



# Some results on the stabilization of elastic / viscoelastic transmission problems with Kelvin-Voigt or fractional Kelvin-Voigt damping

Ibtissam Issa

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# THÈSE DE DOCTORAT

Soutenue à Aix-Marseille Université  
le 07 Décembre 2021 par

**Ibtissam ISSA**

Some results on the stabilization of elastic / viscoelastic  
transmission problems with Kelvin-Voigt or fractional  
Kelvin-Voigt damping

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# Affidavit

Je soussigné, Ibtissam Issa, déclare par la présente que le travail présenté dans ce manuscrit est mon propre travail, réalisé sous la direction scientifique des Prof. Michel Mehrenbergerr et Prof. Ali Wehbe, dans le respect des principes d'honnêteté, d'intégrité et de responsabilité inhérents à la mission de recherche. Les travaux de recherche et la rédaction de ce manuscrit ont été réalisés dans le respect à la fois de la charte nationale de déontologie des métiers de la recherche et de la charte d'Aix-Marseille Université relative à la lutte contre le plagiat.

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# Liste de publications et participation aux conférences

## Liste des publications réalisées dans le cadre du projet de thèse:

1. Stability Results of an Elastic/Viscoelastic Transmission Problem of Locally Coupled Waves with Non Smooth Coefficients. In: Acta Applicandae Mathematicae [94](doi: <https://doi.org/10.1007/s10440-021-00384-8>, Ali wehbe, Ibtissam Issa, and Mohammad Akil)
2. Energy Decay of some boundary coupled systems involving wave/ Euler-Bernoulli beam with one locally singular fractional Kelvin-Voigt damping (<https://arxiv.org/abs/2102.12732>, Mohammad Akil, Ibtissam Issa, and Ali Wehbe)
3. A N-dimensional elastic/viscoelastic transmission problem with Kelvin-Voigt damping and non smooth coefficient at the interface (<https://arxiv.org/abs/2107.13785>, Mohammad Akil, Ibtissam Issa, and Ali Wehbe)

## Participation aux conférences et écoles d'été au cours de la période de thèse:

1. Attending mini courses in the Conference of Center for Advanced Mathematical Sciences (CAMS), American University of Beirut, Lebanon.
2. Participation in Discrete Mathematics and Computer Science, DIMACOS'19, Tunis.
3. Participation in the Lebaneses International Conference on Mathematics and Applications (LICMA'19).
4. Participation in Centre International de Mathématiques Pures et Appliquées (CIMPA).



# Résumé

Cette thèse est consacrée à l'étude de la stabilisation de certains systèmes localement couplés. Tout d'abord, nous étudions la stabilité d'équations d'onde couplées unidimensionnelles avec deux amortissements visqueux intérieurs non lisses où nous établissons une stabilité exponentielle. Dans un second temps, nous étudions la stabilisation d'équations d'onde localement couplées avec un seul amortissement viscoélastique interne de type Kelvin-Voigt. L'amortissement et les coefficients de couplage ne sont pas lisses. En utilisant une approche spectrale, nous démontrons la stabilité non uniforme du système. Ensuite, en utilisant une approche du domaine fréquentiel, combinée à une technique de multiplicateur par morceaux et à la construction d'un nouveau multiplicateur satisfaisant quelques équations différentielles ordinaires, nous montrons que l'énergie de la solution lisse du système décroît polynomiale.

Troisièmement, nous étudions la décroissance énergétique de systèmes hyperboliques de type onde-onde, onde-Euler Bernoulli et faisceau-faisceau. En effet, les deux équations sont couplées par liaison limite avec un seul amortissement fractionnaire Kelvin Voigt localisé non lisse. En utilisant l'approche du domaine fréquentiel, combinée à la technique du multiplicateur et à certaines inégalités d'interpolation, nous établissons différents types de taux de décroissance d'énergie polynomiale qui dépendent de l'ordre de la dérivée fractionnaire et du type de l'équation amortie dans le système. Enfin, nous étudions la stabilité d'un système multidimensionnel de deux équations d'onde couplées par des vitesses avec un seul amortissement Kelvin-Voigt localisé non lisse. En utilisant une analyse spectrale, nous prouvons la stabilité non uniforme du système. De plus, en utilisant une approche du domaine fréquentiel combinée à une technique de multiplicateur, nous établissons des résultats de stabilité polynomiale en considérant différentes conditions géométriques sur les domaines de couplage et d'amortissement. De plus, en l'absence de toute condition géométrique, nous établissons deux taux de décroissance polynomiale de l'énergie du système sur un domaine carré où l'amortissement et le couplage sont localisés dans une bande verticale

Mots clés:  $C_0$ -semi-groupe, Stabilité forte, Stabilité exponentielle, Stabilité polynomiale, Méthode du domaine fréquentiel, Méthode du multiplicateur, Analyse spectrale, Condition géométrique

# Abstract

This thesis is devoted to study the stabilization of some locally coupled systems. First, we study the stability of a one-dimensional coupled wave equations with two interior non smooth viscous dampings where we establish exponential stability. Second, we study the stabilization of a locally coupled wave equations with only one internal viscoelastic damping of Kelvin-Voigt type. Both the damping and the coupling coefficients are non smooth. Using a spectrum approach, we prove the non-uniform stability of the system. Next, using a frequency domain approach, combined with a piecewise multiplier technique and the construction of a new multiplier satisfying some ordinary differential equations, we show that the energy of the smooth solution of the system decays polynomially.

Third, we investigate the energy decay of hyperbolic systems of wave-wave, wave-Euler Bernoulli beam and beam-beam types. Indeed, the two equations are coupled through boundary connection with only one localized non smooth fractional Kelvin Voigt damping. By using frequency domain approach, combined with multiplier technique and some interpolation inequalities, we establish different types of polynomial energy decay rate which depend on the order of the fractional derivative and the type of the damped equation in the system.

Finally, we study the stability of a multidimensional system of two wave equations coupled by velocities with only one localized non-smooth Kelvin-Voigt damping. By using a spectral analysis, we prove the non uniform stability of the system. Further, using a frequency domain approach combined with a multiplier technique, we establish some polynomial stability results by considering different geometric conditions on the coupling and the damping domains. In addition, in the absence of any geometric condition, we establish two polynomial energy decay rates of the system on a square domain where the damping and the coupling are localized in a vertical strip.

**Keywords:**  $C_0$ -semigroup, Strong Stability, Exponential Stability, Polynomial Stability, Frequency domain method, Multiplier method, Spectral analysis, Geometric Condition

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# List of Symbols

$\mathbb{R}$	The set of real numbers.
$\mathbb{R}_+$	The set of non negative real numbers.
$\mathbb{R}^*$	The set of non zero real numbers.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}^*$	The set of non zero natural numbers.
$\mathbb{Z}$	The set of integer numbers.
$\mathbb{Z}^*$	The set of non zero integer numbers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Q}_+$	The set of non negative rational numbers.
$\mathbb{Q}^*$	The set of non zero rational numbers.
$\mathbb{C}$	The set of complex numbers.
$i$	The imaginary unit.
$\text{Re}$	The real part.
$\text{Im}$	The imaginary part.
$L^p$	The Lebesgue space.
$H^m$	The sobolev space.
$C^0$	The space of continuous function.
$C^1$	the space of continuously differentiable functions.
$ \cdot $	The modulus.
$\ \cdot\ $	The norm.
$\max$	The maximum.
$\min$	The minimum.
$\sup$	The supreme.
$\inf$	The infimum.
$f_y = \partial_y f$	The partial derivative of $f$ with respect of $y$ .
$f_{yy} = \partial_{yy} f$	The second partial derivative of $f$ with respect of $y$ .
$\partial_t^{\alpha, \eta}$	Fractional Derivative.
$\text{sign}$	The sign function or signum function.



# Asymptotic Notation

Let  $f$  and  $g$  be two functions defined on some subset of the real numbers, we define

- $f(x) = O(g(x))$  as  $x \rightarrow \infty$  to mean that there exists a positive number  $M$  and a real numbers  $x_0$  such that

$$|f(x)| \leq M |g(x)| \quad \forall x \geq x_0.$$

- $f(x) = o(g(x))$  as  $x \rightarrow \infty$  to mean that

$$\lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| = 0.$$

- $f \sim g$  as  $x \rightarrow \infty$  to mean that

$$\lim_{x \rightarrow \infty} \left| \frac{f(x)}{g(x)} \right| = 1.$$

# Introduction

The word control has a double meaning. First, controlling a system can be understood simply as testing or checking that its behavior is satisfactory. In a deeper sense, to control is also to act, to put things in order to guarantee that the system behaves as desired. S. Bennett starts the first volume of his book [25] on the history of Control Engineering quoting the following sentence of Chapter 3, Book 1, of the monograph 'politics' by Aristotle: ". . . if every instrument could accomplish its own work, obeying or anticipating the will of others . . . if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves". This sentence by Aristotle describes in a rather transparent way the guiding goal of Control Theory: the need of automatizing processes to let the human being gain in liberty, freedom, and quality of life.

Control theory is applied in a diverse range of scientific and engineering disciplines, such as the reduction of noise, the vibration of structures like seismic waves and earthquakes, the regulation of biological systems like the human cardiovascular system, and the design of robotic systems.

Stability and controllability are among the fundamental concepts in modern mathematical control theory. They are qualitative properties of control systems and are of particular importance in control theory.

In this thesis, we study the stabilization of certain problems with different types of damping and by considering different positions of the damping by using the semigroup theory. The thesis is divided into five chapters.

**In Chapter 1**, we present some results and theorems in semigroup theory. Also, we present some theorems that acknowledge us about the stabilization of some systems of differential equations including some theorems on strong, exponential, polynomial and analytic stability of a  $C_0$ -semigroup. In addition, we present the definition of some geometric conditions.

**In Chapter 2**, we study the stability of a one-dimensional coupled wave equations

with two interior viscous dampings. The system is given as follows

$$\begin{cases} \varphi_{tt} - a\varphi_{xx} + (\mathbb{1}_{(\alpha,\beta)})(x)\varphi_t + c(x)\psi_t = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ \psi_{tt} - \psi_{xx} + (\mathbb{1}_{(\alpha,\beta)})(x)\psi_t - c(x)\varphi_t = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ \varphi(0, t) = \varphi(L, t) = 0, & t > 0, \\ \psi(0, t) = \psi(L, t) = 0, & t > 0. \end{cases} \quad (0.0.1)$$

where

$$c(x) = \begin{cases} c_0 & \text{if } x \in (\alpha, \gamma) \\ 0 & \text{otherwise} \end{cases} \quad (0.0.2)$$

and  $0 < \alpha < \beta < \gamma < L$ ,  $a > 0$  and  $c_0 \in \mathbb{R}^*$  and with the following initial data

$$\varphi(\cdot, 0) = \varphi_0(\cdot), \varphi_t(\cdot, 0) = \varphi_1(\cdot), \psi(\cdot, 0) = \psi_0(\cdot) \quad \text{and} \quad \psi_t(\cdot, 0) = \psi_1(\cdot). \quad (0.0.3)$$

First, we start by studying the existence, uniqueness and regularity of the solution of the system using the semigroup approach. Let  $(\varphi, \varphi_t, \psi, \psi_t)$  be a regular solution of (0.0.1)-(0.0.3), its associated total energy is defined by

$$E_a(t) = \frac{1}{2} \int_0^L \left( |\varphi_t|^2 + a|\varphi_x|^2 + |\psi_t|^2 + |\psi_x|^2 \right) dx$$

By direct computation, we obtain

$$\frac{d}{dt} E_a(t) = - \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x) |\varphi_t|^2 dx - \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x) |\psi_t|^2 dx \leq 0.$$

Thus, System (0.0.1)-(0.0.3) is dissipative in the sense that its energy is a non-increasing function with respect to the time variable  $t$ . Next, we define the energy space  $\mathcal{H}_a$  by

$$\mathcal{H}_a = (H_0^1(0, L) \times L^2(0, L))^2.$$

We denote by  $\eta = \varphi_t$  and  $\xi = \psi_t$ . The energy space  $\mathcal{H}_a$  is endowed with the following norm

$$\|\Phi\|_{\mathcal{H}_a}^2 = \|\eta\|^2 + a\|\varphi_x\|^2 + \|\xi\|^2 + \|\psi_x\|^2,$$

where  $\|\cdot\|$  denotes the norm of  $L^2(0, L)$  and  $\Phi = (\varphi, \eta, \psi, \xi)$  in  $\mathcal{H}_a$ . We define the unbounded linear operator  $\mathcal{A}_a : D(\mathcal{A}_a) \subset \mathcal{H}_a \longrightarrow \mathcal{H}_a$  by

$$D(\mathcal{A}_a) = (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)^2, \quad (0.0.4)$$

and for all  $\Phi = (\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$ ,

$$\mathcal{A}_a(\varphi, \eta, \psi, \xi) = (\eta, a\varphi_{xx} - (\mathbb{1}_{(\alpha,\beta)})(x)\eta - c(x)\xi, \xi, \psi_{xx} - (\mathbb{1}_{(\alpha,\beta)})(x)\xi + c(x)\eta)^\top. \quad (0.0.5)$$

If  $\Phi = (\varphi, \eta, \psi, \xi)$  is the state of System (0.0.1), then this system is transformed into a first order evolution equation on the Hilbert space  $\mathcal{H}_a$  given by

$$\Phi_t = \mathcal{A}_a \Phi, \quad \Phi(0) = \Phi_0, \quad (0.0.6)$$

where  $\Phi_0 = (\varphi_0, \eta_0, \psi_0, \xi_0)$ .

