describes the relationship between the flow through the hose  $Q_{out}$  and the pressure drop over the hose  $\Delta p := p_{out} - p_{aw}$ . From experiments it is concluded that the hose can be modeled as follows:

$$\begin{aligned} Q_{out} &:= R_{hose}(\Delta p) \\ &= \text{sign}(\Delta p) \frac{-R_1 + \sqrt{R_1^2 + 4R_2|\Delta p|}}{2R_2}, \end{aligned} \quad (4.50)$$

where  $R_1$  and  $R_2$  are the hose-resistance parameters.

Next, the combined patient-leak model  $G_p$  describes the relationship between the outlet flow  $Q_{out}$  and the system output  $y = p_{aw}$ . This patient model is described by

the following first-order state-space model, based on the linear one-compartmental lung model in Bates (2009):

$$\begin{aligned}\dot{p}_{lung} &= a_p p_{lung} + b_p Q_{out} \\ p_{aw} &= c_p p_{lung} + d_p Q_{out}\end{aligned}\tag{4.51}$$

with

$$\begin{aligned}a_p &= -\frac{1}{C_{lung}(R_{leak} + R_{lung})}, \\ b_p &= \frac{R_{leak}}{C_{lung}(R_{leak} + R_{lung})}, \\ c_p &= \frac{R_{leak}}{R_{leak} + R_{lung}}, \\ d_p &= \frac{R_{leak}R_{lung}}{R_{leak} + R_{lung}}.\end{aligned}\tag{4.52}$$

Finally, these separate models are combined to obtain the open-loop plant model, as depicted inside the dashed box in Fig. 4.13. Note that an additional term  $\ell\Delta p$  is added to the nonlinear hose-resistance, i.e.,  $\varphi(\Delta p) := R_{hose}(\Delta p) + \ell\Delta p$ , and subtracted in the parallel path. This is included to ensure that the linear dynamics of the open-loop plant in Lur'e-type form are controllable and observable. The total system's dynamics, i.e., the full Lur'e-type ventilation system, are independent of the choice of  $\ell \in \mathbb{R}$ .

To obtain the open-loop plant model, the blower, hose, and patient models are combined. This gives the open-loop model from  $p_c$  to  $p_{aw}$  in the Lur'e form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{E}\mathbf{w} \\ \mathbf{y} &= \mathbf{M}\mathbf{x} + \mathbf{N}\mathbf{w} \\ \mathbf{w} &= -\varphi(\mathbf{y}) \\ \mathbf{v} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{w},\end{aligned}\tag{4.53}$$

where

$$\begin{aligned}\mathbf{A} &= \begin{bmatrix} A_b & 0 \\ -(1 - \ell d_p)^{-1} \ell C_b b_p & a_p + \ell c_p (1 - \ell d_p)^{-1} b_p \end{bmatrix}, \\ \mathbf{B} &= \begin{bmatrix} B_b \\ 0 \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} 0 \\ -b_p (1 - \ell d_p)^{-1} \end{bmatrix}, \\ \mathbf{M} &= \begin{bmatrix} C_b + d_p (1 - \ell d_p)^{-1} \ell C_b \\ -c_p - d_p (1 - \ell d_p)^{-1} \ell c_p \end{bmatrix}^T, \\ \mathbf{N} &= d_p (1 - \ell d_p)^{-1}, \quad \mathbf{D} = -d_p (1 - \ell d_p)^{-1}, \\ \mathbf{C} &= \begin{bmatrix} -d_p (1 - \ell d_p)^{-1} \ell C_b & c_p + d_p (1 - \ell d_p)^{-1} \ell c_p \end{bmatrix},\end{aligned}\tag{4.54}$$

and the nonlinearity

$$\varphi(\mathbf{y}) := R_{hose}(\mathbf{y}) + \ell \mathbf{y}.\tag{4.55}$$

where  $\ell \in \mathbb{R}$  is a degree of freedom.

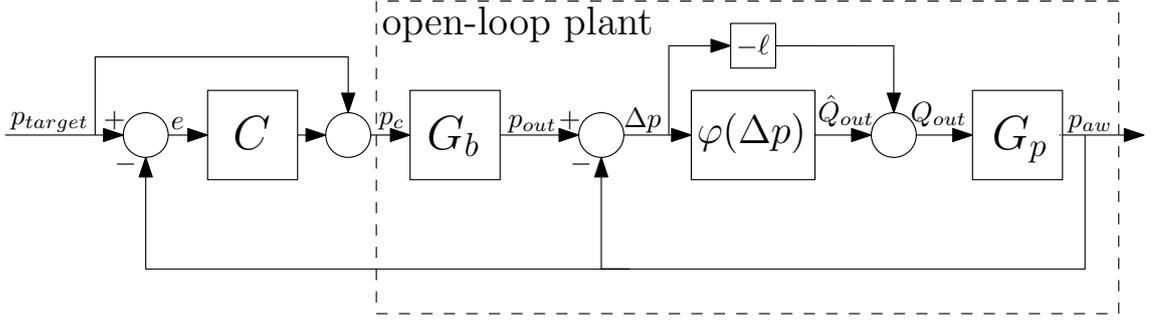


Figure 4.13: Block diagram of the full ventilation system, with  $G_b$  the blower dynamics,  $C$  an arbitrary feedback controller,  $G_p$  the patient-leak dynamics, and  $\varphi(\Delta p) = R_{hose}(\Delta p) + \ell\Delta p$  the nonlinear hose model.

### 4.3.3 Harmonic regulation for Lur'e systems: an incremental circle criterion

The purpose of this section is to design a controller achieving harmonic regulation for the class of Lur'e systems. In particular, we consider a Single-Input Single-Output (SISO) Lur'e-type system of the form (4.53) where  $\mathbf{x} \in \mathbb{R}^{n_x}$  is the state,  $u \in \mathbb{R}$  is the control input,  $\mathbf{w}, \mathbf{y}$  are in  $\mathbb{R}$ ,  $\mathbf{v} \in \mathbb{R}$  is the measured output, and  $\mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{M}, \mathbf{N}, \mathbf{C}$ , and  $\mathbf{D}$  are real matrices of appropriate dimensions. The static nonlinearity  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  satisfies  $\varphi(0) = 0$ , and the following incremental sector bound condition:

$$\underline{\varphi} \leq \frac{\varphi(\mathbf{y}_1) - \varphi(\mathbf{y}_2)}{\mathbf{y}_1 - \mathbf{y}_2} \leq \bar{\varphi}, \quad (4.56)$$

for all  $\mathbf{y}_1 \neq \mathbf{y}_2$ , for some known non-negative constants  $0 \leq \underline{\varphi} \leq \bar{\varphi}$ . The control objective is to regulate the output  $\mathbf{v}$  of the system (4.53) to a  $T$ -periodic bounded reference  $r$ . Hence, the output regulation error is defined as

$$e(t) := r(t) - \mathbf{v}(t). \quad (4.57)$$

By using similar tools as in Section 2.3, the design consists in adding a bunch of  $L$  linear oscillators at the reference's frequencies and its multiples, and a feedback stabilizer for the closed-loop system. In particular, we consider a dynamic controller of the form

$$\dot{\eta} = \phi(\eta, e) := \Phi\eta + \Gamma e \quad (4.58)$$

$$u = \alpha(\eta, e) := K\eta \quad (4.59)$$

where  $\eta = [\eta_0 \ \eta_1^T \ \dots \ \eta_L^T]^T \in \mathbb{R}^{(2L+1) \times 1}$  with  $\eta_0 \in \mathbb{R}$  and  $\eta_k \in \mathbb{R}^{2 \times 1}$  for  $k = 1, \dots, L$ , and where the matrices  $\Phi \in \mathbb{R}^{(2L+1) \times (2L+1)}$ ,  $\Gamma \in \mathbb{R}^{(2L+1) \times 1}$ , and  $K \in \mathbb{R}^{1 \times (2L+1)}$  are defined as

$$\begin{aligned} \Phi &:= \text{blkdiag}(0 \ \phi_1 \ \dots \ \phi_L), \\ \Gamma &:= [\gamma_0 \ \gamma_1^T \ \dots \ \gamma_L^T]^T, \\ K &:= [\kappa_0 \ \kappa_1 \ \dots \ \kappa_L], \end{aligned} \quad (4.60)$$

where

$$\phi_k := k \begin{bmatrix} 0 & \frac{2\pi}{T} \\ -\frac{2\pi}{T} & 0 \end{bmatrix}, \quad k = 1, \dots, L, \quad (4.61)$$

with  $\gamma_0 \neq 0$  the integrator gain such that  $\eta_0$  embeds an integrator, and the matrix  $\gamma_k \in \mathbb{R}^{2 \times 1}$  is chosen such that the pair  $(\phi_k, \gamma_k)$  is controllable for any  $k = 1, \dots, L$ .

Next, the closed-loop system consisting of the plant (4.53), (4.57) and the repetitive controller (4.58), (4.59) is written as a Lur'e-type system

$$\begin{aligned}\dot{\chi} &= A_0\chi + E_0\mathbf{w} + d(t) \\ y_0 &= M_0\chi + N_0\mathbf{w} \\ \mathbf{w} &= -\varphi(y_0)\end{aligned}\tag{4.62}$$

where

$$\begin{aligned}A_0 &:= \begin{bmatrix} \mathbf{A} & \mathbf{BK} \\ -\Gamma\mathbf{C} & \Phi \end{bmatrix}, \quad E_0 := \begin{bmatrix} \mathbf{E} \\ -\Gamma\mathbf{D} \end{bmatrix}, \quad Q_0 := \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, \\ M_0 &:= [\mathbf{M} \ 0], \quad N_0 := \mathbf{N},\end{aligned}\tag{4.63}$$

where  $\chi := [\mathbf{x}^\top, \eta^\top]^\top \in \mathbb{R}^{n_x}$  and  $d(t) := Q_0 r(t)$ , is a periodic, with period time  $T$ , time-varying piece-wise continuous disturbance (induced by the periodic reference).

Next, a loop-transformation as described in (Khalil, 2002, Chapter 7) is applied to the closed-loop dynamics. This loop-transformation gives an equivalent Lur'e-type system where the transformed nonlinearity  $\tilde{\varphi}(y)$  satisfies the incremental sector bound in (4.56) with  $\tilde{\varphi} = 0$  and  $\tilde{\varphi} = \infty$ . This loop-transformation gives the following loop-transformed Lur'e-type system:

$$\begin{aligned}\dot{x} &= Ax + Ew + d(t) \\ y &= Mx + Nw \\ w &= -\tilde{\varphi}(y)\end{aligned}\tag{4.64}$$

where

$$\begin{aligned}A &:= A_0 - (E_0\underline{\varphi}(M_0 + N_0(1 + \underline{\varphi}N_0)^{-1}\underline{\varphi}M_0)), \\ E &:= E_0(1 - \underline{\varphi}D_0(1 + \underline{\varphi}N_0)^{-1}), \\ M &:= \varrho M_0 - \varrho N_0(1 + \underline{\varphi}N_0)^{-1}\underline{\varphi}M_0, \\ N &:= 1 + \varrho N_0(1 + \underline{\varphi}N_0)^{-1}\end{aligned}\tag{4.65}$$

where  $x \in \mathbb{R}^{n_x}$ ,  $\varrho = \bar{\varphi} - \underline{\varphi}$ , and  $\tilde{\varphi}(y)$  satisfies the incremental sector bound in (4.56) with  $\tilde{\varphi} = 0$  and  $\tilde{\varphi} = \infty$ , namely

$$0 \leq \frac{\tilde{\varphi}(y_1) - \tilde{\varphi}(y_2)}{y_1 - y_2} \leq \infty.\tag{4.66}$$

We assume that the following holds.

**Assumption 4.3.1** (Controllability and observability). *Consider system (4.64). The pair  $(A, E)$  is controllable and the pair  $(A, M)$  is observable.*

Controllability and observability imply that the quadruplet  $\{A, E, M, N\}$  admits a transfer function

$$G = M(sI - A)^{-1}E + N\tag{4.67}$$

that represents a minimal realization<sup>1</sup> of the linear part of the system, where  $s$  here indicates the Laplace variables.

**Definition 4.3.1** (Strictly positive real transfer function). *Let  $j$  indicates the imaginary number, i.e.  $j^2 = -1$ . We say that the transfer function (4.67) is strictly positive real if:*

- all the poles of  $G$  have non-positive real part;
- for all real  $\omega$  for which  $j\omega$  is not a pole of any element of  $G$ , the matrix  $G(j\omega) + G^\top(j\omega)$  is positive semi-definite;
- any pure imaginary pole  $j\omega$  of  $G$  is a simple pole and the residue matrix

$$\lim_{s \rightarrow j\omega} (s - j\omega)G(s)$$

is positive semi-definite Hermitian<sup>2</sup>

An important result claiming that a matrix is strictly positive real is the so-called Strictly Positive Real (SPR) lemma (also called the Kalman-Yakubovic-Popov lemma). Such a result is recalled in the following, whose proof can be found in (Khalil, 2002, Lemma 6.3)

**Lemma 4.3.1** (Strictly Positive Real). *Consider the transfer function (4.67) and let Assumption 4.3.1 hold. Then,  $G$  is strictly positive real if and only if there exists a strictly positive matrix  $P = P^\top \succ 0$ , two matrices  $L, W$ , and a strictly positive real number  $\varepsilon > 0$  such that*

$$PA + A^\top P = -\varepsilon P - L^\top L, \quad (4.68a)$$

$$PE = M^\top - L^\top W, \quad (4.68b)$$

$$W^\top W = N + N^\top. \quad (4.68c)$$

In light of the results of Section 2.3 and the design of the internal model unit as in (4.58), (4.60), (4.61), in order to achieve harmonic regulation, it is sufficient to prove that the system is incrementally globally uniformly ISS with respect to  $d$ . To this end, by means of Theorem 1.3.3, it's sufficient to show that system (4.64) with  $d(t) = 0$  is incrementally uniformly globally exponentially stable with respect to a constant metric  $P$ . To have such a result, we will present in the following an incremental version of the Circle Criterion (see (Khalil, 2002, Theorem 7.1) for a non-incremental version). The main intuition is that, if the transfer function (4.67) is strictly positive real, then the matrix  $P$  in (4.68a) can be used as a metric to claim incremental stability by Theorem 1.2.2.

**Proposition 4.3.2** (Incremental Circle Criterion). *Consider system (4.64) with the nonlinearity satisfying (4.66) and let Assumption 4.3.1 hold. If the transfer function (4.67) is strictly positive real, then system (4.64) is incrementally uniformly globally exponentially ISS with respect to  $d$ .*

<sup>1</sup>A transfer function is a minimal realization if it does not exist another realization that has a smaller dimension. It implies that the description of a linear system in state space is equivalent to the one with the transfer function.

<sup>2</sup>A complex matrix  $A \in \mathbb{C}^{n \times n}$  is said to be Hermitian if it's equal to its conjugate transpose

**Proof.** As said, by Theorem 1.3.3, if the system is incrementally globally uniformly exponentially stable with respect to a constant metric  $P$ , then the incremental uniform global ISS property holds. Consider for this the Euclidean metric  $P = P^\top \succ 0$  where  $P$  solves (4.68a). Let  $L_\psi P(x)$  indicate the lie derivative of (4.64) with respect to  $P$ . By using (4.68a) and (4.68b) it follows that

$$\begin{aligned} L_\psi P(x) &= \text{He} \left\{ PA + PE \frac{\partial w}{\partial x}(y) \right\} \\ &= \text{He} \left\{ -\frac{\varepsilon}{2}P - \frac{1}{2}L^\top L + (M^\top - L^\top W) \frac{\partial w}{\partial x}(y) \right\} \end{aligned}$$

By adding and subtracting the term  $\frac{\partial w^\top}{\partial x}(y)N \frac{\partial w}{\partial x}(y)$  and by using (4.68c) we obtain

$$\begin{aligned} L_\psi P(x) &= \text{He} \left\{ -\frac{\varepsilon}{2}P - \frac{1}{2}L^\top L + (M^\top - L^\top W) \frac{\partial w}{\partial x}(y) \pm \frac{\partial w^\top}{\partial x}(y)N \frac{\partial w}{\partial x}(y) \right\} \\ &= \text{He} \left\{ -\frac{\varepsilon}{2}P - \frac{1}{2}L^\top L + \left( M + N \frac{\partial w}{\partial x}(y) \right)^\top \frac{\partial w}{\partial x}(y) \right. \\ &\quad \left. - L^\top W \frac{\partial w}{\partial x}(y) - \frac{1}{2} \frac{\partial w^\top}{\partial x}(y)(N + N^\top) \frac{\partial w}{\partial x}(y) \right\} \\ &= \text{He} \left\{ -\frac{\varepsilon}{2}P + \left( M + N \frac{\partial w}{\partial x}(y) \right)^\top \frac{\partial w}{\partial x}(y) - \frac{1}{2} \left( L + W \frac{\partial w}{\partial x}(y) \right)^\top \left( L + W \frac{\partial w}{\partial x}(y) \right) \right\} \end{aligned}$$

Note however that

$$\begin{aligned} \left( M + N \frac{\partial w}{\partial x}(y) \right)^\top \frac{\partial w}{\partial x}(y) &= \frac{\partial y^\top}{\partial x}(x, w) \frac{\partial w}{\partial x}(y) \\ &= -\frac{\partial y^\top}{\partial x}(x, w) \frac{\partial \tilde{\varphi}}{\partial x}(y) \\ &= -\frac{\partial y^\top}{\partial x}(x, w) \frac{\partial \tilde{\varphi}}{\partial y}(y) \frac{\partial y}{\partial x}(x, w) \end{aligned}$$

Employing the incremental sector bound condition in (4.66), the result follows.

