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Manuscrit

présenté pour l'obtention de

l'Habilitation à Diriger des Recherches

par

Julien Royer

de l'Institut de Mathématiques de Toulouse

The Damped Wave Equation and other Evolution Problems involving Non-selfadjoint Operators

Soutenue le 12 décembre 2022, devant le jury suivant :

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Presentation of the thesis

Avant-propos

This dissertation is about the research I have done since I arrived in Toulouse in 2011, after my Ph.D.

My Ph.D. (2007-2010) was about the high-frequency dissipative Helmholtz equation. More precisely, I proved high-frequency resolvent estimates in [2], and I studied in [3] the semiclassical measure for the solution when the source term concentrates on a bounded submanifold of the Euclidean space. The two papers [4, 5] were published later, but they were already partially contained in my Ph.D. thesis. The work [4] is about the resolvent estimates when the damping can take negative values, and [5] is a generalization of [3] to the case of an unbounded submanifold. These works will not be discussed in this report (except for their links with more recent papers). Neither will be [1], which is an undergraduate work on a completely different subject (modelization of the traffic flow).

The Helmholtz equation is the stationary version of the damped wave equation. After these works, I was motivated by the time-dependent problem. However, to state a result for high frequencies only, one needs a spectral localization. This is an example among many others of an issue completely insignificant in the selfadjoint setting which becomes a real obstacle with damping.

When I arrived in Toulouse, I discussed this question with Jean-Marc Bouclet, and our conclusion was that instead of trying to localize on high-frequencies, we should prove resolvent estimates for low frequencies and then consider the damped wave equation without spectral localization. Since Jean-Marc had just proved low frequency resolvent estimates for the Schrödinger and undamped wave equations at that time, it was a perfect question to begin a collaboration. And it turned out to be a very rich subject.

Il était une fois l'équation des ondes amorties...

An important part of this thesis will be devoted to the damped wave equation. An introduction on the subject will be given in Chapter 1. Several settings will be considered. On the Euclidean space $\mathbb{R}^d$, the damped wave equation reads

$$\partial_t^2 u + Pu + a(x)\partial_t u = 0, \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^d,$$

where P is a general Laplace operator (see (1.8) below) and $a(x) \geqslant 0$ is the absorption index.

I have been interested in the local energy decay for this damped wave equation. In the first results, the purpose was to recover in this setting the results known for the undamped wave equation. As explained above, the main part of the work was the analysis of the contribution of low frequencies.

To see the damped wave equation as a perturbation of the undamped case, we had to assume that the absorption index $a(x)$ decays at infinity. It turned out that when the damping is effective at infinity (for instance if $a(x)$ goes to 1 at infinity) the large time behavior of the solution is different. I studied this new phenomenon in various contexts. Since the technical issues raised by these two aspects are completely different, the discussion about

the damped wave equation is split into two chapters. The case of the wave equation with localized damping is discussed in Chapter 1 while the results about a damping effective at infinity are described in Chapter 2.

Together with the wave equation, I have also considered the local energy decay for the Schrödinger equation. These two problems share many similarities, and it is relevant to discuss them together. In some situations the Schrödinger equation can be simpler, and it has been used as an intermediate step toward the analysis of the wave equation. And finally, the ideas developed for the damped wave equation had an important application to a result about the usual (conservative) Schrödinger equation.

... et d'autres problèmes (pas toujours) linéaires impliquant des opérateurs (pas toujours) non-autoadjoints.

In parallel with my main research work on the damped wave equation, I have also developed various collaborations on other topics. They are often questions which arose from scientific discussions with friends and finally turned into a collaboration.

I would say that my field of research is the analysis of partial differential equations involving non-selfadjoint operators, but this is of course not restrictive. Thus in most of these problems there emerges a non-selfadjoint operator, but in some others only selfadjoint operators are involved. Most results are about a particular PDE, but some are about abstract spectral theory. And most problems are linear, but some are about non-linear equations.

All these various settings will be presented separately in Chapter 3. Some are closely related to my favorite topics, some are quite different, but they all could be the starting points of new perspectives for future research.

Finally, the last chapter will be devoted to a description of my preprints, some works in progress, and some discussions about possible future projects. As for my past research, some are questions about the damped wave equation, and some others go in different directions.

I have chosen to present in this thesis all my papers which are not included in or closely related to my Ph.D. However, since it is not possible to give a detailed description for all of them, some will be emphasized with scientific context, mains ideas of the proofs and comments, and for others I will only briefly describe the results.

A list of my publications and pre-publications, labeled from [1] to [27], is given at page 91. The other references used in the text, labeled from [Aaf21] to [Zwo12], are given at the end of the manuscript.

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Chapter 1

Local Energy Decay for the damped wave equation with localized damping

In this chapter and the following, we discuss the local energy decay for the damped wave equation on unbounded domains.

There are several difficulties. Because of the damping, the corresponding operator is not selfadjoint, which raises some technical issues compared to the undamped case. In the first result (see Theorem 1.1) the purpose was to generalize to this setting some estimates already known without damping. In a second step, I considered problems specific to the dissipative case (see all the results of Chapter 2). And finally, the strategies developed for these dissipative settings gave some new ideas useful even for the selfadjoint setting (see Section 1.7).

The damped wave equation was already well studied on compact domains. In this case, there is no problem with the contribution of low frequencies (from the spectral point of view, 0 does not belong to the spectrum of the corresponding operator). In all my works, I consider the wave equation on unbounded domains. Results about the contribution of high frequencies are essentially contained in my Ph.D. [Roy10] and will not be discussed in details here. My research after my Ph.D. mostly concerns the contribution of low frequencies, and the analysis is completely different. This is the main subject of this thesis.

Finally, an important difficulty is due to the wave equation itself. In many aspects, the problems of the local energy decay for the wave and Schrödinger equations are similar. However, we will see all along this thesis that there are difficulties specific to the wave equation, especially in the dissipative case.

We begin in this chapter with the damped wave equation with localized damping and closely related problems. We will discuss the papers [6, 7, 9, 10, 11, 23]. The earlier results proved in [2, 3, 4, 5], that we do not present in this report, are also related to this setting.

1.1 Local energy decay for the wave equation

The model case. In this first chapter, we consider the wave equation in an asymptotically Euclidean setting. Given f_0 in $H^1(\mathbb{R}^d)$ and g_0 in $L^2(\mathbb{R}^d)$, the model case is the usual free wave equation on $\mathbb{R}^d$

$$\partial_t^2 u_0 - \Delta u_0 = 0, \tag{1.1}$$

with initial condition

$$u_0(0) = f_0, \quad \partial_t u_0(0) = g_0. \tag{1.2}$$

This is the simplest model for the description of waves. In many propagation phenomena, there are also important non-linear effects, but this simple model is already suitable in many concrete situations, such as the propagation of acoustic or electromagnetic waves (if the electromagnetic field does not depend on time then the Maxwell equations can be decoupled in two wave equations for the electric and magnetic fields).

A large part of this thesis deals with the long time behavior of a solution of the wave equation. A relevant way to measure the evolution of a wave is to look at the localization of its energy:

$$E_0(u_0;t) = \int_{\mathbb{R}^d} \left(|\nabla u_0(t,x)|^2 + |\partial_t u_0(t,x)|^2 \right) \mathrm{d}x. \tag{1.3}$$

The global energy is a constant of the motion. However, we can look at the distribution of the quantity $|\nabla u_0(t)|^2 + |\partial_t u_0(t)|^2$ to see where the wave is propagating.

An important property of the wave equation is the finite speed of propagation. With all the physical constants set to 1, a wave propagates at speed at most 1. If f_0 and g_0 are supported in the ball $B(0,R)$ of radius $R>0$, then $u_0(t)$ is supported in $B(0,R+t)$.

In odd dimension, the wave actually propagates exactly at speed 1. This is the strong Huyghens principle. This implies that there is no diffusion of waves. If a light bulb is turned on for one second, someone looking at the bulb will see it for exactly one second, even from very far away.

At least for regular initial data, we have an explicit expression for the solution of (1.1)-(1.2) (see for instance [CH89, Eva98]). For $d=3$, we have

$$u_0(t,x) = \frac{1}{|\partial B(x,t)|} \int_{\partial B(x,t)} \big(f_0(y) + \nabla f_0(y)\cdot(y-x) + t g_0(y) \big) \,\mathrm{d}y.$$

If the initial condition (f_0,g_0) is supported in the ball $B(0,R)$, then $u_0(t)$ is supported in $B(0,t+R)\backslash B(0,t-R)$, and in particular the energy in any compact of $\mathbb{R}^d$ vanishes after finite time. For instance,

$$\forall t \geqslant 2R, \quad \|\nabla u_0(t)\|^2_{L^2(B(0,R))} + \|\partial_t u_0(t)\|^2_{L^2(B(0,R))} = 0. \tag{1.4}$$

The same applies in any odd dimension $d \geqslant 3$. In dimension 1, it is still true that $\partial_x u_0(t)$ and $\partial_t u_0(t)$ are supported in $[-t-R,-t+R]\cup[t-R,t+R]$, but this is not necessarily the case for the solution itself, as can be seen from the classical d'Alembert formula

$$u_0(t,x) = \frac{f_0(x+t)+f_0(x-t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} g_0(s)\,\mathrm{d}s.$$

The behavior is slightly different in even dimension. If one drops at time $t=0$ a pebble in a (two dimensional) pond, a circular wave will emanate outward. The pond is perfectly calm outside this circular wave, but not inside as would be the case with the strong Huyghens principle. Even if the magnitude decays rapidly, the excited state will persist indefinitely. Oral communication would be quite difficult in even dimension.

In this case, the free wave equation can be solved by the descent method (we can see a wave in dimension 2 as the trace on $\mathbb{R}^2\times\{0\}$ of a wave in dimension 3 which does not depend on the last variable). We can see, as expected, that with f_0 and g_0 compactly supported the wave does not vanish in a fixed compact, even for large times. However, the wave mostly propagates at speed close to 1 and the energy on a compact goes to 0 as $t\to+\infty$. More precisely, we can show that for $R>0$ there exists $C_R>0$ such that for (f_0,g_0) supported in $B(0,R)$ the solution u_0 of (1.1)-(1.2) satisfies

$$\|\nabla u(t)\|^2_{L^2(B(0,R))} + \|\partial_t u(t)\|^2_{L^2(B(0,R))} \leqslant C_R \langle t\rangle^{-2d} \left(\|\nabla f_0\|^2_{L^2(\mathbb{R}^d)} + \|g_0\|^2_{L^2(\mathbb{R}^d)} \right). \tag{1.5}$$

This is what we call the local energy decay. In any dimension, the energy of a wave in any compact goes to 0 or, equivalently (since the global energy is constant), the energy of the wave spreads to infinity. Notice that we have to consider localized initial data. Otherwise, a wave coming from far away could reach a fixed compact after a long time and (1.4)-(1.5) could not hold.

It is then natural to wonder if the same phenomenon occurs for waves in more general settings, when explicit descriptions of the solutions are not available. Thus, the first question is whether the energy of the wave on any compact still goes to 0 for any (localized) initial condition in perturbed settings. When this is the case, the next issues are the rate of decay and the uniformity with respect to the initial data (the time $T = 2R$ in (1.4) and the constant C_R in (1.5) depend on R but not on f_0, g_0 supported in $B(0, R)$, and in particular the right-hand side in (1.5) is proportional to the initial energy).

These questions are interesting for themselves, since they give qualitative properties for the long time behavior of the solutions. For instance, if we measure a wave on a bounded subdomain, it can be useful to know for how long there is a relevent information in this region. But the local energy decay, and the resolvent estimates that we develop for the proofs, are also crucial for the mathematical analysis of other important properties (scattering theory, non-linear problems, etc.).

Together with the wave equation, we also consider the Schrödinger equation, which plays a central role in quantum mechanics. The model problem is

$$\begin{cases} -i\partial_t u_0 - \Delta u_0 = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ u_{0|t=0} = f_0, & \text{on } \mathbb{R}^d, \end{cases} \tag{1.6}$$

for some $f_0 \in L^2(\mathbb{R}^d)$. The L^2-norm of the solution is constant:

$$\|u_0(t)\|^2_{L^2(\mathbb{R}^d)} = \|f_0\|^2_{L^2(\mathbb{R}^d)}.$$

There is no analog of the Huyghens principle for the Schrödinger equation. However, the mass also escapes to infinity: for $R > 0$ there exists $C_R > 0$ such that if f_0 is supported in $B(0, R)$ then the solution $u_0(t)$ of (1.6) satisfies

$$\|u(t)\|^2_{L^2(B(0,R))} \leqslant C_R \langle t \rangle^{-d} \|u_0\|^2_{L^2(\mathbb{R}^d)}. \tag{1.7}$$

This is similar to (1.5). The difference is that there is no distinction due to the parity of the dimension d. As for the wave equation, we are interested in this local energy decay for the Schrödinger equation in more general settings.

The asymptotically Euclidean setting. Our main purpose in this chapter is to discuss the local energy decay for the damped wave equation in an aymptotically Euclidean setting. We introduce simultaneously the setting for the Schrödinger equation.

We consider on $\mathbb{R}^d$ a general Laplacian of the form

$$P = -\frac{1}{w(x)} \operatorname{div} G(x)\nabla, \tag{1.8}$$

where the matrix $G(x)$ and the scalar $w(x)$ are positive and bounded. More precisely, we assume that there exist $G_{\min}, G_{\max}, w_{\min}, w_{\max} > 0$ such that, for all $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$,

$$G_{\min} |\xi|^2 \leqslant \langle G(x)\xi, \xi \rangle_{\mathbb{R}^d} \leqslant G_{\max} |\xi|^2 \quad \text{and} \quad w_{\min} \leqslant w(x) \leqslant w_{\max}. \tag{1.9}$$

We also set

$$-\Delta_G = -\operatorname{div} G(x)\nabla.$$

The definition of P includes in particular the case of the standard Laplace operator (with $G(x) = \mathsf{Id}$ and $w(x) = 1$), a Laplacian in divergence form (with $w(x) = 1$, so that $P = -\Delta_G$) or the Laplacian associated with a metric. We recall that the Laplace-Beltrami operator associated with the metric $\mathsf{g}(x) = (g_{j,k}(x))_{1\leqslant j,k\leqslant d}$ is given by

$$\operatorname{div}_{\mathsf{g}} \nabla_{\mathsf{g}} = \frac{1}{|\mathsf{g}(x)|^{\frac{1}{2}}} \sum_{j,k=1}^{d} \frac{\partial}{\partial x_j} |\mathsf{g}(x)|^{\frac{1}{2}} g^{j,k}(x) \frac{\partial}{\partial x_k},$$

where $|\mathsf{g}(x)| = |\det(\mathsf{g}(x))|$ and $(g^{j,k}(x))_{1\leqslant j,k\leqslant d} = \mathsf{g}(x)^{-1}$. Then $P_{\mathsf{g}} = -\operatorname{div}_{\mathsf{g}} \nabla_{\mathsf{g}}$ is of the form (1.8) with $w = |\mathsf{g}|^{\frac{1}{2}}$ and $G = |\mathsf{g}|^{\frac{1}{2}} \mathsf{g}^{-1}$.

We assume that P is associated with a long range perturbation of the flat metric. This means that $G(x)$ and $w(x)$ are long range perturbations of Id and 1, respectively. For some $\rho_0 \in]0, 1]$ there exist constants $C_\alpha > 0$, $\alpha \in \mathbb{N}^d$, such that for all $x \in \mathbb{R}^d$,

$$\left|\partial^\alpha (G(x) - \mathsf{Id})\right| + \left|\partial^\alpha (w(x) - 1)\right| \leqslant C_\alpha \langle x\rangle^{-\rho_0 - |\alpha|}. \tag{1.10}$$

Here and everywhere below we use the standard notation $\langle x\rangle = (1 + |x|^2)^{\frac{1}{2}}$.

For the wave equation we also introduce the absorption index $a(x)$. It is smooth, bounded, takes non-negative values, and it is of short range: choosing the constants C_α larger if necessary, we have

$$\left|\partial^\alpha a(x)\right| \leqslant C_\alpha \langle x\rangle^{-1-\rho_0-|\alpha|}. \tag{1.11}$$

The short range assumption is not just a technical issue. If a decays slower at infinity, then the results are different. This will be discussed in the next chapter.

On the other hand, the case $a = 0$ is allowed in this setting, so the results about the damped wave equation presented in this chapter include in particular the case of the undamped wave equation.

We consider on $\mathbb{R}^d$ the (possibly) damped wave equation

$$\begin{cases} \partial_t^2 u + Pu + a(x)\partial_t u = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \partial_t u)|_{t=0} = (f, g), & \text{on } \mathbb{R}^d, \end{cases} \tag{1.12}$$

where $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For $f \in L^2(\mathbb{R}^d)$ we also consider the Schrödinger equation

$$\begin{cases} -i\partial_t u + Pu = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ u_{|t=0} = f, & \text{on } \mathbb{R}^d. \end{cases} \tag{1.13}$$

In this asymptotically Euclidean setting, we define the energy of the wave by

$$E(u; t) = \int_{\mathbb{R}^d} \left(\langle G(x)\nabla u(t, x), \nabla u(t, x)\rangle_{\mathbb{R}^d} + w(x) \left|\partial_t u(t, x)\right|^2 \right) \mathrm{d}x. \tag{1.14}$$

This is equivalent to the usual energy (1.3), in the sense that there exists $C \geqslant 1$ such that $C^{-1}E_0(u; t) \leqslant E(u; t) \leqslant CE_0(u; t)$. This definition of the energy is adapted to the geometry of the problem, since with this choice a solution of (1.12) has a non-increasing energy. More precisely, we formally have

$$\frac{\mathrm{d}}{\mathrm{d}t} E(u; t) = -\int_{\mathbb{R}^d} a(x)w(x) \left|\partial_t u(t, x)\right|^2 \mathrm{d}x \leqslant 0.$$

For the damped wave equation, the decay of the global energy is already an interesting issue. However, the damping can be 0, and in any case it is small at infinity, so the waves at infinity are not really damped and we cannot expect a uniform decay for the global energy.

Here we are interested in the local energy decay. We are going to prove this property about the time dependent problem via a spectral approach. In particular, we can separate the contributions of high and low frequencies.

The non-trapping condition. The contribution of high frequencies, though highly non-trivial, is now quite well understood. It is known that the propagation of a high frequency wave is well approximated by the classical rays of light (for instance, the geometric optics is a good approximation of wave optics if the wavelength is very small compared to the other lengths of the problem). This is made rigorous with semiclassical analysis (see for instance [Zwo12]). For a free wave, rays of light follow straight lines at constant speed (and necessarily escape to infinity). With an obstacle, they bounce on the obstacle following the laws of geometric optics (there are subtle behaviors for rays tangent to the boundary, which we do not discuss here). For a metric g (corresponding to an inhomogeneous refraction index), the rays of light are the geodesics. In general, for $(x,\xi) \in \mathbb{R}^{2d} \simeq T^*\mathbb{R}^d$ we set

$$p(x,\xi) = \left\langle w(x)^{-1} G(x)\xi, \xi \right\rangle_{\mathbb{R}^d}. \tag{1.15}$$

Then rays of light are the solutions of the Hamiltonian problem associated with p. For $(x_0,\xi_0) \in \mathbb{R}^{2d}$ we denote by $(x(t;x_0,\xi_0), \xi(t;x_0,\xi_0))$ the solution of

$$\begin{cases} x'(t) = \partial_\xi p(x(t),\xi(t)), \\ \xi'(t) = -\partial_x p(x(t),\xi(t)), \\ x(0) = x_0, \quad \xi(0) = \xi_0. \end{cases}$$

Since high frequency waves follow these classical trajectories, one expects that they all escape to infinity if and only if all rays of light go to infinity (for the space variable). This is the non-trapping condition:

$$\forall (x_0,\xi_0) \in p^{-1}(\{1\}), \quad |x(t;x_0,\xi_0)| \xrightarrow[t\to\pm\infty]{} +\infty. \tag{1.16}$$

The motion of rays of light in the phase space is only an approximation and, in fact, the local energy always goes to 0. However, the higher the frequency is, the more accurate the approximation is. So without (1.16) the high frequency wave stays trapped for a long time, and the local energy decay is very slow. Thus, without (1.16), the estimate is not uniform (proportional to $E(u;0)$) but we have a loss of derivative (it is proportional to $\|f\|^2_{H^{s+1}} + \|g\|^2_{H^s}$ for some $s > 0$).

The behavior of the contribution of low frequencies is completely different. Compared to high frequencies, the local energy decay for the contribution of low frequencies is always uniform with respect to the initial data. On the other hand, while the contribution of high frequencies decays very fast (at least for regular initial data) the contribution of low frequencies is responsible for the lack of time decay. For instance, the rate of decay in (1.5) is governed by low frequencies. This question will be the main issue in this and the next chapters.

Literature about the undamped case. There is an important literature about the local energy decay for the wave equation. An early result is [Mor61], where decay at rate $1/t$ is proved by a mutiplier method for the free wave outside a star-shaped obstable in dimension 3 (with a Dirichlet boundary condition). Then exponential decay is proved in [LMP63] via an analysis of the corresponding semigroup. In [LP62] it is proved that the local energy for the wave in a general exterior domain goes to 0. See also [Zac66] for more general hyperbolic equations. In [LP72], the Lax-Phillips method is adapted to even dimensions. We refer to the book [LP67] (or the revised version [LP89]) for a review of this theory. The non-trapping condition is already mentioned there.

It is proved in [Ral69] that this non trapping condition is necessary to have uniform local energy decay. Local energy decay outside non-trapping obstacles is considered in [Mor75, Str75, MRS77], via the existence of an escape function (an escape function is a function on the phase space which is increasing along the Hamiltonian flow). It is proved in [MRS77] that we have decay at rate t^{-1} in even dimension, and exponential decay in dimension 3.

This question is then discussed in [Vai75]. Compared to the previous strategies, the properties of the time-dependent problem are now deduced from the analysis of the stationary problem. Another important step is the analysis of [Mel79], based on the propagation of singularities of [MS78]). In particular, a decay at rate t^{-d} is proved in even dimensions. This has then been improved in [Kaw93]. We also refer to [Vod99] for a more general setting. Finally, it is proved in [Bur98] that outside any compact smooth obstable we have at least logarithmic decay if we allow a loss of regularity.

The stationary problem of the wave equation (the Helmholtz equation) is closely related to the one for the Schrödinger equation. Then local energy decay has also been proved for the latter. See [Rau78] for an exponentially decaying potential, [JK79] for more precise asymptotics (of the resolvent and the propagator) and [Mur82] for more general operators. The case of an exterior domain is discussed in [Tsu84].

Finally, most of the recent papers deal simultaneously with the Schrödinger and wave equations. We refer to [Bou11a, BH12] for estimates with an ε-loss on an asymptotically Euclidean setting (the Laplacian is a Laplace-Beltrami operator associated with a long-range perturbation of the standard metric on $\mathbb{R}^d$). This means that the decay rate for the local energy is of size $O(t^{-d+\varepsilon})$ for the Schrödinger equation and $O(t^{-2d+\varepsilon})$ for the wave equation. The ε-loss has finally been removed in [BB21]. The method does not see the parity of the dimension, so this final result is optimal for Schrödinger or for the wave in even dimension, but not for the wave in odd dimension. However, it is proved [BH13] that if the metric goes faster to the flat metric at infinity then we can recover a better estimate on the local energy in odd dimensions.

In these works, the time decay is proved from resolvent estimates. And the main contributions of these papers is the analysis of low frequency resolvent estimates. High frequency resolvent estimates were already understood for the Schrödinger operator in close settings. See for instance [RT87] for the Schrödinger operator with a potential, [Rob92] for a second order perturbation of the Laplacian and [Bur02] for a general compactly supported perturbation of the Laplacian in an exterior domain (via the contradiction argument using semiclassical defect measures [Gér91], as also used in [Leb96]). Low frequency resolvent estimates were also already discussed in [Bou11b, BH10]. Earlier papers also considered the limiting absorption principle at zero energy in some particular settings (see for instance [Wan06, DS09] and references therein).

About the damped wave equation. Here we are mainly interested in the damped wave equation. The stabilization of the wave equation also has a long history on compact domains. In this case, we consider the global energy. The wave cannot escape to infinity, but it is dissipated and we similarly study the decay to 0. In this setting there is no difficulty with low frequencies, but the analysis of high frequencies is similar. In particular, the analog of the non-trapping condition is the so-called geometric control condition. The energy decays uniformly if and only if all the classical trajectories go through the damping region.

We refer to [RT74] for exponential decay with dissipation in the compact manifold and to [BLR92] for dissipation at the boundary (see also [BG97]). For logarithmic decay with loss of regularity without the control condition we refer to [Leb96] for internal damping and to [LR97] for damping at the boundary. Then there have been several results about intermediate situations, where the geometric control condition does not hold, but the set of undamped rays of light is small in some suitable sense. See for instance [BH07] for the problem on the stadium (giving polynomial decay of the energy), [Chr07] for a situation where the decay is exponential but not uniform, [LL17] for the so-called open book and [BG20] for a rough damping. We also refer to [Sjö00, Ana10] for more results on manifolds without boundary.

In an unbounded domain we have additional difficulties, in particular due to the contribution of low frequencies. As said above, we only consider the local energy decay in this

chapter.

In this setting, the energy on a compact has now two reasons to decay. Either because it escapes to infinity, or because it is dissipated. The expected corresponding condition on classical trajectories is that they should all escape to infinity or go through the damping region. This means that we can allow trapped trajectories if they are damped. We set

$$\Omega_{\mathrm{b}} = \left\{ (x_0, \xi_0) \in p^{-1}(\{1\}) \, : \, \sup_{t \in \mathbb{R}} |x(t; x_0, \xi_0)| < +\infty \right\}.$$

Then the condition on classical trajectories (which we can call geometric damping condition) reads

$$\forall (x_0, \xi_0) \in \Omega_{\mathrm{b}}, \exists t \in \mathbb{R}, \quad a(x(t, x_0, \xi_0)) > 0. \tag{1.17}$$

In [AK02], L. Aloui and M. Khenissi have considered the wave equation in an exterior domain, with a compactly supported damping, via the theory of Lax-Philipps and the contradiction argument with semiclassical measures. They recover, under the assumption analogous to (1.17) in an exterior domain, the exponential decay of the local energy in odd dimension. A polynomial decay is obtained in [Khe03] in even dimension.

The main result. The local energy decay for the damped wave equation (1.12) (in dimension $d \geqslant 3$) has been proved in [6]. This is a collaboration with Jean-Marc Bouclet. It has then been slightly improved in [11].

Instead of considering compactly supported initial data and the energy on a compact, we choose initial data which decay at infinity and consider a weighted energy, which gives slightly better results. For $\delta \in \mathbb{R}$ we denote by $L^{2,\delta}(\mathbb{R}^d)$ the weighted space $L^2(\langle x \rangle^{2\delta} \, \mathrm{d}x)$ and for $k \in \mathbb{N}$ we denote by $H^{k,\delta}(\mathbb{R}^d)$ the corresponding Sobolev space. The main result about the damped wave equation is the following.

Theorem 1.1. *Assume that the geometric damping condition* (1.17) *holds. Let* $\varepsilon > 0$ *and* $\delta > d + \frac{1}{2}$. *There exists* $C > 0$ *such that for* $(f, g) \in H^{1,\delta}(\mathbb{R}^d) \times L^{2,\delta}(\mathbb{R}^d)$ *and* $t \geqslant 0$ *we have*

$$\begin{aligned} &\| \langle x \rangle^{-\delta} \nabla u(t) \|_{L^2(\mathbb{R}^d)} + \| \langle x \rangle^{-\delta} \partial_t u(t) \|_{L^2(\mathbb{R}^d)} \\ &\qquad \leqslant C \langle t \rangle^{-(d-\varepsilon)} \left(\| \langle x \rangle^{\delta} \nabla f \|_{L^2(\mathbb{R}^d)} + \| \langle x \rangle^{\delta} g \|_{L^2(\mathbb{R}^d)} \right), \end{aligned}$$

where u *is the solution of* (1.12).

Notice that in even dimension we have an ε-loss compared to (1.5) (where the squares of the norms are considered). When [6] and [11] were written, this was the best known result even for the undamped case. Now the ε-loss has been removed in the undamped case in [BB21], and it is one of the perspectives to improve this result in the general case (see Section 4.1.1).

We discuss this theorem in the next three sections (notice that the presentation is slightly different than in the original papers), and then we will turn to the Schrödinger equation (see in particular the local energy decay for (1.13) in Theorem 1.25 below).

1.2 Resolvent estimates for the wave operator

Theorem 1.1 is proved from the spectral point of view. More precisely, we deduce estimates on the time-dependent problem from resolvent estimates for the corresponding wave operator.

1.2.1 The wave operator

As usual for a wave equation, we rewrite (1.12) as a first order problem of the form

$$\begin{cases} \partial_t U(t) - \mathcal{W} U(t) = 0, \quad \forall t \geqslant 0, \\ U(0) = F. \end{cases} \tag{1.18}$$

At least formally, $u(t)$ is a solution of (1.12) if and only if $U(t) = \big(u(t), w\partial_t u(t)\big)$ satisfies (1.18) with $F = (f, wg)$ and

$$\mathcal{W} = \begin{pmatrix} 0 & w^{-1} \\ \Delta_G & -a \end{pmatrix}. \tag{1.19}$$

We set $\mathscr{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. It is endowed with the natural norm. We also define the energy space $\mathscr{E}$ as the Hilbert completion of $\mathscr{S} = \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ for the norm defined by

$$\|(u,v)\|_{\mathscr{E}}^2 = \int_{\mathbb{R}^d} \langle G(x)\nabla u(x), \nabla u(x)\rangle_{\mathbb{R}^d}\, \mathrm{d}x + \int_{\mathbb{R}^d} \frac{|v(x)|^2}{w(x)}\, \mathrm{d}x.$$

In particular, $\mathscr{H}$ is dense in $\mathscr{E}$. We set

$$\mathsf{Dom}(\mathcal{W}) = \{U \in \mathscr{E} \,:\, \mathcal{W}U \in \mathscr{E}\}, \tag{1.20}$$

where $\mathcal{W}U$ is understood in the sense of distributions. Then we consider on $\mathscr{E}$ the operator $\mathcal{W}$ defined by (1.19) on the domain $\mathsf{Dom}(\mathcal{W})$.

Notice that if u is a solution of (1.12) then its energy $E(u;t)$ (see (1.14)) is equal to $\|(u(t), w\partial_t u(t))\|_{\mathscr{E}}^2$, while the usual energy E_0 (see (1.3)) corresponds to the square of the usual norm on $\dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. As already observed for the energy, this choice of norm on $\mathscr{E}$ is equivalent to the usual one, but it is adapted to the operator $\mathcal{W}$, in the sense that with this Hilbert structure the operator $\mathcal{W}$ is skew-adjoint if $a = 0$, and $i\mathcal{W}$ is dissipative in the general case $a \geqslant 0$.

There are two possible conventions for dissipativeness. We choose the convention which is usual for Schrödinger operators. We set

$$\mathbb{C}^{\pm} = \{z \in \mathbb{C} \,:\, \pm\mathsf{Re}(z) > 0\} \quad \text{and} \quad \mathbb{C}_{\pm} = \{z \in \mathbb{C} \,:\, \pm\mathsf{Im}(z) > 0\}.$$

Definition 1.2. We say that the operator T on the Hilbert space $\mathcal{H}$ is dissipative if

$$\forall \varphi \in \mathsf{Dom}(T), \quad \mathsf{Im}\,\langle T\varphi, \varphi\rangle_{\mathcal{H}} \leqslant 0.$$

Similarly, T is accretive if

$$\forall \varphi \in \mathsf{Dom}(T), \quad \mathsf{Re}\,\langle T\varphi, \varphi\rangle_{\mathcal{H}} \geqslant 0.$$

Then we say that the dissipative (accretive) operator T is maximal dissipative (maximal accretive) if its resolvent set contains some –hence any– z in $\mathbb{C}_+$ ($z \in \mathbb{C}^-$).

The other possible definition for dissipativeness, satisfied by $\mathcal{W}$ on $\mathscr{E}$, is the property $\mathsf{Re}\,\langle T\varphi, \varphi\rangle_{\mathcal{H}} \leqslant 0$ (in this case T is dissipative if $(-T)$ is accretive).

With the convention of Definition 1.2, $i\mathcal{W}$ is dissipative (or $(-\mathcal{W})$ is accretive) on $\mathscr{E}$ since

$$\forall U = (u,v) \in \mathsf{Dom}(\mathcal{W}), \quad \mathsf{Im}\,\langle i\mathcal{W}U, U\rangle_{\mathscr{E}} = \mathsf{Re}\,\langle \mathcal{W}U, U\rangle_{\mathscr{E}} = -\langle av, v\rangle_{L^2(\mathbb{R}^d)} \leqslant 0.$$

Now let $\zeta \in \mathbb{C}^+$ and $F \in \mathscr{H}$. If we solve formally the equation $(\mathcal{W} - \zeta)U = F$ for $U \in \mathsf{Dom}(\mathcal{W})$ we get

$$(\mathcal{W} - \zeta)^{-1}F = \begin{pmatrix} -R(i\zeta)(aw + \zeta w) & -R(i\zeta) \\ w - wR(i\zeta)(\zeta aw + \zeta^2 w) & -\zeta wR(i\zeta) \end{pmatrix} F, \tag{1.21}$$

where for $z \in \mathbb{C}_+$ we have set

$$R(z) = \big(-\Delta_G - iawz - wz^2\big)^{-1}. \tag{1.22}$$

We also have

$$(\mathcal{W} - \zeta)^{-1}F = \begin{pmatrix} -\zeta^{-1}\big(1 + R(i\zeta)\Delta_G\big) & -R(i\zeta) \\ -wR(i\zeta)\Delta_G & -\zeta R(i\zeta) \end{pmatrix} F. \tag{1.23}$$

For $R(z)$ we have again chosen a convention consistent with the Schrödinger setting (z belongs to the upper half-plane and $R(z)$ looks like the resolvent of the Schrödinger equation if $a = 0$). With this convention we have $R(i\zeta) = (-\Delta_G + aw\zeta + w\zeta^2)^{-1}$.

All this can be made rigorous. We first prove that $R(z)$ is well defined and then we check that the right-hand side of (1.21) or (1.23) defines a bounded inverse of $(\mathcal{W} - \zeta)^{-1}$.

Proposition 1.3. *The operator* $\big(- \Delta_G - iawz - wz^2\big)$ *with domain* $H^2(\mathbb{R}^d)$ *has a bounded inverse on* $L^2(\mathbb{R}^d)$ *for all* $z \in \mathbb{C}_+$. *Moreover, its inverse* $R(z)$ *extends to a bounded operator from* $H^{-1}(\mathbb{R}^d)$ *to* $H^1(\mathbb{R}^d)$ *and there exists* $C > 0$ *such that for* $z \in \mathbb{C}_+$ *and* $\beta_1, \beta_2 \in \mathbb{N}^d$ *with* $|\beta_1| \leqslant 1$, $|\beta_2| \leqslant 1$, *we have*

$$\big\|\partial_x^{\beta_1} R(z) \partial_x^{\beta_2}\big\|_{\mathcal{L}(L^2)} \leqslant \frac{C\,|z|^{|\beta_1|+|\beta_2|-1}}{\mathsf{Im}(z)}. \tag{1.24}$$

Proposition 1.4. *The operator* $i\mathcal{W}$ *is maximal dissipative on* $\mathscr{E}$. *Moreover, for* $\zeta \in \mathbb{C}^+$ *the resolvent* $(\mathcal{W} - \zeta)^{-1}$ *is given by* (1.23), *and for* $F \in \mathscr{H}$ *we also have* (1.21).

By the Hille-Yosida Theorem, we deduce from Proposition 1.4 that $\mathcal{W}$ generates a contractions semigroup on $\mathscr{E}$, and in particular the problem (1.18) is well posed. Then we can rewrite Theorem 1.1 with $\mathcal{W}$ in the weighted energy spaces. For $\delta \in \mathbb{R}$ we denote by $\mathscr{E}^\delta$ the Hilbert completion of $\mathscr{S}$ for the norm given by

$$\|(u,v)\|^2_{\mathscr{E}^\delta} = \big\| \langle x\rangle^\delta \nabla u\big\|^2_{L^2(\mathbb{R}^d)} + \big\| \langle x\rangle^\delta v\big\|^2_{L^2(\mathbb{R}^d)}.$$

Theorem 1.5. *Assume that the geometric damping condition* (1.17) *holds. Let* $\varepsilon > 0$ *and* $\delta > d + \frac{1}{2}$. *There exists* $C > 0$ *such that for* $F \in \mathscr{E}^\delta$ *we have*

$$\big\|e^{t\mathcal{W}} F\big\|_{\mathscr{E}^{-\delta}} \leqslant C\,\langle t\rangle^{-(d-\varepsilon)}\,\|F\|_{\mathscr{E}^\delta}\,.$$

The main ingredients of the proof described below are given in [6]. However, in [6] there is a loss of regularity, and the estimate also depends on the L^2-norm of the initial condition u_0. Then the estimate of [6] is

$$\big\| \langle x\rangle^{-\delta} \nabla u(t)\big\|_{L^2(\mathbb{R}^d)} + \big\| \langle x\rangle^{-\delta} \partial_t u(t)\big\|_{L^2(\mathbb{R}^d)} \leqslant C\,\langle t\rangle^{-(d-\varepsilon)}\,\big(\|u_0\|_{H^{2,\delta}(\mathbb{R}^d)} + \|u_1\|_{H^{1,\delta}(\mathbb{R}^d)}\big).$$

Theorem 1.5 is the version proved in [11].

Remark 1.6. The fact that $\dot{H}^1$ is not included in L^2 raises some difficulties, and one might prefer to work in $\mathscr{H}$ instead of $\mathscr{E}$. This is in some sense more convenient, but this will not give the same results. If we estimate U in (a weighed version of) $\mathscr{H}$, we are not only considering the sizes of the derivatives of the wave (the energy), but also the size of the solution itself, estimated with respect to the initial energy but also the size of the initial condition f in $L^2(\mathbb{R}^d)$. This can be considered –or not– as a drawback. We are also loosing the nice structural properties of the operator $\mathcal{W}$, but the resolvent $(\mathcal{W} - \zeta)^{-1}$ is still defined on $\mathbb{C}^+$ and the propagator $e^{t\mathcal{W}}$ is still defined for $t \geqslant 0$. We will work in $\mathscr{H}$ in the next chapter, see Section 2.2.1.

Of course, this discussion is irrelevant in situations where $\dot{H}^1$ and H^1 are equal as sets with equivalent norms, for instance on a bounded domain with Dirichlet boundary condition. In unbounded domains, the same applies if we work with initial data supported in some fixed compact, as is usually the case in results about the local energy decay. Similarly, if (f, g) is not compactly supported but belongs to some weighted space $H^{1,\delta}(\mathbb{R}^d) \times L^{2,\delta}(\mathbb{R}^d)$, and if we are ready to work in a slightly more restrictive space, then we can use the following generalization of the Hardy inequality (see Lemma 4.1 in [11]). For $\varepsilon > 0$ there exists $C > 0$ such that for all $u_0 \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\| \langle x\rangle^\delta u_0\|_{L^2(\mathbb{R}^d)} \leqslant C\| \langle x\rangle^{\delta+1+\varepsilon} \nabla u_0\|_{L^2(\mathbb{R}^d)}. \tag{1.25}$$

Remark 1.7. In the literature about the undamped wave equation, it is usual to diagonalise the operator $\mathcal{W}$, and the analysis reduces to a problem involving the operator $\sqrt{P}$ on $L^2(\mathbb{R}^d)$. We will not use this possibility, since with the damping it is no longer possible to diagonalise $\mathcal{W}$. We could certainly deal with the non-diagonal terms, but the gain is not clear at all and we prefer to stay in the (possibly) unconvenient but (certainly) natural space $\mathscr{E}$. Moreover, it seems slightly more natural to prove estimates on $\| \langle x \rangle^{-\delta} \nabla u(t)\|_{L^2(\mathbb{R}^d)}$ rather than on $\| \langle x \rangle^{-\delta} \sqrt{P} u(t)\|_{L^2(\mathbb{R}^d)}$ for the local energy decay.

1.2.2 From resolvent estimates to local energy decay

To deduce properties on the wave equation from resolvent estimates for $\mathcal{W}$, we write the propagator $e^{t\mathcal{W}}$ in terms of the resolvent $(\mathcal{W}-\zeta)^{-1}$, $\zeta \in \mathbb{C}^+$.

By density, it is enough to prove Theorem 1.5 for $F \in \mathscr{S}$. Let $\phi \in C^\infty(\mathbb{R}, [0,1])$ be equal to 0 on $]-\infty, 1]$ and equal to 1 on $[2, +\infty[$. For $\zeta \in \mathbb{C}^+$ we set

$$F_\zeta = \int_{\mathbb{R}} \phi'(s) e^{s(\mathcal{W}-\zeta)} F \, \mathrm{d}s. \tag{1.26}$$

We can see F_ζ as a "regularized in time" version of F (we recover F if we replace ϕ by $\mathbb{1}_{\mathbb{R}_+}$, hence ϕ' by δ_0). Then we have

$$(\mathcal{W}-\zeta)^{-1} F_\zeta = -\int_{\mathbb{R}} \phi(s) e^{s(\mathcal{W}-\zeta)} F \, \mathrm{d}s.$$

We still have propagation at finite speed for a wave in our perturbed setting, so for $\delta \geqslant 0$ and $T > 0$ there exists $C \geqslant 0$ such that for $s \in [0, T]$ and $F \in \mathscr{S}$ we have (see [11, Lemma 5.1])

$$\left\| e^{s\mathcal{W}} F \right\|_{\mathscr{E}^\delta} \leqslant C \left\| F \right\|_{\mathscr{E}^\delta}.$$

Thus there exists $C > 0$ such that, for all $F \in \mathscr{S}$ and $\zeta \in \mathbb{C}^+$,

$$\left\| F_\zeta \right\|_{\mathscr{E}^\delta} \leqslant C \left\| F \right\|_{\mathscr{E}^\delta}.$$

Let $\mu > 0$. Given $\tau \in \mathbb{R}$ we write ζ for $\mu - i\tau$. The interest of considering F_ζ instead of F is that $(\mathcal{W}-\zeta)^{-1} F_\zeta$ decays rapidly as $|\tau| \to \infty$. Then the Fourier inversion formula gives

$$\phi(t) e^{t\mathcal{W}} F = \frac{1}{2i\pi} \int_{\mathsf{Re}(\zeta)=\mu} e^{t\zeta} (\mathcal{W}-\zeta)^{-1} F_\zeta \, \mathrm{d}\zeta, \tag{1.27}$$

where the line $\mathsf{Re}(\zeta) = \mu$ is oriented from top to bottom (from $\mu + i\infty$ to $\mu - i\infty$). Since $\left\| e^{t\mathcal{W}} \right\|_{\mathcal{L}(\mathscr{E})} \leqslant 1$, the estimate of Theorem 1.5 is clear for t in a compact. It is then enough to estimate (1.27), which coincides with $e^{t\mathcal{W}} F$ for $t \geqslant 2$.

We deal separately with the contributions of high and low frequencies. We consider $\chi_{\mathsf{low}} \in C_0^\infty(\mathbb{R}, [0,1])$ equal to 1 on a neighborhood of 0, and $\chi_{\mathsf{high}} = 1 - \chi_{\mathsf{low}}$. For $* \in \{\mathsf{low}, \mathsf{high}\}$ and $t \geqslant 2$ we set

$$\mathcal{U}_{*,\mu}(t) F = \frac{1}{2i\pi} \int_{\mathsf{Re}(\zeta)=\mu} \chi_*(\mathsf{Im}(\zeta)) e^{t\zeta} (\mathcal{W}-\zeta)^{-1} F_\zeta \, \mathrm{d}\zeta. \tag{1.28}$$

We set $\mathbb{D} = \{z \in \mathbb{C} : |z| \leqslant 1\}$. To estimate $\mathcal{U}_{\mathsf{high},\mu}(t) F$, we need estimates for $(\mathcal{W}-\zeta)^{-1}$. We have the following result.

Theorem 1.8. *Assume that the geometric damping condition* (1.17) *holds. Let* $n \in \mathbb{N}^*$ *and* $\delta > n - \frac{1}{2}$. *There exists* $C > 0$ *such that for* $\zeta \in \mathbb{C}^+ \backslash \mathbb{D}$ *we have*

$$\left\| (\mathcal{W}-\zeta)^{-n} \right\|_{\mathcal{L}(\mathscr{E}^\delta, \mathscr{E}^{-\delta})} \leqslant C.$$

The uniform estimates of Theorem 1.8 are not enough to estimate directly the integral $\mathcal{U}_{\mathsf{high},\mu}(t)F$. This is the reason why we had a loss of derivative in [6]. This point has been improved in [11], and we finally have the following result.

Proposition 1.9. *Let $\gamma \geqslant 0$ and $\delta > \gamma + \frac{1}{2}$. There exists $C \geqslant 0$ such that for $F \in \mathscr{S}$, $\mu \in]0,1]$ and $t \geqslant 1$ we have*

$$\|\mathcal{U}_{\mathsf{high},\mu}(t)F\|_{\mathscr{E}^{-\delta}} \leqslant C \langle t\rangle^{-\gamma} e^{t\mu} \|F\|_{\mathscr{E}^{\delta}} .$$

Notice that under the geometric condition (1.17), we can estimate uniformly as many derivatives as we wish for the resolvent if the weight is strong enough. Then we get a fast decay for $\mathcal{U}_{\mathsf{high},\mu}(t)F$. This is not the case for the contribution of $\mathcal{U}_{\mathsf{low},\mu}(t)F$, which depends on the resolvent estimates for $\zeta \in \mathbb{C}^+$ close to 0.

