Invariants of the Legendrian lift of an exact Lagrangian submanifold in the circular contactization of a Liouville manifold

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Thèse de doctorat de

NANTES UNIVERSITÉ

École Doctorale N° 601
Mathématiques et Sciences et Technologies de l'Information et de la Communication
Spécialité : *Mathématiques et leurs Interactions*

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Invariants du relevé Legendrien d'une sous-variété Lagrangienne exacte dans la contactisation circulaire d'une variété de Liouville

Thèse présentée et soutenue à Nantes Université, le 16 septembre 2022
Unité de recherche : Laboratoire de Mathématiques Jean Leray (LMJL)

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Remerciements

Ce mémoire de thèse vient clore une longue aventure qui a finalement été tout sauf solitaire. J'aimerais remercier ici toutes les personnes qui y ont participé, et qui l'ont rendu si spéciale.

Évidemment le directeur de thèse joue un rôle essentiel durant ces trois années. Merci Paolo de m'avoir fait découvrir de belles mathématiques, de m'avoir introduit dans la communauté des chercheurs en topologie symplectique, et de m'avoir emmené avec toi jusqu'en Suède où j'ai pu travailler dans un cadre exceptionnel. Merci surtout de m'avoir patiemment écouté, semaine après semaine, sombrer un peu plus dans l'algèbre sans d'autres plaintes que tes légendaires "Povero Paolo". Merci Vincent pour ta disponibilité et ta bienveillance à chaque fois que j'en avais besoin ; merci aussi de nous avoir aidé à surmonter les problèmes administratifs. Je remercie Mihai Damian et Alexandru Oancea d'avoir accepté d'être rapporteurs pour cette thèse. Merci à Claire Amiot, Frédéric Bourgeois, Georgios Dimitroglou Rizell et Agnès Gadbled d'avoir accepté d'être membre du jury.

J'aimerais consacrer quelques lignes aux chercheurs que j'ai rencontré durant ces trois années et qui ont eu un rôle dans cette thèse, à commencer par les rapporteurs et membres du jury. Merci Georgios pour ton intérêt envers mon problème de thèse ; merci d'avoir toujours pris le temps de répondre à mes questions malgré ton emploi du temps surchargé ; merci enfin pour tous les moments moins formels passés à Uppsala et à Nantes. Merci Alexandru de t'être intéressé à mon travail de thèse et de m'avoir plusieurs fois écouté en parler. Merci Mihai pour les remarques qui ont permis d'améliorer le mémoire. Merci Frédéric pour le regard bienveillant que tu as porté sur mon travail, en particulier pendant les réunions du CSI. Merci Agnès pour ton écoute des doctorants pendant les conférences. Passons maintenant aux chercheurs du laboratoire Jean Leray. Merci à Baptiste Chantraine pour les très nombreuses discussions que nous avons eues ; merci en particulier de m'avoir encouragé à étudier la localisation des catégories A_∞, qui joue finalement un rôle très important dans ma thèse. Je remercie aussi Marco Golla pour sa bienveillance et son soutien au laboratoire, Erwan Brugallé pour sa bonne humeur aux déjeuners du séminaire TGA, François Laudenbach pour ses nombreux conseils et sa disponibilité, Stéphane Guillermou pour avoir essayé de m'expliquer à quoi servent les faisceaux en géométrie de contact, Gilles Carron pour l'attention portée à mon CSI, Gabriel Rivière pour la gentillesse qu'il a eu de répondre aux questions un peu "annexes" que Antoine et moi nous sommes posés. Merci aux algébristes du LMJL, en particulier Friedrich Wagemann, Salim Rivière et Johan Leray pour m'avoir donné une vision plus large des objets algébriques que je manipulais. Enfin, je remercie Tobias Ekholm et Vivek Shende pour les conversations enrichissantes que j'ai eu avec eux lorsque j'étais à l'institut Mittag-Leffler.

Durant ces trois années, je ne peux compter le nombre de fois où j'ai dû demander l'aide des secrétaires du laboratoire. Merci à Brigitte, Alexandra, Caroline, Stéphanie, Anaïs, Béatrice pour leur gentillesse et leur incroyable efficacité. Merci à Eric et Saïd pour leur aide lors des soucis informatiques. Merci à Anh et Claude, grâce à qui nous avons accès aux livres et articles du CRDM. Merci enfin aux personnels d'entretien qui nous permettent de travailler dans des locaux propres.

J'aimerais maintenant me livrer aux traditionnels remerciements des doctorants, post-docs et ATER. Tout au long de l'aventure que constitue la thèse, il est évident que ces derniers jouent un rôle très particulier. Ils contribuent à la formidable ambiance qui règne chaque jour au rez-de-chaussée (sous-sol ?) du LMJL, et j'aurais du mal à dire à quel point leur présence a été importante pour moi. Il me semble difficile de ne pas commencer par remercier mon inimitable co-bureau. Merci Antoine d'avoir été un soutien sans faille durant ces trois longues années. Il est probablement impossible de chiffrer le nombre d'heures qu'on a passé à discuter de maths, de cours, de la recherche, de la vie. Tout comme il est impossible de chiffrer le nombre de cafés pris ensemble. J'espère que l'expression "être dans le bidual" aura une postérité dans le labo. Merci aussi à Céline pour avoir magnifiquement décoré notre bureau et avoir amené un peu de folie dans nos soirées entre doctorants. Merci Claire et Arthur d'avoir été mes premiers co-bureaux et de m'avoir si bien accueilli ; vous m'avez tout de suite fait me sentir à l'aise dans ce nouvel environnement. Merci Solène de m'avoir laissé ton bureau et de m'avoir appris que Hanabi est un jeu plus addictif qu'il n'y paraît. Merci Matthieu pour m'avoir appris à jouer à la tarouinche (j'espère que ça s'écrit comme ça) et pour ton esprit critique sur le monde de la recherche. Merci Caroline pour ta bonne humeur et tes histoires invraisemblables du collège où tu enseignais. Merci Hélène pour ton accueil dans ton bureau, c'était toujours très agréable d'arriver le matin et de discuter avec toi (avec seulement la lumière du lampadaire). Merci Zeinab pour tous ces moments où tu nous as fait rire, en particulier autour de la machine à café (on se demande toujours qui a bien pu arrêter le nettoyage de la machine en plein milieu) ou de ton fameux café libanais. Merci Maha pour tous tes conseils, que ce soit pour la thèse ou les cours, c'était toujours très agréable de discuter avec toi. Merci Côme de m'avoir patiemment accompagné dans la découverte du monde de l'homologie de contact Legendrienne, j'espère qu'on continuera longtemps à discuter de maths ; merci surtout pour la phrase introductive de ta thèse dont je suis très jaloux. Merci Germain pour m'avoir tant rassuré sur le déroulement de la thèse, sur les cours à donner et les corrections de copies ; merci pour toutes les discussions qu'on a pu avoir, j'avais toujours envie de venir m'asseoir quelques minutes (ou plus) sur le fauteuil de ton bureau pour échanger avec toi. Merci Thomas G., figure incontournable du labo, pour tous les conseils que tu m'as donné et pour avoir si bien intégré tous les nouveaux doctorants. Merci Mohamad pour ta gentillesse et ton calme rassurant. Merci Meissa pour ton humour et tes anecdotes incroyables. Merci Trung pour les chansons qui parvenaient parfois jusqu'à mon bureau. Merci Mael d'avoir partagé le bureau avec Antoine et moi, merci pour nos discussions souvent drôles sur les copies de nos étudiants. Merci Anh pour ta gentillesse, pour toutes nos discussions aux déjeuners du séminaire TGA ou en dehors, pour t'être confronté avec moi aux aspects administratifs de la thèse ; merci enfin de nous avoir fait découvrir le plumfoot, j'espère que cette tradition restera au labo. Merci Alexandre L. pour tes énigmes mathématiques, ta connaissance encyclopédique des jeux de société et surtout pour ta gentillesse qui rendaient les pauses dans le labo et les soirées en dehors du labo très agréables. Merci Fabien pour ta capacité à t'intéresser à chacun des doctorants du LMJL, je pense que chacun de nous apprécie sincèrement de discuter avec toi ; merci pour les falafels du mercredi soir où on discutait de tout et n'importe quoi, merci surtout de nous avoir fait penser à payer l'addition. Merci Samuel pour avoir imprimé (en 3D !) des objets mathématiques très bizarres, et pour tes jolis exposés au séminaire des doctorants. Merci Anthony pour avoir partagé avec moi tes réflexions sur la thèse, la recherche et la science en général tout au long de ces trois années ; merci aussi de m'avoir fait croire des choses invraisemblables. Merci Flavien pour ta gentillesse, ton humour et ta passion pour Civilization (6 ?). Merci Adri$x$n avec $x \in \{a, e\} \setminus \{a\}$ pour avoir fait vivre le groupe des doctorants en période de Covid, merci pour nos discussions, mathématiques ou non, et pour tes têtes mémorables au plumfoot. Merci aussi à Lisa pour savoir si bien imiter les accents en anglais. Merci Silvère pour avoir si souvent accepté de discuter d'algèbre avec moi ; merci aussi de m'avoir souvent accueilli dans ton bureau lorsque j'étais épuisé par l'écriture de la thèse et que j'avais besoin de parler d'autre chose. Merci Jean et Lucas D. pour les moments agréables passés autour des pots de thèse. Merci Ludovic M. pour ta participation assidue au séminaire des doctorants et pour les discus-

sions qu'on a pu avoir dans notre bureau. Merci Hai pour m'avoir offert des gâteaux très bons mais à la texture surprenante. Merci Ludovic G. pour nous avoir motivé (forcé ?) à aller à la piscine, et pour nous avoir fait faire des exercices aussi improbables que le grand chien et le petit chien. Merci Hanine pour ta gentillesse et ta capacité à si bien expliquer des mathématiques difficiles. Merci Michele pour avoir réfléchi avec moi sur la preuve du théorème de Bennequin. Merci Fabrice pour ton avis éclairé sur l'enseignement et le monde de la recherche, et pour ton aide précieuse avec l'agreg. Merci Destin pour m'avoir appris ce qu'était un réseau de neurones. Merci Charbella pour ton sourire communicatif. Merci Khaled pour ta bonne humeur à chaque fois que l'on se croise dans les couloirs. Merci Gaspard pour ton investissement dans la vie du labo, discuter avec toi est toujours très agréable. Merci Gurvan pour ta sérénité à toute épreuve (qui te donne un style inimitable au baby foot), merci pour m'avoir embarqué dans un problème de dénombrement lorsque j'avais besoin de me changer les idées. Merci Thomas C. pour ton humour, pour nous avoir permis de jouer au baby foot avec une balle dédicacée par un joueur célèbre et pour tes prouesses au plumfoot (j'espère que vous récupérerez celle qui est sur le toit un jour). Merci Damien pour ton calme et ta gentillesse, vous formez avec Thomas un sacré binôme ! Merci de m'avoir accueilli dans votre bureau lorsque je n'en avais plus, j'espère Damien que tu prendras le dessus au tableau des scores échiquéen. Merci Klervi (alias responsable du comité d'animation) pour ta bonne humeur et l'ambiance que tu mets dans le groupe des doctorants, merci aussi de m'avoir prêté ton appartement. Merci Lucile pour m'avoir encouragé dans la dernière ligne droite avant la soutenance, j'étais très content de te revoir après ces trois ans loin de l'ENS Rennes. Malgré le peu de temps que nous avons passé ensemble, merci à tous les nouveaux doctorants que j'ai rencontré : Alexandre P., Lucas M., Clément, Paul, Giovanny, Elric... Merci également Noémie pour avoir accepté de partager ton bureau à Uppsala, et pour les nombreux moments agréables passés avec Paolo et Georgios. Enfin, merci à Cyril, Ella et Simon avec qui j'étais toujours impatient de discuter lors des séminaires Nantes-Orsay.

Ces trois années de thèse auraient sans doute été complètement différentes sans la présence de mes amis et ma famille en dehors du laboratoire. Merci Jérémy pour ton calme et ton humilité qui m'ont toujours impressionné, nos trajets en bus pendant lesquels on discutait de tout et de rien me manquent. Merci Mégane pour ta gentillesse et ton humour, j'espère que j'aurai encore beaucoup d'occasions de te parler de fonctions propres. Il est clair que faire des mathématiques avec vous deux, au dernier étage de l'IRMAR pendant l'agreg, a fortement contribué à ma manière de faire des mathématiques aujourd'hui. Merci

Rudy et Roméo d'avoir été présents depuis la première année de l'ENS Rennes, pendant laquelle on faisait des dîners à Bruz village avec Camille, jusqu'à cette dernière année de thèse où l'on s'est retrouvé encore récemment sur les quais de la Seine. J'espère de tout cœur qu'on arrivera à rallonger notre liste de souvenirs dans les années qui viennent. Merci aussi à Michel, Corentin et Émeric pour tous les bons moments passés, et aussi toutes les mathématiques apprises ensemble. Merci enfin à tous mes autres camarades de l'ENS Rennes qui m'ont fait vivre quatre années formidables et enrichissantes. Merci Titouan d'avoir été le meilleur des compagnons de route lors des trajets Rennes-Nantes que nous faisions en M2, merci surtout pour ta gentillesse et nos discussions sur la science en général que je trouve toujours aussi passionnantes. Merci Raïssa, Medhi, Maryane et Karoline pour avoir formé un groupe d'amis aussi exceptionnel. Le fait d'être aussi soudé après tant d'années est un grand bonheur. Votre soutien constant durant ces trois années, ainsi que nos parties de bowling, de pétanque, de ping-pong (où nous n'avons pas toujours brillé par notre niveau), ont été très important pour moi. Je voudrais enfin remercier les formidables familles que j'ai la chance d'avoir autour de moi. Merci aux Peudevin avec qui j'ai passé d'excellents moments, que ce soit à l'escrime ou ailleurs. Merci surtout pour toute l'aide que vous nous avez apporté dans les moments difficiles. Merci aux Goullencourt avec qui j'ai passé une grande partie de ma vie, et qui m'accueillent encore aujourd'hui pour dîner lorsque je suis seul de l'autre côté de la rue. Nul doute que vous êtes une famille très spéciale pour moi. Merci aux Moreno, avec qui j'ai fêté tant d'évènements importants. C'est toujours un immense plaisir lorsqu'on se retrouve tous pour manger, présence d'Alexis oblige, une raclette. Merci aux Philippon qui n'ont cessé de me soutenir pendant ces trois années de thèse. J'ai toujours trouvé du réconfort chez vous, près de vos inimitables bébés chiens. Merci Maman et Lora pour tout l'amour que vous me donnez (et, il faut bien le dire, pour avoir préparé le pot de thèse). J'aime à penser que Janine et Papa seraient heureux de nous voir tous réunis pour cette occasion.

Enfin, pour toi qui a été près de moi pendant les moments les plus heureux et les plus difficiles, quelques lignes ne suffiront pas à exprimer tout ce que je te dois. En bref, pour m'avoir soutenu, rassuré, et fait grandir à tes côtés, merci Amandine.

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Introduction

L'homologie de contact Legendrienne est un invariant des sous-variétés Legendriennes introduit par Chekanov [19] et Eliashberg [34], qui de plus s'inscrit dans la Théorie Symplectique des Champs de [35]. L'algèbre différentielle graduée de Chekanov-Eliashberg, dont l'homologie est précisément cet invariant, a été rigoureusement définie dans la contactisation d'une variété de Liouville par Ekholm, Etnyre et Sullivan dans [28] et [26]. Son importance va au-delà des applications à la classification des Legendriennes : elle a ainsi été utilisée par Bourgeois, Ekholm et Eliashberg dans [16] pour calculer des invariants symplectiques des variétés de Weinstein, mais aussi d'une manière différente par Chantraine, Dimitroglou Rizell, Ghiggini et Golovko dans [18] afin de prouver un résultat de génération pour la catégorie de Fukaya enroulée des variétés de Weinstein.

La motivation pour cette thèse est l'étude de l'homologie de contact Legendrienne dans les variétés de contact munies d'un remplissage sous-critique et dans les variétés dites de Boothby-Wang, qui sont les espaces totaux des fibrés de préquantifications. Cette étude a été menée en dimension 3 par Ekholm et Ng dans [33] pour le cas sous-critiquement remplissable, et par Sabloff dans [56] pour le cas Boothby-Wang. L'importance des espaces du premier type vient du fait que toute variété de Weinstein est obtenue à partir d'une variété sous-critique (de la forme $\mathbf{C} \times P$ pour une certaine variété de Weinstein P) par attachement d'anses le long de sous-variétés Legendriennes dans son bord à l'infini. L'importance du second type d'espaces vient d'un théorème de Donaldson dans [24], qui énonce que toute variété symplectique entière (X, ω) admet une sous-variété symplectique $D \subset X$ de codimension 2 telle que $X \setminus D$ est une variété de Liouville dont le bord à l'infini est une variété de Boothby-Wang. La première étape pour étudier les deux situations présentées ci-dessus est de s'intéresser à l'homologie de contact Legendrienne dans la contactisation circulaire $S^1 \times P$ d'une variété de Liouville P. En effet, les variétés de contact remplissables par une variété sous-critique et les espaces de Boothby-Wang peuvent être vus comme des compactifications de contactisations circulaires.

Le chapitre 1 contient les définitions des objets algébriques, ainsi que quelques résultats de base, qu'on utilise dans la thèse. Dans le chapitre 2, on utilise les colimites homotopiques de catégories A_∞ introduites par Ganatra, Pardon et Shende dans [37] et

[38] pour définir le tore d'application associé à une quasi-autoéquivalence de catégorie A_∞. On démontre ensuite les Théorèmes A et B, qui sont les contributions algébriques principales de la thèse. Ils permettent de calculer ce tore d'application dans certaines situations. Notons que ces théorèmes ne nécessitent pas de contexte Legendrien, et peuvent être vus comme des résultats algébriques généraux sur les catégories A_∞. Le chapitre 3 est consacré à l'algèbre différentielle graduée (DG) de Chekanov-Eliashberg d'une Legendrienne dans une variété munie d'une forme de contact hypertendue. En adaptant des arguments de [28], et en utilisant la notion de cylindre d'application comme dans [53], on prouve le Théorème C, qui explicite le comportement de cette algèbre DG sous certaines modifications de la Legendrienne et de la structure presque complexe. Dans le chapitre 4, on étudie les relations entre une sous-variété Lagrangienne exacte L dans une variété de Liouville (P, λ) et un relevé Legendrien Λ° de L dans la contactisation circulaire $(S^1 \times P, d\theta - \lambda)$. Contrairement au cas de la contactisation standard $\mathbf{R} \times P$, chaque point d'une Legendrienne dans $S^1 \times P$ donne lieu à une infinité dénombrable de cordes de Reeb pour la forme de contact standard, et on est dans une situation dite dégénérée. Le Théorème D, qui peut être vu comme le résultat central de la thèse, relie l'algèbre A_∞ de Floer de L et l'algèbre DG de Chekanov-Eliashberg de Λ°. Ce résultat a de fortes similitudes avec la Conjecture 6.3 de Seidel dans [57], dont une preuve est esquissée par Ganatra et Maydanskiy dans l'Appendice de [16]. Notre approche est cependant différente de celles qui apparaissent dans les références citées ci-avant. En résumé, on applique une succession d'opérations géométriques à la situation initiale, et on utilise les Théorèmes A, B et C pour comprendre comme les objets algébriques correspondants sont reliés les uns aux autres.

Avant d'énoncer les résultats principaux de la thèse, on donne une introduction aux domaines dans laquelle cette dernière s'inscrit, c'est à dire la géométrie symplectique et la géométrie de contact.

Géométrie symplectique et géométrie de contact

On commence par définir et donner des exemples des objets de base en géométrie symplectique et en géométrie de contact. On présente ensuite brièvement les notions de variétés de Liouville et variétés de Weinstein.

Définitions et exemples

On renvoie à [51] (pour la partie symplectique) et [39] (pour la partie contact) pour un exposé détaillé de ces deux domaines.

Géométrie symplectique

Définition. Une variété symplectique (X, ω) est la donnée d'une variété différentielle X et d'une 2-forme ω fermée et non-dégénérée, appelée forme symplectique.

La condition de non-dégénérescence sur la forme symplectique implique qu'une variété symplectique est nécessairement de dimension paire.

Exemples. On donne quelques exemples de variétés symplectiques.

1. La 2-forme
$$\omega_0 = dx_1 \wedge dy_1 + \cdots + dx_n \wedge dy_n$$
est une forme symplectique sur $\mathbf{C}^n$.

2. La restriction de ω_0 sur la sphère unité $S^{2n-1} \subset \mathbf{C}^n$ est invariante par l'action de S^1 (par multiplication). Elle induit ainsi une forme symplectique ω_{FS} sur le quotient $S^{2n-1}/S^1 = \mathbf{CP}^{n-1}$, appelée forme de Fubini-Study.

3. Soit M une variété différentielle. Notons $\pi : T^*M \to M$ la projection. On appelle forme de Liouville sur T^*M la 1-forme λ_M définie par
$$\lambda_M\,(q,p)\,v = \langle p, D\pi(q)v\rangle$$
pour tout $p \in T_q^*M$ et $v \in T_{(q,p)}T^*M$. Alors $\omega_M = -d\lambda_M$ est une forme symplectique sur T^*M.

Définition. Soit (X_1, ω_1) et (X_2, ω_2) deux variétés symplectiques. Un symplectomorphisme $f : (X_1, \omega_1) \to (X_2, \omega_2)$ est un difféomorphisme $f : X_1 \to X_2$ qui vérifie $f^*\omega_2 = \omega_1$.

Remarque. L'application
$$\begin{array}{ccc} \mathbf{C}^n & \longrightarrow & T^*\mathbf{R}^n \\ ((x_1, y_1), \ldots, (x_n, y_n)) & \mapsto & (\mathbf{x}, \langle \mathbf{y}, \cdot \rangle) \end{array}$$
(où $\langle \cdot, \cdot \rangle$ désigne le produit scalaire usuel sur $\mathbf{R}^n$) définit un symplectomorphisme de $(\mathbf{C}^n, \omega_0)$ vers $(T^*\mathbf{R}^n, \omega_{\mathbf{R}^n})$.

Un aspect frappant de la géométrie symplectique est qu'elle n'a pas d'invariant local. Plus précisément on a le résultat suivant, appelé Théorème de Darboux.

Théorème. *Soit* (X, ω) *une variété symplectique. Tout point de* X *admet un voisinage symplectomorphe à un voisinage de* 0 *dans* $(\mathbf{C}^n, \omega_0)$.

On définit maintenant la notion de Lagrangienne.

Définition. Soit (X, ω) une variété symplectique de dimension $2n$. Une immersion $i : M \to X$ est dite isotrope si $i^*\omega = 0$. Dans ce cas, la dimension de M est nécessairement inférieure ou égale à n. Si $\dim(M) = n$, on dit que i est Lagrangienne. Enfin, une sous-variété L de X est dite Lagrangienne si L est l'image d'un plongement Lagrangien.

Exemples. On donne quelques exemples de sous-variétés Lagrangiennes.

1. Le sous-espace
$$\mathbf{R}^n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n \mid z_1, \ldots, z_n \in \mathbf{R}\}$$
et le tore $(S^1)^n$, où S^1 désigne le cercle unité dans $\mathbf{C}$, sont des sous-variétés Lagrangiennes de $(\mathbf{C}^n, \omega_0)$.

2. L'espace projectif réel
$$\mathbf{RP}^n = \{[z_1 : \ldots : z_{n+1}] \in \mathbf{CP}^n \mid z_1, \ldots, z_n \in \mathbf{R}\}$$
et le tore de Clifford
$$L_{Cl} = \{[z_1 : \ldots : z_{n+1}] \in \mathbf{CP}^n \mid |z_1| = \cdots = |z_n|\}$$
sont des sous-variétés Lagrangiennes de $(\mathbf{CP}^n, \omega_{FS})$.

3. La 0-section $0_M = \{(q, 0) \in T^*M \mid q \in M\}$ est une sous-variété Lagrangienne de (T^*M, ω_M). Plus généralement, l'image d'une 1-forme différentielle $\eta : M \to T^*M$ sur M est une sous-variété Lagrangienne de (T^*M, ω_M) (par exemple, la 0-section 0_M est l'image de la 1-forme nulle).

4. Pour tout point q_0 de M, la fibre $T^*_{q_0}M$ est une sous-variété Lagrangienne de (T^*M, ω_M). Plus généralement, si S est une sous-variété de M, alors son fibré conormal
$$L_S = \{(q, p) \in T^*M \mid q \in S \text{ et } \langle p, T_qS \rangle = 0\}$$

est une sous-variété Lagrangienne de (T^*M, ω_M) (par exemple, la fibre $T^*_{q_0}M$ est le fibré conormal du point q_0).

On termine ce paragraphe en énonçant le théorème de Weinstein sur les voisinages des Lagrangiennes.

Théorème. *Soit (X, ω) une variété symplectique, et $i : M \to X$ un plongement Lagrangien d'image L. Alors il existe un voisinage $\mathcal{U}$ de L dans X, un voisinage $\mathcal{V}$ de 0_M dans T^*M, et un symplectomorphisme $f : \mathcal{U} \to \mathcal{V}$ tel que*

$$(f \circ i)(q) = (q, 0)$$

pour tout $q \in M$.

Géométrie de contact

Définition. Une variété de contact (Y, ξ) est une variété différentielle Y munie d'un champ d'hyperplans $\xi \subset TY$, appelé structure de contact, vérifiant la condition suivante : pour tout point $a \in Y$, il existe un voisinage $\mathcal{V}$ de a dans Y, et une 1-forme α sur $\mathcal{V}$, telle que $\xi_{|\mathcal{V}} = \ker(\alpha)$ et $(d\alpha)_{|\ker(\alpha)}$ est non-dégénérée.

Une forme de contact sur une variété différentielle Y est une 1-forme sur Y dont le noyau est une structure de contact. Etant donné une structure de contact ξ sur Y, il existe une forme de contact sur Y dont le noyau est ξ si et seulement si ξ est co-orientable (c'est à dire que le fibré en droites TY/ξ est orientable).

Remarque. Notons que si f est une fonction qui ne s'annule jamais, et si $d\alpha$ est non-dégénérée sur $\ker(\alpha)$, alors $d(f\alpha)$ est non-dégénérée sur $\ker(f\alpha) = \ker(\alpha)$.

Comme en géométrie symplectique, la condition de non-dégénérescence implique que les hyperplans de la structure de contact sont nécessairement de dimension paire. Autrement dit, une variété de contact est nécessairement de dimension impaire.

Définition. Soit α une forme de contact sur une variété Y. Le champ de Reeb de α est le champ de vecteurs R_α défini par les relations

$$\iota_{R_\alpha} d\alpha = 0 \text{ et } \alpha(R_\alpha) = 1.$$

Exemples. On donne quelques exemples de variétés de contact.

1. La 1-forme
$$\alpha_0 = d\theta - (y_1 dx_1 + \cdots + y_n dx_n)$$
est de contact sur $\mathbf{R}_\theta \times \mathbf{C}^n$. Le champ de Reeb de α_0 est $R_{\alpha_0} = \partial_\theta$.

2. Considérons la 1-forme
$$\lambda_0 = \frac{1}{2}\left((y_1 dx_1 - x_1 dy_1) + \cdots + (y_n dx_n - x_n dy_n)\right)$$
sur $\mathbf{C}^n$. Alors $\alpha := (-\lambda_0)_{|S^{2n-1}}$ est une forme de contact sur la sphère unité S^{2n-1} de $\mathbf{C}^n$. Le champ de Reeb de α est donné par
$$R_\alpha\left(z_1, \ldots, z_n\right) = 2\left(iz_1, \ldots, iz_n\right).$$
De plus, la fibration de Hopf
$$\Pi_{Hopf} : S^{2n-1} \to \mathbf{CP}^{n-1}, \quad (z_1, \ldots, z_n) \mapsto [z_1 : \ldots : z_n]$$
vérifie
$$\Pi^*_{Hopf}\omega_{FS} = d\alpha.$$

3. Soit M une variété différentielle, et Σ l'une des variétés $\mathbf{R}_\theta$ ou $\mathbf{R}_\theta/\mathbf{Z}$. La 1-forme
$$\alpha_M = d\theta - \Pi^*_{T^*M}\lambda_M$$
(où $\Pi_{T^*M} : \Sigma \times T^*M \to T^*M$ est la projection) est de contact sur $\Sigma \times T^*M$. Le champ de Reeb de α_M est $R_{\alpha_M} = \partial_\theta$. La variété $(\mathbf{R} \times T^*M, \alpha_M)$ est l'espace des 1-jets de M.

4. Soit M une variété Riemannienne. La restriction de $(-\lambda_M)$ sur le fibré cotangent unitaire
$$U^*M = \{(q,p) \in T^*M \mid \|p\| = 1\}$$
est une forme de contact sur U^*M. Le champ de Reeb associé engendre le flot géodésique sur U^*M.

5. Soit M une variété différentielle. Notons $\mathbf{P}T^*M$ le projectivisé du fibré cotangent, et $\pi : \mathbf{P}T^*M \to M$ la projection. Si (q, ℓ) est un élément de $\mathbf{P}T^*M$ (c'est à dire que

ℓ est une droite dans T_q^*M), on note $H(q,\ell) \subset T_qM$ le noyau d'une forme linéaire non nulle dans ℓ (elles définissent toutes le même hyperplan) et on pose

$$\xi\,(q,\ell) := D\pi(q)^{-1}H\,(q,\ell) \subset T_{(q,\ell)}\mathbf{P}T^*M.$$

Alors $\xi = (\xi(q,\ell))_{(q,\ell)}$ est une structure de contact sur $\mathbf{P}T^*M$, qui en général n'est pas co-orientable. La variété $(\mathbf{P}T^*M, \xi)$ est l'espace des éléments de contact de M.

Définition. Soit (Y_1,ξ_1) et (Y_2,ξ_2) deux variétés de contact. Un contactomorphisme $f : (Y_1,\xi_1) \to (Y_2,\xi_2)$ est un difféomorphisme $f : Y_1 \to Y_2$ qui vérifie $f^*\xi_2 = \xi_1$. Si (Y_1,ξ_1) et (Y_2,ξ_2) sont munies de formes de contact α_1 et α_2, un contactomorphisme strict $f : (Y_1,\alpha_1) \to (Y_2,\alpha_2)$ est un contactomorphisme $f : (Y_1,\xi_1) \to (Y_2,\xi_2)$ qui satisfait $f^*\alpha_2 = \alpha_1$.

Remarque. L'application

$$\begin{array}{ccc} \mathbf{R}\times\mathbf{C}^n & \longrightarrow & \mathbf{R}\times T^*\mathbf{R}^n \\ (\theta,(x_1,y_1),\ldots,(x_n,y_n)) & \mapsto & (\theta,(\mathbf{x},\langle\mathbf{y},\cdot\rangle)) \end{array}$$

(où $\langle\cdot,\cdot\rangle$ désigne le produit scalaire usuel sur $\mathbf{R}^n$) définit un contactomorphisme strict de $(\mathbf{R}\times\mathbf{C}^n,\alpha_0)$ vers $(\mathbf{R}\times T^*\mathbf{R}^n,\alpha_{\mathbf{R}^n})$.

Exactement comme en géométrie symplectique, il n'y a pas d'invariant local en géométrie de contact.

Théorème. *Soit* (Y,ξ) *une variété de contact. Tout point de* Y *admet un voisinage contactomorphe à un voisinage de* 0 *dans* $(\mathbf{R}\times\mathbf{C}^n,\ker(\alpha_0))$.

On définit maintenant la notion de Legendrienne.

Définition. Soit (Y,ξ) une variété de contact de dimension $2n+1$. Une immersion $i : M \to Y$ est dite isotrope si $\mathrm{im}(Di) \subset \xi$. Dans ce cas la dimension de M est nécessairement inférieure ou égale à n. Si $\dim(M) = n$, on dit que i est Legendrienne. Enfin, une sous-variété Λ de Y est dite Legendrienne si Λ est l'image d'un plongement Legendrien.

Exemples. On donne quelques exemples de sous-variétés Legendriennes.

1. Le sous-espace

$$\{0\}\times\mathbf{R}^n = \{(0,z_1,\ldots,z_n)\in\mathbf{R}\times\mathbf{C}^n \mid z_1,\ldots,z_n\in\mathbf{R}\}$$

est une sous-variété Legendrienne de $(\mathbf{C}^n, \ker(\alpha_0))$.

2. Le sous-espace

$$\mathbf{R}^n \cap S^{2n-1} = \left\{(z_1, \dots, z_n) \in S^{2n-1} \mid z_1, \dots, z_n \in \mathbf{R}\right\}$$

est une sous-variété Legendrienne de $(S^{2n+1}, \ker(\lambda_0))$.

3. La 0-section $\{0\} \times 0_M$ est une sous-variété Legendrienne de $\Sigma \times T^*M$ (rappelons que Σ est l'une des variétés $\mathbf{R}_\theta$ ou $\mathbf{R}_\theta/\mathbf{Z}$). Plus généralement, le 1-jet

$$j^1(f) = \{(f(q), (q, df(q))) \in \Sigma \times T^*M \mid q \in M\}$$

d'une fonction $f : M \to \Sigma$ est une sous-variété Legendrienne de $(\Sigma \times T^*M, \ker(\alpha_M))$ (par exemple, la 0-section est le 1-jet de la fonction nulle).

4. La fibre $U^*_{q_0}M$ est une sous-variété Legendrienne de $(U^*M, \ker(\lambda_M))$. Plus généralement, si S est une sous-variété de M, alors son fibré conormal unitaire

$$\Lambda_S = \{(q,p) \in U^*M \mid q \in S \text{ et } \langle p, T_qS\rangle = 0\}$$

est une sous-variété Legendrienne de $(U^*M, \ker(\lambda_M))$ (par exemple, la fibre $U^*_{q_0}M$ est le fibré conormal du point q_0).

5. Pour tout point q_0 de M, la fibre $\mathbf{P}T^*_{q_0}M$ est une sous-variété Legendrienne de $\mathbf{P}T^*M$. Plus généralement, si S est une sous-variété de M, alors le projectivisé du conormal

$$\Lambda_S = \{(q,\ell) \in \mathbf{P}T^*M \mid q \in S \text{ et } \langle p, T_qS\rangle = 0 \text{ pour tout } p \in \ell\}$$

est une sous-variété Legendrienne de $\mathbf{P}T^*M$ (la fibre $\mathbf{P}T^*_{q_0}M$ est le projectivisé du conormal de $\{q_0\}$).

On présente maintenant une autre classe d'exemples de Legendriennes, généralisant celle des 1-jets de fonctions.

Exemple. Soit $\pi : E \to M$ une submersion. Soit $f : E \to \mathbf{R}$ une fonction sur E telle que

$df : E \to T^*E$ est transverse à l'application

$$\pi^{-1}T^*M \to T^*E, \quad p \in T^*_{\pi(x)}M \mapsto \langle p, D\pi(x)\rangle \in T^*_x E.$$

Alors l'application

$$E \underset{T^*E}{\times} \pi^{-1}T^*M \to \mathbf{R} \times T^*M, \quad (x, p) \mapsto (f(x), (\pi(x), p))$$

est une immersion Legendrienne, et on dit que f est une famille génératrice pour cette Legendrienne.

Notons que si $x_0 \in \pi^{-1}\{q_0\}$ est un extremum local de $f_{|\pi^{-1}\{q_0\}}$, alors il existe un unique $p_0 \in T^*_{q_0}M$, appelé multiplicateur de Lagrange, tel que

$$df(x_0) = \langle p_0, D\pi(x)\rangle,$$

c'est à dire tel que $(x_0, p_0) \in E \underset{T^*E}{\times} \pi^{-1}T^*M$.

On termine ce paragraphe en énonçant un théorème sur les voisinages des Legendriennes.

Théorème. *Soit (Y, ξ) une variété de contact, et $i : M \to Y$ un plongement Legendrien d'image Λ. Alors il existe un voisinage $\mathcal{U}$ de Λ dans Y, un voisinage $\mathcal{V}$ de $\{0\} \times 0_M$ dans $\mathbf{R} \times T^*M$, et un contactomorphisme $f : \mathcal{U} \to \mathcal{V}$ tel que*

$$(f \circ i)(q) = (0, (q, 0))$$

pour tout $q \in M$. Si de plus $\xi_{|\mathcal{U}}$ admet une forme de contact, le contactomorphisme peut être pris strict (avec la forme de contact standard α_M sur $\mathcal{V}$).

Variétés de Liouville et variétés de Weinstein

On présente ici les notions de variétés de Liouville et variétés de Weinstein. On renvoie au livre [21] pour un exposé détaillé.

Variétés de Liouville

Définition. Une variété symplectique exacte (X, λ) est une variété différentielle X munie d'une 1-forme λ telle que $(X, -d\lambda)$ est une variété symplectique. On appelle alors champ de Liouville de (X, λ) le champ de vecteurs Z_λ défini par la relation

$$\iota_{Z_\lambda} d\lambda = \lambda.$$

Un symplectomorphisme exact $f : (X_1, \lambda_1) \to (X_2, \lambda_2)$ est un difféomorphisme $f : X_1 \to X_2$ tel que la 1-forme $(f^*\lambda_2 - \lambda_1)$ est exacte.

Exemple. Si Y est une variété munie d'une forme de contact α, alors $(\mathbf{R}_\sigma \times Y, e^\sigma \alpha)$ est une variété symplectique exacte, appelée symplectisation de (Y, α).

Définition. Un domaine de Liouville est une variété symplectique exacte (X, λ) qui est compacte à bord et dont le champ de Liouville est transversalement sortant au bord. Dans ce cas, la restriction de $(-\lambda)$ au bord de X est une forme de contact sur ∂X.

Définition. Une variété de Liouville est une variété symplectique exacte (X, λ) (non compacte et sans bord) dans laquelle il existe un compact $X_0 \subset X$ tel que $(X_0, \lambda_{|X_0})$ est un domaine de Liouville, et le flot du champ de Liouville sur X induit un difféomorphisme

$$\mathbf{R}_{>0} \times \partial X_0 \to X \setminus X_0, \quad (t, x_0) \mapsto \varphi^t_{Z_\lambda}(x_0).$$

Si X_0 et X_0' satisfont les propriétés ci-dessus, le flot de Liouville induit un contactomorphisme entre ∂X_0 et $\partial X_0'$. Il y a donc une variété de contact bien définie $(\partial_\infty X, \xi)$, appelée "bord à l'infini de X".

Exemples. On donne des exemples de variétés de Liouville.

1. L'espace $(\mathbf{C}^n, \lambda_0)$ est une variété de Liouville, et

   $$Z_{\lambda_0} = \frac{1}{2}\left(x_1\partial_{x_1} + y_1\partial_{y_1} + \cdots + x_n\partial_{x_n} + y_n\partial_{y_n}\right).$$

   Par ailleurs, le bord à l'infini de $(\mathbf{C}^n, \lambda_0)$ est contactomorphe à la sphère (S^{2n-1}, λ_0).

2. Soit M une variété différentielle. Rappelons qu'on a définit une 1-forme λ_M sur T^*M dans la section précédente. Alors (T^*M, λ_M) est une variété de Liouville, et Z_{λ_M} est le champ de vecteurs défini par

   $$Z_{\lambda_M}(f) = \langle \partial_p f, p \rangle$$

pour tout $f : T^*M \to \mathbf{R}$ (notons que $\partial_p f(q,p) \in \left(T^*_q M\right)^*$). Par ailleurs, le bord à l'infini de (T^*M, λ_M) est contactomorphe au fibré cotangent unitaire (U^*M, λ_M) (une fois qu'on a choisi une métrique sur M).

Variétés de Weinstein

Définition. Une variété de Weinstein, est une variété de Liouville (X, λ) munie d'une fonction de Morse $\phi : X \to \mathbf{R}$ qui est propre, minorée, et Lyapunov pour Z_λ, c'est à dire qu'il existe une métrique sur X et $\delta > 0$ tels que

$$d\phi\left(Z_\lambda\right) \geq \delta\left(\|Z_\lambda\|^2 + \|d\phi\|^2\right).$$

En particulier, ϕ est croissante le long du flot de Z_λ, et les points critiques de ϕ sont exactement les zéros de Z_λ.

Exemple. Les espaces $(\mathbf{C}^n, \lambda_0)$ et (T^*M, λ_M) sont des variétés de Weinstein.

La proposition suivante montre que la topologie d'une variété de Weinstein est contrainte.

Proposition ([21] Proposition 11.9)**.** *Soit (X, λ, ϕ) une variété de Weinstein de dimension $2n$. Les variétés stables des points critiques de ϕ sont isotropes. En particulier, l'indice d'un point critique de ϕ est toujours inférieur ou égal à n.*

Le résultat suivant, dû à K. Cieliebak, décrit la topologie symplectique des variétés de Weinstein sans anse d'indice maximal.

Théorème ([20] Théorème 1.1)**.** *Soit (X, λ, ϕ) une variété de Weinstein de dimension $2n$. Si ϕ n'a pas de point critique d'indice n, alors il existe une variété de Weinstein (X', λ', ϕ') et un symplectomorphisme exact*

$$(X, \lambda) \simeq \left(\mathbf{C} \times X', \lambda_0 \oplus \lambda'\right).$$

Pour finir, mentionnons le fait important suivant. Si (X, λ) est une variété de Weinstein, et si Λ est une sous-variété Legendrienne de $\partial_\infty X$, alors on peut construire une nouvelle variété de Weinstein en attachant une anse d'indice maximal sur Λ.

Invariants algébriques

Dans toute cette section, le corps de base est $\mathbf{F} = \mathbf{Z}/2\mathbf{Z}$.

Invariants Legendriens classiques

On commence par présenter les invariants dits “classiques” des Legendriennes. On suit l'exposition donnée par T. Ekholm, J. Etnyre et M. Sullivan dans [27]. Dans la suite, on se donne une variété Y de dimension $(2n+1)$ munie d'une forme de contact α.

Classe de rotation On commence par introduire la notion de structure presque complexe sur $\xi = \ker(\alpha)$.

Définition. Une structure presque complexe sur ξ est une section J de $\mathrm{End}\,(\xi) \to Y$ telle que

$$J(a)^2 = -\mathrm{id}_{\xi(a)}$$

pour tout $a \in Y$. On dit que J est compatible avec $(d\alpha)_{|\xi}$ si $(d\alpha)_{|\xi}\,(\cdot, J\cdot)$ est un produit scalaire sur ξ.

On a le résultat classique suivant.

Proposition. *L'espace des structures presque complexes sur ξ compatibles avec $(d\alpha)_{|\xi}$ est non-vide et contractile.*

On peut maintenant définir la classe de rotation d'une immersion Legendrienne.

Définition. Soit $i : M \to Y$ une immersion Legendrienne. Soit J une structure presque complexe sur ξ compatible avec $(d\alpha)_{|\xi}$. On note $Di_{\mathbf{C}} : \mathbf{C} \otimes TM \to i^{-1}\xi$ l'isomorphisme de fibrés complexes défini par

$$Di_{\mathbf{C}}(q)\left((a+ib)\otimes v\right) = \left(a + bJ\left(i(q)\right)\right) Di(q)v.$$

La classe d'homotopie de $Di_{\mathbf{C}}$ dans l'espace des isomorphismes de fibrés complexes de $\mathbf{C} \otimes TM$ vers $i^{-1}\xi$ est appelée classe de rotation de i et notée $r(i)$ (ou $r(\Lambda)$ si i est l'inclusion d'une sous-variété Legendrienne Λ dans (Y, ξ)).

En raison du h-principe pour les immersions Legendriennes (voir [41]), deux immersions Legendriennes ont la même classe de rotation si et seulement s'il existe un chemin d'immersions Legendriennes entre les deux.

Remarque. Examinons ce qu'est la classe de rotation d'une immersion Legendrienne dans $(\mathbf{R} \times \mathbf{C}^n, \xi_0 = \ker(\alpha_0))$. Choisissons la structure presque complexe J sur ξ_0 définie par

$$J\left(\theta, (\mathbf{x}, \mathbf{y})\right)\left(\partial x_j + y_j \partial_\theta\right) = \partial_{y_j} \text{ et } J\left(\theta, (\mathbf{x}, \mathbf{y})\right)\left(\partial y_j\right) = -\left(\partial x_j + y_j \partial_\theta\right).$$

Alors la projection Lagrangienne $\Pi_{\mathbf{C}^n} : \mathbf{R} \times \mathbf{C}^n \to \mathbf{C}^n$ induit un isomorphisme complexe $\xi_0 \xrightarrow{\sim} \mathbf{C}^n$. On peut donc voir $Di_{\mathbf{C}}$ comme une trivialisation $\mathbf{C} \otimes TM \to \mathbf{C}^n$. De plus, on peut choisir des métriques hermitiennes sur $\mathbf{C} \otimes TM$ et $\mathbf{C}^n$ de sorte que $Di_{\mathbf{C}}$ soit une isométrie complexe. On peut vérifier que l'espace $[M, U(n)]$ des applications continues de M dans le groupe unitaire $U(n)$ agit librement et transitivement sur l'espace des isométries complexes de $\mathbf{C} \otimes TM$ dans $\mathbf{C}^n$, donc la classe de rotation de i peut être vue comme un élément de $[M, U(n)]$. Dans le cas où $M = S^n$, $r(i)$ est un élément de $\pi_n(U(n))$, qui est trivial si n est pair, et isomorphe à $\mathbf{Z}$ si n est impair. On parle alors de nombre de rotation.

Invariant de Thurston-Bennequin On définit maintenant l'invariant de Thurston-Bennequin d'une sous-variété Legendrienne. Il a été définit indépendamment par Bennequin (dans sa thèse [11]) et Thurston lorsque $n = 1$, puis généralisé par Tabachnikov dans [61].

Notons que, comme $\xi = \ker(\alpha)$ est une structure de contact, $\alpha \wedge (d\alpha)^n$ ne s'annule jamais, et donc définit une orientation sur Y.

Définition. Soit Λ une sous-variété Legendrienne connexe et orientée dans Y. On suppose que la classe d'homologie dans $H_*(Y)$ définie par Λ est triviale. Il existe alors une chaine C telle que $\partial C = \Lambda$. L'invariant de Thurston-Bennequin $tb(\Lambda)$ de Λ est le nombre d'intersection de C avec une petite perturbation de Λ dans la direction du champ de Reeb de α. Autrement dit, $tb(\Lambda)$ est le nombre d'enlacement de Λ et d'une variété obtenue en poussant Λ par R_α. Si $(\Lambda_t)_{0 \leq t \leq 1}$ est un chemin lisse de sous-variétés Legendriennes, alors $tb(\Lambda_0) = tb(\Lambda_1)$.

Dans le cas où (Y, α) est l'espace standard $(\mathbf{R} \times \mathbf{C}^n, \alpha_0)$, l'invariant de Thurston-Bennequin peut être calculé de la manière suivante. Soit Λ une sous-variété Legendrienne de $\mathbf{R} \times \mathbf{C}^n$. Une corde de Reeb de Λ est un chemin $c : [0, T] \to \mathbf{R} \times \mathbf{C}^n$ tel que

$$c(0), c(T) \in \Lambda \text{ et } c'(t) = R_{\alpha_0}(c(t))$$

pour tout $t \in [0, T]$. Supposons que Λ soit corde générique, c'est à dire que pour toute corde de Reeb $c : [0, T] \to \mathbf{R} \times \mathbf{C}^n$ de Λ, $D\varphi^T_{R_{\alpha_0}}(c(0))\, T_{c(0)}\Lambda$ est un sous-espace de $\xi(c(T))$ transverse à $T_{c(T)}\Lambda$. Ici, cela équivaut à dire que les sous-espaces Lagrangiens

$$V_{c(0)} = D\Pi_{\mathbf{C}}\left(T_{c(0)}\Lambda\right) \text{ et } V_{c(T)} = D\Pi_{\mathbf{C}}\left(T_{c(T)}\Lambda\right)$$

sont transverses dans $\mathbf{C}^n$. On définit le nombre $signe(c)$ comme étant $(+1)$ si l'orientation de $V_{c(0)} \oplus V_{c(T)}$ coincide avec celle de $\mathbf{C}^n$, et (-1) sinon. Alors

$$tb\,(\Lambda) = \sum_{c \in \mathcal{R}(\Lambda)} signe(c)$$

où $\mathcal{R}\,(\Lambda)$ désigne l'ensemble des cordes de Reeb de Λ.

Homologie de contact Legendrienne

On présente maintenant un invariant plus "sophistiqué". L'homologie de contact Legendrienne a d'abord été introduite par Chekanov dans [19] et par Eliashberg dans [34], et fait plus généralement partie de la théorie symplectique des champs exposée dans [35]. Elle a été rigoureusement définie dans la contactisation $(\mathbf{R} \times P, d\theta - \Pi_P^* \lambda)$ d'une variété de Liouville (P, λ) dans [26] et [28].

Soit Y une variété différentielle de dimension $(2n+1)$ munie d'une forme de contact α. On suppose que α est hypertendue, c'est à dire que les orbites périodiques du champ de Reeb de α ne sont pas contractiles. Comme expliqué dans [22] section 3, l'homologie de contact Legendrienne est bien définie dans cette situation.

Soit Λ une sous-variété Legendrienne de $(Y, \xi = \ker(\alpha))$. Une corde de Reeb de Λ est un chemin $c : [0, T] \to Y$ tel que

$$c(0), c(T) \in \Lambda \text{ et } c'(t) = R_\alpha(c(t))$$

pour tout $t \in [0, T]$. On suppose que Λ est corde générique, c'est à dire que pour toute corde de Reeb $c : [0, T] \to Y$ de Λ, $D\varphi^T_{R_\alpha}\left(T_{c(0)}\Lambda\right)$ est un sous-espace de $\xi(c(T))$ transverse à $T_{c(T)}\Lambda$.

Indice de Conley-Zehnder On définit l'indice de Conley-Zehnder d'une corde de Reeb de Λ qui commence et termine sur la même composante connexe. On suppose que $H_1(Y)$ est libre, que la première classe de Chern de ξ (munie d'une structure presque complexe compatible) est de 2-torsion, et que le nombre de Maslov $m(\Lambda)$ de Λ (qu'on définira plus bas) est nul.

On commence par définir brièvement l'indice de Maslov d'un lacet dans la Grassmanienne des sous-espaces Lagrangiens de $\mathbf{C}^n$ (voir [51] Théorème 2.3.7 pour une description plus détaillée de l'indice de Maslov). On utilise le résultat suivant.

Lemme. *Tout sous-espace Lagrangien K de $\mathbf{C}^n$ s'écrit*

$$K = \mathrm{im} \begin{pmatrix} X \\ Y \end{pmatrix}$$

avec $X + iY$ dans $U(n)$.

Pour définir l'indice de Maslov, on considère l'application

$$\rho : K \mapsto \det (X + iY)^2 \in S^1$$

(où (X, Y) et associé à K comme dans le lemme) et on pose, pour tout lacet Γ dans la Grassmannienne des sous-espaces Lagrangiens de $\mathbf{C}^n$,

$$\mu (\Gamma) := \deg (\rho \circ \Gamma) .$$

Rappelons qu'on a supposé $H_1 (Y)$ libre. On choisit une famille $(h_1, \ldots, h_r)$ de cercles plongés dans Y qui représentent une base de $H_1(Y)$, et une trivialisation symplectique de ξ au-dessus de chaque h_i. Si γ est un lacet dans Λ, il existe une unique famille d'entiers $(a_1, \ldots, a_r)$ telle que la classe $[\gamma_c - \sum_i a_i h_i]$ est nulle dans $H_1 (Y)$. On choisit une surface Σ_γ dans Y tel que

$$\partial\Sigma_\gamma = \gamma - \sum_i a_i h_i.$$

Il existe une unique trivialisation de ξ au-dessus de Σ_γ qui étend les trivialisations fixées au-dessus des h_i. On obtient ainsi une trivialisation $\gamma^{-1}\xi \simeq S^1 \times \mathbf{C}^n$ (où n est la dimension de Λ). On note Γ le lacet de plans Lagrangiens dans $\mathbf{C}^n$ correspondant, via cette trivialisation, au lacet $t \mapsto T_{\gamma(t)}\Lambda$. L'indice de Maslov de Γ ne dépend pas du choix de Σ_γ parce qu'on a supposé $2c_1(\xi) = 0$. Cette construction définit un morphisme $H_1 (\Lambda, \mathbf{Z}) \to \mathbf{Z}$, et le *nombre de Maslov* $m(\Lambda)$ de Λ est le générateur de son image. Dans la suite, on suppose que $m(\Lambda) = 0$.

Soit maintenant $c : [0, T] \to Y$ une corde de Reeb de Λ. On fixe un chemin γ_c dans Λ qui relie $c(T)$ à $c(0)$. On note $\overline{\gamma}_c$ le lacet obtenu par concaténation de γ et c. Soit $(a_1, \ldots, a_r)$ l'unique famille d'entiers telle que la classe $[\overline{\gamma}_c - \sum_i a_i h_i]$ est nulle dans $H_1 (Y)$. On choisit une surface Σ_c dans Y telle que

$$\partial\Sigma_c = \overline{\gamma}_c - \sum_i a_i h_i.$$

Il existe une unique trivialisation de ξ au-dessus de Σ_c qui étend les trivialisations fixées au-dessus des h_i. On obtient ainsi une trivialisation $\overline{\gamma_c}^{-1}\xi \simeq S^1 \times \mathbf{C}^n$. On note Γ_c le chemin de sous-espaces Lagrangiens de $\mathbf{C}^n$ correspondant à la concaténation de $t \mapsto T_{\gamma(t)}\Lambda$ et $t \mapsto D\varphi^t_{R_\alpha}\left(T_{c(0)}\Lambda\right)$. Comme Λ est corde générique, Γ_c n'est pas un lacet : on le ferme de la manière suivante. On choisit une structure complexe I sur $\mathbf{C}^n$ qui est compatible avec ω_0 et qui envoie $\Gamma_c(1)$ sur $\Gamma_c(0)$. On note alors $\overline{\Gamma}_c$ la concaténation de Γ_c et du chemin $t \mapsto \exp(tI)\,\Gamma_c(1)$. L'indice de Conley-Zehnder de c est l'indice de Maslov de $\overline{\Gamma}_c$:

$$CZ(c) = \mu(\overline{\Gamma}_c).$$

L'indice de Conley-Zehnder d'une corde de Reeb ne dépend pas du choix de Σ_c parce que $2c_1(\xi) = 0$, et il ne dépend pas non plus du choix de γ_c parce que le nombre de Maslov de Λ est nul.

Espaces de modules

Définition. Un disque Riemannien avec $(d+1)$ points marqués est un triplet $(D, j, \boldsymbol{\zeta})$ tel que

1. D est une variété à bord difféomorphe au disque unité fermé de $\mathbf{C}$,
2. j est une structure presque complexe intégrable sur D,
3. $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \ldots, \zeta_d)$ est une famille de points distincts sur ∂D, cycliquement ordonnée par rapport à l'orientation induite par j.

Si $(D, j, \boldsymbol{\zeta})$ est un disque Riemannien, on note $\Delta := D \setminus \{\zeta_0, \ldots, \zeta_d\}$. Un isomorphisme $\phi : (D, j, \boldsymbol{\zeta}) \to (D', j', \boldsymbol{\zeta}')$ est un biholomorphisme $\phi : (D, j) \to (D', j')$ qui envoie $\boldsymbol{\zeta}$ sur $\boldsymbol{\zeta}'$.

Définition. Soit $(D, j, \boldsymbol{\zeta})$ un disque Riemannien avec $(d+1)$ points marqués. Un choix de coordonnées près des perçures pour $(D, j, \boldsymbol{\zeta})$ est une famille d'applications

$$\epsilon_0 : \mathbf{R}_{\geq 0} \times [0,1] \to \Delta, \quad (\epsilon_k : \mathbf{R}_{\leq 0} \times [0,1] \to \Delta)_{1 \leq k \leq d}$$

telle que pour tout $1 \leq k \leq d$

1. ϵ_0 et ϵ_k sont des plongements tels que

$$\epsilon_0\left(\mathbf{R}_{\geq 0} \times \{0,1\}\right) \subset \partial D \text{ et } \epsilon_k\left(\mathbf{R}_{\leq 0} \times \{0,1\}\right) \subset \partial D$$

2. ϵ_0 et ϵ_k sont holomorphes,

3. on a

$$\epsilon_0(s,t) \underset{s\to+\infty}{\longrightarrow} \zeta_0 \text{ et } \epsilon_k(s,t) \underset{s\to-\infty}{\longrightarrow} \zeta_k.$$

Dans la suite on fixe un choix de coordonnées près des perçures pour chaque disque Riemannien avec points marqués. On peut maintenant définir les espaces de modules qui nous intéressent.

Définition. Soit J une structure presque complexe sur ξ compatible avec $(d\alpha)_{|\xi}$. Soit $a : [0, T_0] \to Y$ et $\mathbf{b} = (b_k : [0, T_k] \to Y)_{1\le k\le d}$ des cordes de Reeb de Λ. On note $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha)$ l'ensemble des classes d'équivalences de familles $(D, j, \boldsymbol{\zeta}, u)$ telles que

1. $(D, j, \boldsymbol{\zeta})$ est un disque Riemannien avec $(d+1)$ points marqués et u est une application lisse de $\Delta = D \setminus \{\zeta_0, \ldots, \zeta_d\}$ dans $\mathbf{R}\times Y$ qui envoie le bord de Δ dans $\mathbf{R}\times\Lambda$,

2. u est pseudo-holomorphe pour la structure presque complexe sur $\mathbf{R}_\sigma\times Y$ qui envoie ∂_σ sur R_α et dont la restriction à ξ est égale à J,

3. il existe des coordonnées près des perçures $\epsilon_0, (\epsilon_k)_{1\le k\le d}$ pour $(D, j, \boldsymbol{\zeta})$ telles que

$$(u\circ\epsilon_0)(s,t) \underset{s\to+\infty}{\longrightarrow} (+\infty, a(T_0t)) \text{ et } (u\circ\epsilon_k)(s,t) \underset{s\to-\infty}{\longrightarrow} (-\infty, b_k(T_kt)),$$

où deux familles $(D, j, \boldsymbol{\zeta}, u)$ et $(D', j', \boldsymbol{\zeta}', u')$ comme ci-dessus sont dites équivalentes s'il existe un isomorphisme $\phi : (D, j, \boldsymbol{\zeta}) \to (D', j', \boldsymbol{\zeta}')$ tel que $u\circ\phi^{-1} = u'$. Notons que $\mathbf{R}$ agit sur $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha)$ par translation dans la coordonnée $\mathbf{R}_\sigma$. On pose

$$\mathcal{M}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha) := \widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha)/\mathbf{R}.$$

On a le fait important suivant : pour des structures presque complexe génériques dites régulières (par rapport à Λ et α), les espaces de modules $\mathcal{M}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha)$ sont des variétés lisses de dimension

$$\dim \mathcal{M}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha) = CZ(a) - \left(\sum_{k=1}^{d} CZ(b_k)\right) + d - 2.$$

De plus, $\mathcal{M}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha)$ est compacte lorsque sa dimension est nulle, donc consiste en un nombre fini de points.

Algèbre différentielle graduée de Chekanov-Eliashberg On fixe, pour chaque corde de Reeb de Λ, un chemin dans Λ reliant le point d'arrivée de la corde à son point de départ. On choisit aussi une structure presque complexe J sur ξ compatible avec $(d\alpha)_{|\xi}$ qui soit régulière. Rappelons que cela implique que les espaces de modules $\mathcal{M}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha)$ sont des variétés lisses, et qu'ils sont compacts lorsqu'ils sont de dimension 0. On note $CE_*(\Lambda) = CE_*(\Lambda, J, \alpha)$ l'algèbre graduée unitaire libre engendrée par les cordes de Reeb de Λ, où le degré d'une corde de Reeb c de Λ est donné par

$$|c| = CZ(c) - 1.$$

Par ailleurs, si a est une corde de Reeb de Λ, on pose

$$\partial a = \sum_{\mathbf{b}} \#\mathcal{M}_{a,\mathbf{b}}(\mathbf{R}\times\Lambda, J, \alpha)\,\mathbf{b}$$

où $\#\mathcal{M} \in \mathbf{F}$ désigne le nombre d'éléments modulo 2 dans $\mathcal{M}$ si $\dim(\mathcal{M}) = 0$, et 0 sinon. Finalement, on étend ∂ à $CE_*(\Lambda)$ de sorte que

$$\partial(xy) = (\partial x)\,y + x\,(\partial y)$$

pour tous $x, y \in CE_*(\Lambda)$.

Résumons : étant donné une variété de contact (Y, ξ) avec $H_1(Y)$ libre, $2c_1(\xi) = 0$, et

1. une sous-variété Legendrienne Λ de (Y, ξ) avec $m(\Lambda) = 0$,
2. une forme de contact hypertendue α sur (Y, ξ) pour laquelle Λ est corde générique,
3. une structure presque complexe J sur ξ compatible avec $(d\alpha)_{|\xi}$ et régulière (par rapport à Λ et α),

on définit une algèbre graduée $CE_*(\Lambda)$ et une application linéaire $\partial : CE_*(\Lambda) \to CE_*(\Lambda)$. Le résultat essentiel de la construction est que $\partial \circ \partial = 0$ (voir [26] pour le cas des contactisations et [25], [22] pour le cas général), et donc $CE_{-*}(\Lambda)$ est une algèbre différentielle graduée (algèbre DG). Son homologie, qui est une algèbre graduée, est appelée homologie de contact Legendrienne de Λ.

De plus, au moins dans le cas des contactisations (voir [26]), un chemin $(\Lambda_t)_{0\le t\le 1}$ de sous-variétés Legendriennes induit un morphisme d'algèbres DG $CE_{-*}(\Lambda_0) \to CE_{-*}(\Lambda_1)$ inversible à homotopie près. L'algèbre DG de Chekanov-Eliashberg a permis d'exhiber des

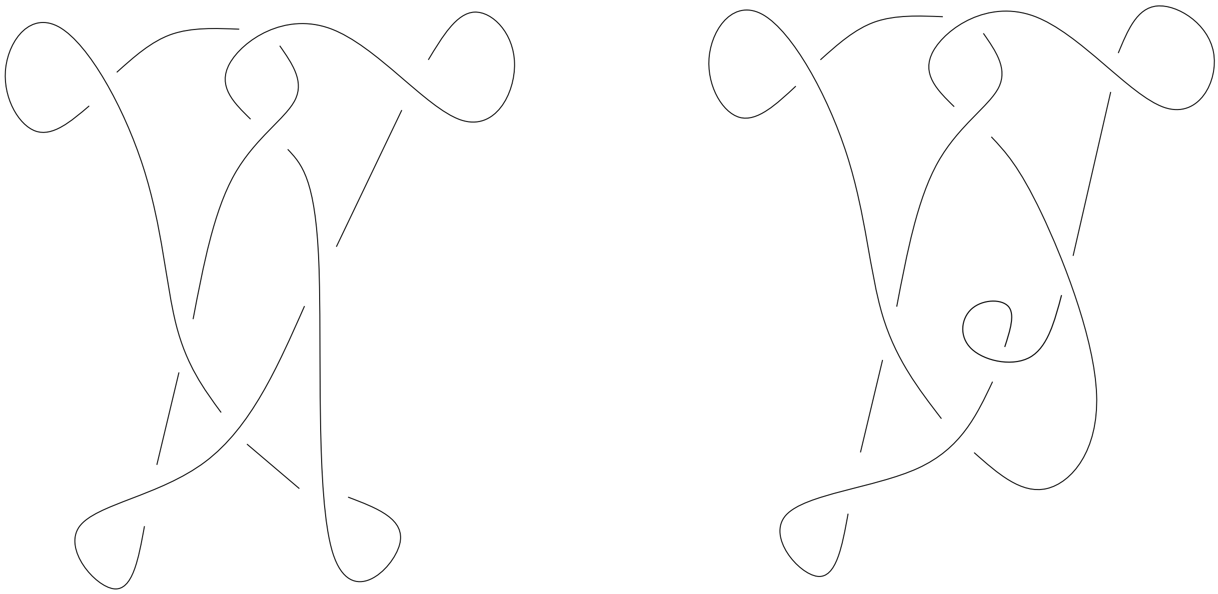

FIGURE 1 : Projections Lagrangiennes des noeuds de Chekanov-Eliashberg

sous-variétés Legendriennes non-isotopes mais avec des invariants classiques identiques. Le premier exemple a été donné par Chekanov dans [19] : les noeuds Legendriens dans $\mathbf{R} \times \mathbf{C}$ dont les images par la projection $\Pi_{\mathbf{C}} : \mathbf{R} \times \mathbf{C} \to \mathbf{C}$ sont représentées sur la Figure 1 ont mêmes invariants classiques, mais leurs algèbres DG ne sont pas équivalentes. En particulier, il n'existe pas d'isotopie Legendrienne reliant l'un à l'autre.

Catégories de Fukaya

On présente ici très brièvement les catégories de Fukaya compactes et enroulées associées à un domaine de Liouville X. On renvoie à [59], [3] et [37] pour un exposé détaillé de ces notions. L'objectif est de montrer un lien entre ces catégories A_∞ et l'algèbre DG de Chekanov-Eliashberg qui motive le travail de cette thèse.

On commence par définir ce qu'est une catégorie A_∞. Une présentation plus complète de cette notion est donnée dans le chapitre 1.

Définition. Une catégorie A_∞ $\mathcal{C}$ est la donnée de

1. une collection d'objets $\operatorname{ob}\mathcal{C}$,

2. pour tout couple d'objets (X, Y), un espace vectoriel gradué de morphismes $\mathcal{C}(X, Y)$,

3. une famille d'applications linéaires de degré $(2-d)$

$$\mu^d : \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{d-1}, X_d) \to \mathcal{C}(X_0, X_d)$$

indéxée par les suites d'objets $(X_0, \ldots, X_d)$, telle que

$$\sum_{0 \leq i < j \leq d} \mu^{d-(j-i)+1} \circ \left(\mathbf{1}^i \otimes \mu^{j-i} \otimes \mathbf{1}^{d-j}\right) = 0$$

pour tout $d \geq 1$.

Une algèbre A_∞ est une catégorie A_∞ avec un seul objet.

Remarque. Soit $\mathcal{C}$ une catégorie A_∞.

1. La relation A_∞ pour $d = 1$ s'écrit

$$\mu^1 \circ \mu^1 = 0,$$

donc $(\mathcal{C}(X, Y), \mu^1)$ est un complexe de chaines pour tout couple d'objets (X, Y).

2. La relation A_∞ pour $d = 2$ s'écrit

$$\mu^1 \circ \mu^2 + \mu^2 \circ \left(\mathbf{1} \otimes \mu^1\right) + \mu^2 \circ \left(\mu^1 \otimes \mathbf{1}\right) = 0,$$

donc μ^1 satisfait la règle de Leibniz par rapport à μ^2.

3. La relation A_∞ pour $d = 3$ s'écrit

$$\begin{aligned} &\mu^1 \circ \mu^3 + \mu^2 \circ \left(\mu^2 \otimes \mathbf{1}\right) + \mu^2 \circ \left(\mathbf{1} \otimes \mu^2\right) \\ &\qquad + \mu^3 \circ \left(\mu^1 \otimes \mathbf{1} \otimes \mathbf{1}\right) + \mu^3 \circ \left(\mathbf{1} \otimes \mu^1 \otimes \mathbf{1}\right) + \mu^3 \circ \left(\mathbf{1} \otimes \mathbf{1} \otimes \mu^1\right) = 0, \end{aligned}$$

donc μ^2 est “associatif à homotopie près”.

