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Christopher Cabezas

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Thèse de Doctorat

Mention Mathématiques

présentée à l'École Doctorale en Sciences Technologie et Santé (ED 585)

de l'Université de Picardie Jules Verne

par

Christopher Cabezas

pour obtenir le grade de Docteur de l'Université de Picardie Jules Verne

On the study of homomorphisms between \mathbb{Z}^d -topological dynamical systems

Soutenue le 13 septembre 2022, après avis des rapporteurs, devant le jury d'examen:

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Contents

Introduction	7
Résumé	19
1 Definitions and background	31
1.1 Basics on discrete, convex and fractal geometry	32
1.1.1 Discrete geometry	32
1.1.2 Convex geometry	33
1.1.3 Fractal Geometry	35
1.2 Topological dynamical systems	36
1.2.1 Basic definitions	36
1.2.2 Homomorphisms between topological dynamical systems	37
1.3 Measure-preserving systems	41
1.4 Symbolic Dynamics	42
1.5 Nondeterministic directions of a topological dynamical system	44
1.6 Odometer systems	45
1.7 Multidimensional constant-shape substitutions	48
2 The symmetry semigroup of \mathbb{Z}^2-odometers	55
2.1 The universal odometer	55
2.2 Bifurcation phenomenon of the normalizer semigroup for constant base \mathbb{Z}^2 - odometer systems	56
2.3 The diagonal case	58
2.4 The triangular case	59
2.5 Proof of Theorem 2.2	61
3 The recognizability property of constant-shape substitutions	67
3.1 The recognizability property of aperiodic symbolic factors of substitutive subshifts	68
3.2 Invariant orbits of substitutive subshifts	71
3.3 The repetitivity function of substitutive subshifts	72

3.4	Substitutive subshifts as extension of d -dimensional odometers	74
3.4.1	Substitutive subshifts as a finite-to-1 extension of a d -dimensional odometer by the recognizability property	75
3.4.2	The maximal equicontinuous factor of a substitutive subshift	79
3.5	Aperiodic symbolic factors of substitutive subshifts are conjugate to substitutive subshifts	82
4	Measurable morphisms between substitutive subshifts	87
4.1	Measurable factors implies continuous ones for substitutive subshifts	88
4.2	Applications of rigidity results on endomorphisms of substitutive subshifts	95
4.2.1	Coalescence of substitutive subshifts	95
4.2.2	The automorphism group of substitutive subshifts.	96
4.3	Rigidity properties for homomorphisms between substitutive subshifts and applications	96
5	Precisions on bijective constant-shape substitutions	101
5.1	The automorphism group of substitutive subshifts from bijective constant-shape substitutions	102
5.2	Nondeterministic directions of substitutive subshifts of bijective on extremities constant-shape substitutions	103
5.3	The polytope case	107
5.4	Dynamical properties of substitutive subshifts of polytope substitutions	110
6	Some examples of constant-shape substitutions	119
6.1	The table substitution	119
6.2	The half-hex substitution	122
	Perspectives	127

Introduction

The central objects of study of this thesis are *homomorphisms* between topological \mathbb{Z}^d -actions $T : X \times \mathbb{Z}^d \rightarrow X$ on a compact metric space X . A homomorphism is a continuous surjective map $\phi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ such that for some \mathbb{Z}^d -automorphism $M \in GL(d, \mathbb{Z})$, we have $\phi \circ T^n = T^{Mn} \circ \phi$ for all $n \in \mathbb{Z}^d$. This notion extends the classical dynamical one of morphism like *factor*, when M is the identity and *conjugacy* when ϕ is an invertible factor. Invertible homomorphisms, which will be called *isomorphisms*, are then conjugacies of \mathbb{Z}^d -actions, up to a $GL(d, \mathbb{Z})$ -transformation.

For \mathbb{Z} -actions, isomorphisms are nothing else than *flip-conjugacies*, i.e., homeomorphisms ϕ such that $\phi \circ T$ is equal to $T \circ \phi$ or $T^{-1} \circ \phi$. Factors and conjugacies are referred to as *endomorphisms* and *automorphisms*, respectively when the dynamical systems are the same. While endomorphisms represent a kind of internal symmetries in the system, such as permutations, homomorphisms represent furthermore symmetries of the orbits, such as rotations and reflections. That is why they are also sometimes called *extended symmetries* (see for example [6, 10, 20, 21]). From an algebraic point of view, while the automorphism group is the centralizer of the action group $\langle T \rangle$ in the group $\text{Homeo}(X)$ of self-homeomorphisms in X , the isomorphism group is the normalizer of the action group $\langle T \rangle$ in $\text{Homeo}(X)$, that is, the set of self-homeomorphisms ϕ such that $\phi \langle T \rangle \phi^{-1} = \langle T \rangle$.

The study of homomorphisms of a dynamical system (X, T, \mathbb{Z}^d) is a classical problem. The elements in the group $\langle T \rangle$ generated by the action are automorphisms of the system, hence the automorphism group is always nonempty. However the existence of homomorphisms for a particular matrix $M \in GL(d, \mathbb{Z})$ is generally an open problem. Classical questions concern their dynamical and algebraic properties in relation with the dynamical ones of (X, T, \mathbb{Z}^d) . The description for these isomorphisms of their possible subgroups, their quotients, their amenability or their action on the T -invariant measures depending on the properties of (X, T, \mathbb{Z}^d) are natural problems. In a general context these questions are widely open. Another classical question is the determination of topological factors of a particular system. Their explicit description can be used to unravel the structure of the system. For certain aspects, they carry relevant information and enable to do concrete calculations or to study some specific structures (for example in spectral theory [37]).

The study of isomorphisms of the particular class of minimal systems where X is a

Cantor set is motivated by algebraic reasons. Indeed, to any \mathbb{Z} -minimal Cantor system (X, T, \mathbb{Z}) there is an associated countable group called the *topological full group* and denoted by $[[T]]$. The elements of this group are self-homeomorphisms of the space X that are locally power of the transformation T . Homomorphisms of the system (X, T, \mathbb{Z}) induce group homomorphisms in $[[T]]$. Conversely, the topological full group is a complete invariant of flip-conjugacy [59]. It appears such full groups $[[T]]$ present remarkable properties. H. Matui proved in [87] that its commutator subgroup is *simple*. Moreover, he also showed that this group is *finitely generated* if and only if (X, T, \mathbb{Z}) is conjugated to a minimal subshift. Thus, these groups enable to construct finitely generated simple ones with unexpected algebraic properties. For instance any topological full group is *amenable* [72], giving the first examples of finitely generated simple groups in this class. Using isomorphisms of some *low complexity* \mathbb{Z} -subshifts (namely linearly recurrent ones with arbitrarily long palindromic words like the substitutive Fibonacci subshift) V. Nekrashevych in [92] succeed to construct with full groups a finitely generated simple group *with intermediate growth*, i.e., which is neither of polynomial nor of exponential growth. Moreover it is of *Burnside type*, that is, an infinite group where each element is periodic.

Another motivation for the study of isomorphisms comes from theoretical physics. The discovery in 1984 by D. Schechtman et.al. [105] of a metal alloy structure similar to ideal crystal ones deeply influenced the study of multidimensional aperiodic structure. This alloy presented a discrete diffraction pattern, like for crystals, but had a five-fold rotational symmetry which is forbidden for ideal crystals. The term *quasicrystal* was then invented to describe these new classes of crystals with “forbidden” symmetry, although there is little agreement on the precise definition of a quasicrystal. Roughly speaking, a quasicrystal is a solid, which exhibits sharp bright spots (called *Bragg peaks*) in their X-ray diffraction pattern but has an aperiodic structure (usually manifested by the presence of a non-quasicrystallographic symmetry). The presence of Bragg peaks indicates the presence of “long-range order” in the structure (see [83]). This work earned D. Schechtman the Wolf Prize in Physics in 1999 and the Nobel Prize in Chemistry in 2011. One can assume that (at least approximately) a quasicrystal consists of atoms located at the vertices of an almost periodic tiling. It is possible to recover quantitative physical properties by studying a dynamical system associated with the tiling. Such systems were first introduced by D. Rudolph in [102]. Since then, a series of articles are devoted to their study (see [8, 55, 100, 106] for an extensive bibliography on this subject). The diffraction pattern is then essentially the point spectrum of the corresponding translation action [83]. Moreover its symmetries are reflected by the isomorphisms of its dynamical system [98]. In 1982, A. Mackay [85] published the diffraction pattern of a tiling created by R. Penrose several years before [95], which has very similarities with the ones discovered by D. Schechtman.

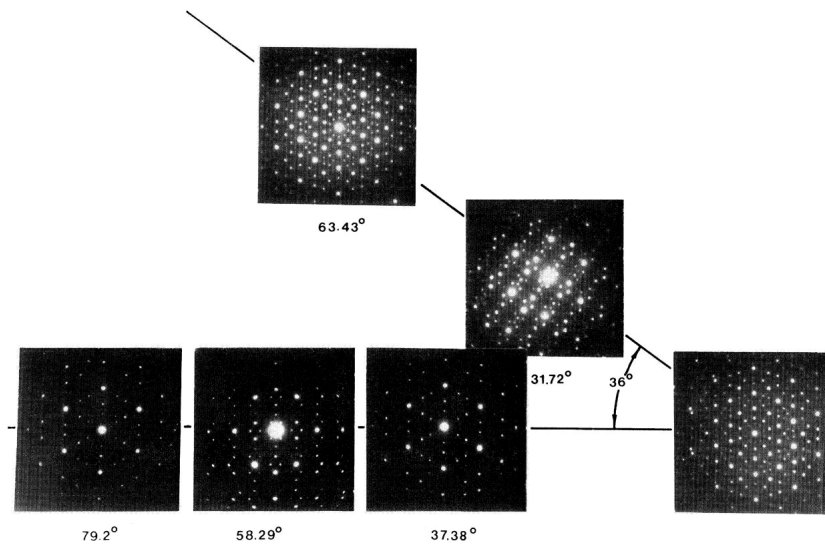


Figure 1: The quasicrystal diffraction images appearing in the original article of D. Shechtman et.al. [105]

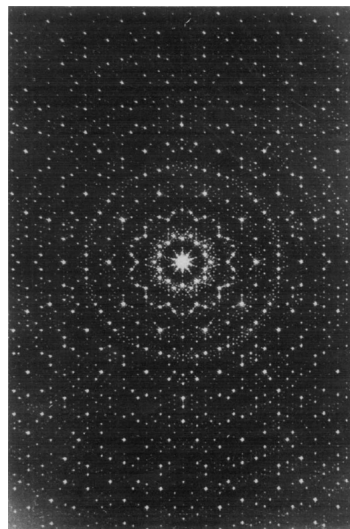


Figure 2: The diffraction pattern of the Penrose tiling as it appears in the original article of A. Mackay [85]

The *Penrose tiling* is then a good mathematical model of quasicrystals. It is build with only 2 tiles, up to rotations and translations. The generation of patterns is obtained by means of an algorithmic method so-called *substitution*. Roughly speaking, this process

consists in substituting tiles by a union of tiles and applying the same rule to this new pattern. Such construction provides, at the limit, most of the simplest aperiodic tilings, in the sense that they have the lowest complexity [80].

Another interesting property of the Penrose tiling, meaningful in the crystallographic context of short range interaction, is that all the allowed tilings of the associated system are the ones verifying a finite set of *local rules*. This is a geometrical analogue of one-dimensional *subshifts of finite type*. However as a difference with one-dimensional subshifts of finite type that always contains periodic points, Penrose tiling system is aperiodic. This highlights a fundamental difference between the one and two dimensional combinatorial properties which are linked with logic and computability. The seminal work of H. Wang [112] already established relation between decidability of certain first-order logic formulas and domino problems. His student R. Berger [12] showed the undecidability of the domino problem by exhibiting a (huge) family of tiles with adjancies rules, called *Wang tiles*, allowing only aperiodic tilings. Later, this example was simplified by R. Robinson [101] with 20 Wang tiles (5 up to rotation and reflection). More recently by E. Jeandel and M. Rao [71] provided a similar example with only 11 Wang tiles, and showed it is the optimal bound. A powerful method to generate aperiodic Wang tiles is through the use of so-called *constant-shape substitution*, where the shape of the image of tiles are the same. The result of S. Mozes [90] illustrates this procedure giving sufficient conditions for a (constant-shape) substitutive subshift to be a factor of a subshift of finite type. Let us also mention that both examples of R. Robinson and Jeandel-Rao contain minimal systems that are substitutive subshifts [56, 79].

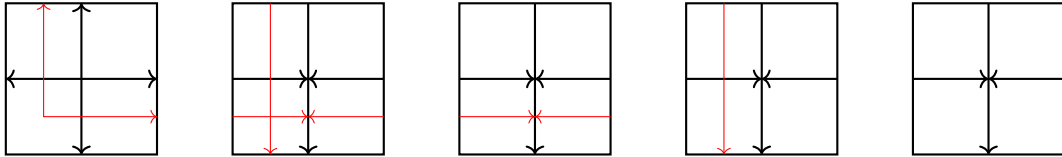


Figure 3: The five basic Robinson tiles (up to rotation and reflection).

Motivated by all these reasons, in this thesis we will focus on the study of homomorphisms between some specific multidimensional subshifts: the substitutive ones, generated by multidimensional constant-shape substitutions. Let us recall some historical results, first in the one-dimensional context. For \mathbb{Z} -actions, isomorphisms are closely related with automorphisms. They are the same, or the isomorphism group is an index-2 group extension of the automorphism group. So most of the properties of isomorphisms can be deduced by these ones. Automorphisms of symbolic systems present rigidity properties already in the one-dimensional case. For instance, the famous Curtis-Hedlund-Lyndon theorem [65], ensures that any factor between subshifts is a *sliding block code* or a *cellular automata*. Actually, homomorphisms are also induced by local maps, but the center changed according to the matrix of the homomorphism (Theorem 1.9). This shows both the automorphism

group and the isomorphism group of a subshift are countable and discrete subgroups on the group of self-homeomorphisms of the phase space.

The automorphism group of symbolic systems was initially studied for subshifts of finite type by G. Hedlund in [65]. This group is infinitely generated and is very large. It contains all the finite ones, free groups, the direct sum of a countable number of copies of \mathbb{Z} , any countable collection of finite groups, etc. In particular it is not an amenable group. However, it is residually finite, so does not contain divisible groups (like \mathbb{Q}) or the infinite symmetric group [18, 74]. Nevertheless, there is still no general description of the automorphism group for a given subshift nor of their generators. For example, whether the automorphism groups of the two-letter full-shift and the three-letter full-shift are algebraically isomorphic is still an open problem.

Large complexity is not enough to have a large automorphism group. In [19, 40] the authors gave a family of Toeplitz subshifts with arbitrarily large positive entropy with trivial automorphism group. Also, the size of the automorphism group imposes no restrictions on the entropy as shown in [40]. A large class of infinite finitely generated abelian groups can be realized as the automorphism group of arbitrarily large or zero entropy Toeplitz subshift.

At the opposite, low complexity of the subshift restricts algebraic properties of the automorphism group. In [31, 33, 39] it was proved the automorphism group is virtually \mathbb{Z} for minimal subshifts with *non super-linear complexity*, i.e., such that $\liminf_{n \rightarrow \infty} p_X(n)/n < \infty$ where $p_X(n)$ denote the number of words of length n . This hypothesis implies the subshift has finitely many *asymptotic pairs*, i.e., two different points $x, y \in X$ with a common past. The strategy in [39] is based under the property that automorphisms permutes asymptotic pairs and non super-linear complexity subshifts have a finite number of asymptotic pairs. For higher complexity subshifts, the growth rate of the automorphism group is bounded by the complexity of the subshift, In particular it is amenable for a large class of zero entropy subshift as proved in [35, 36]. Beyond quantitative properties, other algebraic restrictions do exist for zero-entropy subshifts. For example, in [32] was provided the first examples of countable groups that cannot embed (as the Baumslag-Solitar groups $BS(1, n)$) into the automorphism group of any zero-entropy subshift. But there still some countable groups, such as the discrete Heisenberg group, that it is not known whether they can embed into the automorphism group of a one-dimensional subshift.

Let us recall some examples of subshifts having a non trivial isomorphism, i.e., that is not an automorphism (see [10]): The full-shift, any palindromic subshift, such as, the *sturmian* shifts, the *period doubling shift* and the *Thue-Morse* shift. The first one has a huge automorphism group, not even amenable, whereas the automorphism group of the second and third ones are trivial. The automorphism group of the last one is isomorphic to $\mathbb{Z} \oplus C_2$. These examples suggest that algebraic properties of the automorphism group does not imply the existence of non trivial isomorphisms.

Few general results are known in the multidimensional context: In [67] M. Hochman

proved that most of the one-dimensional properties of the automorphism group are preserved for the class of subshifts of finite type with positive entropy. Nevertheless, he presented a remarkable example: A subshift of finite type with automorphism group isomorphic to $\mathbb{Z}^2 \oplus G$, where G is locally finite and factors onto a virtually simple group. This is in contrast to the one-dimensional setting, where the group has to be residually finite. Isomorphisms of the chair tiling has been studied in [10]. It appears its automorphism group is trivial and its isomorphism group is a semi-direct product of \mathbb{Z}^2 by the symmetry group of the square.

The study of automorphisms is also a classical subject in the context of ergodic theory. In the measure-theoretic framework we consider (X, μ, T, \mathbb{Z}) , where (X, \mathcal{F}, μ) is a standard probability space, T represent the action generated by a measure-preserving transformation and automorphism are defined almost everywhere and preserved the measure μ . Let us recall some important results (we refer the reader to [52] for an overview of this theme). D. Ornstein [93] proved that a mixing rank one dynamical system has a trivial (measurable) automorphism group. Later, A. del Junco [38] showed that the example given by Chacon [25], also has this property. Then, J. King and J.-P. Thouvenot [76] proved that for mixing systems of finite rank, its measurable automorphism group is virtually \mathbb{Z} .

In the class of substitutive subshifts, homomorphisms have presented stronger rigidity properties than the ones mentioned above. Let us make a brief description of substitutions. These are combinatorial objects which produce infinite sequences by an iteration process. Their deep understanding took several decades. We refer [53, 97] for extensive bibliographies on the earlier developments of the subject. Using the shift as the action on these infinite sequences, we obtain the substitutive subshifts, which are the simplest nontrivial zero-entropy symbolic systems. They were introduced by W.H. Gottschalk in [60]. Their simplicity makes them appear in many different fields of mathematics, such as, combinatorics on words (see [13]), number theory (especially in transcendental number theory [1]), numeration systems (see [26]), diophantine approximations (see [2]), and computer science (especially automata theory [3]).

B. Host and F. Parreau in [69] gave a complete description of factors between subshifts arising from certain constant-length substitutions like for instance the Thue-Morse substitution defined by $0 \mapsto 01, 1 \mapsto 10$. They proved that any measurable factor induces a continuous one, and the automorphism group is isomorphic to a direct product of \mathbb{Z} with a finite group. Moreover, any finite group can be realized as a quotient group $\text{Aut}(X, S, \mathbb{Z}) / \langle S \rangle$ for these subshifts as proved by M. Lemańczyk and M. K. Mentzen in [82]. Later, I. Fagnot [51] proved that the problem of whether there exists a factor map between two constant-length substitutions subshifts is *decidable*, using the first-order logic framework of Presburger arithmetic. Some years later, F. Durand in [45] showed that *linearly recurrent subshifts* (in particular substitutive subshifts) have finitely many subshift factors, up to conjugacy. Also in [45] it was proved that topological Cantor factors of substitutive subshifts are either substitutive subshifts or odometer systems. V. Salo

and I. Törmä provided in [103] a renormalization process of the factor maps to extend the description obtained in [69]. Next, C. Müllner and R. Yassawi [91] demonstrated that any aperiodic symbolic factor of a constant-length substitutive subshift is conjugated via a letter-to-letter map to a constant-length substitutive subshift. More recently, F. Durand and J. Leroy [50] showed the decidability of the existence problem of a factor map between two minimal substitutive subshifts.

Presentation of main results

In this thesis, we study homomorphisms between multidimensional substitutive subshifts generated by constant-shape substitutions. In our context, $L \in \mathcal{M}(d, \mathbb{Z})$ is an integer *expansion* matrix, i.e. $\|L\| > 1$ and $\|L^{-1}\| < 1$. A constant-shape substitution ζ is a map $\mathcal{A} \rightarrow \mathcal{A}^F$, where \mathcal{A} is a finite alphabet and F is a fundamental domain of $L(\mathbb{Z}^d)$ in \mathbb{Z}^d . The set F is called the *support* of the substitution. Constant-shape substitutions are a multidimensional analogue of one-dimensional constant-length substitutions. Here the “length” of the substitution is represented by the expansion matrix L . For every $n > 0$, any iteration ζ^n of the substitutions can also be obtained by a constant-shape substitution, with expansion matrix L^n and support F_n . At the differences with the one-dimensional case, these substitutions may not be linearly recurrent (Example 3.6), and can have topological Cantor factors that are neither expansive nor equicontinuous (Example 4.3). Some known results of the one-dimensional case are still preserved for these constant-shape substitutions, such as, they are finite extensions of a specific odometer system given by the data of the substitution (Lemma 3.9) and their maximal equicontinuous factors have a similar structure to the one-dimensional case (Proposition 3.15).

In [21] the authors studied isomorphisms, called extended symmetries, for a class of constant shape substitutions, called *bijective block substitutions*. A constant-shape substitution is *bijective* if for any index $\mathbf{f} \in F$ we have $|\{\zeta(a)_{\mathbf{f}} : a \in \mathcal{A}\}| = |\mathcal{A}|$. Block substitutions are constant-shape substitutions with diagonal expansion matrix and a parallelepiped support. The authors proved in [21] that the set of matrices which define isomorphisms is a finite group. Although it is not proved in the article, it can be deduced that the group of isomorphisms is virtually generated by the shift action.

We extend the study of homomorphisms, by describing the isomorphism group for general constant-shape substitutions. As a difference with [21], our strategy is by describing the set of *nondeterministic directions* (Theorem 5.2). The matrices defining isomorphisms preserve the set of nondeterministic directions (Proposition 1.12). A vector $\mathbf{v} \in \mathbb{S}^{d-1}$ is said to be nondeterministic for the subshift (X, S, \mathbb{Z}^d) if there is two different points $x \neq y \in X$ such that are equal in the half-space $H_{\mathbf{v}} = \{\mathbf{t} \in \mathbb{R}^d : \langle \mathbf{t}, \mathbf{v} \rangle < 0\}$. This is a multidimensional analogue of asymptotic pairs [39]. Nondeterministic directions were introduced in [17] with the notion of *nonexpansive subspaces* to study sub-actions of a given \mathbb{Z}^d -action, for $d > 1$. This notion appears to be meaningful in symbolic dynamics: As an example let us mention

[34] in which these objects were used to prove a weak version of Nivat’s conjecture.

We give a description of the set of nondeterministic directions for bijective constant-shape substitutions.

Theorem A (Theorem 5.2). Let ζ be an aperiodic bijective primitive constant-shape substitution. The set of nondeterministic directions of its substitutive subshift is the intersection of \mathbb{S}^{d-1} with a nonempty union of limits of opposite normal cones of faces of the convex hull support of ζ^n , for integers $n > 0$.

This theorem gives topological constraints on the set of nondeterministic directions for bijective substitutions. Under geometrical conditions of the support we get stronger properties about nondeterministic directions. A bijective constant-shape substitution is *polytope* when the convex hull of the compact set defined as the limit of $L^{-n}(F_n)$ (called *digit tile*), with respect to the Hausdorff metric (see Section 1.7), is a polytope. In this case the set of nondeterministic directions is much more restricted: it is a finite union of closed balls (eventually degenerated). For instance, in the two-dimensional case it cannot be a Cantor set (Corollary 5.3). In contrast with the result of M. Boyle and D. Lind in [17] and M. Hochman in [68], where they proved any compact set of \mathbb{S}^1 can be realized as the set of nonexpansive directions of a subshift. The work in this thesis gives the first descriptions of the set of nondeterministic directions for minimal \mathbb{Z}^d -actions.

When the rank of nondeterministic directions is maximal, thanks to the former description, we get the following constraints on the homomorphisms of substitutive subshifts.

Theorem B (Proposition 5.15 and Theorem 5.17). Let ζ be an aperiodic bijective primitive polytope substitution. If the substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ has d linearly independent nondeterministic directions, then:

1. Any homomorphism on the substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ is invertible.
2. The group of isomorphisms is virtually generated by the shift action.

Block substitutions are particular cases of polytope substitutions and it can be easily to check it satisfies the hypothesis on the rank of the set of nondeterministic directions, so Theorem B generalizes the ones in [21]. Actually we can relax the notion of bijectivity by the one called *bijective on extremities* (see Chapter 5) and *reducibility* (see Chapter 4) and is enough to keep the same conclusion. This hypothesis of reducibility is almost optimal because we provide in Chapter 6 an example of a constant-shape substitutive subshift with an infinite set of matrices defining isomorphisms.

The hypothesis on the rank of the set of nondeterministic directions is weak since we didn’t know a substitution that does not satisfy it. We provide an algorithm to check if this hypothesis is satisfied (Lemma 5.12). Furthermore, given the result in [63], the hypothesis is true for a generic family of two-dimensional bijective constant-shape substitutions. In a

private communication, P. Guillon [62] mentioned this result was already proved for higher dimensions, but nowhere published.

To get Theorem B, we need some control on the block maps defining the isomorphisms. For this, we follow the strategy of B. Host and F. Parreau in [69]. Moreover we also get a strong rigidity property when the matrices commute with the expansion matrix of the substitution. Recall that substitutive subshifts are uniquely ergodic, so any continuous endomorphism induces a measurable one. We provide a partial converse:

Theorem C (Theorem 4.1, simplified version). Let $(X_\zeta, S, \mathbb{Z}^d)$ be a subshift generated by an aperiodic, primitive reduced constant-shape substitution. For every measurable endomorphism ϕ , there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $S^{\mathbf{j}}\phi$ is equal to a continuous endomorphism ψ , satisfying the following two properties:

1. The endomorphism ψ has a bounded radius given by the substitution.
2. There exist an integer $n > 0$ and $\mathbf{p} \in \mathbb{Z}^d$ such that, $S^{\mathbf{p}}\psi\zeta_1^n = \zeta_2^n\psi$.

Theorem C implies that, under reducibility, any measurable endomorphism induced a continuous one. In fact, we prove the set of measurable endomorphisms is countable. Theorem C is a multidimensional analogue of the one proved by B. Host and F. Parreau in [69].

Counterexamples of Theorem C are provided by substitutive subshifts that are metrically isomorphic to their maximal equicontinuous factors. This occurs when a substitution has a combinatorial condition called *coincidence* [97] (see Example 4.3). The set of measurable endomorphisms of odometer systems is then uncountable and any element of the odometer system represent a measurable endomorphism via addition. So, as in the original article [69], reducibility is an optimal hypothesis for Theorem C.

We then get some dynamical consequence of Theorem C. Under the reducibility condition, the substitutive subshifts are coalescent (Proposition 4.7), i.e., any endomorphism on the substitutive subshift is invertible. This was already known for linearly recurrent subshifts, first in the one-dimensional case in [45], then in higher dimensions in [29].

The chair tiling (see [10]) and the half-hex substitutive subshifts, studied in Chapter 6, are examples of substitutive subshifts having a coincidence. Hence their measurable endomorphisms form an uncountable set. Nevertheless both examples are coalescent and their automorphism groups are virtually generated by the shift action. Outside the reducibility condition, we do not know whether all aperiodic substitutive subshifts satisfy these last properties.

The half-hex substitutive subshift, mentioned before, does not satisfy neither the hypothesis of Theorem C nor Theorem B. Nevertheless we are able to describe its maximal equicontinuous factor and characterize its isomorphisms. The *symmetry group* of a subshift is the set of matrices $M \in GL(d, \mathbb{Z})$ defining an isomorphism. Thanks to this example we get the following result.

Theorem D (Theorem 6.3). There exists a minimal aperiodic subshift (in fact a substitutive one) with an infinite symmetry group.

More precisely, the isomorphism group of the half-hex substitutive subshift is isomorphic to a semidirect product between \mathbb{Z}^2 and $GL(2, \mathbb{Z})$, so its symmetry group is the largest possible. Subshifts with infinite symmetry groups have been found before, as in [7], studying their relation with topological entropy. But these examples are far from being minimal.

Finally, concerning the factors of substitutive subshifts we have the following characterization.

Theorem E (Theorem 3.22). Let (Y, S, \mathbb{Z}^d) be an aperiodic symbolic factor of a subshift generated by an aperiodic primitive constant-shape substitution ζ . Then, there exists an aperiodic primitive constant-shape substitution ζ' , with the same structure of a power of ζ , generating a system $(X_{\zeta'}, S, \mathbb{Z}^d)$ conjugated to the symbolic factor (Y, S, \mathbb{Z}^d) .

This is a multidimensional analogue of a result proved by C. Müllner and R. Yassawi [91] for the one-dimensional case, which is a refinement of a result proved in [48]. This result leaves open what can be said about other topological Cantor factors of substitutive subshifts. Example 4.3 gives a substitutive subshift with a Cantor topological factor that is neither expansive nor equicontinuous. Also Example 4.3 has a symbolic factor with a non-trivial period and an infinite phase space. This is in contrast with the one-dimensional dichotomy proved in [45].

Theorem A, Theorem B, Theorem C and Theorem E are published in [22].

Organization of this thesis

This thesis is organized as follows. The basic definitions and background to be used throughout this thesis are introduced in Chapter 1. We recall some classical notions of topological dynamical systems, ergodic theory and symbolic dynamics. We develop the relation between homomorphisms and topological factors of dynamical systems. Then, we present the example of odometer systems and Toeplitz subshifts. We obtain a characterization to define a homomorphism between odometers. We finish this chapter with a brief survey of multidimensional constant-shape substitutions, which is where we mainly study the homomorphisms.

In Chapter 2, we study the symmetry semigroup of two-dimensional constant-base odometer systems given by a matrix L . In this case, we get a description of a bifurcation phenomenon at the level of the symmetry semigroup with respect to arithmetical relations of invariants of the matrix L . The main theorem (Theorem 2.2) shows in most cases the symmetry semigroup is the centralizer of the matrix L . This will help to get a characterization of the isomorphism semigroup of aperiodic primitive constant-shape substitutions using the relation between homomorphisms and their maximal equicontinuous factors (Lemma 1.7).

The main result in Chapter 3 is the characterization about aperiodic symbolic factors of substitutive subshifts given by an aperiodic, primitive constant-shape substitution. They are conjugate to substitutive subshifts generated by aperiodic primitive constant-shape substitutions (Theorem E). Substitutive subshifts are not necessarily linearly repetitive (Example 3.6). Nevertheless, we prove a polynomial growth on the repetitivity function for constant-shape substitutions (Lemma 3.7).

Chapter 4 is devoted to prove rigidity properties about measurable factors and homomorphisms between substitutive subshifts (Theorem C). Then, we deduce that these substitutive subshifts are coalescent (Proposition 4.7) and their automorphism group is virtually generated by the shift action (Proposition 4.8).

In Chapter 5 we describe the isomorphism group for general constant-shape substitutions. We prove it is virtually generated by the shift action (Theorem B). To do this, we relate the symmetry group with different types of supports of the substitution and non-diagonal expansion matrices, via the nondeterministic directions. We characterize the nondeterministic directions through the digit tile for some weaker version of bijective substitutions (Theorem A). Moreover, these directions are computable in terms of the combinatorics of the substitution (Corollary 5.13).

Finally, in Chapter 6 we characterize the isomorphism group for two examples of constant-shape substitutions. The first one, called table substitution, satisfied the hypothesis of the results in Chapter 4 and Chapter 5. The second example does not satisfy the hypothesis for the previous results. Nevertheless, a description of its isomorphism group is provided by Theorem D.

Résumé

Cette thèse traite des *homomorphismes* entre des \mathbb{Z}^d -actions topologiques $T : X \times \mathbb{Z}^d \rightarrow X$ sur des espaces métriques compacts X . Un homomorphisme est une surjection continue $\phi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ telle que pour un \mathbb{Z}^d -automorphisme $M \in GL(d, \mathbb{Z})$, on ait $\phi \circ T^n = T^{Mn} \circ \phi$ pour tout $n \in \mathbb{Z}^d$. Cette notion étend le concept de *facteur*, lorsque M est l'identité et de *conjugaison* lorsque ϕ est un facteur inversible. Les *isomorphismes* (homomorphismes inversibles) sont alors des conjugaisons de \mathbb{Z}^d -actions, à une transformation de $GL(d, \mathbb{Z})$ près.

Pour les \mathbb{Z} -actions, les isomorphismes sont des *flip-conjugaisons*, c'est-à-dire des homéomorphismes ϕ tels que $\phi \circ T$ est égal à $T \circ \phi$ ou $T^{-1} \circ \phi$. Les facteurs et les conjugaisons sont appelés respectivement *endomorphismes* et *automorphismes* lorsque les systèmes dynamiques sont identiques. Alors que les endomorphismes représentent une sorte de symétrie interne du système, comme les permutations, les homomorphismes représentent en plus des symétries des orbites, comme les rotations et les réflexions. C'est pourquoi ils sont aussi parfois appelés *symétries étendues* (voir par exemple [6, 10, 20, 21]). D'un point de vue algébrique, alors que le groupe d'automorphisme est le centralisateur du groupe d'action $\langle T \rangle$ dans le groupe $\text{Homeo}(X)$ des homéomorphismes de X , le groupe d'isomorphisme est le normalisateur du groupe d'action $\langle T \rangle$ dans $\text{Homeo}(X)$, c'est-à-dire l'ensemble des homéomorphismes ϕ tels que $\phi \langle T \rangle \phi^{-1} = \langle T \rangle$.

L'étude des homomorphismes d'un système dynamique (X, T, \mathbb{Z}^d) est un problème classique. Les éléments du groupe $\langle T \rangle$ engendré par l'action sont des automorphismes du système. Le groupe d'automorphisme est donc toujours non-vide. Cependant, l'existence d'homomorphismes pour une matrice particulière $M \in GL(d, \mathbb{Z})$ est généralement un problème ouvert. Les questions classiques concernent leurs propriétés dynamiques et algébriques en relation avec la dynamique de (X, T, \mathbb{Z}^d) . La description pour ces isomorphismes de leurs sous-groupes, de leurs quotients, de leur moyennabilité ou de leur action sur les mesures invariantes de l'action $\langle T \rangle$ en fonction des propriétés de (X, T, \mathbb{Z}^d) sont des problèmes naturels. Dans un contexte général, ces questions sont largement ouvertes. Une autre question classique est la détermination des facteurs topologiques d'un système particulier. Leur description explicite peut être utilisée pour décrire la structure du système. Par certains aspects, ils contiennent des informations pertinentes et permettent de faire

des calculs concrets ou d'étudier certaines structures spécifiques (par exemple en théorie spectrale [37]).

L'étude des isomorphismes de la classe particulière des systèmes minimaux où X est un ensemble de Cantor est motivée par des raisons algébriques. En effet, à tout système minimal de Cantor (X, T, \mathbb{Z}) il existe un groupe dénombrable associé appelé *groupe topologique plein* et noté $[[T]]$. Les éléments de ce groupe sont des homéomorphismes de l'espace X qui sont localement des puissances de la transformation T . Les homomorphismes du système (X, T, \mathbb{Z}) induisent des homomorphismes de groupe dans $[[T]]$. Réciproquement, le groupe topologique plein est un invariant complet de flip-conjugaison [59]. Il apparaît que de tels groupes pleins $[[T]]$ présentent des propriétés remarquables. H. Matui a prouvé dans [87] que son sous-groupe dérivé est *simple*. De plus, il a également montré que ce groupe est *finiment engendré* si et seulement si (X, T, \mathbb{Z}) est conjugué à un sous-shift minimal. Ainsi, ces groupes permettent de construire des groupes simples finiment engendrés avec des propriétés algébriques inattendues. Par exemple, tout groupe topologique plein est *moyennable* [72], ce qui donne les premiers exemples de groupes simples finiment engendrés dans cette classe. En utilisant les isomorphismes de certains sous-shifts de *complexité faible* (à savoir les sous-shifts linéairement récurrentes avec des mots palindromiques arbitrairement longs comme le sous-shift substitutive de Fibonacci), V. Nekrashevych dans [92] réussit à construire avec les groupes pleins un groupe simple finiment engendré *avec une croissance intermédiaire*, c'est-à-dire qui n'est ni de croissance polynomiale ni de croissance exponentielle. De plus il est de *type Burnside*, c'est-à-dire un groupe infini où chaque élément est périodique.

Une autre motivation pour l'étude des isomorphismes provient de la physique théorique. La découverte en 1984 par D. Schechtman et al. [105] d'une structure d'alliage métallique similaire à celle des cristaux idéaux a profondément influencé l'étude de la structure aperiodique multidimensionnelle. Cet alliage présentait un diagramme de diffraction discret, comme les cristaux, mais avait une symétrie rotationnelle d'ordre 5, ce qui est interdit pour les cristaux idéaux. Le terme *quasi-cristal* a alors été inventé pour décrire ces nouvelles classes de cristaux avec des symétries "interdites", s'il n'y a pas consensus sur la définition précise d'un quasi-cristal. En substance, un quasi-cristal est un solide qui présente des points brillants nets (appelés *pics de Bragg*) dans son diagramme de diffraction aux rayons X, mais qui a une structure aperiodique généralement manifestée par la présence d'une symétrie non-quasi-cristallogique. La présence de pics de Bragg indique la présence d'un "ordre à longue portée" dans la structure (voir [83]). Ces travaux ont valu à D. Schechtman le prix Wolf de physique en 1999 et le prix Nobel de chimie en 2011. On peut supposer que (au moins approximativement) un quasi-cristal est constitué d'atomes situés aux sommets d'un pavage presque périodique. Il est possible de retrouver des propriétés physiques quantitatives en étudiant un système dynamique associé au pavage. De tels systèmes ont été présentés pour la première fois par D. Rudolph dans [102]. Dès lors, une série d'articles ont été consacrés à leur étude (voir [8, 55, 100, 106] pour une vaste bibliographie sur ce sujet).

Le diagramme de diffraction est alors essentiellement le spectre ponctuel de l'action de translation correspondante [83]. De plus, ses symétries sont reflétées par les isomorphismes de son système dynamique [98]. En 1982, A. Mackay [85] a publié le schéma de diffraction d'un pavage créé par R. Penrose quelques années plus tôt [95], qui présente de grandes similitudes avec ceux découverts par D. Schechtman (voir figure 1 et figure 2).

Le *pavage de Penrose* est alors un bon modèle mathématique de quasi-cristaux. Il est construit avec seulement 2 tuiles, aux rotations et translations près. La génération des motifs est obtenue par une méthode algorithmique appelé *substitution*. Cette procédure consiste à substituer des tuiles par une union de tuiles et appliquer la même règle à ce nouveau motif. En itérant ce processus ad infinitum, cette construction fournit la plupart des pavages apériodiques les plus simples, dans le sens où ils ont la plus faible complexité possible [80].

Une autre propriété intéressante du pavage de Penrose, est que tous les pavages autorisés du système associé sont ceux qui vérifient un ensemble fini de *règles locales*. C'est l'analogue géométrique des *sous-shifts de type fini* unidimensionnels. Cependant, le système de pavage de Penrose est apériodique, à l'opposé des sous-shifts de type fini unidimensionnelles qui contiennent toujours des points périodiques. Cela met en évidence une différence fondamentale entre les propriétés combinatoires unidimensionnelles et bidimensionnelles qui sont liées à la logique et à la calculabilité. Le travail fondateur de H. Wang [112] a déjà établi une relation entre la décidabilité de certaines formules de la logique du premier ordre et les problèmes de domino. Son étudiant R. Berger [12] a montré l'indécidabilité du problème de domino en exposant une énorme famille de tuiles avec des règles d'adjacence, appelées *tuiles de Wang*, ne permettant que des pavages apériodiques. Plus tard, cet exemple a été simplifié par R. Robinson [101] avec 20 tuiles de Wang (5 à rotation près). Plus récemment, E. Jeandel et M. Rao [71] ont fourni un exemple similaire avec seulement 11 tuiles de Wang et ont montré que c'est la borne optimale. Une méthode puissante pour générer des tuiles de Wang apériodiques est l'utilisation de ce que l'on appelle *substitution de forme constante*, où la forme de l'image des tuiles est la même. Le résultat de S. Mozes [90] illustre cette procédure en donnant des conditions suffisantes pour qu'un sous-shift substitutif (de forme constante) soit un facteur d'un sous-shift de type fini. Nous mentionnons également que les deux exemples de R. Robinson et Jeandel-Rao contiennent des sous-shifts substitutifs minimaux [56, 79].

Motivés par toutes ces raisons, nous nous concentrerons dans cette thèse sur l'étude des homomorphismes entre certains sous-shifts spécifiques: les substitutifs, générés par des substitutions multidimensionnelles de forme constante. Rappelons quelques résultats historiques dans le contexte unidimensionnel. Pour les \mathbb{Z} -actions, les isomorphismes sont étroitement liés aux automorphismes. Le groupe d'isomorphisme est une extension de groupe d'indice au plus 2 du groupe d'automorphisme. Ainsi, la plupart des propriétés des isomorphismes peuvent être déduites de celles-ci. Les automorphismes des systèmes symboliques présentent des propriétés de rigidité déjà dans le cas unidimensionnel. Par

exemple, le célèbre théorème de Curtis-Hedlund-Lyndon [65] assure que tout facteur entre sous-shifts est un *fonction de bloc glissant* ou un *automate cellulaire*. En fait, les homomorphismes sont aussi induits par des fonctions locales, mais le centre change selon la matrice de l'homomorphisme (théorème 1.9). Cela montre que le groupe d'automorphisme et le groupe d'isomorphisme d'un sous-shift sont des sous-groupes dénombrables, discrets dans le groupe des homéomorphismes de l'espace des phases.

Le groupe d'automorphisme des systèmes symboliques a été initialement étudié pour les sous-shifts de type fini par G. Hedlund dans [65]. Ce groupe est infiniment engendré et contient de nombreux sous-groupes: n'importe quel groupe fini, les groupes libres, la somme directe d'un nombre dénombrable de copies de \mathbb{Z} , toute collection dénombrable de groupes finis, etc. Cependant, il est résiduellement fini, et ne contient donc pas de groupes divisibles (comme \mathbb{Q}) ou le groupe symétrique infini [18, 74]. Néanmoins, il n'existe pas de description générale du groupe d'automorphisme ni de leurs générateurs pour un sous-shift donné. Par exemple, la question de savoir si les groupes d'automorphisme du full-shifts sur 2 et 3 lettres sont algébriquement isomorphes est un problème encore ouvert.

Une grande complexité n'est pas suffisante pour avoir un grand groupe d'automorphisme. Dans [19, 40], les auteurs ont donné une famille de sous-shifts Toeplitz d'entropie positive arbitrairement grande présentant un groupe d'automorphisme trivial. De plus, la taille du groupe d'automorphisme n'impose aucune restriction sur l'entropie, comme prouvé dans [40]. Une grande classe de groupes abéliens infinis finiment engendrés peut être réalisée comme le groupe d'automorphisme d'un sous-shift Toeplitz d'entropie arbitrairement grande ou nulle.

À l'inverse, la complexité faible du sous-shift restreint les propriétés algébriques du groupe d'automorphisme. Il est prouvé dans [31, 33, 39] que le groupe d'automorphisme est virtuellement \mathbb{Z} pour les sous-shifts minimaux de *complexité non superlinéaire*, c'est-à-dire tels que $\liminf_{n \rightarrow \infty} p_X(n)/n < \infty$ où $p_X(n)$ correspond au nombre de mots de longueur n . Deux suites différents $x, y \in X$ sont dites *asymptotiques* lorsqu'elles ont le même passé. La stratégie de [39] est basée sur la propriété que les automorphismes permutent les *paires asymptotiques* et les sous-shifts de complexité non super-linéaire ont un nombre fini de paires asymptotiques. Pour les sous-shifts de complexité plus élevée, le taux de croissance du groupe d'automorphisme est limité par la complexité du sous-shift. En particulier, il est moyennable pour une grande classe de sous-shifts à entropie nulle, comme prouvé dans [35, 36]. Au-delà des propriétés quantitatives, d'autres restrictions algébriques existent pour les sous-shifts à entropie nulle. Par exemple, dans [32] a été fourni les premiers exemples de groupes dénombrables qui ne peuvent pas être des sous-groupes d'automorphisme (comme les groupes de Baumslag-Solitar $BS(1, n)$) d'un sous-shift d'entropie nulle. Mais il existe encore quelques groupes dénombrables, comme le groupe discret de Heisenberg, dont on ne sait pas s'il peut être un sous-groupe du groupe d'automorphisme d'un sous-shift unidimensionnel.

Rappelons quelques exemples de sous-shifts ayant un isomorphisme non trivial, c'est-à-

dire qui n'est pas un automorphisme (voir [10]): le full-shift, tout sous-shift palindromique, comme les shifts *sturmien*, le *period-doubling sous-shift* et le sous-shift de *Thue-Morse*. Le premier a un énorme groupe d'automorphisme, même pas moyennable, alors que les groupes d'automorphisme du deuxième et du troisième sont triviaux. Le groupe d'automorphisme du dernier est isomorphe à $\mathbb{Z} \oplus C_2$. Ces exemples suggèrent que les propriétés algébriques du groupe d'automorphisme n'impliquent pas l'existence d'isomorphismes non triviaux.

Il y a peu de résultat généraux existents dans le contexte multidimensionnel. Dans [67] M. Hochman a prouvé que la plupart des propriétés unidimensionnelles des groupes d'automorphisme sont préservées pour la classe des sous-shifts de type fini d'entropie positive. Cependant, il a présenté un exemple remarquable d'un sous-shift de type fini dont le groupe d'automorphisme est isomorphe à $\mathbb{Z}^2 \oplus G$, où G est localement fini et se factorise sur un groupe virtuellement simple. Ceci diffère du cas unidimensionnel où le groupe doit être résiduellement fini. Les isomorphismes du pavage de la chaise ont été étudiés dans [10] où il a été démontré que le groupe d'automorphisme est trivial et le groupe d'isomorphisme est un produit semi-direct de \mathbb{Z}^2 pour le groupe de symétries du carré.

L'étude des homomorphismes est également un sujet classique dans le contexte de la théorie ergodique. Dans le cadre de la théorie de la mesure, nous considérons les systèmes dynamiques (X, μ, T, \mathbb{Z}) , où (X, \mathcal{F}, μ) est un espace de probabilité standard, T est l'action générée par une transformation préservant la mesure μ et les automorphismes sont définis presque partout et préservent la mesure μ . Rappelons quelques résultats importants (nous renvoyons le lecteur à [52] pour un aperçu de ce thème). D. Ornstein [93] a prouvé qu'un système dynamique mélangeant de rang un possède un groupe d'automorphisme mesurable trivial. Plus tard, A. del Junco [38] a montré que l'exemple donné par Chacon [25], possède également cette propriété. Ensuite, J. King et J.-P. Thouvenot [76] ont prouvé que le groupe d'automorphisme mesurable est virtuellement \mathbb{Z} pour les systèmes mélangeant de rang fini.

Dans la famille des sous-shifts substitutifs, les homomorphismes présentent des propriétés de rigidité plus fortes que celles mentionnées ci-dessus. Faisons une brève description des substitutions. Ce sont des objets combinatoires qui produisent des suites infinies par un processus d'itération. Leur compréhension profonde a pris plusieurs décennies. Nous renvoyons à [97, 53] pour des bibliographies complètes sur le sujet. En utilisant le shift comme action sur ces suites infinies, nous obtenons les sous-shifts substitutifs, qui sont les plus simples systèmes symboliques non triviaux d'entropie nulle. Ils ont été introduits par W.H. Gottschalk dans [60]. Leur simplicité les fait apparaître dans de nombreux domaines des mathématiques, tels la combinatoire des mots (voir [13]), la théorie des nombres (en particulier dans la théorie des nombres transcendants [1]), les systèmes de numération (voir [26]), les approximations diophantiennes (voir [2]), et l'informatique (en particulier la théorie des automates [3]).

B. Host et F. Parreau dans [69] ont donné une description complète des facteurs entre les sous-shifts résultant de certaines substitutions de longueur constante, comme par exemple la substitution de Thue-Morse définie par $0 \mapsto 01, 1 \mapsto 10$. Ils ont prouvé que tout facteur

mesurable induit un facteur continu, et que le groupe d'automorphisme est isomorphe à un produit direct de \mathbb{Z} avec un groupe fini. De plus, tout groupe fini peut être réalisé comme un groupe quotient $\text{Aut}(X, S, \mathbb{Z}) / \langle S \rangle$ pour ces sous-shifts comme l'ont prouvé M. Lemańczyk et M. K. Mentzen dans [82]. Plus tard, I. Fagnot [51] a prouvé que le problème de savoir s'il existe un facteur entre deux sous-shifts substitutifs de longueur constante est *décidable*, en utilisant le cadre de la logique du premier ordre de l'arithmétique de Presburger. Quelques années plus tard, F. Durand dans [45] a montré que *les sous-shifts linéairement récurrents* (en particulier les sous-shifts substitutifs) ont un nombre fini de facteurs symboliques, à conjugaison près. De plus, dans [45], il a été prouvé que les facteurs de Cantor topologiques des sous-shifts substitutifs sont soit des sous-shifts substitutifs, soit des odomètres. V. Salo et I. Törmä ont fourni dans [103] un processus de renormalisation des facteurs étendant la description obtenue dans [69]. Ensuite, C. Müllner et R. Yassawi [91] ont démontré que tout facteur symbolique apériodique d'un sous-shift substitutif de longueur constante est conjugué par une fonction lettre à lettre à un sous-shift substitutif de longueur constante. Plus récemment, F. Durand et J. Leroy [50] ont montré la décidabilité du problème d'existence d'un facteur entre deux sous-shifts substitutifs minimaux.

Présentation des principaux résultats

Dans cette thèse, nous étudions les homomorphismes entre les sous-shifts substitutifs multidimensionnels générés par les substitutions à forme constante. Dans notre contexte, $L \in \mathcal{M}(d, \mathbb{Z})$ est une matrice d'*expansion* entière, c'est-à-dire $\|L\| > 1$ et $\|L^{-1}\| < 1$. Une substitution de forme constante est une fonction $\zeta : \mathcal{A} \rightarrow \mathcal{A}^F$, où \mathcal{A} est un alphabet fini et F est un domaine fondamental de $L(\mathbb{Z}^d)$ dans \mathbb{Z}^d . L'ensemble F est appelé le *support* de la substitution. Les substitutions de forme constante sont les analogues multidimensionnels des substitutions de longueur constante unidimensionnel. Ici, la "longueur" de la substitution est représentée par la matrice d'expansion L . Pour chaque $n > 0$, toute itération ζ^n des substitutions peut également être obtenue par une substitution de forme constante de une matrice d'expansion L^n et de support F_n . À la différence du cas unidimensionnel, ces substitutions peuvent ne pas être linéairement récurrentes (exemple 3.6) et peuvent avoir des facteurs de Cantor non expansifs ni équicontinus (exemple 4.3). Des résultats connus du cas unidimensionnel sont encore préservés pour ces substitutions de forme constante: elles sont des extensions finies d'un odomètre spécifique donné par la substitution (lemme 3.9) et leurs facteurs équicontinus maximaux ont une structure similaire au cas unidimensionnel (proposition 3.15).

Dans [21], les auteurs ont étudié les isomorphismes, appelés symétries étendues, pour une classe de substitutions de forme constante, appelée *substitutions par blocs bijectifs*. Une substitution de forme constante est *bijective* si tout indice $\mathbf{f} \in F$ vérifie $|\{\zeta(a)_{\mathbf{f}} : a \in \mathcal{A}\}| = |\mathcal{A}|$. Les substitutions de blocs sont des substitutions de forme constante avec une matrice d'expansion diagonale et un support parallélépipédique. Les auteurs ont prouvé dans [21]

que l'ensemble des matrices qui définissent les isomorphismes est un groupe fini. Bien que cela ne soit pas prouvé dans l'article, on peut en déduire que le groupe des isomorphismes est virtuellement engendré par l'action du shift.

Nous étendons l'étude des homomorphismes en décrivant le groupe d'isomorphisme pour les substitutions générales de forme constante. À la différence de [21], notre stratégie consiste à décrire l'ensemble des *directions non déterministes* (théorème 5.2). Les matrices définissant les isomorphismes préservent l'ensemble des directions non déterministes (proposition 1.12). Un vecteur $\mathbf{v} \in \mathbb{S}^{d-1}$ est dit nondéterministe pour le sous-shift (X, S, \mathbb{Z}^d) s'il existe deux points différents $x \neq y \in X$ qui sont égaux dans le demi-espace $H_{\mathbf{v}} = \{\mathbf{t} \in \mathbb{R}^d : \langle \mathbf{t}, \mathbf{v} \rangle < 0\}$. C'est un analogue multidimensionnel des paires asymptotiques [39]. Les directions non déterministes ont été introduites dans [17] avec la notion de *sous-espace non expansif* pour étudier les sous-actions d'une action \mathbb{Z}^d donnée, pour $d > 1$. Cette notion est important en dynamique symbolique. À titre d'exemple, mentionnons [34] où ces objets ont été utilisés pour prouver une version faible de la conjecture de Nivat.