Theorem 1.10. *Let $n \in \mathbb{N}^*$, $\delta > n - \frac{1}{2}$ and $\varepsilon > 0$. There exists $C > 0$ such that for $\zeta \in \mathbb{C}^+ \cap \mathbb{D}$ we have*

$$\left\|(\mathcal{W} - \zeta)^{-n}\right\|_{\mathcal{L}(\mathscr{E}^{\delta},\mathscr{E}^{-\delta})} \leqslant C |\zeta|^{\min(0,d-n-\varepsilon)} .$$

For low frequencies, it is not difficult to convert the resolvent estimates into time decay. We refer to [6, 11] or [BB21, Section 5].

Proposition 1.11. *Let $\delta > d + \frac{1}{2}$ and $\varepsilon > 0$. There exists $C > 0$ such that for all $F \in \mathscr{S}$ and $\mu > 0$ we have*

$$\|\mathcal{U}_{\mathsf{low},\mu}(t)F\|_{\mathscr{E}^{-\delta}} \leqslant C \langle t\rangle^{-(d-\varepsilon)} e^{t\mu} \|F\|_{\mathscr{E}^{\delta}} .$$

Applying Propositions 1.9 and 1.11 and letting μ go to 0, we deduce Theorem 1.5. Now we focus on the main ingredients for the proofs of the resolvent estimates. In Section 1.3 we introduce the abstract commutators method, and in Section 1.4 we explain how it is used to prove Theorems 1.8 and 1.10.

1.3 The Mourre commutators method

1.3.1 Introduction

A classical argument to prove resolvent estimates for a selfadjoint Schrödinger operator near the real axis is the commutators method of Mourre.

Let H be a selfadjoint operator on a Hilbert space $\mathcal{H}$. Given another selfadjoint operator A (the conjugate operator) we have formally

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\langle Ae^{-itH}\varphi, e^{-itH}\varphi\right\rangle_{\mathcal{H}} = \left\langle i[H,A]e^{-itH}\varphi, e^{-itH}\varphi\right\rangle_{\mathcal{H}} .$$

Thus if $i\mathsf{ad}_A(H) = i[H,A]$ is a positive operator, the observation $\left\langle Ae^{-itH}\varphi, e^{-itH}\varphi\right\rangle_{\mathcal{H}}$ is an increasing function of time. In particular, it is easy to see that if $i[H,A] \geqslant c_0$ for some $c_0 > 0$ then H cannot have an eigenvalue. Before E. Mourre, there were already important results about the nature of the spectrum of H based on a positive commutator assumption (see for instance [Put67]).

The great contribution of Mourre in [Mou81] is that the positive commutator assumption is localized in energy with respect to H. Given an interval J of $\mathbb{R}$, the assumption is then

$$\mathbb{1}_J(H)[H,iA]\mathbb{1}_J(H) \geqslant c_0 \mathbb{1}_J(H), \tag{1.29}$$

where $c_0 > 0$ and $\mathbb{1}_J(H)$ is the spectral projection of H on J.

For the free Laplacian $H = -\Delta$ on $\mathcal{H} = L^2(\mathbb{R}^d)$, we can use the generator of dilations

$$A = -\frac{x \cdot i\nabla + i\nabla \cdot x}{2} = -\frac{id}{2} - x \cdot i\nabla. \tag{1.30}$$

This is the quantization of the symbol $(x,\xi) \mapsto x\cdot\xi$, which is an escape function for the Laplacian (the Poisson bracket $\{\xi^2, x\cdot\xi\}$ is positive on $p^{-1}(\{1\}) = \{(x,\xi) : |\xi| = 1\}$). In this case we have $i[H,A] = 2H$. This is not greater than a positive constant, but (1.29) holds for any $J \subset\subset \mathbb{R}_+^* =]0,+\infty[$ with $c_0 = 2\inf(J) > 0$.

From (1.29), Mourre deduced important properties such as the limiting absorption principle and the absolute continuity of the spectrum of H in J. For this, he proves in particular uniform resolvent estimates of the form

$$\left\| \langle A\rangle^{-\delta} (H-z)^{-1} \langle A\rangle^{-\delta} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C, \tag{1.31}$$

where $\delta \geqslant 1$ (then for $\delta > \frac{1}{2}$ in [Mou83]) and $z \in \mathbb{C}\backslash\mathbb{R}$ has real part in $I \subset\subset J$.

The idea is to prove estimates uniform with respect to z and $\varepsilon > 0$ for

$$F_\varepsilon(z) = \langle A\rangle^{-1} (H_\varepsilon - z)^{-1} \langle A\rangle^{-1}, \quad \text{where} \quad H_\varepsilon = H - i\varepsilon \mathbb{1}_J(H)[H,iA]\mathbb{1}_J(H). \tag{1.32}$$

The operator H_ε is dissipative. The dissipative part is not bounded below by a positive constant, but with the positivity given by (1.29) we can prove an estimate of the form

$$\left\| \mathbb{1}_J(H)(H-z)^{-1} \langle A\rangle^{-1} \right\| \lesssim \frac{\|F_\varepsilon(z)\|^{\frac{1}{2}}}{\sqrt{\varepsilon}}.$$

And for $\mathsf{Re}(z) \in I \subset\subset J$ we have a uniform estimate for $(1-\mathbb{1}_J(H))(H-z)^{-1}\langle A\rangle^{-1}$ by the spectral theorem. Since this does not give an estimate for $F_\varepsilon(z)$ uniform in ε small, we also estimate the derivative of $F_\varepsilon(z)$ with respect to ε. After having removed the factors $\mathbb{1}_J(H)$ we have to estimate

$$\langle A\rangle^{-1} (H_\varepsilon - z)^{-1}[H,A](H_\varepsilon - z)^{-1} \langle A\rangle^{-1}.$$

This is where we use the fact that the dissipative part of H_ε is a commutator of H. The factor H is absorbed by one of the resolvents, and the weight $\langle A\rangle^{-1}$ is used to absorb the factor A. We finally get an estimate for the derivative, from which we deduce that the limit $F_0(z)$ is bounded uniformly in z. We also refer to [Gér08] for an alternative approach.

The flexibility of the assumption (1.29) makes the result applicable in many situations, and the Mourre method has been extended in many directions. We refer to the book [ABG96] for a general overview on the subject.

Here we are only interested in the uniform estimates, but for the resolvent and its derivatives. The multiple resolvent estimates for a selfadjoint operator are given in [JMP84, Jen85] under additional assumptions about the multiple commutators $\mathsf{ad}_A^n(H)$, $n \in \{1,\dots,N+1\}$, for some given $N \geqslant 2$. The idea is to use other resolvent estimates involving the spectral projections $\mathbb{1}_{\mathbb{R}_\pm}(A)$. For $\delta_1, \delta_2 \geqslant 0$ such that $\delta_1 + \delta_2 < N-1$ and $\delta \in \left]\frac{1}{2}, N\right[$ there exists $C > 0$ such that for $z \in \mathbb{C}_+$ with $\mathsf{Re}(z) \in I$ we have

$$\left\| \langle A\rangle^{\delta_1} \mathbb{1}_{\mathbb{R}_-}(A)(H-z)^{-1}\mathbb{1}_{\mathbb{R}_+}(A) \langle A\rangle^{\delta_2} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C, \tag{1.33}$$

$$\left\| \langle A\rangle^{-\delta} (H-z)^{-1}\mathbb{1}_{\mathbb{R}_+}(A) \langle A\rangle^{\delta-1} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C, \tag{1.34}$$

$$\left\| \langle A\rangle^{\delta-1} \mathbb{1}_{\mathbb{R}_-}(A)(H-z)^{-1} \langle A\rangle^{-\delta} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C. \tag{1.35}$$

Then we can deduce similar estimates (with different conditions for the weights) for the powers of $(H-z)^{-1}$. For instance, for $\delta > \frac{3}{2}$ we can write

$$\begin{aligned}
&\left\| \langle A\rangle^{-\delta} (H-z)^{-2} \langle A\rangle^{-\delta} \right\| \\
&\quad \leqslant \left\| \langle A\rangle^{-\delta} (H-z)^{-1} \langle A\rangle^{-(\delta-1)} \right\| \left\| \langle A\rangle^{\delta-1} \mathbb{1}_{\mathbb{R}_-}(A)(H-z)^{-1} \langle A\rangle^{-\delta} \right\| \\
&\quad + \left\| \langle A\rangle^{-\delta} (H-z)^{-1}\mathbb{1}_{\mathbb{R}_+}(A) \langle A\rangle^{\delta-1} \right\| \left\| \langle A\rangle^{-(\delta-1)} (H-z)^{-1} \langle A\rangle^{-\delta} \right\|,
\end{aligned}$$

and use (1.31), (1.34), (1.35) and (1.31) again. We similarly prove analogs of (1.33)-(1.35) for $(H-z)^{-2}$ and then we prove by induction estimates for higher powers of the resolvent.

1.3.2 The dissipative Mourre method

In [2], I generalized (1.31) to a parameter-dependent dissipative operator, of the form

$$H_\lambda = H_{1,\lambda} - iV_\lambda \tag{1.36}$$

where $H_{1,\lambda}$ is selfadjoint and semi-bounded, and V_λ is selfadjoint, non-negative and $H_{1,\lambda}$-bounded with relative bound smaller than 1. There is no convenient and general way to localize with respect to the spectrum of a non-selfadjoint operator, so the assumption of positive commutator is localized with respect to the selfadjoint part $H_{1,\lambda}$. A key observation in [2] is that the positive commutator is used to give some dissipativeness to the operator H_ε in (1.32). Thus, if the operator is already dissipative, we can use the dissipative part of the operator to weaken the assumption (1.29) on the commutator. More precisely, in [2] the estimate (1.31) is proved under an assumption of the form

$$\mathbb{1}_J(H_{1,\lambda})\big([H_{1,\lambda}, iA_\lambda] + \beta_\lambda V_\lambda\big)\mathbb{1}_J(H_{1,\lambda}) \geqslant c_0 \mathbb{1}_J(H_{1,\lambda}), \tag{1.37}$$

for some $\beta_\lambda \in [0,1]$. We do not give too much details here since [2] is already described in [Roy10].

For the proofs of Theorems 1.8 and 1.10 we need resolvent estimates for the powers (or, equivalently, for the derivatives) of the resolvent. Thus, we first have to generalize (1.33)-(1.35) and then the estimates for the powers of the resolvent to a dissipative setting.

But this is not enough. The commutators method cannot be applied directly to the operator $i\mathcal{W}$, for instance because its selfadjoint part is not semibounded. Thus, we first estimate the derivatives of $R(z)$ by the Mourre method, and then we deduce estimates for the derivative of $(\mathcal{W}-\zeta)^{-1}$ by (1.21). Notice that compared to the usual settings, the derivatives of $R(z)$ are not given by its powers. We have

$$R'(z) = R(z)\big(ia(x)w(x) + 2zw(x)\big)R(z),$$

and we see by induction on $m \in \mathbb{N}$ that $R^{(m)}(z)$ can be written as a sum of terms of the form

$$T_{k;j_1,\dots,j_k}(z) = R(z)(iaw + 2zw)^{j_1} R(z) \dots (iaw + 2zw)^{j_k} R(z), \tag{1.38}$$

where $k \in \{0,\dots,m\}$ and $j_1,\dots,j_k \in \{0,1\}$ are such that $2k - \sum_{\ell=1}^k j_\ell = m$. Thus, we also have to take into account the factors $(iaw + 2zw)$ inserted between the factors $R(z)$ (they do not commute with $R(z)$). We proved in [6] that under some suitable behavior between these inserted factors and the conjugate operator (basically, the commutators with the conjugate operator extend to bounded operators), we can generalize to this kind of setting the multiple commutator estimates.

After [6, 11], I have been interested in the wave equation in a domain Ω with boundary, in particular in wave guides, with damping at the boundary. This will be discussed with more details in the next chapter, but this motivated the analysis of a Schrödinger operator with dissipative Robin boundary condition. Given $a \in W^{1,\infty}(\partial\Omega;\mathbb{R}_+)$, we consider on Ω the operator

$$H_a = -\Delta, \quad \mathsf{Dom}(H_a) = \big\{u \in H^2(\Omega), \partial_\nu u = iau \text{ on } \partial\Omega\big\}. \tag{1.39}$$

In this case, we cannot write H_a as the sum of a selfadjoint operator and a dissipative part as in (1.36), so we cannot apply the results of [2, 6]. However, we can write such a sum for the quadratic form corresponding to H_a on $H^1(\Omega)$:

$$Q_\alpha(u) = \int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x - i\int_{\partial\Omega} a(x)\,|u(x)|^2 \,\mathrm{d}\sigma(x).$$

In [9], I generalized the dissipative Mourre method to such operators (note that this includes the previous cases). This is inspired by [ABG88] about a similar question for selfadjoint operators.

This version of the dissipative Mourre method has been applied to wave guides in [9]. It also had an unexpected application to the dissipative Schrödinger equation on $\mathbb{R}^d$ (see Remark 1.24 below).

Finally, in [23] I again had to use the Mourre method in a setting which was not included in the previous versions. More details will be given in Section 1.7 below, but for this problem we have to apply the multiple resolvent estimates simultaneously for two different operators. This means that we have to estimate a product with factors given by different resolvents, and with inserted factors as above.

Moreover, in [6, 11] we have applied the Mourre method to (1.22) with $w = 1$, so it was still possible to see $R(z)$ as the resolvent of the parameter-dependant operator $-\Delta_G - iza$ with spectral parameter z^2. In [23] (see Section 1.7), there is no damping but w is not necessarily equal to 1, so we include the spectral parameter in the operator, and just see the resolvent as the inverse of a parameter-dependent dissipative operator. Thus, we need a dissipative version of the commutators method even if the operator under study is selfadjoint.

1.3.3 The statement

Since it includes all the previous versions, we give here the statement of [23] for the commutators method. Notice that the proofs given in [23] are self-contained.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert spaces. We assume that $\mathcal{K}$ is densely and continuously embedded in $\mathcal{H}$. We denote by $\mathcal{K}^*$ the space of continuous semilinear forms on $\mathcal{K}$ (we have $\varphi(u_1 + \lambda u_2) = \varphi(u_1) + \bar{\lambda}\varphi(u_2)$ for $\varphi \in \mathcal{K}^*$, $u_1, u_2 \in \mathcal{K}$ and $\lambda \in \mathbb{C}$). We refer to [EE87, pp. 3–4] for a discussion about this choice. We identify $\mathcal{H}$ with $\mathcal{H}^*$ (with this convention, the identification is linear). Then $\mathcal{K}$ is naturally (linearly) identified with a subset of $\mathcal{K}^*$.

We consider a selfadjoint operator A on $\mathcal{H}$ with domain $\mathcal{D}_{\mathcal{H}}$. We set

$$\mathcal{D}_{\mathcal{K}} = \{\varphi \in \mathcal{K} \cap \mathcal{D}_{\mathcal{H}} \,:\, A\varphi \in \mathcal{K}\}\,. \tag{1.40}$$

By restriction, A defines an operator $A_{\mathcal{K}}$ on $\mathcal{K}$ with domain $\mathcal{D}_{\mathcal{K}}$. Then $\mathcal{D}_{\mathcal{K}}$ is endowed with the graph norm of $A_{\mathcal{K}}$. We can see $A_{\mathcal{K}}$ as an operator in $\mathcal{L}(\mathcal{D}_{\mathcal{K}}, \mathcal{K})$ and $A_{\mathcal{K}}^*$ maps $\mathcal{K}^*$ to $\mathcal{D}_{\mathcal{K}}^*$.

For $S \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ we set $\mathsf{ad}_A^0(S) = S$. Then, by induction on $n \in \mathbb{N}^*$, if the commutator $\mathsf{ad}_A^{n-1}(S)A_{\mathcal{K}} - A_{\mathcal{K}}^*\mathsf{ad}_A^{n-1}(S) \in \mathcal{L}(\mathcal{D}_{\mathcal{K}}, \mathcal{D}_{\mathcal{K}}^*)$ extends to an operator in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$, then we denote this extension by $\mathsf{ad}_A^n(S)$. We can similarly define commutators in $\mathcal{L}(\mathcal{H}, \mathcal{K})$.

For $Q \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ we set $\mathsf{Re}(Q) = (Q + Q^*)/2$ and $\mathsf{Im}(Q) = (Q - Q^*)/2i$. We say that $Q \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ is non-negative if for all $\varphi \in \mathcal{K}$ we have $\langle Q\varphi, \varphi\rangle_{\mathcal{K}^*,\mathcal{K}} \geqslant 0$.

All this being set, we consider $Q \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ with negative imaginary part: there exists $c_0 > 0$ such that

$$Q_+ := -\mathsf{Im}(Q) \geqslant c_0\mathcal{I},$$

where $\mathcal{I} \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ is the natural embedding. By the Lax-Milgram Theorem, Q has an inverse $Q^{-1} \in \mathcal{L}(\mathcal{K}^*, \mathcal{K}) \subset \mathcal{L}(\mathcal{H})$.

Example 1.12. If H is a selfadjoint semibounded operator and $z \in \mathbb{C}_+$ is a spectral parameter, we recover the usual setting by choosing $Q = Q(z) = H - z$, seen as an operator from the form domain $\mathcal{K}$ of H to $\mathcal{K}^*$. Then $Q_+ = \mathsf{Im}(z)$. If H is a dissipative operator of the form $H = H_1 - iV$, with H_1 selfadjoint and $V \geqslant 0$, then $Q_+ = V + \mathsf{Im}(z)$.

Our purpose is to prove estimates on Q^{-1}. For this, we use the following notion of conjugate operator.

Definition 1.13. Let $N \in \mathbb{N}^*$ and $\Upsilon \geqslant 1$. Let A be a selfadjoint operator on $\mathcal{H}$. We say that A is Υ-conjugate to Q up to order N if the following conditions are satisfied.

(H1) For $\varphi \in \mathcal{K}$ we have $\|\varphi\|_{\mathcal{H}} \leqslant \Upsilon \|\varphi\|_{\mathcal{K}}$.

(H2) For all $\theta \in [-1, 1]$ the propagator $e^{-i\theta A} \in \mathcal{L}(\mathcal{H})$ defines by restriction a bounded operator on $\mathcal{K}$.

(H3) The commutator $\mathsf{ad}_A^n(Q)$ is well defined in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ and satisfies $\|\mathsf{ad}_A^n(Q)\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \leqslant \Upsilon$ for all $n \in \{1, \dots, N+1\}$.

(H4) There exist $Q_\perp \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ with $\mathsf{Im}(Q_\perp) \leqslant 0$, $Q_\perp^+ \in \mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ non-negative and $\Pi \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ such that

(a) $Q = Q_\perp - iQ_\perp^+$,

(b) $\|Q_\perp^+\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \leqslant \Upsilon$, $\|\Pi\|_{\mathcal{L}(\mathcal{H},\mathcal{K})} \leqslant \Upsilon$, $\|\mathsf{ad}_A(\Pi)\|_{\mathcal{L}(\mathcal{H},\mathcal{K})} \leqslant \Upsilon$, and for $\varphi \in \mathcal{H}$ we have $\|\Pi\varphi\|_{\mathcal{K}} \leqslant \Upsilon \|\Pi\varphi\|_{\mathcal{H}}$,

(c) $Q_\perp$ has an inverse $R_\perp \in \mathcal{L}(\mathcal{K}^*, \mathcal{K})$ which satisfies $\|(\mathsf{Id}_{\mathcal{K}} - \Pi)R_\perp\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \leqslant \Upsilon$ and $\|R_\perp(\mathsf{Id}_{\mathcal{K}^*} - \Pi^*)\|_{\mathcal{L}(\mathcal{K}^*,\mathcal{K})} \leqslant \Upsilon$.

(H5) There exists $\beta \in [0, \Upsilon]$ such that if we set

$$M = \mathsf{Re}(i\mathsf{ad}_A(Q)) + \beta Q_+,$$

then $\|M\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \leqslant \Upsilon$, $\|\mathsf{ad}_A(M)\|_{\mathcal{L}(\mathcal{K},\mathcal{K}^*)} \leqslant \Upsilon$, and in the sense of quadratic forms on $\mathcal{H}$ we have

$$\Pi^* M \Pi \geqslant \Upsilon^{-1} \Pi^* \mathcal{I} \Pi. \tag{1.41}$$

Example 1.14. For the dissipative operator $H = H_1 - iV$ (as in Example 1.12) and $\mathsf{Re}(z) \in I \subset\subset J$, we recover the setting of [2] by choosing $\Pi = \chi(H_1)$ (with $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ supported in J and equal to 1 on a neighborhood of I), $Q_\perp = H_1 - z$ and $Q_\perp^+ = V$. Then $Q_\perp^{-1}\Pi_\perp = (H_1 - z)^{-1}(1 - \chi)(H_1)$ is bounded uniformly in z by the spectral theorem.

Now we can state the result for a single resolvent. In these estimates, it is important that the constants C do not depend on Q but only on the parameter Υ.

Theorem 1.15. *Let $N \in \mathbb{N}^*$ and $\Upsilon \geqslant 1$. Assume that A is Υ-conjugate to Q up to order N.*

(i) *Let $\delta > \frac{1}{2}$. There exists $C > 0$ which only depends on Υ and δ such that*

$$\left\| \langle A \rangle^{-\delta} Q^{-1} \langle A \rangle^{-\delta} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C. \tag{1.42}$$

(ii) *Assume that $N \geqslant 2$ and let $\delta_1, \delta_2 \geqslant 0$ such that $\delta_1 + \delta_2 < N - 1$. There exists $C > 0$ which only depends on N, Υ, δ_1, δ_2 and such that*

$$\left\| \langle A \rangle^{\delta_1} \mathbb{1}_{\mathbb{R}_-}(A) Q^{-1} \mathbb{1}_{\mathbb{R}_+}(A) \langle A \rangle^{\delta_2} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C. \tag{1.43}$$

(iii) *Assume that $N \geqslant 2$ and let $\delta \in \left]\frac{1}{2}, N\right[$. There exists $C > 0$ which only depends on N, Υ and δ such that*

$$\left\| \langle A \rangle^{-\delta} Q^{-1} \mathbb{1}_{\mathbb{R}_+}(A) \langle A \rangle^{\delta - 1} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C \tag{1.44}$$

and

$$\left\| \langle A \rangle^{\delta - 1} \mathbb{1}_{\mathbb{R}_-}(A) Q^{-1} \langle A \rangle^{-\delta} \right\|_{\mathcal{L}(\mathcal{H})} \leqslant C. \tag{1.45}$$

Then we use the following abstract lemma to deduce multiple resolvent estimates from the estimates of Theorem 1.15.

Lemma 1.16. *Let $\mathcal{H}$ be a Hilbert space. Let $n \in \mathbb{N}^*$ and $T_1, \dots, T_n \in \mathcal{L}(\mathcal{H})$. For $j \in \{0, \dots, n\}$ we consider on $\mathcal{H}$ a selfadjoint operator $\Theta_j \geqslant 1$, and $\Pi_j^-, \Pi_j^+ \in \mathcal{L}(\mathcal{H})$ such that $\Pi_j^- + \Pi_j^+ = \mathsf{Id}_{\mathcal{H}}$. Let $N \in \mathbb{N}^*$. For $j \in \{1, \dots, n\}$ we assume that there exist $\nu_j \geqslant 0$, $\sigma_j \in [0, \nu_j]$ and a collection $\mathcal{C}_j = \{C_j; (C_{j,\delta_1,\delta_2}); (C_{j,\delta})\}$ of constants such that for $\delta_1, \delta_2 \geqslant 0$ with $\delta_1 + \delta_2 < N - \nu_j$ and $\delta \in [\sigma_j, N]$ we have*

$$\left\|\Theta_{j-1}^{-\sigma_j} T_j \Theta_j^{-\sigma_j}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_j, \tag{1.46}$$

$$\left\|\Theta_{j-1}^{\delta_1} \Pi_{j-1}^- T_j \Pi_j^+ \Theta_j^{\delta_2}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_{j,\delta_1,\delta_2}, \tag{1.47}$$

$$\left\|\Theta_{j-1}^{\delta-\nu_j} \Pi_{j-1}^- T_j \Theta_j^{-\delta}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_{j,\delta}, \tag{1.48}$$

$$\left\|\Theta_{j-1}^{-\delta} T_j \Pi_j^+ \Theta_j^{\delta-\nu_j}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_{j,\delta}. \tag{1.49}$$

Let $T = T_1 \dots T_n$. We set

$$\nu = \sum_{j=1}^{n} \nu_j, \quad \sigma_+ = \sum_{j=1}^{n-1} \nu_j + \sigma_n, \quad \sigma_- = \sigma_1 + \sum_{j=2}^{n} \nu_j.$$

Assume that $N > \nu$. We set $\Pi_- = \Pi_0^-$ and $\Pi_+ = \Pi_n^+$. There exists a collection of constants $\mathcal{C} = \{C; (C_{\delta_-,\delta_+}); (C_\delta^-); (C_\delta^+)\}$ which only depend on the constants $\mathcal{C}_j$, $1 \leqslant j \leqslant n$, and such that

$$\left\|\Theta_0^{-\sigma_+} T \Theta_n^{-\sigma_-}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C, \tag{1.50}$$

for $\delta_-, \delta_+ \geqslant 0$ such that $\delta_- + \delta_+ < N - \nu$ we have

$$\left\|\Theta_0^{\delta_-} \Pi_- T \Pi_+ \Theta_n^{\delta_+}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_{\delta_-,\delta_+}, \tag{1.51}$$

for $\delta \in [\sigma_-, N[$ we have

$$\left\|\Theta_0^{\delta-\nu} \Pi_- T \Theta_n^{-\delta}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_\delta^-, \tag{1.52}$$

and finally, for $\delta \in [\sigma_+, N[$ we have

$$\left\|\Theta_0^{-\delta} T \Pi_+ \Theta_n^{\delta-\nu}\right\|_{\mathcal{L}(\mathcal{H})} \leqslant C_\delta^+. \tag{1.53}$$

It is important that the constants in the conclusions of the lemma only depend on the constants in the assumptions. Thus, if for some operators $T_j(z)$, $1 \leqslant j \leqslant n$, the estimates (1.46)-(1.48) are independent of the parameter z, then so are the estimates (1.50)-(1.53).

Theorem 1.15 allows to apply Lemma 1.16 with $\nu_j = 1$, $\sigma_j \in \left]\frac{1}{2}, 1\right]$, $\Theta_j = \langle A \rangle$, $\Pi_j^- = \mathbb{1}_{\mathbb{R}_-^*}(A)$ and $\Pi_j^+ = \mathbb{1}_{\mathbb{R}_+}(A)$, where A is the conjugate operator. Notice that the assumptions of Definition 1.13 are used to prove Theorem 1.15 but no longer play a role to get the multiple resolvent estimates.

Since Lemma 1.16 is completely abstract, it can also be applied to the inserted factors. Roughly, if the commutators of T_j with A are bounded on $\mathcal{H}$, then the assumptions of Lemma 1.16 hold with $\nu_j = \sigma_j = 0$ (see Proposition 3.11 in [23]).

Thus, we can apply Lemma 1.16 to a product of resolvents and inserted factors as in (1.38). Moreover, the resolvents do not have to be associated with the same operator, and they do not even have to be indeed resolvents.

1.4 Strategy for the proofs of the resolvent estimates

In this paragraph we explain how we use the dissipative commutators method of Section 1.3 to prove Theorems 1.8 and 1.10. This is an occasion to rewrite some arguments of [6, 11] in the spirit of [23].

1.4.1 Contribution of intermediate frequencies

The first step is to prove uniform estimates on $R(z)$ and then on $(\mathcal{W}+iz)^{-1}$ for $\mathsf{Im}(z)$ positive and $\mathsf{Re}(z)$ in a compact subset of $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$. For $z \in \mathbb{C}$ we set

$$P_{\mathsf{R}}(z) = \mathsf{Re}\big(-\Delta_G - iawz - wz^2\big) = -\Delta_G + aw\mathsf{Im}(z) - w\mathsf{Re}(z^2).$$

With the generator of dilations A defined in (1.30) we have

$$[P_{\mathsf{R}}(z), iA] = 2P_{\mathsf{R}}(z) + K(z) + 2\mathsf{Re}(z^2),$$

where

$$\begin{aligned} K(z) = {} & \mathrm{div}(x\cdot\nabla G(x))\nabla - \mathsf{Im}(z)(x\cdot\nabla)(aw) + \mathsf{Re}(z^2)(x\cdot\nabla)w \\ & - 2aw\mathsf{Im}(z) + 2\mathsf{Re}(z^2)(w-1). \end{aligned}$$

Then we apply a compactness argument. Let $\tau \in \mathbb{R}^*$. Since 0 is not an eigenvalue of $P_{\mathsf{R}}(\tau)$ (see [KT06]), the operator $\mathbb{1}_{[-\eta,\eta]}(P_{\mathsf{R}}(\tau))$ goes weakly to 0 as $\eta \to 0$. On the other hand, $\mathbb{1}_{[-1,1]}(P_{\mathsf{R}}(\tau))K(\tau)\mathbb{1}_{[-1,1]}(P_{\mathsf{R}}(\tau))$ is compact, so $\mathbb{1}_{[-\eta,\eta]}(P_{\mathsf{R}}(\tau))K(\tau)\mathbb{1}_{[-\eta,\eta]}(P_{\mathsf{R}}(\tau))$ goes to 0 as $\eta \to 0$. For $\eta > 0$ small enough we get

$$\mathbb{1}_{[-\eta,\eta]}(P_{\mathsf{R}}(\tau))[P_{\mathsf{R}}(\tau), iA]\mathbb{1}_{[-\eta,\eta]}(P_{\mathsf{R}}(\tau)) \geqslant \frac{3\tau^2}{2}\,\mathbb{1}_{[-\eta,\eta]}(P_{\mathsf{R}}(\tau)).$$

Then for $\chi \in C_0^\infty(\mathbb{R},[0,1])$ supported in $]-\eta,\eta[$ and equal to 1 on $\big[-\frac{\eta}{2},\frac{\eta}{2}\big]$ we get for $z \in \mathbb{C}_+$ close to τ

$$\chi(P_{\mathsf{R}}(z))[P_{\mathsf{R}}(z), iA]\chi(P_{\mathsf{R}}(z)) \geqslant \tau^2\,\chi(P_{\mathsf{R}}(z))^2. \tag{1.54}$$

This is the main assumption in Definition 1.13 (see (1.41) with $Q = (-\Delta_G - iawz - wz^2)$, $\Pi = \chi(P_{\mathsf{R}}(z))$ and $\beta = 0$). We get the estimates of Theorem 1.15 with $Q^{-1} = R(z)$.

Then we use (1.21) to deduce estimates on $(\mathcal{W}+iz)^{-1}$. With (1.10) and (1.11) it is classical that the estimates of Theorem 1.15 also hold with Q^{-1} replaced by a or w, for any $\delta, \delta_1, \delta_2 \geqslant 0$, and $\delta - 1$ replaced by δ in (1.44) and (1.45)). Then the assumptions of Lemma 1.16 are satisfied with $\Theta_j = \langle A\rangle$ and $\nu_j = \sigma_j = 0$. Using also the good commutation properties of $\langle D\rangle = (1-\Delta)^{\frac{1}{2}}$ with A, we can finally prove with Lemma 1.16 that the estimates of Theorem 1.15 hold for $Q^{-1} = (\mathcal{W}+iz)^{-1}$ and A replaced by the operator

$$\mathcal{A} = \begin{pmatrix} \langle D\rangle^{-1} A \langle D\rangle & 0 \\ 0 & A \end{pmatrix},$$

which is selfadjoint on $\mathscr{H}$. Applying once more Lemma 1.16, we deduce the multiple resolvent estimates for $(\mathcal{W}+iz)^{-n}$ with weight $\langle\mathcal{A}\rangle^{-\delta}$ in $\mathscr{H}$. Notice that $\mathcal{A}$ does not have to be conjugate to $i\mathcal{W}$ in the sense of Definition 1.13 (and it is not).

Using the regularity given by the resolvents of $\mathcal{W}$ to remove the derivatives in the weights $\langle\mathcal{A}\rangle^{\delta}$ (see the discussion about low frequencies below), we can finally prove the following estimates.

Proposition 1.17. *Let K be a compact subset of $\mathbb{C}^*$. Let $n \in \mathbb{N}^*$ and $\delta > n - \frac{1}{2}$. There exists $C > 0$ such that for $\zeta \in K \cap \mathbb{C}^+$ we have*

$$\big\|(\mathcal{W}-\zeta)^{-n}\big\|_{\mathcal{L}(\mathscr{E}^{\delta},\mathscr{E}^{-\delta})} \leqslant C.$$

1.4.2 Contribution of high frequencies

In this paragraph we briefly discuss the proof of Theorem 1.8. With Proposition 1.17, it is enough to consider spectral parameters $z = i\zeta$ such that $\mathsf{Im}(z) > 0$ and $|\mathsf{Re}(z)| \gg 1$.

As explained in the introduction, the behavior of the contribution of high frequencies depends on the geometries of the domain and the damping region. It was the main motivation of [2] to prove in a slightly different context the following estimate for a single resolvent.

Theorem 1.18. *Assume that the geometric damping condition* (1.17) *holds. Let* $\delta > \frac{1}{2}$. *There exist* $\tau_0 > 0$ *and* $c > 0$ *such that for* $z \in \mathbb{C}_+$ *with* $\mathsf{Re}(z) \geqslant \tau_0$ *we have*

$$\left\| \langle x \rangle^{-\delta} \partial_x^{\beta_1} R(z) \partial_x^{\beta_2} \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leqslant \frac{c}{|z|^{1-|\beta_1|-|\beta_2|}}.$$

In [2] the estimate was given for the semiclassical Schrödinger operator with potential $-h^2\Delta + V_1(x) - ihV_2(x)$ (and without the additional derivatives), but the strategy is the same for a Laplacian associated with a metric (see [6, Section 8]). We do not give details for Theorem 1.18 since the proof is essentially the same as in [Roy10], but we can at least recall that we apply the Mourre method with a conjugate operator given by the Weyl quantization of a symbol f which satisfies for some $\beta \geqslant 0$

$$\{p, f\} + \beta a \geqslant c_0 > 0, \qquad \text{on } p^{-1}(\{1\}) \tag{1.55}$$

(p is the symbol defined in (1.15)). The positivity of the Poisson bracket $\{p, f\}$ means that f is increasing along the Hamiltonian flow associated with p. With the second term in the left-hand side of (1.55) (which corresponds to the second term in the left-hand side of (1.37) or (1.41)), the symbol does not have to be increasing along the flow in the damping region. This is why we can allow bounded classical trajectories if they go through this damping region (see (1.17)).

Then, as explained in Section 1.3, Theorem 1.18 was generalized in [6] to multiple resolvent estimates with inserted factors, to deal with terms of the form (1.38). From this we can deduce Theorem 1.8. We omit the details and refer to [6, Section 8] and [11, Sections 3.1 and 4.2].

1.4.3 Contribution of low frequencies

The main result in [6] is the resolvent estimates for the contribution of low frequencies. The commutators method does not directly give uniform bounds for the resolvent of a Schrödinger operator near 0. We see from (1.54) that the estimate becomes bad when $\tau \to 0$. We can apply the commutators method to the operator $|z|^{-2}(-\Delta - izaw - z^2 w)$ for z small, but we only get an estimate of size $O(|z|^{-2})$ for $R(z)$ in weighted spaces. This is at least uniform with respect to $\mathsf{Im}(z)$ close to 0, but we need an additional argument to get a uniform bound when $\mathsf{Re}(z) \to 0$.

A key argument to recover some smallness in the low frequency analysis is a generalization of the Hardy inequality, which roughly says that in dimension $d \geqslant 3$ the multiplication by $|x|^{-1}$ behaves like a derivative. And for low frequencies, a derivative can be seen as a small operator.

In this report, I present the results of [6, 11] with the point of view developed recently in [23] for the Schrödinger equation. For $r \in]0, 1]$ we set $D_r = \sqrt{-\Delta}/r$. Then for $s \in \mathbb{R}$ we denote by H_r^s and $\dot{H}_r^s$ the usual Sobolev spaces H^s and $\dot{H}^s$, endowed repectively with the norms defined by

$$\|u\|_{H_r^s} = \|\langle D_r \rangle^s u\|_{L^2}, \quad \|u\|_{\dot{H}_r^s} = \|D_r^s u\|_{L^2}. \tag{1.56}$$

In particular $\|u\|_{\dot{H}^s} = r^s \|u\|_{\dot{H}_r^s}$, and for $\alpha \in \mathbb{N}^d$ and $s \in \mathbb{R}$ the derivative ∂_x^α defines an operator of size $r^{|\alpha|}$ from H_r^s to $H_r^{s-|\alpha|}$. Finally, we denote by $\mathscr{H}_r^s$ the Hilbert completion of $\mathscr{S}$ for the norm defined by

$$\|U\|_{\mathscr{H}_r^s}^2 = \|ru\|_{H_r^{s+1}}^2 + \|v\|_{H_r^s}^2, \quad U = (u, v). \tag{1.57}$$

In particular, $\mathscr{H} = \mathscr{H}_1^0$. We write D_z, H_z^s, $\dot{H}_z^s$ and $\mathscr{H}_z^s$ for $D_{|z|}$, $H_{|z|}^s$, $\dot{H}_{|z|}^s$ and $\mathscr{H}_{|z|}^s$.

Let d_0 be a fixed integer greater than $\frac{d}{2}$. For $\kappa \geqslant 0$ we denote by $\mathcal{S}^{-\kappa}$ the set of smooth functions ϕ such that

$$\|\phi\|_{\mathcal{S}^{-\kappa}} = \sup_{|\alpha| \leqslant d_0} \sup_{x \in \mathbb{R}^d} \left| \langle x \rangle^{\kappa + |\alpha|} \partial^\alpha \phi(x) \right| < +\infty. \tag{1.58}$$

We can say that the multiplication by a decaying function behaves like a derivative, and hence is small for low frequencies, in the following sense (see Proposition 7.2 in [6] or Proposition 3.1 in [23]).

Proposition 1.19. *Let $s \in \left]-\frac{d}{2}, \frac{d}{2}\right[$ and $\kappa \geqslant 0$ such that $s-\kappa \in \left]-\frac{d}{2}, \frac{d}{2}\right[$. Let $\eta > 0$. There exists $C \geqslant 0$ such that for $\phi \in \mathcal{S}^{-\kappa-\eta}$, $u \in H^s$ and $r \in]0,1]$ we have*

$$\|\phi u\|_{H_r^{s-\kappa}} \leqslant C r^{\kappa} \|\phi\|_{\mathcal{S}^{-\kappa-\eta}} \|u\|_{H_r^s}.$$

In particular, if $\phi \in \mathcal{S}^{-\eta}$ for some $\eta > 0$, then for any $s \in \left]-\frac{d}{2}, \frac{d}{2}\right[$ the multiplication by $(1+\phi)$ defines a bounded operator on H_r^s uniformly in $r \in]0,1]$.

A first application is that for $s \in \left]-\frac{d}{2}, \frac{d}{2}\right[$ and $\rho \in]0, \rho_0[$ small enough the assumptions (1.10) and (1.11) give

$$\|za(x)w(x)\|_{\mathcal{L}(H_z^{s+1}, H_z^{s-1})} + \left\|z^2(w(x)-1)\right\|_{\mathcal{L}(H_z^{s+1}, H_z^{s-1})} \lesssim |z|^{2+\rho}. \tag{1.59}$$

Thus the corresponding terms are perturbations of $-\Delta - z^2$ in $\mathcal{L}(H_z^{s+1}, H_z^{s-1})$ when z is small. Notice that it is important here that a is of short range.

Even if $G(x) - \mathsf{Id}$ also decays by (1.10), the same does not apply to $(\Delta_G - \Delta)$. Indeed, this term is already of order 2 and we cannot pay more regularity to get some smallness in $\mathcal{L}(H_r^{s+1}, H_r^{s-1})$ for z small. The idea is then to consider first the case where $\|G(x) - \mathsf{Id}\|_{\mathcal{S}^{-\rho}}$ is small enough, and then to add a compactly supported contribution for the metric. In [6] this is done in Section 7.3. In [23] (for the Schrödinger equation) we deal with this compactly supported perturbation in each intermediate result. This is an important technical difficulty that we omit here for simplicity. Thus we proceed as if $\|G(x) - \mathsf{Id}\|_{\mathcal{S}^{-\rho}}$ were already small enough.

Ideas of proof for Theorem 1.10. We see with (1.59) that $(-\Delta_G + raw + r^2 w)$ is a small perturbation of $-\Delta + r^2$ in $\mathcal{L}(H_r^{s+1}, H_r^{s-1})$ for any $s \in \left]-\frac{d}{2}, \frac{d}{2}\right[$, and in particular

$$\|R(ir)\|_{\mathcal{L}(H_r^{s-1}, H_r^{s+1})} \lesssim \frac{1}{r^2}.$$

From (1.23) we deduce, for $s \in \left]-\frac{d}{2}+1, \frac{d}{2}\right[$,

$$\left\|(\mathcal{W} - r)^{-1}\right\|_{\mathcal{L}(\mathscr{H}_r^{s-1}, \mathscr{H}_r^{s})} \lesssim \frac{1}{r}.$$

Thus, for $s_1, s_2 \in \left[0, \frac{d}{2}\right[$ and $m \in \mathbb{N}^*$ such that $s_1 + s_2 \leqslant m$ we have

$$\left\|(\mathcal{W} - r)^{-m}\right\|_{\mathcal{L}(\mathscr{H}_r^{-s_2}, \mathscr{H}_r^{s_1})} \lesssim \frac{1}{r^m}. \tag{1.60}$$

We can similarly estimate $(\mathcal{W} - \zeta)^{-m}$ by $|\zeta|^{-m}$ if $\mathsf{Re}(\zeta) \gtrsim |\zeta|$.

In general, we set $r = |\zeta|$ and we observe that, by the resolvent identity, we can write $(\mathcal{W} - \zeta)^{-n}$ as a sum of terms of the form

$$(\zeta - r)^{m-n} (\mathcal{W} - r)^{-m} \tag{1.61}$$

for some $m \geqslant n$, or

$$(\zeta - r)^{2N+\nu-n} (\mathcal{W} - r)^{-N} (\mathcal{W} - \zeta)^{-\nu} (\mathcal{W} - r)^{-N}, \tag{1.62}$$

for $\nu \leqslant n$ and N as large as we wish.

The idea to estimate a term of the form (1.61) is to use the weight (remember that we need an estimate in $\mathcal{L}(\mathscr{E}^{\delta}, \mathscr{E}^{-\delta})$) to convert the elliptic regularity given by (1.60) into smallness. By Proposition 1.19 and (1.25) we can prove that for $s \in \left[0, \frac{d}{2}\right[$ and $\delta > s$ we have

$$\|U\|_{\mathscr{E}^{-\delta}} \lesssim r^s \|U\|_{\mathscr{H}_r^s} \quad \text{and} \quad \|U\|_{\mathscr{H}_r^{-s}} \lesssim r^s \|U\|_{\mathscr{E}^{\delta}}. \tag{1.63}$$

With (1.60) and (1.63) applied with

$$s_1 = s_2 = s = \frac{1}{2}\min(d-\varepsilon, m),$$

we obtain that (1.61) satisfies the estimate of Theorem 1.10.

Now we turn to the terms of the form (1.62). In such a term we still have a factor $(\mathcal{W}-\zeta)^{-\nu}$ as at the begining, but the interest of the decomposition (1.61)-(1.62) is that in (1.62) we now have the factors $(\mathcal{W}-r)^{-N}$ which give ellipitic regularity. The main step of the proof remains the commutators method. Notice that we have omitted here the fact that $G-\mathsf{Id}$ is only small at infinity. In general, to take this into account for the commutators method, we can as in [BB21] replace the usual generator of dilations (1.30) by a generator of dilations at infinity

$$A_z = -\frac{(1-\chi_z)x\cdot i\nabla + i\nabla\cdot x(1-\chi_z)}{2}, \tag{1.64}$$

where $\chi_z(x) = \chi(|z|\,x)$ for some $\chi \in C_0^\infty(\mathbb{R}^d,[0,1])$ equal to 1 on a neighborhood of 0.