□

A consequence is the following.

**Corollary 4.3.2.1.** *Consider system (4.53) with (4.54) coupled with the regulation error (4.57) and assume that the nonlinearity satisfies (4.56). Consider the dynamic control law (4.58), (4.59), (4.60), (4.61) and perform the coordinates change (4.62), (4.63), (4.64). If Assumption 4.3.1 holds and the transfer function (4.67) is strictly positive real, then the harmonic regulation problem is solved.*

**Remark 4.3.3.** *An important feature of the proposed design is that the Strict Positive Real assumption of the transfer function (4.67) can be visually checked through the Nyquist plot, similarly to frequency-domain designs techniques of linear systems control design. This provides a method to verify the conditions of Proposition 4.3.2*

that is very simple, especially in practical cases such as the ventilation machine.

**Remark 4.3.4.** *On the other hand, a drawback of the proposed design is that it does not provide a constructive method to choose the gain  $K$  in (4.59), resulting in a ‘try-and-check’ condition, in which the control engineer has to choose a-priori the value of  $K$  and see if the conditions of the propositions are satisfied. For the particular case of the ventilation machine, such a constructive design cannot be easily provided because of the non-linearity appearing in the output  $\mathbf{v}$  in (4.53). The reason is easy to see. The  $\mathbf{x}$  dynamics is passive with respect to the nonlinear term. This because of the strictly positive real lemma and by (Khalil, 2002, Lemma 6.4). At the same time, the internal model composed of the oscillators is passive with respect to the regulation error  $e$ . If the output  $\mathbf{v}$  had been linear, it would have resulted in a feedback of passive systems, which is again passive. However, the nonlinearity appears also in the output. This results in a triangular crossed structure, in which it is not in general possible to apply the composition rules for passive systems without explicitly choosing a suitable storage function.*

#### 4.3.4 Experimentations

In this section, we aim to apply the design proposed in Section 4.3.3 to the practical case of the ventilation machine. The main components of the experimental setup used in this case study are depicted in Fig. 4.14. The figure shows the Macawi blower-driven mechanical ventilation module DEMCON macawi respiratory systems (2021). The dSPACE system (dSPACE GmbH, Paderborn, Germany) is used to implement the controls in MATLAB Simulink (MathWorks, Natick, MA). Furthermore, the ASL 5000™ Breathing Simulator (IngMar Medical, Pittsburgh, PA) represents the patient. This lung simulator can be used to emulate a wide variety of patients with a linear resistance and compliance. Furthermore, a typical ventilation hose with a leak is used to attach the ventilation module to the lung simulator. The system parameters that are used for the stability analysis are shown in Table 4.4. The leak and hose parameters are obtained by calibration and the patient parameters are the settings used on the mechanic lung simulator, i.e., the patient emulator in Fig. 4.14.

The analysis in the following section is done using a continuous-time representation of the controller and plant model. However, the controller is implemented in dSPACE using a discrete-time representation of the continuous-time control strategy. The discrete-time controllers are obtained using the zero-order hold discretization scheme at a sampling frequency of 500 Hz. This sampling frequency is significantly higher than the relevant system dynamics, e.g., the blower shows strong roll-off at frequencies above 10 Hz. Furthermore, 500 Hz is significantly higher than the frequency content of the reference signal. Therefore, the continuous-time controller design and stability analysis is deemed relevant for this application.

**Controller design for mechanical ventilation** Next, the controller design for the mechanical ventilation is described. For the design of the feedback controller  $C$  in Fig. 4.13, the control strategy in (4.58), (4.60), and (4.61) with feedback law (4.59) is followed. This means that the feedback controller  $C$  consists of an integrator and  $L$  oscillators from the first up until the  $L^{\text{th}}$  harmonic of the breathing frequency  $\omega_b = \frac{2\pi}{T}$  rad/s. Besides this feedback controller, a unity feedforward controller as depicted in Fig. 4.13 is used. The unity feedforward term is included to improve the overall regulation accuracy. Note that it does not affect stability since

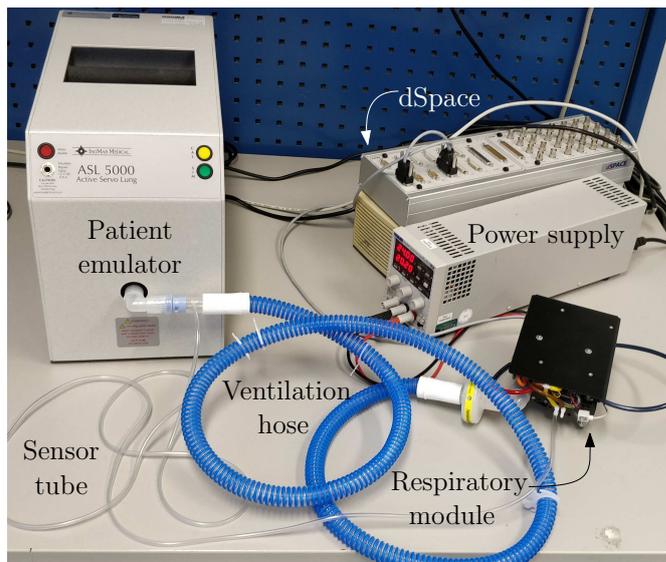


Figure 4.14: The experimental setup with the mechanical patient simulator, the respiratory module, ventilation hose, and dSPACE module.

Table 4.4: The relevant system and experiment parameter for the stability analysis.

Parameter	Value	Unit
$R_{leak}$	24	mbar s/L
$R_1$	2.8	mbar s/L
$R_2$	1.6	mbar s <sup>2</sup> /L <sup>2</sup>
$R_{lung}$	5	mbar s/L
$C_{lung}$	50	mL/mbar
$\ell$	-0.1	mL/s/mbar

it is included in the closed-loop ventilation system through the disturbance term  $d$  in (4.62). The stability analysis is independent of this disturbance in view of the convergence properties of the closed-loop dynamics.

For the final RC design, different controllers are designed to analyze the effect of the number of oscillators, i.e.,  $L \in \{0, 1, 5, 15, 20\}$ . We select the integrator gain as  $\gamma_0 = 2\pi$ , and oscillator gains as  $\gamma_k = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{2}{k^{1+\epsilon}}$  with  $\epsilon = 0.4$ , for  $k = 1, 2, \dots, L$ . The feedback law is chosen as  $K \in \mathbb{R}^{1 \times (2L+1)}$  with all entries 1. Next, the stability properties of the closed-loop ventilation system with the RC controller are analyzed.

**Stability analysis.** To guarantee exponential convergence of the closed-loop ventilation system, Proposition 4.3.2 is verified. First of all, the controlled system is written in the closed-loop form of (4.62), and the upper  $\bar{\varphi}$  and lower  $\underline{\varphi}$  sector bounds of the nonlinearity  $\varphi(y)$  in (4.55) are computed. Using these bounds, the loop-transformation is applied to obtain the system in (4.64). The upper sector bound  $\bar{\varphi}$  is defined by taking the derivative of  $\varphi(\Delta p)$  at the origin, where the slope of  $\varphi$  is the largest, see Fig. 4.15, which gives  $\bar{\varphi} = \frac{1}{R_1} + \ell$ . The lower sector bound  $\underline{\varphi}$  is obtained from visual inspection, such that it holds on a finite domain of  $\Delta p \in [-20, 20]$  mbar; this domain is sufficient for the practical application of ventilation. This leads to the sector  $\varphi \in [\underline{\varphi}, \bar{\varphi}] = [80, \frac{1}{R_1} + \ell]$  for the nonlinearity in (4.55). The nonlinearity and these sector bounds are visualized in Fig. 4.15.

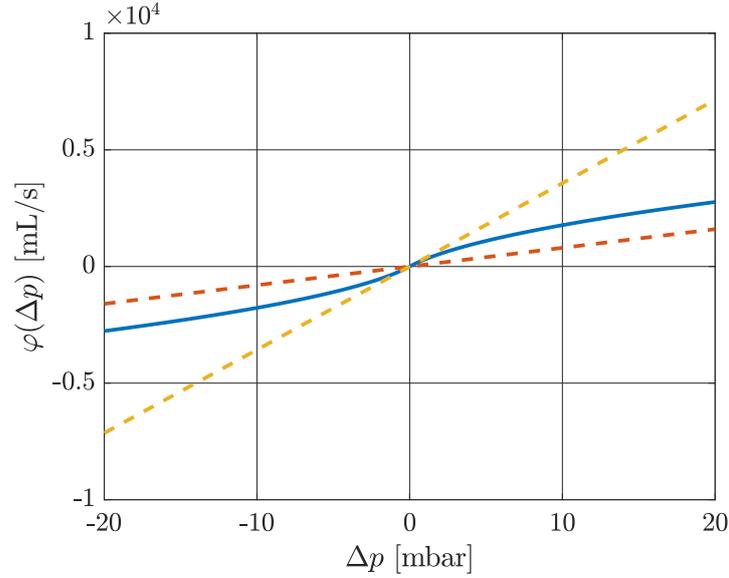


Figure 4.15: Visualization of the nonlinearity  $\varphi(\Delta p)$  (—), and its sector bounds  $\underline{\varphi}\Delta p$  (---) and  $\overline{\varphi}\Delta p$  (---), showing that the incremental sector condition holds for  $[\underline{\varphi}, \overline{\varphi}] = [80, \frac{1}{R_1} + \eta]$ .

Using these sector bounds, the loop-transformation is performed to obtain the system in (4.64), and it is verified that the pair  $(A, E)$  is controllable and the pair  $(A, M)$  is observable for every  $L$ , i.e., Assumption 4.3.1 holds. Thereafter,  $G(s)$  is constructed using the matrices of the loop-transformed system. Then, it is guaranteed that  $G(s)$  is SPR, it is first verified that for all  $L \in \{0, 1, 5, 15, 20\}$  the transfer function  $G(s)$  is Hurwitz, which is verified by computing the poles and checking that they reside in the open left-half plane. Thereafter, it is graphically validated that  $\text{re}(G(j\omega)) > 0 \forall \omega \in [-\infty, \infty]$ . This is validated in Fig. 4.16; it is clearly shown that for all considered values of  $L$  the real part of  $G(j\omega)$  is strictly positive. Finally, it is verified that  $G(\infty) > 0$ . This is also the case for all  $L \in \{0, 1, 5, 15, 20\}$ .

From these results, it is concluded that the nonlinear closed-loop ventilation system is incrementally uniformly globally exponentially stable for  $\Delta p \in [-20, 20]$  mbar and that this controller solves the harmonic regulation problem. Next, the performance of the different controllers is analyzed through experiments.

**Experimental results for mechanical ventilation.** The main experimental results are shown in Fig. 4.17 and 4.18. The time-domain results of the 20<sup>th</sup> breath with the integrator only, i.e.,  $L = 0$ , and the repetitive controller with 20 oscillators, i.e.,  $L = 20$ , are visualized in Fig. 4.17. The top plot shows the reference and the measured outputs and the bottom plot shows the tracking error for both controllers. The figure clearly shows that the tracking error is significantly reduced by the controller. The overshoot is eliminated and the rise-time is significantly shorter. Note that the residual error contains oscillatory behavior, especially during the expiration at the PEEP level, i.e., between 82 and 84 seconds. These oscillations contain mostly frequency content higher than 20 times the breathing frequency, i.e., above 5 Hz. It is observed that the tracking error's frequency content at frequencies above the  $L^{\text{th}}$  harmonic is increased.

The  $\mathcal{L}^2$ -norm of the error per breath for every controller is shown in Fig. 4.18. The

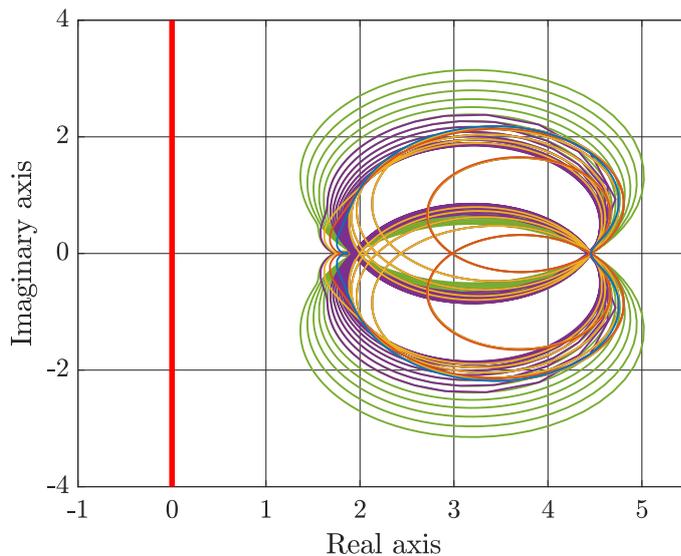


Figure 4.16: Nyquist plot of  $G(s)$  for  $L = 0$  (—),  $L = 1$  (—),  $L = 5$  (—),  $L = 15$  (—), and  $L = 20$  (—). The figure shows that  $\text{re}(G(j\omega)) > 0 \forall \omega \in [-\infty, \infty]$ .

$\mathcal{L}^2$ -norm of the error of a particular breath  $j$  is defined as

$$|e|_2 = \left( \sum_{k=1+(j-1)\rho}^{j\rho} |e(k)|^2 \right)^{\frac{1}{2}}$$

with  $\rho = \frac{T}{\Delta T}$  and  $\Delta T$  the sampling time. The figure clearly shows that increasing the number of oscillators reduces the  $\mathcal{L}^2$ -norm of the error upon convergence. Including 20 oscillators in the loop reduces the  $\mathcal{L}^2$ -norm of the error by more than a factor 3 compared to integral action only. Furthermore, it is observed that the convergence time is longer for an increasing number of oscillators and the controller with 20 oscillators converges in approximately 15 breaths.

Concluding, all controllers show convergent behavior in the experiments, as expected by the analysis. Furthermore, the tracking error is reduced significantly, by more than a factor 3, by including the harmonic control. The Fourier coefficients of the steady-state output error  $e(t)$  are suppressed at the frequencies  $w = k\frac{2\pi}{T}$ ,  $k = 0, 1, \dots, L$ .

**Analysis of conservatism** To analyze how conservative the convergence properties of Proposition 4.3.2 are, an experimental use case is presented where the SPR properties are violated. This is achieved by considering a ventilation use-case with lung parameters that represent a baby patient, i.e.,  $C_{lung} = 3$  mL/mbar and  $R_{lung} = 50$  mbar s/L. The same hose and blower system as for the adult use case is used, hence, the same sector conditions for the linearity can be used. Furthermore, the same harmonic control design as for the adult use case is followed for  $L = 20$ . The transfer function  $G(s)$  is computed for this system and visualized in Fig. 4.19. This figure clearly shows that the second condition for SPR transfer functions is violated for  $L = 20$ . Therefore, the desired convergence properties of the system cannot be guaranteed for this controller design with  $L = 20$  oscillators.

The resulting  $\mathcal{L}^2$ -norm of the error per breath is shown in Fig. 4.20. This figure

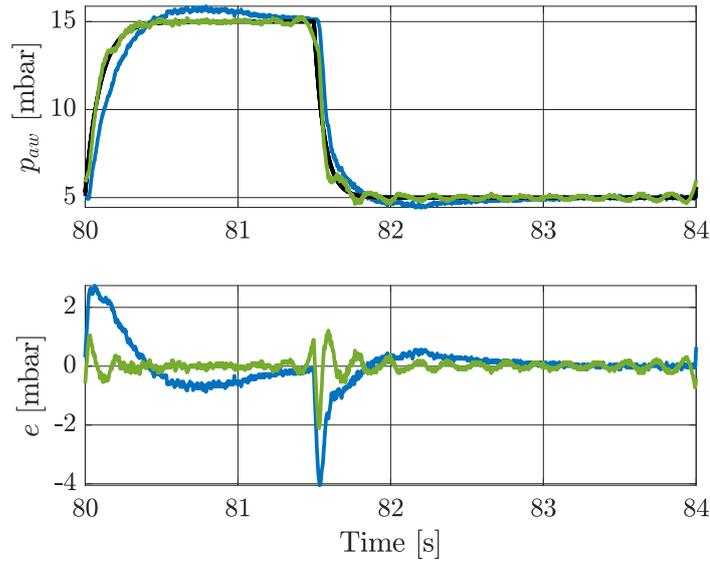


Figure 4.17: The time domain results upon convergence for  $L = 0$  (—), and  $L = 20$  (—), and the target pressure (—). The figure shows that the error is significantly reduced by the harmonic controller.