Remarque. Une algèbre DG est une algèbre A_∞ avec $\mu^d = 0$ pour $d \geq 3$.

Avant de définir les catégories de Fukaya compactes et enroulées, on donne la définition suivante.

Définition. Soit (X, λ) une variété de Liouville.

1. Une sous-variété Lagrangienne exacte de (X, λ) est une sous-variété L de X telle que la 1-forme $\lambda_{|L}$ est exacte (en particulier, L est une sous-variété Lagrangienne de $(X, -d\lambda)$).

2. Une sous-variété Lagrangienne exacte cylindrique de (X, λ) est une sous-variété Lagrangienne exacte de (X, λ) à laquelle le champ de Liouville Z_λ est tangent en dehors d'un compact de X.

La catégorie de Fukaya compacte $\mathcal{F}uk(X)$ d'une variété de Liouville (X, λ) est une catégorie A_∞ dont les objets sont les sous-variétés Lagrangiennes exactes compactes de X. L'espace des morphismes entre deux Lagrangiennes transverses dans X est engendré par leur intersection (l'espace des morphismes entre deux objets non transverses est plus compliqué à définir), et les opérations A_∞ sont définies en considérant des disques pseudo-holomorphes percés dans X, à bord sur les Lagrangiennes considérées, et asymptotiques à des points d'intersections aux perçures. On renvoie à [59] pour une définition détaillée.

La catégorie de Fukaya enroulée $\mathcal{WF}uk(X)$ d'une variété de Liouville (X, λ) est une catégorie A_∞ dont les objets sont les sous-variétés Lagrangiennes exactes cylindriques de X. L'espace des morphismes entre deux objets est plus compliqué à définir, mais les opérations font toujours intervenir des disques pseudo-holomorphes perçés dans X. On renvoie à [3] et [37] pour des définitions précises.

Le résultat que nous voulions présenter est le suivant. Il illustre le lien entre catégories de Fukaya et algèbre de Chekanov-Eliashberg, et motive l'étude de l'homologie de contact Legendrienne dans la contactisation circulaire d'une variété de Liouville.

Théorème ([16], [18], [37]). *Soit P une variété de Weinstein. Considérons la variété de Weinstein $X = \mathbf{C} \times P$. Soit Λ un entrelac de sphères Legendriennes dans $\partial_\infty X$. Notons X_Λ la variété de Weinstein obtenue à partir de X par chirurgie sur Λ. Alors la catégorie de Fukaya enroulée de X_Λ se plonge dans la catégorie des DG-modules sur l'algèbre de Chekanov-Eliashberg de Λ.*

Main results and organization of the thesis

Algebra In chapter 1, we give definitions of the algebraic objects we are using, which basically are Adams-graded DG-algebras, A_∞-(co)algebras and A_∞-categories, and we prove basic facts about them. In addition to standard notions, we discuss some less studied (to our knowledge) objects, such as the mapping cylinder of a DG-map (introduced in [53]), the cone of an A_∞-map (we learned about this in [42]), or the homotopy colimit of A_∞-categories (defined in [38]). Moreover, we introduce two new notions. The first one is called "quasi mapping cylinder of an elementary DG-isomorphism" and is defined only because it appears naturally in the proof of Theorem C. We call the second one "cylinder object for an A_∞-category" because it is supposed to mimic the corresponding notion in homotopy theory. We use it for the proof of Theorem B.

Mapping torus of a quasi-autoequivalence In chapter 2, we introduce the notion of mapping torus for a quasi-autoequivalence of an A_∞-category (see Definition 2.1). Observe that this terminology was also used in [45] but we do not know if the two notions coincide. When considering a quasi-autoequivalence τ of an A_∞-category $\mathcal{A}$, we always assume that $\mathcal{A}$ is equipped with a $\mathbf{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$ compatible with τ, which is a bijection

$$\mathbf{Z} \times \mathcal{E} \xrightarrow{\sim} \mathrm{ob}\left(\mathcal{A}\right), \quad \left(n, E\right) \mapsto X^n\left(E\right).$$

such that

$$\tau\left(X^n\left(E\right)\right) = X^{n+1}\left(E\right)$$

for every $n \in \mathbf{Z}$ and $E \in \mathcal{E}$ (see Definition 2.5).

We give a first result which computes the mapping torus of a quasi-autoequivalence $\tau : \mathcal{A} \to \mathcal{A}$ when τ is strict (see Definition 1.51) and acts bijectively on hom-sets. This allows us to define a new Adams-graded A_∞-category $\mathcal{A}_\tau$ (see Definition 2.7).

Theorem A. *Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbf{Z}$-splitting of $\mathrm{ob}\left(\mathcal{A}\right)$. Assume that τ is strict and acts bijectively on hom-sets. Then the mapping torus of τ is quasi-equivalent to $\mathcal{A}_\tau$.*

Then we give a second result which computes the mapping torus of a quasi-autoequivalence but in a different situation. We denote by $\mathbf{F}\left[t\right]$ the augmented Adams-graded associative algebra generated by a variable t of bidegree $(2,1)$, and by $\overline{\mathbf{F}\left[t\right]}$ its augmentation ideal (or equivalently, the ideal generated by t). Besides, we use the functors $\mathcal{C} \mapsto \mathcal{C}'$ and $\mathcal{D} \mapsto \mathbf{F}\left[t\right] \otimes \mathcal{D}$ introduced in Definitions 1.53 and 1.54 respectively (see also Remark 1.55). Finally, if $\mathcal{C}$ is an A_∞-category equipped with a $\mathbf{Z}$-splitting of $\mathrm{ob}\left(\mathcal{C}\right)$, we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$.

Theorem B. *Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ directed with respect to some compatible $\mathbf{Z}$-splitting of $\mathrm{ob}\left(\mathcal{A}\right)$. Assume that there exists a closed degree 0*

bimodule map $f : \mathcal{A}'(-,-) \to \mathcal{A}'(-,\tau(-))$ (see Definitions 1.63 and 1.64) such that for every $i < j < k$, for every $E, E' \in \mathcal{E}$, the chain maps

$$\begin{cases} \mu^2_{\mathcal{A}'}\left(-, f(e_{X^j(E)})\right) & : \quad \mathcal{A}'\left(X^i(E'), X^j(E)\right) \quad \to \quad \mathcal{A}'\left(X^i(E'), X^{j+1}(E)\right) \\ \mu^2_{\mathcal{A}}\left(f(e_{X^j(E)}), -\right) & : \quad \mathcal{A}'\left(X^{j+1}(E), X^{k+1}(E')\right) \quad \to \quad \mathcal{A}'\left(X^j(E), X^{k+1}(E')\right) \end{cases}$$

are quasi-isomorphisms. Then the mapping torus of τ is quasi-equivalent to the A_∞-category $(\mathcal{A}')^0 \oplus \left(\overline{\mathbf{F}[t]} \otimes \mathcal{A}'\left[f(\text{units})^{-1}\right]^0\right)$.

Remark. In this thesis, we will apply Theorems A and B to A_∞-categories equipped with $\mathbf{Z}$-splittings such that $\mathcal{E}$ has only one element. We chose to state and prove Theorems A and B in the more general case where $\mathcal{E}$ is arbitrary for future applications.

Invariance and homotopies in Legendrian contact homology In chapter 3, we study the Chekanov-Eliashberg DG-algebra of a Legendrian in a hypertight contact manifold. We refer to section 3.1 for the definitions. In section 3.2, we generalize techniques from [28] (for the invariance) and [53] (for the homotopies) in order to prove a result on the behavior of the Chekanov-Eliashberg DG-algebra under change of data. In particular, we use the notion of mapping cylinder associated to a DG-map. However, we restrict ourselves to what we call *good paths* of data (see Definition 3.7). This is unfortunately a strong assumption, as Legendrians are assumed to always have finitely many Reeb chords, and we do not treat birth-death of Reeb chords phenomena.

Let (V,ξ) be a contact manifold equipped with a hypertight contact form α. In order for the Chekanov-Eliashberg algebras of the Legendrians to be $\mathbf{Z}$-graded, we assume that $H_1(V)$ is free and that the first Chern class of ξ (equipped with any compatible almost complex structure) is 2-torsion. We denote by $\mathcal{L}$ the space of chord generic compact Legendrian submanifolds in (V,ξ) with Maslov number 0 and with finitely many Reeb chords with respect to α.

Theorem C. *There is a way to associate, to each good path $(\Lambda_t, J_t)_{0\leq t\leq 1}$ (see Definition 3.7), a tame DG-isomorphism*

$$\varphi : CE_{-*}(\Lambda_0, J_0, \alpha) \xrightarrow{\sim} CE_{-*}(\Lambda_1, J_1, \alpha)$$

such that the following holds:

1. *if φ is the DG-map associated to a good path $(\Lambda_t, J_t)_{0\leq t\leq 1}$, if $\chi_1 : [0,1] \to [0,t_0]$ and*

$\chi_2 : [0,1] \to [t_0, 1]$ are two non-decreasing functions satisfying

$$\chi_1(0) = 0,\ \chi_1(1) = t_0 = \chi_2(0),\ \chi_2(1) = 1,$$

then $\varphi = \varphi_2 \circ \varphi_1$, where φ_k is the DG-map associated to $(\Lambda_{\chi_k(t)}, J_{\chi_k(t)})_{0\leq t\leq 1}$,

2. *if $\varphi^{\pm}$ are the DG-maps associated to good paths $(\Lambda_t^{\pm}, J_t^{\pm})_{0\leq t\leq 1}$ satisfying*

$$(\Lambda_0^-, J_0^-) = (\Lambda_0^+, J_0^+), \quad (\Lambda_1^-, J_1^-) = (\Lambda_1^+, J_1^+),$$

and if there is a family $(\Lambda_{s,t})_{0\leq s,t\leq 1}$ such that for every s, t

$$\Lambda_{s,t} \in \mathcal{L}, \quad (\Lambda_{s,0}, \Lambda_{s,1}) = (\Lambda_0^{\pm}, \Lambda_1^{\pm}), \quad (\Lambda_{0,t}, \Lambda_{1,t}) = (\Lambda_t^-, \Lambda_t^+),$$

then φ^- and φ^+ are DG-homotopic.

Legendrian lift of an exact Lagrangian in the circular contactization In chapter 4, we start with a compact, connected, exact Lagrangian submanifold L in a Liouville manifold (P, λ), and we study a Legendrian lift Λ° of L in the circular contactization $(S^1 \times P, \ker(d\theta - \lambda))$. In order for the Floer complex of L and the Chekanov-Eliashberg algebra of Λ° to be $\mathbf{Z}$-graded, we assume that $H_1(P)$ is free, that the first Chern class of P (equipped with any almost complex structure compatible with $(-d\lambda)$) is 2-torsion, and that the Maslov class of L vanishes. We refer to section 4.1 for a description of the Floer A_∞-algebra $CF^*(L)$, the Chekanov-Eliashberg Adams-graded DG-algebra $CE_{-*}(\Lambda^\circ)$ and the Adams-graded A_∞-algebra $LA^*(\Lambda^\circ)$. The latter is the endomorphism algebra of the trivial augmentation of $CE_{-*}(\Lambda^\circ)$ in the unital version of the augmentation category $\mathcal{A}ug_-(\Lambda^\circ)$ introduced in [17] (see [15] for the non unital version).

As above, we denote by $\mathbf{F}[t]$ the augmented Adams-graded associative algebra generated by a variable t of bidegree $(2,1)$, and by $\overline{\mathbf{F}[t]}$ its augmentation ideal (or equivalently, the ideal generated by t). We also denote by $\mathbf{F}[t_\#]$ the coaugmented Adams-graded associative coalgebra generated by a variable $t_\#$ of bidegree $(-2,1)$, and by $\overline{\mathbf{F}[t_\#]}$ its coaugmentation ideal. Finally, we use the cobar construction $C \mapsto \Omega C$ and the Koszul dual construction $A \mapsto E(A)$ introduced in Definitions 1.32 and 1.39 respectively.

Theorem D. *There is a quasi-isomorphism of augmented Adams-graded A_∞-algebras*

$$LA^*(\Lambda^\circ) \simeq \mathbf{F} \oplus \left(\overline{\mathbf{F}[t]} \otimes CF^*(L)\right).$$

Moreover, Koszul duality holds for $CE_{-*}(\Lambda^{\circ})$*, and* $E(CE_{-*}(\Lambda^{\circ})) \simeq LA^{*}(\Lambda^{\circ})$*. In particular, there are quasi-isomorphisms of Adams-graded DG-algebras*

$$CE_{-*}(\Lambda^{\circ}) \simeq E\left(\mathbf{F} \oplus \left(\overline{\mathbf{F}[t]} \otimes CF^{*}(L)\right)\right) \simeq \Omega\left(\mathbf{F} \oplus \left(\overline{\mathbf{F}[t_{\#}]} \otimes CF_{-*}(L)\right)\right).$$

If B is a (unpointed) space, we consider its one-point compactification B^{*} and view it as a pointed space (with base point the point at infinity). If moreover X is a pointed space, we consider the half-smash product of B and X,

$$X \rtimes B := X \wedge B^{*}$$

(where $\wedge$ denotes the smash product of pointed spaces).

Corollary. *There is a quasi-isomorphism of augmented (non Adams-graded)* A_{∞}*-algebras*

$$LA^{*}(\Lambda^{\circ}) \simeq C^{*}(\mathbf{CP}^{\infty} \rtimes L),$$

and a quasi-isomorphism of augmented (non Adams-graded) DG-algebras

$$CE_{-*}(\Lambda^{\circ}) \simeq C_{-*}(\Omega(\mathbf{CP}^{\infty} \rtimes L)).$$

Chapter 1

Algebra

In this chapter, we give definitions about the algebraic objects we are using. We recall the notions of DG-(co)algebras, A_∞-(co)algebras and A_∞-categories in sections 1.1, 1.2 and 1.3 respectively. In addition to basic facts, we discuss some less studied (to our knowledge) objects, such as the mapping cylinder of a DG-map (that we will use as in [53]), the cone of an A_∞-map (we learned about this in [42]), or the homotopy colimit of A_∞-categories (defined in [38]). Moreover, we introduce two new notions. We call the first one "quasi mapping cylinder of an elementary DG-isomorphism". We define it only because it appears naturally in the proof of Theorem C. We call the second one "cylinder object for an A_∞-category" because it is supposed to mimic the corresponding notion in homotopy theory. We use it for the proof of Theorem B.

In the following, $\mathbf{F}$ denotes the field $\mathbf{Z}/2\mathbf{Z}$.

1.1 DG-(co)algebras

The goal of this section is to give the basic definitions of DG-(co)algebras, and to introduce a mapping cylinder construction considered in [53]. All our DG-(co)algebras and (co)maps will be strictly unital unless otherwise stated. Besides, we give the definitions in the Adams-graded setting, which is a bigraded setting: the first grading is called the cohomological degree while the second grading is called the Adams degree. The non Adams-graded counterpart should be transparent.

1.1.1 Basic definitions

Definition 1.1. An Adams-graded DG-algebra is a bigraded associative algebra A together with a bidegree $(1,0)$ linear map $\partial : A \to A$ (called the differential) such that

$$\partial \circ \partial = 0 \text{ and } \partial\,(ab) = (\partial a)b + a(\partial b)$$

for every $a, b \in A$.

A DG-map $f : A_1 \to A_2$ between two Adams-graded DG-algebras is a bidegree $(0, 0)$ algebra map such that

$$f \circ \partial_{A_1} = \partial_{A_2} \circ f.$$

We say that f is a quasi-isomorphism if it induces an isomorphism in homology.

Definition 1.2. Let $f, g : A_1 \to A_2$ be two DG-maps. A DG-homotopy between f and g is a bidegree $(-1, 0)$ linear map $H : A_1 \to A_2$ such that

$$g = f + H \circ \partial_{A_1} + \partial_{A_2} \circ H \text{ and } H(xy) = H(x)g(y) + f(x)H(y)$$

for every $x, y \in A_1$.

Definition 1.3. Let A be an Adams-graded DG-algebra, and consider $\mathbf{F}$ as an Adams-graded DG-algebra concentrated in bidegree $(0, 0)$ with trivial differential. An augmentation of A is a DG-map $\epsilon : A \to \mathbf{F}$. .

A DG-map $f : (A_1, \epsilon_1) \to (A_2, \epsilon_2)$ between augmented Adams-graded DG-algebras is required to satisfy $\epsilon_1 = \epsilon_2 \circ f$.

Definition 1.4. An Adams-graded DG-coalgebra is an associative coalgebra C together with a bidegree $(1, 0)$ linear map δ (called the codifferential) such that

$$\delta \circ \delta = 0 \text{ and } \Delta \circ \delta = (\delta \otimes \mathbf{1}) \circ \Delta + (\mathbf{1} \otimes \delta) \circ \Delta$$

where $\Delta : C \to C \otimes C$ denotes the coproduct of C.

A DG-comap $f : C_1 \to C_2$ between two Adams-graded DG-coalgebras is a bidegree $(0, 0)$ coalgebra map $f : C_1 \to C_2$ such that

$$f \circ \delta = \delta \circ f.$$

We say that f is a quasi-isomorphism if it induces an isomorphism in homology.

Definition 1.5. Let C be an Adams-graded DG-coalgebra, and consider $\mathbf{F}$ as an Adams-graded DG-coalgebra concentrated in bidegree $(0, 0)$ with trivial codifferential. A coaugmentation of C is a DG-comap $\eta : \mathbf{F} \to C$.

A DG-comap $f : (C_1, \eta_1) \to (C_2, \eta_2)$ between coaugmented Adams-graded DG-coalgebras is required to satisfy $f \circ \eta_1 = \eta_2$.

1.1.2 Triangular DG-algebras

In the following, we follow closely the presentation given in [53] section 2.2.

Definition 1.6. A based algebra is a free associative graded algebra together with a family $(a_1, \ldots, a_m)$ of pairwise distinct homogeneous elements of A which generate A.

An elementary automorphism of a based algebra A with generating family $(a_1, \ldots, a_m)$ is an algebra map $\varphi : A \to A$ for which there exists $1 \leq i \leq m$ and b in the sub-DG-algebra of A generated by $\{a_1, \ldots, a_{i-1}\}$ such that

$$\varphi(a_i) = a_i + b \text{ and } \varphi(a_j) = a_j \text{ if } j \neq i.$$

A tame isomorphism of based algebras $\varphi : A_1 \to A_2$ is an algebra map which can be written as a composition

$$\varphi = \varphi_r \circ \cdots \circ \varphi_1 \circ i$$

where $i : A_1 \to A_2$ extends an order-preserving bijection of generating sets and each φ_i is an elementary automorphism of A_2.

Definition 1.7. A triangular differential on a based algebra $(A, (x_1, \ldots, x_m))$ is a differential ∂ such that ∂x_i belongs to the sub-DG-algebra of A generated by $\{x_1, \ldots, x_{i-1}\}$ for every $1 \leq i \leq m$. We say that (A, ∂) is a triangular DG-algebra.

Let ∂_1 and ∂_2 be two triangular differentials on a based algebra A. An elementary DG-isomorphism $\varphi : (A, \partial_1) \to (A, \partial_2)$ is a DG-isomorphism which is an elementary automorphism of A.

A tame DG-isomorphism of triangular DG-algebras $\varphi : A_1 \to A_2$ is a DG-isomorphism which is a tame isomorphism between the underlying based algebras.

Definition 1.8. A stabilization of a triangular DG-algebra A is a triangular DG-algebra such that

1. as a free associative graded algebra, it is of the form $A \star S$ ($\star$ denotes the coproduct in the category of free associative algebras), where S is a free associative graded algebra generated by $x_1, y_1, \ldots, x_r, y_r$ with $|x_i| = |y_i| - 1$,

2. the differential satisfies
$$\partial_{|A} = \partial_A \text{ and } \partial x_i = y_i.$$

Proposition 1.9. *Let A be a triangular DG-algebra with generating family $(a_1, \ldots, a_m)$. Assume that for some $1 \leq i < j \leq m$,*

$$\partial a_j = a_i + b,$$

with b in the sub-DG-algebra of A generated by $\{a_1, \ldots, a_{i-1}\}$. Denote by I the ideal of A generated by $\{a_j, \partial a_j\}$ and by $p : A \to A/I$ the projection. Then the following holds:

1. *the differential ∂ induces a differential ∂^I on A/I such that $\partial^I \circ p = p \circ \partial$, and $(A/I, \partial^I)$ is a triangular DG-algebra with generating family*

$$(p(a_1), \ldots, p(a_{i-1}), p(a_{i+1}), \ldots, p(a_{j-1}), p(a_{j+1}), \ldots, p(a_m)),$$

2. *there is a tame DG-isomorphism from A to a stabilization of A/I which sends a_k to $p(a_k)$ if $k \neq i, j$.*

Proof. This follows from Proposition 2.3 in [53].

□

1.1.3 Mapping cylinder of a DG-map

In the following, we follow closely the presentation given in [53] sections 2.3-2.6. We also cite results from [30] section 7.

If A is a based algebra with generating family $(a_1, \ldots, a_m)$, we denote by $\widehat{A}$ the based algebra generated by $(\widehat{a_1}, \ldots, \widehat{a_m})$, where $|\widehat{a_i}| = |a_i| - 1$.

Definition 1.10. Let $f : A \to B$ be a DG-map between two triangular DG-algebras A and B with generating families $(a_1, \ldots, a_m)$ and $(b_1, \ldots, b_n)$ respectively. A mapping cylinder DG-algebra for f is a triangular DG-algebra C such that

1. as a based algebra, $C = A \star \widehat{A} \star B$ with generating family $(b_1, \ldots, b_n, a_1, \ldots, a_m, \widehat{a_1}, \ldots, \widehat{a_m})$,

2. the differential satisfies $(\partial_C)_{|A} = \partial_A$, $(\partial_C)_{|B} = \partial_B$, and for every generator $a \in A$,

$$\partial \widehat{a} = a + f(a) + O(1),$$

where $O(1)$ denotes a sum of terms containing at least one hat-generator.

Proposition 1.11. *Let $f : A \to B$ be a DG-map between two triangular DG-algebras. Denote by $\Gamma : A \to A \star \widehat{A} \star B$ the unique linear map such that $\Gamma(a) = \widehat{a}$ for every generator $a \in A$ and*

$$\Gamma(xy) = \Gamma(x)y + f(x)\Gamma(y)$$

for every $x, y \in A$. Define $\partial : A \star \widehat{A} \star B \to A \star \widehat{A} \star B$ to be the unique derivation satisfying $\partial_{|A} = \partial_A$, $\partial_{|B} = \partial_B$, and for every generator $a \in A$,

$$\partial\widehat{a} = a + f(a) + \Gamma(\partial a).$$

Then $\mathrm{Cyl}(f) := \left(A \star \widehat{A} \star B, \partial\right)$ is a mapping cylinder DG-algebra for f, called the standard mapping cylinder of f.

Proof. This is Proposition 2.7 in [53], and also Lemma 7.1 in [30].

□

Proposition 1.12. *Let C be a mapping cylinder for a DG-map $f : A \to B$. There is a tame isomorphism $\psi : C \to \mathrm{Cyl}(f)$ which restricts to the identity on A and B.*

Proof. This follows from Proposition 2.8 in [53].

□

Proposition 1.13. *Let $f : A_- \to A_0$ and $g : A_0 \to A_+$ be two DG-maps between triangular DG-algebras with generating families*

$$\left(a_1^*, \dots, a_{m_*}^*\right), \quad * \in \{-, 0, +\}.$$

Let $C = A_- \star \widehat{A_-} \star A_0$, $D = A_0 \star \widehat{A_0} \star A_+$ be mapping cylinder DG-algebras for f and g respectively. Observe that A_0 is a DG-subalgebra of both C and D. We denote by $C \underset{A_0}{\star} D$ the fibered coproduct (in the category of semi-free DG-algebras) of C and D over A_0: this is the semi-free DG-algebra obtained by first taking the disjoint union of C and D, and then identifying, for each generator $a^0 \in A_0$, the corresponding generators $a^0 \in C$ and $a^0 \in D$. We make $C \underset{A_0}{\star} D$ a triangular DG-algebra by specifying the generating family

$$\left(a_1^+, \dots, a_{m_+}^+, a_1^0, \dots, a_{m_0}^0, a_1^-, \dots, a_{m_-}^- \widehat{a_1^0}, \dots, \widehat{a_{m_0}^0}, \widehat{a_1^-}, \dots, \widehat{a_{m_-}^-}\right).$$

There is a tame isomorphism from $C \underset{A_0}{\star} D$ to a stabilization of a mapping cylinder DG-algebra for $g \circ f$ which restricts to the identity on $A_\pm$.

Proof. According to Proposition 1.12, we can assume that C and D are standard mapping cylinders. Then the result follows from [30] Lemma 7.3.

□

Proposition 1.14. *Let C_1, C_2 be two mapping cylinders for DG-maps $f_1 : A \to B$, $f_2 : A \to B$ respectively. If there is a DG-map $\psi : C_1 \to C_2$ which restricts to the identity on A and B, then f_1 and f_2 are DG-homotopic in the sense of Definition 1.2.*

Proof. This follows from Proposition 2.12 in [53].

□

1.1.4 Quasi mapping cylinder of an elementary DG-isomorphism

As in the previous section, if A is a based algebra with generating family $(a_1, \ldots, a_m)$, we denote by $\widehat{A}$ the free based algebra generated by $(\widehat{a_1}, \ldots, \widehat{a_m})$, where $|\widehat{a_i}| = |a_i| - 1$.

Definition 1.15. Let ∂_- and ∂_+ be two triangular differentials on a based algebra A with generating family $(a_1, \ldots, a_m)$, and let $\varphi : (A, \partial_-) \to (A, \partial_+)$ be an elementary DG-isomorphism. A quasi mapping cylinder DG-algebra for φ is a triangular DG-algebra C such that

1. as a based algebra, $C = A_- \star \widehat{A_-} \star A_+$ (where $A_\pm$ are two copies of A) with generating family is $(a_1^+, \ldots, a_m^+, a_1^-, \ldots, a_m^-, \widehat{a_1^-}, \ldots, \widehat{a_m^-})$,

2. the differential satisfies $(\partial_C)_{|A_-} = \partial_-$, $(\partial_C)_{|A_+} = \partial_+$, and for every generator $a \in A$,

$$\partial \widehat{a^-} = \begin{cases} a^- + a^+ + O(1) & \text{if } \varphi(a) = a \\ a^- + a^+ + \epsilon + O(1) & \text{if } \varphi(a) \neq a \end{cases}$$

   where $O(1)$ denotes a sum of terms containing at least one hat-generator, and ϵ is a sum of terms without hat-generator such that $\varphi(a)$ is obtained from $(a^+ + \epsilon)$ by changing the generators $a^\pm$ into a.

Lemma 1.16. *Let $C_{0+} = (A_0 \star \widehat{A_0} \star A_+, \partial_{0+})$ be a quasi mapping cylinder DG-algebra for an elementary DG-isomorphism $\varphi : (A, \partial_0) \to (A, \partial_+)$, and let $C_{-0} = (A_- \star \widehat{A_-} \star A_0, \partial_{-0})$ be a mapping cylinder DG-algebra for the identity map* $\mathrm{id} : (A, \partial_-) \to (A, \partial_0)$ *(in particular, $\partial_- = \partial_0$). There is a tame DG-isomorphism from $C_{-0} \underset{A_0}{\star} C_{0+}$ to a stabilization of a mapping cylinder DG-algebra $(C_{-+} = A_- \star \widehat{A_-} \star A_+, \partial_{-+})$ for φ which restricts to the identity on $A_\pm$.*

Proof. We inductively cancel the generators $\left(a^0, \widehat{a^0}\right)$ starting from $\left(a^0_m, \widehat{a^0_m}\right)$ until $\left(a^0_1, \widehat{a^0_1}\right)$. See the proof of [30] Lemma 7.3 where the same argument is used.

Since φ is an elementary automorphism, there exists $1 \leq i_0 \leq m$ and b in the sub-DG-algebra of A generated by $\{a_1, \dots, a_{i_0-1}\}$ such that

$$\varphi(a_{i_0}) = a_{i_0} + b \text{ and } \varphi(a_i) = a_i \text{ if } i \neq i_0.$$

Let C_0 be the triangular DG-algebra $C_{-0} \underset{A_0}{\star} C_{0+}$ with generating family

$$\left(a^+_1, \dots, a^+_m, a^0_1, \dots, a^0_m, a^-_1, \dots, a^-_m, \widehat{a^0_1}, \dots, \widehat{a^0_m}, \widehat{a^-_1}, \dots, \widehat{a^-_m}\right)$$

and denote by Δ_0 its differential.

Let I_0 be the ideal of C_0 generated by $\widehat{a^0_m}$ and $\Delta_0\left(\widehat{a^0_m}\right)$, and let

$$C_1 := C_0/I_0.$$

According to Proposition 1.9, C_1 has a natural structure of triangular DG-algebra with generating family

$$\left(a^+_1, \dots, a^+_m, a^0_1, \dots, a^0_{m-1}, a^-_1, \dots, a^-_m, \widehat{a^0_1}, \dots, \widehat{a^0_{m-1}}, \widehat{a^-_1}, \dots, \widehat{a^-_m}\right)$$

and there is a tame DG-isomorphism from C_0 to a stabilization of C_1 which restricts to $\mathrm{id}_{A_\pm}$ on $A_\pm$. Observe that the differential Δ_1 on C_1 basically equals Δ_0 on every generator, except that

$$\Delta_1\left(\widehat{a^-_m}\right) = \begin{cases} a^-_m + a^+_m + O(1) & \text{if} \quad a_m \neq a_{i_0} \\ a^-_m + a^+_m + \epsilon + O(1) & \text{if} \quad a_m = a_{i_0} \end{cases}$$

because

$$\Delta_0\left(\widehat{a^0_m}\right) = \begin{cases} a^0_m + a^+_m + O(1) & \text{if} \quad a_m \neq a_{i_0} \\ a^0_m + a^+_m + \epsilon + O(1) & \text{if} \quad a_m = a_{i_0} \end{cases}$$

and

$$\Delta_0\left(\widehat{a^-_m}\right) = a^-_m + a^0_m + O(1).$$

We inductively build like this DG-algebras (C_k, Δ_k) by setting

$$C_{k+1} := C_k/I_k$$

where I_k is the ideal of C_k generated by $\widehat{a^0_{m-k}}$ and $\Delta_k\left(\widehat{a^0_{m-k}}\right)$. For every k, there is a tame DG-isomorphism from C_0 to a stabilization of C_k which restricts to $\mathrm{id}_{A_\pm}$ on $A_\pm$, and it is not hard to see that

$$(C_{-+}, \partial_{-+}) := (C_m, \Delta_m)$$

actually is a mapping cylinder DG-algebra for φ, as we explain now.

First, the DG-algebra C_k has generating family

$$\left(a_1^+, \dots, a_m^+, a_1^0, \dots, a_{m-k}^0, a_1^-, \dots, a_m^-, \widehat{a_1^0}, \dots, \widehat{a_{m-k}^0}, \widehat{a_1^-}, \dots, \widehat{a_m^-}\right).$$

Then, observe that

$$\forall k \leq m - i_0, \quad \Delta_k\left(\widehat{a_i^-}\right) = \begin{cases} a_i^- + a_i^0 + O(1) & \text{if} \quad i \leq m-k \\ a_i^- + a_i^+ + O(1) & \text{if} \quad i > m-k \end{cases}$$

and

$$\Delta_{m-i_0+1}\left(\widehat{a_i^-}\right) = \begin{cases} a_i^- + a_i^0 + O(1) & \text{if} \quad i < i_0 \\ a_{i_0}^- + a_{i_0}^+ + \epsilon + O(1) & \text{if} \quad i = i_0 \\ a_i^- + a_i^+ + O(1) & \text{if} \quad i > i_0. \end{cases}$$

Then, continuing the induction progressively replaces the generators a^0 into terms of the form $a^+ + O(1)$, so that in the end

$$\Delta_m\left(\widehat{a_i^-}\right) = \begin{cases} a_i^- + a_i^+ + O(1) & \text{if} \quad i \neq i_0 \\ a_{i_0}^- + a_{i_0}^+ + f\left(a_{i_0}^-\right) + O(1) & \text{if} \quad i = i_0. \end{cases}$$

This concludes the proof.

□

1.2 A_∞-(co)algebras

We refer to [47] for a detailed exposition on A_∞-(co)algebras, and to [32] section 2.3, [49] for results about Koszul duality in the context of A_∞-(co)algebras. All our A_∞-(co)algebras and (co)maps will be strictly unital unless otherwise stated. Besides, we give the definitions in the Adams-graded setting, which is a bigraded setting: the first grading is called the cohomological degree while the second grading is called the Adams degree..

The non-Adams-graded counterpart should be transparent.

1.2.1 Basic definitions

Algebras

Definition 1.17. An Adams-graded A_∞-algebra is the data of a bigraded vector space A, a bidegree $(0,0)$ element $e \in A$, and a family of bidegree $(2-d,0)$ linear maps $\mu^d : A^{\otimes d} \to A$ indexed by the positive integers d, such that

$$\sum_{0 \le i < j \le d} \mu^{d-(j-i)+1} \circ \left(\mathbf{1}^i \otimes \mu^{j-i} \otimes \mathbf{1}^{d-j} \right) = 0$$

for all $d \ge 1$, and

$$\begin{cases} \mu^2 \circ (e \otimes \mathbf{1}) = \mu^2 \circ (\mathbf{1} \otimes e) = \mathbf{1} \\ \mu^d \circ \left(\mathbf{1}^i \otimes e \otimes \mathbf{1}^{d-i-1} \right) = 0 \qquad \text{for all } d \ne 2,\ 0 \le i \le d-1. \end{cases}$$

Remark 1.18. An Adams-graded DG-algebra is an Adams-graded A_∞-algebra with $\mu^d = 0$ for all $d \ge 3$.

Definition 1.19. Let A_1, A_2 be two Adams-graded A_∞-algebras. An A_∞-map $f : A_1 \to A_2$ is a family of bidegree $(1-d,0)$ linear maps $f^d : A_1^{\otimes d} \to A_2$ indexed by the positive integers d, such that

$$\begin{aligned} &\sum_{0 \le i < j \le d} f^{d-(j-i)+1} \circ \left(\mathbf{1}^i \otimes \mu_1^{j-i} \otimes \mathbf{1}^{d-j} \right) \\ &\qquad = \sum_{0 = i_0 < \cdots < i_r = d} \mu_2^r \circ \left(f^{i_1 - i_0} \otimes \cdots \otimes f^{i_r - i_{r-1}} \right) \end{aligned}$$

for all $d \ge 1$, and

$$\begin{cases} f^1(e_1) = e_2 \\ f^d \circ \left(\mathbf{1}^i \otimes e_1 \otimes \mathbf{1}^{d-i-1} \right) = 0 \quad \text{for all } d \ge 2,\ 0 \le i \le d-1. \end{cases}$$

We say that f is a quasi-isomorphism if the chain map f^1 is a quasi-isomorphism.

Definition 1.20. Let $f_1 : A_1 \to A_2$ and $f_2 : A_2 \to A_3$ be two A_∞-maps between Adams graded A_∞-algebras. The composition $f_2 \circ f_1 : A_1 \to A_3$ is the A_∞-map defined by

$$(f_2 \circ f_1)^d = \sum_{0=i_0<\cdots<i_r=d} f_2^r \circ \left(f_1^{i_1-i_0} \otimes \cdots \otimes f_1^{i_r-i_{r-1}} \right)$$

for all $d \geq 1$.

Definition 1.21. Let A be an Adams-graded A_∞-algebra, and consider $\mathbf{F}$ as an Adams-graded A_∞-algebra concentrated in bidegree $(0,0)$ with $\mu_{\mathbf{F}}^d = 0$ when $d \neq 2$ and $\mu_{\mathbf{F}}^2$ is the multiplication of $\mathbf{F}$. An augmentation of A is an A_∞-map $\epsilon : A \to \mathbf{F}$. We say that ϵ is strict if $\epsilon^d = 0$ when $d \geq 2$. In this case, the augmentation ideal of A is $\overline{A} := \ker(\epsilon)$.

An A_∞-map $f : A_1 \to A_2$ between augmented Adams-graded A_∞-algebras is required to satisfy $\epsilon_1 = \epsilon_2 \circ f$.

Definition 1.22. Let A be an augmented Adams-graded A_∞-algebra.

1. We say that A is connected if its augmentation ideal $\overline{A}$ is concentrated in positive cohomological degrees, and we say that A is simply connected if its augmentation ideal $\overline{A}$ is concentrated in cohomological degrees strictly greater than 1.

2. We say that A is Adams connected if its augmentation ideal $\overline{A}$ is concentrated in positive Adams degrees.

Coalgebras

Definition 1.23. An Adams-graded quasi-A_∞-coalgebra is the data of a bigraded vector space C, a bidegree $(0,0)$ linear map $\epsilon : C \to \mathbf{F}$, and a family of bidegree $(2-d, 0)$ linear maps $\delta^d : C \to C^{\otimes d}$ indexed by the positive integers d, such that

$$\sum_{0 \leq i < j \leq d} \left(\mathbf{1}^i \otimes \delta^{j-i} \otimes \mathbf{1}^{d-j} \right) \circ \delta^{d-(j-i)+1} = 0$$

for all $d \geq 1$, and

$$\begin{cases} (\epsilon \otimes \mathbf{1}) \circ \delta^2 = (\mathbf{1} \otimes \epsilon) \circ \delta^2 = \mathbf{1} & \\ \left(\mathbf{1}^i \otimes \epsilon \otimes \mathbf{1}^{d-i-1} \right) \circ \delta^d = 0 & \text{for all } d \neq 2,\ 0 \leq i \leq d-1. \end{cases}$$

We say that C is an A_∞-coalgebra if moreover the map

$$C \to \prod_{d \geq 1} C^{\otimes d}, \quad x \mapsto \left(\delta^d (x) \right)_{d \geq 1}$$

factors through the inclusion $\bigoplus_{d\geq 1} C^{\otimes d} \to \prod_{d\geq 1} C^{\otimes d}$.

Remark 1.24. An Adams-graded DG-coalgebra is an Adams-graded A_∞-coalgebra with $\delta^d = 0$ for all $d \geq 3$.

Definition 1.25. Let C_1, C_2 be two Adams-graded quasi-A_∞-coalgebras. An A_∞-comap $f : C_1 \to C_2$ is a family of bidegree $(1-d, 0)$ linear maps $f^d : C_1 \to C_2^{\otimes d}$ indexed by the positive integers d, such that

$$\begin{aligned}\sum_{0\leq i<j\leq d} &\left(\mathbf{1}^i \otimes \delta_D^{j-i} \otimes \mathbf{1}^{d-j}\right) \circ f^{d-(j-i)+1} \\ &= \sum_{0=i_0<\cdots<i_r=d} \left(f^{i_1-i_0} \otimes \cdots \otimes f^{i_r-i_{r-1}}\right) \circ \delta_C^r\end{aligned}$$

for all $d \geq 1$, and

$$\begin{cases} \epsilon_2 \circ f^1 = \epsilon_1 \\ \left(\mathbf{1}^i \otimes \epsilon_2 \otimes \mathbf{1}^{d-i-1}\right) \circ f^d = 0 \quad \text{for all } d \geq 2,\ 0 \leq i \leq d-1. \end{cases}$$

We say that f is a quasi-isomorphism if the map f^1 (which induces a map on homology) is a quasi-isomorphism. If C_1, C_2 are true A_∞-coalgebras, we moreover require that the map

$$C_1 \to \prod_{d\geq 1} C_2^{\otimes d}, \quad x \mapsto \left(f^d(x)\right)_{d\geq 1}$$

factors through the inclusion $\bigoplus_{d\geq 1} C^{\otimes d} \to \prod_{d\geq 1} C^{\otimes d}$.

Definition 1.26. Let $f_1 : C_1 \to C_2$ and $f_2 : C_2 \to C_3$ be two A_∞-comaps between Adams graded quasi-A_∞-coalgebras. The composition $f_2 \circ f_1 : C_1 \to C_3$ is the A_∞-comap defined by

$$(f_2 \circ f_1)^d = \sum_{0=i_0<\cdots<i_r=d} \left(f_2^{i_1-i_0} \otimes \cdots \otimes f_2^{i_r-i_{r-1}}\right) \circ f_1^r$$

for all $d \geq 1$.

Definition 1.27. Let C be an Adams-graded A_∞-coalgebra, and consider $\mathbf{F}$ as an Adams-graded A_∞-coalgebra concentrated in bidegree $(0,0)$. A coaugmentation of C is an A_∞-comap $\eta : \mathbf{F} \to C$. We say that η is strict if $\eta^d = 0$ when $d \geq 2$. In this case, the coaugmentation ideal of C is $\overline{C} := \operatorname{coker}(\eta)$.

An A_∞-comap $f : C_1 \to C_2$ between coaugmented Adams-graded A_∞-coalgebras is required to satisfy $f \circ \eta_1 = \eta_2$.

Definition 1.28. Let C be a coaugmented Adams-graded A_∞-coalgebra.

1. We say that C is cohomologically connected if its coaugmentation ideal $\overline{C}$ is concentrated in negative cohomological degrees, and we say that C is cohomologically simply connected if its coaugmentation ideal $\overline{C}$ is concentrated in cohomological degrees strictly less than -1.

2. We say that C is Adams connected if its coaugmentation ideal $\overline{C}$ is concentrated in positive Adams degrees.

1.2.2 Bar, cobar, graded dual and Koszul dual

Bar construction

Definition 1.29. Let A be an augmented Adams-graded A_∞-algebra. The bar construction BA of A is a coaugmented DG-coalgebra. Its underlying graded vector space is

$$BA = \mathbf{F} \oplus \left(\bigoplus_{d \geq 1} \left(\overline{A}\,[1] \right)^{\otimes d} \right)$$

where $\overline{A} = \ker(\epsilon)$ and

$$\overline{A}\,[1] = \bigoplus_{(n,k) \in \mathbf{Z} \times \mathbf{Z}} \overline{A}\,(n+1,k)$$

(we only shift the cohomological degree). The differential δ^1 is defined by

$$\delta^1 \left(x_0 \otimes \cdots \otimes x_{d-1} \right) = \sum_{0 \leq i < j \leq d} x_0 \otimes \cdots \otimes x_{i-1} \otimes \mu_A^{j-i} \left(x_i, \ldots, x_{j-1} \right) \otimes x_j \otimes \cdots \otimes x_{d-1}$$

and the coproduct δ^2 by

$$\delta^2 \left(x_0 \otimes \cdots \otimes x_{d-1} \right) = \sum_{1 \leq i \leq d-1} \left(x_0 \otimes \cdots \otimes x_{i-1} \right) \otimes \left(x_i \otimes \cdots \otimes x_{d-1} \right).$$

Finally, the strict coaugmentation $\mathbf{F} \to BA$ is given by the inclusion.

Any A_∞-map $f : A_1 \to A_2$ between augmented Adams-graded A_∞-algebras induces a DG-comap $Bf : BA_1 \to BA_2$ defined by

$$Bf \left(x_0 \otimes \cdots \otimes x_{d-1} \right) = \sum_{0 = i_0 < \cdots < i_r = d} f^{i_1 - i_0} \left(x_0, \ldots, x_{i_1 - 1} \right) \otimes \cdots \otimes f^{i_r - i_{r-1}} \left(x_{i_{r-1}}, \ldots, x_{d-1} \right).$$

Proposition 1.30. *Let A_1, A_2 be two augmented Adams-graded A_∞-algebras. If $f : A_1 \to A_2$ is a quasi-isomorphism, then the naturally induced DG-comap $Bf : BA_1 \to BA_2$ is also a quasi-isomorphism.*

Proof. This is Proposition 2.2.4 in [48].

□

Remark 1.31. Let A be a locally finite augmented A_∞-algebra. If A is cohomologically simply connected or Adams connected (see Definition 1.22), then BA is locally finite (each fixed bidegree component is finite dimensional).

Cobar construction

Definition 1.32. Let C be a coaugmented Adams-graded A_∞-coalgebra. The cobar construction ΩC of C is an augmented DG-algebra. Its underlying graded vector space is

$$\Omega C = \mathbf{F} \oplus \left(\bigoplus_{d \geq 1} \left(\overline{C}\,[-1] \right)^{\otimes d} \right)$$

where $\overline{C} = \operatorname{coker}(\eta)$ and

$$\overline{C}\,[-1] = \bigoplus_{(n,k) \in \mathbf{Z} \times \mathbf{Z}} \overline{C}\,(n-1, k)$$

(we only shift the cohomological degree). The differential μ^1 is defined by

$$\mu^1 \left(x_0 \otimes \cdots \otimes x_{d-1} \right) = \sum_{0 \leq i \leq d-1, j \geq 1} x_0 \otimes \cdots \otimes x_{i-1} \otimes \delta_C^j \left(x_i \right) \otimes x_{i+1} \otimes \cdots \otimes x_{d-1}$$

(observe that this is well defined because $\delta_C^j \left(x_i \right)$ vanishes for j large enough) and the product μ^2 by

$$\mu^2 \left(x_0 \otimes \cdots \otimes x_{i-1}, x_i \otimes \cdots \otimes x_{d-1} \right) = x_0 \otimes \cdots \otimes x_{d-1}.$$

Finally, the strict augmentation $\Omega C \to \mathbf{F}$ is given by the natural projection.

Any A_∞-comap $f : C_1 \to C_2$ between coaugmented Adams-graded A_∞-coalgebras induces a DG-map $\Omega f : \Omega C_1 \to \Omega C_2$ defined by

$$\Omega f \left(x_0 \otimes \cdots \otimes x_{d-1} \right) = \sum_{i_0, \ldots, i_{d-1} > 0} f^{i_0} \left(x_0 \right) \otimes \cdots \otimes f^{i_{d-1}} \left(x_{d-1} \right).$$

In general, the cobar functor does not preserve quasi-isomorphisms. However, it does in certain situations: we will consider two of them. The first one is summarized in the following result.

Lemma 1.33. *Let $f : C_1 \to C_2$ be an A_∞-comap between two coaugmented Adams-graded A_∞-coalgebras. Assume that C_1 and C_2 are locally finite and cohomologically simply connected (see Definition 1.28). If f is a quasi-isomorphism, then $\Omega f : \Omega C_1 \to \Omega C_2$ is a quasi-isomorphism.*

Proof. For $i \in \{1, 2\}$, consider the decreasing, exhaustive, bounded above filtration

$$\Omega \mathcal{C}_i = \mathcal{F}_i^0 \supset \mathcal{F}_i^1 \supset \cdots$$

given by

$$\mathcal{F}_i^p = \bigoplus_{d \geq p} \left(\overline{C_i}\,[-1]\right)^{\otimes d}, \quad p \geq 1.$$

$\mathcal{F}_i$ is degree-wide bounded because C_i is assumed to be locally finite and cohomologically simply connected. Therefore, the spectral sequence associated to $\mathcal{F}_i$ converges to $H^*\left(\Omega C_i\right)$ (see for example [13] Theorem 14.6). Moreover, Ωf sends $\mathcal{F}_1$ to $\mathcal{F}_2$ and induces a quasi-isomorphism on the corresponding graded objects when f is a quasi-isomorphism. This concludes the proof.

□

The second situation in which cobar preserved quasi-isomorphisms is the filtered setting, as explained in [47] section 1.3.2 for the DG-coalgebra case.

Definition 1.34. Let C be an A_∞-coalgebra. Let $\mathcal{F} = \left(\mathcal{F}^p\right)_{p \geq 0}$ be an increasing filtration of the graded chain complex $\overline{C}$. Following the terminology of [47] section 1.3.2, we say that

1. $\mathcal{F}$ is admissible if $\mathcal{F}$ is exhaustive and $\mathcal{F}^0 = 0$,
2. $(C, \mathcal{F})$ is a filtered A_∞-coalgebra if

$$\delta_C^d\left(\mathcal{F}^p\right) \subset \bigoplus_{q_1 + \cdots + q_d = p} \mathcal{F}^{q_1} \otimes \cdots \otimes \mathcal{F}^{q_d}$$

for every $d \geq 1$ and $p \geq 0$.

Moreover, we say that an A_∞-comap $f : C_1 \to C_2$ between filtered A_∞-coalgebras $(C_1, \mathcal{F}_1)$ and $(C_2, \mathcal{F}_2)$ is a filtered quasi-isomorphism if f^1 sends $\mathcal{F}_1$ to $\mathcal{F}_2$ and induces a quasi-isomorphism between the corresponding graded objects.

Then we have the following result.

Proposition 1.35. *If f is a filtered quasi-isomorphism between admissible filtered A_∞-coalgebras, then Ωf is a quasi-isomorphism.*

Proof. The exact same proof as that of Lemma 1.3.2.2 in [47], which is the desired result in the DG case, works here as well.

□

Remark 1.36. Let $f : C_1 \to C_2$ be a quasi-isomorphism between two coaugmented Adams-graded A_∞-coalgebras. If C_1 and C_2 are Adams-connected, then the DG-map $\Omega f : \Omega C_1 \to \Omega C_2$ is a quasi-isomorphism. Indeed, we can consider the filtrations

$$\mathcal{F}_i^p := \bigoplus_{k=0}^{p} \{x \in \overline{C_i} \mid |x|_{Adams} = k\}, \quad p \in \mathbf{N},$$

of $\overline{C_i}$ for $i \in \{1, 2\}$. Then

1. for each $i \in \{1, 2\}$, the filtration $\mathcal{F}_i = (\mathcal{F}_i^p)_{p \in \mathbf{N}}$ is admissible because C_i is Adams-connected,

2. for each $i \in \{1, 2\}$, $(C_i, \mathcal{F}_i)$ is a filtered A_∞-coalgebra, and

3. the map f is a filtered quasi-isomorphism from $(C_1, \mathcal{F}_1)$ to $(C_2, \mathcal{F}_2)$ because it preserves the Adams-degree and it is assumed to be a quasi-isomorphism,

and therefore we can apply Proposition 1.35.

Graded dual

Definition 1.37. If V is an Adams-graded vector space, its graded dual is the Adams-graded vector space $V^\#$ whose bidegree (n, k) component is the space of linear forms on the bidegree $(-n, k)$ component of V.

The map $C \mapsto C^\#$ defines a functor from Adams-graded A_∞-coalgebra to Adams-graded A_∞-algebra. Besides, $A \mapsto A^\#$ defines a functor from locally finite (each fixed bidegree component is finite dimensional) Adams-graded A_∞-algebra to Adams-graded quasi-A_∞-coalgebra. Moreover, $A \mapsto A^{\#}$ defines a functor from locally finite, cohomologically simply connected or Adams connected, augmented Adams-graded A_∞-algebra to coaugmented Adams-graded A_∞-coalgebra (the connectivity assumption guarantees that the large order coproducts of an element are zero).

Proposition 1.38. *The functors $C \mapsto C^{\#}$ and $A \mapsto A^{\#}$ preserve quasi-isomorphisms.*

Proof. This follows from the facts that, over a field, cohomology commutes with graded dual and quasi-isomorphisms have quasi-inverse.

□

Koszul dual

Definition 1.39 (See Definition 13, Proposition 14 in [32] and the beginning of section 2 in [49])**.** Let A be an augmented Adams-graded A_∞-algebra. Its Koszul dual is the DG-algebra $E(A) = (BA)^{\#}$.

Proposition 1.40. *Let A_1, A_2 be two quasi-isomorphic augmented Adams-graded A_∞-algebras. Then their Koszul dual $E(A_1)$ and $E(A_2)$ are also quasi-isomorphic.*

Proof. This follows from Propositions 1.30 and 1.38.

□

Remark 1.41. If C is a coaugmented Adams-graded DG-coalgebra, then there is a natural DG-map $\Omega C \to B(C^{\#})^{\#}$ which sends $x \in \overline{C}[-1]$ to

$$(\varphi \mapsto \varphi(x)) \in \left(\overline{C^{\#}}[1]\right)^{\#} \subset B\left(C^{\#}\right)^{\#}.$$

Besides, if A is an augmented Adams-graded A_∞-algebra, there is a natural A_∞-map $A \to \Omega(BA)$ which sends $(x_1, \dots, x_d) \in A^d$ to

$$x_1 \otimes \cdots \otimes x_d \in \overline{BA}[-1] \subset \Omega(BA).$$

Thus, if A is an augmented Adams-graded A_∞-algebra, there is a natural A_∞-map $A \to E(E(A))$ which is the composition

$$A \to \Omega(BA) \to B\left((BA)^{\#}\right)^{\#} = E(E(A)).$$

Definition 1.42 (See Definition 17 in [32])**.** Let A be an augmented Adams-graded A_∞-algebra. We say that Koszul duality holds for A if the natural A_∞-map $A \to E\left(E\left(A\right)\right)$ (see Remark 1.41) is a quasi-isomorphism.

Proposition 1.43. *If A is an augmented Adams-graded A_∞-algebra such that BA is locally finite, then Koszul duality holds for A. In particular, Koszul duality holds for locally finite, cohomologically simply connected or Adams connected, augmented Adams-graded A_∞-algebras according to Remark 1.31.*

Proof. This is Theorem A in [49].

□

Proposition 1.44. *Let A be a locally finite, cohomologically simply connected or Adams connected, augmented Adams-graded A_∞-algebra. Then there is a quasi-isomorphism*

$$\Omega\left(A^{\#}\right) \to E\left(A\right).$$

Proof. This follows from Lemma 10 in [32] and Remark 1.31.

□

1.2.3 Cone of an A_∞-map

Definition 1.45 (See [42] Definition 3.1.3)**.** Let $f : A_1 \to A_2$ be an A_∞-map between two A_∞-algebras. The cone of f is the non-unital A_∞-algebra with underlying graded vector space

$$\mathrm{Cone}\left(f\right) = A_1 \oplus \left(A_2\left[-1\right]\right),$$

and with operations given by

$$\begin{aligned}\mu^d_{\mathrm{Cone}(f)}\left((x_1^0, x_2^0), \ldots, (x_1^{d-1}, x_2^{d-1})\right) = \Bigg(&\mu_1^d\left(x_1^0, \ldots, x_1^{d-1}\right), f^d\left(x_1^0, \ldots, x_1^{d-1}\right) \\ &+ \sum_{0=i_0<\cdots<i_r=d-1} \mu_2^{r+1}\left(f^{i_1-i_0}\left(x_1^0, \ldots, x_1^{i_1-1}\right), \ldots, f^{i_r-i_{r-1}}\left(x_1^{i_{r-1}}, \ldots, x_1^{d-2}\right), x_2^{d-1}\right)\Bigg).\end{aligned}$$

Remark 1.46. In this Remark, we will use the version $\overline{B}$ of the bar construction for non-unital (and thus non-augmented) A_∞-algebras. Its definition is easily adapted from Definition 1.29 (it is also defined in [46] section 3.6). Observe that if A is a non unital

A_∞-algebra, then $(\mathbf{F} \oplus A)$ is naturally an augmented A_∞-algebra and

$$\mathbf{F} \oplus \overline{B}A = B(\mathbf{F} \oplus A).$$

Let $f : A_1 \to A_2$ be an A_∞-map between A_∞-algebras. Consider the projections $p_1 : \overline{B}\mathrm{Cone}(f) \to A_1[1]$, $p_2 : \overline{B}\mathrm{Cone}(f) \to A_2$, $\pi_1 : \overline{B}A_1 \to A_1[1]$, $\pi_2 : \overline{B}A_2 \to A_2[1]$, and the degree (-1) map $s : A_2 \to A_2[1]$. Besides, observe that the projection $p : \mathrm{Cone}(f) \to A_1$ is an A_∞-map, and let $G : \overline{B}\mathrm{Cone}(f) \to \overline{B}A_2$ be the unique linear map such that

$$\pi_2 \circ G = s \circ p_2 \text{ and } \delta^2_{\overline{B}A_2} \circ G = \left(\overline{B}(f \circ p) \otimes G + G \otimes 0\right) \circ \delta^2.$$

Then the A_∞-operations on $C := \mathrm{Cone}(f)$ translate as

$$\begin{cases} p_1 \circ \delta^1_{\overline{B}C} &= \pi_1 \circ \delta^1_{\overline{B}A_1} \circ \overline{B}p \\ s \circ p_2 \circ \delta^1_{\overline{B}C} &= \pi_2 \circ \left(\overline{B}(f \circ p) + \delta^1_{\overline{B}A_2} \circ G\right). \end{cases}$$

Since $p_1 = \pi_1 \circ \overline{B}p$, and $s \circ p_2 = \pi_2 \circ G$, we get

$$\begin{cases} \pi_1 \circ \overline{B}p \circ \delta^1_{\overline{B}C} &= \pi_1 \circ \delta^1_{\overline{B}A_1} \circ \overline{B}p \\ \pi_2 \circ G \circ \delta^1_{\overline{B}C} &= \pi_2 \circ \left(\overline{B}(f \circ p) + \delta^1_{\overline{B}A_2} \circ G\right). \end{cases}$$

It follows that

$$\begin{cases} \overline{B}p \circ \delta^1_{\overline{B}C} &= \delta^1_{\overline{B}A_1} \circ \overline{B}p \\ G \circ \delta^1_{\overline{B}C} &= \overline{B}(f \circ p) + \delta^1_{\overline{B}A_2} \circ G. \end{cases}$$

In particular, G is an homotopy between $\overline{B}(f \circ p)$ and 0 (see [54] Lemma 4.2), which implies that $(f \circ p)$ is homotopic to 0.

Remark 1.47. Let M be a compact manifold with boundary, and let $i : \partial M \hookrightarrow M$ be the inclusion. If we see $\Omega^*(M)$ and $\Omega^*(\partial M)$ as A_∞-algebras, then the cone of $i^* : \Omega^*(M) \to \Omega^*(\partial M)$ (in the sense of Definition 1.45) is the graded vector space

$$C^* = \Omega^*(M) \oplus \Omega^{*-1}(\partial M)$$

with differential

$$\mu^1(\alpha, \beta) = (d\alpha, -d\beta + i^*\alpha)$$

(compare with [13] Relative de Rham theory), product

$$\mu^2\left(\left(\alpha_1,\beta_1\right),\left(\alpha_2,\beta_2\right)\right)=\left(\alpha_1\wedge\alpha_2,(-1)^{|\alpha_1|}\,i^*\alpha_1\wedge\beta_2\right),$$

and zero higher operations, so that C^* can be seen as a non-unital DG-algebra. Moreover, if we consider the non-unital DG-algebra

$$\Omega^*\left(M,\partial M\right)=\{\alpha\in\Omega^*\left(M\right)\mid i^*\alpha=0\},$$

then the map

$$\Omega^*\left(M,\partial M\right)\to C^*,\quad \alpha\mapsto(\alpha,0)$$

is a quasi-isomorphism of non-unital DG-algebras.

Proposition 1.48. *Consider the following commutative diagram of A_∞-algebras*

$$\begin{array}{ccc} A_1 & \xrightarrow{f} & A_2 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} \\ A_1' & \xrightarrow{f'} & A_2' \end{array}.$$

Then the collection of maps $\left(\phi^d:\mathrm{Cone}(f)^{\otimes d}\to\mathrm{Cone}(f')\right)_{d\geq1}$ defined by

$$\begin{aligned}\phi^d\left((x_1^0,x_2^0),\dots,(x_1^{d-1},x_2^{d-1})\right)=\Big(&g_1^d\left(x_1^0,\dots,x_1^{d-1}\right),\\ &\sum_{0=i_0<\dots<i_r=d-1}g_2^{r+1}\left(f^{i_1-i_0}\left(x_1^0,\dots,x_1^{i_1-1}\right),\dots,f^{i_r-i_{r-1}}\left(x_1^{i_{r-1}},\dots,x_1^{d-2}\right),x_2^{d-1}\right)\Big)\end{aligned}$$

defines an A_∞-map $\phi:\mathrm{Cone}(f)\to\mathrm{Cone}(f')$. If moreover g_1 and g_2 are quasi-isomorphisms, then ϕ is a quasi-isomorphism.

Proof. This is a straightforward verification. □

1.3 A_∞-categories

We refer to [59] for a detailed exposition on A_∞-categories, and to [37], [38] for the localization of A_∞-categories and more specific results about modules over A_∞-categories.

All our A_∞-categories will be strictly unital. We give the definitions in the Adams-graded setting, which is a bigraded setting: the first grading is called the cohomological degree while the second grading is called the Adams degree.. The non-Adams-graded counterpart should be transparent.

1.3.1 Basic definitions

Definition 1.49. An Adams-graded A_∞-category $\mathcal{C}$ is the data of

1. a collection of objects $\operatorname{ob}\mathcal{C}$,
2. for every objects X, Y, a bigraded vector space of morphisms $\mathcal{C}(X,Y)$,
3. for every object X, a bidegree $(0,0)$ element $e_X \in \mathcal{C}(X,X)$, and
4. a family of bidegree $(2-d,0)$ linear maps

$$\mu^d : \mathcal{C}(X_0,X_1) \otimes \cdots \otimes \mathcal{C}(X_{d-1},X_d) \to \mathcal{C}(X_0,X_d)$$

indexed by the sequences of objects $(X_0,\dots,X_d)$, such that

$$\sum_{0\le i<j\le d} \mu^{d-(j-i)+1} \circ \left(\mathbf{1}^i \otimes \mu^{j-i} \otimes \mathbf{1}^{d-j}\right) = 0$$

for all $d \geq 1$, and

$$\begin{cases} \mu^2 \circ \left(e_X \otimes \mathbf{1}_{\mathcal{C}(X,Y)}\right) = \mu^2 \circ \left(\mathbf{1}_{\mathcal{C}(X,Y)} \otimes e_Y\right) = \mathbf{1}_{\mathcal{C}(X,Y)} & \\ \mu^d(\dots, e_X, \dots) = 0 & \text{for all } d \neq 2. \end{cases}$$

Remark 1.50. An Adams-graded A_∞-algebra is an Adams-graded A_∞-category with only one object.

Definition 1.51. Let $\mathcal{C}, \mathcal{D}$ be two Adams-graded A_∞-categories. An A_∞-functor $\Phi : \mathcal{C} \to \mathcal{D}$ is the data of

1. a map $\Phi : \operatorname{ob}\mathcal{C} \to \operatorname{ob}\mathcal{D}$, and
2. a family of bidegree $(1-d,0)$ linear maps

$$\Phi^d : \mathcal{C}(X_0,X_1) \otimes \cdots \otimes \mathcal{C}(X_{d-1},X_d) \to \mathcal{D}(\Phi X_0, \Phi X_d)$$

indexed by the sequences of objects $(X_0, \dots, X_d)$, such that

$$\sum_{0 \le i < j \le d} \Phi^{d-(j-i)+1} \circ \left(\mathbf{1}^i \otimes \mu^{j-i} \otimes \mathbf{1}^{d-j}\right) \\ = \sum_{0=i_0<\cdots<i_r=d} \mu^r \circ \left(\Phi^{i_1-i_0} \otimes \cdots \otimes \Phi^{i_r-i_{r-1}}\right)$$

for all $d \geq 1$, and

$$\begin{cases} \Phi^1(e_X) = e_{\Phi X} \\ \Phi^d(\dots, e_X, \dots) = 0 \quad \text{for all } d \geq 2. \end{cases}$$

We say that Φ is strict if $\Phi^d = 0$ for every $d \geq 2$. Moreover, we say that Φ is a quasi-equivalence if the induced functor between the cohomological categories is an equivalence.

Definition 1.52 (See [59] section (1h))**.** Let $\Phi, \Psi : \mathcal{C} \to \mathcal{D}$ be two A_∞-functors acting in the same way on objects. A homotopy between Φ and Ψ is a family of bidegree $(-d, 0)$ linear maps

$$T^d : \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{d-1}, X_d) \to \mathcal{D}(\Phi X_0, \Psi X_d),$$

indexed by the sequences of objects $(X_0, \dots, X_d)$, such that

$$\Phi + \Psi = \sum T(\dots, \mu_{\mathcal{C}}(\dots), \dots) \\ + \sum \mu_{\mathcal{D}}(\Phi(\dots), \dots, \Phi(\dots), T(\dots), \Psi(\dots), \dots, \Psi(\dots)).$$

1.3.2 From Adams-graded to non Adams-graded and back

If V is an Adams-graded vector space, we denote by V' the graded vector space whose degree n component is the direct sum of the bidegree (m, k) components of V, where the sum is over the set of couples $(m, k) \in \mathbf{Z} \times \mathbf{Z}$ such that $m - 2k = n$.

Definition 1.53. If $\mathcal{C}$ is an Adams-graded A_∞-category, we denote by $\mathcal{C}'$ the (non Adams-graded) A_∞-category obtained from $\mathcal{C}$ by changing the grading so that

$$\mathcal{C}'(X_0, X_1) = \mathcal{C}(X_0, X_1)'$$

Observe that any A_∞-functor $\Phi : \mathcal{C}_1 \to \mathcal{C}_2$ between two Adams-graded A_∞-categories induces an A_∞-functor from $\mathcal{C}_1'$ to $\mathcal{C}_2'$ (that we still denote by Φ) which acts exactly as Φ on objects and morphisms. This defines a functor $\mathcal{C} \mapsto \mathcal{C}'$ from the category of Adams-graded A_∞-categories to the category of (non Adams-graded) A_∞-categories.

We denote by $\mathbf{F}[t]$ the augmented Adams-graded associative algebra generated by a variable t of bidegree $(2,1)$, and by $\overline{\mathbf{F}[t]}$ its augmentation ideal (or equivalently, the ideal generated by t).

Definition 1.54. If $\mathcal{D}$ is a (non Adams-graded) A_∞-category, we denote by $\mathbf{F}[t] \otimes \mathcal{D}$ the Adams-graded A_∞-category such that

1. the objects of $\mathbf{F}[t] \otimes \mathcal{D}$ are those of $\mathcal{D}$,

2. the space of morphisms from Y_1 to Y_2 is $\mathbf{F}[t] \otimes \mathcal{D}(Y_1, Y_2)$, and if $y \in \mathcal{D}(Y_1, Y_2)$ is of degree j, $t^k \otimes y$ is of bidegree $(j+2k, k)$,

3. the operations send any sequence $\left(t^{k_0} \otimes y_0, \ldots, t^{k_{d-1}} \otimes y_{d-1}\right)$ of morphisms to

$$\mu_{\mathbf{F}[t]\otimes\mathcal{D}}\left(t^{k_0} \otimes y_0, \ldots, t^{k_{d-1}} \otimes y_{d-1}\right) = t^{k_0+\cdots+k_{d-1}} \otimes \mu_{\mathcal{D}}(y_0, \ldots, y_{d-1}).$$

Observe that any A_∞-functor $\Psi : \mathcal{D}_1 \to \mathcal{D}_2$ between (non Adams-graded) A_∞-categories induces an A_∞-functor $\mathbf{F}[t] \otimes \mathcal{D}_1 \to \mathbf{F}[t] \otimes \mathcal{D}_2$ which acts as Ψ on objects, and which sends any sequence $\left(t^{k_0} \otimes y_0, \ldots, t^{k_{d-1}} \otimes y_{d-1}\right)$ of morphisms to $t^{k_0+\ldots k_{d-1}} \otimes \Psi(y_0, \ldots, y_{d-1})$. This defines a functor $\mathcal{D} \mapsto \mathbf{F}[t] \otimes \mathcal{D}$ from the category of (non Adams-graded) A_∞-categories to the category of Adams-graded A_∞-categories.

