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Manh Tien Nguyen

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Manh Tien Nguyen. Weighted monotonicity theorems and applications to minimal surfaces of the hyperbolic space. Differential Geometry [math.DG]. Université Libre de Bruxelles (Belgique); Aix-Marseille Université, France, 2022. English. NNT : 2022AIXM0273 . tel-03800192

HAL Id: tel-03800192

<https://hal.science/tel-03800192v1>

Submitted on 6 Oct 2022

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THÈSE DE DOCTORAT

Soutenue à l'Université Libre de Bruxelles

dans le cadre d'une cotutelle avec l'Université Libre de Bruxelles

le 31 août 2022 par

Manh Tien NGUYEN

Théorèmes de monotonicité à poids et applications
aux surfaces minimales de l'espace hyperbolique

Discipline

Mathématiques

École doctorale

ED 184

Laboratoire/Partenaires de recherche

Service de Géométrie Différentielle, ULB

Institut de Mathématiques de Marseille

Excellence of Science

grant number 30950721

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Liste de publications et participation aux conférences

Liste des publications réalisées dans le cadre du projet de thèse :

1. Manh Tien Nguyen, *Weighted monotonicity theorems and applications to minimal surfaces in hyperbolic space*, <https://arxiv.org/abs/2105.12625>

Participation aux conférences et écoles d'été au cours de la période de thèse :

1. Mois thématique, *Géométrie complexe*, CIRM, Marseille, 01/2019
2. Introductory workshop, *Microlocal Analysis*, MSRI Berkeley, 09/2019
3. Atelier, *Géométrie différentielle et l'analyse globale*, Montréal, 05/2021
4. École d'été, *Curvature constraints and Spaces of Metrics*, Institut Fourier, Grenoble, 06/2021
5. Conférence, *Geometry and Analysis on Non-Compact Manifolds Géométrie et analyse sur les variétés non compactes*, CIRM, Marseille, 03/2022

Abstract

We study new monotonicity theorems for minimal surfaces in warped geometry. Applications include renormalised versions of the isoperimetric inequality for complete minimal surfaces of the hyperbolic space and a vanishing result for knot/link invariants obtained by counting these surfaces.

Keywords: minimal surface, harmonic map, hyperbolic space, renormalised area, isoperimetric inequality, knot

Résumé

Nous étudions de nouveaux théorèmes de monotonie pour les surfaces minimales en géométrie déformée. Les applications obtenues incluent des versions renormalisées de l'inégalité isopérimétrique pour les surfaces minimales complètes de l'espace hyperbolique et un résultat d'annulation pour les invariants de nœuds/entrelacs obtenus en comptant ces surfaces.

Mots clés : surface minimale, application harmonique, espace hyperbolique, aire renormalisée, inégalité isopérimétrique, nœud

Acknowledgements

Most of the results in this text trace back to a discussion with my supervisor Joel Fine on which later became Theorem 2.40. I am grateful to his availability and his patience during the last four years, which played an even more important role in this thesis than the aforementioned discussion.

I would like to thank Julien Keller for his guidance since 2017. His advices help me extend my mathematical culture and his comments on this text greatly improves its readability.

I thank my parents and my younger brother for being supportive of my career.

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Summary

Monotonicity theorems

The objects studied in this thesis are minimal submanifolds of a Riemannian manifold (X, g) . They are the critical points of the volume functional, and equivalently, have vanishing mean curvature. The most important tools in the study of such submanifolds in the Euclidean space are the Barrier Method and the Monotonicity Theorem. The latter says that the *density* of a minimal submanifold is an increasing function. To define the density, we look at the part of the submanifold inside a ball of radius r and normalise its volume by that of the equatorial disc of the ball. Concretely, for any point $p_0 \in \mathbb{R}^n$ and minimal k -submanifold Σ , the function

$$\Theta(r) = \frac{\text{vol}(\Sigma \cap B(p_0, r))}{\text{vol}(B^k)r^k} \quad (0.1)$$

is increasing in r .

In [And82], Anderson proved that the same result holds in the hyperbolic space \mathbb{H}^n . A different version of the monotonicity theorem for $X = \mathbb{H}^n$ and $X = S^n$ was proved by Choe and Gulliver [CG92a]. In this version, the volume of the submanifold and of the ball are weighted by the function $\cosh r$ (or $\cos r$ in S^n).

The first main result of this thesis is the following monotonicity theorem for minimal k -submanifolds of \mathbb{H}^n . We call a *Minkowskian coordinate* of \mathbb{H}^n a function ξ coming from pulling back a coordinate function of the Minkowski space $\mathbb{R}^{n,1}$ via the embedding of \mathbb{H}^n as the unit hyperboloid. The restricted Lorentz group $SO_+(n, 1)$ acts on these functions and preserves the Minkowskian norm of $d\xi$. We say that ξ is of time/space/null type if this norm is respectively -1 , $+1$ or 0 . Let a and t be positive real numbers, and ξ be a Minkowskian coordinate, we define a function $Q(a, t)$ by

$$Q(a, t) = \int_a^t (t^2 + \delta)^{k/2-1} dt + \frac{1}{ka} (a^2 + \delta)^{k/2}.$$

This function only depends on ξ via a parameter δ , given by $\delta = -1, 0, +1$ if ξ is of time, null, or space type.

Theorem 0.1 (1.33). *Let Σ be a minimal k -submanifold of \mathbb{H}^n and let a be the infimum of ξ on Σ . Suppose that $a > 0$, then the density*

$$\Theta(t) = \frac{\text{vol}(\Sigma \cap \{a < \xi < t\})}{Q(a, t)}$$

is an increasing function in t .

When ξ is of time type and $a = 1$, we recover the result of Anderson mentioned above.

We obtain Theorem 0.1 with the following strategy. We first prove its weighted version, which for a time coordinate is the aforementioned monotonicity of Choe–Gulliver. Then we point out a comparison mechanism, which produces more monotonicity results out of a given one.

We will see in Chapter 1 that each function whose Hessian is a multiple of the metric yields a weighted monotonicity theorem for minimal k -submanifolds of X . Such functions satisfy

$$\text{Hess } h = U \cdot g, \quad U \in C^0(X) \quad (0.2)$$

and such monotonicity theorems concern the volume of the submanifold between level sets of h , weighted by the function U . The denominator of (0.1) will be the weighted volume of a tube formed by flowing along the gradient field of h , a $(k-1)$ -submanifold. A different choice of this submanifold only changes the density $\Theta(r)$ by a constant and so does not affect the monotonicity result. Equation (0.2) is locally equivalent to a way to write X as a warped product. We will prove that:

Theorem 0.2 (1.8). *The weighted density of a minimal submanifold is increasing in the region $U > 0$ and decreasing in the region $U < 0$.*

When $X = \mathbb{R}^n$, the only functions satisfying (0.2) are the coordinates x_i and the distance $\rho := \sum x_i^2$. Equation (0.2) pulls back to the harmonic equation for the coordinates function x_i and therefore yields the Barrier Method. Theorem 0.2 for ρ is the classical version.

The Euclidean coordinates of S^n and the Minkowskian coordinates of \mathbb{H}^n also satisfy (0.2). These coordinates are obtained by embedding S^n as the unit sphere in \mathbb{R}^{n+1} and of \mathbb{H}^n as the hyperboloid in $\mathbb{R}^{n,1}$. Each point \mathbb{H}^n (or S^n) corresponds to a unique time coordinate (or Euclidean coordinate) that it minimises. Choe–Gulliver’s result follows from Theorem 0.2 for these functions. Meanwhile, each totally geodesic hyperplane of \mathbb{H}^n defines a space-coordinate and each point on the sphere at infinity S_∞ defines a null-coordinate. Theorem 0.2 is new in these cases.

The proof of Theorem 0.2 can be adapted for weaker analogues of minimal submanifolds such as stationary currents, varifolds or harmonic maps. We do this in the Appendix A.

Comparison lemma

We can define the P -density of Σ by replacing U with a different function P . The scope of Theorem 0.2 becomes significantly larger due to the following lemma:

Lemma 0.3 (1.15). *There is a transitive relation \ll among the weights such that if $P_1 \ll P_2$ then:*

1. *any submanifold Σ (not necessarily minimal) having increasing P_2 -density automatically has increasing P_1 -density. Moreover, the P_2 -density is always greater than the P_1 -density.*
2. *a submanifold Σ that has increasing P_1 -density may not have increasing P_2 -density. However, its P_2 -density is still always greater than its P_1 -density.*

In the unit ball \mathbb{B}^n , a submanifold can have five different densities, depending on the metric of the ball (the Poincaré metric, the Euclidean metric or the half-sphere metric) and whether we want to weight the volume or not. These densities can be compared in the following way:

$$\text{weighted hyperbolic} \gg \text{hyperbolic} \gg \text{Euclidean} \gg \text{spherical} \gg \text{weighted spherical.} \quad (0.3)$$

The relation (0.3) combines with Lemma 0.3 to give unweighted monotonicity theorems in the hyperbolic space. They are statements on how minimal submanifolds distribute their volume between level sets of Minkowskian coordinates (Theorem 1.33).

On the other hand, the Clifford torus of S^3 is minimal but not monotone in the unweighted density. This shows that we cannot go backwards in the last \gg of (0.3).

Because the Monotonicity Theorem 0.2 is in itself an inequality, it still holds when (0.2) is replaced by $\text{Hess } h \geq Ug$. Such function h appears as the distance function in Riemannian manifold with curvature bounded from above. The extension of Theorem 0.2 to this context will be presented in Appendix B. We will also see a continuous version of the chain (0.3) there.

Isoperimetric inequalities

Each Minkowskian coordinate ξ can be used to normalise the hyperbolic metric to a metric $\xi^{-2}g$ in the conformal infinity of \mathbb{H}^n . These are round metrics on S^{n-1} for time coordinates, flat metrics on $S^{n-1} \setminus \{\text{pt}\} = \mathbb{R}^{n-1}$ for null coordinates and the doubled hyperbolic metrics. The third kind of metric is given by putting the $(n-1)$ -dimensional hyperbolic metric on the two halves of the sphere at infinity S_∞ divided by the totally geodesic hyperplane associated to ξ . In Chapter 2, we will give upper bounds of Graham–Witten’s renormalised area of a minimal surface in term of the length of its boundary under these metrics. Because a complete minimal surface of \mathbb{H}^n has infinite area and perimeter, this can be considered as a renormalised version of the isoperimetric inequality.

In Proposition 0.4, the supremum in (0.4) is taken among round metrics on S_∞ . The metrics $\tilde{g}_0, \tilde{g}_1, \tilde{g}_{\text{light}}$ in (0.5) are the round/ the doubled hyperbolic/ the flat metrics associated to the time coordinate ξ_0 / the space coordinate ξ_1 / the null coordinate ξ_{light} . The estimate (0.4) was independently discovered by Jacob Bernstein [Ber21].

Proposition 0.4 (2.22, 2.23). *Let $\Sigma \subset \mathbb{H}^n$ be a minimal surface with ideal boundary $\gamma \subset S_\infty$ and $\mathcal{A}_R(\Sigma)$ its renormalised area, then*

$$\mathcal{A}_R(\Sigma) + \sup_{\text{round } \tilde{g}} |\gamma|_{\tilde{g}} \leq 0. \quad (0.4)$$

Moreover, if a Minkowskian coordinate ξ_* has minimum value $a > 0$ on Σ , then:

$$\mathcal{A}_R(\Sigma) + \begin{cases} \frac{1}{2} \left(a + \frac{1}{a} \right) |\gamma|_{\tilde{g}_0} \\ \frac{1}{2} \left(a - \frac{1}{a} \right) |\gamma|_{\tilde{g}_1} \\ \frac{1}{2} a |\gamma|_{\tilde{g}_{\text{light}}} \end{cases} \leq 0 \quad (0.5)$$

Minimal surfaces and knot invariants

The counting problem for minimal surfaces bounded by a given curve was initiated by Tomi–Tromba’s resolution of the embedded Plateau problem [TT78] and was later studied systematically by Brian White [Whi87b]. Let Σ be a surface with boundary and

- \mathcal{M} be the quotient space of $C^{k,\alpha}$ minimal immersions from Σ to \mathbb{R}^n by boundary-preserving diffeomorphisms.
- $\mathcal{C} = C^{k,\alpha}(\partial\Sigma, \mathbb{R}^n)$ be the space of boundary curves.

White proved that \mathcal{M} is a Banach manifold and that the boundary map $\Pi : \mathcal{M} \rightarrow \mathcal{C}$ is Fredholm and of index 0. Moreover, when we restrict to the space of embedded minimal surfaces in a convex body of \mathbb{R}^3 , the map Π is proper and an integer-valued degree can be given to a curve by counting with sign the number of minimal surfaces filling it.

A similar counting problem can be set up in \mathbb{H}^n for minimal surfaces with ideal boundary in the sphere at infinity. The case $n = 3$ was studied by Alexakis and Mazzeo [AM10] where \mathcal{M} is the space of properly embedded minimal surfaces of \mathbb{H}^3 and \mathcal{C} is the space of collections of circles in S^2 .

It is of great interest to upgrade Alexakis and Mazzeo’s result to dimension $n = 4$ because the degrees, depending only on the isotopy class of the boundary curve, are by definition knot/link invariants. This direction was recently investigated by Joel Fine [Fin21], who proves that the projection map $\Pi : \mathcal{M} \rightarrow \mathcal{C}$ is Fredholm and of index 0. He also shows that the only way properness can fail is when the complex structure of the minimal surface develops a node. The counting problem of minimal surfaces will be discussed in Chapter 2, Appendix C and Section 2.5. The contribution of this thesis is

Theorem 0.5 (2.40). *Let $L := L_1 \sqcup L_2$ be a separated union of two embedded $(k - 1)$ -submanifolds of S^{n-1} . Then we can rearrange L in its isotopy class so that there is no connected minimal k -submanifolds in \mathbb{H}^n whose ideal boundary is L .*

In particular, all Alexakis–Mazzeo degrees vanish for collections of more than 1 circle.

The proof of Theorem 0.5 is based on the observation that a minimal surface whose ideal boundary is a curve L can only pass by points that "see" L as longer than 2π . The volume of a submanifold of S_∞ seen from an interior point of \mathbb{H}^n is called the *visual volume* by Gromov and the set of such points is called the *visual hull* of γ [Gro83].

On the other hand, let H_a be the Hopf link $zw = a$ in $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, a family of explicit minimal annuli M_a of \mathbb{H}^4 whose ideal boundary is H_a will be constructed in Section 2.2.

It follows from Lemma 0.3 that the standard Hopf link H_0 has no minimal filling other than the union M_0 of two totally geodesic discs. Up to $SO(4)$, the annuli M_a provide a 2-parameter family of perturbations of M_0 . This suggests that links which bound nodal minimal surfaces only occurs in codimension 2.

Visual volume, first eigenvalue of the Laplacian and free boundary minimal surfaces

Each k -dimensional submanifold γ on the sphere S^n induces a positive function V_γ on \mathbb{H}^{n+1} whose value at a point p is given by the volume of γ under the round metric

associated to p . The visual hull is, in particular, an upper level set of this function. Chapter 3 is dedicated to the study of the visual volume function V_γ .

It was proved by Li and Yau [LY82] then El Soufi and Ilyas [EI83], that the maximum value of V_γ is an upper bound for the normalised first eigenvalue of the Laplacian on γ . Here we equip γ with any metric g in the conformal class induced by that of S^n and normalise the first eigenvalue by the volume of γ under g . This upper bound was the motivation behind Li–Yau’s definition of *conformal volume* and was utilised by them to verify the Willmore conjecture for certain tori. A historical account on this can be found in Section 3.1. The new results in Chapter 3 are:

Proposition 0.6 (3.19). *Let γ be a k -dimensional immersed submanifold of S^n and V_γ be its visual volume function. Then the restriction of V_γ to any minimal submanifold of \mathbb{H}^n of dimension at least $k + 1$ is super-harmonic.*

Because the Douglas–Rado Plateau problem can be solved in \mathbb{H}^n , it follows that:

Corollary 0.7 (3.27). *When γ is in S^n , the upper level sets of V_γ are simply-connected.*

It is easy to check if a point is a local maximiser of V_γ but very difficult to prove that it is a global maximiser. This question arises in the computation of the conformal volume [MR86; Bry15], and more recently in the work of Fraiser and Schoen on free boundary minimal submanifolds. For such a submanifold Σ of the Euclidean unit ball, one can check that $V_{\partial\Sigma}$ is locally maximised at the centre of the ball. Whether this is a global maximum is still an open question. I hope that Proposition 0.6 can be useful here.

The appendices

Several parts of this thesis can be safely skipped without affecting the readability of the text as a whole. For this reason, I put them in the appendices. The goal of Appendix A and Appendix B was explained above. Appendix C reviews of the counting problem for minimal surfaces in the Euclidean context. In Appendix D, I define Graham–Witten’s renormalised area and explain why it is an inherent quantity of the surface. It also contains a criteria for a quantity of a surface to be renormalisable. In particular, I proved that Sacks–Uhlenbeck’s α -energies [SU81] are renormalisable for a wide class of surfaces.

Résumé

Théorème de monotonie à poids

Dans cette thèse, nous nous intéressons aux sous-variétés minimales, c-à-d. à courbure moyenne nulle, d'une variété riemannienne (X, g) . Dans l'espace euclidien, les outils les plus importants dans l'étude de ces sous-variétés sont la technique de barrière et le théorème de monotonie. Ce dernier dit que la *densité* d'une telle sous-variété Σ^k , obtenue en normalisant son volume dans une boule de rayon r par le volume d'un k -disque de même rayon, c-à-d.

$$\Theta(r) = \frac{\text{vol}(\Sigma \cap B(p_0, r))}{\text{vol}(B^k)r^2} \quad (0.6)$$

est une fonction croissante en r .

Il était démontré par Anderson [And82] que le résultat est encore valable pour les sous-variétés de l'espace hyperbolique. Une autre version du théorème de monotonie pour $X = \mathbb{H}^n$ (et $X = S^n$) était obtenue par Choe et Gulliver [CG92a] où l'aire (de la sous-variété et de la boule) est pondérée par la fonction $\cosh r$ (respectivement $\cos r$).

Le premier résultat de cette thèse est la version suivante du théorème de monotonie pour les sous-variétés minimales de l'espace hyperbolique. Soit ξ une coordonnée minkowskienne, a et t deux nombres réels positifs, nous définissons

$$Q(a, t) = \int_a^t (t^2 + \delta)^{k/2-1} dt + \frac{1}{ka} (a^2 + \delta)^{k/2}$$

où $\delta = -1, 0$ ou $+1$ quand ξ est de genre temps, lumière ou espace.

Theorem 0.8 (1.33). *Soit Σ une sous-variété minimale à dimension k dans la région $\xi > a$. Alors la densité*

$$\Theta(t) = \frac{\text{vol}(\Sigma \cap \{a < \xi < t\})}{Q(a, t)}$$

est une fonction croissante en t .

Nous retrouvons, lorsque ξ est de genre temps et $a = 1$, le résultat susmentionné d'Anderson.

Pour obtenir le Théorème 0.8, nous démontrons d'abord sa version à poids (qui pour les coordonnées de genre temps redonne la monotonie de Choe et Gulliver), puis nous montrons une lemme de comparaison qui nous permet à redescendre vers la version sans poids.

Nous allons voir dans le chapitre 1 qu'il y a un théorème de monotonie associé à chaque fonction h de X dont le hessien est colinéaire au tenseur de métrique g . Concrè-

tement, une telle fonction satisfait

$$\text{Hess } h = U.g, \quad \text{pour } U \in C^0(X) \quad (0.7)$$

et le théorème associé concerne le volume pondéré par U de la sous-variété entre deux niveaux de h . On normalise ce volume par celui d'un tube, formé par l'image d'une sous-variété à dimension $k - 1$ sous le flot du gradient de h . Chaque fonction h qui satisfait (0.7) correspond à une façon de réaliser X comme produit tordu de variétés riemanniennes.

Theorem 0.9 (1.8). *La densité à poids d'une sous-variété minimale est croissante dans la région $U > 0$ et décroissante dans la région $U < 0$.*

Dans l'espace euclidien, les seules fonctions qui satisfont (0.7) sont les coordonnées x_i , pour lesquelles (0.7) redonne la technique de barrière, et la fonction distance $\rho := \sum x_i^2$ avec laquelle on retrouve la version classique du théorème de monotonie.

Les coordonnées euclidiennes de la sphère et les coordonnées minkowskiennes de l'espace hyperbolique satisfont aussi (0.7). On obtient ces coordonnées en plongeant S^n et \mathbb{H}^n dans \mathbb{R}^{n+1} et $\mathbb{R}^{n,1}$ comme la sphère et l'hyperboloïde unité. Géométriquement, chaque point de \mathbb{H}^n correspond à une coordonnée de genre temps (qui y est minimisée), et chaque point de la sphère S^n correspond à une coordonnée euclidienne. En appliquant le Théorème 0.9 à ces fonctions nous retrouvons le résultat de Choe et Gulliver. D'autre part, les coordonnées de genre espace de \mathbb{H}^n sont caractérisées par les codimensions 1 totalement géodésiques et les coordonnées de genre lumière par les points de la sphère à l'infini. Le Théorème 0.9 est nouveau dans ce cas.

Nous mettons en Appendice A les adaptations du Théorème 0.9 pour les sous-variétés minimales au sens faible. Le lecteur y trouvera les versions pour les courants stationnaires, les varifolds et les applications harmoniques.

Lemme de comparaison

Nous appelons la *densité à poids* P la quantité obtenue en remplaçant U dans la définition de la densité à poids par une autre fonction P . Le lemme suivant élargit le champ d'application du Théorème 0.9.

Lemma 0.10 (1.15). *Il existe une relation transitive \ll parmi les poids avec les propriétés suivantes. Soit Σ une sous-variété (non nécessairement minimale) et supposons que $P_1 \ll P_2$.*

1. *Si la densité à poids P_2 de Σ est croissante, alors la densité à poids P_1 l'est aussi et de plus elle est majorée par la première.*
2. *Si la densité à poids P_1 est croissante, nous ne pouvons rien conclure de la monotonie à poids P_2 . Par contre, nous avons toujours la majoration de la densité à poids P_1 par celle à poids P_2 .*

Il y a, pour une sous-variété de la boule unité, 5 densités différentes, provenant du choix de la métrique (Poincaré/ euclidienne/ sphérique) et du poids (uniforme/ temporel/

euclidien). Ces densités se comparent de manière suivante :

$$\text{hyperbolique à poids} \gg \text{hyperbolique} \gg \text{euclidien} \gg \text{sphérique} \gg \text{sphérique à poids.} \quad (0.8)$$

Cette relation nous permet de retrouver les théorèmes de monotonie à poids uniforme pour les sous-variétés minimales de \mathbb{H}^n . Ils sont des contraintes sur la distribution de volume d'une sous-variété minimale entre les niveaux d'une coordonnée minkowskienne (voir Théorème 1.33).

Étant de nature une inégalité, le Théorème 0.9 est encore valable lorsque nous remplaçons (0.7) par une inégalité de la forme $\text{Hess } h \geq Ug$. Une telle fonction h apparaît naturellement comme la fonction distance dans une variété riemannienne de courbure majorée. Nous adaptons le Théorème 0.9 dans ce contexte en Appendice B. Le lecteur y trouvera aussi la version continue de la chaîne (0.8).

Inégalités isopérimétriques

Chaque coordonnée minkowskienne définit une métrique dans la classe conforme à l'infini de \mathbb{H}^n . Elles sont des métriques rondes (de courbure $+1$) pour les coordonnées de genre temps, des métriques plates sur $S^{n-1} \setminus \{\text{pt}\} = \mathbb{R}^{n-1}$ pour les coordonnées de genre lumière et des métriques hyperboliques doublées le long du bord de la codimension 1 pour les coordonnées de genre espace. Dans le chapitre 2, nous utilisons le Théorème 0.9 pour majorer l'aire renormalisée de Graham–Witten d'une surface minimale en terme de la longueur de son bord idéal pour ces métriques. Nous pouvons regarder ces résultats comme des inégalités isopérimétriques pour les surfaces minimales complètes de \mathbb{H}^n .

Dans la Proposition 0.11 ci-dessous, le supremum de (0.9) est pris parmi les métriques rondes sur S^{n-1} . Les métriques $\tilde{g}_0, \tilde{g}_1, \tilde{g}_{\text{light}}$ dans (0.10) sont respectivement les métriques rondes/ hyperboliques doublées/ plates, associées aux coordonnées de genre temps ξ_0 / de genre espace ξ_1 / de genre lumière ξ_{light} . L'inégalité (0.9) était démontrée indépendamment par Jacob Bernstein [Ber21].

Proposition 0.11 (2.22, 2.23). *Soit $\Sigma \subset \mathbb{H}^n$ une surface minimale dont le bord idéal est donné par une courbe $\gamma \subset S^{n-1}$ et $\mathcal{A}_R(\Sigma)$ l'aire renormalisée de Σ . Alors :*

$$\mathcal{A}_R(\Sigma) + \sup_{\text{round } \tilde{g}} |\gamma|_{\tilde{g}} \leq 0 \quad (0.9)$$

Plus généralement, soit ξ_* une coordonnée minkowskienne et $a > 0$ sa valeur minimale sur Σ , alors :

$$\mathcal{A}_R(\Sigma) + \begin{cases} \frac{1}{2} \left(a + \frac{1}{a} \right) |\gamma|_{\tilde{g}_0} \\ \frac{1}{2} \left(a - \frac{1}{a} \right) |\gamma|_{\tilde{g}_1} \\ \frac{1}{2} a |\gamma|_{\tilde{g}_{\text{light}}} \end{cases} \leq 0 \quad (0.10)$$

Application à la théorie de nœuds

L'idée de compter le nombre de surfaces minimales dont le bord est une courbe donnée est due à Tomi et Tromba dans leur résolution du problème de Plateau pour les surfaces

plongées [TT78]. Cette idée a été développée de manière systématique par Brian White [Whi87b]. Soit Σ une surface à bord et

- \mathcal{M} le quotient de l'espace des immersions minimales de Σ dans \mathbb{R}^n par le sous-groupe des difféomorphismes de Σ qui fixent $\partial\Sigma$.
- $\mathcal{C} = C^{k,\alpha}(\partial\Sigma, \mathbb{R}^n)$ l'espace de courbes dans \mathbb{R}^n .

Alors \mathcal{M} est une variété de Banach et que l'application bord $\Pi : \mathcal{M} \rightarrow \mathcal{C}$ est Fredholm et d'indice 0. De plus, Π est propres si on se restreint aux surfaces plongées (au lieu d'immergées) dans un ensemble convexe de \mathbb{R}^3 . Dans ce cas, nous pouvons associer à chaque courbe dans \mathcal{C} un entier, qui est le degré de l'application Π .

On peut aussi penser à compter le nombre de surfaces minimales de \mathbb{H}^n dont le bord idéal est une courbe donnée dans la sphère à l'infini. Ce problème était étudié dans \mathbb{H}^3 par Alexakis et Mazzeo [AM10] pour l'espace \mathcal{M} des surfaces minimales proprement plongées et \mathcal{C} des collections de cercles dans S^2 .

Il est de grand intérêt d'étendre la théorie des degrés d'Alexakis et Mazzeo à la dimension 4, avec \mathcal{C} l'espace des courbes plongées dans la sphère S^3 . Les degrés, qui ne dépendent que de la classe d'isotopie du bord, sont des invariants de nœuds/entrelacs. Cette piste de recherche était récemment explorée par Joel Fine [Fin21]. Nous allons la discuter en détail dans l'Appendice C et la Section 2.5 du Chapitre 2.

La contribution de cette thèse dans la direction susmentionnée est le théorème suivant.

Theorem 0.12 (2.40). *Soit $L := L_1 \sqcup L_2$ une union séparée de deux sous-variétés plongées à dimension $k - 1$. Alors il est toujours possible de réarranger L dans sa classe d'isotopie de manière qu'il n'existe pas de surface minimale connexe de \mathbb{H}^n dont le bord idéal est L .*

En particulier, les degrés d'Alexakis–Mazzeo s'annulent pour toute collection qui contient plus qu'un cercle.

Le Théorème 0.12 se découle d'une observation clé : une surface minimale de \mathbb{H}^n de bord idéal une courbe γ ne peut contenir que les points intérieurs qui voient γ de longueur supérieure à 2π . L'ensemble de tels points est appelé la *clôture visuelle* de γ [Gro83].

D'autre part, soit H_a l'entrelacs de Hopf donné par l'équation $zw = a$ dans $S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$, on va construire dans la Section 2.2 une couronne minimale $M_a \subset \mathbb{H}^4$ de bord idéal H_a . D'autre part, il n'y a qu'une seule surface minimale admettant H_0 comme bord idéal : l'union de deux disques totalement géodésiques.

Volume visuel, première valeur propre du Laplacien et surfaces minimales au bord libre

Nous pouvons associer à chaque sous-variété γ à dimension k de la sphère S^n une fonction positive V_γ sur \mathbb{H}^{n+1} dont la valeur dans un point p est donnée par la volume de γ sous la métrique ronde associée à p . La clôture visuelle est, en particulier, un super-niveau de V_γ . Le chapitre 3 est dédié à l'étude de cette fonction.

Il a été observé par Li et Yau [LY82], puis El Soufi et Ilyas [EI83], que le maximum de V_γ est une borne supérieure naturelle de la première valeur propre normalisée du laplacien de γ . Ici le laplacien provient d'une métrique g dans la classe conforme de γ induite par celle de S^n et sa première valeur propre est normalisée par le volume de γ

sous g . Cette observation a motivé Li et Yau à définir leur notion de *volume conforme* qu'ils ont utilisée ultérieurement pour vérifier partiellement la conjecture de Willmore. Un résumé de ce développement se trouve dans la Section 3.1. Les nouveaux résultats du chapitre 3 sont :

Proposition 0.13 (3.19). *Soit γ une sous-variété immergée à dimension k de S^n . Alors la restriction de V_γ à une sous-variété minimale de \mathbb{H}^n à dimension au moins $k + 1$ est super-harmonique.*

Nous obtenons, par la résolution du problème de Plateau dans \mathbb{H}^n au sens de Douglas–Rado que

Corollary 0.14 (3.27). *Pour une courbe γ dans S^n , les super-niveaux de V_γ dans \mathbb{H}^{n+1} , y inclus la clôture visuelle, sont simplement connexes.*

Il est facile de vérifier si un point est maximum local de V_γ , mais très difficile de montrer s'il est un maximum global. Cette question se pose dans les calculs du volume conforme, et plus récemment dans le travail de Fraiser et Schoen sur les sous-variétés minimales à bord libre. Pour une telle sous-variété Σ de la boule unité euclidienne, il est démontré par Fraiser et Schoen que la fonction $V_{\partial\Sigma}$ est localement maximisée à l'origine. On ne sait toujours pas s'il s'agit d'un maximum global et j'espère que la Proposition 0.13 sera utile pour avancer sur cette question.

1 Weighted monotonicity theorems and comparison lemma

The goal of this chapter is to prove that:

1. Each function h on a Riemannian manifold (M, g) whose Hessian is a multiple of the metric tensor corresponds to a weighted monotonicity theorem for minimal submanifolds of M .
2. Weighted monotonicity theorems can be compared to each other.

Such a function h corresponds to a way to write M as a warped space.

1.1 Warped spaces and examples

A metric on a Riemannian manifold $M = N \times [a, b]$ is called a *warped product* if it has the form

$$g = dr^2 + f^2(r)g_N \quad (1.1)$$

where $r \in [a, b]$ and g_N is a Riemannian metric on N . It can be checked that an anti-derivative h of the warping function f satisfies $\text{Hess}(h) = f'(r)g$. On the other hand, if such function h exists, the space can be locally written as a warped space by the level sets of h .

Proposition 1.1 (cf. [CC96]). *Let h be a C^2 function on (M, g) with no critical point. Suppose that the level sets of h are connected and*

$$\text{Hess } h = U \cdot g \quad (1.2)$$

for a function $U \in C^0(M)$. Then:

1. U is a function of h , i.e. a composition of h and a function $U : \mathbb{R} \rightarrow \mathbb{R}$. The function $V := |dh|^2 \in C^1(M)$ is also a function of h and we have $U = \frac{1}{2}V'$.
2. The metrics g_a, g_b induced from g on the level sets $h^{-1}(a)$ and $h^{-1}(b)$ are related by $\frac{g_a}{V(a)} = \frac{g_b}{V(b)}$ via the inverse gradient flow of h . This defines a metric \tilde{g} on level sets of h under which the flow is isometric. The inverse gradient flow of h defines a map $h^{-1}(a) \times \text{Range}(h) \rightarrow M$ and the metric g on M pulls back to

$$g = \frac{V(h)}{V(a)}g_a + \frac{dh^2}{V(h)} = V(h)\tilde{g} + \frac{dh^2}{V(h)} \quad (1.3)$$

which is a warped product after a change of variable $dr = \frac{dh}{V(h)^{1/2}}$.

Proof. For any vector field v , we have:

$$v(V) = 2g(\nabla_v \nabla h, \nabla h) = 2 \text{Hess}(h)(v, \nabla h) = 2Ug(v, \nabla h)$$

It follows, by first taking v to be any vector field tangent to level sets of h , then to be the inverse gradient $u := \frac{\nabla h}{|\nabla h|^2}$, that V is constant on the level sets, and as a function of h , $V' = 2U$.

For the second part, let v_t be the vector field of M tangent to level sets $h^{-1}(t)$ given by pushing forward via the flow of u a vector field v_a tangent to $h^{-1}(a)$. By definition of Lie bracket, $[v_t, u] = 0$ for all t , and so

$$\frac{d}{dt}|v_t|^2 = 2g(\nabla_u v_t, v_t) = 2g(\nabla_{v_t} u, v_t) = \frac{2}{|\nabla h|^2} \text{Hess}(h)(v_t, v_t) = \frac{V'}{V}|v_t|^2.$$

Here in the third equality, we used the fact that v_t is orthogonal to the gradient of h . We conclude that $\frac{|v_t|^2}{V(t)}$ is constant along the flow and so $\frac{g_a}{V(a)} = \frac{g_b}{V(b)}$ for all a, b in the range of h . \square

The metric \tilde{g} is the g_N of (1.1) and it is conformal to the restriction of g on level sets of h . We will use \tilde{g} to denote the metric $V(h)^{-1}g$ on M and will call it the *normalised metric*.

In applications, we will only assume that the function h satisfies (1.2) on M and it can have critical points, as in the following examples.

Example 1.2. *In the Euclidean space, it is not difficult to prove that the only functions satisfying (1.2) are*

1. the coordinates $x_i, i = 1, \dots, n$ with $U = 0, V = 1$ and
2. the square of distance $\rho := \frac{1}{2} \sum_{i=1}^n x_i^2$ with $U = 1, V = 2\rho$.

Example 1.3. *In the unit sphere*

$$S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\},$$

the Euclidean coordinates x_i satisfy (1.2) with

$$U = -x_i, \quad V = 1 - x_i^2.$$

Example 1.4. *In the hyperbolic space*

$$\mathbb{H}^n = \{(\xi_0, \dots, \xi_n) \in \mathbb{R}^{n,1} : \xi_0^2 - \sum_{i=1}^n \xi_i^2 = 1, \xi_0 > 0\},$$

the Minkowskian coordinates ξ_α satisfy (1.2) with

$$U = \xi_\alpha, \quad V = \xi_\alpha^2 - |\partial_{\xi_\alpha}|^2,$$

where $|\partial_{\xi_\alpha}|^2$ is the Minkowskian norm, which is $+1$ for time-like unit vectors and -1 for space-like vectors.

1.2. Weighted monotonicity theorems in Warped space

By Proposition 1.1, the hyperbolic space can be written as a warped product in three distinct ways:

1. Each interior point corresponds to a unique time coordinate that it minimises.
2. Each boundary point corresponds to a half-space model, thus defines a null coordinate uniquely up to a multiplicative constant.
3. Each (cooriented) totally geodesic hyperplane corresponds to a unique space coordinate that vanishes on it. Note that no other level set is totally geodesic.

The normalised metrics \tilde{g} in the case (\mathbb{R}^n, ρ) and (S^n, x_i) are the round metric on S^{n-1} . In the hyperbolic space, there are 3 types of normalised metrics in the conformal infinity of \mathbb{H}^n . Those are round metrics for time coordinates, flat metrics for null coordinates (the Euclidean metric in half-space model) and the doubled hyperbolic metrics for space coordinates. The third metrics are obtained by glueing two $(n-1)$ -dimensional hyperbolic metrics on the hemispheres along a totally geodesic hyperplane.

The three examples 1.2, 1.3, 1.4 are summarized in the following table

(M, g)	h	$U(h)$	$V(h)$	\tilde{g}
\mathbb{R}^n	Euclidean coordinate x_i	0	1	flat
	$\rho = \sum_i x_i^2$	1	2ρ	round
$S^n \hookrightarrow \mathbb{R}^{n+1}$	Euclidean coordinate x_i	$-x_i$	$1 - x_i^2$	round
$\mathbb{H}^n \hookrightarrow \mathbb{R}^{1,n}$	time coordinate ξ_0	ξ_0	$\xi_0^2 - 1$	round
	null coordinate ξ_l	ξ_l	ξ_l^2	flat
	space coordinate ξ_1	ξ_1	$\xi_1^2 + 1$	doubled hyperbolic

1.2 Weighted monotonicity theorems in Warped space

Given a function h on a Riemannian manifold M and a submanifold Σ , we write $\int_{\Sigma, h \leq t}$ and $\int_{\Sigma, h=t}$ for the integration over the sub-level $h^{-1}[t, +\infty)$ and the level set $h = t$ in Σ .

The gradient vector of h in M is denoted by ∇h and its projection to the tangent of Σ by $\nabla^\Sigma h$. The rough Laplacian of h on Σ is

$$\Delta^\Sigma h = \operatorname{div}_\Sigma \nabla^\Sigma h$$

The volume of the k -dimensional unit sphere will be denoted by ω_k .

1.2.1 Weighted volume and density

Let h be a C^2 function satisfying (1.2) on M and $V = |dh|^2$, we will define the *weighted volume* of a k -submanifold Σ to be

$$A_h(\Sigma)(t) := \int_{\Sigma, h \leq t} U. \quad (1.4)$$

The weighted density is obtain by normalising the weighted volume of Σ by that of a k -dimensional tube. Let us suppose for a moment that h is bounded from below by h_{\min} . Given a $(k-1)$ -dimensional submanifold γ in a level set of h , the h -tube $T_\gamma(t)$ is built by flowing γ along the gradient field of h among level sets $h_{\min} \leq h \leq t$. The weighted volume of $T_\gamma(t)$ can be computed by (1.3) to be

$$A_h(T_\gamma)(t) = \int_{T_\gamma} U = \left(\int_\gamma d \text{vol}_g^{k-1} \right) \left(\int_{h_{\min}}^t V'(h) V^{\frac{k}{2}-1}(h) dh \right) = \frac{|\gamma|}{k} V(t)^{k/2} \quad (1.5)$$

where $|\gamma|$ is the \tilde{g} -volume of γ . The last equality is because $V(h_{\min}) = 0$.