Following Sections 1.3 and 1.4.1, beging careful with the dependance on z going to 0, we prove that A_z is Υ-conjugate to $|z|^2(-\Delta - izaw - z^2w)$ for some Υ independent of $z\in\mathbb{C}_+$ close to 0 (and with $\mathsf{Re}(z^2)\gtrsim |z|^2$). We get in particular an estimates of the form

$$\left\|\langle A_z\rangle^{-\delta} R(z)\langle A_z\rangle^{-\delta}\right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim \frac{1}{|z|^2}.$$

See [23, Sec. 5.2] for the case $a=0$. Then, as in Section 1.4.1, we can deduce with successive uses of Lemma 1.16 an estimate of the form

$$\left\|\langle \mathcal{A}_z\rangle^{-\delta}(\mathcal{W}+iz)^{-\nu}\langle \mathcal{A}_z\rangle^{-\delta}\right\|_{\mathcal{L}(\mathscr{H}_z^0)} \lesssim \frac{1}{|z|^\nu}, \quad \mathcal{A}_z = \begin{pmatrix}\langle D_z\rangle A_z\langle D_z\rangle^{-1} & 0\\ 0 & A_z\end{pmatrix}. \tag{1.65}$$

It remains to use the factors $(\mathcal{W}-r)^{-N}$ to recover some smallness for (1.62). We also use this regularity to compensate the derivatives used in the weights $\langle\mathcal{A}_z\rangle$. More precisely, for $s\in\left[0,\frac{d}{2}\right[$ and $\delta>s$ we can prove

$$\left\|(\mathcal{W}-r)^{-N}\langle\mathcal{A}_\zeta\rangle^{\delta}\right\|_{\mathcal{L}(\mathscr{H}_z^0,\mathscr{E}^{-\delta})} \lesssim r^{s-N}, \quad \left\|\langle\mathcal{A}_\zeta\rangle^{\delta}(\mathcal{W}-r)^{-N}\right\|_{\mathcal{L}(\mathscr{E}^{\delta},\mathscr{H}_z^0)} \lesssim r^{s-N}. \tag{1.66}$$

Notice that we cannot directly use pseudodifferential calculus to prove these estimates, since the commutators of two operators are usually not smaller than the products. We can do the computation by hand.

Finally, with (1.65) and (1.66) we obtain that (1.62) also satisfies the estimate of Theorem 1.10. □

As said above, the ideas of the proof are not presented here with the point of view of [6, 11]. For instance, in these papers we did the decomposition (1.61)-(1.62) on the expression (1.38), and not directly on the resolvent of $\mathcal{W}$. On the other hand, in [6, 11] we rescaled the operators and not the Sobolev spaces. This is equivalent.

1.5 Optimal resolvent estimate for the Schrödinger equation in an asymptotically conical setting

In this paragraph, we briefly discuss the paper [7] about sharp low frequency resolvent estimates on asymptotically conical manifold. This is a collaboration with Jean-Marc Bouclet. It is related to the other works presented in this chapter since it is about low frequency resolvent estimates for a Laplace operator, but the motivations are of different nature.

The setting is a connected and asymptotically conical manifold $\mathcal{M}$ with Riemannian metric G (possibly with boundary). Asymptotically conical means that there exists a compact subset K of $\mathcal{M}$ such that $(\mathcal{M}\backslash K, G)$ is isometric to $]R_0, +\infty[\times\mathcal{S}$ for some $R_0 > 0$ and some closed Riemannian manifold $\mathcal{S}$. This product is endowed with a metric approaching the metric $dr^2 + r^2 h_0$, where h_0 is a Riemannian metric on $\mathcal{S}$. More precisely, there exists a diffeomorphism

$$\kappa : \left\{ \begin{array}{ccc} \mathcal{M}\backslash K & \to &]R_0, +\infty[\times\mathcal{S} \\ m & \mapsto & (r(m), \omega(m)) \end{array} \right.$$

through which we can write

$$G = \kappa^* \big(a(r)dr^2 + 2rb(r)dr + r^2 h(r)\big),$$

where $a(r)$ is a function on $\mathcal{S}$ going to 1, $b(r)$ is a 1-form on $\mathcal{S}$ going to 0 and $h(r)$ is a Riemannian metric on $\mathcal{S}$ going to h_0, in the sense that there exists $\rho > 0$ such that for $j \in \mathbb{N}$ we have

$$\left\|\partial_r^j (a(r) - 1)\right\|_{\Gamma^0(\mathcal{S})} + \left\|\partial_r^j b(r)\right\|_{\Gamma^1(\mathcal{S})} + \left\|\partial_r^j (h(r) - h_0)\right\|_{\Gamma^2(\mathcal{S})} \lesssim r^{-j-\rho},$$

$\Gamma^k(\mathcal{S})$ being any seminorm on the space of smooth sections of $(T^*\mathcal{S})^{\otimes^k}$.

This setting is more general than the scattering metrics of [Mel95]. Moreover, even if $\mathcal{M} = \mathbb{R}^d$, the assumption on the metric is more general than being asymptotically flat since h_0 does not have to be the usual metric on the sphere $\mathcal{S} = S^{d-1}$.

We denote by P the Laplace-Beltrami operator on $\mathcal{M}$ (with Dirichlet boundary condition if $\partial\mathcal{M} \neq \varnothing$). The main results in [7] are the following resolvent estimates.

Theorem 1.20. *There exist a neighborhood $\mathcal{U}$ of 0 in $\mathbb{C}$ and $C > 0$ such that for $z \in \mathcal{U}\backslash\mathbb{R}$ we have*

$$\left\| \langle r\rangle^{-1} (P - z)^{-1} \langle r\rangle^{-1} \right\|_{\mathcal{L}(L^2(\mathcal{M}))} \leqslant C.$$

Theorem 1.21. *Let $N \in \mathbb{N}^*$ and let J be a compact interval of $]0, +\infty[$. There exists $C > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon \in]0, \varepsilon_0]$ and $z \in \mathbb{C}\backslash\mathbb{R}$ with $\mathsf{Re}(z) \in J$ we have*

$$\left\| \langle \varepsilon r\rangle^{-N} (\varepsilon^{-2} P - z)^{-N} \langle \varepsilon r\rangle^{-N} \right\|_{\mathcal{L}(L^2(\mathcal{M}))} \leqslant C.$$

The two main interests of these results are the general geometric setting and the optimal weight $\langle r\rangle^{-1}$. The best weight in the previous papers about the low frequency resolvent estimates was of the form $\langle r\rangle^{-s}$ for $s > 1$. It was in particular motivated by the Strichartz estimates written later in [BM].

Notice that the second result corresponds to what we would get for the free Laplacian on $\mathbb{R}^d$ from an estimate on $\langle r\rangle^{-N} (-\Delta - z)^{-N} \langle r\rangle^{-N}$ by a scaling argument. Thus Theorem 1.21 says that the same estimate holds in a setting where such a scaling is meaningless.

We do not go into the details of the proofs here. The first important step is to reduce the problem to resolvent estimates for a Schrödinger operator on the pure cone $]0, +\infty[\times\mathcal{S}$. Then the main part of the proof is to develop the Mourre theory on this pure cone for such an operator.

1.6 The damped Schrödinger equation

In [10] we have considered with Moez Khenissi the local energy decay for a damped Schrödinger equation.

Local energy decay for a dissipative Schrödinger equation has already been studied on exterior domains, with some potential damping in the domain ($\alpha = 0$ with the notation of (1.67) below) in [AK07] or at the boundary in [AK10]. See also [BC14] for a Schrödinger equation with non-linear damping at infinity.

Nevertheless, a potential damping is not strong enough to have the smoothing effect typical for the usual Schrödinger equation. For the regularized Schrödinger equation ($\alpha = 1$ in (1.67)), we can recover the usual gain of half a derivative (see [Alo08b, Alo08a]), or a $H^{\frac{1}{2}-\varepsilon}$-smoothing effect if a few classical trajectories fail to satisfy the geometric damping condition (see [AKV13], see also [Bur04] for a similar result in the selfadjoint case).

In [10], we consider this damped Schrödinger equation in the asymptotically Euclidean setting. We consider on $\mathbb{R}^d$ the problem

$$\begin{cases} -i\partial_t u + Pu - ia(x)\langle D\rangle^{\alpha} a(x)u = 0, & \forall t \geqslant 0, \forall x \in \mathbb{R}^d, \\ u_{|t=0} = u_0, \end{cases} \tag{1.67}$$

where $P = -\Delta_G$ is a Laplacian in divergence form, $D = \sqrt{-\Delta}$, $\alpha \in [0, 2[$ and a is of short range as in (1.11). Since a appears twice, the dissipative term actually decays at least like $\langle x\rangle^{-2-2\rho_0}$ with respect to the space variable.

For the Schrödinger equation, it is the L^2-norm of the solution which is constant without damping and non-increasing in general. Formally,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|^2_{L^2(\mathbb{R}^d)} = -2\langle\langle D\rangle^{\alpha} au(t), au(t)\rangle_{L^2(\mathbb{R}^d)} \leqslant 0.$$

We set

$$P_\alpha = P - ia(x)\langle D\rangle^{\alpha} a(x), \quad \mathsf{Dom}(P_\alpha) = H^2(\mathbb{R}^d).$$

The strength of the damping depends on the parameter α. This only plays a role for the contribution of high frequencies (for low frequencies we have $\langle D\rangle \simeq 1$).

Under the non-trapping condition (1.16), the damping should not play an important role. And indeed, we recover in this dissipative setting what was at that time the best result known even in the undamped case for the local energy decay (the ε loss has been removed later in [BB21]), and the smoothing effect.

Now assume that (1.16) does not hold but the geometric damping condition (1.17) does. If the damping is strong enough ($\alpha \geqslant 1$), we recover the same results as under the non-trapping condition. Of course, we cannot get a better estimate even with a stronger damping ($\alpha > 1$) since the classical trajectories at infinity do not see the damping.

If the damping is weak ($\alpha < 1$), then the high-frequency resolvent estimates are weaker in the presence of bounded trajectories. However, if we have a strong weight or if we can allow a loss of regularity, then we can recover the same decay for large times (remember that the rate of decay is actually limited by the contribution of low frequencies). However the regularizing effect (governed by high frequencies) will be weaker than usual in this case.

More precisely, we prove in [10] the following local energy decay and smooting effect for (1.67).

Theorem 1.22 (Local energy decay)**.** *Let $\varepsilon > 0$. Let $\delta > \frac{d+1}{2}$ if d is even and $\delta > \frac{d}{2} + 1$ if δ is odd. Let $N \in \mathbb{N}$ and $\sigma \in [0, 2]$. Assume that*

(i) the non-trapping condition (1.16) *holds,*

(ii) <u>or</u> the geometric damping condition (1.17) *holds, $N\min(1, \alpha) + \sigma \geqslant 2$ and $\delta > N - \frac{1}{2}$.*

Then there exists $C \geqslant 0$ such that for $u_0 \in H^{\sigma,\delta}(\mathbb{R}^d)$ and $t \geqslant 0$ we have

$$\left\|e^{-itP_\alpha}u_0\right\|_{L^{2,-\delta}(\mathbb{R}^d)} \leqslant C\langle t\rangle^{-\frac{d}{2}+\varepsilon}\|u_0\|_{H^{\sigma,\delta}(\mathbb{R}^d)}.$$

The condition $N\min(1, \alpha) + \sigma \geqslant 2$ means that even if α is small, we can apply the high-frequency resolvent estimate (see (1.68) below) with N large, or we can pay a loss of derivative (typically if $\alpha = 0$).

Theorem 1.23 (Global smoothing effect)**.** *Assume that the geometric damping condition* (1.17) *holds. Then there exists* $C \geqslant 0$ *such that for all* $u_0 \in L^2(\mathbb{R}^d)$ *we have*

$$\int_0^{+\infty} \left\| \langle x \rangle^{-1} \langle D \rangle^{\min(1,\alpha)/2} e^{-itP_\alpha} u_0 \right\|^2_{L^2(\mathbb{R}^d)} \mathrm{d}t \leqslant C \left\| u_0 \right\|^2_{L^2(\mathbb{R}^d)} .$$

Moreover, under the non-trapping condition (1.16)*, we can replace* $\min(1,\alpha)$ *by* 1*.*

Notice that to have the weight $\langle x \rangle^{-1}$ in Theorem 1.23 we use the optimal resolvent estimate at low frequencies deduced from Theorem 1.20.

For high frequencies, we use the same strategy as in [2, 6] (see Section 1.4.2), except that the damping is not necessarily strong enough. For $N \in \mathbb{N}^*$ and $\delta > N - \frac{1}{2}$ we get an estimate of the form

$$\left\| \langle x \rangle^{-\delta} (P_\alpha - z)^{-N} \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \lesssim |z|^{-\frac{N \min(1,\alpha)}{2}} . \tag{1.68}$$

For low frequencies, we adapt the proof of [6] to a dissipative perturbation of the Laplacian which is of different nature. In particular, there is no time derivative in the dissipative term, hence no factor z in the corresponding term for the resolvent. This is why we need more spatial decay for the absorption index. Thus we use Proposition 1.19 in an even more crucial way than for the wave equation, and the restriction on the Sobolev indices therein could have been a serious difficulty. We do not discuss all the details and only emphasize the fact that we unexpectedly had to use the dissipative Mourre theory in the sense of forms:

Remark 1.24. To apply the first version of the dissipative commutators method given in [2] to the operator P_α at low frequencies, the dissipative part $a(x) \langle D \rangle^\alpha a(x)$ has to be uniformly relatively bounded with respect to the selfadjoint part P. For this we use the decay of $a(x)$ at infinity and Proposition 1.19. P defines an operator of size $O(|z|^2)$ in $\mathcal{L}(H^2_z, L^2)$. By Proposition 1.19, $a(x)$ can be seen as an operator of size $O(|z|)$ in $\mathcal{L}(H^{s+1}_z, H^s_z)$ and in $\mathcal{L}(H^s_z, H^{s-1}_z)$ (we omit $\langle D \rangle^\alpha$ which does not play an important role). To have a dissipative part of size $O(|z|^2)$ in $\mathcal{L}(H^2_z, L^2)$, we have to apply this with $s = 1$. Because of the restriction of Proposition 1.19, this is only possible if $d \geqslant 5$.

However, for $d \geqslant 3$ we can proceed similarly with $s = 0$, which means that we see the dissipative part as an operator from H^1_z to H^{-1}_z. Thus, even if P_α is the sum of a selfadjoint operator and a dissipative part as in [2] (see also (1.36)), for $d = 3, 4$ we can only apply the Mourre method is the sense of forms as in [9].

Finally, the smoothing effect can be directly deduced from the resolvent estimates via the theory of relatively smooth operators, which is classical for selfadjoint operators (see [RS79, Sec. XIII.7]). For dissipative operators, this relies on the theory of selfadjoint dilations. For this we refer to Proposition 6.2 in [9].

1.7 Asymptotic behavior for the Schrödinger equation

We finish this chapter with the most recent result about low frequency resolvent estimates. In [23], we improve the results known for the usual selfadjoint Schrödinger equation (1.13) about the local energy decay and low frequency resolvent estimates.

The optimal low frequency resolvent estimates and then the optimal local energy decay have been proved in [BB21]. We go beyond and prove that the solution of (1.13) behaves (for low frequencies and hence for large times) like the solution of the free Schrödinger equation (1.6) for some $f_0 \in L^2(\mathbb{R}^d)$. More precisely, we prove that the local energy of the difference $u(t) - u_0(t)$ decays faster than the local energy of $u_0(t)$. This means that $u_0(t)$ is the leading term for the asymptotic expansion of $u(t)$ for large t. We recover in particular the same local energy decay for $u(t)$ as for $u_0(t)$. We denote by P_0 the usual Laplacian on $\mathbb{R}^d$. The precise result is the following.

Theorem 1.25. *Assume that the non-trapping condition* (1.16) *holds. Let* $\rho_1 \in [0, \rho_0[$ *and* $\delta \geqslant \frac{d}{2} + 2$. *There exists* $C \geqslant 0$ *such that for* $t \geqslant 0$ *we have*

$$\left\| \langle x \rangle^{-\delta} \left(e^{-itP} - e^{-itP_0} w \right) \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leqslant C \langle t \rangle^{-\frac{d}{2} - \frac{\rho_1}{2}} .$$

The factor w next to e^{-itP_0} means that for $f \in L^2(\mathbb{R}^d)$ we compare the solution u of (1.13) with the solution u_0 of (1.6) with $f_0 = wf$.

Since the decay at rate $t^{-\frac{d}{2}}$ is optimal for the free Schrödinger equation, Theorem 1.25 indeed says that the difference $e^{-itP} - e^{-itP_0} w$ decays faster than $e^{-itP_0} w$ (in the sense of the local energy). Moreover, as a corollary we recover the optimal decay at rate $t^{-\frac{d}{2}}$ for the solution of (1.13).

We already know that under the non-trapping condition (1.16), the contributions of high frequencies for both problems decay fast, so Theorem 1.25 is again mostly a result about low frequencies. Thus we have to compare the resolvents of P and P_0 near 0.

Theorem 1.26. *Let* $\rho_1 \in [0, \rho_0[$, $n \in \mathbb{N}^*$ *and* $\delta > n + \frac{1}{2}$. *There exists* $C > 0$ *such that for* $\zeta \in \mathbb{D} \backslash \mathbb{R}_+$ *we have*

$$\left\| \langle x \rangle^{-\delta} \left((P - \zeta)^{-n} - (P_0 - \zeta)^{-n} w \right) \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leqslant C |\zeta|^{\min(0, \frac{d+\rho_1}{2} - n)} .$$

Notice that asymptotic expansions for the resolvent of a Schrödinger operator at the low frequency limit, and then the expansion of the propagator for large times, have already been studied for perturbations by a potential. We have already mentioned [JK79]. We also refer to the recent papers [Wan20] and [Aaf21] for complex-valued potentials. The difficulty in these cases is that we can have an eigenvalue or a resonance at the bottom of the spectrum, which gives a singularity for the resolvent. This is why these results require strong decay assumptions on the potential.

We have already more or less used the setting of [23] to explain the results of [6, 11] in Sections 1.3 and 1.4. Therefore, in this section we only add some comments specific to the fact that we not only estimate resolvents but compare the resolvents of two different operators.

In particular, one of the difficulties for the proof is that we have to use the commutators method simultaneously for the two operators P and P_0. We have already explained in Section 1.3 that this is one of the improvements of the method developed for [23].

Another problem is that the operator P and P_0 are selfadjoint, but they are not selfadjoint on the same Hilbert space (even if $L^2(\mathbb{R}^d, \mathrm{d}x)$ and $L^2(\mathbb{R}^d, w\,\mathrm{d}x)$ are equal as sets and have equivalent norms).

We replace $(P - \zeta)^{-1}$ by $(-\Delta_G - \zeta w)^{-1} w$. The operator $(-\Delta_G - \zeta w)^{-1}$ is no longer the resolvent of a selfadjoint operator, but for $\zeta \in \mathbb{C}_+$ it is still the inverse of the dissipative operator $-\Delta_G - \zeta w$. It is dissipative on the usual space $L^2(\mathbb{R}^d)$, as is $-\Delta - \zeta$. Thus we can work in $L^2(\mathbb{R}^d)$ endowed with its usual structure. This is why in Theorem 1.15 we no longer consider resolvents but the inverse of parameter-dependent dissipative operators. In particular, even for this selfadjoint problem we have to use the dissipative version of the commutators method.

Replacing $(P - \zeta)^{-1}$ by $(-\Delta_G - \zeta w)^{-1}$ is not just a technical issue, and it is really $(-\Delta_G - \zeta w)^{-1}$ that is close to $(-\Delta - \zeta)^{-1}$ in a suitable sense. Then $(P - \zeta)^{-1}$ is close to $(-\Delta - \zeta)^{-1} w$, which explains the additional factor w in Theorem 1.26 and then in Theorem 1.25.

Ideas of proof. We have to estimate $(P - \zeta)^{-n} w^{-1} - (P_0 - \zeta)^{-n}$ for $\zeta = z^2$ close to 0. We first multiply this difference by $|z|^{2n}$ to have a spectral parameter of order 1. For $n \in \mathbb{N}^*$ we

set

$$R^{[n]}(z) = \left(\frac{P}{|z|^2} - \frac{z^2}{|z|^2}\right)^{-n} w^{-1} = |z|^{2n} \left((-\Delta_G - z^2 w)^{-1} w\right)^{n-1} (-\Delta_G - z^2 w)^{-1}, \quad (1.69)$$

and for consistancy we also set $R_0^{[n]}(z) = |z|^{2n} (-\Delta - z^2)^{-n}$.

Since the Mourre method is designed to estimate products of resolvents, we use the resolvent identity to rewrite the difference $R^{[n]}(z) - R_0^{[n]}(z)$ as

$$\sum_{k=1}^{n-1} R^{[n-k]}(z)\theta_0(z)R_0^{[k]}(z) - \sum_{k=1}^{n} R^{[n-k+1]}(z)\theta_1(z)R_0^{[k]}(z), \quad (1.70)$$

where we have set

$$\theta_0(z) = w - 1 \quad \text{and} \quad \theta_1(z) = \frac{(-\Delta_G - z^2 w) - (-\Delta - z^2)}{|z|^2}. \quad (1.71)$$

The fact that the difference $R^{[n]}(z) - R_0^{[n]}(z)$ is smaller than $R_0^{[n]}(z)$ for $z \in \mathbb{C}_+$ close to 0 is due to the smallness of $\theta_0(z)$ and $\theta_1(z)$ in the suitable rescaled Sobolev spaces (see (1.56)), given once more by Proposition 1.19.

Notice that we have factors w, $\theta_0(z)$ and $\theta_1(z)$ between the resolvents, but thanks to the analysis of the damped wave equation we know that this is not a problem for the commutators method. We get for instance the following estimate (with A_z being the generator of dilations at infinity defined in (1.64)):

Proposition 1.27. *Let $\rho \in [0, \rho_0[$, $n_1, n_2 \in \mathbb{N}^*$, $\delta > n_1 + n_2 - \frac{1}{2}$ and $\sigma \in \{0, 1\}$. There exists $C > 0$ such that for $z \in \mathbb{D}_+$ we have*

$$\left\| \langle A_z \rangle^{-\delta} R^{[n_1]}(z)\theta_\sigma(z)R_0^{[n_2]}(z) \langle A_z \rangle^{-\delta} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leqslant C\, |z|^\rho .$$

The factor $|z|^\rho$ which appears in this estimate is due the small factor $\theta_\sigma(z)$ and gives the extra smallness in Theorem 1.26 compared to the estimate of $R_0^{[n]}(z)$ or $R^{[n]}(z)$ alone.

However, this is an estimate on the rescaled resolvents and the estimate on the resolvents would be of size $O(|z|^{\rho - 2n_1 - 2n_2})$. As in Section 1.4.3, we recover some smallness with the weights $\langle x \rangle^{-\delta}$ by ellipitic regularity and Proposition 1.19. And this regularity is also used to compensate the derivatives which appear in the powers of the conjugate operator A_z. □

Theorems 1.25 and 1.26 are important for at least two reasons. The first is that ten years ago the motivation of [6] was to recover in a non-selfadjoint setting estimates which were already known in the selfadjoint case. Now we have reached the point where ideas developed for the dissipative case are also used to improve the analysis of the selfadjoint Schrödinger operator.

On the other hand, for the low frequency resolvent estimates, this result re-opens a topic which seemed to be closed by the optimal estimates of [BB21]. Perspectives in that direction will be discussed in Section 4.1.1.

Chapter 2

The damped wave equation with damping at infinity

In this chapter we continue the analysis of the damped wave equation (1.12). However, instead of assuming that the damping is localized, we now consider settings for which the damping is effective at infinity. This gives completely different properties. The results discussed in this chapter are published in [8, 14, 15, 17, 18].

2.1 Introduction

We have seen in Theorem 1.1 that if the damping is small enough at infinity (in the sense of (1.11)), the contribution of low frequencies for the damped wave equation (1.12) behaves as for the undamped case. We will see in this chapter that this is no longer the case if the damping is effective at infinity.

Our model case in this chapter is the free wave equation ($P = -\Delta$) with constant damping $a(x) = a_1 > 0$:

$$\begin{cases} \partial_t^2 u - \Delta u + a_1 \partial_t u = 0, \\ (u, \partial_t u)_{|t=0} = (f, g). \end{cases} \tag{2.1}$$

With damping everywhere, it is clear that the damping condition (1.17) is satisfied, and as expected the local energy of the contribution of high frequencies will decay rapidly. The difference is that *all* the classical trajectories go through the damping region, and not only the bounded ones, so for high frequencies the *global* energy decays *uniformly exponentially*. As one could have guessed, a stronger damping implies stronger decay properties for the energy.

The most interesting part is the contribution of low frequencies, whose behavior is not that simple.

We first observe that for a very slowly oscillating solution u, the damping term $a_1 \partial_t u$ is small, but the second order term $\partial_t^2 u$ is even smaller. Neglecting this term leads to the conjecture that u should behave like a solution of the heat equation

$$\begin{cases} a_1 \partial_t u_0 - \Delta u_0 = 0, \\ u_{0|t=0} = f_0. \end{cases} \tag{2.2}$$

And this is precisely what happens. This is sometimes called the diffusive phenomenon. And since the local energy decay is slower for the heat equation than for the standard wave equation (see Proposition 2.2 below), it turns out that the decay for the contribution of low frequencies is not as fast with damping everywhere as it is without any damping.

There is also a rich literature about the wave equation with damping at infinity. The local energy decay for (2.1) has been discussed in [Mat76]. Matsumura proves $L^p - L^q$ estimates by explicit computations on the Fourier transform. The corresponding semilinear problem is also discussed (see Section 4.5.2 for this question). Then the comparison with the heat equation (2.2) has been studied in many papers. More precisely, it is proved that the solution u of (2.1) behaves for large times like the solution u_0 of (2.2) with

$$f_0 = f + \frac{g}{a_1}. \tag{2.3}$$

Then u_0 is the asymptotic profile of u for large times. We refer to [Nis03] for a result in dimension 3, [MN03] in dimension 1, [HO04] in dimension 2 and a more general discussion in [Nar04].

The same problem has been studied in an exterior domain. See [Ike02] for a result with constant coefficients and [AIK15] for an absorption index equal to 1 outside a compact subset.

A question that will not be discussed here is the case of a slowly decaying damping. This means that a goes to 0 at infinity, but it is not of short range as in (1.11). We refer to [TY09] for the energy decay when $a(x) \simeq \langle x \rangle^{-\alpha}$ for some $\alpha \in [0, 1[$, to [ITY13] for the critical case $\alpha = 1$ and to [Wak14] for the asymptotic profile when $a(x) = \langle x \rangle^{-\alpha}$, $\alpha \in [0, 1[$ (which is as expected a solution of $\langle x \rangle^{-\alpha} \partial_t u_0 - \Delta u_0 = 0$). Finally, we refer to [SW16] for the same question in an exterior domain.

There are also some abstract results in [CH04, RTY11, Nis16, RTY16]. For instance, in [RTY11] the Laplacian is replaced by a general nonnegative selfadjoint operator, while in [RTY16] another operator C acts on $\partial_t^2 u$. In [Nis16], a general damping operator acts on $\partial_t u$, and an application closer to our setting is provided, since the wave equation with a Laplacian in divergence form and an absorption index which can vanish is considered. Only the decay of the full energy is discussed in this setting.

My contributions for this problem are about the wave equation on a wave guide and in an asymptotically periodic setting.

The original motivation was to consider the wave equation on a straight wave guide, with damping at the boundary. See the setting of Section 2.3. I first tried to apply the Mourre theory. This gave the paper [9], already discussed in Section 1.3, but this was not enough to get the local energy decay. Then, as an intermediate step, I considered the case of the Schrödinger equation with a one-dimensional cross-section, which is a much simpler model. This has been published in [8], which is discussed in Section 2.7 below. The problem of a wave guide with constant damping on the boundary has finally been solved in [14]. This is discussed in Section 2.3.

After this, I invited in Toulouse Mohamed Malloug, who was at that time a Ph.D. Student in Sousse with Moez Khenissi. We discussed together the case of a wave guide with internal damping at infinity. This is a simpler setting, but it was an occasion to deal with a damping which does not satisfy the geometric damping condition (1.17). See Section 2.4. It is only then that, as a byproduct of the analysis, we wrote a result about the problem on the full Euclidean space (where $a(x)$ is a long-range perturbation of 1), which is actually much easier.

After wave guides, I considered with Romain Joly the wave equation in an asymptotically periodic setting. This raised different difficulties, which will be detailed Section 2.5.

Since this report is an opportunity to rewrite history, we begin here with the Euclidean case and present in Section 2.2 some ideas on the model problem (2.1). Then, in the following sections, we will discuss the difficulties coming from wave guides and the periodic setting.

2.2 On the Euclidean space

In this paragraph, we discuss some general ideas on the model case (2.1). More precisely, we show from a spectral point of view why the solution of (2.1) behaves for large times like the

solution of (2.2) with f_0 given by (2.3). This is not written like this in any paper, since there were already other proofs for this case in the literature. However, we use this simple setting to show some ideas used in the papers [14, 15, 17], without the difficulties specific to these more sophisticated cases.

2.2.1 The wave operator in the inhomogeneous energy space

We are going to compare a solution of the damped wave equation with a solution of the heat equation. However, for the wave equation we usually consider the first derivatives (with respect to space and time) of the solution, while for the heat equation we estimate the solution itself. Thus, in addition to the energy space $\mathscr{E} = \dot{H}^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, we also consider the Hilbert space $\mathscr{H} = H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to rewrite the wave equation (1.12) in the form (1.18).

We define the wave operator $\mathcal{W}_{\mathscr{E}}$ by (1.19)-(1.20). Notice that we denote by $\mathcal{W}_{\mathscr{E}}$ the operator denoted by $\mathcal{W}$ in the previous chapter. In particular, by Proposition 1.4 the operator $i\mathcal{W}_{\mathscr{E}}$ is maximal dissipative and $\mathcal{W}_{\mathscr{E}}$ generates a contractions semigroup on $\mathscr{E}$. Moreover, the resolvent of $\mathcal{W}_{\mathscr{E}}$ is given by (1.23).

In this chapter, we denote by $\mathcal{W}$ the operator defined by (1.19) on $\mathscr{H}$, with domain $\mathsf{Dom}(\mathcal{W}) = \{U \in \mathscr{H} : \mathcal{W}U \in \mathscr{H}\}$. This is convenient to consider the solution and not only the derivatives, but $i\mathcal{W}$ is no longer a dissipative operator. However, we will still be able to write (1.27) in $\mathscr{H}$.

Proposition 2.1. *The operator $\mathcal{W}$ satisfies on $\mathscr{H}$ the following properties.*

(i) $i\big(\mathcal{W} - \frac{1}{2}\big)$ *is maximal dissipative on* $\mathscr{H}$.

(ii) $\mathbb{C}^+ \subset \rho(\mathcal{W})$, *and for* $\zeta \in \mathbb{C}^+$ *the resolvent* $(\mathcal{W} - \zeta)^{-1}$ *is given by* (1.21) *or* (1.23).

(iii) $\mathcal{W}$ *generates a* C^0*-semigroup on* $\mathscr{H}$. *Moreover, for* $\nu \in]0,1]$ *there exists* $M_\nu \geqslant 0$ *such that* $\|e^{t\mathcal{W}}\|_{\mathcal{L}(\mathscr{H})} \leqslant M_\nu e^{t\nu}$ *for all* $t \geqslant 0$.

2.2.2 Local energy decay for the heat equation

We have said that we compare the solution of (2.1) with a solution of (2.2). Before going further, we recall the local energy decay for a solution of the heat equation. For the solution itself and its first derivatives.

Proposition 2.2. *Let* $s_1, s_2 \in \big[0, \frac{d}{2}\big]$ *and* $s \in [0,1]$. *Let* $\kappa > 1$. *There exists* $C > 0$ *such that for* $t > 0$ *we have*

$$\begin{aligned}
\| \langle x\rangle^{-\kappa s_1} e^{t\Delta} \langle x\rangle^{-\kappa s_2} \|_{\mathcal{L}(L^2(\mathbb{R}^d))} &\leqslant C \langle t\rangle^{-\frac{1}{2}(s_1+s_2)}, \\
\| \langle x\rangle^{-\kappa s_1 - s} \nabla e^{t\Delta} \langle x\rangle^{-\kappa s_2 - s} \|_{\mathcal{L}(L^2(\mathbb{R}^d))} &\leqslant C \langle t\rangle^{-\frac{1}{2}(1+s_1+s_2+s)}, \\
\| \langle x\rangle^{-\kappa s_1} \partial_t e^{t\Delta} \langle x\rangle^{-\kappa s_2} \|_{\mathcal{L}(L^2(\mathbb{R}^d))} &\leqslant C \langle t\rangle^{-\frac{1}{2}(2+s_1+s_2)}.
\end{aligned}$$

In [15] we provided a proof based on the explicit kernel for the heat equation (see Proposition 3.1 therein). Here we describe on this model case a more general strategy which will also be used to estimate the difference between the solutions of (2.1) and (2.2) below, following [17, Prop. 4.12]. See also [14, Prop. 3.3] for a third approach.

In Proposition 2.2 we give estimates in weighted L^2-spaces, but in the proof they are deduced from L^p-L^q estimates. This means that we can also directly state L^p-L^q estimates in this context.

The first estimate can be rewritten as

$$\| \langle x\rangle^{-\kappa s_1} e^{t\Delta}\phi\|_{L^2(\mathbb{R}^d)} \lesssim \langle t\rangle^{-\frac{1}{2}(s_1+s_2)} \| \langle x\rangle^{\kappa s_2} \phi\|_{L^2(\mathbb{R}^d)}, \quad \phi \in L^{2,\kappa s_2}(\mathbb{R}^d).$$

The parameter s_2 measures how localized is the initial data ϕ. With $s_2 = 0$ we are considering a general $\phi \in L^2(\mathbb{R}^d)$, while if ϕ is compactly supported we get a better estimate by choosing

$s_2 = \frac{d}{2}$. Notice that we do not get a better estimate by considering a stronger weight, see Remark 2.3. Similarly, s_1 gives the weight for the energy. If $s_1 = 0$ we consider the L^2 norms of $u(t)$, $\nabla u(t)$ and $\partial_t u(t) = \Delta u(t)$, but if we are only interested in the L^2 norms of these quantities in a compact subset of $\mathbb{R}^d$ we get better estimates with $s_1 = \frac{d}{2}$. All the intermediate situations $s_1, s_2 \in \left]0, \frac{d}{2}\right[$ are also possible. There is a parameter $\kappa > 1$ in the weights since if $s_j \in \left]0, \frac{d}{2}\right]$ we need a weight $\langle x\rangle^{-\delta_j}$ with $\delta_j > s_j$, while no weight is necessary ($\delta_j = 0$) when $s_j = 0$. Finally, as we will see in the last step of the proof, we can get an extra power of $t^{-\frac{1}{2}}$ for the gradient if we can pay an extra weight $\langle x\rangle^{-1}$ on both sides of the estimate. This is the role of the parameter s.

Proof. We first notice that in the third estimate we can replace $\partial_t e^{t\Delta}$ by $\Delta e^{t\Delta}$. Let $f_0 \in \mathcal{S}(\mathbb{R})$. For $t \geqslant 0$ we set $u_0(t) = e^{t\Delta} f_0$. Let

$$p_1 = \frac{2d}{d - 2s_1} \in [2, +\infty] \quad \text{and} \quad p_2 = \frac{2d}{d + 2s_2} \in [1, 2]. \tag{2.4}$$

We denote by $p_1' \in [1, 2]$ and $p_2' \in [2, +\infty]$ the conjugate exponents of p_1 and p_2, respectively. Let $\alpha \in \mathbb{N}^d$. Since $L^{p_1}(\mathbb{R}^d)$ is continuously embedded in $L^{2,-\kappa s_1}(\mathbb{R}^d)$, and by continuity of the (inverse) Fourier transform (with respect to the space variable) from $L^{p_1'}(\mathbb{R}^d)$ to $L^{p_1}(\mathbb{R}^d)$, we have

$$\|D^\alpha u_0(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} \lesssim \|D^\alpha u_0(t)\|_{L^{p_1}(\mathbb{R}^d)} \lesssim \left\|\widehat{D^\alpha u_0}(t)\right\|_{L^{p_1'}(\mathbb{R}^d)}. \tag{2.5}$$

Similarly,

$$\|\hat{f}_0\|_{L^{p_2'}(\mathbb{R}^d)} \lesssim \|f_0\|_{L^{p_2}(\mathbb{R}^d)} \lesssim \|f_0\|_{L^{2,\kappa s_2}(\mathbb{R}^d)}. \tag{2.6}$$

Let $p_0' \in [1, +\infty]$ be such that

$$\frac{1}{p_0'} = \frac{1}{p_1'} - \frac{1}{p_2'} = \frac{1}{p_2} - \frac{1}{p_1} = \frac{s_1 + s_2}{d}.$$

By the Hölder inequality and the change of variable $\eta = \sqrt{t}\xi$ we get

$$\left\|\widehat{D^\alpha u_0}(t)\right\|_{L^{p_1'}(\mathbb{R}^d)} \leqslant \|\xi^\alpha e^{-t|\xi|^2}\|_{L^{p_0'}_\xi(\mathbb{R}^d)} \|\hat{f}_0\|_{L^{p_2'}(\mathbb{R}^d)} \leqslant t^{-\frac{|\alpha|}{2} - \frac{d}{2p_0'}} \|\eta^\alpha e^{-|\eta|^2}\|_{L^{p_0'}_\eta(\mathbb{R}^d)} \|\hat{f}_0\|_{L^{p_2'}(\mathbb{R}^d)}.$$

Finally,

$$\left\|D^\alpha u_0(t)\right\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} \lesssim t^{-\frac{1}{2}(|\alpha| + s_1 + s_2)} \|f_0\|_{L^{2,\kappa s_2}(\mathbb{R}^d)},$$

and the three estimates follow if $s = 0$. Now we assume that $s = 1$. Only the estimate for the gradient is concerned. Let $j \in \{1, \dots, d\}$ and $x \in \mathbb{R}^d$. We compute

$$\begin{aligned}
\partial_{x_j} u_0(t, x) &= \int_{\mathbb{R}^d} e^{ix\cdot\xi} i\xi_j e^{-t|\xi|^2} \hat{f}_0(\xi)\,\mathrm{d}\xi \\
&= -\frac{i}{2t} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \partial_{\xi_j}(e^{-t|\xi|^2}) \hat{f}_0(\xi)\,\mathrm{d}\xi \\
&= -\frac{x_j}{2t} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \hat{f}_0(\xi)\,\mathrm{d}\xi + \frac{i}{2t} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \partial_{\xi_j} \hat{f}_0(\xi)\,\mathrm{d}\xi \\
&= \frac{1}{2t} \left(-x_j (e^{t\Delta} f_0)(x) + e^{t\Delta}(y_j f_0)(x)\right).
\end{aligned}$$

Then, using the first estimate of the proposition,

$$\begin{aligned}
\left\|\partial_{x_j} u_0(t)\right\|_{L^{2,-\kappa s_1 - 1}(\mathbb{R}^d)} &\lesssim t^{-1 - \frac{s_1 + s_2}{2}} \left(\|f_0\|_{L^{2,\kappa s_2}(\mathbb{R}^d)} + \|y_j f_0\|_{L^{2,\kappa s_2}(\mathbb{R}^d)}\right) \\
&\lesssim t^{-1 - \frac{s_1 + s_2}{2}} \|f_0\|_{L^{2,\kappa s_2 + 1}(\mathbb{R}^d)}.
\end{aligned}$$

This gives the estimate for the gradient when $s = 1$. The case $s \in]0, 1[$ follows by interpolation. □

Remark 2.3. The estimates of Proposition 2.2 are optimal in the sense that we do not get better decay even if we consider stronger (for instance compactly supported) weights. Indeed, for $f_0 \in C_0^\infty(\mathbb{R}^d)$, the explicit expression for u_0 gives uniformly for x in a compact

$$u_0(t,x) = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} f_0(y)\,\mathrm{d}y \underset{t\to+\infty}{\sim} \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f_0(y)\,\mathrm{d}y,$$

$$\nabla u_0(t,x) = -\frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} \frac{(x-y)}{2t} e^{-\frac{|x-y|^2}{4t}} f_0(y)\,\mathrm{d}y \underset{t\to+\infty}{\sim} -\frac{1}{2t(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (x-y) f_0(y)\,\mathrm{d}y,$$

and

$$\partial_t u_0(t,x) = -\frac{d}{2t} u_0(t,x) + \mathcal{O}(t^{-\frac{d}{2}-2}) \underset{t\to+\infty}{\sim} -\frac{d}{2t(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} f_0(y)\,\mathrm{d}y.$$

Our purpose is now to prove that the decay of the difference between the solutions of (2.1) and (2.2) is faster than the decay given by Proposition 2.2.

2.2.3 The contribution of high frequencies

We begin with high frequencies. As already said, since we have damping everywhere, all the classical trajectories and not only the bounded ones see the damping. From this we can prove uniform estimates for the resolvent $(W-\zeta)^{-1}$ directly in the energy space, and not in weighted versions. This implies in particular that the imaginary axis (except 0) is included in the resolvent set of $\mathcal{W}$, with uniform estimates for the resolvent (away from a neighborhood of 0). It is classical that if the imaginary axis is included in the resolvent set with uniform estimates for the resolvent, then the propagator decays uniformly exponentially (by the Gearhart-Prüss-Greiner Theorem, see [EN00, Th. V.1.11]). We cannot apply this result directly to $\mathcal{W}$ because of the problem at 0, but it is not surprising to recover a uniform and exponential decay for the contribution of high frequencies.

Theorem 2.4. *There exist $\gamma > 0$ and $C > 0$ such that for all $\zeta \in \mathbb{C}$ with $\mathsf{Re}(\zeta) \geqslant -2\gamma$ and $|\mathsf{Im}(\zeta)| \geqslant 1$ we have $\zeta \in \rho(\mathcal{W})$ and $\left\|(\mathcal{W}-\zeta)^{-1}\right\|_{\mathcal{L}(\mathscr{H})} \leqslant C$.*

Ideas of proof. Since the model case (2.1) is a problem with constant coefficients on $\mathbb{R}^d$, we can use the Fourier transform. For $F \in \mathscr{S}$ and $\xi \in \mathbb{R}^d$ we have

$$\widehat{\mathcal{W}F}(\xi) = M(\xi)\hat{F}(\xi), \quad \text{where} \quad M(\xi) = \begin{pmatrix} 0 & 1 \\ -\xi^2 & -a_1 \end{pmatrix}. \tag{2.7}$$

The eigenvalues of $M(\xi)$ are

$$\lambda_\pm(\xi) = \frac{-a_1 \pm \sqrt{a_1^2 - 4\xi^2}}{2},$$

where we have set $\sqrt{a_1^2-4\xi^2} = i\sqrt{4\xi^2-a_1^2}$ if $4\xi^2 > a_1^2$. Then $\zeta \in i(\mathbb{R}\backslash]-1,1[)$ is not an eigenvalue of $M(\xi)$ and we see that $(M(\xi)-\zeta)^{-1}$ is bounded in $\mathcal{L}(\mathbb{C}^2_\xi)$ uniformly in ζ and ξ, with $\|(\alpha,\beta)\|^2_{\mathbb{C}^2_\xi} = \left(1+|\xi|^2\right)|\alpha|^2 + |\beta|^2$. Then $(\mathcal{W}-\zeta)^{-1}F$ is the inverse Fourier transform of $\xi \mapsto (M(\xi)-\zeta)^{-1}\hat{F}(\xi)$ and we deduce Theorem 2.4 by the Parseval identity. □

2.2.4 A first change of contour - Separation of low and high frequencies

As with a localized damping (see Section 1.2.2), we write the propagator $e^{t\mathcal{W}}$ in terms of the resolvent $(\mathcal{W}-\zeta)^{-1}$. See (1.27). Since the integrand is holomorphic in ζ, we can change the contour of integration $\{\mathsf{Im}(\zeta) = \mu\}$ by Γ_μ defined as follows (see Figure 2.1). Given $\theta \in C_0^\infty(\mathbb{R},[0,1])$ supported in]-2,2[and equal to 1 on [-1,1] we set

$$\Gamma_\mu : \tau \in \mathbb{R} \mapsto -i\tau + \mu\theta(\tau) - \gamma(1-\theta(\tau)). \tag{2.8}$$

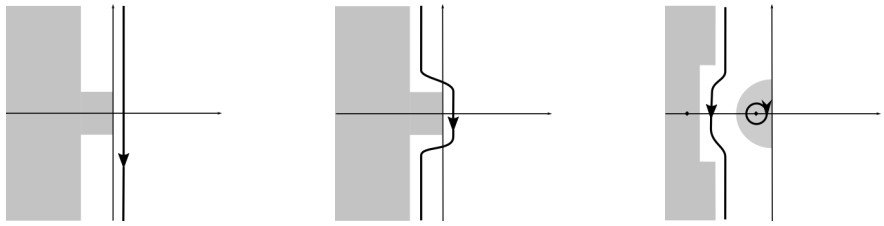

Figure 2.1: Contours $\{\mathsf{Re}(z) = \mu\}$, Γ_μ and $\Gamma_{-a/2} \cup \mathcal{C}(\lambda_+(\xi), r)$.
(in grey, a region which contains the spectrum of $\mathcal{W}$ in the first two pictures, and the eigenvalues of $M(\xi)$ or $\mathcal{W}_\sigma$, for ξ or σ small, in the third)

Then we separate the contributions of high and low frequencies. We consider $\chi_{\mathsf{low}} \in C_0^\infty(\mathbb{R}, [0,1])$ supported in]-3,3[and equal to 1 on [-2,2]. Then we set $\chi_{\mathsf{high}} = 1 - \chi_{\mathsf{low}}$. Finally, for $* \in \{\mathsf{low}, \mathsf{high}\}$ we set

$$\mathcal{U}_*(t)F = \frac{1}{2i\pi} \int_{\Gamma_\mu} e^{t\zeta} \chi_*(\mathsf{Im}(\zeta))(\mathcal{W} - \zeta)^{-1} F_\zeta \, \mathrm{d}\zeta. \tag{2.9}$$

This is analogous to (1.28). In particular, we use the notation F_ζ introduced in (1.26). Notice that it is important to write the integral on the contour Γ_μ before separating the contributions of low and high frequencies, since the integrands in $\mathcal{U}_*(t)F$ are no longer holomorphic in the region $\{2 \leqslant |\mathsf{Im}(\zeta)| \leqslant 3\}$. It is still possible to change the contour for $\mathcal{U}_{\mathsf{low}}(t)$ in the strip $\{|\mathsf{Im}(\zeta)| \leqslant 2\}$. In particular $\mathcal{U}_{\mathsf{low}}(t)F$, as $\mathcal{U}_{\mathsf{high}}(t)F$, does not depend on μ.