Nous donnons une description de l'ensemble des directions non déterministes pour les substitutions bijectives de forme constante.

Théorème A (théorème 5.2). Soit ζ une substitution aperiodique bijective primitive de forme constante. L'ensemble des directions non déterministes de son sous-shift substitutif est l'intersection de \mathbb{S}^{d-1} avec une union non vide des limites des cônes normaux des faces de l'enveloppe convexe du support de ζ^n , pour les entiers $n > 0$.

Ce théorème donne des contraintes topologiques sur l'ensemble des directions non déterministes pour les substitutions bijectives. Sous des conditions géométriques du support, nous obtenons des propriétés plus fortes sur les directions non déterministes. Une substitution bijective de forme constante est *polytope* si l'enveloppe convexe de l'ensemble limite de $L^{-n}(F_n)$ (appelé *digit tile*, voir la section 1.7) est un polytope. Dans ce cas, l'ensemble des directions non déterministes est beaucoup plus restreint: c'est une union finie de boules fermées (éventuellement dégénérées). Par exemple, dans le cas bidimensionnel, il ne peut pas être un ensemble de Cantor (corollaire 5.3). Ceci diffère du le résultat prouvé par M. Boyle et D. Lind [17] et M. Hochman [68]. Ils assurent que tout ensemble compact de \mathbb{S}^1 peut être réalisé comme l'ensemble des directions non expansifs d'un sous-shift. Le travail de cette thèse donne les premiers description de l'ensemble des directions non déterministes pour des \mathbb{Z}^d -actions minimaux.

Lorsque le rang des directions non déterministes est maximal, grâce à la description précédente, nous obtenons les contraintes suivantes sur les homomorphismes des sous-shifts substitutifs.

Théorème B (proposition 5.15 et théorème 5.17). Soit ζ une substitution polytope primitive bijective aperiodique. Si le sous-shift substitutif $(X_\zeta, S, \mathbb{Z}^d)$ a d directions non déterministes linéairement indépendantes, alors:

1. tout homomorphisme du sous-shift substitutif $(X_\zeta, S, \mathbb{Z}^d)$ est inversible.
2. Le groupe d'isomorphisme est virtuellement engendré par l'action du shift.

Les substitutions de blocs sont des cas particulier des substitutions polytopes et il est simple de vérifier qu'elles vérifient l'hypothèse sur le rang de l'ensemble des directions non déterministes. Ainsi le théorème B généralise les résultats de [21]. En fait, nous pouvons affaiblir la notion de bijectivité par les notions de *bijectivité sur les extrémités* (voir le chapitre 5) et *réductibilité* (voir le chapitre 4). Cela suffit pour garder la même conclusion. Cette hypothèse de réductibilité est presque optimale car nous fournissons dans l'exemple 6 un exemple de sous-shift substitutif de forme constante dont l'ensemble des matrices définissant les isomorphismes est infini.

L'hypothèse sur le rang de l'ensemble des directions non déterministes est faible puisque nous ne connaissons pas de substitution qui ne la satisfasse pas. Nous fournissons un algorithme pour vérifier si cette hypothèse est satisfaite (lemme 5.12). De plus, par le résultat dans [63], cette hypothèse est vérifiée pour une famille générique de substitutions bijectives de forme constante bidimensionnelles. Dans une communication privée, P. Guillon [62] a mentionné que ce résultat étant également pour les dimensions supérieures. Malheureusement la preuve n'a jamais été publié.

Pour obtenir le théorème B, nous avons besoin d'un certain contrôle sur les fonctions de blocs définissant les isomorphismes. Pour cela, nous suivons la stratégie de B. Host et F. Parreau dans [69]. Nous obtenons également une propriété remarquable de rigidité lorsque les matrices commuent avec la matrice d'expansion de la substitution. Nous rappelons que les sous-shifts substitutifs sont uniquement ergodiques, donc tout endomorphisme continu induit un endomorphisme mesurable. Nous fournissons une réciproque partielle.

Théorème C (théorème 4.1, version simplifiée). Soit $(X_\zeta, S, \mathbb{Z}^d)$ un sous-shift généré par une substitution apériodique primitive de forme constante. Pour tout endomorphisme mesurable ϕ , il existe $\mathbf{j} \in \mathbb{Z}^d$ tel que $S^{\mathbf{j}}\phi$ est égal à un endomorphisme continu ψ , satisfaisant les deux propriétés suivantes:

1. l'endomorphisme ψ a un rayon borné par la substitution.
2. Il existe des entiers $n > 0$ et $\mathbf{p} \in \mathbb{Z}^d$ tels que $S^{\mathbf{p}}\psi\zeta_1^n = \zeta_2^n\psi$.

Le théorème C implique que pour une substitution réduite tout endomorphisme mesurable induit un endomorphisme continu. Ainsi l'ensemble des endomorphismes mesurés est un ensemble dénombrable. Le théorème C est un analogue multidimensionnel de celui prouvé par B. Host et F. Parreau dans [69].

Des contre-exemples au théorème C sont fournis par des sous-shifts substitutifs qui sont métriquement isomorphes à leur facteur équicontinu maximal. Cela se produit lorsqu'une substitution possède une condition combinatoire appelée *coïncidence* [97] (voir l'exemple

4.3). L'ensemble des endomorphismes mesurés des odomètres est alors indénombrable et tout élément de l'odomètre représente un endomorphisme mesurable par addition. Ainsi, comme dans l'article original [69], la réductibilité est une hypothèse optimale pour le théorème C.

Nous obtenons ensuite quelques conséquences dynamiques du théorème C. Sous la condition de réductibilité, les sous-shifts substitutifs sont coalescents (proposition 4.7), c'est-à-dire que tout endomorphisme du sous-shift est inversible. Ceci était déjà connu pour les sous-shifts linéairement récurrents, d'abord pour le cas unidimensionnel dans [45], puis par les dimensions supérieures dans [29].

Le pavage de la chaise (voir [10]) et les sous-shifts substitutifs du demi-hexagone, étudiés dans le chapitre 6, sont des exemples de sous-shifts substitutifs avec coïncidence. Par conséquent, leurs endomorphismes mesurés forment un ensemble indénombrable. Néanmoins, les deux exemples sont coalescents et leurs groupes d'automorphisme sont virtuellement engendrés par l'action de shift. Sans la condition de réductibilité, nous ne savons pas si tous les sous-shifts substitutifs aperiodiques satisfont ces dernières propriétés.

Le sous-shift substitutif du demi-hexagone, mentionné précédemment, ne satisfait pas les hypothèses des théorèmes C et B. Néanmoins, nous sommes capables de décrire son facteur équicontinu maximal et de caractériser ses isomorphismes. Le *groupe de symétrie* d'un sous-shift est l'ensemble des matrices $M \in GL(d, \mathbb{Z})$ définissant un isomorphisme. Grâce à cet exemple, nous obtenons le résultat suivant.

Théorème D (théorème 6.3). Il existe un sous-shift aperiodique minimal (en fait un sous-shift substitutif) avec un groupe de symétrie infini.

Plus précisément, le groupe d'isomorphisme du sous-shift substitutif du demi-hexagone est isomorphe au produit semidirect de \mathbb{Z}^2 avec $GL(2, \mathbb{Z})$. Son groupe de symétrie est donc le plus grand possible. Des sous-shifts avec des groupes de symétrie infini ont été trouvés auparavant, comme dans [7], en étudiant leurs relations avec l'entropie topologique. Mais ces exemples sont loin d'être minimaux.

Enfin, concernant les facteurs de sous-shifts substitutifs, nous avons la caractérisation suivante.

Théorème E (théorème 3.22). Soit (Y, S, \mathbb{Z}^d) un facteur symbolique aperiodique d'un sous-shift $(X_\zeta, S, \mathbb{Z}^d)$ généré par une substitution ζ aperiodique primitive de forme constante. Alors, il existe une substitution primitive aperiodique de forme constante ζ' , de la même structure qu'une puissance de ζ , dont le système $(X_{\zeta'}, S, \mathbb{Z}^d)$ est conjugué au facteur symbolique (Y, S, \mathbb{Z}^d) .

Ceci est un analogue multidimensionnel d'un résultat prouvé par C. Müllner et R. Yassawi [91] pour le cas unidimensionnel, qui est un raffinement d'un résultat prouvé dans [48]. En revanche, ce résultat ne dit rien sur les autres facteurs topologiques sur un Cantor des sous-shifts substitutifs. L'exemple 4.3 donne un exemple de sous-shift substitutif avec

un facteur topologique de Cantor qui n'est ni expansif ni équicontinu. De même, l'exemple 4.3 possède un facteur symbolique avec une période non triviale et une espace des phases infini. Ceci est en contraste avec la dichotomie unidimensionnelle prouvée dans [45].

Les théorèmes A, B, C et E sont publiés dans [22].

Organisation de cette thèse

Cette thèse est organisée comme suit. Les définitions de base et le contexte qui seront utilisés tout au long de cette thèse sont introduits dans le chapitre 1. Nous rappelons quelques notions classiques de systèmes dynamiques topologiques, de théorie ergodique et de dynamique symbolique. Nous développons la relation entre les homomorphismes et les facteurs topologiques des systèmes dynamiques. Ensuite, nous présentons l'exemple des odomètres et des sous-shifts de Toeplitz. Nous obtenons une caractérisation pour des homomorphismes entre odomètres. Nous terminons ce chapitre par un bref survol des substitutions multidimensionnelles de forme constante, dont nous étudions principalement les homomorphismes.

Dans le chapitre 2, nous étudions le semigroupe de symétrie des odomètres bidimensionnels de base constante donnée par une matrice L . Dans ce cas, nous obtenons une description d'un phénomène de bifurcation au niveau du semigroupe de symétrie par rapport aux relations arithmétiques des invariants de la matrice L . Le théorème principal (théorème 2.2) montre que dans la plupart des cas le semigroupe de symétrie est le centralisateur de la matrice L . Cela aidera à obtenir une caractérisation du semigroupe normalisateur des substitutions aperiodiques primitives de forme constante en utilisant la relation entre les homomorphismes et leurs facteurs équicontinus maximaux (lemme 1.7).

Le résultat principal du chapitre 3 est la caractérisation des facteurs symboliques aperiodiques des sous-shifts substitutifs donnés par une substitution aperiodique primitive de forme constante. Ils sont conjugués aux sous-shifts substitutifs générés par des substitutions aperiodiques primitives de forme constante (théorème E). Les sous-shifts substitutifs ne sont pas nécessairement linéairement répétitifs (exemple 3.6). Néanmoins, nous donnons une croissance polynomiale sur la fonction de répétitivité pour les substitutions de forme constante (lemme 3.7).

Le chapitre 4 est consacré à la preuve de propriétés de rigidité des facteurs mesurés et les homomorphismes entre sous-shifts substitutifs (théorème C). Ensuite, nous déduisons que ces sous-shifts substitutifs sont coalescentes (proposition 4.7) et que leurs groupes d'automorphisme est virtuellement engendrés par l'action de shift (proposition 4.8).

Dans le chapitre 5, nous décrivons le normalisateur des substitutions générales de forme constante. Nous prouvons que le normalisateur est virtuellement engendré par l'action de shift (théorème B). Pour ce faire, nous relierons le normalisateur à différents types de supports des substitutions, via les directions non-déterministes. Nous caractérisons les directions non-déterministes par le digit tile pour une version affaiblie de substitutions

bijectives (théorème A). De plus, ces directions sont calculables en termes de combinatoire de la substitution (théorème 5.13).

Enfin, dans le chapitre 6, nous caractérisons le groupe normalisateur pour deux exemples de substitutions de forme constante. Le premier, appelé substitution de la table, satisfait les hypothèses des résultats des chapitres 4 et 5. Le pavage du demi-hexagone ne satisfait pas les hypothèses des résultats précédents. Néanmoins, une description de son groupe d'isomorphisme est fournie par le théorème D.

Chapter 1

Definitions and background

In this chapter, we fix some notations, definitions and show some general properties to be used throughout this thesis. We start with some notions on discrete, convex and fractal geometry. Then we recall some classical notions of topological dynamical systems and the ones of homomorphisms between them. In particular, we will see some relations between homomorphisms and topological factors of dynamical systems (Lemma 1.6). We also recall some basics of ergodic theory, symbolic dynamics, and nondeterministic directions (also called nonexpansive half-spaces).

In Section 1.2 we define the central objects of study of this thesis, which are homomorphisms between topological \mathbb{Z}^d -actions $T : X \times \mathbb{Z}^d \rightarrow X$ on a compact metric space X . This types of morphisms have been studied in both one and higher dimensions actions. For \mathbb{Z} -actions, the isomorphism group (the group of invertible homomorphisms) of a topological dynamical system is either the automorphism group $\text{Aut}(X, T, \mathbb{Z}^d)$, or an index-2 extension of $\text{Aut}(X, T, \mathbb{Z}^d)$. Therefore, the study of isomorphisms for \mathbb{Z} -actions focuses on the existence of an isomorphism, which are sometimes called *reversors* (see [6] for a brief guide to the study of these isomorphisms). Since $GL(d, \mathbb{Z})$ is an infinite group for $d > 1$, the relation between automorphisms and isomorphisms becomes less clear. Isomorphisms have been studied for particular subshifts [10] and more recently for a class of substitutive subshifts under strong geometrical and combinatorial restrictions [20, 21]. In this thesis we extend the study of homomorphisms by describing the isomorphism group (called the *normalizer group*) for general constant-shape substitutions that are defined in this chapter.

We then describe the nondeterministic directions for a subshift. This notion was introduced in [63] to the study of two-dimensional subshifts. In [34] they were used to prove a weak version of Nivat's conjecture. This notion is motivated by the work of M. Boyle and D. Lind, about *nonexpansive subspaces*. When the space X is infinite such subspaces always exist [17, Theorem 3.7]. In fact, they can be described only by hyperplanes [17, Theorem 3.6], hence the term of *nondeterministic directions*. In this thesis, nondeterministic directions are characterized for bijective substitutions (Theorem 5.2) and used to describe

the isomorphisms for a big family of substitutive subshifts (Theorem 5.17).

We also present the example of odometer systems and Toeplitz sequences in Section 1.6. Odometer systems are the most natural equicontinuous systems in the study of minimal Cantor systems. In fact, they are the maximal equicontinuous factor for a big family of symbolic systems, such as, some substitutions and Toeplitz sequences. Toeplitz subshifts are symbolic systems that are the orbit closures of the regular quasi-periodic points of the subshift. We refer to [42, 28, 30] for the study of odometer systems for different actions. We also obtain a characterization of homomorphisms between two odometer systems (Lemma 1.14), useful to describe isomorphisms for substitutive subshifts.

We finish this chapter with a brief survey of multidimensional constant-shape substitutions. They represent the simplest nontrivial zero-entropy symbolic systems, since they are generated by finite data. By this fact, ergodic and topological properties of substitution dynamical systems have been extensively studied. They were introduced by W.H. Gottschalk in [60] (see [53, 97] for a good bibliography on this subject). We also provide some good finite sets useful to the structure of the sequences in substitutive subshifts. As a corollary, we get a canonical form of symbolic factors of constant-shape substitutions.

1.1 Basics on discrete, convex and fractal geometry

1.1.1 Discrete geometry

If $F \subseteq \mathbb{Z}^d$ is a finite set, it will be denoted by $F \Subset \mathbb{Z}^d$, and we use the notation $\|F\| = \max_{\mathbf{n} \in F} \|\mathbf{n}\|$, where $\|\cdot\|$ is the standard Euclidean norm of \mathbb{R}^d . The standard Cartesian product in \mathbb{R}^d will be denoted by $\langle \cdot, \cdot \rangle$. If $L \in \mathcal{M}(d, \mathbb{R})$ is a matrix, we denote $\|L\| = \max_{\mathbf{x} \in \mathbb{R} \setminus \{0\}} \|L(\mathbf{x})\| / \|\mathbf{x}\|$ as the *matrix norm of L* . We denote $GL(d, \mathbb{Z})$ as the set of $d \times d$ matrices M with integer coefficients such that $|\det(M)| = 1$.

We will call a sequence of finite sets $(F_n)_{n>0} \subseteq \mathbb{Z}^d$ a *Følner sequence* if for all $\mathbf{n} \in \mathbb{Z}^d$ we have that

$$\lim_{n \rightarrow \infty} \frac{|F_n \Delta (\mathbf{n} + F_n)|}{|F_n|} = 0.^1$$

For any $r > 0$ and $F \Subset \mathbb{Z}^d$ we denote F^{or} as the set of all elements $\mathbf{f} \in F$ such that $\mathbf{f} + (B(\mathbf{0}, r) \cap \mathbb{Z}^d) \subseteq F$, i.e.,

$$F^{or} = \{\mathbf{f} \in F : \mathbf{f} + (B(\mathbf{0}, r) \cap \mathbb{Z}^d) \subseteq F\}.$$

Note that the Følner assumption implies for any $r > 0$

$$\lim_{n \rightarrow \infty} \frac{|F_n^{or}|}{|F_n|} = 1. \tag{1.1}$$

¹In the literature, especially Group Theory, is common to also ask that the union of the sequence of sets $(F_n)_{n>0}$ is equal to \mathbb{Z}^d for a sequence to be Følner, but we will not use it in this thesis.

1.1.2 Convex geometry

In the following we will give some basics on convex geometry that we will be used in the rest of the thesis. We refer to [5] for a survey of results about this field.

A set $C \subseteq \mathbb{R}^d$ is said to be *convex* if for all $\mathbf{x}, \mathbf{y} \in C$ the set $[\mathbf{x}, \mathbf{y}] = \{\mathbf{z} \in \mathbb{R}^d : \mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}, t \in [0, 1]\}$ is included in C . Recall that the image of a convex set under an affine map is also a convex set, and the intersection of an arbitrary family of convex sets is also a convex set. This leads to the notion of the *convex hull* of a set.

If $A \subseteq \mathbb{R}^d$ we define the convex hull of A , denoted by $\text{conv}(A)$, as the intersection of all convex sets containing A .

A set $S \subseteq \mathbb{R}^d$ is an *affine set* if for any $\mathbf{x}, \mathbf{y} \in S$ the line $\{t\mathbf{x} + (1-t)\mathbf{y} : t \in \mathbb{R}\}$ is contained in S . For any set $A \subseteq \mathbb{R}^d$ we define the *affine hull* of A , denoted by $\text{Aff}(A)$, as the intersection of all affine sets containing A .

A fundamental characterization of convex sets is provided by Carathéodory's theorem.

Theorem 1.1 (Carathéodory's theorem). *For any $A \subseteq \mathbb{R}^d$, any element of $\text{conv}(A)$ can be represented as a convex combination of no more than $(d+1)$ elements of A .*

We now recall some basic topological concepts associated with convex sets. We say that a point $\mathbf{x} \in A$ is *relative interior* for A , if A contains the intersection of a ball centered at \mathbf{x} with $\text{Aff}(A)$, i.e., $\exists r > 0, B(\mathbf{x}, r) \cap \text{Aff}(A) \subseteq A$. The set of all relative interior points of A is called its *relative interior*, and is denoted by $\text{ri}(A)$. We can also define the *relative boundary* $\partial_{\text{ri}}(A)$ as the set difference of the closure and the relative interior, i.e., $\partial_{\text{ri}}(A) = \text{cl}(A) \setminus \text{ri}(A)$.

An important notion for convex sets are the supporting hyperplanes. If $C \subseteq \mathbb{R}^d$ is a closed convex set, and $\mathbf{x} \in C$ is a point in the relative boundary of C , An affine hyperplane $\partial H[\mathbf{a}; c] = \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{a}, \mathbf{y} \rangle = c\}$, for some $\mathbf{a} \in \mathbb{R}^d \setminus \{0\}$ and $c \in \mathbb{R}$ is *supporting to C at \mathbf{x}* , if $\mathbf{x} \in \partial H[\mathbf{a}; c]$ and

$$\inf_{\mathbf{y} \in C} \langle \mathbf{a}, \mathbf{y} \rangle < \langle \mathbf{a}, \mathbf{x} \rangle = c = \sup_{\mathbf{y} \in C} \langle \mathbf{a}, \mathbf{y} \rangle.$$

We now recall some basic notions about cones and polyhedral sets. A nonempty set $C \subseteq \mathbb{R}^d$ is said to be a *cone* if for every $\mathbf{x} \in C$, the set C contains the positive ray $\mathbb{R}_+\mathbf{x} = \{t\mathbf{x} : t > 0\}$ spanned by \mathbf{x} . A translation of a cone by a non zero vector is called an *affine convex cone*. A cone $C \subseteq \mathbb{R}^d$ is said to be *finitely generated* if it can be written as

$$C = \left\{ \sum_{i=1}^p t\mathbf{u}_i : \mathbf{u}_i \in \mathbb{R}^d, t_i \geq 0, i = 1, \dots, p \right\}.$$

For a given nonempty set $A \subseteq \mathbb{R}^d$, the smallest cone containing the set A is called the *positive hull* (or *conical hull*) of A . It is the smallest cone containing the set A , and is given by

$$\text{cone } A = \{t\mathbf{x} : \mathbf{x} \in A, t > 0\} \cup \{0\}.$$

The positive hull is also said to be the *cone generated by* A .
 A set $P \subseteq \mathbb{R}^d$ is called polyhedral if it has the form

$$P = \left\{ \mathbf{x} \in \mathbb{R}^d : \langle \mathbf{u}_i, \mathbf{x} \rangle \leq a_i, i = 1, \dots, p \right\},$$

where $\mathbf{u}_i \in \mathbb{R}^d$, $a_i \in \mathbb{R}$. The following is characterization of polyhedral sets.

Theorem 1.2 (Minkowski-Weyl Theorem). *A cone C is polyhedral if and only if it is finitely generated.*

Convex sets can be represented, but it requires the notion of extreme points and extreme rays.

A point \mathbf{x} in a convex set C is called an *extreme point*, if can not be written as the convex combination of two different points in C , i.e., if \mathbf{x} is equal to $t\mathbf{u} + (1-t)\mathbf{v}$ for some $0 \leq t \leq 1$, with $\mathbf{u}, \mathbf{v} \in C$, then $\mathbf{u} = \mathbf{v} = \mathbf{x}$. We denote by $\text{Ext}(C)$ the set of the extreme points of a convex set C . Extreme points are special cases of *faces* of a convex set. A compact convex set is called a *polytope* if it has a finite number of extreme points.

A convex set $\mathbf{F} \subseteq C$ is called a *face* of C if for every $\mathbf{x} \in \mathbf{F}$ and every $\mathbf{y}, \mathbf{z} \in C$ such that $\mathbf{x} = t\mathbf{y} + (1-t)\mathbf{z}$, with $0 < t < 1$, we have that $\mathbf{y}, \mathbf{z} \in \mathbf{F}$. The *dimension* of a face \mathbf{F} of C is the dimension of its affine hull. The 0-dimensional faces of C are exactly the extreme points of C , and the bounded 1-dimensional faces are called *segments* or *edges*. An *extreme ray* of a convex set C is the direction of a half-line that is a face of C . A useful result about representation of closed convex sets in \mathbb{R}^d is the following

Theorem 1.3 (Krein-Milman theorem for unbounded convex sets). *If a nonempty closed convex set $C \subseteq \mathbb{R}^d$ has at least one extreme point, i.e., does not have an affine line. Then C can be written as the sum of the convex hull of its extreme points and the cone generated by its extreme rays.*

Some useful notion for convex sets corresponds to the *normal cone*. Let \mathbf{F} be a nonempty face of a closed convex polytope C . The *opposite normal cone*² $\hat{N}_{\mathbf{F}}(C)$ of C at \mathbf{F} is defined as

$$\hat{N}_{\mathbf{F}}(C) = \left\{ \mathbf{v} \in \mathbb{R}^d : \min_{\mathbf{t} \in C} \langle \mathbf{v}, \mathbf{t} \rangle = \langle \mathbf{v}, \mathbf{p} \rangle, \forall \mathbf{p} \in \mathbf{F} \right\}.$$

The *opposite normal fan* of C is the collection of all opposite normal cones of C :

$$\hat{\mathcal{N}}(C) = \{ \hat{N}_{\mathbf{F}}(C) : \mathbf{F} \text{ is a proper face of } C \}.$$

The following are simple statements on the normal fan

- If \mathbf{F} is a face of C , then $\dim(\hat{N}_{\mathbf{F}}(C)) = d - \dim(\mathbf{F})$.

²The word *opposite* comes from the fact that the usual normal cone is related to the outward normal vectors of convex sets, and in this thesis we will use the inward normal vectors.

- If F is a face of G , which is a face of C , then $\hat{N}_G(C)$ is a face of $\hat{N}_F(C)$.
- The set $\bigcup_{F \text{ face of } C} \hat{N}_F(C)$ is equal to \mathbb{R}^d .

Fig. 1.1 illustrate the opposite normal cones of a triangle.

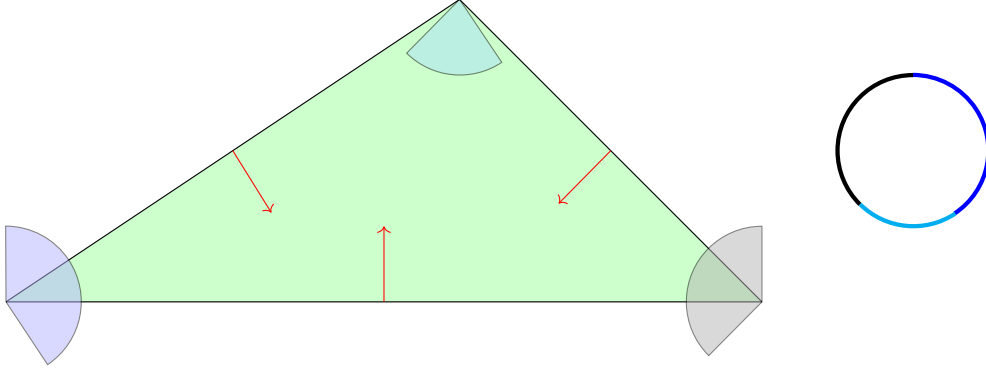


Figure 1.1: Example of the opposite normal cones of a triangle, and the stratification of the circle \mathbb{S}^1 given by them.

1.1.3 Fractal Geometry

In the following we present some definitions and some properties satisfied for some fractals sets which are defined by *iterated function systems* or IFS. We refer to [77, 109, 110] for some results that will be used throughout this thesis.

Let $\mathcal{C}(\mathbb{R}^d)$ denote the collection of all nonempty compact subsets of \mathbb{R}^d . The *Hausdorff metric* \mathcal{H} on $\mathcal{C}(\mathbb{R}^d)$ is defined as follows:

$$\mathcal{H}(A, B) = \inf\{\varepsilon : A \subseteq B_\varepsilon \wedge B \subseteq A_\varepsilon\},$$

where $A_\varepsilon = \{t \in \mathbb{R}^d : \|t - y\| \leq \varepsilon, \text{ for some } y \in A\}$. We have that $(\mathcal{C}(\mathbb{R}^d), \mathcal{H})$ is a complete metric space.

A map $f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is said to be a *contraction* if there exists $0 < c < 1$ such that $\|f(x) - f(y)\| \leq c\|x - y\|$ for all $x, y \in \mathbb{R}^d$. Let $\{f_i\}_{i=1}^N$ be a set of contraction maps on \mathbb{R}^d , and define the map

$$\begin{aligned} F : \mathcal{C}(\mathbb{R}^d) &\rightarrow \mathcal{C}(\mathbb{R}^d) \\ A &\mapsto \bigcup_{i=1}^N f_i(A) \end{aligned}$$

This map is a contraction on $(\mathcal{C}(\mathbb{R}^d), \mathcal{H})$, so by the Banach fixed-point Theorem, there exists a unique set $T \in \mathcal{C}(\mathbb{R}^d)$ (called *digit tile*) such that

$$T = \bigcup_{i=1}^N f_i(T).$$

A way to approximate this set is by iterations

$$T = \lim_{n \rightarrow \infty} F^n(T_0), \quad (1.2)$$

where T_0 is an arbitrary compact set in \mathbb{R}^d and the limit is with respect to the Hausdorff metric.

Since the convex hull of a compact set in \mathbb{R}^d is compact, the map $\text{conv} : \mathcal{C}(\mathbb{R}^d) \rightarrow \mathcal{C}(\mathbb{R}^d)$ which gives for any set $A \in \mathcal{C}(\mathbb{R}^d)$ its convex hull is well defined, and is well known to be continuous.

1.2 Topological dynamical systems

In this section, we will present the basic definitions and some properties of topological dynamical systems. We also define the central object of study of this thesis, which are homomorphisms between topological dynamical systems, and present some basic results about them. We finish this section with a survey on some results about the compatibility of homomorphisms between topological factors of topological dynamical systems. We mention [4] for an extensive bibliography of this area.

1.2.1 Basic definitions

A *topological dynamical system* is a triple (X, T, G) , where (X, ρ) is a compact metric space, G is a group of homeomorphisms of the space X into itself, and $T : X \times G \rightarrow X$ is a continuous map, satisfying $T(x, e) = x$, and $T(T(x, g), h) = T(x, gh)$ for all $x \in X$, and $g, h \in G$. We will denote T^g to the homeomorphism $T(\cdot, g)$.

If (X, ρ) is a compact metric space, we denote $\text{Homeo}(X)$ the group of all homeomorphisms from X to itself, and if $T \in \text{Homeo}(X)$, we use (X, T, \mathbb{Z}) to denote the topological dynamical system $(X, T, \{T^n : n \in \mathbb{Z}\})$. Similarly, if T_1, \dots, T_d are d commuting homeomorphisms on X , we denote (X, T, \mathbb{Z}^d) to denote the topological dynamical system $(X, T, \langle \{T_1, \dots, T_d\} \rangle)$.

For a point $x \in X$, we define its *orbit* as the set $\mathcal{O}(x, G) = \{T^g(x) : g \in G\}$. If $A \subseteq X$, we say that A is *G-invariant* if for all $x \in A$, $\mathcal{O}(x, G)$ is included in A .

If (X, T, G) is a topological dynamical system, a subset $K \subseteq X$ is called a *minimal set* if K is closed, nonempty, G -invariant, and has no proper closed nonempty invariant subsets, i.e., if $N \subseteq K$ is closed and G -invariant, then $N = \emptyset$ or $N = K$. In this case, we say that $(K, T|_K, G)$ is a *minimal system*, where $T|_K : K \times G \rightarrow K$ corresponds to the restriction of T in K . It is easy to see that a system is minimal if and only if it is the closure orbit of all of its points.

An important type of topological dynamical systems are the so-called equicontinuous systems. A topological dynamical system (X, T, \mathbb{Z}^d) is said to be *equicontinuous* if the set of maps $\{T^n : \mathbf{n} \in \mathbb{Z}^d\}$ forms an equicontinuous family of homeomorphisms. The

equicontinuous systems are, in some sense, the simplest dynamical systems. In fact, there exists a complete characterization of them [4].

1.2.2 Homomorphisms between topological dynamical systems

In the following, we define the homomorphisms between topological dynamical systems, which are the central object of study on this thesis. Homomorphisms represent internal symmetries of a given topological dynamical system, such as rotations and reflections. Invertible homomorphisms, which will be called *isomorphisms*, are then conjugacies of \mathbb{Z}^d -actions, up to a $GL(d, \mathbb{Z})$ -transformation. We refer to [6] for a brief guide to homomorphisms both for one-dimensional systems and higher-dimensional dynamical systems.

Notation and basic properties

Definition 1.4. Let (X, T, \mathbb{Z}^d) , (Y, T, \mathbb{Z}^d) be two topological dynamical systems and $M \in GL(d, \mathbb{Z})$. A *homomorphism associated with M* is a surjective continuous map $\phi : X \rightarrow Y$ such that for all $\mathbf{n} \in \mathbb{Z}^d$ we have that $\phi \circ T^{\mathbf{n}} = T^{M\mathbf{n}} \circ \phi$. If ϕ is invertible, then ϕ is an *isomorphism*.

In the following we fix the different notations that we will use throughout this thesis:

- We denote the set of all homomorphisms associated with M between (X, T, \mathbb{Z}^d) and (Y, T, \mathbb{Z}^d) by $\text{Hom}_M(X, Y, T, \mathbb{Z}^d)$.
- The *set of homomorphisms* between two dynamical systems, is defined as the collection of all of homomorphisms, i.e.,

$$\text{Hom}(X, Y, T, \mathbb{Z}^d) = \bigcup_{M \in GL(d, \mathbb{Z})} \text{Hom}_M(X, Y, T, \mathbb{Z}^d).$$

- In the special case M is the identity matrix, homomorphisms are called *factors* and we denote $\text{Fac}(X, Y, T, \mathbb{Z}^d)$ the collection of all factors between (X, T, \mathbb{Z}^d) and (Y, T, \mathbb{Z}^d) . If a factor is invertible, then it is called a *conjugacy*.
- In the case $(X, T, \mathbb{Z}^d) = (Y, T, \mathbb{Z}^d)$, we simply denote these sets as $N_M(X, T, \mathbb{Z}^d)$ and $N(X, T, \mathbb{Z}^d)$ and we call the last one, the *normalizer semigroup of (X, T, \mathbb{Z}^d)* . A factor map is called an *endomorphism*, and a conjugacy is called an *automorphism*. We denote the set of all endomorphisms and automorphisms of a topological dynamical system as $\text{End}(X, T, \mathbb{Z}^d)$ and $\text{Aut}(X, T, \mathbb{Z}^d)$, respectively.
- We define the *symmetry semigroup* $\vec{N}(X, T, \mathbb{Z}^d)$ of (X, T, \mathbb{Z}^d) as the collection of all matrices $M \in GL(d, \mathbb{Z})$ with $N_M(X, T, \mathbb{Z}^d) \neq \emptyset$.

- A topological dynamical system (X, T, \mathbb{Z}^d) is said to be *coalescent* if every endomorphism of (X, T, \mathbb{Z}^d) is an automorphism.

Note that the symmetry semigroup of a topological dynamical system is an invariant under conjugation.

As an example, we can define an isomorphism for the \mathbb{Z} -action $T : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ given by the rotation $T_\alpha(x) = x + \alpha$, $\alpha \in \mathbb{S}^1$, by the map $\phi(x) = -x$. Indeed, we have that $\phi \circ T = T^{-1} \circ \phi$.

For the \mathbb{Z}^2 -action on the torus \mathbb{T}^2 generated by the actions $T_1(x) = x + \alpha$, $T_2(x) = x + \beta$, an isomorphism is given by the map $\psi : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ defined as $\psi(x) = -x$.

Isomorphisms of a dynamical system has been studied before. In [6] they are called as *reversors* and the normalizer group $N^*(X, T, \mathbb{Z})$ (generated by isomorphisms) is called the *reversing symmetry group*. Their study was inspired by the time-reversal symmetry of many fundamental equations in physics. In this case $N^*(X, T, \mathbb{Z}) = \text{Aut}(X, T, \mathbb{Z})$ or $N^*(X, T, \mathbb{Z})$ is an index-2 extension of $\text{Aut}(X, T, \mathbb{Z})$. Isomorphisms are always elements of even or infinite order. The existence of isomorphisms have been studied for particular subshifts. Evidence suggest that algebraic properties of the automorphism group does not affect its existence:

- The full shift $(\mathcal{A}^{\mathbb{Z}}, S, \mathbb{Z})$. The automorphism group is huge (not amenable).
- Any *sturmian* subshift, which is always palindromic [44]. Its automorphism group is trivial, i.e., $\text{Aut}(X, S, \mathbb{Z}) = \langle S \rangle$.
- The *period doubling shift*, defined by the primitive substitution $0 \mapsto 01, 1 \mapsto 00$.
- The *Thue-Morse* shift, defined by $0 \mapsto 01, 1 \mapsto 10$.
- The *square-free* shift, obtained as the orbit closure of the characteristic function of the square-free integers.

For higher-dimensional systems they are sometimes called *extended symmetries* (see for example [10, 20, 21]). Since $GL(d, \mathbb{Z})$ is an infinite group for $d > 2$, the relation between $N^*(X, T, \mathbb{Z}^d)$ and $\text{Aut}(X, T, \mathbb{Z}^d)$ is less clear than in the one-dimensional case.

If $\phi \in N_{M_1}(X, T, \mathbb{Z}^d)$, and $\psi \in N_{M_2}(X, T, \mathbb{Z}^d)$, then $\phi\psi$ is in $N_{M_1M_2}(X, T, \mathbb{Z}^d)$, so the sets $N_M(X, T, \mathbb{Z}^d)$ are not semigroups (except if M is the identity matrix). Now, even though the matrices $M \in GL(d, \mathbb{Z})$ are invertible in \mathbb{Z}^d , the symmetry semigroup $\vec{N}(X, T, \mathbb{Z}^d)$ is not necessarily a group, since the existence of a homomorphism associated with a matrix M does not necessarily imply the existence of a homomorphism associated with M^{-1} . Nevertheless, we get the following result when a dynamical system is coalescent.

Proposition 1.5. *Let (X, T, \mathbb{Z}^d) be a coalescent system. If $\vec{N}(X, T, \mathbb{Z}^d)$ is a group, then any homomorphism in $N(X, T, \mathbb{Z}^d)$ is invertible.*

Proof. Let ϕ, ψ be two homomorphisms onto (X, T, \mathbb{Z}^d) associated with M, M^{-1} , respectively. Then $\phi\psi$ is a factor map onto (X, T, \mathbb{Z}^d) . Since (X, T, \mathbb{Z}^d) is coalescent, then $\phi\psi$ is invertible. We conclude that ϕ and ψ are invertible maps. \square

Equicontinuous systems are examples of coalescent systems [4]. See [94] for an example of a non-coalescent system.

When the action generated by T is free, the *normalizer group* $N^*(X, T, \mathbb{Z}^d)$ (which is the group of isomorphisms of (X, T, \mathbb{Z}^d)) is the normalizer of $\langle T \rangle$ in $\text{Homeo}(X)$. This is because $g \langle T \rangle g^{-1} = \langle T \rangle$, for $g \in \text{Homeo}(X)$ is only possible if the generators of $\langle T \rangle$, which are T^{e_i} for $1 \leq i \leq d$, are conjugated into generators of $\langle T \rangle$. In the particular case $\langle T \rangle$ is not free, then $\text{norm}_{\text{Homeo}}(\langle T \rangle)$ contains $N^*(X, T, \mathbb{Z}^d)$ as a subgroup, but possibly further elements (see [9] for some examples). Nevertheless, in this thesis we are only interested in free actions.

The groups $\langle T \rangle$ and $\text{Aut}(X, T, \mathbb{Z}^d)$ are normal subgroups of $N^*(X, T, \mathbb{Z}^d)$, and the centers of $N^*(X, T, \mathbb{Z}^d)$ and $\text{Aut}(X, T, \mathbb{Z}^d)$ are the same. In fact, we have the following short exact sequences

$$1 \rightarrow \langle T \rangle \rightarrow \text{Aut}(X, T, \mathbb{Z}^d) \rightarrow \text{Aut}(X, T, \mathbb{Z}^d) / \langle T \rangle \rightarrow 1 \quad (1.3)$$

$$1 \rightarrow \text{Aut}(X, T, \mathbb{Z}^d) \rightarrow N^*(X, T, \mathbb{Z}^d) \rightarrow \vec{N}^*(X, T, \mathbb{Z}^d) \rightarrow 1. \quad (1.4)$$

Compatibility properties of homomorphisms

If $\pi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ is a factor map between minimal systems, and there exists $y \in Y$ such that $|\pi^{-1}(\{y\})| = 1$, then this property is satisfied in a G_δ dense subset $Y_0 \subseteq Y$. In this case, we say that π is *almost 1-to-1*. If $|\pi^{-1}(\{y\})| = K$ for all y in a G_δ dense subset of Y , then we say that π is *almost K -to-1*. If $|\pi^{-1}(\{y\})| \leq K < \infty$ for all $y \in Y$, we say that π is *finite-to-1*.

For every topological dynamical system, there exists at least one equicontinuous factor, which is the system given by one point. Furthermore, for every topological dynamical system, there exists its *maximal equicontinuous factor*, i.e., a factor $\pi_{eq} : (X, T, \mathbb{Z}^d) \rightarrow (X_{eq}, T_{eq}, \mathbb{Z}^d)$ such that $(X_{eq}, T_{eq}, \mathbb{Z}^d)$ is an equicontinuous system, and for every equicontinuous factor $\pi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$, there exists a factor map $\phi : (X_{eq}, T_{eq}, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ such that $\pi = \phi \circ \pi_{eq}$. In the one-dimensional case, the maximal equicontinuous factor of a minimal system is a rotation on a compact monothetic topological group [96, Theorem 2.11], which is a group G for which there exists an element $g \in G$ such that the subgroup generated by g is dense. Such groups are always abelian. Also, in the particular case, where $\pi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ is an almost 1-to-1 factor and (Y, T, \mathbb{Z}^d) is an equicontinuous system, then (Y, T, \mathbb{Z}^d) is the maximal equicontinuous factor of (X, T, \mathbb{Z}^d) . For instance, odometer systems (defined in Section 1.6) are almost 1-to-1 factors of Toeplitz systems [28, 42].

We say a factor map $\pi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ is *compatible* if for any endomorphism $\phi \in \text{End}(X, T, \mathbb{Z}^d)$, and every $x, y \in X$, if $\pi(x)$ is equal to $\pi(y)$, then $\pi(\phi(x))$ is equal to $\pi(\phi(y))$. With the same spirit, we say a factor π is *compatible with homomorphisms* if for any homomorphism $\phi \in N(X, T, \mathbb{Z}^d)$, and every $x, y \in X$, if $\pi(x)$ is equal to $\pi(y)$, then $\pi(\phi(x))$ is equal to $\pi(\phi(y))$.

The compatibility property allow us to study homomorphisms of some topological dynamical system as shown in the following result.

Lemma 1.6. *Let (X, T, \mathbb{Z}^d) , (Y, T, \mathbb{Z}^d) be two minimal systems, such that $\pi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ is a compatible factor. Then, there is a semigroup homomorphism $\hat{\pi} : \text{End}(X, T, \mathbb{Z}^d) \rightarrow \text{End}(Y, T, \mathbb{Z}^d)$ such that*

1. $\hat{\pi}(\phi)(\pi(x)) = \pi(\phi(x))$ for all $\phi \in \text{End}(X, T, \mathbb{Z}^d)$ and $x \in X$.
2. $\hat{\pi}(\text{Aut}(X, T, \mathbb{Z}^d)) \leq \text{Aut}(Y, T, \mathbb{Z}^d)$.
3. For all $\psi \in \text{End}(Y, T, \mathbb{Z}^d)$, $|\hat{\pi}^{-1}(\{\psi\})| \leq \min_{y \in Y} |\pi^{-1}(y)|$.

Moreover, if π is compatible with homomorphisms, there is an extension of $\hat{\pi} : N(X, T, \mathbb{Z}^d) \rightarrow N(Y, T, \mathbb{Z}^d)$ defined as in Item 1. for all $\phi \in N(X, T, \mathbb{Z}^d)$, such that $\hat{\pi}(N_M(X, T, \mathbb{Z}^d)) \leq N_M(Y, T, \mathbb{Z}^d)$, for any $M \in GL(d, \mathbb{Z})$.

Furthermore, if $c = \min_{y \in Y} |\pi^{-1}(y)|$, then for each $M \in GL(d, \mathbb{Z})$, the map $\hat{\pi} : N_M(X, T, \mathbb{Z}^d) \rightarrow N_M(Y, T, \mathbb{Z}^d)$ is at most c -to-1.

Proof. Set $\phi \in \text{End}(X, T, \mathbb{Z}^d)$. By definition, the map $\hat{\pi}(\phi) : Y \rightarrow Y$ given by $\hat{\pi}(\phi)(\pi(x)) = \pi(\phi(x))$ is well defined and, by minimality of (Y, T, \mathbb{Z}^d) , it is a surjective map, so $\hat{\pi}(\phi)$ is an endomorphism of (Y, T, \mathbb{Z}^d) . Moreover, if ϕ is an automorphism of (X, T, \mathbb{Z}^d) , then $\hat{\pi}(\phi)$ is invertible. Indeed, $\hat{\pi}(\phi) \circ \hat{\pi}(\phi^{-1}) \circ \pi = \pi \circ \phi \circ \phi^{-1} = \pi$, so we conclude that $\hat{\pi}(\phi) \circ \hat{\pi}(\phi^{-1}) = \text{id}_Y$.

Now, set $\psi \in \text{End}(Y, T, \mathbb{Z}^d)$ and suppose that $\min_{y \in Y} |\pi^{-1}(\{y\})| = c < \infty$ (if not, then there is nothing to prove). Let $x_0 \in X$ and $y_0 \in Y$ be such that $|\pi^{-1}(\{y_0\})| = c$, and $y_0 = \psi(\pi(x_0))$. Assume there exists $c + 1$ endomorphisms ϕ_0, \dots, ϕ_c of (X, T, \mathbb{Z}^d) , in $\hat{\pi}(\{\psi\})^{-1}$. By the pigeonhole principle, $y_0 = \psi(\pi(x_0)) = \pi(\phi_0(x_0)) = \dots = \pi(\phi_c(x_0))$. So, there must exist two different indices $0 \leq i, j \leq c$ such that $\phi_i(x_0) = \phi_j(x_0)$, which, by minimality of (X, T, \mathbb{Z}^d) , implies $\phi_i = \phi_j$.

Finally, note that the proof for homomorphisms use similar arguments. □

We will use Lemma 1.6 to describe isomorphisms of particular examples in Chapter 6.

It is known that factor maps between equicontinuous systems are compatible [4], but as we will see in the next section, they are not necessarily compatible with homomorphisms (see Remark 2.7). Nevertheless, the maximal equicontinuous factor is an example of a factor compatible with homomorphisms as proved in [10, Theorem 5 and Corollary 3]. The next result summarizes the compatibility properties of the maximal equicontinuous factor.

Lemma 1.7. [10, Theorem 5 and Corollary 3] For any minimal topological dynamical system (X, T, \mathbb{Z}^d) such that the action of the maximal equicontinuous factor $\langle T_{eq} \rangle$ is free, the maximal equicontinuous factor $\pi_{eq} : (X, T, \mathbb{Z}^d) \rightarrow (X_{eq}, T_{eq}, \mathbb{Z}^d)$ is compatible with homomorphisms. In this case, there exists a semigroup homomorphism $\theta : (X, T, \mathbb{Z}^d) \rightarrow \text{Homeo}(X_{eq})$, with $\theta(\text{End}(X, T, \mathbb{Z}^d)) = \text{id}$ such that

$$\pi_{eq}(\phi(x)) = \hat{\pi}(\phi) + \theta(\phi)(\pi_{eq}(x)),$$

for all $x \in X$, and $\phi \in N(X, T, \mathbb{Z}^d)$, i.e., any homomorphism $\phi \in N(X, T, \mathbb{Z}^d)$ induces a unique homomorphism in X_{eq} given by $z \mapsto \hat{\pi}(\phi) + \theta(\phi)(z)$.

Moreover, if $c = \min_{z \in X_{eq}} |\phi_{eq}^{-1}(\{z\})|$ is finite, then for all $n \geq c$, and $\phi \in N(X, T, \mathbb{Z}^d)$, we have that

$$\{z \in X_{eq} : |\pi_{eq}^{-1}(\{z\})| = n\} = \theta(\phi)(\{z \in X_{eq} : |\pi_{eq}^{-1}(\{z\})| = n\}) + \hat{\pi}(\phi),$$

In some cases the quantity $\min_{y \in Y} |\pi^{-1}(y)|$ can be computed. We refer to [31] for the class of substitutive systems generated by constant-length substitutions. Also Lemma 3.9 is the analogue for constant-shape substitutions.

1.3 Measure-preserving systems

In the following, we present the basics on ergodic theory. We mention [96, 111] for classical references on this theme.

A *measure-preserving system* is a 4-tuple (X, μ, T, G) , where (X, \mathcal{F}, μ) is a probability space and G is a countable group of measurable and measure-preserving transformations acting on X (where the action is denoted by T), i.e., $\forall A \in \mathcal{F}, \forall g \in G, \mu(T^g A) = \mu(A)$.

We say that (X, μ, T, G) is *ergodic* if for all $A \in \mathcal{F}$ we have that

$$\left[(\forall g \in G) \mu(T^{g^{-1}}(A) \Delta A) = 0 \right] \implies \mu(A) = 0 \vee \mu(A) = 1.$$

We now recall the notions of measurable homomorphisms in the measure-theoretic framework.

Let (X, μ, T, G) and (Y, ν, T, G) be measure-preserving systems and $M \in GL(d, \mathbb{Z})$. A *measurable homomorphism associated with M* is a measure-preserving map $\phi : X' \rightarrow Y'$ where X', Y' are measurable subset of X, Y respectively, $\mu(X') = \nu(Y') = 1$ and for any $g \in G, T^g(X') \subseteq X', T^g(Y') \subseteq Y'$ such that for any $\mathbf{n} \in \mathbb{Z}^d$ we have that $\phi \circ T^{\mathbf{n}} = T^{M\mathbf{n}} \circ \phi$ in X' .

If there is a measurable factor ϕ between X and Y , then X is said to be an *extension* of Y . If ϕ is a bi-measurable bijection, we say that ϕ is a *measurable conjugacy* and in this case (X, μ, T, G) and (Y, ν, T, G) are *metrically isomorphic*.

For \mathbb{Z}^d -actions, we always have at least one invariant probability measure for topological dynamical systems (in fact, at least one ergodic probability measure). We define

$\mathcal{M}(X, T, \mathbb{Z}^d)$ the set of all invariant probability measures. This set is convex and compact on the weak-* topology. We say that (X, T, \mathbb{Z}^d) is *uniquely ergodic* if $|\mathcal{M}(X, T, \mathbb{Z}^d)| = 1$, and *strictly ergodic* if it is minimal and uniquely ergodic.

In the special case of strictly ergodic topological dynamical systems (X, T, \mathbb{Z}^d) , (Y, T, \mathbb{Z}^d) we denote $m\text{Hom}(X, Y, T, \mathbb{Z}^d)$, $m\text{Fac}(X, Y, T, \mathbb{Z}^d)$ the collection of all measurable homomorphisms and factors between X and Y , respectively. We recall that a map ϕ is in $m\text{Hom}_M(X, Y, T, \mathbb{Z}^d)$, for a particular matrix $M \in GL(d, \mathbb{Z})$, if ϕ is measure-preserving and there exists two subsets $X' \subseteq X$, $Y' \subseteq Y$ with $\mu_X(X') = 1$, $\mu_Y(Y') = 1$ such that for all $\mathbf{n} \in \mathbb{Z}^d$, $\phi \circ S^{\mathbf{n}} = S^{M\mathbf{n}} \circ \phi$ for μ_X -a.e in X' .

1.4 Symbolic Dynamics

In this section, we will present the basic definitions and some background about symbolic dynamics that will be used in the rest of this thesis. We refer to [84] for a classical reference in the one-dimensional case, and [24] for actions on more abstract groups.

Let \mathcal{A} be a finite alphabet and $d \geq 1$ be an integer. We define a topology on $\mathcal{A}^{\mathbb{Z}^d}$ by endowing \mathcal{A} with the discrete topology, and considering in $\mathcal{A}^{\mathbb{Z}^d}$ the product topology, which is generated by cylinders. Since \mathcal{A} is finite, $\mathcal{A}^{\mathbb{Z}^d}$ is a metrizable compact space. In this space \mathbb{Z}^d acts by translations, defined for every $\mathbf{n} \in \mathbb{Z}^d$ by:

$$S^{\mathbf{n}}(x)_{\mathbf{k}} = x_{\mathbf{n}+\mathbf{k}}, \quad x \in \mathcal{A}^{\mathbb{Z}^d}, \quad \mathbf{k} \in \mathbb{Z}^d.$$

The \mathbb{Z}^d -action $(\mathcal{A}^{\mathbb{Z}^d}, S, \mathbb{Z}^d)$ is called the *fullshift*.

Let $P \subseteq \mathbb{Z}^d$ be a finite subset. A *pattern* is an element $\mathbf{p} \in \mathcal{A}^P$. We say that P is the *support* of \mathbf{p} , and we denote $P = \text{supp}(\mathbf{p})$. A pattern *occurs in* $x \in \mathcal{A}^{\mathbb{Z}^d}$, if there exists $\mathbf{n} \in \mathbb{Z}^d$ such that $\mathbf{p} = x|_{\mathbf{n}+P}$, in this case we denote it $\mathbf{p} \sqsubseteq x$, and we call this \mathbf{n} an *occurrence in* x of \mathbf{p} .

A *subshift* (X, S, \mathbb{Z}^d) is given by a closed subset $X \subseteq \mathcal{A}^{\mathbb{Z}^d}$ which is invariant by the \mathbb{Z}^d -action. A subshift can also be defined by its *language*. For $P \subseteq \mathbb{Z}^d$ we define

$$\mathcal{L}_P(X) = \{\mathbf{p} \in \mathcal{A}^P : \exists x \in X, \mathbf{p} \sqsubseteq x\}.$$

We define the language of a subshift X by

$$\mathcal{L}(X) = \bigcup_{P \subseteq \mathbb{Z}^d} \mathcal{L}_P(X).$$

Let (X, S, \mathbb{Z}^d) be a minimal subshift and $x \in X$. We say that $\mathbf{p} \in \mathbb{Z}^d$ is a *period* of x if for all $\mathbf{n} \in \mathbb{Z}^d$, $x_{\mathbf{n}+\mathbf{p}} = x_{\mathbf{n}}$. We say that (X, S, \mathbb{Z}^d) is *aperiodic* if there are no nontrivial periods.