Remark 1.55. If $\mathcal{D}$ is a (non Adams-graded) A_∞-category, we can define a non unital Adams-graded A_∞-category $\overline{\mathbf{F}[t]} \otimes \mathcal{D}$ exactly as in the definition above.

Definition 1.56. Let $\mathcal{C}$ be an Adams-graded A_∞-category concentrated in non-negative Adams-degree, and let $\mathcal{D}$ be a (non Adams-graded) A_∞-category. To any A_∞-functor $\Psi' : \mathcal{C}' \to \mathcal{D}$, we associate an A_∞-functor $\Psi : \mathcal{C} \to \mathbf{F}[t] \otimes \mathcal{D}$ which sends a sequence $(x_0, \ldots, x_{d-1})$, where x_j is of bidegree (i_j, k_j), to

$$\Psi(x_0, \ldots, x_{d-1}) = t^{k_0+\cdots+k_{d-1}} \otimes \Psi'(x_0, \ldots, x_{d-1}).$$

This defines an adjunction between the category of Adams-graded A_∞-categories concentrated in non-negative Adams-degree and the category of (non Adams-graded) A_∞-categories.

1.3.3 Quotient and localization

Quotient of A_∞-categories Let $\mathcal{C}, \mathcal{D}$ be two Adams-graded A_∞-categories, and let $\mathcal{A}, \mathcal{B}$ be two full subcategories of $\mathcal{C}, \mathcal{D}$ respectively.

Definition 1.57. The A_∞-quotient $\mathcal{C}/\mathcal{A}$ is the Adams-graded A_∞-category such that

1. the objects of $\mathcal{C}/\mathcal{A}$ are those of $\mathcal{C}$,

2. the space of morphisms from X to Y are

$$\mathcal{C}/\mathcal{A}(X,Y) = \bigoplus_{\substack{p\geq 1 \\ A_1,\dots,A_{p-1}\in\mathcal{A}}} \mathcal{C}(X,A_1)\otimes\mathcal{C}(A_1,A_2)[1]\otimes\cdots\otimes\mathcal{C}(A_{p-1},Y)[1]$$

(by convention, the term $p=1$ is $\mathcal{C}(X,Y)$), and

3. the operations send any sequence

$$\mathbf{x}_i = \left(x_i^0,\dots,x_i^{p_i-1}\right)\in\mathcal{C}/\mathcal{A}(X_i,X_{i+1}) \quad (0\leq i\leq d-1)$$

to

$$\mu_{\mathcal{C}/\mathcal{A}}(\mathbf{x}_0,\dots,\mathbf{x}_{d-1}) = \sum_{\substack{0\leq i\leq p_0, 1\leq j\leq p_{d-1} \\ i<j \text{ if } d=1}} x_0^0\otimes\cdots\otimes x_0^{i-1}\otimes\mu_{\mathcal{C}}\left(x_0^i,\dots,x_{d-1}^{j-1}\right)\otimes x_{d-1}^j\otimes\cdots\otimes x_{d-1}^{p_{d-1}-1}.$$

Remark 1.58. We have $(\mathcal{C}/\mathcal{A})' = \mathcal{C}'/\mathcal{A}'$ (see Definition 1.53 for the meaning of the "prime" symbol).

Definition 1.59 (See [50] section 3)**.** Let $\Phi : \mathcal{C}\to\mathcal{D}$ be an A_∞-functor such that $\Phi(\mathcal{A})$ is contained in $\mathcal{B}$. Then there is an induced A_∞-functor $\widetilde{\Phi} : \mathcal{C}/\mathcal{A}\to\mathcal{D}/\mathcal{B}$ which sends any sequence

$$\mathbf{x}_i = \left(x_i^0,\dots,x_i^{p_i-1}\right)\in\mathcal{C}/\mathcal{A}(X_i,X_{i+1}) \quad (0\leq i\leq d-1)$$

to

$$\widetilde{\Phi}(\mathbf{x}_0,\dots,\mathbf{x}_{d-1}) = \sum_{(i_1,\dots,i_r)\in L(p_0,\dots,p_{d-1})} \Phi\left(x^0,\dots,x^{i_1-1}\right)\otimes\cdots\otimes\Phi\left(x^{i_{r-1}},\dots,x^{i_r-1}\right)$$

where

$$L(p_0, \dots, p_{d-1}) = \bigcup_{r \geq 1} \{(i_1, \dots, i_r) \in \mathbf{Z}^r_{>0} \mid i_s = p_0 + \dots + p_{t-1} \text{ iff } s = r \text{ and } t = d\}$$

and

$$\left(x^0, \dots, x^{p_0 + \dots + p_{d-1} - 1}\right) = \left(x_0^0, \dots, x_0^{p_0 - 1}, \dots, x_{d-1}^0, \dots, x_{d-1}^{p_{d-1}-1}\right).$$

Remark 1.60. Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an A_∞-functor, and let $\Psi : \mathcal{D} \to \mathcal{E}$ be another A_∞-functor towards a third A_∞-category $\mathcal{E}$. Assume that $\Phi(\mathcal{A})$ is contained in $\mathcal{B}$. Then $\widetilde{\Psi} \circ \widetilde{\Phi} = \widetilde{\Psi \circ \Phi} : \mathcal{C}/\mathcal{A} \to \mathcal{E}/\Psi(\mathcal{B})$ as A_∞-functors (see [50] section 3).

Localization of A_∞-categories

Definition 1.61 (See [9] section 3.2)**.** We denote by $\operatorname{Tw}\mathcal{C}$ the Adams-graded A_∞-category such that:

1. the objects of $\operatorname{Tw}\mathcal{C}$ are the twisted complexes $(\mathfrak{X}, \delta)$ of $\mathcal{C}$, which consist in
   (a) a finite family $\mathfrak{X} = (X_i, k_i)_{1 \leq i \leq N}$ where X_i is an object of $\mathcal{C}$ and k_i is an integer,
   (b) a strictly upper triangular matrix $\delta = (\delta_{i,j})_{1 \leq i < j \leq N}$, where $\delta_{i,j}$ is a bidegree $(k_j - k_i + 1, 0)$ morphism in $\mathcal{C}(X_i, X_j)$, such that
   $$\sum_{d \geq 1} \mu^d_{\mathcal{C}}(\delta, \dots, \delta) = 0$$
   (in matrix notation),
2. a bidegree (s, ℓ) morphism from $\mathfrak{X}$ to $\mathfrak{X}'$ is a matrix $(x_{i,j})_{1 \leq i \leq N, 1 \leq j \leq N'}$, where $x_{i,j}$ is a bidegree $\left(s + k'_j - k_i, \ell\right)$ morphism in $\mathcal{C}(X_i, X_j)$,
3. the operations send any sequence
   $$\left(x^0, \dots, x^{d-1}\right) \in \operatorname{Tw}\mathcal{C}(\mathfrak{X}_0, \mathfrak{X}_1) \times \dots \times \operatorname{Tw}\mathcal{C}(\mathfrak{X}_{d-1}, \mathfrak{X}_d)$$
   to
   $$\mu_{\operatorname{Tw}\mathcal{C}}\left(x^0, \dots, x^{d-1}\right) = \sum_{j_0, \dots, j_d \geq 0} \mu_{\mathcal{C}}\left(\delta_0^{j_0}, x^0, \delta_1^{j_1}, x^1, \dots, x^{d-1}, \delta_d^{j_d}\right)$$
   (in matrix notation).

Definition 1.62. Let W be a set of closed bidegree $(0,0)$ morphisms in $\mathcal{C}$. We denote by $\operatorname{Cone} W$ the full subcategory of $\operatorname{Tw}\mathcal{C}$ whose objects are the twisted complexes

$$\operatorname{Cone} w := \left(\left(\left(X_1, 1\right), \left(X_2, 0\right)\right), w : X_1 \to X_2\right)$$

for w in W. The localization $\mathcal{C}\left[W^{-1}\right]$ of $\mathcal{C}$ at W is the full subcategory of $\left(\operatorname{Tw}\mathcal{C}\right) / \left(\operatorname{Cone} W\right)$ whose objects are those of $\mathcal{C}$.

1.3.4 Modules

Let $\mathcal{C}, \mathcal{D}$ be two Adams-graded A_∞-categories, and let $\mathcal{A}, \mathcal{B}$ be two full subcategories of $\mathcal{C}, \mathcal{D}$ respectively.

A_∞-bimodules

Definition 1.63. A $(\mathcal{C}, \mathcal{D})$-bimodule $\mathcal{M}$ consists of the following data:

1. for every pair $(X, Y) \in \operatorname{ob}(\mathcal{C}) \times \operatorname{ob}(\mathcal{D})$, an Adams-graded vector space $\mathcal{M}(X, Y)$,

2. a family of bidegree $(1-p-q, 0)$ linear maps

$$\begin{aligned}\mu_{\mathcal{M}} : \mathcal{C}\left(X_0, X_1\right) \otimes \cdots \otimes \mathcal{C}\left(X_{p-1}, X_p\right) \otimes \mathcal{M}\left(X_p, Y_q\right) \\ \otimes \mathcal{D}\left(Y_q, Y_{q-1}\right) \otimes \cdots \otimes \mathcal{D}\left(Y_1, Y_0\right) \to \mathcal{M}\left(X_0, Y_0\right)\end{aligned}$$

indexed by the sequences

$$\left(X_0, \ldots, X_p, Y_0, \ldots, Y_q\right) \in \operatorname{ob}(\mathcal{C})^{p+1} \times \operatorname{ob}(\mathcal{D})^{q+1},$$

which satisfy the relations

$$\begin{aligned}\sum \mu_{\mathcal{M}}\left(\ldots, \mu_{\mathcal{C}}\left(\ldots\right), \ldots, u \ldots\right) + \sum \mu_{\mathcal{M}}\left(\ldots, \mu_{\mathcal{M}}\left(\ldots, u, \ldots\right), \ldots\right) \\ + \sum \mu_{\mathcal{M}}\left(\ldots, u, \ldots, \mu_{\mathcal{D}}\left(\ldots\right), \ldots\right) = 0.\end{aligned}$$

A degree (s, ℓ) morphism $t : \mathcal{M}_1 \to \mathcal{M}_2$ between two $(\mathcal{C}, \mathcal{D})$-bimodules consists of a family of bidegree $(s-p-q,\ell)$ linear maps

$$\begin{aligned} t : \mathcal{C}\left(X_0, X_1\right) \otimes \cdots \otimes \mathcal{C}\left(X_{p-1}, X_p\right) \otimes \mathcal{M}_1\left(X_p, Y_q\right) \\ \otimes \mathcal{D}\left(Y_q, Y_{q-1}\right) \otimes \cdots \otimes \mathcal{D}\left(Y_1, Y_0\right) \to \mathcal{M}_2\left(X_0, Y_0\right) \end{aligned}$$

indexed by the sequences

$$\left(X_0, \ldots, X_p, Y_0, \ldots, Y_q\right) \in \operatorname{ob}\left(\mathcal{C}\right)^{p+1} \times \operatorname{ob}\left(\mathcal{D}\right)^{q+1}.$$

The differential of such a morphism is defined by

$$\begin{aligned} &\mu^1_{\mathrm{Mod}_{\mathcal{C},\mathcal{D}}}\left(t\right)\left(\ldots, u, \ldots\right) \\ &\quad = \sum t\left(\ldots, \mu_{\mathcal{C}}\left(\ldots\right), \ldots, u, \ldots\right) + \sum t\left(\ldots, \mu_{\mathcal{M}_1}\left(\ldots, u, \ldots\right), \ldots\right) \\ &\quad + \sum t\left(\ldots, u, \ldots, \mu_{\mathcal{D}}\left(\ldots\right), \ldots\right) + \sum \mu_{\mathcal{M}_2}\left(\ldots, t\left(\ldots, u, \ldots\right), \ldots\right). \end{aligned}$$

Definition 1.64. Let $\Phi_1, \Phi_2 : \mathcal{C} \to \mathcal{D}$ be two A_∞-functors. Then there is a $\mathcal{C}$-bimodule $\mathcal{D}\left(\Phi_1(-), \Phi_2(-)\right)$ defined as follows:

1. on the objects, it sends (X_1, X_2) to $\mathcal{D}\left(\Phi_1 X_1, \Phi_2 X_2\right)$,

2. on the morphism, it sends a sequence $(\ldots, y, \ldots)$ in

$$\begin{aligned} \mathcal{C}\left(X_0, X_1\right) \times \cdots \times \mathcal{C}\left(X_{p-1}, X_p\right) \times \mathcal{D}\left(\Phi_1 X_p, \Phi_2 X_{p+1}\right) \\ \times \mathcal{C}\left(X_{p+1}, X_{p+2}\right) \times \cdots \times \mathcal{C}\left(X_{p+q}, X_{p+q+1}\right) \end{aligned}$$

to

$$\begin{aligned} &\mu_{\mathcal{D}(\Phi_1(-),\Phi_2(-))}\left(\ldots, y, \ldots\right) \\ &\qquad = \sum \mu_{\mathcal{D}}\left(\Phi_1\left(\ldots\right), \ldots, \Phi_1\left(\ldots\right), y, \Phi_2\left(\ldots\right), \ldots, \Phi_2\left(\ldots\right)\right). \end{aligned}$$

A_∞-modules In the following, we will focus on *left* $\mathcal{C}$-modules, which correspond to $(\mathcal{C}, \mathbf{F})$-bimodules.

Definition 1.65 (See [59] section (1j))**.** A $\mathcal{C}$-module $\mathcal{M}$ is an A_∞-functor from $\mathcal{C}^{\mathrm{op}}$ to Ch. In other words, there is a chain complex $\mathcal{M}\left(X\right)$ for every object X in $\mathcal{C}$, and, for each sequence of objects $(X_0, \dots, X_d)$, there is a bidegree $(1-d, 0)$ linear map

$$\mu_{\mathcal{M}} : \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{d-1}, X_d) \otimes \mathcal{M}(X_d) \to \mathcal{M}(X_0).$$

Besides, the operations satisfy the relations

$$\sum \mu_{\mathcal{M}}(\dots, \mu_{\mathcal{C}}(\dots), \dots) + \mu_{\mathcal{M}}(\dots, \mu_{\mathcal{M}}(\dots)) = 0.$$

A degree (s, ℓ) morphism $t : \mathcal{M}_1 \to \mathcal{M}_2$ between two $\mathcal{C}$-modules consists of a bidegree $(s-d, \ell)$ linear map

$$\mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{d-1}, X_d) \otimes \mathcal{M}_1(X_d) \to \mathcal{M}_2(X_0)$$

for every sequence $(X_0, \dots, X_d)$ of objects. The differential of such a morphism is defined by

$$\begin{aligned}\mu^1_{\mathrm{Mod}_{\mathcal{C}}}(t) = \sum t(\dots, \mu_{\mathcal{C}}(\dots), \dots) + t(\dots, \mu_{\mathcal{M}_1}(\dots)) \\ + \sum \mu_{\mathcal{M}_2}(\dots, t(\dots)).\end{aligned}$$

Finally, the composition of $t_1 : \mathcal{M}_1 \to \mathcal{M}_2$ and $t_2 : \mathcal{M}_2 \to \mathcal{M}_3$ is given by

$$t_2 \circ t_1 = \mu^2_{\mathrm{Mod}_{\mathcal{C}}}(t_1, t_2) = \sum t_2(\dots, t_1(\dots)).$$

We denote by $\mathrm{Mod}_{\mathcal{C}}$ the Adams-graded DG-category of (left) $\mathcal{C}$-modules.

Definition 1.66. Let $t : \mathcal{M}_1 \to \mathcal{M}_2$ be a degree 0 closed $\mathcal{C}$-module map. We say that t is a quasi-isomorphism if the induced chain map $t : \mathcal{M}_1(X) \to \mathcal{M}_2(X)$ is a quasi-isomorphism for every object X in $\mathcal{C}$. (See [38] section A.2 for a discussion on quasi-isomorphisms between A_∞-modules).

Definition 1.67. Let $t, t' : \mathcal{M}_1 \to \mathcal{M}_2$ be two degree 0 closed morphisms of $\mathcal{C}$-modules. A homotopy between t and t' is a $\mathcal{C}$-module map $h : \mathcal{M}_1 \to \mathcal{M}_2$ such that

$$t + t' = \mu^1_{\mathrm{Mod}_{\mathcal{C}}}(h).$$

Definition 1.68 (See [59] section (1l) and [38] section A.1)**.** There is an A_∞-functor

$$\mathcal{C} \to \mathrm{Mod}_{\mathcal{C}}, \quad Y \mapsto \mathcal{C}(-, Y),$$

called the Yoneda A_∞-functor, defined as follows. For every object X,

$$\mathcal{C}(-,Y)(X)=\mathcal{C}(X,Y).$$

Besides, a sequence

$$(x_0,\ldots,x_{d-1})\in\mathcal{C}(X_0,X_1)\times\cdots\times\mathcal{C}(X_{d-1},X_d)$$

acts on an element u in $\mathcal{C}(X_d,Y)$ via the operations

$$\mu_{\mathcal{C}(-,Y)}(x_0,\ldots,x_{d-1},u)=\mu_{\mathcal{C}}(x_0,\ldots,x_{d-1},u).$$

Finally, let

$$\mathbf{y}=(y_0,\ldots,y_{p-1})\in\mathcal{C}(Y_0,Y_1)\times\cdots\times\mathcal{C}(Y_{p-1},Y_p)$$

be a sequence of morphisms in $\mathcal{C}$. Then the Yoneda functor gives a morphism of $\mathcal{C}$-modules $t_{\mathbf{y}}:\mathcal{C}(-,Y_0)\to\mathcal{C}(-,Y_p)$ which sends every sequence $(x_0,\ldots,x_{d-1},u)$ as above to

$$\mu_{\mathcal{C}}(x_0,\ldots,x_{d-1},u,y_0,\ldots,y_{p-1})\in\mathcal{C}(X_0,Y_p).$$

We have the following important result.

Proposition 1.69 (Yoneda lemma)**.** *The Yoneda A_∞-functor*

$$\mathcal{C}\to\mathrm{Mod}_{\mathcal{C}},\quad Y\mapsto\mathcal{C}(-,Y)$$

is cohomologically full and faithful.

Proof. This is Lemma 2.12 in [59], and also Lemma A.1 in [38].

□

The Yoneda lemma has the following easy consequence. We state it for future reference.

Corollary 1.70. *Every closed $\mathcal{C}$-module map $f:\mathcal{C}(-,X)\to\mathcal{C}(-,Y)$ is homotopic to the $\mathcal{C}$-module map $t_{f(e_X)}$ induced by $f(e_X)\in\mathcal{C}(X,Y)$. (see Definition 1.68).*

Proof. According to the Yoneda lemma, f is homotopic to t_x for some closed x in $\mathcal{C}(X,Y)$. Thus, there exists a $\mathcal{C}$-module map $h:\mathcal{C}(-,X)\to\mathcal{C}(-,Y)$ such that

$$f=t_x+\mu^1_{\mathrm{Mod}_{\mathcal{C}}}(h).$$

Evaluating the latter relation at the unit $e_X \in \mathcal{C}(X, X)$ gives

$$f(e_X) = x + \mu_{\mathcal{C}}^1(he_X).$$

Therefore, x is homotopic to $f(e_X)$, and this implies that t_x is homotopic to $t_{f(e_X)}$ by the Yoneda lemma. Finally, we have that f is homotopic to $t_{f(e_X)}$.

$\square$

Cone of module maps

Definition 1.71. Let $t : \mathcal{M}_1 \to \mathcal{M}_2$ be a degree 0 closed morphism of $\mathcal{C}$-modules. We denote by

$$\operatorname{Cone}\left(\mathcal{M}_1 \xrightarrow{t} \mathcal{M}_2\right) = \begin{bmatrix} \mathcal{M}_1 \\ \big\downarrow t \\ \mathcal{M}_2 \end{bmatrix}$$

the $\mathcal{C}$-module $\mathcal{M}$ defined as follows. For every object X in $\mathcal{C}$,

$$\mathcal{M}(X) = \mathcal{M}_1(X)[1] \oplus \mathcal{M}_2(X)$$

as graded vector space, and any sequence

$$(x_0, \ldots, x_{d-1}) \in \mathcal{C}(X_0, X_1) \times \cdots \times \mathcal{C}(X_{d-1}, X_d)$$

acts on an element $u_1 \oplus u_2$ in $\mathcal{M}(X_d)$ via the operations

$$\begin{aligned} &\mu_{\mathcal{M}}(x_0, \ldots, x_{d-1}, u_1 \oplus u_2) = \\ &\qquad \mu_{\mathcal{M}_1}(x_0, \ldots, x_{d-1}, u_1) \oplus (\mu_{\mathcal{M}_2}(x_0, \ldots, x_{d-1}, u_2) + t(x_0, \ldots, x_{d-1}, u_1)). \end{aligned}$$

If we have two $\mathcal{C}$-module maps $t : \mathcal{M}_1 \to \mathcal{M}_2$ and $t' : \mathcal{M}_1 \to \mathcal{M}_2'$, then we set

$$\left[\begin{tikzcd} & \mathcal{M}_1 \arrow[dl, "t"'] \arrow[dr, "t'"] & \\ \mathcal{M}_2 & & \mathcal{M}_2' \end{tikzcd}\right] := \begin{bmatrix} \mathcal{M}_1 \\ \big\downarrow (t,t') \\ \mathcal{M}_2 \oplus \mathcal{M}_2' \end{bmatrix}.$$

Proposition 1.72. *Consider a diagram of $\mathcal{C}$-modules*

$$\begin{tikzcd} \mathcal{M}_1 \arrow[r, "t_1"] \arrow[d, "t'_1"] & \mathcal{M}_2 \arrow[d, "t_2"] \\ \mathcal{M}'_2 \arrow[r, "t'_2"] & \mathcal{M}_3 \end{tikzcd}$$

where all the morphisms are of degree 0 *and closed. Then any homotopy* $h : \mathcal{M}_1 \to \mathcal{M}_3$ *between*

$$t := \mu^2_{\mathrm{Mod}_\mathcal{C}}(t_1, t_2) \text{ and } t' := \mu^2_{\mathrm{Mod}_\mathcal{C}}(t'_1, t'_2)$$

induces a degree 0 *closed* $\mathcal{C}$*-module map*

$$t_h : \left[\begin{tikzcd} & \mathcal{M}_1 \arrow[dl, "t_1"'] \arrow[dr, "t'_1"] & \\ \mathcal{M}_2 & & \mathcal{M}'_2 \end{tikzcd} \right] \to \mathcal{M}_3$$

defined by

$$\begin{aligned} &t_h(x_0, \ldots, x_{d-1}, u_1 \oplus u_2 \oplus u'_2) = \\ &\qquad h(x_0, \ldots, x_{d-1}, u_1) + t_2(x_0, \ldots, x_{d-1}, u_2) + t'_2(x_0, \ldots, x_{d-1}, u'_2). \end{aligned}$$

Proof. The only thing to check is that $\mu^1_{\mathrm{Mod}_\mathcal{C}}(t_h) = 0$, which is straightforward. □

Pullback of A_∞-modules

Definition 1.73 (See [59] section (1k))**.** Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an A_∞-functor. Then there is a DG-functor

$$\Phi^* : \mathrm{Mod}_\mathcal{D} \to \mathrm{Mod}_\mathcal{C}, \quad \mathcal{N} \mapsto \Phi^*\mathcal{N}.$$

We describe this functor more precisely. Let $\mathcal{N}$ be a $\mathcal{D}$-module. For every object X,

$$\Phi^*\mathcal{N}(X) = \mathcal{N}(\Phi X).$$

Besides, a sequence

$$(x_0, \ldots, x_{d-1}) \in \mathcal{C}(X_0, X_1) \times \cdots \times \mathcal{C}(X_{d-1}, X_d)$$

acts on an element $u \in \Phi^* \mathcal{N}(X_d)$ via the operations

$$\mu_{\Phi^*\mathcal{N}}(x_0, \ldots, x_{d-1}, u) = \sum \mu_{\mathcal{N}}\left(\Phi(x_0, \ldots, x_{i_1-1}), \ldots, \Phi(x_{d-i_r}, \ldots, x_{d-1}), u\right).$$

Finally, let $t : \mathcal{N}_1 \to \mathcal{N}_2$ be a $\mathcal{D}$-module map. Then the above functor gives a $\mathcal{C}$-module map $\Phi^* t : \Phi^* \mathcal{N}_1 \to \Phi^* \mathcal{N}_2$ which sends every sequence $(x_0, \ldots, x_{d-1}, u)$ as above to

$$\Phi^* t(x_0, \ldots, x_{d-1}, u) = \sum t\left(\Phi(x_0, \ldots, x_{i_1-1}), \ldots, \Phi(x_{d-i_r}, \ldots, x_{d-1}), u\right).$$

Remark 1.74. Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an A_∞-functor, and let $\Psi : \mathcal{D} \to \mathcal{E}$ be another A_∞-functor towards a third A_∞-category $\mathcal{E}$. Then $\Phi^* \circ \Psi^* = (\Psi \circ \Phi)^*$ as DG-functors.

Definition 1.75. Let Y be an object of $\mathcal{C}$, and let $\Phi : \mathcal{C} \to \mathcal{D}$ be an A_∞-functor. Then there is a degree 0 closed $\mathcal{C}$-module map $t_\Phi : \mathcal{C}(-, Y) \to \Phi^* \mathcal{D}(-, \Phi(Y))$ which sends any sequence

$$(x_0, \ldots, x_{d-1}, u) \in \mathcal{C}(X_0, X_1) \times \cdots \times \mathcal{C}(X_{d-1}, X_d) \times \mathcal{C}(X_d, Y)$$

to

$$t_\Phi(x_0, \ldots, x_{d-1}, u) = \Phi(x_0, \ldots, x_{d-1}, u) \in \mathcal{D}(\Phi X_0, \Phi Y).$$

Quotient of A_∞-modules

Definition 1.76 (See [37] section 3.1.3)**.** There is a DG-functor

$$\mathrm{Mod}_{\mathcal{C}} \to \mathrm{Mod}_{\mathcal{C}/\mathcal{A}}, \quad \mathcal{M} \mapsto {}_{\mathcal{A}\backslash}\mathcal{M}.$$

We describe this functor more precisely. Let $\mathcal{M}$ be a $\mathcal{C}$-module. For every object X,

$${}_{\mathcal{A}\backslash}\mathcal{M}(X) = $$
$$\mathcal{M}(X) \bigoplus \left(\bigoplus_{\substack{p \geq 1 \\ A_1, \ldots, A_p \in \mathcal{A}}} \mathcal{C}(X, A_1)[1] \otimes \cdots \otimes \mathcal{C}(A_{p-1}, A_p)[1] \otimes \mathcal{M}(A_p) \right).$$

Besides, a sequence

$$\mathbf{x}_i = \left(x_i^0, \ldots, x_i^{p_i-1}\right) \in \mathcal{C}/\mathcal{A}(X_i, X_{i+1}) \quad (0 \leq i \leq d-1)$$

acts on an element

$$\mathbf{u} = \left(x_d^0, \ldots, x_d^{p_d-1}, u\right) \in {}_{\mathcal{A}\backslash}\mathcal{M}\left(X_d\right)$$

via the operations

$$\begin{aligned}
&\mu_{\mathcal{A}\backslash\mathcal{M}}\left(\mathbf{x}_0, \ldots, \mathbf{x}_{d-1}, \mathbf{u}\right) = \\
&\qquad \sum_{\substack{0\leq i\leq p_0, 1\leq j\leq p_d \\ i<j \text{ if } d=0}} x_0^0 \otimes \cdots \otimes x_0^{i-1} \otimes \mu_{\mathcal{C}}\left(x_0^i, \ldots, x_d^{j-1}\right) \otimes x_d^j \otimes \cdots \otimes x_d^{p_d-1} \otimes u \\
&\qquad + \sum_{0\leq i\leq p_0} x_0^0 \otimes \cdots \otimes x_0^{i-1} \otimes \mu_{\mathcal{M}}\left(x_0^i, \ldots, x_d^{p_d-1}, u\right).
\end{aligned}$$

Finally, let $t : \mathcal{M}_1 \to \mathcal{M}_2$ be a $\mathcal{C}$-module map. Then the above functor gives a $\mathcal{C}/\mathcal{A}$-module map ${}_{\mathcal{A}\backslash}t : {}_{\mathcal{A}\backslash}\mathcal{M}_1 \to {}_{\mathcal{A}\backslash}\mathcal{M}_2$ which sends every sequence $\left(\mathbf{x}_0, \ldots, \mathbf{x}_{d-1}, \mathbf{u}\right)$ as above to

$${}_{\mathcal{A}\backslash}t\left(\mathbf{x}_0, \ldots, \mathbf{x}_{d-1}, \mathbf{u}\right) = \sum_{0\leq i\leq p_0} x_0^0 \otimes \cdots \otimes x_0^{i-1} \otimes t\left(x_0^i, \ldots, x_d^{p_d-1}, u\right).$$

Remark 1.77. If $t : \mathcal{M}_1 \to \mathcal{M}_2$ is a degree 0 closed $\mathcal{C}$-module map, then

$${}_{\mathcal{A}\backslash}\mathrm{Cone}\left(\mathcal{M}_1 \xrightarrow{t} \mathcal{M}_2\right) = \mathrm{Cone}\left({}_{\mathcal{A}\backslash}\mathcal{M}_1 \xrightarrow{{}_{\mathcal{A}\backslash}t} {}_{\mathcal{A}\backslash}\mathcal{M}_2\right).$$

Proposition 1.78. *Let $\mathcal{M}$ be a $\mathcal{C}$-module. If A is some object in $\mathcal{A}$, then ${}_{\mathcal{A}\backslash}\mathcal{M}\left(A\right)$ is acyclic.*

Proof. This is Lemma 3.12 in [37].

□

Proposition 1.79. *Let $\mathcal{M}$ be a $\mathcal{C}$-module. If $\mathcal{M}\left(A\right)$ is acyclic for every A in $\mathcal{A}$, then the inclusion $\mathcal{M}\left(X\right) \hookrightarrow {}_{\mathcal{A}\backslash}\mathcal{M}\left(X\right)$ is a quasi-isomorphism for every object X.*

Proof. This is Lemma 3.13 in [37].

□

Relations between pullback and quotient of A_∞-modules

Definition 1.80. Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an A_∞-functor such that $\Phi\left(\mathcal{A}\right)$ is contained in $\mathcal{B}$, and let X be a fixed object of $\mathcal{C}$. Then, for each $\mathcal{D}$-module $\mathcal{N}$, there is a chain map ${}_{\mathcal{A}\backslash}\left(\Phi^*\mathcal{N}\right)\left(X\right) \to {}_{\mathcal{B}\backslash}\mathcal{N}\left(\Phi X\right)$ which sends an element

$$\mathbf{u} = \left(x^0, \ldots, x^{p-1}, u\right) \in {}_{\mathcal{A}\backslash}\left(\Phi^*\mathcal{N}\right)\left(X\right)$$

to

$$\sum \Phi\left(x^0, \ldots, x^{i_1-1}\right) \otimes \cdots \otimes \Phi\left(x^{i_r}, \ldots, x^{p-1}\right) \otimes u \in {}_{\mathcal{B}\backslash}\mathcal{N}(\Phi X).$$

This defines a natural transformation between the functors $\mathcal{N} \mapsto {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N})(X)$ and $\mathcal{N} \mapsto {}_{\mathcal{B}\backslash}\mathcal{N}(\Phi X)$ from $\mathrm{Mod}_{\mathcal{D}}$ to Ch. In other words, for every $\mathcal{D}$-module map $t : \mathcal{N}_1 \to \mathcal{N}_2$, the following diagram of chain complexes commutes

$$\begin{tikzcd} {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N}_1)(X) \arrow[r] \arrow[d, "{}_{\mathcal{A}\backslash}(\Phi^* t)"] & {}_{\mathcal{B}\backslash}\mathcal{N}_1(\Phi X) \arrow[d, "{}_{\mathcal{B}\backslash}t"] \\ {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{N}_2)(X) \arrow[r] & {}_{\mathcal{B}\backslash}\mathcal{N}_2(\Phi X) \end{tikzcd}.$$

Remark 1.81. Let Y be an object of $\mathcal{C}$, and let $\Phi : \mathcal{C} \to \mathcal{D}$ be an A_∞-functor such that $\Phi(\mathcal{A})$ is contained in $\mathcal{B}$. Let $\widetilde{\Phi} : \mathcal{C}/\mathcal{A} \to \mathcal{D}/\mathcal{B}$ be the A_∞-functor induced by Φ (see Definition 1.59). Localize the morphism $t_\Phi : \mathcal{C}(-, Y) \to \Phi^*\mathcal{D}(-, \Phi Y)$ of Definition 1.75 at $\mathcal{A}$ and evaluate at X to get a chain map

$$\mathcal{C}/\mathcal{A}(X, Y) = {}_{\mathcal{A}\backslash}\mathcal{C}(-, Y)(X) \xrightarrow{{}_{\mathcal{A}\backslash}t_\Phi} {}_{\mathcal{A}\backslash}(\Phi^*\mathcal{D}(-, \Phi Y))(X).$$

Then the composition of this map with the chain map

$${}_{\mathcal{A}\backslash}(\Phi^*\mathcal{D}(-, \Phi Y))(X) \to {}_{\mathcal{B}\backslash}\mathcal{D}(-, \Phi Y)(\Phi X) = \mathcal{D}/\mathcal{B}(\Phi X, \Phi Y)$$

of Definition 1.80 is the chain map $\widetilde{\Phi} : \mathcal{C}/\mathcal{A}(X, Y) \to \mathcal{D}/\mathcal{B}(\Phi X, \Phi Y)$.

Proposition 1.82. *Let $\Phi : \mathcal{C} \to \mathcal{D}$ be an A_∞-functor such that $\Phi(\mathcal{A})$ is contained in $\mathcal{B}$, and let $\widetilde{\Phi} : \mathcal{C}/\mathcal{A} \to \mathcal{D}/\mathcal{B}$ be the A_∞-functor induced by Φ (see Definition 1.59).*

Now let Y be a fixed object of $\mathcal{C}$. Assume that we have a $\mathcal{C}$-module $\mathcal{M}_\mathcal{C}$, a $\mathcal{D}$-module $\mathcal{M}_\mathcal{D}$, a degree 0 closed $\mathcal{C}$-module map $t_\mathcal{C} : \mathcal{C}(-, Y) \to \mathcal{M}_\mathcal{C}$, a degree 0 closed $\mathcal{D}$-module map $t_\mathcal{D} : \mathcal{D}(-, \Phi Y) \to \mathcal{M}_\mathcal{D}$ and a degree 0 closed $\mathcal{C}$-module map $t_0 : \mathcal{M}_\mathcal{C} \to \Phi^*\mathcal{M}_\mathcal{D}$ such that the following diagram of $\mathcal{C}$-modules commutes

$$\begin{tikzcd} \mathcal{C}(-, Y) \arrow[r, "t_\Phi"] \arrow[d, "t_\mathcal{C}"] & \Phi^*\mathcal{D}(-, \Phi Y) \arrow[d, "\Phi^* t_\mathcal{D}"] \\ \mathcal{M}_\mathcal{C} \arrow[r, "t_0"] & \Phi^*\mathcal{M}_\mathcal{D} \end{tikzcd}$$

(see Definition 1.75 for the map t_Φ). Then for every object X in $\mathcal{C}$, there is a chain

map $u : {}_{\mathcal{A}\backslash}\mathcal{M}_{\mathcal{C}}(X) \to {}_{\mathcal{B}\backslash}\mathcal{M}_{\mathcal{D}}(\Phi X)$ *such that the following diagram of chain complexes commutes*

$$\begin{tikzcd}
\mathcal{C}/\mathcal{A}(X,Y) \arrow[r, "\widetilde{\Phi}"] \arrow[d, "{}_{\mathcal{A}\backslash}t_{\mathcal{C}}"] & \mathcal{D}/\mathcal{B}(\Phi X,\Phi Y) \arrow[d, "{}_{\mathcal{B}\backslash}t_{\mathcal{D}}"] \\
{}_{\mathcal{A}\backslash}\mathcal{M}_{\mathcal{C}}(X) \arrow[r, "u"] & {}_{\mathcal{B}\backslash}\mathcal{M}_{\mathcal{D}}(\Phi X) \\
\mathcal{M}_{\mathcal{C}}(X) \arrow[r, "t_0"] \arrow[u, hook] & \mathcal{M}_{\mathcal{D}}(\Phi X) \arrow[u, hook]
\end{tikzcd}$$

(the two lowest vertical maps are the inclusions). If moreover the following holds:

1. *for every objects* A, B *in* $\mathcal{A}, \mathcal{B}$*, the complexes* $\mathcal{M}_{\mathcal{C}}(A)$ *and* $\mathcal{M}_{\mathcal{D}}(B)$ *are acyclic,*

2. *the morphisms* ${}_{\mathcal{A}\backslash}t_{\mathcal{C}} : {}_{\mathcal{A}\backslash}\mathcal{C}(-,Y) \to {}_{\mathcal{A}\backslash}\mathcal{M}_{\mathcal{C}}$ *and* ${}_{\mathcal{B}\backslash}t_{\mathcal{D}} : {}_{\mathcal{B}\backslash}\mathcal{D}(-,\Phi Y) \to {}_{\mathcal{B}\backslash}\mathcal{M}_{\mathcal{D}}$ *are quasi-isomorphisms, and*

3. *the morphism* $t_0 : \mathcal{M}_{\mathcal{C}} \to \Phi^*\mathcal{M}_{\mathcal{D}}$ *is a quasi-isomorphism,*

then the map $\widetilde{\Phi} : \mathcal{C}/\mathcal{A}(X,Y) \to \mathcal{D}/\mathcal{B}(\Phi X,\Phi Y)$ *is a quasi-isomorphism for every object* X *in* $\mathcal{C}$*.*

Proof. We apply the functor $\mathcal{P} \mapsto {}_{\mathcal{A}\backslash}\mathcal{P}$ to the first diagram, we evaluate at X and we use the natural map of Definition 1.80 to get the following commutative diagram of chain complexes

$$\begin{tikzcd}
{}_{\mathcal{A}\backslash}\mathcal{C}(-,Y)(X) \arrow[r, "{}_{\mathcal{A}\backslash}t_{\Phi}"] \arrow[d, "{}_{\mathcal{A}\backslash}t_{\mathcal{C}}"] & {}_{\mathcal{A}\backslash}\left(\Phi^*\mathcal{D}(-,\Phi Y)\right)(X) \arrow[r] \arrow[d, "{}_{\mathcal{A}\backslash}(\Phi^* t_{\mathcal{D}})"] & {}_{\mathcal{B}\backslash}\mathcal{D}(-,\Phi Y)(\Phi X) \arrow[d, "{}_{\mathcal{B}\backslash}t_{\mathcal{D}}"] \\
{}_{\mathcal{A}\backslash}\mathcal{M}_{\mathcal{C}}(X) \arrow[r, "{}_{\mathcal{A}\backslash}t_0"] & {}_{\mathcal{A}\backslash}\left(\Phi^*\mathcal{M}_{\mathcal{D}}\right)(X) \arrow[r] & {}_{\mathcal{B}\backslash}\mathcal{M}_{\mathcal{D}}(\Phi X)
\end{tikzcd}.$$

Then we compose the horizontal maps and we use Remark 1.81 to get a commutative diagram of chain complexes

$$\begin{tikzcd}
\mathcal{C}/\mathcal{A}(X,Y) \arrow[r, "\widetilde{\Phi}"] \arrow[d, "{}_{\mathcal{A}\backslash}t_{\mathcal{C}}"] & \mathcal{D}/\mathcal{B}(\Phi X,\Phi Y) \arrow[d, "{}_{\mathcal{B}\backslash}t_{\mathcal{D}}"] \\
{}_{\mathcal{A}\backslash}\mathcal{M}_{\mathcal{C}}(X) \arrow[r, "u"] & {}_{\mathcal{B}\backslash}\mathcal{M}_{\mathcal{D}}(\Phi X)
\end{tikzcd}.$$

This proves the first part of the Proposition because the following diagram of chain complexes commutes

$$\begin{tikzcd} {}_{\mathcal{A}\backslash}\mathcal{M}_{\mathcal{C}}(X) \arrow[r, "u"] & {}_{\mathcal{B}\backslash}\mathcal{M}_{\mathcal{D}}(\Phi X) \\ \mathcal{M}_{\mathcal{C}}(X) \arrow[u, hook] \arrow[r, "t_0"] & \mathcal{M}_{\mathcal{D}}(\Phi X) \arrow[u, hook] \end{tikzcd} .$$

The second part of the Proposition follows directly with Proposition 1.79.

□

1.3.5 Grothendieck construction and homotopy colimit

An exposition on Grothendieck constructions and homotopy colimits in the context of A_∞-categories can be found in [38] appendix A. We recall here definitions and basic facts that will serve us in the following.

Definition 1.83. Consider a diagram of Adams-graded A_∞-categories

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi_1"] \arrow[d, "\Phi_2"'] & \mathcal{D}_1 \\ \mathcal{D}_2 & \end{tikzcd} .$$

The Grothendieck construction of this diagram is the Adams-graded A_∞-category $\mathcal{G}$ such that

1. the set of objects is $\mathrm{ob}(\mathcal{C}) \sqcup \mathrm{ob}(\mathcal{D}_1) \sqcup \mathrm{ob}(\mathcal{D}_2)$,

2. the space of morphisms between two objects X and Y is given by

$$\mathcal{G}(X,Y) = \begin{cases} \mathcal{C}(X,Y) & \text{if } X, Y \in \mathrm{ob}(\mathcal{C}) \\ \mathcal{D}_i(X,Y) & \text{if } X, Y \in \mathrm{ob}(\mathcal{D}_i) \\ \mathcal{D}_i(\Phi_i X, Y) & \text{if } X \in \mathrm{ob}(\mathcal{C}) \text{ and } Y \in \mathrm{ob}(\mathcal{D}_i) \\ 0 & \text{otherwise.} \end{cases}$$

3. the operations involving only objects of $\mathcal{C}$, respectively of $\mathcal{D}_i$, are the same as in $\mathcal{C}$,

respectively in $\mathcal{D}_i$, and for every sequence

$$\begin{aligned}(x_0, \ldots, x_{p-1}, y, z_0, \ldots, z_{q-1}) \in \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{p-1}, X_p) \\ \otimes \mathcal{G}(X_p, Y_0) \otimes \mathcal{D}_i(Y_0, Y_1) \otimes \cdots \otimes \mathcal{D}_i(Y_{q-1}, Y_q),\end{aligned}$$

we have

$$\begin{aligned}&\mu_{\mathcal{G}}(x_0, \ldots, x_{p-1}, y, z_0, \ldots, z_{q-1}) = \\ &\quad \sum \mu_{\mathcal{D}_i}\left(\Phi_i(x_0, \ldots, x_{i_1-1}), \ldots, \Phi_i(x_{p-i_r}, \ldots, x_{p-1}), y, z_0, \ldots, z_{q-1}\right).\end{aligned}$$

We will call *adjacent unit* of $\mathcal{G}$ any morphism in $\mathcal{G}(X, \Phi_i(X))$ which corresponds to the unit in $\mathcal{D}_i(\Phi_i(X), \Phi_i(X))$. The *homotopy colimit* $\mathcal{H}$ of the above diagram is the localization of $\mathcal{G}$ at its adjacent units (see Definition 1.62).

Proposition 1.84. *Let $\mathcal{G}$ be the Grothendieck construction of a diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi_1"] \arrow[d, "\Phi_2"'] & \mathcal{D}_1 \\ \mathcal{D}_2. & \end{tikzcd}$$

Then any strictly commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi_1"] \arrow[d, "\Phi_2"'] & \mathcal{D}_1 \arrow[d, "\Psi_1"] \\ \mathcal{D}_2 \arrow[r, "\Psi_2"] & \mathcal{E} \end{tikzcd}$$

induces a functor $\sigma : \mathcal{G} \to \mathcal{E}$ defined as follows. On the objects, σ acts on $\mathcal{D}_i$ as Ψ_i, and on $\mathcal{C}$ as $\Psi_1 \circ \Phi_1 = \Psi_2 \circ \Phi_2$; on the morphisms, σ acts on $\mathcal{D}_i$ as Ψ_i, on $\mathcal{C}$ as $\Psi_1 \circ \Phi_1 = \Psi_2 \circ \Phi_2$, and it sends any sequence

$$\begin{aligned}(x_0, \ldots, x_{p-1}, y, z_0, \ldots, z_{q-1}) \in \mathcal{C}(X_0, X_1) \otimes \cdots \otimes \mathcal{C}(X_{p-1}, X_p) \\ \otimes \mathcal{G}(X_p, Y_0) \otimes \mathcal{D}_i(Y_0, Y_1) \otimes \cdots \otimes \mathcal{D}_i(Y_{q-1}, Y_q),\end{aligned}$$

to

$$\sigma\left(x_0,\ldots,x_{p-1},y,z_0,\ldots,z_{q-1}\right)=\\ \sum\Psi_i\left(\Phi_i\left(x_0,\ldots,x_{i_1-1}\right),\ldots,\Phi_i\left(x_{p-i_r},\ldots,x_{p-1}\right),y,z_0,\ldots,z_{q-1}\right).$$

Proof. This is a straightforward verification.

□

Proposition 1.85. *A strictly commutative diagram of A_∞-categories*

$$\begin{tikzcd} \mathcal{B}_1 \arrow[d] & \mathcal{A} \arrow[l] \arrow[r] \arrow[d] & \mathcal{B}_2 \arrow[d] \\ \mathcal{D}_1 & \mathcal{C} \arrow[l] \arrow[r] & \mathcal{D}_2 \end{tikzcd}$$

induces an A_∞-functor from the Grothendieck construction of the top line to the Grothendieck construction of the bottom line which preserves adjacent units. If moreover each vertical arrow is a quasi-equivalence, then the induced functor

$$\operatorname{hocolim}\left(\begin{tikzcd} \mathcal{A} \arrow[r] \arrow[d] & \mathcal{B}_1 \\ \mathcal{B}_2 & \end{tikzcd}\right)\to\operatorname{hocolim}\left(\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D}_1 \\ \mathcal{D}_2 & \end{tikzcd}\right)$$

is a quasi-equivalence.

Proof. This is Lemma A.5 in [38].

□

Proposition 1.86. *Consider two diagrams of A_∞-categories*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Phi_1"] \arrow[d, "\Phi_2"'] & \mathcal{D}_1 \\ \mathcal{D}_2 & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathcal{C} \arrow[r, "\Psi_1"] \arrow[d, "\Psi_2"'] & \mathcal{D}_1 \\ \mathcal{D}_2. & \end{tikzcd}$$

If Φ_i and Ψ_i (for $i\in\{1,2\}$) are homotopic (see Definition 1.52), then the homotopy colimits of the diagrams above are quasi-equivalent.

Proof. Let $\mathcal{G}_0$ and $\mathcal{G}_1$ the Grothendieck constructions of the above diagrams.

Let T_i be an homotopy from Φ_i to Ψ_i. This means that

$$\begin{aligned}\Phi_i + \Psi_i = &\sum T_i\left(\dots, \mu_{\mathcal{C}}\left(\dots\right), \dots\right)\\ &+ \sum \mu_{\mathcal{D}_i}\left(\Psi_i\left(\dots\right), \dots, \Psi_i\left(\dots\right), T_i\left(\dots\right), \Phi_i\left(\dots\right), \dots, \Phi_i\left(\dots\right)\right).\end{aligned}$$

We consider the functor $\kappa : \mathcal{G}_0 \to \mathcal{G}_1$ such that

$$\kappa_{|\mathcal{C}} = \mathrm{id}_{\mathcal{C}}, \quad \kappa_{|\mathcal{D}_i} = \mathrm{id}_{\mathcal{D}_i},$$

and which sends every sequence

$$\begin{aligned}\left(\dots, y, \dots\right) \in \mathcal{C}\left(X_0, X_1\right) \times \dots \times \mathcal{C}\left(X_{p-1}, X_p\right) \times \mathcal{G}_0\left(X_p, Y_0\right)\\ \times \mathcal{D}_i\left(Y_0, Y_1\right) \times \dots \times \mathcal{D}_i\left(Y_{q-1}, Y_q\right),\end{aligned}$$

to

$$\begin{aligned}&\kappa\left(\dots, y, \dots\right) =\\ &\sum \mu_{\mathcal{D}_i}\left(\Psi_i\left(\dots\right), \dots, \Psi_i\left(\dots\right), T_i\left(\dots\right), \Phi_i\left(\dots\right), \dots, \Phi_i\left(\dots\right), y, \dots\right)\end{aligned}$$

if p is positive, and to

$$\Phi\left(y, \dots\right) = \mathrm{id}_{\mathcal{D}_i}\left(y, \dots\right)$$

otherwise. Using the facts that Φ_i, Ψ_i are A_∞-functors, that T_i is an homotopy from Φ_i to Ψ_i, and gathering the terms depending on if they contain $T_i^k\left(\dots\right)$ or y, we conclude that κ satisfies the A_∞-relations. This proves the result because κ is a quasi-equivalence sending the adjacent units of $\mathcal{G}_0$ onto those of $\mathcal{G}_1$.

$\square$

1.3.6 Cylinder object and homotopy

Let $\mathcal{A}_\perp$, $\mathcal{A}_I$ and $\mathcal{A}_\top$ be three copies of an A_∞-category $\mathcal{A}$. We denote by $\mathcal{C}$ be the Grothendieck construction of the following diagram

$$\begin{tikzcd} \mathcal{A}_I \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathcal{A}_\top \\ \mathcal{A}_\perp & \end{tikzcd},$$

and we let $\iota_\perp, \iota_I, \iota_\top : \mathcal{A} \to \mathcal{C}$ be the strict inclusions with images $\mathcal{A}_\perp$, $\mathcal{A}_I$, $\mathcal{A}_\top$ respectively. Finally, we denote by $W_\mathcal{C}$ the set of adjacent units in $\mathcal{C}$, and we let $\mathcal{C}yl_\mathcal{A} = \mathcal{C}\left[W_\mathcal{C}^{-1}\right]$ be the homotopy colimit of the diagram above. We say that $\mathcal{C}yl_\mathcal{A}$ is a cylinder object for $\mathcal{A}$.

We denote by $\pi : \mathcal{C} \to \mathcal{A}$ the A_∞-functor induced by the following commutative square

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathcal{A} \arrow[d, "\mathrm{id}"] \\
\mathcal{A} \arrow[r, "\mathrm{id}"] & \mathcal{A}
\end{tikzcd}$$

(see Proposition 1.84).

Proposition 1.87. *The following diagram of A_∞-categories commutes*

$$\begin{tikzcd}
\mathcal{A} \sqcup \mathcal{A} \arrow[r, "\iota_\perp \sqcup \iota_\top"] \arrow[rr, bend right, "\mathrm{id} \sqcup \mathrm{id}"'] & \mathcal{C} \arrow[r, "\pi"] & \mathcal{A}.
\end{tikzcd}$$

Moreover, π sends $W_\mathcal{C}$ to the set of units in $\mathcal{A}$, and the induced A_∞-functor $\widetilde{\pi} : \mathcal{C}yl_\mathcal{A} \to \mathcal{A}\left[\mathrm{units}^{-1}\right]$ (see Definition 1.59) is a quasi-equivalence.

Proof. The facts that $\pi \circ (\iota_\perp \sqcup \iota_\top) = \mathrm{id} \sqcup \mathrm{id}$ and that π sends $W_\mathcal{C}$ to the set of units in $\mathcal{A}$ are clear. We now show that $\widetilde{\pi} : \mathcal{C}yl_\mathcal{A} \to \mathcal{A}\left[\mathrm{units}^{-1}\right]$ is a quasi-equivalence.

We first observe that it is enough to show that the map

$$\widetilde{\pi} : \mathcal{C}yl_\mathcal{A}(X, X') \to \mathcal{A}\left[\{\mathrm{units}\}^{-1}\right](\pi X, \pi X')$$

is a quasi-isomorphism for every object X in $\mathcal{C}$ and X' in $\mathcal{A}_\perp \sqcup \mathcal{A}_\top \subset \mathcal{C}$. Indeed, we can deduce from this that $\widetilde{\pi} : \mathcal{C}yl_\mathcal{A}(X, X') \to \mathcal{A}\left[\{\mathrm{units}\}^{-1}\right](\pi X, \pi X')$ is a quasi-isomorphism when $X' = \iota_I(Z)$ is in $\mathcal{A}_I$ as follows. Consider the commutative diagram of $\mathcal{C}$-modules

$$\begin{tikzcd}
\mathcal{C}(-, \iota_I(Z)) \arrow[r, "t_\pi"] \arrow[d] & \pi^*\mathcal{A}(-, Z) \\
\mathcal{C}(-, \iota_\perp(Z)) \arrow[ur, "t_\pi"] &
\end{tikzcd}$$

(the vertical arrow is composition with the morphism $\iota_I(Z) \to \iota_\perp(Z)$ in $W_\mathcal{C}$, and the other arrows are the ones of Definition 1.75). If we localize at $W_\mathcal{C}$ and then evaluate at some object X, we get a commutative diagram of chain complexes

$$\begin{tikzcd}
\mathcal{C}yl_{\mathcal{A}}\left(X,\iota_I(Z)\right) \arrow[r, "W_{\mathcal{C}}^{-1}t_\pi"] \arrow[d, "\sim"] & W_{\mathcal{C}}^{-1}\left(\pi^*\mathcal{A}\left(-,Z\right)\right)(X) \\
\mathcal{C}yl_{\mathcal{A}}\left(X,\iota_\perp(Z)\right). \arrow[ur, "W_{\mathcal{C}}^{-1}t_\pi"'] &
\end{tikzcd}$$

Now using the natural map of Definition 1.80 and Remark 1.81, we get the following commutative diagram of chain complexes

$$\begin{tikzcd}
\mathcal{C}yl_{\mathcal{A}}\left(X,\iota_I(Z)\right) \arrow[r, "\widetilde{\pi}"] \arrow[d, "\sim"] & {}_{\{\text{units}\}^{-1}}\mathcal{A}\left(-,Z\right)(\pi X) = \mathcal{A}\left[\{\text{units}\}^{-1}\right](\pi X, Z) \\
\mathcal{C}yl_{\mathcal{A}}\left(X,\iota_\perp(Z)\right). \arrow[ur, "\widetilde{\pi}"] &
\end{tikzcd}$$

Thus, knowing that the map $\widetilde{\pi}:\mathcal{C}yl_{\mathcal{A}}\left(X,\iota_\perp(Z)\right)\to\mathcal{A}[\{\text{units}\}^{-1}](\pi X,Z)$ is a quasi-isomorphism implies that $\widetilde{\pi}:\mathcal{C}yl_{\mathcal{A}}\left(X,\iota_I(Z)\right)\to\mathcal{A}[\{\text{units}\}^{-1}](\pi X,Z)$ is also a quasi-isomorphism.

It remains to show that $\widetilde{\pi}:\mathcal{C}yl_{\mathcal{A}}\left(X,\iota_\triangle(Z)\right)\to\mathcal{A}[\{\text{units}\}^{-1}](\pi X,Z)$ is a quasi-isomorphism when $\triangle\in\{\perp,\top\}$. Our strategy is to apply Proposition 1.82. For the $\mathcal{C}$-module we take

$$\mathcal{M}_{\mathcal{C}} = \left[\begin{tikzcd}
 & \mathcal{C}\left(-,\iota_I(Z)\right) \arrow[dl] \arrow[dr] & \\
\mathcal{C}\left(-,\iota_\perp(Z)\right) & & \mathcal{C}\left(-,\iota_\top(Z)\right)
\end{tikzcd}\right],$$

where

$$t_{I\triangle}:\mathcal{C}\left(-,\iota_I(Z)\right)\to\mathcal{C}\left(-,\iota_\triangle(Z)\right),\quad \triangle\in\{\perp,\top\},$$

is the $\mathcal{C}$-module map induced by the adjacent unit in $\mathcal{C}\left(\iota_I(Z),\iota_\triangle(Z)\right)$ (see Definition 1.68)). For the $\mathcal{A}$-module we simply take $\mathcal{A}\left(-,Z\right)$. Besides, we let $t_{\mathcal{C}}:\mathcal{C}\left(-,\iota_\triangle(Z)\right)\to\mathcal{M}_{\mathcal{C}}$ be the $\mathcal{C}$-module inclusion, and we let $t_{\mathcal{A}}:\mathcal{A}\left(-,Z\right)\to\mathcal{A}\left(-,Z\right)$ be the identity map. We now define the morphism $t_0:\mathcal{M}_{\mathcal{C}}\to\pi^*\mathcal{A}\left(-,Z\right)$. Consider the following diagram of

$\mathcal{C}$-modules

$$\begin{tikzcd} \mathcal{C}\left(-,\iota_I(Z)\right) \arrow[r, "t_{I\perp}"] \arrow[d, "t_{I\top}"] & \mathcal{C}\left(-,\iota_\top(Z)\right) \arrow[d, "t_\pi"] \\ \mathcal{C}\left(-,\iota_\perp(Z)\right) \arrow[r, "t_\pi"] & \pi^*\mathcal{A}\left(-,Z\right). \end{tikzcd}$$

Observe that this diagram is commutative, and thus it induces a strict $\mathcal{C}$-module map $t_0 : \mathcal{M}_\mathcal{C} \to \pi^*\mathcal{A}\left(-,Z\right)$ according to Proposition 1.72. It is then easy to see that the following diagram commutes

$$\begin{tikzcd} \mathcal{C}\left(-,\iota_\triangle(Z)\right) \arrow[r, "t_\pi"] \arrow[d, "t_\mathcal{C}"] & \pi^*\mathcal{A}\left(-,Z\right) \arrow[d, "\pi^* t_\mathcal{A}"] \\ \mathcal{M}_\mathcal{C} \arrow[r, "t_0"] & \pi^*\mathcal{A}\left(-,Z\right). \end{tikzcd}$$

To conclude the proof, it suffices to check the three items of Proposition 1.82. Observe that the pair $\left(\mathcal{A}\left(-,Z\right), t_\mathcal{A}\right)$ trivially satisfies the two first items.

We check that $\left(\mathcal{M}_\mathcal{C}, t_\mathcal{C}\right)$ satisfies the first item of Proposition 1.82. Let Y be an object in $\mathcal{A}$ and let w be the adjacent unit in $\mathcal{C}\left(\iota_I(Y),\iota_\perp(Y)\right)$ (the proof is the same for the adjacent unit in $\mathcal{C}\left(\iota_I(Y),\iota_\top(Y)\right) \cap W_\mathcal{C}$). Then

$$\begin{aligned} \mathcal{M}_\mathcal{C}\left(\operatorname{Cone} w\right) &= \operatorname{Cone}\left(\mathcal{M}_\mathcal{C}\left(\iota_\perp(Y)\right) \xrightarrow{\mu^2_\mathcal{C}(w,-)} \mathcal{M}_\mathcal{C}\left(\iota_I(Y)\right)\right) \\ &= \operatorname{Cone}\left(\mathcal{C}\left(\iota_\perp(Y),\iota_\perp(Z)\right) \xrightarrow{\mu^2_\mathcal{C}(w,-)} K\right). \end{aligned}$$

where

$$K = \left[\begin{tikzcd} & \mathcal{C}\left(\iota_I(Y),\iota_I(Z)\right) \arrow[dl, "t_{I\perp}"'] \arrow[dr, "t_{I\top}"] & \\ \mathcal{C}\left(\iota_I(Y),\iota_\perp(Z)\right) & & \mathcal{C}\left(\iota_I(Y),\iota_\top(Z)\right) \end{tikzcd}\right].$$

Observe that $\mu^2_\mathcal{C}\left(w,-\right) : \mathcal{C}\left(\iota_\perp(Y),\iota_\perp(Z)\right) \to K$ is injective so its cone is quasi-isomorphic to its cokernel, which is the cone of $t_{I\top} : \mathcal{C}\left(\iota_I(Y),\iota_I(Z)\right) \to \mathcal{C}\left(\iota_I(Y),\iota_\top(Z)\right)$. The latter map is a quasi-isomorphism, so $\mathcal{M}_\mathcal{C}\left(\operatorname{Cone} w\right)$ is acyclic.

We now check that $\left(\mathcal{M}_\mathcal{C}, t_\mathcal{C}\right)$ satisfies the second item of Proposition 1.82. Observe that

$$ {}_{W_{\mathcal{C}}^{-1}}\mathcal{M}_{\mathcal{C}} = \left[\begin{tikzcd} & {}_{W_{\mathcal{C}}^{-1}}\mathcal{C}\left(-,\iota_I(Z)\right) \arrow[dl, "{}_{W_{\mathcal{C}}^{-1}}t_{I\perp}"'] \arrow[dr, "{}_{W_{\mathcal{C}}^{-1}}t_{I\top}"] & \\ {}_{W_{\mathcal{C}}^{-1}}\mathcal{C}\left(-,\iota_\perp(Z)\right) & & {}_{W_{\mathcal{C}}^{-1}}\mathcal{C}\left(-,\iota_\top(Z)\right) \end{tikzcd} \right] $$

and ${}_{W_{\mathcal{C}}^{-1}}t_{\mathcal{C}} : {}_{W_{\mathcal{C}}^{-1}}\mathcal{C}\left(-,\iota_\triangle(Z)\right) \to {}_{W_{\mathcal{C}}^{-1}}\mathcal{M}_{\mathcal{C}}$ is the inclusion. Thus if X is some object of $\mathcal{C}$, the cone of ${}_{W_{\mathcal{C}}^{-1}}t_{\mathcal{C}} : \mathcal{C}yl_{\mathcal{A}}\left(X,\iota_\triangle(Z)\right) \to {}_{W_{\mathcal{C}}^{-1}}\mathcal{M}_{\mathcal{C}}\left(X\right)$ is quasi-isomorphic to the cone of the multiplication in $\mathcal{C}yl_{\mathcal{A}}$ by an element of $W_{\mathcal{C}}$, which is a quasi-isomorphism. Thus the map ${}_{W_{\mathcal{C}}^{-1}}t_{\mathcal{C}} : {}_{W_{\mathcal{C}}^{-1}}\mathcal{C}\left(-,\iota_\triangle(Z)\right) \to {}_{W_{\mathcal{C}}^{-1}}\mathcal{M}_{\mathcal{C}}$ indeed is a quasi-isomorphism.

It remains to check the third item of Proposition 1.82, which is that the map $t_0 : \mathcal{M}_{\mathcal{C}}\left(X\right) \to \pi^*\mathcal{A}\left(-,Z\right)\left(X\right)$ is a quasi-isomorphism for every object X. This is clear when X is in $\mathcal{A}_\perp \sqcup \mathcal{A}_\top$, and it can be deduced when $X = \iota_I(Y)$ using the following commutative diagram

$$ \begin{tikzcd} \mathcal{M}_{\mathcal{C}}\left(\iota_I(Y)\right) \arrow[r, "t_0"] & \pi^*\mathcal{A}\left(-,Z\right)\left(\iota_I(Y)\right) \\ \mathcal{M}_{\mathcal{C}}\left(\iota_\perp(Y)\right) \arrow[r, "t_0"] \arrow[u, "\sim"] & \pi^*\mathcal{A}\left(-,Z\right)\left(\iota_\perp(Y)\right). \arrow[u, equal] \end{tikzcd} $$

This concludes the proof.

□

Remark 1.88. Proposition 1.87 can be thought as saying that $\mathcal{C}yl_{\mathcal{A}}$ is a cylinder object for $\mathcal{A}$.

Proposition 1.89. *If two A_∞-functors $\Phi, \Psi : \mathcal{A} \to \mathcal{B}$ are homotopic (see Definition 1.52), then there is an A_∞-functor $\eta : \mathcal{C} \to \mathcal{B}$ which sends the adjacent units of $\mathcal{C}$ to the units in $\mathcal{B}$ and such that the following diagram commutes*

$$ \begin{tikzcd} \mathcal{A} \arrow[d, "\iota_\top"'] \arrow[dr, bend left, "\Phi"] & \\ \mathcal{C} \arrow[r, "\eta"] & \mathcal{B} \\ \mathcal{A}. \arrow[u, "\iota_\perp"] \arrow[ur, bend right, "\Psi"'] & \end{tikzcd} $$

Proof. On the objects, we set $\eta\left(X_\triangle\right) = \Phi\left(X\right) = \Psi\left(X\right)$ for every object X of $\mathcal{A}$ and

$\triangle \in \{\perp, I, \top\}$. On the morphisms, we set

$$\eta_{|\mathcal{A}_\perp} = \eta_{|\mathcal{A}_I} = \Psi,\ \eta_{|\mathcal{A}_\top} = \Phi$$

and ask for the restriction of η to

$$\mathcal{A}_I\left(X_0, X_1\right) \otimes \cdots \otimes \mathcal{A}_I\left(X_{p-1}, X_p\right) \otimes \mathcal{C}\left(X_p, X_{p+1}\right) \otimes \mathcal{A}_\perp\left(X_{p+1}, X_{p+2}\right) \otimes \cdots \otimes \mathcal{A}_\perp\left(X_{p+q}, X_{p+q+1}\right)$$

to be Ψ. It remains to define η for

$$\begin{aligned}(\ldots, x, \ldots) \in & \mathcal{A}_I\left(X_0, X_1\right) \otimes \cdots \otimes \mathcal{A}_I\left(X_{p-1}, X_p\right) \\ & \otimes \mathcal{C}\left(X_p, X_{p+1}\right) \otimes \mathcal{A}_\top\left(X_{p+1}, X_{p+2}\right) \otimes \cdots \otimes \mathcal{A}_\top\left(X_{p+q}, X_{p+q+1}\right).\end{aligned}$$

For this we take a homotopy T between Φ and Ψ, which means that

$$\begin{aligned}\Phi + \Psi = & \sum T\left(\ldots, \mu_{\mathcal{A}}\left(\ldots\right), \ldots\right) \\ & + \sum \mu_{\mathcal{B}}\left(\Phi\left(\ldots\right), \ldots, \Phi\left(\ldots\right), T\left(\ldots\right), \Psi\left(\ldots\right), \ldots, \Psi\left(\ldots\right)\right).\end{aligned}$$

Then we let

$$\begin{aligned}&\eta\left(\ldots, x, \ldots\right) = \\ &\quad \sum \mu_{\mathcal{B}}\left(\Phi(\ldots), \ldots, \Phi(\ldots), T(\ldots), \Psi(\ldots), \ldots, \Psi(\ldots), \Psi(\ldots, x, \ldots)\Psi(\ldots), \ldots, \Psi(\ldots)\right)\end{aligned}$$

if p is positive, and $\eta\left(x, \ldots\right) = \Psi\left(x, \ldots\right)$ otherwise.

$\square$

Chapter 2

Mapping torus of a quasi-autoequivalence

In this chapter, we introduce the notion of mapping torus for a quasi-autoequivalence of an A_∞-category, by analogy with the mapping torus associated to an automorphism of a topological space. This terminology was also used in [45], but we do not know if the two notions coincide. The two main theorems of this chapter allow us to compute this mapping torus under different hypotheses. We will then use these results in the next chapter in order to prove Theorem D.

2.1 Definitions and main results

In this section, all the A_∞-categories will be assumed to be small, i.e. to have a set of objects.