We define the *weighted density* of Σ as

$$\Theta_h^A(t) := \frac{A_h(\Sigma)(t)}{\frac{\omega_{k-1}}{k} V^{k/2}(t)}. \quad (1.6)$$

Equation (1.5) shows that, as a function of t , the weighted volume of a tube is, up to a factor, independent of γ . In other words, the density of a tube is constant.

When h is not bounded from below, or when the area of Σ is not finite in the region $h \leq t$, (this happens for example when h is the space and null coordinates of \mathbb{H}^n). We define the *compensated weighted volume* by counting only the volume in the region $h \geq h_0$:

$$B_h(\Sigma)(h_0, t) := \int_{\Sigma, h_0 \leq h \leq t} U(h) + \frac{1}{k} \int_{\gamma_0} |\nabla^\Sigma h|. \quad (1.7)$$

Here γ_0 is the intersection of Σ and the level set $h = h_0$. The *compensated weighted density* is defined as

$$\Theta_h^B(t) := \frac{B_h(t)}{\frac{\omega_{k-1}}{k} V^{k/2}(t)}.$$

It is useful to give a name to the boundary term in (1.7).

Definition 1.5. Let T be a distribution of k -dimensional planes of M along a submanifold γ of a level set of h . The T -parallel volume $|\gamma^T|$ of γ is defined as

$$|\gamma^T| := \int_\gamma \cos \angle(\nabla h, T) \text{vol}_g^{k-1},$$

where the integral was taken with the volume form of the normalised metric \tilde{g} and $\angle(\nabla h, T)$ is the angle between ∇h and T . When ∇h is contained in T at every point of γ , the parallel volume is the \tilde{g} -volume.

The volume forms of g and \tilde{g} on γ are related by $\text{vol}_g^{k-1} = \text{vol}_{\tilde{g}}^{k-1} V^{\frac{k-1}{2}}$ and the angle $\angle(\nabla h, T)$ satisfies $\cos \angle(\nabla h, T) \cdot V^{1/2} = |\nabla^\Sigma h|$. So the compensated volume can

be rewritten as

$$B_h(\Sigma)(t) := \int_{\Sigma, h_0 \leq h \leq t} U(h) + \frac{|\gamma_0^{T\Sigma}|}{k} V(h_0)^{k/2}.$$

The denominator of Θ_h^B is again, up to constant, the compensated weighted volume of a tube $T_\gamma(h_0, t)$ built by the gradient flow of a $(k-1)$ -submanifold γ from level h_0 to level t . This can be seen via a computation similar to (1.5):

$$B_h(T_\gamma(h_0, t)) = \frac{|\gamma|}{k} (V(t)^{k/2} - V(h_0)^{k/2}) + \frac{|\gamma|}{k} V(h_0)^{k/2} = \frac{|\gamma|}{k} V(t)^{k/2}. \quad (1.8)$$

Recall that $|\gamma|$ is the volume of γ under \tilde{g} .

1.2.2 Compensated or Uncompensated?

The condition (1.2) forces all non-degenerate critical points of h to be either local maxima or local minima. The functions in Examples 1.2, 1.3, 1.4 fall into two types:

1. h has no other critical value than its minimum h_{\min} and the sublevel sets $\{h \leq t\}$ are compact. This is the case of (\mathbb{R}^n, ρ) , $(S^n \setminus \{\text{pt}\}, x_i)$ and (\mathbb{H}^n, ξ_0) . Both densities Θ_h^A and Θ_h^B are well-defined.
2. h has no critical point and the sublevel sets are no longer compact, as in the case of \mathbb{H}^n with the null and space coordinates. The integral in (1.4) may not be finite and we are therefore interested in the compensated density Θ_h^B .

We will see that it is useful to look at the compensated density even when the function h is of the first type.

1.2.3 Weighted monotonicity theorem

The following lemma often appears in the literature as

$$\text{div}_\Sigma X^\Sigma = \text{div}_\Sigma X + g(k(\Sigma), X)$$

for a submanifold Σ and a vector field X along Σ . Here $k(\Sigma)$ is the mean curvature and X^Σ is the tangent component of X . We will denote the gradient vector in M of a function h by ∇h and its projection to Σ by $\nabla^\Sigma h$.

Lemma 1.6 (Leibniz rule). *Let $f : (\Sigma^k, g_\Sigma) \rightarrow (M^n, g)$ be a map between Riemannian manifolds and $\tau(f)$ be its tension field¹, then for any C^2 function h on M . We have:*

$$\Delta_\Sigma(h \circ f) = \text{Tr}_\Sigma f^* \text{Hess } h + dh \cdot \tau(f) \quad (1.9)$$

In particular, the Laplacian of h on a submanifold Σ is given by

$$\Delta_\Sigma h = \text{Tr}_\Sigma \text{Hess } h + dh \cdot k(\Sigma). \quad (1.10)$$

In particular, if $k(\Sigma) = 0$ at a point $p \in \Sigma$ or $T_p \Sigma$ contains the gradient of h , then $\Delta_\Sigma h = \text{Tr}_\Sigma \text{Hess } h$ at p .

1. The definition of the tension field can be found in Appendix A. It is the mean curvature field when f is isometric.

Definition 1.7. Let Σ be a k -dimensional submanifold of M whose boundary is a $(k-1)$ -dimensional submanifold γ (not necessarily contained in a level set of h). The h -tube extension of Σ is the union of Σ and the image of γ under the gradient flow of h .

Theorem 1.8 (Weighted Monotonicity). Suppose that h is a C^2 function on a Riemannian manifold M that satisfies (1.2) with U and $V = |\nabla h|^2$ being functions of h such that $U = \frac{1}{2}V'$. Let Σ^k be a minimal submanifold in M . Assume that the integral in (1.4) (respectively (1.7)) is finite, then the derivative of the weighted density $\frac{d}{dt}\Theta_h^A$ (respectively $\frac{d}{dt}\Theta_h^B$) has the same sign as U .

Moreover, if Σ is a minimal submanifold whose boundary γ is a smooth $(k-1)$ -submanifold of M , the conclusion still holds for a tube extension $\tilde{\Sigma}$ of Σ .

In both cases, $\frac{d}{dt}\Theta_h^A$ and $\frac{d}{dt}\Theta_h^B$ vanish at $t = t_0$ if and only if the gradient of h is tangent to Σ at level $h = t_0$.

Proof. It follows from Lemma 1.6 that $\Delta_\Sigma h = kU$ on Σ . By Stokes' theorem,

$$A_h(\Sigma)(t) = \int_{\Sigma, h \leq t} U(h) = \frac{1}{k} \int_{\Sigma, h \leq t} \Delta_\Sigma h = \frac{1}{k} \int_{\Sigma, h=t} \nabla^\Sigma h \cdot n = \frac{1}{k} \int_{\Sigma, h=t} |\nabla^\Sigma h| \quad (1.11)$$

because the outer normal of the sublevel set $h \leq t$ in Σ is $n = \frac{\nabla^\Sigma h}{|\nabla^\Sigma h|}$. By the coarea formula,

$$\frac{dA_h(t)}{dt} = U(t) \int_{\Sigma, h=t} \frac{1}{|\nabla^\Sigma h|}.$$

Combining this with (1.11) and $|\nabla^\Sigma h|^2 \leq V(h)$, one has $\frac{1}{U} \frac{dA_h}{dt} \geq \frac{kA_h}{V}$, or

$$\frac{1}{U} \frac{d}{dt} \left(\frac{A_h}{V^{k/2}} \right) \geq 0.$$

Similarly, one has $B_h(t) = \frac{1}{k} \int_{\Sigma, h=t} |\nabla^\Sigma h|$, and $\frac{dB_h}{dt} = U(t) \int_{\Sigma, h=t} \frac{1}{|\nabla^\Sigma h|}$ and the same conclusion is drawn for the compensated case.

For tube extension, it suffices to rewrite equation (1.11) as

$$\begin{aligned} kA_h(t) &= \left(\int_{\Sigma, h \leq t} + \int_{T_\gamma, h \leq t} \right) \Delta h \\ &= \int_{\Sigma, h=t} |\nabla^\Sigma h| + \int_{\gamma \cap \{h \leq t\}} |\nabla^\Sigma h| + \int_{T_\gamma, h=t} |\nabla^M h| - \int_{\gamma \cap \{h \leq t\}} |\nabla^M h| \\ &\leq \int_{\Sigma, h=t} |\nabla^\Sigma h| + \int_{T_\gamma, h=t} |\nabla^M h| = \int_{\tilde{\Sigma}, h=t} |\nabla^{\tilde{\Sigma}} h|. \end{aligned}$$

□

Remark 1.9. 1. If Σ contains a multiple of γ , the volume of the tube should be counted with multiplicity.

2. We only need the " \leq " sign in (1.11) and hence it suffices that $\Delta_\Sigma h \geq kU(h)$. Theorem 1.8 still holds if $\text{Hess } h \geq U.g$, provided that U and $V = |dh|^2$ are still functions of h and that $U = \frac{1}{2}V'$. This direction is explored in Appendix B.

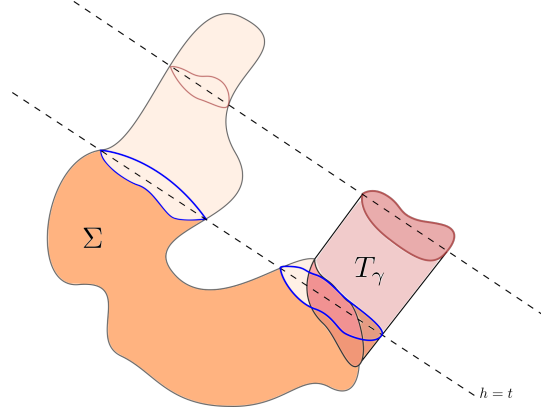


Figure 1.1 – Extension by tube of a surface. The tube does not need to be built over the intersection with a level set of h .

In practice, we are interested in the situations where Σ is contained in a region where U is signed. The U-shape minimal surface in Figure 1.1 for example cannot happen in the region $U \leq 0$ because the function h is super-harmonic on Σ : $\Delta h = kU \leq 0$.

Lemma 1.10. *Let Σ be a minimal submanifold of the region $U \geq 0$ such that $h \leq t_0$ on the boundary of Σ , then $h \leq t_0$ on the entire submanifold.*

1.3 Comparison lemma

We will suppose in this section that the submanifold Σ is contained in the region $U \geq 0$. Theorem 1.8 says that the weighted density is an increasing function. It is useful to weight the volume functional of (1.4) by a function P of h other than U .

The *uncompensated* and *compensated P -volumes* are defined as

$$A_P(\Sigma)(t) := \int_{\Sigma, h \leq t} P(h) \quad \text{or} \quad B_P(\Sigma)(t) := \int_{\Sigma, h_0 \leq h \leq t} P(h) + \frac{c}{k} |\gamma_0^{T\Sigma}| V(h_0)^{k/2} \quad (1.12)$$

where c is a positive real number. The term $|\gamma_0^{T\Sigma}|$ here is the $T\Sigma$ -parallel volume of the intersection of Σ and the level set $h = h_0$.

The uncompensated and compensated P -volumes of tubes are

$$A_P(T_\gamma(t)) = |\gamma| \int_{h \leq t} P(h) V^{\frac{k}{2}-1}(h) dh,$$

$$B_P(T_\gamma(h_0, t)) = |\gamma| \left(\int_{h=h_0}^t P(h) V^{\frac{k}{2}-1}(h) dh + \frac{c}{k} V(h_0)^{k/2} \right)$$

They are still, up to a factor, independent of the choice of γ .

We define the *P -densities* of a submanifold by normalising its P -volume by that of a tube. Concretely,

$$\Theta_P(\Sigma)(t) := \frac{A_P(t)}{Q(t)} \quad \text{or} \quad \frac{B_P(t)}{Q(t)}$$

where

$$Q(t) := \begin{cases} \omega_{k-1} \int_{h \leq t} P(h) V^{\frac{k}{2}-1}(h) dh, & \text{for the uncompensated volume,} \\ \omega_{k-1} \left(\int_{h=h_0}^t P(h) V^{\frac{k}{2}-1}(h) dh + \frac{c}{k} V(h_0)^{k/2} \right) & \text{for the compensated volume} \end{cases} \quad (1.13)$$

When $P = U$ and $c = 1$, these are the weighted volume and weighted density defined in (1.4) and (1.7). We will always assume that the weight function P is positive on Σ .

Remark 1.11. *The compensated P -density of any submanifold Σ has limit*

$$\lim_{t \rightarrow h_0} \Theta_P(\Sigma)(t) = \frac{\int_{\Sigma, h=h_0} |\nabla^\Sigma h|}{\omega_{k-1} V(h_0)^{k/2}} = \frac{|\gamma_0^{T\Sigma}|}{\omega_{k-1}}, \quad (1.14)$$

provided $\mathcal{H}^k(\Sigma \cap h^{-1}(h_0)) = 0$. This is satisfied for example when Σ meets the level set $h = h_0$ transversely. If Σ is strictly contained in the region $h > h_0$, then

$$\lim_{t \rightarrow h_0} \Theta_P(\Sigma)(t) = 0.$$

Therefore, the limit at $t = h_0$ of the density Θ_P does not depend on the weight function P and the constant c .

Example 1.12. *On the round sphere $S^n = \mathbb{R}^n \cup \{\infty\}$, with the metric $g_S = \frac{4}{(1+r^2)^2} g_E$, the function $x := \frac{1-r^2}{1+r^2}$ is the coordinate function of Example 1.3 that is maximised at the origin and minimised at infinity. The g_E -volume functional is a P -volume with $P = (1+x)^{-k}$.*

Example 1.13. *In the Poincaré ball \mathbb{B}^n with metric $g_H = \frac{4}{(1-r^2)^2} g_E$, the function $\xi_0 := \frac{1+r^2}{1-r^2}$ is the time coordinate minimised at the centre. The g_E -volume corresponds to $P = (1+\xi_0)^{-k}$. The volume under the metric g_S above corresponds to $P = \xi_0^{-k}$ and the x -weighted g_S -volume corresponds to $P = \xi_0^{-k-1}$.*

Definition 1.14. *Given 2 weight functions P_1, P_2 (respectively $(P_1, c_1), (P_2, c_2)$) whose tube volumes Q_1, Q_2 are defined by (1.13), we say that P_1 is weaker than P_2 (respectively (P_1, c_1) is weaker than (P_2, c_2)) if*

$$\frac{P_1}{Q_1} \leq \frac{P_2}{Q_2}, \text{ or equivalently } \frac{d}{dt} \frac{Q_1}{Q_2} \leq 0.$$

In other words, the P_2 -volume of a k -dimensional tube increases faster than its P_1 -volume.

Clearly, this is a transitive relation. Moreover, $(P, c_1) \ll (P, c_2)$ if and only if $c_1 > c_2$.

Lemma 1.15 (Comparison). *Let $\Sigma^k \subset M^n$ be a submanifold not necessarily minimal and P_1, P_2 (respectively $(P_1, c_1), (P_2, c_2)$ in the compensated case) be two positive continuous weight functions. Let Θ_1, Θ_2 be the corresponding densities. In the compensated case, we suppose, in addition, that:*

(Non-tangency) Σ meets the level set $h = h_0$ non-tangentially, i.e. the intersection $\gamma_0 = \Sigma \cap h^{-1}(h_0)$ is either empty or has zero \mathcal{H}^k -volume.

1. Suppose that P_1 is weaker than P_2 and that Θ_2 is an increasing function. Then $\Theta_1 \leq \Theta_2$ and

$$\frac{d\Theta_1}{dt} \geq \frac{Q_2}{P_2} \frac{P_1}{Q_1} \frac{d\Theta_2}{dt}.$$

In particular, the density Θ_1 is also increasing.

2. On the other hand, if P_2 is weaker than P_1 and Θ_2 is increasing, then $\Theta_1 \geq \Theta_2$. Moreover, if $\frac{d\Theta_2}{dt}$ vanishes at a certain $t_0 \in \mathbb{R}$, then $\frac{d\Theta_1}{dt}(t_0) \leq 0$.

Proof. One has $P_1^{-1} \frac{dA_{P_1}}{dt} = P_2^{-1} \frac{dA_{P_2}}{dt}$ (or $P_1^{-1} \frac{dB_{P_1}}{dt} = P_2^{-1} \frac{dB_{P_2}}{dt}$ respectively), from coarea formula, therefore

$$\frac{Q_1}{P_1} \frac{d\Theta_1}{dt} + \omega_{k-1} V^{\frac{k}{2}-1} \Theta_1 = \frac{Q_2}{P_2} \frac{d\Theta_2}{dt} + \omega_{k-1} V^{\frac{k}{2}-1} \Theta_2 \quad (1.15)$$

which can be rearranged into

$$P_1^{-1} \frac{d}{dt} (Q_1(\Theta_1 - \Theta_2)) = \left(\frac{Q_2}{P_2} - \frac{Q_1}{P_1} \right) \frac{d\Theta_2}{dt} \quad (1.16)$$

Let us prove the second part of the Lemma. We see from the hypothesis that the RHS of (1.16) is positive, and therefore the function $Q_1(\Theta_1 - \Theta_2)$ is increasing. This function vanishes at $t = h_{\min}$ in the uncompensated case because of Q_1 , and at $t = h_0$ in the compensated case because of $\Theta_1 - \Theta_2$ (Remark 1.11). Therefore one has $\Theta_1 \geq \Theta_2$ at all time. Apply this and the fact that $\frac{dQ_1}{dt} \geq 0$ in (1.16), we see that if $\frac{d\Theta_2}{dt}(t_0) = 0$ then $\frac{d\Theta_1}{dt}(t_0) \leq 0$

For the first part, the RHS of (1.16) is negative and so, for the same reason given above, we have $\Theta_2 \geq \Theta_1$. The conclusion follows by substituting this into (1.15). \square

A more robust proof for rectifiable varifolds and currents can be found in Appendix B.

Remark 1.16. For minimal submanifolds, the non-tangency condition is even less restrictive than transversality to the level $h = h_0$. In particular, it is automatically satisfied if Σ is a minimal submanifold such that $\Sigma \cap \{h \leq h_0\}$ is a submanifold with boundary in the level $h = h_0$. This is because

$$\int_{\Sigma, h=h_0} |\nabla^\Sigma h| = \int_{\Sigma, h \leq h_0} kU. \quad (1.17)$$

and the LHS of (1.17) is positive if Σ is in the region $U > 0$. Similarly, we could not have $\mathcal{H}^k(\Sigma \cap h^{-1}(h_0)) > 0$ because the restriction of h to a minimal submanifold Σ cannot be locally constant.

1.3.1 Volume comparison with tubes

Let Σ be a k -dimensional submanifold of M whose boundary γ is a submanifold of the level set $h^{-1}(t_0)$. Let $\tilde{\Sigma}$ be the extension of Σ by the tube $T_\gamma(t_0, T)$. In the region $t \leq t_0$,

the monotonicity of $\tilde{\Sigma}$ does not reveal any new information other than the monotonicity of Σ .

The monotonicity of $\tilde{\Sigma}$ in the region $t > t_0$ is, on the other hand, equivalent to a comparison between the weighted volume of Σ in the region $h \leq t_0$ and that of a tube (see Figure 1.2).

Lemma 1.17. *Let $t_1 < t_2$ be two numbers in $[t_0, T)$, then the following statements are equivalent:*

1. $\Theta_P^A(\tilde{\Sigma})(t_1) \leq \Theta_P^A(\tilde{\Sigma})(t_2)$ (respectively $\Theta_P^B(\tilde{\Sigma})(t_1) \leq \Theta_P^B(\tilde{\Sigma})(t_2)$).
2. $A_P(\Sigma)(t_0) \leq A_P(T_\gamma(h_{\min}, t_0))$ (respectively $B_P(\Sigma)(t_0) \leq B_P(T_\gamma(h_0, t_0))$)

Here γ is the intersection of Σ and the level set $h = t_0$

Proof. The argument is based on straightforward volume addition/subtraction and the fact that the density of a tube is constant. Without loss of generality, we suppose $t_1 = t_0$ and will only prove the statement for the compensated case. By definition, $\Theta_P^B(\tilde{\Sigma})(t_0) \leq \Theta_P^B(\tilde{\Sigma})(t_2)$ is equivalent to

$$\frac{B_P(\Sigma)(t_0)}{B_P(T_\gamma)(t_0)} \leq \frac{B_P(\Sigma)(t_0) + A_P(T_\gamma)(t_0, t_2)}{B_P(T_\gamma)(t_0) + A_P(T_\gamma)(t_0, t_2)} \quad (1.18)$$

where $A_P(T_\gamma)(t_0, t_2) = \int_{T_\gamma, t_0 \leq h \leq t_2} P$. Here we used the fact that P -volume of T_γ is, up to a factor, independent of γ . Simplify (1.18) and we have $B_P(\Sigma)(t_0) \leq B_P(T_\gamma)(t_0)$. \square

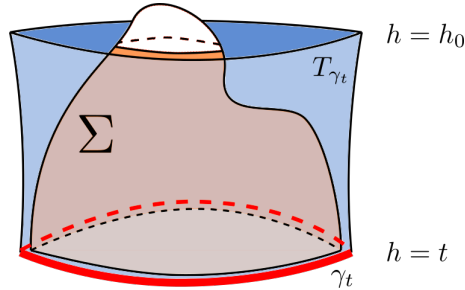


Figure 1.2 – The weighted volume of a minimal submanifold (orange) is less than that of a tube (blue) built upon the same intersection (red).

An immediate consequence of Lemma 1.17 is the following lower bound on the volume of the intersection between a minimal submanifold and the level set $h = t$.

Proposition 1.18 (Estimates of level set). *Let Σ^k be a tube extension of a minimal submanifold in the region $U \geq 0$. Let γ_t , $t \geq h_{\min}$ (respectively $t \geq h_0$) be its intersection with the level set $h = t$. Then*

1. In the uncompensated case,

$$m \frac{\omega_{k-1}}{k} V^{k/2}(t) \leq A_h(\Sigma)(t) \leq \frac{|\gamma_t|}{k} V^{k/2}(t),$$

1.4. Weighted monotonicity in \mathbb{H}^n and S^n

where $m := \lim_{t \rightarrow h_{\min}} \Theta_h(t)$ is the multiplicity of Σ at $h^{-1}(h_{\min})$. In particular,

$$|\gamma_t| \geq m\omega_{k-1}$$

2. Similarly, in the compensated case:

$$\frac{|\gamma_0^{T\Sigma}|}{\omega_{k-1}} Q(t) \leq B_h(\Sigma)(t) \leq \frac{|\gamma_t|}{\omega_{k-1}} Q(t).$$

In particular, $|\gamma_t| \geq |\gamma_0^{T\Sigma}|$. More generally, for any two level sets $\gamma_{t_i} = \Sigma \cap h^{-1}(t_i)$, $i = 1, 2$ with $t_1 < t_2$, the \tilde{g} -volume of γ_{t_2} is greater than the parallel volume of γ_{t_1} .

By Lemma 1.15, we can also compare the P -volume of a minimal submanifold with that of a tube:

Corollary 1.19. *Let P be a weaker weight than U and Σ be a minimal submanifold in the region $U \geq 0$. Then*

1. P -density of Σ is an increasing function
2. The P -volume of Σ in the region $h \leq t$ (respectively the compensated P -volume in the region $h_0 \leq h \leq t$) is less than that of the tube with the same boundary.

1.4 Weighted monotonicity theorems in the hyperbolic space and the sphere

1.4.1 Time Monotonicity and Null Monotonicity

We will restate Theorem 1.8 when M is \mathbb{H}^n and S^n with the function h being a Minkowskian or Euclidean coordinate.

Given a time coordinate ξ_0 of \mathbb{H}^n , the *time-weighted volume* functional is defined as

$$A_{\xi_0}(\Sigma)(t) := \int_{\Sigma, 1 \leq \xi_0 \leq t} \xi_0.$$

For a totally geodesic copy H of \mathbb{H}^k in \mathbb{H}^n passing by the minimum point of ξ_0 , we have $A_{\xi_0}(H)(t) = \frac{\omega_{k-1}}{k} (t^2 - 1)^{k/2}$.

Definition 1.20. *The time-weighted density of a submanifold Σ^k of \mathbb{H}^n is*

$$\Theta_{\xi_0}^A(\Sigma)(t) := \frac{A_{\xi_0}(\Sigma)(t)}{A_{\xi_0}(H)(t)} = \frac{A_{\xi_0}(\Sigma)(t)}{\frac{\omega_{k-1}}{k} (t^2 - 1)^{k/2}}.$$

Theorem 1.21 (Time Monotonicity). *The time-weighted density of a minimal surface (or its extension by ξ_0 -tube) is increasing on $(1, +\infty)$.*

Let ξ_l be a null coordinate and $x = \xi_l^{-1} > 0$ be the corresponding half-space coordinate. The *null-weighted volume* functional is defined as

$$A_{\xi_l}(\Sigma) := \int_{\Sigma, \xi_l \leq t} \xi_l = \int_{\Sigma, x \geq \frac{1}{t}} \frac{1}{x}$$

which is finite for submanifolds not containing the point at infinity $\xi_l^{-1}(0)$ (or equivalently bounded in the corresponding half-space model). For a k -dimensional ξ_l -tube,

$$A_{\xi_l}(T_\gamma(0, t)) = \frac{|\gamma|}{k} t^k.$$

Definition 1.22. *The null-weighted density of a submanifold Σ^k of \mathbb{H}^n is*

$$\Theta_{\xi_l}^A(\Sigma)(t) := \frac{A_{\xi_l}(\Sigma)(t)}{\frac{\omega_{k-1}}{k} t^k}.$$

Theorem 1.23 (Null Monotonicity). *The null-weighted density of a minimal submanifold (or its extension by ξ_l -tube) not containing $\xi_l^{-1}(0)$ is increasing on $(0, +\infty)$.*

It is also useful to write down the compensated version of Theorem 1.21 and Theorem 1.23.

Definition 1.24. 1. *Given any $a > 1$, the compensated time-weighted volume and density are defined as*

$$B_{\xi_0}(\Sigma)(t) := \int_{\Sigma, a \leq \xi_0 \leq t} \xi_0 + \frac{|\gamma_a^{T\Sigma}|}{k} (a^2 - 1)^{k/2}, \quad \Theta_{\xi_0}^B(\Sigma)(t) := \frac{B_{\xi_0}(\Sigma)(t)}{\frac{\omega_{k-1}}{k} (t^2 - 1)^{k/2}},$$

where $|\gamma_a^{T\Sigma}|$ is the parallel volume of the intersection $\gamma_a = \Sigma \cap \xi_0^{-1}(a)$.

2. *Given any $a > 0$, the compensated null-weighted volume and density are defined as*

$$B_{\xi_l}(\Sigma)(t) := \int_{\Sigma, a \leq \xi_l \leq t} \xi_l + \frac{|\gamma_a^{T\Sigma}|}{k} a^k, \quad \Theta_{\xi_l}^B(\Sigma)(t) := \frac{B_{\xi_l}(\Sigma)(t)}{\frac{\omega_{k-1}}{k} t^k},$$

where $|\gamma_a^{T\Sigma}|$ is the parallel volume of the intersection $\gamma_a = \Sigma \cap \xi_l^{-1}(a)$.

Theorem 1.25 (Compensated time and null monotonicity). *Let Σ^k be a minimal submanifold in the region $\xi_0 \geq a > 1$ (respectively $\xi_l \geq a > 0$) that does not meet the a -level set of ξ_0 (respectively ξ_l) tangentially. Then the density $\Theta_{\xi_0}^B$ (respectively $\Theta_{\xi_l}^B$) of any tube extension of Σ is increasing on $(a, +\infty)$.*

1.4.2 Space Monotonicity

To state the weighted monotonicity theorem corresponding to a space coordinate ξ_1 , we assume that the submanifold Σ^k is contained in the region $\xi_1 \geq a$ and that its boundary consists of a (possibly empty) part γ_a in $\xi_1^{-1}(a)$ and a part γ_∞ in the sphere at infinity S_∞ , both of them are disjoint from the equator $\xi_1^{-1}(0) \cap S_\infty$.

The *space-weighted volume* functional is

$$B_{\xi_1}(\Sigma)(t) := \int_{\Sigma, a \leq \xi_1 \leq t} \xi_1 + \frac{|\gamma_a^{T\Sigma}|}{k} (a^2 + 1)^{k/2}$$

and is finite for such submanifold Σ .

In particular, if γ is a $(k-1)$ -submanifold in the interior of $\xi_1^{-1}(0) \cong \mathbb{H}^{n-1}$ with hyperbolic volume $|\gamma|$, the ξ_1 -tube $T_\gamma(a, t)$ built upon γ between level sets $\xi_1 = a$ and

$\xi_1 = t$ has

$$B_{\xi_1}(T_\gamma(a, t)) = \frac{|\gamma|}{k} (t^2 + 1)^{k/2}$$

Definition 1.26. *The space-weighted density of a k -dimensional submanifold Σ of \mathbb{H}^n is defined as*

$$\Theta_{\xi_1}(\Sigma)(t) := \frac{B_{\xi_1}(\Sigma)(t)}{\frac{\omega_{k-1}}{k} (t^2 + 1)^{k/2}}$$

Theorem 1.27 (Space Monotonicity). *The density Θ_{ξ_1} of a minimal submanifold (or its extension by ξ_1 -tube) is increasing on $(a, +\infty)$.*

1.4.3 Weighted Monotonicity theorem in sphere

In the round sphere S^n seen as a warped space using a Euclidean coordinate $x = x_i$, we define the *weighted volume* of a submanifold Σ by:

$$A_x(\Sigma)(t) := \int_{\Sigma, x \geq t} x.$$

A totally geodesic k -sphere S passing by $x^{-1}(1)$ has $A_x(S)(t) = \frac{\omega_{k-1}}{k} (1 - t^2)^{k/2}$.

Definition 1.28. *The weighted density of a submanifold Σ of S^n is*

$$\Theta_x(\Sigma)(t) := \frac{A_x(\Sigma)(t)}{A_x(S)(t)} = \frac{A_x(\Sigma)(t)}{\frac{\omega_{k-1}}{k} (1 - t^2)^{k/2}},$$

Theorem 1.29 (Weighted monotonicity in S^n). *The weighted density of a minimal submanifold (or its extension by x -tube) in S^n , is decreasing on $(0, 1)$ and increasing on $(-1, 0)$.*

1.5 Unweighted monotonicity

The unit ball \mathbb{B}^n can be equipped with 3 metrics g_E, g_S, g_H as in Example 1.12 and Example 1.13. Its submanifolds have 5 different densities: the unweighted densities from the 3 metrics and 2 weighted densities from the time coordinate and Euclidean coordinate associated to the origin.

The time-weighted hyperbolic volume, the hyperbolic volume, the Euclidean volume, the spherical volume, the weighted spherical volume can be written as P -volume with respect to the metric g_S with weights given by

$$P_1 = \xi_0^{-k-1}, \quad P_2 = \xi_0^{-k}, \quad P_3 = (1 + \xi_0)^{-k}, \quad P_4 = 1, \quad P_5 = \xi_0.$$

It can be checked that P_{i+1} is weaker than P_i . Lemma 1.15 says that one has the following chain of monotonicity:

$$\text{time-weighted } g_H \gg \text{unweighted } g_H \gg g_E \gg \text{unweighted } g_S \gg \text{weighted } g_S. \quad (1.19)$$

If a submanifold $\Sigma^k \subset \mathbb{B}^n$ has increasing density in one volume functional of the chain, then

1. it automatically has increasing density in any volume functional on the right.
2. the densities of Σ are comparable, with order given by (1.19).

The chain (1.19) has three implications:

1. The first \gg shows that Choe–Gulliver’s Monotonicity implies Anderson’s monotonicity.
2. The second \gg shows that area-minimising cones in hyperbolic space are exactly those in Euclidean space.
3. The last \gg recovers the volume estimate of Cheng–Li–Yau for minimal submanifold of the sphere [CLY84]

The first point above is immediate. For organisational purpose, the statements of the last two points are delayed until the next chapter.

Remark 1.30. *While minimal surfaces of S^n are weighted-monotone, they may not be monotone with the uniform weight. In fact, the unweighted density of the Clifford torus in S^3 is not an increasing function, even on one hemisphere. This means that the last \gg of (1.19) is strict.*

A continuous version of the chain (1.19) for minimal surfaces in manifolds with sectional curvature bounded from above can be found in the Appendix B. The third implication above becomes the volume estimate of Hoffman and Spruck [HS74].

For the compensated case, we have

Lemma 1.31. *Let ξ be a Minkowskian coordinate of \mathbb{H}^n . Then in the region $\xi \geq a > 0$, the weight $P_1 = \xi$, $c_1 = 1$ is stronger than the weight $P_2 = 1$, $c_2 = a^{-1}$.*

Proof. Because

$$\frac{Q_1}{P_1}(t) = \frac{\omega_{k-1}}{k} \frac{V(t)^{k/2}}{t}, \quad \frac{Q_2}{P_2}(t) = \omega_{k-1} \int_a^t V(\xi)^{\frac{k}{2}-1} d\xi + \frac{\omega_{k-1}}{ka} V(a)^{k/2}$$

and $V'(t) = 2t$ for the three cases of h , we have

$$\frac{d}{dt} \left(\frac{Q_1}{P_1} \right) = \omega_{k-1} \left(V^{\frac{k}{2}-1} - \frac{V^{k/2}}{kt^2} \right) \leq \omega_{k-1} V^{\frac{k}{2}-1} = \frac{d}{dt} \left(\frac{Q_2}{P_2} \right)$$

Therefore the function $\frac{Q_2}{P_2} - \frac{Q_1}{P_1}$ is decreasing. Since it vanishes at $t = a$, one has $\frac{Q_1}{P_1} \leq \frac{Q_2}{P_2}$ for all $t \geq a$, which means (P_1, c_1) is stronger than (P_2, c_2) . \square

It follows from the Comparison lemma 1.15 that the $(1, a^{-1})$ density of a minimal submanifold is increasing. Therefore we have the following Theorem 1.32 and Theorem 1.33.

Theorem 1.32. *Let Σ be a minimal submanifold of the hyperbolic space. We denote by $A(t)$ the volume of Σ inside a geodesic ball $\xi_0 \leq t$, given as a sublevel set of a time coordinate ξ_0 . Then the quantity*

$$\frac{A(t)}{Q(t)}$$

is increasing in t . Here $Q(t)$ is $\frac{1}{\omega_{k-1}}$ the volume of a totally geodesic k -disc in the equator of the ball, which is

$$Q(t) = \int_1^t (s^2 - 1)^{k/2-1} dt + \frac{1}{k}(a^2 - 1)^{k/2}$$

The case $a = 1$ of Theorem ?? was proved by Anderson [And82].

Theorem 1.33. *Let ξ be a Minkowskian coordinate of \mathbb{H}^n and Σ be a minimal submanifold in the region $\xi \geq a$. Suppose that Σ has no boundary except its non-tangential intersection γ with the level set $\xi = a$. We denote by $B(t)$ the compensated volume of Σ in the region $a \leq \xi \leq t$:*

$$B(t) = \text{vol}(\Sigma \cap \{a \leq \xi \leq t\}) + \frac{1}{ka} |\gamma^{T\Sigma}| V(a)^{k/2}$$

and by $Q(t)$ the quantity

$$Q(t) = \int_a^t V(s)^{k/2-1} ds + \frac{1}{ka} V(a)^{k/2}.$$

Then $\frac{A(t)}{Q(t)}$ is increasing in t .

Here $V(s)$ is $s^2 - 1$, $s^2 + 1$, or s^2 depending on the type of ξ . The statement of Theorem 1.33 becomes cleaner in 2 situations:

1. When $k = 2$, the quantity Q is:

$$Q(t) = t - \frac{1}{2} \begin{cases} a + \frac{1}{a}, & \text{for time coordinate} \\ a, & \text{for null coordinate} \\ a - \frac{1}{a}, & \text{for space coordinate} \end{cases}$$

2. When Σ does not intersect the level set $\xi = a$, $B(t)$ is just the area of the surface in the region $a \leq \xi \leq t$.

Recall that by Remark 1.16, the non-tangency condition is automatically satisfied if Σ is the intersection of a minimal submanifold with the region $\xi > a$.

1.6 Appendix A: Weighted monotonicity and comparison lemma for stationary currents, varifolds and harmonic maps. Monotonicity formulae.

Theorem 1.8 also holds when Σ is a stationary rectifiable k -current. The proof can be adapted in the same fashion as [And82] and [EWW02]. We replace the integration by part (1.11) by the first variation formula of current, which reads

$$\int_{\Sigma} \text{div}_{\Sigma} X \, d\|\Sigma\| = 0 \tag{1.20}$$

where X is any smooth vector field and $d\|\Sigma\|$ is the mass measure.

We recover $kA_h - \frac{V}{U}A'_h \leq 0$ by choosing $X := \chi(h)\nabla h$ in (1.20) where χ is a decreasing function that approximates the characteristic function of $[-\infty, t]$:

$$\operatorname{div}_\Sigma X = \chi'|\nabla^\Sigma h|^2 + \chi\Delta_\Sigma h \geq \chi'V + k\chi U.$$

To define the tube extension and the compensated volume B_h , we replace the boundary γ of Σ (and the intersection $\gamma_0 = \Sigma \cap h^{-1}(h_0)$ respectively) by an \mathcal{H}^{k-1} -rectifiable set such that the pair (Σ, γ) is *strongly stationary*. This means that

$$\int_\Sigma \operatorname{div}_\Sigma X \leq \int_\gamma |X^\perp|$$

for any smooth vector field X whose normal component to γ is X^\perp . Equivalently there exists an \mathcal{H}^{k-1} -measurable normal vector field ν on γ with $\sup |\nu| \leq 1$ such that

$$\int_\Sigma \operatorname{div}_\Sigma X = \int_\gamma g(X, \nu)$$

The definition (1.7) should be rewritten for strongly stationary pair (Σ, γ_0) as

$$B_h(\Sigma)(t) := \int_\Sigma U(h) - \frac{1}{k} \int_{\gamma_0} g(\nabla h, \nu_0).$$

Here the normal derivative of h in the boundary term was replaced by $g(\nabla h, \nu_0)$

When Σ is a stationary rectifiable k -varifold, the first variation formula reads

$$\delta\Sigma.X = \int_{T \in Gr(k, M)} \operatorname{div}_T X \, d\Sigma(T)$$

where the integral was taken in the Grassmannian $Gr(k, M)$ of unoriented k -dimensional planes in M . Because $\operatorname{div}_T \nabla h = \operatorname{Tr}_T \operatorname{Hess} h = kU$ for any $T \in Gr(k, M)$, one still has, for any perturbation $X := \chi(h)\nabla h$:

$$\operatorname{div}_T X = \chi'|\nabla^T h|^2 + \chi \operatorname{div}_T \nabla h \geq \chi'V + k\chi U,$$

where χ is a decreasing function and $\nabla^T h$ is the projection of the gradient of h to T .