We finish the analysis of high frequencies. Following the same idea as for Proposition 1.9, we can deduce from Theorem 2.4 that $\mathcal{U}_{\mathsf{high}}(t)F$ decays uniformly exponentially.

Proposition 2.5. *Let $\gamma > 0$ be given by Theorem 2.4. There exists $C > 0$ such that for $t \geqslant 0$ and $F \in \mathscr{H}$ we have*

$$\|\mathcal{U}_{\mathsf{high}}(t)F\|_{\mathscr{H}} \leqslant C e^{-\frac{t\gamma}{2}} \|F\|_{\mathscr{H}}.$$

2.2.5 The contribution of low frequencies - Comparison with the diffusive profile

We turn to the contribution of low frequencies. Replacing F by F_ζ in (2.9) is convenient for the analysis of $\mathcal{U}_{\mathsf{high}}(t)F$, but to obtain a precise expression of the asymptotic profile, we go back to F for low frequencies. For $t \geqslant 1$ we set

$$U_\sharp(t) = \frac{1}{2i\pi} \int_{\Gamma_\mu} e^{t\zeta} \chi_{\mathsf{low}}(\mathsf{Im}(\zeta))(\mathcal{W} - \zeta)^{-1} F \, \mathrm{d}\zeta.$$

For $\zeta \in \mathbb{C}^+$ we have

$$(\mathcal{W} - \zeta)^{-1}(F_\zeta - F) = \int_0^2 \big(1 - \phi(s)\big) e^{s(\mathcal{W} - \zeta)} F \, ds,$$

so with integration by parts we can see that $\mathcal{U}_{\mathsf{low}}(t)F - \mathcal{U}_\sharp(t)F$ is small:

Proposition 2.6. *Let $m \in \mathbb{N}$. There exists $C \geqslant 0$ such that for $F \in \mathscr{H}$ and $t \geqslant 1$ we have*

$$\|\mathcal{U}_{\mathsf{low}}(t)F - \mathcal{U}_\sharp(t)F\|_{\mathscr{H}} \leqslant C \langle t \rangle^{-m} \|F\|_{\mathscr{H}}.$$

To estimate $\mathcal{U}_\sharp(t)F$ we now have several possibilities.

First strategy. The first option is to compare the resolvent

$$R(z) = R_{a_1}(z) = (-\Delta - iza_1 - z^2)^{-1}$$

of the wave equation (2.1) to the resolvent $R_{\mathsf{h}}(z) = (-\Delta - iza_1)^{-1}$ corresponding to the heat equation (2.2). This is what led to the ideas recently developed in [23] and explained in Section 1.7.

Proposition 2.7. *Let $s_1, s_2 \in \left[0, \frac{d}{2}\right[$ and $\kappa > 1$. Let $\beta \in \mathbb{N}^d$ with $|\beta| \leqslant 1$. Let $m \in \mathbb{N}$. There exists $C > 0$ such that for $z \in \mathbb{C}_+ \cap \mathbb{D}$ we have*

$$\left\| \langle x \rangle^{-\kappa s_1} \partial_x^\beta \partial_z^m \big(R(z) - R_{\mathsf{h}}(z)\big) \langle x \rangle^{-\kappa s_2} \right\|_{\mathcal{L}(L^2(\mathbb{R}^d))} \leqslant C\, |z|^{\frac{1}{2}\min(0, s_1+s_2-2m+|\beta|)}.$$

Ideas of proof. Here, for the model case, the resolvent identity gives

$$R(z) - R_{\mathsf{h}}(z) = z^2 R(z) R_{\mathsf{h}}(z). \tag{2.10}$$

As in Section 1.4 we work in the rescaled Sobolev space, except that we now use $H^s_{|z|^{1/2}}$ instead of $H^s_{|z|}$. A derivative is now of size $O(|z|^{\frac{1}{2}})$ in $\mathcal{L}(H^s_{|z|^{1/2}}, H^{s-1}_{|z|^{1/2}})$ for any $s \in \mathbb{R}$. On the other hand, $R(z)$ and $R_{\mathsf{h}}(z)$ are of size $O(|z|^{-1})$ in $\mathcal{L}(H^{s-1}_{|z|^{1/2}}, H^{s+1}_{|z|^{1/2}})$. An important difference with the setting of Chapter 1 is that for $z \in \mathbb{C}_+$ the spectral parameter is iza_1. Since $\mathsf{dist}(iza_1, \mathbb{R}_+) \gtrsim |z|^{-1}$, we do not need the limiting absorption principle and we will not use the commutators method of Section 1.3 in this chapter. The weights $\langle x \rangle^{-\delta}$ are only used to convert the elliptic regularity given by the resolvents into powers of z. By Proposition 1.19 we have

$$\| \langle x \rangle^{-\delta_1} \|_{\mathcal{L}(H^{s_1}_{|z|^{1/2}}, L^2(\mathbb{R}^d))} \lesssim |z|^{\frac{s_1}{2}} \quad \text{and} \quad \| \langle x \rangle^{-\delta_2} \|_{L^2(\mathbb{R}^d), \mathcal{L}(H^{-s_2}_{|z|^{1/2}})} \lesssim |z|^{\frac{s_2}{2}}. \tag{2.11}$$

This is the analog of (1.63) used for (1.61).

We do not go further in the details of the proof, but the rough conclusion is that we have better estimates on $R(z) - R_{\mathsf{h}}(z)$ than on $R_{\mathsf{h}}(z)$. Notice that we can iterate (2.10) and get an expansion of $R(z)$ in terms of $R_{\mathsf{h}}(z)$ up to a rest of any order. With (1.21) we can similarly write an expansion of the resolvent $(\mathcal{W} - \zeta)^{-1}$. □

All the arguments above with the operator $\mathcal{W}$ on $\mathscr{H}$ can be reproduced with the operator Δ/a_1 on $L^2(\mathbb{R}^d)$. Then, up to rapidly decaying terms, the solution $e^{\frac{t\Delta}{a_1}} f_0$ of (2.2) is equal to

$$\frac{1}{2i\pi} \int_{\Gamma_\mu} e^{t\zeta} \chi_{\mathsf{low}}(\mathsf{Im}(\zeta)) \left(\frac{\Delta}{a_1} - \zeta\right)^{-1} f_0 \, \mathrm{d}\zeta = -\frac{a_1}{2i\pi} \int_{\Gamma_\mu} e^{t\zeta} \chi_{\mathsf{low}}(\mathsf{Im}(\zeta)) R_{\mathsf{h}}(i\zeta) f_0 \, \mathrm{d}\zeta.$$

In view of (1.21) (replacing $R(z)$ by $R_{\mathsf{h}}(z)$, removing the terms with extra powers of ζ and removing the constant term in the bottom left coefficient), we see that the leading term for the first component of $U_{\sharp,\mu}(t)$ is the solution of (2.2) with f_0 as given by (2.3) (and the leading term for the second component is its time derivative).

This kind of strategy has been used in [14] and [15] for wave guides. The interest is that we can in particular consider an absorption index $a(x)$ which is a long-range perturbation of the constant a_1, since we can similarly estimate the difference $R_a(z) - R_{a_1}(z)$, using Proposition 1.19 again.

The drawback with this strategy is that we have to know in advance what the asymptotic profile will look like. This can lead to an impasse with a wrong guess. This is what happened for [17]. More important, in the periodic setting the perturbation does not decay at infinity and we can no longer use a result like Proposition 1.19 to recover some smallness for low frequencies. For [17], we had to proceed with a different idea. Here we explain this new strategy on the model case. This is also what gave the idea used in the proof of Proposition 2.2.

Second strategy. The alternative approach uses the symmetries of the problem. Here, with constant coefficients, we use again the Fourier transform. We can write

$$\mathcal{U}_\sharp(t)F = \frac{1}{2i\pi}\int_{\xi\in\mathbb{R}^d}\int_{\zeta\in\Gamma_\mu} e^{ix\cdot\xi}e^{t\zeta}\chi_{\mathsf{low}}(\mathsf{Im}(\zeta))(M(\xi)-\zeta)^{-1}\hat{F}(\xi)\,\mathrm{d}\zeta\,\mathrm{d}\xi.$$

For $\rho > 0$ (to be chosen small enough) there exists $\varepsilon > 0$ such that $\mathsf{Re}(\lambda_\pm(\xi)) \leqslant -2\varepsilon$ if $|\xi| \geqslant \rho$. Then for $|\xi| \geqslant \rho$ we can replace the contour Γ_μ by $\Gamma_{-\varepsilon}$ and the corresponding contribution decays exponentially. It remains the contribution $\mathcal{U}_\rho(t)F$, defined as $\mathcal{U}_\sharp(t)F$ with the integral on ξ over the ball $B(\rho)$ instead of $\mathbb{R}^d$.

For ξ small, $M(\xi)$ has one simple eigenvalue $\lambda_+(\xi)$ close to 0 and the other one close to $-a$. We define $\tilde{\mathcal{U}}_\rho(t)F$ as $\mathcal{U}_\rho(t)F$ with Γ_μ replaced by $\Gamma_{-a/2}$. It decays uniformly exponentially. The difference between $\mathcal{U}_\rho(t)$ and $\tilde{\mathcal{U}}_\rho(t)$ is given by the Residue Theorem (see the third contour in Figure 2.1). We obtain that, up to exponentially decaying terms, $\mathcal{U}_\sharp(t)F$ is equal to

$$\int_{\xi\in B(\rho)} e^{ix\cdot\xi}e^{t\lambda_+(\xi)}\Pi(\xi)\hat{F}(\xi)\,\mathrm{d}\xi, \tag{2.12}$$

where

$$\Pi(\xi) = -\frac{1}{2i\pi}\int_{\zeta\in\mathcal{C}(\lambda_+(\xi),r)}(M(\xi)-\zeta)^{-1}\,\mathrm{d}\zeta$$

is the Riesz projection on the eigenspace of $M(\rho)$ corresponding to the eigenvalue $\lambda_+(\xi)$ (the integral is over a small circle around $\lambda_+(\xi)$, oriented in the direct sense, while the circle in Figure 2.1 is oriented in the clockwise direction). For $G \in \mathbb{C}^2$ we have

$$\Pi(\xi)G = \langle\Psi(\xi), G\rangle_{\mathbb{C}^2}\,\Phi(\xi), \quad \Phi(\xi) = \begin{pmatrix}1\\ \lambda_+(\xi)\end{pmatrix}, \quad \Psi(\xi) = \frac{1}{\sqrt{a_1^2-4\xi^2}}\begin{pmatrix}a_1+\lambda_+(\xi)\\ 1\end{pmatrix}.$$

Notice that

$$\lambda_+(\xi) = -\frac{\xi^2}{a_1} + O(\xi^4), \quad \Phi(\xi) = \begin{pmatrix}1\\ -\frac{\xi^2}{a_1}+O(\xi^4)\end{pmatrix},$$

and

$$\big\langle\Psi(\xi), \hat{F}(\xi)\big\rangle_{\mathbb{C}^2} = \hat{f}(\xi) + \frac{\hat{g}(\xi)}{a_1} + O(\xi^2\|\hat{F}(\xi)\|).$$

In particular, if in (2.12) we replace each factor by the leading term in its Taylor expansion, we expect that $\mathcal{U}_\rho(t)$ should be close to

$$\mathcal{U}_{\mathsf{heat}}(t)F = \int_{B(\rho)} e^{ix\cdot\xi}e^{-\frac{t\xi^2}{a_1}}\,\widehat{f}_0(\xi)\begin{pmatrix}1\\ -\frac{\xi^2}{a_1}\end{pmatrix}\mathrm{d}\xi, \tag{2.13}$$

with f_0 given by (2.3). Up to the contribution of $|\xi| \geqslant \rho$, which is exponentially decaying, we see that the first component of (2.13) is the solution of (2.2)-(2.3) and the second component is its time derivative. To prove that (2.12) is close to (2.13) we apply the idea given in the proof of Proposition 2.2.

Conclusion. We finally obtain the following result, where for $\delta \in \mathbb{R}$ we have set $\mathscr{H}^\delta = H^{1,\delta}(\mathbb{R}^d) \times L^{2,\delta}(\mathbb{R}^d)$.

Theorem 2.8. *Let $s_1, s_2 \in \left[0, \frac{d}{2}\right]$ and $\kappa > 1$. Let $s \in [0,1]$. There exists $C > 0$ such that for $(f,g) \in \mathscr{H}^{\kappa s_2}$ and $t \geqslant 0$ we have*

$$\begin{aligned}
\|u(t)-u_0(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C\,\langle t\rangle^{-\frac{1}{2}(2+s_1+s_2)}\,\|F\|_{\mathcal{H}^{\kappa s_2}}\,,\\
\big\|\nabla\big(u(t)-u_0(t)\big)\big\|_{L^{2,-\kappa s_1-s}(\mathbb{R}^d)} &\leqslant C\,\langle t\rangle^{-\frac{1}{2}(3+s_1+s_2+s)}\,\|F\|_{\mathcal{H}^{\kappa s_2+s}}\,,\\
\big\|\partial_t\big(u(t)-u_0(t)\big)\big\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C\,\langle t\rangle^{-\frac{1}{2}(4+s_1+s_2)}\,\|F\|_{\mathcal{H}^{\kappa s_2}}\,,
\end{aligned}$$

where $u(t)$ is the solution of (2.1) and $u_0(t)$ is the solution of (2.2) with f_0 given by (2.3).

Combining Theorem 2.8 and Proposition 2.2 we deduce in particular that the estimates of Proposition 2.2 also hold for $u(t)$.

We have already observed that the local energy of the wave equation decays slowly with a strong damping compared to the case without damping. We furthermore notice that the decay for the heat equation (2.2) is slower when a_1 becomes large.

2.3 On a wave guide with damping on the boundary

As said in the introduction of the chapter, my first work about a damped wave equation with damping at infinity was not on the Euclidean space but on a wave guide.

More precisely, after a discussion with D. Krejcirik and P. Siegl on my earlier results about the resolvent estimates by the commutator method, I was interested in the analysis of the problem on a wave guide, with damping at the boundary.

On a waveguide we have bounded classical trajectories at infinity, so the geometric damping condition is not satisfied if the damping is localized. It seemed simpler to consider the case of a constant damping. This led to the first question with damping at infinity (even if I finally first wrote [8] about the Schrödinger equation, see Section 2.7).

Notice that there are now recent results [Wanb, Wanc] about the global energy decay for the damped wave equation on (more general) wave guides with damping at the boundary.

2.3.1 The setting

Let $\Omega = \mathbb{R}^d \times \omega$ be a straight wave guide in $\mathbb{R}^{d+n}$. The cross section ω is a smooth, bounded, connected and open subset of $\mathbb{R}^n$. A general point of Ω will be denoted by (x, y) with $x \in \mathbb{R}^d$ and $y \in \omega$.

We consider on Ω the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{on } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u + a\partial_t u = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ (u, \partial_t u)|_{t=0} = (f, g), & \text{on } \Omega, \end{cases} \tag{2.14}$$

where $a > 0$ is a constant absorption index on the boundary and (f, g) belongs to $H^1(\Omega) \times L^2(\Omega)$. The energy is still a non-increasing function of time:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(|\nabla u(t)|^2 + |\partial_t u(t)|^2 \right) = -2a \int_{\partial\Omega} |\partial_t u(t)|^2 \leqslant 0.$$

It is still possible to rewrite the wave equation in the form (1.18). For this we now define on $\mathscr{E} = \dot{H}^1(\Omega) \times L^2(\Omega)$ the operator

$$\mathcal{W} = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \tag{2.15}$$

with domain

$$\mathsf{Dom}(\mathcal{W}) = \{(u, v) \in \mathscr{E} \,:\, (v, \Delta u) \in \mathscr{E} \text{ and } \partial_\nu u + av = 0 \text{ on } \partial\Omega\}. \tag{2.16}$$

We can check that $i\mathcal{W}$ is a dissipative operator on $\mathscr{E}$. Then we introduce the analog of (1.22). It is given by

$$R(z) = (H_{az} - z^2)^{-1}, \tag{2.17}$$

where for $\alpha \in \mathbb{C}$ we have defined on $L^2(\Omega)$ the operator H_α as in (1.39)

$$H_\alpha = -\Delta, \quad \mathsf{Dom}(H_\alpha) = \left\{u \in H^2(\Omega) \,:\, \partial_\nu u = i\alpha u \text{ on } \partial\Omega\right\}. \tag{2.18}$$

One of the difficulty in [6] was that the derivatives of $R(z)$ (as defined by (1.22)) are not its powers. Here it is even worse since the operator $(H_{az} - z^2)$ depends on z via its domain. Thus its derivatives only make sense if it is seen as a form on $H^1(\Omega)$.

In the sense of forms it becomes clearer that $H_{az} - z^2$ is the analog of $(-\Delta - iza(x) - z^2)$ for a damping at the boundary. Indeed, the quadratic form on $H^1(\Omega)$ associated with $H_{az} - z^2$ is $-\Delta - iz\Theta_a - z^2 \in \mathcal{L}(H^1(\Omega), H^1(\Omega)^*)$, where Θ_a is defined by

$$\forall \varphi, \psi \in H^1(\Omega), \quad \langle \Theta_a \varphi, \psi \rangle_{H^1(\Omega)^*, H^1(\Omega)} = a \int_{\partial\Omega} \varphi \bar{\psi}.$$

The fact that the boundary condition is a good model for a damping concentrated at the boundary has been discussed in more details for a quantum graph in [26, Section 4].

Thus, in addition to $R(z)$ we also have to consider, at least for $z \in \mathbb{C}_+$,

$$\tilde{R}(z) = \big(-\Delta - iz\Theta_a - z^2 \big)^{-1} \in \mathcal{L}(H^1(\Omega)^*, H^1(\Omega)).$$

The link between $R(z)$ and $\tilde{R}(z)$ is that for $\varphi \in L^2(\Omega)$ and $\psi \in H^1(\Omega)$ then $u = \tilde{R}(z)(\varphi + \Theta_a \psi)$ is the unique solution in $H^2(\Omega)$ of the problem

$$\begin{cases} (-\Delta - z^2)u = \varphi & \text{on } \Omega, \\ \partial_\nu u = iazu + a\psi & \text{on } \partial\Omega. \end{cases}$$

In particular, $\tilde{R}(z)\varphi = R(z)\varphi$ for all $\varphi \in L^2(\Omega)$.

Then we can check that the operator $i\mathcal{W}$ is maximal dissipative in $\mathscr{E}$, and that for $F \in H^1(\Omega) \times L^2(\Omega)$ its resolvent is given by

$$(\mathcal{W} - \zeta)^{-1} F = \begin{pmatrix} -\tilde{R}(i\zeta)(\Theta_a + \zeta) & -\tilde{R}(i\zeta) \\ 1 - \tilde{R}(i\zeta)(\zeta\Theta_a + \zeta^2) & -\zeta\tilde{R}(i\zeta) \end{pmatrix} F. \tag{2.19}$$

We want to understand the long time behavior of the solution of (2.14), in particular the local energy decay and the asymptotic profile. Compared to (2.1) we cannot just "remove" the term $\partial_t^2 u$ in (2.14) to get a heat equation which should describe the asymptotic behavior of $u(t)$.

2.3.2 Spectral properties of H_α

It is $\tilde{R}(z)$ which appears in (2.19), but since $\tilde{R}(z)$ and $R(z)$ are closely related, we can look at the properties of $R(z)$, hence at the spectral properties of the operator H_α. For this we use the symmetries of Ω. We write the Laplacian H_α as the sum of the second derivatives in the x and y directions. We denote by L the usual Laplacian on $\mathbb{R}^d$ and by T_α the Robin Laplacian on the cross section ω:

$$T_\alpha = -\Delta_\omega, \quad \mathsf{Dom}(T_\alpha) = \left\{ u \in H^2(\omega) \,:\, \partial_\nu u = i\alpha u \text{ on } \partial\omega \right\}. \tag{2.20}$$

Then, if we identify L with $L \otimes \mathsf{Id}_{L^2(\omega)}$ and T_α with $\mathsf{Id}_{L^2(\mathbb{R}^d)} \otimes T_\alpha$, we have

$$H_\alpha = L + T_\alpha.$$

The spectrum of L is the half-line $[0, +\infty[$. On the other hand, T_α has a non-empty resolvent set and its domain $\mathsf{Dom}(T_\alpha)$ is compactly embedded in $L^2(\omega)$, so its spectrum is given by a sequence $(\lambda_k(\alpha))_{k\in\mathbb{N}}$ of isolated eigenvalues of finite multiplicities. Then it is a general result (see for instance [RS79, §XIII.9]) that

$$\sigma(H_\alpha) = \sigma(L) + \sigma(T_\alpha) = \bigcup_{k\in\mathbb{N}} \left\{ \lambda_k(\alpha) + r, k \in \mathbb{N}, r \in \mathbb{R}_+ \right\}. \tag{2.21}$$

Thus $\sigma(H_{az})$ is a sequence of half-lines which depend on the spectral parameter z.

In a selfadjoint setting it is not difficult to combine the properties of the longitudinal and transverse operators to get information on the wave guide. We can proceed similarly if we have a Riesz basis of $L^2(\omega)$ made with eigenvectors of the transverse operator (see (2.54) below, see also [BK08] for another non-selfadjoint context). Here we have to proceed differently.

Let $\tau \in \mathbb{R}^*$. The eigenvalues of $T_{a\tau}$ have negative imaginary parts. Then τ^2 is in the resolvent set of $H_{a\tau}$, so the resolvent $R(\tau)$ is well defined and $-i\tau \in \rho(\mathcal{W})$. The two important steps of the proof are then to show that the resolvent $(\mathcal{W} + i\tau)^{-1}$ is uniformly bounded for $|\tau| \to \infty$ and to understand the behavior of the resolvent for τ close to 0.

2.3.3 Contribution of low frequencies

We begin with the contribution of low frequencies. Notice that T_0 is the Neumann Laplacian on ω, so $\lambda_0(0) = 0$ is a simple eigenvalue of T_0 and the corresponding eigenprojection is the orthogonal projection on constant functions. Seen as an operator on $L^2(\Omega)$ it reads

$$P_\omega u(x, y) = \frac{1}{|\omega|} \int_\omega u(x, \theta)\, \mathrm{d}\theta. \tag{2.22}$$

The other eigenvalues of T_0 are positive.

Since T_α is analytic with respect to α (family of type B in the sense of Kato [Kat80]), the spectrum of T_α is a holomorphic function of α. In particular, there exists a neighborhood $\mathcal{V}$ of 0 such that for z small the operator T_{az} has exactly one (geometrically and algebraically) simple eigenvalue $\lambda_0(az)$ in $\mathcal{V}$.

If λ is another eigenvalue of H_{az} and $r \in \sigma(L) = \mathbb{R}_+$, then the sum $\lambda + r$ (see (2.21)) is "far" from 0, so the intuition is that only the contribution of the "first" transverse eigenvalue $\lambda_0(az)$ of T_{az} should play a role in the behavior of $(H_{az} - z^2)^{-1}$ when z is small.

To make this more precise, we would like to use a kind of spectral localization, but we cannot use spectral projections given by functional calculus (H_{az} is not selfadjoint) or Riesz projections (the spectrum of H_{az} close to 0 is not separated from the rest of the spectrum). However, we can use simultaneously the facts that L is selfadjoint and T_{az} has discrete spectrum.

We can choose $\eta > 0$ in such a way that if we set

$$\mathcal{G} = \{\zeta \in \mathbb{C} \,:\, \mathsf{Re}(\zeta) < \eta, |\mathsf{Im}(\zeta)| < \eta\},$$

then for α small enough we have $\mathcal{G} \cap \sigma(T_\alpha) = \{\lambda_0(\alpha)\}$ and $\partial\mathcal{G} \cap \sigma(T_\alpha) = \varnothing$. Let $\zeta \in \mathcal{G} \cap \rho(H_\alpha)$ and $\sigma \in \partial\mathcal{G}$. We have $\zeta - \sigma \in \mathbb{C}\backslash\mathbb{R}_+ = \rho(L)$, so we can write the resolvent identity

$$(H_\alpha - \zeta)^{-1}(T_\alpha - \sigma)^{-1} = (T_\alpha - \sigma)^{-1}(L - (\zeta - \sigma))^{-1} - (H_\alpha - \zeta)^{-1}(L - (\zeta - \sigma))^{-1}. \tag{2.23}$$

We set

$$P_\alpha := -\frac{1}{2i\pi} \int_{\partial\mathcal{G}} (T_\alpha - \sigma)^{-1}\, \mathrm{d}\sigma.$$

This is the Riesz projection of T_α on its one-dimensional eigenspace $\mathsf{ker}(T_\alpha - \lambda_0(\alpha))$. On the other hand, since L is selfadjoint we have for $\zeta \in \mathcal{G}$

$$-\frac{1}{2i\pi} \int_{\partial\mathcal{G}} (L - (\zeta - \sigma))^{-1}\, \mathrm{d}\sigma = \mathsf{Id}.$$

Finally we can check (see [14, Prop.4.5]) that the operator

$$B_\alpha(\zeta) = \frac{1}{2i\pi} \int_{\partial\mathcal{G}} (T_\alpha - \sigma)^{-1}(L - (\zeta - \sigma))^{-1}\, \mathrm{d}\sigma$$

is a holomorphic function of $\zeta \in \mathcal{G}$. Thus, after integration of (2.23) with respect to $\sigma \in \partial\mathcal{G}$ we have

$$(H_\alpha - \zeta)^{-1} = (H_\alpha - \zeta)^{-1} P_\alpha + B_\alpha(\zeta) = (L + \lambda_0(\alpha) - \zeta)^{-1} P_\alpha + B_\alpha(\zeta),$$

and for z small we get

$$R(z) = (H_{az} - z^2)^{-1} = (L + \lambda_0(az) - z^2)^{-1} P_{az} + B_{az}(z^2).$$

Then we compute

$$\lambda_0'(0) = -i\Upsilon, \quad \Upsilon = \frac{|\partial\omega|}{|\omega|}.$$

At this point, we expect that the wave resolvent $R(z)$ should behave like $(L - ia\Upsilon z)^{-1} P_\omega$ for z small. This is the resolvent corresponding to a heat equation in the x-direction, and the projection on constant functions with respect to y. We can indeed prove the following result.

Theorem 2.9. *For $z \in \mathbb{D}_+$ we have*

$$R(z) = (L - ia\Upsilon z)^{-1} P_\omega + \tilde{R}(z) = -\frac{1}{a\Upsilon}\left(-\frac{L}{a\Upsilon} + iz\right)^{-1} P_\omega + R_1(z),$$

where $R_1(z)$ satisfies the following property. For $m \in \mathbb{N}$, $s \in \left[0, \frac{d}{2}\right[$, $\delta > s$ and $\beta_x \in \mathbb{N}^d$, $\beta_y \in \mathbb{N}^n$ such that $|\beta_x| + |\beta_y| \leqslant 1$ there exists $C > 0$ such that for $z \in \mathbb{D} \cap \mathbb{C}_+$ we have

$$\| \langle x \rangle^{-\delta} \partial_x^{\beta_x} \partial_y^{\beta_y} R_1^{(m)}(z) \langle x \rangle^{-\delta} \|_{L^2(\Omega)} \leqslant C |z|^{\min(0, -m+s+\frac{|\beta_x|}{2})}.$$

We can also write the Taylor expansions of $\lambda_0(az)$ and P_{az} up to any order to get a more precise expansion of $R(z)$. Then from (2.19) we deduce the asymptotics for the resolvent of $\mathcal{W}$ near 0.

2.3.4 Contribution of high frequencies

Since we are not on the Euclidean space as in the previous chapter or in Section 2.2, we also have to discuss the contribution of high frequencies.

The classical trajectories follow straight lines in Ω, bouncing on the boundary according to the laws of geometrical optics. The rays of light parallel to the boundary never see the damping, so we do not expect uniform exponential decay for the global energy. However, all the bounded rays reach the boundary, so we should have uniform decay of the local energy (notice that the set of undamped classical trajectories is unstable, so we should have a good estimate on the global energy decay with loss of regularity, but I did not go in that direction in [14]).

We already know that the resolvent $(H_{a\tau} - \tau^2)^{-1}$ is well defined for any $\tau \neq 0$. To prove uniform estimates for $|\tau| \gg 1$ we again have to combine the properties of L and $T_{a\tau}$. We begin with the transverse operator. We have the following resolvent estimates.

Proposition 2.10. *There exist $\tau_0 > 0$, $\gamma > 0$ and $c > 0$ such that for $|\tau| \geqslant \tau_0$ and $\zeta \in \mathbb{C}$ with $\left|\mathsf{Re}(\zeta) - \tau^2\right| \leqslant \gamma\tau^2$ and $\mathsf{Im}(\zeta) \geqslant -\gamma|\tau|$ the resolvent $(T_{a\tau} - \zeta)^{-1}$ is well defined and we have*

$$\left\|(T_{a\tau} - \zeta)^{-1}\right\|_{\mathcal{L}(L^2(\omega))} \leqslant \frac{c}{|\tau|}.$$

The proof, inspired by [Mil00], is based on the usual contradiction argument with semi-classical measures on the compact domain ω. Compared to my previous results about high frequencies, the damping is at the boundary (as in [BLR92]).

To prove estimates for $R(z)$, we separate the contributions of high frequencies close to longitudinal directions and high frequencies close to transverse directions.

Let $\chi \in C_0^\infty(\mathbb{R},[0,1])$ be supported in $]-\gamma,\gamma[$ and equal to 1 on $\left]-\frac{\gamma}{2},\frac{\gamma}{2}\right[$. We use the functional calculus, available for L, and we estimate separately $\chi(L/\tau^2)R(\tau)$ and $(1-\chi)(L/\tau^2)R(\tau)$.

For the first term, the idea is that with this localization with respect to L, $R(\tau)$ is in some sense not too far from $(T_{a\tau}-\tau^2)^{-1}$. And, indeed, from Proposition 2.10 we can prove that

$$\left\|\chi(L/\tau^2)R(\tau)\right\|_{\mathcal{L}(L^2(\Omega))} \lesssim \frac{1}{\tau}. \tag{2.24}$$

This is Proposition 7.3 in [14]. We gave a proof based on (2.23), but we can also use the Fourier transform with respect to the x variable to conclude. Notice that we are dealing with the contribution of frequencies which are not too large in the direction x. In terms of classical trajectories, this corresponds to rays of light going in a direction transverse to the wave guide, so they see the damping. It is no surprising that there is no weight in (2.24).

This is of course different for $(1-\chi)(L/\tau^2)R(\tau)$, which corresponds to the contribution of rays of light not far from being parallel to the wave guide.

Assume that we can replace $T_{a\tau}$ by one of its eigenvalues $\lambda_k(a\tau)$. Then we have to estimate $(1-\chi)(L/\tau^2)(L+\lambda_k(a\tau)-\tau^2)^{-1}$. If $\tau^2-\mathsf{Re}(\lambda_k(a\tau)) \leqslant \gamma\tau^2/4$, then we can apply the spectral theorem for L. If $\tau^2-\mathsf{Re}(\lambda_k(a\tau)) \geqslant \gamma\tau^2/4$ then, since $\mathsf{Im}(\lambda_k(a\tau))<0$, we can apply the (high-frequency) limiting absorption principle with the spectral parameter $\tau^2-\lambda_k(a\tau)$ for the usual Laplacian L on $\mathbb{R}^d$.

The rigorous proof does not work this way, since we cannot just sum the contributions of each $\lambda_k(a\tau)$. Nevertheless, we can adapt the strategy known for the uniform resolvent estimates for L in the setting of the wave guide to prove directly that for $\delta>\frac{1}{2}$ we have

$$\left\|\langle x\rangle^{-\delta}(1-\chi)(L/\tau^2)R(\tau)\langle x\rangle^{-\delta}\right\|_{\mathcal{L}(L^2(\Omega))} \lesssim \frac{1}{|\tau|}. \tag{2.25}$$

Summing (2.24) and (2.25) gives the following result.

Proposition 2.11. *Let $\delta>\frac{1}{2}$. There exist $\tau_0>0$ and $c>0$ such that for $|\tau|\geqslant\tau_0$ we have*

$$\left\|\langle x\rangle^{-\delta}R(\tau)\langle x\rangle^{-\delta}\right\|_{\mathcal{L}(L^2(\Omega))} \leqslant \frac{c}{|\tau|}.$$

With (2.19) we can get a uniform estimate for $(\mathcal{W}+i\tau)^{-1}$ in weighted space. Because of the weights, we cannot apply the same strategy as in Section 2.2. We apply the same strategy as in Chapter 1 instead. In particular, we have to prove uniform estimates for the derivatives of $R(\tau)$. As explained above, this is not obvious, and this can only be done in the sense of forms. To apply a result like Lemma 1.16, we need for instance estimates of $R(\tau)$ in incoming and outgoing regions. For this, we have to adapt all the proofs known in the Euclidean setting (we omit this part of the proof, see [14, Sec. 7.3]). We finally get the following result.

Proposition 2.12. *Let $n\in\mathbb{N}$ and $\delta>n-\frac{1}{2}$. There exists $C>0$ such that for $|\tau|\geqslant 1$ we have*

$$\left\|(\mathcal{W}+i\tau)^{-n}\right\|_{\mathcal{L}(\mathscr{E}^\delta,\mathscr{E}^{-\delta})} \leqslant C.$$

2.3.5 Local energy decay and asymptotic profile

Now that we have estimated the resolvent, we can deduce the large time behavior for the solution of (2.14). Since the contribution of high frequencies decays fast, the result depends as usual on low frequencies. By Theorem 2.9 we expect that the asymptotic profile should be a solution of

$$a\Upsilon\partial_t u_0 + Lu_0 = 0, \tag{2.26}$$

seen as a problem on Ω. By (2.19) and Theorem 2.9, the initial data for u_0 should be given by

$$f_0 = \frac{1}{a\Upsilon}P_\omega(\Theta_a f+g). \tag{2.27}$$

We can compute

$$P_\omega \Theta_a f = a\Upsilon P_{\partial\omega} f, \quad \text{where} \quad P_{\partial\omega} f(x,y) = \frac{1}{|\partial\omega|}\int_{\partial\omega} f(x,\theta)\,\mathrm{d}\sigma(\theta).$$

This defines a function $P_{\partial\omega} f \in L^2(\Omega)$ which only depends on $x \in \mathbb{R}^d$. Then we set

$$f_0 = P_{\partial\omega} f + \frac{1}{a\Upsilon} P_\omega g. \tag{2.28}$$

This is analogous to (2.3). Since f is "multiplied" by the damping in (2.27), only its values at the boundary play a role in the expression of f_0, while f_0 depends on the values of g on the whole Ω (notice that g is only in $L^2(\Omega)$ and its trace on $\partial\Omega$ does not make sense).

On the other hand, the strength of the damping is given by the product $a\Upsilon$. The coefficient a is the strength of the damping at each point of the boundary, and Υ depends on the shape of the cross-section. If we normalize the section ω by $|\omega| = 1$, then Υ is the size of $\partial\omega$. It is not surprising that for a fixed value of a, the effect of the damping in Ω is stronger when Υ is large.

Compared to Theorem 2.8 we cannot give estimates without weights (case $s_1 = s_2 = s = 0$) since we also need weights for the contribution of high frequencies. Of course, even for high frequencies we can get weaker decay with weaker weights, but for simplicity we do not detail this possibility and only write the statement which gives the best local energy decay in this case.

We observe in particular that the power of t only depends on the number d of longitudinal dimensions.

Theorem 2.13. *Let $\delta > \frac{d}{2} + 2$. There exists $C > 0$ such that for $F = (f,g) \in \mathscr{H}^\delta$ and $t \geqslant 0$ we have*

$$\left\| \langle x\rangle^{-\delta} \nabla\big(u(t) - u_0(t)\big)\right\|_{L^2(\Omega)} + \left\| \langle x\rangle^{-\delta} \partial_t\big(u(t) - u_0(t)\big)\right\|_{L^2(\Omega)} \leqslant C \langle t\rangle^{-\frac{d}{2}-2} \|F\|_{\mathscr{H}^\delta},$$

where $u(t)$ is the solution of (2.14) *and $u_0(t)$ is the solution of* (2.26)-(2.28).

2.4 On a wave guide with damping at infinity

After [14], we discussed with Mohamed Malloug the wave equation with damping inside the waveguide Ω. Since there are less technical problems than with damping at the boundary, it was an occasion to consider other difficulties. Thus, in [15] we consider a situation where the damping is not constant (and in particular breaks the symmetry of the wave guide) and the geometric damping condition is not satisfied.

The setting. Let $\Omega = \mathbb{R}^d \times \omega \subset \mathbb{R}^{d+n}$ be a straight wave guide as in the previous section. We consider on Ω an absorption index $a(x,y) \geqslant 0$ which is a long range perturbation of a positive constant a_1: for some $\rho \in]0,1]$ we have, for $\beta \in \mathbb{N}^d$ and $(x,y) \in \Omega$,

$$\left|\partial_x^\beta (a(x,y) - a_1)\right| \leqslant C_\beta \langle x\rangle^{-\rho-|\beta|}. \tag{2.29}$$

Then we consider on Ω the damped wave equation with Neumann boundary conditions

$$\begin{cases} \partial_t^2 u - \Delta u + a\partial_t u = 0, & \text{on } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = 0, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ (u, \partial_t u)_{|t=0} = (f,g). \end{cases} \tag{2.30}$$

Notice that we can also consider Dirichlet boundary conditions. This is actually much easier since in this case 0 is not in the spectrum of the corresponding operator, so there is no difficulty with low frequencies (see Theorem 4.2 in [15]).

In this problem, a is a perturbation of a constant damping, but it can vanish in a bounded subset of Ω. In particular, we may have undamped bounded classical trajectories. This is only possible for rays of light going in a transverse direction, so the set of undamped rays is unstable. For the contribution of high frequencies, this problem is actually similar to the problem on the stadium with damping on the half-disks (see [BH07]).

In this setting the resolvent of the wave equation is

$$R(z) = \big(-\Delta_N - ia(x,y)z - z^2 \big)^{-1},$$

where $-\Delta_N$ the Neumann Laplacian in Ω.

Resolvent for a damping independent of the transverse variable. Even if a depends on y, we use in the proof the case where it does not. Let α be a bounded function on Ω which only depends on $x \in \mathbb{R}^d$. We can write

$$-\Delta_N - i\alpha(x) = \big(L - i\alpha(x)\big) + T_N,$$

where L is as above the standard Laplacian on $\mathbb{R}^d$ and the transverse operator T_N is the usual non-negative and selfadjoint Neumann Laplacian on $L^2(\omega)$. Compared to the setting of the previous paragraph, we consider here the damping as part of the longitudinal operator.

We denote by $(\lambda_k)_{k\in\mathbb{N}}$ the non-decreasing sequence of eigenvalues (repeated according to multiplicities) of T_N, and we consider a corresponding orthonormal basis of eigenvectors $(\varphi_k)_{k\in\mathbb{N}}$. Compared to the previous section, the fact that we have an orthonormal basis for the transverse operator simplifies the analysis. We have $\lambda_0 = 0$, $\lambda_1 > 0$ and φ_0 is constant.

For $f \in L^2(\Omega)$ we can write $f = \sum_{k\in\mathbb{N}} f_k \otimes \varphi_k$ with $f_k \in L^2(\mathbb{R}^d)$, and for ζ in the resolvent set of $-\Delta_N - i\alpha(x)$ we have

$$\big(-\Delta_N - i\alpha(x) - \zeta \big)^{-1} f = \sum_{k\in\mathbb{N}} \big(L - i\alpha(x) - (\zeta - \lambda_k)\big)^{-1} f_k \otimes \varphi_k. \tag{2.31}$$

Thus we can deduce estimates for $\big(-\Delta_N - i\alpha(x) - \zeta \big)^{-1}$ in $\mathcal{L}(L^2(\Omega))$ from similar estimates for $\big(L - i\alpha(x) - (\zeta - \lambda_k)\big)^{-1}$, $k \in \mathbb{N}$, in $\mathcal{L}(L^2(\mathbb{R}^d))$.

Contribution of low frequencies. Now we go back to the analysis of $R(z)$. We begin with the contribution of low frequencies. If a only depends on x, then we see that in (2.31) (with $\alpha = az$ and $\zeta = z^2$) only the first term should play a role for z small (the other terms are holomorphic around $z = 0$).

We have not assumed that $a(x,y)$ only depends on x, but for low frequencies only the values at infinity play an important role. Thus it is expected that, up to a rest, we can replace $a(x,y)$ by a_1 (and remove the term z^2) in $R(z)$ when z is small. In other words, for z small the resolvent $R(z)$ should be close to the resolvent of the heat equation

$$R_{\mathsf{h}}(z) = (-\Delta_N - iza_1)^{-1}.$$

On the other hand, by (2.31), $R_{\mathsf{h}}(z)$ should be close to $R_{\mathsf{h}}(z)P_\omega$, where P_ω (defined by (2.22)) is the orthogonal projection on $\mathrm{span}(\varphi_0)$. All this holds in the following precise sense.

Proposition 2.14. *Let $s_1, s_2 \in \big[0, \frac{d}{2}\big]$ and $\kappa > 1$. Let $\rho_1 \in]0, \rho[$. Let $\beta \in \mathbb{N}^{d+n}$ with $|\beta| \leqslant 1$. Let $m \in \mathbb{N}^*$. There exists $C > 0$ such that for $z \in \mathbb{D} \cap \mathbb{C}_+$ we have*

$$\Big\| \langle x \rangle^{-\kappa s_1} \, \partial_{x,y}^{\beta} \partial_z^m \big(R(z) - R_{\mathsf{h}}(z) P_\omega \big) \, \langle x \rangle^{-\kappa s_2} \Big\|_{\mathcal{L}(L^2(\Omega))} \leqslant C \, |z|^{\min(0, \frac{1}{2}(s_1+s_2+\rho_1+|\beta|)-m-1)} .$$

Ideas of proof. To estimate the difference between $R(z)$ and $R_{\mathsf{h}}(z)$ we write the resolvent identity

$$R(z) - R_{\mathsf{h}}(z) = R_{\mathsf{h}}(z)\big(iz(a - a_1) + z^2\big)R(z).$$

Then we proceed as in the previous chapter (see Section 1.7) or for Proposition 2.7. Here the rescaled Sobolev spaces (recall (1.56)) are only in the x direction. More precisely, for $s \in \mathbb{R}$ and $z \in \mathbb{C}^*$ we set

$$\|u\|_{H_z^s L^2(\Omega)} = \left\|\langle \sqrt{L/|z|}\rangle^s u\right\|_{L^2(\Omega)}.$$

For any $s \in \mathbb{R}$ and $z \in \mathbb{C}_+$, the resolvent $R_{\mathsf{h}}(z)$ is of size $O(|z|^{-1})$ in $\mathcal{L}(H_z^{s-1}L^2(\Omega), H_z^{s+1}L^2(\Omega))$. On the other hand, by an analog of Proposition 1.19, we see that $(iz(a-a_1)+z^2)$ is of size $O(|z|^{1+\frac{\rho_1}{2}})$ in $\mathcal{L}(H_z^{s_2}L^2(\Omega), H_z^{s_1}L^2(\Omega))$ if $s_1, s_2 \in \left]-\frac{d}{2}, \frac{d}{2}\right[$ are such that $s_2 - s_1 \geqslant \rho_1$. In particular $R(z) - R_{\mathsf{h}}(z)$ is smaller than $R_{\mathsf{h}}(z)$ for z small, and $R(z)$ is of size $O(|z|^{-1})$ in $\mathcal{L}(H_z^{s-1}L^2(\Omega), H_z^{s+1}L^2(\Omega))$ for any $s \in \left]-\frac{d}{2}, \frac{d}{2}\right[$. Notice also that ∇_x is of size $O(|z|^{\frac{1}{2}})$ in $\mathcal{L}(H_z^{s}L^2(\Omega), H_z^{s-1}L^2(\Omega))$ and we can use (2.31) to see that $\nabla_y R_{\mathsf{h}}(z)$ is bounded in a neighborhood of $z=0$. Finally, we use the weights $\langle x\rangle^{-\kappa s_j}$ to convert regularity into powers of $|z|$ as in (2.11). □

Contribution of high frequencies. We turn to the contribution of high frequencies. Since we have damping everywhere at infinity, we can consider the global energy, and hence resolvent estimates without weights. On the other hand, since the geometric damping condition (the analog of (1.17) in Ω) does not necessarily hold, we cannot expect an estimate as good as in the previous settings.