It is easy to prove that the operator  $\mathcal{A}_a$  is m-dissipative on  $\mathcal{H}_a$  and consequently, generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_a})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8 (see in [73] and [82]). Then, the solution of the evolution Equation (0.0.6) admits the following representation

$$\Phi(t) = e^{t\mathcal{A}_a} \Phi_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.6). Using the semi-group theory we have the following existence and uniqueness theorem.

**Theorem 0.0.1.** *Let  $\Phi_0 \in \mathcal{H}_a$ . Then, Problem (0.0.6) admits a unique weak solution  $\Phi$  that satisfies*

$$\Phi(t) \in C^0(\mathbb{R}^+, \mathcal{H}_a).$$

*Moreover, if  $\Phi_0 \in D(\mathcal{A}_a)$  then, Problem (0.0.6) admits a unique strong solution  $\Phi$  that satisfies*

$$\Phi(t) \in C^1(\mathbb{R}^+, \mathcal{H}_a) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_a)).$$

Next, we prove the strong stability of the system by using a general criteria of Arendt and Batty. The result is given by the following theorem

**Theorem 0.0.2.** *The  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_a})_{t \geq 0}$  is strongly stable in  $\mathcal{H}_a$ ; i.e. for all  $\Phi_0 \in \mathcal{H}_a$ , the solution of (0.0.6) satisfies*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_a} \Phi\|_{\mathcal{H}_a} = 0.$$

The proof of this theorem is done by proving that  $\mathcal{A}_a$  has non pure imaginary eigenvalues. Since  $D(\mathcal{A}_a)$  is compactly embeded in  $\mathcal{H}_a$ ,  $\mathcal{A}_a$  is maximal dissipative,  $\mathcal{A}_a$  has no pure imaginary eigenvalues, and  $\mathcal{A}_a$  has a compact resolvent. Then, using Arendt-Batty Theorem 1.3.3, we reach that the  $C_0$ -semigroup  $(e^{t\mathcal{A}_a})_{t \geq 0}$  is strongly stable.

Now, we prove the energy decay rate by using a frequency domain approach combined with a piecewise multiplier technique. The main result is represented in the following theorem.

**Theorem 0.0.3.** *The  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_a})_{t \geq 0}$  is exponentially stable, i.e. there exists constants  $M \geq 1$  and  $\tau > 0$  independent of  $\Phi_0$  such that*

$$\|e^{t\mathcal{A}_a} \Phi_0\|_{\mathcal{H}_a} \leq M e^{-\tau t} \|\Phi_0\|_{\mathcal{H}_a}, \quad t \geq 0, \forall \Phi_0 \in \mathcal{H}_a.$$

The proof of this theorem is done by using the result of Huang and Prüss (see Theorem 1.3.5).

**In Chapter 3**, we study the stabilization of a locally coupled wave equations with only one internal viscoelastic damping of Kelvin-Voigt type. Both the damping and the coupling coefficients are non smooth. The system is described by

$$u_{tt} - (au_x + b(x)u_{tx})_x + c(x)y_t = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (0.0.7)$$

$$y_{tt} - y_{xx} - c(x)u_t = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (0.0.8)$$

with fully Dirichlet boundary conditions,

$$u(0, t) = u(L, t) = y(0, t) = y(L, t) = 0, \quad \forall t \in \mathbb{R}^+, \quad (0.0.9)$$

where

$$b(x) = \begin{cases} b_0 & \text{if } x \in (\alpha_1, \alpha_3) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c(x) = \begin{cases} c_0 & \text{if } x \in (\alpha_2, \alpha_4) \\ 0 & \text{otherwise} \end{cases} \quad (0.0.10)$$

and  $a > 0, b_0 > 0$  and  $c_0 \in \mathbb{R}^*$ , and where we consider  $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L$ . This system is considered with the following initial data

$$u(\cdot, 0) = u_0(\cdot), \quad u_t(\cdot, 0) = u_1(\cdot), \quad y(\cdot, 0) = y_0(\cdot) \quad \text{and} \quad y_t(\cdot, 0) = y_1(\cdot). \quad (0.0.11)$$

The main novelty in this chapter is that it generalizes the work in [50] by considering the damping and the coupling coefficients to be non-smooth, the damping to be internally localized and by improving the polynomial energy decay rate of the system. First, using a semigroup approach we establish the well posedness of the system (0.0.7)-(0.0.11). The energy of System (0.0.7)-(0.0.11) is given by

$$E(t) = \frac{1}{2} \int_0^L (|u_t|^2 + a|u_x|^2 + |y_t|^2 + |y_x|^2) dx.$$

Let  $(u, u_t, y, y_t)$  be a regular solution of (0.0.7)-(0.0.11). Multiplying (0.0.7), (0.0.8) by  $u_t, y_t$ , respectively, then using the boundary conditions in (0.0.9), we get

$$E'(t) = - \int_0^L b(x)|u_{tx}|^2 dx,$$

using the definition of the function  $b(x)$ , we get that  $E'(t) \leq 0$ . Thus, System (0.0.7)-(0.0.11) is dissipative in the sense that its energy is a non-increasing function with respect to the time variable  $t$ . We define the energy space  $\mathcal{H}$  by

$$\mathcal{H} = (H_0^1(0, L) \times L^2(0, L))^2.$$

The energy space  $\mathcal{H}$  is equipped with the inner product defined by

$$\langle U, U_1 \rangle_{\mathcal{H}} = \int_0^L v \bar{v}_1 dx + a \int_0^L u_x (\bar{u}_1)_x dx + \int_0^L z \bar{z}_1 dx + \int_0^L y_x (\bar{y}_1)_x dx,$$

for all  $U = (u, v, y, z)$  and  $U_1 = (u_1, v_1, y_1, z_1)$  in  $\mathcal{H}$ . We use  $\|U\|_{\mathcal{H}}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \longrightarrow \mathcal{H}$  by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, y, z) \in \mathcal{H}; \ y \in H^2(0, L) \cap H_0^1(0, L) \\ v, z \in H_0^1(0, L), (au_x + b(x)v_x)_x \in L^2(0, L) \end{array} \right\}$$

and for all  $U = (u, v, y, z) \in D(\mathcal{A})$ ,

$$\mathcal{A}(u, v, y, z) = (v, (au_x + b(x)v_x)_x - c(x)z, z, y_{xx} + c(x)v)^\top.$$

If  $U = (u, u_t, y, y_t)$  is the state of System (0.0.7)-(0.0.11), then this system is transformed into the first order evolution equation on the Hilbert space  $\mathcal{H}$  given by

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \quad (0.0.12)$$

where  $U_0 = (u_0, u_1, y_0, y_1)$ .

It is easy to check that the operator  $\mathcal{A}$  is m-dissipative on  $\mathcal{H}$  and consequently, it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  following Lumer-Phillips Theorem. Then, the solution of the evolution Equation (0.0.12) admits the following representation

$$U(t) = e^{t\mathcal{A}} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.12). Hence, using the semigroup theory we have the following existence and uniqueness result

**Theorem 0.0.4.** *Let  $U_0 \in \mathcal{H}$  then, problem (0.0.12) admits a unique weak solution  $U$  satisfying*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

*Moreover, if  $U_0 \in D(\mathcal{A})$  then, problem (0.0.12) admits a unique strong solution  $U$  satisfying*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

Next, we study the strong stability of system (0.0.7)-(0.0.11) in the sense that its energy converges to zero when  $t$  tends to infinity for all initial data in  $\mathcal{H}$ . It is easy to see that the resolvent of  $\mathcal{A}$  is not compact, then we use a general criteria of Arendt-Batty Theorem 1.3.3 (see [17]), in which a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  in a Banach space is strongly stable, if  $\mathcal{A}$  has no pure imaginary eigenvalues and  $\sigma(\mathcal{A}) \cap i\mathbb{R}$  contains only a countable number of continuous spectrum of  $\mathcal{A}$ . We have the following stability result :

**Theorem 0.0.5.** *The  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable in  $\mathcal{H}$ ; i.e. for all  $U_0 \in \mathcal{H}$ , the solution of (0.0.12) satisfies*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

The proof of this theorem relies on the following two Lemmas

**Lemma 0.0.6.** For  $\lambda \in \mathbb{R}$ , we have  $i\lambda I - \mathcal{A}$  is injective i.e.

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

**Lemma 0.0.7.** For all  $\lambda \in \mathbb{R}$ , we have

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

Using Lemma 0.0.6, we have that  $\mathcal{A}$  has non pure imaginary eigenvalues. According to Lemmas 0.0.6, 0.0.7 and with the help of the closed graph theorem of Banach, we deduce that  $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ . Thus, we get the result by applying Theorem 1.3.3 of Arendt and Batty.

Now, our aim is to prove that the system (0.0.7)-(0.0.11) is not uniformly stable. Here, we prove the lack of uniform stability using two different approaches: The first approach is when the Kelvin-Voigt damping is global and the other approach when the damping is localized. First, we assume that

$$b(x) = b_0 > 0 \quad \text{and} \quad c(x) = c_0 \in \mathbb{R}^*, \quad \forall x \in (0, L). \quad (0.0.13)$$

Our main result here is the following theorem.

**Theorem 0.0.8.** Under hypothesis (0.0.13), for  $\varepsilon > 0$  small enough, we cannot expect the energy decay rate  $\frac{1}{t^{2-\varepsilon}}$  for all initial data  $U_0 \in D(\mathcal{A})$  and for all  $t > 0$ .

In the second approach, we assume that

$$a = 1, \quad b(x) = \begin{cases} 0 & \text{if } 0 < x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}, \quad \text{and} \quad c(x) = c \in \mathbb{R}^*. \quad (0.0.14)$$

Our main result here is the following theorem.

**Theorem 0.0.9.** Under condition (0.0.14). The semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  generated by the operator  $\mathcal{A}$  is not exponentially stable in the energy space  $\mathcal{H}$ .

This result relies on the fact that there exists a subsequence of eigenvalues of  $\mathcal{A}$  close to the imaginary axis. Thus, under the condition (0.0.14), we prove that there exists  $n_0 \in \mathbb{N}$  sufficiently large and two sequences  $(\lambda_{1,n})_{|n| \geq n_0}$  and  $(\lambda_{2,n})_{|n| \geq n_0}$  satisfying the following asymptotic behavior:

**Case 1.** If  $\sin(\frac{c}{4}) \neq 0$ , then

$$\lambda_{1,n} = 2n\pi i + i\pi - \frac{2\sin^2(\frac{c}{4})(1 - i \operatorname{sign}(n))}{(3 + \cos(\frac{c}{2}))\sqrt{|n|\pi}} + O\left(\frac{1}{n}\right) \quad (0.0.15)$$

and

$$\lambda_{2,n} = 2n\pi i + i \arccos\left(\cos^2\left(\frac{c}{4}\right)\right) - \frac{\gamma}{\sqrt{|n|\pi}} + i \frac{\operatorname{sign}(n)\gamma}{\sqrt{|n|\pi}} + O\left(\frac{1}{n}\right), \quad (0.0.16)$$

where

$$\gamma = \frac{\left( \cos\left(\frac{c}{2}\right) \sin\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right) + \sin\left(\frac{3\arccos(\cos^2(\frac{c}{4}))}{2}\right) \right)}{4\sqrt{1 - \cos^4\left(\frac{c}{4}\right)} \cos\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right)}.$$

**Case 2.** If  $\sin\left(\frac{c}{4}\right) = 0$ , then

$$\lambda_{1,n} = 2n\pi i + i\pi + \frac{i c^2}{32\pi n} - \frac{(4 + i\pi)c^2}{64\pi^2 n^2} + O\left(\frac{1}{|n|^{\frac{5}{2}}}\right) \quad (0.0.17)$$

and

$$\lambda_{2,n} = 2n\pi i + O\left(\frac{1}{n}\right). \quad (0.0.18)$$

Now, in the absence of the uniform stability we aim to establish a polynomial decay rate for the system. Using a frequency domain approach, combined with a piecewise multiplier technique and using the exponential stability result of the system in Chapter 2, we establish a polynomial energy decay rate given in the following theorem.