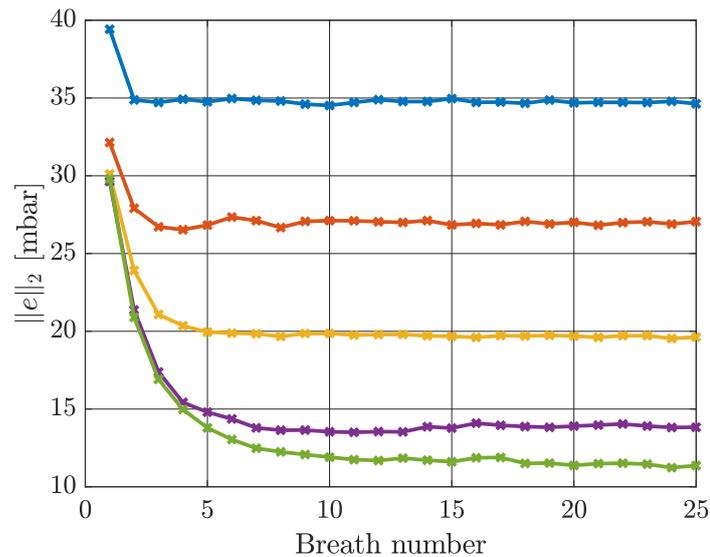


Figure 4.18: The  $\mathcal{L}^2$ -norm of the error for every breath for  $L = 0$  (—),  $L = 1$  (—),  $L = 5$  (—),  $L = 15$  (—), and  $L = 20$  (—). The figure shows that more oscillators results in a smaller error and the controllers converge in approximately 10 breaths.

clearly shows that the system behaves unstable for  $L = 20$ . Concluding, this use-case shows that the sufficient conditions in Proposition 4.3.2 have limited conservatism, which is a desirable property for practical controller design because it allows more design freedom.

**Remarks on the proposed design.** In the experimental analysis, especially in the baby use case, it is observed that the remaining error consists of oscillations at frequencies above the harmonics of the  $L^{\text{th}}$  oscillator. These oscillations in the error are increasing for an increasing number of oscillators, limiting the overall tracking

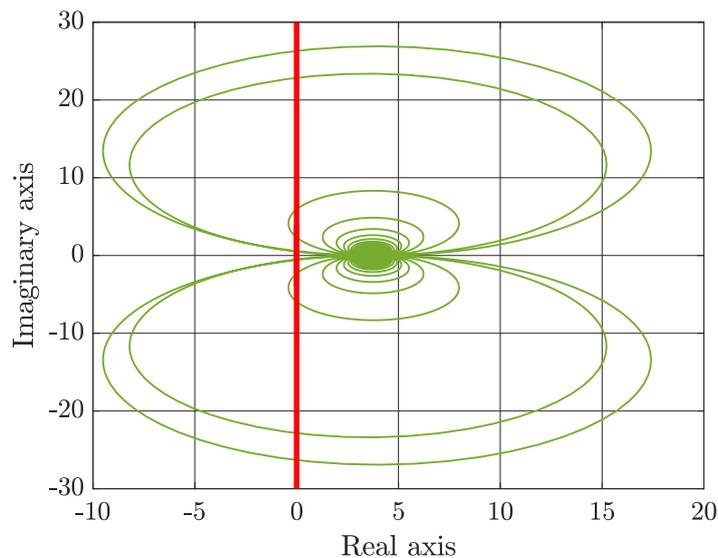


Figure 4.19: Nyquist plot of  $G(s)$  for  $L = 20$  (—) for the baby use-case. The figure shows that  $\text{re}(G(j\omega)) > 0 \forall \omega \in [-\infty, \infty]$  does not hold, hence, convergence is not guaranteed.

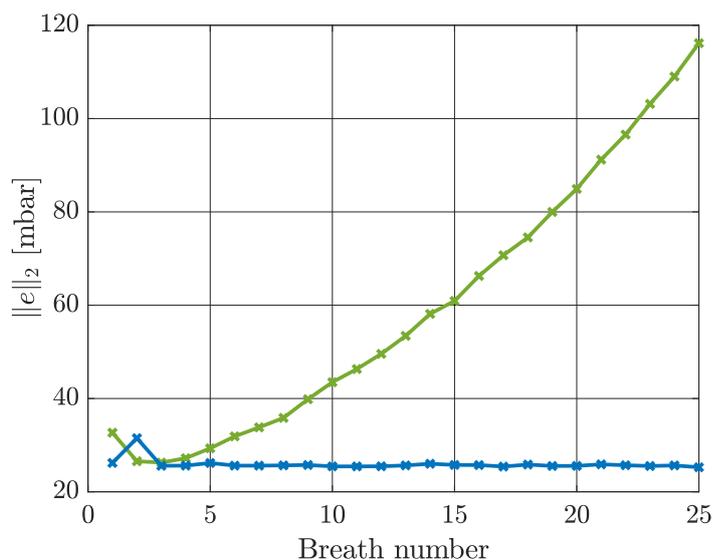


Figure 4.20: The  $\mathcal{L}^2$ -norm of the error for every breath for  $L = 0$  (—), and  $L = 20$  (—) for the baby use-case. The figure shows unstable behavior for that the closed-loop system with  $L = 20$ .

performance. Especially in other use-cases, it is observed that increasing the number of oscillators can significantly deteriorate the system performance. This effect can be explained by analyzing the sensitivity  $S_{re}$ , i.e., the transfer function from the reference  $r$  to the tracking error  $e$ , of a linearization of the closed-loop ventilation system. This linearized closed-loop system is obtained by replacing the nonlinearity in Fig. 4.13 by a linear resistance, i.e.,  $R_{hose}(\Delta p)$  is replaced by  $\frac{\Delta p}{R_{lin}}$  with  $R_{lin} = \frac{2}{\varphi + \bar{\varphi}}$ , and  $\ell = 0$ . The resulting Bode magnitude plot of  $S_{re}$  is shown in Fig. 4.21. This Bode magnitude plot clearly shows that the tracking error is zero at the harmonics of the breathing frequencies. However, it also shows an increase in magnitude at

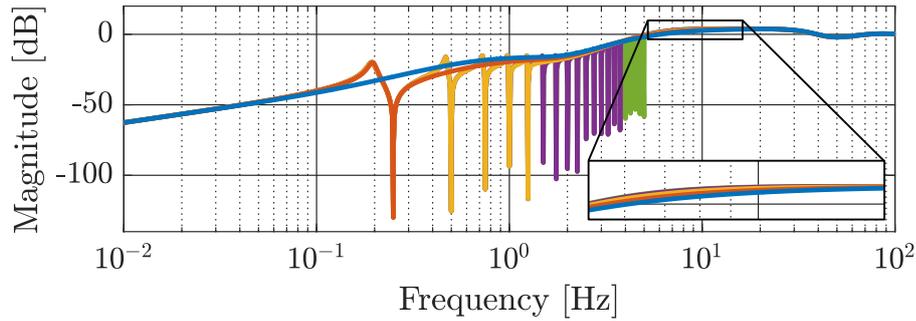


Figure 4.21: Bode magnitude plot of the sensitivity  $S_{re}$  for the linearization of the closed-loop system with  $L = 0$  (—),  $L = 1$  (—),  $L = 5$  (—),  $L = 15$  (—), and  $L = 20$  (—). The figure shows a magnitude increase at frequencies around 8 Hz, causing oscillations at these frequencies.

a frequency above the oscillator frequencies. The magnitude at these frequencies is increasing for an increasing number of oscillators. This increase in magnitude causes the oscillations at these frequencies as shown in the experiments. Therefore, in future work, it should be analyzed how this increase in magnitude at these specific frequencies can be eliminated.

#### 4.4 Conclusions and perspectives

In this chapter, we considered two, separate, practical applications. First, we considered a robust output set-point tracking problem for a power flow controller. Power flow controllers are electrical circuits that have to regulate on a constant value the power of the lines to which they are attached. The tracking task has to be achieved despite the uncertainties that are present in the electrical network. To achieve such a goal, first, we provided a state-space-based model of the power flow controller. This resulted in a bilinear system coupled with a polynomial output. Then, we constructed a dynamic state-feedback control law design. The plant is extended with an integral action processing the regulation error and a saturated stabilizer for the extended dynamics has been provided with forwarding tools specialized for the case of bilinear systems coupled with a polynomial output. In a separate section, we studied the harmonic regulation problem for a ventilation machine. A ventilation machine is a medical device that is used to help patients breathe. For this, a periodic regulation task on the pressure in the patient's lungs has to be achieved. To do so, first, we provided a state-space model for the system. This resulted in a Lur'e system. Then, we constructed a dynamic output-feedback control law. The plant is extended with a bunch of linear oscillators at the frequencies of the periodic signal that has to be tracked. Then, a pure output feedback in the internal model dynamics is used to stabilize the extended system. Stability is claimed using an incremental version of the circle criterion. In both cases, the theoretical results have been validated with experiments.

Future studies will focus on the following aspects.

- For the power flow controller, the focus will be on the meshed role that the device has. In particular, we provided a control action for the single unit. However, power flow controllers have to operate in meshed networks. A possible study will be therefore to analyze the case in which multiple loads and

generators are connected to the network through (or without) other DC/DC converters.

- For the machine ventilation a possible extension can be to look for a more constructive design for the feedback stabilizer, and to try to see whenever passivity properties could be exploited to design alternatives control laws (see Remark 4.3.4).





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# RESUMÉ

## Contexte

Cette thèse traite la notion de stabilité incrémentale et son application dans le contexte de la conception de commandes pour des systèmes non linéaires. Le manuscrit est divisé en quatre chapitres principaux, chacun d'entre eux traitant de sujets différents mais strictement liés entre eux. Dans le premier chapitre, nous étudions la notion de stabilité incrémentale pour les systèmes de contrôle non linéaires. Plus précisément, on dit qu'un système est stable de manière incrémentale si différentes trajectoires du même système convergent asymptotiquement entre elles. Une telle notion est intéressante en raison de certaines propriétés que les systèmes incrémentalement stables partagent, telles que la périodicité des trajectoires, la robustesse par rapport aux perturbations externes et bien d'autres. Parmi les différents outils permettant d'étudier une telle notion, nous nous concentrons sur l'approche dite 'métrique' (Riemannienne). Bien que la théorie des systèmes incrémentalement stables suscite beaucoup d'intérêt de la part de la communauté mondiale des contrôleurs, plusieurs questions ouvertes doivent encore être résolues, concernant l'analyse des propriétés incrémentales et la conception des commandes en boucle fermée permettant d'atteindre la stabilité incrémentale.

Dans le deuxième chapitre, nous nous concentrons sur le problème de la régulation de la sortie. L'objectif est de concevoir une loi de commande (dynamique) telle que la sortie d'un système non linéaire soit capable de suivre asymptotiquement une référence et, en même temps, de rejeter les perturbations externes. En particulier, nous cherchons à obtenir une régulation 'globale' de la sortie, ce qui signifie que la tâche de régulation doit être accomplie indépendamment de la condition initiale et de l'amplitude des signaux externes. Le challenge est de garantir l'existence d'une solution en régime permanent sur laquelle l'erreur de régulation est nulle pour toute valeur des signaux externes, et la convergence des trajectoires d'une telle solution pour toute condition initiale. Si les outils permettant d'obtenir une régulation pour les systèmes à phasage minimal sous forme normale sont bien développés, on en sait beaucoup moins pour des classes de systèmes plus générales, en particulier lorsque la régulation globale est le but recherché. Par conséquent, de nouveaux outils doivent être développés. Dans notre approche en particulier, nous plaçons le problème de la régulation dans le cadre incrémental.

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Dans le troisième chapitre, nous nous concentrons sur le problème de la synchronisation multi-agents. Ici, nous considérons un groupe d'entités identiques qui communiquent entre elles par le biais d'un protocole de communication. L'objectif ici, est la conception d'une loi de contrôle du couplage distribué tel que ces entités parviennent à un consensus. Alors que la théorie des systèmes linéaires est bien développée, de nombreuses questions restent ouvertes pour les systèmes non linéaires ayant une formulation générale. Dans notre approche, nous plaçons le problème de synchronisation dans un cadre incrémental. Ce choix est motivé par le fait que, si les agents sont décrits par le même modèle, alors le problème de synchronisation correspond à la conception d'une loi de contrôle distribuée telle que différentes trajectoires de la même équation différentielle convergent asymptotiquement entre elles.

Dans le quatrième et dernier chapitre, nous nous concentrons sur deux applications pratiques. En particulier, nous considérons deux problèmes distincts. Le premier problème est un problème du suivi du point de consigne pour un contrôleur de flux de puissance. Un contrôleur de flux de puissance est un circuit électrique dont le rôle est de réguler la puissance sur les lignes auxquelles il est attaché, malgré les incertitudes paramétriques de la centrale et sur les références à suivre. Le second problème est un suivi de trajectoire périodique pour une machine de ventilation. Une machine de ventilation est un équipement médical utilisé pour aider les patients à respirer. L'objectif ici, est de concevoir une loi de commande telle que la machine soit capable de suivre un signal périodique de pression représentant la phase de respiration, malgré l'incertitude sur les paramètres.

## Contributions principales

Dans le Chapitre 1, nous étudions les propriétés incrémentales des systèmes non linéaires avec une approche métrique. Nous rappelons quelques résultats de la littérature sur les propriétés que possèdent les systèmes incrémentaux. Nous étudions ensuite les propriétés incrémentales ISS. Pour conclure, nous cherchons à concevoir des lois de contrôle par rétroaction pour que le système en boucle fermée possède des propriétés incrémentales. Les principales contributions de ce chapitre peut être trouvée dans les publications de l'auteur Giaccagli et al. (2022a,b,c,d).

**Contribution 1:** Grâce à la notion de 'champ de Killing', nous dérivons des conditions suffisantes basées sur des métriques pour qu'un système soit incrémental ISS.

**Contribution 2:** Nous fournissons un ensemble de conditions suffisantes pour la conception de lois de contrôle par rétroaction atteignant des propriétés incrémentales pour le système en boucle fermée. En particulier : i) nous déduisons un théorème de petit gain incrémental basé sur des métriques ; ii) nous développons une version incrémentale des techniques de contrôle par forwarding ; iii) nous fournissons un ensemble de conditions LMI pour les systèmes avec des non-linéarités limite secteur.

Dans le Chapitre 2, nous étudions le problème de la régulation de la sortie globale. Nous plaçons le problème dans le cadre de la contraction. Nous étudions deux problèmes séparément : le problème du suivi d'une référence constante et le problème de la cancellation harmonique. Les principales contributions de ce chapitre peut être

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trouvée dans les publications de l’auteur Giaccagli et al. (2020, 2021b, 2022d).

**Contribution 3:** Nous fournissons un ensemble de conditions suffisantes pour réaliser le suivi global du point de consigne à sortie constante et le rejet des perturbations pour les systèmes non linéaires, qui n’admettent pas nécessairement un difféomorphisme globalement défini permettant de les réécrire sous forme normale.

**Contribution 4:** Nous fournissons un ensemble de conditions suffisantes pour obtenir une régulation harmonique globale d’une trajectoire périodique d’un système non linéaire, qui n’admet pas nécessairement un difféomorphisme globalement défini permettant de la réécrire sous forme normale.

Dans le Chapitre 3, nous étudions le problème de la synchronisation multi-agents. Nous formulons le problème avec des outils incrémentaux. Ensuite, nous étudions la synchronisation des réseaux dans le cadre Euclidien et la synchronisation vers un leader dans le cadre Riemannien. Ensuite, dans ce deuxième cas, nous fournissons un algorithme basé sur un réseau neuronal profond pour la mise en œuvre pratique de la loi de contrôle de synchronisation. Les résultats présentés dans ce chapitre ont été obtenus en collaboration avec Samuele Zoboli, LAGEPP, Université de Lyon 1 (Villeurbanne, France) qui s’est occupé de la partie apprentissage. Les principales contributions de ce chapitre peut être trouvée dans les publications de l’auteur Giaccagli et al. (2021a, 2022e).

**Contribution 5:** Nous dérivons un ensemble de conditions suffisantes basées sur une analyse de la métrique Euclidienne pour la synchronisation des réseaux dirigés connectés de systèmes non linéaires.

**Contribution 6:** Nous dérivons un ensemble de conditions suffisantes basées sur une analyse de la métrique de Riemannien pour la synchronisation vers un leader de réseaux non dirigés connectés de systèmes non linéaires. Un algorithme général basé sur des réseaux neuronaux pour la mise en œuvre est présenté.