Let \mathcal{B} be other finite alphabet, and $Y \subseteq \mathcal{B}^{\mathbb{Z}^d}$ be a subshift. For $P \subseteq \mathbb{Z}^d$, we define a *P-block map* as a map of the form $\Phi : \mathcal{L}_P(X) \rightarrow \mathcal{B}$. This map induce a factor $\phi : X \rightarrow Y$

given by

$$\phi(x)_n = \Phi(x|_{n+P}).$$

The map ϕ is called the *sliding block code* induced by Φ , and P is the support of the map ϕ . In most of the cases we may assume the support of the sliding block codes is a ball of the form $B(\mathbf{0}, r)$, for $r \in \mathbb{N}$. We define the *radius* (and we denote by $r(\phi)$) as the infimum of $r \in \mathbb{N}$ such that we can define a $B(\mathbf{0}, r)$ -block map which induced it. The next theorem characterizes the factor maps between two subshifts.

Theorem 1.8 (Curtis-Hedlund-Lyndon theorem). *Let (X, S, \mathbb{Z}^d) and (Y, S, \mathbb{Z}^d) be two subshifts. A map $\phi : (X, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ is a factor if and only if there exists a $B(\mathbf{0}, r)$ -block map $\Phi : \mathcal{L}_{B(\mathbf{0}, r)}(X) \rightarrow \mathcal{L}_1(Y)$, such that $\phi(x)_n = \Phi(x|_{n+B(\mathbf{0}, r)})$, for all $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$.*

For homomorphisms we have a similar characterization, but we need to make a slight variation of this theorem.

Theorem 1.9 (Curtis-Hedlund-Lyndon theorem for homomorphisms). *Let (X, S, \mathbb{Z}^d) and (Y, S, \mathbb{Z}^d) be two subshifts, and $M \in GL(d, \mathbb{Z})$. A map $\phi : (X, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ is a homomorphism associated with M if and only if there exists a $B(\mathbf{0}, r)$ -block map $\Phi : \mathcal{L}_{B(\mathbf{0}, r)}(X) \rightarrow \mathcal{L}_1(Y)$, such that $\phi(x)_n = \Phi(x|_{M^{-1}\mathbf{n}+B(\mathbf{0}, r)})$, for all $\mathbf{n} \in \mathbb{Z}^d$ and $x \in X$.*

Proof. Let $(x^n)_{n \in \mathbb{N}}$ be a convergent sequence to $x \in X$. This implies,

$$(\forall p > 0)(\exists N \in \mathbb{N})(\forall n \geq N) x^n|_{B(\mathbf{0}, p)} = x|_{B(\mathbf{0}, p)}.$$

Set $\mathbf{m} \in \mathbb{Z}^d$. Let $p(\mathbf{m}) \in \mathbb{N}$ be large enough such that $M^{-1}\mathbf{m} + B(\mathbf{0}, r) \subseteq B(\mathbf{0}, p(\mathbf{m}))$. Hence, for any $n \geq N(p(\mathbf{m}))$ we have that

$$\phi(x^n)_m = \Phi(x^n|_{M^{-1}\mathbf{m}+B(\mathbf{0}, r)}) = \Phi(x|_{M^{-1}\mathbf{m}+B(\mathbf{0}, r)}) = \phi(x)_m.$$

We conclude that ϕ is continuous.

Now, set $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^d$. Then for all $x \in X$

$$\phi(S^n x)_m = \Phi((S^n x)|_{M^{-1}\mathbf{m}+B(\mathbf{0}, r)}) = \Phi(x|_{\mathbf{n}+M^{-1}\mathbf{m}+B(\mathbf{0}, r)}),$$

and

$$S^{M\mathbf{n}}\phi(x)_m = \phi(x)_{M\mathbf{n}+\mathbf{m}} = \Phi(x|_{\mathbf{n}+M^{-1}\mathbf{m}+B(\mathbf{0}, r)}).$$

We conclude that $\phi \circ S^n = S^{M\mathbf{n}} \circ \phi$, so ϕ is a homomorphism associated with M .

On the other hand, let $\phi : (X, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ be a homomorphism associated with M , and let $r > 0$ be such that $x|_{B(\mathbf{0}, r)} = y|_{B(\mathbf{0}, r)}$, implies $\phi(x)_\mathbf{0} = \phi(y)_\mathbf{0}$. Then, the local map $\Phi(x|_{B(\mathbf{0}, r)}) = \phi(x)_\mathbf{0}$ is well defined by the very definition of r . Finally, note that $\phi(x)_h = S^h\phi(x)_\mathbf{0} = \phi(S^{M^{-1}h}x)_\mathbf{0} = \Phi(x|_{M^{-1}h+B(\mathbf{0}, r)})$, which proves the claim. \square

This means, for any homomorphism ϕ we can define a *radius* (also denoted by $r(\phi)$), as the infimum of $r \in \mathbb{N}$ such that we can define a $B(\mathbf{0}, r)$ -block map which induced it.

An immediate consequence of Theorem 1.9 is that for any subshift (X, S, \mathbb{Z}^d) , its normalizer group $N^*(X, S, \mathbb{Z}^d)$ is a countable and discrete subset in $\text{Homeo}(X)$.

1.5 Nondeterministic directions of a topological dynamical system

An interesting notion in the study of higher-dimensional dynamical systems, with the objective to study sub-actions of one, is the so-called *nonexpansive subspaces*, introduced by M. Boyle and D. Lind in [17]. When the space X is infinite such subspaces always exist [17, Theorem 3.7]. In fact, nonexpansive subspaces are always subsets of nonexpansive hyperplanes [17, Theorem 3.6]. A key step on the proof of this result, was one proved by S. Schwartzman. S. Schwartzman proved that there are no “one-sided expansive” homeomorphisms, except on finite spaces. Although he never published this result. A proof can be found in [61].

We will only focus on hyperplanes in \mathbb{R}^d , which lead to the notion of deterministic/nondeterministic directions. Let \mathbb{S}^{d-1} be the unit $(d - 1)$ -dimensional sphere. For $\mathbf{v} \in \mathbb{S}^{d-1}$ define $H_{\mathbf{v}} = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{v} \rangle < 0\}$ to be the open half-space with outward unit normal \mathbf{v} . We identify the set \mathbb{H}_d of all half-spaces in \mathbb{R}^d with the sphere \mathbb{S}^{d-1} using the parametrization $\mathbf{v} \longleftrightarrow H_{\mathbf{v}}$.

Definition 1.10. Let (X, S, \mathbb{Z}^d) be a subshift and \mathbf{v} be a unit vector of \mathbb{R}^d . Then \mathbf{v} is *deterministic* for (X, T, \mathbb{Z}^d) if for all $x, y \in X$ we have that

$$x|_{H_{\mathbf{v}} \cap \mathbb{Z}^d} = y|_{H_{\mathbf{v}} \cap \mathbb{Z}^d} \implies x = y.$$

If \mathbf{v} does not satisfies this condition, we say that \mathbf{v} is *nondeterministic* for (X, T, \mathbb{Z}^d) .

In this thesis, we are only interested in the set of nondeterministic directions. In the one-dimensional case, this lead to the notion of *asymptotic pairs*, and in the finitary version to the notion of *special words* [88]. For a subshift (X, S, \mathbb{Z}^d) we denote $\text{ND}(X, S, \mathbb{Z}^d)$ the sets of nondeterministic directions for (X, S, \mathbb{Z}^d) . In [17] it was proved an analogous result of the following, that we mention in our context

Theorem 1.11. [17, Lemma 3.4 and Theorem 3.7] *For any subshift (X, S, \mathbb{Z}^d) , with an infinite phase space X , the set of nondeterministic directions $\text{ND}(X, S, \mathbb{Z}^d)$ is a non-empty compact set.*

The notion of *direction of determinism* was introduced in [63] for two-dimensional subshifts, and in [34] they were used to prove a weak version of Nivat’s conjecture.

The following result establishes a link between nondeterministic directions and the symmetry group of a subshift, which we will use to describe the symmetry group for a big family of substitutive systems (Theorem 5.17).

Proposition 1.12. *Let (X, S, \mathbb{Z}^d) be a subshift. Then, for all $\mathbf{v} \in \text{ND}(X, S, \mathbb{Z}^d)$ and $M \in \vec{N}(X, S, \mathbb{Z}^d)$, we have that $(M^*)^{-1}\mathbf{v}/\|(M^*)^{-1}\mathbf{v}\| \in \text{ND}(X, S, \mathbb{Z}^d)$.*

Proof. If \mathbf{v} is in $\text{ND}(X, S, \mathbb{Z}^d)$, there exists $x \neq y \in X$ with $x|_{H_{\mathbf{v}} \cap \mathbb{Z}^d} = y|_{H_{\mathbf{v}} \cap \mathbb{Z}^d}$. Set $M \in \vec{N}(X, T, \mathbb{Z}^d)$ and $\phi \in N_M(X, S, \mathbb{Z}^d)$. Then, we have that $\phi(x)|_{(MH_{\mathbf{v}})+\mathbf{n}} = \phi(y)|_{(MH_{\mathbf{v}})+\mathbf{n}}$, where \mathbf{n} is a vector of radius $r(\phi)$. We note that $S^n\phi(x)|_{MH_{\mathbf{v}}} = S^n\phi(y)|_{MH_{\mathbf{v}}}$, and we conclude that $(M^*)^{-1}\mathbf{v}/\|(M^*)^{-1}\mathbf{v}\| \in (X, S, \mathbb{Z}^d)$. \square

1.6 Odometer systems

In the following we present the example of odometer systems and Toeplitz sequences. Odometer systems are the most natural equicontinuous systems in the study of minimal Cantor systems. They have been described by algebraic reasons. They present interesting recurrence properties. The return times to clopen sets contain infinite arithmetic progressions. In fact, they are the maximal equicontinuous factor for a big family of symbolic systems, such as, some substitutions and Toeplitz sequences. In this section we describe the symmetry semigroup of odometer systems (Lemma 1.14) which we then use to complete the characterization for two-dimensional constant-base odometer systems (Theorem 2.2). Note that, in [58] there was a first attempt of studying homomorphisms between two-dimensional odometer systems.

Toeplitz sequences have been introduced in dynamical systems by K. Jacobs and M. Keane in [70]. Toeplitz subshifts are symbolic systems that are the orbit closures of the regular quasi-periodic points of the subshift. N. Markley and M. Paul characterize them in [86] as the minimal almost 1-1 extensions of odometer systems. They have been used to provide a series of examples with interesting dynamical properties. We refer to [42] for the study of odometer systems and Toeplitz sequences in the one-dimensional case, [28] for higher dimensional actions, and [30] for more abstract actions given by residually finite groups. Here we will follow the same notation than in [28].

Let $Z_0 \geq Z_1 \geq \dots \geq Z_n \geq Z_{n+1} \geq \dots$ be a nested sequence of finite index subgroups of \mathbb{Z}^d such that $\bigcap_{n \geq 0} Z_n = \{\mathbf{0}\}$, and let $\alpha_n : \mathbb{Z}^d/Z_{n+1} \rightarrow \mathbb{Z}^d/Z_n$ be the function induced by the inclusion map. Following the notation in [30], we consider the inverse limit of these groups

$$\overleftarrow{\mathbb{Z}^d}_{(Z_n)} = \lim_{\leftarrow n} (\mathbb{Z}^d/Z_n, \alpha_n),$$

i.e., $\overleftarrow{\mathbb{Z}^d}_{(Z_n)}$ is the subset of the product $\prod_{n \geq 0} \mathbb{Z}^d/Z_n$ consisting of the elements $\overleftarrow{\mathbf{g}} = (\mathbf{g}_n)_{n \geq 0}$ such that $\alpha_n(\mathbf{g}_{n+1}) = \mathbf{g}_n \pmod{Z_n}$ for all $n \geq 0$. This set is a group equipped with the addition defined coordinate-wise, i.e.,

$$\overleftarrow{\mathbf{g}} + \overleftarrow{\mathbf{h}} = (\mathbf{g}_n + \mathbf{h}_n)_{n \geq 0}.$$

Every group \mathbb{Z}^d/Z_n is endowed with the discrete topology, then $\prod_{n \geq 0} (\mathbb{Z}^d/Z_n)$ is a compact metric space, and $\overleftarrow{\mathbb{Z}}^d_{(Z_n)}$ is a compact topological group whose topology is spanned by the cylinder sets

$$[\mathbf{a}]_n = \left\{ \overleftarrow{\mathbf{g}} \in \overleftarrow{\mathbb{Z}}^d_{(Z_n)} : \mathbf{g}_n = \mathbf{a} \right\},$$

with $\mathbf{a} \in \mathbb{Z}^d/Z_n$, and $n \geq 0$. Now, consider the group homomorphism $\kappa_{(Z_n)} : \mathbb{Z}^d \rightarrow \prod_{n \geq 0} \mathbb{Z}^d/Z_n$ defined for $\mathbf{n} \in \mathbb{Z}^d$ by

$$\kappa_{(Z_n)}(\mathbf{n}) = [\mathbf{n} \pmod{Z_n}]_{n \geq 0}.$$

The image of \mathbb{Z}^d by $\kappa_{(Z_n)}$ is dense in $\overleftarrow{\mathbb{Z}}^d_{(Z_n)}$, so the \mathbb{Z}^d -action $\mathbf{n}(\overleftarrow{\mathbf{g}}) = \kappa_{(Z_n)}(\mathbf{n}) + \overleftarrow{\mathbf{g}}$, with $\mathbf{n} \in \mathbb{Z}^d$, $\overleftarrow{\mathbf{g}} \in \overleftarrow{\mathbb{Z}}^d_{(Z_n)}$, is well defined and $(\overleftarrow{\mathbb{Z}}^d_{(Z_n)}, +_{(Z_n)}, \mathbb{Z}^d)$ is a minimal equicontinuous system as proved in [30]. We call $(\overleftarrow{\mathbb{Z}}^d_{(Z_n)}, +_{(Z_n)}, \mathbb{Z}^d)$ an *odometer system*. From now on, we will denote the odometer system $(\overleftarrow{\mathbb{Z}}^d_{(Z_n)}, +_{(Z_n)}, \mathbb{Z}^d)$ just as $\overleftarrow{\mathbb{Z}}^d_{(Z_n)}$, and in the *constant base* case (where $Z_n = L^n(\mathbb{Z}^d)$, for some matrix $L \in \mathcal{M}(d, \mathbb{Z})$), we will denote it as $\overleftarrow{\mathbb{Z}}^d_{(L^n)}$. Odometer systems have been extensively studied before (see [28, 30, 42]). The next result characterizes the factor odometer systems of a fixed odometer system.

Lemma 1.13. [28, Lemma 1] Let $\overleftarrow{\mathbb{Z}}^d_{(Z_n^1)}$ and $\overleftarrow{\mathbb{Z}}^d_{(Z_n^2)}$ be two odometer systems ($j = 1, 2$). There exists a factor map $\pi : \overleftarrow{\mathbb{Z}}^d_{(Z_n^1)} \rightarrow \overleftarrow{\mathbb{Z}}^d_{(Z_n^2)}$ if and only if for every Z_n^2 there exists some Z_m^1 such that $Z_m^1 \leq Z_n^2$.

The proof of Lemma 1.13 can be modified to provide a characterization for the matrices $M \in GL(d, \mathbb{Z})$ defining a homomorphism $\phi : \overleftarrow{\mathbb{Z}}^d_{(Z_n^1)} \rightarrow \overleftarrow{\mathbb{Z}}^d_{(Z_n^2)}$.

Lemma 1.14. Set $M \in GL(d, \mathbb{Z})$. There exists a homomorphism associated with M from $\overleftarrow{\mathbb{Z}}^d_{(Z_n^1)}$ to $\overleftarrow{\mathbb{Z}}^d_{(Z_n^2)}$, if and only if for all $n \in \mathbb{N}$, there exists $m_M(n) \in \mathbb{N}$ such that

$$MZ_{m_M(n)}^1 \leq Z_n^2. \quad (\text{Normalizer Condition})$$

Proof. First we prove the necessity. Let $\phi : \overleftarrow{\mathbb{Z}}^d_{(Z_n^1)} \rightarrow \overleftarrow{\mathbb{Z}}^d_{(Z_n^2)}$ be a homomorphism associated with a matrix $M \in GL(d, \mathbb{Z})$. By continuity, for any $n \geq 0$ and $\mathbf{g} \in \mathbb{Z}^d/Z_n^2$, there exists $m \geq 0$ and $\mathbf{f} \in \mathbb{Z}^d/Z_m^1$ such that $[\mathbf{f}]_m \subseteq \phi^{-1}([\mathbf{g}]_n)$. Set $\mathbf{h} \in Z_m^1$. Note that for all $\overleftarrow{\mathbf{f}} \in [\mathbf{f}]_m$, we have that $\mathbf{h}(\overleftarrow{\mathbf{f}}) \in [\mathbf{f}]_m$, which implies $\phi(\mathbf{h}(\overleftarrow{\mathbf{f}})) = M\mathbf{h}(\overleftarrow{\mathbf{f}}) \in [\mathbf{g}]_n$. Since $\phi(\overleftarrow{\mathbf{f}})$ is in $[\mathbf{a}]_n$, then $\{\mathbf{m} \in \mathbb{Z}^d : \mathbf{m}(\phi(\overleftarrow{\mathbf{f}})) \in [\mathbf{a}]_n\}$ is equal to Z_n^2 , which implies $M\mathbf{h} \in Z_n^2$.

For the sufficiency, assume that $M \in GL(d, \mathbb{Z})$ satisfies (Normalizer Condition). Since the sequences $\{Z_n^i\}_{n > 0}$, $i = 1, 2$ are decreasing, we may assume that $m(n) \leq m(n+1)$

for all $n > 0$. Thus, we have a homomorphism $\phi_{m(n)} : \mathbb{Z}^d / Z_{m_M(n)}^1 \rightarrow \mathbb{Z}^d / Z_n^2$, given by $\phi_{m(n)}(\mathbf{m}) = M\mathbf{m}$. To finish the proof, we remark that $\phi : \overleftarrow{\mathbb{Z}^d}_{(Z_n^1)} \rightarrow \overleftarrow{\mathbb{Z}^d}_{(Z_n^2)}$ defined as $\phi(\overleftarrow{\mathbf{g}} = (\mathbf{g}_{m(n)})_{n>0}) = (\phi_{m(n)}(\mathbf{g}_{m(n)}))_{n>0}$ is a homomorphism associated with M . \square

In [58] there was a first attempt of studying homomorphisms between two-dimensional odometer systems. Note that for all $n \geq 0$, we may assume that $m_M(n)$ is arbitrarily large. Consider, for all $n \in \mathbb{N}$ a matrix $L_{n,i} \in \mathcal{M}(d, \mathbb{Z})$ such that $L_{n,i}(\mathbb{Z}^d) = Z_n^i$ for $i \in \{1, 2\}$. This matrix is unique, up to a conjugation with a matrix in $GL(d, \mathbb{Z})$. Then, (Normalizer Condition) is equivalent to, for all $n \geq 0$, there exists $m_M(n) \geq 0$ such that $L_{n,2}^{-1} M L_{m_M(n),1}$ is an endomorphism in \mathbb{Z}^d . Since $\det(L)L^{-1} = \text{adj}(L)$, where $\text{adj}(L)$ is the *adjugate matrix* of L , then (Normalizer Condition) is equivalent to

$$\forall n \in \mathbb{N}, \exists m_M(n) \in \mathbb{N}, \text{adj}(L_{n,2}) M L_{m_M(n),1} \equiv 0 \pmod{\det(L_{n,2})}. \quad (\text{NC } 2)$$

In the constant-base case, ignoring the condition $M \in GL(d, \mathbb{Z})$, then L satisfies (NC 2). Moreover, the set of integer matrices $M \in \mathcal{M}(d, \mathbb{Z})$ satisfying (NC 2) is an additive group. In particular, any polynomial in L with integer coefficients also satisfies (NC 2).

A direct corollary of Lemma 1.14 is the following.

Corollary 1.15. *The following are consequences of Lemma 4.10 easily to verified.*

1. *If $L = pM$ is an integer expansion matrix, with $p \in \mathbb{Z}$ and $M \in GL(d, \mathbb{Z})$, then the symmetry semigroup $\vec{N}(\overleftarrow{\mathbb{Z}^d}_{(L^n)})$ is $GL(d, \mathbb{Z})$.*
2. *If $L_1 \in \mathcal{M}(d, \mathbb{Z})$ is an integer expansion matrix and $M \in \vec{N}(\overleftarrow{\mathbb{Z}^d}_{(L_1^n)})$, then for any $P \in GL(d, \mathbb{Z})$, PMP^{-1} is in the symmetry semigroup $\vec{N}(\overleftarrow{\mathbb{Z}^d}_{(PL_1^n P^{-1})})$.*
3. *If $M \in GL(d, \mathbb{Z})$ commutes with some power of the expansion matrix L , then M is in the symmetry semigroup $\vec{N}(\overleftarrow{\mathbb{Z}^d}_{(L^n)})$.*

Now we define Toeplitz sequences. Let \mathcal{A} be a finite alphabet and $Z \subseteq \mathbb{Z}^d$ be a finite index subgroup of \mathbb{Z}^d . For $x \in \mathcal{A}^{\mathbb{Z}^d}$ and $a \in \mathcal{A}$ we define

$$\text{Per}(x, Z, a) = \{\mathbf{n} \in \mathbb{Z}^d : x(\mathbf{n} + \mathbf{m}) = a, \text{ for all } \mathbf{m} \in Z\}.$$

Then $\text{Per}(x, Z) = \bigcup_{a \in \mathcal{A}} \text{Per}(x, Z, a)$ for all $x \in \mathcal{A}^{\mathbb{Z}^d}$. When $\text{Per}(x, \mathbb{Z})$ is non-empty, we say Z is a *group of periods* of x . We say $Z \subseteq \mathbb{Z}^d$ is a *group generated by essential periods* of x if for all finite index subgroup $Z' \subseteq \mathbb{Z}^d$, $\text{Per}(x, \mathbb{Z}) \subseteq \text{Per}(x, Z')$ implies that $Z' \subseteq Z$. We say that x is a \mathbb{Z}^d -*Toeplitz sequence* if for all $\mathbf{n} \in \mathbb{Z}^d$, there exists a finite index subgroup $Z \subseteq \mathbb{Z}^d$ such that $\mathbf{n} \in \text{Per}(x, Z)$. The following is a characterization of Toeplitz sequences

Proposition 1.16. [28, Proposition 14] *The following statements concerning $x \in \mathcal{A}^{\mathbb{Z}^d}$ are equivalent:*

1. x is a Toeplitz sequence.

2. There exists a sequence of positive integer numbers $\{p_n\}_{n \geq 0}$ such that $p_n < p_{n+1}$, p_n divides p_{n+1} and $\llbracket -n, \dots, n \rrbracket^d \subseteq \text{Per}(x, p_n \mathbb{Z}^d)$ for all $n \geq 0$.

Toeplitz subshifts, which are subshifts generated by a Toeplitz sequence, are almost 1-to-1 extensions of odometer systems. In fact

Proposition 1.17. [28, Proposition 20] *Let $x \in \mathcal{A}^{\mathbb{Z}^d}$ be a Toeplitz sequence. If $\{Z_n\}_{n \geq 0}$ is a sequence of groups generated by essential periods of x such that $Z_{n+1} \subseteq Z_n$ and $\bigcup_{n \geq 0} \text{Per}(x, Z_n) = \mathbb{Z}^d$, then $(\overleftarrow{\mathbb{Z}^d}_{Z_n}, +_{Z_n}, \mathbb{Z}^d)$ is the maximal equicontinuous factor of the subshift generated by x .*

1.7 Multidimensional constant-shape substitutions

In this section we will define the main object where we study homomorphisms in this thesis, which are multidimensional substitutive subshifts. They represent the simplest nontrivial zero entropy symbolic systems, since they are generated by finite data. By that, ergodic and topological properties of substitution dynamical systems have been extensively studied. They were introduced by W.H. Gottschalk in [60] (see [97] for a good bibliography on this subject).

Unlike the one-dimensional case, the notion of multidimensional substitutions is not clear. In our case we extend the notion of constant-length substitutions to the multidimensional framework and we call them *constant-shape substitution*. To do this, we use the fact that if $L \in \mathcal{M}(d, \mathbb{Z})$ is an integer matrix with $\det(L) \neq 0$ and F is a fundamental domain of $L(\mathbb{Z}^d)$ (with $\mathbf{0} \in F$), i.e., a set of representative classes of $\mathbb{Z}^d/L(\mathbb{Z}^d)$, then for any $\mathbf{n} \in \mathbb{Z}^d$, there exists unique $\mathbf{p} \in \mathbb{Z}^d$ and $\mathbf{f} \in F$ such that $\mathbf{n} = L(\mathbf{p}) + \mathbf{f}$. This is a multidimensional interpretation of the Euclidean division.

In our context, $L \in \mathcal{M}(d, \mathbb{Z})$ is an integer *expansion* matrix, i.e., $\|L\| > 1$ and $\|L^{-1}\| < 1$, such that $L(\mathbb{Z}^d) \subseteq \mathbb{Z}^d$. Let F be a fundamental domain of $L(\mathbb{Z}^d)$ with $\mathbf{0}$ in F , and \mathcal{A} be a finite alphabet. A *multidimensional constant-shape substitution* ζ is a map $\mathcal{A} \rightarrow \mathcal{A}^F$. The set F is called the *support* of the substitution. An analogue of constant-shape substitutions in the one-dimensional case are the constant-length substitutions. Fig. 1.2 shows an example of a constant-shape substitution with $L = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $F = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}$.

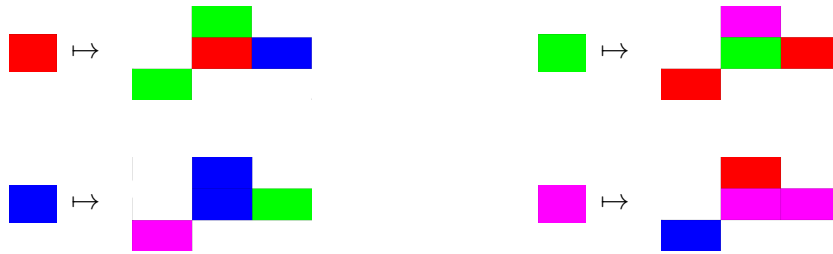


Figure 1.2: An example of a constant-shape substitution over a four-letter alphabet.

Every element of \mathbb{Z}^d can be expressed in a unique way as $\mathbf{p} = L(\mathbf{j}) + \mathbf{f}$, with $\mathbf{j} \in \mathbb{Z}^d$ and $\mathbf{f} \in F$, so we can consider the substitution ζ as a map from $\mathcal{A}^{\mathbb{Z}^d}$ to itself given by

$$\zeta(x)_{L(\mathbf{j})+\mathbf{k}} = \zeta(x(\mathbf{j}))_{\mathbf{k}}.$$

Given a substitution ζ , we will denote by L_ζ to its expansion matrix and F_1^ζ its support. For any $n > 0$ we define the iteration of the substitution $\zeta^n : \mathcal{A} \rightarrow \mathcal{A}^{F_n^\zeta}$, where the supports of these substitutions satisfy the recurrence $F_{n+1}^\zeta = F_1^\zeta + L_\zeta(F_n^\zeta)$ for all $n \geq 1$.

From now on, we may assume that the sequence of supports $(F_n^\zeta)_{n>0}$ is a Følner sequence. Fig. 1.3 shows the first three iterations of the substitution given in Fig. 1.2.

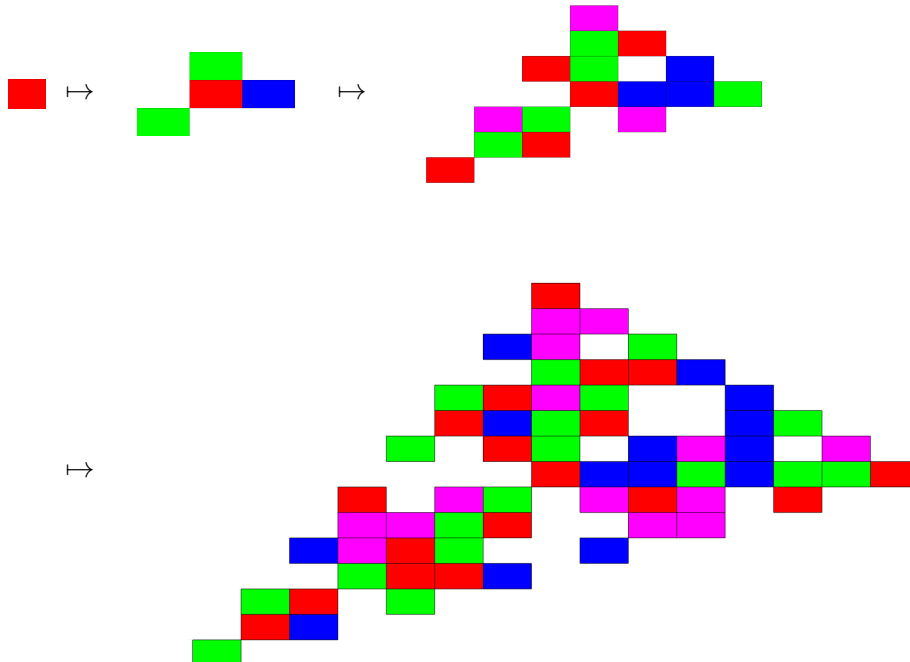


Figure 1.3: First, second and third iterations of Fig. 1.2.

For any $\mathbf{f} \in F_1^\zeta$ we define $p_{\mathbf{f}}$ the restriction of ζ in \mathbf{f} , i.e., for $a \in \mathcal{A}$, we have that $p_{\mathbf{f}}(a) = \zeta(a)_{\mathbf{f}}$. We say the substitution is *bijective* if for all $\mathbf{f} \in F_1^\zeta$ the maps $p_{\mathbf{f}}$ are bijective. A substitution is called *primitive* if there exists a positive integer $n > 0$, such that for every $a, b \in \mathcal{A}$, b occurs in $\zeta^n(a)$. Fig. 1.2 is an example of a bijective and primitive constant-shape substitution. The *language* of a substitution is the set of all patterns that appear in $\zeta^n(a)$, for some $n > 0$, $a \in \mathcal{A}$, i.e.,

$$\mathcal{L}_\zeta = \{\mathbf{p} : \mathbf{p} \sqsubseteq \zeta^n(a), \text{ for some } n > 0, a \in \mathcal{A}\}.$$

Using the language we define the subshift X_ζ associated with a substitution as the set of all sequences $x \in \mathcal{A}^{\mathbb{Z}^d}$ such that every pattern occurring in x is in \mathcal{L}_ζ , and we denote $(X_\zeta, S, \mathbb{Z}^d)$ the *substitutive subshift*. If ζ is a primitive constant-shape substitution, the existence of *periodic points* is well known, i.e., there exists at least one point $x_0 \in X_\zeta$ such that $\zeta^p(x_0) = x_0$ for some $p > 0$. In the primitive case, the subshift is preserved replacing the substitution by a power of it, i.e., X_{ζ^n} is equal to X_ζ for all $n > 0$. Then, we may assume that the substitution possess at least one fixed point, i.e., there exists a point $x \in X_\zeta$ such that $x = \zeta(x)$. It is well known that this subshift is uniquely ergodic (in [81] can be found a proof for substitution tiling systems seen as substitution Delone sets for \mathbb{R}^d -actions that can be adapted for our context with the assumption of the supports $(F_n^\zeta)_{n>0}$ being a Følner sequence). The unique ergodic measure is characterized in terms of the expansion matrix of ζ , and we denote this measure as μ_ζ . For a cylinder set $[\mathbf{p}]_n$, where \mathbf{p} is a pattern in \mathcal{L}_ζ , the quantity $\mu_\zeta([\mathbf{p}])$ represents the frequency of the pattern \mathbf{p} in any sequence in X_ζ .

In the literature, constant-shape substitutions with a positive diagonal expansion matrix $L = \text{diag}(l_i)_{i=1,\dots,d}$ and support equal to the standard d -dimensional parallelepiped $F_1 = \prod_{i=1}^d [0, l_i - 1]$ are called *block substitutions*.

Examples of constant-shape substitutions can be generated via constant-length substitutions as follows: Let $\{\zeta_i\}_{i=1}^d$ be d aperiodic one-dimensional constant-length substitutions with alphabet \mathcal{A}_i and length q_i for $1 \leq i \leq d$. We define the *product substitution of $\{\zeta_i\}_{i=1}^d$* as the constant-shape substitution ζ with alphabet $\mathcal{A} = \prod_{i=1}^d \mathcal{A}_i$,

expansion matrix given by $L_\zeta = \text{diag}(l_i)$ and support $F_1^\zeta = \prod_{i=1}^d [0, l_i - 1]$, defined as

$\zeta(a_1, \dots, a_d)_j = (\zeta_1(a_1)_{j_1}, \dots, \zeta_d(a_d)_{j_d})$. It is straightforward to check that if $\{\zeta_i\}_{i=1}^d$ are primitive, then the product substitution is also primitive. The same is true for bijectivity.

Since L_ζ is an expansion matrix, then L_ζ^{-1} is a contraction map in \mathbb{R}^d . For any $\mathbf{g} \in F_1^\zeta$ define the map $f_{\mathbf{g}}(\cdot) = L_\zeta^{-1}(\cdot + \mathbf{g})$. As mentioned in Section 1.1.3, there exists a nonempty compact subset T_ζ (or denoted $T(L, F_1)$ when there is no substitution defined) in \mathbb{R}^d such

that

$$L_\zeta(T_\zeta) = \bigcup_{g \in F_1^\zeta} T_\zeta + g.$$

As in [110] we call this set the *digit tile* of the substitution. Using $T_0 = \{\mathbf{0}\}$ in (1.2) we get that

$$T_\zeta = \lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} L_\zeta^{-i}(F_1^\zeta) = \lim_{n \rightarrow \infty} L_\zeta^{-n}(F_n^\zeta), \quad (1.5)$$

where the limit is with respect to the Hausdorff metric. Fig. 1.4 shows some examples of approximations of this digit tile.

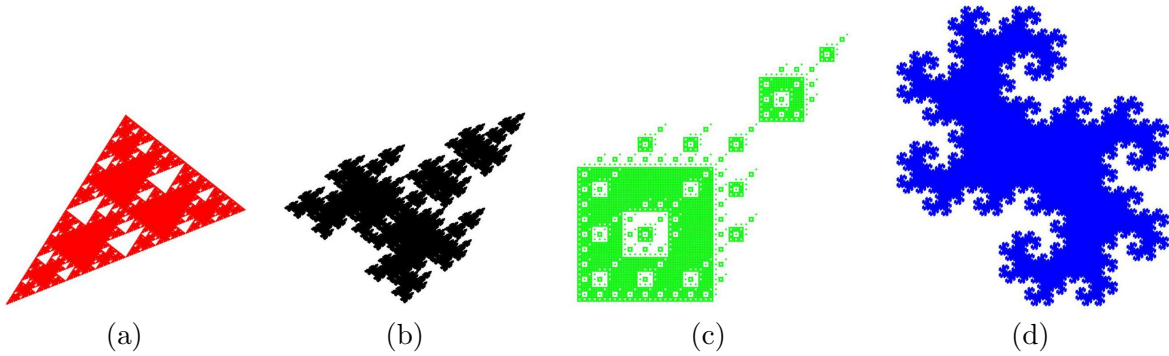


Figure 1.4: Approximation of some digit tiles: (a) Gasket, (b) Rocket, (c) Shooter, (d) Twin Dragon. The names of these tiles comes from [110]

The expansion matrix and fundamental domains of these examples are the following:

$$(a) \quad L_{(a)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \quad F_1^{(a)} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\}.$$

$$(b) \quad L_{(b)} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad F_1^{(b)} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

$$(c) \quad L_{(c)} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad F_1^{(c)} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

$$(d) \quad L_{(d)} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad F_1^{(d)} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

As in the one-dimensional case, the following proposition shows that for any multidimensional constant-shape substitution there exists a finite subset $K \subseteq \mathbb{Z}^d$ whose iteration

under the substitution fill the whole \mathbb{Z}^d . This set determine the fixed points of a primitive constant-shape substitution.

Proposition 1.18. *Let ζ be a multidimensional constant-shape substitution. Then, the set $K_\zeta = \bigcup_{m>0} ((\text{id} - L_\zeta^m)^{-1}(F_m^\zeta) \cap \mathbb{Z}^d)$ is finite and satisfies*

$$\bigcup_{n \geq 0} L_\zeta^n(K_\zeta) + F_n^\zeta = \mathbb{Z}^d,$$

using the notation $F_0^\zeta = \{0\}$.

Proof. Set $\mathbf{n} \in \mathbb{Z}^d$ and consider the sequence $(\mathbf{a}_n)_{n \geq 0} \subseteq \mathbb{Z}^d$ given by $\mathbf{a}_0 = \mathbf{n}$ and for any $n \geq 0$, \mathbf{a}_{n+1} is defined as the unique element in \mathbb{Z}^d such that there exists an element $\mathbf{f}_n \in F_1^\zeta$ with $\mathbf{a}_n = L_\zeta(\mathbf{a}_{n+1}) + \mathbf{f}_n$. Note that for any $n \geq 0$, $\|\mathbf{a}_{n+1}\| \leq \|L_\zeta^{-1}\| \cdot \|\mathbf{a}_n\| + \|L_\zeta^{-1}\| \cdot \|F_1^\zeta\|$, which implies

$$\|\mathbf{a}_n\| \leq \|L_\zeta^{-1}\|^n \|\mathbf{n}\| + \frac{\|L_\zeta^{-1}\| \|F_1^\zeta\| (1 - 2\|L_\zeta^{-1}\|^n)}{1 - \|L_\zeta^{-1}\|},$$

hence $(\mathbf{a}_n)_{n \geq 0}$ is a bounded sequence in \mathbb{Z}^d . By the Pigeonhole Principle there exist $n \geq 0$ and $k > 0$ such that $\mathbf{a}_n = \mathbf{a}_{n+k}$, i.e., $\mathbf{a}_n = L_\zeta^k(\mathbf{a}_n) + \mathbf{f}$, for some $\mathbf{f} \in F_k^\zeta$. It follows the set $K_\zeta = \bigcup_{m>0} ((\text{id} - L_\zeta^m)^{-1}(F_m^\zeta) \cap \mathbb{Z}^d)$ satisfies the property. Now we prove that K_ζ is finite. Note that for any $m > 0$

$$\begin{aligned} \|(\text{id} - L_\zeta^m)^{-1}(F_m^\zeta)\| &= \left\| \sum_{i=0}^{m-1} (\text{id} - L_\zeta^m)^{-1} L_\zeta^i(F_1^\zeta) \right\| \\ &\leq \|F_1^\zeta\| \left\| \sum_{i=0}^{m-1} (\text{id} - L_\zeta^m)^{-1} L_\zeta^i \right\| \\ &\leq \|F_1^\zeta\| \left\| (\text{id} - L_\zeta^m)^{-1} (\text{id} - L_\zeta^m) (\text{id} - L_\zeta)^{-1} \right\| \\ &\leq \|F_1^\zeta\| \|(\text{id} - L_\zeta)^{-1}\|. \end{aligned}$$

We conclude that K_ζ is a finite set. □

Remark 1.19. The following statements can be easily verified.

- (1) In the one-dimensional case for any constant-length substitution ζ , we have that $K_\zeta = \{-1, 0\}$. Moreover, for any d -dimensional block substitution, i.e., L_ζ equal to the diagonal matrix $\text{diag}(l_1, \dots, l_d)$, and F_1^ζ equal to the standard d -dimensional parallelepiped support $\prod_{i=1}^d [0, l_i - 1]$, then $K_\zeta = [-1, 0]^d$.
- (2) If L_ζ is such that $|\det(L_\zeta - \text{id})| = 1$, then K_ζ is equal to $(\text{id} - L_\zeta^{-1})^{-1}(F_1^\zeta)$.

(3) Since K_ζ is a finite set, there exists $j > 0$ such that $K_\zeta = \bigcup_{m=0}^j ((\text{id} - L_\zeta^m)^{-1}(F_m^\zeta) \cap \mathbb{Z}^d)$.

When the substitution ζ is primitive, up to considering a power of ζ , we may assume that the set K_ζ is of the form $(\text{id} - L_\zeta)^{-1}(F_1^\zeta) \cap \mathbb{Z}^d$.

The argument used in the proof of Proposition 1.18 is inspired by the Euclidean Division Algorithm. A similar idea can be used to find different sets satisfying specific statements involving the supports $(F_n^\zeta)_{n>0}$ that will be needed for some proofs.

From now on for any $\mathbf{n} \in \mathbb{Z}^d$ we denote by $\mathbf{d}_\mathbf{n} \in \mathbb{Z}^d$ and $\mathbf{f}_\mathbf{n} \in F_1^\zeta$ the unique elements such that $\mathbf{n} = L_\zeta(\mathbf{d}_\mathbf{n}) + \mathbf{f}_\mathbf{n}$. The following result will be useful in a series of results throughout this thesis.

Proposition 1.20. *Set $A \in \mathbb{Z}^d$ and let $F \in \mathbb{Z}^d$ be such that $F_1^\zeta \subseteq F$. Define $B = \{\mathbf{d}_\mathbf{n}\}_{\mathbf{n} \in F+A}$. Then, there exists a finite subset C of \mathbb{Z}^d satisfying the following conditions:*

1. $B \subseteq C$.
2. $C + F + A \subseteq L_\zeta(C) + F_1^\zeta$.
3. $\|C\| \leq \|B\| + \|L_\zeta^{-1}\| \left(\|A\| + \|F\| + \|F_1^\zeta\| \right) / \left(1 - \|L_\zeta^{-1}\| \right)$.

Proof. We define two sequences of finite sets of \mathbb{Z}^d , $(B_n)_{n \geq 0}$, $(C_n)_{n \geq 0}$, with $B_0 = B$, $C_0 = B + F + A$, and for any $n \geq 0$, set $B_{n+1} \in \mathbb{Z}^d$ such that $B_{n+1} = \{\mathbf{d}_\mathbf{n}\}_{\mathbf{n} \in C_n}$, and $C_{n+1} \in \mathbb{Z}^d$ such that $C_{n+1} = B_{n+1} + F + A$. Note that

$$\begin{aligned} \|B_{n+1}\| &\leq \|L_\zeta^{-1}\| \left(\|C_n\| + \|F_1^\zeta\| \right) \\ &\leq \|L_\zeta^{-1}\| \left(\|B_n\| + \|A\| + \|F\| + \|F_1^\zeta\| \right) \\ &\leq \|L_\zeta^{-1}\| \|B_n\| + \|L_\zeta^{-1}\| \left(\|A\| + \|F\| + \|F_1^\zeta\| \right). \end{aligned}$$

Hence, for any $n > 0$ we have that

$$\|B_n\| \leq \|B\| \|L_\zeta^{-1}\|^n + \frac{1 - \|L_\zeta^{-1}\|^n}{1 - \|L_\zeta^{-1}\|} \left(\|L_\zeta^{-1}\| \left(\|A\| + \|F\| + \|F_1^\zeta\| \right) \right).$$

Since $\|L_\zeta^{-1}\|$ is strictly smaller than 1, then $\|B_n\| \leq \|B\| + \|L_\zeta^{-1}\| \left(\|A\| + \|F\| + \|F_1^\zeta\| \right) / \left(1 - \|L_\zeta^{-1}\| \right)$. This implies there exists $N \in \mathbb{N}$ such that $\bigcup_{n \leq N} B_n = \bigcup_{n \leq N+1} B_n$. We conclude the proof taking $C = \bigcup_{n=0}^N B_n$. \square

Remark 1.21. The following statements will be useful on the rest of the thesis.

- (1) Condition 2. implies $C + A + F_1^\zeta \subseteq L_\zeta(C) + F_1^\zeta$, and a direct induction proves that for all $n \geq 0$, we have that $L_\zeta^n(C + A) + F_n^\zeta \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$.

- (2) Using $F = F_1^\zeta + F_1^\zeta$ and $A = \{\mathbf{0}\}$, we obtain a set $C \Subset \mathbb{Z}^d$ such that $C + F_1^\zeta + F_1^\zeta \subseteq L_\zeta(C) + F_1^\zeta$. Since $\mathbf{0} \in F_1^\zeta$, then $\mathbf{0}$ is in C , which implies $F_1^\zeta + F_1^\zeta \subseteq L_\zeta(C) + F_1^\zeta$. Assume that for some $n > 0$ we have that $F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$. Then, we obtain that

$$\begin{aligned} F_{n+1}^\zeta + F_{n+1}^\zeta &= F_n^\zeta + F_n^\zeta + L_\zeta^n(F_1^\zeta) + L_\zeta^n(F_1^\zeta) \\ &\subseteq L_\zeta^n(C) + F_n^\zeta + L_\zeta^n(F_1^\zeta) + L_\zeta^n(F_1^\zeta) \\ &\subseteq L_\zeta^{n+1}(C) + L_\zeta^n(F_1^\zeta) + F_n^\zeta \\ &= L_\zeta^{n+1}(C) + F_{n+1}^\zeta, \end{aligned}$$

so, by induction we prove that for all $n > 0$, $F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$.

The next result shows that for any aperiodic symbolic factor (Y, S, \mathbb{Z}^d) of $(X_\zeta, S, \mathbb{Z}^d)$ we can change ζ for an appropriate substitution ζ' with the same expansion matrix and fundamental domain such that $(X_{\zeta'}, S, \mathbb{Z}^d)$ and $(X_\zeta, S, \mathbb{Z}^d)$ are conjugate, and there exists a factor $\pi : (X_{\zeta'}, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ induced by a 0-block map.

Lemma 1.22. *Let ζ be an aperiodic primitive constant-shape substitution and $\phi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ be an aperiodic symbolic factor of $(X_\zeta, S, \mathbb{Z}^d)$. Then, there exists a substitution ζ' having the same support and expansion matrix such that $(X_{\zeta'}, S, \mathbb{Z}^d)$ and $(X_\zeta, S, \mathbb{Z}^d)$ are conjugate and a factor map $\pi : (X_{\zeta'}, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ induced by a 0-block map.*

Proof. Suppose that $\phi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ is a factor map via a $B(\mathbf{0}, r)$ -block map. Set $A = B(\mathbf{0}, r)$, by Proposition 1.20 there exists a set $C \Subset \mathbb{Z}^d$ such that $B(\mathbf{0}, r) + F_1^\zeta + C \subseteq L_\zeta(C) + F_1^\zeta$. Set $D = L_\zeta(C) + F_1^\zeta$. We will define a substitution $\zeta^{(D)}$ considering the set $\mathcal{L}_D(X_\zeta)$ as the alphabet with the same expansion matrix and support of ζ in the following way: If $\mathbf{p} \in \mathcal{L}_D(X_\zeta)$, then for any $\mathbf{j} \in F_1^\zeta$ we set $\zeta^{(D)}(\mathbf{p})_{\mathbf{j}} = \zeta(\mathbf{p})|_{\mathbf{j}+D}$. It is straightforward to check that $x \in X_\zeta$ is a fixed point of the substitution ζ , if and only if $y \in \mathcal{L}_D(X_\zeta)^{\mathbb{Z}^d}$ such that $y_{\mathbf{n}} = x|_{\mathbf{n}+D}$ for all $\mathbf{n} \in \mathbb{Z}^d$ is a fixed point of the substitution $\zeta^{(D)}$. With this, we can define the following sliding block codes $\psi_1 : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (X_{\zeta^{(D)}}, S, \mathbb{Z}^d)$ given by the D -block map $\Psi_1(\mathbf{p}) = \mathbf{p}$, and $\psi_2 : (X_{\zeta^{(D)}}, S, \mathbb{Z}^d) \rightarrow (X_\zeta, S, \mathbb{Z}^d)$ given by the 0-block map $\Psi_2(\mathbf{p}) = \mathbf{p}_0$. These maps commute with the shift action and define a conjugacy between X_ζ and $X_{\zeta^{(D)}}$. We then, define a factor map $\phi^{(D)} : (X_{\zeta^{(D)}}, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ given by a 0-block map equal to $\psi_2\phi$. \square

Chapter 2

The symmetry semigroup of \mathbb{Z}^2 -odometers

In this chapter, we will study the symmetry semigroup of some odometer systems. In [58] there is a first attempt to characterize this semigroup for \mathbb{Z}^2 -odometers. We start with the d -dimensional universal odometer system. This odometer is universal in the sense that any d -dimensional odometer system is a topological factor of this one. We get that its symmetry semigroup is the largest one and equal to $GL(d, \mathbb{Z})$. Then, we will restrict on two dimensional constant base odometer systems $\overleftarrow{\mathbb{Z}}^2_{(L^n)}$, where $L \in \mathcal{M}(2, \mathbb{Z})$ is an integer expansion matrix. In this case, we get a description of a bifurcation phenomenon at the level of the symmetry semigroup with respect to arithmetical relations of invariants of the matrix. The main theorem (Theorem 2.2) shows that in most cases the symmetry semigroup is the centralizer of the matrix L . This will help to get a characterization of the normalizer semigroup of aperiodic primitive constant-shape substitutions using the relation between homomorphisms and their maximal equicontinuous factors (Lemma 1.7).

2.1 The universal odometer

Let $(\Gamma_n)_{n \in \mathbb{N}}$ be a enumeration of all finite index subgroups of \mathbb{Z}^d . We define the d -dimensional universal odometer system as follows: Set $Z_0 = \Gamma_0$, and for any $n > 1$ set $Z_n = \Lambda_{n-1} \cap \Gamma_n$. Then, we define the d -dimensional universal odometer as $\overleftarrow{\mathbb{Z}}^d_{(Z_n)}$. This odometer is universal in the sense that by Lemma 1.13 any odometer system is a topological factor of the universal odometer. For instance, the 1-dimensional universal odometer system is equal to $\overleftarrow{\mathbb{Z}}_{(n!)}$. With respect to its symmetry semigroup, (NC 2) implies the following result.

Proposition 2.1. *The symmetry semigroup of the universal odometer is $GL(d, \mathbb{Z})$.*

Proof. Consider $L_n \in \mathcal{M}(d, \mathbb{Z})$ such that $L_n(\mathbb{Z}^d) = \Lambda_n$. A matrix $M \in GL(d, \mathbb{Z})$ is in

$\vec{N}(\overleftarrow{\mathbb{Z}}^d_{(\Lambda_n)})$ if and only if M satisfies (NC 2). Now, for any $n \in \mathbb{N}$ we can choose $m(n) \in \mathbb{N}$ large enough such that $\Lambda_{m(n)} \leq \det(L_n)\mathbb{Z}^d$, so $\text{adj}(L_n)ML_{m(n)} \equiv 0 \pmod{\det(L_n)}$. We then conclude that $\vec{N}(\overleftarrow{\mathbb{Z}}^d_{(\Lambda_n)}) = GL(d, \mathbb{Z})$. \square

2.2 Bifurcation phenomenon of the normalizer semigroup for constant base \mathbb{Z}^2 -odometer systems

To describe the different cases of the normalizer semigroup of odometer systems of the form $\overleftarrow{\mathbb{Z}}^2_{(L^n)}$, where $L \in \mathcal{M}(2, \mathbb{Z})$ is an integer expansion matrix, we need some extra notations. For any positive integer $n > 0$, the *radical* $\text{rad}(n)$ of n is defined as the product of the distinct prime numbers dividing n , if $n < 0$ we define $\text{rad}(n)$ just as $\text{rad}(-n)$. We define the *centralizer of a matrix L on $GL(2, \mathbb{Z})$* $\text{Cent}_{GL(2, \mathbb{Z})}(L)$ as the subgroup of all matrices in $GL(2, \mathbb{Z})$ commuting with L . Recall by Corollary 1.15 the centralizer $\text{Cent}_{GL(2, \mathbb{Z})}(L)$ is always a subgroup of the symmetry semigroup $\vec{N}(\overleftarrow{\mathbb{Z}}^d_{(L^n)})$.

From now on, we will fix the following notation. An integer expansion matrix will be denoted as $L = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ and its powers as $L^n = \begin{pmatrix} p(n) & q(n) \\ r(n) & s(n) \end{pmatrix}$. We will denote a matrix in $GL(2, \mathbb{Z})$ as $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$.

The following theorem summarizes the different cases of the symmetry group $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ depending on the matrix L .