Proposition 2.15. *Let $\beta_1, \beta_2 \in \mathbb{N}^{d+n}$ with $|\beta_1| \leqslant 1$ and $|\beta_2| \leqslant 1$. Let $\tau_0 > 0$. There exists $C \geqslant 0$ such that for $\tau \in \mathbb{R}\backslash[-\tau_0, \tau_0]$ we have*

$$\left\|\partial^{\beta_1} R(\tau)\partial^{\beta_2}\right\|_{\mathcal{L}(L^2(\Omega))} \leqslant C\,|\tau|^{|\beta_1|+|\beta_2|+1}$$

and

$$\left\|(\mathcal{W}+i\tau)^{-1}\right\|_{\mathcal{L}(\mathscr{E})} \leqslant C\tau^2. \tag{2.32}$$

Ideas of proof. As is usual for high frequencies, we can replace $a(x,y)$ by a smaller damping. We choose $\tilde{a}(x)$ which only depends on x and such that $\tilde{a}(x) \leqslant a(x,y)$ for all $(x,y) \in \Omega$. Then we can use (2.31). Since we can take $\tilde{a}$ going to 1 at infinity, it satisfies the damping assumption on $\mathbb{R}^d$, so $(L - i\tau\tilde{a} - \tau^2)^{-1}$ if of size $O(\tau^{-1})$ in $\mathcal{L}(L^2(\mathbb{R}^d))$. We get a similar estimate for $(L - i\tau\tilde{a} - (\tau^2 - \lambda_k))^{-1}$ if $\tau^2 - \lambda_k$ is large. The loss is due to the contributions of k such that $\tau^2 - \lambda_k$ is small. This corresponds to rays of light orthogonal to the boundary, possibly undamped, and explains why we do not have a uniform bound in (2.32). □

Local energy decay. Finally, we have to convert resolvent estimates to large times estimates. For high frequencies we adapt the method of [Leb96, BH07]. Notice that there are abstract results in the case where (2.32) holds for all $\tau \in \mathbb{R}$ (with τ^2 replaced by $\langle\tau\rangle^2$, see for instance [BEPS06, BD08, BT10]), but we cannot apply them because of the problem at 0. For the contribution of low frequencies, the method is similar to the previous cases. From Proposition 2.14 we can deduce that the asymptotic profile $u_0(t)$ for the solution of (2.30) is the function on Ω which does not depend on $y \in \omega$ and is solution with respect to x to the heat equation $a_1\partial_t u_0(t) + Lu_0(t) = 0$, with initial condition $f_0 = P_\omega(f + g/a_1)$:

$$u_0(t) = e^{-\frac{tL}{a_1}} P_\omega\left(f + \frac{g}{a_1}\right). \tag{2.33}$$

Finally, we get the following result.

Theorem 2.16. *Let $s_1, s_2 \in \left[0, \frac{d}{2}\right]$ and $\kappa > 1$. Let $\rho_1 \in]0, \rho[$. Let $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}^{d+n}$ with $\alpha + |\beta| \leqslant 1$. Let $k \in \mathbb{N}^*$. There exists $C > 0$ such that for $F = (f,g) \in \mathscr{H}^{\kappa s_2}(\Omega) \cap \mathsf{Dom}(\mathcal{W}^k)$*

we have

$$\left\|\langle x\rangle^{-\kappa s_1}\,\partial_t^\alpha\partial_{x,y}^\beta\big(u(t)-u_0(t)\big)\right\|_{L^2(\Omega)}$$
$$\leqslant C\left(\langle t\rangle^{-\frac{1}{2}(s_1+s_2+2\alpha+|\beta|+\rho_1)}\,\|F\|_{\mathscr{H}^{\kappa s_2}}+\frac{\ln(t)^{\frac{k}{2}+1}}{t^{\frac{k}{2}}}\,\left\|(\mathcal{W}-1)^kF\right\|_{\mathscr{E}}\right),$$

where $u(t)$ is the solution of (2.30) *and $u_0(t)$ is given by* (2.33).

2.5 In a periodic medium

The motivation for [14] was the analysis of the damped wave equation on a wave guide. It is only while proving this result that I realized that the damped wave equation with damping at infinity was a really exciting problem in itself. When Romain Joly wrote a paper about the damped Klein-Gordon equation with periodic damping (then improved to a more general setting with Nicolas Burq in [BJ16]), giving the behavior of the contribution of high frequencies for the corresponding damped wave equation, it was a natural question to study the contribution of low frequencies for the damped wave equation with periodic damping. This is what we did together in [17].

2.5.1 The asymptotically periodic setting

We consider the damped wave equation (1.12) on the Euclidean space, with a general Laplace operator P as in (1.8), but the coefficients (including the absorption index) are now asymptotically periodic. This means that we can write

$$G(x)=G_{\mathsf{p}}(x)+G_0(x),\quad w(x)=w_{\mathsf{p}}(x)+w_0(x)\quad\text{and}\quad a(x)=a_{\mathsf{p}}(x)+a_0(x),$$

where G_{p}, w_{p} and a_{p} are $\mathbb{Z}^d$-periodic and G_0, w_0 and a_0 are of long range: there exist $\rho_G,\rho_a>0$ (we use two parameters since ρ_G and ρ_a will not play the same role in (2.46) below) and $C_G,C_a\geqslant 0$ such that

$$|G_0(x)|\leqslant C_G\,\langle x\rangle^{-\rho_G}\quad\text{and}\quad|w_0(x)|+|a_0(x)|\leqslant C_a\,\langle x\rangle^{-\rho_a}\,. \tag{2.34}$$

We first consider the purely periodic case, $G_0=0$, $w_0=0$ and $a_0=0$. We denote by $\mathcal{W}_{\mathsf{p}}$ the wave operator (1.19) with coefficients G_{p}, w_{p} and a_{p}.

The damped wave equation with periodic metric had already been studied in [OZP01]. Compared to our setting, the initial condition is localized (at least in L^1) and the absorption index is constant (in particular it does not vanish and we can use some tools of the selfadjoint problem).

Our absorption index a_{p} cannot be identically zero, but it is allowed to vanish. This means that at infinity it takes positive values but it can also be 0. This is an intermediate situation between the setting of Chapter 1, where a is small at infinity, and the settings of Sections 2.2 and 2.4, where a is asymptotically close to a positive constant.

We will see that we recover in this case the diffusive phenomenon typical for the damped wave equation with damping effective at infinity.

Under the geometric damping condition for high frequencies (see (2.45) below), the large time behavior of the wave is as usual given by the contribution of low frequencies. And if the wave length of the solution is very large compared to the period of the medium, this solution only sees, in some sense, the mean value of a_{p}. This mean value is positive.

Thus, as above the solution of the periodic damped wave equation should look like the solution of some heat equation. It is natural to expect that the corresponding heat equation is obtained by removing the second derivative in time in the wave equation. After multiplication by w this gives

$$-\Delta_{G_{\mathsf{p}}}u+a_{\mathsf{p}}(x)w_{\mathsf{p}}(x)\partial_t u=0\quad\text{on }\mathbb{R}_+\times\mathbb{R}^d. \tag{2.35}$$

This is indeed the case, but we want to go further. As said above, the low frequency wave only sees the mean value of $a_{\mathsf{p}} w_{\mathsf{p}}$. We expect that in (2.35) we can replace $a_{\mathsf{p}} w_{\mathsf{p}}$ by its mean value. Similarly, we would like to replace G_{p} by a simpler (constant) matrix. The reader familiar with homogenization (which was not our case when starting this work) knows that the mean value of G_{p} is not the good guess.

2.5.2 The Floquet-Bloch decomposition and spectral properties on the torus

All the coefficients of the equation are $\mathbb{Z}^d$-periodic, but this is not the case for the initial condition $(f, g) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. The solution itself will not be periodic, so the problem does not reduce to a problem on the torus.

In this kind of context it is usual to use the Bloch waves (see for instance [BLP78, Sec. 4.3]). Compared to the plane waves $e^{ix\cdot\xi}$ used for the Fourier transform (suitable for linear problems with constant coefficients), the Bloch waves are of the form $\psi(x) = e^{ix\cdot\sigma}\phi(x)$ where ϕ is $\mathbb{Z}^d$-periodic. They can be chosen in such a way that ψ is an eigenfunction of $-\Delta_{G_{\mathsf{p}}}$. Then ϕ is an eigenfunction for the operator

$$-\Delta_{G_{\mathsf{p}},\sigma} := -(\operatorname{div} + i\sigma^{\intercal}) G_{\mathsf{p}}(x)(\nabla + i\sigma).$$

Compared to the undamped case (or with constant damping), we cannot use a basis of Bloch waves which are eigenfunctions of our operator, but we can still use the Floquet-Bloch decomposition. Let $\mathbb{T}^d = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$. For $u \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$ we write

$$u(x) = \int_{\sigma \in 2\pi\mathbb{T}^d} e^{ix\cdot\sigma} u_\sigma(x)\,\mathrm{d}\sigma, \quad \text{where} \quad u_\sigma(x) = \frac{1}{(2\pi)^d} \sum_{n\in\mathbb{Z}^d} u(x+n) e^{-i(x+n)\cdot\sigma}. \tag{2.36}$$

Notice that u_σ is periodic for all $\sigma \in \mathbb{R}^d$. This decomposition naturally shares some useful properties with the Fourier transform. In particular, we have the Parseval identity for the Floquet-Bloch decompostion: for $u, v \in \mathcal{S}(\mathbb{R}^d)$ we have

$$\langle u, v\rangle_{L^2(\mathbb{R}^d)} = (2\pi)^d \int_{\sigma\in 2\pi\mathbb{T}^d} \langle u_\sigma, v_\sigma\rangle_{L^2_{\mathsf{p}}}\,\mathrm{d}\sigma, \tag{2.37}$$

where the space L^2_{p} of $\mathbb{Z}^d$-periodic and L^2_{loc} functions is endowed with the natural Hilbert structure. We also define the periodic Sobolev spaces H^k_{p} for $k \in \mathbb{N}$.

Now for $U = (u, v) \in \mathscr{S}$ and $x \in \mathbb{R}^d$ we have

$$(\mathcal{W}_{\mathsf{p}} U)(x) = \int_{\sigma\in 2\pi\mathbb{T}^d} e^{ix\cdot\sigma} \mathcal{W}_\sigma U_\sigma(x)\,\mathrm{d}\sigma,$$

where for $\sigma \in \mathbb{R}^d$ we have set $U_\sigma = (u_\sigma, v_\sigma)$ and

$$\mathcal{W}_\sigma = \begin{pmatrix} 0 & w_{\mathsf{p}}^{-1} \\ \Delta_{G_{\mathsf{p}},\sigma} & -a_{\mathsf{p}} \end{pmatrix}.$$

This defines a family $(\mathcal{W}_\sigma)_{\sigma\in 2\pi\mathbb{T}^d}$ of operators on $H^1_{\mathsf{p}} \times L^2_{\mathsf{p}}$ with domain $H^2_{\mathsf{p}} \times H^1_{\mathsf{p}}$. The interest of this decomposition is that each operator $\mathcal{W}_\sigma$ has a compact resolvent, so its spectrum is given by a sequence of isolated eigenvalues. For $U = (u, v) \in H^2_{\mathsf{p}} \times H^1_{\mathsf{p}}$ and $\lambda \in \mathbb{C}$ we have

$$\mathcal{W}_\sigma U = \lambda U \quad \Longleftrightarrow \quad \begin{cases} (-\Delta_{G_{\mathsf{p}},\sigma} + \lambda a_{\mathsf{p}} w_{\mathsf{p}} + \lambda^2 w_{\mathsf{p}}) u = 0, \\ v = \lambda w_{\mathsf{p}} u. \end{cases} \tag{2.38}$$

Then we proceed as explained on the model case in Section 2.2. We first observe that 0 is the only possible purely imaginary eigenvalue of $\mathcal{W}_\sigma$, and this happens if and only if $\sigma = 0$. Moreover, 0 is a (geometrically and algebraically) simple eigenvalue of $\mathcal{W}_0$ and a

corresponding eigenvector is given by $\Phi_0 = (1,0)^\intercal$ (notice that we would have difficulties to work with Φ_0 in the periodic energy space $\dot{H}^1_\mathsf{p} \times L^2_\mathsf{p}$).

By continuity of the spectrum with respect to σ, we deduce that only one simple eigenvalue of $\mathcal{W}_\sigma$ for σ small will play a role for the contribution of low frequencies.

Proposition 2.17. *There exist $r > 0$ and $\gamma_0, \gamma_1, \gamma_2 > 0$ such that*

(i) for $\sigma \in 2\pi\mathbb{T}^d \backslash B(0,r)$ the eigenvalues of $\mathcal{W}_\sigma$ with imaginary parts in $[-3,3]$ have real parts smaller than $-\gamma_0$;

(ii) for $\sigma \in B(0,r)$ the operator $\mathcal{W}_\sigma$ has a unique simple eigenvalue λ_σ such that $|\lambda_\sigma| \leqslant \gamma_1$, and the other eigenvalues of $\mathcal{W}_\sigma$ with imaginary parts in $[-3,3]$ have real parts smaller than $-\gamma_2$ ($\gamma_1 < \gamma_2$).

We denote by Π_σ the Riesz projection of $\mathcal{W}_\sigma$ corresponding to λ_σ and $\Phi_\sigma = \Pi_\sigma \Phi_0 / \|\Pi_\sigma \Phi_\sigma\|$. Then for some $\varphi_\sigma \in H^2_\mathsf{p}$ (smooth with respect to σ) we have $\Phi_\sigma = (\varphi_\sigma, \lambda_\sigma w_\mathsf{p} \varphi_\sigma)^\intercal$.

Then, up to exponentially decaying terms, we see that the important contribution is the analog $\mathcal{U}_{r,\mathsf{p}}(t)F$ of (2.12), defined for $t \geqslant 0$ and $F \in \mathscr{S}$ by

$$\mathcal{U}_{r,\mathsf{p}}(t)F = \int_{\sigma \in B(0,r)} e^{ix\cdot\sigma} e^{t\lambda_\sigma} \Pi_\sigma F_\sigma \, \mathrm{d}\sigma. \tag{2.39}$$

The next step is to replace λ_σ, Φ_σ and Π_σ by their Taylor expansions for σ small. From (2.38) we can prove the following properties.

Proposition 2.18. (i) *Let $i\psi_\sigma$ be the linear term in the Taylor expansion of φ_σ at $\sigma = 0$. Then we have*

$$-\operatorname{div} G_\mathsf{p}(x)(\sigma + \nabla\psi_\sigma) = 0. \tag{2.40}$$

We denote by $W(x)$ the $\mathbb{Z}^d$-periodic matrix such that, for all $\xi \in \mathbb{R}^d$,

$$W(x)\xi = \xi + \nabla\psi_\xi(x). \tag{2.41}$$

(ii) *We have*

$$\lambda_\sigma = -\frac{1}{a_\mathsf{h} w_\mathsf{h}} \langle G_\mathsf{h}\sigma, \sigma\rangle + O(|\sigma|^3),$$

where G_h is the mean value of $W(x)^\intercal \, G_\mathsf{p}(x) \, W(x)$:

$$\langle G_\mathsf{h}\sigma, \sigma\rangle = \int_{\mathbb{T}^d} \langle G_\mathsf{p}(x)(\sigma + \nabla\psi_\sigma(x)), \sigma + \nabla\psi_\sigma(x)\rangle \, \mathrm{d}x. \tag{2.42}$$

(iii) *For $\sigma \in B(0,r)$ we denote by $\Psi_\sigma \in L^2_\mathsf{p} \times L^2_\mathsf{p}$ the vector which satisfies, for all F,*

$$\Pi_\sigma F = \langle \Psi_\sigma, F\rangle_{L^2_\mathsf{p} \times L^2_\mathsf{p}} \, \Phi_\sigma.$$

Then Ψ_σ is a smooth function of σ and we have

$$\Psi_0 = \frac{1}{a_\mathsf{h} w_\mathsf{h}} \begin{pmatrix} a_\mathsf{p} w_\mathsf{p} \\ 1 \end{pmatrix}.$$

The definition of G_h is classical in homogenization theory (the matrix $W(x)$ is the so-called corrector matrix). It is interesting to notice how it appears as the Hessian matrix of the eigenvalue λ_σ at $\sigma = 0$.

2.5.3 Comparison with the asymptotic profile

From the eigenelements of $\mathcal{W}_\sigma$ for σ small we can define the asymptotic profile for the contribution of low frequencies. Since the contribution of high frequencies decays fast under the suitable geometric condition, we finally get a result of local energy decay for the difference between the solution of the damped wave equation and the asymptotic profile.

The asymptotic profile We compare (2.39) with

$$\mathcal{U}_{r,\mathsf{h}}(t)F = \int_{\sigma\in B(0,r)} e^{ix\cdot\sigma} e^{-\frac{t\langle G_{\mathsf{h}}\sigma,\sigma\rangle}{a_{\mathsf{h}}w_{\mathsf{h}}}} \langle F_\sigma, \Psi_0\rangle \Phi_0 \,\mathrm{d}\sigma. \tag{2.43}$$

For this we use the strategy of Proposition 2.2, actually first developed for this point in [17] (see Proposition 4.12 therein). We replace the continuity of the Fourier transform (remember (2.5)-(2.6) with notation (2.4)) by the following property of the Floquet-Bloch transform (see also [17, Cor. 4.2])

$$\|u\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} \lesssim \|u_\sigma\|_{L^{p_1'}(2\pi\mathbb{T}^d;L^2_{\mathsf{p}})}, \quad \|u_\sigma\|_{L^{p_2'}(2\pi\mathbb{T}^d;L^2_{\mathsf{p}})} \lesssim \|u\|_{L^{2,\kappa s_2}(\mathbb{R}^d)}.$$

This can be proved with (2.37) and the following equality for $u \in \mathcal{S}(\mathbb{R}^d)$ and $\psi \in L^2_{\mathsf{p}}$:

$$\langle u_\sigma, \psi\rangle_{L^2_{\mathsf{p}}} = \int_{x\in\mathbb{R}^d} e^{ix\cdot\sigma} u(x)\overline{\psi(x)}\,\mathrm{d}x.$$

The first component of (2.43) gives the asymptotic profile (the second component gives its time derivative, multiplied by w_{p}). For $t \geqslant 0$ we set

$$u_{\mathsf{h}}(t) = e^{-\frac{t}{a_{\mathsf{h}}w_{\mathsf{h}}}P_{\mathsf{h}}} u_{\mathsf{h}}(0), \quad \text{where} \quad u_{\mathsf{h}}(0) = \langle F, \Psi_0\rangle = \frac{a_{\mathsf{p}}w_{\mathsf{p}}f + w_{\mathsf{p}}g}{a_{\mathsf{h}}w_{\mathsf{h}}}. \tag{2.44}$$

To get the asymptotic profile for $\nabla u(t)$, we cannot just take the gradient of $u_{\mathsf{h}}(t)$. Indeed, the leading term for φ_σ is just $\varphi_0 = 1$, but the leading term of $\nabla\varphi_\sigma$ is $i\nabla\psi_\sigma$ (see Proposition 2.18). With the factor $i\sigma$ coming from the factor $e^{ix\cdot\sigma}$ we see that $\nabla u_{\mathsf{p}}(t)$ should be compared with

$$\int_{\sigma\in B(0,r)} e^{ix\cdot\sigma} e^{-\frac{t\langle G_{\mathsf{h}}\sigma,\sigma\rangle}{a_{\mathsf{h}}w_{\mathsf{h}}}} \langle F_\sigma, \Psi_0\rangle i(\sigma + \psi_\sigma)\,\mathrm{d}\sigma.$$

This is why we compare $\nabla u_{\mathsf{p}}(t)$ with $W(x)\nabla u_{\mathsf{p}}(t)$ in Theorem 2.19 below. Notice also that we cannot apply the trick of Proposition 2.2 to get an extra power of $t^{-\frac{1}{2}}$ with stronger weight (case $s = 1$). Then, with a strong weight, the estimate of $\nabla u_{\mathsf{p}}(t) - W\nabla u_{\mathsf{h}}(t)$ in Theorem 2.19 is the same as the estimate of $\nabla u_{\mathsf{h}}(t)$ in Proposition 2.2. Thus, it is not clear that $W\nabla u_{\mathsf{h}}(t)$ is the asymptotic profile for $\nabla u_{\mathsf{p}}(t)$, but this still gives at least the optimal decay for $\nabla u_{\mathsf{p}}(t)$.

Contribution of high frequencies. For the contribution of high frequencies, we can deduce Proposition 2.5 from the high frequency resolvent estimates given in [BJ16]. Notice that we cannot use the assumption (1.17) in this context. In (1.17), it is only assumed that each bounded classical trajectory goes once through the damping region. However, since bounded trajectories live in a compact subset in the context of Chapter 1, the time needed to reach the damping region in uniform, and each ray of light actually goes regularly through the damping region. This is no longer automatic if we consider classical trajectories at infinity, so we have to be more precise about the damping condition. With the same notation as for (1.17), we assume

$$\exists T > 0, \exists \alpha > 0, \quad \forall (x_0, \xi_0) \in p^{-1}(\{1\}), \quad \int_0^T a(x(t; x_0, \xi_0))\,\mathrm{d}t \geqslant \alpha. \tag{2.45}$$

It is clear that (2.45) (for all trajectories, of for the bounded ones) implies (1.17), and on the other hand we can check that if (1.17) holds in the context of Chapter 1 then bounded trajectories satisfy (2.45).

Local energy decay. Now that the contributions of low and high frequencies are understood, we can state the main result of [17].

Theorem 2.19. *Assume that the geometric damping condition* (2.45) *holds. Let* $s_1, s_2 \in \left[0, \frac{d}{2}\right]$ *and* $\kappa > 1$. *There exists* $C > 0$ *such that for* $t \geqslant 0$ *and* $F = (f, g) \in \mathscr{H}^{\kappa s_2}$ *we have*

$$\begin{aligned}
\|u_{\mathsf{p}}(t) - u_{\mathsf{h}}(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C \langle t \rangle^{-\frac{1}{2} - \frac{s_1+s_2}{2}} \|F\|_{\mathscr{H}^{\kappa s_2}} , \\
\|\nabla u_{\mathsf{p}}(t) - W \nabla u_{\mathsf{h}}(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C \langle t \rangle^{-1 - \frac{s_1+s_2}{2}} \|F\|_{\mathscr{H}^{\kappa s_2}} , \\
\|\partial_t u_{\mathsf{p}}(t) - \partial_t u_{\mathsf{h}}(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C \langle t \rangle^{-\frac{3}{2} - \frac{s_1+s_2}{2}} \|F\|_{\mathscr{H}^{\kappa s_2}} ,
\end{aligned}$$

where $u_{\mathsf{p}}(t)$ *is the solution of* (1.12) *with coefficients* $(G_{\mathsf{p}}, w_{\mathsf{p}}, a_{\mathsf{p}})$ *and* $u_{\mathsf{h}}(t)$ *is given by* (2.44).

As for Theorem 2.8, we could expand λ_σ and Π_σ further in (2.39) to get a more precise asymptotics for $u_{\mathsf{p}}(t)$ and its derivatives.

By a perturbative argument, we get the following result for the general case with asymptotically periodic coefficients.

Proposition 2.20. *Assume that the damping condition* (2.45) *holds. Let* $\kappa > 1$ *and* $s_1, s_2, \eta \geqslant 0$ *be such that*

$$\max(s_1, s_2) + \eta < \min\left(\frac{d}{2}, \rho_G, \rho_a + 1\right). \tag{2.46}$$

Then there exists $C \geqslant 0$ *such that for* $U_0 = (u_0, u_1) \in \mathscr{H}^{\kappa s_2}$ *and* $t \geqslant 0$ *we have*

$$\begin{aligned}
\|u(t) - u_{\mathsf{p}}(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C \langle t \rangle^{-\frac{s_1+s_2}{2} - \frac{\eta}{2}} \|U_0\|_{\mathscr{H}^{\kappa s_2}} , \\
\|\nabla u(t) - \nabla u_{\mathsf{p}}(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C \langle t \rangle^{-\frac{1}{2} - \frac{s_1+s_2}{2} - \frac{\eta}{2}} \|U_0\|_{\mathscr{H}^{\kappa s_2}} , \\
\|\partial_t u(t) - \partial_t u_{\mathsf{p}}(t)\|_{L^{2,-\kappa s_1}(\mathbb{R}^d)} &\leqslant C \langle t \rangle^{-1 - \frac{s_1+s_2}{2} - \frac{\eta}{2}} \|U_0\|_{\mathscr{H}^{\kappa s_2}} ,
\end{aligned}$$

where $u(t)$ *and* $u_{\mathsf{p}}(t)$ *are the solutions of* (1.12) *with coefficients* (G, w, a) *and* $(G_{\mathsf{p}}, w_{\mathsf{p}}, a_{\mathsf{p}})$, *respectively.*

Because of the restriction (2.46), this proposition is probably far from being optimal. The purpose was to show that even if the proof of Theorem 2.19 strongly depends on the symmetry of the setting, the result is actually robust with respect to perturbations.

Nonetheless, Proposition 2.20 gives in any case a result for the global energy and for a general initial data ($s_1 = s_2 = 0$). It also provide interesting estimates for a rapidly decaying perturbation (ρ_G and ρ_a not too small).

2.6 The wave equation with highly oscillating damping

The discussions above mostly concern the contribution of low frequencies for the wave equation. However, while preparing [17], a technical detail about the high frequencies estimates in [BJ16] drew my attention.

The analysis of high frequencies is based on semiclassical analysis, which requires quite a lot of regularity for all the coefficients. However, contrary to the case of low frequencies for which the overdamping phenomenon occurs, for high frequencies we have better decay properties with a stronger damping, so if the absorption index a is not smooth, it is enough to prove the estimates with a replaced by a smooth $\tilde{a}$ such that $\tilde{a} \leqslant a$.

However, some regularity is still required on a. In [BJ16] it is assumed that a is at least uniformly continuous. Similarly, in [2] we had to apply the Mourre theory with the original absorption index a, so we assumed existence and boundedness of its first two radial derivatives.

In [18], I have tried to show on a very simple model case that a highly oscillating absorption index should not be a problem, and that the only important quantity is the average damping seen by the classical rays of light, as measured by the integral in (2.45). Note that a more refined analysis of the damped wave equation with rough damping on the torus has then been published in [BG20].

We consider the Klein-Gordon equation with a periodic damping whose period goes to 0. Let $a \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$ be periodic. For $\eta \geqslant 1$ and $x \in \mathbb{R}^d$ we set $a_\eta(x) = a(\eta x)$. For some $m > 0$ and $(f, g) \in \mathscr{H}$ we consider the problem

$$\begin{cases} \partial_t^2 u - \Delta u + mu + a_\eta(x)\partial_t u = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ (u, \partial_t u)_{|t=0} = (f, g), & \text{on } \mathbb{R}^d. \end{cases} \tag{2.47}$$

The interest of considering the Klein-Gordon equation is that it behaves exactly as the wave equation for high frequencies (the additional term mu is much smaller than the others) but there is no problem with low frequencies, since 0 is not in the spectrum of the operator

$$\begin{pmatrix} 0 & 1 \\ \Delta - m & -a_\eta \end{pmatrix},$$

defined on $\mathscr{H}$ with natural domain. Since we consider the free Laplacian in (2.47), we have $p(x, \xi) = |\xi|^2$ and classical trajectories are given by $\phi^t(x_0, \xi_0) = (x_0 + 2t\xi_0, \xi_0)$.

We have said that under the geometric damping condition on all classical trajectories we have uniform exponential decay for the Klein-Gordon equation (see [BJ16]). The first result in [18] is that if (2.45) is uniform in $\eta \geqslant 1$, then the solution u_η of (2.47) decays exponentially, uniformly in the initial condition and uniformly in $\eta \geqslant 1$. Even if the sizes of the derivatives of the absorption index a_η blow up with η. Thus we assume

$$\exists T > 0, \exists \alpha > 0, \forall \eta \geqslant 1, \forall (x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{S}^{d-1}, \quad \int_0^T a_\eta(x + 2t\xi)\,\mathrm{d}t \geqslant \alpha. \tag{2.48}$$

In our particular case, we observe that if the damping condition (2.45) holds for $\eta = 1$, then it holds uniformly in $\eta \geqslant 1$.

Theorem 2.21. *Assume that the uniform geometric damping condition* (2.48) *holds. Then there exist $\gamma > 0$ and $C > 0$ such that for $\eta \geqslant 1$, $(f, g) \in \mathscr{H}$ and $t \geqslant 0$ we have*

$$\|u(t)\|_{H^1(\mathbb{R}^d)} + \|\partial_t u(t)\|_{L^2(\mathbb{R}^d)} \leqslant Ce^{-\gamma t}\big(\|f\|_{H^1(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)}\big),$$

where $u(t)$ is the solution of (2.47).

The second part of the paper is about the case where (2.48) is not satisfied. We know that we cannot have a uniform decay, but we can still have energy decay if we allow a loss of regularity.

Since the damping is regularly distributed for any $\eta \geqslant 1$, we can apply [BJ16, Th. 1.3] to see that we have at least logarithmic decay. The particular case of a periodic damping has been analysed in [Wun17], and in this case we actually have polynomial decay, with the loss of one derivative. This is consistent with the fact that in the periodic setting the set of rays of light which do not see the damping is unstable. We look at the dependence in η for this polynomial decay.

The second result of [18] is again that despite of the sizes of the derivatives of a_η, we recover the same result as in [Wun17], uniformly in $\eta \geqslant 1$. Moreover, the estimate is even better for large η. This is in fact natural, since for large η the classical rays of light see in some sense the same quantity of damping, but this damping is more uniformly distributed in $\mathbb{R}^d$.

Theorem 2.22. *There exists $c > 0$ such that for $\eta \geqslant 1$, $(f, g) \in \mathscr{H}$ and $t \geqslant 0$ we have*

$$\|u(t)\|_{H^1(\mathbb{R}^d)} + \|\partial_t u(t)\|_{L^2(\mathbb{R}^d)} \leqslant \frac{c}{\sqrt{1+t}} \left(\|f\|_{H^1(\mathbb{R}^d)} + \|g\|_{L^2(\mathbb{R}^d)} + \frac{\|\Delta f\|_{L^2(\mathbb{R}^d)} + \|\nabla g\|_{L^2(\mathbb{R}^d)}}{\eta^2} \right),$$

where $u(t)$ is the solution of (2.47).

2.7 The damped Schrödinger equation on a wave guide

In this paragraph we present the results of [8], which was actually written before all the papers discussed in this chapter.

We consider the Schrödinger equation on a straight wave guide with one dimensional cross section. Given $d \geqslant 2$ and $\ell > 0$, we set $\Omega = \mathbb{R}^{d-1} \times]0, \ell[$. Then for $a > 0$ and $f \in L^2(\Omega)$ we consider on Ω the Schrödinger equation

$$\begin{cases} -i\partial_t u - \Delta u = 0, & \text{on } \mathbb{R}_+ \times \Omega, \\ \partial_\nu u = iau, & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u_{|t=0} = f. \end{cases} \tag{2.49}$$

The L^2-norm of the solution is a non-increasing function of time. We have in fact uniform exponential decay:

Theorem 2.23. *There exist $\gamma > 0$ and $C \geqslant 0$ such that for any $f \in L^2(\Omega)$ the solution u of* (2.49) *satisfies*

$$\forall t \geqslant 0, \quad \|u(t)\|_{L^2(\Omega)} \leqslant Ce^{-\gamma t} \|f\|_{L^2(\Omega)}.$$

Instead of a constant damping, we can consider a such that $a_0 \leqslant a \leqslant a_1$ on $\partial\Omega$, for some positive constants a_0 and a_1. The same result also holds if a satisfies this property on one side of the boundary and vanishes on the other side. We can even consider the case where the damping is a positive constant on one side and a negative one on the other side (with a positive sum, so that the damping is positive on average, in the spirit of [4]).

It is remarkable that even if the classical trajectories parallel to the boundary never see the damping, we can have a uniform decay for the global energy of the Schrödinger equation.

The operator corresponding to (2.49) is the Laplacian with Robin boundary condition H_a, as defined by (1.39) or (2.18). To prove Theorem 2.23, we show that we have a spectral gap for H_a.

Theorem 2.24. *There exist $\gamma_1 > 0$ and $C_1 \geqslant 0$ such that any $z \in \mathbb{C}$ with $\mathsf{Im}(z) \geqslant -\gamma_1$ belongs to the resolvent set of H_a and*

$$\left\|(H_a - z)^{-1}\right\|_{\mathcal{L}(L^2(\Omega))} \leqslant C_1.$$

As in Section 2.3, we can deduce the spectral properties of the Laplacian on the wave guide Ω from the properties of the transverse operator T_a (see (2.20)) on $L^2(0, \ell)$. In dimension 1 it reads

$$T_a = -\partial_y^2, \qquad \mathsf{Dom}(T_a) = \left\{u \in H^2(0, \ell) \,:\, u'(0) = -iau(0), u'(\ell) = iau(\ell)\right\}.$$

Then [8] is mainly about the spectral properties of T_a.

Since T_a is a Laplacian in dimension 1, we can get many spectral properties by explicit computations. In particular, for $z \in \mathbb{C}^*$ we can check that z^2 is an eigenvalue of T_a if and only if

$$e^{2iz\ell} = \left(\frac{z-a}{z+a}\right)^2. \tag{2.50}$$

From this, we deduce some qualitative properties for the sequence of eigenvalues and in particular its asymptotic behavior.

We recall that T_0 is the Neumann Laplacian on $]0,\ell[$. The spectrum of T_0 is given by the sequence of simple eigenvalues $(n^2\nu^2)_{n\in\mathbb{N}}$, where

$$\nu = \frac{\pi}{\ell}.$$

Let $n \in \mathbb{N}^*$. By the Implicit Function Theorem there exists an analytic function $a \mapsto z_n(a)$ defined on a neighborhood $\mathcal{V}_n$ of 0, such that $z_n(0) = n\nu$ and $z_n(a)$ is a solution of (2.50) for $a \in \mathcal{V}_n$. Moreover,

$$z_n(a) = n\nu - \frac{2ia}{\pi n} + \gamma a^2 + \underset{a\to 0}{O}(a^3), \quad \text{with} \quad \mathsf{Re}(\gamma) = \frac{4\ell}{n^3\pi^3}.$$

In particular, $\mathsf{Re}(z_n(a)) \in]n\nu,(n+1)\nu[$ for $a > 0$ small. On the other hand, we can check that for a in a fixed compact the solutions of (2.50) are in a horizontal strip of $\mathbb{C}$ and cannot be on the vertical lines $\mathsf{Re}(\zeta) \in \nu\mathbb{N}$. Then, by continuity of the solutions of (2.50) (using for instance the Rouché Theorem), we can deduce that for any $a > 0$ there is exactly one solution of (2.50) with real part in $]n\nu,(n+1)\nu[$. It is denoted by $z_n(a)$. Finally, we check that for $a > 0$ fixed and n large, we have the same asymptotic expansion as above:

$$z_n(a) = n\nu - \frac{2ia}{\pi n} + O(n^{-2}). \tag{2.51}$$

Thus if we set $\lambda_n(a) = z_n(a)^2$, the spectrum of T_a is given by the sequence $(\lambda_n(a))_{n\in\mathbb{N}}$ and we have

$$\lambda_n(a) = n^2\nu^2 - \frac{4ia}{\ell} + \underset{n\to+\infty}{O}(n^{-1}). \tag{2.52}$$

Since the operator T_a depends analytically on a (in the sense of Kato [Kat80]), since the eigenvalues of T_0 are simple, and since the eigenvalues of T_s for $s \in [0,a]$ never meet, we deduce that the eigenvalues of T_a are simple. In particular, there is no Jordan block.

Then we discuss the properties of the eigenfunctions of T_a. An eigenvector corresponding to the eigenvalue $\lambda_n(a)$ is given by

$$\varphi_n(y) = \varphi_n(a;y) = A_n(a)\left(e^{iz_n(a)y} + \frac{z_n(a)+a}{z_n(a)-a}e^{-iz_n(a)y}\right), \quad y \in]0,\ell[,$$

where $A_n(a)$ is for instance chosen positive and such that $\|\varphi_n(a)\|_{L^2(0,\ell)} = 1$. With (2.51) we see that $\varphi_n(a)$ is close to $\varphi_n(0)$ for large n. Since the family $(\varphi_n(0))$ is orthonormal, we deduce from a perturbation argument (see [Kat80, Th. V.2.20]) that the family $(\varphi_n(a))$ is a Riesz basis of $L^2(0,\ell)$.

We recall that a sequence (β_k) in a Hilbert space $\mathcal{H}$ is a Riesz basis if there exist an orthonormal basis (e_k) and a bounded operator $\Theta \in \mathcal{L}(\mathcal{H})$ with bounded inverse such that $\beta_k = \Theta e_k$ for all $k \in \mathbb{N}$. In particular, for $\psi \in \mathcal{H}$ there exists a unique sequence $(\psi_k)_{k\in\mathbb{N}} \in \ell^2(\mathbb{N})$ such that $\psi = \sum_{k=0}^{\infty}\psi_k\beta_k$ and we have, for some $C \geqslant 1$ independent of ψ,

$$C^{-1}\sum_{k=0}^{\infty}|\psi_k|^2 \leqslant \|\psi\|_{\mathcal{H}}^2 \leqslant C\sum_{k=0}^{\infty}|\psi_k|^2. \tag{2.53}$$

With this Riesz basis property, we can use the symmetry of the domain Ω almost the same way as with a selfadjoint Laplacian (or as in (2.31)). In particular, we can write the resolvent of H_a in terms of the resolvent of L. For $u = u(x,y)$ in $L^2(\Omega)$ we write $u(x,y) = \sum_{n\in\mathbb{N}} u_n(x)\varphi_n(y)$ and for $z \in \rho(H_a)$ we have

$$(H_a - z)^{-1}u = \sum_{n\in\mathbb{N}}\left((L-(z-\lambda_n(a)))^{-1}u_n\right)\otimes\varphi_n. \tag{2.54}$$

Theorem 2.23 can be seen as the consequence of the Riesz basis property for (φ_n) (see (2.53)), the spectral gap for the eigenvalues (λ_n) (given by (2.52)) and the fact that all the eigenvalues have negative imaginary parts), and the estimate for the resolvent of the selfadjoint operator L on $L^2(\mathbb{R}^{d-1})$.

Even if it was the original motivation, it was not possible to adapt the ideas of [8] to deal with the wave equation in [14]. First, the Riesz basis property is probably wrong if the cross section has dimension greater than 1. But even with a one dimensional cross section, the difficulty with the wave equation is that we have to deal with a Robin parameter proportional to the frequency. All the results given above are valid for any $a > 0$, but they are not uniform with respect to a going to $+\infty$. In particular, the size of the strip which contains $\sigma(H_a)$ or the constant C_a which appears in the Riesz basis property have no reason to be uniform with respect to a. This is why I finally used other ideas in [14].

All this being said, it is still interesting to understand the behavior of $\sigma(H_a)$ for large values of a.

The first observation is that for a fixed n the map $a \in [0, +\infty[\mapsto \mathsf{Re}(z_n(a))$ is increasing from $n\nu$ to $(n+1)\nu$, while $\mathsf{Im}(z_n(a))$ is 0 for $a = 0$, it is negative for all $a > 0$, and it goes to 0 when $a \to +\infty$. Thus, with a strong damping, we recover the behavior of an undamped problem. More precisely, when a goes from 0 to $+\infty$, $\lambda_n(a)$ goes from the n-th Neumann eigenvalue $n^2\nu^2$ to the n-th Dirichlet eigenvalue $(n+1)^2\nu^2$, through the lower half-plane (it goes through the upper half-plane if $a \leqslant 0$ goes from 0 to $-\infty$).

But all this is not uniform with respect to n. Roughly speaking, $z_n(a)$ is close to $n\nu$ if $a \ll n$ and close to $(n+1)\nu$ if $a \gg n$. But for the wave equation, a is precisely proportional to the frequency, hence of order n. We can be more precise. We have

$$\sup_{a>0} |\mathsf{Im}(z_n(a))| \underset{n\to\infty}{\simeq} \ln(n),$$

so the curve $a \mapsto z_n(a)$ goes deeper and deeper in the lower half-plane for large n. This upper bound is obtained when $a \sim n\nu$. More precisely, for $\beta \in \mathbb{R}$ we have

$$\mathsf{Re}\big(z_n\big(n\nu + \beta \ln(n)\big)\big) - n\nu \xrightarrow[n\to+\infty]{} \frac{1}{\ell}\left(\pi - \arg\left(\beta + \frac{i}{\ell}\right)\right) \quad \in [0, \nu],$$

$$\mathsf{Im}\big(z_n\big(n\nu + \beta \ln(n)\big)\big) \underset{n\to+\infty}{\sim} -\frac{\ln(n)}{\ell},$$

and for $\gamma \in \mathbb{R}_+^* \setminus \{1\}$,

$$\mathsf{Im}\left(z_n(\gamma n\nu)\right) \xrightarrow[n\to+\infty]{} \frac{1}{\ell} \ln\left|\frac{1+\gamma}{1-\gamma}\right| \quad \text{and} \quad \mathsf{Re} z_n(\gamma n\nu) - n\nu \xrightarrow[n\to+\infty]{} \begin{cases} 0 & \text{if } \gamma < 1, \\ \nu & \text{if } \gamma > 1. \end{cases}$$

These results were finally written in an appendix of [14].

These last properties were not used in [8]. They gave an intuition for [14] but could not be used directly since the cross section was no longer of dimension 1. However, it is interesting to have a good picture of the behavior of the eigenvalues in this quite simple setting, since being able to localize (even roughly) the eigenvalues of a Robin-type non-selfadjoint Schrödinger operator for any Robin coefficient will be an important issue for the Schrödinger and wave equations on (one dimensional) quantum graphs. See the discussions in Sections 3.1 and 4.4.2. See also Section 4.1.2 about the generalization of [8] to a system of two coupled equations on the wave guide.

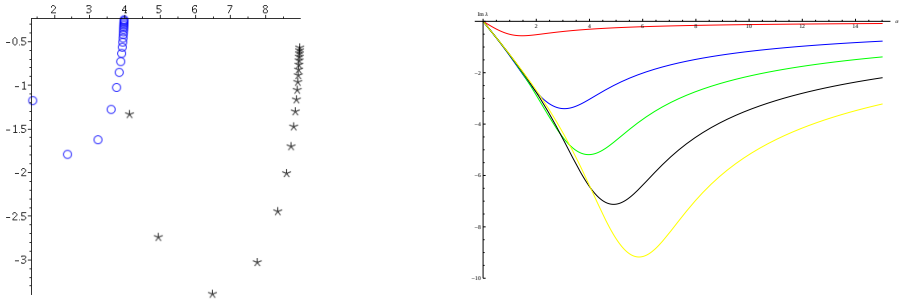

Figure 2.2: The eigenvalues of H_a for a going from 0 to $+\infty$
On the left, the eigenvalues $\lambda_1(a)$ and $\lambda_2(a)$ for various values of a. On the right, the graphs of $a \mapsto \mathsf{Im}(\lambda_n(a))$ for $n \in \{0, \ldots, 4\}$.

Chapter 3

Related topics

The analysis of the damped wave equation has been the main topic of my research, but I have also worked on slightly or completely different problems. In this chapter we will discuss quantum graphs, Agmon estimates, magnetic Laplacians, abstract spectral theory, a problem of control and some non-linear problems. These works opened new directions of research and give interesting perspectives for the future.

3.1 Spectrum of a non-selfadjoint quantum star graph

We begin with the analysis of the spectrum of a non selfadjoint quantum graph. With Gabriel Rivière, we were interested in the spectral properties of quantum graphs. Since we were also both interested by the damped wave equation, we chose to look at the damped wave equation on graphs. In [21] we started with a toy model, namely a non-selfadjoint Schrödinger operator on a star graph.

3.1.1 Quantum graphs

A discrete graph is defined by a set $\mathcal{V}$ of vertices and a set $\mathcal{E}$ of edges, which can be seen as a subset of $\mathcal{V} \times \mathcal{V}$. Two edges v_1 and v_2 are connected if and only if the pair (v_1, v_2) belongs to $\mathcal{E}$. The graph can be oriented (the pairs (v_1, v_2) and (v_2, v_1) are distinct) or not. For spectral properties of discrete graphs, we refer for instance to [Chu97, CdV98] and references therein.

In our context the edges have lengths. Then an edge $e \in \mathcal{E}$ of length $\ell_e > 0$ is identified with $]0, \ell_e[$ and a natural distance can be defined between the points of the graph (vertices or points on edges). We use the terminology *quantum graph* when we consider a (differential) operator on the Hilbert space given by a metric graph. More precisely, the edge e is endowed with the Hilbert structure of $L^2(0, \ell_e)$. Then Γ is endowed with the Hilbert structure defined by

$$L^2(\Gamma) = \bigoplus_{e \in \mathcal{E}} L^2(0, \ell_e).$$

Then we can consider a differential operator on each edge, usually of Schrödinger type, with boundary conditions at the vertices. For a complete introduction about quantum graphs we refer to [BK13].

In [21], we consider the graph Γ with the simplest non-trivial topology. Given $N \in \mathbb{N}^*$, it is defined by a set of $N + 1$ vertices $v, v_1, \ldots, v_N$ and N edges $e_1, \ldots, e_N$. The edge e_j joins the vertex v_j to the central vertex v. Then v has degree N and the other vertices have degree 1.

Our graph is not oriented, but we have to choose a convention for the parametrization of the edges. The edge e_j has length $\ell_j > 0$ and is parametrized by $x_j \in [0, \ell_j]$, with $x_j = 0$

corresponding to the vertex v_j and $x_j = \ell_j$ corresponding to the vertex v.