Dans le Chapitre 4, nous étudions deux applications pratiques. En particulier, nous considérons un problème de suivi robuste de la consigne de sortie pour un contrôleur de flux de puissance et un problème de régulation harmonique pour une machine de ventilation mécanique. Les résultats de ce chapitre ont été obtenus avec Tanguy Simon et ses collaborateurs au Laboratoire Ampère de l’INSA de Lyon (Villeurbanne, France) pour le contrôleur de puissance et avec Joey Reinders et ses collaborateurs du Département d’Ingénierie Mécanique, Université de Technologie d’Eindhoven (Eindhoven, Pays-Bas) pour la machine de ventilation. Dans les deux cas, l’autre équipe s’est occupée de la modélisation et de la partie expérimentale. Les principales contributions de ce chapitre peut être trouvée dans les publications de l’auteur Simon et al. (2021a, 2022); Reinders et al. (2022).

**Contribution 7:** Nous concevons et mettons en œuvre une loi de contrôle dynamique robuste par forwarding pour le contrôleur de flux de puissance permettant un suivi robuste du point de consigne de sortie. La loi de contrôle est validée par des résultats expérimentaux.

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**Contribution 8:** Nous concevons et mettons en œuvre une loi de contrôle dynamique basée sur un critère de cercle incrémentiel pour la machine de ventilation mécanique permettant une régulation harmonique robuste. La loi de contrôle est validée par des résultats expérimentaux.

## Publications

Au cours de cette thèse, le compte total des articles publiés/soumis/en préparation est de 8 articles de journal et 4 articles de conférence. Les publications sont listées ci-dessous.

### Articles journal:

- Giaccagli, M., Astolfi, D., Andrieu, V., and Marconi, L. (2021). Sufficient conditions for global integral action via incremental forwarding for input-affine nonlinear systems. *IEEE Transactions on Automatic Control*.
- Giaccagli, M., Andrieu, V., Tarbouriech, S., and Astolfi, D. (2022). Infinite gain margin, contraction and optimality: an LMI-based design. *European Journal of Control*, 100685.
- Giaccagli, M., Zoboli, S., Astolfi, D., Andrieu, V., and Casadei, G. (2022). Synchronization in networks of nonlinear systems: contraction metric analysis and deep-learning for feedback estimation. Submitted to *IEEE Transactions on Automatic Control*.
- Giaccagli, M., Andrieu, V., Tarbouriech, S., and Astolfi, D. (2022). LMI conditions for contraction, integral action and output feedback stabilization for a class of nonlinear systems. Submitted to *Automatica*.
- Giaccagli, M., Astolfi, D., Andrieu, V., and Marconi, L. (2022). Incremental stabilization of cascade nonlinear systems and harmonic regulation: a forwarding-based design. Submitted to *IEEE Transactions on Automatic Control*.
- Giaccagli, M., Astolfi, D., and Andrieu, V. (2022). A metric approach to incremental stability. In preparation.
- Simon, T., Giaccagli, M., Trégouët, J.F., Astolfi, D., Andrieu, V., Lin-Shi, X., and Morel, H. (2022). Robust regulation of a power flow controller via nonlinear integral action”, Prov. accepted in *IEEE Transaction on Control Systems Technology*.
- Reinders, J., Giaccagli, M., Hunnekens, B., Astolfi, D., Oomen, T., and van de Wouw, N. (2022). Repetitive control for Lur’e-type systems: application to mechanical ventilation. Submitted to *IEEE Transaction on Control Systems Technology*.

### Articles conférence:

- Giaccagli, M., Astolfi, D., Andrieu, V., and Marconi, L. (2020). Sufficient conditions for output reference tracking for nonlinear systems: a contractive approach. In *59th IEEE Conference on Decision and Control (CDC)*, pp. 4580-4585, IEEE.

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# Chapitre 1: stabilité incrémentale

## Introduction

La stabilité est un concept varié. Selon le problème de contrôle à résoudre, différentes notions de stabilité peuvent être étudiées. Les exemples classiques sont la stabilité d'un point d'équilibre dans un problème de stabilisation, la stabilité d'une variété ou la stabilité d'une trajectoire particulière du système. Tous ces exemples ont un point commun : nous étudions le comportement des trajectoires du système par rapport à un élément fixe (point d'équilibre, variété ou trajectoire). Dans ce manuscrit, nous allons étudier une notion différente de la stabilité, appelée stabilité incrémentale. Dans ce contexte, nous ne sommes pas intéressés à étudier comment les trajectoires d'un système se comportent par rapport à un élément fixe, mais plutôt comment les trajectoires se comportent les unes par rapport aux autres. L'intérêt est dû au fait que, dans ce cas, nous n'avons pas besoin de connaître et/ou de supposer l'existence d'éléments fixes. Par conséquent, les propriétés des systèmes incrémentaux sont plus fortes que celles qui seraient obtenues en résolvant un problème de stabilisation.

Considérons un système décrit par

$$\dot{x} = f(x, t) \quad (\text{R.1})$$

où  $x \in \mathbb{R}^{n_x}$  est l'état et  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$  est un champ vectoriel suffisamment lisse. Soit  $\mathcal{X}(x_0, t, t_0)$  la trajectoire du système avec les conditions initiales  $(x_0, t_0) \in \mathbb{R}^{n_x} \times \mathbb{R}$ .

**Définition 1** (Stabilité incrémentale). *Disons que le système (R.1) est*

- *Incrémental Globalement Uniformément Stable ( $\delta GUS$ ) s'il existe une fonction  $\alpha$  de classe- $\mathcal{K}$  telle que*

$$|\mathcal{X}(x_1, t, t_0) - \mathcal{X}(x_2, t, t_0)| \leq \alpha(|x_1 - x_2|); \quad (\text{R.2})$$

- *Incrémental Globalement Uniforme Exponentiellement Stable ( $\delta GUES$ ) s'il existe deux nombres strictement positifs  $\lambda, k > 0$  tels que*

$$|\mathcal{X}(x_1, t, t_0) - \mathcal{X}(x_2, t, t_0)| \leq k |x_1 - x_2| \exp(-\lambda(t - t_0)) \quad (\text{R.3})$$

pour tous  $(x_1, x_2, t) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}$ .

La stabilité incrémentale n'est pas un concept nouveau (voir Giesl et al. (2022)). Dans la littérature, il y a 3 méthodes pour étudier les propriétés incrémentales. La première consiste à utiliser les normes logarithmiques (voir Sontag (2010)). La seconde est une approche de Lyapunov (voir Forni and Sepulchre (2013a); Angeli (2002)). La troisième approche est basée sur l'étude des métriques riemanniennes (voir Lohmiller and Slotine (1998); Andrieu et al. (2016)). Dans ce manuscrit, nous utiliserons cette dernière méthode. Il est en effet connu qu'un système est incrémental s'il existe une métrique Riemannienne telle que le champ de vecteurs génère des trajectoires dont la distance diminue de façon monotone dans le temps (i.e. le mapping  $t \mapsto \mathcal{X}(x_0, t, t_0)$  est une contraction).

**Théorème 1** (Métrique implique incrémentale). *Considérez le système (R.1). S'il existe une fonction matricielle  $C^1$  symétrique et définie positive  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto$*

$\mathbb{R}^{n_x \times n_x}$ , deux nombres strictement positifs  $\underline{p}, \bar{p}$  et une fonction continue  $q : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  tels que

$$L_f P(x, t) \preceq -q(x)P(x, t), \quad \underline{p}I \preceq P(x, t) \preceq \bar{p}I \quad (\text{R.4})$$

pour tous  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ . Alors:

- si  $q(x) \geq 0$  pour tous  $x \in \mathbb{R}^{n_x}$ , le système est  $\delta GUS$ ;
- s'il existe un nombre strictement positif  $\lambda > 0$  tel que  $q(x) \geq \lambda$  pour tous  $x \in \mathbb{R}^{n_x}$ , le système est  $\delta GUES$ .

Les mêmes conditions s'appliquent dans un ensemble compact et invariant  $\mathcal{D}$ . Dans ce cas, le système est incrémental en  $\mathcal{D}$ .

### Propriétés des systèmes incrémentaux

L'intérêt pour les systèmes incrémentaux est dû aux propriétés que ces systèmes possèdent. En particulier, nous nous concentrons sur la périodicité des trajectoires (et l'existence de points d'équilibre) et l'invariance par rapport aux difféomorphismes. Les résultats suivants sont valables.

**Lemme 1** (Trajectoires périodiques et points d'équilibre). *Considérons le système (R.1). Donc:*

1. si le système est  $\delta GUES$  et  $f$  est périodique, c'est-à-dire qu'il existe  $T > 0$  tel que  $f(x, t) = f(x, t + T)$  pour tous  $t \geq t_0$ , alors les trajectoires convergent asymptotiquement vers une solution périodique avec la même période  $T$ ;
2. s'il existe un point d'équilibre  $x^*$  tel que  $f(x^*, 0) = 0$  et le système est  $\delta GUS$  (respectivement  $\delta GUES$ ), alors  $x^*$  est Globalement Stable (respectivement, Globalement Exponentiellement Stable);
3. si le système est  $\delta GUES$  et  $f$  est invariant dans le temps, alors il existe un point d'équilibre  $x^*$  Globalement Exponentiellement Stable

L'incrémentalité est une propriété des trajectoires et est donc invariante par rapport aux difféomorphismes. Le résultat suivant l'affirme.

**Lemme 2** (Invariance aux difféomorphismes). *Considérons le système (R.1). Soit  $\varphi$  un difféomorphisme globalement Lipschitz. Alors si (R.1) est  $\delta GUS$  (respectivement,  $\delta GUES$ ), alors le système  $\dot{z} = f_z(z, t)$  avec  $z = \varphi(x)$  est  $\delta GUS$  (respectivement,  $\delta GUES$ ).*

**Remarque 1.** *Si les propriétés incrémentales ont été obtenues par une métrique  $P$  avec le Théorème 1, alors*

$$P_z(z, t) = \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-\top} P(\varphi^{-1}(z), t) \left( \frac{\partial \varphi}{\partial x}(\varphi^{-1}(z)) \right)^{-1}$$

*est une métrique pour le système dans les nouvelles coordonnées  $z$ .*

### Systèmes non-autonomes

Dans cette section, nous étudions les propriétés incrémentales pour les systèmes du type

$$\dot{x} = f(x, t) + g(x, t)u \quad (\text{R.5})$$

où  $x \in \mathbb{R}^{n_x}$  est l'état,  $u \in \mathcal{U} \subset \mathbb{R}^{n_u}$  est une entrée et  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$ ,  $g : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times \mathbb{R}^{n_u}}$  sont des champs vectoriels suffisamment lisses. Soit  $\mathcal{X}(x, u, t, t_0)$  la trajectoire du système avec la condition initiale  $(x, t_0) \in \mathbb{R}^{n_x} \times \mathbb{R}$  et l'entrée  $u = u(t)$ .

**Définition 2** (Incrémentalement entrée-sortie stable). *Nous disons que le système (R.5) est:*

- *Incrémentalement Globalement Uniformément Entrée-Sortie Stable ( $\delta GUESS$ ) par rapport à  $u$  s'il existe une fonction  $\alpha$  de classe- $\mathcal{K}$  telle que*

$$|\mathcal{X}(x_1, u_1, t, t_0) - \mathcal{X}(x_2, u_2, t, t_0)| \leq \alpha(|x_1 - x_2|) + \sup_{s \in [t_0, t]} [\alpha_d(|u_1 - u_2|)]; \quad (\text{R.6})$$

- *Incrémentalement Globalement Uniformément Entrée-Sortie Exponentiellement Stable ( $\delta GUESES$ ) s'il existe deux nombres strictement positifs  $\lambda, k > 0$  tels que*

$$|\mathcal{X}(x_1, u_1, t, t_0) - \mathcal{X}(x_2, u_2, t, t_0)| \leq k |x_1 - x_2| \exp(-\lambda(t-t_0)) + \sup_{s \in [t_0, t]} [\alpha_d(|u_1 - u_2|)]; \quad (\text{R.7})$$

pour tous  $(x_1, x_2, u_1, u_2, t) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathcal{U} \times \mathcal{U} \times \mathbb{R}$ , et pour une certaine fonction  $\alpha_d$  de classe- $\mathcal{K}$ .

Cette notion a déjà été étudiée en Angeli (2002), mais pas par une approche métrique. Pour étudier la stabilité incrémentale entrée-état, nous introduisons la définition suivante.

**Définition 3** (Champ de Killing). *Soit  $g : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_u}$  un champ vectoriel et  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  une fonction matricielle symétrique définie positive, les deux  $C^1$ . Nous disons que  $g$  est un champ de Killing pour  $P$  si  $L_g P(x, t) = 0$  pour tous  $(x, t) \in \mathbb{R}^{n_x} \times \mathbb{R}$ .*

Nous avons donc le résultat suivant, qui fournit des conditions suffisantes pour être incrémentalement entrée-état stable.

**Théorème 2** (Métrique + Killing implique incremental ESS). *Considérons le système (R.5). Admetons qu'il existe une fonction matricielle  $C^1$  symétrique et définie positive  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$ , deux nombres strictement positifs  $\underline{p}, \bar{p}$  et une fonction continue  $q : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{\geq 0}$  tels que (R.4) est valide. Supposons également que  $g$  est un champ Killing pour  $P$ . Alors:*

- *si  $q(x) \geq 0$  pour tous  $x \in \mathbb{R}^{n_x}$ , le système est  $\delta GUESS$ ;*
- *s'il existe un nombre strictement positif  $\lambda > 0$  tel que  $q(x) \geq \lambda$  pour tous  $x \in \mathbb{R}^{n_x}$ , le système est  $\delta GUESES$ .*

## Conception de la commande en boucle fermée

En conséquence des propriétés des systèmes incrémentaux, de nombreux résultats existent dans la littérature sur la façon de concevoir des lois de contrôle de rétroaction pour un système afin d'obtenir des propriétés incrémentales (voir Zamani and Tabuada (2011); Pavlov et al. (2006); Manchester and Slotine (2017); Tsukamoto et al. (2021) et références). Dans cette section, nous proposons trois conceptions basées sur des métriques.

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## Commande par petit gain incrémental pour des systèmes en rétroaction

Considérons un système du type

$$\dot{x}_1 = f_1(x_1, t) + g_1(x_1, t)x_2, \quad \dot{x}_2 = f_2(x_2, t) + g_2(x_2, t)x_1 \quad (\text{R.8})$$

avec état  $x = (x_1, x_2)$  et  $x_1 \in \mathbb{R}^{n_{x_1}}$ ,  $x_2 \in \mathbb{R}^{n_{x_2}}$ . Dans cette section, nous voulons dériver un théorème du petit-gain incrémental. Ce résultat a déjà été donné en Angeli (2002), mais sans approche métrique. Dans ce qui suit, nous donnons le résultat par une analyse métrique.

**Lemme 3** (Petit gain incrémental). *Considérons le système (R.8). Supposons qu'il existe deux fonctions matricielles  $C^1$  symétriques et positives définies  $P_1 : \mathbb{R}^{n_{x_1}} \times \mathbb{R} \mapsto \mathbb{R}^{n_{x_1} \times n_{x_1}}$ ,  $P_2 : \mathbb{R}^{n_{x_2}} \times \mathbb{R} \mapsto \mathbb{R}^{n_{x_2} \times n_{x_2}}$  et 6 nombres positifs  $\underline{p}_1, \bar{p}_1, \lambda_1, \underline{p}_2, \bar{p}_2, \lambda_2$  tels que*

$$\begin{aligned} L_{f_1}P_1(x_1, t) &\preceq -\lambda_1 I, & \underline{p}_1 I &\preceq P_1(x_1, t) \preceq \bar{p}_1 I, & L_{g_1}P_1(x_1, t) &= 0 \\ L_{f_2}P_2(x_2, t) &\preceq -\lambda_2 I, & \underline{p}_2 I &\preceq P_2(x_2, t) \preceq \bar{p}_2 I, & L_{g_2}P_2(x_2, t) &= 0. \end{aligned}$$

*S'il existe également deux nombres positifs  $\bar{g}_1, \bar{g}_2$  tels que  $\|g_1(x_1, t)\| \leq \bar{g}_1$  et  $\|g_2(x_2, t)\| \leq \bar{g}_2$  et*

$$\lambda_1 \lambda_2 \leq 8 \bar{p}_1 \bar{p}_2 \bar{g}_1 \bar{g}_2,$$

*alors le système est  $\delta GUES$ .*

## Commande par forwarding incrémentale pour des systèmes en cascade

Considérons un système du type

$$\dot{\chi} = f_\chi(\chi) + g_\chi(\chi)(u + w(t)), \quad \dot{\eta} = \Phi\eta + v(\chi) + w(t) \quad (\text{R.9})$$

avec état  $x = (\chi, \eta)$  et  $\chi \in \mathbb{R}^{n_\chi}$ ,  $\eta \in \mathbb{R}^{n_\eta}$ . Nous avons que  $w : \mathbb{R} \mapsto \mathcal{W} \subseteq \mathbb{R}^{n_w}$  est une perturbation et  $f(0) = 0$  et  $v(0) = 0$ . Un système de cette forme est dit en 'feedforward'. Nous voulons développer une version incrémentale de la loi de contrôle par forwarding (voir Mazenc and Praly (1996)). Nous avons donc les hypothèses suivantes.