Lemma 1.10 can be stated and proved in the same manner:

Lemma 1.34. *Let Σ be a rectifiable k -varifold and γ be a $(k-1)$ -rectifiable set such that the pair (Σ, γ) is strongly stationary. Suppose that γ is contained in the region $h \leq t_0$ and Σ is supported in the region $U \geq 0$. Then Σ is supported in the region $h \leq t_0$.*

Proof. By definition of strong stationary pair, one has for any vector field X

$$\int_{Gr(k, M)} \operatorname{div}_T X \, d\Sigma(T) \leq \int_\gamma |X^\perp|.$$

With $X = f(h)\nabla h$, one has $\operatorname{div}_T X = f'|\nabla^T h| + fkU$ for any $T \in \operatorname{Gr}(k, M)$, so

$$\int_\gamma |f|\|\nabla h^\perp\| \geq \int_{\operatorname{Gr}(k, M)} f'|\nabla^T h| d\Sigma(T) + k \int_M fU d\|\Sigma\|. \quad (1.21)$$

Now we choose the function f to be any non-negative, increasing function in h with support in (t_0, ∞) , the LHS of (1.21) vanishes while the RHS is non-negative. Equality in (1.21) happens only if f vanishes $\|\Sigma\|$ -almost everywhere, which is the conclusion. \square

Theorem 1.8 can also be extended for harmonic maps. Given a map $f : (\Sigma, g_\Sigma) \rightarrow (M, g)$, we define its *dimension at a point* $p \in \Sigma$ to be the ratio $\frac{|df_p|^2}{|df_p|_g^2}$ of the tensor norm of the derivative at p (the *energy density*) and its operator norm. It is $+\infty$ if $df_p = 0$. Note that when df_p is non-zero and conformal, this is the dimension of Σ . The *dimension* of f , defined as the smallest dimension among all points of Σ . It will play the role of k in our argument.

The *weighted Dirichlet energy* of f in the region $h \leq t$ is defined as

$$E_h(t) := \int_{\Sigma, h \circ f \leq t} U |df|^2$$

or in the compensated case,

$$E_h(t) := \int_{\Sigma, h_0 \leq h \circ f \leq t} U |df|^2 + \int_{\Sigma, h \circ f = h_0} |d(h \circ f)|.$$

The *weighted density* is defined as $\Theta_h(t) := \frac{E_h(t)}{V(t)^{k/2}}$ where k is the dimension of f .

Theorem 1.35. *Let h, U, V be as in Theorem 1.8 and $f : \Sigma \rightarrow M$ be a harmonic map. Then $\frac{d}{dt}\Theta_h(f)$ has the same sign as U .*

Proof. By Lemma 1.6 one has $\Delta(h \circ f) = U|df|^2$ and by integration by part,

$$E_h(t) = \int_{\Sigma, h \circ f = t} |d(h \circ f)|. \quad (1.22)$$

One then compares E_h with its derivative obtained from coarea formula

$$\frac{dE_h}{dt} = U(t) \int_{\Sigma, h \circ f = t} \frac{|df|^2}{|d(h \circ f)|}.$$

The definition of k guarantees

$$\frac{|df|^2}{|d(h \circ f)|} \geq k \frac{|d(h \circ f)|}{|dh|^2}$$

and therefore $U^{-1} \frac{dE_h(t)}{dt} \geq \frac{k}{V} E_h$. \square

The harmonic map version of Proposition 1.18 is:

Corollary 1.36. *Let h, U, V be as in Theorem 1.8. Let (Σ^2, g_Σ) be a surface with boundary $\gamma = \partial\Sigma$ and $f : \Sigma \rightarrow M$ be a harmonic map. Suppose that f sends γ to a level set $h = t$. Then the weighted Dirichlet energy of f is bounded between:*

$$2m\pi V(t) \leq E_h(f) \leq |f(\gamma)| \cdot V(t) \quad (1.23)$$

where $|f(\gamma)|$ is the length of the image of γ in M and m is the multiplicity of f at $h^{-1}(h_{\min})$.

In particular, if h is minimised at a point and m is the multiplicity of f at his point, then $|f(\gamma)| \geq m \cdot 2\pi$.

Proof. The first half of (1.23) follows from Theorem 1.35. For the second half, since the RHS of (1.22) does not depend on the metric on γ , we have:

$$E_h(f)(t) \leq \int_\gamma |dh|_g \text{vol}_{f^*g}^1 = |f(\gamma)|V(t).$$

□

By carefully writing down all the estimate in the proof of Theorem 1.8, we obtain monotonicity formulae for submanifolds and maps

Theorem 1.37 (Monotonicity formula for submanifolds). *Let $(M, g), h, U, V$ be as in Theorem 1.8 and Σ be a k -dimensional submanifold of M with mean curvature vector H , then*

$$\frac{\omega_{k-1}}{k} \left[\Theta_h^A(\Sigma)(t_2) - \Theta_h^A(\Sigma)(t_1) \right] = \int_{\Sigma, t_1 \leq h \leq t_2} |\nabla h^\perp|^2 \frac{U}{V^{\frac{k}{2}+1}} + \int_{t=t_1}^{t_2} \frac{U}{V^{\frac{k}{2}+1}} \int_{\Sigma, h \leq t} dh \cdot H,$$

where $\nabla^\perp h$ is the part of ∇h orthogonal to Σ .

Theorem 1.38 (Monotonicity formula for maps). *Let $(M, g), h, U, V$ be as in Theorem 1.8 and $f : \Sigma \rightarrow M$ be a C^2 map of dimension k with density $e(f)$ and tension field $\tau(f)$, then*

$$\begin{aligned} \Theta_h(f)(t_2) - \Theta_h(f)(t_1) &= \int_{\Sigma, t_1 \leq h \circ f \leq t_2} \left(e(f) - \frac{|f^*dh|^2}{|dh|^2} k \right) U \\ &\quad + k \int_{t=t_1}^{t_2} \frac{U}{V^{\frac{k}{2}+1}}(t) \int_{h \circ f \leq t} dh \cdot \tau(f) dt \end{aligned}$$

The proof of the Comparison Lemma 1.15 can also be adapted for rectifiable varifolds and currents. We will write down the details for the compensated case here. The relation $P_1^{-1} \frac{dB_{P_1}}{dt} = P_2^{-1} \frac{dB_{P_2}}{dt}$, hence equation (1.15), remains true in distribution sense because of Fubini theorem

$$\int_{t=h_0}^T \int_M \mathbf{1}_{h_0 \leq h \leq t} \cdot f(h(m)) \varphi(t) d\|\Sigma\|(m) dt = \int_M f(h(m)) \int_{t=h_0}^{h(m)} \varphi(t) dt d\|\Sigma\|(m)$$

for all $\forall \varphi \in C_c^\infty(0, T)$. So it follows from (1.16) that $\frac{d}{dt} [Q_1(\Theta_1 - \Theta_2)] \geq 0$ as distribution, i.e.

$$\int_{t=h_0}^T Q_1(\Theta_1 - \Theta_2) \varphi' dt \leq 0 \quad \forall \varphi \in C_c^\infty(h_0, T), \varphi \geq 0 \quad (1.24)$$

We will prove that (1.24) and the non-tangency condition $\|\Sigma\|(h^{-1}(h_0)) = 0$ imply that $\Theta_1 \geq \Theta_2$ on $[h_0, T)$. The monotonicity of B_{P_i} implies that B_{P_i} and Θ_i are continuous except at countably many points. Moreover, we have

$$\limsup_{t \rightarrow a^-} \Theta_i(t) \leq \Theta(a) = \liminf_{t \rightarrow a^+} \Theta_i(t) \quad \forall a \in (0, T) \quad (1.25)$$

Let $t_1 \in (h_0, T)$ be a point where both Θ_i are continuous, we will prove that $\Theta_1(t_1) \geq \Theta_2(t_1)$. It follows from (1.25) that $\Theta_1 \geq \Theta_2$ everywhere. We choose the test function φ in (1.24) so that φ' is a negative bell curve of width ϵ around t_1 and positive bell curve around $h_0 + \frac{\epsilon}{2}$, as in Figure 1.3. It suffices to see now that the negative bell part of the integral (1.24) converges to $[Q_1(\Theta_1 - \Theta_2)](t_1)$ while the positive bell part converges to 0. The former is due to continuity at t_1 . Because Q_1, Q_2, P_1, P_2 are continuous, to prove

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{t=0}^{\epsilon} Q_1(\Theta_1 - \Theta_2) = 0,$$

we only need:

$$\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \int_{t=h_0}^{\epsilon} \int_{\Sigma, h_0 \leq h \leq t} 1 = 0,$$

which is true because of Fubini and Monotonic convergence:

$$\frac{1}{\epsilon} \int_{t=0}^{\epsilon} \int_{\Sigma, h_0 \leq h \leq t} 1 = \frac{1}{\epsilon} \int_{\Sigma, h_0 \leq h \leq h_0 + \epsilon} (h_0 + \epsilon - h) \leq \|\Sigma\|(\{h_0 \leq h \leq h_0 + \epsilon\}) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Now substitute $\Theta_1 \geq \Theta_2$ into (1.15), we have $\Theta_2' \geq 0$ as distribution. By the same choice of test function φ as before, one has $\Theta_2(t_1) \leq \Theta_2(t_2)$ for continuous points $t_1 < t_2$, and then by (1.25) for all $t_1 < t_2$.

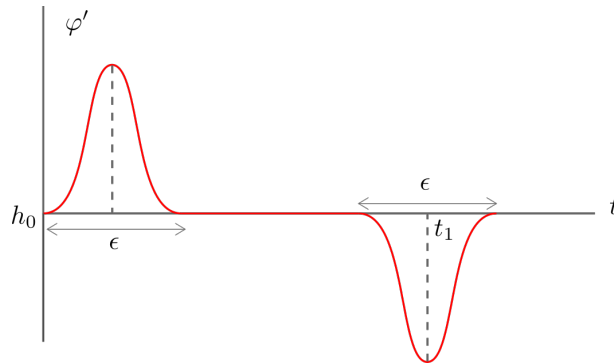


Figure 1.3 – The derivative of the test function used in (1.24).

1.7 Appendix B: Monotonicity theorems and comparison lemma in space of bounded curvature.

Fix a point O in a Riemannian manifold (M^n, g) , and let r_{inj} be the injectivity radius at O . The Hessian of the distance function r to O is given by:

$$\text{Hess}_p r(\partial_r, \cdot) = 0, \quad \text{Hess}_p r(v, v) =: I(v), \quad \forall p \in B(O, r_{\text{inj}}), \quad \forall v \perp \partial_r \quad (1.26)$$

where

$$I(v) = \int_{\Gamma} (|\dot{V}|^2 - K_M(\dot{\gamma}, V)|V|^2)$$

is the index form of the Jacobi field V along the geodesic Γ between O and p that interpolates 0 at O and v at p .

When the sectional curvature satisfies $K_M \leq -a^2$ (respectively b^2), we can check that

$$I(v) \geq a \coth(ar)|v|^2 \quad (\text{respectively } b \cot(br)|v|^2).$$

This gives an estimate of the Hess r on the directions orthogonal to ∂_r . By a change of variable, we can estimate the Hessian in a more isotropic way.

Proposition 1.39. *Inside $B(O, r_{\text{inj}})$, one has*

1. $\text{Hess}(a^{-2} \cosh ar) \geq \cosh ar \cdot g$ if $K_M \leq -a^2$.
2. $\text{Hess}(-b^{-2} \cos br) \geq \cos br \cdot g$ if $K \leq b^2$ and $r \leq \frac{\pi}{b}$.

This means that the functions $h = a^{-2} \cosh ar$ and $h = -b^{-2} \cos br$ satisfy $\text{Hess } h \geq U \cdot g$. We note that the functions U, V defined as in Proposition 1.1 are still functions of h :

$$U = a^2 h \quad (\text{respectively } -b^2 h), \quad V = |\nabla h|^2 = a^2 h^2 - a^{-2} \quad (\text{respectively } -b^2 h^2 + b^{-2})$$

and one still has $U = \frac{1}{2}V'$.

We define the *eligible interval* $[0, r_{\text{max}})$ to be $[0, r_{\text{inj}})$ when $K_M \leq -a^2$ and $[0, \min(r_{\text{inj}}, \frac{\pi}{2b}))$ when $K_M \leq b^2$.

Remark 1.40. 1. *When M is \mathbb{H}^n or S^n , the function h is the time-coordinate ξ_0 and the Euclidean coordinate x in Example 1.4 and Example 1.3.*

2. *It follows from maximum principle and Proposition 1.39 that there exists no closed minimal submanifold in $B(O, r_{\text{max}})$. If M is Cartan–Hadamard ($r_{\text{max}} = +\infty$) and if the boundary of a minimal submanifold is contained in a geodesic ball, the entire submanifold stays inside that ball.*

1.7.1 Weighted monotonicity theorem

As explained in Remark 1.9, we can still have weighted monotonicity theorem for the function h . The *weighted volume* and *weighted density* of a submanifold Σ^k are defined as

$$\bar{A}(\Sigma)(t) := \int_{\Sigma, r \leq t} U, \quad \bar{\Theta}(t) := \frac{\bar{A}(\Sigma)(t)}{Q(t)}, \quad Q(t) := \begin{cases} \frac{\omega_{k-1}}{k} \frac{\sinh^k at}{a^k}, & \text{when } K_M \leq -a^2 \\ \frac{\omega_{k-1}}{k} \frac{\sin^k bt}{b^k}, & \text{when } K_M \leq b^2 \end{cases}$$

Note that Q is the weighted volume of a ball of radius t in the k -dimensional space-forms of curvature $-a^2$ or b^2 and that the density converges to 1 as t decreases to 0.

Theorem 1.41. *Let M be a Riemannian manifold with sectional curvature $K_M \leq -a^2$ or $K_M \leq b^2$ and $\Sigma^k \subset M$ be an extension of a minimal submanifold by geodesic cone, then the density $\Theta(\Sigma)(t)$ is an increasing function on the eligible interval.*

We can also prove that the intersection of a minimal submanifold with a geodesic sphere of M has bigger $(k-1)$ -volume than the great k -sphere of the sphere of same radius in space-form.

Corollary 1.42. *Let $\Sigma^k \subset M$ be a minimal submanifold containing the point O with multiplicity m and $l_t := \mathcal{H}^{k-1}(\Sigma \cap r^{-1}(t))$. For all t in the eligible interval, we have*

$$Q(t) \leq \bar{A}(\Sigma)(t) \leq \frac{l_t}{k} V(t)^{1/2}$$

In particular,

$$l_t \geq \begin{cases} m\omega_{k-1} \left(\frac{\sinh at}{a} \right)^{k-1}, & \text{if } K_M \leq -a^2 \\ m\omega_{k-1} \left(\frac{\sin bt}{b} \right)^{k-1}, & \text{if } K_M \leq b^2 \end{cases}$$

Proof. Instead of Lemma 1.17 (see Remark 1.47), the upper estimate of \bar{A} follows from (1.11):

$$\bar{A}(\Sigma)(t) \leq \frac{1}{k} \int_{\Sigma, h=t} |\nabla^\Sigma h| \leq \frac{V(t)^{1/2}}{k} l_t.$$

□

1.7.2 Comparison lemma

It is more convenient see a weight P as a non-negative continuous function on r instead of h . The P -volume is defined as $A_P(\Sigma)(t) := \int_{\Sigma, r \leq t} P(r)$ and the P -density is $\Theta_P := \frac{A_P}{Q}$. Here Q is P -volume of a ball of radius r in space-form:

$$Q(t) := \begin{cases} \omega_{k-1} \int_{r \leq t} P(r) \frac{\sinh^{k-1} ar}{a^{k-1}} dr, & \text{when } K_M \leq -a^2 \\ \omega_{k-1} \int_{r \leq t} P(r) \frac{\sin^{k-1} br}{b^{k-1}} dr, & \text{when } K_M \leq b^2 \end{cases} \quad (1.27)$$

Lemma 1.43 (Comparison). *Let $\Sigma^k \subset M$ be any submanifold not necessarily minimal and P_1, P_2 be two non-negative, continuous weight functions. Define Q_1, Q_2 from P_1, P_2 as in (1.27).*

1. *If P_1 is weaker than P_2 , i.e. $\frac{P_1}{Q_1} \leq \frac{P_2}{Q_2}$, and $\frac{d}{dt} \Theta_2 \geq 0$ in the eligible interval, then $\Theta_1 \leq \Theta_2$ and*

$$\frac{d\Theta_1}{dt} \geq \frac{Q_2}{P_2} \frac{P_1}{Q_1} \frac{d\Theta_2}{dt} \geq 0$$

2. *If P_2 is weaker than P_1 and $\frac{d}{dt} \Theta_2 \geq 0$ in the eligible interval, then $\Theta_1 \geq \Theta_2$*

We note that it is necessary to mention a or b in order to compare two weights. However, it can be checked that

Lemma 1.44. *For any $a, b \geq 0$ and $u \geq v \geq 0$,*

1. $P_1 = \cosh vr$ is weaker than $P_2 = \cosh ur$ when $K_M \leq -a^2$,
2. $P_1 = \cos ur$ is weaker than $P_2 = \cos vr$ in the interval $t \leq \frac{\pi}{2u}$ when $K_M \leq b^2$.

Remark 1.45. 1. *It follows from Lemma 1.44 and Theorem 1.41 that for negatively curved space $K_M \leq -a^2$, the monotonicity theorem holds for any weight $P_u = \cosh ur$ with $u \in [0, a)$ and in particular the uniform weight $P_0 = 1$. One recovers the Theorem 1 of [And82].*

2. *When $K_M \leq b^2$, the monotonicity theorem holds for any weight $P_u = \cos ur$ with $u \in [b, \infty)$ but not for the uniform weight. However, one can still lower bound the unweighted volume of a minimal submanifold inside a geodesic ball by the volume of a great k -disc in space form.*

Proposition 1.46. *Suppose that $K \leq b^2$, the minimal surface Σ^k contains a point O with multiplicity m and it has no boundary in the interior of $B(O, t)$ for certain $t < r_{\max}$. Then*

$$A(\Sigma \cap B(O, t)) \geq m\omega_{k-1} \int_{r=0}^t \frac{\sin^{k-1}(br)}{b^{k-1}} dr$$

In particular, if M is simply connected, with curvature pinched between $\frac{b^2}{4}$ and b^2 and $\Sigma \subset M$ is a closed minimal surface, then

$$A(\Sigma) \geq \frac{1}{2}\omega_k b^{-k}. \tag{1.28}$$

A weaker version of inequality (1.28), with $\frac{1}{2}\omega_k$ replaced by the volume of the *unit* k -ball, was proved in [HS74].

Remark 1.47. *Lemma 1.17 does not generalise because the P -volume of a geodesic cone in M is no longer proportional to Q .*

2 Harmonic maps and Minimal surfaces in the hyperbolic space and the sphere

2.1 A few properties of minimal surfaces and harmonic maps in \mathbb{H}^n

In this section, we are interested in minimal submanifolds of the hyperbolic space that are asymptotic to a properly embedded submanifold γ of the sphere at infinity. Those are submanifolds-with-boundary $\bar{\Sigma}$ of the compactification \mathbb{B}^n of \mathbb{H}^n , whose boundary is γ . We will denote the interior by Σ and call γ its ideal boundary. The Poincaré ball model provides a C^∞ compactification of \mathbb{H}^n . This is the smooth structure we use to talk about boundary regularity.

2.1.1 Properness, Maximum principle and Convex hull

Recall that a k -dimensional minimal submanifold Σ^k of \mathbb{H}^n is minimal if and only if the Minkowskian coordinates satisfy

$$\Delta \xi_\alpha = k \xi_\alpha \tag{2.1}$$

as functions on Σ . This is obtained by pulling back (1.2) to Σ . The same result also holds for maps. A map $f : \Sigma^k \rightarrow \mathbb{H}^n$ is harmonic if and only if

$$\Delta(\xi_\alpha \circ f) = \lambda \xi_\alpha$$

for all Minkowskian coordinate ξ_α and a certain $\lambda \geq 0$. In that case, λ is the energy density of f .

It was observed by Anderson [And82] that if the boundary of a minimal submanifold lies on one half of the hyperbolic space cut out by a totally geodesic hyperplane H then the entire submanifold also lies on that side. One can see this by applying maximum principle to the space coordinate ξ_1 whose zero set is H , which satisfies (2.1). This argument shows that a minimal submanifold of \mathbb{H}^n is contained in the *convex hull* of its boundary. This convex hull property can be proved for weaker notion of "minimal submanifold", such as stationary currents or varifolds with the technique of Appendix A.

Definition 2.1. *Let γ be a $(k-1)$ -dimensional submanifold of S_∞ . The convex hull of γ , denoted by $\text{ConvHull}(\gamma)$, is the intersection of all half spaces of \mathbb{H}^n containing γ . Here a half space is a connected component of the complement of a totally geodesic hyperplane.*

Equation (2.1) also shows that the time coordinate ξ_0 is sub-harmonic on a minimal

submanifold. Therefore, a minimal submanifold of \mathbb{H}^n is proper if it is proper near the boundary.

Definition 2.2. *Let $f : \Sigma \rightarrow M$ be a map between the interiors of manifolds with boundary $\overline{\Sigma}$ and \overline{M} . We say that f is proper near boundary if there exists an open neighbourhood U at infinity of Σ , i.e. an open subset of Σ such that $\Sigma \setminus U$ is a compact, such that $f|_U : U \rightarrow M$ is proper.*

The previous discussion on the convex hull can be stated as:

Proposition 2.3. *Let Σ^k be a manifold with boundary γ and $f : (\Sigma^k, \gamma) \rightarrow (\mathbb{H}, S_\infty)$ be a harmonic map that is proper near boundary. Then f is proper on Σ and its image is contained in $\text{ConvHull}(\gamma)$.*

It also follows from convex hull property is that minimal submanifolds of the hyperbolic space meets the sphere at infinity at a right angle: the normal vector of S_∞ in $\overline{\mathbb{H}^n}$ is tangent to Σ .

Using the convex hull and the unweighted monotonicity theorem, Anderson solved the asymptotic Plateau problem in \mathbb{H}^n .

Theorem 2.4 (Anderson cf. [And82]). *Let γ be an immersed closed $(k-1)$ -submanifold of S_∞ . There exists an area-minimising locally integral k -current of \mathbb{H}^n asymptotic to γ in the following sense:*

$$\overline{\text{supp } \Sigma} \cap S_\infty = \gamma.$$

For harmonic maps, the initial value problem was solved by Li and Tam using the tension field flow.

Theorem 2.5 (Li–Tam [LT93]). *Let $f : S^{k-1} \rightarrow S^{n-1}$ be a $C^{j,\alpha}$ map, $1 \leq j \leq m-1$ with nowhere vanishing energy density. Then there exists a unique $C^{j,\alpha}$ harmonic extension $F : \mathbb{H}^k \rightarrow \mathbb{H}^n$ of f .*

There is also a notion of convex hull in Lawson’s work [Law70] on minimal submanifolds of S^n that could be contained in one hemisphere. This exploits the fact that the restriction of any Euclidean coordinate of \mathbb{R}^{n+1} to a minimal submanifold is super harmonic when it is positive, as seen by the equation $\Delta x_i = -kx_i$.

2.1.2 Renormalised area

A *boundary defining function* is a non-negative function ρ on the compactification $\overline{\mathbb{H}^n}$ that vanishes exactly on S_∞ and exactly to first order, that is $d\rho \neq 0$ at all points of S_∞ . Such function is called *special* if $|d \ln \rho|_{g_{\mathbb{H}^n}} = 1$ on a neighbourhood of the boundary.

It was proved by Graham and Witten [GW99] that:

Theorem 2.6 (Graham–Witten). *Let Σ^2 be a minimal surface in an asymptotically hyperbolic manifold M that is C^2 up to boundary and ρ be a special boundary defining function. Then the area of Σ has the expansion*

$$A(\Sigma \cap \{\rho \geq \epsilon\}) = \frac{|\gamma|_{\bar{g}}}{\epsilon} + \mathcal{A}_R + o(1) \tag{2.2}$$

where $\bar{g} = \rho^2 g$. Moreover, the coefficient \mathcal{A}_R , called the renormalised area of Σ , is independent of the choice of ρ .

2.1. A few properties of minimal surfaces

Remark 2.7. *The expansion (2.2) will also be valid if ρ is replaced by a boundary defining function ρ_1 that is equal to ρ up to third order: $\rho_1 = \rho + o(\rho^2)$ or equivalently $\rho = \rho_1 + o(\rho_1^2)$. This is because $\rho_1^2 g$ and $\rho^2 g$ induce the same metric on the ideal boundary and*

$$\begin{aligned} A(\Sigma \cap \{\rho_1 \geq \epsilon\}) &= A(\Sigma \cap \{\rho \geq \epsilon + o(\epsilon^2)\}) \\ &= \frac{|\gamma|_{\tilde{g}}}{\epsilon + o(\epsilon^2)} + \mathcal{A}_R + o(1) = \frac{|\gamma|_{\tilde{g}}}{\epsilon} + \mathcal{A}_R + o(1). \end{aligned}$$

The natural boundary defining functions coming from Minkowskian coordinates ξ_0, ξ_1, ξ_l of the hyperbolic space are all special up to third order.

1. The function $\rho_l = \xi_l^{-1}$ is a special boundary defining function. It is the half-space coordinate corresponding to ξ_l . The metric \tilde{g} in (2.2) is the flat metric on boundary.
2. The function $\rho_0 = \xi_0^{-1}$ is also a boundary defining function. It is third order close to a special boundary defining function: In the associated Poincaré model, the function $\rho = 2\frac{1-r}{1+r}$, where r is the Euclidean distance to the centre, is special. It is related to ρ_0 by

$$\rho_0 = \frac{\rho}{1 + \frac{\rho^2}{4}} = \rho + O(\rho^3).$$

Graham–Witten expansion (2.2) becomes

$$A(\Sigma \cap \{\xi_0 \leq t\})(t) = |\gamma|_{\tilde{g}} t + \mathcal{A}_R + O(t^{-1}) \quad (2.3)$$

where \tilde{g} is the round metric associated to ξ_0 .

3. The function $\rho_1 = |\xi_1|^{-1}$ is a boundary defining function except on the equator where it is undefined. It is special up to third order. The function $l = |\operatorname{arsinh} \xi_1|$ which computes the distance to the totally geodesic hyperplane $\xi_1^{-1}(0)$ satisfies $|dl| = 1$, and so $\rho = 2 \exp(-l)$ is special. The functions ρ_1 and ρ are related by

$$\rho_1 = \frac{\rho}{1 - (\rho/2)^2} = \rho + O(\rho^3).$$

The Graham–Witten expansion (2.3) can be rewritten as

$$A(\Sigma \cap \{\xi_1 \leq t\}) = |\gamma_\infty|_{\tilde{g}} t + \mathcal{A}_R + O(t^{-1}) \quad (2.4)$$

where \tilde{g} is the doubled hyperbolic metric associated to ξ_1 (see Example 1.4).

It was proved by Alexakis and Mazzeo [AM10] that the renormalised area and the Willmore energy are, up to a constant, the same for minimal surfaces of \mathbb{H}^n . The *Willmore energy* of a surface is defined as the total integral of its trace-free second fundamental form:

$$\mathcal{W}(\Sigma) = \frac{1}{2} \int_{\Sigma} |\hat{\Pi}|^2.$$

It is a conformal invariant of the metric in the ambient space. For this reason, it is finite if the surface extends smoothly at infinity.

Proposition 2.8 (cf. [AM10]). *Let Σ be a minimal surface of \mathbb{H}^n that extend in a C^2 way up to its ideal boundary γ . Then its renormalised area and Willmore energy are*

related by

$$\mathcal{A}_R(\Sigma) = -\mathcal{W}(\Sigma) - 2\pi\chi(\Sigma) \quad (2.5)$$

Here $\chi(\Sigma)$ is the Euler characteristic.

Moreover, if we double Σ along its boundary to obtain a closed surface 2Σ of S^n , then for any round metric g_S on S^n ,

$$2\mathcal{A}_R(\Sigma) = -A_{g_S}(2\Sigma) - \int_{2\Sigma} H^2. \quad (2.6)$$

Here A_{g_S} and H are the area and mean curvature of 2Σ under g_S .

Proof. It follows from the Gauss–Codazzi formula in \mathbb{H}^n that

$$-1 + H^2 = \kappa + \frac{1}{2}|\hat{\Pi}|^2 \quad (2.7)$$

where κ is the Gauss curvature of the surface Σ and H is its mean curvature. Now we substitute $H = 0$, integrate (2.7) over the region $\Sigma_\epsilon := \Sigma \cap \{\rho \geq \epsilon\}$, and apply Gauss–Bonnet formula to obtain:

$$A(\Sigma_\epsilon) = -2\pi\chi(\Sigma_\epsilon) + \int_{\partial\Sigma_\epsilon} k - \frac{1}{2} \int_{\Sigma_\epsilon} |\hat{\Pi}|^2 \quad (2.8)$$

Here k is the curvature of the curve $\partial\Sigma_\epsilon$ inside Σ . It follows from minimal surface equation of Σ that

$$|\partial\Sigma_\epsilon|_g = |\partial\Sigma| + O(\epsilon^2), \quad k = 1 + O(\epsilon^2)$$

We plug these into (2.8) and compare with Graham–Witten expansion to obtain (2.5).

The equation (2.5) can be rewritten as $\mathcal{A}_R(\Sigma) = -\frac{1}{2}\mathcal{W}(2\Sigma) - 2\pi\chi(\Sigma)$. Now the Gauss–Codazzi formula for the surface 2Σ in S^n says

$$1 + H^2 = \kappa + \frac{1}{2}|\hat{\Pi}|^2$$

Integrate this and we have

$$A_{g_S}(2\Sigma) + \int_{2\Sigma} H^2 = \mathcal{W}(2\Sigma) + 2\pi\chi(2\Sigma) = -2\mathcal{A}_R(\Sigma).$$

□

2.2 Explicit minimal surfaces

2.2.1 Minimal annuli of \mathbb{H}^3

There are three families of minimal surfaces of genus 0 in \mathbb{H}^3 that fill a pair of circles, with three different relative positions. They were documented under different names by do Carmo–Dajczer [CD83] and then later by Krtouš–Zelnikov [KZ14]. Each of these

three families were also separately found by Wang and Wei [BOM83], Mori [Mor81], and recently Martin–White [MW14]. It is perhaps useful to summarise here the results proved by these authors.

- **Two disjoint circles.** Up to isometry, there is a one-parameter family of annuli filling this configuration. These surfaces were first discovered by Mori [Mor81]. They were called surfaces *of spherical type* in [CD83] and *rotational symmetric* in [KZ14]. By a Möbius transform, we can assume that the boundary is a pair of concentric circles of radius $R_1 < R_2$ on the boundary plane of the half-space. The surfaces are rotationally symmetric and the number of solutions with a given boundary modulus can be seen from Figure 2.1. When the two circles are sufficiently far from each other, there is no minimal annulus filling them. As they get closer, we start to see exactly one minimal annulus, which then splits into two: one tends to the two-discs solution and one escapes to infinity. The annulus that escapes to infinity is stable while the other has Morse index 1. This stability question was settled by Bérard and Sa Earp [BS10], with partial results due to Mori and do Carmo–Dajczer.
- **Two circles intersecting at 2 points.** The problem can be reduced further to finding minimal discs filling two circular arcs intersecting each other non-tangentially (see Figure 2.3 and Figure 2.4). Up to isometry, the family is one-parameter. These surfaces were called *of parabolic type* in [CD83] and *translational* in [KZ14]. They were first found by Wang and Wei as counter-examples for the Bernstein conjecture in the hyperbolic space [BOM83]. If we assume, after a Möbius transform, that the boundary curve is the union of two rays starting at the origin of the half-space model, then these surfaces are scale-invariant. Wang and Wei proved that these surfaces are area-minimising. It is worth noting that although the boundary is not C^1 , the minimal disc is smoothly embedded everywhere in the interior.
- **Two circles tangent to each other.** Up to Möbius transform, there is only one configuration of the boundary: two parallel lines of distance 1 to each other on the boundary of the half-space model. The minimal surface filling these two lines is invariant by translation along the lines. It was called *hyperbolic type* in [CD83] and *horocyclic* in [KZ14]. The surface was proved to be stable by do Carmo and Dajczer and later area-minimising by Martin and White when they rediscovered it.

The overlapping efforts and terminologies are summarised in the following table. Figure 2.3 and Figure 2.5 show a parabolic/translational surface and a hyperbolic/horocyclic surface in different models of the hyperbolic space. The cylindrical model is obtained by removing two points on the sphere at infinity. Each circular slice corresponds to a totally geodesic disc.

Boundary circles	Name in [CD83]	Name in [KZ14]	also known to
Disjoint	spherical	rotational	Mori (1981) [Mor81]
Intersecting	parabolic	translational	Wang–Wei (1979–1980) [BOM83]
Tangent	hyperbolic	horocyclic	Martin–White (2014) [MW14]

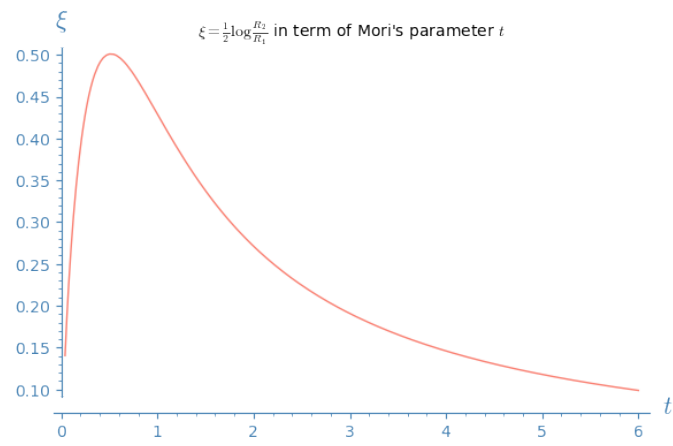


Figure 2.1 – The boundary modulus in term of the parameter of the spherical/rotational family.

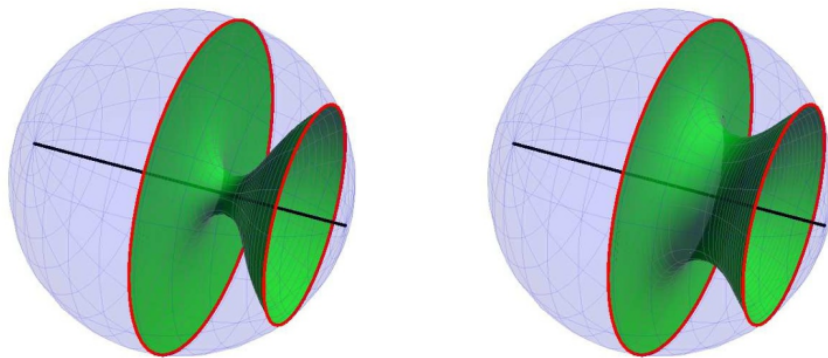


Figure 2.2 – Two spherical/rotational minimal annulus filling a same pair of circles. Image from [KZ14].

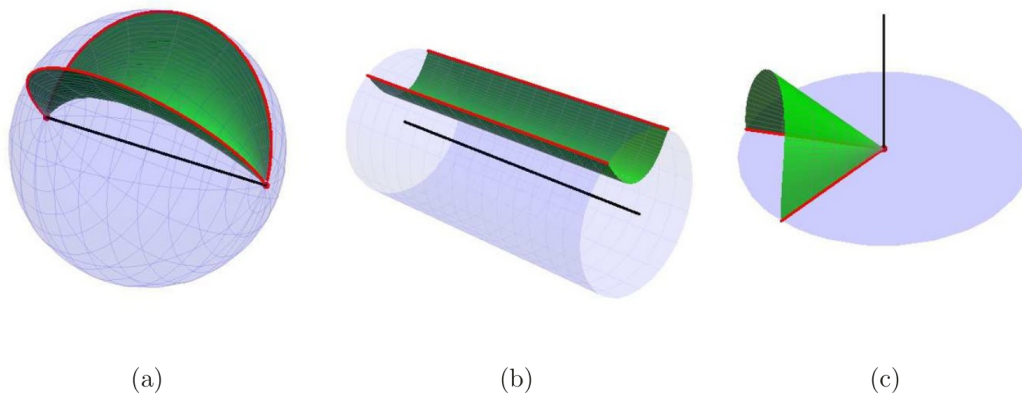


Figure 2.3 – A parabolic type/translational minimal disk viewed in (a) the Poincaré model, (b) the cylindrical model and (c) the half-space model. Image from [KZ14].

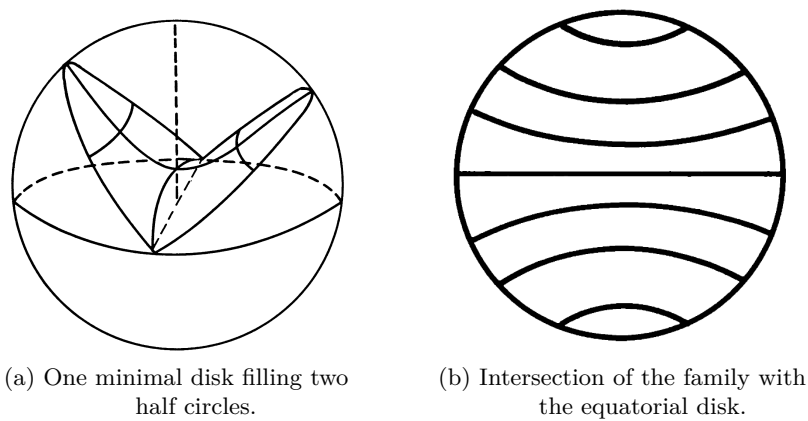


Figure 2.4 – The parabolic type/translational minimal disks illustrated by Wang and Wei. Image from [BOM83].