**Theorem 0.0.10.** There exists a constant  $C > 0$  independent of  $U_0$ , such that the energy of system (0.0.7)-(0.0.11) satisfies the following estimation:

$$E(t) \leq \frac{C}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}). \quad (0.0.19)$$

We prove this theorem using the result of Borichev-Tomilov 1.3.10 (see [27]), by taking  $\ell = 2$ , the polynomial energy decay (0.0.19) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad (H1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^2} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (H2)$$

are satisfied. Condition (H1) is already proved in Lemma 0.0.6. We prove condition (H2) using an argument of contradiction. For this purpose, we suppose that (H2) is false, then there exists  $\{(\lambda_n, U_n = (u_n, v_n, y_n, z_n))\}_{n \geq 1} \subset \mathbb{R} \times D(\mathcal{A})$  and

$$\lambda_n \rightarrow +\infty, \quad \|U_n\|_{\mathcal{H}} = 1, \quad (0.0.20)$$

such that

$$\lambda_n^2 (i\lambda_n U_n - \mathcal{A} U_n) = (f_{1,n}, g_{1,n}, f_{2,n}, g_{2,n}) := F_n \rightarrow 0 \text{ in } \mathcal{H}. \quad (0.0.21)$$

We check the condition (H2) by finding a contradiction with (0.0.20) by showing that  $\|U\|_{\mathcal{H}} = o(1)$ . Hence, we reach that the condition (H2) holds and, consequently, that the polynomial energy decay (0.0.19) holds.

**In Chapter 4**, the main interest in this chapter is that the results here improve that in



[47] in the case when the damping acts on the wave equation only. Also, the result, in this chapter, when considering the coupled wave and Euler-Bernoulli beam equations with the damping acting on the beam equation shows that the boundary conditions play a critical role in the energy decay rate of the system (compared to that in [47]). In this chapter, we investigate the stability results of five models of systems with a non-smooth localized fractional Kelvin-Voigt damping where the coupling is made via boundary connection. In the first model (EBB)- $W_{FKV}$ , we investigate the stability of coupled Euler-Bernoulli beam and wave equations. The coupling is via boundary connections with localized non-regular fractional Kelvin-Voigt damping where the damping acts through the wave equation only. The system that describes this model is as follows

$$\left\{ \begin{array}{ll} u_{tt} - (au_x + d(x)\partial_t^{\alpha,\eta}u_x)_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ y_{tt} + by_{xxxx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ u(L, t) = y(-L, t) = y_x(-L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) + by_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, L), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (-L, 0). \end{array} \right. \quad ((\text{EBB})-W_{FKV})$$

The coefficients  $a, b$  are strictly positive constant numbers,  $\alpha \in (0, 1)$  and  $\eta \geq 0$ . We suppose that there exists  $0 < l_0 < l_1 < L$  and a strictly positive constant  $d_0$ , such that

$$d(x) = \begin{cases} d_0, & x \in (l_0, l_1), \\ 0, & x \in (0, l_0) \cup (l_1, L). \end{cases} \quad (0.0.22)$$

The Caputo's fractional derivative  $\partial_t^{\alpha,\eta}$  of order  $\alpha \in (0, 1)$  with respect to time variable  $t$  defined by

$$[D^{\alpha,\eta}\omega](t) = \partial_t^{\alpha,\eta}\omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s) ds, \quad (0.0.23)$$

where  $\Gamma$  denotes the Gamma function.

First, we need to reformulate the system into an augmented model. For this purpose, we recall theorem 2 stated in [3, 79].

**Theorem 0.0.11.** *Let  $\alpha \in (0, 1)$ ,  $\eta \geq 0$  and  $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$  be the function defined almost everywhere on  $\mathbb{R}$ . The relation between the 'input'  $V$  and the 'output'  $O$  of the following*

system

$$\partial_t \omega(x, \xi, t) + (\xi^2 + \eta) \omega(x, \xi, t) - V(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+^*, \quad (0.0.24)$$

$$\omega(x, \xi, 0) = 0, (x, \xi) \in (0, L) \times \mathbb{R}, \quad (0.0.25)$$

$$O(x, t) - \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi = 0, (x, t) \in (0, L) \times \mathbb{R}_+^*, \quad (0.0.26)$$

is given by

$$O = I^{1-\alpha, \eta} V, \quad (0.0.27)$$

where

$$[I^{\alpha, \eta} V](x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} V(s) ds \quad \text{and} \quad \kappa(\alpha) = \frac{\sin(\alpha\pi)}{\pi}.$$

By taking the input  $V(x, t) = \sqrt{d(x)} u_{xt}(x, t)$  in Theorem 0.0.11, we deduce that system ((EBB)- $\mathbf{W}_{FKV}$ ) can be recast into the following augmented model

$$\begin{cases} u_{tt} - \left( a u_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_+^*, \\ y_{tt} + b y_{xxxx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_+^*, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} u_{xt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_+^*, \end{cases} \quad (0.0.28)$$

with the following transmission and boundary conditions

$$\begin{cases} u(L, t) = y(-L, t) = y_x(-L, t) = 0, & t \in (0, \infty), \\ a u_x(0, t) + b y_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{cases} \quad (0.0.29)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \omega(x, \xi, 0) = 0 \quad x \in (0, L), \xi \in \mathbb{R}, \\ y(x, 0) &= y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (-L, 0). \end{aligned} \quad (0.0.30)$$

We define the energy of the system (0.0.28)-(0.0.30) by

$$E_1(t) = \frac{1}{2} \int_0^L (|u_t|^2 + a |u_x|^2) dx + \frac{1}{2} \int_{-L}^0 (|y_t|^2 + b |y_{xxx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

Let  $U = (u, u_t, y, y_t, \omega)$  be a regular solution of the System (0.0.28)-(0.0.30). By some direct computations we deduce that the energy  $E_1(t)$  satisfies the following estimation

$$\frac{d}{dt} E_1(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (0.0.31)$$

Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and then  $\frac{d}{dt} E_1(t) \leq 0$ . Thus, system (0.0.28)-(0.0.30) is dissipative in the sense that its energy is a non-increasing function with respect to

time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_1$  by

$$\mathcal{H}_1 = \{(u, v, y, z, \omega) \in H_R^1(0, L) \times L^2(0, L) \times H_L^2(-L, 0) \times L^2(-L, 0) \times W; \ u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_R^1(0, L) = \{u \in H^1(0, L); u(L) = 0\}, \\ H_L^2(-L, 0) = \{y \in H^2(-L, 0); y(-L) = y_x(-L) = 0\}. \end{cases} \quad (0.0.32)$$

We note that the space  $\mathcal{H}_1$  is a closed subspace of  $H_R^1(0, L) \times L^2(0, L) \times H_L^2(-L, 0) \times L^2(-L, 0) \times W$ .

The energy space  $\mathcal{H}_1$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_1} &= \int_0^L v \overline{v_1} dx + a \int_0^L u_x (\overline{u_1})_x dx + \int_{-L}^0 z \overline{z_1} dx + b \int_{-L}^0 y_{xx} (\overline{y_1})_{xx} dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1}(x, \xi) d\xi dx, \end{aligned} \quad (0.0.33)$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_1$ . We use  $\|U\|_{\mathcal{H}_1}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_1 : D(\mathcal{A}_1) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_1$  by

$$D(\mathcal{A}_1) = \left\{ U = (u, v, y, z, \omega) \in \mathcal{H}_1; \begin{aligned} &(v, z) \in H_R^1(0, L) \times H_L^2(-L, 0), \ y \in H^4(-L, 0), \\ &\left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \in L^2(0, L), \\ &-(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ &au_x(0) + by_{xxx}(0) = 0, \ y_{xx}(0) = 0, \text{ and } v(0) = z(0) \end{aligned} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$ ,

$$\mathcal{A}_1(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ \left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \\ z \\ -by_{xxxx} \\ -( |\xi|^2 + \eta ) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (0.0.28)-(0.0.30), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_1$  given by

$$U_t = \mathcal{A}_1 U, \quad U(0) = U_0, \quad (0.0.34)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ . It's easy to prove that the operator  $\mathcal{A}_1$  is m-dissipative on  $\mathcal{H}_1$ , consequently it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_1})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8 (see in [82] and [73]). Then, the solution of the evolution Equation (0.0.34) admits the following representation

$$U(t) = e^{t\mathcal{A}_1} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.34). Hence, we have the following result.

**Theorem 0.0.12.** *Let  $U_0 \in \mathcal{H}_1$ , then problem (0.0.34) admits a unique weak solution  $U$  that satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}_1).$$

*Moreover, if  $U_0 \in D(\mathcal{A}_1)$ , then problem (0.0.34) admits a unique strong solution  $U$  that satisfies*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}_1) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_1)).$$

Second, we prove the strong stability of the system in the sense that its energy converges to zero when  $t$  goes to infinity for all initial data in  $\mathcal{H}_1$ . It is easy to see that the resolvent of  $\mathcal{A}_1$  is not compact. For this aim, we use a general criteria of Arendt-Batty [17], in which a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_1})_{t \geq 0}$  in a Banach space is strongly stable, if  $\mathcal{A}_1$  has no pure imaginary eigenvalues and  $\sigma(\mathcal{A}_1) \cap i\mathbb{R}$  contains only a countable number of continuous spectrum of  $\mathcal{A}_1$ . Our main result here is the following theorem.

**Theorem 0.0.13.** *Assume that  $\eta \geq 0$ , then the  $C_0$ -semigroup of contractions  $e^{t\mathcal{A}_1}$  is strongly stable on  $\mathcal{H}_1$  in the sense that*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_1} U_0\|_{\mathcal{H}_1} = 0 \quad \forall U_0 \in \mathcal{H}_1.$$

In order to proof Theorem 0.0.13 we need the following Lemmas

**Lemma 0.0.14.** *Assume that  $\eta \geq 0$ . Then, for all  $\lambda \in \mathbb{R}$ , we have  $i\lambda I - \mathcal{A}_1$  is injective, i.e.*

$$\ker(i\lambda I - \mathcal{A}_1) = \{0\}.$$

**Lemma 0.0.15.** *Assume that  $\eta = 0$ . Then, the operator  $-\mathcal{A}_1$  is not invertible and consequently  $0 \in \sigma(\mathcal{A}_1)$ .*

**Lemma 0.0.16.** *If  $(\eta > 0 \text{ and } \lambda \in \mathbb{R})$  or  $(\eta = 0 \text{ and } \lambda \in \mathbb{R}^*)$ , then  $i\lambda I - \mathcal{A}_1$  is surjective.*

Thus, using Lemma 0.0.14, we directly deduce that  $\mathcal{A}_1$  has no pure imaginary eigenvalues. Next, using Lemmas 0.0.15, 0.0.16 and with the help of the closed graph theorem of Banach, we deduce that  $\sigma(\mathcal{A}_1) \cap i\mathbb{R} = \{\emptyset\}$  if  $\eta > 0$  and  $\sigma(\mathcal{A}_1) \cap i\mathbb{R} = \{0\}$  if  $\eta = 0$ . Hence, we get the conclusion by applying Arendt-Batty Theorem 1.3.3. Third, we study the polynomial stability of the system (0.0.28)-(0.0.30) in the case  $\eta > 0$ . For this purpose, we will use a frequency domain approach method, and the result of Borichev-Tomilov in [27]. Our main result in this part is the following theorem.

**Theorem 0.0.17.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_1})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_1 > 0$  such that for every  $U_0 \in D(\mathcal{A}_1)$ , we have*

$$E_1(t) \leq \frac{C_1}{t^{\frac{4}{2-\alpha}}} \|U_0\|_{D(\mathcal{A}_1)}^2, \quad t > 0, \forall U_0 \in D(\mathcal{A}_1). \quad (0.0.35)$$

According to Theorem 1.3.10 of Borichev-Tomilov by taking  $\ell = 1 - \frac{\alpha}{2}$ , the polynomial energy decay (0.0.35) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_1), \quad (\text{H}_1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{1-\frac{\alpha}{2}}} \|(i\lambda I - \mathcal{A}_1)^{-1}\|_{\mathcal{L}(\mathcal{H}_1)} < \infty, \quad (\text{H}_2)$$

are satisfied. Since Condition (H<sub>1</sub>) is already proved in Lemma 0.0.6. We will prove condition (H<sub>2</sub>) by an argument of contradiction. For this purpose, we suppose that (H<sub>2</sub>) is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))^T)\} \subset \mathbb{R}^* \times D(\mathcal{A}_1)$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_1} = \|(u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))\|_{\mathcal{H}_1} = 1, \quad (0.0.36)$$

such that

$$(\lambda_n)^{1-\frac{\alpha}{2}} (i\lambda_n I - \mathcal{A}_1) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}(\cdot, \xi))^T \rightarrow 0 \text{ in } \mathcal{H}_1. \quad (0.0.37)$$

We check the condition (H<sub>2</sub>) by finding a contradiction with (0.0.36) by showing that  $\|U\|_{\mathcal{H}_1} = o(1)$ . Hence, we reach that the condition (H<sub>2</sub>) holds and thus we establish the desired polynomial energy decay rate.