2.1.1 Definitions

Definition 2.1. Let τ be a quasi-autoequivalence of an Adams-graded A_∞-category $\mathcal{A}$. The mapping torus of τ is the homotopy colimit of the following diagram

$$\begin{tikzcd} \mathcal{A} \sqcup \mathcal{A} \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\mathrm{id}\sqcup\mathrm{id}"'] & \mathcal{A} \\ \mathcal{A} & \end{tikzcd}$$

(see Definition 1.83).

Remark 2.2. We use the terminology “mapping torus” by analogy with the analogous situation in the category of topological spaces. Indeed, if f is an automorphism of some topological space X, then the mapping torus of f

$$M_f = (X \times [0,1]) \,/\, ((x,0) \sim (f(x),1))$$

is the homotopy colimit of the following diagram

$$\begin{tikzcd} X \sqcup X \arrow[r, "\mathrm{id} \sqcup f"] \arrow[d, "\mathrm{id} \sqcup \mathrm{id}"'] & X \\ X. & \end{tikzcd}$$

Remark 2.3. The terminology "mapping torus of an autoequivalence of A_∞-categories" also appears in [45], and it is used in [44] in order to distinguish open symplectic mapping tori. We do not know if the two notions (the one of [45] and the one of Definition 2.1) coincide.

Remark 2.4. The mapping torus of a quasi-autoequivalence is also Adams-graded, because it is the localization of an Adams-graded A_∞-category at morphisms of Adams-degree 0.

Definition 2.5. Let $\mathcal{A}$ be an A_∞-category. A $\mathbf{Z}$-splitting of $\mathrm{ob}\,(\mathcal{A})$ is a bijection

$$\mathbf{Z} \times \mathcal{E} \xrightarrow{\sim} \mathrm{ob}\,(\mathcal{A})\,, \quad (n, E) \mapsto X^n\,(E)\,.$$

If such a splitting has been chosen, we define the Adams-grading of an homogeneous element $x \in \mathcal{A}\,(X^i(E), X^j(E))$ to be $(j-i)$. This turns $\mathcal{A}$ into an Adams-graded A_∞-category.

Let τ be a quasi-autoequivalence of $\mathcal{A}$. We say that a $\mathbf{Z}$-splitting of $\mathrm{ob}\,(\mathcal{A})$ is compatible with τ if

$$\tau\,(X^n\,(E)) = X^{n+1}\,(E)$$

for every $n \in \mathbf{Z}$ and $E \in \mathcal{E}$.

We say that $\mathcal{A}$ is weakly directed with respect to a $\mathbf{Z}$-splitting of $\mathrm{ob}\,(\mathcal{A})$ if

$$\mathcal{A}\left(X^j(E), X^i(E')\right) = 0$$

for every $i < j$ and $E, E' \in \mathcal{E}$ (we use the term "weakly directed" A_∞-category because the notion is slightly more general than that of directed A_∞-category defined by Seidel in [59] section (5m)).

Remark 2.6. Compatible $\mathbf{Z}$-splittings naturally arise in the context of $\mathbf{Z}$-actions. A strict $\mathbf{Z}$-action on an A_∞-category $\mathcal{A}$ is a family of A_∞-endofunctors $(\tau_n)_{n\in\mathbf{Z}}$ such that $\tau_0 = \mathrm{id}_{\mathcal{A}}$ and $\tau_{i+j} = \tau_i \circ \tau_j$ (see [59] Paragraph (10b)). If the induced $\mathbf{Z}$-action on $\mathrm{ob}\,(\mathcal{A})$ is free, then any section σ of the projection $\mathrm{ob}(\mathcal{A}) \to \mathcal{E}$, where $\mathcal{E}$ is the set of equivalence classes of objects in $\mathcal{A}$ under the $\mathbf{Z}$-action, gives a $\mathbf{Z}$-splitting

$$\mathbf{Z} \times \mathcal{E} \xrightarrow{\sim} \mathrm{ob}\,(\mathcal{A})\,, \quad (n, E) \mapsto \tau_n\,(\sigma\,(E))$$

which is compatible with the automorphism τ_1.

2.1.2 Main results

First result

Definition 2.7. Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbf{Z}$-splitting of $\mathrm{ob}\,(\mathcal{A})$. Assume that τ is strict (see Definition 1.51) and acts bijectively on hom-sets. In this case, we define an Adams-graded A_∞-category $\mathcal{A}_\tau$ as follows:

1. the set of objects of $\mathcal{A}_\tau$ is $\mathcal{E}$,
2. the space of morphisms $\mathcal{A}_\tau\,(E, E')$ is the Adams-graded vector space given by

$$\mathcal{A}_\tau\,(E, E') = \left(\bigoplus_{i,j\in\mathbf{Z}} \mathcal{A}\left(X^i\,(E)\,, X^j\,(E')\right)\right) / \left(\tau(x) \sim x\right)$$

3. the operations are the unique linear maps such that for every sequence

$$(x_0, \ldots, x_{d-1}) \in \mathcal{A}\left(X^{i_0}\,(E_0)\,, X^{i_1}\,(E_1)\right) \times \cdots \times \mathcal{A}\left(X^{i_{d-1}}\,(E_{d-1})\,, X^{i_d}\,(E_d)\right),$$

   we have

$$\mu_{\mathcal{A}_\tau}\,([x_0], \ldots, [x_{d-1}]) = [\mu_{\mathcal{A}}\,(x_0, \ldots, x_{d-1})]\,,$$

   where $[\cdot] : \mathcal{A}\,(X^i\,(E)\,, X^j\,(E')) \to \mathcal{A}_\tau\,(E, E')$ denotes the projection. (It is not hard to see that such operations exist and satisfy the A_∞-relations.)

Theorem 2.8 (Theorem A in the Introduction)**.** *Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbf{Z}$-splitting of* $\mathrm{ob}\,(\mathcal{A})$*. Assume that τ is strict and acts bijectively on hom-sets. Then the mapping torus of τ is quasi-equivalent to $\mathcal{A}_\tau$.*

Remark 2.9. In Theorem 2.8, the A_∞-category $\mathcal{A}_\tau$ is the (ordinary) colimit of the following diagram

$$\begin{tikzcd} \mathcal{A} \sqcup \mathcal{A} \arrow[r, "\mathrm{id} \sqcup \tau"] \arrow[d, "\mathrm{id} \sqcup \mathrm{id}"'] & \mathcal{A} \\ \mathcal{A}. & \end{tikzcd}$$

Thus Theorem 2.8 can be seen as a “ homotopy colimit equals colimit” result.

Second result We denote by $\mathbf{F}[t]$ the augmented Adams-graded associative algebra generated by a variable t of bidegree $(2,1)$, and by $\overline{\mathbf{F}[t]}$ its augmentation ideal (or equivalently, the ideal generated by t). Besides, we use the functors $\mathcal{C} \mapsto \mathcal{C}'$ and $\mathcal{D} \mapsto \mathbf{F}[t] \otimes \mathcal{D}$ introduced in Definitions 1.53 and 1.54 respectively (see also Remark 1.55).

Finally, if $\mathcal{C}$ is an A_∞-category equipped with a $\mathbf{Z}$-splitting of $\mathrm{ob}(\mathcal{C})$, we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$.

Theorem 2.10 (Theorem B in the Introduction)**.** *Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbf{Z}$-splitting of* $\mathrm{ob}(\mathcal{A})$*. Assume that the following holds:*

1. *$\mathcal{A}$ is weakly directed with respect to the $\mathbf{Z}$-splitting of* $\mathrm{ob}(\mathcal{A})$ *(see Definition 2.5),*

2. *there exists a closed degree* 0 *bimodule map $f : \mathcal{A}'(-,-) \to \mathcal{A}'(-,\tau(-))$ (see Definitions 1.63 and 1.64) such that for every $i < j < k$ and for every $E, E' \in \mathcal{E}$, the chain maps*

$$\begin{cases} \mu^2_{\mathcal{A}'}\left(-, f(e_{X^j(E)})\right) & : \quad \mathcal{A}'(X^i(E'), X^j(E)) \quad \to \quad \mathcal{A}'(X^i(E'), X^{j+1}(E)) \\ \mu^2_{\mathcal{A}'}\left(f(e_{X^j(E)}), -\right) & : \quad \mathcal{A}'\left(X^{j+1}(E), X^{k+1}(E')\right) \quad \to \quad \mathcal{A}'\left(X^j(E), X^{k+1}(E')\right) \end{cases}$$

are quasi-isomorphisms.

Then the mapping torus of τ is quasi-equivalent to $(\mathcal{A}')^0 \oplus \left(\overline{\mathbf{F}[t]} \otimes \mathcal{A}'\left[f(\mathrm{units})^{-1}\right]^0\right)$.

Remark 2.11. In [36], the chain complex of $\mathcal{A}$-bimodule maps from the diagonal bimodule $\mathcal{A}(-,-)$ to some $\mathcal{A}$-bimodule $\mathcal{B}$ is called the two-pointed complex for Hochschild cohomology of $\mathcal{A}$ with coefficients in $\mathcal{B}$. Proposition 2.5 in [36] shows that this complex is quasi-isomorphic to the (ordinary) Hochschild cochain complex of $\mathcal{A}$ with coefficients in $\mathcal{B}$. In particular, the bimodule map f in Theorem 2.10 defines a class in the Hochschild cohomology of $\mathcal{A}'$ with coefficients in $\mathcal{A}'(-,\tau(-))$.

Remark 2.12. The A_∞-category which computes the mapping torus in Theorem 2.10 is very similar to the curved A_∞-categories studied by Seidel in [57].

Outline of the chapter In section 2.2, we consider an A_∞-category $\mathcal{A}$ equipped with a $\mathbf{Z}$-splitting of $\mathrm{ob}\,(\mathcal{A})$ and a choice of a closed degree 0 morphism $c_n(E) \in \mathcal{A}\left(X^n(E), X^{n+1}(E)\right)$ for every $n \in \mathbf{Z}$ and every $E \in \mathcal{E}$. We give technical results about specific modules associated to this data. This will be used in the proof of Theorem 2.10 with $c_n\left(E\right) = f\left(e_{X^n(E)}\right)$.

In section 2.3, we consider the Grothendieck construction $\mathcal{G}$ of a slightly different diagram than the one in Definition 2.1, together with a set $W_{\mathcal{G}}$ of closed degree 0 morphisms. The idea is that the localization $\mathcal{H} = \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]$ is the homotopy colimit of a diagram obtained from the one in Definition 2.1 by a cofibrant replacement of the diagonal functor $\mathcal{A} \sqcup \mathcal{A} \to \mathcal{A}$. Thus it is not surprising that $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ. Moreover, we prove technical results about specific modules over $\mathcal{G}$ that will be used in the proofs of Theorems 2.8 and 2.10.

In section 2.4, we prove Theorem 2.8. We first define an A_∞-functor $\Phi : \mathcal{G} \to \mathcal{A}_\tau$ which sends $W_{\mathcal{G}}$ to the set of units in $\mathcal{A}_\tau$. Then we prove that the induced A_∞-functor $\widetilde{\Phi} : \mathcal{H} \to \mathcal{A}_\tau\left[\{\text{units}\}^{-1}\right]$ is a quasi-equivalence. To do that, our strategy is to apply Proposition 1.82 using the results of section 2.3 about the specific $\mathcal{G}$-modules.

In section 2.5, we prove Theorem 2.10. We use the fact that $\mathcal{G}$ is "big enough" (there are more objects and morphisms than in the Grothendieck construction of the diagram in Definition 2.1) in order to define an A_∞-functor $\Psi' : \mathcal{G}' \to \mathcal{A}'$ (see Definition 1.53 for the meaning of the "prime" symbol). This induces an A_∞-functor $\widetilde{\Psi} : \mathcal{H} \to \mathbf{F}\left[t\right] \otimes \mathcal{A}'\left[f\left(\{\text{units}\}\right)^{-1}\right]$. Then we prove that for every Adams degree $j \geq 1$, and for every objects X, Y in $\mathcal{H}$, the map

$$\widetilde{\Psi} : \mathcal{H}\left(X, Y\right)_j \to \left(\mathbf{F}\left[t\right] \otimes \mathcal{A}'\left[f\left(\{\text{units}\}\right)^{-1}\right]\right)\left(\Psi X, \Psi Y\right)_j$$

is a quasi-isomorphism (if V is an Adams-graded vector space, V_j denotes the subspace of Adams degree j elements). To do that, we apply once again Proposition 1.82 using the results of sections 2.2 and 2.3 about the specific modules over $\mathcal{A}'$ and $\mathcal{G}$ respectively. This allows us to finish the proof of Theorem 2.10.

2.2 Results about specific modules

In this section, we give technical results that will allow us to apply Proposition 1.82 in the proof of Theorem 2.10.

Let $\mathcal{A}$ be an A_∞-category equipped with a $\mathbf{Z}$-splitting of ob $(\mathcal{A})$. Assume that we chose, for every $n \in \mathbf{Z}$ and every $E \in \mathcal{E}$, a closed degree 0 morphism $c_n(E) \in \mathcal{A}\left(X^n(E), X^{n+1}(E)\right)$. Moreover, assume that we chose a set $W_{\mathcal{A}}$ of closed degree 0 morphisms which contains the morphisms $c_n(E)$.

Remark 2.13. According to Definition 2.5, the $\mathbf{Z}$-splitting of ob $(\mathcal{A})$ naturally induces an Adams-grading on $\mathcal{A}$. However in this section, we do not consider $\mathcal{A}$ as being Adams-graded.

In the following, we fix some element $E_\diamond \in \mathcal{E}$. When we write an object X^n or a morphism c_n without specifying the element of $\mathcal{E}$, we mean $X^n(E_\diamond)$ or $c_n(E_\diamond)$ respectively. Recall that $t_{c_n} : \mathcal{A}\left(-, X^n\right) \to \mathcal{A}\left(-, X^{n+1}\right)$ denotes the $\mathcal{A}$-module map induced by $c_n \in \mathcal{A}\left(X^n, X^{n+1}\right)$ (see Definition 1.68).

Definition 2.14. We set $\mathcal{M}_{\mathcal{A}}$ to be the $\mathcal{A}$-module

$$\mathcal{M}_{\mathcal{A}} := \left[\begin{tikzcd} \cdots \arrow[dr, "t_{c_{-1}}"] & \mathcal{A}\left(-, X^0\right) \arrow[d, "\mathrm{id}"] \arrow[dr, "t_{c_0}"] & \mathcal{A}\left(-, X^1\right) \arrow[d, "\mathrm{id}"] \arrow[dr, "t_{c_1}"] & \cdots \\ \cdots & \mathcal{A}\left(-, X^0\right) & \mathcal{A}\left(-, X^1\right) & \cdots \end{tikzcd}\right]$$

(see Definition 1.71). Besides, we set $t_{\mathcal{A}} : \mathcal{A}\left(-, X^n\right) \to \mathcal{M}_{\mathcal{A}}$ to be the $\mathcal{A}$-module inclusion for every $n \in \mathbf{Z}$.

The first result highlights a key property of the module $\mathcal{M}_{\mathcal{A}}$.

Lemma 2.15. *For every $n \in \mathbf{Z}$, the closed $\mathcal{A}$-module map $t_{\mathcal{A}} \circ t_{c_n} : \mathcal{A}\left(-, X^n\right) \to \mathcal{M}_{\mathcal{A}}$ is homotopic to $t_{\mathcal{A}} : \mathcal{A}\left(-, X^n\right) \to \mathcal{M}_{\mathcal{A}}$.*

Proof. Consider the degree (-1) strict $\mathcal{A}$-module map $s : \mathcal{A}\left(-, X^n\right) \to \mathcal{M}_{\mathcal{A}}$ which sends a morphism in $\mathcal{A}\left(X, X^n\right)$ to the corresponding shifted element in $\mathcal{A}\left(X, X^n\right)[1]$. Then an easy computation gives

$$\mu^1_{\mathrm{Mod}_{\mathcal{A}}}(s) = t_{\mathcal{A}} \circ t_{c_n} + t_{\mathcal{A}}.$$

This concludes the proof. □

In the proof of the two results below, we will use specific $\mathcal{A}$-modules. If p is a fixed non negative integer, we set

$$K_p = \left[\begin{tikzcd} \cdots \arrow[dr, "t_{c_{p-2}}"] & \mathcal{A}\left(-, X^{p-1}\right) \arrow[d, "\mathrm{id}"] \arrow[dr, "t_{c_{p-1}}"] & \mathcal{A}\left(-, X^{p}\right) \arrow[d, "\mathrm{id}"] \\ \cdots & \mathcal{A}\left(-, X^{p-1}\right) & \mathcal{A}\left(-, X^{p}\right) \end{tikzcd}\right]$$

and

$$\widetilde{K}_p = \left[\begin{tikzcd} \mathcal{A}\left(-, X^{p}\right) \arrow[dr, "t_{c_{p}}"] & \mathcal{A}\left(-, X^{p+1}\right) \arrow[d, "\mathrm{id}"] \arrow[dr, "t_{c_{p+1}}"] & \cdots \\ & \mathcal{A}\left(-, X^{p+1}\right) & \cdots \end{tikzcd}\right].$$

Moreover, we will consider the sequences of $\mathcal{A}$-modules $\left(F_p^q\right)_{q\geq 0}$, $\left(\widetilde{F}_p^q\right)_{q\geq 0}$ starting at $F_p^0 = \widetilde{F}_p^0 = 0$ and with

$$F_p^q = \left[\begin{tikzcd} \mathcal{A}\left(-, X^{p-q+1}\right) \arrow[d, "\mathrm{id}"] \arrow[dr, "t_{c_{p-q+1}}"] & \cdots \arrow[dr, "t_{c_{p-1}}"] & \mathcal{A}\left(-, X^{p}\right) \arrow[d, "\mathrm{id}"] \\ \mathcal{A}\left(-, X^{p-q+1}\right) & \cdots & \mathcal{A}\left(-, X^{p}\right) \end{tikzcd}\right]$$

and

$$\widetilde{F}_p^q = \left[\begin{tikzcd} \mathcal{A}\left(-, X^{p}\right) \arrow[dr, "t_{c_{p}}"] & \cdots \arrow[dr, "t_{c_{p+q-1}}"] & \\ & \cdots & \mathcal{A}\left(-, X^{p+q}\right) \end{tikzcd}\right]$$

for $q \geq 1$.

The following Lemma is mostly technical. It will be used in the proofs of Lemmas 2.17 and 2.30.

Lemma 2.16. *Assume that for every $i < j$, for every $E \in \mathcal{E}$, the chain map*

$$\mu^2_{\mathcal{A}}\left(-, c_j\right) : \mathcal{A}\left(X^i(E), X^j\right) \to \mathcal{A}\left(X^i(E), X^{j+1}\right)$$

is a quasi-isomorphism. Then for every $k < n$, for every $E \in \mathcal{E}$, the inclusion $\mathcal{A}\left(X^k(E), X^n\right) \hookrightarrow \mathcal{M}_{\mathcal{A}}\left(X^k(E)\right)$ is a quasi-isomorphism.

Proof. The cone of the inclusion $\mathcal{A}\left(X^k(E), X^n\right) \hookrightarrow \mathcal{M}_{\mathcal{A}}\left(X^k(E)\right)$ is quasi-isomorphic to its cokernel, which is $K_{n-1}\left(X^k(E)\right) \oplus \widetilde{K}_n\left(X^k(E)\right)$.

We have to show that these complexes are acyclic. Observe that $\left(F^q_{n-1}\left(X^k(E)\right)\right)_{q\geq 0}$ and $\left(\widetilde{F}^q_n\left(X^k(E)\right)\right)_{q\geq 0}$ are increasing, exhaustive, and bounded from below filtrations of $K_{n-1}\left(X^k(E)\right)$ and $\widetilde{K}_n\left(X^k(E)\right)$ respectively. For every $q \geq 1$, we have

$$F^q_{n-1}\left(X^k(E)\right)/F^{q-1}_{n-1}\left(X^k(E)\right) = \left[\begin{array}{c} \mathcal{A}\left(X^k(E), X^{n-q}\right) \\ \Big\downarrow{\scriptstyle \mathrm{id}} \\ \mathcal{A}\left(X^k(E), X^{n-q}\right) \end{array}\right]$$

and

$$\widetilde{F}^q_n\left(X^k(E)\right)/\widetilde{F}^{q-1}_n\left(X^k(E)\right) = \left[\begin{array}{cc} \mathcal{A}\left(X^k(E), X^{n+q-1}\right) & \\ & \searrow^{t_{c_{n+q-1}}} \\ & \mathcal{A}\left(X^k(E), X^{n+q}\right) \end{array}\right].$$

The first of the two latter complexes is clearly acyclic, and the second one is acyclic by assumption on the morphisms c_j. Thus the entire complex $K_{n-1}\left(X^k(E)\right) \oplus \widetilde{K}_n\left(X^k(E)\right)$ is acyclic, which is what we needed to prove.

□

The following Lemma will be used later in order to apply Proposition 1.82.

Lemma 2.17. *Assume that for every $i < j < k$, for every $E \in \mathcal{E}$, the chain maps*

$$\begin{cases} \mu^2_{\mathcal{A}}(-, c_j) & : & \mathcal{A}(X^i(E), X^j) & \to & \mathcal{A}(X^i(E), X^{j+1}) \\ \mu^2_{\mathcal{A}}(c_j(E), -) & : & \mathcal{A}\left(X^{j+1}(E), X^{k+1}\right) & \to & \mathcal{A}\left(X^j(E), X^{k+1}\right) \end{cases}$$

are quasi-isomorphisms. Then the following holds for every $n \in \mathbf{Z}$:

1. *the chain complex $\mathcal{M}_{\mathcal{A}}(\mathrm{Cone}\, c_n(E))$ is acyclic for every $E \in \mathcal{E}$, and*

2. *the $\mathcal{A}$-module map ${}_{W_{\mathcal{A}}^{-1}}t_{\mathcal{A}} : {}_{W_{\mathcal{A}}^{-1}}\mathcal{A}(-, X^n) \to {}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}$ is a quasi-isomorphism.*

Proof. We have

$$\mathcal{M}_{\mathcal{A}}(\mathrm{Cone}\, c_n(E)) = \mathrm{Cone}\left(\mathcal{M}_{\mathcal{A}}\left(X^{n+1}(E)\right) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E), -)} \mathcal{M}_{\mathcal{A}}(X^n(E))\right),$$

so we have to prove that $\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E), -) : \mathcal{M}_{\mathcal{A}}(X^{n+1}(E)) \to \mathcal{M}_{\mathcal{A}}(X^n(E))$ is a quasi-isomorphism. Observe that we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{A}}(X^{n+1}(E)) \arrow[r, "\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E),-)"] & \mathcal{M}_{\mathcal{A}}(X^n(E)) \\
\mathcal{A}(X^{n+1}(E), X^{n+2}) \arrow[u, hook] \arrow[r, "\mu^2_{\mathcal{A}}(c_n(E),-)"] & \mathcal{A}(X^n(E), X^{n+2}) \arrow[u, hook].
\end{tikzcd}$$

The bottom horizontal map is a quasi-isomorphism by assumption on the morphisms $c_j(E)$. Moreover, the vertical maps are quasi-isomorphisms according to Lemma 2.16. This implies that $\mu^2_{\mathcal{M}_{\mathcal{A}}}(c_n(E), -)$ is indeed a quasi-isomorphism.

Now let X be some object of $\mathcal{A}$. We want to prove that the chain map ${}_{W_{\mathcal{A}}^{-1}}t_{\mathcal{A}} : {}_{W_{\mathcal{A}}^{-1}}\mathcal{A}(X, X^n) \to {}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}(X)$ is a quasi-isomorphism. Observe that

$${}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}(X) = \left[\begin{tikzcd}
\cdots \arrow[dr] & \mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^0) \arrow[d, "\mathrm{id}"] \arrow[dr, "{}_{W_{\mathcal{A}}^{-1}}t_{c_0}"] & \mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^1) \arrow[d, "\mathrm{id}"] \arrow[dr, "{}_{W_{\mathcal{A}}^{-1}}t_{c_1}"] & \cdots \\
\cdots & \mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^0) & \mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^1) & \cdots
\end{tikzcd}\right]$$

and the chain map ${}_{W_{\mathcal{A}}^{-1}}t_{\mathcal{A}} : {}_{W_{\mathcal{A}}^{-1}}\mathcal{A}(X, X^n) \to {}_{W_{\mathcal{A}}^{-1}}\mathcal{M}_{\mathcal{A}}(X)$ is the inclusion. The cone of the latter is then quasi-isomorphic to its cokernel, which is ${}_{W_{\mathcal{A}}^{-1}}K_{n-1}(X) \oplus {}_{W_{\mathcal{A}}^{-1}}\widetilde{K}_n(X)$. Observe that $\left({}_{W_{\mathcal{A}}^{-1}}F^q_{n-1}(X)\right)_{q\geq 0}$, $\left({}_{W_{\mathcal{A}}^{-1}}\widetilde{F}^q_n(X)\right)_{q\geq 0}$ are increasing, exhaustive, and bounded from below filtrations of ${}_{W_{\mathcal{A}}^{-1}}K_{n-1}(X)$, ${}_{W_{\mathcal{A}}^{-1}}\widetilde{K}_n(X)$ respectively. For every $q \geq 1$, we have

$${}_{W_{\mathcal{A}}^{-1}}F^q_{n-1}(X) / {}_{W_{\mathcal{A}}^{-1}}F^{q-1}_{n-1}(X) = \left[\begin{tikzcd}
\mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^{n-q}) \arrow[d, "\mathrm{id}"] \\
\mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^{n-q})
\end{tikzcd}\right]$$

and

$${}_{W_{\mathcal{A}}^{-1}}\widetilde{F}^q_n(X) / {}_{W_{\mathcal{A}}^{-1}}\widetilde{F}^{q-1}_n(X) = \left[\begin{tikzcd}
\mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^{n-1+q}) \arrow[dr, "{}_{W_{\mathcal{A}}^{-1}}t_{c_{n-1+q}}"] & \\
& \mathcal{A}\left[W_{\mathcal{A}}^{-1}\right](X, X^{n+q})
\end{tikzcd}\right].$$

The first of the two latter complexes is clearly acyclic, and the second one is acyclic because c_{n-1+q} belongs to the set $W_{\mathcal{A}}$ by which we localized. Thus the entire complex ${}_{W_{\mathcal{A}}^{-1}}K_{n-1}(X) \oplus {}_{W_{\mathcal{A}}^{-1}}\widetilde{K}_n(X)$ is acyclic, which is what we needed to prove.

□

2.3 The A_∞-category and modules for the mapping torus

In this section, we consider an A_∞-category $\mathcal{G}$, together with a set $W_{\mathcal{G}}$ of closed degree 0 morphisms. We prove that $\mathcal{H} = \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]$ is quasi-equivalent to the mapping torus of τ, and we prove technical results about specific $\mathcal{G}$-modules that will allow us to apply Proposition 1.82 in the proofs of Theorems 2.8 and 2.10.

Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbf{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$. If $\mathcal{A}_\triangle$ is a copy of $\mathcal{A}$, we denote by $X_\triangle^n(E)$ the object of $\mathcal{A}_\triangle$ corresponding to $(n, E) \in \mathbf{Z} \times \mathcal{E}$.

2.3.1 The A_∞-category $\mathcal{G}$

The A_∞-category $\mathcal{G}$ will be the Grothendieck construction of a slightly different diagram than the one in Definition 2.1. The idea is to introduce an A_∞-category $\mathcal{C}$ together with a set of closed degree 0 morphisms $W_{\mathcal{C}}$ such that the localization $\mathcal{C}\left[W_{\mathcal{C}}^{-1}\right]$ is a cylinder object for $\mathcal{A}$. If this enlargement might looks weird at first sight, we will need it in the proof of Theorem 2.10.

Definition 2.18. Let $\mathcal{A}_\perp$, $\mathcal{A}_I$ and $\mathcal{A}_\top$ be three copies of $\mathcal{A}$. We denote by $\mathcal{C}$ be the Grothendieck construction (see Definition 1.83) of the following diagram

$$\begin{tikzcd} \mathcal{A}_I \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathcal{A}_\top \\ \mathcal{A}_\perp & \end{tikzcd}$$

and we let $\iota_\perp, \iota_I, \iota_\top : \mathcal{A} \to \mathcal{C}$ be the strict inclusions with images $\mathcal{A}_\perp$, $\mathcal{A}_I$, $\mathcal{A}_\top$ respectively. Finally, we denote by $W_{\mathcal{C}}$ the set of adjacent units in $\mathcal{C}$, and we let $\mathcal{C}yl_{\mathcal{A}} = \mathcal{C}\left[W_{\mathcal{C}}^{-1}\right]$ be the homotopy colimit of the diagram above.

Definition 2.19. Let $\mathcal{A}_-$, $\mathcal{A}_+$, $\mathcal{A}_\bullet$ be three copies of $\mathcal{A}$. We denote by $\mathcal{G}$ the Grothendieck construction of the following diagram

$$\begin{tikzcd} \mathcal{A}_- \sqcup \mathcal{A}_+ \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\iota_\perp\sqcup\iota_\top"'] & \mathcal{A}_\bullet \\ \mathcal{C}. & \end{tikzcd}$$

Besides, we denote by $W_\mathcal{G}$ the union of $W_\mathcal{C}$ and the set of adjacent units in $\mathcal{G}$, and we set

$$\mathcal{H} := \mathcal{G}\left[W_\mathcal{G}^{-1}\right].$$

According to Proposition 1.87, $\mathcal{C}yl_\mathcal{A}$ can be thought as a cylinder object for $\mathcal{A}$. Therefore, the following result should not be surprising.

Lemma 2.20. *The mapping torus of τ is quasi-equivalent to $\mathcal{H}$.*

Proof. Let $\pi : \mathcal{C} \to \mathcal{A}$ be the A_∞-functor induced by the following commutative diagram

$$\begin{tikzcd} \mathcal{A}_I \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathcal{A}_\top \arrow[d, "\mathrm{id}"] \\ \mathcal{A}_\perp \arrow[r, "\mathrm{id}"] & \mathcal{A} \end{tikzcd}$$

(see Proposition 1.84). We get a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"] & \mathcal{A}_- \sqcup \mathcal{A}_+ \arrow[l, "\iota_\perp\sqcup\iota_\top"'] \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\mathrm{id}"] & \mathcal{A}_\bullet \arrow[d, "\mathrm{id}"] \\ \mathcal{A} & \mathcal{A}_- \sqcup \mathcal{A}_+ \arrow[l, "\mathrm{id}\sqcup\mathrm{id}"] \arrow[r, "\mathrm{id}\sqcup\tau"'] & \mathcal{A}_\bullet \end{tikzcd}$$

which induces an A_∞-functor χ from $\mathcal{G}$ to the Grothendieck construction of the bottom line (see Proposition 1.85). Observe that χ sends $W_\mathcal{C}$ to the set U of units in $\mathcal{A}$. Now, according to Proposition 1.87, the A_∞-functor $\widetilde{\pi} : \mathcal{C}yl_\mathcal{A} = \mathcal{C}\left[W_\mathcal{C}^{-1}\right] \to \mathcal{A}\left[U^{-1}\right]$ is a quasi-equivalence. According to Lemma A.6 in [38] (called "localization and homotopy colimits commute"), this implies that the A_∞-functor induced by χ

$$\operatorname{hocolim}\left(\begin{tikzcd} \mathcal{A}_- \sqcup \mathcal{A}_+ \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\iota_\perp\sqcup\iota_\top"] & \mathcal{A}_\bullet \\ \mathcal{C} & \end{tikzcd}\right)\left[W_\mathcal{C}^{-1}\right] \xrightarrow{\widetilde{\chi}} \operatorname{hocolim}\left(\begin{tikzcd} \mathcal{A}_- \sqcup \mathcal{A}_+ \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\mathrm{id}\sqcup\mathrm{id}"] & \mathcal{A}_\bullet \\ \mathcal{A} & \end{tikzcd}\right)\left[U^{-1}\right]$$

is a quasi-equivalence. This completes the proof because the source of $\widetilde{\chi}$ is exactly $\mathcal{H}$, and its target is quasi-equivalent to the mapping torus of τ.

$\square$

2.3.2 Modules over $\mathcal{G}$

In the following, we fix some element $E_\diamond \in \mathcal{E}$. When we write an object $X^n_\triangle$ without specifying the element of $\mathcal{E}$, we mean $X^n_\triangle(E_\diamond)$. Moreover, we denote by

$$t^n_{\triangle\square} : \mathcal{G}\left(-, X^n_\triangle\right) \to \mathcal{G}\left(-, X^{n+\delta_{\triangle\square}}_\square\right)$$

the $\mathcal{G}$-module map induced by the adjacent unit in $\mathcal{G}\left(X^n_\triangle, X^{n+\delta_{\triangle\square}}_\square\right)$ (see Definition 1.68), where

$$\delta_{\triangle\square} = \begin{cases} 1 & \text{if } (\triangle, \square) = (+, \bullet) \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.21. We denote by $\mathcal{M}_{\mathcal{G}}$ the $\mathcal{G}$-module defined by

$$\mathcal{M}_{\mathcal{G}} = \left[\begin{tikzcd} \cdots \arrow[dr, "t^{-1}_{+\bullet}"] & \mathcal{G}\left(-, X^0_-\right) \arrow[d, "t^0_{-\bullet}"] \arrow[dr, "t^0_{-\perp}"] & \mathcal{G}\left(-, X^0_I\right) \arrow[d, "t^0_{I\perp}"] \arrow[dr, "t^0_{I\top}"] & \mathcal{G}\left(-, X^0_+\right) \arrow[d, "t^0_{+\top}"] \arrow[dr, "t^0_{+\bullet}"] & \mathcal{G}\left(-, X^1_-\right) \arrow[d, "t^1_{-\bullet}"] \arrow[dr, "t^1_{-\perp}"] & \cdots \\ \cdots & \mathcal{G}\left(-, X^0_\bullet\right) & \mathcal{G}\left(-, X^0_\perp\right) & \mathcal{G}\left(-, X^0_\top\right) & \mathcal{G}\left(-, X^1_\bullet\right) & \cdots \end{tikzcd} \right]$$

(see Definition 1.71). For practical reasons, we also consider the $\mathcal{G}$-modules

$$\mathcal{M}^n_\star := \left[\begin{tikzcd} & \mathcal{G}\left(-, X^n_I\right) \arrow[dl, "t^n_{I\perp}"'] \arrow[dr, "t^n_{I\top}"] & \\ \mathcal{G}\left(-, X^n_\perp\right) & & \mathcal{G}\left(-, X^n_\top\right) \end{tikzcd} \right], \quad n \in \mathbf{Z}.$$

Besides, we denote by $t_{\mathcal{G}} : \mathcal{G}\left(-, X^n_\triangle\right) \to \mathcal{M}_{\mathcal{G}}$ the $\mathcal{G}$-module inclusion for every $\triangle \in \{\perp, \top, \bullet\}$ and $n \in \mathbf{Z}$.

Remark 2.22. We can write

$$\mathcal{M}_{\mathcal{G}} = \left[\begin{tikzcd} & \bigoplus_{n\in\mathbf{Z}} \left(\mathcal{G}\left(-, X^n_-\right) \oplus \mathcal{G}\left(-, X^n_+\right)\right) \arrow[dl, "\bigoplus_{n\in\mathbf{Z}} \left(t^n_{-\perp}\oplus t^n_{+\top}\right)"'] \arrow[dr, "\bigoplus_{n\in\mathbf{Z}} \left(t^n_{-\bullet}\oplus t^n_{+\bullet}\right)"] & \\ \bigoplus_{n\in\mathbf{Z}} \mathcal{M}^n_\star & & \bigoplus_{n\in\mathbf{Z}} \mathcal{G}\left(-, X^n_\bullet\right) \end{tikzcd}\right].$$

The following Lemma is an analog of Lemma 2.16. It will be used later in order to apply Proposition 1.82.

Lemma 2.23. *The pair $(\mathcal{M}_{\mathcal{G}}, t_{\mathcal{G}})$ satisfies the following*

1. *for every w in $W_{\mathcal{G}}$, the chain complex $\mathcal{M}_{\mathcal{G}}(\mathrm{Cone}\, w)$ is acyclic, and*
2. *the $\mathcal{H}$-module map ${}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}} : {}_{W_{\mathcal{G}}^{-1}}\mathcal{G}\left(-, X^n_\triangle\right) \to {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}$ is a quasi-isomorphism for every $\triangle \in \{\perp, \top, \bullet\}$ and $n \in \mathbf{Z}$.*

Proof. We begin to prove the first item of the Lemma. We have to treat different cases.

Let w be the morphism in $W_{\mathcal{G}} \cap \mathcal{G}\left(X^k_I(E), X^k_\top(E)\right)$ (the proof is analogous for the morphism in $W_{\mathcal{G}} \cap \mathcal{G}\left(X^k_I(E), X^k_\perp(E)\right)$). Then

$$\begin{aligned}\mathcal{M}_{\mathcal{G}}(\mathrm{Cone}\, w) &= \mathrm{Cone}\left(\mathcal{M}_{\mathcal{G}}\left(X^k_\top(E)\right) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)} \mathcal{M}_{\mathcal{G}}\left(X^k_I(E)\right)\right)\\ &= \bigoplus_n \mathrm{Cone}\left(\mathcal{G}\left(X^k_\top(E), X^n_\top\right) \xrightarrow{\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)} \mathcal{M}^n_\star\left(X^k_I(E)\right)\right).\end{aligned}$$

We want to prove that $\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-) : \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \to \mathcal{M}^n_\star\left(X^k_I(E)\right)$ is a quasi-isomorphism for every n. Observe that the following diagram of chain complexes is commutative

$$\begin{tikzcd} \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \arrow[r, "\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-)"] \arrow[d, equal] & \mathcal{M}^n_\star\left(X^k_I(E)\right) \\ \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \arrow[r, "\mu^2_{\mathcal{G}}(w,-)"] & \mathcal{G}\left(X^k_I(E), X^n_\top\right) \arrow[u, hook] \end{tikzcd}.$$

The rightmost vertical arrow is injective, so its cone is quasi-isomorphic to its cokernel, which is the cone of $t^n_{I\perp} : \mathcal{G}\left(X^k_I(E), X^n_I\right) \to \mathcal{G}\left(X^k_I(E), X^n_\perp\right)$. Since the latter map is a quasi-isomorphism, the cone of $\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-) : \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \to \mathcal{M}^n_\star\left(X^k_I(E)\right)$ is quasi-isomorphic to the cone of $\mu^2_{\mathcal{G}}(w,-) : \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \to \mathcal{G}\left(X^k_I(E), X^n_\top\right)$. The latter map is a quasi-isomorphism, so we conclude that $\mu^2_{\mathcal{M}_\mathcal{G}}(w,-) : \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \to \mathcal{M}^n_\star\left(X^k_I(E)\right)$ is a quasi-isomorphism for every n, and thus $\mathcal{M}_\mathcal{G}(\operatorname{Cone} w)$ is acyclic.

Now let w be the morphism in $W_\mathcal{G} \cap \mathcal{G}\left(X^k_+(E), X^k_\top(E)\right)$ (the proof is analogous for the morphism in $W_\mathcal{G} \cap \mathcal{G}\left(X^k_-(E), X^k_\perp(E)\right)$). Then

$$\begin{aligned}\mathcal{M}_\mathcal{G}(\operatorname{Cone} w) &= \operatorname{Cone}\left(\mathcal{M}_\mathcal{G}\left(X^k_\top(E)\right) \xrightarrow{\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)} \mathcal{M}_\mathcal{G}\left(X^k_+(E)\right)\right) \\ &= \bigoplus_n \operatorname{Cone}\left(\mathcal{G}\left(X^k_\top(E), X^n_\top\right) \xrightarrow{\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)} K^n\right).\end{aligned}$$

where

$$K^n = \left[\begin{tikzcd} & \mathcal{G}\left(X^k_+(E), X^n_+\right) \arrow[dl, "t^n_{+\top}"'] \arrow[dr, "t^n_{+\bullet}"] & \\ \mathcal{G}\left(X^k_+(E), X^n_\top\right) & & \mathcal{G}\left(X^k_+(E), X^{n+1}_\bullet\right)\end{tikzcd}\right].$$

Observe that $\mu^2_{\mathcal{M}_\mathcal{G}}(w,-) : \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \to K^n$ is injective, so its cone is quasi-isomorphic to its cokernel, which is the cone of $t^n_{+\bullet} : \mathcal{G}\left(X^k_+(E), X^n_+\right) \to \mathcal{G}\left(X^k_+(E), X^{n+1}_\bullet\right)$. The latter map is a quasi-isomorphism because τ is a quasi-equivalence. This implies that the cone of $\mu^2_{\mathcal{M}_\mathcal{G}}(w,-) : \mathcal{G}\left(X^k_\top(E), X^n_\top\right) \to K^n$ is acyclic for every n, and thus $\mathcal{M}_\mathcal{G}(\operatorname{Cone} w)$ is acyclic.

It remains to consider a morphism w in $W_\mathcal{G} \cap \mathcal{G}\left(X^k_+(E), X^{k+1}_\bullet(E)\right)$ (the proof is analogous for the morphism in $W_\mathcal{G} \cap \mathcal{G}\left(X^k_-(E), X^k_\bullet(E)\right)$). Then

$$\begin{aligned}\mathcal{M}_\mathcal{G}(\operatorname{Cone} w) &= \operatorname{Cone}\left(\mathcal{M}_\mathcal{G}\left(X^{k+1}_\bullet(E)\right) \xrightarrow{\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)} \mathcal{M}_\mathcal{G}\left(X^k_+(E)\right)\right) \\ &= \bigoplus_n \operatorname{Cone}\left(\mathcal{G}\left(X^{k+1}_\bullet(E), X^n_\bullet\right) \xrightarrow{\mu^2_{\mathcal{M}_\mathcal{G}}(w,-)} K^n\right).\end{aligned}$$

where

$$K^n = \left[\begin{tikzcd} & \mathcal{G}\left(X^k_+(E), X^{n-1}_+\right) \arrow[dl, "t^{n-1}_{+\top}"'] \arrow[dr, "t^{n-1}_{+\bullet}"] & \\ \mathcal{G}\left(X^k_+(E), X^{n-1}_\top\right) & & \mathcal{G}\left(X^k_+(E), X^n_\bullet\right)\end{tikzcd}\right].$$

Observe that $\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-) : \mathcal{G}\left(X^{k+1}_{\bullet}(E), X^n_{\bullet}\right) \to K^n$ is injective, so its cone is quasi-isomorphic to its cokernel, which is the cone of $t^{n-1}_{+\top} : \mathcal{G}\left(X^k_+(E), X^{n-1}_+\right) \to \mathcal{G}\left(X^k_+(E), X^{n-1}_\top\right)$. The latter map is a quasi-isomorphism, so we conclude that the cone of $\mu^2_{\mathcal{M}_{\mathcal{G}}}(w,-) : \mathcal{G}\left(X^{k+1}_{\bullet}(E), X^n_{\bullet}\right) \to K^n$ is acyclic for every n, and thus $\mathcal{M}_{\mathcal{G}}(\operatorname{Cone} w)$ is acyclic.

We now prove the second part of the Lemma. A priori, we also have to treat different cases depending on if $\triangle$ equals $\perp$, $\top$ or $\bullet$, but it turns out that the proofs are almost identical in every cases, so we will only do one, for example when $\triangle = \top$. We fix an object X in $\mathcal{G}$, and we want to prove that ${}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}} : {}_{W_{\mathcal{G}}^{-1}}\mathcal{G}(X, X^n_\top) \to {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X)$ is a quasi-isomorphism. Observe that

$$
{}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}} := \left[\begin{tikzcd}
\cdots \arrow[dr] & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(-, X^0_+\right) \arrow[d, "{}_{W_{\mathcal{G}}^{-1}}t^0_{+\top}"] \arrow[dr, "{}_{W_{\mathcal{G}}^{-1}}t^0_{+\bullet}"'] & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(-, X^1_-\right) \arrow[d, "{}_{W_{\mathcal{G}}^{-1}}t^1_{-\bullet}"] \arrow[dr, "{}_{W_{\mathcal{G}}^{-1}}t^1_{-\perp}"] & \cdots \\
\cdots & {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}^0_{\star} & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(-, X^1_\bullet\right) & \cdots
\end{tikzcd}\right]
$$

and that the chain map ${}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}} : {}_{W_{\mathcal{G}}^{-1}}\mathcal{G}(X, X^n_\top) \to {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X)$ is the inclusion. The cone of the latter is then quasi-isomorphic to its cokernel, which can be written $K' \oplus K''$ with

$$
K' = \left[\begin{tikzcd}
\cdots \arrow[dr] & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(X, X^n_-\right) \arrow[d, "{}_{W_{\mathcal{G}}^{-1}}t^n_{-\bullet}"] \arrow[dr, "{}_{W_{\mathcal{G}}^{-1}}t^n_{-\perp}"'] & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(X, X^n_I\right) \arrow[d, "{}_{W_{\mathcal{G}}^{-1}}t^n_{I\perp}"] \\
\cdots & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(X, X^n_\bullet\right) & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(X, X^n_\perp\right)
\end{tikzcd}\right]
$$

and

$$
K'' = \left[\begin{tikzcd}
\mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(X, X^n_+\right) \arrow[dr, "{}_{W_{\mathcal{G}}^{-1}}t^n_{+\bullet}"'] & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(X, X^{n+1}_-\right) \arrow[d, "{}_{W_{\mathcal{G}}^{-1}}t^{n+1}_{-\bullet}"'] \arrow[dr, "{}_{W_{\mathcal{G}}^{-1}}t^{n+1}_{-\perp}"] & \cdots \\
 & \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right]\left(X, X^{n+1}_\bullet\right) & \cdots
\end{tikzcd}\right].
$$

Observe that the maps defining the chain complexes structures in K' and K'' are all quasi-isomorphisms. Thus it is not difficult to show using an increasing exhaustive and bounded from below filtration of K' and K'' that these complexes are acyclic (compare the end of the proof of Lemma 2.17). This implies that the map ${}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}} : {}_{W_{\mathcal{G}}^{-1}}\mathcal{G}(X, X^n_\top) \to {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X)$ is a quasi-isomorphism.

□

2.4 Proof of the first result

Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbf{Z}$-splitting of $\operatorname{ob}(\mathcal{A})$. Assume that τ is strict and acts bijectively on hom-sets.

Observe that there is a strict A_∞-functor $\sigma : \mathcal{A} \to \mathcal{A}_\tau$ which sends $X^n(E)$ to E, and which sends $x \in \mathcal{A}\left(X^i(E_1), X^j(E_2)\right)$ to $[x] \in \mathcal{A}_\tau\left(E_1, E_2\right)$. Besides, let $\pi : \mathcal{C} \to \mathcal{A}$ be the A_∞-functor induced by the following commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathcal{A} \arrow[d, "\mathrm{id}"] \\
\mathcal{A} \arrow[r, "\mathrm{id}"] & \mathcal{A}
\end{tikzcd}$$

(see Proposition 1.84). Then the diagram of Adams-graded A_∞-categories

$$\begin{tikzcd}
\mathcal{A}_- \sqcup \mathcal{A}_+ \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\iota_\perp \sqcup \iota_\top"'] & \mathcal{A}_\bullet \arrow[d, "\sigma"] \\
\mathcal{C} \arrow[r, "\sigma\circ\pi"] & \mathcal{A}_\tau
\end{tikzcd}$$

is commutative because $\sigma \circ \tau = \sigma$. Moreover, the induced A_∞-functor $\Phi : \mathcal{G} \to \mathcal{A}_\tau$ is strict, and it sends $W_{\mathcal{G}}$ to the set of units in $\mathcal{A}_\tau$. Let

$$\widetilde{\Phi} : \mathcal{H} \to \mathcal{A}_\tau\left[\{\text{units}\}^{-1}\right]$$

be the A_∞-functor induced by Φ (see Definition 1.59).

According to Lemma 2.20, $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ. Moreover, $\mathcal{A}_\tau\left[\{\text{units}\}^{-1}\right]$ is quasi-equivalent to $\mathcal{A}_\tau$. Thus, Theorem 2.8 will follow if we prove that $\widetilde{\Phi}$ is a quasi-equivalence. Our strategy is to apply Proposition 1.82.

In the following, we fix some element $E_\diamond \in \mathcal{E}$. When we write an object $X^n_\triangle$ without specifying the element of $\mathcal{E}$, we mean $X^n_\triangle(E_\diamond)$. We consider the corresponding $\mathcal{G}$-module $\mathcal{M}_{\mathcal{G}}$, and the $\mathcal{G}$-module maps

$$t_{\mathcal{G}} : \mathcal{G}\left(-, X^n_\triangle\right) \to \mathcal{G}, \quad \triangle \in \{\perp, \top, \bullet\},$$

of Definition 2.21. Moreover, we set

$$\mathcal{M}_{\mathcal{A}_\tau} := \mathcal{A}_\tau\left(-, E_\diamond\right) \text{ and } t_{\mathcal{A}_\tau} := \mathrm{id} : \mathcal{A}_\tau\left(-, E_\diamond\right) \to \mathcal{M}_{\mathcal{A}_\tau}.$$

Lemma 2.24. *There exists a quasi-isomorphism $t_0 : \mathcal{M}_{\mathcal{G}} \to \Phi^* \mathcal{M}_{\mathcal{A}_\tau}$ (see Definition 1.73 for the pullback functor) such that the following diagram of $\mathcal{G}$-modules commutes*

$$\begin{tikzcd}
\mathcal{G}(-,Y) \arrow[r, "t_\Phi"] \arrow[d, "t_{\mathcal{G}}"] & \Phi^* \mathcal{A}_\tau(-,E_\diamond) \arrow[d, "\Phi^* t_{\mathcal{A}_\tau} = \mathrm{id}"] \\
\mathcal{M}_{\mathcal{G}} \arrow[r, "t_0"] & \Phi^* \mathcal{M}_{\mathcal{A}_\tau}
\end{tikzcd}$$

(see Definition 1.75 for the map t_Φ) for every object $Y = X^{n_0}_\triangle$ with $\triangle \in \{\perp, \top, \bullet\}$.

Proof. Observe that the diagram of $\mathcal{G}$-modules

$$\begin{tikzcd}
\mathcal{G}(-,X^n_I) \arrow[r, "t^n_{I\top}"] \arrow[d, "t^n_{I\perp}"] & \mathcal{G}(-,X^n_\top) \arrow[d, "t_\Phi"] \\
\mathcal{G}(-,X^n_\perp) \arrow[r, "t_\Phi"] & \Phi^* \mathcal{M}_{\mathcal{A}_\tau}
\end{tikzcd}$$

is commutative, so that it induces a morphism of $\mathcal{G}$-modules $\mathcal{M}^n_\star \to \Phi^* \mathcal{M}_{\mathcal{A}_\tau}$ (see Proposition 1.72). Now observe that the following diagram of $\mathcal{G}$-modules commutes

$$\begin{tikzcd}
\bigoplus_{n\in\mathbf{Z}} \left(\mathcal{G}\left(-,X^n_-\right) \oplus \mathcal{G}\left(-,X^n_+\right)\right) \arrow[r, "\bigoplus_{n\in\mathbf{Z}} \left(t^n_{-\bullet} \oplus t^n_{+\bullet}\right)"] \arrow[d, "\bigoplus_{n\in\mathbf{Z}} \left(t^n_{-\perp} \oplus t^n_{+\top}\right)"] & \bigoplus_{n\in\mathbf{Z}} \mathcal{G}(-,X^n_\bullet) \arrow[d, "t_\Phi"] \\
\bigoplus_{n\in\mathbf{Z}} \mathcal{M}^n_\star \arrow[r] & \Phi^* \mathcal{M}_{\mathcal{A}_\tau}
\end{tikzcd}.$$

We let $t_0 : \mathcal{M}_{\mathcal{G}} \to \Phi^* \mathcal{M}_{\mathcal{A}_\tau}$ be the induced $\mathcal{G}$-module map. It is then easy to verify that the following diagram of $\mathcal{G}$-modules is commutative

$$\begin{tikzcd}
\mathcal{G}(-,Y) \arrow[r, "t_\Phi"] \arrow[d, "t_{\mathcal{G}}"] & \Phi^* \mathcal{A}_\tau(-,E_\diamond) \arrow[d, "\Phi^* t_{\mathcal{A}_\tau} = \mathrm{id}"] \\
\mathcal{M}_{\mathcal{G}} \arrow[r, "t_0"] & \Phi^* \mathcal{M}_{\mathcal{A}_\tau}
\end{tikzcd}$$

for every object $Y = X^{n_0}_\triangle$ with $\triangle \in \{\perp, \top, \bullet\}$.

It remains to prove that t_0 is a quasi-isomorphism. First observe that it is easy to check that $t_0 : \mathcal{M}_{\mathcal{G}}\left(X^k_\square(E)\right) \to \Phi^* \mathcal{M}_{\mathcal{A}_\tau}\left(X^k_\bullet(E)\right) = \mathcal{A}_\tau(E, E_\diamond)$ is a quasi-isomorphism when $\square \in \{\perp, \top, \bullet\}$ because in this case

$$\mathcal{M}_{\mathcal{G}}\left(X^k_{\square}(E)\right) = \bigoplus_n \mathcal{G}\left(X^k_{\square}(E), X^n_{\square}\right) = \bigoplus_n \mathcal{A}\left(X^k(E), X^n\right)$$

and

$$t_0 : \mathcal{M}_{\mathcal{G}}\left(X^k_{\square}(E)\right) = \bigoplus_n \mathcal{A}\left(X^k(E), X^n\right) \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}\left(X^k_{\bullet}(E)\right) = \mathcal{A}_\tau\left(E, E_\diamond\right)$$

is the projection. Then we deduce that

$$t_0 : \mathcal{M}_{\mathcal{G}}\left(X^k_{+}(E)\right) \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}\left(X^k_{+}(E)\right) = \mathcal{A}_\tau\left(E, E_\diamond\right)$$

is a quasi-isomorphism using the commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{G}}\left(X^k_{+}(E)\right) \arrow[r, "t_0"] & \Phi^*\mathcal{M}_{\mathcal{A}_\tau}\left(X^k_{+}(E)\right) \\
\mathcal{M}_{\mathcal{G}}\left(X^{k+1}_{\bullet}(E)\right) \arrow[u, "\sim"] \arrow[r, "t_0"] & \Phi^*\mathcal{M}_{\mathcal{A}_\tau}\left(X^{k+1}_{\bullet}(E)\right) \arrow[u, equal]
\end{tikzcd}$$

(the leftmost vertical arrow is the action of the adjacent unit in $\mathcal{G}\left(X^k_{+}(E), X^{k+1}_{\bullet}(E)\right)$, which is a quasi-isomorphism by Lemma 2.23). Finally, when $X = X^k_{-}(E)$ or $X = X^k_{I}(E)$, we prove that $t_0 : \mathcal{M}_{\mathcal{G}}(X) \to \Phi^*\mathcal{M}_{\mathcal{A}_\tau}(X)$ is a quasi-isomorphism using the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{G}}(X) \arrow[r, "t_0"] & \Phi^*\mathcal{M}_{\mathcal{A}_\tau}(X) \\
\mathcal{M}_{\mathcal{G}}\left(X^k_{\perp}(E)\right) \arrow[u, "\sim"] \arrow[r, "t_0"] & \Phi^*\mathcal{M}_{\mathcal{A}_\tau}\left(X^k_{\perp}(E)\right) . \arrow[u, equal]
\end{tikzcd}$$

This concludes the proof.

□

Lemma 2.25. *For every objects X and $Y = X^{n_0}_{\triangle}$ in $\mathcal{H}$, the chain map*

$$\widetilde{\Phi} : \mathcal{H}(X, Y) \to \mathcal{A}_\tau\left[\{\text{units}\}^{-1}\right](\Phi X, \Phi Y)$$

is a quasi-isomorphism.

Proof. We first observe that it is enough to show that the chain map $\widetilde{\Phi} : \mathcal{H}(X,Y) \to \mathcal{A}_\tau[\{\text{units}\}^{-1}](\Phi X, \Phi Y)$ is a quasi-isomorphism for every object X in $\mathcal{H}$ and $Y = X^{n_0}_{\triangle}$ with $\triangle \in \{\perp, \top, \bullet\}$. Indeed, we can deduce from this that $\widetilde{\Phi} : \mathcal{H}(X,Y) \to \mathcal{A}_\tau[\{\text{units}\}^{-1}](\Phi X, \Phi Y)$ is a quasi-isomorphism when $Y = X^{n_0}_{\pm}$ or $Y = X^{n_0}_{I}$ as follows. First, let

$$Z = \begin{cases} X^{n_0}_{\perp} & \text{if } Y = X^{n_0}_{-} \text{ or } Y = X^{n_0}_{I} \\ X^{n_0+1}_{\bullet} & \text{if } Y = X^{n_0}_{+}. \end{cases}$$

Then consider the commutative diagram of $\mathcal{G}$-modules

$$\begin{tikzcd} \mathcal{G}(-,Y) \arrow[r, "t_\Phi"] \arrow[d] & \Phi^*\mathcal{A}_\tau(-,E_\diamond) \\ \mathcal{G}(-,Z) \arrow[ur, "t_\Phi"'] & \end{tikzcd}$$

(the vertical arrow is composition with the adjacent morphism $Y \to Z$ in $W_\mathcal{G}$, and the other arrows are the ones of Definition 1.75). If we localize at $W_\mathcal{G}$ and then evaluate at some object X, we get a commutative diagram of chain complexes

$$\begin{tikzcd} \mathcal{H}(X,Y) \arrow[r, "W_\mathcal{G}^{-1} t_\Phi"] \arrow[d, "\sim"] & W_\mathcal{G}^{-1}\left(\Phi^*\mathcal{A}_\tau(-,E_\diamond)\right)(X) \\ \mathcal{H}(X,Z)\,. \arrow[ur, "W_\mathcal{G}^{-1} t_\Phi"'] & \end{tikzcd}$$

Now, using the natural map of Definition 1.80 and Remark 1.81, we get the following commutative diagram of chain complexes

$$\begin{tikzcd} \mathcal{H}(X,Y) \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\sim"] & {}_{\{\text{units}\}^{-1}}\mathcal{A}_\tau(-,E_\diamond)(\Phi X) = \mathcal{A}_\tau\left[\{\text{units}\}^{-1}\right](\Phi X, E_\diamond) \\ \mathcal{H}(X,Z)\,. \arrow[ur, "\widetilde{\Phi}"] & \end{tikzcd}$$

Thus, knowing that the map $\widetilde{\Phi} : \mathcal{H}(X,Z) \to \mathcal{A}_\tau[\{\text{units}\}^{-1}](\Phi X, E_\diamond)$ is a quasi-isomorphism implies that $\widetilde{\Phi} : \mathcal{H}(X,Y) \to \mathcal{A}_\tau[\{\text{units}\}^{-1}](\Phi X, E_\diamond)$ is also a quasi-isomorphism.

We now fix an object $Y = X^{n_0}_{\triangle}$ with $\triangle \in \{\perp, \top, \bullet\}$, and we want to prove that the map

$$\widetilde{\Phi} : \mathcal{H}(X,Y) = \mathcal{G}\left[W_\mathcal{G}^{-1}\right] \to \mathcal{A}_\tau[\{\text{units}\}^{-1}](\Phi X, E_\diamond)$$

is a quasi-isomorphism for every object X in $\mathcal{H}$. According to Lemmas 2.23 and 2.24, the assumptions of Proposition 1.82 are satisfied. This concludes the proof.

□

As explained above, Theorem 2.8 follows from Lemma 2.25 since $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ (see Lemma 2.20) and $\mathcal{A}_\tau\left[\{\text{units}\}^{-1}\right]$ is quasi-equivalent to $\mathcal{A}_\tau$.

2.5 Proof of the second result

Let τ be a quasi-autoequivalence of an A_∞-category $\mathcal{A}$ equipped with a compatible $\mathbf{Z}$-splitting of $\operatorname{ob}(\mathcal{A})$. Assume that the following holds:

1. $\mathcal{A}$ is weakly directed with respect to the $\mathbf{Z}$-splitting of $\operatorname{ob}(\mathcal{A})$ (see Definition 2.5),

2. there exists a closed degree 0 bimodule map $f : \mathcal{A}'(-,-) \to \mathcal{A}'(-,\tau(-))$ (see Definitions 1.63 and 1.64) such that for every $i < j < k$, for every $E, E' \in \mathcal{E}$, the chain maps
$$\begin{cases} \mu^2_{\mathcal{A}'}\left(-, f(e_{X^j(E)})\right) & : \quad \mathcal{A}'\left(X^i(E'), X^j(E)\right) \quad \to \quad \mathcal{A}'\left(X^i(E'), X^{j+1}(E)\right) \\ \mu^2_{\mathcal{A}'}\left(f(e_{X^j(E)}), -\right) & : \quad \mathcal{A}'\left(X^{j+1}(E), X^{k+1}(E')\right) \quad \to \quad \mathcal{A}'\left(X^j(E), X^{k+1}(E')\right) \end{cases}$$
are quasi-isomorphisms.

In the following, we set
$$c_n(E) := f\left(e_{X^n(E)}\right)$$
for every $n \in \mathbf{Z}$, $E \in \mathcal{E}$, and
$$W_{\mathcal{A}'} := \{c_n(E) \mid n \in \mathbf{Z}, E \in \mathcal{E}\} \cup \{\text{units of } \mathcal{A}'\}.$$

2.5.1 Generalized homotopy

Recall that we introduced a functor $\mathcal{D} \mapsto \mathcal{D}'$ from the category of Adams-graded A_∞-categories to the category of (non Adams-graded) A_∞-categories. Also, recall that we introduced Adams-graded A_∞-categories $\mathcal{C}$ and $\mathcal{G}$ in Definitions 2.18 and 2.19 respectively.

Observe that $\mathcal{C}'$ and $\mathcal{G}'$ are the Grothendieck constructions of the diagrams

$$\begin{tikzcd} \mathcal{A}'_I \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \mathcal{A}'_\top \\ \mathcal{A}'_\perp & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{A}'_- \sqcup \mathcal{A}'_+ \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\iota_\perp \sqcup \iota_\top"'] & \mathcal{A}'_\bullet \\ \mathcal{C}' & \end{tikzcd}$$

respectively. We denote by $W_{\mathcal{C}'}$ the set of adjacent units in $\mathcal{C}'$, and by $W_{\mathcal{G}'}$ the union of $W_{\mathcal{C}'}$ and the set of adjacent units in $\mathcal{G}'$.

We would like to define an A_∞-functor $\Psi' : \mathcal{G}' \to \mathcal{A}'$ which sends $W_{\mathcal{G}'}$ to $W_{\mathcal{A}'}$. According to Proposition 1.84, it is enough to prove the following result.

Lemma 2.26. *There exists an A_∞-functor $\eta : \mathcal{C}' \to \mathcal{A}'$ which sends $W_{\mathcal{C}'}$ to $W_{\mathcal{A}'}$, and such that*

$$\eta \circ \iota_I = \eta \circ \iota_\perp = \mathrm{id}, \quad \eta \circ \iota_\top = \tau.$$

Proof. We first define η to be id on $\mathcal{A}'_\perp$, $\mathcal{A}'_I$. and to be τ on $\mathcal{A}'_\top$. Observe that this completely defines η on the objects. Besides, we ask for η to act as the identity on the sequences involving an adjacent morphism from $\mathcal{A}_I$ to $\mathcal{A}_\perp$.

It remains to define η on the sequences involving an adjacent morphism from $\mathcal{A}_I$ to $\mathcal{A}_\top$. Consider a sequence of morphisms

$$\begin{aligned}(x_0, \dots, x_{p+q}) \in\ & \mathcal{C}'\left(X_I^{i_0}(E_0), X_I^{i_1}(E_1)\right) \times \cdots \times \mathcal{C}'\left(X_I^{i_{p-1}}(E_{p-1}), X_I^{i_p}(E_p)\right) \\ & \times \mathcal{C}'\left(X_I^{i_p}(E_p), X_\top^{i_{p+1}}(E_{p+1})\right) \\ & \times \mathcal{C}'\left(X_\top^{i_{p+1}}(E_{p+1}), X_\top^{i_{p+2}}(E_{p+2})\right) \times \cdots \times \mathcal{C}'\left(X_\top^{i_{p+q}}(E_{p+q}), X_\top^{i_{p+q+1}}(E_{p+q+1})\right).\end{aligned}$$

Observe that

$$\mathcal{C}'\left(X_I^i(E), X_I^j(E')\right) = \mathcal{C}'\left(X_I^i(E), X_\top^j(E')\right) = \mathcal{C}'\left(X_\top^i(E), X_\top^j(E')\right) = \mathcal{A}'\left(X^i(E), X^j(E)\right).$$

Then we set

$$\eta\left(x_0, \dots, x_{p+q}\right) := f\left(x_0, \dots, x_{p-1}, x_p, x_{p+1}, \dots, x_{p+q+1}\right) \in \mathcal{A}'\left(X^{i_0}(E_0), \tau X^{i_{p+q+1}}(E_{p+q+1})\right).$$

The functor η we defined satisfies the A_∞-relations because $f : \mathcal{A}'(-,-) \to \mathcal{A}'(-,\tau(-))$ is a closed degree 0 bimodule map. Moreover, η sends $W_{\mathcal{C}'}$ to $W_{\mathcal{A}'}$ by construction.

□

Remark 2.27. First observe that, according to Remark 1.58, we have

$$\mathcal{C}yl_{\mathcal{A}'} = \mathcal{C}'\left[W_{\mathcal{C}'}^{-1}\right] = \left(\mathcal{C}\left[W_{\mathcal{C}}^{-1}\right]\right)' = \left(\mathcal{C}yl_{\mathcal{A}}\right)'.$$

According to Lemma 2.26, the functor η induces an A_∞-functor $\widetilde{\eta} : \mathcal{C}yl_{\mathcal{A}'} \to \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]$. Moreover, Lemma 2.26 implies that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}'_+ \arrow[d, "\lambda_{\mathcal{C}'} \circ \iota_\top"'] \arrow[dr, bend left, "\lambda_{\mathcal{A}'} \circ \tau"] & \\
\mathcal{C}yl_{\mathcal{A}'} \arrow[r, "\widetilde{\eta}"] & \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right] \\
\mathcal{A}'_- \arrow[u, "\lambda_{\mathcal{C}'} \circ \iota_\perp"] \arrow[ur, bend right, "\lambda_{\mathcal{A}'}"] &
\end{tikzcd}$$

($\lambda_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]$ and $\lambda_{\mathcal{C}'} : \mathcal{C}' \to \mathcal{C}'\left[W_{\mathcal{C}'}^{-1}\right]$ denote the localization functors). Since $\mathcal{C}yl_{\mathcal{A}'}$ should be thought as a cylinder object for $\mathcal{A}'$ (see Proposition 1.87), we should think that the functors $\lambda_{\mathcal{A}'}$ and $\lambda_{\mathcal{A}'} \circ \tau$ are homotopic (even if they do not act the same way on objects) and that $\widetilde{\eta}$ is a generalized homotopy between them (see Proposition 1.89 for a justification of this terminology).