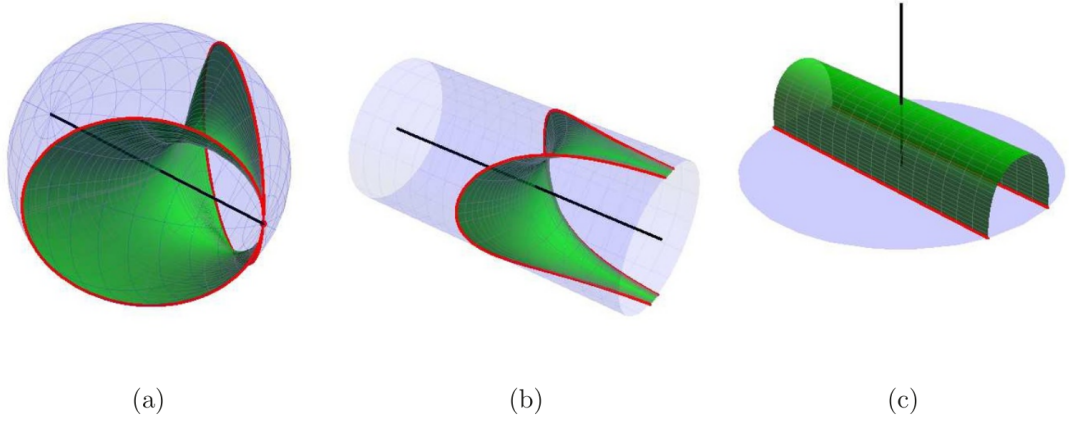


Figure 2.5 – A hyperbolic type/horocyclic minimal strip viewed in (a) the Poincaré model, (b) the cylindrical model and (c) the half-space model. Image from [KZ14].

2.2.2 Minimal annuli filling Hopf link

We will point out an explicit one-parameter family of annuli of \mathbb{H}^4 whose boundaries are Hopf links of S^3 . We see a Hopf link as a pair of orbits of the S^1 -action

$$(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{-i\theta} z_2) \quad (2.9)$$

on the 3-sphere $\rho := |z|^2 + |w|^2 = 1$ in \mathbb{C}^2 . If \mathbb{C}^2 is identified with the space of quaternions by $(z, w) \mapsto z + jw$, this action corresponds to the multiplication on the left by $e^{i\theta}$.

Let $\varphi = \varphi(\rho)$ be any radial function on \mathbb{R}^4 , we will construct surfaces that are invariant by (2.9) and minimal under the metric $g = e^{2\varphi} g_E$. When φ is constant, these surfaces are the complex curves $z_1 z_2 = a$, $a \in \mathbb{C}$. In the general case, they are obtained by rotating a curve

$$z_1 z_2 = F(\rho)$$

in the real plane $\text{Im } z_1 = \text{Im } z_2 = 0$. Here F is a real function on ρ . The minimal surface equation can be rewritten as the following second order ODE of F

$$\frac{X'}{X} - \frac{Y'}{Y} + \frac{1}{\rho} + \frac{\varphi'}{2} \left[8 + \rho \left(\frac{X^2}{Y^2} - 4 \right) \frac{F'}{F} \right] = 0, \quad \text{where } X = F - F'\rho, \quad Y = \frac{F'}{2} \sqrt{\rho^2 - 4F^2}.$$

This can be reduced to a first order ODE using a symmetry of the problem. Starting from a solution curve, we can obtain more solution by rotating it in the real plane. This second rotation corresponds to multiplying on the right of $z_1 + jz_2$ by $e^{j\alpha}$ and it commutes with the left multiplication by $e^{i\theta}$.

Concretely, by a change of variable $F = \frac{\rho}{2} \sin \theta(\rho)$ the ODE above reduces either to the first order Bernoulli equation $\theta'^2 = -\rho^2 + C^2 \rho^4 e^{4\varphi}$ for a parameter $C > 0$, or to $\theta' = 0$ which corresponds to pairs of 2-planes. The profile curve can be described in a more geometric fashion, as in Proposition 2.9. This description allows us to see that these surfaces are the complex curves $z_1 z_2 = a$ when g is the Euclidean metric.

Proposition 2.9. *Let M_C be the surface in \mathbb{R}^4 given by rotating the following real plane*

2.2. Explicit minimal surfaces

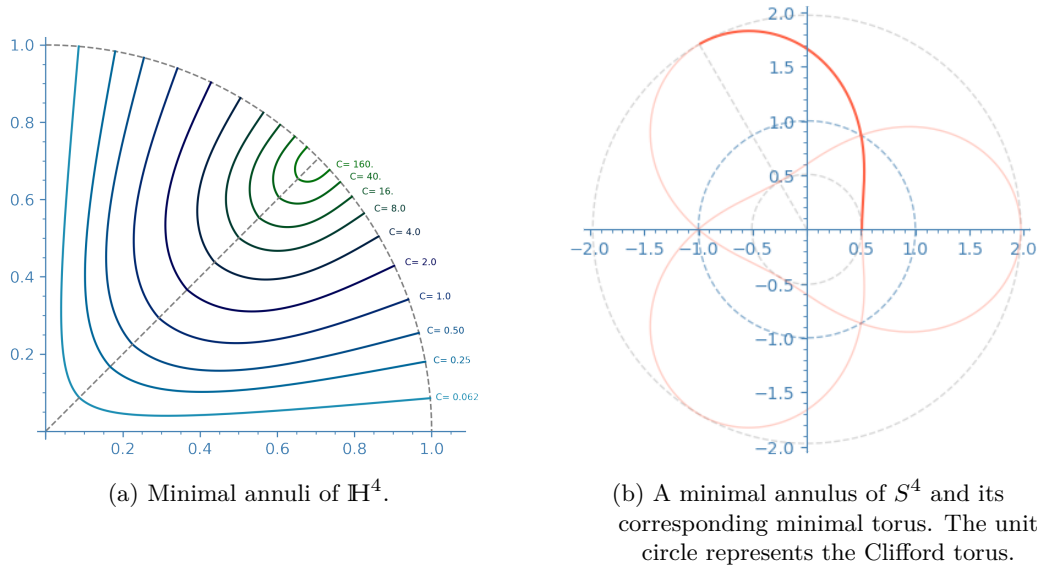


Figure 2.6 – The profile curves of a few rotational minimal surfaces in \mathbb{H}^4 and S^4 .

curve:

$$\sin^2 \psi = C^2 \frac{e^{-4\varphi}}{\rho^2}, \quad C > 0 \quad (2.10)$$

Here ψ is the angle formed by the tangent of the curve at a point p and the radial direction \overrightarrow{Op} . Then M_C is minimal under the metric $g = e^{2\varphi} g_E$. Up to $SO(4)$, the annuli M_C and the 2-planes are the only minimal surfaces obtained as orbit of a real plane curve by the rotation (2.9).

For the hyperbolic space, $e^\varphi = \frac{2}{1-\rho}$. Seen from the origin, the profile curve spans an angle θ_C that varies between 0 ($C = +\infty$) and $\frac{\pi}{2}$ ($C = 0$). The profile curves are drawn in Figure 2.6 (a).

For the round sphere, $e^\varphi = \frac{2}{1+\rho}$ and the parameter C can only be chosen in $(0, 1)$. The angle θ_C can take any value between $\frac{\pi}{2}$ ($C = 0$) and $\frac{\pi}{\sqrt{2}}$ ($C = 1$). In the case $C = 0$, M_C is a totally geodesic S^2 and in the case $C = 1$, M_C is (part of) the Clifford torus of the equatorial S^3 . In particular, if θ_C is a rational multiple of π in this interval, we can close the surface by repeating the profile curve. This produces a countable family of immersed tori in S^4 that are invariant by the quaternionic rotation. The dynamical system description of the family M_C in the next section will show that the resulting tori are smooth at the circles where the annuli are glued together. It was pointed out to the author by Benjamin Aslan that the annuli M_C in the case of S^4 are already known. Hsiang and Lawson [[Hsiang.H.BlaineLawson71_MinimalSubmanifoldsLow](#)] constructed a family $\tau_{p,q,\alpha}$ of minimal annuli of S^3 invariant by the (p, q) -rotation

$$(z, w) \longrightarrow (e^{ip\theta} z, e^{iq\theta} w)$$

where we have identified the Euclidean space \mathbb{R}^4 , where S^3 is embedded as the unit sphere, with \mathbb{C}^2 . Lawson [[Lawson70_CompleteMinimalSurfaces](#)] defined a trans-

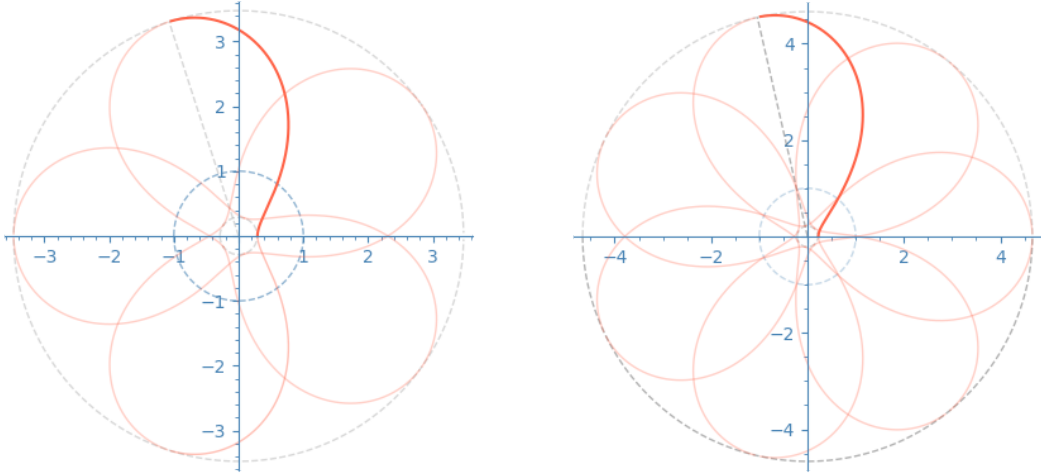


Figure 2.7 – More minimal tori obtained by the method.

form, called bipolar transform, that turns a minimal surface in S^3 into a minimal surface in S^5 . This transformation is defined by wedging a conformal harmonic map $f : \Sigma \rightarrow S^3 \subset \mathbb{R}^4$ with its Gauss map (valued in \mathbb{R}^4). The result is a map from Σ to the unit sphere of $\Lambda^2 \mathbb{R}^4 \cong \mathbb{R}^6$, which is conformal and harmonic. It is easy to check that if the map f is (p, q) -invariant, then its bipolar transform is contained in a subsphere S^4 of S^5 and is invariant by a $(p + q, p - q)$ -rotation. In case $p = 0, q = 1$, one obtains the annuli M_C .

2.2.3 From dynamical system point of view

In this subsection, we will write the minimal surface equation of M_C as a dynamical system. We start with the following Lemma which computes how the second fundamental form changes under conformal transform. Recall that for a C^2 map $f : (\Sigma, h) \rightarrow (M, g)$, the *second fundamental form* is a 2-form on Σ with value in f^*TM given by the covariant derivative $\Pi(f) = \nabla df$. Its trace $\tau(f)$ is called the *tension field*. Here we see df as a section of $\Omega^1(f^*TM)$ and the connection on f^*TM is obtained by the Levi-Civita connections of M and Σ .

Lemma 2.10. *Let $f : (\Sigma^k, h) \rightarrow (M^n, g)$ be a C^2 map and denote by $\Pi(f, h, g)$ its second fundamental form. Then*

1. *For all function φ on M*

$$\Pi(f, h, e^{2\varphi}g) = \Pi(f, h, g) + df \otimes d(\varphi \circ f) + d(\varphi \circ f) \otimes df - \nabla^g \varphi f^*g$$

2. *For all function φ on Σ*

$$\Pi(f, e^{2\varphi}h, g) = \Pi(f, h, g) - df \otimes d\varphi - d\varphi \otimes df + (f_* \nabla^h \varphi) \otimes h$$

$$\text{In particular, } \tau(f, e^{2\varphi}h) = e^{-2\varphi} [\tau(f, e^{2\varphi}h) + (k - 2)f_* \nabla^h \varphi].$$

2.2. Explicit minimal surfaces

Proof. Both equations follow from Leibniz rule $\Pi(u \circ v) = v^* \Pi(u) + u_* \Pi(v)$ and the formula of the second fundamental form of the identity map $\iota : (M, g) \rightarrow (M, e^{2\varphi}g)$:

$$\Pi(\iota) = \iota \otimes d\varphi + d\varphi \otimes \iota + \nabla\varphi \otimes g,$$

where $\nabla\varphi$ denotes the g -gradient. \square

Now let us look for harmonic maps $f : (S^1 \times \mathbb{R}, d\theta^2 + ds^2) \rightarrow (\mathbb{R}^4, g_\varphi)$ given by the equation

$$z_1 = e^{iq\theta}u(s), \quad z_2 = e^{-ip\theta}v(s), \quad p, q \in \mathbb{Z} \quad (2.11)$$

Here u and v are complex-valued functions on \mathbb{R} .

The Euclidean tension field of f is

$$\tau(f) = \left(\frac{u''}{u} - q^2\right) z_1 \partial_{z_1} + \left(\frac{\bar{u}''}{\bar{u}} - q^2\right) \bar{z}_1 \partial_{\bar{z}_1} + \left(\frac{v''}{v} - p^2\right) z_2 \partial_{z_2} + \left(\frac{\bar{v}''}{\bar{v}} - p^2\right) \bar{z}_2 \partial_{\bar{z}_2}.$$

So f is harmonic into the Euclidean space if and only if

$$u = A_1 e^{qs} + \bar{A}_2 e^{-qs}, \quad v = B_1 e^{ps} + \bar{B}_2 e^{-ps}, \quad (A_1, A_2, B_1, B_2) \in \mathbb{C}^4 \quad (2.12)$$

Among these maps, those that are conformal have

$$q^2 A_1 A_2 + p^2 B_1 B_2 = 0 \quad (2.13)$$

Most of the minimal surfaces found this way are not complex curves under any complex structure of \mathbb{R}^4 compatible with the Euclidean metric. By definition, a surface is complex under any such complex structure if and only if one of its two Gauss lifts is constant. Let J_0 be the complex structure that gives the coordinates (z_1, z_2) .

Proposition 2.11. *The surfaces given by the equation (2.11) and (2.12) with parameters $(A_1, A_2, B_1, B_2) \in \mathbb{C}^4$ satisfying (2.13) are minimal in the Euclidean 4-space.*

1. *When $p = q = 1$, all of these surfaces are complex curves under compatible complex structures of \mathbb{R}^4 .*
2. *When $p \neq q$, the surfaces are complex curves under a complex structure homotopic to J_0 if and only if $A_1 B_1 = A_2 B_2 = 0$, and a complex structure not homotopic to J_0 if and only if $A_1 B_2 = A_2 B_1 = 0$.*

The tension field under the metric g_φ can be computed using Lemma 2.10. The map f is g_φ -harmonic if and only if

$$\begin{cases} u'' - q^2 u = 2u\varphi' \left[q|u|^2 + p|v|^2 + |v'|^2 - \frac{u'^2 \bar{u}}{u} - (v'\bar{v} + v\bar{v}') \frac{u'}{u} \right] \\ v'' - p^2 v = 2v\varphi' \left[q|u|^2 + p|v|^2 + |u'|^2 - \frac{v'^2 \bar{v}}{v} - (u'\bar{u} + u\bar{u}') \frac{v'}{v} \right] \end{cases} \quad (2.14)$$

Here φ' denotes $\frac{d\varphi}{ds}$ while all other derivatives are taken in s .

By holomorphicity of the Hopf differential, a map f given by (2.14) is conformal everywhere if it is conformal at one value $s = s_0$. For this reason, the conformality

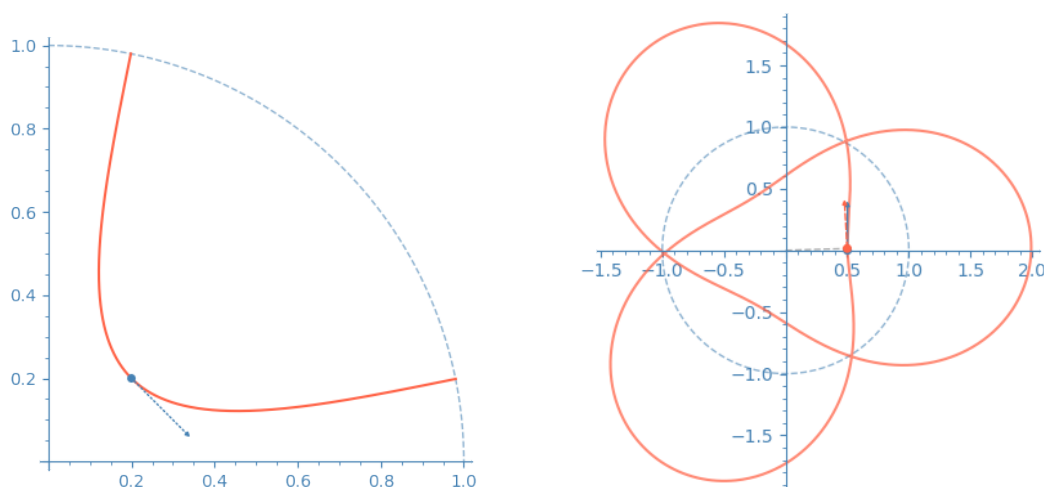


Figure 2.8 – Approximation of the profile curve in Figure 2.8 using the system (2.14). Initial state is described with a blue dot and arrow.

condition is an initial condition of the system. Concretely, f is conformal if and only if

$$\begin{cases} q^2|u|^2 + p^2|v|^2 = |u'|^2 + |v'|^2 \\ q(u\bar{u}' - u'\bar{u}) = p(v\bar{v}' - v'\bar{v}) \end{cases} \quad (2.15)$$

It can be checked that (2.15) holds for all s if it holds for one value $s = s_0$.

When u, v are real-valued, the initial condition (2.15) is equivalent to $H := \frac{1}{2}(u'^2 + v'^2 - q^2u^2 - p^2v^2) = 0$. Under this condition, (2.14) becomes

$$\begin{cases} u'' = q^2u + 4\varphi'v'(uv' - u'v) \\ v'' = p^2v - 4\varphi'u'(uv' - u'v) \end{cases} \quad (2.16)$$

It can be checked that H is a first integral.

When $p = q = 1$, the system (2.16) has a symmetry as described in subsection 2.2.2. This corresponds to the first integral

$$C = e^{2\varphi}(uv' - u'v).$$

which is the parameter C in Proposition 2.9.

When $p \neq q$, the system no longer has the quaternionic symmetry and I cannot solve (2.14) explicitly. A few numerical approximations are shown in Figure 2.8 and Figure 2.9.

2.3 Applications of monotonicity theorems

In this section, we will be looking at submanifolds of the Euclidean unit ball \mathbb{B}^n . We denote by $B(r)$ the closed n -ball centred at the origin with radius $r \in (0, 1)$ and $S(r)$ its boundary sphere.

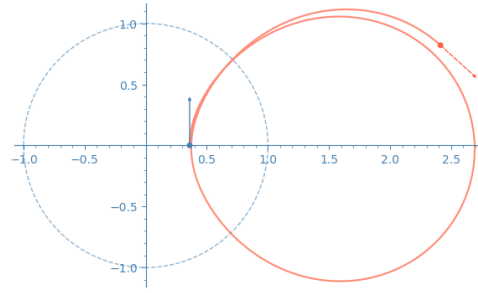


Figure 2.9 – A solution of the system (2.14) in S^4 when $(p, q) = (2, 5)$ that seems to be periodic.

2.3.1 Intersection curves

Recall that Proposition 1.18 allows us to estimate the intersection of a minimal submanifold with the level set $h = t$. One immediate consequence is:

Proposition 2.12. *Let Σ^k be a submanifold of the unit ball \mathbb{B}^n that contains the origin with density m . Suppose that the intersection of Σ with $B(r)$ has no other boundary than its intersection γ_r with the sphere. If Σ is minimal, either under the Poincaré metric, the Euclidean metric or the half sphere metric of \mathbb{B}^n , then the radial projection of γ_r on $S(1)$ has Euclidean volume least $m\omega_{k-1}$.*

The Propositions 2.13 and 2.14 are the application of Proposition 1.18 to null and space coordinates of \mathbb{H}^n :

Proposition 2.13. *Let Σ be a minimal k -submanifold of \mathbb{H}^n that is bounded in the half space model and C^2 up to its ideal boundary γ_∞ . Let γ_t be the intersection of Σ and the horizontal hyperplane $\xi_l = t$, or equivalently $x = \frac{1}{t}$ for the half space coordinate x . Then the Euclidean volume of γ_∞ is greater than the $T\Sigma$ -parallel volume of γ_t : $|\gamma_\infty| \geq |\gamma_t^{T\Sigma}|$.*

More generally, the Euclidean volume of a level set $\gamma_{t_1} = \Sigma \cap \{x = \frac{1}{t_1}\}$ is greater than the parallel volume of any level set $\gamma_{t_2} = \Sigma \cap \{x = \frac{1}{t_2}\}$ higher than it (that is, with $t_2 < t_1$).

Recall that here the parallel volume of γ_t (see Definition 1.5) is the Euclidean volume weighted by the angle between $T\Sigma$ at the vertical direction. Proposition 2.13 is not at all obvious when we look at a neighbourhood of the smaller boundary circle of a Mori surface (as shown in Figure 2.10). The circumference of horizontal slices of this collar increases as we get higher and Proposition 2.13 becomes a statement about how horizontal the surface has to be as it goes into the interior of \mathbb{H}^n . In other words, it is an estimate of the angle θ formed by the surface and the vertical direction.

We will illustrate Proposition 2.13 by computing the parallel length of the *waist* of the Mori's surfaces. Recall that in the half-space model, these are rotational annuli bounded by a pair of concentric circles. The waist is the circle closest to the rotational axis in hyperbolic distance. The green curve in Figure 2.11 is the parallel length of this circle. The orange curve represents its Euclidean length, of which we have no control. Proposition 2.13 says that the green curve has to stay below 1.

For the space coordinates, we have:

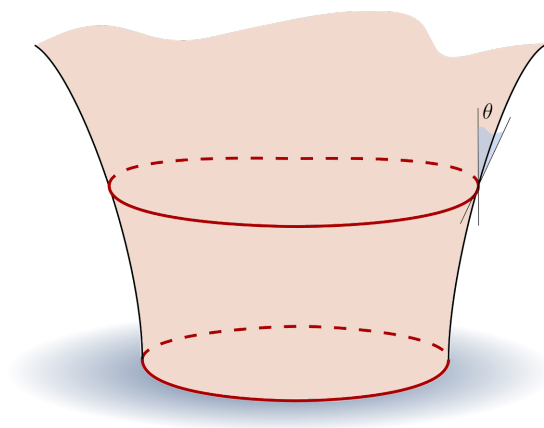


Figure 2.10 – A collar neighborhood of the smaller boundary circle of a Mori surface.

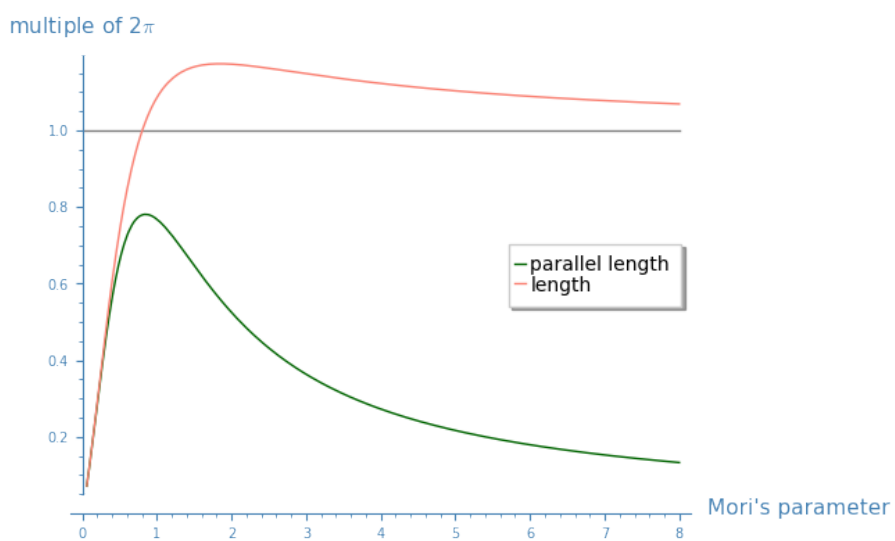


Figure 2.11 – The length and parallel length of the waist of Mori surfaces. We normalised the radius of the smaller boundary circle to be 1.

Proposition 2.14. *Let Σ be a minimal k -submanifold of \mathbb{H}^n in the region $\xi_1 \geq a > 0$. Suppose that the boundary of Σ is the union of a $(k-1)$ -submanifold γ_a of the level set $\xi_1 = a$ and a $(k-1)$ -submanifold γ_t of $\xi_1 = t > a$. Then the normalised volume of γ_t is not less than the $T\Sigma$ -parallel volume of γ_a .*

In particular, when $t = \infty$ and γ_∞ is a submanifold of S_∞ , we have $|\gamma_\infty| \geq |\gamma_a^{T\Sigma}|$.

2.3.2 Lower bound of area of minimal surface in S^n

The Comparison Lemma 1.15 can be useful even when we only have monotonicity in a weaker weight (in order words, how to go upstream in the chain (1.19)). We illustrate this with the following result, originally proved by Cheng, Li and Yau using by their estimate of the heat kernel of space forms.

Proposition 2.15 (Cheng–Li–Yau [CLY84]). *Let Σ^k be a submanifold of the unit ball that contains the origin with density m . Suppose that Σ is minimal under the metric g that is either the Poincaré, the Euclidean or the half sphere metric of \mathbb{B}^n . Then the g -volume of Σ inside any ball $B(r)$ is not less than m times the g -volume of a totally geodesic k -disc containing the origin.*

Corollary 2.16. *Any k -dimensional closed minimal submanifold of the sphere has volume at least ω_k . Equality happens only for totally geodesic subspheres.*

Proof of Proposition 2.15. It suffices to prove that the unweighted density functional with respect to the metric g is not less than m . When g is the Euclidean metric, this is Proposition 1.18. When g is the hyperbolic metric, this is Corollary 1.19. When g is the sphere metric, we know by Theorem 1.8 that the weighted density in the half sphere is not less than m . We do not know if the unweighted density is an increasing function, but the second half of the Comparison Lemma 1.15 says that it is also not less than m . \square

Proof of Corollary 2.16. We pick any point p on the submanifold and apply Proposition 2.15 to prove that the unweighted volume in the half sphere containing p is at least $\frac{\omega_k}{2}$. We do not know whether the submanifold contains the antipodal $-p$, but we know that the weighted volume in the half sphere centred at $-p$ is equal to the weighted volume in the half sphere centred at p . This is because any Euclidean coordinate function x of $\mathbb{R}^{n+1} \supset S^n$ satisfies

$$\int_{\Sigma} x - \frac{1}{k} \int_{\Sigma} \Delta x = 0.$$

Now repeat the argument of Proposition 2.15 in the half sphere containing $-p$. \square

2.3.3 Area-minimising cone

As we claimed in Section 1.5, the second \gg of the chain (1.19) recovers the following result, originally due to Anderson:

Proposition 2.17 (cf. Theorem 9 of [And82]). *In the unit ball \mathbb{B}^n , minimising cones in the Poincaré metric are exactly minimising cones in the Euclidean metric.*

It is not difficult to see that a cone is hyperbolic minimising only if it is Euclidean minimising.¹

Proposition 2.18. *Let C_γ be a k -dimensional radial cone, constructed over a submanifold γ on the sphere at infinity of the Poincaré model. If C_γ is Euclidean-minimising, then there is no other minimal surface asymptotic to γ .*

Proposition 2.18 is stronger than Proposition 2.17. Knowing that the pair of 2-planes $zw = 0$ in \mathbb{C}^2 is hyperbolic-minimising only allows us to rule out minimal surfaces of \mathbb{H}^4 that agree with the planes on a neighbourhood of infinity.

Proof of Proposition 2.18. We know from the monotonicity chain (1.19) that the Euclidean volume functional is weaker than the hyperbolic volume. Suppose by contradiction that there is a minimal submanifold Σ^k of \mathbb{H}^n filling γ . Then Σ^k is Euclidean monotone and by Corollary 1.19, it has Euclidean volume strictly less than that of C_γ . This means that C_γ was not Euclidean-minimising. \square

In \mathbb{R}^4 , the cone whose section is the Hopf link $zw = \epsilon, |z|^2 + |w|^2 = 1$ is not Euclidean-minimising because this link bounds the complex curve $zw = \epsilon$. This cone is therefore not hyperbolic minimising either. This is why in Section 2.2, we were able to find another minimal surface of \mathbb{H}^4 filling it.

2.3.4 Isoperimetric inequality for minimal surfaces in \mathbb{H}^n . Visual volume.

In the Euclidean plane, the round disk maximises area among shapes with the same perimeter. More precisely, a curve of length L encloses a region of area at most $\frac{L^2}{4\pi}$. In general, if Ω is a domain in the 2-dimensional space form of curvature K , then its area A and perimeter L are related by:

$$4\pi A \leq L^2 + KA^2 \tag{2.17}$$

Equality happens if and only if the domain is a geodesic disk.

Because the space forms $\mathbb{R}^2, S^2, \mathbb{H}^2$ can be isometrically and minimally embedded in higher dimensional space forms, we may hope to generalise (2.17) for minimal surfaces in $\mathbb{R}^n, S^n, \mathbb{H}^n$. Even in the Euclidean case, the history of this problem is fascinating.

— The case of minimal disk was proved by Carleman in 1921 [Car21].

From the late 50s, the restriction on the topological type of the surface was removed:

— The case of minimal surfaces with connected boundary was proved by Reid in 1959 [Rei59] and Hsiung in 1961 [Chi61].

— The case of minimal annuli was proved by Osserman–Schiffer in 1975 [OS75] and Feinberg in 1977 [Fei77].

The late 80s saw more relaxation on the boundary curve:

— Li, Schoen and Yau [LSY84] proved in that the inequality holds for minimal surfaces with *weakly connected* boundary. This means that there is a orthogonal coordinates system $\{x_i\}$ of \mathbb{R}^n such that $\partial\Sigma$ cannot be separated by any plane $x_i = c$. Any curve with 2 components fall into this type.

1. Suppose that it is not, then by zooming in at the tip of the cone, the hyperbolic metric looking more and more Euclidean, we could just replace a small neighbourhood of the tip by a submanifold having strictly smaller volume.

- In 1990, Choe [Cho90] proved that the inequality holds for minimal surfaces with *radially connected boundary*. This means that there is a point O in the surface such that the set $\{d(a, O) : a \in \partial\Sigma\}$ of distances from O to boundary points is a connected interval. This condition is satisfied for connected surfaces with two boundary components (it suffices to choose O to have equal distance to each boundary component).
- Very recently, Brendle [Bre20] proved the isoperimetric inequality for minimal submanifolds of codimension at most 2 in \mathbb{R}^n .

The isoperimetric inequality (2.17) for minimal surfaces in the hyperbolic space was proved by Choe and Gulliver [CG92b] when the boundary curve is radially connected. The crucial idea here, also featured in Choe’s work on the Euclidean case, is that:

1. the area of a minimal surface in the Poincaré ball is less than the radial cone built upon its boundary and²
2. the quantity $4\pi A - KA^2$ is increasing in A when $K = -1$ or $K = 0$.

Therefore, it suffices to verify the inequality (2.17) for radial cones.

Recall that each interior point of the hyperbolic space defines a round metric on the sphere at infinity S_∞ and that the volume of a submanifold of S_∞ under this metric is by definition its visual volume at O .

Lemma 2.19 (Choe–Gulliver [CG92b]). *Let γ be a closed embedded, radially connected curve in the interior of the Poincaré model. Suppose that the visual volume of γ from the centre O of the ball is at least 2π , then the hyperbolic length L of γ and the hyperbolic area A of the radial cone of section γ satisfy*

$$4\pi A + A^2 \leq L^2. \tag{2.18}$$

Theorem 2.20 (Choe–Gulliver [CG92b]). *Let $\Sigma \subset \mathbb{H}^n$ be a minimal surface whose boundary $\partial\Sigma$ is radially connected from a point of Σ . Then the area A of Σ and the length L of its boundary also satisfy (2.18). Equality happens if and only if Σ is a geodesic disk in a totally geodesic copy of \mathbb{H}^2 .*

Here the *visual volume* from O of a submanifold $\gamma \subset \overline{\mathbb{H}^n}$ is the Euclidean volume of its radial projection to S_∞ . When γ lies on a geodesic sphere centred at O , this is exactly its volume under the normalised metric.

To prove Theorem 2.20 from Lemma 2.19, it suffices to choose O on Σ . Proposition 1.18 guarantees that the visual volume is at least 2π .

The visual volume was called the *angle* of γ by Choe and Gulliver. We adopt here the terminology due to Gromov [Gro83], which was defined in a larger context.

Definition 2.21. *Let X^n be a complete, connected, simply connected Riemannian manifold without conjugate points and Y^k be an immersed submanifold of X . Let p be a point in $X \setminus Y$ and $E_p : X \setminus \{p\} \rightarrow S^{n-1}$ be the geodesic projection onto the unit sphere of the tangent space of X at p . The visual volume $\text{Vis}(Y, p)$ of Y from p is the Euclidean volume of its image via E_p .*

2. Note that Lemma 1.17 requires the boundary curve to stay on a sphere centred at the origin, but we can always reduce to this case by radially extend the surface.

A priori, the point p has to be outside of Y , but when Y is smoothly immersed, the function $\text{Vis}(Y, \cdot)$ extends continuously to the interior of Y (except when it is $+\infty$ everywhere). For example, if Y is a k -dimensional plane of the Euclidean space \mathbb{R}^n , then $\text{Vis}(Y, \cdot)$ is $\frac{1}{2}\omega_k$ everywhere. More generally, if Y is smoothly immersed, then the visual volume from an interior point of Y is at least $\frac{1}{2}\omega_k$. We will discuss more on the visual volume in the next Chapter.

2.3.5 Renormalised Isoperimetric inequality.

The most technical part of Choe–Gulliver’s proof of Theorem 2.20 was to show that the area of a minimal surface is less than a cone built upon its boundary. We have seen in the previous subsection that the comparison lemma provides a shortcut to this. We will now use the same argument to prove a renormalised version of isoperimetric inequality for complete minimal surfaces of \mathbb{H}^n .

Since the area of a complete surface is infinite, we need to replace it by the Graham–Witten’s renormalised area (see Theorem 2.6 for definition). The perimeter also needs to be normalised. We have already seen three ways to do that: with the round metric, the flat metric or the doubled hyperbolic metric in the conformal class at infinity.

Our first estimate of the renormalised area is a consequence of Corollary 1.19 for a time coordinate ξ_0 and the uniform weight. This corollary says that the area of a minimal surface Σ in the region $\xi_0 \leq t$ is less than the area of a radial cone built over the intersection $\gamma_t = \Sigma \cap \xi_0^{-1}(t)$:

$$A(\Sigma \cap \{\xi_0 \leq t\}) \leq |\gamma_t| \int_1^t dh = |\gamma_t|(t-1) = |\gamma|t - |\gamma| + O(t^{-1}) \quad (2.19)$$

where $|\gamma_t|$ and $|\gamma|$ are the length under the normalised metric \tilde{g} of the intersection γ_t and the ideal boundary γ of Σ . For the last equality of (2.19), we have used

$$|\gamma_t| = |\gamma| + O(t^{-2}) \quad (2.20)$$

which is because Σ meets S_∞ at right angle.

Now we replace the LHS of (2.19) by the Graham–Witten series development (2.3) to obtain

$$\mathcal{A}_R(\Sigma) + |\gamma| \leq 0.$$

By doing this argument for all the time coordinate functions, we have the following renormalised version of the isoperimetric inequality. Theorem 2.22 was also independently proved by Jacob Bernstein [Ber21].

Theorem 2.22. *Let Σ be a complete minimal surface in \mathbb{H}^n whose ideal boundary is a curve γ in the sphere infinity. Suppose that Σ is C^2 near boundary, then*

$$\mathcal{A}_R(\Sigma) + \sup_{\text{round } \tilde{g}} |\gamma|_{\tilde{g}} \leq 0 \quad (2.21)$$

where the supremum was taken among metrics of sectional curvature $+1$ in the conformal infinity of \mathbb{H}^n . In other words, $\mathcal{A}_R(\Sigma) + \sup_{p \in \mathbb{H}^n} \text{Vis}(\gamma, p) \leq 0$.

Although the inequality (2.21) is sharp, it does not give us good estimate of the

2.3. Applications of monotonicity theorems

renormalised area of the Mori's family or of the family M_C in Proposition 2.9. Because the boundaries of these surfaces are union of two round circles on the conformal sphere, the boundary term in (2.21) is at most 4π , and so the best we can get is $\mathcal{A}_R \leq -4\pi$. The renormalised area of these families however, tends to $-\infty$ as the surfaces "escape" to infinity.

One way to see this without any computation of the renormalised area is by the version of (2.19) for compensated area. Recall that for any Minkowskian coordinate ξ_* , with $*$ = 0, 1, l and any number $a > 0$, the uniform weight ($P = 1, c = \frac{1}{a}$) is weaker than the weight ($P = \xi_*, c = 1$) in the region $\xi \geq a$ (Lemma 1.31). Let Σ_t be a minimal surface in the region $a < \xi_* \leq t$ with boundary γ_t in $\xi_*^{-1}(t)$. By Corollary 1.19, the area of Σ_t is less than that of the tube built upon γ_t , that is

$$B_1(\Sigma)(t) = A(\Sigma \cap \{\xi \leq t\}) \leq B_1(T_\gamma(a, t)) \quad (2.22)$$

For a surface meeting S_∞ at right angle, the expansion (2.20) also holds for null and space coordinates. Combine this with (2.22), we have the following estimates, which generalise Theorem 2.22.

Theorem 2.23 (Renormalised isoperimetric inequalities). *Let ξ_0, ξ_1, ξ_l be Minkowskian coordinates with the corresponding round/ double hyperbolic/ flat metrics on S_∞ denoted by $\tilde{g}_0, \tilde{g}_1, \tilde{g}_l$. Let Σ be a complete minimal surface that is embedded and C^2 near its ideal boundary γ .*

1. *If Σ is in the region $\xi_0 \geq a \geq 1$ then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}|\gamma|_{\tilde{g}_0} \left(a + \frac{1}{a} \right) \leq 0 \quad (2.23)$$

2. *If Σ is in the region $\xi_1 \geq a > 0$ then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}|\gamma|_{\tilde{g}_1} \left(a - \frac{1}{a} \right) \leq 0 \quad (2.24)$$

3. *If Σ is in the region $\xi_l \geq a > 0$ then*

$$\mathcal{A}_R(\Sigma) + \frac{1}{2}|\gamma|_{\tilde{g}_l} a \leq 0 \quad (2.25)$$

Remark 2.24. 1. *The null estimate (2.25) is the limit case of (2.23) and (2.24) when the point associated to ξ_0 and the codimension 1 associated to ξ_1 are sent to infinity.*

2. *Both the time and space estimates imply that the renormalised area of a minimal surface is at most -2π (see Corollary 2.25 below). The null estimate (2.25) on the other hand only implies $\mathcal{A}_R \leq -\pi$.*

Corollary 2.25. *The renormalised area of a minimal surface of \mathbb{H}^n is at most -2π .*

Proof. We can choose a time coordinate associated to a point on the minimal surface, then use (2.23) and Proposition 2.12. \square

There are two other proofs of Corollary 2.25, both use rescaling argument. The first one uses the space estimate (2.24) and the second one uses the relation between renormalised area and the Willmore energy (Proposition 2.8).