Next, in the second part of this chapter we consider the model W-W<sub>FKV</sub>, which represents the system of two wave equations coupled through boundary connection with a localized non-regular fractional Kelvin-Voigt damping acting through one wave equation only. The system that describes this model is as follows

$$\left\{ \begin{array}{ll} u_{tt} - (au_x + d(x)\partial_t^{\alpha, \eta} u_x)_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ y_{tt} - by_{xx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ u(L, t) = y(-L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) = by_x(0, t), & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, L), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (-L, 0). \end{array} \right. \quad (\text{W-W}_{FKV})$$

We reformulate the system (W-W<sub>FKV</sub>) into the following augmented model

$$\begin{cases} u_{tt} - \left( au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ y_{tt} - by_{xx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} u_{xt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{cases} \quad (0.0.38)$$

with the following transmission and boundary conditions

$$\begin{cases} u(L, t) = y(-L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) = by_x(0, t), & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{cases} \quad (0.0.39)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \omega(x, \xi, 0) = 0 \quad x \in (0, L), \xi \in \mathbb{R}, \\ y(x, 0) &= y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (-L, 0). \end{aligned} \quad (0.0.40)$$

The energy of the system (0.0.38)-(0.0.40) is given by

$$E_2(t) = \frac{1}{2} \int_0^L (|u_t|^2 + a|u_x|^2) dx + \frac{1}{2} \int_{-L}^0 (|y_t|^2 + b|y_x|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

By doing some computations, we obtain

$$\frac{d}{dt} E_2(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (0.0.41)$$

Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and therefore  $\frac{d}{dt} E_2(t) \leq 0$ . Thus, system (0.0.38)-(0.0.40) is dissipative in the sense that its energy is a non-increasing function with respect to time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_2$  by

$$\mathcal{H}_2 = \{(u, v, y, z, \omega) \in H_R^1(0, L) \times L^2(0, L) \times H_L^1(-L, 0) \times L^2(-L, 0) \times W; \ u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_R^1(0, L) = \{u \in H^1(0, L); u(L) = 0\}, \\ H_L^1(-L, 0) = \{y \in H^1(-L, 0); y(-L) = 0\}. \end{cases} \quad (0.0.42)$$

The energy space  $\mathcal{H}_2$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_2} &= \int_0^L v \overline{v_1} dx + a \int_0^L u_x (\overline{u_1})_x dx + \int_{-L}^0 z \overline{z_1} dx + b \int_{-L}^0 y_x (\overline{y_1})_x dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1(x, \xi)} d\xi dx, \end{aligned}$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_2$ . We use  $\|U\|_{\mathcal{H}_2}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_2 : D(\mathcal{A}_2) \subset \mathcal{H}_2 \rightarrow \mathcal{H}_2$  by

$$D(\mathcal{A}_2) = \left\{ \begin{aligned} &U = (u, v, y, z, \omega) \in \mathcal{H}_2; \ (v, z) \in H_R^1(0, L) \times H_L^1(-L, 0), \ y \in H^2(-L, 0), \\ &\left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \in L^2(0, L), \\ &-(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ &au_x(0) = by_x(0), \text{ and } v(0) = z(0) \end{aligned} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_2)$ ,

$$\mathcal{A}_2(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ \left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \\ z \\ by_{xx} \\ -( |\xi|^2 + \eta ) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (0.0.38)-(0.0.40), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_2$  given by

$$U_t = \mathcal{A}_2 U, \quad U(0) = U_0, \quad (0.0.43)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ .

It is easy to see that the unbounded linear operator  $\mathcal{A}_2$  is m-dissipative in the energy space  $\mathcal{H}_2$ . Also, the  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_2})_{t \geq 0}$  is strongly stable on  $\mathcal{H}_2$  in the sense that  $\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_2} U_0\|_{\mathcal{H}_2} = 0$ . Further, we establish a polynomial energy decay rate presented in the following theorem.

**Theorem 0.0.18.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_2})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_2 > 0$  such that for every  $U_0 \in D(\mathcal{A}_2)$ , we have*

$$E_2(t) \leq \frac{C_2}{t^{\frac{4}{2-\alpha}}} \|U_0\|_{D(\mathcal{A}_2)}^2, \quad t > 0, \forall U_0 \in D(\mathcal{A}_2). \quad (0.0.44)$$

Next, in the third part of this chapter we consider the third model W-(EBB)<sub>FKV</sub>, where we consider a system of coupled Euler-Bernouli beam and wave equations. These two equations are coupled through boundary connection. In this case, the localized non-smooth fractional Kelvin-Voigt damping acts only on the Euler-Bernoulli beam. The system that represents this model is given as follows

$$\left\{ \begin{array}{ll} u_{tt} - au_{xx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha, \eta} y_{xx})_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\ u(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) + by_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (-L, 0), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (0, L). \end{array} \right. \quad (\text{W-(EBB)}_{FKV})$$

Firstly, we recast the system into the following augmented model

$$\left\{ \begin{array}{ll} u_{tt} - au_{xx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_*^+, \\ y_{tt} + \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} y_{xxt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{array} \right. \quad (0.0.45)$$

with the following transmission and boundary conditions

$$\left\{ \begin{array}{ll} u(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) + by_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{array} \right. \quad (0.0.46)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) & x \in (-L, 0) \\ y(x, 0) &= y_0(x), \quad y_t(x, 0) = y_1(x), \quad \omega(x, \xi, 0) = 0 & x \in (0, L), \xi \in \mathbb{R}. \end{aligned} \quad (0.0.47)$$

We define the energy of the system (0.0.45)-(0.0.47) by

$$E_3(t) = \frac{1}{2} \int_{-L}^0 (|u_t|^2 + a|u_x|^2) dx + \frac{1}{2} \int_0^L (|y_t|^2 + b|y_{xx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

By doing some computations, it is easy to see that the energy  $E_3(t)$  satisfies the following estimation

$$\frac{d}{dt} E_3(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (0.0.48)$$



Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and therefore  $\frac{d}{dt} E_3(t) \leq 0$ . Thus, system (0.0.45)-(0.0.47) is dissipative in the sense that its energy is a non-increasing function with respect to time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_3$  by

$$\mathcal{H}_3 = \{(u, v, y, z, \omega) \in H_L^1(-L, 0) \times L^2(-L, 0) \times H_R^2(0, L) \times L^2(0, L) \times W; \ u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_L^1(-L, 0) = \{u \in H^1(-L, 0); u(-L) = 0\}, \\ H_R^2(0, L) = \{y \in H^2(0, L); y(L) = y_x(L) = 0\}. \end{cases} \quad (0.0.49)$$

The energy space  $\mathcal{H}_3$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_3} &= \int_{-L}^0 v \overline{v_1} dx + a \int_{-L}^0 u_x (\overline{u_1})_x dx + \int_0^L z \overline{z_1} dx + b \int_0^L y_{xx} (\overline{y_1})_{xx} dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1(x, \xi)} d\xi dx, \end{aligned}$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_3$ . We use  $\|U\|_{\mathcal{H}_3}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_3 : D(\mathcal{A}_3) \subset \mathcal{H}_3 \rightarrow \mathcal{H}_3$  by

$$D(\mathcal{A}_3) = \left\{ \begin{aligned} &U = (u, v, y, z, \omega) \in \mathcal{H}_3; \ (v, z) \in H_L^1(-L, 0) \times H_R^2(0, L), \ u \in H^2(-L, 0), \\ &\left( by_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \in L^2(0, L), \\ &- (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ &au_x(0) + by_{xxx}(0) = 0, \ y_{xx}(0) = 0, \text{ and } v(0) = z(0) \end{aligned} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_3)$ ,

$$\mathcal{A}_3(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ au_{xx} \\ z \\ - \left( by_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \\ - (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (0.0.45)-(0.0.47), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_3$  given by

$$U_t = \mathcal{A}_3 U, \quad U(0) = U_0, \quad (0.0.50)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ .

It is easy to prove that the operator  $\mathcal{A}_3$  is m-dissipative on  $\mathcal{H}_3$ , consequently it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_3})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8 (see in [73] and [82]). Then, the solution of the evolution Equation (0.0.50) admits the following representation

$$U(t) = e^{t\mathcal{A}_3} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.50). Hence, we have the following result.

**Theorem 0.0.19.** *Let  $U_0 \in \mathcal{H}_3$ , then problem (0.0.50) admits a unique weak solution  $U$  that satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}_3).$$

*Moreover, if  $U_0 \in D(\mathcal{A}_3)$ , then problem (0.0.50) admits a unique strong solution  $U$  that satisfies*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}_3) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_3)).$$

**Theorem 0.0.20.** *Assume that  $\eta \geq 0$ , then the  $C_0$ -semigroup of contractions  $e^{t\mathcal{A}_3}$  is strongly stable on  $\mathcal{H}_3$  in the sense that  $\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_3} U_0\|_{\mathcal{H}_3} = 0$ .*

The proof of this result is based on Arendt-Batty Theorem 1.3.3 and similar to that in the first model given in (0.0.28). Now, we study the polynomial stability of system (0.0.45)-(0.0.47) in the case  $\eta > 0$ . This is established by using specific multipliers, some interpolation inequalities and by solving differential equations of order 4. Our main result here is the following theorem.

**Theorem 0.0.21.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_3})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_3 > 0$  such that for every  $U_0 \in D(\mathcal{A}_3)$ , we have*

$$E_3(t) \leq \frac{C_3}{t^{\frac{2}{3-\alpha}}} \|U_0\|_{D(\mathcal{A}_3)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_3). \quad (0.0.51)$$

According to Theorem 1.3.10 of Borichev-Tomilov, by taking  $\ell = 3 - \alpha$ , the polynomial energy decay (0.0.51) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_3), \quad (\mathbf{R}_1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{3-\alpha}} \|(i\lambda I - \mathcal{A}_3)^{-1}\|_{\mathcal{L}(\mathcal{H}_3)} < \infty, \quad (\mathbf{R}_2)$$

are satisfied. Since condition  $(\mathbf{R}_1)$  is already proved, we still need to prove condition  $(\mathbf{R}_2)$ . For this purpose we will use an argument of contradiction. Suppose that  $(\mathbf{R}_2)$  is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))^T)\} \subset \mathbb{R}^* \times D(\mathcal{A}_3)$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_3} = \|(u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))\|_{\mathcal{H}_3} = 1, \quad (0.0.52)$$

such that

$$(\lambda_n^{3-\alpha}) (i\lambda_n I - \mathcal{A}_3) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}(\cdot, \xi))^T \rightarrow 0 \quad \text{in } \mathcal{H}_3. \quad (0.0.53)$$

Here we will check the condition (R<sub>2</sub>) by finding a contradiction with (0.0.52) by showing that  $\|U\|_{\mathcal{H}_3} = o(1)$ . Hence, we reach that the condition (R<sub>2</sub>) holds and that the energy of smooth solution of system (0.0.45)-(0.0.47) decays polynomial to zero as  $t$  goes to infinity.