2.5.2 Relation between $\mathcal{G}$ and $\mathcal{A}'$

Using the A_∞-functor $\eta : \mathcal{C}' \to \mathcal{A}'$ of Lemma 2.26, we get a commutative diagram of (non Adams-graded) A_∞-categories

$$\begin{tikzcd}
\mathcal{A}'_- \sqcup \mathcal{A}'_+ \arrow[r, "\mathrm{id} \sqcup \tau"] \arrow[d, "\iota_\perp \sqcup \iota_\top"'] & \mathcal{A}'_\bullet \arrow[d, "\mathrm{id}"] \\
\mathcal{C}' \arrow[r, "\eta"] & \mathcal{A}'
\end{tikzcd}$$

and the induced A_∞-functor $\Psi' : \mathcal{G}' \to \mathcal{A}'$ (see Proposition 1.84) sends $W_{\mathcal{G}'}$ to $W_{\mathcal{A}'}$ (see Lemma 2.26). Let

$$\widetilde{\Psi'} : \mathcal{H}' = \mathcal{G}'\left[W_{\mathcal{G}'}^{-1}\right] \to \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]$$

be the A_∞-functor induced by Ψ' (see Definition 1.59). Observe that, since $\mathcal{A}$ is assumed to be weakly directed and since the Adams-degree of $\mathcal{A}$ comes from the $\mathbf{Z}$-splitting of $\mathrm{ob}\,(\mathcal{A})$ (see Definition 2.5), $\mathcal{H}$ is concentrated in non-negative Adams-degree. In particular, we can apply the adjunction of Definition 1.56 to $\widetilde{\Psi'}$, which gives an A_∞-functor

$$\widetilde{\Psi} : \mathcal{H} = \mathcal{G}\left[W_{\mathcal{G}}^{-1}\right] \to \mathbf{F}[t] \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right].$$

We would like to prove that for every Adams degree $j \geq 1$, and for every objects X, Y in $\mathcal{H}$, the map

$$\widetilde{\Psi} : \mathcal{H}(X,Y)_j \to \left(\mathbf{F}[t] \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]\right)(\Psi X, \Psi Y)_j = t^j \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right](\Psi X, \Psi Y)$$

is a quasi-isomorphism (if V is an Adams-graded vector space, V_j denotes the subspace of Adams degree j elements). Our strategy is once again to apply Proposition 1.82.

In the following we fix some element $E_\diamond \in \mathcal{E}$. When we write an object $X^n_\triangle$ or c_n without specifying the element of $\mathcal{E}$, we mean $X^n_\triangle(E_\diamond)$ or $c_n(E_\diamond)$ respectively. We consider the corresponding $\mathcal{G}$-module $\mathcal{M}_{\mathcal{G}}$, and the $\mathcal{G}$-module maps

$$t_{\mathcal{G}} : \mathcal{G}\left(-, X^n_\triangle\right) \to \mathcal{G}, \quad \triangle \in \{\perp, \top, \bullet\},$$

of Definition 2.21. Moreover, we consider the $\mathcal{A}'$-module $\mathcal{M}_{\mathcal{A}'}$ and the $\mathcal{A}'$-module maps

$$t_{\mathcal{A}'} : \mathcal{A}'(-, X^n) \to \mathcal{M}_{\mathcal{A}'}, \quad n \in \mathbf{Z},$$

associated to the morphisms $(c_n)_{n\in\mathbf{Z}}$ as in Definition 2.14.

The following result is a first step in order to define a $\mathcal{G}'$-module map $t'_0 : \mathcal{M}'_{\mathcal{G}} \to \Psi'^*\mathcal{M}_{\mathcal{A}'}$ as in Proposition 1.82.

Lemma 2.28. *The diagram of $\mathcal{G}'$-modules*

$$\begin{tikzcd}
\mathcal{G}'(-, X^n_I) \arrow[r, "t^n_{I\top}"] \arrow[d, "t^n_{I\perp}"] & \mathcal{G}'(-, X^n_\top) \arrow[d, "\Psi'^* t_{\mathcal{A}} \circ t_{\Psi'}"] \\
\mathcal{G}'(-, X^n_\perp) \arrow[r, "\Psi'^* t_{\mathcal{A}} \circ t_{\Psi'}"] & \Psi'^* \mathcal{M}_{\mathcal{A}'}
\end{tikzcd}$$

(see Definition 1.73 for the pullback functor) commutes up to homotopy.

Proof. First observe that $\iota_I^*\mathcal{G}'(-, X^n_I) = \mathcal{A}'(-, X^n)$, and $\iota_I^*\Psi'^*\mathcal{M}_{\mathcal{A}'} = \mathcal{M}_{\mathcal{A}'}$ because $\Psi' \circ \iota_I = \mathrm{id}$ (see Remark 1.74). Moreover, it suffices to show that the $\mathcal{A}'$-modules maps

$$\iota_I^*\left(\Psi'^* t_{\mathcal{A}'} \circ t_{\Psi'} \circ t^n_{I\perp}\right) = t_{\mathcal{A}'} \circ \iota_I^*\left(t_{\Psi'} \circ t^n_{I\perp}\right) : \mathcal{A}'(-, X^n) \to \mathcal{M}_{\mathcal{A}'}$$

and

$$\iota_I^* \left(\Psi'^* t_{\mathcal{A}'} \circ t_{\Psi'} \circ t_{I\top}^n\right) = t_{\mathcal{A}'} \circ \iota_I^* \left(t_{\Psi'} \circ t_{I\top}^n\right) : \mathcal{A}' \left(-, X^n\right) \to \mathcal{M}_{\mathcal{A}'}$$

are homotopic because

$$\mathcal{G}' \left(X_\triangle^k, X_I^n\right) = 0 \text{ if } \triangle \neq I.$$

On the one hand,

$$t_{\mathcal{A}'} \circ \iota_I^* \left(t_{\Psi'} \circ t_{I\perp}^n\right) = t_{\mathcal{A}'}.$$

On the other hand, $\iota_I^* \left(t_{\Psi'} \circ t_{I\top}^n\right) : \mathcal{A}' \left(-, X^n\right) \to \mathcal{A}' \left(-, X^{n+1}\right)$ is closed (as composition and pullback of closed module maps), and

$$\iota_I^* \left(t_{\Psi'} \circ t_{I\top}^n\right) \left(e_{X^n}\right) = \eta \left(e_{X^n}\right) = c_n$$

according to Lemma 2.26. Therefore, $\iota_I^* \left(t_{\Psi'} \circ t_{I\top}^n\right)$ is homotopic to t_{c_n} according to Corollary 1.70, and thus $t_{\mathcal{A}'} \circ \iota_I^* \left(t_{\Psi'} \circ t_{I\top}^n\right)$ is homotopic to $t_{\mathcal{A}'} \circ t_{c_n}$. But according to Lemma 2.15, $t_{\mathcal{A}'} \circ t_{c_n}$ is homotopic to $t_{\mathcal{A}'}$. This concludes the proof.

□

We can now state the result establishing the existence of a $\mathcal{G}'$-module map $t_0' : \mathcal{M}_{\mathcal{G}}' \to \Psi'^* \mathcal{M}_{\mathcal{A}'}$ as in Proposition 1.82.

Lemma 2.29. *There exists a $\mathcal{G}'$-module map $t_0' : \mathcal{M}_{\mathcal{G}}' \to \Psi'^* \mathcal{M}_{\mathcal{A}'}$ such that the following holds. The following diagram of $\mathcal{G}'$-modules commutes*

$$\begin{tikzcd}
\mathcal{G}' \left(-, Y\right) \arrow[r, "t_{\Psi'}"] \arrow[d, "t_{\mathcal{G}'}"] & \Psi'^* \mathcal{A}' \left(-, \Psi' Y\right) \arrow[d, "\Psi'^* t_{\mathcal{A}'}"] \\
\mathcal{M}_{\mathcal{G}}' \arrow[r, "t_0'"] & \Psi'^* \mathcal{M}_{\mathcal{A}'}.
\end{tikzcd}$$

for every object $Y = X_\triangle^{n_0}$ with $\triangle \in \{\perp, \top, \bullet\}$, and the $\mathcal{G}$-module map $t_0 : \mathcal{M}_{\mathcal{G}} \to$ $\mathbf{F}[t] \otimes \Psi'^* \mathcal{M}_{\mathcal{A}'}$ (see Definition 1.56) is a quasi-isomorphism in each positive Adams-degree.

Proof. Using Lemma 2.28 and Proposition 1.72, we get a $\mathcal{G}'$-modules map $t_\star^n : \left(\mathcal{M}_\star^n\right)' \to \Psi'^* \mathcal{M}_{\mathcal{A}'}$ for every $n \in \mathbf{Z}$ (see Definition 2.21 for the $\mathcal{G}$-modules $\mathcal{M}_\star^n$). Observe that the diagram of $\mathcal{G}'$-modules

$$\begin{tikzcd}
\bigoplus_{n\in\mathbf{Z}} \left(\mathcal{G}'\left(-,X_-^n\right) \oplus \mathcal{G}'\left(-,X_+^n\right)\right) \arrow[rr, "\bigoplus_{n\in\mathbf{Z}} \left(t_{-\bullet}^n \oplus t_{+\bullet}^n\right)"] \arrow[d, "\bigoplus_{n\in\mathbf{Z}} \left(t_{-\perp}^n \oplus t_{+\top}^n\right)"] & & \bigoplus_{n\in\mathbf{Z}} \mathcal{G}'\left(-,X_\bullet^n\right) \arrow[d, "\bigoplus_{n\in\mathbf{Z}} \Psi'^* t_{\mathcal{A}} \circ t_{\Psi'}"] \\
\bigoplus_{n\in\mathbf{Z}} \left(\mathcal{M}_\star^n\right)' \arrow[rr, "\bigoplus_{n\in\mathbf{Z}} t_\star^n"] & & \Psi'^* \mathcal{M}_{\mathcal{A}'}
\end{tikzcd}$$

is commutative (the composition is id for $-$-terms and τ for $+$-terms), so that it induces a morphism of $\mathcal{G}'$-modules $t_0' : \mathcal{M}_{\mathcal{G}}' \to \Psi'^* \mathcal{M}_{\mathcal{A}'}$. It is then easy to verify that the following diagram of $\mathcal{G}'$-modules is commutative

$$\begin{tikzcd}
\mathcal{G}'\left(-,Y\right) \arrow[r, "t_{\Psi'}"] \arrow[d, "t_{\mathcal{G}'}"] & \Psi'^* \mathcal{A}'\left(-,\Psi' Y\right) \arrow[d, "\Psi'^* t_{\mathcal{A}'}"] \\
\mathcal{M}_{\mathcal{G}}' \arrow[r, "t_0'"] & \Psi'^* \mathcal{M}_{\mathcal{A}'}.
\end{tikzcd}$$

It remains to show that the map $t_0 : \mathcal{M}_{\mathcal{G}}\left(X\right)_j \to t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(\Psi' X\right)$ is a quasi-isomorphism for every object X and every $j \geq 1$. We will first do it when $X = X_\square^k(E)$ is in $\mathcal{A}_\bullet \sqcup \mathcal{A}_\perp \sqcup \mathcal{A}_\top$. Observe that

$$\mathcal{M}_{\mathcal{G}}\left(X_\square^k(E)\right)_j = \mathcal{G}\left(X_\square^k(E), X_\square^{k+j}\right) = \mathcal{A}\left(X^k(E), X^{k+j}\right)$$

and the maps

$$\left\{\begin{array}{lcl}
\mathcal{A}\left(X^k(E), X^{k+j}\right) & = & \mathcal{M}_{\mathcal{G}}\left(X_\bullet^k(E)\right)_j \xrightarrow{t_0} t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^k(E)\right) \\
\mathcal{A}\left(X^k(E), X^{k+j}\right) & = & \mathcal{M}_{\mathcal{G}}\left(X_\perp^k(E)\right)_j \xrightarrow{t_0} t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^k(E)\right)
\end{array}\right.$$

are the inclusions, whereas the map

$$\mathcal{A}\left(X^k(E), X^{k+j}\right) = \mathcal{M}_{\mathcal{G}}\left(X_\top^k(E)\right)_j \xrightarrow{t_0} t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^{k+1}(E)\right)$$

is induced by τ. We will only prove that the latter is a quasi-isomorphism, the proof for the two former being almost the same (and easier). Observe that the following diagram of chain complexes commutes

$$\begin{tikzcd}
\mathcal{A}'\left(X^k(E), X^{k+j}\right) \arrow[r, "t_0"] \arrow[d, equal] & t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^{k+1}(E)\right) \\
\mathcal{A}'\left(X^k(E), X^{k+j}\right) \arrow[r, "t^j \otimes \tau"] & t^j \otimes \mathcal{A}'\left(X^{k+1}(E), X^{k+j+1}\right) . \arrow[u, hook]
\end{tikzcd}$$

The map $\tau : \mathcal{A}'\left(X^k(E), X^{k+j}\right) \to \mathcal{A}'\left(X^{k+1}(E), X^{k+j+1}\right)$ is a quasi-isomorphism by assumption, and the inclusion $\mathcal{A}'\left(X^{k+1}(E), X^{k+j+1}\right) \hookrightarrow \mathcal{M}_{\mathcal{A}'}\left(X^{k+1}(E)\right)$ is a quasi-isomorphism according to Lemma 2.16 (observe that it is important here that j is *strictly* greater than 0). Therefore the map $t_0 : \mathcal{A}'\left(X^k(E), X^{k+j}\right) \to t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^{k+1}(E)\right)$ is a quasi-isomorphism, which is what we needed to prove. Finally, we deduce that the map $t_0 : \mathcal{M}_{\mathcal{G}}\left(X^k_{\pm}\right)_j \to t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^k_{\pm}\right)$ and the map $t_0 : \mathcal{M}_{\mathcal{G}}\left(X^k_I\right)_j \to t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^k_I\right)$ are quasi-isomorphisms using the commutative diagrams

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{G}}\left(X^k_{-}\right)_j \arrow[r, "t_0"] & t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^k\right) \\
\mathcal{M}_{\mathcal{G}}\left(X^k_{\perp}\right)_j \arrow[u, "\sim"] \arrow[ur, "t_0"] &
\end{tikzcd} , \quad
\begin{tikzcd}
\mathcal{M}_{\mathcal{G}}\left(X^k_{+}\right)_j \arrow[r, "t_0"] & t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^k\right) \\
\mathcal{M}_{\mathcal{G}}\left(X^k_{\top}\right)_j \arrow[u, "\sim"] \arrow[ur, "t_0"] &
\end{tikzcd} ,$$

and

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{G}}\left(X^k_I\right)_j \arrow[r, "t_0"] & t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(X^k_I\right) \\
\mathcal{M}_{\mathcal{G}}\left(X^k_{\perp}\right)_j \arrow[u, "\sim"] \arrow[ur, "t_0"] &
\end{tikzcd}$$

respectively (each vertical arrow is the action of the corresponding element in $W_{\mathcal{G}}$, which is a quasi-isomorphism by Lemma 2.23).

$\square$

Lemma 2.30. *Let $j \geq 1$ be a fixed positive Adams degree. For every objects X and $Y = X^{n_0}_{\triangle}$ in $\mathcal{H}$, the map*

$$\widetilde{\Psi} : \mathcal{H}(X,Y)_j \to \left(\mathbf{F}[t] \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]\right)(\Psi X, \Psi Y)_j = t^j \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right](\Psi X, \Psi Y)$$

is a quasi-isomorphism (if V is an Adams-graded vector space, V_j denotes the subspace of

Adams degree j elements).

Proof. We first observe that the same type of argument given at the beginning of the proof of Lemma 2.25 shows that it is enough to prove that the map $\widetilde{\Psi} : \mathcal{H}(X,Y)_j \to t^j \otimes \mathcal{A}'[W_{\mathcal{A}'}^{-1}](\Psi X, \Psi Y)$ is a quasi-isomorphism for every object X and $Y = X_{\triangle}^{n_0}$ with $\triangle \in \{\perp, \top, \bullet\}$.

We now fix an object $Y = X_{\triangle}^{n_0}$ with $\triangle \in \{\perp, \top, \bullet\}$, and we want to prove that the map $\widetilde{\Psi} : \mathcal{H}(X,Y)_j \to t^j \otimes \mathcal{A}'[W_{\mathcal{A}'}^{-1}](\Psi X, \Psi Y)$ is a quasi-isomorphism for every object X in $\mathcal{H}$. Using Lemma 2.29 and Proposition 1.82, we know that for each object X of $\mathcal{G}'$, there exists a chain map $u' : {}_{W_{\mathcal{G}'}^{-1}}\mathcal{M}'_{\mathcal{G}}(X) \to {}_{W_{\mathcal{A}'}^{-1}}\mathcal{M}_{\mathcal{A}'}(\Psi' X)$ such that the following diagram of chain complexes commutes

$$\begin{tikzcd}
\mathcal{H}'(X,Y) \arrow[r, "\widetilde{\Psi}'"] \arrow[d, "{}_{W_{\mathcal{G}'}^{-1}}t_{\mathcal{G}'}"] & \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right](\Psi X, \Psi Y) \arrow[d, "{}_{W_{\mathcal{A}'}^{-1}}t_{\mathcal{A}'}"] \\
{}_{W_{\mathcal{G}'}^{-1}}\mathcal{M}'_{\mathcal{G}}(X) \arrow[r, "u'"] & {}_{W_{\mathcal{A}'}^{-1}}\mathcal{M}_{\mathcal{A}'}(\Psi' X) \\
\mathcal{M}'_{\mathcal{G}}(X) \arrow[r, "t'_0"] \arrow[u, hook] & \mathcal{M}_{\mathcal{A}'}(\Psi' X). \arrow[u, hook]
\end{tikzcd}$$

Now observe that

$$\begin{cases} \mathcal{H}'(X,Y) & = & \mathcal{H}(X,Y)' \\ {}_{W_{\mathcal{G}'}^{-1}}\mathcal{M}'_{\mathcal{G}}(X) & = & {}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X)' \\ \mathcal{M}'_{\mathcal{G}}(X) & = & \mathcal{M}_{\mathcal{G}}(X)'. \end{cases}$$

Applying the adjunction of Definition 1.56 to the last diagram, we get the following commutative diagram of Adams-graded chain complexes

$$\begin{tikzcd}
\mathcal{H}(X,Y) \arrow[r, "\widetilde{\Psi}"] \arrow[d, "{}_{W_{\mathcal{G}}^{-1}}t_{\mathcal{G}}"] & \mathbf{F}[t] \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right](\Psi X, \Psi Y) \arrow[d, "t^j \otimes {}_{W_{\mathcal{A}'}^{-1}}t_{\mathcal{A}'}"] \\
{}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}(X) \arrow[r, "u"] & \mathbf{F}[t] \otimes {}_{W_{\mathcal{A}'}^{-1}}\mathcal{M}_{\mathcal{A}'}(\Psi' X) \\
\mathcal{M}_{\mathcal{G}}(X) \arrow[r, "t_0"] \arrow[u, hook] & \mathbf{F}[t] \otimes \mathcal{M}_{\mathcal{A}'}(\Psi' X). \arrow[u, hook]
\end{tikzcd}$$

Specializing to the components of fixed Adams degree j, we get the following commutative diagram of chain complexes

$$\begin{tikzcd}
\mathcal{H}\left(X,Y\right)_j \arrow[r, "\widetilde{\Psi}"] \arrow[d, "W_{\mathcal{G}}^{-1}t_{\mathcal{G}}"] & t^j \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]\left(\Psi X, \Psi Y\right) \arrow[d, "t^j \otimes_{W_{\mathcal{A}'}^{-1}} t_{\mathcal{A}'}"] \\
{}_{W_{\mathcal{G}}^{-1}}\mathcal{M}_{\mathcal{G}}\left(X\right)_j \arrow[r, "u"] & t^j \otimes_{W_{\mathcal{A}'}^{-1}} \mathcal{M}_{\mathcal{A}'}\left(\Psi' X\right) \\
\mathcal{M}_{\mathcal{G}}\left(X\right)_j \arrow[r, "t_0"] \arrow[u, hook] & t^j \otimes \mathcal{M}_{\mathcal{A}'}\left(\Psi' X\right). \arrow[u, hook]
\end{tikzcd}$$

Using Lemma 2.23, respectively Lemma 2.17, and Proposition 1.79, we know that all the vertical maps on the left, respectively on the right, are quasi-isomorphisms. Besides, Lemma 2.29 implies that the lowest horizontal map is a quasi-isomorphism. Thus, the chain map $\widetilde{\Psi} : \mathcal{H}\left(X,Y\right)_j \to t^j \otimes \mathcal{A}'[W_{\mathcal{A}'}^{-1}]\left(\Psi X, \Psi Y\right)$ is a quasi-isomorphism.

□

2.5.3 End of the proof

We end the section with the proof of Theorem 2.10. Now that we have proved Lemma 2.30 which takes care of the *positive* Adams-degrees, we have to treat the zero Adams-degree part (recall that $\mathcal{H}$ is concentrated in non-negative Adams-degree because $\mathcal{A}$ is assumed to be weakly directed).

Let $\mathcal{I}$ be the full A_∞-subcategory of $\mathcal{H}$ with

$$\mathrm{ob}\left(\mathcal{I}\right) = \left\{X^0_\bullet(E) \mid E \in \mathcal{E}\right\}$$

and

$$\mathcal{I}\left(X^0_\bullet\left(E\right), X^0_\bullet\left(E'\right)\right) = \mathcal{G}\left(X^0_\bullet\left(E\right), X^0_\bullet\left(E'\right)\right) \oplus \left(\bigoplus_{j\geq 1} \mathcal{H}\left(X^0_\bullet\left(E\right), X^0_\bullet\left(E'\right)\right)_j\right)$$

(recall that if V is an Adams-graded vector space, we denote by V_j its component of Adams-degree j).

Lemma 2.31. *The inclusion $\mathcal{I} \hookrightarrow \mathcal{H}$ is a quasi-equivalence.*

Proof. Observe that the inclusion $\mathcal{I} \hookrightarrow \mathcal{H}$ is cohomologically essentially surjective because every object of $\mathcal{H}$ can be related to one of $\mathcal{I}$ by a zigzag of morphisms in $W_{\mathcal{G}}$, which are quasi-isomorphisms in $\mathcal{H}$. Therefore, it suffices to show that the inclusion

$$\mathcal{G}\left(X_\bullet^0(E), X_\bullet^0(E_\diamond)\right) \hookrightarrow \mathcal{H}\left(X_\bullet^0(E), X_\bullet^0(E_\diamond)\right)_0$$

is a quasi-isomorphism for every $E, E_\diamond \in \mathcal{E}$.

Let $E_\diamond$ be an element of $\mathcal{E}$. When we write an object $X_\bullet^n$ without specifying the element of $\mathcal{E}$, we mean $X_\bullet^n(E_\diamond)$. Recall that we introduced a pair $(\mathcal{M}_\mathcal{G}, t_\mathcal{G})$ in Definition 2.21. According to Lemma 2.23 and Proposition 1.79, the inclusion $\mathcal{M}_\mathcal{G}(X_\bullet^0(E)) \hookrightarrow {}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}(X_\bullet^0(E))$ and the map ${}_{W_\mathcal{G}^{-1}}t_\mathcal{G} : \mathcal{H}(X_\bullet^0(E), X_\bullet^0) \to {}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}(X_\bullet^0(E))$ are quasi-isomorphisms for every $E \in \mathcal{E}$. Besides, observe that

$$\mathcal{M}_\mathcal{G}\left(X_\bullet^0(E)\right)_0 = \mathcal{G}\left(X_\bullet^0(E), X_\bullet^0\right).$$

The result then follows from the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{G}(X_\bullet^0(E), X_\bullet^0) \arrow[r, equal] \arrow[d, equal] & \mathcal{G}(X_\bullet^0(E), X_\bullet^0) \arrow[r, equal] \arrow[d, hook] & \mathcal{G}(X_\bullet^0(E), X_\bullet^0) \arrow[d, hook] \\
\mathcal{M}_\mathcal{G}(X_\bullet^0(E))_0 \arrow[r, hook, "\sim"] & {}_{W_\mathcal{G}^{-1}}\mathcal{M}_\mathcal{G}(X_\bullet^0(E))_0 & \mathcal{H}(X_\bullet^0(E), X_\bullet^0)_0 . \arrow[l, "\sim"', "{}_{W_\mathcal{G}^{-1}}t_\mathcal{G}"]
\end{tikzcd}$$

□

Recall that if $\mathcal{C}$ is an A_∞-category such that there is a splitting $\mathrm{ob}(\mathcal{C}) \simeq \mathbf{Z} \times \mathcal{E}$, then we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$. The following diagram of Adams-graded A_∞-categories is commutative

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "\widetilde{\Psi}"] & \mathbf{F}[t] \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right] \\
\mathcal{I} \arrow[u, hook, "\sim"] \arrow[r, "\widetilde{\Psi}"] & (\mathcal{A}')^0 \oplus \left(\overline{\mathbf{F}[t]} \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]^0\right). \arrow[u, hook]
\end{tikzcd}$$

Moreover, since $\mathcal{A}$ is assumed to be weakly directed with respect to the $\mathbf{Z}$-splitting of $\mathrm{ob}(\mathcal{A})$, Lemma 2.30 implies that the bottom horizontal A_∞-functor is a quasi-equivalence. Therefore we have

$$\mathcal{H} \simeq (\mathcal{A}')^0 \oplus \left(\overline{\mathbf{F}[t]} \otimes \mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right]^0\right).$$

Recall that $W_{\mathcal{A}'} = f\left(\text{units}\right) \cup \{\text{units}\}$, so that

$$\mathcal{A}'\left[W_{\mathcal{A}'}^{-1}\right] \simeq \mathcal{A}'\left[f\left(\text{units}\right)^{-1}\right].$$

This concludes the proof of Theorem 2.10, since $\mathcal{H}$ is quasi-equivalent to the mapping torus of τ (see Lemma 2.20).

Chapter 3

Legendrian invariants

In this chapter, we study the Chekanov-Eliashberg DG-algebra of a Legendrian in a hypertight contact manifold. We begin to give definitions in section 3.1. Then in section 3.2, we generalize techniques from [28] and [53] in order to prove results on the behavior of the Chekanov-Eliashberg DG-algebra under change of data. However, we restrict ourselves to what we call *good paths* of data (see Definition 3.7). This is unfortunately a strong assumption, as Legendrians are assumed to always have finitely many Reeb chords, and we do not treat birth-death of Reeb chords phenomena. We will use Theorem C in the proof of Theorem D.

In the following, (V, ξ) is a contact manifold of dimension $(2n+1)$ equipped with a hypertight contact form α, which means that the Reeb vector field of α has no contractible periodic orbits.

3.1 Definitions

Let Λ be a Legendrian submanifold in (V, ξ). We assume that Λ is *chord generic* with respect to α, which means that the following holds:

1. for every Reeb chord $c : [0, T] \to V$ of Λ, the space $D\varphi^T_{R_\alpha}\left(T_{c(0)}\Lambda\right)$ is transverse to $T_{c(T)}\Lambda$ in ξ,

2. different Reeb chords belong to different Reeb trajectories.

As explained in [22] Section 3.2 for example, chord genericity can be achieved after an arbitrarily C^1-small Legendrian perturbation.

3.1.1 Conley-Zehnder index

In the following, we define the Conley-Zehnder index of a Reeb chord of Λ starting and ending on the same connected component (such chords are called *pure*). We will assume that $H_1(V)$ is free, that the first Chern class of ξ (equipped with any compatible almost complex structure) is 2-torsion, and that the Maslov number $m(\Lambda)$ of Λ (which we will define below) vanishes.

We briefly recall what is the Maslov index of a loop in the Grassmanian of Lagrangian subspaces in $\mathbf{C}^n$. We refer to [55] for a precise exposition. Fix a Lagrangian subspace K, and denote by $\Sigma_k(K)$ the set of Lagrangian subspaces in $\mathbf{C}^n$ whose intersection with K is k dimensional. Consider the *Maslov cycle*

$$\Sigma = \Sigma_1(K) \cup \cdots \cup \Sigma_n(K).$$

This is an algebraic variety of codimension one in the Lagrangian Grassmanian. Now if Γ is a loop in the Lagrangian Grassmanian, its Maslov index $\mu(\Gamma) \in \mathbf{Z}$ is the intersection number of Γ with Σ. The contribution of an intersection instant t_0 is computed as follows. Choose a Lagrangian complement W of K in $\mathbf{C}^n$. Then for each v in $\Gamma(t_0) \cap K$, there exists a vector $w(t)$ in W such that $v + w(t)$ is in $\Gamma(t)$ for every t near t_0. Consider the quadratic form

$$Q(v) = \frac{d}{dt}\omega(v, w(t))_{|t=t_0}$$

on $\Gamma(t_0) \cap K$. Without loss of generality, Q can be assumed non-singular and the contribution of t_0 to $\mu(\Gamma)$ is the signature of Q.

Recall that $H_1(V)$ is assumed to be free. We choose a family $(h_1, \ldots, h_r)$ of embedded circles in V which represent a basis of $H_1(V)$, and a symplectic trivialization of ξ over each h_i. If γ is some loop in Λ, there is a unique family $(a_1, \ldots, a_r)$ of integers such that $[\gamma_c - \sum_i a_i h_i]$ is zero in $H_1(V)$. Choose a surface Σ_γ in V such that

$$\partial\Sigma_\gamma = \gamma - \sum_i a_i h_i.$$

There is a unique trivialization of ξ over Σ_γ which extends the chosen trivializations over h_i. Thus we get a trivialization $\gamma^{-1}\xi \simeq S^1 \times \mathbf{C}^n$ (where n is the dimension of Λ). We denote by Γ the loop of Lagrangian planes in $\mathbf{C}^n$ corresponding, via the latter trivialization, to the loop $t \mapsto T_{\gamma(t)}\Lambda$. The Maslov index of Γ does not depend on the choice of the surface Σ_γ because we assumed $2c_1(\xi) = 0$. This construction defines a morphism $H_1(\Lambda, \mathbf{Z}) \to \mathbf{Z}$, and the *Maslov number* $m(\Lambda)$ of Λ is the generator of its image. In the following, we assume that the Maslov number of Λ is zero.

Now, let c be a pure Reeb chord of Λ. We choose a path $\gamma_c : [0,1] \to \Lambda$ which starts at the endpoint of c, and ends at its starting point (γ_c is called a *capping path* of c). We denote by $\overline{\gamma}_c$ the loop obtained by concatenating γ and c. Let $(a_1, \dots, a_r)$ be the unique family of integers such that $[\overline{\gamma}_c - \sum_i a_i h_i]$ is zero in $H_1(V)$, and choose a surface Σ_c in V such that

$$\partial \Sigma_c = \overline{\gamma}_c - \sum_i a_i h_i.$$

There is a unique trivialization of ξ over Σ_c which extends the chosen trivializations over h_i. Thus we get a trivialization $\overline{\gamma}_c^{-1}\xi \simeq S^1 \times \mathbf{C}^n$ (where n is the dimension of Λ). We denote by Γ_c the path of Lagrangian planes in $\mathbf{C}^n$ corresponding, via the latter trivialization, to the concatenation of $t \mapsto T_{\gamma(t)}\Lambda$ and $t \mapsto D\varphi^t_{R_\alpha}\left(T_{c(0)}\Lambda\right)$. Since Λ is chord generic, Γ_c is not a loop: we close it in the following way. Let I be a complex structure on $\mathbf{C}^n$ which is compatible with the standard symplectic form on $\mathbf{C}^n$ and such that $I\left(\Gamma_c(1)\right) = \Gamma_c(0)$. Then we let $\overline{\Gamma}_c$ be the loop of Lagrangian subspaces obtained by concatenating Γ_c and the path $t \mapsto e^{tI}\Gamma_c(1)$. The Conley-Zehnder index of c is the Maslov index of $\overline{\Gamma}_c$:

$$CZ(c) := \mu\left(\overline{\Gamma}_c\right).$$

The Conley-Zehnder index of a Reeb chord does not depend on the choice of Σ_c because the first Chern class of ξ is 2-torsion, and it does not depend on the choice of γ_c because the Maslov number of Λ vanishes.

3.1.2 Moduli spaces

In the following, we introduce the moduli spaces needed to define the Chekanov-Eliashberg algebra of Λ.

Definition 3.1. A Riemann disk with $(d+1)$ marked points is a triple $(D, j, \boldsymbol{\zeta})$ such that

1. D is a smooth manifold-with-boundary diffeomorphic to the closed unit disk in $\mathbf{C}$,
2. j is an integrable almost complex structure on D,
3. $\boldsymbol{\zeta} = (\zeta_0, \zeta_1, \dots, \zeta_d)$ is a family of distinct points on ∂D, cyclically ordered with respect to the orientation induced by j.

If $(D, j, \boldsymbol{\zeta})$ is a Riemann disk, we denote by $\Delta := D \setminus \{\zeta_0, \dots, \zeta_d\}$ the corresponding punctured disk.

Definition 3.2. Let $(D, j, \boldsymbol{\zeta})$ and $(D', j', \boldsymbol{\zeta'})$ be two Riemann discs with $(d+1)$ marked points. An isomorphism $(D, j, \boldsymbol{\zeta}) \to (D', j', \boldsymbol{\zeta'})$ is a map $\phi : D \to D'$ such that

1. ϕ is a diffeomorphism,
2. $\phi_* j = j'$,
3. $(\phi(\zeta_0), \ldots, \phi(\zeta_d)) = (\zeta'_0, \ldots, \zeta'_d)$.

Definition 3.3. Let $(D, j, \boldsymbol{\zeta})$ be a Riemann disk with $(d+1)$ marked points. A choice of strip-like ends for $(D, j, \boldsymbol{\zeta})$ is a collection of maps

$$\epsilon_0 : \mathbf{R}_{\geq 0} \times [0,1] \to \Delta, \quad (\epsilon_k : \mathbf{R}_{\leq 0} \times [0,1] \to \Delta)_{1 \leq k \leq d}$$

such that

1. ϵ_0 and ϵ_k are embeddings such that
$$\epsilon_0 \left(\mathbf{R}_{\geq 0} \times \{0,1\}\right) \subset \partial D \text{ and } \epsilon_k \left(\mathbf{R}_{\leq 0} \times \{0,1\}\right) \subset \partial D,$$
2. ϵ_0 and ϵ_k are (i,j)-holomorphic, where i is the standard complex structure on $\mathbf{C}$,
3. we have
$$\epsilon_0 (s,t) \underset{s \to +\infty}{\longrightarrow} \zeta_0 \text{ and } \epsilon_k (s,t) \underset{s \to -\infty}{\longrightarrow} \zeta_k.$$

In the following, we fix a choice of strip-like ends for each Riemann disk.

Definition 3.4. Let J be an almost complex structure on ξ compatible with $(d\alpha)_{|\xi}$. Let $a : [0, T_0] \to V$ and $\mathbf{b} = (b_k : [0, T_k] \to V)_{1 \leq k \leq d}$ be Reeb chords of Λ. We denote by $\widetilde{\mathcal{M}}_{a,\mathbf{b}} (\mathbf{R} \times \Lambda, J, \alpha)$ the set of equivalence classes of tuples $(D, j, \boldsymbol{\zeta}, u)$ such that

1. $(D, j, \boldsymbol{\zeta})$ is a Riemann disk with $(d+1)$ marked points and u is a smooth map from $\Delta = D \setminus \{\zeta_0, \ldots, \zeta_d\}$ to $\mathbf{R} \times V$ which maps the boundary of Δ to $\mathbf{R} \times \Lambda$,
2. u is pseudo-holomorphic with respect to the unique almost complex structure on $\mathbf{R}_\sigma \times V$ which sends ∂_σ to R_α and which restricts to J on ξ,
3. with the choice of strip-like ends $\epsilon_0, (\epsilon_k)_{1 \leq k \leq d}$ for $(D, j, \boldsymbol{\zeta})$ we have
$$(u \circ \epsilon_0)(s,t) \underset{s \to +\infty}{\longrightarrow} (+\infty, a(T_0 t)) \text{ and } (u \circ \epsilon_k)(s,t) \underset{s \to -\infty}{\longrightarrow} (-\infty, b_k(T_k t)),$$

where two tuples $(D, j, \boldsymbol{\zeta}, u)$, and $(D', j', \boldsymbol{\zeta'}, u')$ as above are said to be equivalent if there exists an isomorphism $\phi : (D, j, \boldsymbol{\zeta}) \to (D', j', \boldsymbol{\zeta'})$ such that $u \circ \phi^{-1} = u'$. Observe that $\mathbf{R}$ acts on $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ by translation in the $\mathbf{R}_\sigma$-coordinate. We set

$$\mathcal{M}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha) := \widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)/\mathbf{R}.$$

The moduli spaces $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ might not be compact, but their lack of compactness can be understood. Since we will not use this notion in detail, we refer to [14] and [1] for the definitions and main results about compactness in symplectic field theory. What we need can be roughly stated as follows.

Proposition 3.5. *Any sequence in $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ has a subsequence converging to a broken holomorphic curve.*

Proof. This follows from the compactness results in [14] and [1]. See in particular Theorem 3.20 in [1].

□

The moduli spaces $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ can be seen as the zero-set of a section $\overline{\partial} : \mathcal{B} \to \mathcal{E}$ of a Banach bundle $\mathcal{E} \to \mathcal{B}$ (see for example [26]). We say that $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ is transversely cut out if $\overline{\partial}$ is transverse to the 0-section.

Proposition 3.6. *There is a Baire set of almost complex structures J on ξ for which the moduli spaces $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ are transversely cut out smooth manifolds of dimension*

$$\dim \widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha) = CZ(a) - \left(\sum_{k=1}^{d} CZ(b_k) \right) + d - 1$$

(such almost complex structures will be called regular). Moreover, the moduli spaces $\widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ may be supposed to be transversely cut out after a perturbation of J supported in an arbitrarily small neighborhood of the chord a.

Proof. This is Proposition 3.13 in [22].

□

In particular, for J regular, the moduli spaces $\mathcal{M}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)$ are smooth manifolds of dimension

$$\dim \widetilde{\mathcal{M}}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha) = CZ(a) - \left(\sum_{k=1}^{d} CZ(b_k) \right) + d - 2,$$

and they are compact if their dimension is 0.

3.1.3 Chekanov-Eliashberg DG-algebra

Choose a regular almost complex structure J on ξ. We denote by $CE_*(\Lambda) = CE_*(\Lambda, J, \alpha)$ the free unital graded algebra generated by the Reeb chords of Λ, where the grading of a Reeb chord c is given by

$$|c| = CZ(c) - 1.$$

Besides, if a is a Reeb chord in $CE_*(\Lambda)$, we set

$$\partial a = \sum_{\mathbf{b}, A} \#\mathcal{M}_{a,\mathbf{b}}(\mathbf{R} \times \Lambda, J, \alpha)\, \mathbf{b}$$

where $\#\mathcal{M} \in \mathbf{F}$ denotes the number of elements modulo 2 in $\mathcal{M}$ if $\dim(\mathcal{M}) = 0$, and 0 otherwise. Finally, we extend ∂ to $CE_*(\Lambda)$ so that it satisfies the Leibniz rule with respect to the concatenation product.

Let us sum up: given a contact manifold (V, ξ) with $H_1(V)$ free, $2c_1(\xi) = 0$, and

1. a Legendrian submanifold Λ of (V, ξ) with $m(\Lambda) = 0$,
2. a hypertight contact form α on (V, ξ) for which Λ is chord generic,
3. an almost complex structure on ξ compatible with $(d\alpha)_{|\xi}$ and regular (with respect to Λ and α),

we defined a graded algebra $CE_*(\Lambda) = CE_*(\Lambda, J, \alpha)$, and a linear map $\partial : CE_*(\Lambda) \to CE_*(\Lambda)$ satisfying the Leibniz rule. It follows from Propositions 3.6, 3.5 (and a gluing result) that $\partial \circ \partial = 0$, and ∂ decreases the grading by 1. As a result, $CE_{-*}(\Lambda)$ is a DG-algebra (see Definition 1.1).

Augmentations and Legendrian A_∞-(co)algebra Recall from Definition 1.21 that an augmentation of an A_∞-algebra A is an A_∞-map from A to the base field $\mathbf{F}$. Assume now that we have an augmentation $\varepsilon : CE_{-*}(\Lambda) \to \mathbf{F}$ of the Chekanov-Eliashberg DG-algebra. Denote by ϕ_ε the algebra automorphism of $CE_{-*}(\Lambda)$ defined by

$$\phi_\varepsilon(c) = c + \varepsilon(c)$$

for every Reeb chord c of Λ. We denote by $CE^{\varepsilon}_{-*}(\Lambda)$ the DG-algebra whose underlying graded algebra is the same as for $CE_{-*}(\Lambda)$, but the differential is $\partial_{\varepsilon} = \phi_{\varepsilon} \circ \partial \circ \phi_{\varepsilon}^{-1}$. Now let $\overline{LC^{\varepsilon}_{*}(\Lambda)}$ be the graded vector space generated by the Reeb chords of Λ, where the grading of a chord c is $(-CZ(c))$. Observe that, as a graded algebra, we have

$$CE^{\varepsilon}_{-*}(\Lambda) = \mathbf{F} \oplus \left(\bigoplus_{d \geq 1} \left(\overline{LC^{\varepsilon}_{*}(\Lambda)}[-1] \right)^{\otimes d} \right).$$

If we write

$$(\partial_{\varepsilon})_{|\overline{LC^{\varepsilon}_{*}(\Lambda)}} = \sum_{d \geq 0} \partial^{d}_{\varepsilon} \text{ with } \partial^{d}_{\varepsilon} : \overline{LC^{\varepsilon}_{*}(\Lambda)} \to \overline{LC^{\varepsilon}_{*}(\Lambda)}^{\otimes d},$$

then $\partial^{0}_{\varepsilon} = \varepsilon \circ \partial = 0$. Moreover, the operations $\left(\partial^{d}_{\varepsilon}\right)_{d \geq 1}$ make $\overline{LC^{\varepsilon}_{*}(\Lambda)}$ a non-counital A_{∞}-coalgebra (see Definition 1.23). We define the coaugmented A_{∞}-coalgebra of (Λ, ε) to be

$$LC^{\varepsilon}_{*}(\Lambda) := \mathbf{F} \oplus \overline{LC^{\varepsilon}_{*}(\Lambda)}$$

(the A_{∞}-cooperations are naturally extended so that $e_{\mathbf{F}}$ is a counit). Observe that, as a DG-algebra,

$$CE^{\varepsilon}_{-*}(\Lambda) = \Omega \left(LC^{\varepsilon}_{*}(\Lambda) \right)$$

(see Definition 1.32 for the cobar construction). Finally, we define the augmented A_{∞}-algebra (see Definition 1.17) of (Λ, ε) to be the graded dual (see Definition 1.37) of $LC^{\varepsilon}_{*}(\Lambda)$:

$$LA^{*}_{\varepsilon}(\Lambda) = LC^{\varepsilon}_{*}(\Lambda)^{\#}.$$

3.2 Invariance and homotopies

3.2.1 Main result

Let (V, ξ) be a contact manifold equipped with a hypertight contact form α. Assume that $H_1(V)$ is free and that the first Chern class of ξ (equipped with any compatible almost complex structure) is 2-torsion. In the following, we denote by $\mathcal{L}$ the space of chord generic compact Legendrian submanifolds in (V, ξ) with Maslov number 0 and with finitely many Reeb chords with respect to α. In order to state the main result of this section, we give the following definition.

Definition 3.7. We say that a path $(\Lambda_t, J_t)_{0 \leq t \leq 1}$, where Λ_t is a Legendrian submanifold of (V, ξ) and J_t is an almost complex structure on ξ, is *good* if the following conditions hold:

1. for every $0 \leq t \leq 1$, Λ_t is in $\mathcal{L}$,
2. there exists a partition $0 = t_0 < \cdots < t_{N+1} = 1$ such that for every $0 \leq i \leq N$, either $\Lambda_t = \Lambda_{t_i}$ for every $t \in [t_i, t_{i+1}]$, or $J_t = J_{t_i}$ for every $t \in [t_i, t_{i+1}]$.

The rest of the section is devoted to the proof of the following result.

Theorem 3.8. *There is a way to associate, to each good path $(\Lambda_t, J_t)_{0 \leq t \leq 1}$, a tame DG-isomorphism (see Definition 1.7)*

$$\varphi : CE_{-*}(\Lambda_0, J_0, \alpha) \xrightarrow{\sim} CE_{-*}(\Lambda_1, J_1, \alpha)$$

such that the following holds:

1. *if φ is the DG-map associated to a good path $(\Lambda_t, J_t)_{0 \leq t \leq 1}$, if $\chi_1 : [0,1] \to [0, t_0]$ and $\chi_2 : [0,1] \to [t_0, 1]$ are two non-decreasing functions satisfying*

$$\chi_1(0) = 0,\ \chi_1(1) = t_0 = \chi_2(0),\ \chi_2(1) = 1,$$

then $\varphi = \varphi_2 \circ \varphi_1$, where φ_k is the DG-map associated to $(\Lambda_{\chi_k(t)}, J_{\chi_k(t)})_{0 \leq t \leq 1}$,

2. *if $\varphi^{\pm}$ are the DG-maps associated to good paths $(\Lambda_t^{\pm}, J_t^{\pm})_{0 \leq t \leq 1}$ satisfying*

$$(\Lambda_0^-, J_0^-) = (\Lambda_0^+, J_0^+), \quad (\Lambda_1^-, J_1^-) = (\Lambda_1^+, J_1^+),$$

and if there is a family $(\Lambda_{s,t})_{0 \leq s,t \leq 1}$ such that for every s, t

$$\Lambda_{s,t} \in \mathcal{L}, \quad (\Lambda_{s,0}, \Lambda_{s,1}) = (\Lambda_0^{\pm}, \Lambda_1^{\pm}), \quad (\Lambda_{0,t}, \Lambda_{1,t}) = (\Lambda_t^-, \Lambda_t^+),$$

then φ^- and φ^+ are DG-homotopic (see Definition 1.2).

3.2.2 Constructions in $V \times T^*\mathbf{R}$

We describe constructions of Legendrians and almost complex structures in the manifold $V \times T^*\mathbf{R}_x$ equipped with the contact form

$$\widehat{\alpha} := \alpha - y dx.$$

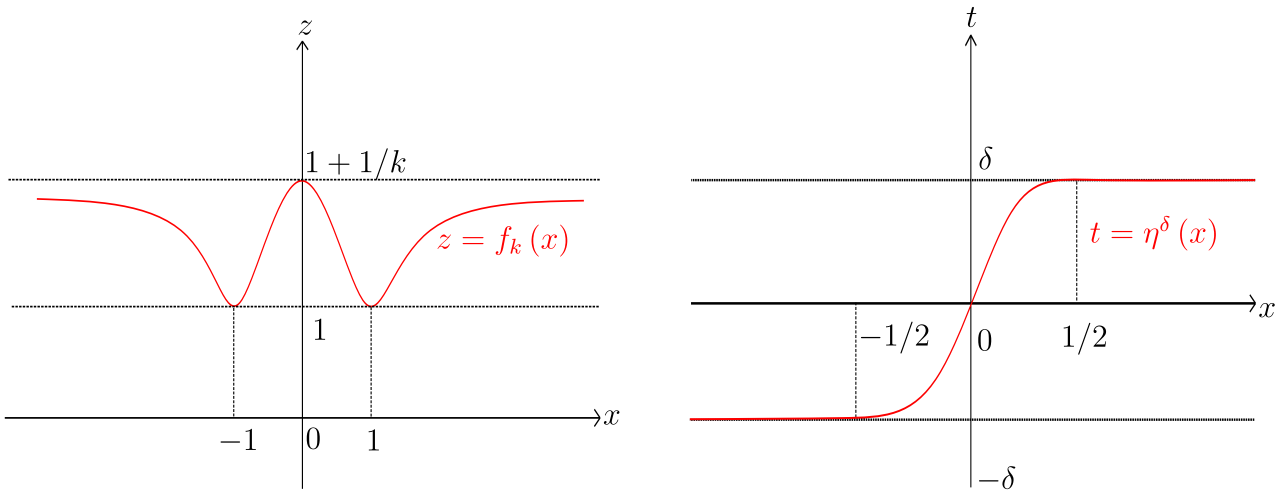

Figure 3.1: Functions f_k (on the left) and η^δ (on the right)

We fix a family $\left(\eta^\delta : \mathbf{R}_x \to \mathbf{R}_t\right)_{\delta \geq 0}$ of non-decreasing functions such that

$$\eta^\delta(0) = 0 \text{ and } \eta^\delta(\pm x) = \pm\delta \text{ for } x \geq 1/2.$$

We also consider a family $(f_k : \mathbf{R}_x \to \mathbf{R}_z)_{k \geq 1}$ such that the only two local minima of f_k are at ± 1, the only local maximum of f_k is at 0, the image of f_k is contained in $[1, 1+1/k]$, and

$$f_k(\pm 1) = 1, f_k(0) = 1 + 1/k.$$

We moreover require that all the derivatives of f_k vanish at ± 1 (we will need this assumption later, see Remark 3.31). See Figure 3.1.

In the following, we fix a good path $(\Lambda_t, J_t)_{0 \leq t \leq 1}$ and some instant $t^0 \in (0,1)$.

Legendrians Let $(\phi_t : M \to V)_{t \in [0,1]}$ be a family of Legendrian embeddings such that $\mathrm{im}(\phi_t) = \Lambda_t$. We describe Legendrian embeddings $\Phi^{k,\delta}_{t^0} : M \times \mathbf{R} \to V \times T^*\mathbf{R}$ obtained from $\left(\phi_{t^0 + \eta^\delta(x)}\right)_{x \in \mathbf{R}}$ and f_k.

We first choose a function $h : M \to \mathbf{R}$ such that for every Reeb chord of Λ_{t^0} going from $\phi_{t^0}(q_-)$ to $\phi_{t^0}(q_+)$, we have

$$h(q_-) = 0, h(q_+) > 0 \text{ and } dh(q_\pm) = 0$$

(observe that such a function exists because we assumed Λ_{t^0} has finitely many Reeb chords). We let $\psi : (\mathcal{N}(\Lambda_{t^0}), \alpha) \hookrightarrow (J^1(M), dz - pdq)$ be a strict contact embedding of a neighborhood of Λ_{t^0} inside the one-jets of M such that $\psi \circ \phi_{t^0}$ is the one-jet of h.

For δ small enough, $\left(\psi \circ \phi_{t^0+\eta^\delta(x)}\right)(q)$ is in $\mathcal{N}(\Lambda_{t^0})$ for every (q,x), and we set

$$\left(z_x^\delta(q), \left(a_x^\delta(q), b_x^\delta(q)\right)\right) := \left(\psi \circ \phi_{t^0+\eta^\delta(x)}\right)(q) \in \mathbf{R} \times T^*M.$$

Observe that

$$(q,x) \mapsto \left(z_x^\delta(q), \left(a_x^\delta(q), b_x^\delta(q)\right), \left(x, \frac{d}{dx} z_x^\delta(q) - \langle b_x^\delta(q), \frac{d}{dx} a_x^\delta(q)\rangle\right)\right)$$

defines a Legendrian embedding of $M \times \mathbf{R}$ inside $\mathbf{R} \times T^*M \times T^*\mathbf{R}$ when equipped with the contact form $(dz - pdq - ydx)$. Moreover,

$$\begin{aligned}(q,x) \mapsto \Big(& f_k(x) z_x^\delta(q), \left(a_x^\delta(q), f_k(x) b_x^\delta(q)\right), \\ & \left(x, f_k(x)\left(\frac{d}{dx} z_x^\delta(q) - \langle b_x^\delta(q), \frac{d}{dx} a_x^\delta(q)\rangle\right) + f_k'(x) z_x^\delta(q)\right)\Big)\end{aligned}$$

also defines a Legendrian embedding of $M \times \mathbf{R}$ inside $\mathbf{R} \times T^*M \times T^*\mathbf{R}$.

Then we define

$$\begin{aligned}\Phi_{t^0}^{k,\delta} : (q,x) \mapsto \Big(& \psi^{-1}\left(f_k(x) z_x^\delta(q), \left(a_x^\delta(q), f_k(x) b_x^\delta(q)\right)\right), \\ & \left(x, f_k(x)\left(\frac{d}{dx} z_x^\delta(q) - \langle b_x^\delta(q), \frac{d}{dx} a_x^\delta(q)\rangle\right) + f_k'(x) z_x^\delta(q)\right)\Big)\end{aligned}$$

which is a Legendrian embedding of $M \times \mathbf{R}$ inside $V \times T^*\mathbf{R}$. Observe that

$$\Phi_{t^0}^{k,0}(q,x) = \left(\psi^{-1}\left(f_k(x)h(q), (q, f_k(x)dh(q))\right), (x, f_k'(x)h(q))\right)$$

and

$$\Phi_{t^0}^{k,\delta}(q,\pm 1) = \left(i_{t^0\pm\delta}(q), (\pm 1, 0)\right).$$

Example 3.9. Assume that $V = \mathbf{R}_z$, $M = \{q_-, q_+\}$, and $\phi_t = \phi : M \to V$ with

$$\phi(q_-) = 0 \text{ and } \phi(q_+) = 1.$$

Choose $h : M \to \mathbf{R}$ such that

$$h(q_-) = 0 \text{ and } h(q_+) = 1$$

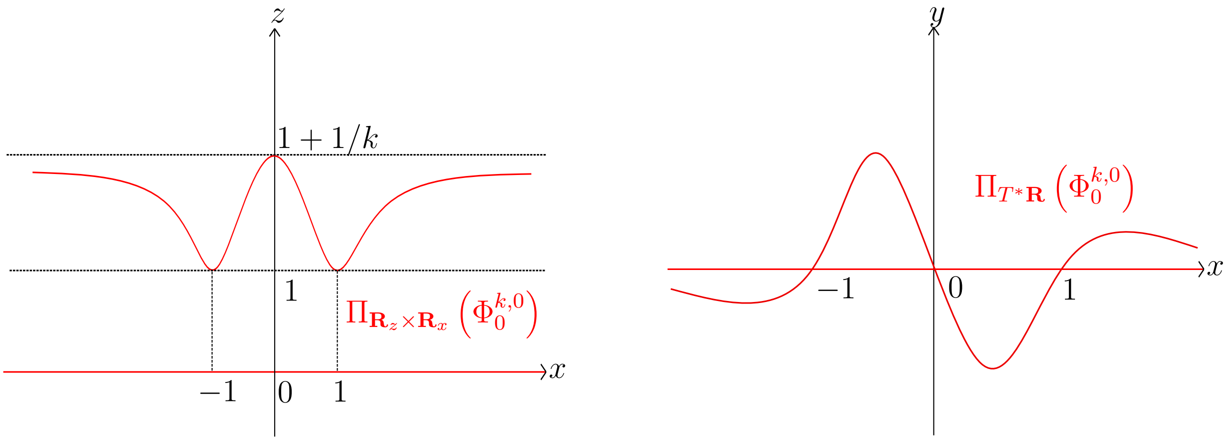

Figure 3.2: Front (on the left) and Lagrangian (on the right) projections of $\Phi_0^{k,0}$

and $\psi : \mathcal{N}(\Lambda) \to J^1(M) = \mathbf{R} \times M$ such that

$$\psi(z) = (z, q_\pm) \text{ if } z \in \mathcal{N}(\phi(q_\pm)).$$

Then

$$\Phi_0^{k,0}(q_-, x) = (0, (x, 0)) \text{ and } \Phi_0^{k,0}(q_+, x) = (f_k(x), (x, f_k'(x))).$$

See Figure 3.2.

Lemma 3.10. *For every Reeb chord $c : [0,T] \to V$ of Λ_{t^0} going from $\phi_{t^0}(q_-)$ to $\phi_{t^0}(q_+)$, for every $x \in \mathbf{R}$ and for every k large enough, there is a Reeb chord $c_{x,k}$ of the Legendrian*

$$\left\{\psi^{-1}(f_k(x)h(q), (q, f_k(x)dh(q))) \mid q \in M\right\} \subset V$$

which starts at $\phi_{t^0}(q_-) = \psi^{-1}(f_k(x)h(q_-), (q_-, f_k(x)dh(q_-)))$ and which ends at the point $\psi^{-1}(f_k(x)h(q_+), (q_+, f_k(x)dh(q_+)))$.

Proof. Let $\gamma : \mathbf{R} \to V$ be the Reeb trajectory such that $\gamma_{|[0,T]} = c$. We have

$$\begin{cases} \gamma(0) &= \phi_{t^0}(q_-) &= \psi^{-1}(h(q_-), (q_-, dh(q_-))) &= \psi^{-1}(0, (q_-, 0)) \\ \gamma(T) &= \phi_{t^0}(q_+) &= \psi^{-1}(h(q_+), (q_+, dh(q_+))) &= \psi^{-1}(h(q_+), (q_+, 0)). \end{cases}$$

First observe that, since $h(q_-) = 0$ and $dh(q_-) = 0$, we have

$$\gamma(0) = \psi^{-1}(f_k(x)h(q_-), (q_-, f_k(x)dh(q_-))).$$

Then observe that, since $h(q_+) > 0$ and $dh(q_+) = 0$, and because ψ sends the Reeb vector field of V to the one of $J^1(M)$, it follows that for every $x \in \mathbf{R}$ and for every k large enough, there exists $T_{x,k} > 0$ such that

$$\gamma(T_{x,k}) = \psi^{-1}\left(f_k(x)h(q_+),(q_+,0) = \psi^{-1}\left(f_k(x)h(q_+),(q_+,f_k(x)dh(q_+))\right)\right).$$

The result follows by setting $c_{x,k} := \gamma_{|\left[0,T_{x,k}\right]}$.

□

Almost complex structures We describe compatible almost complex structures $J^\delta_{t^0}$ on $\widehat{\xi} = \ker(\widehat{\alpha})$ obtained from $\left(J_{t^0+\eta^\delta(x)}\right)_{x\in\mathbf{R}}$. Observe that there is a symplectic isomorphism

$$\widehat{\xi}\left(a,(x,y)\right) \xrightarrow{\sim} \xi(a) \oplus \mathbf{C}, \quad (v,\zeta) \mapsto \left(v - ydx\left(\zeta\right)R_\alpha, \zeta\right).$$

We denote by $J^\delta_{t^0}\left(a,(x,y)\right)$ the almost complex structure on $\widehat{\xi}\left(a,(x,y)\right)$ which is the pullback of $\left(J_{t^0+\eta^\delta(x)}\left(a\right) \oplus i\right)$ by this isomorphism.

Remark 3.11. Let Δ be a punctured Riemann disk (see Definition 3.1). A map

$$u : \Delta \to \mathbf{R} \times V \times T^*\mathbf{R}, \quad z \mapsto \left(\tau\left(z\right), v(z), (x(z), y(z))\right)$$

is pseudo-holomorphic for the the structure induced by $J^\delta_{t^0}$ and $\widehat{\alpha} = \alpha - ydx$ if and only if

$$\begin{cases} d\tau \circ i & = & ydx - v^*\alpha \\ (dv - R_\alpha \otimes v^*\alpha) \circ i & = & J_{t^0+\eta^\delta(x)} \circ (dv - R_\alpha \otimes v^*\alpha) \\ (dx, dy) \circ i & = & i \circ (dx, dy)\,. \end{cases}$$

In particular, the projection of such a map to $T^*\mathbf{R}$ is holomorphic. Moreover, if the map $z \mapsto x(z)$ constantly equals ± 1, then the map

$$\Delta \to \mathbf{R} \times V, \quad z \mapsto \left(\tau\left(z\right), v(z)\right)$$

is pseudo-holomorphic for the structure induced by $J_{t^0\pm\delta}$ and α.

Lemma 3.12. *Let Δ be a punctured Riemann disk (see Definition 3.1). A map*

$$u : \Delta \to \mathbf{R} \times V \times T^*\mathbf{R}, \quad z \mapsto \left(\tau\left(z\right), v(z), (x(z), y(z))\right)$$

is pseudo-holomorphic for the the structure induced by $J^0_{t^0}$ and $\widehat{\alpha} = \alpha - ydx$ if and only if the following two conditions hold:

1. *the map*
$$\Delta \to \mathbf{R} \times V, \quad z \mapsto \left(\tau(z) - y(z)^2/2, v(z)\right)$$
is pseudo-holomorphic for the structure induced by J_{t^0} and α,

2. *the map*
$$\Delta \to T^*\mathbf{R}, \quad z \mapsto (x(z), y(z))$$
is holomorphic.

Proof. The map u is pseudo-holomorphic if and only if

$$\begin{cases} d\tau \circ i & = & ydx - v^*\alpha \\ (dv - R_\alpha \otimes v^*\alpha) \circ i & = & J_{t^0} \circ (dv - R_\alpha \otimes v^*\alpha) \\ (dx, dy) \circ i & = & i \circ (dx, dy) \,. \end{cases}$$

Besides, a map
$$\Delta \to \mathbf{R} \times V, \quad z \mapsto (\sigma(z), v(z))$$
is pseudo-holomorphic if and only if

$$\begin{cases} d\sigma \circ i & = & -v^*\alpha \\ (dv - R_\alpha \otimes v^*\alpha) \circ i & = & J \circ (dv - R_\alpha \otimes v^*\alpha) \,. \end{cases}$$

The result follows from the fact that

$$d\left(y^2/2\right) + ydx \circ i = 0$$

when the map
$$\Delta \to T^*\mathbf{R}, \quad z \mapsto (x(z), y(z))$$
is holomorphic.

□

3.2.3 Construction of the tame DG-isomorphism

We mimic the construction made in [28]. Observe that if $(\Lambda_t)_{0 \leq t \leq 1}$ is a path in $\mathcal{L}$, then the Reeb chords of Λ_t are naturally identified with those of Λ_0 for every t. Therefore, we will consider that the DG-algebras $CE_{-*}(\Lambda_t)$ and $CE_{-*}(\Lambda_0)$ have the same underlying associative graded algebra.

If $(\Lambda_t, J_t)_{0\leq t\leq 1}$ is a good path, we consider the parametrized moduli space

$$\mathcal{M}^{[0,1]}_{a,\mathbf{b}} := \{(u,t) \mid u \in \mathcal{M}_{a,\mathbf{b}}\left(\mathbf{R}\times\Lambda_t, J_t, \alpha\right)\}$$

(see Definition 3.4).

Lemma 3.13 (Analog of Proposition 2.9 in [28])**.** *The following holds for generic good paths $(\Lambda_t, J_t)_{0\leq t\leq 1}$. If a, $\mathbf{b}$ are such that*

$$d := |a| - |\mathbf{b}| \leq 1,$$

then $\mathcal{M}^{[0,1]}_{a,\mathbf{b}}$ is a transversely cut out manifold of dimension d. Besides, there exists $t_1 < \cdots < t_r$, called handle slide instants*, and Reeb chords $(a_i, \mathbf{b}_i)_{1\leq j\leq r}$ such that*

$$\bigsqcup_{|a|-|\mathbf{b}|=0} \mathcal{M}^{[0,1]}_{a,\mathbf{b}} = \bigsqcup_{i=1}^{r} \mathcal{M}_{a_i,\mathbf{b}_i}\left(\mathbf{R}\times\Lambda_{t_i}, J_{t_i}\right)\times\{t_i\}\,.$$

Moreover, $\mathcal{M}_{a_i,\mathbf{b}_i}\left(\mathbf{R}\times\Lambda_{t_i}, J_{t_i}\right)$ contains exactly one point, and $\mathcal{M}_{c,\mathbf{d}}\left(\mathbf{R}\times\Lambda_{t_i}, J_{t_i}\right)$ is a transversely cut out manifold of dimension 0 whenever $|c| - |\mathbf{d}| = 1$.

Proof. This follows from Lemma B.8 in [25] (recall from Definition 3.7 that in a good path, we can partition $[0,1]$ so that in each piece either the Legendrian or the almost complex structure is changing but not both).

□

Lemma 3.14 (Analog of Proposition 2.10 in [28])**.** *Let $(\Lambda_t, J_t)_{0\leq t\leq 1}$ be a good path. Any sequence in $\mathcal{M}^{[0,1]}_{a,\mathbf{b}}$ has a subsequence that converges to a broken holomorphic curve.*

Proof. This follows from the compactness results of [14] and [1]. See also [25] Appendix B.

□

Lemma 3.15 (Analog of Lemma 2.11 in [28])**.** *Let $(\Lambda_t, J_t)_{0\leq t\leq 1}$ be a generic path of compatible almost complex structures on ξ. Let $t < t'$ be such that there is no handle slide instants in $[t, t']$. Then the identity map induces a DG-isomorphism:*

$$CE_{-*}\left(\Lambda_t, J_t\right) \simeq CE_{-*}\left(\Lambda_{t'}, J_{t'}\right).$$

Proof. As in [28], this follows from the smoothness and compactness of the parametrized moduli space, which here correspond to Lemmas 3.13 and 3.14.

□

We now fix a generic path $(J_t)_{0\leq t\leq 1}$ of compatible almost complex structures on ξ. We denote by $t_1 < \cdots < t_r$ the handle slide instants of Lemma 3.13, and we look at what happens at an handle slide instant t_i. To do that, we consider the Legendrians embeddings $\left(\Phi_{t_i}^{k,\delta}\right)_{k\geq 1}$ and the almost complex structures $\left(J_{t_i}^{\delta}\right)_{\delta\geq 0}$ described in Section 3.2.2.

Observe at this point that the analogs of Lemmas 10.1, 10.2 and 10.3 in [28] remain true in our context using the same arguments. We recall them for completeness.

Lemma 3.16 (Analog of Lemma 10.1 in [28])**.** *Let u be a disk in $\mathcal{M}\left(\mathbf{R}\times\Phi_{t_i}^{k,\delta}, J_{t_i}^{\delta}\right)$ with one positive puncture converging to a Reeb chord $t\mapsto(c(t),(x(t),y(t)))$. If $t\mapsto x(t)$ constantly equals ± 1, then the image of u lies in $\{x=\pm 1\}$.*

Proof. The same proof as in [28] works here because the projection to $T^*\mathbf{R}$ of a disc in $\mathcal{M}\left(\mathbf{R}\times\Phi_{t_i}^{k,\delta}, J_{t_i}^{\delta}\right)$ is holomorphic (see Remark 3.11).

□

Lemma 3.17 (Analog of Lemma 10.2 in [28])**.** *The image of every disk in $\mathcal{M}\left(\mathbf{R}\times\Phi_{t_i}^{k,\delta}, J_{t_i}^{\delta}\right)$ is contained in the region $\{|x|\leq 1\}$.*

Proof. Once again, the same proof as in [28] works here because the projection to $T^*\mathbf{R}$ of a disc in $\mathcal{M}\left(\mathbf{R}\times\Phi_{t_i}^{k,\delta}, J_{t_i}^{\delta}\right)$ is holomorphic (see Remark 3.11).

□

Lemma 3.18 (Analog of Lemma 10.3 in [28])**.** *There is a well defined Chekanov-Eliashberg DG-algebra associated to $\left(\Phi_{t_i}^{k,\delta}, J_{t_i}^{\delta}\right)$.*

Proof. According to Proposition 3.6, transversality of the moduli spaces can be achieved by a perturbation of the almost complex structure near the positive puncture of a disk.

Moreover, although the Legendrian $\Phi_{t_i}^{k,\delta}$ is not compact, Lemma 3.17 implies that the differential ∂ defined as in Section 3.1.3 satisfies $\partial\circ\partial=0$.

□

We also state and prove the analog of Lemma 10.4 in our context.