Alternative proof #1 of Corollary 2.25. We first note that the function $y := \frac{\xi_1}{\xi_0}$ foliates \mathbb{H}^n into totally geodesic codimension 1. So by a Möbius transform, we can put the boundary of Σ , and by convexity the entire minimal surface, between level sets $y = \beta$ and $y = \beta + \epsilon$, $\beta > 0$. This guarantees that $\xi_1 \geq \alpha := \frac{\beta}{\sqrt{1-\beta^2}}$ on the surface. The Möbius transform can be chosen so that the normalised length of the boundary curve is ϵ -near to that of a great circle on $\{y = \beta\} \cap S_\infty$, which is $2\pi \frac{\sqrt{1-\beta^2}}{\beta}$. Apply (2.24) and send ϵ to 0, we have

$$\mathcal{A}_R + 2\pi\left(1 - \frac{1}{2\beta^2}\right) \leq 0$$

for all $\beta > 0$, which means $\mathcal{A}_R \leq -2\pi$. Figure 2.12 illustrates the level sets of $y = \frac{\xi_1}{\xi_0}$ and those of ξ_1 in the Poincaré model. \square

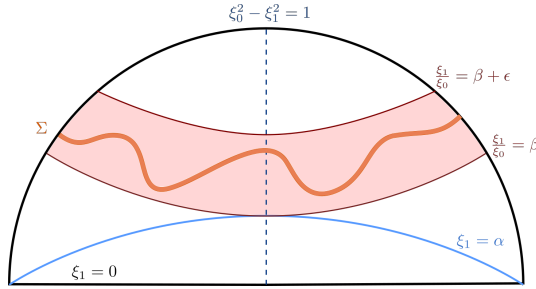


Figure 2.12 – Level sets of y (totally geodesic, in red) and those of ξ_1 (in blue) in the proof of Corollary 2.25. The $(n-1)$ -discs $y = \beta$ and $\xi_1 = \alpha$ touch each other at the center.

Alternative proof #2 of Corollary 2.25. It follows from (2.6) that

$$\mathcal{A}_R(\Sigma) + \frac{1}{2} \sup_g A_g(2\Sigma) \leq 0$$

where the supremum is taken among round metric on the sphere S^n containing the doubled surface 2Σ . In other words:

$$\mathcal{A}_R(\Sigma) + \frac{1}{2} \sup_{p \in \mathbb{H}^{n+1}} \text{Vis}(2\Sigma, p) \leq 0 \tag{2.26}$$

As p tends to a point on 2Σ , the visual area converges to 4π . The estimate (2.26) implies that $\mathcal{A}_R \leq -2\pi$. \square

Remarks on the time estimate (2.23) and renormalised area of the family M_C

We see from the time estimate (2.23) that the renormalised area of the annuli M_C of Proposition 2.9 is unbounded as the surfaces escape to infinity (as $C \rightarrow +\infty$). This is because the two boundary circles have Euclidean length 4π while the distance of the annuli towards the center increases to infinity. More precisely, the minimum value of the time coordinate (based at the origin) on M_C is

$$a = a(C) := \min_{M_C} \xi_0 = (C + 1)^{1/2}.$$

This means that the boundary term $b(C) := \frac{1}{2}(a + a^{-1}) \cdot 4\pi$ increases like $\frac{1}{2}C^{1/2}$ as the surfaces escape. The renormalised area can be computed directly to be

$$\mathcal{A}_R(M_C) = \int_a^\infty \left[\frac{\xi^2 - 1}{(\xi^2 - 1)^2 - C^2} - 1 \right] d\xi - 4\pi a$$

The comparison between these two quantities can be seen from Figure 2.13. The graph is drawn in log scale and the two lines are asymptotically parallel. It can be proved that $-\mathcal{A}_R(M_C) \approx 1.198 b(C)$ as C tends to infinity.

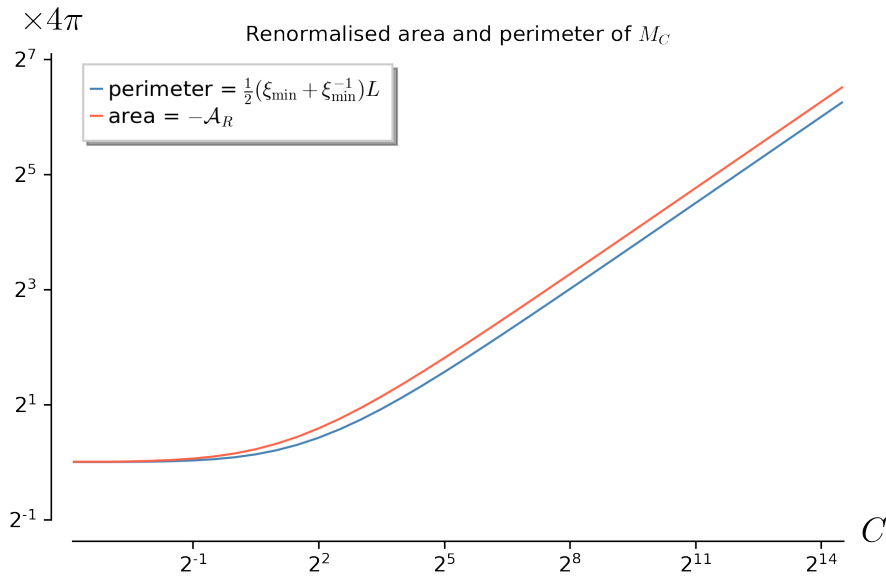


Figure 2.13 – The boundary term (blue) and the renormalised area (orange) in the estimate (2.23) applied on the family M_C .

Remarks on the null estimate (2.25) and renormalised area of Mori's surfaces.

The null estimate (2.25) can be rewritten in the half space model as

$$\mathcal{A}_R(\Sigma) + \frac{1}{2} \frac{|\gamma|}{x_{\max}} \leq 0$$

where $|\gamma|$ is the Euclidean length of the boundary and x_{\max} is the maximal height of the surface. For totally geodesic disc, $\mathcal{A}_R = -2\pi$ and $\frac{1}{2} \frac{|\gamma|}{x_{\max}} = \pi$ and so one may think that perhaps the constant $\frac{1}{2}$ can be removed. We will see that if it is replaced by any constant bigger than 0.599 then the resulting estimate fails for a Mori annulus.

Proposition 2.26. *Let a be any number bigger than*

$$a_0 := E(0.5)\sqrt{2} - \frac{K(0.5)}{\sqrt{2}} \approx 0.59907,$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of first and second type. Then there is a Mori surface with

$$\mathcal{A}_R(\Sigma) + a \frac{|\gamma|}{x_{\max}} > 0$$

Proof. The renormalised area of Mori's family was computed by Krtouš and Zelnikov [KZ14] in term of their parameter t of the family

$$\mathcal{A}_R = 4\pi \left[\frac{t^2}{\sqrt{1+2t^2}} K\left(\frac{1+t^2}{1+2t^2}\right) - \sqrt{1+2t^2} E\left(\frac{1+t^2}{1+2t^2}\right) \right].$$

The parameter t is related to the Mori's parameter θ_0 by $t = \tan \theta_0$. (When $t = 0$, the annulus degenerates to a pair of disc). The complete elliptic integrals are defined as

$$K(m) := \int_0^{\pi/2} \frac{dx}{\sqrt{1-m\sin^2 x}}, \quad E(m) := \int_0^{\pi/2} \sqrt{1-m\sin^2 x}.$$

The maximal height x_{\max} is difficult to write down explicitly, but can be estimated in a relatively simple way. Suppose that the Mori surface is placed so that its boundary is a pair of concentric circles of radius 1 and $R > 1$. The maximal height can be bounded between:

$$\cos \theta_0 \leq x_{\max} \leq R \cos \theta_0.$$

Because R decreases to 1 as $t \rightarrow +\infty$, we have:

$$\frac{|\gamma|}{x_{\max}} = 4\pi t + o(t) \quad \text{as } t \rightarrow +\infty.$$

The conclusion follows from:

$$\lim_{t \rightarrow \infty} \frac{\mathcal{A}_R(\Sigma_t)}{t} = 4\pi \left[\frac{K(0.5)}{\sqrt{2}} - E(0.5)\sqrt{2} \right] = -4\pi a_0.$$

□

2.4 Appendix C: Degree theory for minimal surfaces in \mathbb{R}^n

The counting problem for minimal surfaces bounded by a curve was investigated in many contexts. In the works mentioned below, the general expectation is that

1. the space \mathcal{M} of minimal surfaces is a Banach manifold,
2. the boundary map $\Pi : \mathcal{M} \rightarrow \mathcal{C}$, which associates to a surface in \mathcal{M} its boundary, which lies in a certain space \mathcal{C} of curves is Fredholm and of index 0.

If we can prove that Π is proper, then the number of surfaces bounded by a generic curve is finite. This number defines a degree modulo 2 for each curve in \mathcal{C} . If the index bundle of Π is trivial, this degree can be chosen to be integer-valued.

It is important to specify the space \mathcal{M} and \mathcal{C} carefully. In addition to the regularity, several details need to be given, such as whether we consider at surfaces or maps, and in the second case, which reparametrisation we need to mod out, whether the surfaces need to be immersed, embedded, orientable, which topological type they are etc. . .

2.4.1 Tomi–Tromba’s solution to the embedded Plateau problem

The counting problem for minimal surfaces was first considered by Tomi and Tromba [TT78] in their resolution of the embedded Plateau problem. Let $f : S^1 \rightarrow \mathbb{R}^n$ be a map with nowhere vanishing density $|df|$ on the circle and that can be extended to the disk D . It was proved by Douglas and Rado in the early 30s that f can be extended to a conformal harmonic map $F : D \rightarrow \mathbb{R}^n$. The embedded Plateau problem asks whether the extension F can be chosen to be embedded:

Theorem 2.27 (Tomi–Tromba). *Let $f : S^1 \rightarrow \mathbb{R}^3$ be a C^2 curve on the boundary of a convex body B of \mathbb{R}^3 , then there exists an embedded conformal harmonic map $F : D \rightarrow \mathbb{R}^3$ such that $F(S^1) = f(S^1)$.*

The space of curves \mathcal{C} in Tomi and Tromba’s context is space of H^5 immersions from S^1 to \mathbb{R}^3 and the space of "surfaces" is

$$\mathcal{S} := \mathcal{C} \times \mathcal{D}$$

where \mathcal{D} is the space of H^2 diffeomorphisms of S^1 that fixes 3 points $e^{ik\frac{2\pi}{3}}, k = 0, 1, 2$.³ Each element (f, φ) of \mathcal{S} defines an H^2 map $f \circ \varphi$ from S^1 to \mathbb{R}^3 and is identified with its harmonic extension F to the disk. The boundary map $\Pi : \mathcal{S} \rightarrow \mathcal{C}$ is just the first projection. The space \mathcal{M} of "minimal surfaces" is defined as the set of $(f, \varphi) \in \mathcal{S}$ such that F is conformal. Because the Hopf’s quadratic differential of a harmonic map is holomorphic, it suffices to check that F is conformal at all points on S^1 and that dF does not degenerate at any point of D . The defining equation of \mathcal{M} in \mathcal{S} is

$$q(f, \varphi) = 0 \quad \text{on } S^1, \quad dF \neq 0 \quad \text{on } D$$

Tomi and Tromba proved that

1. \mathcal{M} is a Banach submanifold of \mathcal{S} ,
2. the boundary map $\Pi|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{C}$ is Fredholm and of index 0.

Even though the space \mathcal{M} here contains immersed surfaces, the two facts above can be used to solve the embedded Plateau problem.

³ This "three-point property" is critical to the resolution of the Plateau problem. The famous *Courant–Lebesgue Lemma* says that a family of maps from D with bounded energy and the three-point property is equicontinuous.

Tomi and Tromba call a curve that lies on the boundary of a convex body *extreme*. Any minimal surface bounded by such a curve is embedded on the boundary. In other word, an extreme curve satisfies the following property

(P) every conformal harmonic immersion $F : D \rightarrow \mathbb{R}^3$ filling γ has no branch point on boundary and the boundary does not overlap with the interior, i.e. $F(D) \cap F(S^1) = \emptyset$.

It was proved in [TT78] that this property is open. On the other hand, it can be proved by maximum principle that a limit of a family of embedded minimal surface is automatically embedded in the interior if it is embedded on the boundary.

Lemma 2.28 (Gulliver–Spruck [GS76]). *Let (F_n) be a sequence of conformal harmonic embeddings in \mathcal{M} that C^1 -converge to a limit $F_\infty \in \mathcal{M}$. Suppose that the boundary of F_n C^2 -converges to the boundary $f_\infty \in \mathcal{C}$ of F_∞ . Then if f_∞ satisfies (P) then F_∞ is embedded.*

We can now prove Theorem 2.27 by the continuity method.

Proof of Theorem 2.27. Given a curve f_1 on the boundary of the convex body of \mathbb{R}^3 , we can isotope it to a plane curve f_0 via a path $\{f_t\}$ of embedded curve on ∂B . Now if the family $\{f_t\}$ is already transversal to the map $\Pi|_{\mathcal{M}}$, we can lift it to one-dimensional submanifold of \mathcal{M} . Let $(F_t)_{t \in [0,1]}$ be its connected component containing the plane $F_0 = \Pi^{-1}(f_0)$. By Lemma 2.28, the set $\{t \in [0,1] : F_t \text{ is embedding}\}$ is both closed and open in $[0,1]$ and so is $[0,1]$. This means that there is at least one embedded conformal harmonic map F_1 whose boundary is by f_1 .

If the family $\{f_t\}$ is not transversal to $\Pi|_{\mathcal{M}}$, we can perturb it by Sard–Smale without changing the fact that the f_t satisfy condition (P) (they may no longer be extreme). The previous argument gives us embedded minimal disc bounded by generic curves arbitrarily closed to f_1 . The conclusion follows from the compactness result for maps satisfying the three-point property, then from Lemma 2.28. \square

2.4.2 Brian White’s integral degree

The previously results of Tomi and Tromba was generalised by White [Whi87b], [Whi89], then Hoffman and White [HW08] to minimal immersions of a manifold with boundary Σ into a Riemannian manifold M . The properness of $\Pi : \mathcal{M} \rightarrow \mathcal{C}$ is not always guaranteed, but when this is the case, the modulo 2 degree can be lifted to an integer-valued degree. The sign associated to each minimal surface is given by the parity of its stability index.

The space \mathcal{S} in this case is the quotient of the space of $C^{j,\alpha}$ maps from $(\Sigma, \partial\Sigma)$ to $(M, \partial M)$, by the group of diffeomorphisms of Σ that is identity on $\partial\Sigma$. The space \mathcal{C} of curves is $C^{j,\alpha}(\partial\Sigma, \partial M)$ and $\mathcal{M} \subset \mathcal{S}$ is the space of minimal immersions.

The *Jacobi operator* (or *stability operator*) of a minimal immersion $f : (\Sigma, \partial\Sigma) \rightarrow (M, \partial M)$ is a self-adjoint, second-order differential operator on the normal bundle Nf

of f given by

$$\begin{aligned} J : C_0^{j,\alpha}(Nf) &\longrightarrow C^{j-2,\alpha}(Nf) \\ s &\longmapsto J(s) := \Delta^N s + \text{Tr}_\Sigma(R^N(s, \cdot)\cdot) + \tilde{A}(s) \end{aligned} \quad (2.27)$$

where Δ^N is the connection Laplacian on the normal bundle, R^N is the normal projection of the Riemann curvature of M and $\tilde{A} = A^*A$ is the Simons' operator. This operator is obtained by composing the second fundamental form with its adjoint. Here we see the second fundamental form as a map $A : Nf \longrightarrow T\Sigma^* \otimes T\Sigma$.

The *stability index* of a minimal immersion is the number of negative eigenvalues of J . The kernel of J is called the *nullity* of f and its elements are called *Jacobi fields*.

Theorem 2.29 (White [Whi87b]). *1. The space*

$$\mathcal{M} = \frac{\{f \in C^{j,\alpha}(\Sigma, \mathbb{R}^n) : f \text{ is minimal immersion}\}}{\{\varphi : \Sigma \longrightarrow \Sigma, C^{j,\alpha} \text{ diffeomorphism, } \varphi|_{\partial\Sigma} = \text{Id}\}}$$

is a smooth Banach manifold. The boundary map $\Pi : \mathcal{M} \longrightarrow C^{j,\alpha}(\partial\Sigma, \mathbb{R}^n)$ is a smooth Fredholm map of index 0.

2. The previous statement also holds if the numerator of \mathcal{M} is replaced by

$$\left\{ \begin{array}{l} f \in C^{j,\alpha}(\Sigma, M) : f \text{ is minimal immersion,} \\ f(\partial\Sigma) \subset \partial M, f(\text{int } \Sigma) \subset \text{int } M, f \text{ is nowhere tangential to } \partial M \end{array} \right\}$$

where M is a smooth manifold with boundary M , and $\Pi : \mathcal{M} \longrightarrow C^{j,\alpha}(\Sigma, \partial M)$.

3. Suppose that the restriction $\Pi|_{\mathcal{M}_0} : \mathcal{M}_0 \longrightarrow \mathcal{C}_0$ of the boundary map to an open set \mathcal{M}_0 of \mathcal{M} , with image in an open set \mathcal{C}_0 of \mathcal{C} is proper. Then the following degree is well-defined for a generic curve $\gamma \in \mathcal{C}_0$ and only depends on the connected component of \mathcal{C}_0 containing γ :

$$d(\Sigma, \gamma) := \sum_{[f] \in \Pi^{-1}(\gamma)} (-1)^{\text{index}[f]}. \quad (2.28)$$

Here $\text{index}[f]$ is the stability index of $[f]$. It only depends on the equivalent class of f .

It was proved by White [Whi87b] that the properness condition is satisfied when M is a strictly convex region of \mathbb{R}^3 , \mathcal{M}_0 is the space of embedded minimal surfaces and \mathcal{C}_0 is the space of embedded curve. It is based on a compactness result proved by White [Whi87a] in a larger context.

We also note that instead of minimal surfaces, White studied immersions that are stationary to a more general functional

$$A_\Theta(f) := \int_\Sigma \Theta(f(x), \text{Im } df(x)) \text{vol}_{f^*g}.$$

Here Θ is a function on the Grassmannian bundle of k -dimensional tangent planes in M and is supposed to satisfy an elliptic condition. When $\Theta = 1$, this is just the area

functional.

Properness fails, as pointed out in [Whi89, Theorem 3], in any 3-manifold with boundary M that admits a closed minimal surface, and in particular in any 3-manifold that is not homeomorphic to a ball. In such manifold, there exists a sequence of embedded discs with boundary in ∂M that converge to a smoothly embedded curve, but the area of these disc remains unbounded.

On the other hand, when properness does hold (for example in a strictly convex region of \mathbb{R}^3), the corresponding degree can be computed explicitly. It is $+1$ when Σ is a union of discs. We can see this by isotoping the boundary components to flat curves, which can only be filled by planes. The embedded Plateau problem follows directly from this. When Σ is not a union of discs, White proved that the degree vanishes. This result can be generalised for any weight function $\Theta(x, T)$ on $\mathbb{R}^3 \times Gr(2, 3)$ that is independent of the first variable.

In general, we can think of the degree as the Euler characteristic of the fibre $\Pi^{-1}(\gamma)$ in \mathcal{S} . The volume functional defines a function on the fibre $\Pi^{-1}(\gamma)$ whose critical points are elements of $\Pi^{-1}(\gamma) \cap \mathcal{M}$. Their Morse index is, by definition, their stability index and Morse equality says that the RHS of (2.28) is the Euler characteristic of the fibre. This explains why the degree is independent of Θ .

The choice of sign in (2.28) is closely related to the proof that \mathcal{M} is a Banach manifold. One way to prove that the space of minimal immersions is a Banach manifold is to see it as a subset of the space of $C^{j,\alpha}$ maps defined by the equation $H = 0$. The mean curvature H is a section of the vector bundle $\mathcal{B} \rightarrow \mathcal{S}$ whose fibre over each map $f \in \mathcal{S}$ is the Banach space of $C^{j-2,\alpha}$ sections of the normal bundle of f . We need to check that H is transversal to the 0-section.

This is not how Theorem 2.29 was proved in [Whi87b]. We will instead use \mathcal{C} as a model to construct an atlas of \mathcal{M} . The benefit of this is that we can relate the degree of Π and the stability index of minimal immersion.

Let us recall a few facts about Fredholm maps between Banach manifolds. Suppose that \mathcal{M}, \mathcal{C} are Banach manifolds and $\Pi : \mathcal{M} \rightarrow \mathcal{C}$ is a Fredholm map. If the differential $d\Pi(f_0)$ at a point $f_0 \in \mathcal{M}$ is a bijection, then there exists an inversion $F : \mathcal{C} \rightarrow \mathcal{M}$ of Π that maps diffeomorphically a neighbourhood of f_0 in \mathcal{M} onto a neighbourhood of $\Pi(f_0)$ in \mathcal{C} . If $d\Pi(f_0)$ has a k -dimensional kernel K and a k' -dimensional cokernel K' , then we can still invert Π up to finite dimension. This means that there exists a diffeomorphism $F : \mathcal{C} \oplus K \rightarrow \mathcal{M} \oplus K'$ and so the space \mathcal{M} can be embedded into $\mathcal{C} \oplus K$ as a codimension k' submanifold given by the equation $g = 0$. Here $g : \mathcal{C} \oplus K \rightarrow K'$ is the composition of F and the second projection. There are of course several ways to choose F (and g correspondingly).

Now let \mathcal{M} be the space of conformal harmonic maps as before (to be proved a Banach manifold) and $\mathcal{C} = C^{j,\alpha}(\partial\Sigma, \mathbb{R}^n)$. We also denote by $\Pi : C^{j,\alpha}(\Sigma, \mathbb{R}^n) \rightarrow C^{j,\alpha}(\partial\Sigma, \mathbb{R}^n)$ the restriction map that induces the boundary map $\mathcal{M} \rightarrow \mathcal{C}$. Given a minimal immersion $f_0 \in C^{j,\alpha}(\Sigma, \mathbb{R}^n)$ with boundary $\gamma_0 : \partial\Sigma \rightarrow \mathbb{R}^n$, we will construct the "inverse" F of the previous paragraph around γ_0 with the following properties:

- (F1) $F : \mathcal{C} \rightarrow C^{j,\alpha}(\Sigma, \mathbb{R}^n)$ if the nullity of f_0 is 0. The map F is C^∞ , well-defined on a neighbourhood of γ_0 in \mathcal{C} and satisfies $\Pi \circ F = Id_{\mathcal{C}}$ and $F(\gamma_0) = f_0$. The image of F are minimal immersions, and contains all minimal immersions near $[f_0]$ in \mathcal{S} .

- (F2) $F : \mathcal{C} \oplus K \longrightarrow C^{j,\alpha}(\Sigma, \mathbb{R}^n)$ if the nullity of f_0 is a k -dimensional vector space K . The map F is C^∞ , satisfies $F(\gamma_0, 0) = f_0$ and $\Pi \circ F : \mathcal{C} \oplus K \longrightarrow \mathcal{C}$ is the first projection. The image of F in \mathcal{S} contains all conformal harmonic immersions near $[f_0]$ and there exists a smooth function $g : \mathcal{C} \oplus K \longrightarrow K$ such that $F(\gamma, \kappa)$ is a minimal immersion if and only if $g(\gamma, \kappa) = 0$. Moreover, $D_2F(\gamma_0, 0)$ is the inclusion of K into $C^{j,\alpha}(\Sigma, \mathbb{R}^n)$.

This automatically gives \mathcal{M} a Banach chart around each point $f_0 \in \mathcal{M}$. Then, we need to prove that the transition maps between overlapping charts are smooth.

The Fredholm property of Π and its vanishing index follow from the identification of the kernel and cokernel of Π as the vector space K . This identification implies that determinant bundle of F is trivial, and thus the preimage $\Pi^{-1}(\gamma)$ of a generic $\gamma \in \mathcal{C}$ is orientable.

There is an extra consequence of the existence of such F . By definition, the kernel of $d\Pi$ contains vector fields that can be integrated to a one-parameter family of minimal immersions. The nullity, on the other hand, contains all Jacobi fields (defined as solutions of a differential equation). Because the two are the same space K , we have

Corollary 2.30 (White [Whi87b]). *Any Jacobi field along a minimal immersion f_0 can be realised as the velocity of a one-parameter family⁴ $\{f_t\}$ of conformal harmonic immersions.*

The map $F : \mathcal{C} \oplus K \longrightarrow \mathcal{S}$ is constructed via the Implicit Function Theorem, and is uniquely characterised by the condition

$$p(H(F(\gamma, \kappa))) \subset K \quad \forall (\gamma, \kappa) \text{ near } (\gamma_0, 0)$$

where $H(F(\gamma, \kappa))$ is the mean curvature of $F(\gamma, \kappa)$ and p is the projection of \mathbb{R}^n onto the normal vector space of f_0 . The map g will be then given by

$$\begin{aligned} g : \mathcal{C} \oplus K &\longrightarrow K \\ (\gamma, \kappa) &\longmapsto pH(F(\gamma, \kappa)) \end{aligned} \tag{2.29}$$

When (γ, κ) is near $(\gamma_0, 0)$, the condition $g = 0$ is equivalent to $F(\gamma, \kappa)$ being minimal.

Proof of part 1 of Theorem 2.29. As discussed above, we will construct the coordinate map $F : \mathcal{C} \times K \longrightarrow \mathcal{S}$ near a minimal immersion $f_0 : \Sigma \longrightarrow \mathbb{R}^n$ where K is the nullity of f_0 . We will regard the normal projection p as a map from Σ to $Gr(n, k)$ and a section s of the normal bundle Nf_0 of f_0 as $s : \Sigma \longrightarrow \mathbb{R}^n$ with image $s(a) \in \text{Im } p_a$.

We consider the easier case first, when the nullity K is 0. Given any map $\gamma : \partial\Sigma \longrightarrow \mathbb{R}^n$ near the boundary γ_0 of f_0 , we denote by $\phi(\gamma)$ the harmonic extension of γ to Σ . The metric used on Σ is the pullback of the Euclidean metric via f_0 , so $\phi(\gamma_0) = f_0$, but $\phi(\gamma)$ is not conformal in general. We define

$$\begin{aligned} \Psi : C^{j,\alpha}(\partial\Sigma, \mathbb{R}^n) \times C_0^{j,\alpha}(Nf_0) &\longrightarrow C^{j-2,\alpha}(Nf_0) \\ (\gamma, s) &\longmapsto pH(\phi(\gamma) + s) \end{aligned} \tag{2.30}$$

4. The family f_t is allowed to move $\partial\Sigma$ in second order or higher, so Corollary 2.30 does not contradict the notion of "non-integrable" Jacobi field.

Clearly $\Psi(\gamma_0, 0) = 0$ and $D_2\Psi(\gamma_0, 0)$ is the Jacobi operator, which is invertible in this case. The Inverse Function Theorem says that we can find a neighbourhood $U \subset C^{j,\alpha}(\partial\Sigma, \mathbb{R}^n)$ of γ_0 , a neighbourhood $V \subset C^{j,\alpha}(Nf_0)$ of 0 and a map $F_0 : U \rightarrow V$ such that solutions of $\Psi = 0$ in $U \times V$ are of form $(\gamma, F_0(\gamma))$. Then $F(\gamma) = F_0 + \phi(\gamma)$ satisfies (F1). Moreover, since any immersion f that is $C^{j,\alpha}$ -close to f_0 can be reparameterised as $\phi(\gamma_0) + s$, the image of F in \mathcal{S} contains the equivalent classes of all minimal immersions near $[f_0]$.

Now let us consider the case when the nullity of f_0 is non-trivial. Let K^\perp and K_0^\perp be the L^2 -orthogonal complement of K in $C^{j-2,\alpha}(Nf_0)$ and $C_0^{j,\alpha}(Nf_0)$ respectively. Because $C^{0,\alpha} \hookrightarrow L^2$, these are closed subspaces and the projection π_{K^\perp} onto K^\perp is continuous. We define

$$\begin{aligned} \Psi : C^{j,\alpha}(\partial\Sigma, \mathbb{R}^n) \times K \times K_0^\perp &\longrightarrow K^\perp \\ (\gamma, \kappa, s) &\longmapsto \pi_{K^\perp} pH(\phi(\gamma) + \kappa + s) \end{aligned} \quad (2.31)$$

As before, we remark that $\Psi(\gamma_0, 0, 0) = 0$ and that by elliptic theory, $D_3\Psi(\gamma_0, 0, 0)$ is invertible. Therefore we can find an open neighbourhood U of $(\gamma_0, 0)$ in $C^{j,\alpha}(\partial\Sigma, \mathbb{R}^n) \times K$, and V of 0 in K_0 and a map $F_0 : U \rightarrow V$ such that solutions of $\Psi = 0$ in $U \times V$ are of form $(\gamma, \kappa, F_0(\gamma, \kappa))$. The map $F(\gamma, \kappa) = \phi(\gamma) + \kappa + F_0(\gamma, \kappa)$ satisfies condition (F2).

Now that we have a smooth Banach chart around each minimal immersion f_0 of \mathcal{M} , the remaining of the proof is to show that the transition maps between overlapping charts are smooth. Let $F_i : \mathcal{C} \oplus K_i \rightarrow C^{j,\alpha}(\Sigma, \mathbb{R}^n)$ be two overlapping charts centred around minimal immersions f_1 and f_2 . This means that each $F_i^{-1}(\mathcal{M}) = g_i^{-1}(0)$ is a smooth k_i -codimensional submanifold of $\mathcal{C} \oplus K_i$ and that $F_1(\gamma_1, \kappa_1) = F_2(\gamma_2, \kappa_2) = f$ is a minimal immersion. The transition map $\tau_{12} : g_1^{-1}(0) \rightarrow g_2^{-1}(0)$ is well-defined on a neighbourhood of (γ_1, κ_1) .

The trick here is construct smooth maps $\Omega_i : C^{j,\alpha}(\Sigma, \mathbb{R}^n) \rightarrow \mathcal{C} \oplus \mathbb{R}^{k_i}$ such that $\Omega_i \circ F_i$ is a diffeomorphism. If we can do this, then

$$\tau_{12} = (F_2 \circ \Omega_2)^{-1} \circ \Omega_2 \circ F_1$$

is smooth. The maps are illustrated in the following diagram.

$$\begin{array}{ccc} \mathcal{C} \oplus K_2 & \xrightarrow{\cong} & \mathcal{C} \oplus \mathbb{R}^{k_2} \\ & \searrow F_2 & \uparrow \Omega_2 \\ \mathcal{C} \oplus K_1 & \xrightarrow{F_1} & C^{j,\alpha}(\Sigma, \mathbb{R}^n) \end{array}$$

Let us construct Ω_1 . If $k_1 = \dim K_1 = 0$, then we can just use $\Omega_1 = \Pi$. If not, we will choose

$$\Omega_1(f) := \left(\Pi(f), \int_\Sigma f^* \omega_1, \dots, \int_\Sigma f^* \omega_{k_1} \right), \quad f \in C^{j,\alpha}(\Sigma, \mathbb{R}^n).$$

where $\omega_1, \dots, \omega_{k_1}$ are differential forms of \mathbb{R}^n with degree equal to the dimension of Σ . It suffices to choose the form ω_i so that $d(\Omega_1 \circ F_1)$ is invertible at $(\gamma_1, 0)$ and

then shrink the chart F_1 so that $\Omega_1 \circ F_1$ is invertible. Recall that $D_2F_1(\gamma_1, 0) \cdot \kappa = \kappa$ for any Jacobi field κ of f_1 and that for any perturbation $\{u_t\}$ of a map u_0 along a vector field X , we have $\left. \frac{d}{dt} \right|_{t=0} \int_{\Sigma} u_t^* \omega = \int_{\Sigma} u_0^* \mathcal{L}_X \omega$. We extend any vector field $\kappa \in K$ along the image of F_1 by $\frac{d}{dt} F_1(\gamma_0, t\kappa)$ and use the same notation for the extension. Then $d(\Omega_1 \circ F_1)$ is invertible if and only if for a basis $\kappa_1, \dots, \kappa_{k_1}$ of K , the $k_1 \times k_1$ matrix $\left(\int_{\Sigma} f_1^* \mathcal{L}_{\kappa_j} \omega_i \right)_{ij}$ is invertible.

Since for any vector field $\kappa \in K$, there exists a form ω such that

$$\int_{\Sigma} f_1^* \mathcal{L}_{\kappa} \omega \neq 0.$$

It suffices to apply the following linear algebra result:

Lemma 2.31. *Let V and W be vector spaces over a field R and $A : V \times W \rightarrow R$ be a bilinear map such that:*

For any $v \in V \setminus \{0\}$, there exists $w \in W$ such that $A(v, w) \neq 0$.

Then for any set of linearly independent vectors v_1, \dots, v_n in V , there exist w_1, \dots, w_n in W such that the $n \times n$ matrix $(A(v_i, w_j))_{ij}$ is invertible.

□

The following formula relates the stability index of a minimal immersion in a chart around f_0 and the index of f_0 .

Lemma 2.32 (cf. Proposition 4 of [Whi87b]). *Let $F(\gamma, \kappa)$ be a minimal immersion near f_0 and \tilde{K} be the k -dimensional space of normal vector fields along $F(\gamma, \kappa)$ given by the normal part of elements in the image of $D_2F(\gamma, \kappa)$. The Jacobi operator of $F(\gamma, \kappa)$ defines a symmetric bilinear form on \tilde{K} . Let d be its index (maximal dimension of a negative definite subspace). Then the stability index of $F(\gamma, \kappa)$ and that of f_0 are related by*

$$\text{index } F(\gamma, \kappa) = \text{index } f_0 + d \tag{2.32}$$

In particular, when the nullity of f_0 is 1, then d is given by the sign of $D_2g(\gamma, \kappa)$:

$$d = \begin{cases} 1, & \text{if } D_2g \text{ preserves orientation of } K \\ 0, & \text{if } D_2g = 0 \\ -1, & \text{if } D_2g < 0 \text{ reverses orientation of } K \end{cases}$$

Corollary 2.33. *Let $f_0 : \Sigma \rightarrow M$ be an immersed minimal surface of nullity k , then any immersed minimal surface near f_0 has stability index in the range $[\text{index } f_0, \text{index } f_0 + k]$.*

We will prove using the relation (2.32) that the RHS of (2.28) is constant on connected component of \mathcal{C} , and so it can be used to defined an integer-valued degree of Π . Let $L := \{\gamma_t\}$ be a path in \mathcal{C} connecting two generic curves γ_0, γ_1 and that is transversal to Π . We can suppose that $d\Pi$ has at most 1 dimensional kernel for every surface in $L' := \Pi^{-1}(L)$. It suffices to give L' an orientation that is compatible with (2.28). Relation (2.32) proves that the following choice works:

- For each $f_t \in \Pi^{-1}(\gamma_t)$ where $d\Pi$ has no kernel, its neighbourhood U in L' is mapped diffeomorphically to a neighbourhood of γ_t in L . We orient U so that Π is orientation-preserving if and only if the index of f_t is even.
- If the kernel of $d\Pi$ at f_t has dimension one, then a neighbourhood U of f_t in L' can be embedded as a codimension 1 of $L \times K$, given by an equation $g(t, \kappa) = 0$ with $D_1g \neq 0$. We pick any orientation of $K \cong \mathbb{R}$ and use the counter-clockwise orientation of $L \times K \cong [0, 1] \times \mathbb{R}$. We orient $U = g^{-1}(0)$ as the boundary of $g > 0$ if and only if the index of f_t is even.

Proof of Lemma 2.32. By differentiating g in (2.29), we have

$$D_2g(\gamma, \kappa) = p.DH(F(\gamma, \kappa)).D_2F(\gamma, \kappa)$$

We will denote by B_0 and B_1 the normal bundles of f_0 and $F(\gamma, \kappa)$. By choosing (γ, κ) near $(\gamma_0, 0)$, we can suppose that the image of $D_2F(\gamma, \kappa)$ is k -dimensional and that its projection to B_0 , the space \tilde{K} in the statement, is also k -dimensional.

It suffices to apply the following Lemma 2.34 with J_0 being the Jacobi operator of f_0 , $J_1 = pJ_{(\gamma, \kappa)}p^*$ where $J_{(\gamma, \kappa)}$ is the Jacobi operator of $F(\gamma, \kappa)$ and $K_1 = p^{*-1}(\tilde{K})$. Here $p^* : B_0 \rightarrow B_1$ is the fibre-wise adjoint of projection p from B_1 to B_0 . It is exactly the projection to B_1 . Because

$$\langle J_1u, u \rangle = \langle J_{(\gamma, \kappa)}p^*u, p^*u \rangle \quad \forall u \in C_0^{k, \alpha}(NF(\gamma, \kappa)),$$

the operators J_1 and $J_{(\gamma, \kappa)}$ have the same index. □

Lemma 2.34 (Lemma 4 of [Whi87b]). *Let $J_0, J_1 : C_0^{h, \alpha}(B_0) \rightarrow C^{h-2, \alpha}(B_0)$ be two Schrödinger operators on a vector bundle B_0 over Σ such that:*

- *they are ϵ -closed to each other, i.e.*

$$\|J_0 - J_1\|_{L^2} \leq \epsilon \tag{2.33}$$

- *there exists a k -dimensional vector space $K_1 \subset C_0^\infty(B_0)$ such that $J_1(K_1) \subset K$ where $K = \ker J_0$ also has dimension k .*

Then for ϵ sufficiently small, one has $\text{index } J_1 = \text{index } J_0 + d$ where d is the index of J_1 as a symmetric bilinear form on K .

Proof. The spectra of J_i are discrete subset of \mathbb{R} whose intersection with any interval $(-\infty, c)$ is a finite set. Now we decompose

$$C^{k, \alpha}(B_0) = N \oplus^\perp K \oplus^\perp P$$

where N is spanned by negative eigenfunctions of J_0 and P is the closure of the subspace spanned by positive eigenfunctions. We claim that

$$C^{k, \alpha}(B_0) = N \oplus K_1 \oplus P. \tag{2.34}$$

For (co)dimensional reason, it suffices to check that there is no element a in $K_1 \cap N \oplus P$. Such a would have $J_1a \in K$ and $J_0a \perp K$, so $J_1a \perp J_0a$. But these two

elements had to be closed to each other due to (2.33). A contradiction would happen if ϵ is small enough so that there is no non-zero eigenvalue of J_0 in $(-2\epsilon, 2\epsilon)$.