Next, in the fourth part of this chapter, we consider the model ((EBB)<sub>FKV</sub>) where a system of Euler-Bernoulli beam with a non-regular localized fractional Kelvin-Voigt damping is considered. The system is as follows

$$\begin{cases} y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha,\eta} y_{xx})_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\ y(0, t) = y_x(0, t) = y_{xx}(L, t) = y_{xxx}(L, t) = 0, & t \in (0, \infty), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (0, L). \end{cases} \quad ((\text{EBB})_{FKV})$$

We recast the system into the following augmented model

$$\begin{cases} y_{tt} + \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta)\omega(x, \xi, t) - \sqrt{d(x)}y_{xxt}(x, t)|\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}^+, \end{cases} \quad (0.0.54)$$

with the boundary conditions

$$y(0, t) = y_x(0, t) = 0, y_{xx}(L, t) = 0, y_{xxx}(L, t) = 0, \quad t \in (0, \infty), \quad (0.0.55)$$

and with the following initial conditions

$$y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), \omega(x, \xi, 0) = 0, x \in (0, L), \xi \in \mathbb{R}. \quad (0.0.56)$$

The energy of the system (0.0.54)-(0.0.56) is given by

$$E_4(t) = \frac{1}{2} \int_0^L (|y_t|^2 + b|y_{xx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

By doing some computations, we get that the energy  $E_4(t)$  satisfies the following estimation

$$\frac{d}{dt} E_4(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx \leq 0. \quad (0.0.57)$$

Now, we define the following Hilbert energy space  $\mathcal{H}_4$  by

$$\mathcal{H}_4 = H_L^2(0, L) \times L^2(0, L) \times W,$$

where  $W = L^2((0, L) \times \mathbb{R})$  and  $H_L^2(0, L) = \{y \in H^2(0, L); y(0) = y_x(0) = 0\}$ .

The energy space  $\mathcal{H}_4$  is equipped with the inner product defined by

$$\langle U, U_1 \rangle_{\mathcal{H}_4} = \int_0^L z \overline{z_1} dx + b \int_0^L y_{xx}(\overline{y_1})_{xx} dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1}(x, \xi) d\xi dx,$$

for all  $U = (y, z, \omega)$  and  $U_1 = (y_1, z_1, \omega_1)$  in  $\mathcal{H}_4$ . We use  $\|U\|_{\mathcal{H}_4}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_4 : D(\mathcal{A}_4) \subset \mathcal{H}_4 \rightarrow \mathcal{H}_4$  by

$$D(\mathcal{A}_4) = \left\{ \begin{array}{l} U = (y, z, \omega) \in \mathcal{H}_4; z \in H_L^2(0, L), \\ \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} \in L^2(0, L), \\ -(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ y_{xx}(L) = 0, \text{ and } y_{xxx}(L) = 0. \end{array} \right\},$$

and for all  $U = (y, z, \omega) \in D(\mathcal{A}_4)$ ,

$$\mathcal{A}_4(y, z, \omega)^\top = \begin{pmatrix} z \\ - \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} \\ -(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (y, y_t, \omega)$  is a regular solution of system (0.0.54)-(0.0.56), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_4$  given by

$$U_t = \mathcal{A}_4 U, \quad U(0) = U_0, \quad (0.0.58)$$

where  $U_0 = (y_0, y_1, 0)$ .

It is easy to see that the unbounded linear operator  $\mathcal{A}_4$  is m-dissipative in the energy space  $\mathcal{H}_4$ . Also, the  $C_0$ -semigroup of contractions  $e^{t\mathcal{A}_4}$  is strongly stable on  $\mathcal{H}_4$  in the sense that  $\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_4} U_0\|_{\mathcal{H}_4} = 0$ . Now, we establish a polynomial energy decay rate presented in the following theorem

**Theorem 0.0.22.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_4})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_4 > 0$  such that for every  $U_0 \in D(\mathcal{A}_4)$ , we have*

$$E_4(t) \leq \frac{C_4}{t^{\frac{2}{1-\alpha}}} \|U_0\|_{D(\mathcal{A}_4)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_4). \quad (0.0.59)$$

Finally, we study the stability of the model (EBB)-(EBB) $_{FKV}$ , that represents a system of two Euler-Bernoulli beam equations coupled through boundary connection. The localized non-smooth fractional Kelvin-Voigt damping acts only on one of the two

equations. The system that represents this model is as follows

$$\left\{ \begin{array}{ll} u_{tt} + au_{xxxx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha, \eta} y_{xx})_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\ u(-L, t) = u_x(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_{xxx}(0, t) - by_{xxx}(0, t) = 0, u_{xx}(0, t) = y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (-L, 0), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (0, L). \end{array} \right. \quad ((\text{EBB})-(\text{EBB})_{FKV})$$

We recast the system into the following augmented model

$$\left\{ \begin{array}{ll} u_{tt} + au_{xxxx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_*^+, \\ y_{tt} + \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} y_{xxt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{array} \right. \quad (0.0.60)$$

with the following transmission and boundary conditions

$$\left\{ \begin{array}{ll} u(-L, t) = u_x(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_{xxx}(0, t) - by_{xxx}(0, t) = 0, u_{xx}(0, t) = y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{array} \right. \quad (0.0.61)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x) & \quad x \in (-L, 0) \\ y(x, 0) = y_0(x), \quad y_t(x, 0) = y_1(x), \quad \omega(x, \xi, 0) = 0 & \quad x \in (0, L), \xi \in \mathbb{R}. \end{aligned} \quad (0.0.62)$$

The energy of the system (0.0.60)-(0.0.62) is given by

$$E_5(t) = \frac{1}{2} \int_{-L}^0 (|u_t|^2 + a|u_{xx}|^2) dx + \frac{1}{2} \int_0^L (|y_t|^2 + b|y_{xx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

It is easy to see that the energy  $E_5(t)$  satisfies the following estimation

$$\frac{d}{dt} E_5(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx \leq 0. \quad (0.0.63)$$

We define the Hilbert energy space  $\mathcal{H}_5$  by

$$\mathcal{H}_5 = \{(u, v, y, z, \omega) \in H_L^2(-L, 0) \times L^2(-L, 0) \times H_R^2(0, L) \times L^2(0, L) \times W; \quad u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_L^2(-L, 0) = \{u \in H^2(-L, 0); u(-L) = u_x(-L) = 0\}, \\ H_R^2(0, L) = \{y \in H^2(0, L); y(L) = y_x(L) = 0\}. \end{cases}$$

The energy space  $\mathcal{H}_5$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_5} &= \int_{-L}^0 v \overline{v_1} dx + a \int_{-L}^0 u_{xx}(\overline{u_1})_{xx} dx + \int_0^L z \overline{z_1} dx + b \int_0^L y_{xx}(\overline{y_1})_{xx} dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1}(x, \xi) d\xi dx, \end{aligned}$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_5$ . We use  $\|U\|_{\mathcal{H}_5}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_5 : D(\mathcal{A}_5) \subset \mathcal{H}_5 \rightarrow \mathcal{H}_5$  by

$$D(\mathcal{A}_5) = \left\{ \begin{aligned} &U = (u, v, y, z, \omega) \in \mathcal{H}_5; (v, z) \in H_L^2(-L, 0) \times H_R^2(0, L), u \in H^4(-L, 0), \\ &\left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \in L^2(0, L), \\ &- (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ &au_{xxx}(0) - by_{xxx}(0) = 0, u_{xx}(0) = y_{xx}(0) = 0, \text{ and } v(0) = z(0) \end{aligned} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_5)$ ,

$$\mathcal{A}_5(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ -au_{xxxx} \\ z \\ - \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \\ - (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (0.0.60)-(0.0.62), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_5$  given by

$$U_t = \mathcal{A}_5 U, \quad U(0) = U_0, \quad (0.0.64)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ .

The operator  $\mathcal{A}_5$  is m-dissipative on  $\mathcal{H}_5$  (in a similar way as in the first part of this chapter), consequently it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_5})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8 (see in [73] and [82]). Then, the solution of the evolution

Equation (0.0.64) admits the following representation

$$U(t) = e^{t\mathcal{A}_5} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.64). Using the frequency domain approach we have the following polynomial energy decay result.

**Theorem 0.0.23.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_5})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_5 > 0$  such that for every  $U_0 \in D(\mathcal{A}_5)$ , we have*

$$E_5(t) \leq \frac{C_5}{t^{\frac{2}{3-a}}} \|U_0\|_{D(\mathcal{A}_5)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_5). \quad (0.0.65)$$

In Chapter 5, we study a multidimensional system of coupled wave equations with a localized Kelvin-Voigt damping under different types of geometric conditions. Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set with Lipschitz boundary  $\Gamma$ . The system is described as follows:

$$\begin{cases} u_{tt} - \operatorname{div}(a \nabla u + b(x) \nabla u_t) + c(x) y_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ y_{tt} - \Delta y - c(x) u_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \quad (0.0.66)$$

with the following initial conditions:

$$u(\cdot, 0) = u_0(\cdot), \quad y(\cdot, 0) = y_0(\cdot), \quad u_t(\cdot, 0) = u_1(\cdot), \quad y_t(\cdot, 0) = y_1(\cdot) \quad \text{in } \Omega, \quad (0.0.67)$$

and the following boundary conditions:

$$u(x, t) = y(x, t) = 0 \quad \text{on } \Gamma \times \mathbb{R}^+. \quad (0.0.68)$$

The functions  $b, c \in L^\infty(\Omega)$  such that  $b : \Omega \rightarrow \mathbb{R}_+$  is the viscoelastic damping and  $c : \Omega \rightarrow \mathbb{R}^*$  is the coupling function. The constant  $a$  is a strictly positive constant.

The main interest in this chapter is that we consider the system of wave equations coupled through velocity with non-smooth coupling and damping coefficients and by considering several geometric conditions (see (H1), (H2), (H3), (H4), and (H5) below). In addition, this chapter is a generalization of the third chapter ([94]) in this thesis. Moreover, in the absence of any geometric condition, we study the stability of the system on the 2-dimensional square domain.

First, using semigroup approach we study the existence, uniqueness and regularity of the solution of the system. Let  $(u, u_t, y, y_t)$  be a regular solution of the system (0.0.66)-(0.0.68). The energy of the System (0.0.66)-(0.0.68) is given by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + |y_t|^2 + a|\nabla u|^2 + |\nabla y|^2) dx. \quad (0.0.69)$$

Then, a straightforward computation gives

$$E'(t) = - \int_{\Omega} b(x) |\nabla u_t|^2 dx \leq 0.$$

Thus, the System (0.0.66)-(0.0.68) is dissipative in the sense that its energy is a non increasing function with respect to the time variable  $t$ . We define the energy Hilbert space  $\mathcal{H}$  by

$$\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$$

equipped with the following inner product

$$\langle U, \tilde{U} \rangle = \int_{\Omega} (a \nabla u \cdot \nabla \bar{\tilde{u}} + \nabla y \cdot \nabla \bar{\tilde{y}} + v \bar{\tilde{v}} + z \bar{\tilde{z}}) dx,$$

for all  $U = (u, v, y, z)^\top \in \mathcal{H}$  and  $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})^\top \in \mathcal{H}$ . Finally, we define the unbounded linear operator  $\mathcal{A}$  by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega) \in \mathcal{H}; \operatorname{div}(a(x) \nabla u + b(x) \nabla v) \in L^2(\Omega), \\ v, z \in H_0^1(0, L), y \in H^2(\Omega) \cap H_0^1(\Omega) \end{array} \right\}$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A})$ ,

$$\mathcal{A}(u, v, y, z)^\top = \begin{pmatrix} v \\ \operatorname{div}(a \nabla u + b(x) \nabla v) - c(x)z \\ z \\ \Delta y + c(x)v \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t)^\top$  is a regular solution of system (0.0.66)-(0.0.68), then we rewrite this system as the following evolution equation

$$U_t = \mathcal{A}U, \quad U(0) = U_0 \quad (0.0.70)$$

where  $U_0 = (u_0, u_1, y_0, y_1)^\top$ . It is easy to check that the operator  $\mathcal{A}$  is m-dissipative on  $\mathcal{H}$  and consequently, it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  following Lumer-Phillips Theorem. Then, the solution of the evolution Equation (0.0.70) admits the following representation

$$U(t) = e^{t\mathcal{A}} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (0.0.70). Hence, using the semigroup theory we have the following existence and uniqueness result.

**Theorem 0.0.24.** *Let  $U_0 \in \mathcal{H}$  then, problem (0.0.70) admits a unique weak solution  $U$  that satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

*Moreover, if  $U_0 \in D(\mathcal{A})$  then, problem (0.0.70) admits a unique strong solution  $U$  that satisfies*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$



Second, we study the strong stability of system (0.0.66)-(0.0.68) in the sense that its energy tends to zero when  $t$  tends to infinity for all initial data in  $\mathcal{H}$ . We study the strong stability in the absence of the compactness of the resolvent and without any geometric condition. The proof of the stability result is done by using the unique continuation theorem based on a Carleman estimate and a general criteria of Arendt-Batty Theorem 1.3.3 ([17]). For this aim, we assume that there exists constants  $b_0 > 0$  and  $c_0 > 0$  and two nonempty open sets  $\omega_b \subset \Omega$  and  $\omega_c \subset \Omega$ , such that

$$b(x) \geq b_0 > 0, \quad \forall x \in \omega_b, \quad (0.0.71)$$

$$c(x) \geq c_0 > 0, \quad \forall x \in \omega_c. \quad (0.0.72)$$

Here, we prove that the energy of the system (0.0.66)-(0.0.68) decays to zero as  $t$  tends to infinity if one of the following assumptions hold:

(A1) Assume that  $\omega_b$  and  $\omega_c$  are non-empty open subsets of  $\Omega$  such that  $\omega_c \subset \omega_b$  and  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$  (see Figures 5.1, 5.2, 5.3).