Then, given $\alpha \in \mathbb{C}$, we define a Schrödinger operator H_α on $L^2(\Gamma)$ with (possibly) non-selfadjoint Robin condition at the central vertex. Its domain $\mathsf{Dom}(H_\alpha)$ is the set of functions $u = (u_j)_{1 \leqslant j \leqslant N}$ in $\bigoplus_{j=1}^N H^2(0, \ell_j)$ which satisfy the Dirichlet boundary condition at the exterior vertices

$$\forall j \in \{1, \dots, N\}, \quad u_j(0) = 0, \tag{3.1}$$

continuity at the central vertex v

$$\forall j, k \in \{1, \dots, N\}, \quad u_j(\ell_j) = u_k(\ell_k), \tag{3.2}$$

and the Robin condition

$$\sum_{j=1}^N u_j'(\ell_j) + \alpha u(v) = 0, \tag{3.3}$$

where $u(v)$ stands for the common value $u_j(\ell_j)$, $j \in \{1, \dots, N\}$. Finally, for $u = (u_j)_{1 \leqslant j \leqslant N}$ in $\mathsf{Dom}(H_\alpha)$ we define $H_\alpha u \in L^2(\Gamma)$ by

$$(H_\alpha u)_j = -u_j'', \qquad \forall j \in \{1, \dots, N\}. \tag{3.4}$$

The quadratic form associated with the operator H_α is given by

$$\langle H_\alpha u, u \rangle_{L^2(\Gamma)} = \sum_{j=1}^N \|u_j'\|^2_{L^2(0,\ell_j)} + \alpha \left| u(v) \right|^2. \tag{3.5}$$

The vertex condition (3.3) is the analog of the usual Robin boundary condition $\partial_\nu u + \alpha u = 0$ for Schrödinger operators on domains of $\mathbb{R}^d$ (see (2.18), with a different convention for α). When $\alpha = 0$ we recover the usual Kirchhoff (or Neumann) condition, and we see from (3.5) that the perturbation when $\alpha \neq 0$ corresponds to a Dirac potential at the central vertex v.

Notice that the particular case $N = 1$ corresponds to the Schrödinger operator on a bounded interval with mixed Dirichlet and Robin conditions at the boundary, while the case $N = 2$ is the Dirichlet Schrödinger operator on a bounded interval with a singular potential in the interval.

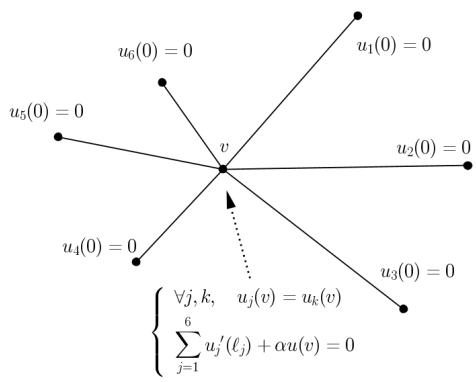

Figure 3.1: The quantum star graph with $N = 6$ edges.

3.1.2 Main spectral results

Our purpose in [21] is to discuss some spectral properties of the operator H_α. We are mainly motivated by the non-selfadjoint case $\alpha \notin \mathbb{R}$. Contrary to the selfadjoint case (see for instance

[BK13]), little is known about the spectral properties of Schrödinger operators on graphs in the non-selfadjoint case (see however [Hus14, HKS15] for some general properties).

Here we only consider a model case, and considering more general graphs will be a natural perspective. However, this allows us to give results without any restriction on the lengths $(\ell_j)_{1\leqslant j\leqslant N}$, when many results about quantum graphs are only given for rational lengths or for a generic family of lengths.

We are interested in the qualitative behavior of the large eigenvalues. The main motivation is the behavior of the imaginary parts, which are related to the decay for the corresponding time-dependent problem, but the results also give information about the real parts (and in particular we obtain new information even in the selfadjoint case). We describe the spectrum of H_α by comparison with the spectrum of the Kirchhoff operator H_0. We denote by $(\lambda_n(0))_{n\in\mathbb{N}^*}$ the non-decreasing sequence (with $\lambda_1(0)>0$) of eigenvalues of H_0 repeated according to their multiplicities.

We say that an operator T on the Hilbert space $\mathcal{H}$ is sectorial if there exists $\gamma\in\mathbb{R}$ and $\theta\in\left[0,\frac{\pi}{2}\right[$ such that the numerical range of T is included in the sector $\Sigma_{\gamma,\theta}=\{\zeta\in\mathbb{C}\,:\,|\arg(\zeta-\gamma)|\leqslant\theta\}$. Then T is said to be maximal sectorial if $\sigma(T)\subset\Sigma_{\gamma,\theta}$.

We begin with a rough localization of the eigenvalues of H_α.

Proposition 3.1. *Let $\alpha\in\mathbb{C}$. The operator H_α is maximal sectorial and its spectrum is given by an infinite sequence $(\lambda_n(\alpha))_{n\in\mathbb{N}^*}$ of isolated eigenvalues whose geometric and algebraic multiplicities are finite and coincide. These eigenvalues (repeated according to multiplicities) can be labeled in such a way that for $n\in\mathbb{N}^*$ large enough we have*

$$|\lambda_n(\alpha)-\lambda_n(0)|\leqslant \mathsf{dist}\big(\lambda_n(0),\sigma(H_0)\setminus\{\lambda_n(0)\}\big).$$

Moreover, if $\tau\in\frac{\pi\mathbb{Z}}{\ell_j}\cap\frac{\pi\mathbb{Z}}{\ell_k}$ (non-empty if $\ell_j/\ell_k\in\mathbb{Q}$) for some $j,k\in\{1,\dots,N\}$, then τ^2 is a common eigenvalue of H_0 and H_α, with the same multiplicities.

This proposition says that the Robin eigenvalues are in some sense not so far from the corresponding eigenvalues of the Kirchhoff operator. This gives in particular a Weyl law, already known for the latter. For $R>0$ we set

$$\mathcal{N}_\alpha(R)=\mathsf{Card}\,\{n\in\mathbb{N}^*\,:\,\mathsf{Re}(\lambda_n(\alpha))\leqslant R\}\,.$$

The Weyl law for a Laplacian on a domain Ω of $\mathbb{R}^d$ (or a manifold) depends on the dimension d and the size of the domain:

$$N_\Omega(R)\underset{R\to+\infty}{\sim}\frac{|\Omega|\,\omega_d R^{\frac{d}{2}}}{(2\pi)^d},$$

where ω_d is the volume of the unit ball. Here the size of the metric graph is the sum of the lengths of the edges: $|\Gamma|=\sum_{j=1}^N\ell_j$. The following result agrees with the result already known on manifold or for selfadjoint quantum graphs.

Theorem 3.2 (Weyl Law)**.** *Let $\alpha\in\mathbb{C}$. We have*

$$\mathcal{N}_\alpha(R)=\frac{|\Gamma|\sqrt{R}}{\pi}+\underset{R\to+\infty}{O}(1).$$

For $\alpha\in\mathbb{C}$ and $n\in\mathbb{N}^*$ we set $\eta_n(\alpha)=\lambda_n(\alpha)-\lambda_n(0)$. For Theorem 3.2 we only need a uniform bound on $\eta_n(\alpha)$. The main result in [21] is a refined analysis of this difference. We set $\ell=(\ell_1,\dots,\ell_N)$.

Theorem 3.3. *There exists a probability measure μ_ℓ on $\mathbb{C}$ such that for any $\alpha\in\mathbb{C}^*$ we have in the weak sense for measures*

$$\frac{1}{n}\sum_{k=1}^n\delta_{\frac{\eta_k(\alpha)}{\alpha}}\xrightarrow[n\to+\infty]{}\mu_\ell.$$

Moreover μ_ℓ satisfies the following properties.

(i) *μ_ℓ is supported in* $[0, 2\,|\Gamma|^{-1}]$.

(ii) *If $\ell_j/\ell_k \in \mathbb{Q}$ for all $j, k \in \{1, \dots, N\}$ then μ_ℓ is a linear combination of Dirac distributions, including the one at 0.*

(iii) *If $\kappa \cdot \ell \neq 0$ for all $\kappa \in \mathbb{Z}^d \setminus \{0\}$ then μ_ℓ is absolutely continuous with respect to the Lebesgue measure on $[0, 2\,|\Gamma|^{-1}]$ and its support is exactly $[0, 2\,|\Gamma|^{-1}]$.*

If s belongs to the support of μ_ℓ there exists an increasing sequence (n_k) in $\mathbb{N}^*$ such that $\eta_{n_k}(\alpha)$ goes to $s\alpha$ as $k \to \infty$. The last result is about the rate of convergence for this limit.

Proposition 3.4. *Let $\alpha \in \mathbb{C}$.*

(i) There exists an increasing sequence $(n_k)_{k\in\mathbb{N}^*}$ such that

$$\eta_{n_k}(\alpha) = \underset{k\to\infty}{O}\left(\frac{1}{n_k^2}\right).$$

(ii) There exists $\Omega \subset (\mathbb{R}_+^*)^N$ of Lebesgue measure 0 such that for $\ell \in (\mathbb{R}_+^*)^N \setminus \Omega$, $\alpha \in \mathbb{C}$, $s \in [0, 2\,|\Gamma|^{-1}]$ and $\varepsilon > 0$ we can find an increasing sequence $(n_k)_{k\in\mathbb{N}^*}$ such that

$$\eta_{n_k}(\alpha) = s\alpha + \underset{k\to\infty}{O}\left(\frac{1}{n_k^{\frac{1}{N}-\varepsilon}}\right).$$

By the last statement of Proposition 3.1, the sequence $(\eta_n(\alpha))$ has a vanishing subsequence except in the last case of Theorem 3.3, so the first statement essentially concerns the purely irrational case. Notice in particular that 0 always belongs to the support of μ_ℓ.

In these results, we only describe the eigenvalues of H_α by comparison with those of H_0. However, since $\mathsf{Im}(\lambda_n(\alpha)) = \mathsf{Im}(\eta_n(\alpha))$, this gives a direct description of the imaginary parts, which was our original motivation. The spacing between the real parts of $\lambda_0(\alpha)$ and $\lambda_n(\alpha)$ is nevertheless also an important result.

3.1.3 Ideas of proof

In the rest of this section, we comment some ideas of the proof of Theorem 3.3.

Comparison with the Dirichlet eigenvalues. We denote by H_D the operator defined as H_α with (3.3) replaced by the Dirichlet condition $u(v) = 0$ at the central vectex (this corresponds to $\alpha = \infty$). Then the edges are disconnected, and it is easy to identify the non-decreasing sequence $(\lambda_{D,n})_{n\in\mathbb{N}^*}$ of eigenvalues for H_D (repeated according to multiplicities). By a standard argument based on the Min-max Theorem (see for instance [BK13, Th. 1.3.8]) we have

$$\lambda_1(0) \leqslant \lambda_{D,1} \leqslant \lambda_2(0) \leqslant \lambda_{D,2} \leqslant \ldots \leqslant \lambda_n(0) \leqslant \lambda_{D,n} \leqslant \ldots \tag{3.6}$$

Moreover, we have a similar result with $\lambda_n(\alpha)$ instead of $\lambda_n(0)$ if α is real. Since the eigenvalues of the Dirichlet problem are easily identified, this gives a rough localization of $\lambda_n(\alpha)$ when α is real.

We cannot use this idea when $\alpha \notin \mathbb{R}$. Nevertheless, we can check that the two following facts remain valid in general. If λ is an eigenvalue of H_D of multiplicity $m \in \mathbb{N}^*$, then it is an eigenvalue of H_α of (geometric and algebraic) multiplicity $m - 1$. On the other hand, the eigenvalues of H_α which do not coincide with an eigenvalue of H_D are of (geometric and algebraic) multiplicity 1. The difficulty is then to localize these other eigenvalues.

The secular equation. Let $z \in \mathbb{C}^*$. Assume that z^2 is an eigenvalue of H_α and that u is a corresponding eigenvector. By the Dirichlet condition (3.1), u_j is necessarily of the form

$$u_j(x_j) = \beta_j \sin(zx_j), \quad x_j \in [0, \ell_j], \tag{3.7}$$

for some $\beta_j \in \mathbb{C}$. With this information only, we can already prove that

$$\limsup_{\substack{\lambda \in \sigma(H_\alpha) \\ \mathsf{Re}(\lambda) \to +\infty}} |\mathsf{Im}(\lambda)| \leqslant \frac{2}{|\Gamma|} |\mathsf{Im}(\alpha)| . \tag{3.8}$$

This implies in particular that $\sigma(H_\alpha)$ is included in a horizontal strip of $\mathbb{C}$.

The conditions (3.2) and (3.3) at the central vertex v give N linear equations with respect to the N coefficients β_j, $j \in \{1, \dots, N\}$. Then z^2 is an eigenvalue of H_α if and only if the corresponding determinant is 0. In general, the determinant is some polynomial in $\sin(z\ell_j)$ and $\cos(z\ell_j)$, $j \in \{1, \dots, N\}$. Here, with a star-graph, everything can be explicited. We obtain the secular equation

$$F_0(z\ell) + \frac{\alpha}{z} F_D(z\ell) = 0, \tag{3.9}$$

where for $y = (y_1, \dots, y_d) \in \mathbb{C}^d$ we have set

$$F_0(y) = \sum_{j=1}^{N} \cos(y_j) \prod_{k \neq j} \sin(y_k) \quad \text{and} \quad F_D(y) = \prod_{j=1}^{N} \sin(y_j).$$

The determinant $F_0(z\ell)$ corresponds to the case $\alpha = 0$. This is the determinant for the Kirchhoff operator. On the other hand, $F_D(z\ell)$ is the determinant corresponding to the Dirichlet condition at v. For $y \in \mathbb{C}^N$ and $z \in \mathbb{C}$ such that $F_D(y) \neq 0$ and $F_D(z\ell) \neq 0$ we set

$$\Psi(y) = -\frac{F_0(y)}{F_D(y)} = -\sum_{j=1}^{N} \operatorname{cotan}(y_j) \quad \text{and} \quad \psi(z) = \Psi(z\ell).$$

Then $z^2 \in \mathbb{C} \backslash \sigma(H_D)$ (remember that $\sigma(H_\alpha) \cap \sigma(H_D)$ is already understood) is an eigenvalue of H_α if and only if

$$\psi(z) = \frac{\alpha}{z}. \tag{3.10}$$

Perturbation of the Kirchhoff eigenvalues. The family of operators $(H_\alpha)_{\alpha \in \mathbb{C}}$ is analytic with respect to α (family of type B in the sense of [Kat80]). At least for α small the eigenvalues of H_α are in some sense close to the eigenvalues of H_0. The next step is to show that the eigenvalues of H_0 cannot move too far as long as α stays in a compact set of $\mathbb{C}$.

Let $n \in \mathbb{N}^*$ such that $\lambda_n(0) \neq \lambda_{D,n}$ (in particular, $\lambda_n(0)$ is a simple eigenvalue of H_0). The idea is to find a bounded domain ω_n such that $\omega_n \cap \sigma(H_0) = \{\lambda_n(0)\}$ and $\partial\omega_n \cap H_{t\alpha} = \varnothing$ for $t \in [0, 1]$. By continuity of the spectrum, this will imply that H_α has exactly one simple eigenvalue in ω_n. For the star graph, we can use the explicit expression for ψ. Using the Taylor expansion around $\tau_n = \sqrt{\lambda_n(0)}$, we can prove that (3.10) with α replaced by $t\alpha$ cannot have a solution on the circle $\mathcal{C}\big(\tau_n, \frac{8|\alpha|}{\tau_n \psi'(\tau_n)}\big)$ when n is large. In particular there is an eigenvalue $\lambda_n(\alpha)$ of H_α which remains "close" to $\lambda_n(0)$. Then, by Taylor expansion again, we can deduce that

$$\eta_n(\alpha) = \frac{2\alpha}{\psi'(\tau_n)} + O(\tau_n^{-1}). \tag{3.11}$$

At this stage, we have roughly localized some eigenvalues of H_α. For $n \in \mathbb{N}^*$, if $\lambda_n(0)$ coincides with a Dirichlet eigenvalue, then it is also an eigenvalue of H_α and we set $\lambda_n(\alpha) = \lambda_n(0)$ (the multiplicities agree), otherwise there is an eigenvalue $\lambda_n(\alpha)$ close to $\lambda_n(0)$ (both are simple). We have used the continuity of the eigenvalues of $H_{t\alpha}$ to see how the eigenvalues of H_0 move to those of H_α. It could also happen that some new eigenvalues appear at infinity. From the

analysis of ψ again, we check that the eigenvalues of $H_{t\alpha}$ for $t \in [0,1]$ all stay in a region of $\mathbb{C}$ which only has bounded connected components, so this phenomenon cannot occur here. Thus, we have identified *all* the eigenvalues of H_α. For some of them we have $\eta_n(\alpha) = 0$ (this happens if some lengths are commensurable), and for the others the difference η_n satisfies (3.11).

The Barra-Gaspard measure and definition of μ_ℓ. We observe that the leading term in (3.11) is simply linear with respect to α. For a better understanding of the sequence $(\eta_n(\alpha))_{n\in\mathbb{N}^*}$, the next step is the analysis of the sequence $(\psi'(\tau_n))_{n\in\mathbb{N}^*}$. This is a quantity which only depends on the τ_n, $n \in \mathbb{N}^*$.

For this we use the measure introduced by Barra and Gaspard in [BG00] to study the level spacing of H_0. See also [KMW03, BW10]. We see F_0 as a function on the N-dimensional torus $\mathbb{T}^N$. We set $\mathcal{Z} = \{y \in \mathbb{T}^N : F_0(y) = 0\}$, so that τ^2 is an eigenvalue of H_0 if and only if $\tau\ell \in \mathcal{Z}$. For $t \geqslant 0$ we set $\varphi_\ell^t = [t\ell]$ in $\mathbb{T}^N$.

Here we focus on the case where $\kappa\cdot\ell \neq 0$ for all $\kappa \in \mathbb{Z}^d\setminus\{0\}$ (otherwise φ_ℓ^t lives in a subtorus of $\mathbb{T}^N$, the general case is considered in [21]). In this case $\mathcal{Z}_0 = \mathcal{Z}\setminus\{0\}$ is a submanifold of dimension $N-1$ in $\mathbb{T}^N$ to which any ℓ with positive components is transverse. For this last property we use again the explicit expressions available for the case of a star graph. The contribution of a small neighborhood of 0 in $\mathcal{Z}$ will be small and will not play any role. On $\mathcal{Z}_0$ we consider the Barra-Gaspard measure

$$\mu_{\mathsf{BG}} = |\ell \cdot \nu(y)|\, \mu_{\mathcal{Z}_0}, \tag{3.12}$$

where $\mu_{\mathcal{Z}_0}$ is the Lebesgue measure on $\mathcal{Z}_0$ and $\nu(y)$ is a normal unit vector. Using in particular unique ergodicity, we see that if g is a continuous and compactly supported function on $\mathcal{Z}_0$ we have

$$\frac{1}{n}\sum_{k=1}^{n} g(\varphi_\ell^{\tau_k}) \xrightarrow[n\to\infty]{} \frac{\pi}{|\Gamma|\,|\mathbb{T}^N|}\int_{\mathcal{Z}_0} g\,\mathrm{d}\mu_{\mathsf{BG}}.$$

Then the measure μ_ℓ of Theorem 3.3 is given by

$$\mu_\ell = \frac{\pi}{|\Gamma|\,|\mathbb{T}^N|}\Phi_*\mu_{\mathsf{BG}}, \quad \text{where} \quad \Phi(y) = \frac{2}{\nabla\Psi(y)\cdot\ell}.$$

When some lengths are commensurable, $|\mathbb{T}^N|$ has to be replaced by the size of the sub-torus in which the flow φ_ℓ^t is dense, and there is in μ_ℓ a Dirac mass at 0, corresponding of the eigenvalues which do not move with α.

From the expression of μ_ℓ we can then prove the properties given in Theorem 3.3. We do not discuss this part of the proof here.

It is natural to wonder if we can recover the same results on a more general quantum graph. Some possible extansions of the results of [21] will be discussed in Section 4.4.2.

3.2 Agmon estimates for non-accretive Schrödinger operators

In this paragraph we briefly discuss the result of [13] about some non-accretive Schrödinger operators. This is a joint work with David Krejčiřík, Nicolas Raymond and Petr Siegl.

The motivation for this paper was the Agmon estimates for a large class of non-selfadjoint Schrödinger operators. The so-called Agmon estimates measure the decay of the eigenfunctions of a Schrödinger operator in the classically forbidden region.

If we consider on $\mathbb{R}$ the case of a confining real valued potential V with minimum at 0 (for instance, $V(x) = x^2$), then a classical particle of energy E will necessarily stay in the neighborhood of 0 defined by $\{x \in \mathbb{R} : V(x) \leqslant E\}$. This is not the case for a quantum particle, but an eigenfunction of the operator $-\partial_x^2 + V(x)$ corresponding to the eigenvalue E

decays exponentially fast outside this region, and the rate of decay is given by the distance to the classical region as defined by the Agmon (pseudo) metric

$$\sqrt{(V(x)-E)_+}\,\mathrm{d}x^2.$$

More generally, in a non-confining setting and for more general operators, the decay of eigenfunctions was studied for eigenvalues smaller than the bottom of the essential spectrum. We refer for instance to [Agm82]. See also [HS96, Ch. 3]. The Agmon estimates have then been extended in many settings. For a review, see for instance the talk [Hel19] and references therein.

The purpose of [13] is to show that we have a similar phenomenon when the particle is confined by the imaginary part of the potential. We also consider magnetic Laplacians, and it turns out that the magnetic field actually plays exactly the same role. We can even go further and show that if the imaginary part of the potential or the magnetic field becomes large, then we can allow a (smaller) negative real part for the potential. As a typical example, we can consider on $L^2(\mathbb{R})$ the operator

$$-\frac{\mathrm{d}^2}{\mathrm{d}x^2}-x^2+ix^3,$$

whose numerical range covers the whole complex plane.

In fact, much wilder potentials are allowed. The setting of [13] is the following. We consider an open and connected subset Ω of $\mathbb{R}^d$, $V\in C^1(\bar{\Omega},\mathbb{C})$ and $A\in C^2(\bar{\Omega},\mathbb{R}^d)$. The magnetic matrix is $B=(\partial_jA_k-\partial_kA_j)_{1\leqslant j,k\leqslant d}$. We set

$$\langle B;V\rangle=\sqrt{1+|B|+|V|}.$$

Conditions sufficient for all the results of the paper are

$$(\mathsf{Re}V)_-=\underset{|x|\to+\infty}{o}\big(\langle B;V\rangle\big)$$

$$|\nabla V|+|\nabla B|=\underset{|x|\to+\infty}{o}\big(\langle B;V\rangle^{\frac{3}{2}}\big).$$

The second assumption prevents strong osclillations for the coefficients. The power $\frac{3}{2}$ is an improvement compared to previous papers on related subjects. The first condition says that the negative real part has to be small at infinity compared to the imaginary part or the magnetic matrix, as discussed above. The purpose is then to define and give spectral properties for the electromagnetic operator

$$\mathscr{L}=(-i\nabla+A)^2+V.$$

We are thus considering Schrödinger operators which are not even accretive. We first define the operator. The form domain is

$$\mathcal{V}=\Big\{u\in L^2(\Omega)\,:\,(-i\nabla+A)u\in L^2(\Omega),\langle B;V\rangle^{\frac{1}{2}}\,u\in L^2(\Omega)\Big\}.$$

We cannot apply a standard version of the representation theorem, so we use a generalized version given in [AH15]. This defines a closed operator $\mathscr{L}$ with domain

$$\mathsf{Dom}(\mathscr{L})=\big\{u\in\mathcal{V}\,:\,(-i\nabla+A)^2u+Vu\in L^2(\Omega)\big\}.$$

The next step is to prove that we have separation of the spectrum, namely

$$\mathsf{Dom}(\mathscr{L})=\big\{u\in\mathcal{V}\,:\,(-i\nabla+A)^2u\in L^2(\Omega)\ \textit{and}\ Vu\in L^2(\Omega)\big\}.$$

The main result is then the following. For $E\in\mathbb{R}$ we set

$$\Sigma_E=\{\zeta\in\mathbb{C}\,:\,\mathsf{Re}(\zeta)+|\mathsf{Im}(\zeta)|<E\}.$$

Theorem 3.5. (i) *There exist $\gamma_1 > 0$ and $\gamma_2 \in \mathbb{R}$ such that for*

$$\mu < \liminf_{|x|\to+\infty} \langle B; V\rangle (x)$$

then any $\zeta \in \Sigma_{\gamma_1\mu-\gamma_2}$ is such that $(\mathscr{L} - \zeta)$ is Fredholm with index 0 and the spectrum of $\mathscr{L}$ in $\Sigma_{\gamma_1\mu-\gamma_2}$ consists of isolated eigenvalues with finite algebraic multiplicities.

(ii) *Given $\lambda \in \Sigma_{\gamma_1\mu-\gamma_2}$ we define the corresponding Agmon distance on Ω by*

$$d_{\mathsf{Ag},\lambda}(x,y) = \inf_{\gamma\in C^1(x,y)} \int_0^1 \sqrt{(\gamma_1 \langle B; V\rangle (\gamma(t)) - \gamma_2 - \mathsf{Re}(\lambda) - |\mathsf{Im}(\lambda)|)_+}\, \left|\gamma'(t)\right| \,\mathrm{d}t,$$

where $C^1(x,y)$ is the set of continuous and piecewise C^1 path $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$ and $\gamma(1) = y$. Let $x_0 \in \Omega$ be fixed. If λ is an eigenvalue of $\mathscr{L}$ then for any ψ in the corresponding algebraic eigenspace we have

$$e^{\frac{1-\varepsilon}{3} d_{\mathsf{Ag}(\cdot,x_0)}}\psi \in L^2(\Omega).$$

3.3 Absence of embedded eigenvalues for a magnetic Laplacian

In this paragraph we present [20]. This joint work with Nicolas Raymond is about the absence of embedded eigenvalues in the essential spectrum of a magnetic Laplacian on $\mathbb{R}^2$.

We consider a magnetic field B which is invariant with respect to one variable. We set $B(x,y) = b(x)$ for some $b \in C^1(\mathbb{R};\mathbb{R})$. A corresponding vector potential $A = (A_1, A_2)$ is given by $A_1(x,y) = 0$ and, for some $a_0 \in \mathbb{R}$,

$$A_2(x,y) = a(x) = a_0 + \int_0^x b(s)\,\mathrm{d}s. \tag{3.13}$$

Then we consider the (selfadjoint) operator

$$\mathscr{L} = (-i\nabla - A)^2 = -\partial_x^2 + \big(-i\partial_y - a(x)\big)^2,$$

defined on the domain

$$\mathsf{Dom}(\mathscr{L}) = \left\{u \in L^2(\mathbb{R}^2) \,:\, (-i\nabla - A)u, (-i\nabla - A)^2 u \in L^2(\mathbb{R}^2)\right\}.$$

The spectrum of $\mathscr{L}$ is purely essential. The purpose is to prove, under some assumptions on b, that there are no eigenvalues in (some parts of) the spectrum of $\mathscr{L}$. In some cases, the spectrum of $\mathscr{L}$ can even be purely absolutely continuous. Such results cannot be true in general since for a constant and non-zero magnetic field B, the spectrum of $\mathscr{L}$ consists of eigenvalues of infinite multiplicities, given by the Landau levels $(2n-1)\,|B|$, $n \in \mathbb{N}^*$ (so $\sigma(\mathscr{L}) = \sigma_{\mathsf{ess}}(\mathscr{L}) = \sigma_{\mathsf{p}}(\mathscr{L})$ in this case).

Let

$$\phi_\pm = \lim_{x\to\pm\infty} a(x).$$

It is already known [Iwa85] that if $\phi_+ = \phi_- = \pm\infty$ or if $b(x)$ has two distinct finite and non-zero limits at $-\infty$ and $+\infty$, then $\mathscr{L}$ has an absolutely continuous spectrum (and in particular, no eigenvalues). Our two main results in [20] are the following.

Theorem 3.6. *Assume that the limit ϕ_- exists in $\mathbb{R} \cup \{\pm\infty\}$. Assume that there exist $\alpha \in]-1, +\infty[\setminus\{0\}$ and $c_1, C > 0$ such that*

$$b(x) \underset{x\to+\infty}{\sim} c_1 x^\alpha \quad \textit{and} \quad \left|b'(x)\right| \leqslant C\,\langle x\rangle^{\alpha-1}, \quad x \geqslant 0.$$

Then $\mathscr{L}$ has no eigenvalue.

We could similarly consider the cases where b is replaced by $-b$ or where the behavior of b is prescribed near $-\infty$. Notice that Theorem 3.6 covers in particular situations where $b(x)$ goes to 0 or to $+\infty$ for $x \to +\infty$.

In the second theorem, we consider situations where b goes faster to 0 at infinity. We assume that ϕ_- and ϕ_+ are finite but with $\phi_+ - \phi_- \gg 1$ (the case $\phi_- - \phi_+ \gg 1$ is similar). For instance, we assume $\phi_- = 0$ and $\phi = \phi_+ \gg 1$.

Theorem 3.7. *Assume that b takes positive values, that it is integrable and that $b'(x) = \mathcal{O}(x^N)$ for some $N \geqslant 0$ as $x \to +\infty$. We choose $a_0 = \int_{-\infty}^0 b(s)\,\mathrm{d}s$ in (3.13), so that $\phi_- = 0$ and $\phi := \phi_+ = \|b\|_{L^1(\mathbb{R})}$. Then we assume that $a \in L^1(\mathbb{R}_-)$ and $a - \phi \in L^1(\mathbb{R}_+)$.*

(i) $\sigma(\mathscr{L}) = [0, +\infty[$.

(ii) *$\mathscr{L} = \mathscr{L}_\phi$ has no eigenvalue in $\left[\frac{\phi^2}{4}, +\infty\right[$.*

(iii) *Let η_ϕ be such that $\eta_\phi = o(\ln(\phi)^{-6})$ as $\phi \to +\infty$. Then there exists $\phi_0 > 0$ such that for $\phi \geqslant \phi_0$ the operator $\mathscr{L} = \mathscr{L}_\phi$ has no eigenvalue smaller than $\eta_\phi \phi^2$.*

The question of existence of eigenvalues in $\left[\eta_\phi \phi^2, \frac{\phi^2}{4}\right[$ remains open in this case.

The proofs of these two results are based on a Fourier transform with respect to y and a careful analysis of the family of one-dimensional Schrödinger operators

$$\mathscr{L}_\xi = -\partial_x^2 + \big(\xi - a(x)\big)^2.$$

In particular, a refined version of the harmonic approximation is provided to give a precise description of the eigenvalues of $\mathscr{L}_\xi$.

3.4 Reduction of dimension in an abstract setting

This paragraph is devoted to the paper [16], written with David Krejčiřík, Nicolas Raymond and Petr Siegl.

This work was motivated by a Schrödinger operator with non-selfadjoint Robin condition in a shrinking layer. More precisely, we consider a closed, orientable, smooth hypersurface Σ in $\mathbb{R}^d$ without boundary, and we denote by $\nu : \Sigma \to \mathbb{S}^{d-1}$ a unit normal vector field which specifies the orientation of Σ. We assume that Σ has a tubular neighborhood: for some $\varepsilon > 0$ the map $\Theta_\varepsilon : (s,t) \mapsto s + \varepsilon t \nu(s)$ is injective on $\Omega = \Sigma \times [-1,1]$ and defines a diffeomorphism from Ω to its image $\Omega_\varepsilon = \Theta_\varepsilon(\Omega)$.

We set $\Sigma_{\pm,\varepsilon} = \Theta_\varepsilon(\Sigma \times \{\pm 1\})$. Given a smooth function $\alpha : \Sigma \to \mathbb{C}$ we define $\alpha_{\pm,\varepsilon}$ on $\Sigma_{\pm,\varepsilon}$ by $\alpha_{\pm,\varepsilon}(s \pm \varepsilon\nu(s)) = \alpha(s)$. Then we consider on $L^2(\Omega_\varepsilon)$ the operator P_ε defined as the usual Laplace operator on Ω_ε subject to the Robin boundary condition

$$\partial_\nu u + \alpha_{\pm,\varepsilon} u = 0 \quad \text{ on } \Sigma_{\pm,\varepsilon}.$$

This is a very particular (PT-symmetric) choice of Robin coefficients, but a more general setting could be similarly considered. Then the purpose is to prove that when $\varepsilon \to 0$ the spectral properties of P_ε are close to those of an effective operator $P_{\mathsf{eff}} = -\Delta_\Sigma + V_{\mathsf{eff}}$ on $L^2(\Sigma)$. Here $-\Delta_\Sigma$ is the usual Laplace-Beltrami operator on Σ and V_{eff} is a potential which depends on the geometry and the Robin coefficient α:

$$V_{\mathsf{eff}} = |\alpha| - 2\alpha \mathsf{Re}(\alpha) - \alpha(\kappa_1 + \cdots + \kappa_{d-1}),$$

where $\kappa_1, \ldots, \kappa_{d-1}$ are the principal curvatures.

The convergence of P_ε to P_{eff} is in the sense of the norm of the resolvent. More precisely, we denote by Π the projection of $L^2(\Omega)$ on functions which do not depend on t. For $u \in L^2(\Omega)$

and $(s,t) \in \Omega$ we set $(\Pi u)(s,t) = \frac{1}{2}\int_{-1}^{1} u(s,\theta)\,\mathrm{d}\theta$. Moreover, for $\varepsilon > 0$ small enough, an explicit unitary operator $U_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega, w_\varepsilon\,\mathrm{d}s\ dt)$ is defined, for some $w_\varepsilon : \Omega \to \mathbb{R}_+$ bounded and bounded away from 0 uniformly in ε. In the following result, P_{eff} is seen as an operator on $\Pi L^2(\Omega)$.

Theorem 3.8. *Let K be a compact subset of $\rho(P_{\mathsf{eff}})$. There exist $\varepsilon_0 > 0$ and $C \geqslant 0$ such that for $z \in K$ and $\varepsilon \in]0,\varepsilon_0]$ we have $z \in \rho(P_\varepsilon)$ and*

$$\left\|(P_\varepsilon - z)^{-1} - U_\varepsilon^{-1}(P_{\mathsf{eff}} - z)^{-1}\Pi U_\varepsilon\right\|_{\mathcal{L}(L^2(\Omega_\varepsilon))} \leqslant C\varepsilon.$$

Notice that the convergence in the sense of the norm-resolvent is very strong and we can deduce important properties for P_ε with $\varepsilon > 0$ small from the properties of P_{eff}. We say that we have reduced the dimension since a d-dimensional problem is reduced to a $(d-1)$-dimensional problem, which can be significantly simpler.

There were already many results of this kind in the literature for selfadjoint operators. The common aspect is that the spectral properties of a parameter-dependent operator are well described by those of an effective operator when the parameter goes to some limit. However, in the previous literature, this similarity is hidden in the various technical steps specific to each situation. In [16], we give a unified abstract result, described below. This includes various settings such as the semiclassical Born-Oppenheimer approximation, a problem on a shrinking tubular neighborhood as above, or the similar problem on a fixed domain but for a large Robin parameter α. Reduction of dimension is already known in some of these contexts, but the abstract result also provides new results. Here we only describe the abstract result, but these examples of applications are discussed in [16].

The abstract setting is the following. Given a measure space Σ and a family of Hilbert spaces $\mathcal{H}_s$, $s \in \Sigma$, we set $\mathcal{H} = \bigoplus_{s\in\Sigma}\mathcal{H}_s$. A vector in $\mathcal{H}$ is a family $(\phi_s)_{s\in\Sigma}$, where each ϕ_s belongs to $\mathcal{H}_s$, and $\int_{s\in\Sigma}\|\phi_s\|^2_{\mathcal{H}_s}\,\mathrm{d}\sigma(s) < +\infty$. We consider on $\mathcal{H}$ a selfadjoint operator of the form

$$\mathscr{L} = S^*S + \bigoplus_{s\in\Sigma} T_s,$$

where S is densely defined and T_s is selfadjoint and non-negative for all $s \in \Sigma$.

Typically, if we consider on $\mathbb{R}^2$ an operator of the form $-h^2\Delta + V$ with $V \geqslant 0$ then we can take $\Sigma = \mathbb{R}$, $S = -ih\partial_s$ and $T_s = -\partial_t^2 + V(s,t)$ for each $s \in \mathbb{R}$.

Then we consider a function $s \in \Sigma \to \gamma_s$ such that $\gamma_s \geqslant \gamma$ for some $\gamma > 0$. For $s \in \Sigma$ we denote by Π_s the spectral projection of T_s on $[0,\gamma_s[$. We define $\Pi \in \mathcal{L}(\mathcal{H})$ by $\Pi\phi = (\Pi_s\phi_s)_{s\in\Sigma}$ for $\phi = (\phi_s)_{s\in\Sigma}$, and we set $\Pi^\perp = \mathsf{Id} - \Pi$. We assume that $\mathsf{Dom}(S)$ is invariant under Π.

In a typical example, the bottom of the spectrum of T_s is given by a simple eigenvalue and γ_s is the second eigenvalue of T_s, so that Π_s is the orthogonal projection on the eigenspace corresponding to the first eigenvalue. After having chosen a corresponding eigenvector, $\Pi_s\phi_s$ is characterized by a scalar and $\Pi\phi$ can be identified with a function on Σ.

We set $\mathscr{L}_{\mathsf{eff}} = \Pi\mathscr{L}\Pi$. This is an operator on $\Pi\mathcal{H}$ with domain $\Pi\mathcal{H}\cap\mathsf{Dom}(\mathscr{L})$. The purpose is to compare $\mathscr{L}$ with the simpler operator $\mathscr{L}_{\mathsf{eff}}$. As an intermediate step we consider the operator $\widehat{\mathscr{L}} = \mathscr{L}_{\mathsf{eff}} + \mathscr{L}^\perp$, where $\mathscr{L}^\perp = \Pi^\perp\mathscr{L}\Pi^\perp$ on $\mathcal{H}$. Then we have $\sigma(\widehat{\mathscr{L}}) = \sigma(\mathscr{L}_{\mathsf{eff}}) \cup \sigma(\mathscr{L}^\perp)$ and for $z \in \rho(\widehat{\mathscr{L}})$ such that $z \notin [\gamma,+\infty[$ we have

$$\left\|(\widehat{\mathscr{L}} - z)^{-1} - (\mathscr{L}_{\mathsf{eff}} - z)^{-1}\Pi\right\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{\mathsf{dist}(z,[\gamma,+\infty[)}.$$

Thus the main result is about the difference between $\mathscr{L}$ and $\widehat{\mathscr{L}}$. We consider $\eta_1(z)$, $\eta_2(z)$, $\eta_3(z)$ and $\eta_4(z)$ which are explicit functions of $z \in \mathbb{C}$, γ and $\|[S,\Pi]\|_{\mathcal{L}(\mathcal{H})}$ (see (1.5) in [16]). Then we have the following estimate.

Theorem 3.9. *Let $z \in \rho(\widehat{\mathscr{L}})$. Assume that*

$$1 - \eta_1(z)\|(\widehat{\mathscr{L}} - z)^{-1}\| - \eta_2(z) > 0.$$

Then $z \in \rho(\mathscr{L})$ and

$$\begin{aligned}&\|(\mathscr{L} - z)^{-1} - (\widehat{\mathscr{L}} - z)^{-1}\| \\ &\quad \leqslant \eta_1(z)\|(\mathscr{L} - z)^{-1}\|\|(\widehat{\mathscr{L}} - z)^{-1}\| + \eta_2(z)\|(\mathscr{L} - z)^{-1}\| + \eta_3(z)\|(\widehat{\mathscr{L}} - z)^{-1}\| + \eta_4(z).\end{aligned}$$

Combining these two results we get information on the resolvent of $\mathscr{L}$ from properties on the resolvent of $\mathscr{L}_{\mathsf{eff}}$.

The regret with Theorem 3.9 is that even if it is quite general, it does not include the non-selfadjoint setting of Theorem 3.8. However, the proofs follow the same ideas and use common intermediate results. It would be interesting to be able to prove an abstract theorem which also includes non-selfadjoint settings.

3.5 Observability for a Kolmogorov equation

In this paragraph we consider a question of control for a degenerate evolution equation. This is a joint work with Jérémi Dardé [22].

3.5.1 Setting and main result

We consider a two-dimensional domain Ω of the form $\mathbb{T} \times I$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $I =]-\ell_-, \ell_+[$ for some $\ell_\pm > 0$. A general point in Ω is denoted by (x, y) with $x \in \mathbb{T}$ and $y \in I$. Let $q \in C^3(\bar{I}, \mathbb{R})$ such that $q(0) = 0$ and $\min(q') > 0$. The model case is $q(y) = y$. Let $T > 0$. We consider a Kolmogorov equation of the form

$$\begin{cases} \partial_t u + q(y)^2 \partial_x u - \partial_{yy} u = 0, & \text{on }]0, T[\times\Omega, \\ u = 0, & \text{on }]0, T[\times\partial\Omega, \\ u_{|t=0} = u_0 \in L^2(\Omega). \end{cases} \tag{3.14}$$

We say that the Kolmogorov equation (3.14) is observable through $\partial\Omega$ in time $T > 0$ if there exists $C > 0$ such that for any $u_0 \in L^2(\Omega)$ the solution u of (3.14) satisfies

$$\|u(T)\|^2_{L^2(\Omega)} \leqslant C \int_0^T \|\partial_\nu u(t)\|^2_{L^2(\partial\Omega)} \, \mathrm{d}t. \tag{3.15}$$

We can replace $\partial\Omega$ by any non-empty open subset of $\partial\Omega$. We can also consider observability through a non-empty open subset ω of Ω. In this case, we replace (3.15) by

$$\|u(T)\|^2_{L^2(\Omega)} \leqslant C \int_0^T \|u(t)\|^2_{L^2(\omega)} \, \mathrm{d}t. \tag{3.16}$$

The critical time $\mathcal{T}_c \in [0, +\infty]$ for the observability of (3.14) through $\partial\Omega$ is the infimum of the times $T > 0$ for which we have observability. In particular, (3.14) is not observable in time $T > 0$ through $\partial\Omega$ if $T < \mathcal{T}_c$. Moreover, since (3.14) is dissipative (the norm of $u(t)$ is a non-increasing function of time), (3.14) is observable in any time $T > \mathcal{T}_c$.

The main motivation is the null-controllability by the boundary for the adjoint problem. Given $f \in L^2(0, T; L^2(\partial\Omega))$ we consider the problem

$$\begin{cases} \partial_t u - q(y)^2 \partial_x u - \partial_{yy} u = 0, & \text{on }]0, T[\times\Omega, \\ u = f, & \text{on }]0, T[\times\partial\Omega, \\ u_{|t=0} = u_0 \in L^2(\Omega). \end{cases} \tag{3.17}$$

Then we say that the Kolmogorov equation (3.17) is null-controllable by $\partial\Omega$ in time $T > 0$ if for any $u_0 \in L^2(\Omega)$ there exists $f \in L^2(0,T;L^2(\partial\Omega))$ such that the solution u satisfies $u(T) = 0$. It is classical in control theory (see for instance [Cor07, Th. 2.44]) that the null-controllability for (3.17) is equivalent to the observability for (3.14).

This problem is similar to the better understood Grushin equation (see Section 4.2 below). For the Kolmogorov equation, it is proved in [Bea14] that (3.14) is observable through any open subset of Ω if $q(y)^2$ is replaced by y, and that there is a critical time $\mathcal{T}_c \geqslant \frac{a^2}{2}$ for the observability when $q(y)^2 = y^2$ and $\omega = \mathbb{T} \times [a,b]$ for some $0 < a < b < \ell_+$. With the same ω, it is also proved in [BHHR15] that (3.14) cannot be observable in any time for $q(y)^2$ replaced by y^n with $n \geqslant 3$. There are also many results about parabolic equations generated by ge neral quadratic operators, including the usual Kolmogorov equation (see for instance [BPS18] and references therein).

Our purpose in [22] is to prove for the Kolmogorov equation a result similar to the result of [BDE20] about the Grushin equation. The main result is the following.

Theorem 3.10. *Let*

$$T_- = \frac{1}{q'(0)} \int_{-\ell_-}^{0} |q(s)| \, \mathrm{d}s \quad \textit{and} \quad T_+ = \frac{1}{q'(0)} \int_{0}^{\ell_+} q(s) \, \mathrm{d}s.$$

There exists

$$\mathcal{T}_c \in \big[\min(T_-, T_+), \max(T_-, T_+) \big]$$

such that the Kolmogorov equation (3.14) *is observable through* $\partial\Omega$ *in any time* $T > \mathcal{T}_c$*, but not in time* $T < \mathcal{T}_c$*.*

We observe that we get exactly the critical time $\mathcal{T}_c$ in any configuration for which $T_- = T_+$. This is in particular the case in a symmetric setting ($\ell_- = \ell_+$ and q is odd).

Since the coefficients in (3.14) do not depend on x, we can consider the Fourier series of the solutions with respect to x. Setting

$$u(t,x,y) = \sum_{n \in \mathbb{Z}} u_n(t,y) e^{inx}, \quad u_n(t,y) = \frac{1}{2\pi} \int_{\mathbb{T}} e^{-inx} u(t,x,y) \, \mathrm{d}x,$$

the Fourier coefficients $u_n(t,y)$ are solutions of

$$\begin{cases} \partial_t u_n - \partial_{yy} u_n + inq(y)^2 u_n = 0, & \text{on }]0,T[\times I, \\ u_n(t,-\ell_-) = u_n(t,\ell_+) = 0, & \text{for } t \in]0,T[, \\ u_n(0) \in L^2(I). \end{cases} \tag{3.18}$$

Then (3.15) holds if and only if we have observability for u_n uniformly in $n \in \mathbb{Z}$. This means that there exists $C > 0$ such that for $n \in \mathbb{N}$ (the case n negative follows by complex conjugation of (3.18)) and a solution u_n of (3.18) we have

$$\|u_n(T)\|^2_{L^2(I)} \leqslant C \int_0^T \big(|\partial_y u_n(t,-\ell_-)|^2 + |\partial_y u_n(t,\ell_+)|^2 \big) \, \mathrm{d}t. \tag{3.19}$$

3.5.2 Spectral properties of the Kolmogorov operator

The analysis of (3.18) depends on the spectral properties of the non-selfadjoint operator

$$K_n = -\partial_{yy} + inq(y)^2, \quad \text{with domain } \mathsf{Dom}(K_n) = H^2(I) \cap H^1_0(I) \subset L^2(I).$$

In particular we use the decay of the corresponding semigroup.