Theorem 2.2. *Let $L \in \mathcal{M}(2, \mathbb{Z})$ be an integer expansion matrix.*

- *If $\text{rad}(\det(L))$ divides $\text{trace}(L)$, then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is equal to $GL(2, \mathbb{Z})$.*
- *Otherwise*
 1. *If the spectrum of the matrix L is disjoint from the integers, then the symmetry semigroup $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is the centralizer $\text{Cent}_{GL(2, \mathbb{Z})}(L)$. Moreover, if the spectrum of L is disjoint from the real line, then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is a finite group.*
 2. *When the spectrum of L contains an integer value, then the matrix coefficients of elements in $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n(\mathbb{Z}^2))})$ are characterized by linear relations with respect to the coefficients given by the one of the matrix L . In particular, under explicit arithmetical properties of the coefficients of L , $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n(\mathbb{Z}^2))})$ can be isomorphic to $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}^2/2\mathbb{Z} \times 2\mathbb{Z}$, or $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n(\mathbb{Z}^2))})/(\text{Cent}_{GL(2, \mathbb{Z})}(L))$ can be virtually to \mathbb{Z} .*

Along the proof of Theorem 2.2 we will get more precise information about the symmetry semigroup, when we have more restrictions on the matrix L .

Since odometer systems are coalescent systems, then as a consequence of the proof of Theorem 2.2 and Proposition 1.5 we get the following result.

Proposition 2.3. *Let $L \in \mathcal{M}(2, \mathbb{Z})$ be an integer expansion matrix. Then, the symmetry semigroup of $\overleftarrow{\mathbb{Z}^2}_{(L^n)}$ is a group. In particular, any homomorphism in $N(\overleftarrow{\mathbb{Z}^2}_{(L^n)})$ is invertible.*

Proposition 2.3 will be proved in Section 2.5.

The following examples illustrate the different consequences of Theorem 2.2 according to the expansion matrix L .

Example 2.4 (Different results for Theorem 2.2). (1) Consider the matrix $L_1 = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$. We have that $\text{trace}(L_1) = 5$, and $\det(L_1) = 7$, so by Theorem 2.2 a matrix $M \in GL(2, \mathbb{Z})$ is in $\vec{N}(\overleftarrow{\mathbb{Z}^2}_{(L_1^n)})$ if and only if M commutes with L_1 . Note that $\text{trace}(L_1)^2 - 4\det(L_1) = -3$, so L_1 has complex eigenvalues (which are $5/2 \pm i\sqrt{3}/2$), and $\text{Cent}_{GL(2, \mathbb{Z})}(L_1)$ is equal to

$$\left\{ \begin{array}{l} \left(\begin{array}{cc} 1 & 1 \\ -1 & 0 \end{array} \right), \left(\begin{array}{cc} -1 & -1 \\ 1 & 0 \end{array} \right), \left(\begin{array}{cc} 0 & -1 \\ 1 & 1 \end{array} \right) \\ \left(\begin{array}{cc} 0 & -1 \\ 1 & -1 \end{array} \right), \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right), \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) \end{array} \right\}.$$

(2) Consider the matrix $L_2 = \begin{pmatrix} 2 & -1 \\ 1 & 5 \end{pmatrix}$. In this case $\text{trace}(L_2) = 7$ and $\det(L_2) = 11$, hence by Theorem 2.2 the matrices in $\vec{N}(\overleftarrow{\mathbb{Z}^2}_{(L_2^n)})$ are the ones commuting with L_2 . Note that L_2 has real eigenvalues (which are equal to $7/2 \pm \sqrt{5}/2$) and $\text{Cent}_{GL(2, \mathbb{Z})}(L_2)$ is an infinite group containing $\begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$.

(3) Set $L_3 = \begin{pmatrix} 0 & 6 \\ -3 & 9 \end{pmatrix}$. This matrix has integer eigenvalues, which are 3 and 6. We will get that a matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ is in $\vec{N}(\overleftarrow{\mathbb{Z}^2}_{(L_3^n)})$ if and only if $|m_{11}m_{22} - m_{12}m_{21}| = 1$ and $m_{12} = 4m_{21} + 2m_{22} - 2m_{11}$. This set of matrices is conjugate, via the matrix $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, to the set of matrices $\left\{ \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix} : |m_{11}m_{22}| = 1, m_{12} \in \mathbb{Z} \right\}$. It is easy to see that $\vec{N}(\overleftarrow{\mathbb{Z}^2}_{(L_3^n)}) / \text{Cent}_{GL(2, \mathbb{Z})}(L_3)$ is virtually \mathbb{Z} .

Remark 2.5. Theorem 2.2 implies the factor map between equicontinuous systems is not necessarily compatible with homomorphisms. Consider X as the 2-dimensional universal odometer, and $Y = \overleftarrow{\mathbb{Z}}^2_{(L_1^n)}$, where $L_1 = \begin{pmatrix} 2 & -1 \\ 1 & 3 \end{pmatrix}$, so Y is an equicontinuous factor of X . Now, by Proposition 2.1, we can define a homomorphism associated with the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ in X , but by Theorem 2.2 and Lemma 1.6, this is not possible in Y .

2.3 The diagonal case

We will decompose the proof of Theorem 2.2 starting with the case where the expansion matrix has integer eigenvalues. First we will study the diagonal case, i.e., $q = r = 0$, since we will get more precise results about the symmetry semigroup. Note that $\det(L) = ps$ and $\text{trace}(L) = p + s$, so the condition $\text{rad}(\det(L))$ divides $\text{trace}(L)$ is equivalent to $\text{rad}(p) = \text{rad}(s)$. The following result is a particular case of Theorem 2.2 for diagonal matrices.

Proposition 2.6. *Let $L = \text{diag}(p, s)$ be an expansion 2×2 diagonal matrix such that $\text{rad}(\det(L))$ does not divide $\text{trace}(L)$. We have the following:*

1. *If $(\text{rad}(p)$ does not divide s and $\text{rad}(s)$ divides p), or $(\text{rad}(p)$ divides s and $\text{rad}(s)$ does not divide p), then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})/(\text{Cent}_{GL(2, \mathbb{Z})}(L))$ is virtually \mathbb{Z} .*
2. *If $\text{rad}(p)$ does not divide s and $\text{rad}(s)$ does not divide p , then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $M \in GL(2, \mathbb{Z})$ satisfying (NC 2). Then, for any $n > 0$, we get the following equations

$$\text{adj}(L^n)ML^{m(n)} = \begin{pmatrix} m_{11}p^{m(n)}s^n & m_{12}s^{m(n)+n} \\ m_{21}p^{m(n)+n} & m_{22}s^{m(n)}p^n \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{p^n s^n}, \quad (2.1)$$

Assuming $m(n) > 0$ is large enough, we simplify the equations in the anti-diagonal, and we get

$$m_{12}s^{m(n)} \equiv 0 \pmod{p^n}, \quad (2.2)$$

$$m_{21}p^{m(n)} \equiv 0 \pmod{s^n} \quad (2.3)$$

If $\text{rad}(p)$ does not divide s , there exists a prime number t dividing p such that for all $m > 0$, and all $n > 0$, s^m is an invertible element in $\mathbb{Z}/t^n\mathbb{Z}$. This implies $m_{12} \equiv 0 \pmod{t^n}$ for all $n > 0$, hence $m_{12} = 0$. On the other hand, if $\text{rad}(p)$ divides s , then for any $n > 0$, there exists $m(n) > 0$ large enough such that p^n divides $s^{m(n)}$. In this case any $m_{12} \in \mathbb{Z}$

is solution of (2.2). Using the same arguments we have the same conclusion for m_{21} with respect to $\text{rad}(s)$ and p .

Assuming $\text{rad}(p)$ does not divide s and $\text{rad}(s)$ does not divide p , we get $m_{12} = 0$ and $m_{21} = 0$, so the only matrices satisfying (NC 2) are diagonal matrices. We conclude $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is isomorphic to $\mathbb{Z}^2/(2\mathbb{Z} \times 2\mathbb{Z})$. \square

Remark 2.7. The proof of Proposition 2.6 can be easily generalized to higher dimensions in the following way: Suppose that $L = \text{diag}(p_1, \dots, p_d)$ is an expansion $d \times d$ diagonal matrix. If $\text{rad}(p_i)$ does not divide p_j for some $1 \leq i, j \leq d$, then for any $M \in \vec{N}(\overleftarrow{\mathbb{Z}}^d_{(L^n)})$, we have that $m_{i,j} = 0$. In particular, if for any pair of distinct indices $1 \leq i \neq j \leq d$, $\text{rad}(p_i)$ does not divide p_j , then by (NC 2), $\vec{N}(\overleftarrow{\mathbb{Z}}^d_{(L^n)})$ is isomorphic to $\mathbb{Z}^d/(2\mathbb{Z} \times \dots \times 2\mathbb{Z})$, as can be deduced by the proof of Theorem 28 in [21].

2.4 The triangular case

We will next consider the case where the expansion matrix is a triangular matrix. We will focus on the upper triangular case, i.e., $q \neq 0$ and $r = 0$, since the lower triangular case can be deduce by this one via conjugation with the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For all $n \in \mathbb{Z}$ we have that

$$L^n = \begin{pmatrix} p^n & q(n) \\ 0 & s^n \end{pmatrix},$$

where $q(n) = q(p^n - s^n)/(p - s) = q \sum_{i=0}^{n-1} p^i s^{n-1-i}$. As in the diagonal case, $\det(L) = ps$ and $\text{trace}(L) = p + s$, so the condition $\text{rad}(\det(L))$ divides $\text{trace}(L)$ is equivalent to $\text{rad}(p) = \text{rad}(s)$. In this case, there is a similar result to the one obtained for the diagonal case, as it is shown in the following proposition:

Proposition 2.8. *Let $L \in \mathcal{M}(2, \mathbb{Z})$ be an expansion upper triangular matrix such that $\text{rad}(\det(L))$ does not divide $\text{trace}(L)$. Then, we have the following:*

1. *If $\text{rad}(p)$ does not divide s and $\text{rad}(s)$ divides p , then a matrix $M \in GL(2, \mathbb{Z})$ is in $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{L^n})$ if and only if $(p - s)^2 m_{12} = m_{21} q^2 + (p - s)(m_{11} - m_{22})q$. In particular, if $(p - s) | q$, then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{L^n}) / (\text{Cent}_{GL(2, \mathbb{Z})}(L))$ is virtually \mathbb{Z} .*
2. *Assume that $\text{rad}(p)$ divides s and $\text{rad}(s)$ does not divide p . Then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)}) / (\text{Cent}_{GL(2, \mathbb{Z})}(L))$ is virtually \mathbb{Z} .*
3. *If $\text{rad}(p)$ does not divide s and $\text{rad}(s)$ does not divide p , we have two cases:*
 - *If $2q \in (p - s)\mathbb{Z}$ then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

- Otherwise, $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof. Let M be in $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$. Define the matrix $\overline{M} = (p-s)M - (m_{11} - m_{22})L - (p \cdot m_{22} - m_{11} \cdot s) \text{id}_{\mathbb{R}^2}$. Then, \overline{M} satisfies (Normalizer Condition). Note that \overline{M} has the form

$$\overline{M} = \begin{pmatrix} 0 & \overline{m}_{12} \\ \overline{m}_{21} & 0 \end{pmatrix},$$

where $\overline{m}_{12} = (p-s)m_{12} - (m_{11} - m_{22})q$ and $\overline{m}_{21} = (p-s)m_{21}$, with $\overline{m}_{12}, \overline{m}_{21} \in \mathbb{Z}$. Now, for all $n, m > 0$, (NC 2) implies

$$\begin{pmatrix} -\overline{m}_{21}p^m q(n) & \overline{m}_{12}s^{n+m} - \overline{m}_{21}q(m)q(n) \\ \overline{m}_{21}p^{n+m} & \overline{m}_{21}pq(m) \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{p^n s^n}. \quad (2.4)$$

Suppose $\text{rad}(s)$ does not divide p . Then, there exists a prime number t dividing s such that for all $n > 0$ and $m > 0$, p^m is an invertible element in $\mathbb{Z}/t^n\mathbb{Z}$. Hence, $\overline{m}_{21} \equiv 0 \pmod{t^n}$, which implies $\overline{m}_{21} = 0$, so $m_{21} = 0$. Now, by (2.4) we get that

$$\overline{m}_{12}s^m \equiv 0 \pmod{p^n}. \quad (2.5)$$

We have two cases:

- If $\text{rad}(p)$ does not divide s , then (2.5) implies $\overline{m}_{12} = 0$. We conclude that $\overline{M} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, i.e., $(p-s)M = (m_{11} - m_{22})L + (p \cdot m_{22} - m_{11} \cdot s) \text{id}_{\mathbb{R}^2}$. Since $m_{21} = 0$, then M has the form

$$M = \begin{pmatrix} m_{11} & m_{12}(m_{11}, m_{22}) \\ 0 & m_{22} \end{pmatrix}.$$

where $m_{12}(m_{11}, m_{11})$ satisfies $(p-s)m_{12}(m_{11}, m_{11}) = (m_{11} - m_{22})q$.

- Note that $m_{11} = m_{22}$ if and only if $m_{12} = 0$.
- If $m_{11} \neq m_{22}$, then $m_{11} - m_{22} \in \{-2, 2\}$, so $(p-s)m_{12} = \pm 2q$. Since M has integer coefficients, this necessarily implies $2q \in (p-s)\mathbb{Z}$. If this condition is satisfied, then M has the form

$$M = \begin{pmatrix} m_{11} & \frac{(m_{11}-m_{22})q}{p-s} \\ 0 & m_{22} \end{pmatrix}.$$

It is not difficult to see that M^2 is the identity matrix. We conclude that $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. If $2q \notin (p-s)\mathbb{Z}$, then $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

- If $\text{rad}(p)$ divides s , then any $\overline{m}_{12} \in \mathbb{Z}$ satisfies (2.5). Thus, any matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ 0 & m_{22} \end{pmatrix}$ with $|m_{11}m_{22}| = 1$ satisfies (NC 2).

Finally, if $\text{rad}(s)$ divides p , then for any $n > 0$, and any m large enough s^n divides p^m and $q(m)$. Let t be a prime number dividing p that does not divide s . Then, by (2.4) we obtain

$$(p-s)^2 \overline{m}_{12} s^{n+m} \equiv \overline{m}_{21} q^2 s^{n+m} \pmod{t^n}. \quad (2.6)$$

Since t does not divide s , then for any $n, m > 0$, s^{n+m} is an invertible element in $\mathbb{Z}/t^n\mathbb{Z}$, so (2.6) is reduced to

$$(p-s)^2 \overline{m}_{12} \equiv \overline{m}_{21} q^2 \pmod{t^n}, \quad (2.7)$$

which implies $(p-s)^2 \overline{m}_{12} = \overline{m}_{21} q^2$. Thus, we get that

$$(p-s)^2 m_{12} = m_{21} q^2 + (p-s)(m_{11} - m_{22})q. \quad (2.8)$$

Now, if $(p-s)$ divides q , we write $q = k(p-s)$ for some $k \in \mathbb{Z}$. By (2.8), we have that

$$m_{12} = m_{21} k^2 + k(m_{11} - m_{22}).$$

Since $|\det(M)| = 1$, and $\det(M) = (m_{11} + m_{21} \cdot k)(m_{22} - m_{21} \cdot k)$, we get that $|m_{11} + m_{21} \cdot k| = 1$ and $|m_{22} - m_{21} \cdot k| = 1$. We then can parameterize the matrices in $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ as follows:

$$\left\{ \begin{array}{l} \left(\begin{array}{cc} 1 - m \cdot k & -mk^2 \\ m & 1 + m \cdot k \end{array} \right), \left(\begin{array}{cc} 1 - m \cdot k & 2k - mk^2 \\ m & m \cdot k - 1 \end{array} \right) \\ \left(\begin{array}{cc} -1 - m \cdot k & -2k - mk^2 \\ m & 1 + m \cdot k \end{array} \right), \left(\begin{array}{cc} -1 - m \cdot k & -mk^2 \\ m & -1 + m \cdot k \end{array} \right) \end{array} : m \in \mathbb{Z} \right\}.$$

In this case, L is conjugate to the matrix $\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}$, via the matrix $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$, so by the diagonal case (Proposition 2.6) we conclude that $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})/(\text{Cent}_{GL(2,\mathbb{Z})}(L))$ is virtually \mathbb{Z} . □

2.5 Proof of Theorem 2.2

Now we will prove Theorem 2.2. Since we already proved the triangular case, we assume that $q \neq 0$ and $r \neq 0$.

Proof of Theorem 2.2. Let $L \in \mathcal{M}(2, \mathbb{Z})$ be an integer expansion matrix. The Cayley-Hamilton theorem implies $L^2 = \text{trace}(L) \cdot L - \det(L) \cdot \text{id}_{\mathbb{R}^2}$. If $\text{rad}(\det(L))$ divides $\text{trace}(L)$, then $L^2 \equiv 0 \pmod{\text{rad}(\det(L))}$. Hence, for all $n > 0$ and any $m(n) > 0$ large enough, we get that $L^{m(n)} \equiv 0 \pmod{\det(L)^n}$. So any matrix in $GL(2, \mathbb{Z})$ satisfies (NC 2), and we conclude that $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)}) = GL(2, \mathbb{Z})$.

Suppose now that $\text{rad}(\det(L))$ does not divide $\text{trace}(L)$. By induction, for any $n > 0$ we have that $L^n \equiv \text{trace}(L)^{n-1}L \pmod{\det(L)}$. Since $\text{rad}(\det(L))$ does not divide $\text{trace}(L)$, there exists a prime number t dividing $\det(L)$ that does not divide p or s . Without loss of generality (up to a conjugation with $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$) we may assume that t does not divide s . Since $s(n) \equiv \text{trace}(L)^{n-1}s \pmod{\det(L)}$, then, for all $n > 0$, and $m > 0$, $s(m)$ is an invertible element in $\mathbb{Z}/t^n\mathbb{Z}$.

Let $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ be in $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$. Define the matrix $\overline{M} = rM - m_{21}L - (r \cdot m_{11} - p \cdot m_{21}) \text{id}_{\mathbb{R}^2}$. Then, the matrix \overline{M} satisfies (Normalizer Condition) and has the form

$$\overline{M} = \begin{pmatrix} 0 & \overline{m_{12}} \\ 0 & \overline{m_{22}} \end{pmatrix},$$

where $\overline{m_{12}} = r \cdot m_{12} - q \cdot m_{21}$, $\overline{m_{22}} = r \cdot m_{22} - s \cdot m_{21} - r \cdot m_{11} + p \cdot m_{21}$, with $\overline{m_{12}}, \overline{m_{21}} \in \mathbb{Z}$. Now, for all $n, m > 0$

$$\begin{aligned} \text{adj}(L^n)ML^m &= \begin{pmatrix} r(m)(\overline{m_{12}}s(n) - \overline{m_{22}}q(n)) & s(m)(\overline{m_{12}}s(n) - \overline{m_{22}}q(n)) \\ r(m)(-\overline{m_{12}}r(n) + \overline{m_{22}}p(n)) & s(m)(-\overline{m_{12}}r(n) + \overline{m_{22}}p(n)) \end{pmatrix} \\ &\equiv \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \pmod{\det(L)^n}. \end{aligned} \quad (2.9)$$

Since $s(m)$ is an invertible element in $\mathbb{Z}/t^n\mathbb{Z}$, (2.9) implies

$$\overline{m_{12}}s(n) - \overline{m_{22}}q(n) \equiv 0 \pmod{t^n}, \quad (2.10)$$

$$-\overline{m_{12}}r(n) + \overline{m_{22}}p(n) \equiv 0 \pmod{t^n}, \quad (2.11)$$

which is equivalent to

$$\text{adj}(L^n) \begin{pmatrix} \overline{m_{12}} \\ \overline{m_{22}} \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pmod{t^n}. \quad (2.12)$$

Consider the set $E = \left\{ \begin{pmatrix} \overline{m_{12}} \\ \overline{m_{22}} \end{pmatrix} \in \mathbb{Z}^2 : \text{satisfying (2.12) for all } n > 0 \right\}$. This set is $\text{adj}(L)$ -invariant, and if $\overline{m_{22}} = 0$, then by (2.10), we get that $\overline{m_{12}}s(n) \equiv 0 \pmod{t^n}$, which implies $\overline{m_{12}} = 0$.

Now, set $\left(\frac{\overline{m_{12}^1}}{\overline{m_{22}^1}}\right), \left(\frac{\overline{m_{12}^2}}{\overline{m_{22}^2}}\right) \in E$. Note that

$$\overline{m_{22}^2} \left(\frac{\overline{m_{12}^1}}{\overline{m_{22}^1}}\right) - \overline{m_{22}^1} \left(\frac{\overline{m_{12}^2}}{\overline{m_{22}^2}}\right) = \begin{pmatrix} \overline{m_{22}^2} \overline{m_{12}^1} - \overline{m_{22}^1} \overline{m_{12}^2} \\ 0 \end{pmatrix},$$

hence $\overline{m_{22}^2} \left(\frac{\overline{m_{12}^1}}{\overline{m_{22}^1}}\right) = \overline{m_{22}^1} \left(\frac{\overline{m_{12}^2}}{\overline{m_{22}^2}}\right)$, so E is an $\text{adj}(L)$ -invariant \mathbb{Z} -module of rank at most 1.

If L does not have integer eigenvalues, E must have rank 0. This implies $\overline{m_{12}} = \overline{m_{22}} = 0$.

Hence $\overline{M} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and then rM (so M) commutes with L .

Since $\overline{m_{12}} = \overline{m_{22}} = 0$, we have that

$$\begin{aligned} r \cdot m_{12} - q \cdot m_{21} &= 0 \\ r \cdot m_{22} - s \cdot m_{21} - r \cdot m_{11} + p \cdot m_{21} &= 0. \end{aligned} \tag{2.13}$$

- Suppose $p = s$. In this case, (2.13) implies $m_{11} = m_{22}$ and $m_{21} = m_{11} \cdot r/q$. Note that L has complex eigenvalues if and only if $(2p)^2 - 4(p^2 - qr) < 0$, i.e., $qr < 0$. Since $|\det(M)| = 1$, then $m_{11}^2 - m_{12}^2 \cdot r/q = \pm 1$, so the condition $qr < 0$ implies there exists a finite number of points $(m_{11}, m_{12}) \in \mathbb{Z}^2$ satisfying (2.13).
- If $p \neq s$, then (2.13) implies $m_{12} = q(m_{11} - m_{22})/(p - s)$ and $m_{21} = r(m_{11} - m_{22})/(p - s)$. Since $M \in GL(2, \mathbb{Z})$, we get that

$$m_{11}m_{22} - (m_{11} - m_{22})^2 \frac{qr}{(p - s)^2} = \pm 1. \tag{2.14}$$

In this case, there is a finite number of solutions if $(p - s)^2 - 4qr < 0$, which is equivalent to $\text{trace}(L)^2 - 4 \det(L) < 0$. Since $\text{trace}(L)^2 - 4 \det(L)$ is the discriminant of the characteristic polynomial of L , this is equivalent to L having complex eigenvalues. \square

Now we will prove Proposition 2.3

Proof of Proposition 2.3. First note that if $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ is in $GL(2, \mathbb{Z})$, then

$$M^{-1} = \begin{cases} \begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix} & \text{if } \det(M) = 1, \\ \begin{pmatrix} -m_{22} & m_{12} \\ m_{21} & -m_{11} \end{pmatrix} & \text{if } \det(M) = -1. \end{cases}$$

Then, we will prove that if M satisfies the arithmetical relations given by the proof of Theorem 2.2, then M^{-1} also satisfies them. We will do it by cases as in the proof of Theorem 2.2:

- The statement is easily to verifies when L is a diagonal matrix.
- If $L = \begin{pmatrix} p & q \\ 0 & s \end{pmatrix}$ is an upper triangular matrix. We only need to prove it for Item 1 in Proposition 2.8. The rest of the cases are easily to verify.

We recall that a matrix M satisfies (Normalizer Condition) in this case if and only if $(p-s)^2 m_{12} = m_{21} q^2 + (p-s)(m_{11} - m_{22})q$. We will see that either of the cases $\det(M) = 1$ or $\det(M) = -1$, M^{-1} also satisfies this arithmetical relation. Indeed

- Assume that $\det(M) = 1$, then

$$\begin{aligned} (p-s)^2(-m_{12}) &= -m_{21}q^2 + (p-s)(m_{22} - m_{11})q \\ (p-s)^2 m_{12} &= m_{21}q^2 + (p-s)(m_{11} - m_{22})q. \end{aligned}$$

- If $\det(M) = -1$, we have that

$$\begin{aligned} (p-s)^2 m_{12} &= m_{21}q^2 + (p-s)((-m_{22}) - (-m_{11}))q \\ (p-s)^2 m_{12} &= m_{21}q^2 + (p-s)(m_{11} - m_{22})q. \end{aligned}$$

We conclude that the statement is true when L is an upper triangular case. We recall the lower triangular case is deduced by this one.

- In the general case, we need to separate it in two cases:
 - If L does not have integer eigenvalues, by Theorem 2.2 the symmetry semi-group $\vec{N}(\overleftarrow{\mathbb{Z}}^2_{(L^n)})$ is exactly $\text{Cent}_{GL(2,\mathbb{Z})}(L)$. Since $\text{Cent}_{GL(2,\mathbb{Z})}(L)$ is a group, we conclude the statement.
 - Suppose now that $L = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ has integer eigenvalues. By the proof of Theorem 2.2 we have a matrix $M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$ satisfies (Normalizer Condition) if and only if the vector $\begin{pmatrix} r \cdot m_{12} - q \cdot m_{21} \\ r \cdot m_{22} - s \cdot m_{21} - r \cdot m_{11} + p \cdot m_{21} \end{pmatrix}$ is an eigenvector of $\text{adj}(L)$ (with associated eigenvalue having a prime divisor not dividing $\text{trace}(L)$). A direct computation shows that in both cases ($\det(M) = 1$ or $\det(M) = -1$), if $M \in GL(2,\mathbb{Z})$ satisfies this property, then the inverse matrix M^{-1} also satisfies it.

We conclude that the statement is true for all the different cases. \square

Remark 2.9. 1. In the particular case $\gcd(\text{trace}(L), \det(L)) = 1$, we can simplify the proof noting that (2.10), (2.11) imply the existence of two sequences

$(k_n^1)_{n>0}, (k_n^2)_{n>0} \subseteq \mathbb{Z}$ such that $\det(L)^n k_n^1 = \overline{m_{12}}s(n) - \overline{m_{22}}q(n)$ and $\det(L)^n k_n^2 = -\overline{m_{12}}r(n) + \overline{m_{22}}p(n)$, i.e.,

$$\begin{aligned} k_n^1 &= \overline{m_{12}} \frac{s(n)}{\det(L)^n} - \overline{m_{22}} \frac{q(n)}{\det(L)^n} \\ k_n^2 &= -\overline{m_{12}} \frac{r(n)}{\det(L)^n} + \overline{m_{22}} \frac{p(n)}{\det(L)^n}. \end{aligned} \tag{2.15}$$

Since L is an expansion matrix, then L^{-1} is a contraction, so we have that

$$\lim_{n \rightarrow \infty} \frac{p(n)}{\det(L)^n} = \lim_{n \rightarrow \infty} \frac{q(n)}{\det(L)^n} = \lim_{n \rightarrow \infty} \frac{r(n)}{\det(L)^n} = \lim_{n \rightarrow \infty} \frac{s(n)}{\det(L)^n} = 0,$$

this implies for all n large enough, $k_n^1 = k_n^2 = 0$, and we conclude that $\overline{m_{12}} = \overline{m_{22}} = 0$.

Chapter 3

The recognizability property of constant-shape substitutions

The recognizability property of substitutions is a combinatorial property that provides a form of invertibility of the morphism, to uniquely decompose points in the substitutive subshift. For \mathbb{Z} -actions, this property has a dynamical interpretation, since it implies the presence of *refining Kakutani–Rokhlin partitions* which allows encoding the dynamics in an infinite graph [66] (called *Bratteli diagrams*). This coding enabled, among other results, the study of topological orbital equivalence of minimal Cantor systems [57], to characterize continuous and measurable eigenvalues [47] and to analyze their invariant measures [16]. It was first proved for any aperiodic primitive substitution by B. Mossé in [89], and then in [15] for the non-primitive case. Also, V. Berthé, W. Steiner, J. M. Thuswaldner, and R. Yassawi [14] studied a recognizability property for different types of one-dimensional morphisms $\zeta : \mathcal{A} \rightarrow \mathcal{B}^{\mathbb{Z}}$, where \mathcal{A} and \mathcal{B} can be different alphabets. In the multidimensional context, B. Solomyak showed in [107] that aperiodic translationally finite self-affine tilings of \mathbb{R}^d satisfy a recognizability property (called *unique composition property*).

In this chapter, we will get a similar result as [107], but for aperiodic symbolic factors (Proposition 3.3). We then present several consequences on invariants of factors and homomorphisms of substitutive subshifts: There exists a finite number of orbits in X_ζ which are invariant by the substitution map (Proposition 3.4). They are not necessarily linearly repetitive (Example 3.6). Nevertheless, there is a polynomial growth of the repetitivity function of substitutive subshifts (Lemma 3.7). We determine their maximal equicontinuous factors, which is an explicit odometer system (Proposition 3.15). Thanks to these last descriptions, we get that any aperiodic symbolic factor of a substitutive subshift is conjugate to a substitutive subshift via a letter-to-letter map (Theorem 3.22). This extends the result proved in [91]. In the original article, a key part is a characterization of periodic sequences by their complexity function. Our proof does not require this characterization, which is known as Nivat’s conjecture in the two-dimensional case (and it is known to be

false for higher dimensions [23]).

3.1 The recognizability property of aperiodic symbolic factors of substitutive subshifts

The substitution ζ seen as a map from X_ζ to $\zeta(X_\zeta)$ is continuous. Moreover, when the constant-shape substitution is aperiodic and primitive this map is actually a homeomorphism. This comes from the notion of *recognizability* of a substitution.

Definition 3.1. A constant-shape substitution ζ is said to be *recognizable*, if there exists $R > 0$ such that for any $x, y \in X_\zeta$ satisfying $x|_{B(\mathbf{0}, R) \cap \mathbb{Z}^d} = y|_{B(\mathbf{0}, R) \cap \mathbb{Z}^d}$, then x is in $\zeta(X_\zeta)$ if and only if y is in $\zeta(X_\zeta)$.

This implies for every $x \in X_\zeta$ there exist a unique $x' \in X_\zeta$ and a unique $\mathbf{j} \in F_1^\zeta$ such that $x = S^{\mathbf{j}}\zeta(x')$. With this, the set $\zeta(X_\zeta)$ is a clopen subset of X_ζ and $\{S^{\mathbf{j}}\zeta(X_\zeta) : \mathbf{j} \in F_1^\zeta\}$ is a clopen partition of X_ζ (a proof for the one-dimensional case can be found in [97], which can be adapted to our case). This property is also satisfied for the iterations ζ^n , for all $n > 0$. The recognizability property was first proved for any aperiodic primitive substitution by B. Mossé in [89] for the one-dimensional case, and in the multidimensional case by B. Solomyak in [107]. In this section we will prove it for aperiodic symbolic factors of substitutive subshifts (Proposition 3.3). This property will allow us to determine its maximal equicontinuous factors. The proof of Proposition 3.3 is a multidimensional analogue of the one given by P. Kůrka in [78] with the use of the following repulsion property for constant-shape substitutions proved in [107].

Proposition 3.2 (Repulsion Property). *[107, Lemma 3.2] Let ζ be an aperiodic primitive constant-shape substitution, and $x \in X_\zeta$. Then, there exists $N > 0$ such that, for all $n > 0$, and every neighborhood $V \subseteq \mathbb{R}^d$ of the origin, if a pattern $\mathbf{p} \sqsubseteq x$, with $(L_\zeta^n(V) \cap \mathbb{Z}^d) + \mathbf{s} \subseteq \text{supp}(\mathbf{p})$ for some $\mathbf{s} \in \mathbb{Z}^d$, has two occurrences $\mathbf{j}_1, \mathbf{j}_2 \in \mathbb{Z}^d$ such that $\mathbf{j}_1 - \mathbf{j}_2 \in L_\zeta^{n-N}(V)$, then \mathbf{j}_1 is equal to \mathbf{j}_2 .*

As proved in Lemma 1.22, we may assume that an aperiodic symbolic factor of a substitutive subshift is induced by a letter-to-letter map.

Proposition 3.3 (Recognizability property of aperiodic symbolic factors of substitutive subshifts). *Let \mathcal{A}, \mathcal{B} be two finite alphabets, ζ be an aperiodic primitive constant-shape substitution from the alphabet \mathcal{A} , and $\mathcal{T} : \mathcal{A} \rightarrow \mathcal{B}$ be a map such that $(\tau(X_\zeta), S, \mathbb{Z}^d)$ is an aperiodic subshift. Let $x \in X_\zeta$ be a fixed point of ζ and $y = \tau(x)$. Then, there exists $R > 0$ such that if $y|_{\mathbf{i}+B(\mathbf{0}, R)} = y|_{\mathbf{j}+B(\mathbf{0}, R)}$ and $\mathbf{i} \in L_\zeta(\mathbb{Z}^d)$, then $\mathbf{j} \in L_\zeta(\mathbb{Z}^d)$.*

Proof. Let N given by Proposition 3.2 and $V \subseteq \mathbb{R}^d$ be a neighborhood of $\mathbf{0}$ such that $L_\zeta^N([-1, 1]^d) \subseteq V$ and let $B = \prod_{i=1}^d I_i$ be a box containing $(V + [-1, 1]^d)$, where I_i is a

finite interval of \mathbb{Z} . Assume the contrary, then for every $n > 0$ there exist $\mathbf{i}_n \in L_\zeta(\mathbb{Z}^d)$, $\mathbf{j}_n \notin L_\zeta(\mathbb{Z}^d)$ and a pattern $\mathbf{u}_n \in \mathcal{L}_B(X_\zeta)$ such that

$$y|_{\mathbf{i}_n + L_\zeta^n(B \cap \mathbb{Z}^d)} = \tau(\zeta^n(\mathbf{u}_n)) = y|_{\mathbf{j}_n + L_\zeta^n(B)}.$$

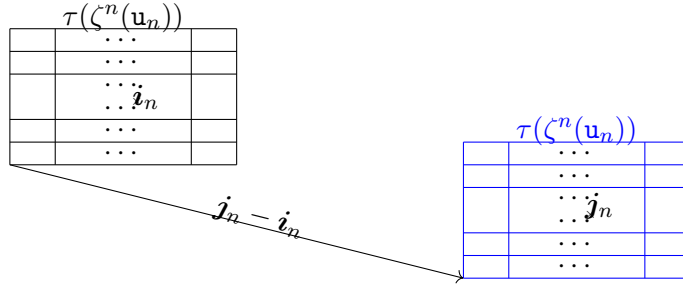


Figure 3.1: Illustration of the patterns $\tau(\zeta^n(\mathbf{u}_n))$. The pattern in \mathbf{i}_n (black) comes from the fact that $x = \zeta(x)$. We will prove that the pattern in \mathbf{j}_n (blue) also comes from this fact.

Since $x = \zeta(x)$, there exist a finite subset $C^{(n)} \subseteq \mathbb{Z}^d$, a pattern $\mathbf{v}_n \in \mathcal{L}_{C^{(n)}}(X_\zeta)$ and an occurrence $\mathbf{c}_n \in L_\zeta(\mathbb{Z}^d)$ of $\zeta(\mathbf{v}_n)$ with $\mathbf{j}_n + (L_\zeta^n(B) \cap \mathbb{Z}^d) \subseteq \mathbf{c}_n + L_\zeta^n(C^{(n)}) \subseteq \bigcup_{\substack{(F_n^\zeta + \mathbf{n}) \cap (\mathbf{j}_n + L_\zeta^n(B)) \neq \emptyset \\ \mathbf{n} \in \mathbb{Z}^d}} F_n^\zeta + \mathbf{n}$. In particular, $\zeta^n(\mathbf{u}_n) \sqsubseteq \zeta^n(\mathbf{v}_n)$, as illustrated in Fig. 3.2:

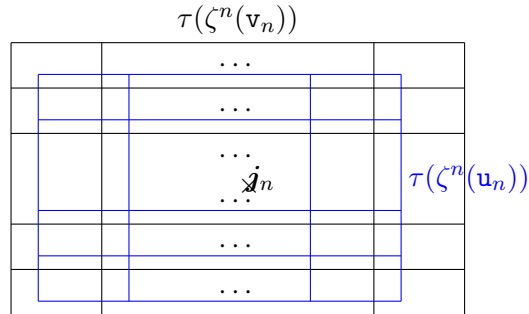


Figure 3.2: Illustration of the patterns $\tau(\zeta^n(\mathbf{v}_n))$ and $\tau(\zeta^n(\mathbf{u}_n))$ in \mathbf{j}_n .

By the Pigeonhole Principle, there are an infinite set $E \subseteq \mathbb{N}$, a finite set $C \subseteq \mathbb{Z}^d$ and patterns $\mathbf{u} \in \mathcal{L}_B(X_\zeta)$, $\mathbf{v} \in \mathcal{L}_C(X_\zeta)$ such that for all $n \in E$, $C^{(n)}$ is equal to C , $\mathbf{u}_n = \mathbf{u}$, and $\mathbf{v}_n = \mathbf{v}$. Consider $D = \{\mathbf{n} \in C : \mathbf{n} + \llbracket -1, 1 \rrbracket^d \subseteq C\}$, and let $\mathbf{w} = x|_{\mathbf{k}_n + D}$ where $\mathbf{k}_n \in \mathbb{Z}^d$ is such that $L_\zeta^n(\mathbf{k}_n + D)$ is strictly contained in $\mathbf{j}_n + (L_\zeta^n(B) \cap \mathbb{Z}^d)$ and set

$\mathbf{a}_n = x|_{(j_n + L_\zeta^n(B)) \setminus (L_\zeta^n(\mathbf{k}_n + D))}$. We have that $\text{supp}(\mathbf{a}_n) \subseteq (\partial(j_n + L_\zeta^n(B)) + L_\zeta^n([-1, 1]^d))$ as illustrated in Fig. 3.3:

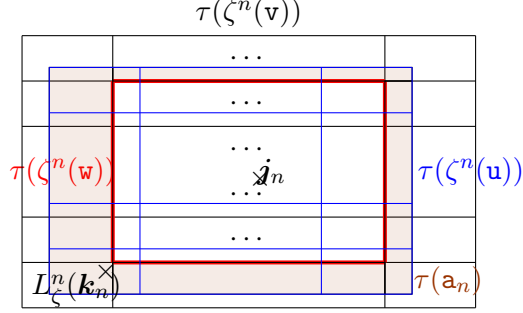


Figure 3.3: Illustration of the patterns $\zeta^n(\mathbf{w})$ and \mathbf{a}_n in \mathbf{j}_n .

Set $m > n \in E$. Applying ζ^{m-n} to $\zeta^n(\mathbf{u})$, we obtain the patterns $\zeta^m(\mathbf{a}_n)$ and $\zeta^{m-n}(\zeta^n(\mathbf{w})) = \zeta^m(\mathbf{w})$. If $\zeta^{m-n}(\mathbf{a}_n) \neq \mathbf{a}_m$, there is two occurrences of $\zeta^m(\mathbf{w})$ in $\zeta^m(\mathbf{u})$. Since the supports of $\zeta^{m-n}(\mathbf{a}_n)$ and \mathbf{a}_m are in the same set, the distance between these two occurrences is smaller than $\max_{t \in [-1, 1]^d} \|L_\zeta^m(t)\|$. If these occurrences are not the same, the repulsion property (Proposition 3.2) gives a contradiction, so $\zeta^{m-n}(\mathbf{a}_n) = \mathbf{a}_m$ as illustrated in Fig. 3.4:

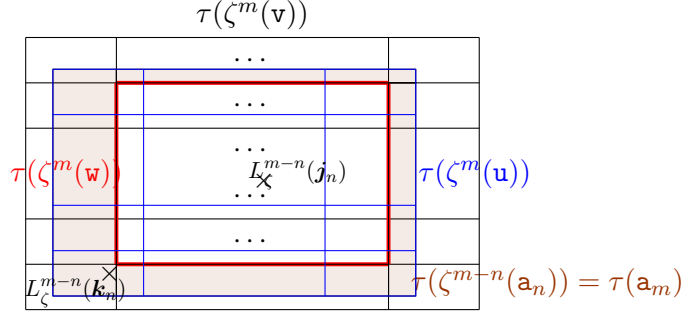


Figure 3.4: Illustration of the patterns $\zeta^{m-n}(\mathbf{a}_n)$ in $L_\zeta^{m-n}(\mathbf{j}_n)$.

Now, since $x = \zeta^{m-n}(x)$, $y|_{L_\zeta^{m-n}(j_n + L_\zeta^n(B))}$ is equal to $\tau(\zeta^m(\mathbf{u}))$. Hence, we have $L_\zeta^{m-n}(\mathbf{j}_n) - L_\zeta^{m-n}(\mathbf{k}_n) = \mathbf{j}_m - L_\zeta^m(\mathbf{k}_m)$. This implies that $\mathbf{j}_m \in L_\zeta(\mathbb{Z}^d)$ which gives a contradiction. \square

3.2 Invariant orbits of substitutive subshifts

As mentioned in the last proof, we assume that primitive constant-shape substitutions admit at least one fixed point for the map $\zeta : X_\zeta \rightarrow X_\zeta$. The orbits of these fixed points lead to the notion of ζ -invariant orbits. An orbit $\mathcal{O}(x, \mathbb{Z}^d)$ is called ζ -invariant if there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $\zeta(x) = S^{\mathbf{j}}x$, i.e., the orbit is invariant under the action of ζ in X_ζ . Since for every $\mathbf{n} \in \mathbb{Z}^d$ we have $\zeta \circ S^{\mathbf{n}} = S^{L_\zeta \mathbf{n}} \circ \zeta$, the definition is independent of the choice of the point in the \mathbb{Z}^d -orbit of x . The orbit of a fixed point of the substitution is an example of an invariant orbit. In the following, we will prove that for aperiodic primitive constant-shape substitutions there exist finitely many ζ -invariant orbits. This property will be used to prove other properties about some constant-shape substitutions such as coalescence (Proposition 4.7) and the automorphism group of some substitutive subshifts is virtually generated by the shift action (Proposition 4.8).

Proposition 3.4. *Let ζ be an aperiodic primitive constant-shape substitution. Then, there exist finitely many ζ -invariant orbits in the substitutive subshift X_ζ . The bound is explicit and depends only on d , $|\mathcal{A}|$, $\|L_\zeta^{-1}\|$, $\|F_1^\zeta\|$ and $\det(L_\zeta - \text{id})$.*

Proof. Let $x \in X_\zeta$ be such that $\zeta(x) = S^{\mathbf{j}_x}x$, for some $\mathbf{j}_x \in \mathbb{Z}^d$. For any $\mathbf{m} \in \mathbb{Z}^d$, we have

$$\zeta(S^{\mathbf{m}}x) = S^{L_\zeta \mathbf{m}}\zeta(x) = S^{L_\zeta \mathbf{m} + \mathbf{j}_x}x = S^{(L_\zeta - \text{id})\mathbf{m} + \mathbf{j}_x}S^{\mathbf{m}}x,$$

and thus $\mathbf{j}_x - \mathbf{j}_{S^{\mathbf{m}}x} \in (L_\zeta - \text{id})\mathbb{Z}^d$. Let $H \Subset \mathbb{Z}^d$ be a fundamental domain of $(L_\zeta - \text{id})(\mathbb{Z}^d)$ in \mathbb{Z}^d with $\mathbf{0} \in H$. We may assume that $x \in X_\zeta$ is in a ζ -invariant orbit with $\mathbf{j}_x \in H$. Set $K_\zeta \Subset \mathbb{Z}^d$ be from Proposition 1.18. Using $-H$ as the set A and $F = F_1^\zeta$ in Proposition 1.20 we obtain a set $C \Subset \mathbb{Z}^d$ such that $L_\zeta^n(-H + C) + F_n^\zeta \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$ for all $n > 0$. Define $D \Subset \mathbb{Z}^d$ be such that $D = C + K_\zeta - H$. Suppose that there are more than $|\mathcal{A}|^{|D|} \cdot |H|$ ζ -invariant orbits. By the Pigeonhole Principle, there exist $\mathbf{j} \in H$ and two different points $x \neq y \in X_\zeta$ such that $x|_D = y|_D$ and $\zeta(x) = S^{\mathbf{j}}x$, $\zeta(y) = S^{\mathbf{j}}y$. Note that

$$\zeta(x|_D) = \zeta(x)|_{L_\zeta(D) + F_1^\zeta} = x|_{\mathbf{j} + L_\zeta(D) + F_1^\zeta}$$

Hence, we have $x|_{\mathbf{j} + L_\zeta(D) + F_1^\zeta} = y|_{\mathbf{j} + L_\zeta(D) + F_1^\zeta}$. Inductively, we obtain for every $n \geq 0$

$$x|_{\left(\sum_{k=0}^n L_\zeta^k \mathbf{j}\right) + L_\zeta^{n+1}(D) + F_{n+1}^\zeta} = y|_{\left(\sum_{k=0}^n L_\zeta^k \mathbf{j}\right) + L_\zeta^{n+1}(D) + F_{n+1}^\zeta}.$$

Let E_0 be equal to D and for all $n > 0$, define $E_n = \left(\sum_{k=0}^{n-1} L_\zeta^k \mathbf{j}\right) + L_\zeta^n(D) + F_n^\zeta$. We will prove that $\bigcup_{n \geq 0} E_n = \mathbb{Z}^d$. This implies $x = y$, which is a contradiction. To do this, we will prove that for every $n \geq 0$ that $L_\zeta^n(K_\zeta) + F_n^\zeta \subseteq E_{n+1}$ and we conclude by Proposition 1.18.

Note that $L_\zeta^n(K) + F_n^\zeta \subseteq E_{n+1}$ if and only if $L_\zeta^n(K - \mathbf{j}) + \left(\sum_{k=0}^{n-1} L_\zeta^k(F_1^\zeta - \mathbf{j}) \right) \subseteq L_\zeta^{n+1}(D) + F_{n+1}^\zeta$.

Claim 1. For every $n \geq 0$ we have $\sum_{k=0}^n L_\zeta^k(F_1 - \mathbf{j}) \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$.

Proof of Claim. For $n = 0$, note that $F_1 - \mathbf{j}$ is included in $L_\zeta(C) + F_1^\zeta$ by Proposition 1.20.

Assume that for some $n \geq 0$ $\sum_{k=0}^n L_\zeta^k(F_1 - \mathbf{j}) \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$. We have

$$\begin{aligned} \sum_{k=0}^{n+1} L_\zeta^k(F_1 - \mathbf{j}) &= \left(\sum_{k=0}^n L_\zeta^k(F_1 - \mathbf{j}) \right) + L_\zeta^{n+1}(F_1 - \mathbf{j}) \\ &\subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta + L_\zeta^{n+1}(F_1 - \mathbf{j}) \\ &\subseteq L_\zeta^{n+1}(C + F_1^\zeta - \mathbf{j}) + F_{n+1}^\zeta \\ &\subseteq L_\zeta^{n+2}(C) + L_\zeta^{n+1}(F_1^\zeta) + F_{n+1}^\zeta \quad (\text{by Proposition 1.20}) \\ &= L_\zeta^{n+2}(C) + F_{n+2}^\zeta \end{aligned}$$

We conclude that for every $n \geq 0$, $\sum_{k=0}^n L_\zeta^k(F_1 - \mathbf{j}) \subseteq L_\zeta^{n+1}(C) + F_{n+1}^\zeta$ □

By Claim 1 $L_\zeta^n(K - \mathbf{j}) + \left(\sum_{k=0}^{n-1} L_\zeta^k(F_1 - \mathbf{j}) \right)$ is included in $L_\zeta^n(K - \mathbf{j}) + L_\zeta^n(C) + F_n^\zeta$, and $L_\zeta^n(K + C - \mathbf{j})$ is a subset of $L_\zeta^{n+1}(D) + L_\zeta^n(F_1^\zeta)$, so we have $L_\zeta^n(K - \mathbf{j}) + L_\zeta^n(C) + F_n^\zeta \subseteq L_\zeta^{n+1}(D) + L_\zeta^n(F_1^\zeta) + F_n^\zeta = L_\zeta^{n+1}(D) + F_{n+1}^\zeta$ and we conclude the proof. □

Remark 3.5. Let ζ be an aperiodic primitive substitution with an expansion matrix L_ζ such that $|\det(L_\zeta - \text{id}_{\mathbb{R}^d})| = 1$. This implies $(L_\zeta - \text{id}_{\mathbb{R}^d})(\mathbb{Z}^d) = \mathbb{Z}^d$. Let $x \in X_\zeta$ be a point in a ζ -invariant orbit, i.e., there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $\zeta(x) = S^{\mathbf{j}}x$ and set $\mathbf{m} \in \mathbb{Z}^d$ such that $(L_\zeta - \text{id}_{\mathbb{R}^d})(\mathbf{m}) = -\mathbf{j}$. Then $\zeta(S^{\mathbf{m}}x) = S^{L_\zeta \mathbf{m} + \mathbf{j}} \zeta(x) = S^{\mathbf{m} + (L_\zeta - \text{id}_{\mathbb{R}^d})(\mathbf{m}) + \mathbf{j}} x = S^{\mathbf{m}}x$. Hence $S^{\mathbf{m}}x$ is a fixed point of ζ . We conclude that the only ζ -invariant orbits in this case are the ones given by the fixed points of the substitution.

3.3 The repetitivity function of substitutive subshifts

Let ζ be an aperiodic primitive constant-shape substitution, and assume that $x \in X_\zeta$ is a fixed point of the substitution. The minimality property implies the substitutive subshift is *repetitive*, i.e., for every pattern $\mathbf{p} \sqsubseteq x$ there is a radius $R > 0$ such that for every $\mathbf{n} \in \mathbb{Z}^d$, the ball $B(\mathbf{n}, R)$ contains an occurrence of \mathbf{p} in x . The *repetitivity function* is the map $M_{X_\zeta} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ defined for $R > 0$ as the smallest radius such that every ball $B(\mathbf{n}, M_{X_\zeta}(R))$ contains an occurrence of every pattern with $\text{diam}(\text{supp}(\mathbf{p})) \leq 2R$. We say the substitution

is *linearly recurrent* or *linearly repetitive* if the repetitivity function has a linear growth, i.e., there exists $C > 0$ such that $M_{X_\zeta}(R) \leq C \cdot R$. The notion of linearly recurrent was introduced by F. Durand in [45], to study the relations between the substitutive dynamical systems and the stationary dimension groups. For the multidimensional context, the same notion appears in [80] for Delone sets of \mathbb{R}^d . In fact, according to a result in [80, Theorem 2.3], the linear growth of the repetitivity function is the slowest possible for an aperiodic Delone set. F. Durand proved one-dimensional substitutive subshift are linearly recurrent [48], but in the multidimensional case this is no longer true, as we can see in Example 3.6.

Example 3.6 (A nonlinearly repetitive constant-shape substitution). Consider the substitution σ_1 , given by $L_{\sigma_1} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, and $F_1^{\sigma_1} = \llbracket 0, 1 \rrbracket \times \llbracket 0, 2 \rrbracket$ defined by

$$\sigma_1 : \begin{array}{ccc} & b & c & & a & c & & c & b \\ a & \mapsto & c & b & b & \mapsto & c & b & c & \mapsto & a & c \\ & & a & b & & & b & c & & & c & b. \end{array}$$

For $p > 1$, we consider the pattern $\mathbf{w}_p = \sigma_1^p(a)|_{\llbracket 0, 2^{p-1} \rrbracket \times \{0\}} \in \mathcal{L}_{\llbracket 0, 2^{p-1} \rrbracket \times \{0\}}(X_\zeta)$. A direct induction enable to prove that if $\mathbf{w}_p \sqsubseteq \sigma_1^p \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, for $\alpha, \beta, \gamma, \delta \in \{a, b, c, \varepsilon\}$ (where ε denote the empty pattern), then one of the letters must be a . Moreover, \mathbf{w}_p only appears in the lower left corner of the pattern $\sigma_1^p(a)$. These properties imply that there is only one occurrence of \mathbf{w}_p in $\sigma_1^p(\mathbf{w}_p)$, which is in the lower left corner of $\sigma_1^p(\mathbf{w}_p)$ as seen in Fig. 3.5:

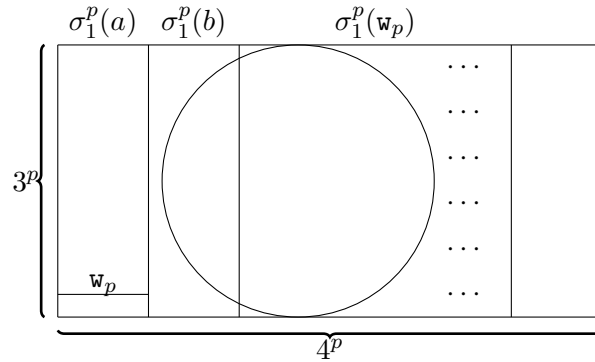


Figure 3.5: Decomposition of $\sigma_1^p(\mathbf{w}_p)$.

Then, there is a ball of radius $3^p/2$ in the support of $\sigma_1^p(\mathbf{w}_p)$ with no occurrences of \mathbf{w}_p . Since this is true for any p , this implies that this substitution is not linearly recurrent.

However, the repetitivity function has at most polynomial growth, with exponent depending only on the expansion map of the substitution.