(A2) Assume that  $\omega_b$  and  $\omega_c$  are non-empty open subsets of  $\Omega$  such that  $\omega = \omega_b \cap \omega_c \neq \emptyset$ . Also, assume that  $\omega$  satisfies  $\text{meas}(\overline{\omega} \cap \Gamma) > 0$  (see Figures 5.4, 5.5, 5.6).

(A3) Assume that  $\omega_b$  and  $\omega_c$  are non-empty open subset of  $\Omega$  such that  $\omega = \omega_b \cap \omega_c \neq \emptyset$ ,  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$  and  $\omega_c$  not near the boundary (see Figure 5.7).

(A4) Assume that  $\omega_b$  is non-empty open subsets of  $\Omega$  and  $c(x) = c_0 \in \mathbb{R}^*$  in  $\Omega$ . Also, assume that  $\omega_b$  is not near the boundary (see Figure 5.8).

We give the strong stability result in the following theorem.

**Theorem 0.0.25.** *Assume that either (A1), (A2), (A3) or (A4) holds. Then, the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable in  $\mathcal{H}$ ; i.e, for all  $U_0 \in \mathcal{H}$ , the solution of (0.0.70) satisfies*

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 0.0.25, the resolvent of  $\mathcal{A}$  is not compact. Then, in order to prove this Theorem we will use a general criteria of Arendt-Batty. We need to prove that the operator  $\mathcal{A}$  has no pure imaginary eigenvalues and  $\sigma(\mathcal{A}) \cap i\mathbb{R}$  contains only a countable number of continuous spectrum of  $\mathcal{A}$ . The argument for Theorem 0.0.25 relies on the subsequent lemmas.

**Lemma 0.0.26.** *Assume that (A1), (A2), (A3) or (A4) holds. Then, we have*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

**Lemma 0.0.27.** *Assume that either (A1), (A2), (A3) or (A4) holds. Then, we have*

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}, \quad \forall \lambda \in \mathbb{R}.$$

Using Lemma 0.0.26, we have that  $\mathcal{A}$  has non pure imaginary eigenvalues. According to Lemmas 0.0.26, 0.0.27 and with the help of the closed graph theorem of Banach, we deduce that  $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ . Thus, we get the conclusion by applying Arendt Batty Theorem 1.3.3. The proof of the theorem is thus complete. Now, our aim is to prove the non-uniform stability of the system (0.0.66)-(0.0.68). For this aim, assume that

$$b(x) = b \in \mathbb{R}_+ \quad \text{and} \quad c(x) = c \in \mathbb{R}^*. \quad (0.0.73)$$

Our main result in this part is the following theorem.

**Theorem 0.0.28.** *Under condition (0.0.73). Then, the energy of the system (0.0.66)-(0.0.68) does not decay uniformly in the energy space  $\mathcal{H}$ .*

This result relies on the fact that there exists a subsequence of eigenvalues of  $\mathcal{A}$  close to the imaginary axis. Thus, we prove that there exists  $k_0 \in \mathbb{N}^*$  sufficiently large and two sequences  $(\lambda_{1,k})_{|k| \geq k_0}$  and  $(\lambda_{2,k})_{|k| \geq k_0}$  satisfying the following asymptotic behavior:

$$\lambda_{1,k} = i\mu_k - \frac{c^2}{2b\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right) \quad (0.0.74)$$

and

$$\lambda_{2,k} = -i\mu_k - \frac{c^2}{2b\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right). \quad (0.0.75)$$

where  $\mu_k$  is the sequence of the eigenvalues of the Laplace operator with fully Dirichlet boundary conditions in  $\Omega$ , i.e,

$$\begin{cases} -\Delta \varphi_k = \mu_k^2 \varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \Gamma. \end{cases} \quad (0.0.76)$$

In the absence of exponential stability, we aim to establish a polynomial stability for the system. We study the polynomial stability under several geometric conditions. The geometric assumptions on  $\omega_b$ ,  $\omega_c$  and  $\omega = \omega_b \cap \omega_c \neq \emptyset$  are as follows:

(H1) The open subset  $\omega$  verifies the GCC (see Figure 5.6 and Figure 5.9).

(H2) Assume that  $\text{meas}(\overline{\omega_c} \cap \Gamma) > 0$  and  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$ . Also, assume that  $\omega_c \subset \omega_b$  and  $\omega_c$  satisfies the GCC (see Figure 5.3).

(H3) Assume that  $\omega_b \subset \Omega$ ,  $\overline{\omega_c} \subset \omega_b$  such that  $\Omega$  is a non-convex open set and  $\omega_c$  satisfies GCC (see Figure 5.10).

(H4) Assume that  $\Omega = (0, L) \times (0, L)$ ,  $\omega_c \subset \omega_b$  such that  $\omega_b = \{(x, y) \in \mathbb{R}^2; \varepsilon_1 < x < \varepsilon_4 \text{ and } 0 < y < L\}$  and  $\omega_c = \{(x, y) \in \mathbb{R}^2; \varepsilon_2 < x < \varepsilon_3 \text{ and } 0 < y < L\}$ , for  $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < L$  (see Figure 5.11).

(H5) Assume that  $\Omega = (0, L) \times (0, L)$ ,  $\omega_c \subset \omega_b$  such that  $\omega_b = \{(x, y) \in \mathbb{R}^2; 0 < x < \varepsilon_2 \text{ and } 0 < y < L\}$  and  $\omega_c = \{(x, y) \in \mathbb{R}^2; 0 < x < \varepsilon_1 \text{ and } 0 < y < L\}$ , for  $0 < \varepsilon_1 < \varepsilon_2 < L$  (see Figure 5.12).

In order to study the polynomial stability of the system (0.0.66)-(0.0.68) such that the

assumption (H1) holds and such that  $c \in W^{1,\infty}(\Omega)$  we need to use the exponential energy decay of the coupled wave equations via velocities with two viscous dampings. We consider the following system

$$\begin{cases} \varphi_{tt} - a\Delta\varphi + d(x)\varphi_t + c(x)\psi_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \psi_{tt} - \Delta\psi + d(x)\psi_t - c(x)\varphi_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \varphi(x, t) = \psi(x, t) = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ (\varphi(x, 0), \psi(x, 0)) = (\varphi_0, \psi_0) \quad \text{and} \quad (\varphi_t(x, 0), \psi_t(x, 0)) = (\varphi_1, \psi_1) & \text{in } \Omega. \end{cases} \quad (0.0.77)$$

where  $d \in W^{1,\infty}(\Omega)$  such that

$$d(x) \geq d_0 > 0 \quad \text{on } \omega_\epsilon \subset \omega \subset \Omega.$$

The energy of the system (0.0.77) satisfies the following estimate

$$E_{aux}(t) \leq M e^{-\theta t} E(0), \quad \forall t > 0. \quad (0.0.78)$$

such that  $M \geq 1$  and  $\theta > 0$ , for all initial data  $U_0 \in \mathcal{H}_{aux}$ .

Now, we will state the main theorems in this part.

**Theorem 0.0.29.** *Assume that the boundary  $\Gamma$  is of class  $C^3$ . Also, assume that assumption (H1) holds and that  $c \in W^{1,\infty}(\Omega)$ . Then, for all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $C > 0$  independent of  $U_0$ , such that the energy of the system (0.0.66)–(0.0.68) satisfies the following estimation*

$$E(t, U) \leq \frac{C}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (0.0.79)$$

According to Theorem 1.3.10 of Borichev and Tomilov, by taking  $\ell = 2$ , the polynomial energy decay (0.0.79) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad (C1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^2} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (C2)$$

are satisfied. Since Condition (C1) is already given in Lemma 0.0.26. We will prove condition (C2) by an argument of contradiction. For this purpose, suppose that (C2) is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n)^\top)\} \subset \mathbb{R}^* \times D(\mathcal{A})$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1, \quad (0.0.80)$$

such that

$$(\lambda_n)^2 (i\lambda_n I - \mathcal{A}) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^\top \rightarrow 0 \quad \text{in } \mathcal{H}. \quad (0.0.81)$$

By using the exponential stability of the system (0.0.77) and by proving that the solution of the system (0.0.66)-(0.0.68) satisfies several asymptotic behaviors, we reach that  $\|U_n\|_{\mathcal{H}} = o(1)$ . Hence, we reach that the condition (C2) holds and thus we deduce the polynomial energy decay estimate (0.0.79).

Now, one of the main tools to prove the polynomial stability of the system (0.0.66)-(0.0.68) when one of the assumptions (H2), (H3), (H4) or (H5) holds is to use the exponential or polynomial decay of the wave equation with viscous damping. We consider the following system

$$\begin{cases} \varphi_{tt} - \Delta\varphi + \mathbb{1}_{\omega_c}(x)\varphi_t = 0 & \text{in } \Omega \times (0, +\infty) \\ \varphi = 0 & \text{in } \Gamma \times (0, +\infty) \\ \varphi(\cdot, 0) = \varphi_0, \quad \varphi_t(\cdot, 0) = \varphi_1. \end{cases} \quad (0.0.82)$$

Now, we state the polynomial stability results when the assumption (H2), (H3), (H4) or (H5) holds.

**Theorem 0.0.30.** *Assume that assumption (H2) or (H3) holds. Also, assume that the energy of the system (0.0.82) is exponentially stable. Then, for all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $C_2 > 0$  independent of  $U_0$ , the energy of the system (0.0.66)-(0.0.68) satisfies the following estimation*

$$E(t, U) \leq \frac{C_2}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (0.0.83)$$

**Theorem 0.0.31.** *Assume that assumption (H4) or (H5) holds. Then, for all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $C_3 > 0$  independent of  $U_0$ , the energy of the system (0.0.66)-(0.0.68) satisfies the following estimation*

$$E(t, U) \leq \frac{C_3}{t^{\frac{2}{2+4\beta}}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0, \quad (0.0.84)$$

where

$$\beta = \begin{cases} 2 & \text{if (H5) holds,} \\ \frac{3}{2} & \text{if (H6) holds.} \end{cases} \quad (0.0.85)$$

The proof of these two theorems is based on the frequency domain approach, namely we will use Theorem 1.3.10 combined with a multiplier method and by using the stability results of system (0.0.82).

# 1 Preliminaries

As the analysis in this thesis is based on the semigroup and spectral analysis theories, in this chapter we will present some well-known results about semigroups, including some theorems about strong, exponential, and polynomial stability of a  $C_0$ - semigroup. Also, we recall some geometric conditions used in our work . All of the theorems are stated without proofs, but the relevant references are given. The reader may skip this chapter in the first reading, then refer to it as a reference of related. For more details see [17, 24, 27, 28, 38, 53, 55, 57, 71, 75, 85].

## Sommaire

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## 1.1 Bounded and Unbounded linear operators

We start this chapter by giving some well known results about bounded and unbounded operators. We are not trying to give a complete development, but rather review the basic definitions and theorems, mostly without proof ([28, 55, 57]).

Let  $(E, \|\cdot\|_E)$  and  $(F, \|\cdot\|_F)$  be two Banach spaces over  $\mathbb{C}$ , and  $H$  will always denote a Hilbert space equipped with the scalar product  $\langle \cdot, \cdot \rangle_H$  and the corresponding norm  $\|\cdot\|_H$ .

A linear operator  $T : E \longrightarrow F$  is a transformation which maps linearly  $E$  in  $F$ , that is

$$T(\alpha u + \beta v) = \alpha T(u) + \beta T(v), \quad \forall u, v \in E \text{ and } \alpha, \beta \in \mathbb{C}.$$

**Definition 1.1.1.** A linear operator  $T : E \rightarrow F$  is said to be bounded if there exists  $C \geq 0$  such that

$$\|Tu\|_F \leq C\|u\|_E \quad \forall u \in E.$$

The set of all bounded linear operators from  $E$  into  $F$  is denoted by  $\mathcal{L}(E, F)$ . Moreover, the set of all bounded linear operators from  $E$  into  $E$  is denoted by  $\mathcal{L}(E)$ .

**Definition 1.1.2.** A bounded operator  $T \in \mathcal{L}(E, F)$  is said to be compact if for each sequence  $(x_n)_{n \in \mathbb{N}} \in E$  with  $\|x_n\|_E = 1$  for each  $n \in \mathbb{N}$ , the sequence  $(Tx_n)_{n \in \mathbb{N}}$  has a subsequence which converges in  $F$ .