**Hypothèse 1** (Stabilité en boucle ouverte et champ de Killing). *Considérons le système (R.9). On connaît une fonction  $C^1$   $\phi_0 : \mathbb{R}^{n_\chi} \mapsto \mathbb{R}^{n_\eta}$ , une fonction matricielle  $C^1$  prenant des valeurs symétriques et positives  $P_\chi : \mathbb{R}^{n_\chi} \mapsto \mathbb{R}^{n_\chi \times n_\chi}$  et trois nombres réels positifs  $\underline{p}_\chi, \bar{p}_\chi, p_\chi$  tels que la fonction  $f_0(\chi) = f_\chi(\chi) + g_\chi(\chi)\phi_0(\chi)$  satisfait*

$$L_{f_0}P_\chi(\chi) \preceq -p_\chi I \quad \underline{p}_\chi I \preceq P_\chi(\chi) \preceq \bar{p}_\chi I, \quad L_{g_\chi}P_\chi(\chi) = 0$$

*pour tout  $\chi \in \mathbb{R}^{n_\chi}$ .*

**Hypothèse 2** (Stabilité de  $\Phi$ ). *Il existe une matrice symétrique définie positive  $H = H^\top \succ 0$  telle que  $H\Phi + \Phi^\top H \preceq 0$ .*

En bref, le forwarding est basé sur le fait qu'il existe une variété invariante pour la dynamique  $\eta$  qui peut être exprimée par la solution d'une équation différentielle. Dans notre étude, nous développerons le plus général forwarding mod  $\{L_g V\}$  (voir Praly et al. (2001)). Nous avons donc cette dernière hypothèse.

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**Hypothèse 3** (Forwarding mod  $\{L_g V\}$ ). On connaît trois fonctions  $C^1$   $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$ ,  $\Delta : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$  et  $\varrho : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  telles que, pour tout  $\chi \in \mathbb{R}^{n_x}$ , les conditions suivantes sont vérifiées:

- les fonctions  $\bar{M}$  et  $\Delta$  sont solution de  $L_{f_x} \bar{M}(\chi) = \Phi \bar{M}(\chi) + v(\chi) + \Delta(\chi)$ ;
- il existe une matrice  $\Lambda$  telle que  $L_{g_x} \bar{M}(\chi) = \Lambda$  pour tout  $\chi$  et tel que le couple  $(\Phi, (H\Lambda)^\top)$  est détectable ;
- la fonction  $\varrho$  satisfait  $\Lambda \frac{\partial \varrho}{\partial \chi}(\chi) = -\frac{\partial \Delta}{\partial \chi}(\chi)$ ;
- l'inégalité suivante est vérifiée  $L_{f_x} P_x(\chi) + \text{He} \left\{ P_x(\chi) g_x(\chi) \frac{\partial \varrho}{\partial \chi}(\chi) \right\} \preceq -\lambda I$ , pour un certain  $\lambda > 0$ .

Nous avons donc le résultat suivant.

**Théorème 3** (Forwarding incremental). Considérons le système (R.9) et supposons que les hypothèses 1, 2 et 3 soient vérifiées. De plus, supposons qu'il existe  $L_M \geq 0$  tel que  $\left\| \frac{\partial \bar{M}}{\partial \chi}(\chi) \right\| \leq L_M$  pour tout  $\chi$ . Alors, pour tout gain  $\kappa > 0$ , le système (R.9) en boucle fermée avec la loi de commande

$$u = \phi_0(\chi) + \kappa (H\Lambda)^\top (\eta - \bar{M}(\chi)) + \varrho(\chi)$$

est  $\delta$ GUESES par rapport à  $w$ .

### Contrôle avec des LMIs pour des systèmes monotones/sectoriels

Considérons maintenant un système du type

$$\dot{x} = Ax + G\varphi(z) + Rw(t), \quad z = Hx \quad (\text{R.10})$$

où  $A, G, R, H$  sont des matrices et où  $\varphi$  satisfait une des conditions suivantes.

**Hypothèse 4.** La fonction  $\varphi$  satisfait à l'une des conditions suivantes:

- (Limite secteur) il existe une matrice définie positive symétrique  $S = S^\top \succ 0$  et une matrice  $\Omega$  telle que

$$\text{He} \left\{ \frac{\partial \varphi}{\partial z}(z)^\top S \left[ \frac{\partial \varphi}{\partial z}(z) + \Omega \right] \right\} \preceq 0;$$

- (Monotone) il existe une matrice définie positive symétrique  $\Gamma = \Gamma^\top \succ 0$  telle que

$$\frac{\partial \varphi}{\partial z}(z) = \frac{\partial \varphi^\top}{\partial z}(z), \quad 0 \preceq \text{He} \left\{ \frac{\partial \varphi}{\partial z}(z) \right\} \preceq \Gamma.$$

L'idée est de ne considérer que des métriques Euclidiennes (matrice constante  $P$ ) et de dériver des conditions que nous exprimerons sous la forme d'inégalités matricielles linéaires (LMIs) pour l'analyse et la conception du contrôle afin d'obtenir des propriétés incrémentales pour cette classe de systèmes.

**Lemme 4** (LMIs pour l'analyse). Considérons le système (R.10) et supposons qu'il existe une matrice  $P = P^\top \succ 0$  et un nombre  $\lambda > 0$  tels que

$$\left[ \begin{array}{cc} A^\top P + PA + \lambda I & PG - H^\top \Omega^\top S \\ (PG - H^\top \Omega^\top S)^\top & -2S \end{array} \right] \preceq 0 \quad \left| \quad \left[ \begin{array}{cc} A^\top P + PA + \lambda I & PG + H^\top \\ (PG + H^\top)^\top & -4\Gamma^{-1} \end{array} \right] \preceq 0 \right.$$

$\varphi$  est limite secteur  $\varphi$  est monotone

alors le système est  $\delta$ GUESES par rapport à  $w$ .

Considérons maintenant

$$\dot{x} = Ax + Bu + G\varphi(z) + Rw(t), \quad z = Hx \quad (\text{R.11})$$

et cherchons des conditions pour faire un contrôleur. Nous nous concentrons sur des contrôleurs du type

$$u = \alpha(x) = Kx + N\varphi(z). \quad (\text{R.12})$$

Ceci est dû au fait que le système en boucle fermée est encore une fois composé d'un terme linéaire et d'un terme non linéaire satisfaisant à la condition de monotonie/limite secteur. Nous avons l'ensemble des résultats suivants.

**Lemme 5** (LMIs pour la conception du contrôle). *Considérons le système (R.11) avec une non-linéarité  $\varphi$  qui satisfait à l'une des conditions monotone/limite secteur. Ensuite:*

- nous pouvons construire des LMIs telles que, si elles admettent une solution, alors le système en boucle fermée avec (R.12) est  $\delta$ GUESES;
- nous pouvons construire des LMIs telles que, si elles admettent une solution, alors le système en boucle fermée avec  $u = \kappa\alpha(x)$ , où  $\alpha$  est défini dans (R.12), est  $\delta$ GUESES pour tous  $\kappa \geq 1$  (contrôleur à marge de gain infinie). Dans ce cas, nous pouvons construire une fonction de coût qui est minimisée par une telle loi de commande;
- nous pouvons construire des LMIs telles que, si elles admettent une solution, le système en boucle fermée avec le contrôleur en rétroaction de sortie donné par (R.12) et l'observateur en Arcak et al. (2003) à l'origine qui est globalement exponentiellement stable.

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## Chapitre 2: réglage de la sortie globale

Le problème dans lequel la sortie d'un système dynamique doit être ajustée asymptotiquement vers une référence et, en même temps, annuler les perturbations externes, est appelé le problème de réglage de la sortie. Pour les systèmes linéaires, le problème a été complètement résolu. La solution implique un contrôleur dynamique composé de deux éléments. Le premier est le modèle interne : un système qui traite l'erreur de contrôle et dont la dynamique est équivalente à celle de l'exosystème générant les références et les perturbations. Le second est un stabilisateur pour le système étendu. Voir, par exemple, Francis and Wonham (1976); Isidori et al. (2003). La solution permet une conception séparée des deux unités, grâce au principe du modèle interne. Le problème pour les systèmes non linéaires n'a pas encore été résolu. Pour cette raison, la littérature sur le problème du réglage de la sortie est vaste, voir Serani et al. (2001); Isidori and Byrnes (1990); Astolfi et al. (2022a) et références. Dans ce chapitre, nous traiterons le problème de la régulation de la sortie globale. Par 'globale', nous entendons que nous voulons résoudre le problème indépendamment des conditions initiales du système et de l'amplitude des signaux externes. De plus, nous voulons résoudre le problème sans faire appel aux formes normales. Pour cette raison, nous encadrons le problème par des techniques incrémentales. Nous allons aborder deux problèmes séparément. Le premier est le problème de la régulation de la sortie constante. Le second est le problème de la régulation harmonique.

### Régulation globale de la sortie constante

Dans ce chapitre, nous considérons un système du type

$$\dot{x} = f(x) + g(x)(u + d), \quad e = h(x) - r \quad (\text{R.13})$$

où  $x \in \mathbb{R}^{n_x}$  est l'état,  $u \in \mathbb{R}^{n_u}$  est la commande,  $e \in \mathbb{R}^{n_e}$  est l'erreur de régulation entre une sortie et une référence constante  $r$  et  $d$  est une perturbation constante. Motivés par le cas linéaire, nous étendons le système avec une action intégrale

$$\dot{\eta} = e \quad (\text{R.14})$$

qui traite l'erreur de régulation et nous cherchons un contrôleur en rétroaction  $u = \alpha(x, \eta)$  pour le système en boucle fermée. Nous avons donc le problème suivant.

**Problème 1** (Régulation globale constante de la sortie). *Considérons le système (R.13) étendu avec (R.14). Trouver un contrôleur du type  $u = \alpha(x, \eta)$  tel qu'il existe deux ensembles  $\mathcal{S} \subseteq \mathbb{R}^{n_x+n_e}$  et  $\mathcal{W} \subseteq \mathbb{R}^{n_u+n_e}$ , contenant leurs origines respectives tels que :*

1. si  $(d, r) = (0, 0)$  alors l'origine du système étendu est globalement asymptotiquement stable ;
2. pour chaque  $(d, r) \in \mathcal{W}$  constantes et pour chaque condition initiale  $(x_0, \eta_0) \in \mathcal{S}$ , le système en boucle fermée a des trajectoires bornées et, asymptotiquement  $\lim_{t \rightarrow +\infty} e(t) = 0$ .

Alors disons que:

- si  $\mathcal{S}$  et  $\mathcal{W}$  sont non vides, le problème de la régulation régionale constante est résolu;

- Si  $\mathcal{S} = \mathbb{R}^{n_x \times n_e}$  et  $\mathcal{W} = \mathbb{R}^{n_u \times n_e}$  le problème de régulation constante globale est résolu;
- Si  $\mathcal{S}$  n'est pas vide et que  $\mathcal{W}$  est de la forme  $\mathcal{W} = \emptyset \times \mathcal{R}$  pour un ensemble non vide  $\mathcal{R} \subseteq \mathbb{R}^{n_e}$ , le problème de suivi du point de consigne de la sortie constante globale est résolu.

Nous utilisons des propriétés incrémentales pour résoudre le problème. En fait, l'objectif est de garantir l'existence et la stabilité d'un point d'équilibre  $x^*$  pour chaque  $w = (d, r)$ . Cela peut être garanti si le système en boucle fermée est stable de manière incrémentielle. En fait, en réécrivant le système à boucle fermée comme

$$\dot{\chi} = \varphi(\chi) + \Gamma(\chi)w, \quad \varphi(\chi) := \begin{pmatrix} f(x) + g(x)\alpha(x, \eta) \\ h(x) \end{pmatrix}, \quad \Gamma(\chi) := \begin{pmatrix} \Gamma_d(\chi) & \Gamma_r \end{pmatrix},$$

avec  $\Gamma_d^\top(\chi) = (g(x), 0)$  et  $\Gamma_r = (0, -I)$ , grâce à l'action intégrale nous avons que  $0 = \dot{\eta} = h(x^*) - r \implies h(x^*) = r$ . Nous avons donc le résultat suivant.

**Théorème 4** (Contraction implique action intégrale globale). *Considérons le système (R.13), (R.14). S'il existe une fonction  $C^2$   $\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_e} \mapsto \mathbb{R}^{n_u}$ , une fonction de matrice  $C^1$  symétrique définie positive  $P : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_x}$ , une fonction continue positive  $q : \mathbb{R}^{n_x} \mapsto \mathbb{R}_{>0}$  et deux nombres réels positifs  $\underline{p}, \bar{p}$  tels que:*

$$\underline{p}I \preceq P(x) \preceq \bar{p}I \quad L_\varphi P(\chi) \preceq -q(\chi)I, \quad L_\Gamma P(\chi) = 0. \quad (\text{R.15})$$

Alors:

- s'il existe un nombre réel positif  $\lambda > 0$  tel que  $q(x) \geq \lambda$  pour tout  $x$ , alors le problème de régulation globale à sortie constante est résolu;
- Sinon, nous avons les propriétés suivantes:
  - (a) Soit  $\mathcal{W}$  un sous-ensemble non vide de  $\mathbb{R}^{n_u \times n_e}$  tel que, pour tout  $w \in \mathcal{W}$ , il existe une condition initiale  $x_0$  telle que la trajectoire correspondante du système en boucle fermée  $\mathcal{X}(x_0, t)$  est bornée. Alors, le problème de régulation régionale à sortie constante est résolu pour  $\mathcal{S} = \mathbb{R}^{n_x \times n_e}$  et un tel  $\mathcal{W}$ ;
  - (b) Il existe  $\bar{w} > 0$  tel que, le problème de régulation régionale à sortie constante est résolu pour  $\mathcal{S} = \mathbb{R}^{n_x \times n_e}$  et  $\mathcal{W} = \{w \in \mathbb{R}^{n_u \times n_e}, |w| \leq \bar{w}\}$

En plus, si  $L_{\Gamma_d} P(\chi) = 0$  n'est pas vérifiée, alors les résultats 1) et 2) ci-dessus sont valables pour le problème de suivi de consigne à sortie constante, en particulier avec  $\mathcal{W}$  de la forme  $\mathcal{W} = \emptyset \times \mathcal{R}$ , pour un certain  $\mathcal{R} \subseteq \mathbb{R}^{n_e}$ .

Motivés par la structure en boucle fermée (R.13), (R.14), nous utilisons une loi de contrôle par forwarding incrémental. Nous avons donc les hypothèses suivantes.

**Hypothèse 5** (Stabilité boucle ouverte). *Considérons le système (R.13). On connaît une fonction  $C^1$   $\phi_0 : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$ , une fonction matricielle  $C^1$  prenant des valeurs symétriques et positives  $P_0 : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_x \times n_x}$  et trois nombres réels positifs  $\underline{p}_0, \bar{p}_0, p_0$  tels que la fonction  $f_0(x) = f(x) + g(x)\phi_0(x)$  satisfait*

$$L_{f_0} P_0(x) \preceq -p_0 I, \quad \underline{p} I \preceq P_0(x) \preceq \bar{p} I, \quad L_g P_0(x) = 0$$

pour tout  $x \in \mathbb{R}^{n_x}$ .

Puisque la dynamique  $x$  satisfait Hypothèse 5, l'origine est un point d'équilibre qui est globalement asymptotiquement stable et localement exponentiellement stable. Ainsi, pour (Mazenc and Praly, 1996, Lemma IV.2), il existe une fonction  $M$  qui satisfait

$$L_{f_0}M(x) = h(x).$$

**Hypothèse 6** (Contrôlabilité étendue). *Il existe un nombre positif  $b > 0$  qui satisfait  $L_gM(x)L_gM^\top(x) \succeq bI$  pour tout  $x \in \mathbb{R}^{n_x}$ .*

Cette deuxième hypothèse correspond à une hypothèse de contrôlabilité sur le système étendu  $(x, \eta)$ . Nous avons donc ce résultat.

**Théorème 5** (Suivi du point de consigne de sortie régional). *Considérons le système (R.13) et supposons que les Hypothèses 5 et 6 sont vérifiées. Soit*

$$\Psi(x) := L_gM(x)^\top (L_gM(x)L_gM(x)^\top)^{-1},$$

*et supposons également qu'il existe deux nombres  $L_M > 0$  et  $k_1 \geq 0$  tels que  $\|\frac{\partial M}{\partial x}(x)\| \leq L_M$  et  $\|\frac{\partial \Psi v}{\partial x}(x)\| \leq k_1|v|$ ,  $\forall v \in \mathbb{R}^{n_e}$ . Définissons  $\beta : \mathbb{R}^{n_e} \mapsto \mathbb{R}^{n_e}$  comme toute fonction  $C^1$  satisfaisant*

$$|\beta(s)| \leq \frac{1}{k_1}, \quad \frac{\partial \beta}{\partial s}(s) = \frac{\partial \beta^\top}{\partial s}(s), \quad 0 \prec \frac{\partial \beta}{\partial s}(s) \preceq I.$$

*Alors, il existe un nombre positif  $\kappa^*$  tel que, pour chaque  $\kappa \in (0, \kappa^*]$ , le système en boucle fermée avec (R.14) et*

$$u = \phi_0(x) + \kappa \Psi(x)\beta(\eta - M(x))$$

*il résout le problème régional de suivi de consigne à sortie constante pour un  $\mathcal{R} \subset \mathbb{R}^{n_e}$  non vide.*

Le résultat précédent ne permet pas de faire un réglage global. La principale limite se trouve dans le fait que le terme  $L_gM(x)$  n'est pas constant. Pour surmonter ce problème, nous changeons nos hypothèses et utilisons le forwarding mod  $\{L_gV\}$ .