Now we decompose

$$K_1 = N' \oplus K' \oplus P'$$

using the symmetric bilinear form associated to J_1 : $\langle J_1 u, u \rangle$ is negative (respectively positive) if u lies in N' (respectively P') and $J_1 u \perp K_1$ if $u \in K'$. This gives a finer decomposition of $C_0^{k,\alpha}(B_0)$ as

$$C_0^{k,\alpha}(B_0) = N \oplus N' \oplus K' \oplus P' \oplus P$$

We will prove that J_1 is negative definite on $N \oplus N'$, positive definite on $P \oplus P'$ and zero on K' . Lemma 2.34 clearly follows from this:

$$\text{index } J_1 = \dim N + \dim N' =: \text{index } J_0 + d.$$

Given any $a \in K'$, we have $J_1 a \perp N \oplus P$ because $J_1 a \in K$. Also $J_1 a \perp K_1$ by the definition of K' . We conclude, by (2.34), that $J_1 a = 0$. This means that J_1 vanishes on K' .

Given $n \in N$ and $n' \in N'$, we have

$$\langle J_1(n + n'), n + n' \rangle = \langle J_1 n, n \rangle + \langle J_1 n', n' \rangle + 2\langle J_1 n', n \rangle,$$

where the first two term of the RHS are strictly negative. The third term vanishes because $J_1 n' \in K_1 \perp N$. Therefore J_1 is negative definite on $N \oplus N'$. Similar argument proves that J_1 is positive definite on $P \oplus P'$. \square

2.5 Degree theory for minimal surfaces in the hyperbolic space

In this subsection, we consider the counting problem of minimal surfaces in the hyperbolic space. The surfaces considered here are complete and smoothly embedded near their ideal boundary. We also suppose that the surfaces are sufficiently smooth on a neighbourhood of their ideal boundary.

The Fredholm part of White's Theorem 2.29 becomes more delicate because the minimal surface equation is no longer elliptic. Its symbol is positive in the interior, but tends to 0 on the boundary. We can see this by writing down the Jacobi operator (see (2.27))

$$J(s) := \Delta^N s - 2s + \tilde{A}(s), \quad s \in \Omega^0(N\Sigma)$$

Recall that here $\tilde{A} = A^* A$ is the Simons' operator on the normal bundle and the second fundamental form is written as a map $A : N\Sigma \rightarrow \text{End}(T\Sigma)$.

It follows from Lemma 2.10 that the second fundamental form tends uniformly to 0 if the surface is C^2 up to the boundary and meets the sphere at infinity orthogonally. Concretely, in the half space model $\{(x, y_1, \dots, y_{n-1}) \in \mathbb{R}_{>0} \times \mathbb{R}^{n-1}\}$, the norms of the Euclidean second fundamental form $\tilde{\text{II}}$ and the hyperbolic second fundamental form II are related by

$$|\text{II}|^2 = x^2 |\tilde{\text{II}}|^2 + 2|\partial_x^\perp|^2.$$

Here $|\partial_x^\perp|$ is the Euclidean norm of the orthogonal projection of ∂_x to the normal bundle.

Therefore, near infinity, the Jacobi operator looks like

$$(\Delta - 2)s_i = \left[x^2(\partial_x^2 + \partial_y^2) - 2 \right] s_i, \quad i = 1, 2, \dots, n-2 \quad (2.35)$$

where $(x, y) \in \mathbb{R}_{>0} \times \mathbb{R}$ are the half-space coordinates of \mathbb{H}^2 . Here (s_1, \dots, s_{n-2}) are the coordinates of s in a hyperbolic frame of \mathbb{H}^n . The operator $\Delta - 2$ is elliptic in the interior, but its symbol fails to be invertible on the boundary. The Fredholm property of such operators was studied by Mazzeo and Melrose [MM87], [Maz91].

Let (M, g) be a conformally compact⁵ $(n+1)$ -manifold, x be a boundary defining function and \bar{g} be the normalised metric on \bar{M} given by $\bar{g} = x^2g$. An operator of the form

$$P = \sum_{i+|j|\leq m} a_{ij}(x, y)(x\partial_x)^i(y\partial_y)^j \quad (2.36)$$

where $i \in \mathbb{N}, j \in \mathbb{N}^n, (x, y) \in \mathbb{R}_{>0} \times \mathbb{R}^n$ are local coordinates in a neighbourhood of infinity, is called a *0-differential operator* of order m . It is *0-elliptic* if the symbol

$$\sum_{i+|j|=m} a_{ij}(x, y)t^i s^j$$

is invertible for any non-zero s, t at any interior point (x, y) .

The Fredholm property of such operators is studied as maps between weighted Hölder and Sobolev spaces. The weighted Hölder space $x^\mu \Lambda^{k, \alpha}(M)$ is defined as the vector space of all functions $f = x^\mu u$ where u is a $C^{k, \alpha}$ function on M . This means that k^{th} derivatives of u with respect to vector fields $x\partial_x$ and $x\partial_{y_i}$ are α -Hölder under the metric g . The norm of f is defined as the Hölder norm of u and this turns $x^\mu \Lambda^{k, \alpha}(M)$ into a Banach space. The letter Λ is used here to remind us that the Hölder norm was taken in (M, g) and not (\bar{M}, \bar{g}) . In the same manner, we define the Sobolev spaces $x^\mu H^k(M)$ to be the Banach space of functions $f = x^\mu u$ with $u \in H^k(M)$.⁶

The operator P can be seen as a bounded map between Hölder and Sobolev spaces:

$$P : x^\mu \Lambda^{k, \alpha}(M) \longrightarrow x^\mu \Lambda^{k-m, \alpha}, \quad P : x^\mu H^k(M) \longrightarrow x^\mu H^{k-m}$$

To prove that P is Fredholm, we have to check that its normal operator is invertible. Suppose that P is given locally by (2.36). Its *normal operator* N_q at point q on the boundary of M is obtained by freezing the coordinates (x, y) of q and replacing the derivatives $x\partial_x, x\partial_{y_i}$ by the derivatives $t\partial_t$ and $t\partial_{s_i}$ of \mathbb{H}^{n+1} . For example, if q is $(0, 0)$ then

$$N_q = \sum_{i+|j|\leq m} a_{ij}(0, 0)(t\partial_t)^i(t\partial_s)^j, \quad (t, s) \in \mathbb{R}_{>0} \times \mathbb{R}^n.$$

When P is the operator between vector bundles E and F , N_q is an operator on the trivial bundle $\mathbb{H}^{n+1} \times E_q \longrightarrow \mathbb{H}^{n+1} \times F_q$.

5. See Appendix D for definition.

6. Here we use the g -volume form $\frac{dx}{x} \frac{dy_1}{x} \dots \frac{dy_n}{x}$ to define the L^2 norm. Several authors use the \bar{g} -volume form $dx dy_1 \dots dy_n$. The two space are equivalent, up to a shift in the weight:

$$t^\mu H^k(\text{vol}_g) = t^{\mu - \frac{n+1}{2}} H^k(\text{vol}_{\bar{g}}).$$

The *indicial roots* of P are solutions of the equation

$$\det\left(\sum_i a_{i0}(0,0)X^i\right) = 0.$$

This equation is obtained by replacing $t\partial_s$ by 0 and $t\partial_t$ by a variable X in the formula of N_q .

What we did in (2.35) was to compute the normal operator of the Jacobi operator. Since

$$x^2(\partial_x^2 + \partial_y^2) - 2 = (x\partial_x)^2 - x\partial_x + (x\partial_y)^2 - 2,$$

the indicial roots solve the equation $X^2 - X - 2 = 0$. So they are -1 and 2 .

Theorem 2.35 (Mazzeo [Maz91]). *Let P be a second-order 0-elliptic operator and N_q be its normal operator at q . Suppose that the indicial roots of P are real and constant on ∂M . If there is a certain ν_0 such that the maps $N_q : x^{\nu_0 - \frac{n}{2}} H^k(\mathbb{H}^{n+1}) \rightarrow x^{\nu_0 - \frac{n}{2}} H^{k-2}(\mathbb{H}^{n+1})$ are isomorphism for all $q \in \partial M$, then:*

1. *The operator $P : x^{\nu_0 - \frac{n}{2}} H^k(M) \rightarrow x^{\nu_0 - \frac{n}{2}} H^{k-2}(M)$ is Fredholm.*
2. *Let (μ_1, μ_2) be an interval that contains no indicial root. Suppose that $\nu_0 \in (\mu_1, \mu_2)$, then for all $\nu \in (\mu_1, \mu_2)$, $N_q : x^{\nu - \frac{n}{2}} H^k \rightarrow x^{\nu - \frac{n}{2}} H^{k-2}$ is invertible and the operators*

$$\begin{aligned} P : x^{\nu - \frac{n}{2}} H^k(M) &\rightarrow x^{\nu - \frac{n}{2}} H^{k-m}(M) \\ P : x^\nu \Lambda^{k,\alpha}(M) &\rightarrow x^\nu \Lambda^{k-m,\alpha}(M) \end{aligned}$$

are Fredholm. Their kernels are the same vector subspace and are contained in $x^{\mu_2} \Lambda^{j,\alpha}(M)$ for any j .

Because the Laplacian Δ^N is negative, the operator $\Delta^N - 2$ is invertible on $H^2 \rightarrow L^2$. It follows from Theorem 2.35 with $\nu_0 = \frac{1}{2} \in (-1, 2)$ that the Jacobi operator

$$J : x^\mu \Lambda^{j,\alpha}(Nf) \rightarrow x^\mu \Lambda^{j-2,\alpha}(Nf) \tag{2.37}$$

is Fredholm for every $\mu \in (-1, 2)$.

There is a small subtlety here. The space \mathcal{S} (or \mathcal{M}) contains immersions that are $C^{j,\alpha}$ up to the boundary while the Fredholm property (2.37) is between the weighted Hölder spaces. These function spaces are related in the following way. On a conformally compact manifold (M, g) , we denote by $C^{j,\alpha}(\overline{M})$ the Hölder space on $(\overline{M}, \overline{g})$ and $C_r^{j,\alpha}(\overline{M})$, $r \leq j$ the closed subspace of functions that vanish up to order r , that is

$$(\partial_x)^j f = 0 \quad \text{on } \partial M \quad \forall j = 0, 1, \dots, r$$

We equip $C_r^{j,\alpha}(\overline{M})$ with the Banach structure of $C^{j,\alpha}(\overline{M})$. It turns out that the Banach spaces $x^{j+\alpha} \Lambda^{j,\alpha}(M)$ and $C_j^{j,\alpha}(\overline{M})$ are equivalent (see [Lee01]).

Now if we are to set the problem in the space of $C^{2,\alpha}$ minimal immersions, the perturbations field s (as in (2.30) and (2.31)) should be at least $x^{2+\alpha} \Lambda^{2,\alpha}$ when written under a \overline{g} -frame, or in other words $x^{1+\alpha} \Lambda^{2,\alpha}$ under a g -frame. So the Jacobi operator goes between Hölder spaces of weight $1 + \alpha$, which indeed lies in the range $(-1, 2)$.

The Fredholm part of Theorem 2.29 was established in \mathbb{H}^4 in a recent work of Joel Fine [Fin21]. In his work, the space of maps is modded out by all diffeomorphisms and

not just those preserving the boundary. Also, by the Eells–Salamon’s correspondence [ES85], he studies holomorphic curves in the twistor space instead. This means that the minimal surfaces considered there are allowed to have branched points.

Theorem 2.36 (Fine [Fin21]). *Let Σ be a compact, connected, oriented surface with c boundary components and \mathcal{C} be the space of $C^{2,\alpha}$ oriented links of c components in S^3 .*

The space \mathcal{M} is defined as $\{(f, j)\} / \sim$ where j is a complex structure on Σ and $f : (\bar{\Sigma}, j) \rightarrow \bar{\mathbb{H}}^4$ such that:

- *The map f is $C^{2,\alpha}$ and conformal on $\bar{\Sigma}$. In the interior, f is harmonic,*
- *$f(\Sigma) \subset \mathbb{H}^4$, f meets S_∞ transversally,*
- *f embeds the boundary $\partial\Sigma$ to a $C^{2,\alpha}$ link in S_∞ ,*
- *The relation \sim identifies pairs (f, j) that are related by a diffeomorphism of $\bar{\Sigma}$.*

The boundary map $\Pi : \mathcal{M} \rightarrow \mathcal{C}$ associates to each class $[f, j]$ the link $[f|_{\partial\Sigma}]$.

Then \mathcal{M} is a smooth Banach manifold, Π is Fredholm and has index 0. Moreover, the determinant bundle of $d\Pi$ has a distinguished trivialisation over the complement of a codimension 2 in \mathcal{M} .

The proof of Theorem 2.36 is more complicated than the sketch we did above. This is due to the quotient by the full diffeomorphism group as well as the inclusion of branched minimal surfaces in the setup. The prize to gain from this extra work lies in the "properness part". Recall that in Theorem 2.29, properness of Π was a hypothesis. In the hyperbolic setting, one may hope that it is a feature. The precise statement and a sketch of proof for discs will be given in Section 2.5.1.

The determinant bundle of Π is given by top power of its kernel and cokernel bundles:

$$\det \Pi = \det(\ker d\Pi) \otimes (\det \operatorname{coker} d\Pi)^*.$$

So a trivialisation of $\det \Pi$ near a generic curve γ is a choice of volume form, therefore an orientation, of the fibre $\Pi^{-1}(\gamma)$. These orientations are consistent in the sense that for any Sard–Smale curve $\{\gamma_t\}_{t \in [0,1]}$ transversal to Π , the union $\Pi^{-1}(\gamma_0) \cup \Pi^{-1}(\gamma_1)$ with the opposite orientation on $\Pi^{-1}(\gamma_1)$ has the boundary orientation of $\Pi^{-1}(\{\gamma_t\})$. In the context of Theorem 2.36, this means (with properness supposed) we can have an integer-valued degree of Π . This integer is an invariant of each connected component of \mathcal{C} , and so is a knot/link invariant.

We can have more invariants by just counting minimal surfaces in a component of \mathcal{S} , for example those with a given Euler characteristic. There is another topological invariant that describes how an immersed oriented surface sits in a four-dimensional space. For closed surfaces, this is the self-intersection number, or equivalently, the Chern number of its normal bundle. When the surface is open, any complex line bundle over it is trivialisable but we still have an analogue of this, given by the following instruction:

1. First trivialisise the normal bundle of the surface with a non-zero normal vector field \vec{n} . Because the surface meets the sphere S^3 orthogonally at a curve γ , \vec{n} is tangent to S^3 and is a normal vector field of γ . Such choice of \vec{n} is often called a *framing* of γ
2. In S^3 , compute the number τ of turns \vec{n} makes along γ . It is the linking number between γ and its push-off $\gamma + \epsilon\vec{n}$ for a small, positive ϵ .

The number τ is in fact independent of \vec{n} . To see this, take any other normal vector field \vec{n}' of the surface. This new vector field has to be the rotation of \vec{n} by an angle θ in the normal bundle. Here θ is a function on the surface with value in S^1 . The difference between τ and the new linking numbers is just the degree of the map $\theta|_{\gamma} : \gamma \rightarrow S^1$. Since θ extends to the surface, this degree is zero.

In \mathbb{H}^3 , the properness of the boundary map Π was proved by Alexakis and Mazzeo [AM10]. In their setup, \mathcal{S} is the space of embedded $C^{k,\alpha}$ surfaces ($k \geq 2$) and \mathcal{C} is the space of embedded disjoint circles in S^2 . This is enough to define a $\mathbb{Z}/2\mathbb{Z}$ -valued degree that counts the number of minimal surfaces of Euler characteristic χ that fill a collection of disjoint circles.

A way to lift these degrees to \mathbb{Z} was also proposed in [AM10] using the stability index as White did in Theorem 2.29. It is unclear how this sign convention is well defined and consistent. The reason is as follows. Although we can prove that certain⁷ eigenspaces of the Jacobi operator J are finite-dimensional, Lemma 2.34 needs the spectrum of J to be discrete. On a compact manifold, this is due to the fact that any minimising sequence of the Rayleigh quotient has a converging subsequence. This may fail in the hyperbolic space because the inclusion $H^1 \hookrightarrow L^2$ is no longer compact.

On the other hand, the minimal surfaces that are C^2 up to their ideal boundary have finite index in a more relaxed sense. First, the classical argument shows that any open region Ω compactly contained in the interior of the surface has a finite stability index d_Ω . If a minimal surface of \mathbb{H}^3 has finite total curvature

$$\int_{\Sigma} |\Pi|^2 < +\infty,$$

it was proved by Bérard, do Carmo and Santos [BCS97] that $\sup_{\Omega} d_\Omega < +\infty$. The finitude of the index in this sense was first studied by Fischer-Colbrie [Fis85], who proved that a minimal surface of \mathbb{R}^3 has finite index if and only if its total curvature is finite. The total curvature of a minimal surface in \mathbb{H}^n is its Willmore energy and thus is finite if the surface is C^2 up to its ideal boundary.

2.5.1 Properness of the boundary map

Fine proves that the only way the properness of Π can fail is when the complex structures of the surface develops node.

Theorem 2.37 (Fine [Fin21]). *Let $f_n : (\Sigma, j_n) \rightarrow \mathbb{H}^4$ be a sequence of conformal harmonic maps in the space \mathcal{M} of Theorem 2.36. Suppose that the complex structures j_n converge to smooth limit j_∞ and that the boundary links $f_n(\partial\Sigma)$ converge to a $C^{2,\alpha}$ link K . Then a subsequence of f_n converges to a map $(f_\infty, j_\infty) \in \mathcal{M}$ with boundary K .*

We will sketch out a few ideas by proving Theorem 2.37 for the disc. By acting with diffeomorphisms on Σ , we can assume that the j_n are the same complex structure j .

The proof is divided into two pieces: in the *interior piece* and the *collar piece*. In the interior, we will upper-bound all higher derivatives $\nabla^k f$ of a conformal harmonic map $f : (\Sigma, j) \rightarrow \mathbb{H}^4$. Here we equip Σ with the unique hyperbolic metric in the conformal class given by j . In the collar piece of the proof, we prove that the images of f_n are

⁷. This is because the indicial roots of $J - \lambda$ depend on λ .

embedded as graphs uniformly over a cylinder built upon K and more importantly, in this region, their Euclidean second fundamental form are uniformly bounded.

In the interior, to control all derivatives of a conformal harmonic map f , it suffices to control its energy density $e(f) = |\nabla f|^2$. The *Bochner formula* for a harmonic map $f : \Sigma \rightarrow M$ between Riemannian manifolds Σ and M reads

$$\Delta e(f) = |\text{II}(f)|^2 + \langle \text{Ric}(\Sigma) \nabla_v f, \nabla_v f \rangle - \langle \text{Riem}(M)(\nabla_v f, \nabla_w f) \nabla_v f, \nabla_w f \rangle \quad (2.38)$$

where $\text{II}(f)$ is the second fundamental form of f , $\text{Ric}(\Sigma)$ and $\text{Riem}(M)$ are the Ricci and Riemann curvatures of Σ and M . The subscripts u, v indicate how the trace is taken. The Laplacian here is non-positive: $\Delta = \text{Tr} \nabla^2$. When f is conformal, $M = \mathbb{H}^4$ and the metric on Σ is hyperbolic, (2.38) becomes

$$\Delta e \geq -e + \frac{e^2}{2}. \quad (2.39)$$

It follows from the harmonic map equation that the energy density $e(f)$ converge to 2 on the boundary (cf. Corollary 1.4 of [LT93]). Combine this with (2.39) and apply Maximum Principle, we have $e(f) \leq 2$ on Σ .

Now let us move to the collar piece of the proof. Let $f_n : (\Sigma, j_n) \rightarrow \mathbb{H}^4$ be a sequence of conformal harmonic maps in the space \mathcal{M} of Theorem 2.36 and suppose that their boundary $K_n = f_n(\partial\Sigma)$ converges in C^2 to a link K_∞ . We fix a half-space model of \mathbb{H}^4 where K_n are bounded and denote the coordinates by (x, y_1, y_2, y_3) .

Proposition 2.38 (Fine [Fin21]). *Let $f_n : (\Sigma, j_n) \rightarrow \mathbb{H}^4$ be a sequence of conformal harmonic maps whose boundary $K_n = f_n(\partial\Sigma)$ converges in $C^{2,\alpha}$ to a link K_∞ . Let $\overline{M}_n := f_n(\Sigma)$, $\overline{M}_{n,\epsilon} := \overline{M}_n \cap x^{-1}([0, \epsilon])$ and $\overline{\Sigma}_{n,\epsilon} = f_n^{-1}(\overline{M}_{n,\epsilon})$. Then there exists ϵ sufficiently small such that:*

1. *For sufficiently large n , the maps f_n are embedding in $\overline{\Sigma}_{n,\epsilon}$ and the surfaces $\overline{M}_{n,\epsilon}$ are graph over a cylinder $C_K := [0, \epsilon] \times K$ built over a smooth link K near to K_∞ . The graphing functions s_n are $C^{2,\alpha}$ sections of the normal bundle of C_K .*
2. *The Euclidean second fundamental form of $\overline{M}_{n,\epsilon}$ are uniformly bounded and after passing to subsequence, $\{s_n\}$ converges in $C^{2,\alpha}$.*

To prove Proposition 2.38, we assume that the conclusion is false, then try to find a contradiction by rescaling M_n using the homotheties $(x, y_i) \mapsto (\lambda x, \lambda y_i)$, $\lambda > 0$. The argument can be sketched as follows.

Let Q be a geometric quantity of the surfaces M_n that we want to control. Suppose that:

1. Q is invariant by homothety, i.e. for any surface M of \mathbb{H}^4 and any point $p \in M$, $Q(M, p) = Q(\lambda M, \lambda p)$.
2. $Q = 0$ everywhere on any vertical plane H (copy of \mathbb{H}^2 in \mathbb{H}^4), and $Q = 0$ on each boundary K_n of M_n .

By geometric quantity, we assume that if a family of pointed surfaces (M_n, p_n) converge in C^∞ to (H, p_∞) on an open set of p_∞ , then $Q(M_n, p_n) \rightarrow Q(H, p_\infty) = 0$.

Examples of such Q are the hyperbolic second fundamental form $|\text{II}|$, the product $x|\tilde{\text{II}}|$ of the half space coordinate and the Euclidean second fundamental form, and the

(Euclidean) gradient norm $|\nabla s_n|$ of the graphing function s_n in Proposition 2.38. We claim that for these examples, $Q(M_n)$ converges uniformly to 0 as $x \rightarrow 0$, i.e.

$$\lim_{\epsilon \rightarrow 0} \sup_n \sup_{M_{n,\epsilon}} Q = 0.$$

Suppose by contradiction that there are points $p_n \in M_n$ with $x(p_n) \rightarrow 0$ and $Q(M_n, p_n)$ is bigger than, say, 1. Let $b_n \in K$ be the point closest to the projection $(y_1, y_2, y_3)(p_n)$. We translate and rotate M_n so that b_n is at the origin and that the tangent of K there is in y_1 -direction. Then we rescale M_n by ratio $\lambda_n = \frac{1}{x(p_n)}$ to obtain a new family of surfaces Y_n . After rescaling, the bad points p_n become points q_n on the level set $x = 1$ that converge to $q_\infty = (1, 0, 0, 0)$. The rescaled surfaces Y_n converge as sets to the vertical plane $H = \{y_2 = y_3 = 0\}$. This notion of convergence is due to White and means that

$$H = \{p \in \mathbb{H}^4 : \limsup_{n \rightarrow \infty} d(p, Y_n) = 0\} = \{p \in \mathbb{H}^4 : \liminf_{n \rightarrow \infty} d(p, Y_n) = 0\}$$

Both of these convergences can be justified by Anderson's convex hull and the fact that the links K_n has the same osculating radius: there exists a number $r > 0$ independent of n such that each link K_n lies outside of any 3-ball of radius r tangent to it.

Now we have $Q(Y_n, q_n) > 1$ and the pointed surfaces (Y_n, q_n) converge as sets to (H, q_∞) which has $Q = 0$. A contradiction will be produced if we can upgrade this convergence to C^2 . This is where we need the following version of Allard regularity, due to Brian White.

Theorem 2.39 (White, Theorem 5.1 of [Whi16]). *Let M_n be a sequence of stationary integral k -varifolds in a Riemannian manifold Ω . Suppose that M_n converges as set to a subset of a connected, smoothly embedded submanifold H of Ω and that at some point of H , there is a neighbourhood $U \subset \Omega$ such that $M_n \cap U$ converge weakly to $H \cap U$ with multiplicity 1. Then M_n converges smoothly to M with multiplicity 1 everywhere.*

As pointed out by White, the key difference between Theorem 2.39 and Allard's Regularity Theorem is that one only needs weak convergence on a small open set.

Back to our situation, we can assume that p_n are the lowest points of M_n such that $Q(M_n, p_n) \geq 1$. This mean that $Q(Y_n, \cdot) \leq 1$ in the region $x \leq 1$. In all three instances of Q , this bound gives us $C^{0,\alpha}$ convergence (thus weak convergence) of Y_n to H in a neighbourhood U of $(\frac{1}{2}, 0, 0, 0)$. Theorem 2.39 upgrades this to C^∞ convergence at q_∞ and produces a contradiction. The case $Q = |\nabla s|$ in particular proves part 1 of Proposition 2.38.

For part 2 of Proposition 2.38, we need to bound the Euclidean second fundamental form of M_n uniformly in the region $M_{n,\epsilon}$ for sufficiently small ϵ . To do this, we repeat the above rescaling argument: denote by $p_n \in M_n$ the bad points with $x(p_n) \rightarrow 0$ and $|\tilde{\Pi}(M_n, p_n)| \rightarrow \infty$, then rescale (M_n, p_n) to new surfaces (Y_n, q_n) with $|\tilde{\Pi}(Y_n, q_n)| = 1$. Since we have proved that $Q = x|\tilde{\Pi}(M_n)|$ converges uniformly to 0 as $x \rightarrow 0$, we have $\lim_{n \rightarrow \infty} x(q_n) = 0$.

Now let us show that the points p_n (thus q_n) can be chosen so that $|\tilde{\Pi}(Y_n, q)| \leq 2$ for all $q \in Y_n$ with $x(q) \leq 1$. Let us fix n . If we cannot choose such p_n , there must be a point p_n^1 with

$$x(p_n^1) \leq |\tilde{\Pi}(M_n, p_n)|^{-1}, \quad |\tilde{\Pi}(M_n, p_n^1)| \geq 2|\tilde{\Pi}(M_n, p_n)|$$

and similarly a point p_n^2 with

$$x(p_n^2) \leq |\tilde{\Pi}(M_n, p_n^1)|^{-1} \leq \frac{1}{2} |\tilde{\Pi}(M_n, p_n)|^{-1}, \quad |\tilde{\Pi}(M_n, p_n^2)| \geq 2 |\tilde{\Pi}(M_n, p_n^1)|$$

This procedure goes on and produces a sequence $\{p_n^k\}_{k \in \mathbb{N}} \subset M_n$ with

$$|\tilde{\Pi}(M_n, p_n^k)| \geq 2^k |\tilde{\Pi}(M_n, p_n)|, \quad x(p_n^k) \leq 2^{-k} |\tilde{\Pi}(M_n, p_n)|^{-1}$$

which is clearly a contradiction as k tends to infinity.

As before, White's theorem says that the rescaled surfaces Y_n converge smoothly to H in the interior of \mathbb{H}^4 . This does not immediately produce a contradiction because the points q_n where $|\tilde{\Pi}(Y_n, q_n)| = 1$ converge to $q_\infty = 0$ on the boundary of \bar{H} . To finish, we need the graphing functions w_n of \bar{Y}_n over \bar{H} to C^2 -converge up to boundary. By the point-picking argument above, we already have C^0 control on the Euclidean second fundamental form. We use this to upper bound the $C^{0,\alpha}$ norm of the Euclidean mean curvature \tilde{k}_n of Y_n in a neighbourhood of q_∞ . Recall that the mean curvature \tilde{k}_n is an elliptic operator $L_n(w_n)$ on the graphing function w_n . The coefficients of L_n depend on ∇w_n , but since we have arranged so that $|\tilde{\Pi}(Y_n)|$ is bounded in the region $x \leq 1$, these are bounded. Therefore a uniform $C^{0,\alpha}$ bound on \tilde{k}_n produces uniform $C^{2,\alpha}$ bound on w_n and thus a contradiction.

The Euclidean mean curvature can be estimated via the *Willmore equation*

$$\Delta_n \tilde{k}_n = \tilde{F}_n$$

Here \tilde{F}_n is an algebraic expression built up from the Euclidean second fundamental form $\tilde{\Pi}(Y_n)$, and so $\|\tilde{F}_n\|_{C^0}$ is bounded uniformly in n . The Δ_n is the normal bundle Laplacian of Y_n and so its coefficients depend on first order of the surface. Because of this, the bootstrap argument requires extra care. We refer to [Fin21] for a detailed treatment.

To prove the convergence of f_n , we need to find a collar neighbourhood V of Σ such that $f_n(V)$ is contained in $\bar{M}_{n,\epsilon}$ for all n . Then we apply the collar argument on V and interior argument on its complement. The existence of such V is also a delicate argument. One first proves that the length of $\partial \bar{\Sigma}_{n,\epsilon}$ under the hyperbolic metric of Σ is upper bounded by a constant l independent of n . This boils down to a uniform lower bound on the energy density of f_n along this curve, which requires another rescaling argument. When this is done, it suffices to apply an isometry of \mathbb{H}^2 so that a point on this curve is placed at the origin and choose V to be the complement of the geodesic ball of radius l .

2.5.2 Minimal surfaces and splitness of their boundary

It is difficult to compute the degree of a given link γ from its definition. The only case where this can be done is when γ is an unlinked union of c circles. When $c = 1$, we can isotope γ to a round circle which, by Maximum Principle, can only be filled by a totally geodesic disc.

When $c > 1$, after isotoping the components of γ into round circles, we see that there is at least one minimal surface filling it, namely the union of totally geodesic discs. Theorem 2.40 below says that it is the only solution.

Theorem 2.40. *Let $L := L_1 \sqcup L_2$ be a separated union of two $(k-1)$ -dimensional immersed submanifolds L_1, L_2 in S^{n-1} . There is a way to rearrange L in its isotopy class such that any minimal submanifold Σ^k in \mathbb{H}^n that fills L and C^1 up to boundary is a disjoint union of minimal submanifolds filling each L_i .*

When $k = 2$, the same rearrangement also rules out harmonic maps: There is no connected, oriented surface $\bar{\Sigma}$ and harmonic map $f : (\Sigma, \partial\Sigma) \rightarrow (\mathbb{H}^n, S^{n-1})$ that is C^1 up to boundary, and meets S^{n-1} transversally at L .

Proof. Let us just consider the case of minimal surface ($k = 2$). For higher k , one replaces 2π by ω_{k-1} . For harmonic maps, it suffices to use Corollary 1.36 instead of Proposition 2.12.

Recall that a minimal surface of \mathbb{H}^n passing by the origin O of the Poincaré model has to intersect any 3-sphere of radius r at a curve longer than $2\pi r$ (see Proposition 2.12). If the surface extends in C^1 up to its ideal boundary, we can send r to 1 and conclude that its ideal boundary has Euclidean length at least 2π .

We isotope L so that L_1 (respectively L_2) is contained in a small ball centred at the North (respectively South) pole of the Poincaré ball and so that its Euclidean length is less than π . It suffices to prove that Σ has no intersection with the equatorial hyperplane. By convexity, such intersection is contained in a small ball centred at the origin O . If it was non-empty, by a small Möbius transform we could suppose that Σ contains O while keeping the Euclidean length of L less than 2π . This is a contradiction. \square

Corollary 2.41. *All Alexakis–Mazzeo degrees vanish, except in the case of one circle.*

2.6 Appendix D: Renormalisation in Asymptotically Hyperbolic manifold

In this Appendix, we will prove the area development (2.3) of Graham and Witten [GW99]. In fact, we will give a criteria for a quantity Q defined on a surface Σ (not necessarily minimal) in an asymptotically hyperbolic manifold to be *renormalisable* in the sense that

$$Q(\Sigma)(\epsilon) := \int_{\rho \geq \epsilon} Q = \frac{LQ}{\epsilon} + Q_R + O(\epsilon).$$

Here ρ is a special boundary defining function and Q_R is a constant independent of the choice of ρ .

We will also see that under a natural hypothesis, Sack–Uhlenbeck’s α -energy of a map can be renormalised. This hypothesis concerns the growth of the Hopf differential of the map at infinity and is automatically satisfied if the map is conformal or harmonic.

2.6.1 Asymptotically hyperbolic manifold and Graham–Witten lemma

Definition 2.42. *A Riemannian manifold (M^{n+1}, g) is called conformally compact if the following two conditions are satisfied:*

1. M is the interior of a compact manifold \bar{M} with boundary $X^n = \partial M$

2. There exists a boundary defining function ρ such that $\rho > 0$ in M , $\rho = 0$ and $d\rho \neq 0$ at every point of ∂M . Moreover the metric $\bar{g} := \rho^2 g$ extends over \bar{M} .

We will always fix a compactification \bar{M} in advance. The restriction of the normalised metric \bar{g} to X varies in a same conformal class independent of ρ . This class is called the conformal infinity of g . A collar neighbourhood is an open set of \bar{M} containing X .

The sectional curvatures R of g and \bar{R} of \bar{g} are related by:

$$R_{ijij} = \rho^{-2} \bar{R}_{ijij} + \rho^{-1} [\text{Hess } \rho(i, i) + \text{Hess } \rho(j, j)] - |d \ln \rho|_g^2$$

where the Hessian is taken with \bar{g} . It follows from this that:

Lemma 2.43. *Sectional curvatures of a conformally compact manifold with boundary defining function ρ converge uniformly to $-|d\rho|_{\bar{g}}^2 = -|d \log \rho|_g^2$. In particular, a conformally compact manifold is negatively curved near infinity.*

Definition 2.44. *A conformally compact manifold (M, g) is called asymptotically hyperbolic (AH) if the limit of all sectional curvatures at infinity is -1 . In other words, any boundary defining function ρ satisfies*

$$|d \log \rho|_g = |d\rho|_{\bar{g}} = 1 \text{ on } X.$$

A boundary-defining function ρ is called special if $|d\rho|_{\bar{g}} = 1$ on an open neighbourhood of X .

Using a special boundary defining function ρ , we can rewrite the metric g in a collar neighbourhood $X \times [0, \epsilon)_\rho$ as

$$g = \frac{d\rho^2 + h(\rho)}{\rho^2} = \frac{d\rho^2 + h_0}{\rho^2} + \frac{h_1}{\rho} + h_2 + \dots \quad (2.40)$$

where $h(\rho)$ is a path of metrics on X and $h_0 + \rho h_1 + h_2 \rho^2$ is its Taylor expansion in $S^2 T^* X$.

Lemma 2.45 (Graham–Witten [GW99]). *Let (M^{n+1}, g) be an AH manifold.*

1. *Let h_0 be any metric in the conformal infinity of g . Then there exists a special boundary defining function ρ such that $\rho^2 g$ restricts to h_0 on X . Moreover, ρ is unique in a collar neighbourhood.*
2. *Let ρ_1, ρ_2 be two special boundary defining functions. Then*

$$\rho_2 = \rho_1 e^\varphi \quad (2.41)$$

where φ is a function on the collar $X \times [0, \epsilon)$ that satisfies

$$\left. \frac{\partial \varphi}{\partial \rho_i} \right|_{\rho_i=0} = 0.$$

Proof. For the first part, we pick any boundary defining function r such that the metric $\bar{g} := r^2 g$ restricts to h_0 on X and construct ρ of the form $\rho = r e^\varphi$ where φ is a function on a collar neighbourhood. The condition $|d \log \rho|_g = 1$ is equivalent to

$$2\langle dr, d\varphi \rangle_{\bar{g}} + r |d\varphi|_{\bar{g}}^2 = \frac{1 - |dr|_{\bar{g}}^2}{r} \quad (2.42)$$

The equation (2.42) is a first order, non-characteristic PDE of φ and can be solved for arbitrarily prescribed boundary data.

For the second part, we can suppose that the boundary defining function r in (2.42) is special, and so the RHS vanishes, therefore

$$2 \frac{d\varphi}{dr} + r |d\varphi|_{\bar{g}}^2 = 0$$

This means that $\varphi = O(r^2)$. □

Fefferman and Graham proved that if the metric g is Einstein, then h is determined up to order $n - 1$ by h_0 . This means that the tensors h_1, h_2, \dots, h_{n-1} and the trace of h_n are completely determined by h_0 . Moreover, all the odd order term h_{2i+1} with $2i + 1 < n$ vanish. When n is even, there is an obstruction to determine the trace-free part of h_n but it can be cancelled by adding a term of order $\rho^n \log \rho$ in the Taylor expansion of g .

Theorem 2.46 (Fefferman–Graham [FG07]). *Let (M^{n+1}, g) be both AH and Einstein and ρ be a special boundary defining function. Then the metric g has the following normal form*

$$g = \frac{d\rho^2 + h_0 + h_2 \rho^2 + (\text{even power}) + (n-1)^{\text{th-term}} + h_n \rho^n}{\rho^2}. \quad (2.43)$$

where the

$$(n-1)^{\text{th-term}} = \begin{cases} h_{n-1} \rho^{n-1}, & \text{if } n \text{ is odd} \\ h_{n-1} \rho^n \log \rho, & \text{if } n \text{ is even} \end{cases}$$

We will not need the full expansion (2.43), but only the fact that h_1 vanishes. So we will only assume that g is Einstein up to first order. The proof of Theorem 2.46 is quite simple in this case and can be seen via the following Lemma.

Lemma 2.47 (Fefferman–Graham [FG07]). *Let (M^{n+1}, g) be an AH manifold and ρ be a special boundary defining function. Let h_0, h_1 be the coefficient in the extension (2.40) of g . Then the trace-free tensor $\text{Ric}^g + ng$ of g extends to \bar{M} and is given by*

$$(\text{Ric}^g + ng)_{ab} = \begin{cases} 0 & \text{if } ab = ir \\ \frac{\rho}{2} \text{Tr}_{h_0} h_1 + \rho^2 \text{Ric}_{\rho\rho}^{\bar{g}} & \text{if } ab = \rho\rho \\ -\frac{\rho}{2} [(n-1)h_1 + (\text{Tr}_{h_0} h_1)h_0] + \rho^2 \left(\text{Ric}_{ij}^{h(\rho)} + R_{\rho ij \rho}^{\bar{g}} \right) & \text{if } ab = ij \end{cases} \quad (2.44)$$

Here i, j denote directions in X and a, b denote directions in X and along the gradient of ρ . The tensor $R^{\bar{g}}$ is the Riemann curvature of \bar{g} .