The set of all compact operators from  $E$  into  $F$  is denoted by  $\mathcal{K}(E, F)$ . For simplicity one writes  $\mathcal{K}(E) = \mathcal{K}(E, E)$ .

**Definition 1.1.3.** Let  $T \in \mathcal{L}(E, F)$ , we define

- Range of  $T$  by

$$\mathcal{R}(T) = \{Tu : u \in E\} \subset F.$$

- Kernel of  $T$  by

$$\ker(T) = \{u \in E : Tu = 0\} \subset E.$$

**Theorem 1.1.4. (Fredholm alternative)** see Theorem 6.6 in [28]. If  $T \in \mathcal{K}(E)$ , then

- $\ker(I - T)$  is finite dimension ( $I$  is the identity operator on  $E$ ).
- $\mathcal{R}(I - T)$  is closed.
- $\ker(I - T) = 0 \Leftrightarrow \mathcal{R}(I - T) = E$ .

**Definition 1.1.5.** An unbounded linear operator  $T$  from  $E$  into  $F$  is a pair  $(T, D(T))$ , consisting of a subspace  $D(T) \subset E$  (called the domain of  $T$ ) and a linear transformation.

$$T : D(T) \subset E \longrightarrow F.$$

In the case when  $E = F$  then we say  $(T, D(T))$  is an unbounded linear operator on  $E$ . If  $D(T) = E$  then  $T \in \mathcal{L}(E, F)$ .

**Definition 1.1.6.** Let  $T : D(T) \subset E \longrightarrow F$  be an unbounded linear operator.

- The range of  $T$  is defined by

$$\mathcal{R}(T) = \{Tu : u \in D(T)\} \subset F.$$

- The Kernel of  $T$  is defined by

$$\ker(T) = \{u \in D(T) : Tu = 0\} \subset E.$$

- The graph of  $T$  is defined by

$$G(T) = \{(u, Tu) : u \in D(T)\} \subset E \times F.$$

**Definition 1.1.7.** A map  $T$  is said to be closed if  $G(T)$  is closed in  $E \times F$ . The closedness of an unbounded linear operator  $T$  can be characterized as following

if  $u_n \in D(T)$  such that  $u_n \rightarrow u$  in  $E$  and  $Tu_n \rightarrow v$  in  $F$ , then  $u \in D(T)$  and  $Tu = v$ .

**Definition 1.1.8.** Let  $T : D(T) \subset E \longrightarrow F$  be a closed unbounded linear operator.

## 1 Preliminaries – 1.1 Bounded and Unbounded linear operators

- The resolvent set of  $T$  is defined by

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is bijective from } D(T) \text{ onto } F\}.$$

- The resolvent of  $T$  is defined by

$$R(\lambda, T) = \{(\lambda I - T)^{-1} : \lambda \in \rho(T)\}.$$

- The spectrum set of  $T$  is the complement of the resolvent set in  $\mathbb{C}$ , denoted by

$$\sigma(T) = \mathbb{C} / \rho(T).$$

**Definition 1.1.9.** Let  $T : D(T) \subset E \rightarrow F$  be a closed unbounded linear operator. We can split the spectrum  $\sigma(T)$  of  $T$  into three disjoint sets, given by

- The ponctuel spectrum of  $T$  is defined by

$$\sigma_p(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\}\}$$

in this case  $\lambda$  is called an eigenvalue of  $T$ .

- The continuous spectrum of  $T$  is defined by

$$\sigma_c(T) = \left\{ \lambda \in \mathbb{C} : \ker(\lambda I - T) = 0, \overline{\mathcal{R}(\lambda I - T)} = F \text{ and } (\lambda I - T)^{-1} \text{ is not bounded} \right\}.$$

- The residual spectrum of  $T$  is defined by

$$\sigma_r(T) = \{\lambda \in \mathbb{C} : \ker(\lambda I - T) = 0 \text{ and } \mathcal{R}(\lambda I - T) \text{ is not dense in } F\}.$$

**Definition 1.1.10.** Let  $T : D(T) \subset E \rightarrow F$  be a closed unbounded linear operator and let  $\lambda$  be an eigenvalue of  $A$ . A non-zero element  $e \in E$  is called a generalized eigenvector of  $T$  associated with the eigenvalue  $\lambda$ , if there exists  $n \in \mathbb{N}^*$  such that

$$(\lambda I - T)^n e = 0 \quad \text{and} \quad (\lambda I - T)^{n-1} e \neq 0.$$

If  $n = 1$ , then  $e$  is called an eigenvector.

**Definition 1.1.11.** Let  $T : D(T) \subset E \rightarrow F$  be a closed unbounded linear operator. We say that  $T$  has compact resolvent, if there exist  $\lambda_0 \in \rho(T)$  such that  $(\lambda_0 I - T)^{-1}$  is compact.

**Theorem 1.1.12.** (see [55]). Let  $(T, D(T))$  be a closed unbounded linear operator on  $H$  then the space  $(D(T), \|\cdot\|_{D(T)})$  where  $\|u\|_{D(T)} = \|Tu\|_H + \|u\|_H \quad \forall u \in D(T)$  is Banach space.

**Theorem 1.1.13.** (see [57]). Let  $(T, D(T))$  be a closed unbounded linear operator on  $H$  then,  $\rho(T)$  is an open set of  $\mathbb{C}$ .

## 1.2 Semigroups, Existence and uniqueness of solution

In this section, we introduce some basic concepts concerning the semigroups. The vast majority of the evolution equations can be reduced to the form

$$\begin{cases} U_t &= AU, \quad t > 0, \\ U(0) &= U_0, \end{cases} \quad (1.2.1)$$

where  $A$  is the infinitesimal generator of a  $C_0$ -semigroup  $S(t)$  over a Hilbert space  $H$ . Let's start by basic definitions and theorems (see [28, 82]). Let  $(X, \|\cdot\|_X)$  be a Banach space, and  $H$  be a Hilbert space equipped with the inner product  $\langle \cdot, \cdot \rangle_H$  and the induced norm  $\|\cdot\|_H$ .

**Definition 1.2.1.** A family  $(S(t))_{t \geq 0}$  of bounded linear operators in  $X$  is called a strong continuous semigroup (in short, a  $C_0$ -semigroup) if

- $S(0) = I$  ( $I$  is the identity operator on  $X$ ).
- $S(t+s) = S(t)S(s)$ ,  $\forall t, s \geq 0$ .
- For each  $u \in H$ ,  $S(t)u$  is continuous in  $t$  on  $[0, +\infty[$ .

Sometimes we also denote  $S(t)$  by  $e^{tA}$ .

**Definition 1.2.2.** For a semigroup  $(S(t))_{t \geq 0}$ , we define an linear operator  $A$  with domain  $D(A)$  consisting of points  $u$  such that the limit

$$Au := \lim_{t \rightarrow 0^+} \frac{S(t)u - u}{t}, \quad u \in D(A)$$

exists. Then  $A$  is called the infinitesimal generator of the semigroup  $(S(t))_{t \geq 0}$ .

**Proposition 1.2.3.** (See Theorem 2.2 in [82]). Let  $(S(t))_{t \geq 0}$  be a  $C_0$ -semigroup in  $X$ . Then there exist a constant  $M \geq 1$  and  $\omega \geq 0$  such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \forall t \geq 0.$$

If  $\omega = 0$  then the corresponding semigroup is uniformly bounded. Moreover, if  $M = 1$  then  $(S(t))_{t \geq 0}$  is said to be a  $C_0$ -semigroup of contractions.

**Definition 1.2.4.** An unbounded linear operator  $(A, D(A))$  on  $H$ , is said to be dissipative if

$$\operatorname{Re} \langle Au, u \rangle_H \leq 0, \quad \forall u \in D(A).$$

**Definition 1.2.5.** An unbounded linear operator  $(A, D(A))$  on  $X$ , is said to be  $m$ -dissipative if

- $A$  is a dissipative operator.
- $\exists \lambda_0 > 0$  such that  $\mathcal{R}(\lambda_0 I - A) = X$ .



**Theorem 1.2.6.** (See Theorem 4.5 in [82]). Let  $A$  be a  $m$ -dissipative operator, then

- $\mathcal{R}(\lambda I - A) = X, \quad \forall \lambda > 0.$
- $]0, \infty[ \subseteq \rho(A).$

**Theorem 1.2.7.** (Hille-Yosida see Theorem 3.1 in [82]). An unbounded linear operator  $(A, D(A))$  on  $X$ , is the infinitesimal generator of a  $C_0$ -semigroup of contractions  $(S(t))_{t \geq 0}$  if and only if

- $A$  is closed and  $\overline{D(A)} = X.$
- The resolvent set  $\rho(\mathcal{A})$  of  $\mathcal{A}$  contains  $\mathbb{R}^+$ , and for all  $\lambda > 0,$

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \lambda^{-1}.$$

**Theorem 1.2.8.** (Lumer-Phillips see Theorem 4.3 in [82]). Let  $(A, D(A))$  be an unbounded linear operator on  $X$ , with dense domain  $D(A)$  in  $X$ .  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions if and only if it is a  $m$ -dissipative operator.

**Theorem 1.2.9.** (see Theorem 4.6 in [82]). Let  $(A, D(A))$  be an unbounded linear operator on  $X$ . If  $A$  is dissipative with  $\mathcal{R}(I - A) = X$ , and  $X$  is reflexive then  $\overline{D(A)} = X.$

**Corollary 1.2.10.** Let  $(A, D(A))$  be an unbounded linear operator on  $H$ .  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions if and only if  $A$  is a  $m$ -dissipative operator.

**Theorem 1.2.11.** Let  $A$  be a linear operator with dense domain  $D(A)$  in a Hilbert space  $H$ . If  $A$  is dissipative and  $0 \in \rho(A)$ , then  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions on  $H$ .

**Theorem 1.2.12.** (Hille-Yosida see Theorem 7.4 in [28]). Let  $(A, D(A))$  be an unbounded linear operator on  $H$ . Assume that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions  $(S(t))_{t \geq 0}.$

1. For  $U_0 \in D(A)$ , the problem (1.2.1) admits a unique strong solution

$$U(t) = S(t)U_0 \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, H).$$

2. For  $U_0 \in H$ , the problem (1.2.1) admits a unique weak solution

$$U(t) \in C^0(\mathbb{R}^+, H).$$

## 1.3 Stability of semigroup

In this section, we introduce some definitions about strong, exponential, polynomial and analytic stability of a  $C_0$ -semigroup. Then, we give some results about the stability of  $C_0$ -semigroup. For more details see [17, 24, 27, 38, 53, 71, 75, 85].

Let  $(X, \|\cdot\|_X)$  be a Banach space, and  $H$  be a Hilbert space equipped with the inner product  $\langle \cdot, \cdot \rangle_H$  and the induced norm  $\|\cdot\|_H.$

**Definition 1.3.1.** Assume that  $A$  is the generator of a strongly continuous semigroup of contractions  $(S(t))_{t \geq 0}$  on  $X$ . We say that the  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  is

- Strongly stable if

$$\lim_{t \rightarrow +\infty} \|S(t)u\|_X = 0, \quad \forall u \in X.$$

- Uniformly stable if

$$\lim_{t \rightarrow +\infty} \|S(t)\|_{\mathcal{L}(X)} = 0.$$

- Exponentially stable if there exist two positive constants  $M$  and  $\epsilon$  such that

$$\|S(t)u\|_X \leq Me^{-\epsilon t} \|u\|_X, \quad \forall t > 0, \forall u \in X.$$

- Polynomially stable if there exist two positive constants  $C$  and  $\alpha$  such that

$$\|S(t)u\|_X \leq Ct^{-\alpha} \|u\|_X, \quad \forall t > 0, \forall u \in X.$$

The  $C_0$ -semigroup  $(S(t))_{t \geq 0}$  is said to be polynomially stable with optimal decay rate  $t^{-\alpha}$  (with  $\alpha > 0$ ) if it is polynomially stable with decay rate  $t^{-\alpha}$  and, for any  $\epsilon > 0$  small enough, the semigroup  $(S(t))_{t \geq 0}$  does not decay at a rate  $t^{-(\alpha-\epsilon)}$ .

**Proposition 1.3.2.** Assume that  $A$  is the generator of a strongly continuous semigroup of contractions  $(S(t))_{t \geq 0}$  on  $X$ . The following statements are equivalent

- $(S(t))_{t \geq 0}$  is uniformly stable.
- $(S(t))_{t \geq 0}$  is exponentially stable.