Lemma 3.19 (Analog of Lemma 10.4 in [28]). *For every k large enough, the Reeb chords of $\Phi^{k,\delta}_{t_i}$ are of the form*

$$c\,[x_0] := (c_{x_0,k}, (x_0, 0))$$

where c is a Reeb chord of Λ_{t_i}, $x_0 \in \{-1, 0, 1\}$, and $c_{x_0,k}$ is the chord defined in Lemma 3.10. Besides, for every Reeb chord c we have

$$|c[0]| = |c| + 1 \text{ and } |c[\pm 1]| = |c|.$$

Proof. Since the Legendrians

$$\left\{\psi^{-1}\left(f_k(x)h(q), (q, f_k(x)dh(q))\right) \mid q \in M\right\} \subset V$$

converge to Λ_{t_i} when k goes to $+\infty$, their Reeb chords are exactly the ones described in Lemma 3.10 for k large enough. Now let γ be a Reeb chord of $\Phi^{k,\delta}_{t_i}$. There exists a Reeb chord c of Λ_{t_i} and (x_0, y_0) in $T^*\mathbf{R}$ such that

$$\gamma(t) = (c_{x_0,k}(t), (x_0, y_0))\,.$$

If c goes from $\phi_{t_i}(q_-)$ to $\phi_{t_i}(q_+)$ we get

$$y_0 = f_k'(x_0)h(q_-) = f_k'(x_0)h(q_+).$$

Since $h(q_-) = 0$ and $h(q_+) > 0$, it follows that $f_k'(x_0) = 0$ and thus $y_0 = 0$.

□

We now deal with the important Lemmas 10.5, 10.6 and 10.7 of [28].

Lemma 3.20 (Analog of Lemma 10.5 in [28]). *There exists k_0 such that for all $k \geq k_0$, there exists a positive δ_k such that for every δ in $[0, \delta_k]$ and any Reeb chord c the following holds. The moduli space $\mathcal{M}_{c[0],c[\pm 1]}\left(\mathbf{R} \times \Phi^{k,\delta}_{t_i}, J^{\delta}_{t_i}\right)$ consists of exactly one point which is a transversely cut out rigid disk.*

Proof. We first prove that the moduli space $\mathcal{M}_{c[0],c[+1]}\left(\mathbf{R} \times \Phi^{k,0}_{t_i}, J^0_{t_i}\right)$ contains an element. In the following, $c : [0, T] \to V$ goes from $\phi_{t_i}(q_-)$ to $\phi_{t_i}(q_+)$. According to Lemma 3.12, it suffices to find a pseudo-holomorphic map

$$\mathbf{R} \times [0, 1] \to \mathbf{R} \times V, \quad (s, t) \mapsto (\sigma(s, t), v(s, t))$$

and an holomorphic map

$$\mathbf{R} \times [0,1] \to T^*\mathbf{R}, \quad (s,t) \mapsto (x(s,t), y(s,t))$$

such that the map

$$u : \mathbf{R} \times [0,1] \to \mathbf{R} \times V \times T^*\mathbf{R}, \quad (s,t) \mapsto \left(\sigma(s,t) + \frac{y(s,t)^2}{2}, v(s,t), (x(s,t), y(s,t)) \right)$$

has the appropriate asymptotic and boundary properties. We first pick the only (up to reparametrization) holomorphic map

$$\mathbf{R} \times [0,1] \to T^*\mathbf{R}, \quad (s,t) \mapsto (x_k(s,t), y_k(s,t))$$

such that

$$\begin{cases} (x_k(s,0), y_k(s,0)) \in [0,1] \times \{0\}, & (x_k(s,1), y_k(s,1)) \in \{y = f_k'(x) h(q_+)\} \\ (x_k(s,t), y_k(s,t)) \underset{s \to -\infty}{\longrightarrow} (\pm 1, 0)\,, & (x_k(s,t), y_k(s,t)) \underset{s \to +\infty}{\longrightarrow} (0,0)\,. \end{cases}$$

Then we look for the map

$$\mathbf{R} \times [0,1] \to \mathbf{R} \times V, \quad (s,t) \mapsto (\sigma(s,t), v(s,t))\,.$$

Let $\gamma : \mathbf{R} \to V$ be the Reeb trajectory satisfying $\gamma(0) = \phi_{t_i}(q_-)$. We let $\rho_k : \mathbf{R} \times [0,1] \to \mathbf{R}$ be the unique harmonic function such that

$$\rho_k(s,0) = 0 \text{ and } \rho_k(s,1) = T_{x(s,1),k}$$

where $T_{x_k(s,1),k}$ is the unique positive number satisfying

$$\gamma(T_{x_k(s,1),k}) = \psi^{-1}\left(f_k(x_k(s,1)) h(q_+), (q_+, f_k(x_k(s,1)) dh(q_+)) \right)$$

(see Lemma 3.10). Let $\sigma_k : \mathbf{R} \times [0,1] \to \mathbf{R}$ be a conjugate of ρ_k (meaning that $d\sigma_k = d\rho_k \circ i$). Then the map

$$\mathbf{R} \times [0,1] \to \mathbf{R} \times V, \quad (s,t) \mapsto (\sigma_k(s,t), v_k(s,t)) = (\sigma_k(s,t), \gamma(\rho_k(s,t)))$$

is pseudo-holomorphic because

$$dv_k = R_\alpha(v_k) \otimes d\rho_k \text{ and } v_k^*\alpha = d\rho_k = -d\sigma_k \circ i.$$

Moreover, the map

$$u_k : \mathbf{R} \times [0,1] \to \mathbf{R} \times V \times T^*\mathbf{R}, \quad (s,t) \mapsto \left(\sigma_k(s,t) + \frac{y_k(s,t)^2}{2}, v_k(s,t), (x_k(s,t), y_k(s,t))\right)$$

has the appropriate asymptotic and boundary properties.

We now explain why the latter disk is transversely cut out. Recall that the maps

$$(x_k, y_k) : \mathbf{R} \times [0,1] \to \mathbf{C} \text{ and } (\sigma_k, v_k) : \mathbf{R} \times [0,1] \to \mathbf{R} \times V$$

are (pseudo-)holomorphic. Besides, (σ_k, v_k) goes to the standard strip over the Reeb chord c when k goes to $+\infty$. The latter is transversely cut out because Λ is chord generic. This implies that u_k is transversely cut out.

We now explain why u_k is the unique element of $\mathcal{M}_{c[0],c[+1]}\left(\mathbf{R} \times \Phi_{t_i}^{k,0}, J_{t_i}^0\right)$ for k large enough. Recall from Lemma 3.12 that if

$$u : \Delta \to \mathbf{R} \times V \times T^*\mathbf{R}, \quad z \mapsto (\tau(z), v(z), (x(z), y(z)))$$

is in $\mathcal{M}_{c[0],c[+1]}\left(\mathbf{R} \times \Phi_{t_i}^{k,0}, J_{t_i}^0\right)$, then

$$\widetilde{u} : \Delta \to \mathbf{R} \times V, \quad z \mapsto (\widetilde{\tau}(z), \widetilde{v}(z))$$

is pseudo-holomorphic for the structure induced by J_{t_i} and α. Besides, observe that $\Phi_{t^0}^{k,0}$ goes to $\Lambda_{t^0} \times 0_{\mathbf{R}}$ when k goes to $+\infty$. Therefore, there exists $\nu > 0$ such that for every k large enough, for every u in $\mathcal{M}_{c[0],c[+1]}\left(\mathbf{R} \times \Phi_{t_i}^{k,0}, J_{t_i}^0\right)$, the area

$$A(\widetilde{u}) = \int_\Delta \widetilde{u}^* d(e^\sigma \alpha) = \int_{c[0]} \alpha - \int_{c[+1]} \alpha$$

of $\widetilde{u}$ is either greater than ν or 0. Since the lengths of $c[0]$ and $c[\pm 1]$ converge to the length of c when k goes to ∞, it follows that $A(\widetilde{u})$ is actually 0 for every k large enough and $u \in \mathcal{M}_{c[0],c[+1]}\left(\mathbf{R} \times \Phi_{t_i}^{k,0}, J_{t_i}^0\right)$. This implies that u_k is the unique element of $\mathcal{M}_{c[0],c[+1]}\left(\mathbf{R} \times \Phi_{t_i}^{k,0}, J_{t_i}^0\right)$ for k large enough.

□

Let $(a_i, \mathbf{b}_i)$ be the chords such that $\mathcal{M}_{a_i,\mathbf{b}_i}(\mathbf{R}\times\Lambda_{t_i}, J_{t_i})$ contains a handle slide disk (see Lemma 3.13).

Lemma 3.21 (Analog of Lemma 10.6 in [28])**.** *There exists k_0 such that for all $k \geq k_0$, there exists a positive δ_k such that for every δ in $[0, \delta_k]$ and any Reeb chord $c \neq a_i$ the following holds. If $\mathbf{e} \neq c[\pm 1]$ is a word without $[0]$-generator, and if $|c[0]| - |\mathbf{e}| - 1 = 0$, then the moduli space $\mathcal{M}_{c[0],\mathbf{e}}\left(\mathbf{R}\times\Phi^{k,\delta}_{t_i}, J^{\delta}_{t_i}\right)$ is empty.*

Proof. The proof is almost the same as in [28]. We reproduce it here with the necessary modifications.

We show that the following holds for every $c \neq a_i$ and $\mathbf{e}$ without $[0]$-generator: if $|c[0]| - |\mathbf{e}| - 1 = 0$, then the moduli space $\mathcal{M}_{c[0],\mathbf{e}}\left(\mathbf{R}\times\Phi^{k,0}_{t_i}, J^{0}_{t_i}\right)$ is empty for k large enough. The result will follow because emptiness of a moduli space is an open condition.

We prove the contraposition. Consider a sequence $(u_k)_k$ such that

$$u_k \in \mathcal{M}_{c[0],\mathbf{e}}\left(\mathbf{R}\times\Phi^{k,0}_{t_i}, J^{0}_{t_i}\right)$$

for every k. Write $u_k = (\tau_k, v_k, (x_k, y_k)) : \Delta \to \mathbf{R}\times V \times T^*\mathbf{R}$. According to Lemma 3.12, the map

$$w_k : \Delta \to \mathbf{R}\times V, \quad z \mapsto \left(\tau_k(z) - y_k(z)^2/2, v_k(z)\right)$$

is pseudo-holomorphic with respect to the almost complex structure induced by J_{t_i} and α. Taking $k \to +\infty$, we have $\Phi^{k,0}_{t_i} \to \Lambda_{t_i}\times 0_{\mathbf{R}}$ and (a subsequence of) $(w_k)_k$ converges to a broken disk $(w^j_\infty)_{1\leq j\leq m}$ with boundary on $\mathbf{R}\times\Lambda_{t_i}$ (see Remark 3.22). For a disk w^j_∞, let γ^j be its positive puncture, and $\boldsymbol{\varepsilon}^j$ the word of its negative punctures. Since the positive puncture of u_k is its only $[0]$-puncture, we have

$$|c[0]| - |\mathbf{e}| - 1 = |c| - |\mathbf{e}| = \sum_{j=1}^{m}\left(|\gamma^j| - |\boldsymbol{\varepsilon}^j|\right).$$

Now observe that w^j_∞ either has formal dimension $|\gamma^j| - |\boldsymbol{\varepsilon}^j| - 1$ at least 0, or equals the handle slide disk (in this case recall that $|a_i| - |\mathbf{b}_i| = 0$). Besides, there is exactly one j_0 such that $\gamma^{j_0} = c$. Since $c \neq a$, w^{j_0} is not the handle slide disk and thus

$$|\gamma^{j_0}| - |\boldsymbol{\varepsilon}^{j_0}| - 1 \geq 0.$$

Finally we have

$$|c[0]| - |\mathbf{e}| - 1 = \sum_{j=1}^{m} \left(|\gamma^j| - |\boldsymbol{\varepsilon}^j| \right) \geq 1.$$

□

Remark 3.22. The convergence of the sequence $(w_k)_k$ in the proof of Lemma 3.21 might seem unusual because w_k has boundary on $\mathbf{R} \times \Pi_V \left(\Phi^{k,0}_{t_i} \right)$, and $\Pi_V \left(\Phi^{k,0}_{t_i} \right)$ typically is *not* a Legendrian submanifold of V. However, a careful reading of [1] section 3.3 shows that the SFT compactness result holds in our context because the limit Λ_{t_i} of $\left(\Pi_V \left(\Phi^{k,0}_{t_i} \right) \right)_k$ is a chord generic Legendrian submanifold of (V, α).

Lemma 3.23 (Analog of Lemma 10.7 in [28])**.** *There exists k_0 such that for all $k \geq k_0$, there exists a positive δ_k such that for every δ in $[0, \delta_k]$ the following holds. If $\mathbf{e} \neq a_i\,[\pm 1]$ is a word without $[0]$-generators, and if $|a_i[0]| - |\mathbf{e}| - 1 = 0$, then the moduli space $\mathcal{M}_{a_i[0], \mathbf{e}} \left(\mathbf{R} \times \Phi^{k,\delta}_{t_i}, J^{\delta}_{t_i} \right)$ is either empty, or $\mathbf{b}_i$ is obtained from $\mathbf{e}$ by changing the letters $c\,[\pm 1]$ into c.*

Proof. As for the proof of Lemma 3.21, the proof of Lemma 3.23 is almost the same as in [28] using Lemma 3.12.

□

We choose $k \geq 1$ large enough and $\delta > 0$ small enough so that the last three Lemmas hold. We denote by Δ the differential of $CE_* \left(\Phi^{k,\delta}_{t_i}, J^{\delta}_{t_i} \right)$. We have

$$\Delta a_i[0] = a_i[1] + a_i[-1] + \epsilon + O(1),$$

where $O(1)$ denotes a sum of terms with at least one $[0]$-generator, and ϵ is a sum of terms without $[0]$-generator. Besides, there exists $m_i \in \mathbf{F}$ such that $m_i \mathbf{b}_i$ is obtained from ϵ by changing the generators $c[\pm 1]$ into c. We denote by $\varphi_i : CE_* \left(\Lambda_{t_i - \delta} \right) \to CE_* \left(\Lambda_{t_i + \delta} \right)$ the elementary isomorphism (see Definition 1.6) defined on generators by

$$\varphi_i(a_i) = a_i + m_i \mathbf{b}_i \text{ and } \varphi(c) = c \text{ if } c \neq a_i.$$

We can now write the analog of Lemma 10.8 in [28].

Lemma 3.24 (Analog of Lemma 10.8 in [28])**.** *The map φ_i is an elementary DG-isomorphism*

$$\varphi_i : CE_{-*} \left(\Lambda_{t_i - \delta}, J_{t_i - \delta} \right) \xrightarrow{\sim} CE_{-*} \left(\Lambda_{t_i + \delta}, J_{t_i + \delta} \right).$$

Proof. In [28], this is an algebraic consequence of the Lemmas numbered from 10.1 to 10.7. The proof in our context is identical, using the corresponding Lemmas numbered from 3.16 to 3.23.

□

We can now define the DG-isomorphism of Theorem 3.8.

Definition 3.25. Let $(\Lambda_t, J_t)_{t\in[0,1]}$ be a generic good path. Let $t_1 < \cdots < t_r$ be the handle slide instants of Lemma 3.13. Choose $k \geq 1$ large enough and $\delta > 0$ small enough so that Lemmas 3.19, 3.20, 3.21 and 3.23 hold for every handle slide instant. For every $1 \leq j \leq r$, we denote by $\varphi_i : CE_{-*}(\Lambda_{t_i-\delta}, J_{t_i-\delta}) \xrightarrow{\sim} CE_{-*}(\Lambda_{t_i+\delta}, J_{t_i+\delta})$ the elementary DG-isomorphism of Lemma 3.24. According to Lemma 3.15, $CE_{-*}(\Lambda_t, J_t)$ and $CE_{-*}(\Lambda_{t'}, J_{t'})$ are trivially identified when there is no handle slide instants between t and t'. Thus, the map $\varphi = \varphi_r \circ \cdots \circ \varphi_1$ defines a tame DG-isomorphism from $CE_{-*}(\Lambda_0, J_0)$ to $CE_{-*}(\Lambda_1, J_1)$.

3.2.4 Mapping cylinder as Chekanov-Eliashberg algebra

This section gathers results in order to prove that the DG-isomorphisms defined in Definition 3.25 satisfy the second item of Theorem 3.8.

Let $(J_t)_{t\in[0,1]}$ be a generic path of compatible almost complex structures on ξ. We use the notations of Definition 3.25. Our goal is to show that the mapping cylinder DG-algebra (see Definition 1.10) of the tame DG-isomorphism $\varphi = \varphi_r \circ \cdots \circ \varphi_1$ can be seen as the Chekanov-Eliashberg DG-algebra of some Legendrian. The starting point is the following observation.

Lemma 3.26. *For every $1 \leq i \leq r$, $CE_{-*}\left(\Phi^{k,\delta}_{t_i}, J^{\delta}_{t_i}\right)$ is a quasi mapping cylinder DG-algebra for the elementary DG-isomorphism*

$$\varphi_i : CE_{-*}(\Lambda_{t_i-\delta}, J_{t_i-\delta}) \to CE_{-*}(\Lambda_{t_i+\delta}, J_{t_i+\delta})$$

(see Definition 1.15).

Proof. We denote by Δ the differential of $CE_*\left(\Phi^{k,\delta}_{t_i}, J^{\delta}_{t_i}\right)$. Observe that $CE_{-*}(\Lambda_{t_i-\delta}, J_{t_i-\delta})$ and $CE_{-*}(\Lambda_{t_i+\delta}, J_{t_i+\delta})$ are sub-DG-algebras of $CE_{-*}\left(\Phi^{k,\delta}_{t_i}, J^{\delta}_{t_i}\right)$ using Remark 3.11 and Lemma 3.16. Moreover, according to Lemmas 3.20 and 3.21, we have

$$\Delta c[0] = c[1] + c[-1] + O(1)$$

(where $O(1)$ denotes a sum of terms with at least one [0]-generator) for every Reeb chord $c \neq a_i$, and

$$\Delta a_i[0] = a_i[-1] + a_i[+1] + \epsilon + O(1),$$

where ϵ is a sum of terms without [0]-generator. Moreover, as explained in the paragraph above Lemma 3.24, $\varphi_i(c) = c$ if $c \neq a$, and $\varphi_i(a)$ is obtained from $(a[+1] + \epsilon)$ by changing the generators $c[\pm 1]$ into c. Thus, $CE_{-*}\left(\Phi_\ell, J^{\varepsilon_i}_{t_{i,j}}\right)$ is a quasi mapping cylinder for the elementary DG-isomorphism

$$\varphi_i : CE_{-*}\left(\Lambda_{t_i-\delta}, J_{t_i-\delta}\right) \to CE_{-*}\left(\Lambda_{t_i+\delta}, J_{t_i+\delta}\right).$$

□

We now have to treat the non handle slide instants. To do that we need two preliminary Lemmas.

Lemma 3.27. *There exists ℓ_0 such that for all $\ell \geq \ell_0$, there exists a positive ε_ℓ such that for every ε in $[0, \varepsilon_\ell]$, for every t in $[0,1] \setminus \bigcup_{i=1}^{r} (t_i - \delta, t_i + \delta)$ and for every Reeb chord c the following holds. The moduli space $\mathcal{M}_{c[0],c[\pm 1]}\left(\mathbf{R} \times \Phi^{\ell,\varepsilon}_t, J^{\varepsilon}_t\right)$ consists of exactly one point which is a transversely cut out rigid disk.*

Proof. The proof is the same as for Lemma 3.20 using that $[0,1] \setminus \bigcup_{i=1}^{r} (t_i - \delta, t_i + \delta)$ is compact.

□

Lemma 3.28. *There exists ℓ_0 such that for all $\ell \geq \ell_0$, there exists a positive ε_ℓ such that for every ε in $[0, \varepsilon_\ell]$, for every t in $[0,1] \setminus \bigcup_{i=1}^{r} (t_i - \delta, t_i + \delta)$ and for every Reeb chord c the following holds. If $\mathbf{e} \neq c[\pm 1]$ is a word without [0]-generators, and if $|c[0]| - |\mathbf{e}| - 1 = 0$, then the moduli space $\mathcal{M}_{c[0],\mathbf{e}}\left(\mathbf{R} \times \Phi^{\ell,\varepsilon}_t, J^{\delta}_t\right)$ is empty.*

Proof. Here, the proof is the same as for Lemma 3.21 using that $[0,1] \setminus \bigcup_{i=1}^{r} (t_i - \delta, t_i + \delta)$ is compact.

□

Choose $\ell_0 \geq k$ so that for every $\ell \geq \ell_0$, there exists $\varepsilon_\ell > 0$ such that Lemmas 3.27 and 3.28 hold for every $\varepsilon \in [0, \varepsilon_\ell]$. Fix $\ell \geq \ell_0$, and choose s_0 large enough so that

$$\frac{(t_{i+1} - \delta) - (t_i + \delta)}{2s} \leq \varepsilon$$

for every $s \geq s_0$ and $0 \leq i \leq r$. Now we fix $s \geq s_0$, and for every $0 \leq i \leq r$, $1 \leq j \leq s$ we set

$$\varepsilon_i = \frac{(t_{i+1} - \delta) - (t_i + \delta)}{2s}, \quad t_{i,j} := (t_i + \delta) + (2j-1)\varepsilon_i$$

(by convention, $t_0 + \delta = 0$ and $t_{r+1} - \delta = 1$).

Lemma 3.29. *For every $0 \leq i \leq r$, $1 \leq j \leq s$, $CE_{-*}\left(\Phi^{\ell,\varepsilon_i}_{t_{i,j}}, J^{\varepsilon_i}_{t_{i,j}}\right)$ is a mapping cylinder DG-algebra for the identity map*

$$\mathrm{id} : CE_{-*}\left(\Lambda_{t_{i,j}-\varepsilon_i}, J_{t_{i,j}-\varepsilon_i}\right) \to CE_{-*}\left(\Lambda_{t_{i,j}+\varepsilon_i}, J_{t_{i,j}+\varepsilon_i}\right).$$

Proof. We denote by Δ the differential of $CE_*\left(\Phi^{\ell,\varepsilon_i}_{t_{i,j}}, J^{\varepsilon_i}_{t_{i,j}}\right)$. Observe that $CE_{-*}\left(\Lambda_{t_{i,j}-\varepsilon_i}, J_{t_{i,j}-\varepsilon_i}\right)$ and $CE_{-*}\left(\Lambda_{t_{i,j}+\varepsilon_i}, J_{t_{i,j}+\varepsilon_i}\right)$ are sub-DG-algebras of $CE_{-*}\left(\Phi^{\ell,\varepsilon_i}_{t_{i,j}}, J^{\varepsilon_i}_{t_{i,j}}\right)$ using Remark 3.11 and Lemma 3.16. Moreover, according to Lemmas 3.27 and 3.28, we have

$$\Delta c[0] = c[1] + c[-1] + O(1)$$

(where $O(1)$ denotes a sum of terms with at least one [0]-generator) for every Reeb chord c. Thus, $CE_{-*}\left(\Phi^{\ell,\varepsilon_i}_{t_{i,j}}, J^{\varepsilon_i}_{t_{i,j}}\right)$ is a mapping cylinder for the identity map

$$\mathrm{id} : CE_{-*}\left(\Lambda_{t_{i,j}-\varepsilon_i}, J_{t_{i,j}-\varepsilon_i}\right) \to CE_{-*}\left(\Lambda_{t_{i,j}+\varepsilon_i}, J_{t_{i,j}+\varepsilon_i}\right).$$

□

Now that we have treated handle slide instants (see Lemma 3.26) and non handle slide instants (see Lemma 3.29), we want to "glue" the data together.

Definition 3.30. To each couple $(i,j) \in (\{0,\dots,r\} \times \{0,\dots,s\}) \setminus \{0,0\}$, we associate the positive integer

$$x_{i,j} := i(s+1) + j$$

(observe that $(i,j) \mapsto x_{i,j}$ preserves the lexicographic order). Now we define a Legendrian embedding $\Phi : M \times [0, 2x_{r,s}] \to V \times T^*[0, 2x_{r,s}]$ and an almost complex structure J on the restriction of $\widehat{\xi}$ to $V \times T^*[0, 2x_{r,s}]$ as follows. For every $(i,j) \in (\{0,\dots,r\} \times \{0,\dots,s\}) \setminus \{0,0\}$, for every $x \in [2(x_{i,j}-1), 2x_{i,j}]$, we set

$$\Phi(q,x) = \begin{cases} \Phi^{\ell,\varepsilon_i}_{t_{i,j}}(q, x - (2x_{i,j}-1)) & \text{if } j \neq 0 \\ \Phi^{k,\delta}_{t_i}(q, x - (2x_{i,0}-1)) & \text{if } j = 0 \end{cases}$$

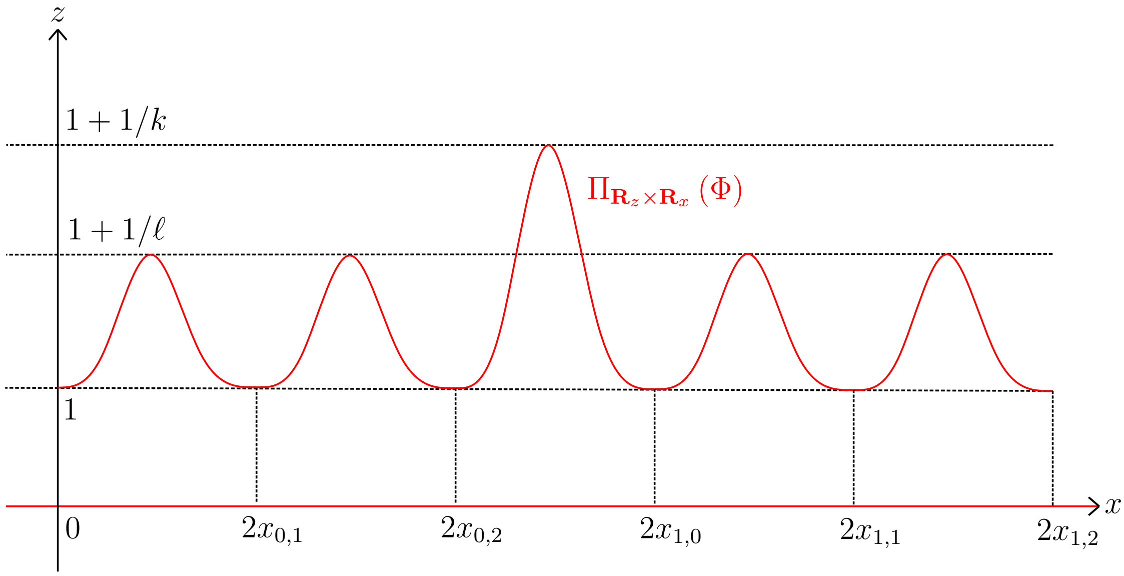

Figure 3.3: Front projection of Φ when $r=1$ and $s=2$

and

$$J(a,(x,y)) = \begin{cases} J^{\varepsilon_i}_{t_{i,j}}(a,(x-(2x_{i,j}-1),y)) & \text{if } j \neq 0 \\ J^{\delta}_{t_i}(a,(x-(2x_{i,0}-1),y)) & \text{if } j = 0. \end{cases}$$

Remark 3.31. Observe that Φ is smooth because we required all the derivatives of f_k to vanish at ± 1 in Section 3.2.2.

Example 3.32. Assume that $V=\mathbf{R}_z$, $M=\{q_-,q_+\}$, and $\phi_t=\phi : M \to V$ with

$$\phi(q_-) = 0 \text{ and } \phi(q_+) = 1.$$

Make the same choices as in Example 3.9. Then for every $(i,j)\in(\{0,\dots,r\}\times\{0,\dots,s\})\setminus\{0,0\}$, for every $x\in[2(x_{i,j}-1),2x_{i,j}]$,

$$\Phi(q_-,x) = (0,(x,0))$$

and

$$\Phi(q_+,x) = \begin{cases} (f_\ell(x),(x,f'_\ell(x))) & \text{if } j \neq 0 \\ (f_k(x),(x,f'_k(x))) & \text{if } j = 0. \end{cases}$$

See Figure 3.3.

Lemma 3.33. *Let (Φ, J) be as in Definition 3.30. We have*

$$\begin{aligned} CE_*(\Phi, J) = & CE_*\left(\Phi^{\ell,\varepsilon_0}_{t_{0,1}}, J^{\varepsilon_0}_{t_{0,1}}\right) \star \cdots \star CE_*\left(\Phi^{\ell,\varepsilon_0}_{t_{0,s}}, J^{\varepsilon_0}_{t_{0,s}}\right) \star CE_*\left(\Phi^{k,\delta}_{t_1}, J^{\delta}_{t_1}\right) \\ & \star CE_*\left(\Phi^{\ell,\varepsilon_1}_{t_{1,0}}, J^{\varepsilon_1}_{t_{1,0}}\right) \star \cdots \star CE_*\left(\Phi^{\ell,\varepsilon_1}_{t_{1,s}}, J^{\varepsilon_1}_{t_{1,s}}\right) \star CE_*\left(\Phi^{k,\delta}_{t_2}, J^{\delta}_{t_2}\right) \\ & \star \cdots \\ & \star CE_*\left(\Phi^{\ell,\varepsilon_{r-1}}_{t_{r-1,0}}, J^{\varepsilon_{r-1}}_{t_{r-1,0}}\right) \star \cdots \star CE_*\left(\Phi^{\ell,\varepsilon_{r-1}}_{t_{r-1,s}}, J^{\varepsilon_{r-1}}_{t_{r-1,s}}\right) \star CE_*\left(\Phi^{k,\delta}_{t_r}, J^{\delta}_{t_r}\right) \\ & \star CE_*\left(\Phi^{\ell,\varepsilon_r}_{t_{r,1}}, J^{\varepsilon_r}_{t_{r,1}}\right) \star \cdots \star CE_*\left(\Phi^{\ell,\varepsilon_r}_{t_{r,s}}, J^{\varepsilon_r}_{t_{r,s}}\right) \end{aligned}$$

where $\star$ always denotes the fibered coproduct (in the category of semi-free DG-algebras) over the common DG-subalgebra of both terms.

Proof. This follows from Lemmas 3.16 and 3.17.

□

We can now state the desired result.

Lemma 3.34. *Let (Φ, J) be as in Definition 3.30. There is a tame DG-isomorphism from $CE_{-*}(\Phi, J)$ to a stabilization of a mapping cylinder DG-algebra for φ which restricts to the identity on $CE_{-*}(\Lambda_0, J_0)$ and $CE_{-*}(\Lambda_1, J_1)$.*

Proof. Look at the decomposition of $CE_{-*}(\Phi, J)$ given by Lemma 3.33. According to Lemmas 3.29 and 3.26, each DG-algebra in this decomposition is either a mapping cylinder for a DG-map given by an identity map, or a quasi mapping cylinder for an elementary DG-isomorphism φ_i. Therefore, the result follows using Lemma 1.16 and Proposition 1.13.

□

3.2.5 Proof of the main result

It is easy to see that the DG-isomorphisms defined in Definition 3.25 satisfy the first item in Theorem 3.8. We now prove that they satisfy the second item.

Let Λ_0, Λ_1 be two chord generic Legendrian submanifolds, and let J_0, J_1 be two compatible almost complex structures on ξ. Let $\left(\Lambda^{\pm}_t, J^{\pm}_t\right)_{t\in[0,1]}$ be two generic good paths with

$$\left(\Lambda^{\pm}_0, J^{\pm}_0\right) = (\Lambda_0, J_0), \quad \left(\Lambda^{\pm}_1, J^{\pm}_1\right) = (\Lambda_1, J_1).$$

Let $t_1^- < \cdots < t_{r^-}^-$ and $t_1^+ < \cdots < t_{r^+}^+$ be the corresponding handle slide instants of Lemma 3.13. Adding non handle slide instants if necessary, we can assume that

$$r^- = r^+ = r \text{ and } \left(t_1^-, \ldots, t_r^-\right) = \left(t_1^+, \ldots, t_r^+\right) = (t_1, \ldots, t_r).$$

Choose $k \geq 1$ large enough and $\delta > 0$ small enough so that Lemmas 3.19, 3.20, 3.21 and 3.23 hold for both $\left(J_t^\pm\right)_{t\in[0,1]}$ at every instant t_i. For every $1 \leq i \leq r$, we denote by $\varphi_i^\pm : CE_{-*}\left(\Lambda_{t_i-\delta}^\pm, J_{t_i-\delta}^\pm\right) \xrightarrow{\sim} CE_{-*}\left(\Lambda_{t_i+\delta}^\pm, J_{t_i+\delta}^\pm\right)$ the DG-isomorphism of Lemma 3.24 (if t_i is not an handle slide instant of $\left(\Lambda_t^\pm, J_t^\pm\right)_{t\in[0,1]}$, then $\varphi_i^\pm$ is the identity map). The tame DG-isomorphisms $CE_{-*}(\Lambda_0, J_0) \to CE_{-*}(\Lambda_1, J_1)$ induced by $\left(\Lambda_t^\pm, J_t^\pm\right)_{t\in[0,1]}$ are $\varphi^\pm = \varphi_r^\pm \circ \cdots \circ \varphi_1^\pm$ (see Definition 3.25).

Now choose ℓ_0 so that for every $\ell \geq \ell_0$, there exists $\varepsilon_\ell > 0$ such that Lemmas 3.27 and 3.28 hold for every $\varepsilon \in [0, \varepsilon_\ell]$ for both $\left(\Lambda_t^\pm, J_t^\pm\right)_{t\in[0,1]}$. Fix $\ell \geq \ell_0$, and choose s_0 large enough so that

$$\frac{(t_{i+1} - \delta) - (t_i + \delta)}{2s} \leq \varepsilon$$

for every $s \geq s_0$ and $0 \leq i \leq r$. Now we fix $s \geq s_0$, and for every $0 \leq i \leq r$, $1 \leq j \leq s$ we set

$$\varepsilon_i = \frac{(t_{i+1} - \delta) - (t_i + \delta)}{2s}, \quad t_{i,j} := (t_i + \delta) + (2j - 1)\varepsilon_i$$

(by convention, $t_0 + \delta = 0$ and $t_{r+1} - \delta = 1$). We denote by $\Phi^\pm$, $J^\pm$ the Legendrian embeddings and almost complex structures of Definition 3.30.

Assume that there is a family $(\Lambda_{s,t})_{0\leq s,t\leq 1}$ such that for every s, t

$$\Lambda_{s,t} \in \mathcal{L}, \quad (\Lambda_{s,0}, \Lambda_{s,1}) = (\Lambda_0^\pm, \Lambda_1^\pm), \quad (\Lambda_{0,t}, \Lambda_{1,t}) = (\Lambda_t^-, \Lambda_t^+).$$

Since moreover the space of compatible almost complex structure is contractible, this implies that there exists a generic good path from (Φ^-, J^-) to (Φ^+, J^+) which is fixed on the extremities. According to the construction made in Section 3.2.3, we get a tame DG-isomorphism $CE_{-*}(\Phi^-, J^-) \to CE_{-*}(\Phi^+, J^+)$. Moreover, since the homotopy has fixed extremities, the DG-map $CE_{-*}(\Phi^-, J^-) \to CE_{-*}(\Phi^+, J^+)$ restricts to the identity on $CE_{-*}(\Lambda_0, J_0)$ and $CE_{-*}(\Lambda_1, J_1)$. We conclude the proof using Lemma 3.34 and Proposition 1.14.

Chapter 4

Legendrian lift of an exact Lagrangian in the circular contactization

In this chapter, we start with a compact, connected, exact Lagrangian submanifold L in a Liouville manifold (P, λ), and we study a Legendrian lift Λ° of L in the circular contactization $S^1 \times P$. For the standard contact form $(d\theta - \lambda)$, each point of Λ° gives rise to a (countable) infinite set of Reeb chords, and thus the situation is degenerate. In section 4.1, we explain how we perturb the contact form, and we state our central result, which computes invariants of Λ° in term of the Floer A_∞-algebra of L. In order to prove Theorem D, we apply a sequence of geometric modifications to the situation. Each of the section numbered from 4.2 to 4.5 explains one of these modifications and describes how the algebraic invariants change accordingly. Observe that we use Theorem 2.8 (Theorem A in the introduction) in section 4.2, and Theorem 3.8 (Theorem C in the introduction) in section 4.4. Finally, we end the proof of the main result in section 4.6, mainly by showing that we can apply Theorem 2.10 (Theorem B in the introduction).

4.1 Setting and Legendrian invariants

Let (P, λ) be a Liouville manifold, and let L be a compact, connected and exact Lagrangian submanifold of P. We fix $f : L \to \mathbf{R}$ such that

$$\lambda_{|L} = df.$$

We consider the manifold

$$V^\circ = S^1 \times P, \text{ where } S^1 = \mathbf{R}_\theta / \mathbf{Z},$$

with contact structure

$$\xi^\circ = \ker \alpha^\circ, \text{ where } \alpha^\circ = d\theta - \lambda,$$

and the Legendrian submanifold

$$\Lambda^\circ = \{(f(q), q) \mid q \in L\} \subset (V^\circ, \xi^\circ).$$

In order for the Floer complex of L and the Chekanov-Eliashberg algebra of Λ° to be $\mathbf{Z}$-graded, we assume that $H_1(P)$ is free, that the first Chern class of P (equipped with any almost complex structure compatible with $(-d\lambda)$) is 2-torsion, and that the Maslov class of L vanishes.

4.1.1 Reeb chords

Observe that Λ° is not chord generic for α° (see section 3.1). We will choose a compactly supported function $H : P \to \mathbf{R}$, and consider the perturbed contact form

$$\alpha_H^\circ = e^H \alpha^\circ.$$

The Reeb vector field of α_H° is then

$$R_{\alpha_H^\circ} = e^{-H} \begin{pmatrix} 1 + \lambda(X_H) \\ X_H \end{pmatrix},$$

where X_H is the unique vector field on P satisfying $\iota_{X_H} d\lambda = -dH$.

We fix a compact neighborhood K of L which is contained in a Weinstein neighborhood of L in P. It is not hard to see that for every positive integer N, the space of smooth functions H on P supported in K, such that the $R_{\alpha_H^\circ}$-chords of Λ° with action less than N are generic, is open and dense in $C_K^\infty(P)$. Therefore, the space of functions $H \in C_K^\infty(P)$ such that Λ° is chord generic with respect to α_H° is a Baire subset of $C_K^\infty(P)$. In particular, the latter is dense in $C_K^\infty(P)$. In the following, we choose $H \in C_K^\infty(P)$ such that

1. Λ° is chord generic with respect to α_H°,

2. H is sufficiently close to 0 so that

$$d\theta\left(R_{\alpha_H^\circ}\right) = e^{-H}(1 + \lambda(X_H)) \geq 1/2.$$

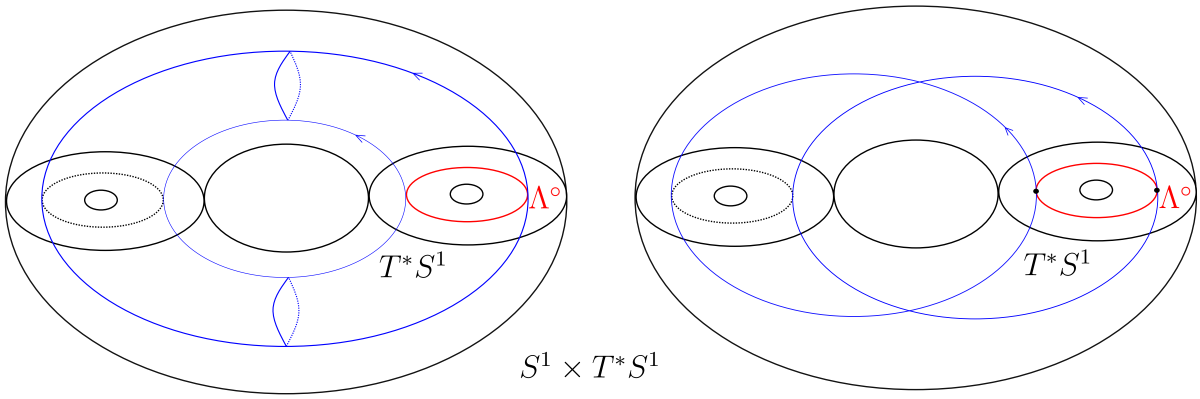

Figure 4.1: Reeb chords (in blue) of $\Lambda^\circ = \{0\} \times 0_{S^1}$ for α° (on the left) and for α_H° (on the right)

Example 4.1. Assume that we are in the case

$$(P, \lambda) = (T^*M, pdq)\,,\ L = 0_M,\ \text{and } H\,(q,p) = h(q),$$

where $h : M \to \mathbf{R}$ is a Morse function (we present this example in order to see what happens, even if H is not compactly supported in T^*M). The Reeb vector field of α_H° is

$$R_{\alpha_H^\circ} = e^{-h} \begin{pmatrix} 1 \\ 0 \\ -dh \end{pmatrix},$$

and therefore the Reeb flow satisfies

$$\varphi^t_{R_{\alpha_H^\circ}}\,(\theta, (q,p)) = \left(\theta + te^{-h(q)}, \left(q, p - te^{-h(q)} dh\,(q)\right)\right).$$

Thus, the $R_{\alpha_H^\circ}$-chords of Λ° are the paths $c : [0, T] \to S^1 \times T^*M$ of the form

$$c(t) = \left(te^{-h(q_0)}, (q_0, 0)\right),\ \text{with } Te^{-h(q_0)} \in \mathbf{Z}_{\geq 1} \text{ and } q_0 \in \operatorname{Crit} h.$$

Observe that these Reeb chords are transverse but lie on top of each others. See Figure 4.1, where we illustrate this perturbation when $M = S^1$.

4.1.2 Conley-Zehnder index

In order to define the Conley-Zehnder index (see section 3.1.1), we need to choose a family $(h_0, h_1, \dots, h_r)$ of embedded circles in $V^\circ = S^1 \times P$ which represent a basis of $H_1(V^\circ)$, and a symplectic trivialization of ξ° over each h_i. We let $h_0 = S^1 \times \{a_0\}$ be some fiber of $S^1 \times P \to P$, and we fix $(h_1, \dots, h_r)$ to be any family of embedded circles in P which represent a basis of $H_1(P)$. We choose a symplectic isomorphism $\psi : (T_{a_0}P, -d\lambda_{a_0}) \xrightarrow{\sim} (\mathbf{C}^n, dx \wedge dy)$, and then we choose the symplectic trivialization

$$\left(\xi^\circ_{|h_0}, d\alpha^\circ\right) \xrightarrow{\sim} (h_0 \times \mathbf{C}^n, dx \wedge dy), \quad ((\theta, a_0), (\lambda_{a_0}(v), v)) \mapsto \left((\theta, a_0), e^{2i\pi\theta}\psi(v)\right).$$

Finally, we choose some trivialization of ξ° over each h_i, $1 \le i \le r$.

Remark 4.2. We chose this trivialization of $h_0 = S^1 \times \{a_0\}$ because it is compatible with the one induced by the filling $D^2 \times P$ of $S^1 \times P$.

Example 4.3. We compute the Conley-Zehnder index of a Reeb chord in the case of example 4.1, i.e. when

$$(P, \lambda) = (T^*M, pdq), \ L = 0_M, \text{ and } H(q,p) = h(q),$$

where $h : M \to \mathbf{R}$ is a Morse function . In this case, the Reeb flow is given by

$$\varphi^t_{R_{\alpha^\circ_H}}(\theta, (q,p)) = \left(\theta + te^{-h(q)}, \left(q, p - te^{-h(q)}dh(q)\right)\right).$$

Let $c : [0,T] \to V^\circ$ be a Reeb chord of Λ°. Then there exists a positive integer k and a critical point q_0 of h such that

$$c(t) = \left(te^{-h(q_0)}, (q_0, 0)\right) \text{ and } Te^{-h(q_0)} = k.$$

Observe that $c(0) = c(T)$, and thus there is no need to choose a capping path for c. Besides, for every u in $T_{q_0}M$, we have

$$D\varphi^t_{R_{\alpha^\circ_H}}(c(0))(0, u, 0) = \left(0, u, -te^{-h(q_0)}D^2h(q_0)u\right).$$

In order to compute the index of c, we first choose coordinates $(x_1, \dots, x_n)$ around $q_0 \in M$

in which

$$h = h(q_0) + \frac{1}{2}\sum_{j=1}^{\dim(M)} \sigma_j x_j^2, \text{ where } \sigma_j = \pm 1,$$

and we extend it to symplectic coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ around $(q_0, 0) \in T^*M$ by setting

$$y_j(q,p) = \langle p, \frac{\partial}{\partial x_j}(q)\rangle.$$

Our choice of trivialization for a fiber of $S^1 \times P \to P$ induces the trivialization

$$e^{2i\pi kt/T}(dx + idy) : c^{-1}\xi^\circ \xrightarrow{\sim} (\mathbf{R}/T\mathbf{Z}) \times \mathbf{C}^n$$

(observe that $\xi^\circ_{c(t)} = \{0\} \times T_{(q_0,0)}(T^*M)$). Accordingly, the path $t \mapsto D\varphi^t_{R_{\alpha^\circ_H}}\left(T_{c(0)}\Lambda^\circ\right)$ induces a path of Lagrangians

$$\Gamma_c : t \in [0,T] \mapsto \left\{\left(e^{2i\pi kt/T}\left(u_j - ite^{-h(q_0)}\sigma_j u_j\right)\right)_{1\leq j\leq n} \mid u \in \mathbf{R}^n\right\} \subset \mathbf{C}^n.$$

We close this path using a counterclockwise rotation Γ, and call the resulting loop $\overline{\Gamma_c}$. In order to compute the Conley-Zehnder index of c, we have to look at how $\overline{\Gamma_c}$ intersects the Lagrangian $i\mathbf{R}^n$ (as explained in [28] section 2.2). Observe that Γ_c intersects $i\mathbf{R}^n$ positively $2k$ times, so that Γ_c contributes $2k$ to the Conley-Zehnder index of c. Moreover, since Γ is a counterclockwise rotation bringing

$$\left\{\left(u_j - iTe^{-h(q_0)}\sigma_j u_j\right)_{1\leq j\leq n} \mid u \in \mathbf{R}^n\right\} \text{ to } \mathbf{R}^n,$$

the contributions to the intersection between Γ and $i\mathbf{R}^n$ come from the negative eigenvalues σ_j. The computation done in [27] Lemma 3.4 implies that Γ contributes $\mathrm{ind}(q_0)$ to the Conley-Zehnder index of c. We conclude that the Conley-Zehnder index of c is

$$CZ(c) = \mu\left(\overline{\Gamma_c}\right) = 2k + \mathrm{ind}(q_0).$$

4.1.3 Main result

Legendrian invariants Let j be an almost complex structure on P compatible with $(-d\lambda)$, and let J° be its lift to a complex structure on ξ°. Recall from section 3.1.3 the definition of the Chekanov-Eliashberg DG-algebra of a Legendrian. In our situation, $CE_{-*}(\Lambda^\circ) = CE_{-*}(\Lambda^\circ, J^\circ, \alpha^\circ_H)$ is an Adams-graded DG-algebra (see Definition 1.1), where the Adams-degree of a Reeb chord c is the number of times c winds around the fiber. Besides, the algebra map $CE_{-*}(\Lambda^\circ) \to \mathbf{F}$ which sends every Reeb chord to zero defines an augmentation of $CE_{-*}(\Lambda^\circ)$. We denote by $LC_*(\Lambda^\circ)$ the corresponding coaugmented A_∞-coalgebra of Λ° (see the last paragraph of section 3.1.3). $LC_*(\Lambda^\circ)$ inherits an Adams-grading from $CE_{-*}(\Lambda^\circ)$ (the same), and we denote by $LA^*(\Lambda^\circ)$ its graded dual with respect to the bigrading (see Definition 1.37). In our situation, $LA^*(\Lambda^\circ)$ is an augmented Adams-graded A_∞-algebra (see Definition 1.17) whose augmentation ideal is generated by the Reeb chords of Λ° (and the Adams-degree of a Reeb chord c is the number of times c winds around the fiber).

Remark 4.4. In the case of example 4.1, the cohomological degree of a Reeb chord c corresponding to a positive integer k and a critical point q_0 is

$$1 - CZ(c) = 1 - 2k - \mathrm{ind}(q_0)$$

when viewed in $CE_{-*}(\Lambda^\circ)$, and $CZ(c) = 2k + \mathrm{ind}(q_0)$ when viewed in $LA^*(\Lambda^\circ)$ (see example 4.3).

Lagrangian invariants We denote by $CF^*(L)$ a finite-dimensional model for the Floer A_∞-algebra of L, as defined for example in [59]. We denote by $CF_{-*}(L)$ its graded dual. Observe that $CF_{-*}(L)$ in general *is not* an A_∞-coalgebra, but only a quasi-A_∞-coalgebra in the sense of Definition 1.23. As pointed out in [32] section 4, $CF_{-*}(L)$ is a true A_∞-coalgebra when L is simply connected.

Result Let $\mathbf{F}[t]$ be the augmented Adams-graded associative algebra generated by a variable t of bidegree $(2,1)$. We denote by

$$\overline{\mathbf{F}[t]} := \mathrm{Vect}_{\mathbf{F}}\{t^k \mid k \geq 1\}$$

its augmentation ideal (or equivalently, the ideal generated by t). Besides, we let $\mathbf{F}[t_\#]$ be the graded dual of $\mathbf{F}[t]$, which is a coaugmented Adams-graded associative coalgebra generated by a variable $t_\#$ of bidegree $(-2,1)$. Finally, we denote by $\overline{\mathbf{F}[t_\#]}$ its coaugmentation ideal.

If A is an augmented A_∞-algebra, then $\mathbf{F} \oplus \left(\overline{\mathbf{F}[t]} \otimes A\right)$ is naturally an augmented Adams-graded A_∞-algebra, where the Adams degree of $t^k \otimes x$ is defined to be k. Moreover, if C is a coaugmented quasi-A_∞-coalgebra, then $\mathbf{F}\oplus\left(\overline{\mathbf{F}\left[t_\#\right]}\otimes C\right)$ is a coaugmented Adams-graded A_∞-coalgebra (the finiteness condition is forced by the operations on $\overline{\mathbf{F}\left[t_\#\right]}$).

The rest of the section is devoted to the proof of the following result.

Theorem 4.5 (Theorem D in the Introduction)**.** *There is a quasi-isomorphism of augmented Adams-graded A_∞-algebras*

$$LA^*\left(\Lambda^\circ\right) \simeq \mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t\right]} \otimes CF^*\left(L\right)\right).$$

Moreover, Koszul duality holds for $CE_{-*}\left(\Lambda^\circ\right)$ (see Definition 1.42), and $E\left(CE_{-*}\left(\Lambda^\circ\right)\right) \simeq LA^*\left(\Lambda^\circ\right)$. In particular there are quasi-isomorphisms of Adams-graded DG-algebras

$$CE_{-*}\left(\Lambda^\circ\right) \simeq E\left(\mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t\right]} \otimes CF^*\left(L\right)\right)\right) \simeq \Omega\left(\mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t_\#\right]} \otimes CF_{-*}\left(L\right)\right)\right).$$

Remark 4.6. Let $f : C_1 \to C_2$ be a quasi-isomorphism between two quasi-A_∞-coalgebras (see Definition 1.25). It naturally induces a quasi-isomorphism of coaugmented Adams-graded A_∞-coalgebras

$$F : \mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t_\#\right]} \otimes C_1\right) \to \mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t_\#\right]} \otimes C_2\right)$$

defined as follows. Choose a basis $\mathcal{B}$ of C_2 and, if $x \in C_1$, write

$$f^d(x) = \sum_{y_1,\ldots,y_d\in\mathcal{B}} m\left(x; y_1, \ldots, y_d\right) y_1 \otimes \cdots \otimes y_d \in C_2^{\otimes d},$$

where $m\left(x; y_1, \ldots, y_d\right) \in \mathbf{F}$. Then

$$F^d\left(t_\#^k \otimes x\right) = \sum_{\substack{y_1,\ldots,y_d\in\mathcal{B}\\ j_1+\cdots+j_d=k}} m\left(x; y_1, \ldots, y_d\right)\left(t_\#^{j_1} \otimes y_1\right) \otimes \cdots \otimes \left(t_\#^{j_d} \otimes y_d\right).$$

Moreover, the coaugmented Adams-graded A_∞-coalgebras $\mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t_\#\right]} \otimes C_i\right)$ are Adams connected (see Definition 1.28). Thus, according to Remark 1.36, F induces a quasi-isomorphism

$$\Omega F : \Omega\left(\mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t_\#\right]} \otimes C_1\right)\right) \to \Omega\left(\mathbf{F} \oplus \left(\overline{\mathbf{F}\left[t_\#\right]} \otimes C_2\right)\right).$$

If B is a (unpointed) space, we consider its one-point compactification B^* and view it as a pointed space (with base point the point at infinity). If moreover X is a pointed space, we consider the half-smash product of B and X,

$$X \rtimes B := X \wedge B^*,$$

where $\wedge$ denotes the smash product of pointed spaces.

Corollary 4.7. *There is a quasi-isomorphism of augmented (non Adams-graded) A_∞-algebras*

$$LA^*(\Lambda^\circ) \simeq C^*(\mathbf{CP}^\infty \rtimes L),$$

and a quasi-isomorphism of augmented (non Adams-graded) DG-algebras

$$CE_*(\Lambda^\circ) \simeq C_*(\Omega(\mathbf{CP}^\infty \rtimes L)).$$

Proof. Let x_0 be the base point of $\mathbf{CP}^\infty$, and set $P := \mathbf{CP}^\infty \setminus \{x_0\}$. Observe that

$$(P \times L)^* = P^* \wedge L^* = \mathbf{CP}^\infty \wedge L^* = \mathbf{CP}^\infty \rtimes L.$$

On the one hand, we have

$$\mathbf{F} \oplus \left(\overline{\mathbf{F}[t]} \otimes CF^*(L)\right) \simeq \mathbf{F} \oplus \left(\overline{\mathbf{F}[t]} \otimes C^*(L)\right) \simeq C^*((P \times L)^*) \simeq C^*(\mathbf{CP}^\infty \rtimes L).$$

Thus, it follows from Theorem 4.5 that

$$LA^*(\Lambda^\circ) \simeq C^*(\mathbf{CP}^\infty \rtimes L).$$

On the other hand, we have

$$\mathbf{F} \oplus \left(\overline{\mathbf{F}[t_\#]} \otimes CF_{-*}(L)\right) \simeq \mathbf{F} \oplus \left(\overline{\mathbf{F}[t_\#]} \otimes C_{-*}(L)\right) \simeq C_{-*}((P \times L)^*) \simeq C_{-*}(\mathbf{CP}^\infty \rtimes L).$$

Moreover, $\mathbf{F} \oplus \left(\overline{\mathbf{F}[t_\#]} \otimes CF_{-*}(L)\right)$ and $C_{-*}(\mathbf{CP}^\infty \rtimes L)$ are locally finite and cohomologically simply connected (see Definition 1.28). Therefore, Theorem 4.5 and Proposition 1.33 imply that

$$CE_{-*}(\Lambda^\circ) \simeq \Omega(C_{-*}(\mathbf{CP}^\infty \rtimes L)).$$

Now, since $\mathbf{CP}^\infty \rtimes L$ is simply connected, Adams result (see [4], [5] and also [32]) yields

$$\Omega(C_{-*}(\mathbf{CP}^\infty \rtimes L)) \simeq C_{-*}(\Omega(\mathbf{CP}^\infty \rtimes L)).$$

□

4.2 Lift to $\mathbf{R} \times P$

In the following we will consider the manifold

$$V = \mathbf{R}_\theta \times P$$

with contact structure

$$\xi = \ker \alpha, \text{ where } \alpha = d\theta - \lambda,$$

and the Legendrian submanifold

$$\mathbf{\Lambda} = \bigsqcup_{n \in \mathbf{Z}} \Lambda^n, \text{ where } \Lambda^\theta = \{(f(q) + \theta, q) \mid q \in L\} \subset (V, \xi).$$

Recall from section 4.1.1 that we chose a compactly supported function $H : P \to \mathbf{R}$ such that

1. Λ° is chord generic with respect to α_H°,
2. H is sufficiently close to 0 so that

$$d\theta \left(R_{\alpha_H^\circ} \right) = e^{-H} \left(1 + \lambda \left(X_H\right)\right) \geq 1/2.$$

We consider the contact form

$$\alpha_H = e^H \alpha,$$

with Reeb vector field

$$R_{\alpha_H} = e^{-H} \begin{pmatrix} 1 + \lambda\left(X_H\right) \\ X_H \end{pmatrix}.$$

Moreover, we denote by J the lift of J° to an almost complex structure on ξ.

4.2.1 The A_∞-category $\mathcal{A}$

Definition 4.8. We consider the A_∞-category $\mathcal{A}$ defined as follows

1. the objects of $\mathcal{A}$ are the Legendrians Λ^n, $n \in \mathbf{Z}$,

2. the space of morphisms from Λ^i to Λ^j is either generated by the R_{α_H}-chords from Λ^i to Λ^j if $i < j$, or $\mathbf{F}$ if $i = j$, or 0 if $i > j$, and

3. the operations are such that $e_{\Lambda^n} \in \mathcal{A}(\Lambda^n, \Lambda^n)$ is a strict unit, and for every sequence of integers $i_0 < \cdots < i_d$, for every sequence of Reeb chords

$$(c_1, \ldots, c_d) \in \mathcal{R}\left(\Lambda^{i_0}, \Lambda^{i_1}\right) \times \cdots \times \mathcal{R}\left(\Lambda^{i_{d-1}}, \Lambda^{i_d}\right),$$

we have

$$\mu_{\mathcal{A}}(c_1, \ldots, c_d) = \sum_{a \in \mathcal{R}\left(\Lambda^{i_0}, \Lambda^{i_d}\right)} \# \mathcal{M}_{a, c_d \ldots c_1}(\mathbf{R} \times \mathbf{\Lambda}, J, \alpha_H)\, a$$

(see Definition 3.4 for the moduli spaces).

4.2.2 The quasi-autoequivalence τ

We introduce an A_∞-endofunctor of $\mathcal{A}$ that will be important in the following.

Definition 4.9. We denote by $\tau : \mathcal{A} \to \mathcal{A}$ the A_∞-functor defined as follows:

1. on objects, τ sends Λ^n to Λ^{n+1},

2. on morphisms, the map

$$\tau^d : \mathcal{A}\left(\Lambda^{i_0}, \Lambda^{i_1}\right) \otimes \cdots \otimes \mathcal{A}\left(\Lambda^{i_{d-1}}, \Lambda^{i_d}\right) \to \mathcal{A}\left(\Lambda^{i_0+1}, \Lambda^{i_d+1}\right)$$

is obtained by dualizing the components of the tame DG-isomorphism

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda^{n+1}, J, \alpha_H\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda^{n}, J, \alpha_H\right)$$

induced by the good path $\left(\bigsqcup_{n=i_0}^{i_d} \Lambda^{n+1-t}, J\right)_{0 \leq t \leq 1}$ (see Theorem 3.8).

Lemma 4.10. *The A_∞-functor $\tau : \mathcal{A} \to \mathcal{A}$ is a quasi-equivalence.*

Proof. Consider the A_∞-functor $\overline{\tau} : \mathcal{A} \to \mathcal{A}$ defined as follows:

1. on objects, $\overline{\tau}$ sends Λ^n to Λ^{n-1},

2. on morphisms, the map

$$\overline{\tau} : \mathcal{A}\left(\Lambda^{i_0}, \Lambda^{i_1}\right) \otimes \cdots \otimes \mathcal{A}\left(\Lambda^{i_{d-1}}, \Lambda^{i_d}\right) \to \mathcal{A}\left(\Lambda^{i_0-1}, \Lambda^{i_d-1}\right)$$

is obtained by dualizing the components of the *inverse* of the tame DG-isomorphism

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda^{n+1}, J, \alpha_H\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda^{n}, J, \alpha_H\right)$$

induced by the path $\left(\bigsqcup_{n=i_0}^{i_d} \Lambda^{n+1-t}, J\right)_{0 \leq t \leq 1}$.

Then $\tau \circ \overline{\tau} = \overline{\tau} \circ \tau = \mathrm{id}_{\mathcal{A}}$.

□

Here the $\mathbf{Z}$-splitting

$$\mathbf{Z} \xrightarrow{\sim} \mathrm{ob}\left(\mathcal{A}\right), \quad n \mapsto \Lambda^n$$

is compatible with the quasi-autoequivalence τ in the sense of Definition 2.5. As explained there, this turns $\mathcal{A}$ into an Adams-graded A_∞-category: the Adams-degree of a morphism $c \in \mathcal{A}\left(\Lambda^i, \Lambda^j\right)$ is $(j-i)$.

4.2.3 Relation between $LA^*\left(\Lambda^\circ\right)$ and $\left(\mathcal{A}, \tau\right)$

We now explain how $LA^*\left(\Lambda^\circ\right)$ and $\left(\mathcal{A}, \tau\right)$ are related. See Figure 4.2, where we illustrate the action of the projection $\Pi_{S^1 \times P}$ in the case

$$(P, \lambda) = \left(T^* S^1, pdq\right), \; L = 0_{S^1}, \text{ and } H\left(q, p\right) = h(q),$$

where $h : S^1 \to \mathbf{R}$ is a Morse function.

Lemma 4.11. *The A_∞-functor τ is strict, and it sends a Reeb chord $t \mapsto \left(\theta\left(t\right), x\left(t\right)\right)$ in $\mathcal{A}\left(\Lambda^i, \Lambda^j\right)$ to the Reeb chord $t \mapsto \left(\theta\left(t\right)+1, x\left(t\right)\right)$ in $\mathcal{A}\left(\Lambda^{i+1}, \Lambda^{j+1}\right)$. In particular, τ acts bijectively on hom-sets.*

Proof. Recall that $\alpha_H = e^H \alpha$, with H a function defined on the base manifold P. In particular, the flow $\varphi^t_{\partial_\theta}$ of ∂_θ is a strict contactomorphism of $\left(V, \alpha_H\right)$. Moreover, since J is the lift of an almost complex structure j on P, the map

$$u = \left(\sigma, v\right) \mapsto \left(\sigma, \varphi^t_{\partial_\theta} \circ v\right)$$

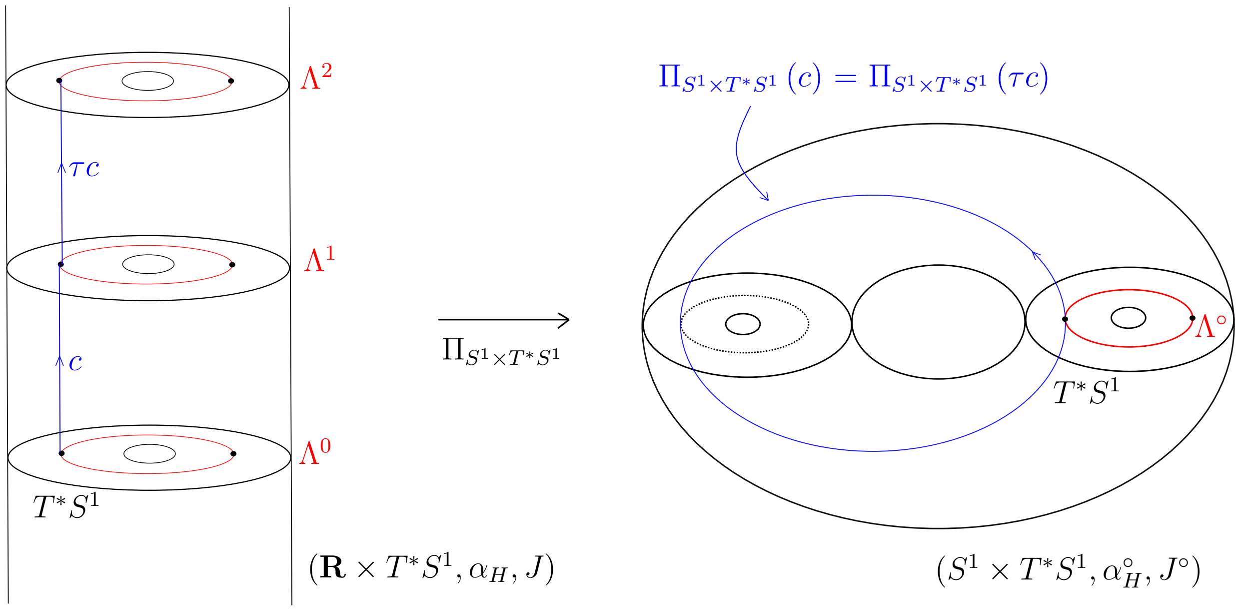

Figure 4.2: Action of the projection $\Pi_{S^1\times T^*S^1}$

induces a bijection

$$\mathcal{M}_{a,c_d\dots c_1}\left(\mathbf{R}\times\bigsqcup_{n=i_0}^{i_d}\Lambda^{n+1-t}, J, \alpha_H\right) \xrightarrow{\simeq} \mathcal{M}_{\varphi^t_{\partial_\theta}(a),\varphi^t_{\partial_\theta}(c_d)\dots\varphi^t_{\partial_\theta}(c_1)}\left(\mathbf{R}\times\bigsqcup_{n=i_0}^{i_d}\Lambda^{n+1}, J, \alpha_H\right)$$

for every t. Thus, there is no handle slide moment to consider for the isomorphism of Theorem 3.8. This implies the result.

□

We denote by $\mathcal{A}_\tau$ the Adams-graded A_∞-category associated to τ as in Definition 2.7. Observe that here $\mathcal{A}_\tau$ has only one object, and thus can be seen as an Adams-graded A_∞-algebra.

Lemma 4.12. *There is a quasi-isomorphism of Adams-graded A_∞-algebras*

$$LA^*(\Lambda^\circ) \simeq \mathcal{A}_\tau.$$

Proof. Consider the map which sends a Reeb chord $c \in \mathcal{R}(\Lambda^i, \Lambda^j)$ to the corresponding chord $\Pi_{S^1\times P}(c) \in \mathcal{R}(\Lambda^\circ)$ (where $\Pi_{S^1\times P} : \mathbf{R}\times P \to S^1 \times P$ is the projection). According to Lemma 4.11, $\Pi_{S^1\times P}(\tau c) = \Pi_{S^1\times P}(c)$, and thus the map $c \mapsto \Pi_{S^1\times P}(c)$ induces a linear map $\psi : \mathcal{A}_\tau \to LA^*(\Lambda^\circ)$. Moreover, observe that ψ is a bijection. It remains to prove that ψ is an A_∞-map. This follows from the fact that the map

$$u = (\sigma, v) \mapsto (\sigma, \Pi_{S^1 \times P} \circ v)$$

induces a bijection

$$\mathcal{M}_{a,c_d \dots c_1}(\mathbf{R} \times \mathbf{\Lambda}, J, \alpha_H) \xrightarrow{\sim} \mathcal{M}_{\psi(a),\psi(c_d)\dots\psi(c_1)}(\mathbf{R} \times \Lambda^\circ, J^\circ, \alpha_H^\circ).$$

□

Lemma 4.13. *The Adams-graded A_∞-algebra $LA^*(\Lambda^\circ)$ is quasi-equivalent to the mapping torus of $\tau : \mathcal{A} \to \mathcal{A}$ (see Definition 2.1).*

Proof. This follows directly from Theorem 2.8 using Lemmas 4.11 and 4.12.

□

4.3 Rectification of the contact form

Now that we are in the usual contactization, we have the following result.