Proposition 3.11. *Let*

$$\gamma < \frac{q'(0)}{\sqrt{2}}.$$

There exists $C > 0$ such that for $n \in \mathbb{N}$, a solution u_n of (3.18) and $\theta_1, \theta_2 \in [0, T]$ with $\theta_1 \leqslant \theta_2$, one has

$$\|u_n(\theta_2)\|^2_{L^2(I)} \leqslant Ce^{-2\gamma\sqrt{n}(\theta_2-\theta_1)} \|u_n(\theta_1)\|^2_{L^2(I)}.$$

We notice that the semigroup e^{-tK_n} is regularizing, even if it is not as small as the propagator of the heat equation for high frequencies.

We also give spectral properties for the operator $K = -\partial_y^2 + q(y)^2\partial_x$ on Ω, and deduce that (3.14) is well posed, dissipative and regularizing (in particular the normal trace in (3.15) makes sense).

Near $y = 0$ (where the problem is degenerate) the coefficient $q(y)^2$ looks like $q'(0)^2y^2$. To prove Proposition 3.11, we compare K_n with the Davies operator H_n on $L^2(\mathbb{R})$:

$$H_n = -\partial_y^2 + inq'(0)^2y^2, \quad \mathsf{Dom}(H_n) = \{u \in H^2(\mathbb{R}) \,:\, y^2u \in L^2(\mathbb{R})\}.$$

The basic spectral properties of H_n are well known (see for instance [Hel13, §1.3]). The spectrum of H_n is given by a sequence $(\lambda_{n,k})_{k\in\mathbb{N}^*}$ of (geometrically and algebraically) simple eigenvalues given by $\lambda_{n,k} = (2k-1)\sqrt{n}q'(0)e^{\frac{i\pi}{4}}$. As is often the case for a non-selfadjoint operator, the size of the resolvent of H_n is far from being controled by the distance between the spectral parameter and the spectrum (see [PS06, HSV13, KSTV15]). However, given $\gamma < \frac{q'(0)}{\sqrt{2}}$ there exists $c > 0$ such that

$$\sup_{\mathsf{Re}(z)\leqslant\gamma\sqrt{n}} \left\|(H_n - z)^{-1}\right\|_{\mathcal{L}(L^2(\mathbb{R}))} \leqslant \frac{c}{\sqrt{n}}. \tag{3.20}$$

To recover similar properties for K_n, we compare the resolvents of K_n and H_n. More precisely, with the natural restriction operator $\mathbb{1}_I : L^2(\mathbb{R}) \to L^2(I)$ we have the following estimate.

Proposition 3.12. *Let $\gamma < \frac{q'(0)}{2}$. Then for n large enough the eigenvalues of K_n have real parts greater that $\gamma\sqrt{n}$. Moreover,*

$$\sup_{\mathsf{Re}(z)\leqslant\gamma\sqrt{n}} \left\|(K_n - z)^{-1} - \mathbb{1}_I(H_n - z)^{-1}\mathbb{1}_I^*\right\|_{\mathcal{L}(L^2(I))} = \underset{n\to+\infty}{o}\left(\frac{1}{\sqrt{n}}\right).$$

We deduce Proposition 3.11 from Proposition 3.12 and the standard semigroup theory (see for instance [EN00, Th. V.1.11]).

Another consequence of Proposition 3.12 is that the spectrum of K_n is in some sense "close" to the spectrum of H_n. More precisely, we are interested in the "smallest" eigenvalue. By perturbation theory, we get the following result.

Proposition 3.13. *There exists a sequence $(\lambda_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that λ_n is an eigenvalue of K_n for all n and*

$$\lambda_n = \sqrt{n}q'(0)e^{\frac{i\pi}{4}} + \underset{n\to+\infty}{o}(\sqrt{n}). \tag{3.21}$$

3.5.3 Positive result: observability in large time

To prove that we have observability in time $T > \max(T_-, T_+)$ we proceed in two steps. The first step consists in proving that (3.19) actually holds in arbitrarily small time. But not uniformly in n. For large n we have the following estimate.

Proposition 3.14. *Let $\tau_1, \tau_2 \in]0, T]$ with $\tau_1 < \tau_2$ and*

$$\kappa > \max\left(\frac{1}{\sqrt{2}}\int_0^{\ell_+} q(s)\,\mathrm{d}s, \frac{1}{\sqrt{2}}\int_{-\ell_-}^0 |q(s)|\,\mathrm{d}s\right) = \frac{q'(0)}{\sqrt{2}}\max(T_-, T_+).$$

There exist $n_0 \in \mathbb{N}$ and $C > 0$ such that for $n \geqslant n_0$ and a solution u_n of (3.18) we have

$$\|u_n(\tau_2)\|^2_{L^2(I)} \leqslant Ce^{2\kappa\sqrt{n}}\int_{\tau_1}^{\tau_2}\left(|\partial_y u_n(t, -\ell_-)|^2 + |\partial_y u_n(t, \ell_+)|^2\right)\mathrm{d}t.$$

The estimate is not uniform for high frequencies, but we control the dependence in n. Then, the second step consists in... waiting. We see from Proposition 3.11 that for large times the contribution of high frequencies is actually small. The time $\max(T_-, T_+)$ is precisely how long we have to wait to ensure that the smallness in Proposition 3.11 (applied with $\theta_2 = T$ and $\theta_1 = \tau_2$) compensates the bad estimate of Proposition 3.14 (applied with τ_2 small enough).

Proposition 3.14 can be seen as a quantified version of a unique continuation result (in particular, if u is a solution of (3.18) with $\partial_\nu u = 0$ on $[0, T] \times \partial I$ then $u(t) = 0$ for all $t \in [0, T]$). It is usual to use a Carleman estimate for this kind of result.

Ideas of proof. Given $n \in \mathbb{N}$ and a solution u of (3.14) (we omit the index n) we set $w = e^{-\phi}u$ for some $\phi \geqslant 0$. We prove estimates on w and then deduce estimates on u (then ϕ should be chosen as small as possible).

If ϕ is large near $t = \tau_1$ and $t = \tau_2$, then w satisfies the assumptions of the following proposition with $[a, b] = [-\ell_-, \ell_+]$ and $g = 0$.

Proposition 3.15. *Let $n \in \mathbb{N}$, $\tau_1, \tau_2 > 0$ with $\tau_1 < \tau_2$, $a, b \in \mathbb{R}$ with $a < b$, and $g \in L^2(]\tau_1, \tau_2[\times]a, b[)$. Let $\phi \in C^4(]\tau_1, \tau_2[\times[a, b], \mathbb{R}_+)$. We consider $w \in C^0([\tau_1, \tau_2], H^2(a, b)) \cap C^1([\tau_1, \tau_2], L^2(a, b))$ such that*

$$\partial_t w - \partial_y^2 w + inq(y)^2 w + \partial_t\phi\, w - 2\partial_t\phi\,\partial_t w - (\partial_y\phi)^2 w - \partial_y^2\phi\, w = g. \tag{3.22}$$

We assume that w also satisfies the Dirichlet boundary condition $w(t, a) = w(t, b) = 0$ for $t \in]\tau_1, \tau_2[$, and the initial and final conditions $w(\tau_1, y) = w(\tau_2, y) = 0$ and $\partial_y w(\tau_1, y) = \partial_y w(\tau_2, y) = 0$ for $y \in]a, b[$. Then we have

$$\int_{\tau_1}^{\tau_2}\int_a^b \left(\Phi_0\,|w|^2 + \Phi_1\,|\partial_y w|^2\right)\mathrm{d}y\,\mathrm{d}t \leqslant -\int_{\tau_1}^{\tau_2}\left[\partial_y\phi\,|\partial_y w|^2\right]_a^b\mathrm{d}t + \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_a^b |g|^2\,\mathrm{d}y\,\mathrm{d}t, \tag{3.23}$$

where

$$\Phi_0 = -2(\partial_y\phi)^2\partial_y^2\phi - \frac{\partial_t^2\phi}{2} + \frac{\partial_y^4\phi}{2} + 2\partial_t\partial_y\phi\,\partial_y\phi - \frac{n^{\frac{3}{2}}q^2q'}{\sqrt{2}}, \qquad \Phi_1 = -2\partial_y^2\phi - \sqrt{2n}q'. \tag{3.24}$$

With $g = 0$ we can deduce an observability estimate for w if Φ_0 and Φ_1 are positive. Thus the purpose is to construct $\phi \geqslant 0$ as small as possible and such that Φ_0 and Φ_1 are bounded away from 0. We construct ϕ in such a way that the first terms in Φ_0 and Φ_1 are positive. Because of the last terms in the expressions of Φ_0 and Φ_1, ϕ should be at least of size $\sqrt{n}$. We construct ϕ of the form $\phi(t, y) = \sqrt{n}\theta(t)\psi(y)$. The second to fourth terms in Φ_0 will be smaller for large n.

The function θ is chosen in such a way that $\theta \geqslant 1$ (otherwise the first term in each expression would be too small), θ goes to $+\infty$ near τ_1 and τ_2 (for the boundary conditions in Proposition 3.15) and $\theta = 1$ on $\left[\frac{2\tau_1+\tau_2}{3}, \frac{\tau_1+2\tau_2}{3}\right]$ (see (3.25) below). Then it remains to define ψ to have, for some $\varepsilon > 0$,

$$\psi \geqslant \varepsilon, \quad -2(\psi')^2\psi'' - \frac{q^2q'}{\sqrt{2}} \geqslant \varepsilon, \quad -2\psi'' - \sqrt{2}q' \geqslant \varepsilon.$$

We see that ψ'' has to be negative. Moreover, $|\psi'|$ has to be bounded away from 0, but the sign of ψ' is not important. We also see from (3.23) that if ψ' is positive it is enough to observe on the left, and that it is enough to observe from the right if ψ' is negative. We could indeed observe from one side only, but it would be expansive to go through 0. Since we observe from both sides, we proceed as follows. Given $\varepsilon > 0$ and then $\delta > 0$ small enough, we apply Proposition 3.15 with $\psi' > 0$ on $[-\ell_-, \delta]$ and with $\psi' < 0$ on $[-\delta, \ell_+]$. More precisely, for $y \in [-\delta, \ell_+]$ we set

$$\psi_+(y) = \left(\frac{1}{\sqrt{2}} + \varepsilon\right) \int_y^{\ell_+} \big(q(s) + \varepsilon\big)\, \mathrm{d}s + c_+,$$

and for $y \in [-\ell_-, \delta]$

$$\psi_-(y) = \left(\frac{1}{\sqrt{2}} + \varepsilon\right) \int_{\ell_-}^{y} \big(-q(s) + \varepsilon\big)\, \mathrm{d}s + c_-.$$

The constants $\varepsilon > 0$ and $c_\pm \geqslant 0$ are chosen in such a way that $\psi_-(0) = \psi_+(0) \leqslant \kappa$. Since w does not vanish at $\pm\delta$, we have to use cut-off functions to apply Proposition 3.15 on $[-\ell_-, \delta]$ and $[-\delta, \ell_+]$. Then g is no longer 0 in (3.23), but the corresponding term on the right can be absorbed by the left-hand side. Gluing the two estimates, we get for u

$$\int_{\tau_1}^{\tau_2} \int_I \big(n^{\frac{3}{2}} |u|^2 + \sqrt{n}\, |u_y|^2\big) e^{-2\sqrt{n}\varphi}\, \mathrm{d}y\, \mathrm{d}t \lesssim \sqrt{n} \int_{\tau_1}^{\tau_2} \big(|u_y(t, -\ell_-)|^2 + |u_y(t, \ell_+)|^2\big)\, \mathrm{d}t.$$

What we need is an estimate on $u(\tau_2)$. However, since $u(\tau_2)$ is smaller that $u(s)$ for any $s \in [\tau_1, \tau_2]$, we have

$$\|u(\tau_2)\|^2_{L^2(I)} \leqslant \frac{3e^{\kappa\sqrt{n}}}{\tau_2 - \tau_1} \int_{\frac{2\tau_1+\tau_2}{3}}^{\frac{\tau_1+2\tau_2}{3}} \left\|e^{-\sqrt{n}\varphi} u(s)\right\|^2_{L^2(I)}\, \mathrm{d}s. \tag{3.25}$$

We only use the estimate for $s \in \left[\frac{2\tau_1+\tau_2}{3}, \frac{\tau_1+2\tau_2}{3}\right]$ since otherwise θ (and hence ϕ) can be large. Proposition 3.14 follows. □

3.5.4 Negative result: non-observability in small time

In this paragraph, we discuss the non-observability for (3.14) when $T < \min(T_-, T_+)$. For this, we construct a sequence of solutions which contradicts (3.15). We need a solution of (3.14) which is small at the boundary. Since the (imaginary) potential $nq(y)^2$ is large away from 0 when n is large, we use the Agmon estimates described in Section 3.2. Following [13], we prove the following estimate.

Proposition 3.16. *Let $E > 0$ and $\varepsilon \in]0, 1[$. For $n \in \mathbb{N}$ and $y \in \bar{I}$ we set*

$$W_{n,\varepsilon}(y) = \frac{1-\varepsilon}{\sqrt{2}} \left| \int_0^y \sqrt{\big(nq(s)^2 - \sqrt{n}(E+\varepsilon)\big)_+}\, \mathrm{d}s \right|. \tag{3.26}$$

There exists $C > 0$ such that for $n \in \mathbb{N}$, $u \in \mathsf{Dom}(K_n)$ and $\lambda \in \mathbb{C}$ with $|\mathsf{Re}(\lambda)| + |\mathsf{Im}(\lambda)| \leqslant E\sqrt{n}$, we have

$$\left\|e^{W_{n,\varepsilon}} u'\right\|^2_{L^2(I)} + \sqrt{n} \left\|e^{W_{n,\varepsilon}} u\right\|^2_{L^2(I)} \leqslant C\sqrt{n}\, \|u\|^2_{L^2(I)} + \frac{C}{\sqrt{n}} \left\|e^{W_{n,\varepsilon}} (K_n - \lambda) u\right\|^2_{L^2(I)}.$$

For large n we see that $W_{n,\varepsilon}(y)$ is close to $\sqrt{n}\kappa_\varepsilon(y)$, where

$$\kappa_\varepsilon(y) = \frac{(1-\varepsilon)}{\sqrt{2}} \int_0^y q(s)\, \mathrm{d}s.$$

We consider the eigenvalue λ_n of K_n given by Proposition 3.13 and a corresponding normalized eigenfunction ψ_n. Since $\psi_n'' = -\lambda_n \psi_n + inq(y)^2$, we can deduce the following estimate.

Proposition 3.17. *There exists $C > 0$ such that for $n \in \mathbb{N}^*$ we have*

$$\left\| e^{\sqrt{n}\kappa_\varepsilon} \psi_n' \right\|^2_{L^\infty(I)} \leqslant Cn \left\| \psi_n \right\|^2_{L^2(I)}.$$

Finally, we set $u_m(t,x,y) = e^{-\lambda_m t} e^{imx} \psi_m(y)$, and we get a sequence of solutions of (3.14) for which (3.15) cannot hold if $T < \min(T_-, T_+)$.

3.6 Nonlinear evolution equations with singular potentials

3.6.1 A singularly perturbed Gross-Pitaevskii equation

Most of my works concern spectral theory and linear PDEs. I have also developed with Stefan Le Coz a collaboration around some non-linear problems. The starting point was a question of Stefan about the spectral properties of the linearized operator which appeared in one of his problems. We discussed more and more about it and, one thing leading to another, I was aware of the whole problem and I finally joined the project.

The purpose of [12] (with Isabella Ianni and Stefan Le Coz) is the analysis of the *black solitons* of the Gross-Pitaevskii equation with Dirac potential

$$-i\partial_t u - \partial_x^2 u + \gamma \delta u = (1 - |u|^2) u, \tag{3.27}$$

subject to the boundary condition

$$|u(t,x)| \xrightarrow[|x| \to \infty]{} 1, \quad \forall t \in \mathbb{R}. \tag{3.28}$$

The difficulty is then twofold. Technical problems arise from the presence of the singular potential and the non standard condition at infinity. For instance, one of the difficulties due to (3.28) is that we have to work in a functional space which is not even a vector space:

$$\mathcal{E} = \left\{ u \in H^1_{\mathsf{loc}}(\mathbb{R}) \,:\, \partial_x u \in L^2(\mathbb{R}), (1 - |u|^2) \in L^2(\mathbb{R}) \right\}.$$

It can however be endowed with a complete metric structure with the distance

$$d_0(u,v) = \left\| \partial_x u - \partial_x v \right\|_{L^2(\mathbb{R})} + \left\| \, |u|^2 - |v|^2 \, \right\|_{L^2(\mathbb{R})} + |u(0) - v(0)|.$$

This work has been an occasion to get familiar with questions and technics of non-linear analysis. The question here is the stability of the stationary solutions (or black solitons), solutions of

$$-\partial_x^2 u + \gamma \delta u = (1 - |u|^2) u.$$

The solutions can be explicited. Up to phase shift, they are given by

$$\kappa(x) = \tanh\left(\frac{x}{\sqrt{2}}\right), \quad b_\gamma(x) = \tanh\left(\frac{|x| - c_\gamma}{\sqrt{2}}\right), \quad \tilde{b}_\gamma(x) = \coth\left(\frac{|x| + c_\gamma}{\sqrt{2}}\right),$$

where c_γ is an explicit parameter ($\tilde{b}_\gamma$ is only defined for $\gamma < 0$). We get a first result of stability by minimization of the energy

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} |\partial_x u|^2 \, \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}} \left(1 - |u|^2\right)^2 \mathrm{d}x + \frac{\gamma}{2} |u(0)|^2. \tag{3.29}$$

Theorem 3.18. *Let $\gamma \in \mathbb{R}^*$.*

(i) *The energy E reaches its (finite) minimum on*

$$\mathcal{G}_\gamma = \begin{cases} \{e^{i\theta} b_\gamma, \theta \in \mathbb{R}\} & \text{if } \gamma > 0, \\ \{e^{i\theta} \tilde{b}_\gamma, \theta \in \mathbb{R}\} & \text{if } \gamma < 0. \end{cases}$$

(ii) *The set $\mathcal{G}_\gamma$ is stable for the flow of* (3.27)*:*

$$\forall \varepsilon > 0, \exists \delta > 0, \forall u_0 \in \mathcal{E}, \quad d_0(u_0, \mathcal{G}_\gamma) < \delta \quad \Longrightarrow \quad \sup_{t\in\mathbb{R}} d_0(u(t), \mathcal{G}_\gamma) < \varepsilon.$$

The other stationary solutions are expected to be unstable. In [12] we prove the linear instability of κ. If we write $u = \kappa + \eta$, then the perturbation η satisfies an equation of the form $\partial_t \eta + L\eta + N(\eta) = 0$ where $N(\eta)$ is non-linear in η and

$$L\eta = -i\big(\partial_x^2 \eta - \gamma\delta\eta + (1-\kappa^2)\eta - 2\kappa^2 \mathsf{Re}(\eta)\big).$$

Then κ is said to be linearly unstable if 0 is an unstable solution of the linear equation $\partial_t \eta + L\eta = 0$.

Theorem 3.19. *If $\gamma > 0$ then κ is linearly unstable.*

The proof relies on the spectral analysis of the $\mathbb{R}$-linear operator L. We can rewrite L as a $\mathbb{R}$-linear matrix operator acting on $(\mathsf{Re}(\eta), \mathsf{Im}(\eta))$:

$$\mathcal{L} = \begin{pmatrix} 0 & -L_-^\gamma \\ L_+^\gamma & 0 \end{pmatrix}, \quad L_-^\gamma = -\partial_x^2 - (1-\kappa^2), \quad L_+^\gamma = -\partial_x^2 + 2 - 3(1-\kappa^2).$$

The operators L_-^γ and L_+^γ are defined on the same domain

$$\mathcal{D}_\gamma = \left\{ u \in H^2(\mathbb{R}^*) \cap H^1(\mathbb{R}) \,:\, u'(0^+) - u'(0^-) = \gamma u(0) \right\}. \tag{3.30}$$

Then a careful spectral analysis of L_-^γ and L_+^γ shows that L has a negative eigenvalue whenever $\gamma > 0$.

Unexpectedly, the main difficulty and hence the main part in [12] turned out to be the analysis of the Cauchy problem (see already [Gér06] without singular potential). Before all the discussion above, we check that the problem (3.27) is globally well posed and that the energy E is conserved. The first step is the analysis of the propagator of the linear part, generated by the operator associated with the quadratic form $u \mapsto \|u'\|^2_{L^2(\mathbb{R})} + \gamma\,|u(0)|^2$. It is given by

$$H_\gamma = -\partial_x^2, \quad \mathsf{Dom}(H_\gamma) = \mathcal{D}_\gamma.$$

The difficulty is to deal with this not so usual operator in the completely unusual space $\mathcal{E}$. By Fourier transform we can see that $e^{-itH_0} - \mathsf{Id}$ maps $\mathcal{E}$ to $H^1(\mathbb{R})$. Since $\mathcal{E} + H^1(\mathbb{R}) = \mathcal{E}$, we deduce that e^{-itH_0} maps $\mathcal{E}$ into itself.

We cannot use the Fourier transform for e^{-itH_γ} when $\gamma \neq 0$. Instead, we compute explicitely the kernel of $\Gamma(t) = e^{-itH_\gamma} - e^{-itH_0}$. For instance, for $\gamma < 0$ it is given by

$$\Gamma(t; x, y) = -\frac{|\gamma|}{2} \int_0^{+\infty} e^{-\frac{|\gamma| s}{2}} K_0(t, s - |x| - |y|)\, \mathrm{d}s + \frac{|\gamma|}{2} e^{\frac{i\gamma^2 t}{4}} e^{-\frac{|\gamma|(|x|+|y|)}{2}},$$

where $K_0(t, \zeta)$ is the kernel of the free Schrödinger equation. By tedious computation, we can show in particular that $\Gamma(t)$ maps $\mathcal{E}$ to $H^1(\mathbb{R})$ (and $\|\Gamma(t)u\|_{H^1(\mathbb{R})}$ is controled by $\|\partial_x u\|_{L^2(\mathbb{R})} + |u(0)|$).

Conservation of the energy is also a non-trivial issue. Finally, the analysis of the Cauchy problem follows the standard strategy but, again, this standard strategy had to be adapted to our non standard setting. We finally have a well-posedness result.

Theorem 3.20. *For $u_0 \in \mathscr{E}$ there exists a unique solution $u \in C^0(\mathbb{R}, \mathscr{E})$ of* (3.27) *such that $u(0) = u_0$. Moreover $E(u(t)) = E(u_0)$ for all $t \in \mathbb{R}$.*

Here, a solution is in the sense given by the Duhamel formula. For a more regular u_0 we have a solution in a stronger sense (in particular with $\partial_t u(t) \in C^0(\mathbb{R}, L^2(\mathbb{R}))$). We also have a result of continuity with respect to the initial condition, but for a slightly different distance

on $\mathcal{E}$.

Notice that a problem on $\mathbb{R}$ with singular potential can be seen has a problem on a graph with two edges of infinite length. The jump condition which appears in (3.30) is then the analog of the Robin condition at the unique vertex (see (3.3)).

3.6.2 A nonlinear Klein-Gordon equation with delta potentials

After [12], we started to discuss with François Genoud and Elek Csobo (who was at that time Ph.D. student under the supervision of F. Genoud and S. Le Coz) another problem of stability for *standing waves*, suggested by a question of Masahito Ohta.

We now consider a non-linear Klein-Gordon equation, again on the real line with a singular potential. We also consider a singular term involving $\partial_t u$. This looks like a singular damping, but with a purely imaginary (conservative) coefficient. More precisely, given $m > 0$ and $\gamma, \alpha \in \mathbb{R}$, the problem considered in [19] is

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + m^2 u + \gamma\delta u + i\alpha\delta\partial_t u - |u|^{p-1} u = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}, \\ u(t,x) \xrightarrow[|x|\to\infty]{} 0, & \forall t > 0, \\ (u, \partial_t u)_{|t=0} = (f, g) \in H^1(\mathbb{R}) \times L^2(\mathbb{R}). \end{cases} \tag{3.31}$$

The first step in the analysis of (3.31) is again the Cauchy problem. Here there is a subtility due to the term $\gamma\delta u$, which is not usual for a wave equation. The operator corresponding to the linear part is

$$\mathcal{W} = \begin{pmatrix} 0 & 1 \\ \partial_x^2 - m^2 & 0 \end{pmatrix},$$

with domain

$$\mathsf{Dom}(\mathcal{W}) = \left\{(u,v) \in \left(H^2(\mathbb{R}^*) \cap H^1(\mathbb{R})\right) \times H^1(\mathbb{R}) \,:\, u'(0^+) - u'(0^-) = \gamma u(0) + i\alpha v(0)\right\}.$$

This defines a skew-adjoint operator on the space $\mathscr{H} = H^1(\mathbb{R}) \times L^2(\mathbb{R})$, endowed with the norm defined by

$$\|(u,v)\|^2_{\mathscr{H},m,\gamma} = \|u'\|^2_{L^2(\mathbb{R})} + m^2 \|u\|^2_{L^2(\mathbb{R})} + \gamma |u(0)|^2 + \|v\|^2_{L^2(\mathbb{R})}.$$

The problem is that if γ is too negative, this norm is not a norm. In this case, we replace m by some $\mu \geqslant m$ large enough. We loose the skew-adjointness, but $\mathcal{W}$ still generates on $(\mathscr{H}, \|\cdot\|_{\mathscr{H},\mu,\gamma})$ a strongly continuous semigroup. Then we can proceed with the local well-posedness for (3.31). Moreover, the following energy is a constant of the motion for $(u,v) = (u, \partial_t u)$:

$$E(u,v) = \frac{1}{2}\|\partial_x u\|^2_{L^2(\mathbb{R})} + \frac{m^2}{2}\|u\|^2_{L^2(\mathbb{R})} + \frac{1}{2}\|v\|^2_{L^2(\mathbb{R})} + \frac{\gamma}{2}|u(0)|^2 - \frac{1}{p+1}\int_{\mathbb{R}} |u|^{p+1}\,\mathrm{d}x.$$

The charge is another constant of the motion:

$$Q(u,v) = \mathsf{Im}\int_{\mathbb{R}} u\bar{v}\,\mathrm{d}x - \frac{\alpha}{2}|u(0)|^2.$$

The main issue of [19] is then the question of orbital stability for the standing waves. For this we follow the general theory of Grillakis, Shatah and Strauss [GSS87].

A standing wave is a solution of (3.31) of the form $e^{i\omega t}\Phi_\omega(x)$, where $\Phi_\omega = (\varphi_\omega, i\omega\varphi_\omega)$ is a solution of the stationary equation

$$E'(\Phi_\omega) + \omega Q'(\Phi_\omega) = 0.$$

Non-trivial solutions exist if and only if $m^2 - \omega^2 > (\gamma - \alpha\omega)^2/4$. A standing wave $e^{i\omega t}\Phi_\omega(t)$ is said to be orbitally stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the following property holds for any solution $U(t)$ of (3.31):

$$\|U(0) - \Phi_\omega\|_{\mathscr{H}} \leqslant \delta \Longrightarrow \sup_t \inf_{\theta\in\mathbb{R}} \left\|U(t) - e^{i\theta}\Phi_\omega\right\|_{\mathscr{H}} \leqslant \varepsilon.$$

This property implies in particular that a solution starting close to Φ_ω is defined for all $t \in \mathbb{R}$.

The method is based on the symplectic structure of the problem (see [DBGRN15]). We can rewrite (3.31) in the form

$$\mathcal{J}U'(t) = E'(U(t)), \quad U(t) = (u(t), \partial_t u(t)),$$

where $\mathcal{J}(u, v) \in \mathscr{H} \mapsto (-i\alpha\delta v, u) \in \mathscr{H}^*$ is skew-symmetric.

Then the criterion for orbital stability of the standing wave $e^{it\omega_0}\Phi_{\omega_0}$ depends on two quantities. The first is the sign of

$$\left.\frac{\mathrm{d}Q(\Phi_\omega)}{\mathrm{d}\omega}\right|_{\omega=\omega_0}.$$

Since Φ_ω is explicitely known, this can be analysed by tedious but straightforward computations. To conclude to orbital stability or instability (we also discuss linear stability and instability), we also need some spectral properties for the operator $\tilde{R}^{-1}L''_\omega(\Phi_\omega)$, where $\tilde{R} : \mathscr{H} \to \mathscr{H}^*$ is defined by $(-\partial_x^2 + 1, \mathsf{Id}_{L^2})$ and $L_\omega = E + \omega Q$. I do not detail the assumptions here, but it concerns the positiveness of essential spectrum and the number of negative eigenvalues.

This is the main part of the paper [19]. However, all these properties depend on the parameters γ, α and ω (the strength p of the non-linearity also plays a role), so the results of stability and instability have to be splitted into many different cases and I choose not to give the precise statements here. See Section 4 in [19].

Chapter 4

Projects and perspectives

After having discussed my past research in the previous chapters, the remaining part of this thesis is devoted to the present and the future. In this last chapter I present unpublished (submitted) results, works in progress and some longer term perspectives.

Some perspectives discussed here are parts of ongoing projects and can be continuations of results already mentioned above, but this thesis is also an occasion to propose new directions of research.

4.1 New questions about local energy decay

We begin with the projects which are close to the main subject of this thesis, namely the local energy decay for wave or Schrödinger equations.

4.1.1 Asymptotic profile for the wave equation

In [23] (see the discussion in Section 1.7), it is proved that the solution of the Schrödinger equation (1.13) behaves for large times like a solution of the free problem (1.6), in the sense of Theorem 1.25. There are two natural continuations for this work.

These two questions, though quite different, are expected to be related to the same difficulty. This difficulty is in fact already present in [23]. The time decay in Theorem 1.25 depends on the parameter ρ_0 which measures the spatial decay of the coefficients in (1.10). This parameter is assumed to be not greater than 1. It is not the purpose of this report to go into technical details, but the reason is that the smallness given by the decay of the coefficients of the operators $\theta_\sigma(z)$ in (1.71) is limited (as always because of the restriction of the Sobolev indices in Proposition 1.19), and our proof would not give a better result with $\rho_0 > 1$. In [23] this is not a problem since we are mostly interested in the case $\rho_0 > 0$ small, but this becomes a more serious problem for the following two perspectives.

The main perspective in that direction is to prove an analog of Theorem 1.25 for the wave equation (1.12). The proof of Theorem 1.25 is written in such a way that it is robust with respect to dissipative perturbations, and adding a damping should not be a major problem. However, even for the undamped case, the asymptotic profile is not completely clear, and the problem comes from the gradient of the solution.

The method of [23] should work for the solution itself and its time derivative. We expect that they should look like a solution of the free wave equation (1.1). From (1.21) we can guess what should be the initial condition (f_0, g_0) for the asymptotic profile. However, there is an additional difficulty for the gradient of the solution. We have said above that we cannot gain as much as we wish with the inserted factors. Here, because of the gradient, the method does not give anything at all. There is something more to understand. In particular, we could be

in the same situation as for Theorem 2.19, where the asymptotic profile for the gradient is not the gradient of the asymptotic profile.

The motivation for having such a result for the wave equation is twofold. In even dimension, we already have the optimal decay in [BB21] for the undamped case (for the solution itself but not for its derivatives), so a result like Theorem 1.25 would give the leading term of the asymptotic expansion (which would be new in any case) and in particular the optimal decay (which is not known in the dissipative case). This would already be an important improvement.

In odd dimension, the situation is different. The proofs of resolvent estimates and local energy decay based on the Mourre method do not see the parity of the dimension, so the results in odd dimensions are actually limited. Comparing the solution of the perturbed problem to the solution of the model case (which decays much faster in odd dimensions) is a way to reintroduce the difference of parity in the method, which is necessary to improve the results in odd dimensions. A generalization of [23] to the wave equation would improve the estimates of [BB21] in odd dimensions, even for the undamped equation.

Another natural continuation of [23] is to give a more precise asymptotic expansion for the solution of the Schrödinger equation (1.13). In [23], we only give the first term of the expansion. It could be interesting to go further.

The leading term for the resolvent is simply given by the resolvent identity (1.70). We can apply the resolvent identity once more to get rests with two factors $\theta_\sigma(z)$. The same strategy should work to estimate these rests, and we can iterate as many times as we wish. In fact, the restriction in the proof of [23] is not about the number of terms in the asymptotic expansion, but about the size of the rest. As above, we could problably consider N factors $\theta_\sigma(z)$ in the rests, but we would not get an optimal result if $N\rho_0 \geqslant 1$. Going further would require a new improvement in the proof.

Finally, a related question is to consider the settings of Chapter 2. With damping at infinity we have already worked by comparison with a model problem, this is even what motivated [23] and the perspectives above. But I have never considered a general metric perturbation of the Laplacian. We have proved a result for a perturbed setting in [17] (see Proposition 2.20), but the assumptions on the perturbation are too strong. This is something I have not tried yet, but considering a long range perturbation of (2.1) would probably raise the same kind of difficulties as the long range perturbation of (1.12).

We could also consider all the intermediate situations (see the introduction of Chapter 2). In particular, not much is known and I am still curious about the critical case $a(x) \simeq C \langle x \rangle^{-1}$.

We can also consider problems with a damping going to infinity (see [FST18] for some basic properties of the wave operator in this case).

4.1.2 Systems on non-selfadjoint equations

I have recently started to consider questions of energy decay for a system of two equations. So far I have only considered model problems, but this kind of systems naturally appear in physics. For instance, there are already results about the energy decay for the Lamé system in [BL01, DDK10].

System of Schrödinger equations on a wave guide In 2019, I invited Radhia Ayechi and Ilhem Boukhris for two months in Toulouse. They were at that time Ph.D. students in Sousse with Moez Khenissi. I suggested to look together at a generalization of [8] (see Section 2.7). Then 2020 was a complicated time, but Ilhem came one more month. We finally finished the work at the end of 2021, which gave the preprint [24].

The setting in [24] is as in [8] a straight wave guide with one-dimensional cross section: $\Omega = \mathbb{R}^{d-1} \times]0, \ell[\subset \mathbb{R}^d$ for some $\ell > 0$. We consider on Ω a system of two Schrödinger equations with damping and coupling at the boundary. Here they are only effective on $\mathbb{R}^{d-1} \times \{0\}$, and

we consider a Neumann boundary condition on the other side (as in [8], various situations could be similarly considered). Given $a > 0$ (the absorption coefficient) and $b \in \mathbb{R}^*$ (the coupling coefficient), we consider the problem

$$\begin{cases} -i\partial_t u - \Delta u = 0, \\ -i\partial_t v - \Delta v = 0, \end{cases} \qquad \text{on } \mathbb{R}_+ \times \Omega, \tag{4.1}$$

with boundary conditions

$$\begin{cases} \partial_\nu u(t;x,0) = iau(t;x,0) + ibv(t;x,0), \\ \partial_\nu v(t;x,0) = -ibu(t;x,0), \\ \partial_\nu u(t;x,\ell) = \partial_\nu v(t;x,\ell) = 0, \end{cases} \qquad \forall t > 0, \forall x \in \mathbb{R}^{d-1}, \tag{4.2}$$

and with initial data

$$(u,v)_{|t=0} = (f,g) \in L^2(\Omega) \times L^2(\Omega). \tag{4.3}$$

Notice that only the first component u is dissipated at the boundary. However, the second component v is indirectly dissipated through the coupling terms. The masses of u and v separately have no reasons to be non-increasing, but it is the case for their sum:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)} \big) = -2a \int_{\mathbb{R}^{d-1}} |u(t;x,0)|^2 \, \mathrm{d}x \leqslant 0.$$

The question is the decay of this total mass. The answer is that even if neither the damping nor the coupling satisfies the geometric condition (the rays of light parallel to the boundary do not see the boundary), and even if the second component v is not dissipated, we have exponential decay for both components.

Theorem 4.1. *Let $a > 0$ and $b \in \mathbb{R}^*$. There exist $\gamma > 0$ and $C > 0$ such that for $(f,g) \in L^2(\Omega) \times L^2(\Omega)$ and $t > 0$ we have*

$$\|u(t)\|^2_{L^2(\Omega)} + \|v(t)\|^2_{L^2(\Omega)} \leqslant Ce^{-\gamma t}\big(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\Omega)} \big),$$

where (u,v) is the solution of (4.1)-(4.3).

As in [8], the analysis on the wave guide is related to the properties on $L^2(0,\ell) \times L^2(0,\ell) \simeq L^2(0,\ell;\mathbb{C}^2)$ of the transverse operator

$$\mathcal{T}_{a,b} = \begin{pmatrix} -\partial^2 & 0 \\ 0 & -\partial^2 \end{pmatrix},$$

defined on the domain

$$\big\{ U \in H^2(0,\ell;\mathbb{C}^2) \,:\, U'(0) + iM_{a,b}U(0) = 0, \quad U'(\ell) = 0 \big\}, \quad M_{a,b} = \begin{pmatrix} a & b \\ -b & 0 \end{pmatrix}.$$

We prove that we have a spectral gap for the eigenvalues of $\mathcal{T}_{a,b}$, and that we have a Riesz basis of $L^2(0,\ell;\mathbb{C}^2)$ made with corresponding (generalized) eigenfunctions.

In [8], the eigenvalues $(\lambda_n(0))$ of T_0 are simple, and a square root $z_n(a)$ of $\lambda_n(a)$ cannot cross the lines $\mathsf{Re}(z) \in \mathbb{N}\nu$ (remember that $\nu = \pi/\ell$), so by continuity it is possible to conclude that the eigenvalues of T_a remain simple with a rough localization.

Here it is no longer that simple. The model operator $\mathcal{T}_{0,0}$ has double eigenvalues $(n\nu)^2$, $n \in \mathbb{N}$. We can check that for (a,b) small the eigenvalue $(n\nu)^2$ splits into two simple eigenvalues if $a^2 \neq 4b^2$ and it stays a double eigenvalue if $a^2 = 4b^2$ (in this case we have a Jordan block if $a \neq 0$). Moreover, the square roots of the eigenvalues can cross the lines $\mathsf{Re}(z) \in \nu\mathbb{N}$ and even for large n the (generalized) eigenfunctions are not close to being orthogonal. However, despite of these additional difficulties, we can prove that we indeed have a spectral gap

and the Riesz basis property.

As a byproduct of this analysis, we also deduce the Weyl Law for the eigenvalues of $\mathcal{T}_{a,b}$. We can see $\mathcal{T}_{a,b}$ as a Schrödinger operator on a graph with two edges of same length ℓ, with a non-standard vertex condition between them. We denote by $\mathcal{N}_{a,b}(R)$ the number of eigenvalues of $\mathcal{T}_{a,b}$ (counted with multiplicities) with real part smaller than R. Then the following result is analogous to Theorem 3.2 with $|\Gamma| = 2\ell$:

$$\mathcal{N}_{a,b}(R) = \frac{2\ell\sqrt{R}}{\pi} + \underset{R\to+\infty}{O}(1). \tag{4.4}$$

Even if it was not necessary for the proof of Theorem 4.1 or for (4.4), we have continued the analysis of the localization of the eigenvalues of $\mathcal{T}_{a,b}$, and in particular the question of the number of square roots of eigenvalues in the region $\mathbb{C}_n = \{\mathsf{Re}(z) \in]n\nu, (n+1)\nu[\}$. We prove in particular the following observations.

- $(n\nu)^2$ is a double eigenvalue of $\mathcal{T}_{0,0}$. Its square root $n\nu$ splits into $z_+(a,b)$ and $z_-(a,b)$ such that $z_+(a,b)^2$ and $z_-(a,b)^2$ are eigenvalues of $\mathcal{T}_{a,b}$.
- One (say $z_+(a,b)$) always belongs to $\mathbb{C}_n$.
- If $a^2 > 4b^2 > 0$, then it is also the case for $z_-(a,b)$ and $z_-(a,b) \neq z_+(a,b)$. The eigenvalues of $\mathcal{T}_{a,b}$ are simple.
- If $a^2 = 4b^2$, then $z_+(a,b) = z_-(a,b)$ and the eigenvalues of $\mathcal{T}_{a,b}$ have algebraic multiplicities 2 (but geometric mutiplicities 1).
- If $a^2 < 4b^2$, then $z_-(a,b)$ can go to $\mathbb{C}_n$ or $\mathbb{C}_{n-1}$ depending on the value of $\theta \in \left]\frac{a}{2\nu}, +\infty\right[$ such that
$$4b^2 = a^2 + \frac{a^2}{4\theta^2\pi^2}\ln\left(\frac{2\theta\nu + a}{2\theta\nu - a}\right)^2.$$
Then $z_-(a,b)$ is in $\mathbb{C}_{n-1}$ if $\theta < n$, in $\mathbb{C}_n$ if $\theta > n$ and $\mathsf{Re}(z_-(a,b)) = n$ if $\theta = n$.

This kind of additional remarks is motivated by curiosity (this is already a good reason), but not only. For Theorem 4.1 it is enough to understand the eigenvalues $\lambda_n(a,b)$ for $n \gg a + |b|$, but this is not the case for the wave equation. Thus such a localization of *all* the eigenvalues is an interesting step toward the analysis of the wave equation.

A natural continuation of this work would be to consider a system of wave equations on a wave guide. We will discuss this issue in the Euclidean space in the next paragraph. But, as will be discussed in Section 4.4.2, we are also interested in the wave equation in general quantum graphs.

It would be interesting to prove rough localization of the eigenvalues on a general compact quantum graphs. To get a better intuition, we could begin by computing what happens for the same problem with N equations, or considering only 2 edges but with a general non-selfadjoint vertex. We will continue this discussion in Section 4.4.2.

System of wave equations In Chapter 2, we have discussed several problems about the wave equation with damping at infinity. Each situation (Euclidean space, wave guide, periodic setting) raised different and challenging difficulties. A new question, which turns out to be again different and challenging, is the case of a system of several equations with coupling.

With Lassaad Aloui and Moez Khenissi, we have started to discuss the simplest case, namely a system of two wave equations with constant coupling and damping on $\mathbb{R}^d$. More precisely, given $a > 0$ and $b \in \mathbb{R}^*$ as above, we consider on $\mathbb{R}^d$ the system

$$\begin{cases} \partial_t^2 u - \Delta u + m_1 u + b\partial_t v + a\partial_t u = 0, \\ \partial_t^2 v - \Delta v + m_2 v - b\partial_t u = 0, \end{cases} \tag{4.5}$$

with initial conditions $(u, \partial_t u)|_{t=0} = (f_0, f_1)$ and $(v, \partial_t v)|_{t=0} = (g_0, g_1)$ in $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. The coefficients m_1 and m_2 are non-negative. When $m_j = 0$ the corresponding equation is a wave equation, while $m_j > 0$ gives a Klein-Gordon equation.

Compared to the previous case, we have damping and coupling everywhere. However, as above, only the first component is dissipated and v can only be dissipated via the coupling.

We look at the decay of the energy

$$\begin{aligned} E(u, v; t) = &\|\partial_t u(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla u(t)\|^2_{L^2(\mathbb{R}^d)} + m_1 \|u(t)\|^2_{L^2(\mathbb{R}^d)} \\ &+ \|\partial_t v(t)\|^2_{L^2(\mathbb{R}^d)} + \|\nabla v(t)\|^2_{L^2(\mathbb{R}^d)} + m_2 \|v(t)\|^2_{L^2(\mathbb{R}^d)} . \end{aligned}$$

The first result is that the contributions of high frequencies for the global energy decay uniformly and exponentially, so both components are dissipated.

For low frequencies, the wave and Klein-Gordon equations have different properties, and the four situations ($m_1 = 0/m_1 \neq 0$ and $m_2 = 0/m_2 \neq 0$) will give four completely different behaviors.

When $m_1 > 0$ and $m_2 > 0$, there is no difficulty with low frequencies and the global energy of u and v decays exponentially. On the other hand, if $m_1 = 0$ or $m_2 = 0$ then 0 belongs to the spectrum of the corresponding operator

$$\mathcal{W} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \Delta - m_1 & -a & 0 & -b \\ 0 & 0 & 0 & 1 \\ 0 & b & \Delta - m_2 & 0 \end{pmatrix},$$

defined on the Hilbert completion $\mathcal{E}$ of $\mathcal{S}(\mathbb{R}^d)^4$ for the norm

$$\|(u_0, u_1, v_0, v_1)\|^2_{\mathcal{E}} = \|\nabla u_0\|^2_{L^2} + m_1 \|u_0\|^2_{L^2} + \|u_1\|^2_{L^2} + \|\nabla v_0\|^2_{L^2} + m_2 \|v_0\|^2_{L^2} + \|v_1\|^2_{L^2} .$$

We have to understand the contribution of low frequencies.