In particular, the trace-free Ricci curvature of an AH metric extends to 0 on X .

Note that in (2.44), we write the curvature of a metric in its orthonormal frame. This means that Ric_{ij}^g and $\text{Ric}_{ij}^{\bar{g}}$ are written using two different frames.

Corollary 2.48 (Fefferman–Graham [FG07]). *Suppose in addition that metric g is Einstein up to first order in ρ , that is, the tensor norm of $\text{Ric}^g + ng$ vanishes in first order at infinity. Then the tensor h_1 of (2.40) vanishes.*

From now on, we will suppose that the metric g is Einstein up to first order and so (2.40) becomes

$$g = \frac{d\rho^2 + h_0}{\rho^2} + O(1) \quad (2.45)$$

for a special boundary defining function ρ .

2.6.2 Renormalisability

Let (Σ, g_Σ) be a conformally compact surface. We call a boundary defining function ρ *eligible* if g_Σ has the expansion

$$g_\Sigma = \frac{dh_1 + h_0}{\rho^2} + O(1). \quad (2.46)$$

in other words, if there is no $\frac{h_1}{\rho}$ term. Here h_0 is a metric on $\partial\Sigma$ and $O(1)$ denotes a finite metric on $\bar{\Sigma}$.

We note that any boundary defining function r third order close to an eligible boundary defining function ρ in the sense that $r = \rho + O(\rho^3)$ or equivalently $\rho = r + O(r^3)$, is also eligible.

Lemma 2.49. *Let (M, g) be an AH manifold and $\bar{\Sigma}$ be a surface in M meeting its boundary transversally, then the pullback of any boundary defining function of M is a boundary defining function of Σ .*

Moreover, if the surface Σ meets the boundary X of M at right angle, then the pullback of any special boundary defining function is eligible to the induced metric on Σ .

Proof. The second statement is true because we can write the map inclusion map $\Sigma \hookrightarrow M$ in collar neighbourhoods $\partial\Sigma \times [0, \epsilon)$ and $X \times [0, \epsilon)$ as $(p, t) \mapsto (p_t, t)$. The point p_t is determined by flowing along the inverse gradient of ρ in Σ from level $\rho = 0$ to $\rho = t$, then flowing backwards to level $\rho = 0$ by the inverse gradient in M . If Σ meets X at right angle, the point p_t is $O(t^2)$ close to p . This leads to (2.46).

The angle condition is necessary because the collar neighbourhoods of M and Σ are identified $X \times [0, \epsilon)$ and $\partial\Sigma \times [0, \epsilon)$ using two different gradient fields. \square

The goal of this section is to study the behaviour at 0 of the function

$$\mathcal{Q}(\epsilon) = \int_{\Sigma \cap \{\rho \geq \epsilon\}} Q \text{vol}_{g_\Sigma} \quad (2.47)$$

associated to each function Q on $\bar{\Sigma}$. Theorem 2.50 gives a condition of the growth of Q at infinity so that the integral (2.47) can be renormalised and that its renormalisation is independent on the choice of ρ . The proof will be delayed until the next Section.

Theorem 2.50 (Renormalisation Criteria). *Let ρ be an eligible boundary defining function of Σ and Q be a function on $\bar{\Sigma}$ such that $\frac{\partial Q}{\partial \rho} = 0$ on the boundary. Then*

$$\mathcal{Q}(\epsilon) = \frac{L_Q}{\epsilon} + \mathcal{Q}_R + O(\epsilon) \quad (2.48)$$

where $L_Q = \int_{\Sigma} Q ds_{\bar{g}}$ is the boundary length of Σ under the metric $\bar{g} = \rho^2 g$ weighted by the function Q .

Moreover, let $\{\rho_t\}_{t \in [0,1]}$ be a family of eligible boundary defining functions such that $\frac{d\rho_t}{dt} = O(\rho_t^2)$, or equivalently $\rho_t = \rho_0 e^{\varphi_t}$ where φ_t has no first order term in ρ_0 . Then the quantity $\mathcal{Q}_R(t)$ computed using ρ_t in (2.47) and (2.48) is independent of t .

Corollary 2.51 (Renormalised Q -area). *Let (M, g) be an AH manifold that is Einstein up to first order and Σ be a surface meeting its boundary at right angle. Let ρ be a special boundary defining function of M and Q be a function on $\bar{\Sigma}$ such that $\frac{\partial Q}{\partial \rho} = 0$ on $\partial\Sigma$. Then the quantity $\mathcal{Q}(\epsilon)$ defined by*

$$\mathcal{Q}(\epsilon) = \int_{\Sigma, \rho \geq \epsilon} Q \text{vol}_{g_{\Sigma}}.$$

has the expansion

$$\mathcal{Q}(\epsilon) = \frac{L_Q}{\epsilon} + \mathcal{Q}_R + O(\epsilon).$$

Moreover, the quantity \mathcal{Q}_R is independent of the choice of ρ .

Proof. By Lemma 2.49 the restriction of any special boundary defining function to Σ is eligible. It is always possible to join two special boundary defining functions ρ_0, ρ_1 of M by a path of special boundary defining function $\{\rho_t\}_{t \in [0,1]}$. This can be done by joining the two normalised metrics $\rho_1^2 g$ and $\rho_0^2 g$ in the conformal infinity by their convex combination, then applying Lemma 2.45. The second half of Lemma 2.45 assures that the resulting family satisfies the condition of Theorem 2.50. \square

When $Q = 1$, Corollary 2.51 was proved by Graham and Witten and \mathcal{Q}_R is their renormalised area.

We can also formulate a version of Theorem 2.50 for maps between a conformally compact surface (Σ, g_{Σ}) and an AH manifold (M, g) . We are interested in maps that send the boundary to boundary and interior to interior. Given such a map f , the pullback $\rho \circ f$ of a boundary defining function ρ of M is a boundary defining function of Σ if and only if f meets X transversally. This latter condition means that the tangent of f on $\partial\Sigma$ is not contained in X at any point.

As before, we assume that g is Einstein up to first order, ρ is special and we want the pullback $\rho \circ f$ to be eligible to the metric f^*g in a collar of Σ . For this, we need f to be immersion in a collar neighbourhood and, similar to Lemma 2.49, the tangent of f contains the gradient of $\nabla \rho$ at all points of $\partial\Sigma$. We say that the map f *meets X at right angle* if these two conditions are satisfied.

We can define a mapping version of (2.47) as follows. Let Q be a function on $\bar{\Sigma}$ and

ρ be a special boundary defining function of M , then we define $\mathcal{Q}(\epsilon)$ by

$$\mathcal{Q}(\epsilon) = \int_{\Sigma, \rho \circ f \geq \epsilon} Q \operatorname{vol}_{g_\Sigma} = \int_{\Sigma, \rho \circ f \geq \epsilon} \frac{Q}{|\det_{g_\Sigma} f^*g|^{1/2}} \operatorname{vol}_{f^*g} \quad (2.49)$$

where $|\det_{g_\Sigma} f^*g|^{1/2}$ is the Jacobian of f . Now it follows from Theorem 2.50 that:

Proposition 2.52. *Let (M, g) be AH manifold, Einstein up to first order and ρ be a special boundary defining function. Let $f : (\Sigma, g_\Sigma) \rightarrow (M, g)$ be a map that meets X at right angle and Q be a function on Σ that satisfies*

$$\frac{\partial}{\partial \rho} \left(\frac{Q}{|\det_{g_\Sigma} f^*g|^{1/2}} \right) = 0 \quad \text{on } X.$$

Then the quantity $\mathcal{Q}(\epsilon)$ defined by (2.49) admits the following expansion

$$\mathcal{Q}(\epsilon) = \frac{L_Q}{\epsilon} + \mathcal{Q}_R + O(\epsilon)$$

where $L_Q = \int_{\partial\Sigma} Q ds_{\bar{g}}$ is the length of $f(\partial\Sigma)$ under the metric $\bar{g} = \rho^2 g$ weighted by Q .

Moreover, the normalisation \mathcal{Q}_R is independent of the choice of the special boundary defining function ρ .

We will use Proposition 2.52 to prove, under a natural boundary condition of f , that the Dirichlet energy and Sacks–Uhlenbeck’s α -energy can be renormalised and that their renormalisations are independent of the choice of boundary defining function.

Recall that the energy density $e(f)$ is defined as the tensor norm of df under the metric g on M and g_Σ on Σ

$$e(f) := |df|^2 = \operatorname{Tr}_{g_\Sigma} f^*g.$$

The *Dirichlet energy* is the total integral of $e(f)$ over Σ and the α -energy is the total integral of $|df|^\alpha = e(f)^{\alpha/2}$.

Definition 2.53. *Given a map f between a Riemann surface and a Riemannian manifold (M, g) , the quadratic differential of f is the 2-tensor*

$$q(f, g) := \left\langle \frac{\partial f}{\partial z}, \frac{\partial f}{\partial z} \right\rangle_g dz \otimes dz$$

where z is a complex coordinate of Σ and the inner product on $TM \otimes \mathbf{C}$ is complexified to be \mathbf{C} -linear on both factors. The map f is conformal if and only if $q(f, g)$ vanishes.

The quadratic differential relates the energy density of a map f and its Jacobian by:

$$\operatorname{Tr}_{g_\Sigma} (f^*g)^2 - 4 \det_{g_\Sigma} (f^*g) = \frac{1}{4} |q(f, g)|_{g_\Sigma}^2 \quad (2.50)$$

Definition 2.54. *A map $f : (\Sigma, \partial\Sigma) \rightarrow (M, X)$ is called renormalisable if its image meets X at right angle and its quadratic differential vanishes on $\partial\Sigma$:*

$$|q(f, g)|_{g_\Sigma} = O(\rho) \quad (2.51)$$

It follows from (2.50) that a map f is renormalisable then $\frac{\text{Tr}_{g_\Sigma} f^*g}{|\det_{g_\Sigma} f^*g|^{1/2}} = 2 + O(\rho^2)$, thus the energy density satisfies the condition of Proposition 2.52. Moreover, direct computations show that:

Lemma 2.55. *Let g_Σ be the hyperbolic metric on Σ , g be an AH metric, Einstein up to first order on M . Let $f : (\Sigma, \partial\Sigma) \rightarrow (M, \partial M)$ be a C^2 , renormalisable map. Then*

$$\det_{g_\Sigma} g = 1 + O(\rho^2), \quad \text{Tr}_{g_\Sigma} f^*g = 2 + O(\rho^2), \quad (\text{Tr}_{g_\Sigma} f^*g)^\alpha = 2^\alpha + O(\rho^2).$$

Corollary 2.56 (Renormalised energies). *Suppose that (M, g) is AH, Einstein up to first order. Let $f : (\Sigma, g_\Sigma) \rightarrow (M, g)$ be a C^1 , renormalisable map. Then the Dirichlet energy of f can be renormalised and its renormalisation is independent of the choice of the boundary defining function.*

Moreover, if g_Σ is the hyperbolic metric, then the same conclusion holds for the α -energy.

We conclude this section with a remark on condition (2.51) on vanishing order of the Hopf differential.

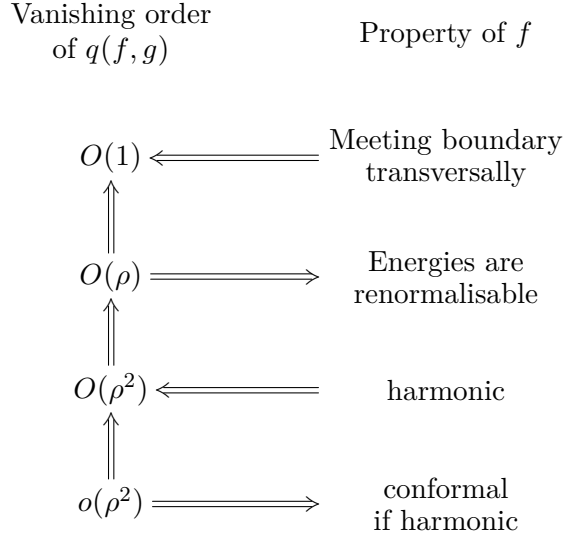
- It follows from just the transversality of f on the boundary that the function $|q(f, g)|_{g_\Sigma}$ extends to $\bar{\Sigma}$, that is $|q(f, g)|_{g_\Sigma} = O(1)$.
- If $|q(f, g)|_{g_\Sigma} = O(\rho)$, we have just seen that the Dirichlet energy and Sacks–Uhlenbeck’s α -energies can be renormalised.
- If we suppose that f is harmonic, then the condition (2.51) is satisfied automatically. In fact, we even have $|q(f, g)|_{g_\Sigma} = O(\rho^2)$. This was proved by Li and Tam (cf. Lemma 1.3 of [LT93]). We give here a much easier proof when the map f extends in C^3 manner to $\bar{\Sigma}$: The quadratic differential $q(f, g)$ is a holomorphic section of the double holomorphic cotangent bundle $T_{2,0}^*\Sigma$. Its singularity on $\partial\Sigma$ is of order ρ^{-2} . It suffices to prove the following Lemma.

Lemma 2.57. *Let F be a holomorphic function on the half disc $\{x^2 + y^2 < 1, y > 0\}$ of \mathbb{C} . Suppose that y^2F extends in C^2 manner over the real segment $(-1, 1)$, then F also extends in C^0 manner over that segment.*

Proof. It is because $\frac{\partial}{\partial \bar{z}}(y^2F) = -\frac{1}{2}F$. □

- If f is harmonic and $|q(f, g)|_{g_\Sigma} = o(\rho^2)$, then f is conformal. This is because the holomorphic section q extends to 0 on the boundary of $\bar{\Sigma}$.

The previous discussion is summarised in the following diagram.



2.6.3 Proof of the Renormalisation Criteria (Theorem 2.50)

We will sketch here the proof of Theorem 2.50. It uses repeatedly the following formulation of the co-area formula. When ρ_t is a linear function in t , this becomes the standard form of the co-area formula.

Lemma 2.58. *Let (Ω, h) be an open domain in a k -dimensional Riemannian manifold and $\{\rho_t\}_{t=0,1}$ be a family of functions on Ω such that $d\rho_t$ are nowhere-zero. Let Ω_t be subdomains cut out by the condition $\rho_t \geq \delta$. Then for any function $F \in C^0(\Omega)$, one has*

$$\frac{d}{dt} \Big|_{t=0} \int_{\Omega_t} F \operatorname{vol}_h^k = \int_{\partial\Omega_0} \frac{d\rho_t}{dt} \Big|_{t=0} \frac{F}{|\nabla\rho_0|_h} \operatorname{vol}_h^{k-1}.$$

To simplify the notation, we will denote the metric on Σ by h . Let ρ be a h -eligible boundary defining function, that is

$$h = \frac{d\rho^2 + h_0}{\rho^2} + O(1) \tag{2.52}$$

and $\bar{h} = \rho^2 h$ be the corresponding normalised metric on $\bar{\Sigma}$. Instead of proving that

$$\mathcal{Q}(\epsilon) = \frac{LQ}{\epsilon} + \mathcal{Q}_R + O(\epsilon),$$

we will prove that

$$\frac{d\mathcal{Q}(\epsilon)}{d\epsilon} + \frac{LQ}{\epsilon^2} = O(1) \tag{2.53}$$

By Lemma 2.58 (in fact, the standard form of the coarea formula is enough), we have

$$\frac{d\mathcal{Q}(\epsilon)}{d\epsilon} = - \int_{\Sigma, \rho=\epsilon} \frac{Q}{|\nabla\rho|} ds_h = -\epsilon^{-2} \int_{\Sigma, \rho=\epsilon} \frac{Q}{|d\rho|_{\bar{h}}} ds_{\bar{h}}. \tag{2.54}$$

By identify the level set $\rho = \epsilon$ with the boundary via the gradient flow of ρ , we have $Q = Q|_{\rho=0} + O(\epsilon^2)$ and

$$\frac{ds_{\bar{h}}|_{\rho=\epsilon}}{ds_{\bar{h}}|_{\rho=0}} = 1 + O(\epsilon^2), \quad |d\rho|_{\bar{h}} = 1 + O(\epsilon^2)$$

by hypothesis on Q and (2.52). Plug all of these into (2.54), we have

$$\frac{d\mathcal{Q}(\epsilon)}{d\epsilon} = -\epsilon^{-2} \int_{\partial\Sigma} Q ds_{\bar{h}} + O(1),$$

which is just (2.53).

Now we will prove that \mathcal{Q}_R is independent of the choice of ρ . Let ρ_t be an eligible family as in Theorem 2.50, that is $\rho_t = \rho_0 e^{\varphi_t}$ with φ_t having no first order term in ρ_0 . We denote by $\mathcal{Q}(\epsilon, t)$ the corresponding functional computed in (2.47). To prove that the renormalisation does not depend on ρ_t , it suffices to prove that $\frac{d}{dt}\mathcal{Q}(\epsilon, t)$ has no $O(1)$ -term.

The derivative $\frac{d}{dt}\mathcal{Q}(\epsilon, t)$ can be computed with Lemma 2.58:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \mathcal{Q}(\epsilon, t) &= - \int_{\rho_0=\epsilon} \frac{\frac{d\rho_t}{dt}\Big|_{t=0}}{|d\rho_0|_h} Q ds_h \\ &= -\epsilon^{-1} \int_{\rho_0=\epsilon} \frac{\frac{d\varphi_t}{dt}\Big|_{t=0}}{|d\rho_0|_{\bar{h}}} Q ds_{\bar{h}} \\ &= -\epsilon^{-1} \left[\int_{\partial\Sigma} \frac{d\varphi_t}{dt}\Big|_{t=0} Q ds_{\bar{h}} + O(\epsilon^2) \right] \end{aligned}$$

where in the last equality, we used the fact that $\frac{d\varphi_t}{dt}$ and Q have no first-order term in ρ_0 and that ρ_0 is eligible.

3 Gromov's Visual volume and Li–Yau's Conformal volume

We have seen that each point p in the interior of the ball \mathbb{B}^{n+1} correspond to a unique round metric g_p on the boundary sphere. This metric is obtained by pulling back the standard Euclidean metric g_0 via the hyperbolic homothety (or *Lorentz boost*) φ_p along the axis Op that sends p to the centre of the ball.

Definition 3.1. 1. The visual volume $V_\Sigma(p)$ of a measurable subset $\Sigma \subset S^n$ at a point $p \in \mathbb{B}^{n+1}$ is its volume under the metric g_p . Given a conformal manifold $(\Sigma, [h])$, the conformal volume of Σ is given by

$$V_c(\Sigma, n) := \inf_f \sup_p V_{f(\Sigma)}(p) \quad (3.1)$$

where the infimum is taken among branched conformal immersions $f : \Sigma \rightarrow S^n$.

2. The Möbius energy $E_M(f, p)$ at a point $p \in \mathbb{B}^{n+1}$ of a map $f : (\Sigma, h) \rightarrow S^n$ is defined as the Dirichlet energy of f under the metric g_p , i.e. $\int_\Sigma \text{Tr}_h[f^*g_p] \text{vol}_h$. The Möbius energy $E_M(f)$ is defined as

$$E_M(f) = \sup_{p \in \mathbb{B}^{n+1}} E_M(f, p) = \sup_{\varphi \in SO_+(1, n+1)} E(\varphi \circ f)$$

We have seen the role played by the visual volume (and its superlevel sets) in the study of minimal surfaces in the hyperbolic space. The conformal volume, on the other hand, was studied by Li and Yau [LY82] as a natural upper bound for the first eigenvalue λ_1 of M . It is an extension of a topological argument by Hersch [Her70] in case of S^2 and of Yang–Yau [YY80] in case of Riemann surfaces.

3.1 Conformal volume and Willmore conjecture

3.1.1 Hersch topological argument and upper bound of λ_1

We call a map $f : X^k \rightarrow Y^n$ between Riemannian manifolds (*measure-theoretically*) *non-constant* if the preimage of any point $y \in Y$ has zero k -dimensional measure on X .

Lemma 3.2 (Hersch–Li–Yau [Her70],[LY82]). *Let $f : (\Sigma^k, h) \rightarrow (S^n, g_0)$ be a measure-theoretically non-constant map. Then there exists a Möbius transform φ that h -balances f , i.e.*

$$\int_\Sigma x_i \circ \varphi \circ f \text{vol}_h = 0$$

for all Euclidean coordinate x_i .

Proof. We define a map $F : \mathbb{B}^{n+1} \rightarrow \mathbb{B}^{n+1}$ by

$$F(p) := \frac{1}{V(\Sigma)} \sum_{i=1}^{n+1} \left(\int_{\Sigma} x_i \circ \varphi_p \operatorname{vol}_h \right) \partial_{x_i}$$

where φ_p is the hyperbolic homothety sending p to the centre of the ball. As p approaches a point p_{∞} on the boundary sphere, the map φ_p send $S^n \setminus \{p_{\infty}\}$ to $-p_{\infty}$ and thus by Dominated Convergence theorem,

$$\lim_{p \rightarrow p_{\infty}} F(p) = -p_{\infty}.$$

So the map F extends continuously to the antipodal map on the boundary sphere and therefore has to be surjective. \square

Using Lemma 3.2, Hersch proved that

Theorem 3.3 (Hersch [Her70]). *The normalised first eigenvalue of any metric h on S^2 is at most 8π .*

Proof. Since there is only one conformal class up to diffeomorphism on S^2 , we can suppose that h is conformal to the standard metric g_0 . By choosing $\Sigma = S^2$ in Lemma 3.2, one has a Möbius transform φ and 3 pull-back coordinate functions $x_i, i = 1, 2, 3$ such that $\int_{S^2} x_i \operatorname{vol}_h = 0$. By the variational characterisation of λ_1 one has

$$\lambda_1 \int_{S^2} x_i^2 \operatorname{vol}_h \leq \int_{S^2} |\nabla^h x_i|_h^2 \operatorname{vol}_h = \int_{S^2} |\nabla x_i|_{g_0}^2 \operatorname{vol}_{g_0} \quad (3.2)$$

where the last equality is due to the conformal nature of the integrand. The RHS sums up, for $i = 1, 2, 3$ to $2.4\pi = 8\pi$ while the LHS is the normalised first eigenvalue of h . \square

Theorem 3.3 was generalised to Riemann surfaces by Yang and Yau [YY80].

Theorem 3.4 (Yang–Yau). *Let (Σ, h) be a Riemannian surface of genus g and of area V . Then*

$$\lambda_1 V \leq 8\pi(g + 1).$$

Proof. It suffices to immerse the surface to S^2 via holomorphic map π of degree $d \leq g + 1$, which always exists due to Riemann–Roch theorem. We then h -balance it via a Möbius transform and rewrite (3.2) as

$$\lambda_1 \int_{\Sigma} x_i^2 \operatorname{vol}_h \leq \int_{\Sigma} |\nabla^h(x_i \circ \pi)|_h^2 \operatorname{vol}_h = d \int_{S^2} |\nabla x_i|_{g_{\text{std}}}^2 \operatorname{vol}_{g_{\text{std}}} = 8\pi d.$$

\square

3.1.2 Conformal volume as upper bound of λ_1

With a same proof, Theorem 3.4 can be stated in a slightly different way using the conformal volume. The case $m = 2$ of the following result was due to Li and Yau [LY82] and the general case to El Soufi and Ilias [EI83].

Theorem 3.5 (Li–Yau , El Soufi–Ilias). *Let (Σ, h) be a Riemannian k -manifold and n be a sufficiently big number so that Σ admits a branched conformal immersion into S^n . Then one has*

$$\lambda_1 V(\Sigma)^{2/k} \leq k V_c(\Sigma, n)^{2/k}.$$

Equality happens if and only if (Σ, h) admits an homothetic minimal immersion into S^n by first eigenfunctions.

An immersion $f : (X, g_X) \rightarrow (Y, g_Y)$ is called homothetic if $f^*g_Y = \lambda g_X$ for a constant λ . We will prove the following stronger statement.

Theorem 3.6. *Let $f : (\Sigma^k, h) \rightarrow (S^n, g_0)$ be any measure-theoretically non-constant map. Then there exists a Möbius transform $\varphi \in SO_+(1, n+1)$ such that*

$$\lambda_1(\Sigma).V(\Sigma) \leq E(\varphi \circ f). \quad (3.3)$$

In particular $\lambda_1 V(\Sigma)$ is not bigger than the Möbius energy of f .

Moreover, if we define the m -energy of f by $E_m(f) := \int_{\Sigma} |df|^m \text{vol}_h$, then

$$\lambda_1 V(\Sigma)^{2/k} \leq E_k(\varphi \circ f)^{2/k}. \quad (3.4)$$

Equality in (3.4) happens if and only if equality in (3.3) happens, and it is exactly when f is a harmonic map with constant energy density $e(f) = \lambda_1$.

Proof. Let φ be the Möbius transform that balances f and $\tilde{x}_i = x_i \circ \varphi \circ f$, it suffices to sum up the following estimate for $i = 1, 2, \dots, n+1$ to obtain (3.3).

$$\lambda_1 \int_{\Sigma} \tilde{x}_i^2 \text{vol}_h \leq \int_{\Sigma} |\nabla^h \tilde{x}_i|^2 \text{vol}_h = \int_{\Sigma} \text{Tr}_h(d\tilde{x}_i)^2 \text{vol}_h.$$

When $\dim \Sigma = k > 2$, (3.4) follows from Hölder inequality:

$$E(\varphi \circ f) \leq \left(\int_{\Sigma} [\text{Tr}_h(\varphi \circ f)^* g_0]^{k/2} \text{vol}_h \right)^{2/k} V(\Sigma)^{1-\frac{2}{k}} = E_k(\varphi \circ f)^{2/k} V(\Sigma)^{1-\frac{2}{k}}$$

□

Remark 3.7. 1. *The k -energy E_k only depends on the conformal class of h . When f is (weakly) conformal it is the mapping volume, $E_k(\varphi \circ f) = k^{k/2} V(\varphi \circ f)$. One obtains Theorem 3.5 by taking supremum among conformal immersion f .*

2. *The two sides of (3.3) are not scale-invariant unless when $k = 2$. This is why we could not just take the supremum directly there.*

3.1.3 Li–Yau’s approach to the Willmore conjecture

Theorem 3.5 can also be seen as a lower bound of the conformal volume in term of the first eigenvalue. This point of view is particularly useful when we have better understanding of the spectrum of *one* metric on Σ than the area of its conformal image in the sphere. This is the case of tori, where we have know very well the spectrum of the flat metric (by Fourier series). The conformal area, on the other hand, is a lower bound of the Willmore energy:

$$V_c(\Sigma) \leq \int_{f(\Sigma)} \text{vol}_{g_p} \leq \int_{f(\Sigma)} (1 + |H_p|^2) \text{vol}_{g_p} =: \mathcal{W}(f(\Sigma)) \quad (3.5)$$

Here $f : \Sigma^2 \rightarrow S^n$ is an arbitrary conformal immersion, g_p is a round metric on the sphere and $|H_p|^2$ is the norm of mean curvature of $f(\Sigma)$ under g_p . The famous Willmore conjecture reads

Theorem 3.8 (Marques–Neves, Willmore conjecture). *For every smooth immersed torus T in S^3 , one has*

$$\mathcal{W}(T) = \int_T |\hat{\Pi}|^2 = \int_T (1 + |H|^2) \geq 2\pi^2$$

It follows from (3.5) that Theorem 3.8 is true when the conformal structure on T satisfies $V_c \geq 2\pi^2$.

The spectrum of the flat metric on tori can be compute explicitly as follows. The conformal structure of a torus T_τ is given by a lattice of \mathbf{C} generated by 1 and a complex number $\tau = a + ib$. The eigenvalues of the flat metric in this case are

$$\lambda_{pq} = 4\pi^2 \left[q^2 + \left(\frac{p - qa}{b} \right)^2 \right]$$

with eigenfunctions

$$f_{pq} = \cos 2\pi \left(qx + \frac{p - qa}{b} y \right), \quad g_{pq} = \sin 2\pi \left(qx + \frac{p - qa}{b} y \right).$$

It can be proved using Fourier series that:

Lemma 3.9 (cf. [MR86], [Bry15]). *Let $\psi : (T_\tau, g_{\text{flat}}) \rightarrow S^n$ be a conformal immersion from the flat torus T_τ into S^n such that $\int_{T_\tau} \psi \text{vol}_{g_{\text{flat}}} = 0$, then the mapping area of ψ is bounded from below by*

$$V(\psi) \geq \frac{4\pi^2 b}{1 + b^2 + a^2 - a}$$

It follows from Lemma 3.2) that:

Theorem 3.10 (Bryant–Montiel–Ros [MR86], [Bry15]). *If $\tau = a + ib$ then*

$$V_c(T_\tau, n) \geq \frac{4\pi^2 b}{1 + b^2 + a^2 - a}. \quad (3.6)$$

In particular, the Willmore conjecture is true for tori of modulus τ in the disk $|\tau - (\frac{1}{2} + i)| < \frac{1}{2}$.

There are two difficulties when we try to compute the conformal volume from the definition, using (3.1): Firstly, we need good test conformal immersion f to estimate for the infimum. Then for such f , we need to compute the maximum of $V_{f(\Sigma)}$. Related to the first difficulty is the following open question by Li and Yau [LY82]

Question 3.11 (Li–Yau). *Is the infimum in the definition of conformal volume (equation (3.1)) attainable by a branched conformal immersion?*

In the case of T_τ , a good candidate $\psi_\tau : T_\tau \rightarrow S^5$ was pointed out in [MR86] using eigenfunctions $f_{10}, f_{01}, f_{11}, g_{10}, g_{01}, g_{11}$. The image of ψ_τ is a surface of constant mean curvature and the immersion is not only conformal but *homothetic*:

$$\psi_\tau^* g_0 = \frac{4\pi^2}{1 + b^2 + a^2 - a} g_E.$$

This means that the value of $V_{\psi_\tau(T_\tau)}$ at O is exactly the RHS of (3.6). Moreover the map ψ_τ is g_0 -balanced, so¹ O is a critical point this function. Equality in (3.6) will be obtained if one can prove that O is the global maximiser of V_M . It is however not true in general, as one can check from the second variation of V_M that O is local maximiser if and only if

$$\left(a - \frac{1}{2}\right)^2 + b^2 < \frac{9}{4} \tag{3.7}$$

It was conjectured by Montiel and Ros in [MR86] and proved by Bryant [Bry15] that

Theorem 3.12 (Bryant–Montiel–Ros). *Under the condition (3.7), the origin O is the global maximiser of $V_M(T_\tau)$. Therefore, the conformal volume of T_τ is:*

$$V_c(T_\tau, n) = \frac{4\pi^2 b}{1 + b^2 + a^2 - a} \quad \text{for all } \tau \text{ with } \left|\tau - \frac{1}{2}\right| < \frac{3}{2}$$

3.1.4 Visual volume of minimal submanifolds

Let Σ^2 be a minimal surface of (S^n, g_0) . The Willmore energy

$$\mathcal{W}(\Sigma) := \int_{\Sigma} (|H|^2 + 1)$$

is a conformal invariant. It is bigger than the area of Σ under any round metric g_p and is exactly its area under g_0 . This means that the function V_Σ is maximised at the origin. This result was known to Li–Yau [LY82] and was generalised to higher dimension by El Soufi and Ilias [EI86].

Theorem 3.13 (Li–Yau [LY82], El Soufi–Ilias [EI86]). *Let $\Sigma \hookrightarrow (S^n, g_0)$ be a minimally immersed submanifold, then the visual volume function V_Σ is decreasing in radial directions.*

In particular, we have

$$V_\Sigma(0) = \max_{\mathbb{B}^{n+1}} V_\Sigma \geq V_c(\Sigma).$$

1. We will verify this in Proposition 3.15 the next section.

Moreover, if there is a point p distinct from the origin such that $V_M(\Sigma, 0) = V_M(\Sigma, p)$, then Σ is a totally geodesic subsphere of S^n .

Therefore, the normalised first eigenvalue of any metric h in the induced conformal class on Σ satisfies

$$\lambda_1(h)V(\Sigma, h)^{2/k} \leq V(\Sigma, g_0)^{2/k}.$$

The proof of Theorem 3.13 can be described in geometric term. It relies on the collinearity of two vector fields on S^n that one can associate to the family of Lorentz boosts in \mathbb{H}^{n+1} . The first vector field is their infinitesimal action on S^n . Each Lorentz boost moves the standard round metric g_0 of S^n inside its conformal class, and the second vector field is the gradient of the conformal factor.

Proof. Let $\{p_t\}_{t \in \mathbb{R}}$ be the arc-length parametrisation of a geodesic line in the Poincaré ball \mathbb{B}^{n+1} along the x_1 -axis, passing by the centre p_0 and connecting 2 antipodal points $p_{\pm\infty} = x_1^{-1}(\pm 1)$. Denote by φ_t the hyperbolic homothety that sends p_t to the centre. Because the orbits of $\{\varphi_t\}$ on S^n are the gradient lines of x_1 , the conformal vector field $v_T = \left. \frac{d\varphi}{dt} \right|_{t=T}$ at a fixed time $T \in (0, \infty)$ is of the form $v_T = -u_1 \partial_{x_1}$ where u_1 is a positive function on S^n depending on the coordinate x_1 , and ∂_{x_1} is the gradient on S^n of the x_1 .

The vector v_T lives in the Poincaré model centred at p_T in which the surface $\varphi_T(\Sigma)$ is minimal under $g_{p_{-T}}$. The conformal factor $\theta \in C^\infty(S^n)$ given by $g_{p_{-T}} = \varphi_{-T}^* g_0 = e^{2\theta} g_0$, is a decreasing function in x_1 and its g_0 -gradient is of the form $\nabla \theta = -u_2 \partial_{x_1}$ for a positive function $u_2 = u_2(x_1)$. Now one has

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=T} V_M(\Sigma, p_t) &= - \int_{\varphi_T(\Sigma)} H_0 \cdot v_T \operatorname{vol}_{g_0} \\ &= -k \int_{\varphi_T(\Sigma)} (\nabla \theta)^\perp \cdot v_T \operatorname{vol}_{g_0} = -k \int_{\varphi_T(\Sigma)} u_1 u_2 |\partial_{x_1}^\perp|^2 \operatorname{vol}_{g_0} \end{aligned} \quad (3.8)$$

Here H_0 is the mean curvature of $\varphi_T(\Sigma)$ under the metric g_0 and is related to the mean curvature under $g_{p_{-T}}$ by:

$$0 = H_{p_{-T}} = e^{-2\theta} \left(H_0 - k(\nabla \theta)^\perp \right).$$

The \perp sign denoting projection to the normal bundle of $\varphi_T(\Sigma)$. So $H_0 = k(\nabla \theta)^\perp$ and this explains the second equality of (3.8). We conclude that $V_\Sigma(p_t)$ is decreasing in t on $(0, +\infty)$.

When there is a point $p \neq O$ with $V_\Sigma(p) = V_\Sigma(0)$, equation (3.8) assures that the S^n -gradient of x_1 at any point of Σ lies inside Σ and so all S^n -gradient lines of x_1 intersecting Σ belongs to it. It follows that Σ contains $p_{\pm\infty}$ and is a totally geodesic subsphere of S^n . \square

One can compute directly that in the proof of Theorem 3.13

$$u_1 = 1, \quad u_2 = \frac{\sinh T}{\cosh T + x_1 \sinh T}.$$

This boils down to the computation of the conformal factor, which will be done in the

3.2. Properties of the visual volume

next section.

It may be tempting to generalise Theorem 3.13 for harmonic maps, the same approach however does not work because the conformal transform for the tension field of a map $f : (\Sigma, h) \rightarrow S^n$ is

$$\tau(f, e^{2\theta}g_0) = \tau(f, g_0) - |df|_{g_0}^2 \nabla^{g_0}\theta + 2f_* \nabla^h(\theta \circ f)$$

It is not always true that $|df|^2 |\nabla\theta|^2 \geq 2g_0(f_* \nabla^h(\theta \circ f), \nabla\theta)$, unless when f is conformal and the dimension of Σ is at least 2.

3.2 Properties of the visual volume

One simple way to compute the conformal factor corresponding to the action of a Lorentz boost on the sphere at infinity is the following. The metric g_p is given by $g_p = (\xi_{0,p} + 1)^2 g_{\mathbb{H}}$ where $\xi_{0,p}$ is the time-coordinate minimising at p , therefore $\frac{g_p}{g_0} = \left(\frac{\xi_{0,O}+1}{\xi_{0,p}+1}\right)^2$. When p lies in the x_1 -axis in the Poincaré model centred at O , the famous Lorentz boost formula reads

$$\xi_{0,p} = \cosh \phi \xi_{0,O} - \sinh \phi \xi_{1,O}, \quad \xi_{1,p} = -\sinh \phi \xi_{0,O} + \cosh \phi \xi_{1,O}$$

where ϕ is the hyperbolic distance from p to O and ξ_1 is the space coordinate pointing to the x_1 -direction. Therefore,

$$\begin{aligned} \frac{\xi_{0,p} + 1}{\xi_{0,O} + 1} &= \cosh \phi \frac{\xi_{0,O}}{\xi_{0,O} + 1} - \sinh \phi \frac{\xi_{1,O}}{\xi_{0,O} + 1} + \frac{1}{\xi_{0,O} + 1} && \text{on } \mathbb{B}^{n+1} \\ &= \cosh \phi - \sinh \phi x_1 && \text{on } S^n \end{aligned}$$

The second equality is because $\frac{\xi_{1,O}}{\xi_{0,O}+1} = x_1$ on S^n and $\xi_{0,O}$ tends to infinity as one approaches the boundary.

Recall that each point $s \in S^n$ defines a null coordinate $\xi_s \in C^\infty(\mathbb{H}^{n+1})$ uniquely up to a multiplicative constant. The conformal factor, as function on S^n can be rewritten as

Lemma 3.14. *Let p and q be interior points of \mathbb{B}^{n+1} and $s \in S^n$ is a point on its boundary, then*

$$\frac{g_p}{g_q}(s) = \left[\frac{\xi_s(q)}{\xi_s(p)} \right]^2$$

In particular, if we choose $q = O$ to be the centre of \mathbb{B}^{n+1} and normalise the null coordinate ξ_s so that $\xi_s(O) = 1$, then

$$\frac{g_p}{g_0}(s) = \xi_s(p)^{-2}.$$

3.2.1 Derivatives

It can be checked that the powers ξ_s^k of the null coordinate ξ_s associated to a boundary point s are eigenfunctions of the hyperbolic Laplacian. By Lemma ??, the visual volume

and the Möbius energy are just combinations of these functions and therefore are also eigenfunctions. We will compute in this subsection the first derivative and the hyperbolic Hessian of the visual volume function. Note that we can always assume the point where we differentiate is the centre O of \mathbb{B}^{n+1} . Let Σ be a k -dimensional submanifold of S^n . In the following statements, the volume form vol_0 is given by the Euclidean metric g_0 and the light coordinates ξ_s are normalised so that $\xi_s(O) = 1$.