**Hypothèse 7** (Forwarding mod  $\{L_gV\}$  incrémental pour l'action intégrale). *On connaît deux fonctions  $C^1$   $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$  et  $\Delta : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_e}$  telles que, pour tout  $x \in \mathbb{R}^{n_x}$ , les conditions suivantes sont vérifiées:*

- *les fonctions  $\bar{M}$  et  $\Delta$  sont solution de  $L_{f_0}\bar{M}(x) = h(x) + \Delta(x)$ ;*
- *il existe une matrice  $\Lambda$  telle que  $L_g\bar{M}(x) = \Lambda$  pour tout  $x$  et tel que  $\text{rank}(\Lambda) = n_e$  ;*
- *l'inégalité suivante est vérifiée  $L_{f_0}P_0(x) + \text{He} \{P_0(x)g(x)\frac{\partial \Delta}{\partial x}(x)\} \preceq -2\lambda_1 I$ , pour un certain  $\lambda_1 > 0$ .*

Dans ce cas, le terme  $L_g\bar{M}$  peut être compris comme une approximation du gain DC entre la commande et la sortie. Nous avons donc le résultat suivant.

**Théorème 6** (Retour d'état pour régulation globale de la sortie constante). *Considérons le système (R.13) et supposons que les Hypothèses 5 et 7 sont vérifiées. Supposons également qu'il existe un nombre positif  $L_M > 0$  tel que  $\|\frac{\partial \bar{M}}{\partial x}(x)\| \leq L_M$ . Alors, pour chaque  $\kappa > 0$ , le système en boucle fermée avec (R.14) et*

$$u = \phi_0(x) + \kappa \Lambda^\top (\Lambda \Lambda^\top)^{-1} (\eta - \bar{M}(x)) - \Lambda^\top (\Lambda \Lambda^\top)^{-1} \Delta(x)$$

*résout le problème de la régulation globale à sortie constante.*

La méthode proposée est en retour d'état. Sous des hypothèses supplémentaires, une rétroaction de sortie statique est possible. Dans ce cas, l'hypothèse supplémentaire est que le système en boucle ouverte est suffisamment robuste. Le contrôleur devient un pur contrôleur I avec un petit gain. Nous avons l'hypothèse suivante.

**Hypothèse 8** (Robustesse pour rétroaction de la sortie). *Il existe deux nombres positifs  $a$  et  $\lambda_2$  tels que l'Hypothèse 7 est vérifiée et tels que*

$$\left( aP_0(x)g(x)\Psi - \frac{1}{a} \frac{\partial \Delta^\top}{\partial x}(x) \right) \left( aP(x)g(x)\Psi - \frac{1}{a} \frac{\partial \Delta^\top}{\partial x}(x) \right)^\top + L_{f_0}P_0(x) \preceq -2\lambda_2 I$$

est valide avec  $\Psi = \Lambda^\top (\Lambda \Lambda^\top)^{-1}$ .

Sous l'hypothèse ci-dessus, nous pouvons concevoir une commande en rétroaction de sortie pure résolvant le problème de régulation de sortie constante globale.

**Théorème 7** (Retour de sortie pour régulation globale de la sortie constante). *Considérons le système (R.13) et supposons que les Hypothèses 5 et 8 sont valables.*

*Supposons également qu'il existe un nombre  $L_M > 0$  tel que  $\left\| \frac{\partial \bar{M}}{\partial x}(x) \right\| \leq L_M$ . Alors il existe un  $\kappa^*$  tel que, pour chaque  $\kappa \in (0, \kappa^*]$ , le système à boucle fermée avec (R.14) et  $u = \phi_0(x) + \kappa \Psi \eta$  résout le problème de régulation globale à sortie constante.*

Nous soulignons les aspects suivants :

- pour les systèmes en forme normale à phase minimale, les hypothèses du Théorème 7 sont toujours vérifiées. Dans ce cas, le contrôleur est linéaire et consiste en un grand gain dans la variable  $e$  et un petit gain dans la variable  $\eta$ . Cela montre que notre résultat se rapproche des résultats existants dans la littérature pour cette classe de systèmes;
- pour les systèmes linéaires avec une non-linéarité Lipschitz, le contrôleur proposé peut être réécrit comme des LMIs.

### Régulation harmonique globale de la sortie

Dans cette section, nous considérons le problème de la régulation harmonique globale. En d'autres termes, nous considérons le système R.13 et faisons l'hypothèse suivante.

**Hypothèse 9** (Exosystème périodique). *Les signaux  $r$  et  $d$  sont variables dans le temps, lisses et périodiques, c'est-à-dire qu'il existe  $T > 0$  tel que  $r(t) = r(t + T)$  et  $d(t) = d(t + T)$  pour chaque  $t \geq 0$ .*

L'objectif est de construire un contrôleur dynamique tel que le contenu harmonique de l'erreur de contrôle à certaines fréquences soit nul. Nous formulons notre problème comme suit.

**Problème 2** (Réglage harmonique global). *Considérons le système (R.13). Trouvez un contrôleur du type*

$$\dot{\eta} = \phi(\eta, e), \quad u = \alpha(x, \eta) \tag{R.16}$$

tel que :

1. si  $(d, r) = (0, 0)$  alors l'origine du système est globalement asymptotiquement stable ;

2. pour chaque  $(d, r) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_e}$  qui satisfait l'Hypothèse 9, le système en boucle fermée a des trajectoires périodiques et bornées et l'erreur  $e(t)$  converge asymptotiquement vers une solution sans contenu harmonique aux fréquences  $\omega_\ell = \ell \frac{2\pi}{T}$  pour  $\ell = 0, 1, \dots, L$  pour un certain  $L \geq 0$ , c'est-à-dire que les premiers  $L$ -coefficients de Fourier sont nuls

$$c_\ell := \frac{1}{T} \int_0^T e(t) \exp\left(2i\ell\pi \frac{t}{T}\right) dt = 0, \quad \forall \ell = 0, \dots, L.$$

Comme dans le cas précédent, nous voulons résoudre le problème sans utiliser de formes normales et obtenir des résultats globaux en termes de domaine d'attraction des conditions initiales et d'amplitude des signaux externes. Nous encadrons à nouveau le problème avec des outils incrémentaux. En effet, si un système incrémental est forcé par un signal périodique, la trajectoire est elle-même périodique avec la même période. Motivés par Ghosh and Paden (2000); Astolfi et al. (2022a), nous construisons le modèle interne (c'est-à-dire la dynamique  $\eta$ ) comme une cascade d'oscillateurs aux fréquences à éliminer et qui traitent l'erreur d'ajustement. En d'autres termes, nous prenons

$$\dot{\eta} = \Phi\eta + \Gamma e$$

avec

$$\Phi = \text{blkdiag}\left(0 \quad \omega_1\Phi_1 \quad \dots \quad \omega_L\Phi_1\right), \quad \Gamma = \text{blkdiag}\left(\Gamma_0 \quad \Gamma_1 \quad \dots \quad \Gamma_L\right)$$

et

$$\Phi_1 = \text{blkdiag}\left(\phi, \dots, \phi\right) = \begin{pmatrix} \phi & & & \\ & \phi & & \\ & & \ddots & \\ & & & \phi \end{pmatrix}, \quad \phi = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

avec  $\Gamma_\ell = (\gamma, \dots, \gamma)$  où la paire  $(\omega_\ell\phi, \gamma)$  est contrôlable. Nous recherchons ensuite un stabilisateur pour le système en boucle fermée. Compte tenu de la structure du système étendu, nous proposons un contrôleur basé sur le forwarding mod $\{L_g V\}$  incrémental. Nous avons donc l'hypothèse suivante.

**Hypothèse 10** (Forwarding mod $\{L_g V\}$  incrémental pour le réglage harmonique). *On connaît trois fonctions  $C^1$   $\bar{M} : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$ ,  $\Delta : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_\eta}$  et  $\varrho : \mathbb{R}^{n_x} \mapsto \mathbb{R}^{n_u}$  telles que, pour tout  $x \in \mathbb{R}^{n_x}$ , les conditions suivantes sont vérifiées:*

- les fonctions  $\bar{M}$  et  $\Delta$  sont solution de  $L_{f_0}\bar{M}(x) = \Phi\bar{M}(x) + \Gamma h(x) + \Delta(x)$ ;
- il existe une matrice  $\Lambda$  telle que  $L_g\bar{M}(x) = \Lambda$  pour tout  $x$  et tel que le couple  $(\Phi, \Lambda^\top)$  est détectable ;
- la fonction  $\varrho$  satisfait  $\Lambda \frac{\partial \varrho}{\partial x}(x) = -\frac{\partial \Delta}{\partial x}(x)$ ;
- l'inégalité suivante est vérifiée  $L_{f_0}P_0(x) + \text{He} \left\{ P_0(x)g(x) \frac{\partial \varrho}{\partial x}(x) \right\} \preceq -\lambda I$ , pour un certain  $\lambda > 0$ .

Nous avons donc le résultat suivant.

**Théorème 8** (Régulation harmonique globale). *Considérons le système (R.13) étendu avec (R.16). Soit les Hypothèses 5 et 10 vérifiées. Supposons en outre qu'il*

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existe un nombre  $L_M > 0$  tel que  $\left\| \frac{\partial \bar{M}}{\partial x}(x) \right\|$ . Ensuite, pour chaque  $(d, r) \in \mathbb{R}^{n_u} \times \mathbb{R}^{n_e}$  satisfaisant l'Hypothèse 9, le système en boucle fermée avec le contrôleur dynamique

$$\dot{\eta} = \Phi\eta + \Gamma e, \quad \phi_0(x) + \kappa \Lambda^\top (\eta - \bar{M}(x)) + \varrho(x)$$

pour chaque  $\kappa > 0$ , résout le Problème 2.

Nous soulignons les aspects suivants :

- pour les systèmes en forme normale à phase minimale, le résultat proposé récupère les résultats existants dans la littérature. Le contrôleur est un grand gain dans la variable  $e$  et un petit gain dans la variable  $(\eta - \bar{M}(x))$ ;
- pour les systèmes linéaires avec une non-linéarité Lipschitz, nous proposons un test qui, s'il est vérifié, satisfait automatiquement toutes les hypothèses du Théorème 8.

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## Chapitre 3: synchronisation multi-agents

Dans ce chapitre, nous traitons du problème de la synchronisation multi-agents. Dans ce cadre, nous avons  $N$  entités distinctes appelées agents qui communiquent entre elles via un protocole de communication. L'objectif est de construire un contrôleur distribué de manière à obtenir un consensus, c'est-à-dire un accord entre les états des différents agents. Dans notre cas, nous considérons un réseau homogène, c'est-à-dire un réseau dans lequel le modèle décrivant chaque agent est le même. Le problème n'est pas nouveau dans la littérature. Elle a été complètement résolue pour les systèmes linéaires, voir (Isidori, 2017, Chapitre 5). Dans ce cas, le problème peut être considéré comme un problème de stabilisation robuste (ou simultané). Pour les systèmes non linéaires, plusieurs solutions ont été proposées, voir Isidori et al. (2014); Pavlov et al. (2022); Arcak (2007); Stan and Sepulchre (2007); Casadei et al. (2019a) et références. Dans notre cas, nous encadrons le problème de synchronisation avec des outils incrémentaux. Cela est dû au fait que le modèle est le même pour chaque agent. Ils ne diffèrent que par leur condition initiale. Le rôle du contrôleur est donc de faire oublier à l'agent individuel ses conditions initiales et de faire en sorte que, asymptotiquement, deux trajectoires d'un même système dynamique soient coïncidentes. On retrouve en ce sens le concept de base de la stabilité incrémentale. Dans nos résultats, nous considérons deux cadres différents. Tout d'abord, nous considérerons les réseaux connectés et dirigés et encadrerons le problème par des outils incrémentaux Euclidiens. Ensuite, nous considérons les réseaux connectés par un leader et les réseaux indirects et nous encadrons le problème par des outils incrémentaux Riemanniens.

### Synchronisation des réseaux avec contraction Euclidienne

Dans cette section, nous considérons un réseau où chaque agent est décrit par

$$\dot{x}_i = f(x_i, t) + Bu_i \quad (\text{R.17})$$

où  $x_i \in \mathbb{R}^{n_x}$  est l'état de l'agent  $i$ ,  $u_i \in \mathbb{R}^{n_u}$  est le contrôleur de l'agent  $i$ ,  $f : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x}$  est une fonction lisse et  $B$  est une matrice. Soit  $\mathbf{x}$  l'état de l'ensemble du réseau et  $\ell_{ij}$  les éléments du Laplacien  $\mathcal{L}$ . Dans cette section, nous considérons l'hypothèse suivante.

**Hypothèse 11** (Graphe pour la synchronisation du réseau). *Le graphe est connecté et dirigé.*

En d'autres termes, il existe un chemin entre chaque agent et chaque autre agent. En outre, le graphe est dirigé, c'est-à-dire que les liens de communication peuvent être unidirectionnels. Nous formulons notre problème comme suit.

**Problème 3** (Synchronisation du réseau). *Soit  $\varphi$  une fonction telle que la variété de synchronisation*

$$\mathcal{D} := \{\mathbf{x} \in \mathbb{R}^{Nn_x} \mid x_i = x_j, \text{ for all } i, j \in \{1, \dots, N\}\}, \quad (\text{R.18})$$

*est globalement uniformément exponentiellement stable pour le système en boucle fermée*

$$\dot{x}_i = f(x_i, t) + Bu_i, \quad u_i = \sum_{j=1}^N \ell_{ij} \varphi(x_j, t) \quad (\text{R.19})$$

c'est-à-dire qu'il existe deux nombres strictement positifs  $\lambda, k > 0$  tels que

$$|\mathcal{X}(\mathbf{x}^\circ, t, t_0)|_{\mathcal{D}} \leq k \exp(-\lambda(t - t_0)) |\mathbf{x}^\circ|_{\mathcal{D}} \quad (\text{R.20})$$

pour chaque  $t \geq t_0$  dans le domaine d'existence des solutions. Nous disons alors que le problème de synchronisation multi-agent global est résolu.

Motivés par le cas linéaire, nous recherchons un contrôleur robuste qui stabilise l'origine de (R.17). Pour cette raison, nous nous concentrons sur une structure impliquant un contrôleur avec une marge de gain infinie qui peut être obtenue en résolvant une inégalité de Riccati. Dans ce cas, on associe chaque agent à une sortie de type

$$y_i = Cx_i. \quad (\text{R.21})$$

Nous avons donc le résultat suivant.

**Théorème 9** (Synchronisation du réseau avec retour de sortie). *Considérons un réseau connecté où chaque agent est décrit par (R.17), (R.21). S'il existe une matrice symétrique définie positive  $P = P^\top \succ 0$ , deux matrices  $D, E$  et deux nombres réels positifs  $\rho, \varepsilon > 0$  tels que*

$$L_f P(x, t) - \rho C^\top E^\top E C \preceq -\varepsilon P, \quad PBD = C^\top E^\top$$

pour tout  $(x, t)$ , alors pour tout  $\kappa$  suffisamment grand, le système en boucle fermée (R.19) avec  $\varphi(x_j, t) = -\kappa DEy_j$  résout le problème de synchronisation multi-agent global.

### Synchronisation vers le leader avec contraction Riemannienne

Dans cette section, nous essayons d'étendre les résultats à une classe plus générale de systèmes définis par

$$\dot{x}_i = f(x_i, t) + g(x_i, t)u_i, \quad y_i = h(x_i, t) \quad (\text{R.22})$$

avec  $x_i \in \mathbb{R}^{n_x}$ ,  $u_i \in \mathbb{R}^{n_u}$  et  $y_i \in \mathbb{R}^{n_y}$ . Dans ce cas, supposons que nous ayons un graphe leader connecté. En d'autres termes, il existe un agent, appelé 'leader', dont le contrôleur est nul et pour lequel il existe un chemin entre le leader et chaque autre agent. En outre, considérons un graphe indirect, c'est-à-dire que les liens de communication sont bidirectionnels.

**Hypothèse 12** (Graphe pour la synchronisation vers le leader). *Le graphe est non dirigé et connecté au leader. En d'autres termes, il contient au moins un arbre couvrant avec le leader comme racine.*

Nous formulons notre problème comme suit.