As in Section 2.2 we can use the Fourier transform. Let $M(\xi)$, $\xi \in \mathbb{R}^d$, be defined as $\mathcal{W}$ with $-\Delta$ replaced by $|\xi|^2$ (as in (2.7)). For a system Wave–Klein-Gordon ($m_1 = 0$, $m_2 > 0$), 0 is a simple eigenvalue of $M(0)$ and we recover a behavior similar to the case of a single equation. In particular, we can observe that u behaves like a solution of the heat equation. If $m_2 = 0$ then 0 is a double eigenvalue of $M(0)$. The behavior of the contribution of low frequencies is governed by the behavior of these two eigenvalues (counted with multiplicities) for $\xi \neq 0$ small (as well as the corresponding eigenprojections). And this strongly depends on the parameters m_1, m_1, a and b. We should get the asymptotic profile in each case, but with a much wilder behavior than with a single equation.

We have already four parameters in all the computations, but we could (should) consider another one. Everywhere, we have considered the speed of propagation c normalized to 1. This is harmless for a single equation, but for (4.5) this means that we consider the same speed for the two components u and v. It would be interesting to introduce the quotient of these two speeds and see how the results would depend on this new parameter.

The problem (4.5) is the simplest possible system of wave equations on an unbounded domain. It will then be natural to consider more evolved settings, as is done for a single equation, and systems of equations which appear in concrete problems.

4.1.3 Local energy decay in more general geometric settings

Most of my works about the damped wave equation are set on asymptotically Euclidean settings. I also considered the problem on a wave guide and on an asymptotically periodic setting, and this highlighted new interesting phenomena.

In [7] (see Section 1.5) we have proved resolvent estimates for the Schrödinger equation on an asymptotically conical manifold. It would be interesting to generalize the results about the wave equation to this setting.

Recently, V. Grasselli has proved in this setting all the results of [BB21]. See [Gra]. Thus, she proved in this context low frequency resolvent estimates and the corresponding local energy decay for the Schrödinger and wave equations. In a similar context, R. Wang [Wana] has also generalized the results of [BJ16] about the exponential decay for the Klein-Gordon equation.

Concerning the Schrödinger equation, the obvious question is the adaptation of the result of [23] to this setting. Is it true that solution of the Schrödinger equation on an aymptotically conical manifold $\mathcal{M}$ looks like (in the sense of Theorem 1.25) the solution of the free Schrödinger equation on the corresponding conical model $\mathcal{M}_0$ for large times ? And how does the geometry affect the initial condition for the asymptotic profile ?

Similarly, a result analogous to those of Chapter 2 could be proved on asymptotically conical manifolds with damping at infinity, or on more general wave guides. And, again, it would be interesting to see what would be the asymptotic profiles in these settings.

The results on the asymptotically conical manifold are expected to be proved with the technics developed in this setting on the one hand, and the strategy developed for the Euclidean setting on the other hand.

A less marked path would be to consider more general space-time geometries, in the spirit of the recent papers [MST20] and [Kof] (the later being in some sense a generalization of [6] on a Lorentzian manifold).

4.2 Null-controllability for a Grushin equation

In 2021, a few months after the publication of [22] about observability for a Kolmogorov equation (see Section 3.5), Armand Koenig arrived as a post-doc in Toulouse. Armand had already several results on similar problems (see [Koe17, DK20]), so it was natural to continue this study together with Jérémi Dardé. In this paragraph, we discuss the preprint [25] and further perspectives on the subject.

Let $T > 0$, $\ell_\pm > 0$, $I =]-\ell_-, \ell_+[$ and $\Omega = I \times \mathbb{T}$. Given an open subset ω of Ω, we are interested in the null-controllability for the following Baouendi-Grushin equation

$$\begin{cases} \partial_t u - \partial_x^2 u - q(x)^2 \partial_y^2 u = \mathbb{1}_\omega f, & \text{on }]0,T[\times\Omega, \\ u(t,\cdot) = 0, & \text{on } \partial\Omega, \text{ for all } t \in]0,T[, \\ u_{|t=0} \in L^2(\Omega). \end{cases} \tag{4.6}$$

This is analogous to (3.17), except that the degenerate term is now a diffusive term. Less important, the control is in the domain and not at the boundary (and not important at all, we have also switched the roles of x and y). The coefficient q is again regular and vanishes at and only at 0. Then (4.6) is a heat equation which is degenerate on the line $x = 0$.

In the first results about the Grushin equation, the control (or observability) region ω is usually rectangular (of the form $[a,b] \times \mathbb{T}$, for $[a,b] \subset I$). In [BCG14], for $q(x)^2 = |x|^\gamma$ and $0 < a < b < \ell_+$, it is proved that there is always observability if $\gamma < 2$ (for any open ω in this case), never if $\gamma > 2$, and that there is a finite critical time $\mathcal{T}_c \geqslant \frac{a^2}{2}$ if $\gamma = 2$. In [BMM15], it is proved that $\mathcal{T}_c = 0$ if $a = 0$ and that the critical time is exactly $\frac{a^2}{2}$ if $\omega = \mathbb{T} \times ([-b,-a] \cup [a,b])$. Finally, [BDE20] deals with the case where q is general and the observation is on one side of the boundary, which is essentially equivalent to observing throught a rectangular neighborhood of the boundary (this is the result that we have proved for the Kolmogorov equation in [22]).

In [Koe17, DK20], results are given for more general control domains. However, these results are only proved for the model case $q(x) = x$. Our purpose in [25] is to extend this analysis for more general coefficients (see the precise assumptions for each result).

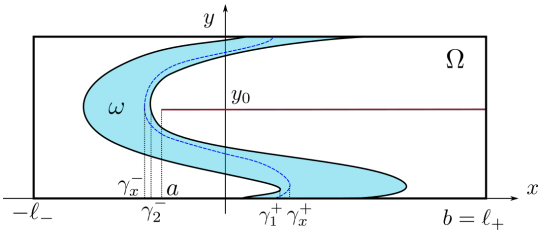

Figure 4.1: A domain of control and the corresponding critical time.
In this picture, a domain of control ω as in Theorem 4.4, a path γ as in Theorem 4.2 (dotted line, with $\delta(\gamma_x^-) = \max(\delta(\gamma_x(s)))$) and a line segment $[a,b] \times \{y_0\}$ as in Theorem 4.3. We can choose a and γ_x^- arbitrarily close to γ_2^-, which gives the critical time $\mathcal{T}_c = \delta(\gamma_2^-)/q'(0)$.

Theorem 4.2 is a positive result of null-controllability. It gives a sufficient condition on ω to prove observability for large times, which gives an upper bound for the critical time of null-controllability.

On the contrary, Theorem 4.3 provides a condition on ω for which we have a positive lower bound for the critical time, which means than we cannot have null-controllability in small times.

Finally, Theorem 4.4 gives a family of domains ω which satisfy the assumptions of both results and for which the lower and upper bounds for the critical time coincide. In this case, we have the precise critical time for the null-controllability of the Baouendi-Grushin equation (4.6).

For these statements we set

$$\delta(x) = \int_0^x q(s)\,\mathrm{d}s, \quad \forall x \in I, \qquad \text{and} \qquad \delta(\pm\ell_\pm) = +\infty.$$

Theorem 4.2. *Assume that $q \in C^3(\overline{I})$ is such that $q(0) = 0$ and $\inf_I q' > 0$. Assume that there exists a closed path $\gamma = (\gamma_x, \gamma_y) \in C^0(\mathbb{T};\omega)$ such that $\{-\ell_-\} \times \mathbb{T}$ and $\{\ell_+\} \times \mathbb{T}$ are included in different connected components of $(\overline{I} \times \mathbb{T}) \backslash \gamma(\mathbb{T})$. Then the Baouendi-Grushin equation (4.6) is null-controllable on ω in any time T such that*

$$T > \frac{1}{q'(0)} \max_{s \in \mathbb{T}} \delta\big(\gamma_x(s)\big).$$

The main result in [25] is the following.

Theorem 4.3. *Assume that $q \in C^2(\overline{I})$ is such that $q(0) = 0$, $q'(0) > 0$ and $q(x) \neq 0$ whenever $x \neq 0$. Let ω be an open subset of $I \times \mathbb{T}$. Assume that there exist $a \in [-\ell_-, 0[$, $b \in]0, \ell_+]$ and $y_0 \in \mathbb{T}$ such that, in $\overline{I} \times \mathbb{T}$,*

$$\big([a,b] \times \{y_0\}\big) \cap \overline{\omega} = \varnothing.$$

Then the generalized Baouendi-Grushin equation (4.6) is not null-controllable on ω in time T such that

$$T < \frac{1}{q'(0)} \min\big(\delta(a), \delta(b)\big).$$

In particular, we never have null-controllability if $]-\ell_-, \ell_+[\times \{y_0\} \cap \overline{\omega} = \varnothing$ for some $y_0 \in \mathbb{T}$.

Theorem 4.4. *Assume that $q \in C^3(\overline{I})$ is such that $q(0) = 0$ and $\inf_I q' > 0$. Let $\gamma_1, \gamma_2 \in C^0(\mathbb{T}; I)$ such that $\gamma_1(y) < \gamma_2(y)$ for all $y \in \mathbb{T}$. We set*

$$\gamma_1^+ = \max\Big(0, \max_{\mathbb{T}} \gamma_1\Big) \quad \text{and} \quad \gamma_2^- = \min\Big(0, \min_{\mathbb{T}} \gamma_2\Big).$$

Then the critical time for the null-controllability of (4.6) *on* $\omega = \{(x,y) \in I \times \mathbb{T} : \gamma_1(y) < x < \gamma_2(y)\}$ *is*

$$\mathcal{T}_c = \frac{1}{q'(0)} \max\big(\delta(\gamma_2^-), \delta(\gamma_1^+)\big).$$

We do not discuss the general strategy of the proof but only emphasize one argument of the proof of Theorem 4.3. Instead of the (non-selfadjoint) operator K_n which appeared in the analysis of the Kolmogorov equation in Section 3.5, we get after Fourier transform the (selfadjoint) operator

$$P_n = -\partial_x^2 + n^2 q(x)^2 \quad (\text{with } \mathsf{Dom}(P_n) = H_0^1(I) \cap H^2(I)).$$

To prove Theorem 4.3 we contradict the corresponding observability estimate (see (3.16)) if T is too small.

A key argument depends on the first eigenvalue λ_n of P_n (which is again close to the eigenvalue $q'(0)n$ of the harmonic oscillator), and a corresponding eigenfunction φ_n. Setting, for some $\varepsilon > 0$,

$$\gamma_{t,x}(n) = e^{-t(\lambda_{n+1} - q'(0)(n+1))} \varphi_{n+1}(x) e^{(n+1)\delta(x)(1-\varepsilon)}$$

we have to prove the following estimate on polynomials

$$\Big\| \sum \gamma_{t,x}(n) a_n z^n \Big\|_{L^\infty(X)} \lesssim \Big\| \sum a_n z^n \Big\|_{L^\infty(V)},$$

uniformly in $t \in]0,T[$ and $x \in I$, where X is a compact subset of $\mathbb{C}$ and V is an open and star-shaped (with respect to 0) neighborhood of X. For this, we use a result (see [Koe17, Th. 18]) which requires estimates on a holomorphic extension of $\gamma_{t,x}$ on some domain of $\mathbb{C}$. In particular, we need properties on the analytic extensions of λ_ν and φ_ν for $\mathsf{Re}(\nu) > 0$. Thus we have to study P_ν for some non-real parameters ν and, even in this setting, spectral properties of non-selfadjoint operators play an important role.

The results about null-controllability for the Grushin (discussed here) or the Kolmogorov (see Section 3.5) equations still only concern particular situations. Even if more and more general control domains ω and coefficients $q(x)$ are considered, much remains to be understood.

In [25], we have provided examples of control domains ω which are not covered by our results. Some geometries are not so complicated and it is frustrating not to be able to conclude in these cases.

Similarly, another assumption that we would like to relax is the increasingness of q in Theorem 4.2. We could also consider the case where q vanishes twice (then q^2 would form a double-well potential). The difficulty does not come from the arguments based on spectral theory, since these more general settings are more or less already understood. The main obstacle is probably the Carleman estimate used for the positive result (Proposition 3.14 in the context of [22]).

Of course, we would like to go further. For me, this first means a better understanding of the classical technics in control theory (Carleman estimates, the method of moments, how we can use the classical tools of microlocal analysis in this kind of context, etc.). This is a very interesting possible perspective.

4.3 Discrete spectrum at the strong magnetic field limit on a curved wave guide

In this section we discuss the preprint [27], written with Engerran Bon-Lavigne, Loïc Le Treust and Nicolas Raymond. This started with a discussion at the CIRM in June 2021.

The question is about the existence of discrete spectrum under the essential spectrum for a magnetic Laplacian on a two dimensional curved wave guide, at the strong magnetic field limit.

We consider a smooth and injective curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with $|\gamma'| = 1$. We assume that the algebraic curvature κ of γ is compactly supported. Then our curved wave guide is defined by

$$\Omega = \Omega_{\gamma,\delta} = \Theta(\Omega_{0,\delta}), \quad \Omega_{0,\delta} = \mathbb{R}\times]-\delta,\delta[, \quad \Theta(s,t) = \gamma(s) + tN(s),$$

where $\delta > 0$ is small enough and $N(s) = \gamma'(s)^\perp$ (with $(a,b)^\perp = (-b,a)$).

We consider on $\Omega_{\gamma,\delta}$ a uniform magnetic field of intensity h^{-1}, with $0 < h \ll 1$. If A is a well chosen vector potential corresponding to a field equal to 1, we consider on $L^2(\Omega)$ the operator

$$P_{h,\gamma,\delta} = (-ih\nabla - A)^2, \quad \mathsf{Dom}(P_{h,\gamma,\delta}) = H_0^1(\Omega) \cap H^2(\Omega).$$

Without magnetic field, the Dirichlet Laplacian always has discrete spectrum if the wave guide is not straight [DE95]. However, it is also known that the magnetic field plays against the existence of discrete spectrum. It has even been conjectured by P. Duclos and P. Exner in the mid-nineties that the discrete spectrum of $P_{h,\gamma,\delta}$ should be empty when h is small enough. In that direction, it is proved in [KR14] that if the magnetic field is compactly supported then there is no discrete spectrum in the strong field limit.

However, we give in [27] a sufficient condition to have discrete spectrum for a strong uniform magnetic field.

Theorem 4.5. *Assume that κ^2 has a unique maximum, which is non-degenerate. There exist $\delta_0 > 0$ and $h_0 > 0$ such that for $\delta \in]0,\delta_0]$ and $h \in]0,h_0]$ we have*

$$\inf \sigma(P_{h,\gamma,\delta}) < \inf \sigma_{\mathsf{ess}}(P_{h,\gamma,\delta}).$$

In particular, $P_{h,\gamma,\delta}$ has non-empty discrete spectrum.

For the essential spectrum, we prove that it is equal to the essential spectrum of the straight wave guide $\Omega_{0,\delta}$, for which we can compute a lower bound. On the other hand, we use the Min-max Theorem to get an upper bound for the minimum of the spectrum. The argument is based on the equality

$$\begin{aligned}\inf \sigma(P_{h,\gamma,\delta}) - h &= \inf_{\psi \in H_0^1(\Omega)\setminus\{0\}} \frac{\|(-ih\nabla - A)\psi\|^2_{L^2(\Omega)} - h\,\|\psi\|^2_{L^2(\Omega)}}{\|u\|^2} \\ &= \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{4h^2 \int_\Omega e^{-\frac{2\phi}{h}} \left|\partial_{\bar z} u\right|^2 \mathrm{d}x}{\int_\Omega e^{-\frac{2\phi}{h}} \left|u\right|^2 \mathrm{d}x}.\end{aligned}$$

The main parts of the proof are the construction of a good function ϕ and then the definition of a suitable fonction u which is in particular holomorphic on Ω except near the boundary (to satisfy the Dirichlet condition). This function ϕ already plays an important role in the choice of the vector field A.

In addition to its own interest for the Duclos-Exner conjecture, this result is a step toward the understanding of a similar property for the magnetic Dirac operator

$$\mathcal{P}_{h,\gamma,\delta} = \big((-ih\partial_{x_1} - A_1)\sigma_1 + (-ih\partial_{x_2} - A_2)\sigma_2\big)^2 = \begin{pmatrix} |-ih\nabla - A|^2 - h & 0 \\ 0 & |-ih\nabla - A|^2 - h \end{pmatrix}.$$

This is the model Hamiltonian for a non-relativistic spin-$\frac{1}{2}$ particle, constrained to move in Ω, interacting with a magnetic field that is perpendicular to the plane. Spectral properties of this operator are studied in [BLTRS21] when Ω is bounded. Our project is to consider as above the operator on a wave guide.

A first step could be as in [27] the existence of discrete spectrum, but the perspective is to prove as in [BLTRS21] more precise asymptotics for the first eigenvalues. For the Duclos-Exner conjecture an upper bound for the bottom of the spectrum was enough, but we would like to prove asymptotic properties for the k-th eigenvalue at the strong field limit, both for the magnetic Laplacian and the Dirac operator. This is much more precise and would require in particular a better understanding on our wave guides of the functional spaces used in [BLTRS21] and wave-guide versions of classical results such as the Paley-Wiener theorem.

4.4 Spectral theory on discrete and quantum graphs

4.4.1 Spectrum of the wave equation on a non-compact star-graph

In Section 3.1, we have discussed the spectrum of a non-selfadjoint compact star graph. Here we discuss the result of the preprint [26] about the damped wave equation on a non-compact star-graph. This is a joint work with David Krejčiřík.

We have already seen that the energy of the wave does not necessarily decrease faster with a stronger damping. A natural question is then the optimal damping which will give the best decay.

This question has been addressed in [BRT82] in the case of a wave on a finite interval with singular damping in the middle. This was motivated by the analysis of stringed instruments. They argue that there is indeed an optimal strength for the damping. The problem of the wave on a finite string with Dirac damping has also been analysed in [ATH00, AHT01, CH08] (see also [AN15, §4.1.1]). The damped wave equation has also been discussed in star-shaped graphs in [AJ04] (damping at the central vertex), [AJK05] (damping at a boundary vertex) or [AJK16] (local energy decay with one infinite edge).

In [26] we consider a star-graph with N infinite edges and damping at the only vertex. The graph can be identified with $(\mathbb{R}_+^*)^N$ (the edges are parametrized from the vertex to infinity). Given any $\alpha \in \mathbb{C}$ we consider the wave equation

$$\partial_t^2 u_j(t,x) - \partial_x^2 u_j(t,x) = 0, \quad 1 \leqslant j \leqslant N, t \geqslant 0, x > 0, \tag{4.7}$$

with continuity at the vertex

$$u_j(t,0) = u_k(t,0), \quad 1 \leqslant j,k \leqslant N, t \geqslant 0, \tag{4.8}$$

(we denote by $u(t,0)$ this common value) and the damping vertex condition

$$\sum_{j=1}^N \partial_x u_j(t,0) + \alpha \partial_t u(t,0) = 0. \tag{4.9}$$

We really have damping when $\alpha < 0$ (the case $\alpha > 0$ would correspond to a supply of energy, while the case $\alpha \in i\mathbb{R}$ is conservative and has an interpretation in quantum mechanics, see Section 7 in [26]). Notice that the case $N = 1$ corresponds to a semi-infinite string with damping at the end, while the case $N = 2$ is an infinite string with singular damping.

The corresponding operator is defined as follows. We denote by $\mathscr{E}$ the set of pairs (u,v) such that $u_j', v_j \in L^2(0,\infty)$ for all j and $u_j(0) = u_k(0)$ for all j,k (we denote by $u(0)$ this common value). This is a Hilbert space if the first component is quotiented by the space of constant functions on the graph. For $u \in \dot{H}^2(\mathbb{R}_+^*)^N$ we set $u'' = (u_j'')$. Then we define the operator $\mathcal{W}$ by

$$\mathcal{W}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ u'' \end{pmatrix}, \quad \mathsf{Dom}(\mathcal{W}) = \left\{ U = (u,v) \in \mathscr{E} \,:\, \mathcal{W}U \in \mathscr{E} \text{ and } \sum_{j=1}^N u_j'(0) + \alpha v(0) = 0 \right\}.$$

We are interested in the spectral properties of this operator $\mathcal{W}$, and the corresponding counterparts for the time dependent problem (4.7)-(4.9). We can check that $\pm\mathcal{W}$ is maximal accretive if $\pm\mathsf{Re}(\alpha) \geqslant 0$ (it is skew-adjoint if $\mathsf{Re}(\alpha) = 0$). In particular, if $\mathsf{Re}(\zeta) \neq 0$ and $\mathsf{Re}(\zeta)\mathsf{Re}(\alpha) \leqslant 0$ then $\zeta \in \rho(\mathcal{W})$ and

$$\left\|(\mathcal{W} - \zeta)^{-1}\right\|_{\mathcal{L}(\mathscr{E})} \leqslant \frac{1}{|\mathsf{Re}(\zeta)|}.$$

We are interested in the spectrum in the half-plane where $\mathsf{Re}(\zeta)$ and $\mathsf{Re}(\alpha)$ have the same sign. This is a difficult question in general and little is known in that direction, but for this model setting we can proceed with explicit computation.

As already observed for a finite string, there is an abrupt change of properties for particular values of α. More precisely, we have the following result about the spectrum of $\mathcal{W}$.

Theorem 4.6. *The spectrum of $\mathcal{W}_\alpha$ is*

$$\begin{cases} i\mathbb{R} & \text{if } \alpha \in \mathbb{C}\backslash\{\pm N\}, \\ \overline{\mathbb{C}_+} & \text{if } \alpha = N, \\ \overline{\mathbb{C}_-} & \text{if } \alpha = -N. \end{cases}$$

Moreover, $i\mathbb{R}$ contains no eigenvalue nor residual spectrum of $\mathcal{W}_\alpha$, and if $\alpha = \pm N$ then any $\zeta \in \mathbb{C}_\pm$ is an eigenvalue of $\mathcal{W}_\alpha$ of geometric multiplicity 1 and infinite algebraic multiplicity.

To understand this behavior, we compute the resolvent of $\mathcal{W}$. As expected, for $\zeta \in \mathbb{C}_\pm$ the norm of $(\mathcal{W} - \zeta)^{-1}$ in $\mathcal{L}(\mathscr{E})$ goes to $+\infty$ when $\alpha \to \pm N$.

Proposition 4.7. *There exist $c_0, C > 0$ such that for $\alpha \in \mathbb{C}_\pm \backslash \{\pm N\}$ and $\zeta \in \mathbb{C}_\pm$ we have*

$$\max\left(\frac{1}{|\mathsf{Re}(\zeta)|}, \frac{c_0}{|\zeta|\,|\alpha \mp N|}\right) \leqslant \left\|(\mathcal{W} - \zeta)^{-1}\right\|_{\mathcal{L}(\mathscr{E})} \leqslant \frac{C}{|\mathsf{Re}(\zeta)|}\left(1 + \frac{1}{|\alpha \mp N|}\right).$$

The values $\alpha = \pm N$ also have a particular role for the wave equation (4.7)-(4.9). Notice that on each edge, the wave has an outgoing part which never sees the damping and behaves as the free wave equation on $\mathbb{R}$. We are interested in the incoming part of the wave, which hits the vertex. The vertex conditions describe how the waves coming from all edges are reflected. In particular, the sum of the energies of all the reflected waves is equal (smaller, larger, respectively) than the energies of the incident waves if $\mathsf{Re}(\alpha) = 0$ ($\mathsf{Re}(\alpha) < 0$, $\mathsf{Re}(\alpha) > 0$, respectively). It turns out that in the particular case $\alpha = -N$ the incident waves are completely absorbed. And the problem is ill posed in the case $\alpha = N$ (we have no solution or an infinite number of solutions for positive times, depending on the initial conditions). Of course, these properties are reversed if we go backward in time. We refer to [26, Th. 2.4] for a more quantitative statement.

4.4.2 Other model problems on non-compact quantum graphs

The paper [21] and the preprint [26] were about particular cases of non-selfadjoint quantum graphs, and it would be interesting to go further in various directions.

The first obvious question is the generalization of [21] to more general graphs. The difficulties will come from the steps of the proof where we have used the explicit expression for the secular equation (3.9). For instance, it has been used to prove that the Neumann eigenvalues cannot escape too far (see the discussion before (3.11)) or to see that ℓ is always transverse to the submanifood $\mathcal{Z}_0$ of $\mathbb{T}^N$ (see before (3.12)). We have already thought about this while writing [21]. Considering general compact quantum graphs will not be trivial, but we can certainly go beyond the star-graph.

Another natural continuation of [21] is to consider the spectrum of the wave operator on a compact star graph (and then in a more general situation).

The main difference between the Schrödinger and wave equations is that instead of the operator H_α defined by (3.1)-(3.4), we have to consider the operator H_{az} where a is the absorption coefficient and z is the spectral parameter (this is completely similar to the wave with damping at the boundary, see (2.18)). Thus we have to work with the parameter $\alpha = az$ which goes to infinity when the spectral parameter z itself goes to infinity. Then the key argument that the eigenvalues of H_α cannot go too far from those of H_0 is no longer valid.

We have seen as a byproduct of studies on wave guides that for one edge (see Section 1.6) or for two edges with particular vertex conditions (see Section 4.1.2) we can prove that eigenvalues stay in some bounded region of $\mathbb{C}$ for any α, but these results were again based on explicit computation (and we already had more difficulties with two edges than with one), so we have to find a more general argument to deal with more general graphs.

Notice that when all the edges have the same length, it is proved in [FL17] that there is a finite number of so-called spectral abscissas. With the terminology of Theorem 3.3, this means that the corresponding measure μ_ℓ is a linear combination of Dirac masses (this is the analog of the second case therein). We expect that a similar result will hold with damping at the vertices. The difficulty, as in [21], is to give a precise statement for any set of lengths (or at least a generic set of lengths, but including the case of irrationally dependent lengths). For a review of some known results about the stabilization of the wave equation on quantum graphs, we also refer to [Zua13].

All the questions around [21] concern the contribution of high frequencies. If we consider non-compact quantum graphs, there will also be interesting phenomena for the contribution of low frequencies, as is the case for the wave on non-compact manifolds.

A non compact graph can be a graph with a finite number of infinite edges, as in [26], or a graph with an infinite number of edges. We give several examples (all with finite edges of equal length) in Figure 4.2.

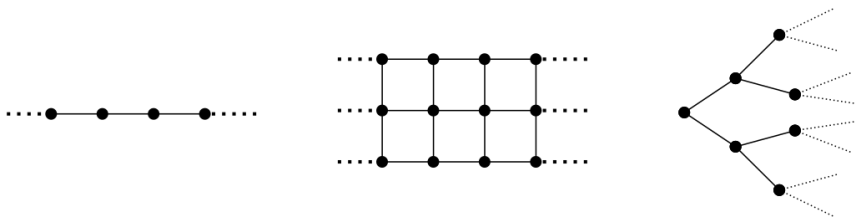

Figure 4.2: Examples of non-compact graphs

Let us briefly discuss for instance the case of trees. This is a particular example of quantum graph which has many applications.

Again, there is less literature on the spectral properties of quantum graphs than on the discrete ones, but this is now an active field. We refer for instance to [Car97, Sol04] (and references therein) for early works about the spectrum of the Laplacian on quantum trees, and to [AISW21] for an example of recent development.

It would be interesting to see what happens if we add some damping on the edges or at the vertices. Since I have not started the analysis, I have to remain vague about what can be expected, but this is a setting that I find exciting and that I would like to explore.

Another question which could be tested on a simple graph is the wave equation with random damping. Consider for instance the case of a wave equation on the line $\mathbb{R}$ with random singular damping at each integer. The damping coefficient a_n at $n \in \mathbb{Z}$ could be independent random variables (for instance they could all follow a Bernouilli distribution, so that there is damping with probability p and nothing with probability $(1-p)$). As usual, we could start with the Schrödinger operator, in which case we are closer to the classical Anderson model (see for instance [Kir08] for an introduction to random Schrödinger operators).

4.4.3 Resolvent estimates for the Schrödinger operator on a discrete graph

When I mention graphs, I mainly think about quantum graphs. However, even if discrete graphs have already been extensively studied, there are still exciting questions to be discussed.

As for quantum graphs, we can consider many different interesting settings, but I will only consider one problem in this parapraph. This is a project that I have started with Olivier Bourget and Diomba Sambou.

With Amal Taarabt, they have studied in [BST20] various spectral properties (limiting absorption principle, resonances, etc.) for a non-selfadjoint Schrödinger operator on $\ell^2(\mathbb{Z})$.

We recall that the one-dimensional discrete Laplacian on $\ell^2(\mathbb{Z})$ is defined by

$$(H_0 u)_n = 2u_n - u_{n-1} - u_{n+1}, \quad n \in \mathbb{Z},$$

for $u \in \ell^2(\mathbb{Z})$ (we could also consider the analog on $\ell^2(\mathbb{Z}^d)$ for any dimension $d \in \mathbb{N}^*$). Its spectrum is

$$\sigma(H_0) = \sigma_{\mathsf{ac}}(H_0) = [0, 4].$$

In [BST20] they add a complex potential V (a potential simply acts as a diagonal matrix on $\ell^2(\mathbb{Z})$: $(Vu)_n = V_n u_n$).

In particular, they prove the limiting absorption principle with the Mourre commutators method.

We recall that this method requires a suitable conjugate operator. In the papers discussed in Chapter 1, the operator was always a perturbation of the usual Laplacian on $\mathbb{R}^d$, and the conjugate operator was always a perturbation of the generator of dilations. Dilations have no obvious analog in the discrete setting. However, if we set $A_0 = \mathsf{Im}(S)X + XIm(S)$, where X is the position operator (defined by $(Xu)_n = nu_n$) and S is the shift operator ($(Su)_n = u_{n+1}$), then we have

$$\mathsf{ad}_{iA_0}(H_0) = 4H_0 - H_0^2.$$

With this conjugate operator, they are able to apply the Mourre theory away from the thresholds 0 and 4 (notice that there are more thresholds in higher dimensions).

This naturally raises the question of the properties (such as resolvent estimates) near the thresholds. This is precisely the analog of studying the resolvent of usual Schrödinger operators near 0.

The discussion is only at its early stage, but it is natural to try to adapt the strategy described in Chapter 1. As for the use of the generator of dilations, many arguments used in the Euclidean setting are meaningless in the discrete case, but there is a reasonnable hope that we can adapt the ideas to get resolvent estimates for a discrete Schrödinger operator. Of course it is also possible to consider the wave equation in this setting.

4.5 Non-linear problems

4.5.1 A non linear problem on a quantum graph

After [12] and [19] (see Section 3.6), I have started a new project with François Genoud and Stefan Le Coz. It is again about a non-linear problem in dimension 1. Another common feature between our two previous works was the singular potential. As observed above, a problem on the real line with a singular potential can be seen as a problem on a graph with two infinite edges, the singular potential being interpreted as a "boundary condition" at the vertex.

Here we really consider a graph. Since our analysis should be essentially local, we consider the model case: a star graph Γ, with N infinite edges (as in Section 4.4.1). We recall that Γ can be seen as N copies of $\mathbb{R}_+^*$, and we define $L^2(\Gamma)$ accordingly.

Given $\gamma \in \mathbb{R}$, we consider on Γ the non-linear Schrödinger equation

$$-i\partial_t u_j - \partial_{x_j}^2 u_j + \gamma\delta u - |u_j|^4 u_j = 0, \quad 1 \leqslant j \leqslant N, \tag{4.10}$$

where δ is the Dirac distribution at the central vertex. A solution on the interval I of $\mathbb{R}$ is a function $u = (u_j)_{1\leqslant j\leqslant N} \in C^0(I, H^1(\mathbb{R}_+^*)^N) \cap C^1(I, (H^1(\mathbb{R}_+^*)^*)^N)$ such that u_j is a solution on each edge, $u(t)$ is continuous at the vertex (as in (4.8)) for all $t \in I$, and it satisfies the Robin condition

$$\sum_{j=1}^N u_j'(t, 0) = \gamma u(t, 0).$$

The Cauchy problem is completed by an initial condition u_0 at time $t = 0$. Notice that since γ is real, the generator of the linear part (the Laplacian on each edge, with these vertex conditions in the domain) is selfadjoint. Two constants of the motion are given by the mass and the energy:

$$M(u) = \frac{\|u\|^2_{L^2(\Gamma)}}{2}, \quad E(u) = \frac{\|\partial_x u\|^2_{L^2(\Gamma)}}{2} + \frac{\gamma\,|u(0)|^2}{2} - \frac{\|u\|^6_{L^6(\Gamma)}}{6}.$$

Our question on this model is the existence of blowing-up solutions. It is known that for the equation on the line,

$$-i\partial_t u - \partial_x^2 u - |u|^4 u = 0, \quad u_{|t=0} = u_0, \tag{4.11}$$

the threshold between global existence of all the solutions and existence of blowing-up solutions is the mass of the ground state $Q : x \mapsto 3^{\frac{1}{4}} \operatorname{sech}(2x)^{\frac{1}{2}}$ (solution of $Q'' - Q + Q^5 = 0$). If $M(u_0) < M(Q)$ then the solution of (4.11) is well defined for all times, and there exists u_0 with $\|u_0\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})}$ such that the corresponding solution of (4.11) blows up in finite time. Indeed, given $T > 0$ we can consider the solution given by pseudo-conformal transform of the standing wave $e^{it}Q(x)$, given by

$$u(t,x) = \frac{1}{\sqrt{T-t}} e^{\frac{i}{T-t}} e^{-\frac{ix^2}{4(T-t)}} Q\left(\frac{x}{T-t}\right).$$

Then we have

$$\|u(t)\|_{L^2(\mathbb{R})} = \|Q\|_{L^2(\mathbb{R})} \quad \text{and} \quad \|\partial_x u(t)\|_{L^2(\mathbb{R})} \underset{t\to T^-}{\sim} \frac{1}{T-t}.$$

On the graph, it is expected that the blowing-up solution with minimal mass has a similar behavior with blow-up at some point of one of the edges. If we only consider radial solutions ($u_j(t,x)$ does not depend on j), then the threshold for global existence is increased. With $\gamma = 0$, we can construct a similar solution by replacing Q by Q_Γ, defined by considering a copy of $Q_{|\mathbb{R}_+}$ on each edge. This is no longer a solution if $\gamma \neq 0$. Our purpose is to construct a blowing-up solution at the minimal mass $M(Q_\Gamma)$. The idea is still to construct a blowing-up solution based on the profile Q_Γ. More precisely, we apply the method of [RS11, LMR16], where the existence of minimal mass blowing-up solutions is proved in other contexts. We also refer to the recent preprint [TX], dealing with the problem on the line.

The expected result is that given $E^\star \in \mathbb{R}$ there exists $T > 0$ and a radial solution u on $[0, T[$ such that $M(u) = M(Q_\Gamma)$, $E(u) = E^\star$, and

$$\|\partial_x u(t)\|_{L^2(\Gamma)} \underset{t\to T^-}{\sim} \frac{1}{(T-t)^{\frac{2}{3}}}.$$

4.5.2 Nonlinear damped wave equations

All the works about the damped wave equation presented in Chapters 1 and 2 concern in various settings the linear equation (1.12). I have also worked on some non-linear problems (see Sections 3.6 and 4.5.1), but it was not about the damped wave equation.

As said in introduction, in many physical situations, propagation of waves is modeled by an equation involving nonlinear terms. A possible perspective for the future is to consider some questions about non-linear damped wave equations. I have not started anything in that direction yet, so I am essentially ignorant, but one can at least discuss some natural questions on the subject.

A first perspective is to continue the analysis of one-dimensional problems with singular potential. In particular, the paper [19] discussed in Section 3.6 is about the non-linear

Klein-Gordon equation. It would be quite natural to consider the same kind of setting, for Klein-Gordon or the wave equation, but with dissipation. This corresponds to replacing $i\alpha$ by $a > 0$ in (3.31). Beginning with one-dimensional settings would be quite convenient, and the difficulties due to the singular potentials are, as already discussed, related to quantum graphs which is another aspect that I would like to develop. The natural first questions are the analysis of the local and global existence of solutions, and the decay of the energy. One remembers from Section 4.4.1 that there are already strange behaviors for the linear problem for some particular values of the damping, and it would be interesting to know what happens if we add a non-linear perturbation.

Nevertheless, the long term perspective is to start to add non-linear terms to the general damped wave equation (1.12). The typical example is to consider a source term of the form

$$f(u) = bu\,|u|^{p-1}\,, \tag{4.12}$$

for some $p > 1$. Another possibility is to consider a non-linear dissipative term. The model example with both non-linearities is then an equation of the form

$$\partial_t^2 u - \Delta u + a\partial_t u\,|\partial_t u|^{\rho-1} = f(u), \tag{4.13}$$

for some $\rho \geqslant 1$.

The first question is the local well-posedness for the Cauchy problem. Then comes the question of global well-posedness or existence of blowing-up solutions. When the solution is globally defined, we can look at the long time behavior of the energy

$$E(u;t) = \frac{1}{2}\,\|\partial_t u\|^2_{L^2(\Omega)} + \frac{1}{2}\,\|\nabla u\|^2_{L^2(\Omega)} - \frac{b}{p+1}\,\|u\|^{p+1}_{L^{p+1}(\Omega)}\,.$$

As for the linear problem, it is natural to wonder if it goes to 0 and, in this case, what is the rate of decay.

It is too early to state a conjecture here, but this report is an occasion to look at the literature. It is of course impossible to be exhaustive about the non-linear wave equation (or the related Schrödinger equation). We refer to the books [Tao06, Caz03, Sog08] for introductions to nonlinear dispersive equations and many references, and in this paragraph we only mention some papers dealing with the damped wave equation.

As for the linear setting, the problem is much better understood on compact domains (typically with Dirichlet boundary conditions), or for the Klein-Gordon equation (with an additional term mu on the left in (4.13)). We refer to [HZ88] for the inhomogeneous equation. In [GT94] it is proved that for

$$1 < p < \begin{cases} \frac{d}{d-2} & \text{if } d \geqslant 3, \\ +\infty & \text{if } d \leqslant 2, \end{cases} \tag{4.14}$$

then (4.13)-(4.12) is globally well-posed if $p \leqslant \rho$ (in the sense that $u \in C^0(\mathbb{R}_+, H^1_0(\Omega))$ and $\partial_t u \in C^0(\mathbb{R}_+, L^2(\Omega)) \cap L^{\rho+1}_{\mathsf{loc}}(\Omega)$), while there exists a solution which blows up in $L^\infty(\Omega)$ if $p > \rho$.

Concerning the decay of the energy, we refer for instance to [Har85, HZ88, Zua88, Zua90, JL13, JL20] for results on a bounded domain and [Zua91, Deh01, DLZ03, AIN11] for the Klein-Gordon equation. Typically, these works use multipliers method or the method based on semiclassical defect measures. Notice that in addition to the geometric condition, the proofs rely on an assumption of unique continuation. Moreover, an important observation used in many of these articles is that once the solution is small, the non-linear term no longer play any important role and the solution of the non-linear problem has the same decay properties as a solution of the linear problem (see [Gér96]).

As for the linear problem, it is a different issue to consider the wave equation in an unbounded domain. Moreover, if the damping is effective at infinity, it is natural to wonder if we have a diffusive phenomenon as described in Chapter 2 for the linear equation. As for the linear equation, there are many results about the problem with constant coefficients ($a(x) = 1$ in (4.13)). The results are different in the defocusing (typically, (4.12) with $b = -1$) of focusing ($b = 1$) cases.

We begin with the case where $f(u)$ is equal to or of the form $-|u|^{p-1} u$. The paper [Mat76], which is one of the firsts about the linear problem (see the discussion in Chapter 2), also dealt with the semilinear equation. The results have been refined in [KNO95] (in particular, general initial data are allowed). In [Kar00] (see Theorem 2.3 therein), the diffusion phenomenon has been proved for initial data in $(H^1 \cap L^1) \times (L^2 \cap L^1)$, when $d \leqslant 3$ and for $p > 1 + \frac{4}{d}$ such that (4.14) holds. Then the asymptotic profile is given by a solution of the heat equation as in the linear case. In dimension 1, a better result is given in [HKN07a]. In [INZ06], the large time asymptotics for the supercritical case $p > 1 + \frac{2}{d}$ is proved in dimension $d \leqslant 3$. The solution is actually close to a solution of the semilinear heat equation

$$\partial_t v - \Delta v + v\,|v|^{p-1} = 0,$$

which behaves itself as a solution of the linear heat equation in the supercritical case (the critical exponent $p = 1 + \frac{2}{d}$ is known as the Fujita's critical exponent [Fuj66]). The result has then been improved for $d = 3$ and extended to $d = 4$ in [Nis06]. Less is known in the critical and subcritical cases. For results about the critical exponent we refer to [HKN07b, HKN06, HN17]. For the subcritical case we refer to [Ham10]. Notice that in these cases the problem is not necessarily globally well posed.

Now we turn to the focusing case, where $f(u)$ is typically of the form $|u|^p$ or $u\,|u|^{p-1}$. In general, the solution blows up in finite times for large initial data (see [Lev74, Section IV]). Results about the existence of global solutions or blow-up (for small initial data) can be found in [NO93, LZ95, TY01, Zha01]. It is proved that for $p > 1 + \frac{2}{d}$ the solution with "small" initial data is defined for all time, while there are solutions which blow up when $p \leqslant 1 + \frac{2}{d}$. These results were proved under strong decay assumption in the initial data (typically, compactly supported). We refer to [Nak01] for the analysis on an exterior domain with a damping which can vanish.

Then there have been several papers with results about local well-posedness for large initial data and global well-posedness for small initial data in $(H^1 \cap L^r) \cap (L^2 \cap L^r)$ under some conditions on r and p. We do not mention them all and refer to the introduction of the recent paper [IIW17], which study the large data local well-posedness and the small data global well-posedness with slowly decaying initial data.

The asymptotic behavior in terms of solutions of the heat equation is also proved in [IIW17]. There are many references dealing with the large time asymptotics in more and more general settings. We mention for instance an early result in dimension 1 [GR98], [KU13] ($d \leqslant 3$), [HKN04] (any dimension, small solutions in particular in L^1) or [NN08] (slowly decaying data). The recent paper [IIOW19] improves the results for the linear and the non-linear problems. Higher order expansions are also available [KT16].

All the papers mentioned above concern the semilinear wave equation with linear damping ($\rho = 1$ in (4.13)). If $\rho > 1$, we notice that if the energy of the wave becomes small, the damping term becomes smaller than in the linear case. Here re-emerges the question whether the energy goes to 0 as $t \to +\infty$ (even with $f = 0$). Depending on the precise setting and on the initial conditions, there are results of decay (the energy goes to 0) and results of non-decay (it does not) for this kind of problem, on the Euclidean space or an exterior domain. It is expected that in the nondecay case, that is when the damping term becomes very small, the solution of (4.13) behaves like a solution of the undamped wave equation. This is indeed the case and it is proved under suitable assumptions in the literature. For

more precise statements we refer to [MM95, MM96, Mat03, TY07, TUY09, Dao18]. See also [Nak13, TY15] for results with non-linear damping and a non-linear potential.

There are actually many other variants of the non-linear damped wave equation in the literature. For instance, various systems of wave equations [GR14, BD13, BR20], friction and viscoelastic damping [MMR20], wave equation with non-linear boundary damping [CDCL07], higher order problems [CX20], time-dependent damping [IW20], delayed damping [Kaf21],variable exponents for the non-linear terms [MTAS17], etc.

We also refer to results about more or less general damped hyperbolic equations: see [LS97, LPS98, CTR21] (global existence of solutions), [GGH16c] (decay estimates), [GGH16a] (asymptotic behavior), [GGH16b] (smoothing effect of solutions with strong damping). See also the (older) lecture notes [Har81] and the book [Har87].

Finally, to make the link with the quantum graphs, we mention the paper [ABM21] about the non-linear damped wave equation on a star-graph, with damping at infinity on one edge.

Even if this bibliography is far from being complete, we already see that there is a very wide variety of problems about the non-linear damped wave equation. Of course, for future works, it would be reasonable to begin with questions closely related to my centers of interest.

The most natural perspective about the non-linear damped wave equation is to continue with the settings introduced in Chapters 1 and 2. As for the linear problem, the difficulty compared to the previous literature is to deal with the contributions of low frequencies in quite general geometric settings. The papers mentioned above about the wave equation on an unbounded domain essentially consider the equation with constant coefficients, sometimes in an exterior domain. It would be interesting to generalized these results, as is done for the linear problem.

An important tool for the analysis of non-linear dispersive equations are the Strichartz estimates, which are space-time integrability properties. They are estimates on the solution of the linear problem, but they are used to estimate the integral term given by the Duhamel formula, which can be crucial for the well-posedness of the Cauchy problem for the non-linear equation. See again [Tao06, Caz03, Sog08] for an introduction to the Strichartz estimates. There are again many papers on this question, for the Schrödinger or the (undamped) wave equations. We can mention for instance the original work [Str77] and the proof of the important endpoint cases in [KT98]. See also [BGT04] in an exterior domain and [BT08] in the asymptotically Euclidean setting. We finally refer to the recent work [BM].

It seems that not much is known about the Strichartz estimates for the damped wave or Schrödinger equations. Some estimates about the regularized Schrödinger equation (1.67) are given in [Kar15]. There are also recent results about the damped wave equation (2.1), see [Wat17, Inu19, IW].

These Strichartz estimates are dispersive estimates, in the spirit of the local energy decay presented in Chapters 1 and 2. Before considering general non-linear problems, it would be completely natural to begin with Strichartz estimates for the damped wave equation (1.12).

It is already a problem which is far from simple, since the standard arguments for the Strichartz estimates as in [KT98] do not apply to a non-selfadjoint setting.

Publications and preprints

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[26] D. Krejčiřík and J. Royer. Spectrum of the wave equation with Dirac damping on a non-compact star graph. Preprint HAL-03652993.

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