Lemma 3.7. *Let ζ be an aperiodic primitive constant-shape substitution.*

$$\text{Then } M_{X_\zeta}(R) \text{ is } \mathcal{O}\left(R^{-\frac{\log(\|L_\zeta\|)}{\log(\|L_\zeta^{-1}\|)}}\right).$$

Proof. Using $A = K_\zeta + K_\zeta$, and $F = F_1^\zeta + F_1^\zeta$ in Proposition 1.20, where K_ζ is given by Proposition 1.18, we obtain a subset $C \Subset \mathbb{Z}^d$ such that for all $n > 0$, $L_\zeta^n(K_\zeta + K_\zeta) + F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$. The recognizability property implies for every pattern $\mathbf{p} \in \mathcal{L}(X_\zeta)$ such that its support is contained in $L_\zeta^n(K_\zeta) + F_n^\zeta$, there exists a pattern $\mathbf{w} \in \mathcal{L}_C(X_\zeta)$ with $\mathbf{p} \sqsubseteq \zeta^n(\mathbf{w})$. Set $T = M_{X_\zeta}(\text{diam}(C))$. By definition, any ball of radius T contains an occurrence of every pattern in $\mathcal{L}_{L_\zeta^n(K_\zeta) + F_n^\zeta}^\zeta(X_\zeta)$.

Set $R > 0$. Fix $n > 0$ such that $L_\zeta^{n-1}(K_\zeta) + F_{n-1}^\zeta \subseteq B(\mathbf{0}, R) \subseteq L_\zeta^n(K_\zeta) + F_n^\zeta$. By definition of n , we get that $\inf_{\|\mathbf{t}\|=1} L_\zeta^{n-1}(\mathbf{t}) \leq 2R$, hence $1/\|L_\zeta^{-1}\|^{n-1} \leq 2R$. Thus

$$M_{X_\zeta}(R) \leq \|L_\zeta^n\|T \leq \|L_\zeta\|^n T \leq C(\zeta)R^{-(\log(\|L_\zeta\|))/(\log(\|L_\zeta^{-1}\|))},$$

for some constant $C(\zeta)$ independent of R . \square

Remark 3.8. The following statements can be easily verified.

- (1) In the case of a symmetric expansion map for the substitution, a bound for $M_{X_\zeta}(R)$ is given by the eigenvalues of the expansion matrix: $M_{X_\zeta}(R) = \mathcal{O}\left(R^{(\log(|\lambda_1|))/(\log(|\lambda_d|))}\right)$, where $|\lambda_1|$, $|\lambda_d|$ are the maximum and minimum of the absolute values of the eigenvalues of L_ζ , respectively.
- (2) In the case of a self-similar tiling (where the expansion map satisfies $\|L_\zeta(\mathbf{t})\| = \lambda\|\mathbf{t}\|$, for some $\lambda > 0$), the norm matrix satisfies $\|L_\zeta\| = (\|L_\zeta^{-1}\|)^{-1} = \lambda$, so the repetitivity function is $\mathcal{O}(R)$, i.e., has a linear growth. Hence self-similar substitutions are linearly recurrent, as it was proved in [107].
- (3) The sufficiency of the previous case is not true, there exist constant-shape substitutions that are not self-similar, but are linearly recurrent.

3.4 Substitutive subshifts as extension of d-dimensional odometers

In this section, we will describe substitutive subshifts as symbolic extensions of odometer systems, which are given by the data of the substitution. Actually, the maximal equicontinuous factor of a substitutive subshift is an explicit odometer system (Proposition 3.15). This was first made by F. Dekking in [37] for the one-dimensional case, where he introduced the notion of *height* to describe the return times of a letter which are coprime with the

length of the substitution. Then, N. P. Frank in [54] followed the same ideas to describe the maximal equicontinuous factor of block substitutions. Here, we follow the same ideas in [54], using the notion of *height lattice*. It is a lattice of \mathbb{Z}^d containing the set of return times of a letter, satisfying a specific property with the expansion matrix of the constant-shape substitution. Using a characterization of the eigenvalues of a substitutive subshift proved in [108] we characterized the maximal equicontinuous factor of substitutive subshift.

3.4.1 Substitutive subshifts as a finite-to-1 extension of a d -dimensional odometer by the recognizability property

The recognizability property establishes a factor map from the substitutive subshift to a d -dimensional odometer as follows. For every $n > 0$, let $\pi_n : X_\zeta \rightarrow F_n^\zeta$ be the map satisfying $x \in S^{\pi_n(x)}\zeta^n(X_\zeta)$. This map is well defined (mod $L_\zeta^n(\mathbb{Z}^d)$). Using basic group theory arguments, we have $\pi_{n+1}(x) = \pi_n(x)$ (mod $L_\zeta^n(\mathbb{Z}^d)$).

Consider the odometer $\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n)}$. The map $\pi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n)}, +_{(L_\zeta^n)}, \mathbb{Z}^d)$ given by $(\pi_n(x))_{n>0}$ is a factor map between $(X_\zeta, S, \mathbb{Z}^d)$ and $(\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n)}, +_{(L_\zeta^n)}, \mathbb{Z}^d)$. Moreover it satisfies the following property.

Lemma 3.9. *Let $(X_\zeta, S, \mathbb{Z}^d)$ be the substitutive subshift from an aperiodic constant-shape substitution ζ . Then, for any $\overleftarrow{g} = (\mathbf{g}_n)_{n>0}$ in $\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n)}$ two different cases occur:*

K1: Assume that $\bigcup_{n>0} (-\mathbf{g}_n + F_n^\zeta) = \mathbb{Z}^d$. Then, there is at most $|\mathcal{A}|$ elements in $|\pi^{-1}(\{\overleftarrow{g}\})|$.

K2: On the other hand, $|\pi^{-1}(\{\overleftarrow{g}\})|$ is no greater than $|\mathcal{A}|^{|\overline{K}_\zeta|}$, where $\overline{K}_\zeta = K_\zeta + C$, with $C + F_1^\zeta + F_1^\zeta \subseteq L_\zeta(C) + F_1^\zeta$ is obtained by Proposition 1.20 using $A = \{\mathbf{0}\}$, and $F = F_1^\zeta + F_1^\zeta$, and K_ζ is given by Proposition 1.18.

In particular, the factor map $\pi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n)}, +_{(L_\zeta^n)}, \mathbb{Z}^d)$ is finite-to-1.

Proof. We separate the proof in these two cases.

K1: Assume that $|\pi^{-1}(\{\overleftarrow{g}\})| > |\mathcal{A}|$. Let $x_0, \dots, x_{|\mathcal{A}|}$ be in $\pi^{-1}(\{\overleftarrow{g}\})$. For all $n > 0$ and all $j \in \{0, \dots, |\mathcal{A}|\}$, there exist $y_j^n \in X_\zeta$ such that $x_j = S^{\mathbf{g}_n} \zeta^n(y_j^n)$. By the Pigeonhole Principle, there exist $j_1 \neq j_2 \in \{0, \dots, |\mathcal{A}|\}$ and an infinite set $E \subseteq \mathbb{N}$ with $y_{j_1}^n|_{\mathbf{0}} = y_{j_2}^n|_{\mathbf{0}}$. This implies for any $n \in E$, $x_{j_1}|_{-\mathbf{g}_n + F_n^\zeta} = x_{j_2}|_{-\mathbf{g}_n + F_n^\zeta}$. To complete the proof, note that for any $n > 0$, we have that $\mathbf{g}_{n+1} - \mathbf{g}_n$ is in $L_\zeta^n(F_1^\zeta)$, so $\mathbf{g}_{n+1} - \mathbf{g}_n + F_n^\zeta$ is included in $F_n^\zeta + L_\zeta^n(F_1^\zeta)$ which is F_{n+1}^ζ by definition. This implies $-\mathbf{g}_n + F_n^\zeta \subseteq -\mathbf{g}_{n+1} + F_{n+1}^\zeta$ and we conclude that $x_{j_1} = x_{j_2}$, which is a contradiction.

K2: Suppose that \overleftarrow{g} does not satisfy K1. Proceeding as in the previous case, we assure the existence of two indices $j_1 \neq j_2$ such that $y_{j_1}^n|_{\overline{K}_\zeta} = y_{j_2}^n|_{\overline{K}_\zeta}$ for any n in an infinite subset $E \subseteq \mathbb{N}$. This will imply that $x_{j_1}|_{-\mathbf{g}_n + L_\zeta^n(\overline{K}_\zeta) + F_n} = x_{j_2}|_{-\mathbf{g}_n + L_\zeta^n(\overline{K}_\zeta) + F_n}$ for any $n \in E$. Now, note that for any $n \in \mathbb{N}$, $\mathbf{g}_{n+1} - \mathbf{g}_n \in L^n(F_1^\zeta)$, and a direct induction prove that $-\mathbf{g}_n + L^n(\overline{K}_\zeta) + F_n^\zeta$ is included in $-\mathbf{g}_{n+1} + L^{n+1}(\overline{K}_\zeta) + F_{n+1}^\zeta$. Since $K_\zeta \subseteq \overline{K}_\zeta$, we conclude that $x_{j_1} = x_{j_2}$ which will be a contradiction. \square

Remark 3.10. Note that

- (1) The set of points $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}}^d_{(L_\zeta^n)}$ satisfying K1 is a G_δ -set. Indeed, for any $M > 0$ define $U_M = \left\{ \overleftarrow{g} \in \overleftarrow{\mathbb{Z}}^d_{(L_\zeta^n)} : \llbracket -M, M \rrbracket^d \subseteq -\mathbf{g}_n + F_n^\zeta, \text{ for some } n > 0 \right\}$. Note that U_M is an open subset of $\overleftarrow{\mathbb{Z}}^d_{(L_\zeta^n)}$ and $\bigcap_{M>0} U_M$ are exactly the points satisfying K1.

Now, for any \mathbf{n} in \mathbb{Z}^d and $M > 0$ we have that U_M is included in $U_{M+|\mathbf{n}|}$, hence the set of points on the odometer satisfying K1 is invariant by the \mathbb{Z}^d -action.

- (2) We will prove that the existence of a point $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}}^d_{(L_\zeta^n(\mathbb{Z}^d))}$ satisfying K1 is equivalent to the existence of $N \in \mathbb{N}$ and $\mathbf{f} \in F_N^\zeta$ such that $B(\mathbf{0}, R_{L_\zeta}) \cap \mathbb{Z}^d \subseteq F_N^\zeta - \mathbf{f}$, with

$$R_{L_\zeta} = \max \left\{ \frac{\|L_\zeta\|}{2}, \frac{\|L_\zeta^{-1}\|}{1 - \|L_\zeta^{-1}\|} \left(1 + \frac{\|L_\zeta\|}{2} \right) \right\}.$$

Which is always true by the Følner assumption. We prove the nontrivial implication. We will show by induction that for every $r \geq R_{L_\zeta}$ there exists $n \in \mathbb{N}$ and $\mathbf{f}_r \in F_n^\zeta$ such that $B(\mathbf{0}, r) \cap \mathbb{Z}^d \subseteq F_n^\zeta - \mathbf{f}_r$. By hypothesis, this is true for $r = R_{L_\zeta}$. Now, assume that for some $r \geq R_{L_\zeta}$ there exists $n \in \mathbb{N}$ and $\mathbf{f}_r \in F_n^\zeta$ such that $B(\mathbf{0}, r) \cap \mathbb{Z}^d \subseteq F_n^\zeta - \mathbf{f}_r$. Then, $L_\zeta(B(\mathbf{0}, r) \cap \mathbb{Z}^d)$ is included in $L_\zeta(F_n^\zeta) - L_\zeta(\mathbf{f}_r)$. This implies

$$L_\zeta(B(\mathbf{0}, r) \cap \mathbb{Z}^d) + \left(B \left(\mathbf{0}, \frac{\|L_\zeta\|}{2} \right) \cap \mathbb{Z}^d \right) \subseteq F_{n+N+1}^\zeta - L_\zeta(\mathbf{f}_r) - \mathbf{f}.$$

Now, we prove that $B(\mathbf{0}, r+1) \cap \mathbb{Z}^d \subseteq L_\zeta(B(\mathbf{0}, r) \cap \mathbb{Z}^d) + (B(\mathbf{0}, \|L_\zeta\|/2) \cap \mathbb{Z}^d)$. Set $\mathbf{n} \in B(\mathbf{0}, r+1) \cap \mathbb{Z}^d$. Since $L_\zeta(\mathbb{Z}^d)$ is $\|L_\zeta\|/2$ -relatively dense, we can write $\mathbf{n} = \mathbf{m} + L_\zeta(\mathbf{p})$, with $\mathbf{m} \in B(\mathbf{0}, \|L_\zeta\|/2) \cap \mathbb{Z}^d$ and $\mathbf{p} \in \mathbb{Z}^d$. We have that

$$\begin{aligned} \|\mathbf{p}\| &\leq \|L_\zeta^{-1}\| \|L_\zeta(\mathbf{p})\| \\ &= \|L_\zeta^{-1}\| \|\mathbf{n} - \mathbf{m}\| \\ &\leq \|L_\zeta^{-1}\| \left(r + 1 + \frac{\|L_\zeta\|}{2} \right). \end{aligned}$$

The last expression is smaller than r whenever $r \geq \|L_\zeta^{-1}\|/(1-\|L_\zeta^{-1}\|) \cdot (1 + \|L_\zeta\|/2)$, which implies $\mathbf{p} \in B(\mathbf{0}, r) \cap \mathbb{Z}^d$. Since $L_\zeta(\mathbf{f}_r) + \mathbf{f}$ is in F_{n+N+1}^ζ , we conclude that there exist $m \in \mathbb{N}$ and $\mathbf{f} \in F_m^\zeta$ such that $B(\mathbf{0}, r+1) \cap \mathbb{Z}^d \subseteq F_m^\zeta - \mathbf{f}$. Finally, using classical compactness arguments we conclude the existence of a point $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}}^d_{(L_\zeta^n(\mathbb{Z}^d))}$ such that $\bigcup_{n>0} F_n^\zeta - \mathbf{g}_n = \mathbb{Z}^d$.

- (3) Let ζ be an aperiodic bijective primitive constant-shape substitution. The map p_0 is a permutation of \mathcal{A} and we consider a power of p_0 such that p_0^n is equal to the identity. Since ζ is primitive, we replace it by ζ^n and with this we may assume that ζ possess at least $|\mathcal{A}|$ fixed points. Now, let \overleftarrow{g} satisfying K1. For any $a \in \mathcal{A}$, consider a fixed point of ζ , denoted as x_a , with $x_a(\mathbf{0}) = a$ and define $x_a^{\overleftarrow{g}} = \lim_{n_m \rightarrow \infty} S^{g_{n_m}} x_a$ for some convergent subsequence. Since the sets $\{S^{g_n}[\zeta^n(a)]\}_{a \in \mathcal{A}}$ are disjoint for any $a \neq b \in \mathcal{A}$, $x_a^{\overleftarrow{g}}$ is different from $x_b^{\overleftarrow{g}}$. Finally, noticing that $\pi(x_a^{\overleftarrow{g}}) = \overleftarrow{g}$, we have that $\pi^{-1}(\{\overleftarrow{g}\}) \geq |\mathcal{A}|$. By Lemma 3.9 we conclude that $\pi^{-1}(\{\overleftarrow{g}\}) = |\mathcal{A}|$ for any \overleftarrow{g} satisfying K1, and then the factor map $\pi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (\overleftarrow{\mathbb{Z}}^d_{(L_\zeta^n)}, +_{(L_\zeta^n)}, \mathbb{Z}^d)$ for aperiodic bijective constant-shape substitutions is almost $|\mathcal{A}|$ -to-1.

In general, this d -dimensional odometer is not the maximal equicontinuous factor of aperiodic constant-shape substitutions.

In some particular cases we can explicitly compute the cardinality of the fibers given by the topological factor in Lemma 3.9, such as in the two examples studied in [99].

The following result shows other examples where we can compute the cardinality of the fibers.

Lemma 3.11. *Let $L \in \mathcal{M}(d, \mathbb{Z})$ be an integer expansion matrix with $\det(L) \geq 3$ and F_1 be a fundamental domain of $L(\mathbb{Z}^d)$ in \mathbb{Z}^d . Let \mathcal{A} be a finite alphabet with cardinality $|\mathcal{A}| = |F_1| - 1$ and $\tau : F_1 \setminus \{\mathbf{0}\} \rightarrow \mathcal{A}$ be a bijection. We define a substitution σ_L as the following:*

$$\forall a \in \mathcal{A}, \quad \sigma(a)_f = \begin{cases} a & \mathbf{f} = \mathbf{0}, \\ \tau(\mathbf{f}) & \mathbf{f} \neq \mathbf{0}. \end{cases}$$

Under the hypothesis that the sequence of supports of the iterations σ_L^n is a Følner sequence, σ_L is an aperiodic primitive constant-shape substitution and the factor map $\pi : (X_{\sigma_L}, S, \mathbb{Z}^d) \rightarrow (\overleftarrow{\mathbb{Z}}^d_{(L^n)}, +_{(L^n)}, \mathbb{Z}^d)$ is almost 1-to-1.

Moreover, we have that

$$|\pi^{-1}(\{\overleftarrow{g}\})| = \begin{cases} |\mathcal{A}| & \overleftarrow{g} \in \mathcal{O}(\overleftarrow{\mathbf{0}}, \mathbb{Z}^d), \\ 1 & \overleftarrow{g} \notin \mathcal{O}(\overleftarrow{\mathbf{0}}, \mathbb{Z}^d). \end{cases}$$

Proof. Since τ is a bijection, by definition σ_L is a primitive substitution. Now, we prove that σ_L is aperiodic. To prove this we prove σ_L is recognizable. We follow similar ideas of the proof of the recognizability property in [78].

Let \bar{x} be a fixed point of σ_L and $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^d$ such that $\mathbf{n} \in L(\mathbb{Z}^d)$ and $\bar{x}|_{\mathbf{n}+F_1} = \bar{x}|_{\mathbf{m}+F_1}$. We prove that $\mathbf{m} \in L(\mathbb{Z}^d)$. Indeed, since \bar{x} is a fixed point of the substitution and $\mathbf{n} \in L(\mathbb{Z}^d)$ we have that for all $\mathbf{f} \in F_1 \setminus \{\mathbf{0}\}$, $\bar{x}_{\mathbf{m}+\mathbf{f}} = \bar{x}_{\mathbf{n}+\mathbf{f}} = \tau(\mathbf{f})$. Set $\mathbf{g} \in F \setminus \{\mathbf{0}\}$ such that $\mathbf{m} + \mathbf{g} \notin L(\mathbb{Z}^d)$ (such $\mathbf{g} \in F_1 \setminus \{\mathbf{0}\}$ exists since $|F_1| \geq 3$). We write $\mathbf{m} + \mathbf{g} = L(\mathbf{p}) + \mathbf{h}$, with $\mathbf{h} \in F_1$. Note that $\mathbf{h} \neq \mathbf{0}$. Hence

$$\bar{x}_{\mathbf{m}+\mathbf{g}} = \bar{x}_{\mathbf{n}+\mathbf{g}} = \tau(\mathbf{g}),$$

but \bar{x} is a fixed point of σ_L so

$$\bar{x}_{\mathbf{m}+\mathbf{g}} = \sigma_L(\bar{x}_{\mathbf{p}})_{\mathbf{h}} = \tau(\mathbf{h}),$$

i.e., $\tau(\mathbf{g}) = \tau(\mathbf{h})$. By bijectivity of τ we have that $\mathbf{g} = \mathbf{h}$, so $\mathbf{m} = L(\mathbf{p})$. We conclude that σ_L is a recognizable substitution, i.e., σ_L is aperiodic.

Now we study the fibers $\pi^{-1}(\{\overleftarrow{g}\})$. Assume that $\overleftarrow{g} = \overleftarrow{0}$. By the proof of Lemma 3.9 we need to compute $\sigma_L^n(\mathbf{w})$ for patterns $\mathbf{w} \in \mathcal{L}_{\overline{K}_{\sigma_L}}(X_{\sigma_L})$ and $n > 0$. Let $\mathbf{w}_1, \mathbf{w}_2$ be two patterns in $\mathcal{L}_{\overline{K}_{\sigma_L}}(X_{\sigma_L})$. Note that, by definition of σ_L , the cardinality of the coordinates where the patterns $\sigma_L^n(\mathbf{w}_1), \sigma_L^n(\mathbf{w}_2)$ differ is constant on $n > 0$ and is at most $|\overline{K}_{\sigma_L}|$. Indeed, if $W = \{\mathbf{k} \in \overline{K}_{\sigma_L} : \mathbf{w}_1(\mathbf{k}) \neq \mathbf{w}_2(\mathbf{k})\}$, then for any $\mathbf{k} \in W$, $\sigma_L^n(\mathbf{w}_1)_{L^n(\mathbf{k})} \neq \sigma_L^n(\mathbf{w}_2)_{L^n(\mathbf{k})}$ and for any $\mathbf{a} \in (L^n(\overline{K}_{\sigma_L}) + F_n^{\sigma_L}) \setminus L^n(W)$ we have that $\sigma_L^n(\mathbf{w}_1)_{\mathbf{a}} = \sigma_L^n(\mathbf{w}_2)_{\mathbf{a}}$. Since the distance between these differences increase (exponentially on $n > 0$) and for any $n > 0$ and $\mathbf{a} \in \mathcal{A}$ the patterns $\sigma_L^n(\mathbf{a})|_{F_n^{\sigma_L}}$ and $\sigma_L^{n+1}(\mathbf{a})|_{F_n^{\sigma_L}}$ are the same, we get two cases:

- If $\mathbf{w}_1(\mathbf{0}) \neq \mathbf{w}_2(\mathbf{0})$, the patterns $\sigma_L^n(\mathbf{w}_1), \sigma_L^n(\mathbf{w}_2)$ converge to two points $x_1, x_2 \in X_{\sigma_L}$ with $x_1(\mathbf{0}) \neq x_2(\mathbf{0})$ and for any $\mathbf{n} \in \mathbb{Z}^d$, $x_1(\mathbf{n}) = x_2(\mathbf{n})$.
- At the opposite, if $\mathbf{w}_1(\mathbf{0}) = \mathbf{w}_2(\mathbf{0})$, the patterns $\sigma_L^n(\mathbf{w}_1), \sigma_L^n(\mathbf{w}_2)$ converge to the same point $x \in X_{\sigma_L}$.

This implies $|\pi^{-1}(\{\overleftarrow{0}\})| \leq |\mathcal{A}|$ and since for any letter $a \in \mathcal{A}$ we have a fixed point x_a such that $x_a(\mathbf{0}) = a$, then $|\pi^{-1}(\{\overleftarrow{0}\})| \geq |\mathcal{A}|$. We conclude that $|\pi^{-1}(\{\overleftarrow{0}\})| = |\mathcal{A}|$. Since this property is invariant under translation, we conclude that for any $\overleftarrow{g} \in \mathcal{O}(\overleftarrow{0}, \mathbb{Z}^d)$ that $|\pi^{-1}(\{\overleftarrow{g}\})|$ is equal to $|\mathcal{A}|$.

For the other case, consider $\overleftarrow{g} = (\mathbf{g}_n)_{n>0} \notin \mathcal{O}(\overleftarrow{0}, \mathbb{Z}^d)$. This implies for every $n > 0$ exists $m > n$ such that $\mathbf{g}_m \neq \mathbf{0}$. Set $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{L}_{\overline{K}_{\sigma_L}}(X_{\sigma_L})$ and let $W = \{\mathbf{k} \in \overline{K}_{\sigma_L} : \mathbf{w}_1(\mathbf{k}) \neq \mathbf{w}_2(\mathbf{k})\}$ be their set of differences. By definition of σ_L , for any $n > 0$ the coordinates where $\sigma_L^n(\mathbf{w}_1)$ and $\sigma_L^n(\mathbf{w}_2)$ differ is $L^n(W)$. Then, for any $M > 0$ we can find $n > 0$ such that $\mathbf{g}_n + \llbracket -M, M \rrbracket^d \subseteq L^n(\overline{K}_{\sigma_L}) + F_n^{\sigma_L}$ and $\mathbf{g}_n + \llbracket -M, M \rrbracket^d \cap L^n(\mathbb{Z}^d) = \emptyset$. Hence $\sigma_L^n(\mathbf{w}_1)|_{\mathbf{g}_n + \llbracket -M, M \rrbracket^d} = \sigma_L^n(\mathbf{w}_2)|_{\mathbf{g}_n + \llbracket -M, M \rrbracket^d}$. Moreover, a direct computation shows that for any $n > 0$, the patterns $\sigma_L^n(\mathbf{w}_1)|_{\mathbf{g}_n + \llbracket -M, M \rrbracket^d}, \sigma_L^{n+1}(\mathbf{w}_1)|_{\mathbf{g}_{n+1} + \llbracket -M, M \rrbracket^d}$ are the same. This implies, taking $M > 0$ arbitrarily large, for any $\mathbf{w} \in \mathcal{L}_{\overline{K}_{\sigma_L}}(X_{\sigma_L})$, the patterns $\sigma_L^n(\mathbf{w})|_{\mathbf{g}_n + \llbracket -M, M \rrbracket^d}$ are the same and then converge to a unique point $x \in X_{\sigma_L}$ such that $\pi(x) = \overleftarrow{g}$. We conclude that $|\pi^{-1}(\overleftarrow{g})| = 1$. \square

We will use this result to study the fibers of a particular case in Chapter 6.

3.4.2 The maximal equicontinuous factor of a substitutive subshift

In this section, we will describe the maximal equicontinuous factor of substitutive subshifts (see [54] for the case with diagonal expansion matrices and [37] for the one-dimensional case).

A subgroup $\mathfrak{L} \leq \mathbb{Z}^d$ is called a *lattice* if it is isomorphic to \mathbb{Z}^d , i.e., it has finite index. For a lattice \mathfrak{L} of \mathbb{Z}^d , we define the *dual lattice* of \mathfrak{L} , as the subgroup

$$\mathfrak{L}^* = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{x}, \mathbf{n} \rangle \in \mathbb{Z}, \forall \mathbf{n} \in \mathfrak{L}\}.$$

We have that $(\mathbb{Z}^d)^* = \mathbb{Z}^d$, and for any $R \in \mathcal{M}(d, \mathbb{Z})$ the set $R(\mathbb{Z}^d)$ is a lattice of \mathbb{Z}^d with dual lattice equal to $(R^*)^{-1}(\mathbb{Z}^d)$, where R^* stands for the *algebraic adjoint* of R . Let $\mathfrak{L}_1, \mathfrak{L}_2$ be two lattices of \mathbb{Z}^d . We denote by $\mathfrak{L}_1 \vee \mathfrak{L}_2$ the smallest lattice that contains \mathfrak{L}_1 and \mathfrak{L}_2 , i.e., if a lattice \mathfrak{L} containing \mathfrak{L}_1 and \mathfrak{L}_2 , then must contain $\mathfrak{L}_1 \vee \mathfrak{L}_2$.

Fix $x \in X_\zeta$. We define the *set of return times* as

$$\mathcal{R}(X_\zeta) = \{\mathbf{j} \in \mathbb{Z}^d : \exists \mathbf{k} \in \mathbb{Z}^d, x_{\mathbf{k}+\mathbf{j}} = x_{\mathbf{k}}\}.$$

By minimality, this set is well-defined independently of $x \in X_\zeta$, and it is syndetic, i.e., there exists a finite subset $A \subseteq \mathbb{Z}^d$ such that $\mathcal{R}(X_\zeta) + A = \mathbb{Z}^d$. We define $\mathfrak{L}(\mathcal{R}(X_\zeta))$ as the smallest lattice containing $\mathcal{R}(X_\zeta)$. The *height lattice* $\mathcal{H}(X_\zeta)$ of a constant-shape substitution ζ is the smallest lattice containing $\mathfrak{L}(\mathcal{R}(X_\zeta))$ such that $\mathcal{H}(X_\zeta) \cap L_\zeta(\mathbb{Z}^d) \leq L_\zeta(\mathcal{H}(X_\zeta))$. Notice that the last property is equivalent to $\mathcal{H}(X_\zeta) \cap L_\zeta^n(\mathbb{Z}^d) \leq L_\zeta^n(\mathcal{H}(X_\zeta))$, for any $n > 0$. The height lattice is *trivial* whenever $\mathcal{H}(X_\zeta) = \mathbb{Z}^d$.

In the following, we will give a description for the height lattice. For $\mathbf{k} \in \mathbb{Z}^d$, we define $\mathcal{R}_{\mathbf{k}}(X_\zeta) = \{\mathbf{j} \in \mathbb{Z}^d : x_{\mathbf{j}+\mathbf{k}} = x_{\mathbf{k}}\}$. Let $\mathfrak{L}(\mathcal{R}_{\mathbf{k}}(X_\zeta))$ be the smallest lattice containing $\mathcal{R}_{\mathbf{k}}(X_\zeta)$ and $\mathcal{H}_{\mathbf{k}}(X_\zeta)$ be the smallest lattice containing $\mathfrak{L}(\mathcal{R}_{\mathbf{k}}(X_\zeta))$ such that $\mathcal{H}_{\mathbf{k}}(X_\zeta) \cap L_\zeta(\mathbb{Z}^d) \leq L_\zeta(\mathcal{H}_{\mathbf{k}}(X_\zeta))$. We adapt the proof for the one-dimensional case [37, 97] and obtain the following result.

Lemma 3.12. *Let ζ be an aperiodic primitive constant-shape substitution. Then, for any $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{Z}^d$ the sets $\mathcal{H}_{\mathbf{k}_1}(X_\zeta), \mathcal{H}_{\mathbf{k}_2}(X_\zeta)$ are the same. In particular, for any $\mathbf{k} \in \mathbb{Z}^d$, $\mathcal{H}_{\mathbf{k}}(X_\zeta)$ is equal to $\mathcal{H}(X_\zeta)$.*

Proof. Let $x \in X_\zeta$ be a fixed point of the substitution, $\mathbf{k}_1, \mathbf{k}_2 \in \mathbb{Z}^d$ and N be large enough such that $x_{\mathbf{k}_2} \sqsubseteq \zeta^N(x_{\mathbf{k}_1})$. Set \mathbf{m} such that $x_{\mathbf{k}_2} = \zeta^N(x_{\mathbf{k}_1})_{\mathbf{m}}$. Since $x = \zeta^N(x)$ for any $\mathbf{j} \in \mathcal{R}_{\mathbf{k}_1}(X_\zeta)$, $L_\zeta^N(\mathbf{k}_1 + \mathbf{j}) + \mathbf{m} \in \mathcal{R}_{\mathbf{k}_2}(X_\zeta)$. Hence $L_\zeta^N(\mathbf{j})$ is in $\mathfrak{L}(\mathcal{R}_{\mathbf{k}_2}(X_\zeta))$ and therefore in $\mathcal{H}_{\mathbf{k}_2}(X_\zeta)$. By definition of $\mathcal{H}_{\mathbf{k}_2}(X_\zeta)$ and invertibility of L_ζ^N , we conclude that $\mathbf{j} \in \mathcal{H}_{\mathbf{k}_2}(X_\zeta)$. Since it is the smallest lattice satisfying the property, $\mathcal{H}_{\mathbf{k}_1}(X_\zeta)$ is a subgroup of $\mathcal{H}_{\mathbf{k}_2}(X_\zeta)$, and by reciprocity we have that these sets are the same. We conclude the second equality by observing that $\mathcal{H}(X_\zeta) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \mathcal{H}_{\mathbf{k}}(X_\zeta)$. \square

As in the one-dimensional case, to study the maximal equicontinuous factor of substitutive subshifts, we study their *eigenvalues*. A vector $\mathbf{x} \in \mathbb{R}^d$ is said to be an eigenvalue for the topological dynamical system $(X_\zeta, S, \mathbb{Z}^d)$ (the measure-preserving system $(X_\zeta, \mu, S, \mathbb{Z}^d)$) if there exists a continuous function $f : X_\zeta \rightarrow \mathbb{C}$ ($f \in L^2(X_\zeta, \mu_\zeta)$) such that for every $\mathbf{n} \in \mathbb{Z}^d$, $f \circ S^\mathbf{n} = e^{2\pi i \langle \mathbf{x}, \mathbf{n} \rangle} f$ in X_ζ ($f \circ S^\mathbf{n} = e^{2\pi i \langle \mathbf{x}, \mathbf{n} \rangle} f$ in X_ζ , μ_ζ -a.e. in X_ζ). In [108] it was proved the following result, generalizing the characterization of eigenvalues for the one-dimensional case [37].

Theorem 3.13. [108, Theorem 3.13] *Let ζ be an aperiodic primitive tiling substitution with expansion matrix L_ζ , which has a fixed point. Then the following are equivalent for $\mathbf{x} \in \mathbb{R}^d$:*

1. *The vector \mathbf{x} is a continuous eigenvalue for the topological dynamical system $(X_\zeta, S, \mathbb{R}^d)$.*
2. *The vector \mathbf{x} is a measurable eigenvalue for the measure-preserving system $(X_\zeta, \mu_\zeta, S, \mathbb{R}^d)$.*
3. *The vector \mathbf{x} satisfies the following condition:*

$$\lim_{n \rightarrow \infty} e^{2\pi i \langle L_\zeta^n \mathbf{j}, \mathbf{x} \rangle} = 1, \quad \forall \mathbf{j} \in \mathcal{R}(X_\zeta).$$

4. *The vector \mathbf{x} satisfies the following condition:*

$$\lim_{n \rightarrow \infty} e^{2\pi i \langle L_\zeta^n \mathbf{j}, \mathbf{x} \rangle} = 1, \quad \forall \mathbf{j} \in \mathcal{H}(X_\zeta).$$

Remark 3.14. We have that

- (1) Condition 4. in Theorem 3.13 is not proved in [108] but it can be easily checked noticing the set of points satisfying $\langle (L_\zeta^*)^N \alpha, \mathbf{j} \rangle \in \mathbb{Z}$ is a lattice.
- (2) The same results is satisfied for the topological dynamical system $(X_\zeta, S, \mathbb{Z}^d)$ with the same arguments given in [108].

This implies the set of continuous (and measurable) eigenvalues of $(X_\zeta, S, \mathbb{Z}^d)$ corresponds to the set $\bigcup_{n \geq 0} (L_\zeta^*)^{-n}(\mathcal{H}^*(X_\zeta))$. In particular the set of eigenvalues $E(X_\zeta, S, \mathbb{Z}^d)$ of $(X_\zeta, S, \mathbb{Z}^d)$ is a subset of \mathbb{Q}^d . A direct consequence of Theorem 3.13 is a description of the maximal equicontinuous factor of $(X_\zeta, S, \mathbb{Z}^d)$.

Proposition 3.15. *Let ζ be an aperiodic primitive constant-shape substitution. The maximal equicontinuous factor of the substitutive subshift of $(X_\zeta, S, \mathbb{Z}^d)$ is the odometer system*

$$\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n(\mathcal{H}(X_\zeta))), +_{L_\zeta^n(\mathcal{H}(X_\zeta))}, \mathbb{Z}^d) = \left(\lim_{\leftarrow n} (\mathbb{Z}^d / L_\zeta^n(\mathcal{H}(X_\zeta)), \alpha_n), +_{(L_\zeta^n(\mathcal{H}(X_\zeta))), \mathbb{Z}^d} \right).$$

For completeness, we provide another description for the maximal equicontinuous factor in a general setting. Let (X, T, \mathbb{Z}^d) be a topological dynamical system, where X is a Cantor set, and $\Gamma \leq \mathbb{Z}^d$ a subgroup with finite index. We say (X, T, \mathbb{Z}^d) admits a Γ -minimal partition if there exists a closed partition $\bigcup_{\mathbf{g} \in \mathbb{Z}^d/\Gamma} X_{\mathbf{g}} = X$ such that for all $\mathbf{g} \in \mathbb{Z}^d/\Gamma$ the set

of return times $\mathcal{RT}(X_{\mathbf{g}}) = \{\mathbf{n} \in \mathbb{Z}^d : T^{-\mathbf{n}}(X_{\mathbf{g}}) \cap X_{\mathbf{g}} \neq \emptyset\}$ is equal to Γ and the topological dynamical system $(X_{\mathbf{g}}, \Gamma)$ is a minimal system. Note that, since its a finite partition, the sets $X_{\mathbf{g}}$ are clopen. The minimality of the induced actions implies there is at most one Γ -minimal partition up to a permutation of the sets $\{X_{\mathbf{g}}\}_{\mathbf{g} \in \mathbb{Z}^d/\Gamma}$. A Γ -minimal partition is associated with an equicontinuous factor of (X, T, \mathbb{Z}^d) . To see this we enumerate $X_{\mathbf{g}}$ such that for all $\mathbf{g} \in \mathbb{Z}^d/\Gamma$ and $\mathbf{n} \in \mathbb{Z}^d$ we have that $T^{\mathbf{n}}X_{\mathbf{g}} = X_{\mathbf{g}+\mathbf{n} \bmod \Gamma}$. Then, the map $\pi : (X, T, \mathbb{Z}^d) \rightarrow (\mathbb{Z}^d/\Gamma, +_{\Gamma}, \mathbb{Z}^d)$, such that $\pi(x) = \mathbf{g}$ if and only if $x \in X_{\mathbf{g}}$, is a factor map onto $(\mathbb{Z}^d/\Gamma, +_{\Gamma}, \mathbb{Z}^d)$, where \mathbb{Z}^d acts by quotient translations onto \mathbb{Z}^d/Γ . The following proposition shows the connection between Γ -minimal partitions and eigenvalues of a topological dynamical system.

Proposition 3.16. *Let (X, T, \mathbb{Z}^d) be a minimal topological dynamical system and $\Gamma \leq \mathbb{Z}^d$ be a finite index subgroup. The system (X, T, \mathbb{Z}^d) admits a Γ -minimal partition, if and only if $\Gamma^* \subseteq E(X, T, \mathbb{Z}^d)$.*

Proof. Let $R \in \mathcal{M}(d, \mathbb{Z})$ be such that $\Gamma = R(\mathbb{Z}^d)$.

Let $\{X_{\mathbf{g}}\}_{\mathbf{g} \in R([0,1)^d) \cap \mathbb{Z}^d}$ be a Γ -minimal partition satisfying $X_{\mathbf{g}} = T^{\mathbf{g}}(X_0)$ for all $\mathbf{g} \in M([0,1)^d) \cap \mathbb{Z}^d$. Let \mathbf{x} be in $(R^*)^{-1}(\mathbb{Z}^d)$. Define f as the map $f = \sum_{\mathbf{g} \in M([0,1)^d) \cap \mathbb{Z}^d} e^{2\pi i \langle \mathbf{x}, \mathbf{g} \rangle} \mathbf{1}_{X_{\mathbf{g}}}$.

Since the sets $\{X_{\mathbf{g}}\}_{\mathbf{g} \in R([0,1)^d) \cap \mathbb{Z}^d}$ are clopen, the map f is continuous. Let $x \in X_0$, $\mathbf{m} \in \mathbb{Z}^d$ and $\mathbf{m}_1 \in \mathbb{Z}^d$, $\mathbf{m}_2 \in R([0,1)^d) \cap \mathbb{Z}^d$ be such that $\mathbf{m} = R(\mathbf{m}_1) + \mathbf{m}_2$. Note that $T^{\mathbf{m}}x$ is in $X_{\mathbf{m}_2}$ and since \mathbf{x} is in $(R^*)^{-1}(\mathbb{Z}^d)$, we have that

$$e^{2\pi i \langle \alpha, \mathbf{m} \rangle} = e^{2\pi i \langle \alpha, R(\mathbf{m}_1) + \mathbf{m}_2 \rangle} = e^{2\pi i (\langle \alpha, R(\mathbf{m}_1) \rangle + \langle \alpha, \mathbf{m}_2 \rangle)} = e^{2\pi i \langle \alpha, \mathbf{m}_2 \rangle}.$$

We find a continuous map such that $f(T^{\mathbf{m}}x) = e^{2\pi i \langle \mathbf{x}, \mathbf{m} \rangle} f(x)$ for all $x \in X$ and all $\mathbf{m} \in \mathbb{Z}^d$. We conclude that $\mathbf{x} \in E(X, T, \mathbb{Z}^d)$.

On the other hand, let \bar{x} be in X . For $j \in \{1, \dots, d\}$ we denote $\mathbf{x}_j = (R^*)^{-1}(e_i)$. Since $\Gamma^* \subseteq E(X, T, \mathbb{Z}^d)$ there exists a map $f_j : X \rightarrow \mathbb{C}$ such that $f_j(T^{\mathbf{m}}\bar{x}) = e^{2\pi i \langle \mathbf{x}_j, \mathbf{m} \rangle} f_j(\bar{x})$ for all $\mathbf{m} \in \mathbb{Z}^d$. Since the eigenspaces are one-dimensional we choose f_j such that $f_j(\bar{x}) = 1$. By the previous formula, the values of $e^{2\pi i \langle \mathbf{x}_j, \mathbf{m} \rangle}$ only depend on $\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d$. Now, for any $j \in \{1, \dots, d\}$ and $\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d$ we denote $X_{\mathbf{m}}^j = f_j^{-1}(\{e^{2\pi i \langle \alpha_j, \mathbf{m} \rangle}\})$. For each $j \in \{1, \dots, d\}$, the set X is equal to $\bigcup_{\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d} X_{\mathbf{m}}^j$. We define $X_{\mathbf{m}} = \bigcap_{j \in \{1, \dots, d\}} X_{\mathbf{m}}^j$. We

will prove that $\{X_{\mathbf{m}}\}_{\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d}$ is a Γ -minimal partition. Note that $X_{\mathbf{m}}$ is Γ -invariant. First, assume that there exist $\mathbf{n}_1, \mathbf{n}_2 \in R([0,1)^d) \cap \mathbb{Z}^d$ such that $\mathbf{n}_1 \neq \mathbf{n}_2$ and $X_{\mathbf{n}_1} \cap X_{\mathbf{n}_2} \neq \emptyset$. Using the \mathbb{Z}^d -action on X , this is equivalent to the existence of $\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d$ with

$\mathbf{m} \neq \mathbf{0}$ and $X_{\mathbf{m}} \cap X_{\mathbf{0}} \neq \emptyset$. This means that for all $j \in \{1, \dots, d\}$, $e^{2\pi i \langle \mathbf{x}_j, \mathbf{m} \rangle}$ is equal to 1, i.e., $\langle \mathbf{x}_j, \mathbf{m} \rangle \in \mathbb{Z}$. Since $\Gamma^* = \langle \{\mathbf{x}_1, \dots, \mathbf{x}_d\} \rangle$, we have that $\mathbf{m} \in (\Gamma^*)^* = \Gamma$ which is a contradiction, so all of these sets are disjoint.

By minimality, we have that $X = \bigcup_{\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d} \overline{\mathcal{O}(T^{\mathbf{m}}\bar{x}, \Gamma)}$. Since $\overline{\mathcal{O}(T^{\mathbf{m}}x, \Gamma)}$ is included in $X_{\mathbf{m}}$, we conclude that $\{X_{\mathbf{m}}\}_{\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d}$ is a clopen partition of X and $\overline{\mathcal{O}(T^{\mathbf{m}}\bar{x}, \Gamma)} = X_{\mathbf{m}}$, so the action of Γ on $X_{\mathbf{m}}$ is minimal. A direct computation shows that the set of return times of each $X_{\mathbf{m}}$ is Γ . Hence $\{X_{\mathbf{m}}\}_{\mathbf{m} \in R([0,1)^d) \cap \mathbb{Z}^d}$ is a Γ -minimal partition. \square

Remark 3.17. The following statements can be easily verified.

- (1) In the case of an aperiodic primitive constant-shape substitution ζ , the recognizability property implies for all $n > 0$, the sets $\{S^j \zeta^n(X_{\zeta})\}_{j \in F_n^{\zeta}}$ are a $L_{\zeta}^n(\mathbb{Z}^d)$ -minimal partition.
- (2) By Theorem 3.13 and Proposition 3.16 for any aperiodic primitive constant-shape substitution there exists a $\mathcal{H}(X_{\zeta})$ -minimal partition.

3.5 Aperiodic symbolic factors of substitutive subshifts are conjugate to substitutive subshifts

In the following we will prove Theorem E. This is a multidimensional analogue of a result proved by C. Müllner and R. Yassawi [91], which is a refinement of a result proved in [48] for constant-length substitutions. We follow the same strategy of [91], with a slight difference. By Proposition 3.3 aperiodic symbolic factors of substitutive subshifts are recognizable. In the original article, this is not mentioned. They proved an odometer system (the one defined in Section 3.4) is an equicontinuous system of aperiodic symbolic factors of substitutive subshifts, following the ideas developed by T. Kamae [73] about *minimal partitions*. To get this, they used a characterization of periodic sequences and the complexity function, known as Morse-Hedlund theorem [88]. Until now, this characterization is not known to be true in the two-dimensional case, and it is called as the *Nivat's conjecture*. However, in higher dimensions is known to be false [23]. Then, using the fact that we can always assume that the factor between a substitutive subshift and an aperiodic symbolic system is induced by a letter-to-letter map (Lemma 1.22), we define an equivalence relation on the alphabet, calling two letters equivalent if and only if they have the same image via the local map given by the factor map. We consider the set of equivalence classes as a new alphabet, and we define a new constant-shape substitution on it. We finally prove that the substitutive subshift generated by this new substitution is conjugate to the aperiodic symbolic factor (Theorem 3.22).

Now, we proceed to the proof. First, we have the following consequence of Proposition 3.3.

Remark 3.18. It is straightforward to check that Proposition 3.3 implies if $x, x' \in X_\zeta$ are such that $\tau(x) = \tau(x')$, then $\pi(x)$ is equal to $\pi(x')$, where $\pi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (\overleftarrow{\mathbb{Z}^d}_{(L_\zeta^n)}, +_{(L_\zeta^n)}, \mathbb{Z}^d)$ is the factor map.

To prove the result we will introduce some notions as in [91].

Let ζ be an aperiodic primitive constant-shape substitution. We consider a labeled directed graph G_ζ with vertex set $E = \mathcal{A}^2$ and there exists an edge (a, b) to (c, d) with label $\mathbf{f} \in F_1^\zeta$ if $\zeta(a)_\mathbf{f} = c$, $\zeta(b)_\mathbf{f} = d$. Note that the diagonal $\Delta_{\mathcal{A}} = \{(a, a) : a \in \mathcal{A}\}$ is a stable set, i.e., $E(\Delta_{\mathcal{A}}) = \Delta_{\mathcal{A}}$. Let $P = (a_0, b_0)(a_1, b_1)(a_2, b_2)$ be a path, by definition, there is an edge from (a_0, b_0) to (a_1, b_1) with a label \mathbf{f}_0 and an edge (a_1, b_1) to (a_2, b_2) with label \mathbf{f}_1 , i.e., $\zeta(a_0)_{\mathbf{f}_0} = a_1$, $\zeta(b_0)_{\mathbf{f}_0} = b_1$, and $\zeta(a_1)_{\mathbf{f}_1} = a_2$, $\zeta(b_1)_{\mathbf{f}_1} = b_2$. Then, we have that $\zeta^2(a_0)_{L_\zeta \mathbf{f}_0 + \mathbf{f}_1} = a_2$, $\zeta^2(b_0)_{L_\zeta \mathbf{f}_0 + \mathbf{f}_1} = b_2$. This means, the paths indicate the simultaneous positions of the letters in the iterates of the substitution.

Definition 3.19. Let ζ be an aperiodic primitive constant-shape substitution.

1. We say that a pair $(a, b) \in \mathcal{A}^2 \setminus \Delta_{\mathcal{A}}$ is a *periodic pair* if there is a cycle in G_ζ which starts and ends in (a, b) . We define $n(a, b) = \min\{|P| : P \text{ is a cycle in } (a, b)\}$ and we denote

$$n(\zeta) = \text{lcm}\{n(a, b) : (a, b) \text{ is a periodic pair}\},$$

we call the substitution *pair-aperiodic* if $n(\zeta) = 1$.

2. We call a pair $(a, b) \in \mathcal{A} \setminus \Delta_{\mathcal{A}}$ an *asymptotic disjoint pair* if for any $k > 0$, there exists a path $P = (a_0, b_0) \dots (a_k, b_k)$ in G_ζ of length k with $(a_0, b_0) = (a, b)$ and $(a_k, b_k) \notin \Delta_{\mathcal{A}}$.

Remark 3.20. The following statements can be easily verified

- (1) As for the case of periodic points for the substitution, we can replace the substitution ζ for an appropriate power, i.e., $\zeta^{n(\zeta)}$, so we may assume that the substitution is pair-aperiodic.
- (2) If the substitution ζ is bijective, every $(a, b) \in \mathcal{A}^2 \setminus \Delta_{\mathcal{A}}$ is an asymptotic disjoint pair.
- (3) Assume that ζ is pair-aperiodic. If $(a, b) \in \mathcal{A}^2$ is not an asymptotic disjoint pair, let \bar{k} be the minimum length of a path from (a, b) such that any path of length \bar{k} has an end in $\Delta_{\mathcal{A}}$. If $\bar{k} > |\mathcal{A}|^2$, there exists a cycle as a subpath in P , i.e., one of the vertex (c, d) is a periodic pair. Since ζ is pair-aperiodic, there exists $\mathbf{f} \in F_1^\zeta$ such that $\zeta(c)_\mathbf{f} = c \neq d = \zeta(d)_\mathbf{f}$. So we can create a path of length arbitrarily large from (a, b) that not reach $\Delta_{\mathcal{A}}$, which is a contradiction of not being asymptotic disjoint pair. We then have that $\bar{k} \leq |\mathcal{A}|^2$, which implies $\zeta^{|\mathcal{A}|^2}(a) = \zeta^{|\mathcal{A}|^2}(b)$.

Definition 3.21. Let \mathcal{A}, \mathcal{B} be two finite alphabets, ζ be an aperiodic primitive constant-shape substitution with alphabet \mathcal{A} and $\tau : \mathcal{A} \rightarrow \mathcal{B}$. We say that $a, b \in \mathcal{A}$ with $a \neq b$ are *indistinguishable* (by (ζ, τ)) if for all $n \geq 0$ we have that $\tau(\zeta^n(a)) = \tau(\zeta^n(b))$.

With these definitions we are ready to prove the next result which is the multidimensional analogue to Theorem 22 in [91].

Theorem 3.22. *Let (Y, S, \mathbb{Z}^d) be an aperiodic symbolic factor (with alphabet \mathcal{B}) of a substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$, with ζ being an aperiodic primitive constant-shape substitution with alphabet \mathcal{A} . Then, there exists an aperiodic primitive constant-shape substitution ζ' with alphabet \mathcal{C} having the same expansion matrix and support of a power of ζ and a conjugacy $\tau' : (X_{\zeta'}, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ via a 0-block map.*

Proof. By Lemma 1.22 we can assume that the factor map $\tau : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ is induced via a 0-block map and by Remark 3.20 we can assume that ζ is a pair-aperiodic substitution.

We define an equivalence relation $a \sim b$ in \mathcal{A} , such that $a \sim b$ if a, b are indistinguishable. By definition, the substitution $\zeta'([a])_{\mathbf{f}} = [\zeta(a)_{\mathbf{f}}]$, $\mathbf{f} \in F_1^\zeta$ in \mathcal{A}/\sim and the map $\mathcal{T}' : \mathcal{A}/\sim \rightarrow \mathcal{B}$, given by $\mathcal{T}'([a]) = \tau(a)$ are well defined. These maps satisfy the following property:

$$\text{Every pair in } \mathcal{A}/\sim \text{ is distinguishable.} \quad (3.1)$$

It is straightforward to check that primitivity of ζ implies primitivity of ζ' .

Assume now that $([a], [b])$ is a periodic pair, i.e., there exists a cycle $P_1 = ([a_0], [b_0]) \dots ([a_k], [b_k])$ in $G_{\zeta'}$ with $([a_0], [b_0]) = ([a_k], [b_k]) = ([a], [b])$. We can consider a path $P_2 = (c_0, d_0) \dots (c_k, d_k)$ in G_ζ with $[c_i] = [a_i]$ and $[d_i] = [b_i]$ for $0 \leq i \leq k$ having the same label of edges as P_1 . Now, repeating this process we get a path P_3 in G_ζ of length $(\max\{|[a]|, |[b]|\} + 1)k$ from (c, d) repeating the labels of the path P_1 with a period k . By the Pigeonhole Principle, there exist two subpaths $P_4 = (e_0, f_0) \dots (e_{l_1 k}, f_{l_1 k})$, $P_5 = (g_0, h_0) \dots (g_{l_2 k}, h_{l_2 k})$ of P_3 , having the same labels of the edges as P_1 repeating with period k , such that $e_0 = e_{l_1 k}$, $h_0 = h_{l_2 k}$ and $[e_0] = [a_0]$, $[h_0] = [b_0]$. Now consider the cycle in G_ζ $(u_0, v_0) \dots (u_{l_1 l_2 k^2}, v_{l_1 l_2 k^2})$ where $u_{l_1 k j} \dots u_{l_1 k (j+1)} = e_0, \dots, e_{l_1 k}$ and $v_{l_2 k m} \dots v_{l_2 k (m+1)} = h_0 \dots h_{l_2 k}$ for all $0 \leq j < l_2 k$, $0 \leq m < l_1 k$. Since ζ is pair-aperiodic, there exists $\mathbf{f} \in F_1^\zeta$ such that $\zeta(e_0)_{\mathbf{f}} = e_0$, $\zeta(h_0)_{\mathbf{f}} = h_0$. We then conclude that $\zeta'([a])_{\mathbf{f}} = [a]$, $\zeta'([b])_{\mathbf{f}} = [b]$, i.e., ζ' is pair-aperiodic.