**Problème 4** (Synchronisation vers le leader). *Soit  $\varphi$  une fonction telle que la variété de synchronisation (R.18) est globalement uniformément exponentiellement stable pour le système en boucle fermée*

$$\dot{x}_i = f(x_i, t) + g(x_i, t)u_i, \quad y_i = h(x_i, t), \quad u_i = \sum_{j=1}^N \ell_{ij} \varphi(y_j, t) \quad (\text{R.23})$$

c'est-à-dire qu'il existe deux nombres strictement positifs  $\lambda, k > 0$  tels que (R.20) est valide pour chaque  $t \geq t_0$  dans le domaine d'existence des solutions du leader. Nous disons alors que le problème de synchronisation multi-agent global vers le leader est résolu.

Dans ce qui suit, nous considérons trois cas différents pour (R.23).

1.

**Théorème 10** (Retour d'état complet). *Considérons le système (R.23) avec  $y_i = x_i$ . Supposons que l'Hypothèse 12 soit vérifiée. Supposons également qu'il existe une fonction matricielle  $C^1$  symétrique et définie positivement  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  et 4 nombres positifs  $\underline{p}, \bar{p}, \varepsilon, \rho$  tels que:*

- l'inégalité de type Riccati suivante est satisfaite

$$L_f P(x, t) - \rho P(x, t) g(x, t) g^\top(x, t) P(x, t) \preceq -\varepsilon P(x, t), \quad \underline{p}I \preceq P(x, t) \preceq \bar{p}I, \quad (\text{R.24})$$

pour tous  $(x, t)$ ;

- Le champ vectoriel  $g$  est un champ de Killing pour  $P$ , c'est-à-dire  $L_g P(x, t) = 0$  pour tous  $(x, t)$ ;

- il existe une fonction  $C^2$   $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_u}$  telle que  $\frac{\partial \alpha^\top}{\partial x}(x, t) = P(x, t)g(x, t)$  pour tous  $(x, t)$ .

Alors, pour tout  $\kappa > 0$  suffisamment grand, le système en boucle fermée (R.23) avec  $\varphi(y_j, t) = -\kappa \alpha(x_j, t)$  résout le problème de synchronisation globale vers le leader.

2.

**Théorème 11** (Retour de sortie incrémental passif). *Considérons le système (R.23). Supposons que l'Hypothèse 12 soit vérifiée. Supposons également qu'il existe une fonction matricielle  $C^1$  symétrique et définie positivement  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  et 4 nombres positifs  $\underline{p}, \bar{p}, \varepsilon, \rho$  tels que:*

- l'inégalité de type Riccati suivante est satisfaite

$$L_f P(x, t) - \rho \frac{\partial h^\top}{\partial x}(x, t) \frac{\partial h}{\partial x}(x, t) \preceq -\varepsilon P(x, t), \quad \underline{p}I \preceq P(x, t) \preceq \bar{p}I,$$

pour tous  $(x, t)$ ;

- Le champ vectoriel  $g$  est un champ de Killing pour  $P$ , c'est-à-dire  $L_g P(x, t) = 0$  pour tous  $(x, t)$ ;

- on a que  $\frac{\partial h^\top}{\partial x}(x, t) = P(x, t)g(x, t)$  pour tous  $(x, t)$ .

Alors, pour tout  $\kappa > 0$  suffisamment grand, le système en boucle fermée (R.23) avec  $\varphi(y_j, t) = -\kappa y_j$  résout le problème de synchronisation globale vers le leader.

3.

**Théorème 12** (Retour de sortie avec entrée pleine). *Considérons le système (R.23) avec  $g(x_i, t) = I$ . Supposons que l'Hypothèse 12 soit vérifiée. Supposons également qu'il existe une fonction matricielle  $C^1$  symétrique et définie positivement  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$  et 4 nombres positifs  $\underline{p}, \bar{p}, \varepsilon, \rho$  tels que:*

- l'inégalité de type Riccati suivante est satisfaite

$$L_f P(x, t) - \rho \frac{\partial h^\top}{\partial x}(x, t) \frac{\partial h}{\partial x}(x, t) \preceq -\varepsilon P(x, t), \quad \underline{p}I \preceq P(x, t) \preceq \bar{p}I,$$

pour tous  $(x, t)$ ;

- *Le champ vectoriel*

$$\alpha(x, t) = P^{-1}(x, t) \frac{\partial h}{\partial x}(x, t)$$

est un champ de Killing pour  $P$ , c'est-à-dire  $L_\alpha P(x, t) = 0$  pour tous  $(x, t)$ .

Alors, pour tout  $\kappa > 0$  suffisamment grand, le système en boucle fermée (R.23) avec  $\varphi(y_j, t) = -\kappa\alpha(x_j, t)y_j$  résout le problème de synchronisation globale vers le leader.

**Remarque 2.** *Nous notons que : les contrôleurs proposés peuvent être étendus aux classes de systèmes en forme normale à phase minimale et aux systèmes linéaires avec des non-linéarité sectorielle. Dans le premier cas, le régulateur est un grand gain dans la variable  $y_j$ . Dans le second cas, les conditions peuvent être réécrites comme des LMIs. Dans les deux cas, nous récupérons les conditions existantes dans la littérature, voir Chopra and Spong (2008); Zhang et al. (2014).*

### Apprentissage profond pour l'estimation de la métrique

Un des inconvénients de la méthode proposée dans la section précédente est le fait qu'il est souvent difficile de trouver une métrique  $P$  non linéaire qui satisfasse les conditions des théorèmes proposés. Dans cette section, nous cherchons des méthodes pour compenser ce défaut. En particulier, nous donnons deux résultats. Tout d'abord, nous montrons comment la synchronisation peut être obtenue dans un domaine plus restreint mais avec des hypothèses plus légères (sans l'hypothèse du champ de Killing). Ensuite, nous proposons un algorithme général basé sur les réseaux neuronaux profonds pour l'apprentissage de la métrique.

**Théorème 13** (Synchronisation sans champ de Killing). *Considérons le système (R.22). Supposons que l'Hypothèse 12 soit vérifiée et qu'il existe une fonction matricielle  $C^1$  prenant des valeurs définies positives symétriques  $P : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_x \times n_x}$ , des constantes strictement positives  $\underline{p}, \bar{p}, \rho, \varepsilon > 0$  et une fonction  $C^2$   $\alpha : \mathbb{R}^{n_x} \times \mathbb{R} \mapsto \mathbb{R}^{n_u}$  telle que (R.24) soit vérifiée et  $\frac{\partial \alpha^\top}{\partial x}(x, t) = P(x, t)g(x, t)$ . Supposons que  $X \subset \mathbb{R}^{N n_x}$  soit un ensemble compact et que  $\kappa$  soit suffisamment grand. Alors, il existe un nombre réel strictement positif  $\epsilon_X > 0$  tel que, si  $\|L_g P(x, t)\| \leq \epsilon_X$  pour tout  $(x, t) \in X \times \mathbb{R}$ , la trajectoire du réseau en boucle fermée (R.23) avec la loi de commande distribuée à retour d'état  $\varphi(x_j, t) = -\kappa\alpha(x_j, t)$  à une convergence exponentielle vers la variété de synchronisation (R.18) pour toute existence temporelle de solutions en  $X$ , c'est-à-dire pour tout  $t_f \geq t_0$  tel que  $\mathcal{X}_i(x_i^\circ, t, t_0) \in X$  pour tout  $t \in [t_0, t_f]$  et pour tout  $i = 1, \dots, N$ .*

Nous proposons maintenant un algorithme général basé sur des Réseaux Neuronaux Profonds (RNP) pour résoudre les hypothèses du Théorème 13 (des conditions similaires peuvent être dérivées pour les Théorèmes 11 et 12). L'utilisation de techniques d'apprentissage automatique combinées à la théorie du contrôle n'est pas un sujet nouveau, voir Zoboli et al. (2021); Sanchez-Escalonilla et al. (2021). Les mêmes idées ont également été utilisées pour obtenir des propriétés incrémentales, voir Tsukamoto et al. (2021); Wei et al. (2021). L'algorithme que nous proposons est le suivant.

1. Nous construisons la métrique comme

$$P(x_i, \vartheta') = \begin{pmatrix} p_1(x_i, \vartheta') & p_2(x_i, \vartheta') & \cdots & p_n(x_i, \vartheta') \\ p_2(x_i, \vartheta') & p_{n+1}(x_i, \vartheta') & \cdots & p_{2n}(x_i, \vartheta') \\ \vdots & \ddots & & \vdots \\ p_n(x_i, \vartheta') & p_{2n}(x_i, \vartheta') & \cdots & p_M(x_i, \vartheta') \end{pmatrix},$$

où  $M$  est le nombre d'entrées à apprendre,  $p = (p_0(x_i, \vartheta'), \dots, p_M(x_i, \vartheta'))^\top$  est la sortie du RNP $_P : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\vartheta'}} \mapsto \mathbb{R}^M$  et  $\vartheta' \in \mathbb{R}^{n_{\vartheta'}}$  est le vecteur de paramètres de la RNP.

2. Nous entraînons le système neuronal en essayant de minimiser le coût

$$J_P(x, \vartheta') = \sum_{i=1}^4 w_i J_i(x, \vartheta'),$$

où  $w = (w_1, \dots, w_4)$  est un vecteur de poids scalaires et

$$M_1 = L_f P(x, \vartheta') - \rho P(x, \vartheta') g(x) g^\top(x) P(x, \vartheta') + \varepsilon I$$

$$M_2 = L_g P(x, \vartheta') - \epsilon I$$

$$M_3 = -L_g P(x, \vartheta') - \epsilon I$$

$$M_4 = -P(x, \vartheta') + \underline{p} I$$

$$J_i(x, \vartheta') = \ln \left( \max \left( \Re \left\{ \lambda_M(M_i) \right\}, 0 \right) + 1 \right), \quad i = 1, \dots, 4$$

Notons comment chacun des coûts  $J_i$  sert à satisfaire l'une des conditions du Théorème 10. En parallèle avec le RNP, nous entraînons un estimateur de paramètres qui fournit les valeurs de  $\rho, \underline{p}, \epsilon, \varepsilon$ .

3. Nous formons maintenant un deuxième réseau neuronal RNP $_\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_{\vartheta'}} \mapsto \mathbb{R}^{n_u}$  pour satisfaire l'hypothèse d'intégrabilité. Soit  $\vartheta''$  l'ensemble des paramètres de RNP $_\alpha$ . Essayons de minimiser le coût

$$J_\alpha(x, \vartheta'') = \left\| \frac{\partial \text{RNP}_\alpha}{\partial x}(x, \vartheta'') - g(x)^\top P(x, \vartheta') \right\|^2.$$

4. En conclusion, nous appliquons au réseau le contrôleur défini par

$$u_i = -\kappa \sum_{j=0}^N \ell_{ij} \text{RNP}_\alpha(x_j, \vartheta'').$$

L'algorithme a été mis en œuvre avec succès dans un problème de synchronisation vers un leader pour un réseau d'oscillateurs de Lorentz.

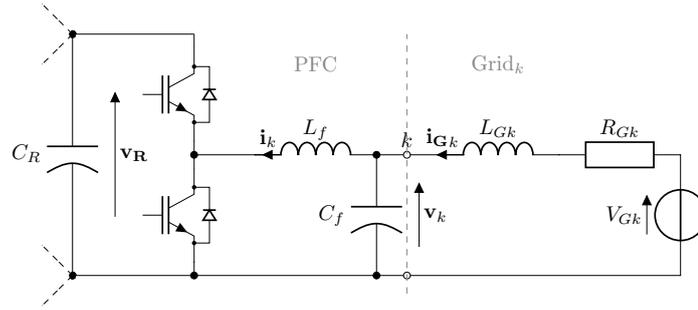


Figure 22: Détail de la  $k$ -ième branche du PFC (à gauche) et le modèle de grille de Thévenin proposé tel que vu par cette borne (à droite).

## Chapitre 4: Applications

Dans ce chapitre, nous traitons deux applications pratiques. Le premier est un problème de suivi robuste du point de consigne pour un Convertisseur de Flux de Puissance (PFC). Ce dispositif est un circuit électrique utilisé dans les réseaux électriques maillés dont le rôle est de réguler la puissance dans les lignes auxquelles il est rattaché. Ceci doit être réalisé malgré l'incertitude des paramètres du système. Le deuxième problème que nous abordons est un problème de régulation harmonique robuste pour une Machine de Ventilation Mécanique (MVM). Ces dispositifs sont utilisés dans le domaine médical pour aider les patients qui ne peuvent pas respirer par eux-mêmes. Dans ce cas, l'objectif est le suivi d'un signal périodique représentant la pression dans les poumons du patient. Ceci doit être réalisé sans une connaissance parfaite du modèle du système. Les contrôleurs proposés pour les deux applications ont été validés avec succès dans la pratique. Les deux applications ont été réalisées en collaboration avec des équipes d'autres universités (Lab. Ampere, INSA, Lyon pour le PFC et Université de Eindhoven, Pays-Bas pour la MVM).

### Suivi robuste du point de consigne pour un Convertisseur de Flux de Puissance

Un contrôleur de flux de puissance DC est un convertisseur DC-DC multi-terminal situé à un noeud d'un réseau maillé. Son objectif est la régulation de la puissance dans chaque ligne du noeud, malgré la forte intermittence des générateurs renouvelables. Plusieurs solutions ont été proposées pour résoudre une telle tâche, voir par exemple Simon et al. (2021b); Natori et al. (2018); Barara et al. (2017). Cependant, les résultats existants présentent plusieurs inconvénients : de l'absence d'un modèle dynamique, à l'absence de preuve de stabilité, de la conception de la loi de contrôle basée sur une connaissance complète des paramètres, à des domaines d'attraction uniquement locaux. Dans cette section, un PFC avec  $n_T$ -terminal est considéré. Supposons un schéma de commutation PWM synchrone sur chaque branche, et supposons que la dynamique du réseau soit partiellement inconnue pour tenir compte de sa grande variabilité. Le circuit considéré est composé de  $n_T$  convertisseurs buck-boost identiques dont le côté haut est connecté en parallèle à un unique condensateur réservoir  $C_R$ . Le réseau connecté à la borne  $k$  est modélisé par un circuit équivalent de Thevenin, voir Fig. 22. Le système est modélisé dans l'espace d'état comme un système bilinéaire couplé à une sortie polynomiale. L'état du système a une dimension  $3n_T + 1$  et est constitué des courants et des tensions dans chaque ligne, ainsi que de la tension sur le condensateur réservoir. Le vecteur d'entrée a une

dimension de  $n_T$  et est constitué par le rapport cyclique de chaque borne et, pour cette raison, il est saturé dans l'ensemble  $\mathcal{U} = [0, 1]^{n_T}$ . La sortie est polynomiale et elle est constituée par la tension sur le condensateur réservoir et les puissances sur  $n_T - 1$  lignes (la dernière est automatiquement réglée à la valeur correcte). Par conséquent, le modèle du CFP est

$$\dot{\mathbf{x}} = \mathbf{A}(\theta)\mathbf{x} + \mathbf{N}(\mathbf{x})\mathbf{u} + \mathbf{q}(\theta), \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{H}(\mathbf{x})\mathbf{x}, \quad \mathbf{u} \in \mathcal{U}, \quad (\text{R.25})$$

où  $\mathbf{x}$  est l'état,  $\mathbf{u}$  est la commande,  $\mathbf{y}$  est la sortie,  $\mathbf{A}(\theta)$ ,  $\mathbf{C}$ ,  $\mathbf{q}(\theta)$ , sont des matrices dépendant d'un vecteur paramètre inconnu  $\theta$  prenant des valeurs dans un ensemble compact  $\Theta$  et  $\mathbf{N}(\mathbf{x})$ ,  $\mathbf{H}(\mathbf{x})$  sont des matrices de fonctions polynomiales du second ordre. Soit  $\mathbf{r} \in \mathcal{R} \subset \mathbb{R}^{n_r}$  le vecteur de référence sur lequel la sortie doit être réglée et  $\delta = (\theta, \mathbf{r})$ . Désignons par  $\delta_{nom}$  la valeur nominale de  $\delta$ . Le problème à résoudre est le suivant.

**Problème 5** (Réglage robuste du PFC). *Étant donné un ensemble nominal de paramètres  $(\Theta, \mathcal{R})$  et  $\delta_{nom} \in (\Theta, \mathcal{R})$ , trouvons des fonctions  $\phi : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \rightarrow \mathbb{R}^{n_\eta}$  et  $\alpha : \mathbb{R}^{n_x} \times \mathbb{R}^{n_\eta} \rightarrow \mathcal{U}$  telles que pour tout ensemble compact arbitrairement grand de conditions initiales il existe un  $\bar{\delta} > 0$  telle que, pour tous  $\delta$  tal que  $|\delta - \delta_{nom}| \leq \bar{\delta}$ , les trajectoires résultantes du système en boucle fermée avec le régulateur*

$$\begin{aligned} \dot{\boldsymbol{\eta}} &= \phi(\mathbf{x}, \boldsymbol{\eta}), \\ \mathbf{u} &= \alpha(\mathbf{x}, \boldsymbol{\eta}) \end{aligned}$$

sont bornés pour les temps positifs et  $\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{r}$ .