Lemma 4.14. *There exists a contactomorphism ϕ_H of (V, ξ) such that*

$$\phi_H^* \alpha_H = \alpha.$$

Proof. Recall that $\alpha_H = e^H \alpha$, with H a compactly supported function on the base manifold P such that $e^{-H}(1 + \lambda(X_H)) \geq 1/2$.

Assume that there is a contact isotopy $(\phi_t)_{0 \leq t \leq 1}$ such that $\phi_0 = \mathrm{id}$ and

$$\phi_t^* \alpha_{tH} = \alpha \tag{4.1}$$

for every t. Let $(F_t)_t$ be the family of functions on V such that

$$\frac{d}{dt}\phi_t = Y_{F_t} \circ \phi_t,$$

where Y_F is the contact vector field on V satisfying

$$\begin{cases} \alpha(Y_F) &= F \\ \iota_{Y_F} d\alpha &= dF(R_\alpha)\alpha - dF. \end{cases}$$

Taking the derivative of equation (4.1) with respect to t, we get

$$H + d\left(e^{tH} F_t\right)\left(R_{\alpha_{tH}}\right) = 0 \tag{4.2}$$

because Y_F satisfies

$$\begin{cases} \alpha_{tH}\left(Y_F\right) & = & e^{tH} F \\ \iota_{Y_F} d\alpha_{tH} & = & d\left(e^{tH} F\right)\left(R_{\alpha_{tH}}\right)\alpha_{tH} - d\left(e^{tH} F\right). \end{cases}$$

Besides, we deduce from

$$R_{\alpha_{tH}} = e^{-tH}\left(\begin{array}{c} 1 + t\lambda\left(X_H\right) \\ tX_H \end{array}\right), \quad \iota_{X_H} d\lambda = -dH,$$

that

$$dH\left(R_{\alpha_{tH}}\right) = 0.$$

Then equation (4.2) gives

$$dF_t\left(R_{\alpha_{tH}}\right) = -He^{-tH}. \tag{4.3}$$

Conversely, if $\left(F_t\right)_t$ is a family of functions on V satisfying equation (4.3), then the contact isotopy $\left(\phi_t\right)_t$ defined by

$$\phi_0 = \mathrm{id} \text{ and } \frac{d}{dt}\phi_t = Y_{F_t} \circ \phi_t,$$

satisfies

$$\frac{d}{dt}\left(\phi_t^* \alpha_{tH}\right) = 0,$$

and thus $\phi_H := \phi_1$ gives the desired result.

Therefore, it remains to find a family $\left(F_t\right)_t$ satisfying equation (4.3). First recall that

$$R_{\alpha_{tH}} = e^{-tH}\left(\begin{array}{c} 1 + t\lambda\left(X_H\right) \\ tX_H \end{array}\right).$$

By assumption on H, the function $d\theta\left(R_{\alpha_{tH}}\right)$ is greater than $1/2$ for every $t \in [0,1]$. Thus, for every $t \in [0,1]$ and every (θ, x) in V, there exists a unique real number $\rho_t(\theta, x)$ such that

$$\varphi_{R_{\alpha_{tH}}}^{-\rho_t(\theta,x)}(\theta, x) \in \{0\} \times P.$$

Then we let
$$F_t := -\rho_t H e^{-tH}.$$
For every real number t, we have
$$F_t \circ \varphi^t_{R_{\alpha_{tH}}} = -\left(\rho_t \circ \varphi^t_{R_{\alpha_{tH}}}\right) H e^{-tH} \text{ because } dH\left(R_{\alpha_{tH}}\right) = 0.$$
But the map $\varphi_{R_{\alpha_{tH}}}^{-\rho_t \circ \varphi^t_{R_{\alpha_{tH}}}+t}$ takes its values in $\{0\} \times P$ by definition of ρ_t, so by uniqueness we have
$$\rho_t \circ \varphi^t_{R_{\alpha_{tH}}} = \rho_t + t.$$
Then we have
$$F_t \circ \varphi^t_{R_{\alpha_{tH}}} = -\left(\rho_t + t\right) H e^{-tH},$$
and thus
$$dF_t\left(R_{\alpha_{tH}}\right) = -He^{-tH}.$$
This concludes the proof.

$\square$

Example 4.15. Assume that we are in the case
$$(P, \lambda) = (T^*M, pdq)\,,\ L = 0_M,\ \text{ and } H\left(q,p\right) = h(q),$$
where $h : M \to \mathbf{R}$ is a Morse function. Then the diffeomorphism ϕ_H defined by
$$\phi_H^{-1}\left(\theta, (q,p)\right) = \left(\theta e^{h(q)}, \left(q, e^{h(q)}p + \theta e^{h(q)} dh(q)\right)\right)$$
satisfies $\phi_H^* \alpha_H = \alpha$. With this choice of ϕ_H, we have in particular
$$\phi_H^{-1}\left(\{\theta\} \times 0_M\right) = j^1\left(\theta e^h\right) \subset \mathbf{R} \times T^*M.$$

4.3.1 The A_∞-category $\mathcal{A}_1$

In the following, we fix a contactomorphism ϕ_H as in Lemma 4.14. We define an A_∞-category $\mathcal{A}_1$, which is roughly obtained by pulling back the data of $\mathcal{A}$ by ϕ_H.

Definition 4.16. Let

$$\Lambda_H^\theta := \phi_H^{-1}\left(\Lambda^\theta\right), \; \mathbf{\Lambda}_H := \bigsqcup_{n\in\mathbf{Z}} \Lambda_H^n, \text{ and } J_H := \phi_H^* J.$$

We consider the A_∞-category $\mathcal{A}_1$ defined as follows

1. the objects of $\mathcal{A}_1$ are the Legendrians Λ_H^n, $n \in \mathbf{Z}$,
2. the vector space $\mathcal{A}_1\left(\Lambda_H^i, \Lambda_H^j\right)$ is either generated by the R_α-chords from Λ_H^i to Λ_H^j if $i < j$, or $\mathbf{F}$ if $i = j$, or 0 if $i > j$, and
3. the operations are such that $e_{\Lambda_H^n} \in \mathcal{A}_1\left(\Lambda_H^n, \Lambda_H^n\right)$ is a strict unit, and for every sequence of integers $i_0 < \cdots < i_d$, for every sequence of Reeb chords

$$(c_1, \ldots, c_d) \in \mathcal{R}\left(\Lambda_H^{i_0}, \Lambda_H^{i_1}\right) \times \cdots \times \mathcal{R}\left(\Lambda_H^{i_{d-1}}, \Lambda_H^{i_d}\right)$$

we have

$$\mu_{\mathcal{A}_1}(c_1, \ldots, c_d) = \sum_{a\in\mathcal{R}\left(\Lambda_H^{i_0}, \Lambda_H^{i_d}\right)} \#\mathcal{M}_{a, c_d\cdots c_1}\left(\mathbf{R}\times\mathbf{\Lambda}_H, J_H, \alpha\right) a$$

(see Definition 3.4 for the moduli spaces).

4.3.2 The quasi-autoequivalence τ_1

Definition 4.17. We denote by $\tau_1 : \mathcal{A}_1 \to \mathcal{A}_1$ the A_∞-functor defined as follows:

1. on the objects, $\tau_1\left(\Lambda_H^n\right) = \Lambda_H^{n+1}$,
2. on the morphisms, the map

$$\tau_1^d : \mathcal{A}_1\left(\Lambda_H^{i_0}, \Lambda_H^{i_1}\right) \otimes \cdots \otimes \mathcal{A}_1\left(\Lambda_H^{i_{d-1}}, \Lambda_H^{i_d}\right) \to \mathcal{A}_1\left(\Lambda_H^{i_0+1}, \Lambda_H^{i_d+1}\right)$$

is obtained by dualizing the components of the tame DG-isomorphism

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1}, J_H, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n}, J_H, \alpha\right)$$

induced by the good path $\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1-t}, J_H\right)_{0\leq t\leq 1}$ (see Theorem 3.8).

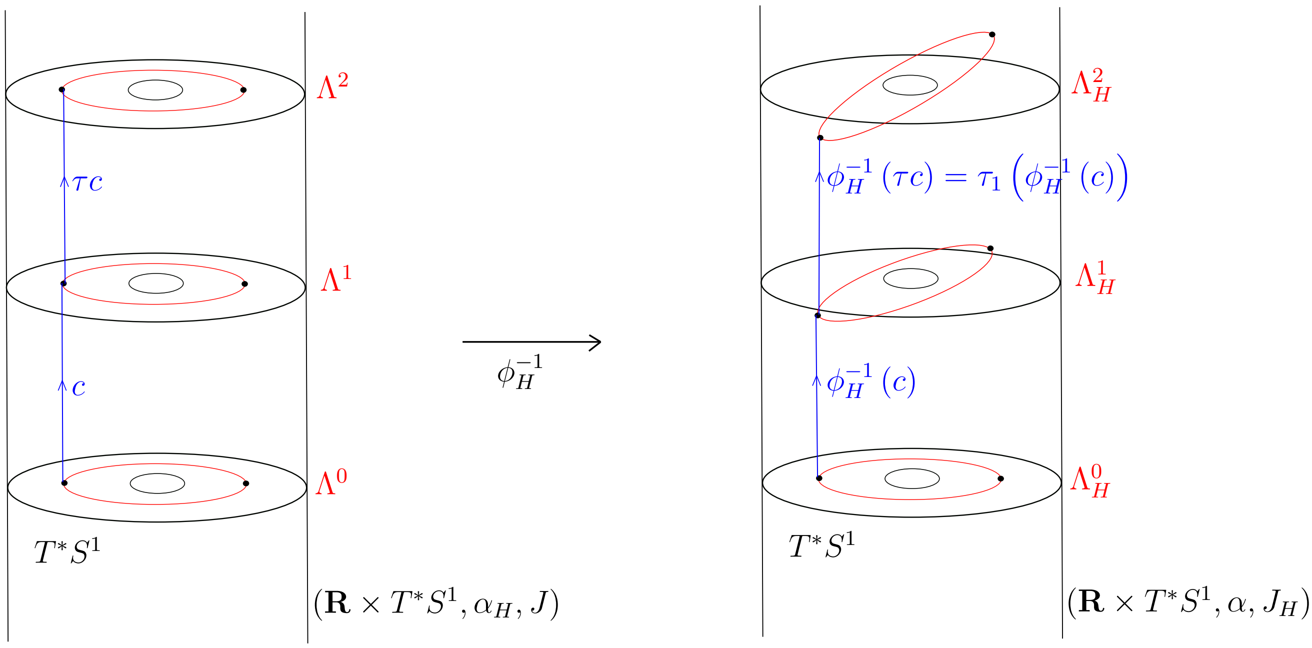

Figure 4.3: Action of the contactomorphism ϕ_H^{-1}

Lemma 4.18. *The A_∞-functor $\tau_1 : \mathcal{A}_1 \to \mathcal{A}_1$ is a quasi-equivalence.*

Proof. This follows because τ_1 is defined by dualizing the components of a DG-isomorphism (see the proof of Lemma 4.10).

□

Here the $\mathbf{Z}$-splitting

$$\mathbf{Z} \xrightarrow{\sim} \mathrm{ob}\left(\mathcal{A}_1\right), \quad n \mapsto \Lambda_H^n,$$

is compatible with the quasi-autoequivalence τ_1 in the sense of Definition 2.5. As explained there, this turns $\mathcal{A}$ into an Adams-graded A_∞-category.

4.3.3 Relation between $(\mathcal{A}, \tau)$ and $(\mathcal{A}_1, \tau_1)$

We now explain how the pairs $(\mathcal{A}, \tau)$ and $(\mathcal{A}_1, \tau_1)$ are related. See Figure 4.3 where we illustrate the action of the contactomorphism ϕ_H^{-1} in the case

$$(P, \lambda) = \left(T^*S^1, pdq\right), \ L = 0_{S^1}, \text{ and } H(q,p) = h(q),$$

where $h : S^1 \to \mathbf{R}$ is a Morse function.

Lemma 4.19. *There is a strict A_∞-isomorphism $\zeta_1 : \mathcal{A} \to \mathcal{A}_1$ defined as follows:*

1. *on the objects, $\zeta_1(\Lambda^n) = \Lambda^n_H$,*

2. *on the morphisms, ζ_1 sends a Reeb chord c in $\mathcal{A}(\Lambda^i, \Lambda^j)$ to the Reeb chord*

$$\zeta_1(c) = \phi_H^{-1} \circ c$$

in $\mathcal{A}_1\left(\Lambda^i_H, \Lambda^j_H\right)$.

Proof. This follows from the fact that the map

$$u = (\sigma, v) \mapsto \left(\sigma, \phi_H^{-1} \circ v\right)$$

induces a bijection

$$\mathcal{M}_{a,c_d\dots c_1}(\mathbf{R} \times \mathbf{\Lambda}, J, \alpha_H) \xrightarrow{\sim} \mathcal{M}_{\phi_H^{-1}(a),\phi_H^{-1}(c_d)\dots\phi_H^{-1}(c_1)}(\mathbf{R} \times \mathbf{\Lambda}_H, J_H, \alpha).$$

□

Lemma 4.20. *We have*

$$\tau_1 = \zeta_1 \circ \tau \circ \zeta_1^{-1}.$$

Proof. This follows from the fact that the map

$$u = (\sigma, v) \mapsto \left(\sigma, \phi_H^{-1} \circ v\right)$$

induces a bijection

$$\mathcal{M}_{a,c_d\dots c_1}\left(\mathbf{R} \times \bigsqcup_{n=i_0}^{i_d} \Lambda^{n+1-t}, J, \alpha_H\right) \xrightarrow{\sim} \mathcal{M}_{\phi_H^{-1}(a),\phi_H^{-1}(c_d)\dots\phi_H^{-1}(c_1)}\left(\mathbf{R} \times \bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1-t}, J_H, \alpha\right)$$

for every $t \in [0, 1]$.

□

Lemma 4.21. *The mapping torus of $\tau : \mathcal{A} \to \mathcal{A}$ is quasi-equivalent to the mapping torus of $\tau_1 : \mathcal{A}_1 \to \mathcal{A}_1$ (see Definition 2.1).*

Proof. According to Lemma 4.20 the following diagram of Adams-graded A_∞-categories is commutative

$$\begin{tikzcd}
\mathcal{A} \arrow[d, "\zeta_1"] & \mathcal{A} \sqcup \mathcal{A} \arrow[l, "\mathrm{id}\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup\tau"] \arrow[d, "\zeta_1\sqcup\zeta_1"] & \mathcal{A} \arrow[d, "\zeta_1"] \\
\mathcal{A}_1 & \mathcal{A}_1 \sqcup \mathcal{A}_1 \arrow[l, "\mathrm{id}\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup\tau_1"] & \mathcal{A}_1.
\end{tikzcd}$$

Moreover, each vertical arrow is a quasi-equivalence according to Lemma 4.19. Thus the result follows from Proposition 1.85.

□

4.4 Back to the original almost complex structure

In this section, we introduce an A_∞-category $\mathcal{A}_2$ which has same objects and morphisms as $\mathcal{A}_1$, but whose operations count punctured discs in $\mathbf{R} \times V$ which are pseudo-holomorphic for the almost complex structure induced by α and J (instead of J_H).

4.4.1 The A_∞-category $\mathcal{A}_2$

Recall that we chose a contactomorphism ϕ_H as in Lemma 4.14, and recall that

$$\Lambda_H^\theta = \phi_H^{-1}\left(\Lambda^\theta\right), \quad \mathbf{\Lambda}_H = \bigsqcup_{n\in\mathbf{Z}} \Lambda_H^n.$$

Definition 4.22. We consider the A_∞-category $\mathcal{A}_2$ defined as follows

1. the objects of $\mathcal{A}_2$ are the Legendrians Λ_H^n, $n \in \mathbf{Z}$,

2. the vector space $\mathcal{A}_2\left(\Lambda_H^i, \Lambda_H^j\right)$ is either generated by the R_α-chords from Λ_H^i to Λ_H^j if $i < j$, or $\mathbf{F}$ if $i = j$, or 0 if $i > j$, and

3. the operations are such that $e_{\Lambda_H^n} \in \mathcal{A}_2\left(\Lambda_H^n, \Lambda_H^n\right)$ is a strict unit, and for every sequence of integers $i_0 < \cdots < i_d$, for every sequence of Reeb chords

$$(c_1, \ldots, c_d) \in \mathcal{R}\left(\Lambda_H^{i_0}, \Lambda_H^{i_1}\right) \times \cdots \times \mathcal{R}\left(\Lambda_H^{i_{d-1}}, \Lambda_H^{i_d}\right)$$

   we have

$$\mu_{\mathcal{A}_2}(c_1, \ldots, c_d) = \sum_{a\in\mathcal{R}\left(\Lambda_H^{i_0},\Lambda_H^{i_d}\right)} \#\mathcal{M}_{a,c_d\ldots c_1}\left(\mathbf{R}\times\mathbf{\Lambda}_H, J, \alpha\right) a.$$

4.4.2 The quasi-autoequivalence τ_2

Definition 4.23. We denote by $\tau_2 : \mathcal{A}_1 \to \mathcal{A}_1$ the A_∞-functor defined as follows:

1. on the objects, $\tau_2 \left(\Lambda_H^n\right) = \Lambda_H^{n+1}$,
2. on the morphisms, the map

$$\tau_2^d : \mathcal{A}_1 \left(\Lambda_H^{i_0}, \Lambda_H^{i_1}\right) \otimes \cdots \otimes \mathcal{A}_1 \left(\Lambda_H^{i_{d-1}}, \Lambda_H^{i_d}\right) \to \mathcal{A}_1 \left(\Lambda_H^{i_0+1}, \Lambda_H^{i_d+1}\right)$$

is obtained by dualizing the components of the tame DG-isomorphism

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1}, J, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n}, J, \alpha\right)$$

induced by the good path $\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1-t}, J\right)_{0\leq t\leq 1}$ (see Theorem 3.8 or [26] Proposition 2.6).

Lemma 4.24. *The A_∞-functor $\tau_2 : \mathcal{A}_2 \to \mathcal{A}_2$ is a quasi-equivalence.*

Proof. This follows from the fact that τ_2 is defined by dualizing the components of a DG-isomorphism (see the proof of Lemma 4.10).

□

Here the $\mathbf{Z}$-splitting

$$\mathbf{Z} \xrightarrow{\sim} \mathrm{ob}\left(\mathcal{A}_2\right), \quad n \mapsto \Lambda_H^n$$

is compatible with the quasi-autoequivalence τ_2 in the sense of Definition 2.5. As explained there, this turns $\mathcal{A}_2$ into an Adams-graded A_∞-category.

4.4.3 Relation between $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$

Lemma 4.25. *Choose a generic path $(J_t^{12})_{0\leq t\leq 1}$ such that $J_0^{12} = J$ and $J_1^{12} = J_H$. There is an A_∞-isomorphism $\zeta_{12} : \mathcal{A}_1 \to \mathcal{A}_2$ defined as follows*

1. *on the objects, $\zeta_{12}\left(\Lambda_H^n\right) = \Lambda_H^n$,*
2. *on the morphisms, the map*

$$\zeta_{12} : \mathcal{A}_1 \left(\Lambda_H^{i_0}, \Lambda_H^{i_1}\right) \otimes \cdots \otimes \mathcal{A}_1 \left(\Lambda_H^{i_{d-1}}, \Lambda_H^{i_d}\right) \to \mathcal{A}_2 \left(\Lambda_H^{i_0}, \Lambda_H^{i_d}\right)$$

is obtained by dualizing the components of the tame DG-isomorphism

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J_H, \alpha\right)$$

induced by the good path $(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J_t^{12})_{0\leq t\leq 1}$ *(see Theorem 3.8).*

Proof. We have to prove that ζ_{12} is an isomorphism. This follows from the fact that it is defined by dualizing the components of a DG-isomorphism (see the proof of Lemma 4.10).

□

Lemma 4.26. *The A_∞-functor τ_2 is homotopic to $\zeta_{12} \circ \tau_1 \circ \zeta_{12}^{-1}$ (see Definition 1.52).*

Proof. First recall that τ_1 is obtained by dualizing the components of the DG-map

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1}, J_H, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J_H, \alpha\right)$$

induced by the path $\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1-t}, J_H\right)_{0\leq t\leq 1}$. Thus, $\zeta_{12} \circ \tau_1$ is obtained by dualizing the components of the composition

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1}, J, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1}, J_H, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J_H, \alpha\right).$$

On the other hand, τ_2 is obtained by dualizing the components of the DG-map

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1}, J, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J, \alpha\right)$$

induced by the path $\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1-t}, J\right)_{0\leq t\leq 1}$. Thus, $\tau_2 \circ \zeta_{12}$ is obtained by dualizing the components of the composition

$$CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^{n+1}, J, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J, \alpha\right) \to CE_{-*}\left(\bigsqcup_{n=i_0}^{i_d} \Lambda_H^n, J_H, \alpha\right).$$

According to Theorem 3.8, the DG-maps used to define $\zeta_{12} \circ \tau_1$ and $\tau_2 \circ \zeta_{12}$ are DG-homotopic. This implies that the A_∞-functors $\zeta_{12} \circ \tau_1$ and $\tau_2 \circ \zeta_{12}$ are homotopic in the sense of Definition 1.52. This implies the result.

$\square$

Lemma 4.27. *The mapping torus of $\tau_1 : \mathcal{A}_1 \to \mathcal{A}_1$ is quasi-equivalent to the mapping torus of $\tau_2 : \mathcal{A}_2 \to \mathcal{A}_2$ (see Definition 2.1).*

Proof. Let $\tau_{12} := \zeta_{12} \circ \tau_1 \circ \zeta_{12}^{-1}$. Consider the following commutative diagram of Adams-graded A_∞-categories

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "\zeta_{12}"] & \mathcal{A}_1 \sqcup \mathcal{A}_1 \arrow[l, "\mathrm{id}\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup\tau_1"] \arrow[d, "\zeta_{12}\sqcup\zeta_{12}"] & \mathcal{A}_1 \arrow[d, "\zeta_{12}"] \\
\mathcal{A}_2 & \mathcal{A}_2 \sqcup \mathcal{A}_2 \arrow[l, "\mathrm{id}\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup\tau_{12}"] & \mathcal{A}_2.
\end{tikzcd}$$

Each vertical arrow is a quasi-equivalence according to Lemma 4.25, so it follows from Proposition 1.85 that the mapping torus of τ_1 is quasi-equivalent to the mapping torus of τ_{12}. Now according to Lemma 4.26, τ_{12} is homotopic to τ_2. Thus the result follows from Proposition 1.86.

$\square$

4.5 Projection to P

4.5.1 The A_∞-category $\mathcal{O}$

In order to define the A_∞-category $\mathcal{O}$, we need to introduce moduli spaces of pseudo-holomorphic discs in P. We fix a choice of strip-like ends for each Riemann disk (see Definition 3.3).

Definition 4.28. Let $\mathbf{L} = (L^n)_{n\in\mathbf{Z}}$ be a family of pairwise transverse exact Lagrangian submanifolds in P. Consider a sequence of integers $i_0 < \cdots < i_d$, and a family of intersection points $(x_0, x_1, \ldots, x_d)$, where

$$x_0 \in L^{i_0} \cap L^{i_d} \text{ and } x_k \in L^{i_{k-1}} \cap L^{i_k},\ 1 \leq k \leq d.$$

We denote by $\mathcal{M}_{x_0,x_d\cdots x_1}(\mathbf{L}, j)$ the set of equivalence classes of tuples $(D, i, \boldsymbol{\zeta}, u)$ such that

1. $(D, i, \boldsymbol{\zeta})$ is a Riemann disk with $(d+1)$ marked points and u is a smooth map from $\Delta = D \setminus \{\zeta_0, \ldots, \zeta_d\}$ to P which maps the boundary arc (ζ_{k-1}, ζ_k) of Δ to $L^{i_{d-k+1}}$,

2. u is pseudo-holomorphic with respect to (i, j),

3. with the choice of strip-like ends $\epsilon_0, (\epsilon_k)_{1\leq k\leq d}$ for $(D, i, \boldsymbol{\zeta})$ we have

$$(u \circ \epsilon_0)(s,t) \underset{s\to+\infty}{\longrightarrow} x_0 \text{ and } (u \circ \epsilon_k)(s,t) \underset{s\to-\infty}{\longrightarrow} x_{d-k+1},$$

where two tuples $(D, i, \boldsymbol{\zeta}, u)$, and $(D', i', \boldsymbol{\zeta'}, u')$ as above are said to be equivalent if there exists an isomorphism $\psi : (D, i, \boldsymbol{\zeta}) \to (D', i', \boldsymbol{\zeta'})$ such that $u \circ \psi^{-1} = u'$.

Recall that we chose a contactomorphism ϕ_H as in Lemma 4.14. We set

$$L^n_H := \Pi_P(\Lambda^n_H) \subset P \text{ and } \mathbf{L}_H := (L^n_H)_{n\in\mathbf{Z}}.$$

Definition 4.29. We denote by $\mathcal{O}$ the A_∞-category defined as follows:

1. the objects of $\mathcal{O}$ are the Lagrangians L^n_H, $n \in \mathbf{Z}$,

2. the space of morphisms from L^i_H to L^j_H is either generated by $L^i_H \cap L^j_H$ if $i < j$, or $\mathbf{F}$ if $i = j$, or 0 if $i > j$, and

3. the operations are such that $e_{L^n_H} \in \mathcal{O}(L^n_H, L^n_H)$ is a strict unit, and for every sequence of integers $i_0 < \cdots < i_d$, for every sequence of intersection points

$$(x_1, \ldots, x_d) \in \left(L^{i_0}_H \cap L^{i_1}_H\right) \times \cdots \times \left(L^{i_{d-1}}_H \cap L^{i_d}_H\right),$$

we have

$$\mu_{\mathcal{O}}(x_1, \ldots, x_d) = \sum_{x_0 \in L^{i_0}_H \cap L^{i_d}_H} \#\mathcal{M}_{x_0, x_d \ldots x_1}(\mathbf{L}_H, j)\, x_0.$$

4.5.2 The quasi-autoequivalence γ

In order to define the A_∞-functor $\gamma : \mathcal{O} \to \mathcal{O}$, we use Legendrian contact homology as defined in [26]. To each generic Legendrian Λ in $\mathbf{R} \times P$, the authors associate a semi-free DG-algebra $A = A(\Lambda, j)$ generated by the self-intersection points of $\Pi_P(\Lambda)$, with a differential $\partial : A \to A$ defined using j-holomorphic discs in P. In our case, the differential of $A\left(\bigsqcup_k \Lambda^k_H, j\right)$ on a generator $x_0 \in L^{i_0}_H \cap L^{i_d}_H$ is given by

$$\partial x_0 = \sum_{(x_1,\ldots,x_d)} \#\mathcal{M}_{x_0, x_d \cdots x_1}(\mathbf{L}_H, j)\, x_d \cdots x_1$$

where the sum is over the sequences

$$(x_1, \ldots, x_d) \in \left(L_H^{i_0} \cap L_H^{i_1}\right) \times \cdots \times \left(L_H^{i_{d-1}} \cap L_H^{i_d}\right).$$

Moreover, for each generic Legendrian isotopy $(\Lambda_t)_{0 \leq t \leq 1}$ such that Λ_t is chord generic for every t, there is a tame DG-isomorphism

$$A(\Lambda_0, j) \xrightarrow{\sim} A(\Lambda_1, j)$$

defined by studying handle slide instants in a family of moduli spaces. In our case, the tame DG-isomorphism

$$A\left(\bigsqcup_k \Lambda_H^{k+1}, j\right) \to A\left(\bigsqcup_k \Lambda_H^k, j\right)$$

induced by the Legendrian isotopy $\left(\bigsqcup_k \Lambda_H^{k+1-t}\right)_{0 \leq t \leq 1}$ is defined by studying handle slide instants in the family

$$\mathcal{M}_{x_0, x_d \ldots x_1}\left(\left(L_H^{k+1-t}\right)_k, j\right), \quad 0 \leq t \leq 1.$$

Remark 4.30. According to [23] Theorem 2.1, Legendrian contact homology as defined in [26] coincides with the version exposed in chapter 3. Here we introduce the framework of [26] for two reasons: the first one is that it makes clearer the fact that the A_∞-functor $\gamma : \mathcal{O} \to \mathcal{O}$ is defined using pseudo-holomorphic discs in the base symplectic manifold P, and the second one is that we will use later analytical results from [29] which are formulated in this setting.

Definition 4.31. We denote by $\gamma : \mathcal{O} \to \mathcal{O}$ the A_∞-functor defined as follows

1. on the objects, $\gamma(L_H^n) = L_H^{n+1}$,

2. on the morphisms, the map

$$\gamma : \mathcal{O}\left(L_H^{i_0}, L_H^{i_1}\right) \otimes \cdots \otimes \mathcal{O}\left(L_H^{i_{d-1}}, L_H^{i_d}\right) \to \mathcal{O}\left(L_H^{i_0+1}, L_H^{i_d+1}\right)$$

   is obtained by dualizing the components of the tame DG-isomorphism

$$A\left(\bigsqcup_{k=i_0}^{i_d} \Lambda_H^{k+1}, j\right) \to A\left(\bigsqcup_{k=i_0}^{i_d} \Lambda_H^k, j\right)$$

induced by the Legendrian isotopy $\left(\bigsqcup_{k=i_0}^{i_d} \Lambda_H^{k+1-t}\right)_{0\leq t\leq 1}$ (see [26] Proposition 2.6).

Lemma 4.32. *The A_∞-functor $\gamma : \mathcal{O} \to \mathcal{O}$ is a quasi-equivalence.*

Proof. This follows from the fact that γ is defined by dualizing the components of a DG-isomorphism (see the proof of Lemma 4.10).

□

Here the **Z**-splitting

$$\mathbf{Z} \xrightarrow{\sim} \mathrm{ob}\left(\mathcal{O}\right), \quad n \mapsto L_H^n$$

is compatible with the quasi-autoequivalence γ in the sense of Definition 2.5. As explained there, this turns $\mathcal{A}$ into an Adams-graded A_∞-category.

4.5.3 Relation between $(\mathcal{A}_2, \tau_2)$ and $(\mathcal{O}, \gamma)$

We now explain how the pairs $(\mathcal{A}_2, \tau_2)$ and $(\mathcal{O}, \gamma)$ are related. See Figure 4.4, where we illustrate the action of the projection Π_P in the case

$$(P, \lambda) = \left(T^*S^1, pdq\right), \; L = 0_{S^1}, \text{ and } H\left(q, p\right) = h(q),$$

where $h : S^1 \to \mathbf{R}$ is a Morse function.

Lemma 4.33. *There is a strict A_∞-isomorphism $\zeta_2 : \mathcal{A}_2 \to \mathcal{O}$ defined as follows:*

1. *on the objects, $\zeta_2\left(\Lambda_H^n\right) = L_H^n$,*
2. *on the morphisms, ζ_2 sends a Reeb chord c in $\mathcal{A}_2\left(\Lambda_H^i, \Lambda_H^j\right)$ to the intersection point*

$$\zeta_2\left(c\right) = \Pi_P\left(c\right)$$

in $\mathcal{O}\left(L_H^i, L_H^j\right)$.

Proof. Since $J = \left(D\Pi_P\right)^*_{|\xi} j$, it follows from [23] Theorem 2.1 that the map

$$u = (\sigma, v) \mapsto \Pi_P \circ v$$

induces a bijection

$$\mathcal{M}_{a,c_d\dots c_1}\left(\mathbf{R}\times\mathbf{\Lambda}_H, J, \alpha\right) \xrightarrow{\sim} \mathcal{M}_{\Pi_P(a),\Pi_P(c_d)\dots\Pi_P(c_1)}\left(\mathbf{L}_H, j\right).$$

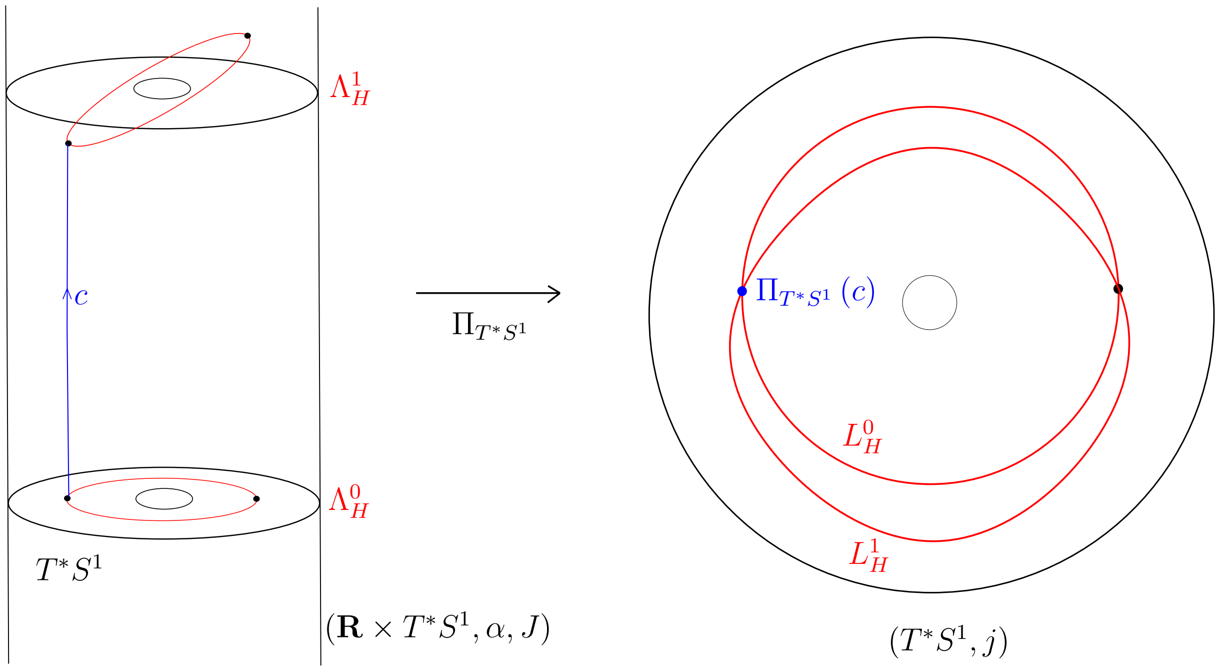

Figure 4.4: Action of the projection $\Pi_{T^*S^1}$

This implies the result.

□

Lemma 4.34. *We have*

$$\gamma = \zeta_2 \circ \tau_2 \circ \zeta_2^{-1}.$$

Proof. Since $J = (D\Pi_P)^*_{|\xi} j$, it follows from [23] Theorem 2.1 that the map

$$u = (\sigma, v) \mapsto (\sigma, \Pi_P \circ v)$$

induces a bijection

$$\mathcal{M}_{a,c_d \dots c_1}\left(\mathbf{R} \times \bigsqcup_{k=i_0}^{i_d} \Lambda_H^{k+1-t}, J, \alpha\right) \xrightarrow{\sim} \mathcal{M}_{\Pi_P(a),\Pi_P(c_d)\dots\Pi_P(c_1)}\left(\left(L_H^{k+1-t}\right)_{i_0 \le k \le i_d}, j\right)$$

for every $t \in [0,1]$. This implies the result.

□

Lemma 4.35. *The mapping torus of $\tau_2 : \mathcal{A}_2 \to \mathcal{A}_2$ is quasi-equivalent to the mapping torus of $\gamma : \mathcal{O} \to \mathcal{O}$ (see Definition 2.1).*

Proof. According to Lemma 4.34 the following diagram of Adams-graded A_∞-categories is commutative

$$\begin{tikzcd}
\mathcal{A}_2 \arrow[d, "\zeta_2"] & \mathcal{A}_2 \sqcup \mathcal{A}_2 \arrow[l, "\mathrm{id}\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup\tau_2"] \arrow[d, "\zeta_2\sqcup\zeta_2"] & \mathcal{A} \arrow[d, "\zeta_2"] \\
\mathcal{O} & \mathcal{O} \sqcup \mathcal{O} \arrow[l, "\mathrm{id}\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup\gamma"] & \mathcal{O}.
\end{tikzcd}$$

Moreover, each vertical arrow is a quasi-equivalence according to Lemma 4.33. Thus, the result follows from Proposition 1.85.

□

4.6 Mapping torus of γ

In this section, we show that we can apply Theorem 2.10 (Theorem B in the introduction) in order to compute the mapping torus of $\gamma : \mathcal{O} \to \mathcal{O}$. This allows us to finish the proof of Theorem 4.5.

Recall that we fixed a contactomorphism ϕ_H of V such that $\phi_H^* \alpha_H = \alpha$. Also recall that if θ is some real number, then

$$\Lambda^\theta = \{(f(q)+\theta, q) \mid q \in L\}, \ \Lambda_H^\theta = \phi_H^{-1}\left(\Lambda^\theta\right), \text{ and } L_H^\theta = \Pi_P\left(\Lambda_H^\theta\right).$$

4.6.1 Continuation elements

We denote by $\mathcal{O}'$ the A_∞-category obtained from $\mathcal{O}$ by applying the functor of Definition 1.53. Besides, we denote by Γ the set of continuation elements in $\mathcal{O}'$ (see for example [37] section 3.3).

Lemma 4.36. *For every $i < j < k$, the chain maps*

$$\begin{cases} \mu^2_{\mathcal{O}'}(c_j, -) & : & \mathcal{O}'\left(L_H^{j+1}, L_H^{k+1}\right) & \to & \mathcal{O}'\left(L_H^{j}, L_H^{k+1}\right) \\ \mu^2_{\mathcal{O}'}(-, c_j) & : & \mathcal{O}'\left(L_H^{i}, L_H^{j}\right) & \to & \mathcal{O}'\left(L_H^{i}, L_H^{j+1}\right) \end{cases}$$

are quasi-isomorphisms.

Proof. This follows from Lemma 3.26 in [37].

□

Recall from section 4.1.3 that we fixed a finite-dimensional model $CF^*(L)$ for the Floer A_∞-algebra, as defined in [58].

Lemma 4.37. *The A_∞-algebra* $\mathrm{End}_{\mathcal{O}'[\Gamma^{-1}]}\left(L_H^0\right)$ *is quasi-isomorphic to* $CF^*(L)$.

Proof. This is explained in [60] Lecture 10.

□

4.6.2 The $\mathcal{O}'$-bimodule map

In order to apply Theorem 2.10, we need a degree 0 closed $\mathcal{O}'$-module map $f : \mathcal{O}'(-,-) \to \mathcal{O}'(-,\gamma(-))$ such that the elements in f (units) satisfy certain hypotheses. As usual, we would like to find such an f "geometrically", i.e. using some Lagrangian (or Legendrian) isotopy. However, here the unit $e_{L_H^k} \in \mathcal{O}\left(L_H^k, L_H^k\right)$, which is not a "geometric" morphism, is supposed to be sent by f to something in $\mathcal{O}\left(L_H^k, L_H^{k+1}\right)$, which is generated by the "geometric" elements of $L_H^k \cap L_H^{k+1}$. Therefore, we need to somehow replace this unit by some intersection point between Lagrangians. The idea is that we will slightly perturb L_H^k to $L_H^{k+\delta}$, and then replace $e_{L_H^k}$ by the continuation element in the vector space generated by $L_H^k \cap L_H^{k+\delta}$.

Observe that if δ is small enough, $L_H^{k+\delta}$ is a small perturbation of L_H^k. Therefore, in a Weinstein neighborhood of L_H^k, the Lagrangian $L_H^{k+\delta}$ is the graph of $dh_{\delta,k}$, where $h_{\delta,k}$ is some Morse function on L. In particular, the intersection points between L_H^k and $L_H^{k+\delta}$ correspond to the critical points of $h_{\delta,k}$. Moreover, the continuation element in the vector space generated by $L_H^k \cap L_H^{k+\delta}$ corresponds to the sum of the minima of $h_{\delta,k}$.

Example 4.38. Assume that we are in the case

$$(P,\lambda) = (T^*M, pdq)\,,\ L = 0_M,\ \text{and } H(q,p) = h(q),$$

where $h : M \to \mathbf{R}$ is a Morse function. As explained in example 4.15, in this case we have

$$L_H^\theta = \Pi_{T^*M}\left(j^1\left(\theta e^h\right)\right) = \operatorname{graph}\left(d\left(\theta e^h\right)\right).$$

Thus, $L_H^{k+\delta}$ is the graph of $d\left(\delta e^h\right)$ over L_H^k.

The following result will allow us to "replace" the units by geometric morphisms as explained above. We denote by $g = -d\lambda(-, j-)$ the metric on P induced by j and $(-d\lambda)$.

Lemma 4.39. *For every positive integer n, there exists $\delta_n > 0$ such that the following* *holds for every $\delta \in]0, \delta_n]$. For every sequence of integers*

$$-n \leq j_0 < \cdots < j_p \leq \ell_0 < \cdots < \ell_q \leq n, \quad p, q \geq 0,$$

the rigid j-holomorphic discs in P with boundary on

$$L_H^{j_0} \cup \cdots \cup L_H^{j_p} \cup L_H^{\ell_0+\delta} \cup \cdots \cup L_H^{\ell_q+\delta}$$

are

1. *in bijection with the rigid j-holomorphic discs in P with boundary on*

$$L_H^{j_0} \cup \cdots \cup L_H^{j_p} \cup L_H^{\ell_0} \cup \cdots \cup L_H^{\ell_q}$$

   if $j_p < \ell_0$, or

2. *in bijection with the rigid j-holomorphic discs in P with boundary on*

$$L_H^{j_0} \cup \cdots \cup L_H^{j_{p-1}} \cup L_H^{\ell_0} \cup L_H^{\ell_1} \cup \cdots \cup L_H^{\ell_q}$$

   with a flow line of $(-\nabla_g h_{\delta,k})$ attached on the component in $L_H^{\ell_0}$ if $j_p = \ell_0$.

Proof. The case $j_p < \ell_0$ follows from transversality of the moduli spaces in consideration. The case $j_p = \ell_0$ follows from the main analytic Theorem of [29] (Theorem 3.6).

□

In order to define the $\mathcal{O}'$-bimodule map f properly, we will use Lemma 4.39 to modify the A_∞-category $\mathcal{O}'$. In the following, we fix a *decreasing* sequence of positive numbers $(\delta_n)_{n\geq 1}$ such that, for every n,

1. Lemma 4.39 holds with δ_n, and

2. δ_n is small enough so that there is no handle slide instant in the Legendrian isotopy

$$\bigsqcup_{\ell=-n}^{n} \Lambda_H^{\ell+\delta_n t}, t \in [0,1] .$$

We define two families of A_∞-categories $(\mathcal{O}_{n,k})_{n,k}$ and $\left(\widetilde{\mathcal{O}}_{n,k}\right)_{n,k}$ indexed by the couples (n, k), where $n \geq 1$ and $-n \leq k \leq n$. The A_∞-category $\mathcal{O}_{n,k}$ is basically obtained from $\mathcal{O}'$ by restricting to objects L_H^i, $-n \leq i \leq n$, and adding a copy of the object L_H^k.

Definition 4.40. For every $j \in \mathbf{Z}$, let $\overline{L_H^j}$ be a copy of L_H^j. We denote by $\mathcal{O}_{n,k}$ the A_∞-category defined as follows:

1. the set of objects of $\mathcal{O}_{n,k}$ is

$$\mathring{ob}\left(\mathcal{O}_{n,k}\right) = \left\{L_H^j \mid -n \leq j \leq k\right\} \cup \left\{\overline{L_H^\ell} \mid k \leq \ell \leq n\right\},$$

2. the spaces of morphisms in $\mathcal{O}_{n,k}$ are the corresponding spaces of morphisms in $\mathcal{O}'$ when we replace $\overline{L_H^\ell}$, $k \leq \ell \leq n$, by L_H^ℓ, except that

$$\mathcal{O}_{n,k}\left(\overline{L_H^k}, L_H^k\right) = \{0\},$$

3. the operations are the same as in $\mathcal{O}'$.

The A_∞-category $\widetilde{\mathcal{O}}_{n,k}$ is obtained from $\mathcal{O}_{n,k}$ by perturbing the objects $\overline{L_H^\ell}$, $k \leq \ell \leq n$, to $L_H^{\ell+\delta_n}$.

Definition 4.41. Let

$$\Theta_{n,k} := \{-n, \dots, k\} \cup \{\ell + \delta_n \mid k \leq \ell \leq n\} \subset \mathbf{R}, \text{ and } \widetilde{\mathbf{L}_H} := \left(L_H^\theta\right)_{\theta \in \Theta_{n,k}}.$$

We denote by $\widetilde{\mathcal{O}}_{n,k}$ the A_∞-category defined as follows:

1. the objects of $\widetilde{\mathcal{O}}_{n,k}$ are the Lagrangians L_H^θ, $\theta \in \Theta_{n,k}$,

2. the space of morphisms from L_H^θ to $L_H^{\theta'}$ is either generated by $L_H^\theta \cap L_H^{\theta'}$ if $\theta < \theta'$, or $\mathbf{F}$ if $\theta = \theta'$, or 0 if $\theta > \theta'$,

3. the operations are such that $e_{L_H^\theta} \in \widetilde{\mathcal{O}}_{n,k}\left(L_H^\theta, L_H^\theta\right)$ is a strict unit, and for every sequence $\theta_0 < \dots < \theta_d$ in $\Theta_{n,k}$, for every sequence of intersection points

$$(x_1, \dots, x_d) \in \left(L_H^{\theta_0} \cap L_H^{\theta_1}\right) \times \dots \times \left(L_H^{\theta_{d-1}} \cap L_H^{\theta_d}\right),$$

we have

$$\mu_{\widetilde{\mathcal{O}}_{n,k}}(x_1, \dots, x_d) = \sum_{x_0 \in L_H^{\theta_0} \cap L_H^{\theta_d}} \#\mathcal{M}_{x_0, x_d \cdots x_1}\left(\widetilde{\mathbf{L}_H}, j\right) x_0.$$

These A_∞-categories being defined, Lemma 4.39 implies the following result.

Lemma 4.42. *There is a strict A_∞-functor $\rho_{n,k} : \mathcal{O}_{n,k} \to \widetilde{\mathcal{O}}_{n,k}$ defined as follows:*

1. *on the objects, $\rho_{n,k}\left(L_H^j\right) = L_H^j$ if $-n \leq j \leq k$ and $\rho_{n,k}\left(\overline{L_H^\ell}\right) = L_H^{\ell+\delta_n}$ if $k \leq \ell \leq n$,*
2. *on the morphisms, $\rho_{n,k}$ sends the unit of $\mathcal{O}_{n,k}\left(L_H^k, \overline{L_H^k}\right) = \mathbf{F}$ to the continuation element in $\widetilde{\mathcal{O}}_{n,k}\left(L_H^k, L_H^{k+\delta_n}\right)$, and it sends any other morphism of $\mathcal{O}_{n,k}$ to the corresponding one in $\widetilde{\mathcal{O}}_{n,k}$.*

Proof. Consider a sequence $(x_0, \ldots, x_{d-1})$ of morphisms in $\mathcal{O}_{n,k}$. If in this sequence there is no morphism from L_H^k to $\overline{L_H^k}$, then the relation

$$\mu_{\widetilde{\mathcal{O}}_{n,k}}\left(\rho_{n,k}x_0, \ldots, \rho_{n,k}x_d\right) = \rho_{n,k}\mu_{\mathcal{O}_{n,k}}\left(x_0, \ldots, x_d\right)$$

follows directly from the first item of Lemma 4.39. Now assume that there is $p \in \{0, \ldots, d-1\}$ such that $x_p = e_{L_H^k} \in \mathcal{O}_{n,k}\left(L_H^k, \overline{L_H^k}\right)$. Recall that the continuation element in $\widetilde{\mathcal{O}}_{n,k}\left(L_H^k, L_H^{k+\delta_n}\right)$ corresponds to the sum of the minima of $h_{\delta_n,k}$. Then the second item of Lemma 4.39 implies that

$$\mu_{\widetilde{\mathcal{O}}_{n,k}}\left(\rho_{n,k}x_0, \ldots, \rho_{n,k}x_d\right) = \begin{cases} \rho_{n,k}x_1 & \text{if } d = 1 \text{ and } p = 0 \\ \rho_{n,k}x_0 & \text{if } d = 1 \text{ and } p = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the A_∞-relation for $\rho_{n,k}$ is still satisfied according to the behavior of the operations $\mu_{\mathcal{O}_{n,k}}$ with respect to the unit $e_{L_H^k}$.

□

Now that we have in some sense "replace" the unit $e_{L_H^k} \in \mathcal{O}_{n,k}\left(L_H^k, \overline{L_H^k}\right)$ by the continuation element in $\widetilde{\mathcal{O}}_{n,k}\left(L_H^k, L_H^{k+\delta_n}\right)$, we can define geometrically an A_∞-functor that will finally allow us to define the $\mathcal{O}'$-bimodule map f.

Definition 4.43. We denote by $\nu_{n,k} : \widetilde{\mathcal{O}}_{n,k} \to \mathcal{O}'$ the A_∞-functor defined as follows:

1. on the objects, $\nu_{n,k}\left(L_H^j\right) = L_H^j$ if $-n \leq j \leq k$, and $\nu_{n,k}\left(L_H^{\ell+\delta_n}\right) = L_H^{\ell+1}$ if $k \leq \ell \leq n$,
2. on the morphisms, $\nu_{n,k}$ is obtained by dualizing the components of the tame DG-isomorphism
$$A\left(\bigsqcup_{i=-n}^{n+1} \Lambda_H^i\right) \xrightarrow{\sim} A\left(\bigsqcup_{j=-n}^{k} \Lambda_H^j \sqcup \bigsqcup_{\ell=k}^{n} \Lambda_H^{\ell+\delta_n}\right).$$

induced by the Legendrian isotopy

$$\left(\bigsqcup_{j=-n}^{k} \Lambda_H^j\right) \sqcup \left(\bigsqcup_{\ell=k}^{n} \Lambda_H^{\ell+1-t(1-\delta_n)}\right), \quad t \in [0,1]$$

(see [26] Proposition 2.6).

Remarks 4.44. We point out some properties of the A_∞-functors

$$\sigma_{n,k} := \nu_{n,k} \circ \rho_{n,k} : \mathcal{O}_{n,k} \to \mathcal{O}'.$$

1. Let $m \leq n$ be two positive integers, and let $k \in \{-m, \dots, m\}$. Recall that we chose δ_m small enough so that there is no handle slide instant in the Legendrian isotopy

$$\bigsqcup_{\ell=-m}^{m} \Lambda_H^{\ell+\delta_m t}, 0 \leq t \leq 1.$$

Since $\delta_n \leq \delta_m$, neither is there any handle slide instant in the Legendrian isotopy

$$\bigsqcup_{\ell=-m}^{m} \Lambda_H^{\ell+\delta_n t}, 0 \leq t \leq 1.$$

Therefore, $\sigma_{n,k}$ agrees with $\sigma_{m,k}$ on $\mathcal{O}_{m,k} \subset \mathcal{O}_{n,k}$.

2. Consider a sequence of integers

$$-n \leq j_0 < \cdots < j_p \leq k_1 < k_2 \leq \ell_0 < \cdots < \ell_q \leq n,$$

and a sequence of morphisms

$$\begin{aligned}(x_0, \dots, x_{p-1}, u, y_0, \dots, y_{q-1}) \in \mathcal{O}_{n,k_i}\left(L_H^{j_0}, L_H^{j_1}\right) \times \cdots \times \mathcal{O}_{n,k_i}\left(L_H^{j_{p-1}}, L_H^{j_p}\right) \\ \times \mathcal{O}_{n,k_i}\left(L_H^{j_p}, \overline{L_H^{\ell_0}}\right) \times \mathcal{O}_{n,k_i}\left(\overline{L_H^{\ell_0}}, \overline{L_H^{\ell_1}}\right) \times \cdots \times \mathcal{O}_{n,k_i}\left(\overline{L_H^{\ell_{q-1}}}, \overline{L_H^{\ell_q}}\right).\end{aligned}$$

Since the Legendrian isotopy defining ν_{n,k_i} is

$$\left(\bigsqcup_{j=-n}^{k_i} \Lambda_H^j\right) \sqcup \left(\bigsqcup_{\ell=k_i}^{n} \Lambda_H^{\ell+1-t(1-\delta_n)}\right), \quad t \in [0,1],$$

we have

$$\begin{cases} \sigma_{n,k_1}\left(x_0,\ldots,x_{p-1}\right)=\delta_{1p}x_0 \\ \sigma_{n,k_2}\left(y_0,\ldots,y_{q-1}\right)=\gamma\left(y_0,\ldots,y_{q-1}\right) \\ \sigma_{n,k_2}\left(x_0,\ldots,x_{p-1},u,y_0,\ldots,y_{q-1}\right)=\sigma_{n,k_1}\left(x_0,\ldots,x_{p-1},u,y_0,\ldots,y_{q-1}\right). \end{cases}$$

3. By construction, the A_∞-functor $\nu_{n,k}$ sends the continuation element in $\widetilde{\mathcal{O}}_{n,k}\left(L_H^k, L_H^{k+\delta_n}\right)$ (corresponding to the sum of the minima of $h_{\delta_n,k}$) to the continuation element c_k in $\mathcal{O}'\left(L_H^k, L_H^{k+1}\right)$. In other words, $\sigma_{n,k}$ sends the unit $e_{L_H^k}\in\mathcal{O}_{n,k}\left(L_H^k,\overline{L_H^k}\right)$ to c_k.

We can now state and prove the desired result.

Lemma 4.45. *There exists a degree* 0 *closed* $\mathcal{O}'$*-bimodule map* $f:\mathcal{O}'(-,-)\to\mathcal{O}'(-,\gamma(-))$ *(see Definitions 1.63 and 1.64) which sends the unit* $e_{L_H^k}\in\mathcal{O}'\left(L_H^k,L_H^k\right)$ *to the continuation element* $c_k\in\mathcal{O}'\left(L_H^k,L_H^{k+1}\right)\cap\Gamma$.

Proof. Consider a sequence of integers

$$j_0<\cdots<j_p\leq k=\ell_0<\cdots<\ell_q,$$

and a sequence of morphisms

$$\begin{aligned}(x_0,\ldots,x_{p-1},u,y_0,\ldots,y_{q-1})\in\mathcal{O}'\left(L_H^{j_0},L_H^{j_1}\right)\times\cdots\times\mathcal{O}'\left(L_H^{j_{p-1}},L_H^{j_p}\right)\\ \times\mathcal{O}'\left(L_H^{j_p},L_H^{k}\right)\times\mathcal{O}'\left(L_H^{k},L_H^{\ell_1}\right)\times\cdots\times\mathcal{O}'\left(L_H^{\ell_{q-1}},L_H^{\ell_q}\right).\end{aligned}$$

We choose $n\geq 1$ such that $-n\leq j_0\leq\ell_q\leq n$, and we set

$$f\left(x_0,\ldots,x_{p-1},u,y_0,\ldots,y_{q-1}\right):=\sigma_{n,k}\left(x_0,\ldots,x_{p-1},u,y_0,\ldots,y_{q-1}\right)\in\mathcal{O}'\left(L_H^{j_0},\gamma L_H^{\ell_q}\right),$$

where on the right hand side we consider that

$$\begin{aligned}(x_0,\ldots,x_{p-1},u,y_0,\ldots,y_{q-1})\in\mathcal{O}_{n,k}\left(L_H^{j_0},L_H^{j_1}\right)\times\cdots\times\mathcal{O}_{n,k}\left(L_H^{j_{p-1}},L_H^{j_p}\right)\\ \times\mathcal{O}_{n,k}\left(L_H^{j_p},\overline{L_H^{k}}\right)\times\mathcal{O}_{n,k}\left(\overline{L_H^{k}},\overline{L_H^{\ell_1}}\right)\times\cdots\times\mathcal{O}_{n,k}\left(\overline{L_H^{\ell_{q-1}}},\overline{L_H^{\ell_q}}\right).\end{aligned}$$

Observe that f is well defined (it does not depend on the choice of n) according to the first item of Remark 4.44.

We now verify that f is closed. According to Definition 1.63, we have

$$\begin{aligned}
&\mu^1_{\mathrm{Mod}_{\mathcal{C},\mathcal{C}}}(f)(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1}) \\
&\qquad = \sum \sigma_{n,k}(\dots,\mu_{\mathcal{O}'}(\dots),\dots,u,\dots) \\
&\qquad + \sum \sigma_{n,\ell_s}(\dots,\mu_{\mathcal{O}'}(x_r,\dots,x_{p-1},u,y_0,\dots,y_{s-1}),\dots) \\
&\qquad + \sum \sigma_{n,k}(\dots,u,\dots,\mu_{\mathcal{O}'}(\dots),\dots) \\
&\qquad + \sum \mu_{\mathcal{O}'}(\dots,\sigma_{n,k}(\dots,u,\dots),\gamma(\dots),\dots,\gamma(\dots)).
\end{aligned}$$

Now according to the second item of Remark 4.44, we have

$$\begin{aligned}
&\sum \sigma_{n,\ell_s}(\dots,\mu_{\mathcal{O}'}(x_r,\dots,x_{p-1},u,y_0,\dots,y_{s-1}),\dots) \\
&\qquad = \sum \sigma_{n,k}(\dots,\mu_{\mathcal{O}'}(x_r,\dots,x_{p-1},u,y_0,\dots,y_{s-1}),\dots)
\end{aligned}$$

and

$$\begin{aligned}
&\sum \mu_{\mathcal{O}'}(\dots,\sigma_{n,k}(\dots,u,\dots),\gamma(\dots),\dots,\gamma(\dots)) \\
&\quad = \sum \mu_{\mathcal{O}'}(\sigma_{n,k}(\dots),\dots,\sigma_{n,k}(\dots),\sigma_{n,k}(\dots,u,\dots),\sigma_{n,k}(\dots),\dots,\sigma_{n,k}(\dots)).
\end{aligned}$$

Therefore, we get

$$\mu^1_{\mathrm{Mod}_{\mathcal{C},\mathcal{C}}}(f)(x_0,\dots,x_{p-1},u,y_0,\dots,y_{q-1}) = 0$$

from the fact that $\sigma_{n,k}$ is an A_∞-functor. Finally, f sends the unit $e_{L^k_H} \in \mathcal{O}'\left(L^k_H, L^k_H\right)$ to the continuation element $c_k \in \mathcal{O}'\left(L^k_H, L^{k+1}_H\right) \cap \Gamma$ according to the third item of Remark 4.44.

□

4.6.3 Proof of the main result

We end the section with the proof of Theorem 4.5 (Theorem D in the Introduction).

Recall that we denote by $\mathbf{F}[t]$ the augmented Adams-graded associative algebra generated by a variable t of bidegree $(2,1)$, and by $\overline{\mathbf{F}[t]}$ its augmentation ideal (or equivalently, the ideal generated by t). The key result is the following.

Lemma 4.46. *The mapping torus of γ is quasi-equivalent to the Adams-graded A_∞-category $\mathbf{F} \oplus \left(\overline{\mathbf{F}[t]} \otimes CF^*(L)\right)$.*

Proof. Let $f : \mathcal{O}'(-,-) \to \mathcal{O}'(-,\gamma(-))$ be the degree 0 closed bimodule map of Lemma 4.45. Recall that f sends the units of $\mathcal{O}'$ to the set Γ. According to Lemma 4.36, the continuation elements $c_n = f(e_{L_H^n})$ satisfy the assumptions of Theorem 2.10. Thus the mapping torus of γ is quasi-equivalent to the Adams-graded A_∞-algebra $(\mathcal{O}')^0 \oplus \left(\overline{\mathbf{F}[t]} \otimes \mathcal{O}'[\Gamma^{-1}]^0\right)$ (recall that if $\mathcal{C}$ is an A_∞-category equipped with a $\mathbf{Z}$-splitting $\mathbf{Z} \times \mathcal{E} \simeq \mathrm{ob}(\mathcal{C})$, we denote by $\mathcal{C}^0$ the full A_∞-subcategory of $\mathcal{C}$ whose set of objects corresponds to $\{0\} \times \mathcal{E}$). Here we have

$$(\mathcal{O}')^0 = \mathrm{End}_{\mathcal{O}'}\left(L_H^0\right) = \mathbf{F} \text{ and } \mathcal{O}'\left[\Gamma^{-1}\right]^0 = \mathrm{End}_{\mathcal{O}'[\Gamma^{-1}]}\left(L_H^0\right).$$

The result follows from Lemma 4.37, which gives

$$\mathrm{End}_{\mathcal{O}'[\Gamma^{-1}]}\left(L_H^0\right) \simeq CF^*(L).$$

□

The quasi-equivalence

$$LA^*(\Lambda^\circ) \simeq \mathbf{F} \oplus \left(\overline{\mathbf{F}[t]} \otimes CF^*(L)\right)$$

now follows directly from Lemmas 4.13, 4.21, 4.27, 4.35 and 4.46.

Now observe that, according to Proposition 1.43, Koszul duality holds for the augmented Adams-graded DG-algebra $CE_{-*}(\Lambda^\circ)$ because it is locally finite and Adams connected (see Definition 1.22). Indeed, recall from section 4.1.3 that the Adams-degree in $CE_{-*}(\Lambda^\circ)$ of a Reeb chord c is the number of times c winds around the fiber. Thus, there is a quasi-isomorphism

$$CE_{-*}(\Lambda^\circ) \simeq E(E(CE_{-*}(\Lambda^\circ))).$$

Besides, recall from section 4.1.3 that there is a coaugmented Adams-graded A_∞-coalgebra $LC_*(\Lambda^\circ)$ such that

$$CE_{-*}(\Lambda^\circ) = \Omega(LC_*(\Lambda^\circ)) \text{ and } LA^*(\Lambda^\circ) = LC_*(\Lambda^\circ)^\#,$$

where the graded dual (see Definition 1.37) is taken with respect to the bigrading. Since there is a quasi-isomorphism $B(\Omega C) \simeq C$ for every A_∞-coalgebra C (see [32] section

2.2.2), it follows that

$$E\left(CE_{-*}\left(\Lambda^{\circ}\right)\right)=B\left(CE_{-*}\left(\Lambda^{\circ}\right)\right)^{\#}\simeq LC_{*}\left(\Lambda^{\circ}\right)^{\#}=LA^{*}\left(\Lambda^{\circ}\right)$$

(graded dual preserves quasi-isomorphisms, see Proposition 1.38). Moreover, the Koszul dual construction preserves quasi-isomorphisms (see Proposition 1.40), so we deduce

$$CE_{-*}\left(\Lambda^{\circ}\right)\simeq E\left(E\left(CE_{-*}\left(\Lambda^{\circ}\right)\right)\right)\simeq E\left(LA^{*}\left(\Lambda^{\circ}\right)\right)\simeq E\left(\mathbf{F}\oplus\left(\overline{\mathbf{F}\left[t\right]}\otimes CF^{*}\left(L\right)\right)\right).$$

Finally, recall that $CF^{*}\left(L\right)$ is a finite-dimensional model for the Floer A_{∞}-algebra of L. Moreover, recall that $\overline{\mathbf{F}[t]}$ is the ideal of $\mathbf{F}[t]$ generated by the variable t of bidegree $(2,1)$. Therefore, the augmented A_{∞}-algebra $\mathbf{F}\oplus\left(\overline{\mathbf{F}\left[t\right]}\otimes CF^{*}\left(L\right)\right)$ is locally finite and Adams connected. According to Proposition 1.44, we get the remaining quasi-isomorphism

$$E\left(\mathbf{F}\oplus\left(\overline{\mathbf{F}\left[t\right]}\otimes CF^{*}\left(L\right)\right)\right)\simeq\Omega\left(\mathbf{F}\oplus\left(\overline{\mathbf{F}\left[t_{\#}\right]}\otimes CF_{-*}\left(L\right)\right)\right).$$

Chapter 5

Perspectives

In this chapter, we present some possible applications of the techniques developed during the thesis to other problems.

5.1 First generalizations

Several exact Lagrangians I believe that the proof of Theorem D can be easily adapted to the case of several Lagrangians. Let $L_1, \ldots, L_d$ be connected exact Lagrangian submanifolds in a Liouville manifold P, and let $\Lambda_1^\circ, \ldots, \Lambda_d^\circ$ be corresponding Legendrian lifts in $S^1 \times P$. Assume that their projections to S^1 are cyclically ordered disjoint arcs and set

$$\Lambda^\circ := \Lambda_1^\circ \sqcup \cdots \sqcup \Lambda_d^\circ$$

Let $\mathcal{B} \subset \mathcal{F}uk\,(P)$ be the full A_∞-subcategory with objects $L_1, \ldots, L_d$, and let $\mathcal{A} \subset \mathcal{B}$ be its directed version.

Conjecture 5.1. *There is a quasi-isomorphism of augmented Adams-graded A_∞-algebras*

$$LA^*\left(\Lambda^\circ\right) \simeq \mathcal{A} \oplus \left(\overline{\mathbf{F}\left[t\right]} \otimes \mathcal{B}\right).$$

Moreover, Koszul duality holds for $CE_{-*}\left(\Lambda^\circ\right)$, and $E\left(CE_{-*}\left(\Lambda^\circ\right)\right) \simeq LA^*\left(\Lambda^\circ\right)$. In particular,

$$CE_{-*}\left(\Lambda^\circ\right) \simeq E\left(\mathcal{A} \oplus \left(\overline{\mathbf{F}\left[t\right]} \otimes \mathcal{B}\right)\right).$$

Extended invariants We could also consider Legendrian invariants with loop space coefficients as defined in [32] (such coefficients were used in the context of Lagrangian Floer cohomology before in [10]). Let L be a connected exact Lagrangian submanifold in a Liouville manifold P, and let Λ° be a Legendrian lift to $S^1 \times P$. Denote by $CE_{-*}^{+}\left(\Lambda^\circ\right)$ the Chekanov-Eliashberg DG-algebra of Λ° with coefficients in $C_{-*}\left(\Omega\Lambda\right)$. In our setting, it is augmented and Adams-graded. We also consider the augmented A_∞-algebra $LA^*_+(\Lambda^\circ)$, which is Adams-graded in our setting. It should be thought of as the endomorphism algebra of the trivial augmentation of $CE_{-*}(\Lambda^\circ)$ in the positive augmentation category $\mathcal{A}ug_+(\Lambda^\circ)$ introduced by [52] in dimension 3 and by [17] in higher dimensions. See Appendix A for a precise definition. I believe that the proof of Theorem D can be easily adapted in order to compute $LA^*_+(\Lambda^\circ)$, yielding the following conjecture.

Conjecture 5.2. *There is a quasi-isomorphism of augmented Adams-graded A_∞-algebras*

$$LA^*_+(\Lambda^\circ) \simeq \mathbf{F}[t] \otimes CF^*(L).$$

If moreover L is simply connected, Koszul duality holds for $CE^+_{-*}(\Lambda^\circ)$, and $E\left(CE^+_{-*}(\Lambda^\circ)\right) \simeq LA^*_+(\Lambda^\circ)$. In particular,

$$CE^+_{-*}(\Lambda^\circ) \simeq E(\mathbf{F}[t] \otimes CF^*(L)) \simeq \Omega(\mathbf{F}[t_\#] \otimes CF_{-*}(L)).$$

Remark 5.3. Observe that using the same arguments as in the proof of Corollary 4.7, Conjecture 5.2 would implies that

$$LA^*_+(\Lambda^\circ) \simeq C^*(\mathbf{CP}^\infty \times L).$$

and, when L is simply connected, that

$$CE^+_*(\Lambda^\circ) \simeq C_*\left(S^1 \times \Omega L\right).$$

We expect that, using different arguments than those developed for Theorem D, one can prove the following result.