On the other hand, for all $n > 0$, we have that $\tau(\zeta^n(a)) = \tau'(\zeta'^n([a]))$, hence Y has the same language as $\tau'(X_{\zeta'})$, so they are equal, since subshifts are uniquely determined by their language.

Finally, we prove that $\tau' : (X_{\zeta'}, S, \mathbb{Z}^d) \rightarrow (Y, S, \mathbb{Z}^d)$ is a conjugacy. Let $x, x' \in X_{\zeta'}$, with $\tau'(x) = \tau'(x')$. By the recognizability property of $X_{\zeta'}$, we can write $x = S\mathbf{f}_1 \zeta^{2|\mathcal{A}|^2}(\bar{x})$, $x' = S\mathbf{f}_2 \zeta^{2|\mathcal{A}|^2}(\bar{x}')$. By Remark 3.18 we have that $\mathbf{f}_1 = \mathbf{f}_2$.

For every $\mathbf{n} \in \mathbb{Z}^d$, $(\bar{x}_{\mathbf{n}}, \bar{x}'_{\mathbf{n}}) \in \mathcal{A}^2$ are not asymptotic disjoint pair. Assume the contrary, i.e., there exists $\mathbf{n} \in \mathbb{Z}^d$ such that $(\bar{x}_{\mathbf{n}}, \bar{x}'_{\mathbf{n}})$ is an asymptotic disjoint pair. Then we can find a periodic pair $([a], [b])$ and a path $P = ([a_0], [b_0]) \dots ([a_k], [b_k])$ in $G_{\zeta'}$ with $([a_0], [b_0]) = (\bar{x}_{\mathbf{n}}, \bar{x}'_{\mathbf{n}})$ and $([a_k], [b_k]) = ([a], [b])$. with $k \leq |\mathcal{A}|^2$. Since ζ' is pair-aperiodic, we have that there exists $\mathbf{f} \in F_{2|\mathcal{A}|^2}^{\zeta}$ such that $(\zeta')^{2|\mathcal{A}|^2}(\bar{x}_{\mathbf{n}})_{\mathbf{f}} = [a]$, $(\zeta')^{2|\mathcal{A}|^2}(\bar{x}'_{\mathbf{n}})_{\mathbf{f}} = [b]$. Since $\tau'(\zeta'^{2|\mathcal{A}|^2}(\bar{x}_{\mathbf{n}})) = \zeta'^{2|\mathcal{A}|^2}(\tau'(\bar{x}_{\mathbf{n}})) = \zeta'^{2|\mathcal{A}|^2}(\tau'(\bar{x}'_{\mathbf{n}})) = \tau'(\zeta'^{2|\mathcal{A}|^2}(\bar{x}'_{\mathbf{n}}))$, we have that $\tau'(a) = \tau'(b)$, then $([a], [b])$ are indistinguishable, which contradicts 3.1.

Thus we have that $(\bar{x}_{\mathbf{n}}, \bar{x}'_{\mathbf{n}})$ is not an asymptotic disjoint pair for any $\mathbf{n} \in \mathbb{Z}^d$. By Remark 3.20 we have that $\zeta^{2|\mathcal{A}|^2}(\bar{x}_{\mathbf{n}}) = \zeta^{2|\mathcal{A}|^2}(\bar{x}'_{\mathbf{n}})$, i.e., $x = x'$. \square

For the one-dimensional case, F. Durand proved [45] that Cantor topological factors of substitutive subshifts are either substitutive subshifts (when the action is expansive) or odometer systems (when the action is equicontinuous). This dichotomy is no longer true in the multidimensional context. Example 4.3 shows an example of substitutive subshift with a Cantor topological factor that is neither expansive nor equicontinuous. Also the example in Example 4.3 has a symbolic factor (in fact substitutive subshift) which has a non-trivial period, and the phase space is still infinite.

Chapter 4

Measurable morphisms between substitutive subshifts

In this chapter, we study different types of homomorphisms between substitutive subshifts. Note that since these subshifts are uniquely ergodic, any topological endomorphism is also a measurable endomorphism preserving the ergodic measure. First, we will extend a result of B. Host and F. Parreau [69] to the multidimensional case (Theorem 4.1), showing some rigidity properties: Any measurable factor between two substitutive subshifts given by two aperiodic constant-shape substitutions with some combinatorial property induces a continuous one. We follow the same strategy for the proof. It is based on the property of these substitutive subshifts being *self-induced systems*, i.e., there exists non-empty clopen proper subsets (in our case $\zeta^n(X_\zeta)$, for all $n > 0$) such that the induced system $(\zeta^n(X_\zeta), S_{\zeta^n(X_\zeta)}, \mathbb{Z}^d)$ is conjugate to the system $(X_\zeta, S, \mathbb{Z}^d)$. Here, the action $\langle S_{\zeta^n(X_\zeta)} \rangle$ is given by $\{S^{L_\zeta^n(\mathbf{m})} : \mathbf{m} \in \mathbb{Z}^d\}$. This implies, any measurable endomorphism ϕ of $(X_\zeta, S, \mathbb{Z}^d)$ is associated with an induced measurable endomorphism ϕ_n of $(\zeta^n(X_\zeta), S_{\zeta^n(X_\zeta)}, \mathbb{Z}^d)$. We prove that these induced measurable endomorphisms are stationary, i.e., there exists $n \neq m > 0$ such that $\phi_n = \phi_m$. Then, we prove that we can approximate these induced measurable endomorphisms by endomorphisms of radius $\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|)\right)\right)$. The finiteness of sliding block codes of a specific radius will let us conclude the theorem (Theorem 4.1).

The result of B. Host and F. Parreau was then extended by V. Salo and Törmä in [103], for topological factors between constant-length substitutions and Pisot substitutions whose associated incidence matrices have the same dominant eigenvalue. As in [69], there exists a bound R such that any factor map is the composition of a sliding block code of radius R with a power of the shift map. They used a renormalization process, where they could reduce the radius of a given factor map. Then, F. Durand and J. Leroy extended this result for any pair of aperiodic substitutions [50]. This was one of the key steps to

prove the decidability of the isomorphism problem between substitutive subshifts.

Our result shows the decidability of the isomorphism problem between multidimensional substitutive subshifts, when both constant-shape substitutions has the same expansion matrix and same support, and satisfy a combinatorial condition (called *reducibility*). This condition is always satisfied for bijective substitutions. Nevertheless, this result is far from representing the complete picture about the decidability of the isomorphism problem between multidimensional substitutive subshifts.

Then, we will deduce restrictions on endomorphisms and homomorphisms, using the finiteness of the number of invariant orbits (Proposition 3.4). Every substitutive subshift given by an aperiodic constant-shape substitution satisfying the combinatorial property is coalescent (Proposition 4.7). This was already proved in the one-dimensional case in [45]. Also we proved the automorphism group of substitutive subshift is virtually generated by the shift action (Proposition 4.8). It was already known in the one-dimensional context by [69] and [82]. In the multidimensional framework was proved under more restrictive geometrical and combinatorial properties [21]. We also give some conditions to get that the automorphism group of a substitutive subshift is isomorphic to a direct product of \mathbb{Z}^d with a finite group (Corollary 4.9). Finally, we extend Theorem 4.1 to homomorphisms associated with matrices commuting with a power of the expansion matrix of the substitution (Theorem 4.12). This leads to the same rigidity properties about these homomorphisms (Proposition 4.16) and for a restricted normalizer group (Proposition 4.16). Notice that in the next chapter we will give sufficient conditions to ensure the former result is a complete characterization of the normalizer group.

4.1 Measurable factors implies continuous ones for substitutive subshifts

In this section, we will extend the result proved in [69] about measurable factors between substitutive subshifts in the multidimensional context. As in [69], we will use the notion of *reducibility* of a substitution. Let ζ be a constant-shape substitution. For any pair of letters $a, b \in \mathcal{A}$, and $n > 0$, we consider the sequence

$$d_n(\zeta^n(a), \zeta^n(b)) = \frac{\left| \left\{ \mathbf{f} \in F_n^\zeta : \zeta^n(a)_{\mathbf{f}} \neq \zeta^n(b)_{\mathbf{f}} \right\} \right|}{|F_n^\zeta|}.$$

This sequence is decreasing for all of the pairs $a, b \in \mathcal{A}$. We say the constant-shape substitution is *reduced* if $\min_{\substack{n \in \mathbb{N} \\ a \neq b \in \mathcal{A}}} d_n(\zeta^n(a), \zeta^n(b)) > 0$. For instance, every bijective constant-shape substitution is reduced.

As mentioned in Section 1.7, the substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ is uniquely ergodic. Denote μ_ζ its unique ergodic invariant measure. Using the recognizability property, as in

the one-dimensional case [97], this unique ergodic measure satisfies

$$\forall U \in \mathcal{F}_{X_\zeta}, \mu_\zeta(U) = \frac{1}{|F_n^\zeta|} \int_{X_\zeta} \left| \left\{ \mathbf{f} \in F_n^\zeta : S^{\mathbf{f}} \zeta^n(x) \in U \right\} \right| d\mu_\zeta(x),$$

where \mathcal{F}_{X_ζ} corresponds to the Borel sets of X_ζ .

The following theorem is a multidimensional analogue of Theorem 1.3 in [69]:

Theorem 4.1. *Let $(X_{\zeta_1}, S, \mathbb{Z}^d)$, $(X_{\zeta_2}, S, \mathbb{Z}^d)$ be two substitutive subshift from two aperiodic primitive constant-shape substitutions ζ_1, ζ_2 from finite alphabets \mathcal{A} and \mathcal{B} , with the same expansion matrix L and same support F_1 . If $(X_{\zeta_2}, S, \mathbb{Z}^d)$ is reduced, then for every measurable factor $\phi : (X_{\zeta_1}, \mu_{\zeta_1}, S, \mathbb{Z}^d) \rightarrow (X_{\zeta_2}, \mu_{\zeta_2}, S, \mathbb{Z}^d)$, there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $S^{\mathbf{j}}\phi$ is equal μ_{ζ_1} -a.e. to a continuous factor $\psi : (X_{\zeta_1}, S, \mathbb{Z}^d) \rightarrow (X_{\zeta_2}, S, \mathbb{Z}^d)$, satisfying the following two properties:*

1. ψ is a sliding block code of radius $\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|) \right) \right)$.
2. There exist an integer $n > 0$ and $\mathbf{p} \in F_n^\zeta$ such that, $S^{\mathbf{p}}\psi\zeta_1^n = \zeta_2^n\psi$.

In Chapter 6 we present an example where Theorem 4.1 can be applied, describing the automorphisms of it.

Remark 4.2. The following statements can be easily verified.

- (1) If L_ζ is a diagonal matrix, then $\|L_\zeta^{-1}\| \leq 1/2$, so ψ is a sliding block code of radius $3\|F_1^\zeta\|$.
- (2) Since the set of sliding block codes of radius $\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|) \right) \right)$ between X_{ζ_1} and X_{ζ_2} is finite, we may consider an appropriate iteration of ζ_1 and ζ_2 such that any factor ψ satisfying 2. in Theorem 4.1 satisfies $S^{\mathbf{p}}\psi\zeta_1 = \zeta_2\psi$.
- (3) As also mentioned in [69], we may assume that $\mathbf{p} \notin (L_\zeta^n - \text{id})(\mathbb{Z}^d)$, because it is equivalent to find a factor map commuting with the substitution map, i.e., $\psi\zeta_1^n = \zeta_2^n\psi$, with $\mathbf{p} \in F_1^\zeta$.

We follows the same strategy of [69]. Substitutive subshifts are self-induced systems, i.e., there exists non-empty clopen proper subsets (in our case $\zeta^n(X_\zeta)$, for all $n > 0$) such that the induced system $(\zeta^n(X_\zeta), S_{\zeta^n(X_\zeta)}, \mathbb{Z}^d)$ is conjugate to the system $(X_\zeta, S, \mathbb{Z}^d)$. We recall, the action $\langle S_{\zeta^n(X_\zeta)} \rangle$ is given by $\left\{ S^{L_\zeta^n(\mathbf{m})} : \mathbf{m} \in \mathbb{Z}^d \right\}$. This implies, any measurable endomorphism ϕ of $(X_\zeta, S, \mathbb{Z}^d)$ is associated with an induced measurable endomorphism ϕ_n of $(\zeta^n(X_\zeta), S_{\zeta^n(X_\zeta)}, \mathbb{Z}^d)$. We prove that these induced measurable endomorphisms are stationary, i.e., there exists $n \neq m > 0$ such that $\phi_n = \phi_m$. Then, we prove that we can approximate these induced measurable endomorphisms by endomorphisms of radius

$\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|)\right)\right)$. The finiteness of sliding block codes of a specific radius will let us conclude the theorem (Theorem 4.1).

If a substitution is not reduced, we consider an equivalence relation calling two letters a, b equivalent when $d_n(\zeta^n(a), \zeta^n(b)) \rightarrow 0$. If two letters $a, b \in \mathcal{A}$ are equivalent, then $(\zeta(a))_{\mathbf{f}} \sim (\zeta(b))_{\mathbf{f}}$ for all $\mathbf{f} \in F_1^\zeta$. We define a substitution $\tilde{\zeta}$ on \mathcal{A}/\sim given by $(\tilde{\zeta}([a]))_{\mathbf{f}} = [\zeta(a)_{\mathbf{f}}]$ for $\mathbf{f} \in F_1^\zeta$. This substitution is reduced. We have a natural letter-to-letter factor map $\phi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (X_{\tilde{\zeta}}, S, \mathbb{Z}^d)$, and is called the *reduced substitution* of ζ .

In the one-dimensional case if (X_ζ, S, \mathbb{Z}) does not have purely discrete spectrum, it can be proved using the results in [37] that $(X_{\tilde{\zeta}}, S, \mathbb{Z})$ is aperiodic. In the multidimensional case this is not true in general, as we can see in Example 4.3.

Example 4.3 (An aperiodic constant-shape substitution, with a periodic reduced substitution). Consider the substitution σ_2 with $L_{\sigma_2} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $F_1^{\sigma_2} = \llbracket 0, 1 \rrbracket^2$, given by

$$\sigma_2 : \quad 0 \mapsto \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix}, \quad 1 \mapsto \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}, \quad 2 \mapsto \begin{pmatrix} 3 & 1 \\ 2 & 0 \end{pmatrix}, \quad 3 \mapsto \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix}.$$

This substitution corresponds to the product substitution between the Thue-Morse substitution ($\sigma_3 : 0 \mapsto 01, 1 \mapsto 10$) and the doubling sequence substitution ($\sigma_4 : a \mapsto ab, b \mapsto aa$). The substitution does not have purely discrete spectrum, since $(X_{TM} \times \overleftarrow{\mathbb{Z}}_{(2^n\mathbb{Z}), S \times +_{(2^n\mathbb{Z})}, \mathbb{Z}^2)$ is a factor of $(X_{\sigma_2}, S, \mathbb{Z}^2)$, where (X_{TM}, S, \mathbb{Z}) corresponds to the one-dimensional Thue-Morse substitutive subshift. The reduced substitution for σ_2 is defined with the same expansion matrix and support, given by:

$$\tilde{\sigma}_2 : \quad a \mapsto \begin{pmatrix} a & b \\ a & b \end{pmatrix}, \quad b \mapsto \begin{pmatrix} b & a \\ b & a \end{pmatrix},$$

where every element in $\{0\} \times \mathbb{Z}$ is a nontrivial period of $\tilde{\sigma}_2$.

However, as proved in [69] for the one-dimensional case, if the reduced substitution system is aperiodic, then $(X_{\tilde{\zeta}}, \mu_\zeta, S, \mathbb{Z}^d)$ is metrically isomorphic to $(X_\zeta, \mu_\zeta, S, \mathbb{Z}^d)$.

Proposition 4.4. *Let ζ be an aperiodic primitive constant-shape substitution. If $\tilde{\zeta}$ is aperiodic, the natural factor between $(X_\zeta, S, \mathbb{Z}^d)$ and $(X_{\tilde{\zeta}}, S, \mathbb{Z}^d)$ is a metric isomorphism.*

Proof. Let $\pi : (X_\zeta, S, \mathbb{Z}^d) \rightarrow (\overleftarrow{\mathbb{Z}}_{(L_\zeta^n)}, +_{(L_\zeta^n)}, \mathbb{Z}^d)$ defined in Section 3.4.1. By Proposition 3.3, for every $n > 0$, $\pi_n(x)$ is equal to $\pi_n(\tilde{\phi}(x))$. In particular, if $x, y \in X_\zeta$ satisfies $\tilde{\phi}(x) = \tilde{\phi}(y)$, then $\pi_n(x), \pi_n(y)$ are equal for any $n > 0$.

Set $U = \{x \in X_\zeta : \exists y \in X_\zeta, \tilde{\phi}(x) = \tilde{\phi}(y), x_0 \neq y_0\}$. It is enough to prove that U is a null-set.

Let $n > 0$, $\mathbf{f} \in F_n^\zeta$ and $x \in X_\zeta$ be such that $S^{\mathbf{f}}\zeta^n(x) \in U$. Then, there exists $y \in X_\zeta$ with $\tilde{\phi}(y) = \tilde{\phi}(S^{\mathbf{f}}\zeta^n(x))$ and $y_0 \neq \zeta^n(x_0)_{\mathbf{f}}$. Then $\pi_n(y) = \pi_n(S^{\mathbf{f}}\zeta^n(x))$ and is equal to \mathbf{f} . Moreover, there exists $z \in X_\zeta$ with $y = S^{\mathbf{f}}\zeta^n(z)$, so $\tilde{\phi}(x)$ is equal to $\tilde{\phi}(z)$. This implies $(\zeta^n z_0)_j, (\zeta^n x_0)_j$ are equivalent for all $j \in F_n^\zeta$. Note that $(\zeta^n z_0)_{\mathbf{f}} = y_0$, so is different from $(\zeta^n x_0)_{\mathbf{f}}$. We define the set

$$G_n = \bigcup_{a,b \in \mathcal{A}} \left\{ \mathbf{f} \in F_n^\zeta : [(\zeta^n a)_{\mathbf{f}}] = [(\zeta^n b)_{\mathbf{f}}], (\zeta^n a)_{\mathbf{f}} \neq (\zeta^n b)_{\mathbf{f}} \right\}.$$

We deduce from the previous paragraph that

$$\begin{aligned} \mu_\zeta(U) &= \frac{1}{|F_n^\zeta|} \int \left| \left\{ \mathbf{f} \in F_n^\zeta : S^{\mathbf{f}}\zeta^n(x) \in U \right\} \right| d\mu(x) \\ &\leq \frac{|G_n|}{|F_n^\zeta|}. \end{aligned}$$

For any $a, b \in \mathcal{A}$ we denote $\mathcal{D}_n^{a,b} = \left\{ (c, d) \in \mathcal{A}^2 : \exists \mathbf{f} \in F_n^\zeta, (\zeta^n a)_{\mathbf{f}} = c, (\zeta^n b)_{\mathbf{f}} = d, [c] \neq [d] \right\}$ and $\mathcal{E}_n^{a,b} = \left\{ (c, d) \in \mathcal{A}^2 : \exists \mathbf{f} \in F_n^\zeta, (\zeta^n a)_{\mathbf{f}} = c, (\zeta^n b)_{\mathbf{f}} = d, [c] = [d] \right\}$. Set $\varepsilon > 0$ and let $j > 0$ be large enough such that for any $a, b \in \mathcal{A}$ we have that

$$d_j(\zeta^j(a), \zeta^j(b)) \leq \lim_{k \rightarrow \infty} d_k(\zeta^k(a), \zeta^k(b)) + \varepsilon.$$

Fix $a, b \in \mathcal{A}$. Note that

$$\begin{aligned} d_{n+j}(\zeta^{n+j}(a), \zeta^{n+j}(b)) &= \frac{1}{|F_{n+j}^\zeta|} \sum_{(c,d) \in \mathcal{D}_n^{a,b}} |F_j^\zeta| d_j(\zeta^j(c), \zeta^j(d)) + \sum_{(c,d) \in \mathcal{E}_n^{a,b}} |F_j^\zeta| d_j(\zeta^j(c), \zeta^j(d)) \\ &= \frac{1}{|F_n^\zeta|} \left(\sum_{(c,d) \in \mathcal{D}_n^{a,b}} d_j(\zeta^j(c), \zeta^j(d)) + \sum_{(c,d) \in \mathcal{E}_n^{a,b}} d_j(\zeta^j(c), \zeta^j(d)) \right) \\ &\leq \frac{1}{|F_n^\zeta|} \left(\sum_{(c,d) \in \mathcal{D}_n^{a,b}} (\lim_{k \rightarrow \infty} d_k(\zeta^k(c), \zeta^k(d)) + \varepsilon) + \sum_{(c,d) \in \mathcal{E}_n^{a,b}} \varepsilon \right) \\ &\leq \frac{\varepsilon(|\mathcal{D}_n^{a,b}| + |\mathcal{E}_n^{a,b}|)}{|F_n^\zeta|} + \frac{1}{|F_n^\zeta|} \sum_{(c,d) \in \mathcal{D}_n^{a,b}} \lim_{k \rightarrow \infty} d_k(\zeta^k(c), \zeta^k(d)). \end{aligned}$$

Since this is for every $\varepsilon > 0$ and $\lim_{k \rightarrow \infty} d_k(\zeta^k(c), \zeta^k(d)) \leq 1$ we have that $d_{n+j}(\zeta^{n+j}(a), \zeta^{n+j}(b)) \leq |\mathcal{D}_n^{a,b}|/|F_n^\zeta|$ and this is true for every j large enough, so

$$\lim_{k \rightarrow \infty} d_k(\zeta^k(a), \zeta^k(b)) \leq \frac{|\mathcal{D}_n^{a,b}|}{|F_n^\zeta|},$$

hence

$$\mu_\zeta(U) \leq \sum_{a,b \in \mathcal{A}} \left(d_n(\zeta^n(a), \zeta^n(b)) - \lim_{k \rightarrow \infty} d_k(\zeta^k(a), \zeta^k(b)) \right).$$

When $n \rightarrow \infty$, the right expression goes to zero, and we conclude $\mu_\zeta(U) = 0$. \square

It is known that if the substitutive subshift has a coincidence, then it is metrically isomorphic to its maximal equicontinuous factor [97]. A constant-length substitution has a coincidence, if there exists $n > 0$ and an index j on the support of ζ^n such that for all pair of letters $a, b \in \mathcal{A}$, $\zeta^n(a)_j = \zeta^n(b)_j$. The doubling sequence substitutive subshift (see Example 4.3) is an example of a substitution having a coincidence. If a substitution has a coincidence, then cannot be reduced. In fact its reduced substitution is trivial, i.e., has an alphabet of cardinality 1. The set of measurable endomorphisms of odometer systems is not discrete. In fact it is uncountable and any element of the odometer system represent a measurable endomorphism via addition. So, as in the original article [69], reducibility is an optimal hypothesis for Theorem C.

Now, to prove Theorem 4.1, we assume that ζ_2 is an aperiodic primitive reduced constant-shape substitution. We denote by $\eta = \min_{\substack{n \in \mathbb{N} \\ a \neq b \in \mathcal{B}}} d_n(\zeta_2^n(a), \zeta_2^n(b))$ and R the radius

from the recognizability property of X_{ζ_2} . Recall that the recognizability property implies the substitutions maps are injective. Let ϕ be in $m \text{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$. The map $\pi_n(x) - \pi_n(\phi x) \pmod{L^n(\mathbb{Z}^d)}$ is invariant under the \mathbb{Z}^d -action, so is constant μ_{ζ_1} -a.e. We denote this constant by $\mathbf{p}_n(\phi)$ in $F_n^{\zeta_1}$. The set $S^{\mathbf{p}_n(\phi)} \phi_{\zeta_1^n}(X_{\zeta_1})$ is included in $\zeta_2^n(X_{\zeta_2})$ up to a μ_{ζ_2} -null set. Since ζ_1^n is a homeomorphism from X_{ζ_1} to $\zeta_1^n(X_{\zeta_1})$, for μ_{ζ_1} -almost all $x \in X_{\zeta_1}$ there exists a unique point $y \in X_{\zeta_2}$ such that $S^{\mathbf{p}_n(\phi)} \phi_{\zeta_1^n}(x) = \zeta_2^n(y)$, which we denote $\phi_n(x)$. So, for every $\phi \in m \text{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ we consider a sequence $(\mathbf{p}_n(\phi))_{n \geq 0}$ and a sequence of maps $(\phi_n)_n \in m \text{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ such that

$$\mathbf{p}_n(\phi) \in F_n^{\zeta_1}, \quad S^{\mathbf{p}_n(\phi)} \phi_{\zeta_1^n}(x) = \zeta_2^n(\phi_n(x)).$$

It is straightforward to check that the sequence satisfies the recurrence $\mathbf{p}_{n+1}(\phi) = \mathbf{p}_n(\phi) + L_{\zeta_1^n}^n \mathbf{p}_1(\phi_n) \pmod{L_{\zeta_1}^{n+1}(\mathbb{Z}^d)}$. We also have the recurrence $(\phi_n)_1 = \phi_{n+1}$.

Now, for $\phi, \psi \in m \text{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$, we denote $d(\phi, \psi) = \mu_{\zeta_1}(\{x \in X_{\zeta_1} : (\phi x)_0 \neq (\psi x)_0\})$. We also denote for any $r > 0$ the quantity $C(r) = |B(\mathbf{0}, r) \cap \mathbb{Z}^d|$.

Lemma 4.5. *If $d(\phi, \psi)$ is smaller than $\eta/C(R)$, then ϕ, ψ are equal μ_{ζ_1} -a.e. in X_{ζ_1} .*

Proof. For all $n \geq 0$, we denote $U_n = \{x \in X_{\zeta_1} : (\phi_n x)_0 \neq (\psi_n x)_0\}$. We will prove by induction on $n \geq 0$ that $\mathbf{p}_n(\phi) = \mathbf{p}_n(\psi)$ and $\mu_{\zeta_1}(U_n) < 1/C(R)$. By hypothesis this is true for $n = 0$. Now, suppose that $\mathbf{p}_n(\phi) = \mathbf{p}_n(\psi)$ and $\mu_{\zeta_1}(U_n) < 1/C(R)$ for some $n \geq 0$. We note the map $\pi_1(\phi_n x) - \pi_1(\psi_n x)$ is equal to $(\mathbf{p}_1(\psi_n) - \mathbf{p}_1(\phi_n)) \pmod{L_{\zeta_1}^n(\mathbb{Z}^d)}$ for μ_{ζ_1} -a.e x in X_{ζ_1} . By the recognizability property, this map vanishes on the set $\{x \in X_{\zeta_1} : (\phi_n x)|_{B(\mathbf{0}, R)} = (\psi_n x)|_{B(\mathbf{0}, R)}\}$ which has a positive measure by hypothesis, and we conclude that $\mathbf{p}_{n+1}(\phi) = \mathbf{p}_{n+1}(\psi)$.

Now, let x be in U_{n+1} . Then, there exist at least $\eta|F_{n+1}^{\zeta_1}|$ indices $\mathbf{f} \in F_{n+1}^{\zeta_1}$ such that $(\zeta_2^{n+1} \phi_{n+1}(x))_{\mathbf{f}} \neq (\zeta_2^{n+1} \psi_{n+1}(x))_{\mathbf{f}}$, i.e., $(S^{\mathbf{p}_{n+1}(\phi)} \phi_{\zeta_1^{n+1}}(x))_{\mathbf{f}} \neq (S^{\mathbf{p}_{n+1}(\psi)} \psi_{\zeta_1^{n+1}}(x))_{\mathbf{f}}$, so we have that

$$\eta|F_{n+1}^{\zeta_1}| \mu_{\zeta_1}(U_{n+1}) \leq \int |\{\mathbf{g} \in F_{n+1}^{\zeta_1} + \mathbf{p}_{n+1}(\phi) : S^{\mathbf{g}} \zeta_1^{n+1} x \in U_0\}| d\mu_{\zeta_1}(x) = |F_{n+1}^{\zeta_1}| \mu_{\zeta_1}(U_0)$$

Hence, $\mu_{\zeta_1}(U_{n+1})$ is less than $1/C(R)$.

Now, we will prove that $\phi = \psi$ for μ_{ζ_1} -a.e. $x \in X_{\zeta_1}$. Let $r > 0$ be an integer. Since $(F_n^{\zeta_1})_{n>0}$ is Følner, we choose $n > 0$ large enough such that $|(F_n^{\zeta_1})^{\text{or}}|/|F_n^{\zeta_1}| \geq 1/2$. Set $\mathbf{p} = \mathbf{p}_n(\phi)$. If $x \in U_0^c$, then $(S^{\mathbf{p}} \phi_{\zeta_1^n} x)|_{F_n^{\zeta_1}} = (S^{\mathbf{p}} \psi_{\zeta_1^n} x)|_{F_n^{\zeta_1}}$, so we have that $(S^{\mathbf{p}+\mathbf{f}} \phi_{\zeta_1^n} x)_{B(\mathbf{0}, r)} = (S^{\mathbf{p}+\mathbf{f}} \psi_{\zeta_1^n} x)_{B(\mathbf{0}, r)}$ for $\mathbf{f} \in (F_n^{\zeta_1})^{\text{or}}$. This implies

$$\begin{aligned} \mu_{\zeta_1}(\{x \in X : (\phi x)|_{B(\mathbf{0}, r)} = (\psi x)_{B(\mathbf{0}, r)}\}) &\geq \frac{|(F_n^{\zeta_1})^{\text{or}}|}{|F_n^{\zeta_1}|} \mu_{\zeta_1}(U_0^c) \\ &\geq \frac{1}{2} \left(1 - \frac{1}{C(R)}\right) > 0. \end{aligned}$$

Finally, the set $\{x \in X : \phi(x) = \psi(x)\}$ is the decreasing intersection of these sets, so has a positive measure. By ergodicity, ϕ, ψ are equal μ_{ζ_1} -almost everywhere in X_{ζ_1} . \square

Lemma 4.6. *Let $\phi \in m \text{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$. Then there exists a sequence (ψ_n) of sliding block codes of radius $\|F_1^{\zeta_1}\| \left(1 + \|L_{\zeta_1}^{-1}\| \left(2 + 1/(1 - \|L_{\zeta_1}^{-1}\|)\right)\right)$ such that $d(\phi_n, \psi_n) \rightarrow 0$.*

Proof. Set $\varepsilon > 0$. By Lusin's theorem, there exist an integer $\ell > 0$ and a continuous map $f : X_{\zeta_1} \rightarrow \mathcal{B}$ such that $f(x)$ only depends on $x|_{B(\mathbf{0}, \ell)}$ and the measure of the set $V = \{x \in X_{\zeta_1} : (\phi x)_0 \neq f(x)\}$ is less than $\varepsilon\eta/6$. Since $(F_n^{\zeta_1})_{n>0}$ is Følner, we choose $n > 0$ large enough such that

$$\frac{|(F_n^{\zeta_1})^{\text{or}}|}{|F_n^{\zeta_1}|} > 1 - \frac{\eta}{3}.$$

Set $\mathbf{p} = \mathbf{p}_n(\phi)$. For every $x \in X_{\zeta_1}$, we denote $J(x) = \{\mathbf{f} \in (F_n^{\zeta_1})^{\text{or}} : S^{\mathbf{f}+\mathbf{p}} \zeta_1^n x \notin V\}$ and $W = \{x \in X_{\zeta_1} : |J(x)| > (1 - \eta/2)|F_n^{\zeta_1}|\}$. Note that

$$\begin{aligned}
\frac{\varepsilon\eta}{6} &> \mu_{\zeta_1}(V) \\
&\geq \frac{1}{|F_n^{\zeta_1}|} \int_{W^c} |(F_n^{\zeta_1})^{\circ\ell}| - |J(x)| d\mu(x) \\
&\geq (1 - \mu_{\zeta_1}(W)) \frac{\eta}{6}.
\end{aligned}$$

So we have that $\mu_{\zeta_1}(W) > 1 - \varepsilon$.

Using Proposition 1.20 with $F = F_1^\zeta + F_1^\zeta$ and $A = \{\mathbf{0}\}$ we find a set $C \Subset \mathbb{Z}^d$ such that $\mathbf{0} \in C$, $\|C\| \leq \|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|)\right)\right)$ and by Remark 1.21 we have that $F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$. If $x, y \in W$ with $x|_C = y|_C$, then for every $\mathbf{f} \in (F_n^{\zeta_1})^{\circ\ell}$, we have that $(S^{\mathbf{f}+\mathbf{p}}\zeta_1^n x)|_{B(\mathbf{0},\ell)} = (S^{\mathbf{f}+\mathbf{p}}\zeta_1^n y)|_{B(\mathbf{0},\ell)}$, so $f(S^{\mathbf{f}+\mathbf{p}}\zeta_1^n x) = f(S^{\mathbf{f}+\mathbf{p}}\zeta_1^n y)$. Moreover, we note for \mathbf{f} in $J(x) \cap J(y)$ that, $(\zeta_2^n \phi_n x)_\mathbf{f} = (S^{\mathbf{p}}\phi\zeta_1^n x)_\mathbf{f} = f(S^{\mathbf{f}+\mathbf{p}}\phi\zeta_1^n x) = f(S^{\mathbf{f}+\mathbf{p}}\phi\zeta_1^n y) = (\zeta_2^n \phi_n y)_\mathbf{f}$. Since $x, y \in W$, there is strictly more than $(1 - \eta)|F_n^{\zeta_1}|$ elements in $J(x) \cap J(y)$, so $(\phi_n x)_\mathbf{0}$ is equal to $(\phi_n y)_\mathbf{0}$ by definition of η . Hence, for every x in W , $(\phi_n x)_\mathbf{0}$ only depends on $x|_C$. \square

Finally, to prove Theorem 4.1 we use similar arguments given in [69] that we describe for completeness.

Proof of Theorem 4.1. For the fixed alphabets \mathcal{A} and \mathcal{B} , there exist a finite number of sliding block codes of radius $\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|)\right)\right)$. By Lemma 4.6, there exist two different integers $m, k \geq 0$ such that $d(\phi_m, \phi_{m+k}) < \eta/C(R)$, so by Lemma 4.5, we have that $\phi_m = \phi_{m+k}$, μ_{ζ_1} -a.e..

Let $n \geq m$ be a multiple of k . We have that $(\phi_n)_k = \phi_{n+k} = (\phi_{m+k})_{n-m} = (\phi_m)_{n-m} = \phi_n$, μ_{ζ_1} -a.e. This implies for all $r \in \mathbb{N}$, ϕ_n is equal to $(\phi_n)_{rk}$, μ_{ζ_1} -a.e. Since the number of sliding block codes of radius $\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|)\right)\right)$ is finite, by Lemma 4.6 ϕ_n is equal to a sliding block code of radius $\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|)\right)\right)$, μ_{ζ_1} -a.e. in X_{ζ_1} . Note that ϕ_n is equal to ϕ_{2n} , μ_{ζ_1} -a.e. We denote $\psi = \phi_n$ and $\mathbf{p} = \mathbf{p}_n(\psi)$. By definition of \mathbf{p} , we have that $S^{\mathbf{p}}\psi\zeta_1^n = \zeta_2^n\psi$.

Set $\mathbf{j} = \mathbf{p}_n(\phi) - \mathbf{p}$, then

$$S^{\mathbf{j}}\phi\zeta_1^n = S^{\mathbf{p}_n(\phi) - \mathbf{p}}\phi\zeta_1^n = S^{-\mathbf{p}}\zeta_2^n\psi = \psi\zeta_1^n, \quad \mu_{\zeta_1} - \text{a.e.},$$

this implies that $S^{\mathbf{j}}\phi$ and ψ coincides in $\zeta_1^n(X_{\zeta_1})$ μ_{ζ_1} -almost everywhere, and by ergodicity in the whole set X_{ζ_1} μ_{ζ_1} -almost everywhere. \square

4.2 Applications of rigidity results on endomorphisms of substitutive subshifts

In this section, we prove two consequences of Theorem 4.1 on endomorphisms of substitutive subshifts. First, we prove that substitutive subshifts given by aperiodic primitive reduced constant-shape substitutions are coalescent. This was first proved in [45] for one-dimensional linearly recurrent subshifts (in particular aperiodic subshifts). Linearly recurrent substitutive subshifts (such as the self-similar ones) are also coalescent as a consequence of a result in [29].

Then, we deduce the automorphism group of a substitutive subshift of an aperiodic primitive reduced constant-shape substitution is virtually generated by the shift action. As a corollary, we get a condition for the automorphism group to be isomorphic to a direct product of \mathbb{Z}^d (given by the shift action) and a finite group. As we will see in the next chapter, this condition is always true for aperiodic bijective primitive constant-shape substitutions.

From now on, during this section we will always assume, up to consider a power of the substitution, that any factor map $\psi \in \text{End}(X_\zeta, S, \mathbb{Z}^d)$ satisfying Property 2. in Theorem 4.1 is true for $n = 1$.

4.2.1 Coalescence of substitutive subshifts

In [45] it was proved that one-dimensional linearly recurrent subshifts (in particular aperiodic substitutions) are coalescent. In the multidimensional context, linearly recurrent substitutive subshifts (such as the self-similar ones) are also coalescent as a consequence of a result in [45]. Here we will use Theorem 4.1 and the finiteness on the number of invariant orbits under the action of ζ (Proposition 3.4), to obtain that substitutive subshifts are also coalescent, for aperiodic primitive reduced constant-shape substitutions.

Proposition 4.7. *Let ζ be an aperiodic primitive reduced constant-shape substitution. Then $(X_\zeta, S, \mathbb{Z}^d)$ is coalescent.*

Proof. Set $\phi \in \text{End}(X_\zeta, S, \mathbb{Z}^d)$. Theorem 4.1 ensures there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $S^{\mathbf{j}}\phi$ is equal to a sliding block code ψ of fixed radius satisfying $S^{\mathbf{p}}\psi\zeta = \zeta\psi$, for some $\mathbf{p} \in F_1^\zeta$. Let $\bar{x} \in X_\zeta$ be in a ζ -invariant orbit, i.e., there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $\zeta(\bar{x}) = S^{\mathbf{j}}\bar{x}$. Note that

$$S^{\mathbf{p}}\psi\zeta(\bar{x}) = S^{\mathbf{p}+\mathbf{j}}\psi\bar{x} = \zeta\psi\bar{x},$$

so, if the orbit of x is in a ζ -invariant orbit, then ψx satisfies the same property. By Proposition 3.4, there exist finitely many ζ -invariant orbits, hence for n large enough, we can find $x \in X_\zeta$ with x and $\psi^n(x)$ being in the same orbit, i.e., there exists $\mathbf{m} \in \mathbb{Z}^d$ such that $S^{\mathbf{m}}\psi^n(x) = x$. By minimality of X_ζ , $\psi^n = S^{-\mathbf{m}}$, hence ψ is invertible, which implies ϕ is invertible. \square

4.2.2 The automorphism group of substitutive subshifts.

Since the set of sliding block codes of radius $\|F_1^\zeta\| \left(1 + \|L_\zeta^{-1}\| \left(2 + 1/(1 - \|L_\zeta^{-1}\|)\right)\right)$ between X_ζ to itself is finite, we have the following result as a direct corollary of Theorem 4.1.

Proposition 4.8. *Let $(X_\zeta, S, \mathbb{Z}^d)$ be a subshift from an aperiodic primitive reduced constant-shape substitution ζ . Then, the quotient group $\text{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is finite. A bound for $|\text{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle|$ is given by an explicit formula depending only on d , $|\mathcal{A}|$, $\|L_\zeta^{-1}\|$, $\|F_1^\zeta\|$.*

In the special case, where any automorphism of $(X_\zeta, S, \mathbb{Z}^d)$ satisfies Property 2. of Theorem 4.1 with $\mathbf{p} = \mathbf{0}$, we have a more rigid result.

Corollary 4.9. *Let ζ be an aperiodic primitive reduced substitution. The group generated by the shift action and $\text{Cent}_{\text{Aut}(X_\zeta, S, \mathbb{Z}^d)}(\zeta)$ is isomorphic to a direct product of \mathbb{Z}^d (generated by the shift action) with a finite group.*

Proof. Note that an automorphism ϕ commutes with the substitution map if and only if ϕ is equal to ϕ_n , for all $n > 0$.

Now, by Lemma 4.6, the Property 2. implies Property 1. of Theorem 4.1. So, the group of automorphisms commuting with the substitution map is finite. To conclude we just need to observe that the pair $(\mathbf{j}_\phi, \psi_\phi)$ in Theorem 4.1 associated with any automorphism ϕ is unique. Indeed, set $\phi \in \text{Aut}(X_\zeta, S, \mathbb{Z}^d)$ and $\mathbf{j}_1, \mathbf{j}_2 \in \mathbb{Z}^d$, $\psi_1, \psi_2 \in \text{Aut}(X_\zeta, S, \mathbb{Z}^d)$ commuting with the substitution map such that $S^{j_i} \phi = \psi_i$, for $i \in \{1, 2\}$. Then, $S^{j_2 - j_1} \psi_1 = \psi_2$. Hence, for any $n > 0$

$$\begin{aligned} S^{j_2 - j_1} \psi_1 \zeta^n &= \zeta^n (S^{j_2 - j_1} \psi_1) \\ &= S^{L_\zeta^n (j_2 - j_1)} \zeta^n \psi_1, \end{aligned}$$

which implies $(\text{id} - L_\zeta^n)(\mathbf{j}_2 - \mathbf{j}_1) = \mathbf{0}$, so $\mathbf{j}_2 = \mathbf{j}_1$ and then $\psi_1 = \psi_2$. \square

We will show in Proposition 5.1 that for bijective substitutions, the automorphism group is isomorphic to a direct product of the shift map and a finite group, generated by permutation of letters.

4.3 Rigidity properties for homomorphisms between substitutive subshifts and applications

This section is devoted to homomorphisms between substitutive subshifts. We recall that for $M \in GL(d, \mathbb{Z})$, the map $\phi : (X, T, \mathbb{Z}^d) \rightarrow (Y, T, \mathbb{Z}^d)$ between two topological dynamical systems is said to be a homomorphism associated with M if for all $\mathbf{m} \in \mathbb{Z}^d$ we have that $\phi \circ S^{\mathbf{m}} = S^{M\mathbf{m}} \circ \phi$. We can also define measurable homomorphisms in the measure-theoretic setting. First, we establishes a necessary condition for the matrices M with

$m \text{Hom}_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ being non empty, whenever ζ_1, ζ_2 are two aperiodic primitive constant-shape substitutions with the same expansion matrix and support. This condition is similar to the one for odometer systems (Lemma 1.14), although the proof is very different. In this case we use a relation between measurable eigenvalues and homomorphisms (which we will prove in the proof of Lemma 4.10). Then we conclude by the fact that measurable eigenvalues and continuous eigenvalues are the same for substitutive subshifts (Theorem 3.13). In some cases this condition implies the matrix M commutes with the expansion matrix L (as shown for constant-base odometer systems in Theorem 2.2).

Then, we prove an analogue of Theorem 4.1 (Theorem 4.12) establishing that measurable homomorphisms induced continuous ones for homomorphisms associated with matrices in the centralizer of some power of the expansion matrix. Finally, we give an explicit bound on the norm of these matrices for the quotient group of a restricted normalizer semigroup with respect to the shift action (Proposition 4.16).

Lemma 4.10. *Let ζ_1, ζ_2 be two aperiodic primitive constant-shape substitutions having the same expansion matrix L and same support F . If $M \in GL(d, \mathbb{Z})$ is such that $m \text{Hom}_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d) \neq \emptyset$, then for all $n > 0$ there exists $m(n) > 0$ such that*

$$ML^{m(n)}(\mathcal{H}(X_{\zeta_1})) \leq L^n(\mathcal{H}(X_{\zeta_2})). \quad (\text{Normalizer Condition for substitutions})$$

Proof. Let ϕ be in $m \text{Hom}_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$, and $\mathbf{x} \in E(X_{\zeta_2}, \mu_{\zeta_2}, S, \mathbb{Z}^d)$. We will prove that $M^* \mathbf{x} \in E(X_{\zeta_1}, \mu_{\zeta_1}, S, \mathbb{Z}^d)$. Indeed, let $f \in L^2(X_{\zeta_2}, \mu_{\zeta_2})$ be such that for all $\mathbf{m} \in \mathbb{Z}^d$, $f \circ S^{\mathbf{m}} = e^{2\pi i \langle \mathbf{x}, \mathbf{m} \rangle} \cdot f$, μ_{ζ_2} -a.e. in X_{ζ_2} . Then, we have that

$$(f \circ \phi) \circ S^{\mathbf{m}} = (f \circ S^{M\mathbf{m}}) \circ \phi = e^{2\pi i \langle \mathbf{x}, M\mathbf{m} \rangle} \cdot f \circ \phi = e^{2\pi i \langle M^* \mathbf{x}, \mathbf{m} \rangle} \cdot f \circ \phi, \quad \mu_{\zeta_1}\text{-a.e. in } X_{\zeta_1}.$$

By Theorem 3.13 and Proposition 3.16, for any $n > 0$, the system $(\mathbb{Z}^d / M^{-1}L^n(\mathcal{H}(X_{\zeta_2})), +, \mathbb{Z}^d)$ is a finite factor of the odometer system $(\overleftarrow{\mathbb{Z}^d}_{(L^n(\mathcal{H}(X_{\zeta_1})))}, +, \mathbb{Z}^d)$, which implies the odometer system $(\overleftarrow{\mathbb{Z}^d}_{(M^{-1}L^n(\mathcal{H}(X_{\zeta_2})))}, +, \mathbb{Z}^d)$ is a factor of $(\overleftarrow{\mathbb{Z}^d}_{(L^n(\mathcal{H}(X_{\zeta_1})))}, +, \mathbb{Z}^d)$. By Lemma 1.13, we conclude that for any $n > 0$, there exists $m(n) > 0$ such that $L^{m(n)}(\mathcal{H}(X_{\zeta_1})) \leq M^{-1}L^n(\mathcal{H}(X_{\zeta_2}))$. \square

A consequence of Proposition 4.7 is that a homomorphism is associated with a matrix with finite order, then is an isomorphism.

Proposition 4.11. *Let ζ be an aperiodic primitive reduced constant-shape substitution. If $M \in GL(d, \mathbb{Z})$ has finite order, then any homomorphism $\phi \in N_M(X_{\zeta}, S, \mathbb{Z}^d)$ is invertible.*

Proof. Since M has finite order, there exists $n > 0$ such that $M^n = \text{id}_{\mathbb{R}^d}$. This implies $\phi^n \in \text{End}(X_{\zeta}, S, \mathbb{Z}^d)$. By Proposition 4.7, ϕ^n is invertible, which implies ϕ is invertible. \square

Now, we will prove an analogue of Theorem 4.1 for homomorphisms associated with matrices commuting with a power of the expansion matrix L . As mentioned before a priori this does not cover all the homomorphisms between substitutive subshifts.

Theorem 4.12. *Let $(X_{\zeta_1}, S, \mathbb{Z}^d)$, $(X_{\zeta_2}, S, \mathbb{Z}^d)$ be two substitutive subshifts from two aperiodic primitive constant-shape substitutions ζ_1, ζ_2 from finite alphabets \mathcal{A} and \mathcal{B} , with the same support F_1 and expansion matrix L . Let $M \in GL(d, \mathbb{Z})$ be a matrix commuting with a power of L , i.e., there exists $n > 0$ such that $ML^n = L^n M$. If $(X_{\zeta_2}, S, \mathbb{Z}^d)$ is reduced, then for every measurable homomorphism associated with M , $\phi : (X_{\zeta_1}, \mu_{\zeta_1}, S, \mathbb{Z}^d) \rightarrow (X_{\zeta_2}, \mu_{\zeta_2}, S, \mathbb{Z}^d)$, there exists $\mathbf{j} \in \mathbb{Z}^d$ such that $S^{\mathbf{j}}\phi$ is equal μ_{ζ_1} -a.e. to a homomorphism associated with M $\psi \in N_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$, satisfying the following two properties:*

1. ψ is given by a block map of radius $\|F_1^\zeta\| \|L_{\zeta_1}^{-1}\| (1 + \|M\|) (2 - \|L_{\zeta_1}^{-1}\|) / (1 - \|L_{\zeta_1}^{-1}\|)$.
2. There exist an integer $n > 0$ and $\mathbf{q} \in F_n^\zeta$ such that $S^{\mathbf{q}}\psi\zeta_1^n = \zeta_2^n\psi$.

In Chapter 6 we provide an example where the hypothesis are not satisfied, so we cannot apply Theorem 4.12. Nevertheless, we are able to describe its normalizer semigroup.

Remark 4.13. Let $\psi \in \text{Hom}_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ satisfying property (2) of Theorem 4.12. For any \mathbf{m} in \mathbb{Z}^d , we have that $S^{\mathbf{q}}\psi(\zeta_1^n(S^{\mathbf{m}}x)) = \zeta_2^n(\psi(S^{\mathbf{m}}x))$, and $S^{\mathbf{q}}\psi(\zeta_1^n(S^{\mathbf{m}}x)) = S^{\mathbf{q}+ML_{\zeta_1}^n\mathbf{m}}\psi(\zeta_1^n(x))$, $\zeta_2^n(\psi(S^{\mathbf{m}}x)) = S^{L_{\zeta_1}^n M\mathbf{m}}\zeta_2^n(\psi(x))$, it follows, $ML_{\zeta_1}^n\mathbf{m} = L_{\zeta_1}^n M\mathbf{m}$, i.e., M and $L_{\zeta_1}^n$ commute, hence this hypothesis is optimal to obtain property (2). Note that if L is an integer multiple of the identity, then any matrix $M \in GL(d, \mathbb{Z})$ commutes with L .

The proof of Theorem 4.12 follows the same strategy as the one of Theorem 4.1, except some small modifications. Since the substitution ζ_1 is primitive, we can replace the substitution by some power ζ_1^n , so we may assume that M commutes with the expansion matrix of ζ_1 . We will replace the term $\mathbf{p}_n(\phi)$ by the map $\pi_n(x) - M^{-1}\pi_n(\phi x) \pmod{L_{\zeta_1}^n(\mathbb{Z}^d)}$, with $\pi_n(x)$ and $M^{-1}\pi_n(\phi x)$ being the representative classes in F_n^ζ . The commutation assumption implies, for any $n > 0$ the map M defines a bijection in $\mathbb{Z}^d/L^n(\mathbb{Z}^d)$, also denoted by M , i.e., $\mathbf{n} = \mathbf{m} \pmod{L^n(\mathbb{Z}^d)}$, if and only if $M\mathbf{n} = M\mathbf{m} \pmod{L^n(\mathbb{Z}^d)}$. With this, the map $\mathbf{p}_n(\phi)$ is invariant under the shift action. Since $(X_{\zeta_1}, \mu_{\zeta_1}, S, \mathbb{Z}^d)$ is ergodic, the map $\mathbf{p}_n(\phi) \in F_n^\zeta$ is a constant map μ_{ζ_1} -a.e. in X_{ζ_1} and the set $S^{M\mathbf{p}_n(\phi)}\phi_{\zeta_1^n}(X_{\zeta_1})$ is included, up to a μ_{ζ_2} -null set, in $\zeta_2^n(X_{\zeta_2})$. We can define the map ϕ_n for μ_{ζ_1} -a.e. in X_{ζ_1} as the unique point $y \in X_{\zeta_2}$ such that $S^{M\mathbf{p}_n(\phi)}\phi_{\zeta_1^n}x = \zeta_2^n y$, where $M\mathbf{p}_n(\phi)$ is the representative element in F_n^ζ . It is straightforward to check that $\phi_n \circ S^{\mathbf{n}} = S^{M\mathbf{n}} \circ \phi_n$ for all $\mathbf{n} \in \mathbb{Z}^d$, so ϕ_n is in $mN_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$. The sequences $\mathbf{p}_n(\phi)$ and (ϕ_n) satisfies the same recurrences given in Section 4.1:

$$\mathbf{p}_{n+1}(\phi) = \mathbf{p}_n(\phi) + L_{\zeta_1}^n \mathbf{p}_1(\phi_n), \quad (\phi_n)_1 = \phi_{n+1}.$$

As in Theorem 4.1 we need the following adaptations of Lemma 4.5 and Lemma 4.6 for homomorphisms. The proof are the same, so we omit them.

Lemma 4.14. *If $\phi, \psi \in mN_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ are such that $d(\phi, \psi)$ is smaller than $\eta/C(R)$, then ϕ, ψ are equal μ_{ζ_1} -a.e in X_{ζ_1} .*

Lemma 4.15. *Let $\phi \in mN_M(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$. Then there exists a sequence (ψ_n) of homomorphisms associated with M of radius $\|F_1^\zeta\| \|L_{\zeta_1}^{-1}\| (1 + \|M\|) (2 - \|L_\zeta^{-1}\|) / (1 - \|L_{\zeta_1}^{-1}\|)$ such that $d(\phi_n, \psi_n) \rightarrow 0$.*

To finish the proof of Theorem 4.12, we proceed exactly as in the proof of Theorem 4.1

Proof of Theorem 4.12. For the fixed alphabets \mathcal{A} and \mathcal{B} , there exists a finite number of homomorphisms associated with M of radius $\|F_1^\zeta\| \|L_{\zeta_1}^{-1}\| (1 + \|M\|) (2 - \|L_\zeta^{-1}\|) / (1 - \|L_{\zeta_1}^{-1}\|)$. By Lemma 4.15, there exist two different integers $m, k \geq 0$ such that $d(\phi_m, \phi_{m+k}) < \eta/C(R)$, so by Lemma 4.14, we have that $\phi_m = \phi_{m+k}$, μ_{ζ_1} -a.e..