Le contrôleur proposé est le suivant. Tout d'abord, nous étendons le système avec une action intégrale qui traite l'erreur de contrôle

$$\dot{\boldsymbol{\eta}} = \mathbf{y} - \mathbf{r} \quad (\text{R.26})$$

avec  $\boldsymbol{\eta} \in \mathbb{R}^{n_\eta}$ . Grâce à l'action intégrale, si nous pouvons garantir l'existence d'un point d'équilibre  $(\mathbf{x}^*, \boldsymbol{\eta}^*)$ , alors nous avons que  $0 = \dot{\boldsymbol{\eta}} = \mathbf{y} - \mathbf{r} \implies \mathbf{y} = \mathbf{r}$ . Nous réalisons ensuite un stabilisateur pour le système étendu. Le contrôleur proposé est basé sur une rétroaction constante (réalisé par inversion du modèle) pour déplacer le point d'équilibre vers l'origine et un second terme réalisé par des techniques par forwarding spécialisées pour les systèmes bilinéaires à sortie polynomiale. En particulier, nous pouvons montrer que les hypothèses suffisantes pour la mise en œuvre de cette technique sont toujours satisfaites pour le système considéré et pour les valeurs des paramètres et des références dans  $(\Theta, \mathcal{R})$ . Enfin, nous ajoutons une action anti-windup  $\zeta$  réalisée par une deadzone sur le contrôleur due au fait que  $\mathbf{u}$  est saturé en  $\mathcal{U}$ . L'anti-windup nous permet d'atténuer les effets incontrôlés de l'action intégrale causés par la saturation. Le schéma de commande final est illustré en Fig. 23. Ce résultat s'exprime de la manière suivante.

**Théorème 14** (Suivi robuste du point de consigne de sortie pour le PFC). *Considérons l'ensemble des références et des paramètres  $\mathcal{S}$  pour lesquels il existe une solution au problème. Soit  $\delta_{nom} = (\theta_{nom}, \mathbf{r}_{nom}) \in \text{int}\{\mathcal{S}\}$  et  $(\mathbf{x}_{nom}^*, \mathbf{u}_{nom}^*)$  le point d'équilibre correspondant où  $\mathbf{y} = \mathbf{r}$ . Ensuite, pour tout  $d > 0$  suffisamment grand, le Problème 5 est résolu avec le contrôleur dynamique*

$$\dot{\boldsymbol{\eta}} = \phi(\mathbf{x}, \boldsymbol{\eta}) := \mathbf{y} - \mathbf{r} - \text{Dz}_d(\boldsymbol{\eta} - M(\mathbf{x} - \mathbf{x}_{nom}^*))$$

$$\mathbf{u} = \alpha(\mathbf{x}, \boldsymbol{\eta}) := \mathbf{u}_{nom}^* + \text{Sat}_{-\mathbf{u}_{nom}^*}^{\mathbf{1}-\mathbf{u}_{nom}^*}(\psi(\mathbf{x} - \mathbf{x}_{nom}^*, \boldsymbol{\eta}))$$

$$\psi(x, z) = - (x^\top P(N(x) + B) - (\eta - M(x))^\top \Omega (M_0 + 2R(x)) (N(x) + B))^\top,$$

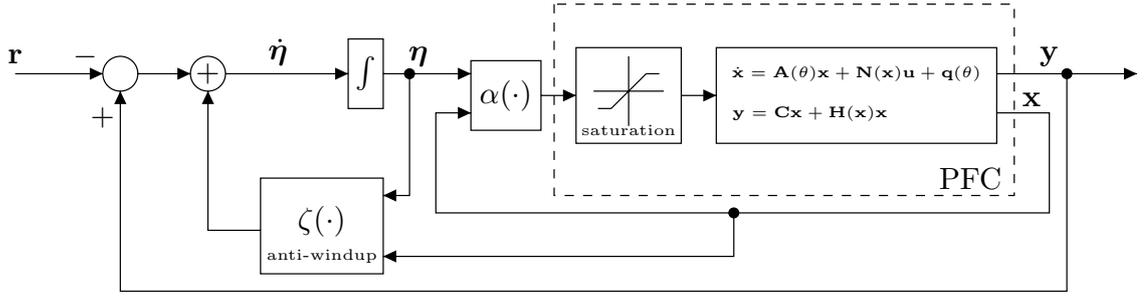


Figure 23: Structure de contrôle proposée pour le PFC

où les matrices  $A, B, C$  et les fonctions  $N(x), H(x)$  sont définies comme suit

$$A := \mathbf{A}(\theta_{nom}) + \sum_{j=1}^{n_T} \mathbf{N}_j \mathbf{u}_{nom,j}^*, \quad B := \mathbf{N}(\mathbf{x}_{nom}^*)$$

$$C := \mathbf{C} + 2\mathbf{H}(\mathbf{x}_{nom}^*), \quad N(x) := \mathbf{N}(x),$$

$$H(x) := [H_1^\top x \quad \dots \quad H_{n_\eta}^\top x]^\top = \mathbf{H}(x), \text{ pour } P, M \text{ satisfaisant}$$

$$PA + A^\top P \prec 0, \quad M_i A + A^\top M_i = \frac{1}{2} (H_i + H_i^\top), \quad \forall i \in \{1, \dots, n_\eta\},$$

$$M_0 = CA^{-1} \text{ avec } M(x) = M_0 x + [M_1 x, \dots, M_{n_\eta} x]^\top \text{ et chaque } \Omega = \Omega^\top \succ 0.$$

La solution proposée a été validée avec succès avec un montage expérimental avec un PFC attaché à 3 terminaux et avec un montage en simulation avec un PFC attaché à 5 terminaux.

## Régulation harmonique pour une Machine de Ventilation Mécanique

Nous examinons ici un appareil de ventilation mécanique (MVM). Les ventilateurs mécaniques sont des équipements essentiels dans les unités de soins intensifs pour aider les patients qui ne peuvent pas respirer par eux-mêmes ou qui ont besoin d'une assistance pour respirer suffisamment. Une ventilation à pression contrôlée (PVC) est considérée. En raison de son importance critique, la modélisation et le contrôle des dispositifs de PVC ont été intensivement étudiés. Un résumé des techniques de modélisation et de contrôle des machines de ventilation mécanique peut être trouvé dans Borrello (2005). En ce qui concerne les stratégies de contrôle, les techniques de base font appel à des lois de contrôle linéaires. Il en résulte généralement des comportements sous-optimaux, notamment en termes de performances de suivi. D'autres stratégies de contrôle ont été proposées. Cependant, ces solutions nécessitent généralement la connaissance d'un ensemble de paramètres représentant la dynamique du patient, une information qui n'est généralement pas parfaitement connue en raison des différentes typologies de patients.

D'un point de vue structurel, le système PVC est entraîné par une soufflerie. Ce ventilateur comprime l'air pour obtenir le profil de référence de pression souhaité dans les poumons du patient. La pression des voies aériennes est augmentée pendant l'inspiration (IPAP), afin d'atteindre la pression positive inspiratoire des voies aériennes, remplissant les poumons du patient d'air. Ensuite, le ventilateur diminue la pression jusqu'à la pression positive de fin d'expiration (PEEP), de sorte que l'air

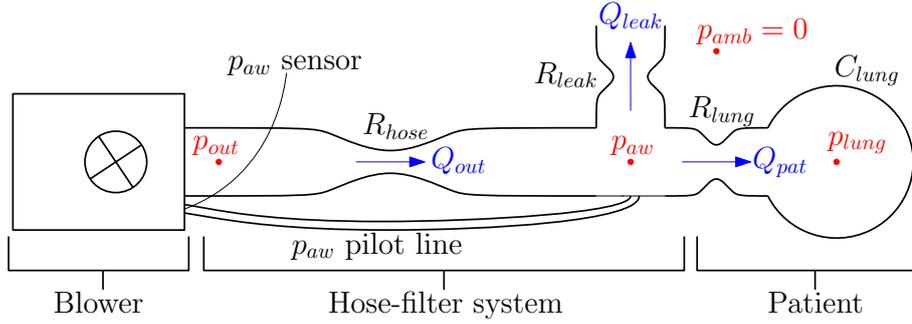


Figure 24: Représentation schématique du système souffleur-tuyau-patient, avec les résistances, la compliance pulmonaire, les pressions et les débits correspondants.

contenu dans les poumons soit évacué. Du point de vue du contrôle, l'objectif d'un MVM est d'assurer le suivi d'un signal. Un tel signal représente la pression dans les poumons du patient pendant la phase de respiration. Il s'agit donc d'un signal périodique.

Le système de machine de ventilation considéré peut être modélisé comme un système dynamique non linéaire de forme Lur'e. Les principaux composants de ce système sont la soufflante, le système tuyau-filtre et le patient. Un ventilateur centrifuge comprime l'air ambiant pour obtenir la pression de sortie souhaitée  $p_{out}$ . La différence entre  $p_{out}$  et la pression des voies respiratoires  $p_{aw}$  résulte du débit de sortie  $Q_{out}$  à travers le tuyau. Ce tuyau est modélisé à l'aide d'un modèle de tuyau non linéaire. Le flux à travers le tuyau, c'est-à-dire le flux de sortie  $Q_{out}$ , est divisé en un flux patient  $Q_{pat}$  et un flux de fuite  $Q_{leak}$ . La fuite prévue près du patient est utilisée pour évacuer l'air riche en  $CO_2$  du système. Enfin, les poumons du patient sont gonflés et dégonflés par le flux du patient. La Fig. 24 présente une structure en bloc. Dans le système PCV, la pression près de la bouche du patient, la pression des voies respiratoires  $p_{aw}$ , doit suivre une référence de pression cible souhaitée  $p_{target}$ , c'est-à-dire  $r := p_{target}$ . Sur un intervalle périodique prédéfini, de longueur  $T$ , le niveau de pression est augmenté jusqu'à l'IPAP, et par conséquent abaissé jusqu'à la PEEP. Ces niveaux de pression variables garantissent le flux d'air souhaité dans et hors des poumons du patient. La durée totale de la respiration  $T$  se compose du temps d'inspiration  $T_i$  et du temps d'expiration  $T_e$ , c'est-à-dire  $T = T_i + T_e$ . Le but de la commande du système PCV est d'obtenir une petite erreur de suivi  $e := r - p_{aw}$ , où la référence  $r(t)$  est un signal variant dans le temps qui est parfaitement périodique avec une longueur d'intervalle  $T$ , c'est-à-dire  $r(t) = r(t + T)$  pour un  $T > 0$  connu et tous les  $t \geq 0$ .

Le modèle de la MVM est défini par

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u + \mathbf{E}\mathbf{w} \\ \mathbf{y} &= \mathbf{M}\mathbf{x} + \mathbf{N}\mathbf{w} \\ \mathbf{w} &= -\varphi(\mathbf{y}) = R_{hose}(\mathbf{y}) + \ell\mathbf{y} \\ \mathbf{v} &= \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{w},\end{aligned}$$

où  $\mathbf{x} \in \mathbb{R}^7$  est l'état représentant la dynamique du patient  $G_b$  (obtenue par des mesures de fréquence) et la dynamique des pertes  $G_p$ ,  $u \in \mathbb{R}$  est le contrôle,  $\mathbf{y}, \mathbf{w} \in \mathbb{R}$ ,  $\mathbf{v} \in \mathbb{R}$  est la sortie mesurée et  $\mathbf{A}, \mathbf{B}, \mathbf{E}, \mathbf{M}, \mathbf{N}, \mathbf{C}$ , et  $\mathbf{D}$  sont des matrices réelles de dimensions appropriées. Le modèle en boucle ouverte de la machine est présenté dans la Fig. 25 La non-linéarité statique  $\varphi : \mathbb{R} \mapsto \mathbb{R}$  satisfait  $\varphi(0) = 0$ , et la

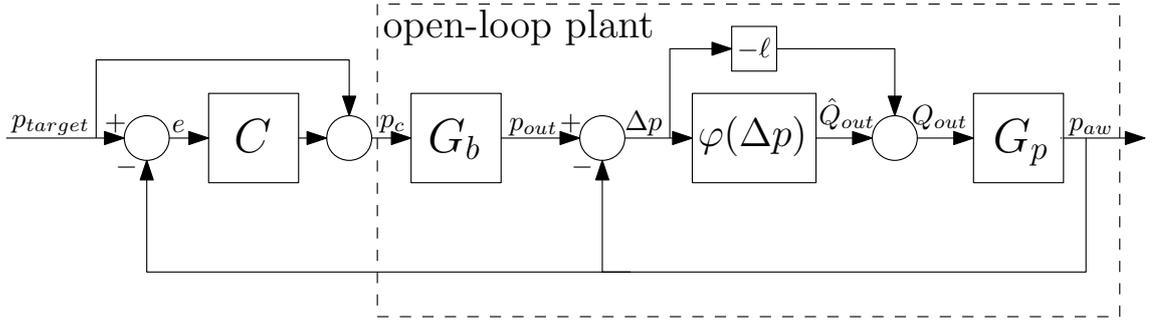


Figure 25: Schéma fonctionnel du système de ventilation complet, avec  $G_b$  la dynamique du ventilateur,  $C$  un contrôleur de rétroaction arbitraire,  $G_p$  la dynamique de la fuite du patient, et  $\varphi(\Delta p) = R_{hose}(\Delta p) + \ell \Delta p$  le modèle non linéaire du tuyau.

condition de limite secteur incrémental suivante :

$$\underline{\varphi} \leq \frac{\varphi(\mathbf{y}_1) - \varphi(\mathbf{y}_2)}{\mathbf{y}_1 - \mathbf{y}_2} \leq \bar{\varphi}, \quad (\text{R.27})$$

pour tout  $\mathbf{y}_1 \neq \mathbf{y}_2$ , pour certaines constantes non négatives connues  $0 \leq \underline{\varphi} \leq \bar{\varphi}$ . L'objectif de la commande est de réguler la sortie  $\mathbf{v}$  du système à une référence  $r$  bornée et périodique. Par conséquent, l'erreur de régulation de la sortie est définie comme

$$e(t) := r(t) - \mathbf{v}(t). \quad (\text{R.28})$$

La solution proposée consiste à utiliser un contrôleur dynamique de la forme suivante

$$\dot{\eta} = \phi(\eta, e) := \Phi \eta + \Gamma e, \quad u = \alpha(\eta, e) := K \eta$$

où  $\eta = [\eta_0 \ \eta_1^T \ \dots \ \eta_L^T]^T \in \mathbb{R}^{(2L+1) \times 1}$  avec  $\eta_0 \in \mathbb{R}$  et  $\eta_k \in \mathbb{R}^{2 \times 1}$  pour  $k = 1, \dots, L$ , et où les matrices  $\Phi \in \mathbb{R}^{(2L+1) \times (2L+1)}$ ,  $\Gamma \in \mathbb{R}^{(2L+1) \times 1}$ , et  $K \in \mathbb{R}^{1 \times (2L+1)}$  sont définies par

$$\Phi := \text{blkdiag}(0, \phi_1, \dots, \phi_L), \quad \Gamma := [\gamma_0, \gamma_1^T, \dots, \gamma_L^T]^T, \quad K := [\kappa_0, \kappa_1, \dots, \kappa_L], \quad (\text{R.29})$$

où

$$\phi_k := k \begin{bmatrix} 0 & \frac{2\pi}{T} \\ -\frac{2\pi}{T} & 0 \end{bmatrix}, \quad k = 1, \dots, L, \quad (\text{R.30})$$

avec  $\gamma_0 \neq 0$  le gain d'intégrateur tel que  $\eta_0$  embarque un intégrateur, et la matrice  $\gamma_k \in \mathbb{R}^{2 \times 1}$  est choisie de telle sorte que la paire  $(\phi_k, \gamma_k)$  soit contrôlable pour tout  $k = 1, \dots, L$ . Nous obtenons les résultats suivants :

- le choix de  $K$  et donc la stabilité du système en boucle fermée est obtenue par une version incrémentale du critère circulaire. Cela garantit que le système en boucle fermée est incrémentalement ESS par rapport à la référence  $r$ ;
- puisque  $r$  est périodique, les trajectoires du système en boucle fermée seront périodiques avec la même période;
- en raison du choix des matrices  $\Phi$  et  $\Gamma$ , les coefficients de Fourier de l'erreur associée aux fréquences  $\omega = k \frac{2\pi}{T}$  pour  $k = 1, \dots, L$  sont asymptotiquement nuls

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Les résultats expérimentaux montrent également que non seulement le contenu fréquentiel de l'erreur est nul aux pulsations définies par  $\Phi$ , mais qu'en outre la norme d'erreur tend asymptotiquement vers zéro d'une manière directement proportionnelle au nombre d'harmoniques annulées, c'est-à-dire à la valeur de  $L$ . Ces résultats ont été obtenus expérimentalement.



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