Conjecture 5.4. *There is a quasi-isomorphism of augmented DG-algebras*

$$CE^+_*(\Lambda^\circ) \simeq C_*\left(S^1 \times \Omega L\right)$$

(even if L is not simply connected).

Sketch of proof. We first observe that it should be enough to prove the statement when

$$(P, \lambda) = (T^*M, pdq) \text{ and } L = 0_M,$$

thus when

$$V^\circ = S^1 \times T^*M \text{ and } \Lambda^\circ = \{pt\} \times 0_M.$$

Then, we see $V^\circ = S^1 \times T^*M$ as part of the boundary at infinity of the Liouville manifold $W = T^*S^1 \times T^*M$, and we view $\Lambda^\circ = \{pt\} \times 0_M$ as a Legendrian stop. Following the surgery perspective described in [32] section 1, we denote by W_{Λ° the completion of the Liouville sector obtained from W by attaching a cotangent cone $T^*\left(\mathbf{R}_{\geq 0} \times \Lambda^\circ\right)$ along Λ°. Now, according to Conjecture 3 in [32] (see also Theorems 1.1 and 1.2 in [8]), $CE^+_{-*}(\Lambda^\circ)$ is quasi-isomorphic to the wrapped Floer cohomology of the Lagrangian disk $D_{\Lambda^\circ} := T^*_{(t_0,x_0)}\left(\mathbf{R}_{\geq 0} \times \Lambda^\circ\right) \subset W_{\Lambda^\circ}$, where $x_0 = (pt, q_0) \in \Lambda^\circ$ and $t_0 > 0$. But here we have

$$W_{\Lambda^\circ} = \left(T^*S^1\right)_{pt} \times T^*M,$$

and

$$\begin{aligned} D_{\Lambda^\circ} &= T^*_{(t_0,pt,q_0)}\left(\mathbf{R}_{\geq 0} \times \{pt\} \times 0_M\right) \\ &= T^*_{(t_0,pt)}\left(\mathbf{R}_{\geq 0} \times \{pt\}\right) \times T^*_{q_0}M \\ &= D_{pt} \times T^*_{q_0}M \end{aligned}$$

Thus we get

$$\begin{aligned} CE^+_{-*}(\Lambda^\circ) &\simeq CW^*\left(D_{\Lambda^\circ}, D_{\Lambda^\circ}\right)_{W_{\Lambda^\circ}} \\ &\simeq CW^*\left(D_{pt} \times T^*_{q_0}M, D_{pt} \times T^*_{q_0}M\right)_{(T^*S^1)_{pt} \times T^*M} \\ &\simeq CW^*\left(D_{pt}, D_{pt}\right)_{(T^*S^1)_{pt}} \otimes CW^*\left(T^*_{q_0}M, T^*_{q_0}M\right)_{T^*M} \end{aligned}$$

where the last quasi-isomorphism follows from the the Künneth formula of [38] (Theorem 1.5). By a famous result of Abouzaid ([2]), we have

$$CW^*\left(T^*_{q_0}M, T^*_{q_0}M\right)_{T^*M} \simeq C_{-*}(\Omega M),$$

and $CW^*\left(D_{pt}, D_{pt}\right)_{(T^*S^1)_{pt}}$ is easily seen to be $C_{-*}(S^1)$ (see Figure 5.1). Finally, we indeed get

$$CE^+_{-*}(\Lambda^\circ) \simeq C_{-*}\left(S^1\right) \otimes C_{-*}(\Omega M) \simeq C_{-*}\left(S^1 \times \Omega M\right).$$

□

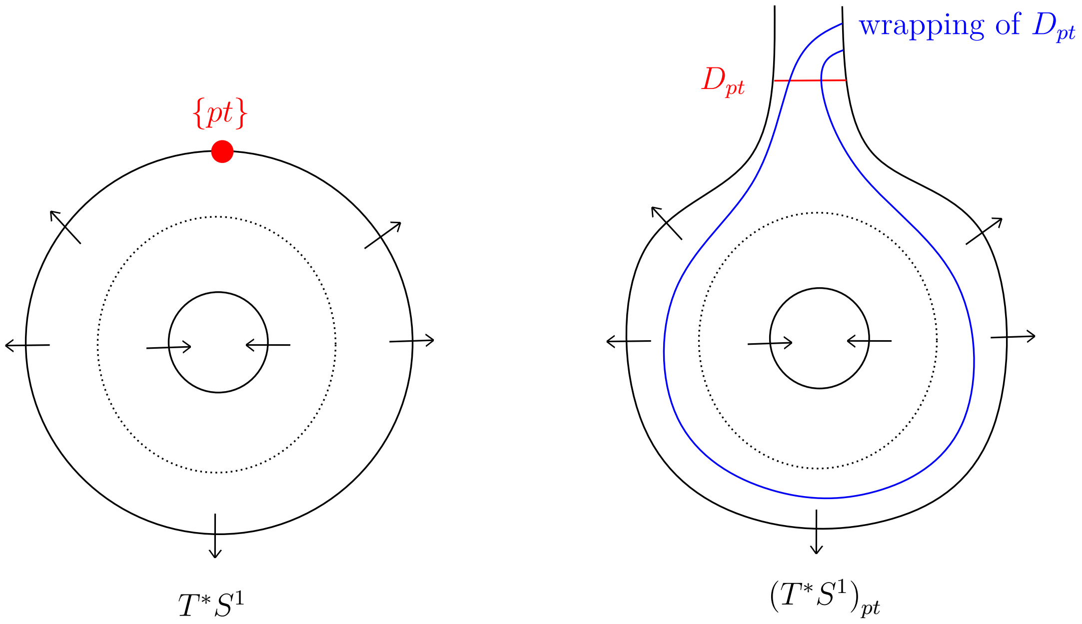

Figure 5.1: Before (on the left) and after (on the right) surgery on the stop $\{pt\}$ in T^*S^1

5.2 Circular contactizations

After studying the Legendrian lift of an exact Lagrangian submanifold, it seems natural to look at the Legendrian lift of an exact Lagrangian *immersion* in the circular contactization. I already started to think at this case, and I believe that the methods developed during my PhD thesis will work as well in this setting. Indeed, in the case of the Legendrian lift Λ° of an exact Lagrangian immersion $\iota : L \to P$, we can still lift the picture to $\mathbf{R} \times P$ and use the technology of localization and homotopy colimits. However, this time the A_∞-categories involved should be related to the immersed Lagrangian Floer theory of ι defined by Akaho and Joyce in [6], and also studied by Alston and Bao in [7]. I can make the following conjecture.

Conjecture 5.5. *Let L be a compact connected manifold, P a Liouville manifold, and $\iota : L \to P$ an exact Lagrangian immersion. Let $\Lambda \subset \mathbf{R} \times P$ and $\Lambda^\circ \subset S^1 \times P$ be Legendrian lifts of ι. Assume that the projection of Λ° to S^1 is an arc. Then $LA^*(\Lambda)$ can be seen as an A_∞-subalgebra of $CF^*(\iota)$, and*

$$CE_{-*}(\Lambda^\circ) \simeq E\left(LA^*(\Lambda) \oplus \left(\overline{\mathbf{F}[t]} \otimes CF^*(\iota)\right)\right).$$

In another direction, I plan to study Legendrian lifts of *Bohr-Sommerfeld* Lagrangian submanifolds in the circular contactization, which are exactly those who lift to Legendrian submanifolds in the circular contactization (they are not necessarily exact). One first case to look at might be the one of *monotone* Lagrangian submanifolds in the Liouville manifold P, which seems to be easier than the general case.

Of course, the goal is to understand Legendrian contact homology of all Legendrian submanifolds in the circular contactization, which in general are lifts of Bohr-Sommerfeld Lagrangian immersion in P. Observe that it should also be possible to study the case when the Legendrians are not compact but have a suitable behavior at infinity using the *sutured* Legendrian contact homology defined by Côme Dattin in his Ph.D. thesis.

5.3 Subcritically fillable contact manifolds

A subcritically fillable contact manifold is the boundary at infinity of a Weinstein manifold $X = \mathbf{C} \times P$, where P is a Weinstein manifold. Legendrian contact homology in subcritically fillable contact manifolds of dimension 3 has been studied combinatorially by Ekholm and Ng in [33]. More recently, Karlsson studied the higher dimensional analog in [43].

Observe in particular that a subcritically fillable contact manifold V splits as

$$V = \left(S^1 \times P\right) \bigcup_{S^1 \times \partial_\infty P} \left(D^2 \times \partial_\infty P\right).$$

Thus, it seems likely that Legendrian contact homology in circular contactization can be used to understand Legendrian contact homology in the subcritically fillable case. I recently observed that this is the kind of strategy which has been used by Ganatra and Maydanskiy in the Appendix of [16], where they show how the results of Bourgeois, Ekholm and Eliashberg imply Seidel's Conjectures 6.2 and 6.3 of [57]. To illustrate the relation with my results, I reproduce here the following conjecture.

Conjecture 5.6. *(Conjecture 6.3 in [57])*

Let $L_1, \ldots, L_d$ be exact Lagrangian spheres in a Weinstein manifold P. Let $\Lambda_1, \ldots, \Lambda_d$ be corresponding Legendrian lifts in $(\partial_\infty X) \cap (S^1 \times P)$ (recall that $X = \mathbf{C} \times P$), and assume that their projections to S^1 are cyclically ordered disjoint arcs. Let $\mathcal{B} \subset \mathcal{F}uk\,(P)$ be the full A_∞-subcategory with objects $L_1, \ldots, L_d$, and let $\mathcal{A} \subset \mathcal{B}$ be its directed version. Let $\mathcal{D}$ be the curved A_∞-category obtained from $\mathcal{A} \oplus \left(\overline{\mathbf{F}\,[[t]] \otimes \mathcal{B}}\right)$ by adding a curvature *term* $\mu^0 = te_{\mathcal{A}}$. *Then*

$$CE_{-*}\left(\Lambda_1 \sqcup \cdots \sqcup \Lambda_d\right) \simeq E\left(\mathcal{D}\right).$$

Since the methods I used to prove Theorem D should be easily adapted to the case of several exact Lagrangians (and thus Legendrians), I expect to be able to prove Conjecture 5.6 with the techniques of my thesis. According to the philosophy of [16], the Chekanov-Eliashberg algebra of $\Lambda = \Lambda_1 \sqcup \cdots \sqcup \Lambda_d$ knows about the wrapped Fukaya category of the Weinstein manifold X_Λ obtained from X by Legendrian surgery on Λ. In particular, there is a cohomologically fully faithful embedding of $\mathcal{W}\left(X_\Lambda\right)$ into the DG-category of A_∞-modules over $CE_*\left(\Lambda\right)$. Since the categories of A_∞-modules over an A_∞-algebra A and over its Koszul dual $E(A)$ are closely related (see for example [49]), Conjecture 5.6 allows us to relate the wrapped Fukaya category of X_Λ to the DG-category of A_∞-modules over $\mathcal{A} \oplus \left(\overline{\mathbf{F}\left[[t]\right]} \otimes \mathcal{B}\right)$, which has been especially studied in [57].

Here are some directions that I would like to explore. The first one is that there is probably a generalization of Conjecture 5.6 for Lagrangians which are not necessarily spheres, and the methods of Theorem D should apply to this case. The second is that Conjecture 5.2 can possibly be used to give an analog of Conjecture 5.6 relating the Chekanov-Eliashberg algebra $CE_*^+\left(\Lambda\right)$ of $\Lambda = \Lambda^- \sqcup \Lambda^+$ (components of Λ^- should be spheres while components of Λ^+ can be arbitrary) with coefficients in $C_*\left(\Omega\Lambda^+\right)$ to the categories $\mathcal{A}$ and $\mathcal{B}$ (I have no specific Conjecture yet). According to the philosophy of [31], this could be used to study the partially wrapped Fukaya category of X_{Λ^-} with stop Λ^+. The third is that Conjecture 5.5 might be upgraded to the case of subcritically fillable contact manifolds, and that this would involves an A_∞-algebra obtained from the pair $\left(LA^*\left(\Lambda\right), CF^*\left(\iota\right)\right)$ as in [57]. The last (and most speculative) is that a generalization of Theorem D to monotone Lagrangians submanifolds might have a counterpart in subcritically fillable contact manifolds, which will introduce the monotone Fukaya category of P in the story.

5.4 Boothby-Wang contact manifolds.

A Boothby-Wang contact manifold is a pair (V, α), where V is the total space of a circle bundle $\pi : V \to W$ over an integral symplectic manifold (W, ω), and α is a contact form on V whose Reeb vector field generates the fibers of π and such that $d\alpha = \pi^*\omega$. Legendrian contact homology in Boothby-Wang manifolds of dimension 3 has been studied combinatorially by Sabloff in [56].

The strategy to understand Legendrian contact homology in a Boothby-Wang manifold $V \xrightarrow{\pi} W$ from the circular contactization case is to use a theorem of Donaldson in [24], which allows us to find in W a codimension 2 symplectic submanifold $K \subset W$ such that $P := W \setminus \mathcal{N}(K)$ is a Liouville manifold. Since $\pi^{-1}(P) \simeq S^1 \times P$, the contact manifold V splits as

$$V = \left(S^1 \times P\right) \bigcup_{S^1 \times \partial P} \pi^{-1}(\mathcal{N}(K)).$$

Thus, we can hope to use the methods of the circular contactization case to understand the Boothby-Wang case. More precisely, I would have to understand how the pseudo-holomorphic discs passing through the divisor K induce a deformation of the Chekanov-Eliashberg algebra. can make the following conjecture.

Conjecture 5.7. *Let $V \to W$ be a Boothby-Wang contact manifold over a monotone symplectic manifold (W, ω). Let K be a codimension 2 symplectic submanifold of W whose Poincaré dual is an integral multiple of ω. Let L be a Bohr-Sommerfeld monotone Lagrangian submanifold of W which doesn't intersect K and which is exact in the Liouville manifold $W \setminus K$. Let Λ° be a Legendrian lift of L in V. Then*

$$CE_{-*}(\Lambda^\circ) \simeq E\left(\mathbf{F} \oplus \ker\left(CF^*_{(W,K)}(L) \to CF^*_{W \setminus K}(L)\right)\right)$$

where $CF^*_{(W,K)}(L)$ is the endomorphism algebra of L in the relative Fukaya category $\mathcal{F}uk(W,K)$ (see [58]), and $CF^*_{W\setminus K}(L)$ is the endomorphism algebra of L in the Fukaya category $\mathcal{F}uk(W \setminus K)$ (there is a strict A_∞-functor $\mathcal{F}uk(W,K) \to \mathcal{F}uk(W \setminus K)$).

A possible application of the Boothby-Wang case is the following. Let (X, ω) be a symplectic manifold whose symplectic form belongs to the integer lattice of $H^2(X, \mathbf{R})$, and let $D \subset X$ be a Donaldson divisor such that $X \setminus D$ is Weinstein. Then the wrapped Fukaya category of $X \setminus D$ is generated by the cocores of its critical handles. The boundary at infinity of a cocore is a Legendrian submanifold of $\partial_\infty(X \setminus D)$, which is a Boothby-Wang contact manifold over D, and thus it is the lift of a Bohr-Sommerfeld Lagrangian immersion in D. Besides, the wrapped Floer cohomology of a cocore C is closely related to the Legendrian contact homology of its boundary at infinity Λ (at the chain level, $CW^*(C)$ is the cone of a map from the linearized Legendrian contact homology of Λ to $CF_*(C)$). Therefore, this will allow us to relate the wrapped Fukaya category of $X \setminus D$ to the Fukaya category of D.

5.5 Contact mapping tori and contact manifolds with open book decomposition

Let ϕ be an exact symplectomorphism of a Liouville manifold (P, λ) which is the identity outside a compact subset of P. The contact mapping torus V_ϕ is obtained from the usual contactization of P by identifying $(t, x) \in \mathbf{R} \times P$ with $(t + f(x), \phi(x))$, where $f : P \to \mathbf{R}$ is a primitive of $(\phi^*\lambda - \lambda)$ (the contact form $(dt - \lambda)$ on $\mathbf{R} \times P$ descends to a contact form on this manifold). Observe that the contact mapping torus of id_P is exactly the circular contactization $S^1 \times P$. I believe that my results can be adapted to this more general situation by considering the A_∞-subcategory of $\mathcal{F}uk(P)$ formed by the Lagrangians $\phi^n(L), n \in \mathbf{N}$, instead of $CF^*(L)$ only.

The importance of contact mapping tori comes from the work of Giroux [40], who developed the use of open books in contact topology. Indeed, an open book decomposition of a contact manifold V allows us to split it as

$$V = \left(D^2 \times B\right) \bigcup_{S^1 \times B} V_\phi$$

where B is a contact submanifold of V of codimension 2 (called the binding), and V_ϕ is the contact mapping torus associated to the monodromy ϕ, which is a symplectomorphism of a page P of the open book. Then, we could use Legendrian contact homology in contact mapping tori to study Legendrian contact homology in contact manifolds endowed with an open book decomposition if one can understand the pseudo-holomorphic curves passing through the symplectization of the binding.

Appendix A

Extended Legendrian A_∞-algebra

In this appendix, we study the endomorphism A_∞-algebra $LA_+^*(\Lambda)$ of an augmentation $\varepsilon : CE_{-*}(\Lambda) \to \mathbf{F}$ in the positive augmentation category $\mathcal{A}ug_+(\Lambda)$ defined by Chantraine in [17] (it was first introduced by Ng, Rutherford, Shende, Sivek and Zaslow in [52] for Legendrians in $\mathbf{R}^3$). The goal is to prove Proposition A.5, which shows that there is a quasi-isomorphism of non-unital A_∞-algebras between $\overline{LA^*(\Lambda)}$ (which is the endomorphism algebra of ε in the augmentation category $\mathcal{A}ug_-(\Lambda)$ defined in [15]) and the cone of an A_∞-map from $LA_+^*(\Lambda)$ to some A_∞-algebra $LA_0^*(\Lambda)$ generated by Morse type generators.

A.1 Setting

Let (V, ξ) be a contact manifold of dimension $(2n+1)$ equipped with a hypertight contact form α, and let Λ be a connected compact Legendrian submanifold in (V, ξ) which is chord generic with respect to α (see chapter 3). In the following we fix an augmentation $\varepsilon : CE_{-*}(\Lambda) \to \mathbf{F}$.

Choose a positive Morse function $f_1 : \Lambda \to \mathbf{R}$ with a unique minimum. We denote by $\mathcal{U}$ a small open neighborhood of f_1 in $C^\infty(\Lambda)$ with the following property: for every f in $\mathcal{U}$, f is positive, Morse, and its critical points and gradient trajectories are in bijection with those of f_1. Then we choose a family $(f_n)_{n\geq 2}$ of functions in $\mathcal{U}$. We also choose a decreasing sequence $(\delta_k)_{k\geq 1}$ of small enough positive numbers so that the family

$$h_n = \sum_{k=1}^{n} \delta_k f_k, \quad n \in \mathbf{N},$$

has the following property: for every $i < j$, $(h_j - h_i)$ lies in the cone over $\mathcal{U}$ (this can be easily achieved by induction on n). Finally, we let $\Lambda_n \subset V$ be the Legendrian corresponding to the one-jet of h_n in a Weinstein neighborhood of $\Lambda = \Lambda_0$. Observe that if the numbers δ_k are small enough, the Chekanov-Eliashberg DG-algebra of Λ_n is naturally identified with $CE_{-*}(\Lambda)$ for every n. In particular, $\varepsilon = \varepsilon_0$ induces an augmentation $\varepsilon_n : CE_{-*}(\Lambda_n) \to \mathbf{F}$ for every n.

Remark A.1. Let η be the length of the smallest Reeb chord of Λ. The set $\mathcal{R}(\Lambda_i, \Lambda_j)$ of Reeb chords starting on Λ_i and ending on Λ_j splits as

$$\mathcal{R}(\Lambda_i, \Lambda_j) = \mathcal{R}_{long}(\Lambda_i, \Lambda_j) \sqcup \mathcal{R}_{short}(\Lambda_i, \Lambda_j),$$

where $\mathcal{R}_{long}(\Lambda_i, \Lambda_j)$ is the set of chords with length greater than $\eta/2$ (and $\mathcal{R}_{short}(\Lambda_i, \Lambda_j)$ is the complementary set). In the following, we choose the numbers δ_k small enough so that the maximum of h_n is very small compare to η. In particular, the set $\mathcal{R}_{short}(\Lambda_i, \Lambda_j)$ is empty when $i \geq j$, and it is in bijection with $\mathrm{Crit}(h_j - h_i)$ when $i < j$.

A.2 Some A_∞-categories and A_∞-algebras

In the following, we fix a regular almost complex structure J on ξ (see section 3.1.2). We define three A_∞-categories $\mathcal{A}_\triangle$, for $\triangle \in \{++, +, 0\}$. $\mathcal{A}_{++}$ corresponds to the directed A_∞-category used in [17] section 3.3 in order to define the endomorphism algebra of ε in $\mathcal{A}ug_+(\Lambda)$, $\mathcal{A}_+$ is a non-directed version of $\mathcal{A}_{++}$, and $\mathcal{A}_0$ is an A_∞-category whose spaces of morphisms are generated by short Reeb chords.

Definition A.2. Let

$$\mathbf{\Lambda} := \bigsqcup_{n \in \mathbf{N}} \Lambda_n.$$

We consider the A_∞-categories $\mathcal{A}_\triangle$, $\triangle \in \{++, +, 0\}$, defined as follows:

1. the set of objects of $\mathcal{A}_\triangle$ is $\{\Lambda_n \mid n \in \mathbf{N}\}$,

2. the space of morphisms from Λ_i to Λ_j is

$$\mathcal{A}_\triangle(\Lambda_i, \Lambda_j) = \begin{cases} \langle \mathcal{R}(\Lambda_i, \Lambda_j) \rangle & \text{if } \triangle \in \{+, ++\} \\ \langle \mathcal{R}_{short}(\Lambda_i, \Lambda_j) \rangle & \text{if } \triangle = 0 \end{cases}$$

if $i < j$,

$$\mathcal{A}_\triangle(\Lambda_i, \Lambda_i) = \begin{cases} \mathbf{F} & \text{if } \triangle \in \{++, 0\} \\ \mathbf{F} \oplus \langle \mathcal{R}(\Lambda_i, \Lambda_i) \rangle & \text{if } \triangle = +, \end{cases}$$

and $\mathcal{A}_\triangle(\Lambda_i, \Lambda_j) = 0$ if $i > j$.

3. the operations are such that $e_{\Lambda_n} \in \mathcal{A}_\triangle(\Lambda_n, \Lambda_n)$ is a strict unit, and for every sequence of generators

$$(x_1, \ldots, x_d) \in \mathcal{A}_\triangle(\Lambda_{i_0}, \Lambda_{i_1}) \times \cdots \times \mathcal{A}_\triangle\left(\Lambda_{i_{d-1}}, \Lambda_{i_d}\right),$$

we have

$$\begin{aligned}&\mu_{\mathcal{A}_\triangle}(x_1, \ldots, x_d)\\&\quad= \sum_{a, \boldsymbol{\gamma_0}, \ldots, \boldsymbol{\gamma_d}} \#\mathcal{M}_{a, \boldsymbol{\gamma_d} x_d \boldsymbol{\gamma_{d-1}} x_{d-1} \ldots x_1 \boldsymbol{\gamma_0}}(\mathbf{R} \times \boldsymbol{\Lambda}, J)\, \varepsilon_{i_0}(\boldsymbol{\gamma_0}) \ldots \varepsilon_{i_d}(\boldsymbol{\gamma_d})\, a,\end{aligned}$$

where in the sum, a is a generator of $\mathcal{A}_\triangle(\Lambda_{i_0}, \Lambda_{i_d})$ and $\boldsymbol{\gamma_j}$ is a word of Reeb chords in $\mathcal{R}\left(\Lambda_{i_j}, \Lambda_{i_j}\right)$ (see Definition 3.4 for the moduli spaces).

Moreover, we let $W_\triangle$ be the set made of the generators $c_n \in \mathcal{A}_\triangle(\Lambda_n, \Lambda_{n+1})$ corresponding to the minimum of $(h_{n+1} - h_n)$ for $n \in \mathbf{N}$. We define

$$LA^*_\triangle(\Lambda) := \mathrm{End}_{\mathcal{A}_\triangle\left[W_\triangle^{-1}\right]}(\Lambda).$$

Lemma A.3. *For every integer j, for every $\triangle \in \{++, +, 0\}$ the morphism $c_j \in \mathcal{A}_\triangle(\Lambda_j, \Lambda_{j+1}) \cap W_\triangle$ satisfies the following. For every $i < j < k$, the chain maps*

$$\begin{cases} \mu^2_{\mathcal{A}_\triangle}(-, c_j) & : & \mathcal{A}_\triangle(\Lambda_i, \Lambda_j) & \to & \mathcal{A}_\triangle(\Lambda_i, \Lambda_{j+1}) \\ \mu^2_{\mathcal{A}_\triangle}(c_j, -) & : & \mathcal{A}_\triangle(\Lambda_{j+1}, \Lambda_{k+1}) & \to & \mathcal{A}_\triangle(\Lambda_j, \Lambda_{k+1}) \end{cases}$$

are quasi-isomorphisms. Moreover, if we denote by $\mathcal{A}_{long}(\Lambda_m, \Lambda_n)$ the subcomplex of $\mathcal{A}_+(\Lambda_m, \Lambda_n)$ generated by $\mathcal{R}_{long}(\Lambda_m, \Lambda_n)$, then the induced chain maps

$$\begin{cases} \mu^2_{\mathcal{A}_+}(-, c_j) & : & \mathcal{A}_{long}(\Lambda_i, \Lambda_j) & \to & \mathcal{A}_{long}(\Lambda_i, \Lambda_{j+1}) \\ \mu^2_{\mathcal{A}_+}(c_j, -) & : & \mathcal{A}_{long}(\Lambda_{j+1}, \Lambda_{k+1}) & \to & \mathcal{A}_{long}(\Lambda_j, \Lambda_{k+1}) \end{cases}$$

also are quasi-isomorphisms.

Proof. When the contact manifold is $V = \mathbf{R} \times P$, the result follows from the main analytic theorem of [29] (Theorem 3.6). Indeed, this result implies that (when the perturbation functions h_n are small enough) the 3-punctured pseudo-holomorphic discs with boundary on $\mathbf{R} \times \bigsqcup_n \Lambda_n$, asymptotic to c_j at one of their punctures, correspond to trivial cylinders with boundary on $\mathbf{R} \times \Lambda$.

The general case is part of a work in progress of Chantraine, Dimitroglou-Rizell and Ghiggini.

$\square$

A.3 Relations between A_∞-categories and A_∞-algebras

Observe that the inclusion $\iota : \mathcal{A}_{++} \to \mathcal{A}_+$ defines a strict A_∞-functor which sends W_{++} to W_+.

Lemma A.4. *The A_∞-functor*

$$\widetilde{\iota} : \mathcal{A}_{++}\left[W_{++}^{-1}\right] \xrightarrow{\sim} \mathcal{A}_+\left[W_+^{-1}\right]$$

induced by ι (see Definition 1.59) is a quasi-equivalence.

Proof. Fix an object $Y = \Lambda_m$ in $\mathcal{A}_{++}$. We want to prove that the map

$$\widetilde{\iota} : \mathcal{A}_{++}\left[W_{++}^{-1}\right](X, Y) \to \mathcal{A}_+\left[W_+^{-1}\right](\widetilde{\iota}X, \widetilde{\iota}Y)$$

is a quasi-isomorphism for every object X. Our strategy is to apply Proposition 1.82. We consider the modules $\mathcal{M}_{\mathcal{A}_+}$, $\mathcal{M}_{\mathcal{A}_{++}}$ and module maps $t_{\mathcal{A}_{++}} : \mathcal{A}_{++}(-, \Lambda_m) \to \mathcal{M}_{\mathcal{A}_{++}}$, $t_{\mathcal{A}_+} : \mathcal{A}_+(-, \Lambda_m) \to \mathcal{M}_{\mathcal{A}_+}$ of Definition 2.14. Observe that the diagram of $\mathcal{A}_{++}$-modules

$$\begin{tikzcd}
\bigoplus\limits_{n\geq 0} \mathcal{A}_{++}(-, \Lambda_n) \arrow[r, "\bigoplus\limits_{n\geq 0} t_{c_n}"] \arrow[d, "\mathrm{id}"] & \bigoplus\limits_{n\geq 0} \mathcal{A}_{++}(-, \Lambda_{n+1}) \arrow[d, "t_\iota"] \\
\bigoplus\limits_{n\geq 0} \mathcal{A}_{++}(-, \Lambda_n) \arrow[r, "t_\iota"] & \iota^*\mathcal{M}_{\mathcal{A}_+}
\end{tikzcd}$$

commutes up to the homotopy $h : \bigoplus\limits_{n\geq 0} \mathcal{A}_{++}(-, \Lambda_n) \to \iota^*\mathcal{M}_{\mathcal{A}_+}$ which is the strict module map sending $x \in \mathcal{A}_{++}(\Lambda_k, \Lambda_n)$ to

$$h(x) = sx \in \mathcal{A}_+(\Lambda_k, \Lambda_n)[1] \subset \iota^*\mathcal{M}_{\mathcal{A}_+}(\Lambda_k)$$

($s : \mathcal{A}_+(\Lambda_k, \Lambda_n) \to \mathcal{A}_+(\Lambda_k, \Lambda_n)[1]$ is the usual degree (-1) map). According to Proposition 1.72, this induces a closed $\mathcal{A}_{++}$-module map $t_0 : \mathcal{M}_{\mathcal{A}_{++}} \to \iota^*\mathcal{M}_{\mathcal{A}_+}$. Moreover, the following diagram of $\mathcal{A}_{++}$-modules is commutative

$$\begin{tikzcd}
\mathcal{A}_{++}(-,Y) \arrow[r, "t_\iota"] \arrow[d, "t_{\mathcal{A}_{++}}"] & \iota^*\mathcal{A}_{+}(-,\iota Y) \arrow[d, "\iota^* t_{\mathcal{A}_+}"] \\
\mathcal{M}_{\mathcal{A}_{++}} \arrow[r, "t_0"] & \iota^*\mathcal{M}_{\mathcal{A}_+}.
\end{tikzcd}$$

It remains to check the three items of Proposition 1.82. The first two items are satisfied according to Lemmas A.3 and 2.17. We now have to prove that $t_0 : \mathcal{M}_{\mathcal{A}_{++}} \to \iota^*\mathcal{M}_{\mathcal{A}_+}$ is a quasi-isomorphism. Let $X = \Lambda_k$ be some object in $\mathcal{A}_{++}$. Observe that the following diagram of chain complexes is commutative

$$\begin{tikzcd}
\mathcal{M}_{\mathcal{A}_{++}}(\Lambda_k) \arrow[r, "t_0"] & \iota^*\mathcal{M}_{\mathcal{A}_+}(\Lambda_k) = \mathcal{M}_{\mathcal{A}_+}(\Lambda_k) \\
\mathcal{A}_{++}(\Lambda_k,\Lambda_{k+1}) \arrow[r, "\iota"] \arrow[u] & \mathcal{A}_{+}(\Lambda_k,\Lambda_{k+1}). \arrow[u]
\end{tikzcd}$$

Each vertical map is a quasi-isomorphism according to Lemma 2.16, and the bottom horizontal map is the identity map. This implies that t_0 is a quasi-isomorphism, which is what we needed to prove.

$\square$

Observe that the projection $\pi : \mathcal{A}_+ \to \mathcal{A}_0$ defines a strict A_∞-functor which sends W_+ to W_0. Thus it induces a strict A_∞-functor $\widetilde{\pi} : \mathcal{A}_+\left[W_+^{-1}\right] \to \mathcal{A}_0\left[W_0^{-1}\right]$ (see Definition 1.59). Besides, let $\overline{LA^*}(\Lambda)$ be the augmented ideal of

$$LA^*(\Lambda) = \mathrm{End}_{\mathcal{A}_+}(\Lambda)$$

($\overline{LA^*}(\Lambda)$ is the endomorphism algebra of ε in the A_∞-category $\mathcal{A}ug_-(\Lambda)$ defined in [15], see section 3.1.3). The inclusion $\overline{LA^*}(\Lambda) \hookrightarrow \mathcal{A}_+(\Lambda,\Lambda)$ induces a strict A_∞-map $\overline{LA^*}(\Lambda) \to LA^*_+(\Lambda)$ whose image is contained in $\ker(\widetilde{\pi})$. Since $\ker(\widetilde{\pi})$ is an A_∞-subalgebra of $\mathrm{Cone}\left(LA^*_+(\Lambda) \xrightarrow{\widetilde{\pi}} LA^*_0(\Lambda)\right)$ (see Definition 1.45), we get a strict A_∞-map

$$\phi : \overline{LA^*}(\Lambda) \to \mathrm{Cone}\left(LA^*_+(\Lambda) \xrightarrow{\widetilde{\pi}} LA^*_0(\Lambda)\right).$$

Proposition A.5. *The A_∞-map*

$$\phi : \overline{LA^*}(\Lambda) \to \mathrm{Cone}\left(LA_+^*(\Lambda) \xrightarrow{\widetilde{\pi}} LA_0^*(\Lambda)\right)$$

is a quasi-isomorphism.

Proof. We will apply Proposition 1.82 to the A_∞-functor $\pi : \mathcal{A}_+ \to \mathcal{A}_0$ and the object $\Lambda_0 = \Lambda$ of $\mathcal{A}_+$. We consider the modules $\mathcal{M}_{\mathcal{A}_0}$, $\mathcal{M}_{\mathcal{A}_+}$ and module maps $t_{\mathcal{A}_+} : \mathcal{A}_+(-,\Lambda) \to \mathcal{M}_{\mathcal{A}_+}$, $t_{\mathcal{A}_0} : \mathcal{A}_0(-,\Lambda) \to \mathcal{M}_{\mathcal{A}_0}$ of Definition 2.14. Observe that the diagram of $\mathcal{A}_+$-modules

$$\begin{tikzcd}
\bigoplus_{n\geq 0} \mathcal{A}_+(-,\Lambda_n) \arrow[r, "\bigoplus_{n\geq 0} t_{c_n}"] \arrow[d, "\mathrm{id}"] & \bigoplus_{n\geq 0} \mathcal{A}_+(-,\Lambda_{n+1}) \arrow[d, "t_\pi"] \\
\bigoplus_{n\geq 0} \mathcal{A}_+(-,\Lambda_n) \arrow[r, "t_\pi"] & \pi^*\mathcal{M}_{\mathcal{A}_0}
\end{tikzcd}$$

commutes up to the homotopy $h : \bigoplus_{n\geq 0} \mathcal{A}_+(-,\Lambda_n) \to \pi^*\mathcal{M}_{\mathcal{A}_0}$ which is the strict module map sending $x \in \mathcal{A}_+(\Lambda_k,\Lambda_n)$ to

$$h(x) = s\pi(x) \in \mathcal{A}_0(\Lambda_k,\Lambda_n)[1] \subset \pi^*\mathcal{M}_{\mathcal{A}_0}(\Lambda_k)$$

($s : \mathcal{A}_0(\Lambda_k,\Lambda_n) \to \mathcal{A}_0(\Lambda_k,\Lambda_n)[1]$ is the usual degree (-1) map). According to Proposition 1.72, this induces a closed $\mathcal{A}_+$-module map $t_0 : \mathcal{M}_{\mathcal{A}_+} \to \pi^*\mathcal{M}_{\mathcal{A}_0}$. Moreover, the following diagram of $\mathcal{A}_+$-modules is commutative

$$\begin{tikzcd}
\mathcal{A}_+(-,\Lambda) \arrow[r, "t_\pi"] \arrow[d, "t_{\mathcal{A}_+}"] & \pi^*\mathcal{A}_0(-,\Lambda) \arrow[d, "\pi^* t_{\mathcal{A}_0}"] \\
\mathcal{M}_{\mathcal{A}_+} \arrow[r, "t_0"] & \pi^*\mathcal{M}_{\mathcal{A}_0}.
\end{tikzcd}$$

Now according to Proposition 1.82, there is a chain map $u : {}_{W_+^{-1}}\mathcal{M}_{\mathcal{A}_+}(\Lambda) \to {}_{W_0^{-1}}\mathcal{M}_{\mathcal{A}_0}(\Lambda)$

such that the following diagram of chain complexes commutes

$$\begin{tikzcd}
\mathcal{A}_+\left[W_+^{-1}\right](\Lambda,\Lambda) \arrow[r, "\widetilde{\pi}"] \arrow[d, "W_+^{-1}t_{\mathcal{A}_+}"] & \mathcal{A}_0\left[W_0^{-1}\right](\Lambda,\Lambda) \arrow[d, "W_0^{-1}t_{\mathcal{A}_0}"] \\
{}_{W_+^{-1}}\mathcal{M}_{\mathcal{A}_+}(\Lambda) \arrow[r, "u"] & {}_{W_0^{-1}}\mathcal{M}_{\mathcal{A}_0}(\Lambda) \\
\mathcal{M}_{\mathcal{A}_+}(\Lambda) \arrow[r, "t_0"] \arrow[u, hook] & \mathcal{M}_{\mathcal{A}_0}(\Lambda) \arrow[u, hook]
\end{tikzcd}$$

(the two vertical maps on the bottom are the inclusions). Moreover, the inclusions

$$\overline{LA^*}(\Lambda) \hookrightarrow LA_+^*(\Lambda)\,,\; \overline{LA^*}(\Lambda) \hookrightarrow {}_{W_+^{-1}}\mathcal{M}_{\mathcal{A}_+}(\Lambda)\,,\; \overline{LA^*}(\Lambda) \hookrightarrow \mathcal{M}_{\mathcal{A}_+}(\Lambda)$$

make the following diagram commutes

$$\begin{tikzcd}
\overline{LA^*}(\Lambda) \arrow[r, hook] \arrow[d, equal] & LA_+^*(\Lambda) \arrow[r, "\widetilde{\pi}"] \arrow[d, "W_+^{-1}t_{\mathcal{A}_+}"] & LA_0^*(\Lambda) \arrow[d, "W_0^{-1}t_{\mathcal{A}_0}"] \\
\overline{LA^*}(\Lambda) \arrow[r, hook] \arrow[d, equal] & {}_{W_+^{-1}}\mathcal{M}_{\mathcal{A}_+}(\Lambda) \arrow[r, "u"] & {}_{W_0^{-1}}\mathcal{M}_{\mathcal{A}_0}(\Lambda) \\
\overline{LA^*}(\Lambda) \arrow[r, hook] & \mathcal{M}_{\mathcal{A}_+}(\Lambda) \arrow[r, "t_0"] \arrow[u, hook] & \mathcal{M}_{\mathcal{A}_0}(\Lambda)\,. \arrow[u, hook]
\end{tikzcd}$$

According to Lemma A.3, $\mathcal{A}_+$ and $\mathcal{A}_0$ satisfy the hypotheses of Lemma 2.17. Thus, using Lemma 2.17 and Proposition 1.79, we know that the vertical maps are quasi-isomorphisms. Therefore, we get the following commutative diagram of chain complexes

$$\begin{tikzcd}
\overline{LA^*}(\Lambda) \arrow[r, "\phi"] \arrow[d, equal] & \mathrm{Cone}\left(LA_+^*(\Lambda) \xrightarrow{\widetilde{\pi}} LA_0^*(\Lambda)\right) \arrow[d, "\sim"] \\
\overline{LA^*}(\Lambda) \arrow[r] \arrow[d, equal] & \mathrm{Cone}\left({}_{W_+^{-1}}\mathcal{M}_{\mathcal{A}_+}(\Lambda) \xrightarrow{u} {}_{W_0^{-1}}\mathcal{M}_{\mathcal{A}_0}(\Lambda)\right) \\
\overline{LA^*}(\Lambda) \arrow[r, "\psi"] & \mathrm{Cone}\left(\mathcal{M}_{\mathcal{A}_+}(\Lambda) \xrightarrow{t_0} \mathcal{M}_{\mathcal{A}_0}(\Lambda)\right). \arrow[u, "\sim"]
\end{tikzcd}$$

Therefore, in order to show that

$$\phi : \overline{LA^*}(\Lambda) \to \mathrm{Cone}\left(LA^*_+(\Lambda) \xrightarrow{\widetilde{\pi}} LA^*_0(\Lambda)\right)$$

is a quasi-isomorphism, it suffices to show that

$$\psi : \overline{LA^*}(\Lambda) \to \mathrm{Cone}\left(\mathcal{M}_{\mathcal{A}_+}(\Lambda) \xrightarrow{t_0} \mathcal{M}_{\mathcal{A}_0}(\Lambda)\right)$$

is a quasi-isomorphism. Observe that the inclusion $\ker(t_0) \hookrightarrow \mathrm{Cone}(t_0)$ is a quasi-isomorphism because t_0 is surjective. Thus it suffices to show that the inclusion $\overline{LA^*}(\Lambda) \hookrightarrow \ker(t_0)$ is a quasi-isomorphism. Recall that we denote by $\mathcal{A}_{long}(\Lambda_m, \Lambda_n)$ the subcomplex of $\mathcal{A}_+(\Lambda_m, \Lambda_n)$ generated by $\mathcal{R}_{long}(\Lambda_m, \Lambda_n)$. Then observe that $\overline{LA^*}(\Lambda) = \mathcal{A}_{long}(\Lambda, \Lambda)$ and

$$\ker(t_0) = \left[\begin{tikzcd} \mathcal{A}_{long}(\Lambda, \Lambda_0) \arrow[d, "\mathrm{id}"] \arrow[dr, "t_{c_0}"] & \mathcal{A}_{long}(\Lambda, \Lambda_1) \arrow[d, "\mathrm{id}"] \arrow[dr, "t_{c_1}"] & \dots \\ \mathcal{A}_{long}(\Lambda, \Lambda_0) & \mathcal{A}_{long}(\Lambda, \Lambda_1) & \dots \end{tikzcd}\right].$$

Thus the result follows from Lemmas A.3 and 2.16.

$\square$

Appendix B

Ekholm-Lekili revisited

In this appendix, we revisit some of the results of [32] using localization of A_∞-categories.

B.1 Setting

Let X be a Weinstein manifold. Now let L be a connected cylindrical exact Lagrangian submanifold of X. We denote by $\Lambda = \partial_\infty L \subset Y$ the corresponding Legendrian submanifold. We will assume that Λ is connected and chord generic with respect to α. Finally, we denote by $\varepsilon_L : CE_{-*}(\Lambda) \to \mathbf{F}$ the augmentation induced by L.

We fix two positive Morse functions $F_1 : L \to \mathbf{R}$ and $f_1 : \Lambda \to \mathbf{R}$ with a unique minimum and a unique maximum, and such that in the ends

$$F_1(t,q) = a_1(t + f_1(q)) + b_1 \text{ for } 0 < a_1 \ll 1 \text{ and } b_1 > 0.$$

We denote by $\mathcal{U}$ a small open neighborhood of (F_1, f_1) in $C^\infty(L) \times C^\infty(\Lambda)$ with the following property: for every pair (F, f) in $\mathcal{U}$, the functions F, f are positive, Morse, and their critical points and gradient trajectories are in bijection with those of F_1, f_1 respectively. Then we choose a family $(F_n, f_n)_{n\geq 2}$ of pairs in $\mathcal{U}$ such that in the ends,

$$F_n(t,q) = a_n(t + f_n(q)) + b_n \text{ for } 0 < a_n \ll 1 \text{ and } b_n > 0.$$

We also choose a decreasing sequence $(\delta_k)_{k\geq 1}$ of small enough positive numbers so that the family

$$\left(H_n = \sum_{k=1}^{n} \delta_k F_k, h_n = \sum_{k=1}^{n} \delta_k f_k\right)_{n\in\mathbf{N}}$$

has the following property: for every $i < j$, the pair $(H_j - H_i, h_j - h_i)$ lies in the cone over $\mathcal{U}$ (this can be easily achieved by induction on n). Finally, we let $L_n \subset X$ be the Lagrangian corresponding to the graph of dH_n in a Weinstein neighborhood of $L = L_0$, and we let $\Lambda_n \subset Y$ be the Legendrian corresponding to the one-jet of h_n in a Weinstein neighborhood of $\Lambda = \Lambda_0$. Observe that we *do not* require the copies to be parallel in the sense of the third item of Lemma 31 (or Theorem 76) in [32].

B.2 Some A_∞-categories

We denote by $\mathcal{A}_+$, $\mathcal{A}_{++}$, $\mathcal{A}_0$ the A_∞-categories associated to Λ and ε_L in Definition A.2. We now associate an A_∞-category to L.

Definition B.1. Let

$$\mathbf{L} := (L_n)_{n\in\mathbf{Z}}\,.$$

We denote by $\mathcal{O}$ the A_∞-category defined as follows:

1. the objects of $\mathcal{O}$ are the Lagrangians L_n, $n \in \mathbf{Z}$,
2. the space of morphisms from L_i to L_j is either generated by $L_i \cap L_j$ if $i < j$, or $\mathbf{F}$ if $i = j$, or 0 if $i > j$; the bigrading of an intersection point $c \in \mathcal{O}(\Lambda^i, \Lambda^j)$ corresponding to $q_0 \in \mathrm{Crit}\, h_0$ is $(2(j-i) + \mathrm{ind}(q_0))$, and
3. the operations are such that $e_{L_n} \in \mathcal{O}(L_n, L_n)$ is a strict unit, and for every sequence of integers $i_0 < \cdots < i_d$, for every sequence of intersection points

$$(x_1, \ldots, x_d) \in (L_{i_0} \cap L_{i_1}) \times \cdots \times \left(L_{i_{d-1}} \cap L_{i_d}\right)$$

we have

$$\mu_{\mathcal{O}}(x_1, \ldots, x_d) = \sum_{x_0 \in L_{i_0} \cap L_{i_d}} \#\mathcal{M}_{x_0, x_d \ldots x_1}(\mathbf{L}, j)\, x_0$$

(see Definition 4.28). Besides, let W_L be the set made of the morphisms $u^n \in \mathcal{O}(L_n, L_{n+1})$ corresponding to the minimum of $(H_{n+1} - H_n)$ for $n \in \mathbf{N}$. We define

$$CF^*(L) := \mathrm{End}_{\mathcal{O}\left[W_L^{-1}\right]}(L)\,.$$

There are natural A_∞-functors $\psi : \mathcal{O} \to \mathcal{A}_{++}$ (see [32] section 5), $\iota : \mathcal{A}_{++} \to \mathcal{A}_+$ (the inclusion) and $\pi : \mathcal{A}_+ \to \mathcal{A}_0$ (the projection). The composition $\phi = \pi \circ \iota \circ \psi : \mathcal{O} \to \mathcal{A}_0$ sends W_L to W_0, and thus it induces an A_∞-map $\widetilde{\phi} : CF^*(L) \to LA_0^*(\Lambda)$.

Definition B.2. The Floer complex of L relative to its boundary Λ is the non unital A_∞-algebra

$$CF^*(L,\Lambda) := \mathrm{Cone}\left(CF^*(L) \xrightarrow{\widetilde{\phi}} LA_0^*(\Lambda)\right)$$

(see Definition 1.45).

B.3 Results

Recall that we are considering A_∞ functors $\psi : \mathcal{O} \to \mathcal{A}_{++}$ (as in [32] section 5) and $\iota : \mathcal{A}_{++} \to \mathcal{A}_+$ (the inclusion). Moreover, the composition $\iota \circ \psi : \mathcal{O} \to \mathcal{A}_+$ sends W_L to W_+.

Theorem B.3. *(Part of Theorem 63 in [32])*
If $HW^*(L) = 0$, then the A_∞-map

$$\widetilde{\iota \circ \psi} : CF^*(L) \to LA_+^*(\Lambda)$$

induced by $\iota \circ \psi$ is a quasi-isomorphism.

Proof. On the objects, $\psi(L_n) = \Lambda_n$ and, on the morphisms, ψ is defined using the exact same count of pseudo-holomorphic curves than in [32] Theorem 63 (the corresponding A_∞-map is denoted $\mathfrak{c}$ there). In [32] Theorem 63, the authors prove that for every $i < j$,

$$\mathrm{Cone}\left(\mathcal{O}(L_i, L_j) \xrightarrow{\psi} \mathcal{A}_{++}(\Lambda_i, \Lambda_j)\right)^{\#} \simeq CW^*\left(L_i, \widehat{L_j}\right)$$

where $\widehat{L_j}$ is a negative wrapping of L_j across L_i. By hypothesis, $CW^*(L)$ is acyclic, which implies that ψ^1 is a quasi-isomorphism. Thus, ψ is a quasi-equivalence, and since it sends W_L into W_+^{dir}, the induced functor $\widetilde{\psi} : \mathcal{O}\left[W_L^{-1}\right] \to \mathcal{A}_{++}\left[\left(W_+^{dir}\right)^{-1}\right]$ is also a quasi-equivalence (see Lemma 2.4 in [38]). This concludes the proof according to Lemma A.4.

□

Theorem B.4. *(Part of Theorem 63 in [32])*
If $HW^*(L) = 0$, then there is a quasi-isomorphism of A_∞-algebras

$$\mathbf{F} \oplus CF^*(L,\Lambda) \xrightarrow{\sim} LA^*(\Lambda).$$

Proof. First recall that the projection to the space of short Reeb chords induces an A_∞-functor $\pi : \mathcal{A}_+ \to \mathcal{A}_0$ which sends W_+ to W_0, and Proposition A.5 gives

$$\overline{LA^*}(\Lambda) \simeq \mathrm{Cone}\left(LA^*_+(\Lambda) \xrightarrow{\widetilde{\pi}} LA^*_0(\Lambda)\right).$$

Then, recall that the composition $\phi = \pi \circ \iota \circ \psi : \mathcal{O} \to \mathcal{A}_0$ sends W_L to W_0, and thus it induces an A_∞-map $\widetilde{\phi} : CF^*(L) \to LA^*_0(\Lambda)$. Now observe that we have the following commutative diagram

$$\begin{tikzcd} CF^*(L) \arrow[r, "\widetilde{\iota\circ\psi}"] \arrow[d, "\widetilde{\phi}"] & LA^*_+(\Lambda, \varepsilon_L) \arrow[d, "\widetilde{\pi}"] \\ LA^*_0(\Lambda, \varepsilon_L) \arrow[r, equal] & LA^*_0(\Lambda, \varepsilon_L). \end{tikzcd}$$

The top horizontal arrow is a quasi-isomorphism by Theorem B.3. According to Proposition 1.48, we get a quasi-isomorphism of non unital A_∞-algebras

$$CF^*(L, \Lambda) \xrightarrow{\sim} \overline{LA^*}(\Lambda).$$

This concludes the proof because

$$LA^*(\Lambda) = \mathbf{F} \oplus \overline{LA^*}(\Lambda).$$

$\square$

Theorem B.5. *(Theorem 51 in [32])*

If Λ is simply connected, then

$$E\left(CE^+_{-*}(\Lambda)\right) \simeq LA^*_+(\Lambda).$$

Idea of proof. Assume that we can choose the copies to be parallel. In [32] section 3.5, the authors define a quasi-isomorphism $\phi : CE^+_{-*}(\Lambda) \to CE^{\|}_{-*}(\Lambda)$. Applying first bar and then graded dual, we get a quasi-isomorphism $B\left(CE^{\|}_{-*}(\Lambda)\right)^{\#} \to B\left(CE^+_{-*}(\Lambda)\right)^{\#}$. Besides, recall that

$$CE^{\|}_{-*}(\Lambda) = \Omega LC^{\|}_*(\Lambda).$$

There is a natural quasi-isomorphism $B\left(\Omega LC^{\|}_*(\Lambda)\right) \to LC^{\|}_*(\Lambda)$ (see section 2.2.2 in [32]), which induces a quasi-isomorphism $LA^*_{\|}(\Lambda) \to B\left(\Omega LC^{\|}_*(\Lambda)\right)^{\#}$. As a result, we get a quasi-isomorphism

$$LA^*_{\|}(\Lambda) \to B\left(CE_{-*}(\Lambda)\right)^{\#}.$$

Now going back to the case where the copies are not parallel, we should still be able to define an A_∞-functor $\Phi : \mathcal{A}_+ \to B\left(CE_{-*}\left(\Lambda, C_{-*}(\Omega\Lambda)\right)\right)^{\#}$ which mimic the map we just described. This functor should look like this. Take sequences

$$\begin{cases} (c_0,\ldots,c_{d-1}) & \in & \mathcal{R}\left(\Lambda_{n_0},\Lambda_{n_1}\right)\times\cdots\times\mathcal{R}\left(\Lambda_{n_{d-1}},\Lambda_{n_d}\right) \\ (x_0,\ldots,x_{r-1}) & \in & CE^+_{-*}(\Lambda)^r \end{cases}$$

such that none of the Reeb chords c_i correspond to the minimum of the Morse function f_1, and each x_i is a word in elements of $C_{-*}(\Omega\Lambda)\cup\mathcal{R}(\Lambda)$. If r is positive, then

$$\left\langle \Phi\left(c_0,\ldots,c_{d-1}\right), sx_0\otimes\cdots\otimes sx_{r-1}\right\rangle = 0.$$

Assume now that $r=0$. Let $(\mathbf{y}_1,\ldots,\mathbf{y}_p)$ be the family of short Reeb chords appearing in $c_0\cdots c_{d-1}$. Then

$$\left\langle \Phi\left(c_0,\ldots,c_{d-1}\right), sx_0\right\rangle = \prod_{i=1}^{p}\left|\mathcal{T}\left(\sigma_i,\mathbf{y}_i\right)\right|$$

(the moduli spaces $\mathcal{T}(\sigma_i,\mathbf{y}_i)$ are defined in [32] section 3.5) if x_0 is obtained from $c_0\cdots c_{d-1}$ by replacing each $\mathbf{y}_i$ by σ_i, and 0 otherwise.

It remains to show that Φ sends W_+ to the unit (this should follow from Lemma 38 in [32]), and that the induced functor $\widetilde{\Phi} : \mathcal{A}_+\left[W_+^{-1}\right] \to B\left(CE^+_{-*}(\Lambda)\right)^{\#}\left[\mathbf{1}^{-1}\right]$ is a quasi-equivalence.

$\square$

Bibliography

[1] Casim Abbas. *An introduction to compactness results in symplectic field theory.* Springer, Heidelberg, 2014, pp. viii+252.

[2] Mohammed Abouzaid. “On the wrapped Fukaya category and based loops”. In: *Journal of Symplectic Geometry* 10.1 (2012), pp. 27–79.

[3] Mohammed Abouzaid and Paul Seidel. “An open string analogue of Viterbo functoriality”. In: *Geometry & Topology* 14.2 (2010), pp. 627–718.

[4] J Frank Adams. “On the cobar construction”. In: *Proceedings of the National Academy of Sciences of the United States of America* 42.7 (1956), p. 409.

[5] J Frank Adams and Peter J Hilton. “On the chain algebra of a loop space”. In: *Commentarii Mathematici Helvetici* 30.1 (1956), pp. 305–330.

[6] Manabu Akaho and Dominic Joyce. “Immersed Lagrangian Floer theory”. In: *Journal of differential geometry* 86.3 (2010), pp. 381–500.

[7] Garrett Alston and Erkao Bao. “Exact, graded, immersed Lagrangians and Floer theory”. In: *Journal of Symplectic Geometry* 16.2 (2018), pp. 357–438.

[8] Johan Asplund and Tobias Ekholm. “Chekanov-Eliashberg dg-algebras for singular Legendrians”. In: *arXiv:2102.04858v2* (2021).

[9] Denis Auroux. “A beginner’s introduction to Fukaya categories”. In: *Contact and symplectic topology.* Springer, 2014, pp. 85–136.

[10] Jean-François Barraud and Octav Cornea. “Lagrangian intersections and the Serre spectral sequence”. In: *Ann. of Math. (2)* 166.3 (2007), pp. 657–722.

[11] Daniel Bennequin. “Entrelacements et équations de Pfaff”. PhD thesis. Université de Paris VII, 1982.

[12] William M Boothby and Hsieu-Chung Wang. “On contact manifolds”. In: *Annals of Mathematics* (1958), pp. 721–734.

[13] Raoul Bott and Loring W Tu. *Differential forms in algebraic topology.* Vol. 82. Springer, 1982.

[14] F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki, and E. Zehnder. “Compactness results in symplectic field theory”. In: *Geom. Topol.* 7 (2003), pp. 799–888.

[15] Frédéric Bourgeois and Baptiste Chantraine. “Bilinearized Legendrian contact homology and the augmentation category”. In: *Journal of Symplectic Geometry* 12.3 (2014), pp. 553–583.

[16] Frédéric Bourgeois, Tobias Ekholm, and Yakov Eliashberg. “Effect of Legendrian surgery”. In: *Geometry & Topology* 16.1 (2012), pp. 301–389.

[17] Baptiste Chantraine. “Augmentations des sous-variétés Legendriennes: invariants associés et applications”. In: (2019).

[18] Baptiste Chantraine, Georgios Dimitroglou Rizell, Paolo Ghiggini, and Roman Golovko. “Geometric generation of the wrapped Fukaya category of Weinstein manifolds and sectors”. In: *arXiv preprint arXiv:1712.09126* (2017).

[19] Yuri Chekanov. “Differential algebra of Legendrian links”. In: *Inventiones mathematicae* 150.3 (2002), pp. 441–483.

[20] Kai Cieliebak. “Subcritical Stein manifolds are split”. In: *arXiv preprint math/0204351* (2002).

[21] Kai Cieliebak and Yakov Eliashberg. *From Stein to Weinstein and back: symplectic geometry of affine complex manifolds.* Vol. 59. American Mathematical Soc., 2012.

[22] Georgios Dimitroglou Rizell. “Legendrian ambient surgery and Legendrian contact homology”. In: *J. Symplectic Geom.* 14.3 (2016), pp. 811–901.

[23] Georgios Dimitroglou Rizell. “Lifting pseudo-holomorphic polygons to the symplectisation of $P \times \mathbf{R}$ and applications”. In: *Quantum Topology* 7.1 (2016), pp. 29–105.

[24] Simon Kirwan Donaldson. “Symplectic submanifolds and almost-complex geometry”. In: *Journal of Differential Geometry* 44.4 (1996), pp. 666–705.

[25] Tobias Ekholm. “Rational symplectic field theory over $\mathbb{Z}_2$ for exact Lagrangian cobordisms”. In: *J. Eur. Math. Soc. (JEMS)* 10.3 (2008), pp. 641–704.

[26] Tobias Ekholm, John Etnyre, and Michael Sullivan. “Legendrian contact homology in $P \times \mathbf{R}$”. In: *Transactions of the American Mathematical Society* 359.7 (2007), pp. 3301–3335.

[27] Tobias Ekholm, John Etnyre, and Michael Sullivan. “Non-isotopic Legendrian submanifolds in $\mathbf{R}^{2n+1}$”. In: *Journal of Differential Geometry* 71.1 (2005), pp. 85–128.

[28] Tobias Ekholm, John Etnyre, and Michael Sullivan. "The contact homology of Legendrian submanifolds in $\mathbf{R}^{2n+1}$". In: *Journal of Differential Geometry* 71.2 (2005), pp. 177–305.

[29] Tobias Ekholm, John B Etnyre, and Joshua M Sabloff. "A duality exact sequence for Legendrian contact homology". In: *Duke mathematical journal* 150.1 (2009), pp. 1–75.

[30] Tobias Ekholm and Tamás Kálmán. "Isotopies of Legendrian 1-knots and Legendrian 2-tori". In: *Journal of Symplectic Geometry* 6.4 (2008), pp. 407–460.

[31] Tobias Ekholm and Yanki Lekili. "Duality between Lagrangian and Legendrian invariants". In: *arXiv preprint arXiv:1701.01284* (2017).

[32] Tobias Ekholm and Yanki Lekili. "Duality between Lagrangian and Legendrian invariants". In: *arXiv:1701.01284v5* (2021).

[33] Tobias Ekholm and Lenhard Ng. "Legendrian contact homology in the boundary of a subcritical Weinstein 4-manifold". In: *Journal of Differential Geometry* 101.1 (2015), pp. 67–157.

[34] Yakov Eliashberg. "Invariants in contact topology". In: *Proceedings of the International Congress of Mathematicians*. Vol. 2. Berlin. 1998, pp. 327–338.

[35] Yakov Eliashberg, A Givental, and Helmut Hofer. "Introduction to Symplectic Field Theory". In: *Visions in mathematics*. Springer, 2000, pp. 560–673.

[36] Sheel Ganatra. "Symplectic cohomology and duality for the wrapped Fukaya category". In: *arXiv preprint arXiv:1304.7312* (2013).

[37] Sheel Ganatra, John Pardon, and Vivek Shende. "Covariantly functorial wrapped Floer theory on Liouville sectors". In: *Publications mathématiques de l'IHÉS* 131.1 (2020), pp. 73–200.

[38] Sheel Ganatra, John Pardon, and Vivek Shende. "Sectorial descent for wrapped Fukaya categories". In: *arXiv:1809.03427v2* (2019).

[39] Hansjörg Geiges. *An introduction to contact topology*. Vol. 109. Cambridge University Press, 2008.

[40] Emmanuel Giroux. "Géométrie de contact: de la dimension trois vers les dimensions supérieures". In: *arXiv preprint math/0305129* (2003).

[41] Mikhael Gromov. *Partial differential relations*. Vol. 9. Springer Science & Business Media, 1986.

[42] Jeff Hicks. "Some Practical Constructions with filtered A_∞-algebras". In: (2019).

[43] Cecilia Karlsson. "Legendrian contact homology for attaching links in higher dimensional subcritical Weinstein manifolds". In: *arXiv preprint arXiv:2007.07108* (2020).

[44] Yusuf Barış Kartal. "Distinguishing open symplectic mapping tori via their wrapped Fukaya categories". In: *Geom. Topol.* 25.3 (2021), pp. 1551–1630.

[45] Yusuf Barış Kartal. "Dynamical invariants of mapping torus categories". In: *Adv. Math.* 389 (2021), Paper No. 107882, 95.

[46] Bernhard Keller. "Introduction to A-infinity algebras and modules". In: *Homology, homotopy and applications* 3.1 (2001), pp. 1–35.

[47] Kenji Lefevre-Hasegawa. "Sur les A_∞-catégories". In: *arXiv preprint math/0310337* (2003).

[48] Jean-Louis Loday and Bruno Vallette. *Algebraic operads*. Vol. 346. Springer Science & Business Media, 2012.

[49] D-M Lu, John H Palmieri, Q-S Wu, and James J Zhang. "Koszul equivalences in A_∞-algebras". In: *New York Journal of Mathematics* 14 (2008).

[50] Volodymyr Lyubashenko and Sergiy Ovsienko. "A construction of quotient A_∞-categories". In: *Homology Homotopy Appl.* 8.2 (2006), pp. 157–203.

[51] Dusa McDuff and Dietmar Salamon. *Introduction to symplectic topology*. Third. Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, 2017, pp. xi+623.

[52] Lenhard Ng, Dan Rutherford, Vivek Shende, Steven Sivek, and Eric Zaslow. "Augmentations are sheaves". In: *Geometry & Topology* 24.5 (2020), pp. 2149–2286.

[53] Yu Pan and Dan Rutherford. "Functorial LCH for immersed Lagrangian cobordisms". In: *J. Symplectic Geom.* 19.3 (2021), pp. 635–722.

[54] Alain Prouté. "A_∞-structures. Modèles minimaux de Baues-Lemaire et Kadeishvili et homologie des fibrations". In: *Reprints in Theory and Applications of Categories* 21 (2011), pp. 1–99.

[55] Joel Robbin and Dietmar Salamon. “The Maslov index for paths”. In: *Topology* 32.4 (1993), pp. 827–844.

[56] Joshua M Sabloff. “Invariants of Legendrian knots in circle bundles”. In: *Communications in Contemporary Mathematics* 5.04 (2003), pp. 569–627.

[57] Paul Seidel. “A_∞-subalgebras and natural transformations”. In: *Homology Homotopy Appl.* 10.2 (2008), pp. 83–114.

[58] Paul Seidel. “Fukaya categories and deformations”. In: *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002).* Higher Ed. Press, Beijing, 2002, pp. 351–360.

[59] Paul Seidel. *Fukaya categories and Picard-Lefschetz theory.* Vol. 10. European Mathematical Society, 2008.

[60] Paul Seidel. “Lectures on categorical dynamics and symplectic topology”. In: *Notes, available on the author’s homepage* (2013).

[61] Serge Tabachnikov. “An invariant of a submanifold that is transversal to a distribution”. In: *Uspekhi Mat. Nauk* 43.3 (1988), p. 261.

DOCTORAT BRETAGNE LOIRE / MATHSTIC

Nantes Université

Titre : Invariants du relevé Legendrien d'une sous-variété Lagrangienne exacte dans la contactisation circulaire d'une variété de Liouville

Mot clés : Contactisation circulaire, Homologie de contact Legendrienne Morse-Bott, Algèbre différentielle graduée de Chekanov-Eliashberg, Cylindre d'application d'un morphisme entre algèbres différentielles graduées, Localisation de catégories A_∞, Colimite homotopique de catégories A_∞

Résumé : On étudie les relations entre une sous-variété Lagrangienne exacte L dans une variété de Liouville P et un de ses relevés Legendriens Λ° dans la contactisation circulaire $S^1 \times P$. Contrairement au cas bien étudié de la contactisation standard $\mathbf{R} \times P$, chaque point d'une Legendrienne dans $S^1 \times P$ donne lieu à une infinité (dénombrable) de cordes de Reeb pour la forme de contact standard, et on est dans une situation dite Morse-Bott.

On démontre d'abord des propriétés homotopiques générales de l'homologie de contact Legendrienne en utilisant les cylindres d'applications de morphismes entre algèbres différentielles graduées. Ensuite, on introduit la notion de tore d'application associé à une quasi-autoéquivalence d'une catégorie A_∞ comme colimite homotopique d'un diagramme. On montre comment calculer cette colimite dans certaines circonstances en travaillant avec des localisations de catégories A_∞. Enfin, on utilise les résultats précédents pour relier l'algèbre A_∞ de Floer de L et l'algèbre différentielle graduée de Chekanov-Eliashberg de Λ°.

Title: Invariants of the Legendrian lift of an exact Lagrangian submanifold in the circular contactization of a Liouville manifold

Keywords: Circular contactization, Morse-Bott Legendrian contact homology, Chekanov-Eliashberg differential graded algebra, Mapping cylinder of a morphism between differential graded algebras, Localization of A_∞-categories, Homotopy colimit of A_∞-categories

Abstract: We study the relations between an exact Lagrangian submanifold L in a Liouville manifold P and one of its Legendrian lift in the circular contactization $S^1 \times P$. Unlike the well-studied case of the standard contactization $\mathbf{R} \times P$, each point of a Legendrian in $S^1 \times P$ gives rise to a (countable) infinite set of Reeb chords for the standard contact form, and we are in a so-called Morse-Bott situation.

We first demonstrate general homotopical properties of Legendrian contact homology using mapping cylinders of morphisms between diffrential graded algebras. Then, we introduce the notion of mapping torus associated to a quasi-autoequivalence of an A_∞-category as homotopy colimit of some diagram. We show how to compute this colimit in certain circonstances by working with localizations of A_∞-categories. Finally, we use the previous results in order to relate the Floer A_∞-algebra of L and the Chekanov-Eliashberg differential graded algebra of Λ°.