Let $n \geq m$ be a multiple of k . We have that $(\phi_n)_k = \phi_{n+k} = (\phi_{m+k})_{n-m} = (\phi_m)_{n-m} = \phi_n$, μ_{ζ_1} -a.e. This implies for all $r \in \mathbb{N}$, ϕ_n is equal to $(\phi_n)_{rk}$, μ_{ζ_1} -a.e. Since the number of sliding block codes of radius $\|F_1^\zeta\| \|L_{\zeta_1}^{-1}\| (1 + \|M\|) (2 - \|L_\zeta^{-1}\|) / (1 - \|L_{\zeta_1}^{-1}\|)$ is finite, by Lemma 4.15 we have that ϕ_n is equal to a homomorphism associated with M of radius $\|F_1^\zeta\| \|L_{\zeta_1}^{-1}\| (1 + \|M\|) (2 - \|L_\zeta^{-1}\|) / (1 - \|L_{\zeta_1}^{-1}\|)$, μ_{ζ_1} -a.e. in X_{ζ_1} . Note that ϕ_n is equal to ϕ_{2n} , μ_{ζ_1} -a.e. We denote $\psi = \phi_n$ and $\mathbf{p} = \mathbf{p}_n(\psi)$. By definition of \mathbf{p} , we have that $S^{M\mathbf{p}}\psi\zeta_1^n = \zeta_2^n\psi$.

Set $\mathbf{j} = M(\mathbf{p}_n(\phi) - \mathbf{p})$, then

$$S^{\mathbf{j}}\phi\zeta_1^n = S^{M(\mathbf{p}_n(\phi) - \mathbf{p})}\phi\zeta_1^n = S^{-M\mathbf{p}}\zeta_2^n\psi = \psi\zeta_1^n, \quad \mu_{\zeta_1} - \text{a.e.},$$

this implies that $S^{\mathbf{j}}\phi$ and ψ coincides in $\zeta_1^n(X_{\zeta_1})$ μ_{ζ_1} -almost everywhere, and by ergodicity in the whole set X_{ζ_1} μ_{ζ_1} -almost everywhere. \square

In the case $\zeta_1 = \zeta_2$, we can consider a restricted normalizer group as all the invertible homomorphisms associated with matrices commuting with a power of L_{ζ_1}

$$NC(X_{\zeta_1}, S, \mathbb{Z}^d) = \bigcup_{\substack{M \in GL(d, \mathbb{Z}) \\ ML_{\zeta_1}^n = L_{\zeta_1}^n M, \text{ for some } n}} (N_M(X, T, \mathbb{Z}^d) \cap \text{Homeo}(X)),$$

This set is a group under composition and $\langle S \rangle, \text{Aut}(X_{\zeta_1}, S, \mathbb{Z}^d)$ are normal subgroups of $NC(X_{\zeta_1}, S, \mathbb{Z}^d)$.

Following the same proof as Proposition 4.8, we have the map $\phi \rightarrow (\mathbf{j}_\phi, \psi_\phi)$ is unique, so we obtain the same result with respect to this restricted normalizer group.

Proposition 4.16. *Let $(X_\zeta, S, \mathbb{Z}^d)$ be a subshift from a reduced aperiodic primitive constant-shape substitution ζ from a finite alphabet. If the set of matrices $M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)$*

commuting with a power of the expansion matrix L_ζ is finite, then the quotient group $NC(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is finite. A bound for $|NC(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle|$ is given by an explicit formula depending only on d , $|\mathcal{A}|$, $\|L_\zeta^{-1}\|$, $\|F_1^\zeta\|$, and $\sup_{M \in \text{Cent}_{GL(d, \mathbb{Z})}(L_\zeta)} \|M\|$.

Proof. Let $\psi \in NC(X_\zeta, S, \mathbb{Z}^d)$, satisfying Property 2. of Theorem 4.12. Following the proof of Proposition 4.7, ψ acts as a permutation of the ζ -invariant orbits. Since the set of matrices $M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)$ commuting with a power of L_ζ is finite, there exists $n > 0$ such that ψ^n is an automorphism of X_ζ . By Proposition 4.8, we have that ψ^n has finite order, which implies ψ has finite order. \square

A substitution with expansion matrix equal to L_1 in Example 2.4 is an example where Proposition 4.16 gives a complete characterization for the normalizer group.

Chapter 5

Precisions on bijective constant-shape substitutions

Bijective substitutions are of great interest because of their mixed dynamic spectrum. They are never extensions almost 1-to-1 of its maximal equicontinuous factor. Bijective substitutions were studied before in [54] for block substitutions, where it was proved that the subshift generated by a bijective constant-shape substitution, with a diagonal expansion matrix is measurable-theoretic isomorphic to a skew product of one-dimensional odometers. Also, in [21] it was studied the normalizer group of bijective block substitutions. We extend the study by describing the normalizer group for general constant-shape substitutions. To do this, we relate the symmetry group with different types of supports of the substitution and non-diagonal expansion matrices.

First, we prove that the automorphism group of substitutive subshifts of bijective substitutions are direct products of the shift action and finite groups, given by a permutation of letters (Proposition 5.1). This is a well known result for one-dimensional and block substitutions ([69, 82, 20, 21]). In the rest of the chapter, under some geometrical conditions (called *polytope substitutions*), we prove that the normalizer group is virtually generated by the shift action. We then describe the symmetry group of these substitutive subshifts (Proposition 5.15 and Theorem 5.17). The strategy is to use the fact that nondeterministic directions, viewed as *asymptotic pairs* in the one-dimensional context in [39], are preserved by the symmetry group (Proposition 1.12). We determine the nondeterministic directions of these substitutive subshifts thanks to the supporting hyperplanes to $\text{conv}(F_n^\zeta)$ (Theorem 5.2). As we see in Section 5.3, the polytope assumption imply strong constraints on such supporting hyperplanes. Conversely, we provides a checkable combinatorial condition to ensure a half-space to be nonexpansive for $(X_\zeta, S, \mathbb{Z}^d)$. (Corollary 5.13). This is the first characterization of nondeterministic directions for a family of minimal systems.

5.1 The automorphism group of substitutive subshifts from bijective constant-shape substitutions

Since bijective substitutions are reduced, Proposition 4.8 implies the automorphism group of $(X_\zeta, S, \mathbb{Z}^d)$ is virtually \mathbb{Z}^d . In fact, we have a more rigid result in the bijective case as shown in the following proposition.

Proposition 5.1. *Let ζ_1, ζ_2 be two aperiodic bijective primitive constant-shape substitutions with the same expansion matrix L and support F_1 . Then, any factor $\psi : X_{\zeta_1} \rightarrow X_{\zeta_2}$ satisfying Property 2. in Theorem 4.1 has radius 0. In particular, the automorphism group $\text{Aut}(X_\zeta, S, \mathbb{Z}^d)$ of a substitutive subshift from an aperiodic bijective primitive constant-shape substitution is isomorphic to the direct product of \mathbb{Z}^d , generated by the shift action, with a finite group given by a permutation of letters.*

Proof. Let $\psi \in \text{Fac}(X_{\zeta_1}, X_{\zeta_2}, S, \mathbb{Z}^d)$ satisfying Property 2. in Theorem 4.1, i.e., there exists $\mathbf{p} \in F_1$ such that $S^{\mathbf{p}}\psi\zeta_1 = \zeta_2\psi$. Suppose that $\mathbf{p} \neq \mathbf{0}$. Let $n > 0$ be large enough such that the set $F_n^{\circ C} = \{\mathbf{f} \in F_n : \mathbf{f} + C \subseteq F_n\}$ is nonempty, where C is the set defined in Lemma 4.6. Then for any $x \in X_{\zeta_1}$, the coordinate $x_{\mathbf{0}}$ determines the pattern $\zeta_1^n(x)|_{F_n}$, hence the pattern $(S^{\mathbf{p}_n}\psi\zeta_1^n)|_{\mathbf{p}_n + (F_n)^{\circ C}}$ is also completely determined by $x_{\mathbf{0}}$, where $\mathbf{p}_n = \sum_{i=0}^{n-1} L^i(\mathbf{p})$. Set $\mathbf{m} \in \mathbb{Z}^d$ such that $(\mathbf{p}_n + (F_n)^{\circ C}) \cap (L^n(\mathbf{m}) + F_n) \neq \emptyset$. Since $S^{\mathbf{p}_n}\psi\zeta_1^n = \zeta_2^n\psi$ and ζ_2 is bijective, we have that $x_{\mathbf{0}}$ determines $\psi(x)_{\mathbf{m}}$, which implies $S^{-\mathbf{m}}\psi$ is a factor map of radius 0 (or a letter-to-letter factor map). Set $\phi = S^{-\mathbf{m}}\psi$, we have that

$$S^{\mathbf{p}_n + \mathbf{m} - L^n(\mathbf{m})}\phi\zeta_1^n = \zeta_2^n\phi,$$

and by bijectivity the coordinate $x_{\mathbf{0}}$ determined two coordinates of ψ , unless for any $n \in \mathbb{N}$ large enough $\mathbf{p}_n + \mathbf{m}$ is in $L_\zeta^n(\mathbb{Z}^d)$, i.e., for any n large enough there exists $\mathbf{r}_n \in \mathbb{Z}^d$ such that $\mathbf{p}_n + \mathbf{m} = L^n(\mathbf{r}_n)$. Note that $\mathbf{p} + \mathbf{r}_n = L(\mathbf{r}_{n+1})$, which implies

$$\|\mathbf{r}_{n+1}\| \leq \|L^{-1}\|(\|\mathbf{r}_n\| + \|\mathbf{p}\|),$$

so (\mathbf{r}_n) is a bounded sequence. Hence there exist $n > 0$ and $N > 1$ such that $\mathbf{r}_{n+N} = \mathbf{r}_n$, which implies $\mathbf{p}_{N-1} \in (L^N - \text{id})(\mathbb{Z}^d)$ which is not possible by Remark 4.2.

If $\mathbf{p}_n + \mathbf{m} \notin L_\zeta^n(\mathbb{Z}^d)$, then $x_{\mathbf{0}}$ determines two coordinates $\mathbf{n}_1, \mathbf{n}_2$ of $\psi(x)$ and then the coordinates $\mathbf{0}$ and $\mathbf{n}_2 - \mathbf{n}_1$ of $\psi_1 = S^{-\mathbf{n}_1}\psi$, since ψ_1 is also induced via a 0-block map. Note that the map $\Psi_1 : \mathcal{A} \rightarrow \mathcal{B}$ inducing ψ_1 is bijective, if not two fixed points x, y with $\Psi_1(x_{\mathbf{0}}) = \Psi_2(y_{\mathbf{0}})$ and $x_{\mathbf{0}} \neq y_{\mathbf{0}}$ generate two points with the same image, which is a contradiction. It follows that $x_{\mathbf{0}}$ determines $x_{\mathbf{n}_2 - \mathbf{n}_1}$, and then $x_{k(\mathbf{n}_2 - \mathbf{n}_1)}$ for all $k \in \mathbb{Z}$, so x has a nontrivial period, which is a contradiction. Finally, we conclude by Corollary 4.9. \square

5.2 Nondeterministic directions of substitutive subshifts of bijective on extremities constant-shape substitutions

In this section we give a characterization of the nondeterministic directions (defined in Section 1.5) of a substitutive subshift $(X_\zeta, S, \mathbb{Z}^d)$ in the case ζ is a bijective on extremities constant-shape substitution. A starting remark is that for each $n > 0$, the set of directions \mathbb{S}^{d-1} is stratified by the opposite normal fan $\mathcal{N}(\text{conv}(F_n^\zeta))$ (see Section 1.1.2). Our description of the nondeterministic directions is given in terms of union of these fans.

We say a constant-shape substitution ζ is *bijective on extremities* if the restriction $p_{\mathbf{f}}$ of ζ in \mathbf{f} is bijective for all $\mathbf{f} \in \text{Ext}(\text{conv}(F_1^\zeta))$. Since $\text{Ext}(\text{conv}(A+B)) \subseteq \text{Ext}(\text{conv}(A)) + \text{Ext}(\text{conv}(B))$, a substitution is bijective on extremities if and only if for any $n > 0$ and $\mathbf{f} \in \text{Ext}(\text{conv}(F_n^\zeta))$ the restriction $p_{\mathbf{f}}$ of ζ^n in \mathbf{f} is bijective. Since $(F_n^\zeta)_{n>0}$ is a Følner sequence, there exists $n > 0$ such that $\text{conv}(F_n^\zeta)$ is a nondegenerate polytope, so up to considering a power of ζ , we may assume that $\text{conv}(F_1^\zeta)$ is a nondegenerate polytope. Using the recognizability property of substitutions and some basic results in convex geometry we prove the following result.

Theorem 5.2. *Let ζ be an aperiodic primitive constant-shape substitution which is bijective on extremities. Then, the set of nondeterministic directions $\text{ND}(X_\zeta, S, \mathbb{Z}^d)$ of the substitutive subshift is the intersection of \mathbb{S}^{d-1} with a nonempty union of limits of opposite normal cones of the form $\hat{N}_{\mathbf{G}_n}(\text{conv}(F_n^\zeta))$, with \mathbf{G}_n a face of $\text{conv}(F_n^\zeta)$, for some integer $n > 0$.*

This theorem gives topological constraints on the set of nondeterministic directions. Actually, we will see when $L_\zeta = \lambda \text{id}_{\mathbb{R}^d}$, that the convex hull of any digit tile is a polytope, i.e. it has a finite number of extreme points (Theorem 5.6). In this case by Theorem 5.2, the set of nondeterministic directions $\text{ND}(X_\zeta, S, \mathbb{Z}^d)$ is a finite union of closed balls (eventually degenerated). More explicitly, in the two-dimensional case we have the following corollary, showing in particular, it cannot be a Cantor set. This in contrast with the result proved by M. Boyle and D. Lind in [17], where they proved any compact set with cardinality at least 2 can be realized as the set of nondeterministic directions of some two-dimensional subshift.

Corollary 5.3. *In the two dimensional case, under the hypothesis of Theorem 5.2, either the set $\text{ND}(X_\zeta, S, \mathbb{Z}^2)$ has nonempty interior, either it has at most 2 accumulation points.*

Proof. Assume that the set of nondeterministic directions $\text{ND}(X_\zeta, S, \mathbb{Z}^2)$ has empty interior. By Theorem 5.2, the elements of $\text{ND}(X_\zeta, S, \mathbb{Z}^2)$ are limits of normal vectors to edges of $\text{conv}(F_n^\zeta)$ for some $n > 0$. In [109] it was proved such vectors are normalized vectors of the form $(L_\zeta^*)^{-k} \mathbf{u}_k$, with $\mathbf{u}_k \in \mathbb{S}^1$ a normal vector to an edge of $\text{conv}(F_1^\zeta)$ for some $k > 0$. Hence, their accumulation points are accumulation points of orbits of the projective action L_ζ^* on the circle \mathbb{S}^1 . A standard analysis of this action provides the cardinality of

the accumulation points is at most 2, when a power of one of the L_ζ -eigenvalues is a real number. Otherwise, the projective orbits of L_ζ^* are dense in the circle. Since $\text{ND}(X_\zeta, S, \mathbb{Z}^2)$ is closed, it is the whole circle, which is a contradiction. \square

Proof of Theorem 5.2. Let \mathbf{v} be a nondeterministic direction for $(X_\zeta, S, \mathbb{Z}^d)$, and $x_1 \neq x_2 \in X_\zeta$ such that $x_1|_{H_\mathbf{v}} = x_2|_{H_\mathbf{v}}$. Consider the set $D = \{\mathbf{n} \in \mathbb{Z}^d : x_1(\mathbf{n}) \neq x_2(\mathbf{n})\}$. Since $D \subseteq \mathbb{R}^d \setminus H_\mathbf{v}$, its convex hull is also contained in $\mathbb{R}^d \setminus H_\mathbf{v}$. We have 2 possibilities:

1. The convex hull $\text{conv}(D)$ has at least one extreme point. Since D is closed in \mathbb{R}^d , then all its extreme points of $\text{conv}(D)$ belong to D . Now, if $\mathbf{v} \in \mathbb{Z}^d$ is an extremal ray of $\text{conv}(D)$, then for any extremal point $\mathbf{n} \in D$, the map $\text{dist}(\mathbf{n} + t\mathbf{v}, H_\mathbf{v})$ must be increasing (if not, there exists $t^* > 0$ with $\mathbf{n} + t^*\mathbf{v} \in H$, which is a contradiction). Hence the distance map to H restricted to $\text{conv}(D)$ is minimized in the extreme points of $\text{conv}(D)$. Since the extreme points are in D , we can use the shift action in x_1, x_2 and assume that $x_1(\mathbf{0}) \neq x_2(\mathbf{0})$.
2. If $\text{conv}(D)$ does not have extreme points, then contains a line. In this case the hyperplane $\partial H_\mathbf{v}$ must be parallel to this line contained in $\text{conv}(D)$ and using the same argument, we can assume that $x_1(\mathbf{0}) \neq x_2(\mathbf{0})$.

So we can assume that $\mathbf{0}$ is in a face \mathbf{F}_0 of smallest dimension of $\overline{\text{conv}(D)}$ and then $\mathbf{v} \in \hat{N}_{\mathbf{F}_0}(\overline{\text{conv}(D)})$. In fact any element in $\hat{N}_{\mathbf{F}_0}(\overline{\text{conv}(D)}) \cap \mathbb{S}^{d-1}$ is a nondeterministic direction for $(X_\zeta, S, \mathbb{Z}^d)$.

Now, for any $k > 0$ consider $R^{(k)} > 0$ be the recognizability radius for ζ^k given by Proposition 3.3 and $R = 4R^{(k)}$. Since x_1 and x_2 coincide in an arbitrarily large ball, they have the same image under the maximal equicontinuous factor, hence $\pi_n(x_1) = \pi_n(x_2) \in F_n^\zeta$ for any $n > 0$. By Lemma 3.9, there exist $n > 0$ and two words $\mathbf{w}_1^{(n)}, \mathbf{w}_2^{(n)} \in \mathcal{L}_{\overline{K}_\zeta}(X_\zeta)$ such that $x_i|_{B(0,R)} \sqsubseteq \zeta^n(\mathbf{w}_i^{(n)})$, for $i \in \{1, 2\}$. By the Pigeonhole Principle there exist an infinite set $E \subseteq \mathbb{N}$, $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{L}_{\overline{K}_\zeta}(X_\zeta)$ and $\mathbf{k}_1, \mathbf{k}_2 \in \overline{K}_\zeta$ such that for all $n \in E$ $x_i|_{B(0,R)} = \zeta^n(\mathbf{w}_i)|_{L_\zeta^n(\mathbf{k}_i) + \pi_n(x_1) + B(0,R)}$, $i \in \{1, 2\}$. The recognizability property implies the origin is in the boundary of $\text{conv}(F_k^\zeta - \pi_k(x_1))$. Letting k to infinity we have that for all $n > 0$, the origin is in the boundary of $\text{conv}(F_n^\zeta - \pi_n(x_1))$, and if \mathbf{G}_n is the face of smallest dimension containing $\pi_n(x_1)$ in $\text{conv}(F_n^\zeta)$, then $\hat{N}_{\mathbf{F}_0}(\overline{\text{conv}(D)})$ is included in $\hat{N}_{\mathbf{G}_n}(\text{conv}(F_n^\zeta))$ for all $n > 0$.

We will show now the converse. We separate the proof in two cases.

Suppose that $\text{conv}(D)$ is closed. Let \mathbf{F}_1 be a face of $\text{conv}(D)$ containing $\mathbf{0}$ of codimension 1. We have that $\mathbf{F}_1 = \text{conv}(\mathbf{F}_1 \cap D)$ because D is closed in \mathbb{R}^d . Fix $\mathbf{t} \in \mathbf{F}_1 \cap D$ different from $\mathbf{0}$. Since F_n^ζ is a fundamental domain of $L_\zeta^n(\mathbb{Z}^d)$, there are $\mathbf{h}_n \in F_n^\zeta$ and $\mathbf{z}_n(\mathbf{t}) \in \mathbb{Z}^d$ such that $\mathbf{t} = \mathbf{h}_n(\mathbf{t}) - \pi_n(x_1) + L_\zeta^n(\mathbf{z}_n(\mathbf{t}))$ and \mathbf{t} , being in the boundary of $\text{conv}(D)$, lies

in the boundary of $\text{conv}(F_n^\zeta - \pi_n(x_1) + L_\zeta^n(\mathbf{z}_n(\mathbf{t})))$. Since this last set is a translated of $\text{conv}(F_n^\zeta - \pi_n(x_1))$ and are both subsets of $\text{conv}(D)$, a basic geometrical argument ensures for any $n > 0$, $\mathbf{h}_n(\mathbf{t})$ and $\pi_n(x_1)$ are in the same face of $\text{conv}(F_n^\zeta)$. The same arguments imply that $\mathbf{h}_n(\mathbf{t}_1), \mathbf{h}_n(\mathbf{t}_2)$ are in the same face for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbf{F}_1 \cap D$. Furthermore, $\mathbf{h}_n(\mathbf{t})$ and $\pi_n(x_1)$ are different for large enough $n \in E$. Indeed, assume the converse, taking $R > \|\mathbf{t}\|$, we have that $x_1|_{\mathbf{t}} = \zeta^{n(\mathbf{w}_i)} L_\zeta^n(\mathbf{k}_3) + \pi_n(x_1)$, for some $\mathbf{k}_3 \in \overline{K}_\zeta$ for infinitely many $n \in E$. Since

$$\|\mathbf{t}\| = \|L_\zeta^n(\mathbf{k}_1 - \mathbf{k}_3)\|, \text{ for infinitely many } n \in E,$$

we have that necessarily $\mathbf{k}_1 = \mathbf{k}_3$, which is a contradiction.

Consider the face \mathbf{H}_n of $\text{conv}(F_n^\zeta)$ of smallest dimension generated by $\{\mathbf{h}_n(\mathbf{t})\}_{\mathbf{t} \in \mathbf{F}_1}$. Notice that $\hat{N}_{\mathbf{F}_1}(\text{conv}(D)) \subseteq \bigcap_{n>0} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta))$. We will prove that $\hat{N}_{\mathbf{F}_1}(\text{conv}(D)) = \bigcap_{n>0} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta))$. By construction of x_1, x_2 , the set \mathbf{z}_n is bounded (for all n large enough it belong to $\overline{K}_\zeta - \overline{K}_\zeta$), so there exists $\mathbf{t} \in \mathbf{F}_1$ and $\varepsilon > 0$ small enough such for all $\mathbf{t}' \in B(\mathbf{t}, \varepsilon) \cap \mathbf{F}_1$ we have that $\mathbf{z}_n(\mathbf{t}') = \mathbf{z}_n(\mathbf{t})$ for all $n \in E$ large enough. Hence \mathbf{H}_n is a face of codimension 1 and an argument of dimensions let us conclude that $\hat{N}_{\mathbf{F}_1}(\text{conv}(D)) = \bigcap_{n \in E} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta))$.

Suppose now that $\text{conv}(D)$ is not closed. Set \mathbf{F}_1 be a face of $\overline{\text{conv}(D)}$ of codimension 1 containing $\mathbf{0}$ and $\mathbf{w} \in \hat{N}_{\mathbf{F}_1}(\overline{\text{conv}(D)})$. We will find a sequence of faces \mathbf{H}_n of $\text{conv}(F_n^\zeta)$ such that $\hat{N}_{\mathbf{F}_1}(\overline{\text{conv}(D)}) = \bigcap_{n>0} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta))$. By definition, we have that

$$\forall \mathbf{t} \in \mathbf{F}_1, \langle \mathbf{w}, \mathbf{t} \rangle = \inf_{\mathbf{n} \in D} \langle \mathbf{w}, \mathbf{n} \rangle = 0. \quad (5.1)$$

Set $\mathbf{t} \in \mathbf{F}_1$ and consider a sequence $(\mathbf{t}^m)_{m>0} \subseteq \text{conv}(D)$ converging to \mathbf{t} . By Caratheodory's Theorem for any $m > 0$, we can write $\mathbf{t}^m = \sum_{i=0}^d t_i^m \mathbf{f}_i^m$, with $t_i \geq 0$, $\sum_{i=0}^d t_i^m = 1$ and $\mathbf{f}_i^m \in D$. Then, (5.1) implies for all i , $t_i^m \langle \mathbf{w}, \mathbf{f}_i^m \rangle \xrightarrow{m \rightarrow \infty} 0$. The only difficulty to get the result, concern the indices i such that $\liminf_{m \rightarrow \infty} t_i^m > 0$. For such i , we have that $\langle \mathbf{w}, \mathbf{f}_i^m \rangle \xrightarrow{m \rightarrow \infty} 0$. Using the recognizability property, for all $n \in E$ we write $\mathbf{f}_i^m = \mathbf{h}(m, i, n) - \pi_n(x_1) + L_\zeta^n(\mathbf{z}(m, i, n))$, with $\mathbf{h}(m, i, n) \in F_n^\zeta$ and $\mathbf{z}(m, i, n) \in \mathbb{Z}^d$. Since $\langle \mathbf{w}, \mathbf{h}(m, i, n) - \pi_n(x_1) \rangle \geq 0$ and $\langle \mathbf{w}, L_\zeta^n(\mathbf{z}(m, i, n)) \rangle \geq 0$, we have that for all $n > 0$

$$\langle \mathbf{w}, \mathbf{h}(m, i, n) - \pi_n(x_1) \rangle \xrightarrow{m \rightarrow \infty} 0 \quad \wedge \quad \langle \mathbf{w}, L_\zeta^n(\mathbf{z}(m, i, n)) \rangle \xrightarrow{m \rightarrow \infty} 0.$$

Since F_n^ζ is finite, we conclude that for all $n \in E$, there exists $m(n)$ such that for all $m \geq m(n)$,

$$\langle \mathbf{w}, \mathbf{h}(m, i, n) \rangle = \langle \mathbf{w}, \pi_n(x_1) \rangle = 0. \quad (5.2)$$

The same argument as the former one when $\text{conv}(D)$ is closed, gives $\mathbf{h}(m, i, n) \neq \pi_n(x_1)$ and if $i \neq j$, $\mathbf{h}(m, i, n) \neq \mathbf{h}(m, j, n)$ for all n large enough.

Now, for any $n > 0$, we define \mathbf{H}_n as the face of $\text{conv}(F_n^\zeta)$ of smallest dimension containing $\pi_n(x_1)$ and $\left\{ \mathbf{h}(m, i, n) : \mathbf{t} \in \mathbf{F}_1, 0 \leq i \leq d, m \geq m(n) \text{ with } \liminf_{m \rightarrow \infty} t_i^m > 0 \right\}$. In particular, (5.2) shows that $\hat{N}_{\mathbf{F}_1}(\overline{\text{conv}(D)}) \subseteq \bigcap_{n>0} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta))$. We claim $\bigcap_{n>0} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta)) = \hat{N}_{\mathbf{F}_1}(\overline{\text{conv}(D)})$. First, note that taking subsequences if its necessary, we get that for all $n \in E$ the following limits

$$\lim_{m \rightarrow \infty} \sum_{i=1}^d t_i^m \mathbf{h}(m, i, n) = \mathbf{h}_n(\mathbf{t}) \quad \wedge \quad \lim_{m \rightarrow \infty} \sum_{i=1}^d t_i^m \mathbf{z}(m, i, n) = \mathbf{z}_n(\mathbf{t}).$$

Hence $\mathbf{h}_n(\mathbf{t}) \in \mathbf{H}_n$ for all $n \in E$ and $\mathbf{z}_n(\mathbf{t}) \in \text{conv}(\overline{K_\zeta} - \overline{K_\zeta})$ for all $n \in E$ large enough. A geometric argument shows that there exists $\mathbf{t} \in \mathbf{F}_1$ and $\varepsilon > 0$ small enough such that for all $\mathbf{t}' \in \mathbf{F}_1$, we have that $\mathbf{z}_n(\mathbf{t}') = \mathbf{z}_n(\mathbf{t})$, so \mathbf{H}_n is a face of codimension 1 for all $n \in E$ large enough. We then conclude that $\bigcap_{n \in E} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta)) = \hat{N}_{\mathbf{F}_1}(\overline{\text{conv}(D)})$.

Thus, the extremal rays of $\hat{N}_{\mathbf{F}_0}(\overline{\text{conv}(D)})$ are equal to sets of the form $\bigcap_{n>0} \hat{N}_{\mathbf{H}_n}(\text{conv}(F_n^\zeta))$, with \mathbf{H}_n being faces eventually of codimension 1 of $\text{conv}(F_n^\zeta)$ containing $\pi_n(x_1)$. □

Then, to determine the nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$ we will study the supporting hyperplanes to $\text{conv}(F_n^\zeta)$. To do this we will focus on the convex hull of the digit tile of the substitution. In general, this convex hull is not a polytope, i.e., can have at least a countable number of extreme points, even if the expansion matrix is diagonal, as we see in Example 5.4:

Example 5.4 (A digit tile, with a nonpolytope convex hull). Consider $L = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ and $F_1 = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 2), (1, -2)\}$.

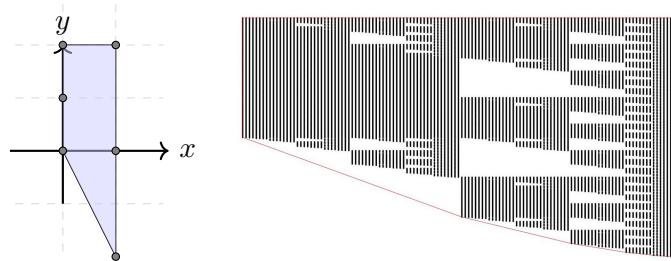


Figure 5.1: The fundamental domain and an approximation of the digit tile of Example 5.4.

A direct computation shows that for any $n > 0$, the extreme points of $\text{conv}(F_n)$ is the set $\{(0, 0), (0, 3^n - 1), (2^n - 1, 3^n - 1)\} \cup \{(2^n - 2^k, 3^k - 3^n) : 0 \leq k \leq n - 1\}$, which implies

$$\text{Ext}(\text{conv}(T(L, F_1))) = \{(0, 0), (0, 1), (1, 1), (1, -1)\} \cup \{(1 - 2^{-k}, -1 + 3^{-k}) : k \geq 0\}.$$

5.3 The polytope case

In this section, we will focus in the case when the convex hull of the digit tile is a polytope. We will present some known results about the convex hull of the digit tile that we will use in the rest of this thesis.

Definition 5.5. We say a substitution ζ is a polytope substitution if it is bijective on extremities, and the convex hull of the digit tile $T_\zeta = T(L_\zeta, F_1^\zeta)$ is a polytope.

From now on, we will only consider polytope substitutions. As we will see, this geometrical hypothesis implies several algebraic restrictions on the expansion matrix L_ζ (Proposition 5.10) and some dynamical consequences for $(X_\zeta, S, \mathbb{Z}^d)$ (Theorem 5.17).

We recall here some results characterizing the polytope case in terms of the extreme points of $\text{conv}(F_n^\zeta)$ [77], and the inward unit normal vectors of the $(d-1)$ -dimensional faces of $\text{conv}(T_\zeta)$ [109].

Theorem 5.6. *Let T be the digit tile for an expansion matrix $L \in M_d(\mathbb{R})$ and a fundamental domain $F_1 \subseteq \mathbb{R}^d$. The following statements are equivalent:*

1. *The convex hull of the digit tile $T(L, F_1)$ is a polytope.*
2. *[109, Theorem 4.2] The inward unit normal vectors of the $(d-1)$ -dimensional faces of $\text{conv}(F_1)$ are eigenvectors of $(L^*)^k$ for some k .*
3. *[77, Theorem 2.2] The cardinality of $\text{Ext}(\text{conv}(F_n))$ and $\text{Ext}(\text{conv}(F_{n+1}))$ are the same for some $n > 0$. In such a case, for any $m > n$, $|\text{Ext}(\text{conv}(F_m))| = |\text{Ext}(\text{conv}(F_n))|$, and then $|\text{Ext}(\text{conv}(T(L, F_1)))| = |\text{Ext}(\text{conv}(F_n))|$.*

Remark 5.7. In the case $L = \lambda \text{id}_{\mathbb{R}^d}$, with $\lambda > 1$, a direct computation shows that the statements (2) and (3) of Theorem 5.6 are satisfied without taking any power of L .

A big family for the polytope case is when a power of L is an integer multiple of the identity, because for any fundamental domain F of L , the convex hull of the digit tile generated by L and F is a polytope. In particular, all the convex hull of the digit tiles of the examples in Fig. 1.4 are polytopes. Although it is not the only case where Theorem 5.6 can be applied.

Example 5.8. (1) (Example of a non self-similar matrix with a polytope digit tile) Consider $L = \begin{pmatrix} 2 & 0 \\ -1 & 3 \end{pmatrix}$ and $F_1 = \{(0, 0), (1, 2), (-2, -1), (-2, -3), (1, 0), (-1, -1)\}$.

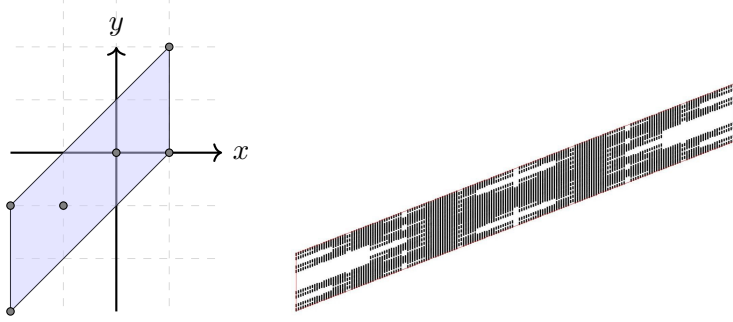


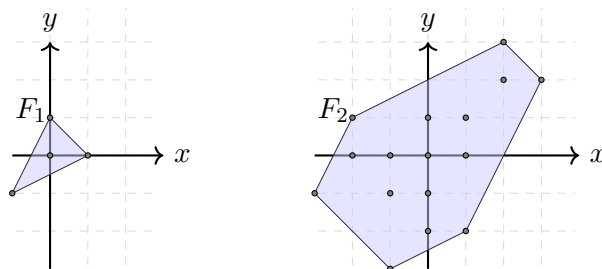
Figure 5.2: The fundamental domain and an approximation of the digit tile of a non self-similar matrix.

We have that $L^* = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}$ with eigenvectors equal to $\{(-1, 1), (1, 0)\}$. A direct computation shows that the extreme points of $\text{conv}(F_n)$ are equal to $\{(2^n - 1, 2^n - (3^n + 1)/2), (2^n - 1, (2^{n+1} + 3^n - 3)/2), (-2(2^n - 1), (3^n + 3 - 2^{n+2})/2), (-2(2^n - 1), (5 - 3^n - 2^{n+2})/2)\}$, so the extreme points of $\text{conv}(T(L, F_1))$ are $\{(1, 1/2), (1, 3/2), (-2, -3/2), (-2, -5/2)\}$.

(2) As an example where the statement (3) in Theorem 5.6 to be applied is not necessarily satisfied in $n = 1$, consider $L = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ and $F_1 = \{(0, 0), (1, 0), (0, 1), (-1, -1)\}$. We have that

$$F_2 = \{(-1, -3), (0, -2), (1, -2), (-3, -1), (-1, -1), (0, -1), (-2, 0), (-1, 0), (0, 0), (1, 0), (-2, 1), (0, 1), (1, 1), (2, 2), (3, 2), (2, 3)\},$$

so $\text{conv}(F_2)$ has 3 extreme points, while $\text{conv}(F_1)$ has 6 extreme points as shown in Fig. 5.3

Figure 5.3: The sets F_1 and F_2 .

In [77] it was proved the following result about the extreme points of $\text{conv}(F_m)$ for any $m > n$, where n is such that $|\text{Ext}(\text{conv}(F_n))| = |\text{Ext}(\text{conv}(F_{n+1}))|$ and $\text{conv}(T(L, F_1))$.

Proposition 5.9. [77, Theorem 4.8] *If $|\text{Ext}(\text{conv}(F_n))| = |\text{Ext}(\text{conv}(F_{n+1}))|$, then all the extreme points of $\text{conv}(T(L, F_1))$ are of the form $\sum_{j>0} L^{-(n+1)j} \left(\sum_{i=0}^n L^i(\mathbf{f}_i) \right)$, with $\sum_{i=0}^n L^i(\mathbf{f}_i)$ being an extreme point of $\text{conv}(F_{n+1})$.*

This implies $\text{conv}(T(L, F_1))$ is equal to $(L^m - \text{id})^{-1} \text{conv}(F_m)$ for all $m > n$.

Now, assume that we are under the condition $|\text{Ext}(\text{conv}(F_1))| = |\text{Ext}(\text{conv}(T(L, F_1)))|$ and for all $n > 0$, $\text{conv}(T(L, F_1)) = (L^n - \text{id})^{-1} \text{conv}(F_n)$. Let \mathbf{u} be an inward unit normal vector of a $(d-1)$ -dimensional face of $\text{conv}(T(L, F_1))$. For each $n > 0$, $((L^n - \text{id})^*)^{-1} \mathbf{u}$ is an inward normal vector of a $(d-1)$ -dimensional face of F_n . By Theorem 5.6 (1), there exists $k > 0$ such that $((L - \text{id})^*)^{-1} \mathbf{u}$ are eigenvectors of $(L^*)^k$. Hence by commutation, \mathbf{u} is an eigenvector of $(L^*)^k$. Since $\text{conv}(T(L, F_1))$ is a polytope, we can take $n > 0$ large enough such that any of the inward unit normal vectors of $\text{conv}(F_1)$ is an eigenvector of the same power $(L^*)^n$. Hence, by the same arguments, up to considering a power of L , we may assume that any of the inward unit normal vectors of the $(d-1)$ -dimensional faces of $\text{conv}(F_1)$ are eigenvector of L^* . This is equivalent to the hyperplane $\partial H[\mathbf{u}] = \{\mathbf{t} \in \mathbb{R}^d : \langle \mathbf{t}, \mathbf{u} \rangle = 0\}$ (the vector space of an affine hull of a face of $\text{conv}(F_1)$) generated by \mathbf{u} being preserved by L , i.e., $L\partial H[\mathbf{u}] = \partial H[\mathbf{u}]$. This implies the normal fan $\mathcal{N}(\text{conv}(F_n))$ is the same for all $n > 0$ and it is equal to the one of $\text{conv}(T(L, F_1))$.

Since for some $n > 0$, $\text{conv}(F_n)$ is nondegenerate (by the Følner condition), it has d linearly independent inward normal vectors (that has integer coordinates with no common divisor) which are eigenvectors of L^* . The polytope condition implies then the following algebraic restrictions on the expansion matrix L . The proof is left to the reader.

Proposition 5.10. *If $|\text{Ext}(\text{conv}(F_1^\zeta))| = |\text{Ext}(\text{conv}(T(L, F_1)))|$, then the eigenvalues of L are integer numbers.*

Moreover, if $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent inward unit normal vectors of $(d-1)$ -dimensional faces of $\text{conv}(T(L, F_1))$, then L restricted to the vector space generated by these vectors acts as a integer multiple of the identity.

In particular, in the two-dimensional case if the digit tile has 3 or at least 5 edges, then it follows that the expansion matrix is an integer multiple of the identity.

Finally, up to taking an appropriate power of a substitution, we may assume the following hypothesis

- (PC 1) The expansion matrix L is diagonalizable with positive eigenvalues.
- (PC 2) The convex set $\text{conv}(F_1)$ is nondegenerate and $|\text{Ext}(\text{conv}(F_1))| = |\text{Ext}(\text{conv}(T(L, F_1)))|$.
- (PC 3) Any inward unit normal vector of a $(d-1)$ -dimensional face of $\text{conv}(F_1)$ is an eigenvector of L^* .
- (PC 4) The set K given by Proposition 1.18 is equal to $(\text{id} - L)^{-1}(F_1) \cap \mathbb{Z}^d$, i.e., for any $\mathbf{k} \in K$, there exists $\mathbf{f} \in F_1$ such that $\mathbf{k} = L_\zeta(\mathbf{k}) + \mathbf{f}$.

5.4 Dynamical properties of substitutive subshifts of polytope substitutions

In this section, we prove the normalizer group of substitutive subshifts given by polytope substitutions is virtually generated by the shift action. We then describe the symmetry group of these substitutive subshifts (Proposition 5.15 and Theorem 5.17). To do this, we determine the nondeterministic directions of these substitutive subshifts thanks to the supporting hyperplanes to $\text{conv}(F_n^\zeta)$ (Theorem 5.2). Conversely, we provides a checkable combinatorial condition to ensure a vector to be nondeterministic for $(X_\zeta, S, \mathbb{Z}^d)$. (Corollary 5.13).

As we saw in the previous section, under the hypothesis (PC 1), (PC 2), (PC 3), and (PC 4), the normal fan is the same for all the supports of ζ^n (for any $n > 0$ large enough) and the digit tile, so we have that the following interpretation of Theorem 5.2 in the polytope case.

Corollary 5.11 (Nondeterministic directions in the polytope case). *Let ζ be an aperiodic primitive polytope substitution. The set of nondeterministic directions $\text{ND}(X_\zeta, S, \mathbb{Z}^d)$ is the intersection of \mathbb{S}^{d-1} with a nonempty union of opposite normal cones of the form $\hat{N}_{\mathbf{G}}(\text{conv}(T_\zeta))$, where \mathbf{G} is a face of $\text{conv}(T_\zeta)$.*

Hence, in the two-dimensional case the former corollary implies strong restrictions on the set of nondeterministic directions. For instance, the number of its connected components is bounded by the number of edges of $\text{conv}(T_\zeta)$.

Now, as shown in the proof of Theorem 5.2, to establish which opposite normal vectors of $\text{conv}(F_1^\zeta)$ appeared as nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$ we will study the convex sets $\text{conv}(L_\zeta^n(\mathbf{k}) + F_n^\zeta)$ generated by the points \mathbf{k} in K_ζ , which depend on the combinatorics of the substitution. We say a subset $W \subseteq K_\zeta$ is a *set of differences* if there exist two patterns $\mathbf{w}_1, \mathbf{w}_2 \in \mathcal{L}_{K_\zeta}(X_\zeta)$ such that $\mathbf{w}_1(\mathbf{k})$ is equal to $\mathbf{w}_2(\mathbf{k})$ if and only if \mathbf{k} is in $K_\zeta \setminus W$.

The next lemma gives a sufficient condition to ensure a vector \mathbf{v} to be a nondeterministic for $(X_\zeta, S, \mathbb{Z}^d)$, seen as the converse of Theorem 5.2 in the polytope case. As in Lemma 3.9, consider a set $C \in \mathbb{Z}^d$ such that for all $n > 0$, $C + F_n^\zeta + F_n^\zeta \subseteq L_\zeta^n(C) + F_n^\zeta$ and $\overline{K}_\zeta = K_\zeta + C$.

Lemma 5.12. *Let $W \subseteq K_\zeta$ be a set of differences, $\mathbf{k} \in W$, $n > 0$, a point $\mathbf{f} \in \partial \text{conv}(L_\zeta^n(\mathbf{k}) + F_n^\zeta)$ and $\mathbf{v} \in \mathbb{S}^{d-1}$ be such that $\mathbf{f} + \partial H_{\mathbf{v}}$ is supporting to $\text{conv}(L_\zeta^n(\mathbf{k}) + F_n^\zeta)$ at \mathbf{f} . Suppose that \mathbf{f} satisfies the following conditions:*

$$H1. \quad \mathbf{f} + \overline{K}_\zeta \subseteq L_\zeta^n(K_\zeta) + F_n^\zeta,$$

$$H2. \quad (\mathbf{f} + \overline{K}_\zeta) \cap (\mathbf{f} + H_{\mathbf{v}}) \subseteq L_\zeta^n(K_\zeta \setminus W) + F_n^\zeta.$$

Then \mathbf{v} is nondeterministic for $(X_\zeta, S, \mathbb{Z}^d)$.

Proof. Let $\mathbf{w}_1, \mathbf{w}_2$ be two patterns such that $\mathbf{w}_1(\mathbf{k}') = \mathbf{w}_2(\mathbf{k}')$ if and only if $\mathbf{k}' \in K_\zeta \setminus W$. Note that Condition H1 is equivalent to for all $m > 0$, $L_\zeta^m(\mathbf{f}) + L_\zeta^m(\overline{K}_\zeta) + F_m^\zeta \subseteq L_\zeta^{n+m}(K_\zeta) + F_{n+m}^\zeta$. Since $\overline{K}_\zeta = K_\zeta + C$, Remark 1.21 (2) implies for all $m > 0$,

$$L_\zeta^m(\mathbf{f}) + F_m^\zeta + (L_\zeta^m(K_\zeta) + F_m^\zeta) \subseteq L_\zeta^{n+m}(K_\zeta) + F_{n+m}^\zeta. \quad (5.3)$$

If \mathbf{f} is an extreme point of $\text{conv}(L_\zeta^n(\mathbf{k}) + F_n^\zeta)$, there exists $\mathbf{g} \in \text{Ext}(\text{conv}(F_1^\zeta))$ such that $\mathbf{f} = L_\zeta^n(\mathbf{k}) + \sum_{i=0}^{n-1} L_\zeta^i(\mathbf{g})$. If \mathbf{f} is in the relative interior of a k -dimensional face of $\text{conv}(L_\zeta^n(\mathbf{k}) + F_n^\zeta)$, ($1 \leq k \leq d-1$), we consider $\mathbf{g} \in \text{Ext}(\text{conv}(F_1^\zeta))$ such that $L_\zeta^n(\mathbf{k}) + \sum_{i=0}^{n-1} L_\zeta^i(\mathbf{g})$ and \mathbf{f} are in the same k -dimensional face of $\text{conv}(L_\zeta^n(\mathbf{k}) + F_n^\zeta)$ as shown in Fig. 5.4.

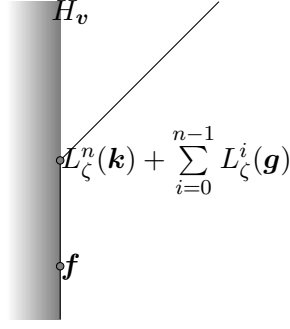


Figure 5.4: The hyperplane ∂H_v supporting to $\text{conv}(L_\zeta^n(\mathbf{k}) + F_n^\zeta)$ at \mathbf{f} and $L_\zeta^n(\mathbf{k}) + \sum_{i=0}^{n-1} L_\zeta^i(\mathbf{g})$.

Now, Condition H2 is equivalent to for all $m > 0$

$$L_\zeta^m((\mathbf{f} + \overline{K_\zeta}) \cap (\mathbf{f} + H_v)) + F_m \subseteq L_\zeta^{n+m}(K_\zeta \setminus W) + F_{n+m}^\zeta. \quad (5.4)$$

We will prove that for all $m > 0$

$$(L_\zeta^m(\mathbf{f}) + \sum_{i=0}^{m-1} L_\zeta^i(\mathbf{g}) + L_\zeta^m(K_\zeta) + F_m^\zeta) \cap (L_\zeta^m(\mathbf{f}) + \sum_{i=0}^{m-1} L_\zeta^i(\mathbf{g}) + H_v) \subseteq (L_\zeta^m(\mathbf{f}) + \overline{K_\zeta}) \cap (\mathbf{f} + H_v) + F_m. \quad (5.5)$$

We have that

$$\langle \mathbf{v}, \mathbf{g} \rangle = \min_{\mathbf{i} \in F_1^\zeta} \langle \mathbf{v}, \mathbf{i} \rangle. \quad (5.6)$$

Fix $\mathbf{n} \in (L_\zeta^m(\mathbf{f}) + \sum_{i=0}^{m-1} L_\zeta^i(\mathbf{g}) + L_\zeta^m(K_\zeta) + F_m^\zeta) \cap (L_\zeta^m(\mathbf{f}) + \sum_{i=0}^{m-1} L_\zeta^i(\mathbf{g}) + H_v)$, there exist $\mathbf{k}_1 \in K_\zeta$, $\mathbf{j} \in F_m^\zeta$ and $\mathbf{h} \in H_v$ such that

$$\mathbf{n} = L_\zeta^m(\mathbf{f}) + \sum_{i=0}^{m-1} L_\zeta^i(\mathbf{g}) + L_\zeta^m(\mathbf{k}_1) + \mathbf{j} = L_\zeta^m(\mathbf{f}) + \sum_{i=0}^{m-1} L_\zeta^i(\mathbf{g}) + \mathbf{h}.$$

Then, there exist $\mathbf{c} \in \overline{K_\zeta}$ and $\mathbf{l} \in F_m^\zeta$ such that

$$\sum_{i=0}^{m-1} L_\zeta^i(\mathbf{g}) + L_\zeta^m(\mathbf{k}_1) + \mathbf{j} = L_\zeta^m(\mathbf{c}) + \mathbf{l}.$$

To prove (5.4), it is enough to show that $\mathbf{c} \in H_{\mathbf{v}}$. Indeed, write $\mathbf{l} = \sum_{i=0}^{m-1} L_{\zeta}^i(\mathbf{l}^{(i)})$, with $\mathbf{l}^{(i)} \in F_1^{\zeta}$ for $0 \leq i \leq m-1$. We have that

$$\begin{aligned} \langle \mathbf{v}, \mathbf{c} \rangle &= \left\langle \mathbf{v}, L_{\zeta}^{-m} \left(\sum_{i=0}^{m-1} L_{\zeta}^i(\mathbf{g} - \mathbf{l}^{(i)}) + \mathbf{h} \right) \right\rangle \\ &= \frac{1}{\lambda^m} \left(\sum_{i=0}^{m-1} \left\langle \mathbf{v}, L_{\zeta}^i(\mathbf{g} - \mathbf{l}^{(i)}) \right\rangle \right) + \frac{1}{\lambda^m} \langle \mathbf{v}, \mathbf{h} \rangle \\ &= \frac{1}{\lambda^m} \left(\sum_{i=0}^{m-1} \lambda^i \langle \mathbf{v}, \mathbf{g} - \mathbf{l}^{(i)} \rangle \right) + \frac{1}{\lambda^m} \langle \mathbf{v}, \mathbf{h} \rangle. \end{aligned}$$

Since $\mathbf{h} \in H_{\mathbf{v}}$, then $\langle \mathbf{v}, \mathbf{h} \rangle \leq 0$, and by (5.6) we have that $\langle \mathbf{v}, \mathbf{g} - \mathbf{l}^{(i)} \rangle \leq 0$, for all $1 \leq i \leq m-1$. Since $\lambda > 0$, we conclude that $\langle \mathbf{v}, \mathbf{c} \rangle \leq 0$, which implies $\mathbf{c} \in H_{\mathbf{v}}$.

By (5.3) and Proposition 1.18, the iterations of the substitution on the patterns $\mathbf{w}_1, \mathbf{w}_2$ leads to two points $x_1 \neq x_2 \in X_{\zeta}$ such that, x_i is in $[\zeta^n(\mathbf{w}_1)]_{-L_{\zeta}^n(\mathbf{f}) - \sum_{i=0}^{n-1} L_{\zeta}^i(\mathbf{g})}$, for $i \in \{1, 2\}$. Finally, (5.4) implies $x_1|_{H_{\mathbf{v}}} = x_2|_{H_{\mathbf{v}}}$, and we conclude that \mathbf{v} is nondeterministic for $(X_{\zeta}, S, \mathbb{Z}^d)$. \square

As described in Theorem 5.2, depending on the faces where the point \mathbf{f} satisfying Condition H1. belongs in $\text{conv}(L_{\zeta}^n(\mathbf{k}) + F_n^{\zeta})$, we may have more nondeterministic directions, obtaining the following corollary:

Corollary 5.13. *Let $W \subseteq K_{\zeta}$ be a set of differences, $\mathbf{k} \in W$, $n > 0$, and a point $\mathbf{f} \in \partial \text{conv}(L_{\zeta}^n(\mathbf{k}) + F_n^{\zeta}) \cap \partial \text{conv}(L_{\zeta}^n(W) + F_n^{\zeta})$ satisfying Condition H1. of Lemma 5.12. Then any \mathbf{v} in $\hat{N}_{\mathbf{F}}(\text{conv}(L_{\zeta}^n(W) + F_n^{\zeta})) \cap \mathbb{S}^{d-1}$ (with \mathbf{F} being the face of smallest dimension where \mathbf{f} belongs in $\text{conv}(L_{\zeta}^n(W) + F_n^{\zeta})$) is nondeterministic for $(X_{\zeta}, S, \mathbb{Z}^d)$.*

In addition to this result, the proof of Theorem 5.2 provides the opposite normal cone $\hat{N}_{\mathbf{F}}(\text{conv}(L_{\zeta}^n(W) + F_n^{\zeta}))$ is equal to an opposite normal cone $\hat{N}_{\mathbf{G}}(\text{conv}(F_n^{\zeta}))$, whenever $\mathbf{f} = L_{\zeta}^n(\mathbf{k}) + \sum_{i=0}^{n-1} L_{\zeta}^i(\mathbf{f}_i)$, with \mathbf{G} is a face of $\text{conv}(F_n^{\zeta})$ such that $\sum_{i=0}^{n-1} L_{\zeta}^i(\mathbf{f}_i) \in \mathbf{G}$.