Proposition 3.15 (First derivative of visual volume). *The first derivative of V_Σ at O is*

$$dV_\Sigma(O) = -k \int_{s \in \Sigma} d \ln \xi_s \text{vol}_0 \quad (3.9)$$

In particular, if $e_i = \frac{1}{2} \frac{\partial}{\partial x_i}$ is the hyperbolic-unit vector at O in the x_i direction then

$$dV_\Sigma(O).e_i = -k \int_{\Sigma} x_i \text{vol}_0$$

Corollary 3.16. *The centre O is a critical point of the visual volume function V_Σ if and only if Σ is balanced, that is if and only if its center of mass is at the origine:*

$$\int_{\Sigma} x_i \text{vol}_0 = 0 \quad \forall i = 1, \dots, n+1.$$

It follows from (3.9) that:

Corollary 3.17. *At any point $p \in \mathbb{H}^{n+1}$,*

$$|dV_\Sigma(p)|_{g_{\mathbb{H}}} \leq kV_\Sigma(p)$$

In particular, given any pair of points p, q of hyperbolic distance D to each other, one has

$$V_\Sigma(q)e^{-kD} \leq V_\Sigma(p) \leq V_\Sigma(q)e^{kD} \quad (3.10)$$

Proposition 3.18 (Hessian of visual volume). *The hyperbolic hessian of V_Σ at O is*

$$\text{Hess } V_\Sigma = -kV_\Sigma g_{\mathbb{H}} + k(k+1) \int_{\Sigma} (d \ln \xi_s)^2 \text{vol}_0 \quad (3.11)$$

In particular, V_Σ is an eigenfunction of the hyperbolic Laplacian: $\Delta V_\Sigma = -k(n-k)V_\Sigma$.

Let e_i be the hyperbolic-unit vector at O pointing to the x_i -direction then by (3.11),

$$\text{Hess } V_\Sigma(e_i, e_i) = -kV_\Sigma + k(k+1) \int_{\Sigma} x_i^2 \text{vol}_0 \quad (3.12)$$

Two immediate consequences of this are:

Corollary 3.19. *Let Σ be a k -dimensional submanifold of S^n .*

1. *The trace of $\text{Hess } V_\Sigma$ on any linear subspace P of $T_O \mathbb{H}^n$, of dimension at least $k+1$ is non-positive, or in other words, $\text{div}_P \nabla V_\Sigma \leq 0$. It is zero if and only if Σ is contained in the intersection of P and S^n .*

2. The restriction of V_Σ on any minimal submanifold of \mathbb{H}^{n+1} of dimension at least $k + 1$ is super-harmonic.

Corollary 3.20. *Given a submanifold $\Sigma^k \subset S^n$, the Morse index of a non-degenerate critical point of V_Σ is at least $n - k$.*

3.2.2 Boundary behavior

Recall that φ_p is our notation for the Lorentz boost sending p to O . Let $\{p_t\}_{t \geq 0}$ be a geodesic ray from O to a boundary point p_∞ which we assume to lie on a k -submanifold Σ . Then $\varphi_{p_t}(\Sigma)$ converges on compact sets of $S^n \setminus \{-p_\infty\}$ to a k -dimensional subsphere of S^n containing p_∞ . If Σ is an immersed submanifold with m sheets passing by p_∞ , the limit will be m copies of S^k whose tangents coincide with sheets of Σ at p_∞ . Therefore we have

$$\lim_{t \rightarrow \infty} V_\Sigma(p_t) = m\omega_k \quad (3.13)$$

It follows from this that an immersed k -submanifold with visual volume less than $2\omega_k$ has to be embedded.

By (3.10), we still have (3.13) for a sequence $\{q_n\}$ of interior points that converges to p_∞ *orthogonally*, i.e. the hyperbolic distance from q_n to the ray $\{p_t\}_{t \geq 0}$ tends to 0.

Lemma 3.21. *Suppose that Σ is an embedded C^2 k -submanifold of S^n and M is a minimal $(k + 1)$ -submanifold of \mathbb{H}^{n+1} filling Σ . Then the restricted function V_Σ on M converges uniformly to ω_k on the boundary.*

In fact, M only needs to be in the convex hull of Σ . To see this, it suffices to prove that

Lemma 3.22. *The hyperbolic distance between a point $p \in \mathbb{H}^{n+1}$ in the convex hull of a C^2 embedded submanifold Σ of S^n and the cone $C_O(\Sigma)$ centred at O built upon Σ tends to zero as p goes to infinity.*

In particular, the function V_Σ can be extended continuously to a function on $\Sigma \cup \text{ConvHull}(\Sigma)$ by imposing its value to be ω_k on Σ .

It is clear that if Lemma 3.22 is true for one choice of the centre O , it is true for all choices. Moreover, if we choose a point ∞ on $S^n \setminus \Sigma$ and denote by $C_\infty(\Sigma)$ the vertical cylinder $\Sigma \times \mathbb{R}_{>0}$ in the half-space model, we only need to prove that the distance $d(p, C_\infty(\Sigma))$ tends to zero as the altitude of p decreases to 0.

The C^2 and embeddedness hypotheses imply the existence of an *osculating radius* $r_0 > 0$. We recall that this means Σ lies on one side of the $(n - 1)$ -sphere of radius r_0 tangent to it. Lemma 3.23 follows from the definition of the convex hull. It proves Lemma 3.22 and thus Lemma 3.21.

Lemma 3.23. *Let $p \in \mathbb{H}^{n+1}$ be a point in the convex hull of Σ , of altitude $x(p) < r_0$ in the half-space model. Then the Euclidean distance between p and the lifted copy $\Sigma \times \{x(p)\}$ of Σ is at most $r_0 - \sqrt{r_0^2 - x(p)^2}$. Therefore the hyperbolic distance satisfies*

$$d(p, C_\infty(\Sigma)) \leq \frac{x(p)}{r_0 + \sqrt{r_0^2 - x(p)^2}} < \frac{x(p)}{r_0}$$

3.2.3 Submanifolds of S^n with visual volume ω_k

Let M^{k+1} be a minimally immersed submanifold of \mathbb{H}^{n+1} with ideal boundary a submanifold Σ^k of S^n . Corollary 3.19 and Lemma 3.21 say that the visual volume function V_Σ is super-harmonic on M with value at least ω_k on the boundary. This gives another proof of Proposition 2.12.

Moreover, if Σ is embedded and $\max_{\mathbb{H}} V_\Sigma = \omega_k$, we can conclude, by Corollary 3.19, that Σ is a totally umbilic subsphere of S^n . This result was proved by Bryant with a different method in his chapter of [Wel88].

Proposition 3.24 (Bryant [Wel88]). *Any immersed submanifold $\Sigma^k \hookrightarrow S^n$, has visual volume at least ω_k . Equality happens only for totally umbilic subspheres.*

Although the above proof requires a C^2 minimal filling M of Σ , the maximum principle argument can be rewritten with the first variation formula for currents. For the claim on equality in Proposition 3.24, we only need a smooth point of M , whose existence is due to Almgren (see for example [Mor16]).

Proposition 3.25. *Let Σ^k be a k -dimensional C^2 embedded submanifold of S^n and M be a stationary, locally integral $(k+1)$ -current of \mathbb{H}^{n+1} asymptotic to Σ . Then $\|M\|$ -almost everywhere, one has $V_M(\Sigma) \geq \omega_k$.*

Proof. The statement is true when Σ is a totally umbilic subsphere of S^n . Suppose that this is not the case, then by Corollary 3.19 $\operatorname{div}_M \nabla V_\Sigma < 0$ everywhere on $\operatorname{supp} M$.

For any C^1 function $f : [0, +\infty) \rightarrow \mathbb{R}$, one has

$$0 = \int_M \operatorname{div}_M [f(V_\Sigma) \nabla V_\Sigma] = \int_M f(V_\Sigma) \operatorname{div}_M \nabla V_\Sigma + \int_M f'(V_\Sigma) |\nabla^M V_\Sigma|^2 \quad (3.14)$$

as long as the vector field $v_f := f(V_\Sigma) \nabla V_\Sigma$ is compactly supported in the interior. When f is a non-negative, decreasing function f_ϵ supported in $[0, \omega_k - \epsilon)$ and is identically 1 on $[0, \omega_k - 2\epsilon)$, the RHS of (3.14) is non-positive. By Lemma 3.22, $\operatorname{ConvHull}(\Sigma) \cap V_\Sigma^{-1}(-\infty, \omega_k - \epsilon]$ lies strictly in the interior and so the vector field v_f is compactly supported. Thus (3.14) implies that $V_\Sigma \geq \omega_k - 2\epsilon$ $\|M\|$ -almost everywhere, and the conclusion as we send ϵ to 0. \square

Bryant's proof [Wel88] of Proposition 3.24 involves writing down the second order development of V_Σ at a point $p \in \Sigma$:

$$V_\Sigma((1-t)p) = \begin{cases} 4\pi + \frac{\pi}{2} |\hat{\Pi}|^2 t^2 \log \frac{1}{t} + O(t^2), & \text{if } k = 2 \\ \omega_k + \frac{2^{k-1} \omega_{k-1}}{k^2 - 4} |\hat{\Pi}|^2 t^2 + O(t^3 \log \frac{1}{t}), & \text{if } k > 2 \end{cases} \quad (3.15)$$

where $|\hat{\Pi}|^2$ is the trace-free second fundamental form of Σ at p under the Euclidean metric.

The following question was raised by Li–Yau [LY82].

Question 3.26 (Li–Yau). *Given a k -dimensional manifold with conformal volume ω_k , can we say that it is conformally a round k -sphere?*

Proposition 3.24 says that if the answer to Question 3.11 is affirmative then so is the answer to Question 3.26.

3.2.4 Superlevel sets

There are two justifications of our interest in superlevel set of the visual volume function. Firstly, it is to understand Gromov’s visual hull, which together with Anderson’s convex hull, are the only restriction for minimal submanifolds of the hyperbolic space in term of their boundary. The second motivation is to understand critical points of the visual volume function, in particular its local maximisers. We have seen in the computation of the conformal volume of tori that this is a delicate issue. One way to prove that a function has a unique global maximiser, is to show that the superlevel sets are eventually connected.

Theorem 3.27. *If Σ^1 is a closed immersed curve in S^n , then any superlevel set of V_Σ is simply-connected. In particular, the visual hull of a closed immersed curve is simply-connected.*

Proof. Let Γ be an arbitrary closed smooth curve in a superlevel set $U_c := V_\Sigma^{-1}[c, \infty)$. Because $\pi_1(\mathbb{B}^{n+1}) = 0$, the space \mathcal{C} of locally Lipschitz maps $F : \overline{\mathbb{B}^2} \rightarrow \mathbb{H}^{n+1}$ that parametrise Γ monotonically on S^1 is non-empty. It was proved by Lonseth [Lon42] and Morrey [Mor48] that \mathcal{C} contains a conformal harmonic map F (that is in fact a minimiser for the mapping area functional) which is a smooth immersion except at isolated branched points.

By Corollary 3.19, $V_\Sigma \circ F$ is a super-harmonic function on \mathbb{B}^2 with value at least c on the boundary circle, and so it is at least c everywhere in \mathbb{B}^2 . We have just constructed a continuous map $F : \mathbb{B}^2 \rightarrow U_c$ filling an arbitrary closed simple curve Γ , and thus proved $\pi_1(U_c) = 0$. \square

In general, the *visual hull* of a k -submanifold Y of a Riemannian manifold X is defined as the set of points $p \in X$ where the visual volume $\text{Vis}(Y, p)$ is at least ω_k , cf. Definition 2.21. It was proved by Gromov [Gro83, Theorem 8.1.B] that a curve is always contractible in its visual hull.² Theorem 3.27 can be seen as a way to improve this in the hyperbolic space.

The only reason why we could not generalise Theorem 3.27 for all dimension of Σ , with π_1 replaced by π_h for any $h \geq k$, is that we no longer have Douglas–Rado–Morrey solution to the Plateau problem.

3.3 Critical metrics and minimal surfaces

We know from Theorem 3.5 that if a metric h on Σ admits an homothetic minimal immersion then its normalised first eigenvalue is the biggest in the conformal class. It was proved by Nadirashvili and El Soufi–Ilias that if h is *critical* to the normalised first eigenvalue functional, then (Σ, h) admits a homothetic minimal immersion to S^n by its first eigenfunctions [EI00; Nad96]. The result was later generalised for other eigenvalues [SI08].

2. This plays a crucial role in his solution of the Gehring’s linking problem.

Remark 3.28. *The eigenvalues $\lambda_k(h)$ is generally not differentiable in h . So to make sense of critical points of λ_k , one needs to take an analytic perturbation $\{h_t\}$ of h , then define left and right derivatives $\delta_+ = \frac{d}{dt}\lambda_k(h)\Big|_{t=0^+}$, $\delta_- = \frac{d}{dt}\lambda_k(h)\Big|_{t=0^-}$. A metric h is called critical (or extremal) if the two derivatives has opposite sign, that is $\delta_+\delta_- \leq 0$, for all analytic perturbations.*

In this section, we will present an expository version of Nadirashvili–El Soufi–Ilias’s argument in the case of surfaces. We will differentiate the eigenvalue and eigenfunctions as if they depend smoothly on the metric. This can be made rigorous by a projection trick due to Fraser and Schoen [FS13], [FS16].

Theorem 3.29 (Nadirashvili [Nad96], El Soufi–Ilias [EI00; SI08]). *Let Σ be a compact surface and h_0 be a critical metric with respect to $\lambda_k(h)V(\Sigma, h)$. Then there exists k^{th} eigenfunctions u_0, \dots, u_n of h_0 such that the map $f = (u_0, \dots, u_n) : \Sigma \rightarrow S^n$ is a homothetic minimal immersion.*

We will denote the k^{th} -eigenspace of h_0 by $E_k(h_0)$.

Assertion 1. *The first variation of the k^{th} eigenvalue λ_k is given by*

$$\dot{\lambda}_k = - \int_{\Sigma} \langle (du)^2 - \frac{1}{2}|\nabla u|^2 h_0 + \frac{\lambda_k}{2}u^2 h_0, \dot{h} \rangle \text{vol}_{h_0} \quad (3.16)$$

for all $u \in E_k(h_0)$ with $\|u\|_{L^2} = 1$.

Proof that Assertion 1 implies Theorem 3.29. Let $\{h_t\}$ be a perturbation of h_0 in a direction $\omega \in S^2T^*\Sigma$ that preserves the area of Σ . This means $\int_{\Sigma} \langle h_0, \omega \rangle \text{vol}_{h_0} = 0$. Because h_0 is critical,

$$0 = \frac{d}{dt}\Big|_{t=0} [\lambda_k(h_t)V(h_t)] = V(h_0) \frac{d}{dt}\Big|_{t=0} \lambda_k(h_t)$$

By Assertion 1, ω is L^2 -orthogonal to $q(u) := (du)^2 - \frac{1}{2}|\nabla u|^2 h_0 + \frac{\lambda_k}{2}u^2 h_0$ in $L^2(S^2T^*\Sigma)$ for all $u \in E_k$ as long as it is L^2 -orthogonal to h_0 . By Hahn–Banach theorem, h_0 lies in the closure of the convex hull of $Q := \{q(u) : u \in E_k(h_0)\}$. The convex hull of Q is already closed, because it is a finite dimensional subspace, at most generated by the images $q(u_i)$ of a basis $\{u_i\}$ of E_k and the polarisations $q(u_i, u_j)$. So one has

$$h_0 = \sum_{i=0}^n q(u_i) = \sum_{i=0}^n (du_i)^2 - \sum_{i=0}^n \frac{1}{2}|\nabla u_i|^2 h_0 + \frac{\lambda_k}{2} \sum_{i=0}^n u_i^2 h_0 \quad (3.17)$$

for certain elements $u_i \in E_k(h_0)$.

By taking inner product of (3.17) with h_0 , we have $\sum u_i^2 = \frac{2}{\lambda_k}$. Substituting this into (3.17), we see that h_0 is conformal to $\sum (du_i)^2$, and the conformal factor is constant because

$$0 = \Delta \left(\sum_{i=0}^n u_i^2 \right) = 2 \sum_{i=0}^n |\nabla u_i|^2 + 2 \sum_{i=0}^n u_i \Delta u_i = 2 \sum_{i=0}^n |\nabla u_i|^2 - 2\lambda_k \sum_{i=0}^n u_i^2 = 2 \sum_{i=0}^n |\nabla u_i|^2 - 4.$$

3.4. Free boundary minimal submanifolds

The theorem follows by normalising the u_i . □

Now it remains to prove the Assertion.

Proof that Assertion 1. Let u_t be a smooth family of k^{th} eigenfunctions of a perturbation h_t in direction $\dot{h} = \frac{d}{dt}\big|_{t=0} h_t$. Suppose also that $\|u_t\|_{L^2(h_t)} = 1$ and differentiate the identity $\int_{\Sigma} |du_t|_{h_t}^2 \text{vol}_{h_t} = \lambda_k(h_t) \int_{\Sigma} u_t^2$, we get

$$\dot{\lambda}_k = \int_{\Sigma} \left[-|du_0|_{h_0^{-1}\dot{h}h_0^{-1}}^2 + 2\langle d\dot{u}, du_0 \rangle + \frac{1}{2}|du_0|_{h_0}^2 \langle h_0, \dot{h} \rangle \right] \text{vol}_{h_0} \quad (3.18)$$

The first term of (3.18) comes from $\frac{d}{dt}(|du_0|_{h_t}^2)$ and the fact that $(h^{-1})^\cdot = -h^{-1}\dot{h}h^{-1}$. The last term comes from $\frac{d}{dt} \text{vol}_{h_t} = \frac{1}{2}\langle \dot{h}, h_0 \rangle \text{vol}_{h_0}$. The identity (3.16) follows from (3.18) by substituting $|du_0|_{h_0^{-1}\dot{h}h_0^{-1}}^2 = \langle (du_0)^2, \dot{h} \rangle_{h_0}$ and

$$\begin{aligned} 2 \int_{\Sigma} \langle d\dot{u}, du_0 \rangle &= -2 \int_{\Sigma} \dot{u} \Delta u_0 = 2\lambda_k \int_{\Sigma} u_0 \dot{u} = \lambda_k \int_{\Sigma} \frac{d}{dt} u_t^2 \text{vol}_{h_0} \\ &= \lambda_k \left(\frac{d}{dt} \bigg|_{t=0} \int_{\Sigma} u_t^2 \text{vol}_{h_t} - \frac{1}{2} \int_{\Sigma} u_0^2 \langle h_0, \dot{h} \rangle \text{vol}_{h_0} \right) = -\frac{\lambda_k}{2} \int_{\Sigma} u_0^2 \langle h_0, \dot{h} \rangle \end{aligned}$$

□

There is a version of Theorem 3.29, also due to El Soufi and Ilias [SI08], that establishes the same relation for metrics that are critical to the normalised eigenvalues in a given conformal class and harmonic maps into sphere with constant energy density. These two relations provide a motivation for the problem of maximising the normalised eigenvalues, either among all metrics or those in a given conformal class. In the case of first eigenvalue of surfaces, the existence and regularity problem for conformally maximal metric was settled by Petrides [Pet13], and for maximal metric Petrides [Pet18], Mathiesen and Siffert [MS19b], [MS19a].

3.4 Free boundary minimal submanifolds

We have seen in the previous section that Riemannian metrics on a surface Σ that are critical to the "normalised eigenvalue" functional are pullback metrics from minimal immersions of Σ into a sphere. The goal of this section is to present a similar relation, discovered by Fraser and Schoen, between critical metrics of the Steklov eigenvalues and free boundary minimal surfaces of the Euclidean unit ball.

3.4.1 Explicit examples

Definition 3.30. A submanifold $(\Sigma^k, \partial\Sigma) \subset (\mathbb{B}^n, S^{n-1})$ is called a free boundary minimal submanifold if Σ is minimal with respect to the Euclidean metric, $\partial\Sigma$ is contained in the sphere S^{n-1} and Σ meets the sphere orthogonally.

Example 3.31 (The critical catenoid). In \mathbb{R}^3 , the catenoid $x^2 + y^2 = \cosh^2 z$ given by rotating the curve $r = \cosh z$ around the z -axis intersects a certain sphere of radius R_0 at right angle. The radius R_0 is approximately 2.1716. To find this, we look for a

point $(z_{\text{crit}}, r_{\text{crit}})$ on the profile curve where the its tangent passes by the origin. Such point is given by $\frac{r_{\text{crit}}}{z_{\text{crit}}} = \sinh z_{\text{crit}}$ and so z_{crit} is the only positive solution of the equation $z = \coth z$.

Rescale (x, y, z) coordinates and we obtain a smaller version of the catenoid, called the critical catenoid which intersects the unit sphere at right angle.

Example 3.32 (The critical Möbius band). One can check that the following map from the Möbius band $M := \mathbb{R} \times S^1 / (t, \theta) \sim (-t, \theta + \pi)$ with metric $h = dt^2 + d\theta^2$ to the Euclidean ball (B^4, g_E) is harmonic

$$F(t, \theta) = (2 \sinh t \cos \theta, 2 \sinh t \sin \theta, \cosh 2t \cos 2\theta, \cosh 2t \sin 2\theta).$$

Moreover because the pull-back metric is $F^*g_E = 4(\cosh^2 t + \sinh^2 2t)h$, the map F is conformal. So its image is an (unorientable) minimal surface of \mathbb{R}^4 .

Because $\sum_{i=1}^4 x_i^2 = 4 \sinh^2 t + \cosh^2 2t$, the surface intersects spheres of \mathbb{R}^4 at level sets of t . We claim that there is a certain level $T_0 > 0$ where the surface meets the sphere at right angle. It is exactly the value of t such that the gradient of $\rho = \sum x_i^2$ as a function on the surface has the same norm its gradient as a function on \mathbb{R}^4 , i.e. $|d(\rho \circ F)|_h^2 \cdot \frac{h}{g_E} = 4\rho$. Because $d(\rho \circ F) = (4 \sinh 2t + 2 \sinh 4t)dt$, one finds $\sinh^2 T_0 = \frac{1}{2}$.

More recent examples can be found in the survey [Li20].

The reason why such submanifolds are called "free boundary" is that they are critical points of the volume functional under free boundary perturbations. These are perturbations of Σ by a family $\{\Sigma_t\} \subset \mathbb{B}^n$ whose boundary $\partial\Sigma_t$ is contained in S^{n-1} at every instance t . The infinitesimal perturbation $v = \frac{d}{dt}\Big|_{t=0} \Sigma_t$ is a vector field along Σ that is tangent to S^{n-1} at boundary. The first variation formula reads

$$\frac{d}{dt}\Big|_{t=0} V(\Sigma_t) = - \int_{\Sigma} H \cdot v + \int_{\partial\Sigma} v \cdot n$$

where H is the mean curvature vector of Σ and n is the normal vector of $\partial\Sigma$ inside Σ . It follows that Σ is critical if and only if its mean curvature vanishes and Σ meets S^{n-1} orthogonally.

3.4.2 Optimisation of Steklov eigenvalues

The free boundary minimal surface equation can be rewritten with the pullback coordinate functions as:

$$\begin{cases} \Delta x_i = 0, & \text{in } \Sigma \\ \frac{\partial x_i}{\partial n} = x_i, & \text{on } \partial\Sigma \end{cases} \quad (3.19)$$

Or equivalently, x_i are eigenfunctions of the Dirichlet-to-Neumann operator:

$$\begin{aligned} L : C^\infty(\partial\Sigma) &\longrightarrow C^\infty(\partial\Sigma) \\ u &\longmapsto Lu = \frac{\partial \tilde{u}}{\partial n} \end{aligned}$$

Here \tilde{u} is the unique harmonic extension of u on Σ . The k^{th} eigenvalue of L , often denoted by σ_k , is called the k^{th} Steklov eigenvalue of Σ .

It was discovered by Fraser and Schoen [FS16] that

Theorem 3.33 (Fraser–Schoen). *Let Σ be a surface with non-empty boundary and h_0 be a metric that maximises the normalised k^{th} Steklov eigenvalue among smooth metrics, i.e.*

$$\sigma_k(h_0)V(\partial\Sigma, h_0) = \max_h \sigma_k(h)V(\partial\Sigma, h)$$

Then there exists a conformal harmonic immersion $f = (u_1, \dots, u_n) : (\Sigma, h_0) \rightarrow \mathbb{B}^n$ formed by k^{th} Steklov eigenfunctions u_i . Moreover, $f|_{\partial\Sigma} : \partial\Sigma \rightarrow S^{n-1}$ is homothetic.

The proof of Theorem 3.33, in sketch, is similar to Theorem 3.29 of Nadirashvili, El Soufi and Ilias. We start with the first variation formula of the k^{th} Steklov eigenvalue.

Lemma 3.34 (cf. [FS16]). *If $\sigma_k(h) > \sigma_{k-1}(h)$ then for any smooth perturbation of h :*

$$\dot{\sigma}_k = - \int_{\Sigma} \langle q(u), \dot{h} \rangle - \frac{\sigma_k}{2} \int_{\partial\Sigma} u^2 \dot{h}(T, T) \quad (3.20)$$

for any k^{th} Steklov eigenfunction u with $\|u\|_{L^2(\partial\Sigma)} = 1$. The vector T is the h -unit tangent of $\partial\Sigma$ and the symmetric 2-tensor $q(u)$ is given by

$$q(u) := (du)^2 - \frac{1}{2} |\nabla u|^2 h.$$

Proof that Lemma 3.34 implies Theorem 3.33. We perturb h_0 in a direction ω that preserves the boundary length of Σ . Such ω satisfies $\int_{\partial\Sigma} \omega(T, T) = 0$. Because h_0 is critical, we have

$$0 = \left. \frac{d}{dt} \right|_{t=0} [\sigma_k V(\partial\Sigma)] = V(\partial\Sigma) \dot{\sigma}_k.$$

So for any ω , $\langle 0_{\Sigma}, \omega \rangle_{L^2(S^2\Sigma)} + \langle 1_{\partial\Sigma}, \omega(T, T) \rangle_{L^2(\partial\Sigma)} = 0$ implies that $\langle q(u), \omega \rangle_{L^2(S^2\Sigma)} + \langle \frac{\sigma_k}{2} u^2, \omega(T, T) \rangle_{L^2(\partial\Sigma)} = 0$ for all function u in the k^{th} -eigenspace \mathcal{E}_k .

This means that in the Hilbert space $\mathcal{H} := L^2(S^2\Sigma) \oplus L^2(\partial\Sigma)$, if a vector $v_{\omega} := (\omega, \omega(T, T))$ is orthogonal to $(0_{\Sigma}, 1_{\partial\Sigma})$, it is also orthogonal to the set $\Lambda := \{(q(u), \frac{\sigma_k}{2} u^2), u \in \mathcal{E}_k\}$. Because the space of all vectors v_{ω} , for smooth symmetric 2-tensor ω , is dense in \mathcal{H} , Hahn–Banach theorem says that $(0_{\Sigma}, 1_{\partial\Sigma})$ lies in the closed convex hull of Λ .

The convex hull of Λ is already closed, for the same finite-dimensional reason given in the proof of Theorem 3.29. This means that $(0_{\Sigma}, 1_{\partial\Sigma})$ can be written as a combination of $(q(u_i), \frac{\sigma_k}{2} u_i^2)$ where $u_i \in \mathcal{E}_k$, that is

$$\sum_{i=1}^n (du_i)^2 = \frac{1}{2} \sum_{i=1}^n |\nabla u_i|^2 h_0 \quad \text{on } \Sigma, \quad \frac{\sigma_k}{2} \sum u_i^2 = 1 \quad \text{on } \partial\Sigma \quad (3.21)$$

It remains to normalise the u_i and the theorem follows. \square

The problem of existence and regularity for metrics maximising the normalised first Steklov eigenvalue on surfaces was solved by Matthiesen and Petrides [MP20].

3.4.3 Boundary volume and Interior volume. Two conjectures by Fraser and Schoen.

We will now investigate geometric property of a free boundary minimal k -submanifold Σ of \mathbb{B}^n . Recall that the pullback Euclidean coordinates are harmonic functions on Σ and there sum of square $\rho = \sum_i x_i^2$ satisfies

$$\Delta\rho = 2k \tag{3.22}$$

In particular it is sub-harmonic on Σ and so interior points of Σ lie strictly in the interior of \mathbb{B}^n .

By integrating (3.22) over Σ , we see that

Proposition 3.35. *Given a free boundary minimal submanifold Σ^k in the Euclidean ball \mathbb{B}^n , its boundary volume and interior volume are related by*

$$V(\partial\Sigma) = kV(\Sigma)$$

A free boundary minimal submanifold and its boundary are balanced:

Proposition 3.36. *Let Σ^k be a free boundary minimal surface in \mathbb{B}^n . Then*

1. *The centre of mass of the $\partial\Sigma$ is at the origin, i.e. for $i = 1, \dots, n$*

$$\int_{\partial\Sigma} x_i = 0. \tag{3.23}$$

2. *The centre of mass of Σ is also at the origin, i.e. for $i = 1, \dots, n$*

$$\int_{\Sigma} x_i = 0. \tag{3.24}$$

Proof. We can see that the boundary is balanced by integrating (3.19) over Σ :

$$0 = \int_{\Sigma} \Delta x_i = \int_{\partial\Sigma} \frac{\partial x_i}{\partial n} = \int_{\partial\Sigma} x_i.$$

To see that Σ is balanced, we use (3.22):

$$2k \int_{\Sigma} x_i = \int_{\Sigma} x_i \Delta\rho = \int_{\partial\Sigma} \left(x_i \frac{\partial\rho}{\partial n} - \rho \frac{\partial x_i}{\partial n} \right) + \int_{\Sigma} \rho \Delta x_i = \int_{\partial\Sigma} x_i + \int_{\Sigma} \rho \Delta x_i = 0.$$

The last equality is due to (3.23) and harmonicity of x_i . □

Recall that (3.23) is the critical point equation of the visual volume (Proposition 3.15). So O is critical to the visual volume function of a submanifold γ of S^{n-1} if γ can be realised as boundary of a free boundary minimal submanifold of \mathbb{B}^n . Fraser and Schoen [FS11; FS13] asked

Conjecture 3.37 (Fraser–Schoen). *If a $(k - 1)$ -dimensional submanifold $\partial\Sigma$ bounds a free boundary minimal submanifold Σ , then its visual volume is maximised at the origin O . In other words, any Möbius transform reduces the volume of $\partial\Sigma$.*

3.4. Free boundary minimal submanifolds

Similarly, if we define the *relative visual volume* V_Σ of a submanifold Σ of \mathbb{B}^n to be the function which associates to each point $p \in \mathbb{B}^n$ the volume of Σ under the metric g_p , then

Conjecture 3.38 (Fraser–Schoen). *The relative visual volume of a free boundary minimal submanifold Σ is maximised at the origin O . In other words, any Möbius transform reduces the volume of Σ .*

The relative visual volume is an upper bound of the normalised first Steklov eigenvalue. The proof of Proposition 3.39 is similar to that of Theorem 3.5.

Proposition 3.39 (Fraser–Schoen). *Let $\sigma_1 > 0$ be the first Steklov eigenvalue of a Riemannian manifold (Σ^k, h) with non-empty boundary. Then for any conformal immersion $\varphi : (\Sigma, \partial\Sigma) \rightarrow (\mathbb{B}^n, S^{n-1})$ we have*

$$\sigma_1(h)V(\partial\Sigma) \leq k(\sup_{\mathbb{B}^n} V_{\varphi(\Sigma)})^{2/k}V(\Sigma)^{1-\frac{2}{k}} \quad (3.25)$$

In a similar way as Li–Yau’s conformal volume, Fraser and Schoen defined the n^{th} *relative conformal volume* of a conformal manifold as:

$$V_{rc}(\Sigma, [h], n) := \inf_{\varphi} \sup_{\mathbb{B}^n} V_{\varphi(\Sigma)}$$

where the infimum is taken among branched conformal immersions from $(\Sigma, \partial\Sigma)$ into (\mathbb{B}^n, S^{n-1}) . By (3.25), we have:

$$\sigma_1(h)V(\partial\Sigma) \leq kV_{rc}(\Sigma, n)^{2/k}V(\Sigma)^{1-\frac{2}{k}}.$$

In particular when $k = 2$, we have $\sigma_1(h)V(\partial\Sigma) \leq 2V_{rc}(\Sigma, n)$.

Because the visual volume $V_{\partial\Sigma}$ converges to ω_k on $\partial\Sigma$, one direct consequence of Conjecture 3.37 would be the following lower bound on the volume of free boundary minimal surfaces and their boundary. This lower bound was proved by Brendle [Bre12].

Theorem 3.40 (Brendle). *Let Σ^k be a free boundary minimal submanifold of \mathbb{B}^n , then*

$$V(\partial\Sigma) = kV(\Sigma) \geq \omega_{k-1}.$$

The case $k = 2$ of Conjecture 3.37 can be checked easily. A proof using a Willmore type argument was first given by Fraser and Schoen in [FS11]. They later provided a different proof in [FS13] using a monotonicity-type of argument. We will present the second proof as it is closely related to Brendle’s proof of Theorem 3.40.

Before doing so, let us rewrite the conformal factor $\frac{g_p}{g_0}$ in a suitable way. In Lemma 3.14, we computed it in term of space coordinates because we were working with minimal surfaces of \mathbb{H}^n . Now to study minimal surfaces of \mathbb{R}^n , we will write the conformal factor as

$$\left(\frac{g_p}{g_0}\right)^{1/2}(q) = \left(\frac{1}{|p|^2} - 1\right) \frac{1}{|q - p_*|^2}, \quad q \in \mathbb{B}^n \quad (3.26)$$

where p_* is image of p via an inversion with pole at the origin and radius 1. Concretely, we suppose that p lies in the x_1 -axis and of distance $s \in [0, 1)$ to the origin, then p_* is

the point of coordinate s^{-1} on the same axis. We will write $\rho_s = |q - p_*|^2$ and so (3.26) becomes

$$\left(\frac{g_p}{g_0}\right)^{1/2} = \left(\frac{1}{s^2} - 1\right) \rho_s^{-1}.$$

Recall that the square-of-distance functions ρ satisfy $\text{Hess } \rho = 2g_E$ and $|\nabla^\Sigma \rho|^2 \leq |d\rho|^2 = 4\rho$. On a minimal submanifold Σ of dimension k ,

$$\Delta \rho^\alpha = \alpha \rho^{\alpha-2} (2k\rho + (\alpha-1)|\nabla \rho|^2) \leq 0 \quad \forall \alpha \leq 1 - \frac{k}{2} \leq 0 \quad (3.27)$$

This means that $\rho^{1-\frac{k}{2}}$ is super-harmonic on Σ .

When $k = 2$, we can look at the function $\log \rho$ instead:

$$\Delta \log \rho = \rho^{-1} \Delta \rho - \rho^{-2} |\nabla \rho|^2 \geq (2k-4)\rho^{-1} \geq 0 \quad \text{when } k \geq 2 \quad (3.28)$$

Because $\rho_s = \rho_0 - \frac{2x_1}{s} + \frac{1}{s^2}$, $\frac{\partial \rho_0}{\partial n} = 2$ and $\frac{\partial x_1}{\partial n} = x_1$ on S^{n-1} , one has

$$\frac{\partial \rho_s}{\partial n} = \rho_s + 1 - \frac{1}{s^2} \quad \text{on } S^{n-1} \quad (3.29)$$

Proof of Conjecture 3.37 when $k = 2$, cf. [FS13]. Because of (3.26),

$$V(\partial \Sigma, g_p) = \left(\frac{1}{s^2} - 1\right) \int_{\partial \Sigma} \rho_s \text{vol}_0.$$

Since $\log \rho_s$ is super-harmonic by (3.28), one has

$$0 \leq \int_{\Sigma} \Delta \log \rho_s = \int_{\partial \Sigma} \frac{\partial \log \rho_s}{\partial n} = \int_{\partial \Sigma} \left[1 + \left(1 - \frac{1}{s^2}\right) \rho_s^{-1}\right] = V(\partial \Sigma, g_0) - V(\partial \Sigma, g_p)$$

□

Proof of Theorem 3.40. Since Conjecture 3.37 was solved when $k = 2$, we can suppose that $k > 2$. It follows from (3.27) and (3.29) that for $\alpha := 1 - \frac{k}{2}$ and $s \in [0, 1)$:

$$0 \geq \int_{\partial \Sigma} \Delta \rho_s^\alpha = \int_{\partial \Sigma} \alpha \rho_s^{\alpha-1} \frac{\partial \rho_s}{\partial n} = 2\alpha s^{k-1} \int_{\partial \Sigma} (s - x_1)(s^2 + 1 - 2sx_1)^{\alpha-1}$$

This means that $0 \geq \alpha \int_{\partial \Sigma} (2s - 2x_1)(s^2 + 1 - 2sx_1)^{\alpha-1} = \frac{d}{ds} g(s)$ where $g(s) = \int_{\partial \Sigma} (s^2 + 1 - 2sx_1)^\alpha$, and so g is a decreasing function. One has

$$g(0) = V(\partial \Sigma) \geq \lim_{s \rightarrow 1^-} g(s) = \int_{\partial \Sigma} \rho_1^\alpha. \quad (3.30)$$

The function ρ_1^α is smooth on $\overline{\mathbb{B}^n}$ except at the point $p_1 := (1, 0, \dots, 0)$ on the boundary. In the interior of Σ , we still have $\Delta \rho_1^\alpha \leq 0$ by (3.27). Let B_ϵ be the ϵ -ball

centred at p_1 , we have

$$0 \leq \frac{1}{\alpha} \int_{\Sigma \setminus B_\epsilon} \Delta \rho_1^\alpha = \int_{\partial \Sigma \setminus B_\epsilon} \rho_1^{\alpha-1} \frac{\partial \rho_1}{\partial n} + \int_{\Sigma \cap \partial B_\epsilon} \rho_1^{\alpha-1} \frac{\partial \rho_1}{\partial \nu} =: I_1 + I_2,$$

Here ν is the outward unit normal vector of $\Sigma \cap B_\epsilon$ in Σ .

By (3.29), $I_1 = \int_{\partial \Sigma \setminus B_\epsilon} \rho_1^\alpha$ and so is smaller than $V(\partial \Sigma)$. On the other hand, the integrand in I_2 is $\rho_1^{\alpha-1} \frac{\partial \rho_1}{\partial \nu} = -\epsilon^{2(\alpha-1)} \cdot 2\epsilon(1 + O(\epsilon))$ due to boundary condition of Σ . Therefore, one has $I_2 = -\omega^{k-1}(1 + O(\epsilon))$. We obtain the conclusion by sending ϵ to zero. \square

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