In the following we present two examples with different behaviors given by Lemma 5.12

Example 5.14 (Different behaviors for the nondeterministic directions). (1) Consider the 2D-Thue Morse substitution with $L_{\zeta_{TM}} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $F_1^{\zeta_{TM}} = \llbracket 0, 1 \rrbracket^2$, given by the substitution

$$\zeta_{TM} : \quad \begin{array}{cc} 0 & 1 \\ 0 \mapsto & 1 \ 0 \quad 1 \mapsto \ 0 \ 1. \end{array}$$

In this case $K_{\zeta} = \llbracket -1, 0 \rrbracket^2$, and we have that

$$\mathcal{L}_{K_{\zeta_{TM}}}(X_{TM}) = \left\{ \begin{array}{cccccccccccc} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right\}.$$

The sets of differences for the 2D-Thue Morse substitution are $\{(0, -1), (0, 0)\}, \{(-1, 0), (0, 0)\}, \{(-1, 0), (0, -1)\}, \{(-1, -1), (0, 0)\}, \{(-1, -1), (-1, 0)\}$.

By Lemma 5.12 it can be proved $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ are the only nondeterministic directions for $(X_{\zeta_{TM}}, S, \mathbb{Z}^2)$.

- (2) Consider the substitution of the table tiling [99], with $L_{\zeta_t} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $F_1^{\zeta_t} = \llbracket 0, 1 \rrbracket^2$, given by ζ_t :

$$0 \mapsto \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}, \quad 1 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad 2 \mapsto \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix}, \quad 3 \mapsto \begin{pmatrix} 0 & 2 \\ 3 & 3 \end{pmatrix}.$$

The set K_{ζ_t} is equal to $\llbracket -1, 0 \rrbracket^2$ and the sets of differences is equal to $2^{K_{\zeta_t}} \setminus \{\emptyset, K_{\zeta_t}, \{(-1, -1), (0, 0)\}, \{(0, -1), (-1, 0)\}\}$. By Lemma 5.12, it can be proved the set of nondeterministic directions for $(X_{\zeta_t}, S, \mathbb{Z}^2)$ is the whole circle \mathbb{S}^1 .

Now, we proceed to determine the normalizer semigroup $N(X_{\zeta}, S, \mathbb{Z}^d)$ of substitutive subshifts of polytope substitutions. Set $M \in \vec{N}(X_{\zeta}, S, \mathbb{Z}^d)$. By Proposition 1.12 if \mathbf{v} is a nondeterministic direction for $(X_{\zeta}, S, \mathbb{Z}^d)$, then $M^*\mathbf{v}/\|M^*\mathbf{v}\|$ is also a nondeterministic direction for $(X_{\zeta}, S, \mathbb{Z}^d)$. Moreover, Theorem 5.2 ensures the matrix M acts on the opposite normal cones of $\text{conv}(T_{\zeta})$ that appeared as nondeterministic directions for $(X_{\zeta}, S, \mathbb{Z}^d)$. In particular, the matrix M^* permutes the hyperplanes defined by the $(d-1)$ -dimensional faces of $\text{conv}(T_{\zeta})$ whose unit opposite normal cones are nondeterministic directions for $(X_{\zeta}, S, \mathbb{Z}^d)$. If there are d linearly independent nondeterministic directions for $(X_{\zeta}, S, \mathbb{Z}^d)$, we have the following result about the matrix M such that $N_M(X_{\zeta}, S, \mathbb{Z}^d)$ being nonempty.

Proposition 5.15. *Let ζ be an aperiodic primitive polytope substitution. If the set of nondeterministic directions for $(X_{\zeta}, S, \mathbb{Z}^d)$ contains d linearly independent vectors, then*

1. *Any homomorphism $\phi \in N(X_{\zeta}, S, \mathbb{Z}^d)$ is invertible.*
2. *Any matrix $M \in \vec{N}(X_{\zeta}, S, \mathbb{Z}^d)$ has finite order.*
3. *The symmetry semigroup $\vec{N}(X_{\zeta}, S, \mathbb{Z}^d)$ is a finite group, and it is isomorphic to a subgroup of $GL(d, \mathbb{Z}/3\mathbb{Z})$.*
4. *The norm of any matrix M in the symmetry group $\vec{N}(X_{\zeta}, S, \mathbb{Z}^d)$ is bounded by an explicit formula only depending on the convex hull of the digit tile T_{ζ} .*

Proof. By assumption and Corollary 5.11 there are at least d inward unit normal vectors to the $(d-1)$ -dimensional faces of $\text{conv}(T_{\zeta})$ that are nondeterministic directions for $(X_{\zeta}, S, \mathbb{Z}^d)$ and d of them are linearly independent. Let n be the maximum of these vectors. So any

matrix M in the symmetry group of $(X_\zeta, S, \mathbb{Z}^d)$ permutes those hyperplanes defined by these $(d-1)$ -dimensional faces of $\text{conv}(T_\zeta)$. By condition (PC 3), the normal vectors of these hyperplanes are invariant by a power of the expansion matrix L_ζ^* . Hence M^* permutes n eigenspaces $\{\mathbb{Q}\mathbf{v}_1, \dots, \mathbb{Q}\mathbf{v}_n\}$ of some power of L_ζ^* . Moreover, we can assume that these vectors have integer coordinates not having common divisors except ± 1 . Note that each vector is unique up to a sign and does not depend on M .

So, $(M^{n!})^*$ leaves invariant these eigenspaces, i.e., $(M^{n!})^*\mathbf{v}_i = \alpha_i\mathbf{v}_i$. Since the vectors \mathbf{v}_i have integer coordinates with no common divisor, we have that $\alpha_i \in \mathbb{Z}$, for all $1 \leq i \leq n$. The same being true for $(M^{-1})^*$, we get that each α_i is invertible in \mathbb{Z} , which implies $|\alpha_i| = 1$ so $M^{2n!}$ is equal to the identity matrix, and then M has a finite order. By Proposition 4.11 any homomorphism of $(X_\zeta, S, \mathbb{Z}^d)$ is invertible.

Furthermore, for any \mathbf{v}_i , we have that $M^*\mathbf{v}_i = \lambda_i\mathbf{v}_j$, for some $1 \leq i, j \leq n$ and $\lambda_i \in \mathbb{Q}$. Since $M \in GL(d, \mathbb{Z})$ and the coordinates of \mathbf{v}_j does not have common divisors, we have that $\lambda_i \in \mathbb{Z}$. Moreover, note that $(M^{2n!})^*\mathbf{v}_i = \lambda_i \cdot \lambda_{i_1} \cdots \lambda_{i_{2n!-1}}\mathbf{v}_i = \mathbf{v}_i$ for some $\lambda_{i_1} \cdots \lambda_{i_{2n!-1}} \in \mathbb{Z}$, hence $|\lambda_i| = 1$ for all $1 \leq i \leq n$. Up to a change of indices, we can assume that $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ is a \mathbb{R}^d -basis, so M^* has the form $M^* = PQ_M P^{-1}$, where Q_M is the matrix with columns equal to the coordinates of $M\mathbf{v}_i$ (which for all $1 \leq i \leq d$ is equal to some \mathbf{v}_j or $-\mathbf{v}_j$) in the basis $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$. We then conclude that the symmetry group $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is finite and by Minkowski's theorem we have that $\vec{N}(X_\zeta, S, \mathbb{Z}^d) \leq GL(d, \mathbb{Z}/3\mathbb{Z})$.

Finally, note that $\|M\| \leq \|P\| \cdot \|Q_M\| \cdot \|P^{-1}\|$, where $\|P\|, \|P^{-1}\|$ and $\sup_{M \in \vec{N}(X_\zeta, S, \mathbb{Z}^d)} \|Q_M\| < \infty$ only depend on the convex hull of the digit tile T_ζ . \square

Remark 5.16. In particular, it follows from the proof that if $n = d$ each matrix Q_M is a permutation matrix, so $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is conjugate to a subgroup of the hyperoctahedral group W_d . These recover results in [21] for block substitutions. By the realization result [21, Theorem 35] these results obtained are optimal.

The following theorem summarizes all the properties satisfied for aperiodic primitive reduced polytope substitutions.

Theorem 5.17. *Let ζ be an aperiodic reduced primitive polytope substitution. Then*

1. *The system $(X_\zeta, S, \mathbb{Z}^d)$ is coalescent, and any homomorphism in $N(X_\zeta, S, \mathbb{Z}^d)$ is invertible.*

If there are d linearly independent vectors that are nondeterministic directions for $(X_\zeta, S, \mathbb{Z}^d)$, we have that

2. *The normalizer group is virtually generated by the shift action.*

3. The symmetry group $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ acts as a permutation group in the set $\{N_{\mathbf{G}}(\text{conv}(T_\zeta)): N_{\mathbf{G}}(\text{conv}(T_\zeta)) \subseteq \text{ND}(X_\zeta, S, \mathbb{Z}^d)\}$. In particular, if $\text{ND}(X_\zeta, S, \mathbb{Z}^d) = \mathbb{S}^{d-1}$, then the symmetry group $\vec{N}(X_\zeta, S, \mathbb{Z}^d)$ is isomorphic to a subgroup of the automorphism group of the normal fan of $\text{conv}(T_\zeta)$.

Proof. The statement 1. is true by Proposition 4.7 and Proposition 4.11.

Now, by the third isomorphism theorem we have that

$$N(X_\zeta, S, \mathbb{Z}^d)/\text{Aut}(X_\zeta, S, \mathbb{Z}^d) \cong \left(N(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle \right) / (\text{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle).$$

Then, by Proposition 5.15 gives the quotient group $N(X_\zeta, S, \mathbb{Z}^d)/\text{Aut}(X_\zeta, S, \mathbb{Z}^d)$ is finite and Proposition 4.8 implies $\text{Aut}(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is also finite, so we conclude that $N(X_\zeta, S, \mathbb{Z}^d)/\langle S \rangle$ is a finite group.

Finally, statement 3. is true by Proposition 5.15. \square

Until now, we didn't find an aperiodic d -dimensional primitive constant-shape substitution with less than d linearly independent nondeterministic directions. In the case of aperiodic primitive block substitutions, it can be easily proved that this hypothesis is true. Moreover, given the result in [63] for the two-dimensional case, by Theorem 5.2, the hypothesis is true for all of the cases where $\text{conv}(T_\zeta)$ does not have two parallel edges. In a private communication, P. Guillon [62] mentioned this result is already proved for higher dimensions, but nowhere published. This implies, we only have to deal in the case $\text{conv}(T_\zeta)$ has two parallel $(d-1)$ -dimensional faces.

Theorem 5.17 deal only with polytope substitutions, and therefore leaves open how to characterize the normalizer semigroup for nonpolytope substitutions, i.e., where the convex hull of the digit tile is nonpolytope. In [109], the authors gives a description of the convex hull of the digit tile on the nonpolytope case. This description may be useful to obtain a characterization for the normalizer semigroup and symmetry semigroup. Nevertheless, until now there are no good descriptions of the convex hull of the digit tile for higher dimensions.

The work in this thesis corresponds to the first examples of realization results about the set of nondeterministic directions for minimal actions. In [17] it was proved that for any compact set of \mathbb{S}^1 that is not a singleton containing one line with irrational slope can be realized as the set of nonexpansive directions of a \mathbb{Z}^2 -action, and the singleton case was after proved by M. Hochmann in [68]. If aperiodic bijective on extremities primitive constant-shape substitutions have d linearly independent nondeterministic directions, then we cannot obtain a unique nondeterministic direction with these substitutions, so we will need to use other types of substitutions, or other type of subshifts (such as Toeplitz sequences) to obtain other realization results with minimal subshifts.

In Chapter 6 we describe the normalizer group for two examples. The first one is a bijective polytope substitution, so we may apply the results proved in Chapter 4 and Chapter 5. The second example is a non-reduced constant-shape substitution, such that

its reduced substitution is trivial, i.e., the subshift generated is conjugate to the one-point system. This implies the techniques developed in Chapter 4 and Chapter 5 does not apply. Nevertheless, we are able to characterize its maximal equicontinuous factor and its normalizer group.

Chapter 6

Some examples of constant-shape substitutions

As we have seen the normalizer group is very restrictive (Theorem 5.17) under strong combinatorial and geometric conditions on the constant-shape substitutions.

In this chapter we characterize the normalizer group for two examples of constant-shape substitutions. The first one is called the *table substitution*, which is a discretization of the table tiling. Its maximal equicontinuous factor was determined in [99]. Here, we prove that the normalizer group of the table tiling is isomorphic to a direct product of the shift action and the dihedral group D_4 , given by the symmetries of the square (Proposition 6.1), using the techniques developed in Chapter 4 and Chapter 5.

The second example is called the *half-hex substitution*, which is a discretization of the so-called half-hex tiling. This is a non-reduced constant-shape substitution, such that its reduced substitution is trivial, i.e., the subshift generated is conjugate to the one-point system. Then, one cannot apply the rigidity results developed in Chapter 4 and Chapter 5 cannot be applied. Nevertheless, we characterize its maximal equicontinuous factor. Furthermore, we prove it is a coalescent system, and its symmetry semigroup is a group. By Proposition 1.5 we have that the normalizer semigroup is actually a group. Moreover, we prove that its normalizer group is a semi-direct product of the shift action and $GL(2, \mathbb{Z})$ (Theorem 6.3). This is a first example of a minimal aperiodic subshift with an infinite symmetry group.

6.1 The table substitution

The table tiling is a well known example of a *rep-tile tiling* (see [8, Section 4.9] for more properties about this tiling), i.e., it is a polygon that can be tiled by a finite number of smaller, congruent copies of itself. The tile-substitution of the table tiling is shown in Fig. 6.1,

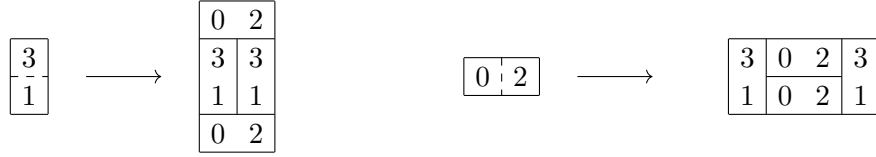


Figure 6.1: Tile-substitution of the table tiling and associated two-dimensional constant-shape substitution to discrete subaction (X_t, S, \mathbb{Z}^d) .

The following is a pattern of the table tiling:

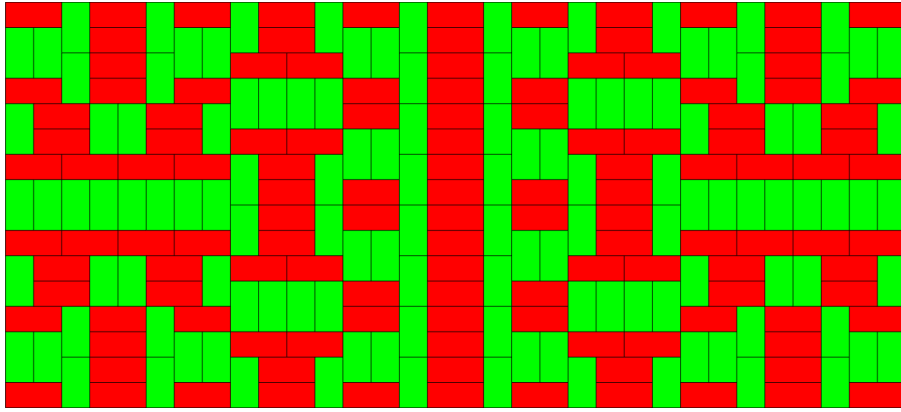


Figure 6.2: A pattern of the table tiling, using red and green rectangles.

The table tiling forms an aperiodic tiling [99]. Also in [99], it was proved that the \mathbb{Z}^2 -subaction of the table tiling is conjugate to the following constant-shape substitution (called *table substitution*) ζ_t with expansion matrix $L_t = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and support $F_1^{\zeta_t} = \llbracket 0, 1 \rrbracket^2$:

$$0 \mapsto \begin{array}{cc} 3 & 0 \\ 1 & 0 \end{array}, \quad 1 \mapsto \begin{array}{cc} 1 & 1 \\ 0 & 2 \end{array}, \quad 2 \mapsto \begin{array}{cc} 2 & 3 \\ 2 & 1 \end{array}, \quad 3 \mapsto \begin{array}{cc} 0 & 2 \\ 3 & 3 \end{array}.$$

This is an aperiodic bijective primitive polytope constant-shape substitution. By Remark 3.10 (3), the factor map $\pi_t : (X_t, S, \mathbb{Z}^2) \rightarrow (\overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z}), +_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, \mathbb{Z}^2)$ is almost 4-to-1. In fact, by [99] we have that $|\pi_t^{-1}(\{\overleftarrow{g}\})| \in \{4, 10, 24\}$ for any $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}$. Moreover, $|\pi_t^{-1}(\overleftarrow{g})| = 24$ if and only if $\overleftarrow{g} \in \mathcal{O}(\overleftarrow{0}, \mathbb{Z}^2)$.

The table substitution has 24 patterns with support $K_{\zeta_t} = \llbracket -1, 0 \rrbracket^2$, which generate the 24 fixed points under the square of the substitution:

1 1	0 3	2 0	3 2	2 0	0 2	2 0	1 1
2 0	0 1	2 0	1 2	0 3	2 0	3 2	0 2
3 1	3 3	2 0	1 3	0 1	3 2	1 3	1 2
1 2	1 1	3 3	3 1	0 3	1 3	0 1	3 2
3 1	0 1	1 2	0 1	1 2	0 2	2 0	0 3
1 3	2 0	0 3	3 2	2 0	3 3	0 2	3 1

In Example 5.14 it was mentioned that the set of nondeterministic directions for (X_t, S, \mathbb{Z}^2) is the whole circle \mathbb{S}^1 . Since the table substitution is a polytope substitution and has 2 linearly independent nondeterministic directions we can apply Theorem 5.17. This implies, its symmetry group $\vec{N}(X_t, S, \mathbb{Z}^2)$ is isomorphic to a subgroup of $\left\langle \left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\} \right\rangle$ which is isomorphic to D_4 . More precisely, we have the following result:

Proposition 6.1. *The normalizer group of the table substitution is isomorphic to $\mathbb{Z}^2 \rtimes D_4$. In particular $\text{Aut}(X_{\zeta_t}, S, \mathbb{Z}^2)$ is the group generated by the shift action.*

Proof. First, we will prove that $\text{Aut}(X_t, S, \mathbb{Z}^2) = \langle S \rangle$. Indeed, the table substitution is bijective. By Proposition 5.1, any automorphism ϕ can be written as $\phi = S^n \tau$, where $\mathbf{n} \in \mathbb{Z}^2$ and τ is defined by a permutation map \mathcal{T} on the alphabet $\{0, 1, 2, 3\}$. In particular, τ acts as a permutation on the set of fixed points of ζ_t^2 . Now, note that $\begin{matrix} 2 & 0 \\ 2 & 0 \end{matrix}$ is the only

pattern of the form $\begin{matrix} i & j \\ i & j \end{matrix}$, $i, j \in \{0, 1, 2, 3\}$. This implies $\mathcal{T}(0) = 0$ and $\mathcal{T}(2) = 2$. In the

same way, $\begin{matrix} 3 & 3 \\ 1 & 1 \end{matrix}$ is the only pattern of the form $\begin{matrix} i & i \\ j & j \end{matrix}$, $i, j \in \{0, 1, 2, 3\}$, which implies $\mathcal{T}(1) = 1$ and $\mathcal{T}(3) = 3$, i.e., $\mathcal{T} = \text{id}_{\{0,1,2,3\}}$. By minimality of (X_t, S, \mathbb{Z}^2) , we have that τ is the identity on X_t . We conclude that $\text{Aut}(X_t, S, \mathbb{Z}^2) = \langle S \rangle$.

Now, for homomorphisms. Set $M_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Consider the permutation $\Phi : 0 \mapsto 3, 1 \mapsto 0, 2 \mapsto 1, 3 \mapsto 2$. This permutation comes by the rotation of the rectangles defining the table tiling. We define the map $\phi : X_t \rightarrow \phi(X_t)$, given by $\phi(x)_{\mathbf{n}} = \Phi(x_{M_1^{-1}\mathbf{n}})$ for all $x \in X_t$, $\mathbf{n} \in \mathbb{Z}^d$. We will prove that ϕ is a homomorphism on X_t . A direct computation shows that the map $S^{(-1,0)}\phi$ permutes the fixed points of ζ_t^2 . By minimality of (X_t, S, \mathbb{Z}^2) we conclude that $S^{(-1,0)}\phi(X_t) = X_t$. Then ϕ is a homomorphism associated with M_1 onto (X_t, S, \mathbb{Z}^2) . Note that, also by minimality, $S^{(-1,0)}\phi\zeta_t^2 = \zeta_t^2(S^{(-1,0)}\phi)$, which implies $S^{(3,0)}\phi\zeta_t^2 = \zeta_t^2\phi$. So ϕ is the homomorphism given by Theorem 4.12.

Set $M_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Consider now the permutation $\Psi : 0 \mapsto 0, 1 \mapsto 3, 2 \mapsto 2,$

$3 \mapsto 1$. This permutation is given by reflecting the rectangles defining the table tiling by the x -axis. We now define the map $\psi : X_t \rightarrow \psi(X_t)$ as $\psi(x)_n = \Psi(x_{M_2^{-1}n})$ for all $x \in X_t$, $n \in \mathbb{Z}^d$. A direct computation shows that $S^{(-1,0)}\phi$ permutes the fixed points of ζ_t^2 . Hence, by minimality of (X_t, S, \mathbb{Z}^d) , we have that $\psi(X_t) = X_t$, so ψ is a homomorphism associated with the matrix M_2 of (X_t, S, \mathbb{Z}^d) . We also have that $S^{(0,3)}\psi\zeta_t^2 = \zeta_t^2\psi$. So ψ is the homomorphism associated with M_2 given by Theorem 4.12. Since $D_4 = \langle M_1, M_2 \rangle$ and the homomorphisms are induced by letter-to-letter maps, the composition of arbitrary maps in $\langle \phi, \psi \rangle$ is a homomorphism induced by a letter-to-letter map. This implies $\langle M_1, M_2 \rangle \cong \langle \phi, \psi \rangle$. To conclude, this last statement let us define a right split on the exact sequence (1.4), i.e., a group homomorphism from D_4 to $N(X_t, S, \mathbb{Z}^2)$. By this, we conclude that $N(X_t, S, \mathbb{Z}^2) = \langle S \rangle \rtimes \langle \phi, \psi \rangle \cong \mathbb{Z}^2 \rtimes D_4$. \square

6.2 The half-hex substitution

Another well known inflation rule is the *half-hex inflation* shown in Fig. 6.3 (see [8, Section 6.4] for more properties about this tiling). In tiling vocabulary, it is a *edge-to-edge inflation*, which means that each inflated tile is precisely dissected into copies of the tiles so the vertices of any tile intersect only the vertices of the adjacent tiles.

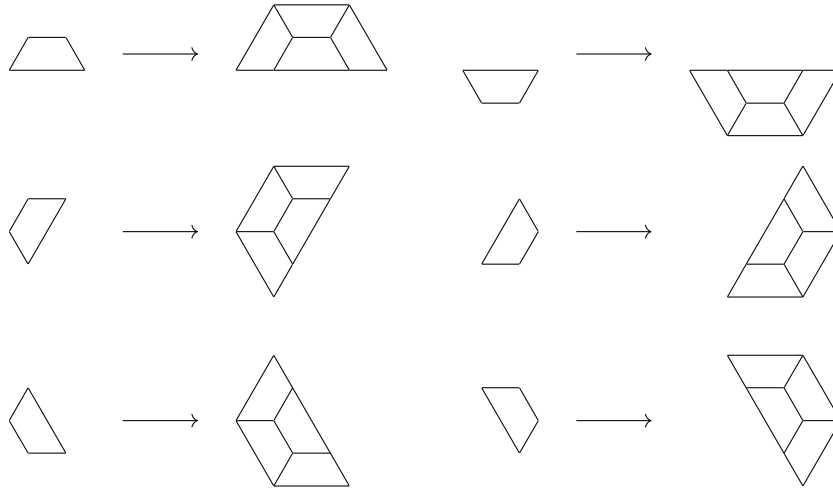


Figure 6.3: Tile-substitution of the half-hex tiling.

This inflation defines an aperiodic tiling of the plane as proved in [8]. The following shows a pattern of the half-hex tiling:

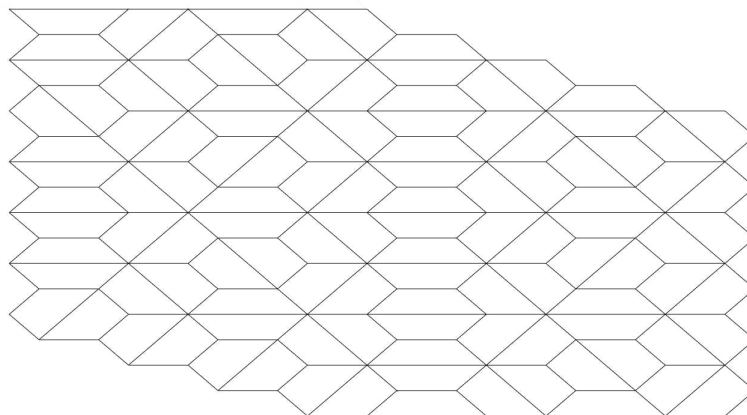


Figure 6.4: A pattern of the half-hex tiling.

Since the largest edge of any half-hex can only meet the largest edge of the adjacent half-hexes, two half-hexes always join to form a regular hexagon, through the largest edge. With this procedure, the half-hex tiling can be decomposed in three types of hexagons which are distinguished by a single diagonal line (see [8]).



Figure 6.5: The three tiles as a new alphabet for the half-hex tiling.

Using this full hexagons, we can define a *pseudo inflation* (using the vocabulary on [8]), which is conjugated to the half-hex tiling as the following.

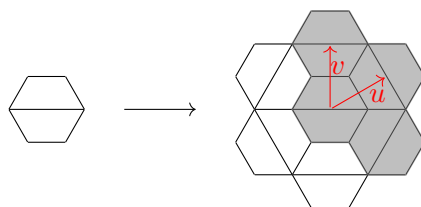


Figure 6.6: New tile-substitution conjugate to the half-hex tiling, with a discrete 2-dimensional substitution in \mathbb{R}^2 .

From this pseudo inflation, we consider an inflation with only the four shaded hexagons in Fig. 6.6. On this tiling there is an invariant lattice (by translation of hexagons) $\Lambda =$

$\langle \{(3/2, \sqrt{3}/2), (0, \sqrt{3})\} \rangle$, generated by the center of these hexagons, using the vectors \mathbf{u} and \mathbf{v} in Fig. 6.6. We can define a constant-shape substitution with expansion matrix $L_{hh} = 2 \cdot \text{id}_{\mathbb{R}^2}$ and $F_1^{hh} = \{(0, 0), (1, 0), (0, 1), (1, -1)\}$

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ 0 & \mapsto & 0 & 2 & 1 & \mapsto & 1 & 2 & 2 & \mapsto & 2 & 2 \\ & & & 1 & & & & 1 & & & & 1, \end{array}$$

where, for convenience, we identify the hexagons in Fig. 6.5 with the letters $\{0, 1, 2\}$. The associated substitutive subshift is conjugated to the Λ -subaction of the half-hex tiling. The set K_{hh} (defined in Section 1.7) is equal to $\{(0, 0), (-1, 0), (0, -1), (-1, 1)\}$. Since this substitution has coincidences (the definition is given in Section 4.1) in all except one coordinate, it has exactly three fixed points $\bar{x}, \bar{y}, \bar{z}$ such that $\bar{x}(0, 0) = 0$, $\bar{y}(0, 0) = 1$, $\bar{z}(0, 0) = 2$ and for all $(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, $\bar{x}(m, n) = \bar{y}(m, n) = \bar{z}(m, n)$. In fact, we can characterize the maximal equicontinuous factor of the half-hex substitutive subshift.

Proposition 6.2. *The half-hex substitutive subshift $(X_{hh}, S, \mathbb{Z}^2)$ is a Toeplitz subshift. Moreover, its maximal equicontinuous factor is the odometer system $(\overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, +_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, \mathbb{Z}^2)$.*

Proof. This is a particular case of Lemma 3.11. Since the factor map $\pi_{hh} : (X_{hh}, S, \mathbb{Z}^2) \rightarrow (\overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, +_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, \mathbb{Z}^2)$ is almost 1-to-1, then $(\overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, +_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, \mathbb{Z}^2)$ is the maximal equicontinuous factor of the half-hex substitutive subshift $(X_{hh}, S, \mathbb{Z}^2)$ and $(X_{hh}, S, \mathbb{Z}^2)$ is a Toeplitz subshift. \square

In fact, by Lemma 3.11, we have that $|\pi_{hh}^{-1}(\{\overleftarrow{g}\})| \in \{1, 3\}$ for any $\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}$, and $|\pi_{hh}^{-1}(\{\overleftarrow{g}\})| = 3$ if and only if $\overleftarrow{g} \in \mathcal{O}(\overleftarrow{0}, \mathbb{Z}^2)$.

We could show, using the rotational symmetries of the substitution generated by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ that $\mathbb{Z}/6\mathbb{Z}$ can embed in $\vec{N}(X_{hh}, S, \mathbb{Z}^2)$. But, as shown in Theorem 6.3 these are not the only isomorphisms this tiling presents.

Theorem 6.3. *The normalizer semigroup of the half-hex substitutive subshift is a group and it is isomorphic to $\mathbb{Z}^2 \rtimes GL(2, \mathbb{Z})$. Moreover, $\text{Aut}(X_{hh}, S, \mathbb{Z}^2)$ is equal to the shift group $\langle S \rangle$.*

Proof. First we will prove that $\text{End}(X_{hh}, S, \mathbb{Z}^2) = \langle S \rangle$. In fact, since the factor map $\pi_{hh} : (X_{hh}, S, \mathbb{Z}^2) \rightarrow (\overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, +_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, \mathbb{Z}^2)$ is almost 1-to-1, then the semigroup homomorphism $\hat{\pi}_{hh} : \text{End}(X_{hh}, S, \mathbb{Z}^2) \rightarrow \text{End}(\overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, +_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})}, \mathbb{Z}^2)$ is injective (Lemma 1.6). Now, by Lemma 1.7, for any endomorphism ϕ , we have that

$$\{\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})} : |\pi_{hh}^{-1}(\{\overleftarrow{g}\})| = 3\} = \{\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^2}_{(2^n \mathbb{Z} \times 2^n \mathbb{Z})} : |\pi_{hh}^{-1}(\{\overleftarrow{g}\})| = 3\} + \hat{\pi}_{hh}(\phi).$$

Since $\{\overleftarrow{g} \in \overleftarrow{\mathbb{Z}^2}_{(2^n\mathbb{Z} \times 2^n\mathbb{Z})} : |\pi_{hh}^{-1}(\{\overleftarrow{g}\})| = 3\}$ is the orbit $\mathcal{O}(\overleftarrow{0}, \mathbb{Z}^2)$, we have that $\hat{\pi}_{hh}(\phi)$ is in $\mathcal{O}(\overleftarrow{0}, \mathbb{Z}^2)$. Moreover, for any $\mathbf{n} \in \mathbb{Z}^2$, $\hat{\pi}_{hh}(S^n)$ is equal to $[\mathbf{n} \pmod{2^n\mathbb{Z} \times 2^n\mathbb{Z}}]$ which is in $\mathcal{O}(\overleftarrow{0}, \mathbb{Z}^2)$. By injectivity of $\hat{\pi}_{hh}$ we conclude that $\text{End}(X_{hh}, S, \mathbb{Z}^2) = \langle S \rangle$. In particular any endomorphism of $(X_{hh}, S, \mathbb{Z}^2)$ is an automorphism. Hence $(X_{hh}, S, \mathbb{Z}^2)$ is a coalescent system.

Now, for homomorphisms.

Claim 2. For any $(m, n) \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2$, the letter in $\overline{x}(m, n)$ (hence $\overline{y}(m, n)$ and $\overline{z}(m, n)$) only depend on the parity of the coordinates.

Proof. Let $(m, n) \in \mathbb{Z}^2 \setminus 2\mathbb{Z}^2$. We can write $(m, n) = 2(a, b) + (f_1, f_2)$ with $(a, b) \in \mathbb{Z}^2$ and $(f_1, f_2) \in F_1^{hh} \setminus \{(0, 0)\}$. Since \overleftarrow{x} is a fixed point of ζ_{hh} we have that

$$\overleftarrow{x}(m, n) = \zeta_{hh}(x(a, b))_{(f_1, f_2)},$$

which only depend on (f_1, f_2) by the very definition of ζ_{hh} . \square

In particular we get that

$$\begin{aligned} (m, n) \equiv (1, 0) \pmod{2} &\implies \overline{x}(m, n) = \overline{y}(m, n) = \overline{z}(m, n) = 2 \\ (m, n) \equiv (0, 1) \pmod{2} &\implies \overline{x}(m, n) = \overline{y}(m, n) = \overline{z}(m, n) = 0 \\ (m, n) \equiv (1, 1) \pmod{2} &\implies \overline{x}(m, n) = \overline{y}(m, n) = \overline{z}(m, n) = 1. \end{aligned}$$

Let $\tau : \mathbb{Z}^2/2\mathbb{Z} \times 2\mathbb{Z} \setminus \{(0, 0)\} \rightarrow \mathcal{A}$ given by above equations. It is important to note that τ is a bijection.

Claim 3. Let $(m, n) \in \mathbb{Z}^2$ with $\text{gcd}(m, n) = 1$. Then for any $k \in \mathbb{Z}$ we have that $\overline{x}(km, kn) = \overline{x}(m, n)$ (and the same for $\overline{y}, \overline{z}$).

Proof of the Claim. Since the sign does not change the parity of the coordinates (m, n) , the statement is true for $k = 1$ and $k = -1$. We only need to prove the claim for $k > 0$. Suppose that for any $1 \leq k' < k$ the statement is true. We separate the proof in two cases:

1. If k is even, we write $k = 2j$. Then $\overline{x}(km, kn) = \overline{x}((2j)m, (2j)n) = \zeta_{hh}(\overline{x})((2jm), 2jn) = \zeta_{hh}(\overline{x}(jm, jn))_{(0,0)} = \overline{x}(jm, jn)$, where the last equality is by the definition of the substitution. We conclude that $\overline{x}(km, kn) = \overline{x}(m, n)$.
2. If k is odd, we write $k = 2j + 1$. Using the Euclidean division we have that

$$\begin{aligned} km &= (2j + 1)m = 2\bar{a} + g_1, & m &= 2a + f_1 \\ kn &= (2j + 1)n = 2\bar{b} + g_2, & n &= 2b + f_2, \end{aligned}$$

with $a, b, \bar{a}, \bar{b} \in \mathbb{Z}^2$ and $(f_1, f_2), (g_1, g_2) \in F_1^{hh}$. Since this decomposition is unique, then $\bar{a} = jm + a$, $\bar{b} = jn + b$, $f_1 = g_1$ and $f_2 = g_2$. By the definition of the substitution we conclude that $\overline{x}(km, kn) = \zeta_{hh}(\overline{x}(2\bar{a} + f_1, 2\bar{b} + f_2)) = \zeta_{hh}(\overline{x}(2\bar{a}, 2\bar{b}))_{(f_1, f_2)} = \overline{x}(m, n)$.

□

Now, note that if $V = \{(m, n) \in \mathbb{Z}^2 : \gcd(m, n) = 1\}$ and $M \in GL(2, \mathbb{Z})$, then $MV = V$. In fact, since M commutes with $2 \cdot \text{id}_{\mathbb{R}^2}$, the matrix M maps the $2\mathbb{Z}^2$ cosets onto $2\mathbb{Z}^2$ cosets. Moreover, any matrix $M \in GL(2, \mathbb{Z})$ induces a bijection in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ via one of the following matrices in $GL(2, \mathbb{Z}/2\mathbb{Z})$:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Each of these matrices define a permutation τ_1^M on $\mathbb{Z}^2/2\mathbb{Z} \times 2\mathbb{Z} \setminus \{(0, 0)\}$. Identifying $(1, -1)$ in F_1^{hh} with $(1, 1) \pmod{2}$, for any matrix $M \in GL(2, \mathbb{Z})$, we have a unique permutation τ_2^M on the alphabet $\mathcal{A} = \{0, 1, 2\}$ given by $\tau_2^M = \tau \circ \tau_1^M \circ \tau^{-1}$. Consider the map $\phi_M : X_{hh} \rightarrow \phi(X_{hh})$ induced by the local map $\phi_M(x)_n = \tau_2^M(x_{M^{-1}n})$, for any $x \in X_{hh}$ and $n = (m, n) \in \mathbb{Z}^2$. We prove that ϕ_M is a homomorphism on $(X_{hh}, S, \mathbb{Z}^2)$ associated with M . Indeed, by Claim 2, since τ is a bijection we have that for any $(m, n) \in V$, if $(m, n) \equiv (f_1, f_2) \pmod{2}$, then

$$\bar{x}_{M(m,n)} = \tau(M(f_1, f_2)) = \tau \circ \tau_1^M(f_1, f_2) = \tau \circ \tau_1^M \circ \tau^{-1}((m, n) \pmod{2}) = \tau_2^M(\bar{x}(m, n)).$$

Then, by Claim 3 we get that the image of a fixed point of ζ_{hh} via ϕ_M is a fixed point of ζ_{hh} . Now, by minimality of $(X_{hh}, S, \mathbb{Z}^2)$, we conclude that $\phi_M(X_{hh}) = X_{hh}$ for any $M \in GL(2, \mathbb{Z})$. So ϕ_M is a homomorphism onto $(X_{hh}, S, \mathbb{Z}^2)$ associated with M . Note that for any $(m, n) \in \mathbb{Z}^2$ we have that $\phi_M(S^{(m,n)}) = S^{M(m,n)}\phi_M$. If ϕ_M is the identity in X_{hh} , this would imply that $S^{(m,n)} = S^{M(m,n)}$ for any $(m, n) \in \mathbb{Z}^2$, i.e., $M = \text{id}$. This implies ϕ_M is a nontrivial isomorphism.

Hence, any matrix $M \in GL(2, \mathbb{Z})$ has a homomorphism ϕ_M induced by a letter-to-letter local map. Since $(X_{hh}, S, \mathbb{Z}^2)$ is a coalescent system, by Proposition 1.5 the normalizer semigroup $N(X_{hh}, S, \mathbb{Z}^2)$ is a group. Furthermore, for any $M_1, M_2 \in GL(2, \mathbb{Z})$ we have that $\phi_{M_1 M_2} = \phi_{M_1} \circ \phi_{M_2}$, so $\{\phi_M : M \in GL(2, \mathbb{Z})\}$ is a subgroup in $N(X_{hh}, S, \mathbb{Z}^2)$. This defines a group homomorphism from $GL(2, \mathbb{Z})$ to $N(X_{hh}, S, \mathbb{Z}^2)$, so using the exact sequence (1.4) we conclude that $N(X_{hh}, S, \mathbb{Z}^2) \cong \mathbb{Z}^2 \rtimes GL(2, \mathbb{Z})$.

□

Perspectives

We present here some perspectives that remain open after the results of this thesis.

On non-deterministic directions of multidimensional subshifts

Beyond polytope substitutions

Until now, we didn't find an aperiodic d -dimensional primitive constant-shape substitution with less than d linearly independent nondeterministic directions. In the case of aperiodic primitive block substitutions, it can be easily proved that this hypothesis is true. Moreover, given the result in [63] for the two-dimensional case, by Theorem 5.2, the hypothesis is true for all of the cases where $\text{conv}(T_\zeta)$ does not have two parallel edges. In a private communication, P. Guillon [62] mentioned this result is already proved for higher dimensions, but nowhere published. This implies, we only have to deal in the case $\text{conv}(T_\zeta)$ has two parallel $(d - 1)$ -dimensional faces.

Another open problem is the study of the normalizer semigroup for nonpolytope substitutions, i.e., where the convex hull of the digit tile is nonpolytope. Using the description of the convex hull of the digit tile given in [109] on the two-dimensional case it may be possible to obtain similar results for the normalizer semigroup and symmetry semigroup. Nevertheless, until now there are no good descriptions of the convex hull of the digit tile for higher dimensions.

Motivated by the classification of full-shifts, in [64] was introduced the notion of *stabilized automorphism group* of a topological dynamical system, which is the group of self-homeomorphisms commuting with a power of T . The authors could distinguished, up to isomorphism, various stabilized automorphism groups of non-trivial mixing shift of finite type. Moreover, in the class of fullshifts, they proved if the stabilized automorphism group of the fullshift on n and m letters are isomorphic, then n and m have the same number of distinct prime divisors. In [104], S. Schmieding studied the relation between topological entropy and the stabilized automorphism group. Moreover, he introduced a certain kind of entropy (in fact a whole family of entropies) for groups which he called *local \mathcal{P} entropy*.

For \mathbb{Z}^d -actions we can define the stabilized automorphism group of a topological dynamical system as the following. Let $M \in \mathcal{M}(d, \mathbb{Z})$ be an invertible integer matrix with

$|\det(M)| > 1$. We define $\text{Aut}_M(X, T, \mathbb{Z}^d)$ as the automorphism group of the system $(X, T|_{M\mathbb{Z}^d}, M\mathbb{Z}^d)$. The stabilized automorphism group $\text{Aut}^{(\infty)}(X, T, \mathbb{Z}^d)$ is then, the collection of these automorphism groups, i.e., $\text{Aut}^{(\infty)}(X, T, \mathbb{Z}^d) = \bigcup_{\substack{M \in \mathcal{M}(d, \mathbb{Z}) \\ |\det(M)| > 1}} \text{Aut}_M(X, T, \mathbb{Z}^d)$.

Note that if $(X_\zeta, S, \mathbb{Z}^d)$ is a substitutive subshift, and $M = L_\zeta$, then, by the recognizability property, $(\zeta(X_\zeta), S|_M, M\mathbb{Z}^d)$ is isomorphic to $(X_\zeta, S, \mathbb{Z}^d)$.

- Question.**
1. *Does any aperiodic bijective on extremities primitive constant-shape substitutions have d linearly independent nondeterministic directions?*
 2. *What can be said about the normalizer semigroup and the symmetry semigroup for nonpolytope substitutions?*
 3. *What can be said about the subgroups of $GL(d, \mathbb{Z})$ that are the symmetry semigroup of substitutive subshifts?*
 4. *What can be said about the stabilized automorphism group of substitutive subshifts? Moreover, substitutive subshifts can be classify by its stabilized automorphism group?*

Realization results of nondeterministic directions for minimal actions

The work in this thesis corresponds to the first examples of realization results about the set of nondeterministic directions for minimal actions. In [17] it was proved that for any compact set of \mathbb{S}^1 that is not a singleton containing one line with irrational slope can be realized as the set of nonexpansive directions of a \mathbb{Z}^2 -action, and the singleton case was after proved by M. Hochmann in [68]. If aperiodic bijective on extremities primitive constant-shape substitutions have d linearly independent nondeterministic directions, then we cannot obtain a unique nondeterministic direction with these substitutions, so we will need to use other types of substitutions, or other type of subshifts (such as Toeplitz sequences) to obtain other realization results with minimal subshifts.

- Question.**
1. *Is there an aperiodic primitive constant-shape substitution with a unique nondeterministic direction?*
 2. *Is there an aperiodic primitive constant-shape substitution such that its set of nondeterministic directions is homeomorphic to a Cantor set?*

Recognizability property of constant-shape substitutions

As mentioned in Section 3.1, the recognizability property is a combinatorial one, useful to prove other properties satisfying the constant-shape substitutions. In this thesis we proved aperiodic symbolic factors of aperiodic primitive constant-shape substitutions are recognizable. This left open the following questions:

- Question.** 1. *Is there a recognizability property satisfied by periodic infinite symbolic factors of a constant-shape substitution? B. Solomyak [107] already proved this result for primitive constant-shape substitutions. An idea would be to see if we can extend this proof for symbolic factors of substitutive subshifts. The answer to this question will help to understand multidimensional substitutive subshifts and their topological Cantor factors.*
2. *Do non-primitive constant-shape substitutions producing an aperiodic subshift satisfy a recognizability property? In the one-dimensional case this property is true [15].*
3. *In the non-minimal case, are the constant-shape substitutions recognizable for aperiodic points as proved in [14] for the one-dimensional case?*

To a Cobham theorem for constant-shape substitutions

An open question of this thesis is how different two substitutions producing the same subshift but with different expanding matrix and/or different fundamental domains can be. For example, if a substitution is defined with a diagonal matrix, but with a triangle support, is there any square substitution (diagonal matrix and the standard square support) producing the same shift?

In [51] it was proved that if ζ_1, ζ_2 are two aperiodic primitive constant-length substitutions, and $(X_{\zeta_2}, S, \mathbb{Z})$ is a symbolic factor of $(X_{\zeta_1}, S, \mathbb{Z})$, then their lengths have a common power (greater than 1). This is still true for multidimensional substitutions with expansion matrix equal to a multiple of the identity and the standard d -dimensional cubic support [46], but there is no generalized version for all constant-shape substitutions. Note that by Theorem 3.22, if ζ_1, ζ_2 are two aperiodic primitive constant-shape substitutions such that $(X_{\zeta_2}, S, \mathbb{Z}^d)$ is a symbolic factor of $(X_{\zeta_1}, S, \mathbb{Z}^d)$, then there is another constant-shape substitution ζ_3 with the same expansion matrix and support of ζ_1 such that $(X_{\zeta_2}, S, \mathbb{Z}^d)$ and $(X_{\zeta_3}, S, \mathbb{Z}^d)$ are conjugate. However, this does not imply combinatorial conditions for ζ_2 . See [50, Section 8.4] for an example of a non constant-length substitution which is conjugate to a constant-length one.

Moreover, it is known that constant-length substitutions are strongly related to *automatic sequences*. Cobham showed [27] that automatic sequences are exactly letter-to-letter projection of fixed points of constant-length substitutions. Automatic sequences are generated by finite automatas, which are one of the most basic models of computation and they have a large number of interesting connections with number theory, such as transcendence theory in positive characteristic (see for example [3]), and expansion in integer bases [1].

- Question.** 1. *Is there a version of Cobham's theorem for constant-shape substitutions, regarding the shape of their supports? Since in the one-dimensional case, substitutions are defined using only intervals as supports of them, there is no known version considering also the geometry of the supports of substitutions.*

2. *Are the letter-to-letter codings of all constant-shape substitutions multidimensional automatic sequences? In particular, are all constant-shape substitutions generated by a DFA?*

Topological factors of constant-shape substitutions

In the one-dimensional case, substitutions (and its topological factors on Cantor sets) are included in a broader class of systems called *finite rank systems*. In [43] it was proved that finite rank systems are either expansive or equicontinuous. This classification result is no longer true in the multidimensional framework (Example 4.3 is an example of a constant-shape substitution with a Cantor factor which neither expansive neither equicontinuous).

In [48] it was proved that expansive Cantor factors of substitutions are conjugate to substitutions, and equicontinuous Cantor factors are conjugate to an odometer. Also, it is shown in [48] that aperiodic substitutions have a finite number of aperiodic Cantor factors up to conjugacy. In this thesis, it was proved that aperiodic symbolic factors of aperiodic primitive constant-shape substitutions are conjugate to aperiodic primitive constant-shape substitutions Theorem 3.22, extending the result proved in [91].

It can be proved using Theorem 4.1 and Theorem 3.22 that if we only consider reduced substitutions, for a substitutive subshift which is conjugate to an aperiodic primitive reduced substitution, there is a finite number of aperiodic symbolic factors, which are conjugate to a substitutive subshift given by an aperiodic primitive reduced substitution. But this does not cover all of the cases. Moreover, it is known the result is true for linearly repetitive primitive constant-shape substitutions as proved in [45].

- Question.**
1. *Substitutive subshifts have a finite number of aperiodic symbolic factors?*
 2. *Does there is a classification theorem for topological factors of constant-shape substitutions, as the one proved in [48] for one-dimensional substitutions?*
 3. *Are all substitutive subshifts coalescent?*
 4. *Is the automorphism group of any substitutive subshift virtually generated by the shift action?*
 5. *Is there a substitutive subshift with a nontrivial topological factor with a connected phase space? Until now, even in the one-dimensional case this question is open.*

Decidability problems on constant-shape substitutions

The *decidability* of a problem corresponds to the existence of an algorithm to give a positive (or negative) answer to the problem. Since substitutions are defined by finite objects, it is natural to ask about the decidability of some properties about them.

F. Durand [49] gives a computable upper bound for the constant of recognizability of one-dimensional primitive constant-length substitutions. In [50] the decidability of the conjugacy problem between one-dimensional substitutions was proved, in contrast to the undecidability of the existence of a conjugacy between two given *subshifts of finite type* ([113, 75]). Recently in [11] the decidability was proved of some properties of the shift space generated by one-dimensional substitutions such as aperiodicity, recognizability, irreducibility, or minimality. These results give rise to the following questions.

- Question.**
1. *Is the aperiodicity of constant-shape substitution decidable?*
 2. *Does there is a computable bound for the recognizability radius of constant-shape substitutions?*
 3. *Does there exists an algorithm to compute the symmetry semigroup of substitutive subshifts?*
 4. *Is it decidable the problem of, given a d -dimensional aperiodic primitive substitution, if the set of nondeterministic directions is equal to $\mathbb{S}^{(d-1)}$?*

On minimal multidimensional S -adic subshifts

All the previous problems can be expressed for multidimensional minimal S -adic subshifts. In the one-dimensional case, this class of minimal subshifts is one of the most natural containing minimal subshifts of sublinear complexity, but it is much broader as was shown in [39, 41]. The class contains several well studied systems, such as substitutive subshifts, symbolic codings of interval exchange transformations, dendric subshifts, and some Toeplitz sequences.

In the multidimensional setting, we can consider the case where the morphisms $\sigma_n : \mathcal{A}_{n+1} \rightarrow \mathcal{A}_n^{F_n^1}$ are given by a constant-shape morphism. In this context the following are open questions.

- Question.**
1. *Is there a recognizability property for minimal multidimensional constant-shape S -adic systems?*
 2. *Is there an analogue definition of finite rank systems in the multidimensional framework?*
 3. *What rigidity properties do homomorphisms satisfy in the context of minimal multidimensional constant-shape S -adic systems? Moreover, if the morphisms are bijective, it is possible to extend Theorem 5.17 in this context?*

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On the study of homomorphisms between \mathbb{Z}^d -topological dynamical systems

We study in this thesis homomorphisms between \mathbb{Z}^d -symbolic dynamical systems generated by constant-shape substitutions. This notion extends the classical dynamical one of morphism like factor and conjugacy. Isomorphisms are conjugacies of \mathbb{Z}^d -actions, up to $GL(d, \mathbb{Z})$ -transformations. We show the class of substitutive subshifts is stable under aperiodic symbolic factors. We prove any measurable factor induces a continuous one. We also get strong restrictions on the homomorphisms of a generic family of substitutive subshifts: they are invertible, the normalizer group is virtually generated by the shift action and its quotient by the automorphisms is limited by the digit tile of the substitution. We prove this by describing their set of nondeterministic directions. Finally, we show the optimality of the hypotheses by exhibiting an example of a minimal subshift with an infinite symmetry group.

Keywords: Symbolic dynamics, substitutive subshift, homomorphism, automorphism group, normalizer group, symmetry group, digit tile, nonexpansive half-space.

Sur l'étude des homomorphismes entre \mathbb{Z}^d -systèmes dynamiques topologiques

Nous étudions les homomorphismes entre des \mathbb{Z}^d -systèmes symboliques engendré par des substitutions de forme constante. Cette notion étend les concepts de facteur et conjugaison. Les isomorphismes sont alors des conjugaisons de \mathbb{Z}^d -actions, à une transformation de $GL(d, \mathbb{Z})$ près. Nous montrons que la classe de sous-shifts substitutifs est stable par les facteur symbolique aperiodique. Nous prouvons que tout facteur mesuré induit un facteur continu. Nous obtenons également des restrictions fortes des homomorphismes d'une famille générique des sous-shifts substitutifs: ils son inversibles, le normalisateur est virtuellement engendré par l'action du shift et son quotient par des automorphismes est limité par le digit tile de la substitution. Nous prouvons ceci en décrivant leurs ensembles des directions non déterministes. Finalement, nous prouvons que nous hypothèses sont optimales en donnant un exemple d'un sous-shift substitutif avec un groupe de symétrie infini.

Mots-clés: dynamique symbolique, sous-shift substitutif, homomorphisme, groupe de automorphisme, normalisateur, groupe de symétrie, digit tile, demi-espace nonexpansif.