To show the strong stability of a  $C_0$ -semigroup of contraction  $(e^{tA})_{t \geq 0}$  we rely on the following result due to Arendt-Batty [17].

**Theorem 1.3.3. (Arendt and Batty)** Assume that  $A$  is the generator of a strongly continuous semigroup of contractions  $(S(t))_{t \geq 0}$  on a reflexive Banach space  $X$ . If

- $A$  has no pure imaginary eigenvalues.
- $\sigma(A) \cap i\mathbb{R}$  is countable.

Then  $S(t)$  is strongly stable.

**Remark 1.3.4.** If the resolvent  $(I - T)^{-1}$  of  $T$  is compact, then  $\sigma(T) = \sigma_p(T)$ . Thus, the state of Theorem 1.3.3 lessens to  $\sigma_p(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ .

Next, when the  $C_0$ -semigroup is strongly stable, we look for the necessary and sufficient conditions of exponential stability of a  $C_0$ -semigroup. In fact, exponential stability results are obtained using different methods like: multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them. In this thesis we will review only two methods. The first method is a frequency domain approach method which was obtained by Huang-Prüss (see [53, 85]).

**Theorem 1.3.5. (Huang-Prüss)** Assume that  $A$  is the generator of a strongly continuous semigroup of contractions  $(S(t))_{t \geq 0}$  on  $H$ .  $S(t)$  is uniformly stable if and only if

- $i\mathbb{R} \subset \rho(A)$ .
- $\sup_{\beta \in \mathbb{R}} \|(i\beta I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty$ .

The second one is a classical method based on the spectrum analysis of the operator  $A$  (see [38, 75]).

**Definition 1.3.6.** Let  $(A, D(A))$  be an unbounded linear operator on  $H$ . Assume that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions  $(S(t))_{t \geq 0}$ .

- The growth bound of  $A$  is defined by

$$\omega_0(A) = \inf \{ \omega \in \mathbb{R} : \exists N_\omega \in \mathbb{R} \text{ such that } \forall t \geq 0 \text{ we have } \|S(t)\| \leq N_\omega e^{\omega t} \}.$$

- The spectral bound of  $A$  is defined by

$$s(A) = \sup \{ \operatorname{Re}(\lambda) : \lambda \in \sigma(A) \}.$$

**Proposition 1.3.7.** Let  $(A, D(A))$  be an unbounded linear operator on  $H$ . Assume that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions  $(S(t))_{t \geq 0}$ . Then  $(S(t))_{t \geq 0}$  is uniformly exponentially stable if and only if its growth bound  $\omega_0(A) < 0$ .

**Proposition 1.3.8.** Let  $(A, D(A))$  be an unbounded linear operator on  $H$ . Assume that  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions  $(S(t))_{t \geq 0}$ . Then, we have

$$s(A) \leq \omega_0(A).$$

**Corollary 1.3.9.** Let  $(A, D(A))$  be an unbounded linear-operator on  $H$ . Assume that  $s(A) = 0$ , then  $(S(t))_{t \geq 0}$  is not uniformly exponentially stable.

In the case when the  $C_0$ -semigroup is not exponentially stable we look for a polynomial one. In general, polynomial stability results are obtained using different methods like: multipliers method, frequency domain approach, Riesz basis approach, Fourier analysis or a combination of them. In this thesis we will review only one method. The frequency domain approach method which was obtained by Batty [24], A.Borichev [27] and Y.Tomilov, Z. Liu and B. Rao [71].

**Theorem 1.3.10.** (**Batty, A.Borichev and Y.Tomilov, Z. Liu and B. Rao.**) Assume that  $A$  is the generator of a strongly continuous semigroup of contractions  $(S(t))_{t \geq 0}$  on  $H$ . If  $i\mathbb{R} \subset \rho(A)$ , then for a fixed  $\ell > 0$  the following conditions are equivalent

$$\lim_{|\lambda| \rightarrow +\infty} \sup \frac{1}{\lambda^\ell} \|(\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty \quad (1.3.1)$$

$$\|S(t) U_0\|_H \leq \frac{C}{t^{\frac{1}{\ell}}} \|U_0\|_{D(A)} \quad \forall t > 0, U_0 \in D(A) \text{ for some } C > 0. \quad (1.3.2)$$

Also, the analytic property of a  $C_0$ -semigroup of contraction  $(S(t))_{t \geq 0}$  is characterized in the following theorem due to Arendt, Batty and Hieber [18].

**Theorem 1.3.11.** Assume that  $A$  is the generator of a strongly continuous semigroup of contractions  $(S(t))_{t \geq 0}$  on  $H$ . Assume that

$$i\mathbb{R} \subset \rho(A). \quad (\text{A1})$$

Then,  $(S(t))_{t \geq 0}$  is analytic if and only if

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{-1}} \|(i\lambda - A)^{-1}\|_{\mathcal{L}(H)} < \infty. \quad (\text{A2})$$

## 1.4 Geometric Conditions

In this section, we recall some definitions of the geometric conditions that we encounter in our work.

**Definition 1.4.1.** For a subset  $\omega$  of  $\Omega$  and  $T > 0$ , we shall say that  $(\omega, T)$  satisfies the **Geometric Control Condition** (GCC in short) if there exists  $T > 0$  such that every geodesic traveling at speed one in  $\Omega$  meets  $\omega$  in time  $t < T$ .

**Definition 1.4.2.** (see [5]) For a subset  $\omega$  of  $\Omega$ , we shall say that  $\omega$  satisfies the **Strictly Geometric Control Condition** (SGCC in short) if there exists an open subset  $\tilde{\omega}$  included strictly in  $\omega$  (i.e.  $\tilde{\omega} \subset \omega$ ) and satisfying the GCC.

**Definition 1.4.3.** Saying that  $\omega$  satisfies the  $\Gamma$ -**condition** if it contains a neighborhood in  $\Omega$  of the set

$$\{x \in \Gamma; (x - x_0) \cdot \nu(x) > 0\},$$

for some  $x_0 \in \mathbb{R}^n$ , where  $\nu$  is the outward unit normal vector to  $\Gamma = \partial\Omega$ .

**Definition 1.4.4.** A subset  $\omega$  satisfies the **Piecewise Multiplier Geometric Condition** (PMGC in short) if there exist:

- $\Omega_j \subset \Omega$  having Lipschitz boundary  $\Gamma_j$ .
- $x_j \in \mathbb{R}^N$ ,  $j = 1, \dots, M$ .

such that

1.  $\Omega_j \cap \Omega_i = \emptyset$  for  $j \neq i$ .
2.  $\omega$  contains a neighborhood in  $\Omega$  of the set

$$\bigcup_{j=1}^M \gamma_j(x_j) \cup \left( \Omega \setminus \bigcup_{j=1}^M \Omega_j \right)$$

where  $\gamma_j(x_j) = \{x \in \Gamma_j; (x - x_j) \cdot \nu_j(x) > 0\}$  and  $\nu_j$  is the outward unit normal vector to  $\Gamma_j$ .

## 2 Stabilization of coupled wave equations with two interior viscous damping

### Sommaire

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### 2.1 Well posedness and strong stability

In this Chapter, we consider a system of coupled wave equations with two interior viscous dampings with fully Dirichlet boundary conditions. The system is given as follows

$$\left\{ \begin{array}{ll} \varphi_{tt} - a\varphi_{xx} + (\mathbb{1}_{(\alpha,\beta)})(x)\varphi_t + c(x)\psi_t = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ \psi_{tt} - \psi_{xx} + (\mathbb{1}_{(\alpha,\beta)})(x)\psi_t - c(x)\varphi_t = 0, & (x, t) \in (0, L) \times \mathbb{R}^+, \\ \varphi(0, t) = \varphi(L, t) = 0, & t > 0, \\ \psi(0, t) = \psi(L, t) = 0, & t > 0. \end{array} \right. \quad (2.1.1)$$

where

$$c(x) = \begin{cases} c_0 & \text{if } x \in (\alpha, \gamma) \\ 0 & \text{otherwise} \end{cases} \quad (2.1.2)$$

and  $0 < \alpha < \beta < \gamma < L$ . This system is considered with the following initial data

$$\varphi(\cdot, 0) = \varphi_0(\cdot), \varphi_t(\cdot, 0) = \varphi_1(\cdot), \psi(\cdot, 0) = \psi_0(\cdot) \quad \text{and} \quad \psi_t(\cdot, 0) = \psi_1(\cdot). \quad (2.1.3)$$

#### 2.1.1 Well-Posedness

Firstly, we reformulate System (2.1.1)-(2.1.3) into an evolution problem in an appropriate Hilbert state space.

The energy of System (2.1.1)-(2.1.3) is given by

$$E_a(t) = \frac{1}{2} \int_0^L \left( |\varphi_t|^2 + a|\varphi_x|^2 + |\psi_t|^2 + |\psi_x|^2 \right) dx.$$

Let  $(\varphi, \varphi_t, \psi, \psi_t)$  be a regular solution of (2.1.1)-(2.1.3). Multiplying the first and the second equations of (2.1.1) by  $u_t$ ,  $y_t$ , respectively, then using the boundary conditions, we get

$$\frac{d}{dt} E_a(t) = - \int_0^L (\mathbb{1}_{(\alpha, \beta)})(x) |\varphi_t|^2 dx - \int_0^L (\mathbb{1}_{(\alpha, \beta)})(x) |\psi_t|^2 dx \leq 0.$$

Thus, System (2.1.1)-(2.1.3) is dissipative in the sense that its energy is a non-increasing function with respect to the time variable  $t$ . Let us define the energy space  $\mathcal{H}_a$  by

$$\mathcal{H}_a = (H_0^1(0, L) \times L^2(0, L))^2.$$

We denote by  $\eta = \varphi_t$  and  $\xi = \psi_t$ . The energy space  $\mathcal{H}_a$  is endowed with the following norm

$$\|\Phi\|_{\mathcal{H}_a}^2 = \|\eta\|^2 + a\|\varphi_x\|^2 + \|\xi\|^2 + \|\psi_x\|^2,$$

where  $\|\cdot\|$  denotes the norm of  $L^2(0, L)$  and  $\Phi = (\varphi, \eta, \psi, \xi)$  in  $\mathcal{H}_a$ . We define the unbounded linear operator  $\mathcal{A}_a : D(\mathcal{A}_a) \subset \mathcal{H}_a \longrightarrow \mathcal{H}_a$  by

$$D(\mathcal{A}_a) = (H^2(0, L) \cap H_0^1(0, L)) \times H_0^1(0, L)^2, \quad (2.1.4)$$

and for all  $\Phi = (\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$ ,

$$\mathcal{A}_a(\varphi, \eta, \psi, \xi) = (\eta, a\varphi_{xx} - (\mathbb{1}_{(\alpha, \beta)})(x)\eta - c(x)\xi, \xi, \psi_{xx} - (\mathbb{1}_{(\alpha, \beta)})(x)\xi + c(x)\eta)^\top. \quad (2.1.5)$$

If  $\Phi = (\varphi, \eta, \psi, \xi)$  is the state of System (2.1.1), then this system is transformed into a first order evolution equation on the Hilbert space  $\mathcal{H}_a$  given by

$$\Phi_t = \mathcal{A}_a \Phi, \quad \Phi(0) = \Phi_0, \quad (2.1.6)$$

where  $\Phi_0 = (\varphi_0, \eta_0, \psi_0, \xi_0)$ .

**Proposition 2.1.1.** *The unbounded linear operator  $\mathcal{A}_a$  is  $m$ -dissipative in the energy space  $\mathcal{H}_a$ .*

*Proof.* For all  $\Phi = (\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$ , we have

$$\operatorname{Re}(\langle \mathcal{A}_a \Phi, \Phi \rangle_{\mathcal{H}_a}) = - \int_0^L (\mathbb{1}_{(\alpha, \beta)})(x) |\varphi_t|^2 dx - \int_0^L (\mathbb{1}_{(\alpha, \beta)})(x) |\psi_t|^2 dx \leq 0,$$

which implies that  $\mathcal{A}_a$  is dissipative. Now, let  $F = (f_1, f_2, f_3, f_4)$ , we prove the existence of  $\Phi = (\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$ , solution of the equation

$$-\mathcal{A}_a \Phi = F. \quad (2.1.7)$$

Equivalently, one must consider the system given by

$$-\eta = f_1, \quad (2.1.8)$$

$$-a\varphi_{xx} + (\mathbb{1}_{(\alpha,\beta)})(x)\eta + c(x)\xi = f_2, \quad (2.1.9)$$

$$-\xi = f_3, \quad (2.1.10)$$

$$-\psi_{xx} + (\mathbb{1}_{(\alpha,\beta)})(x)\xi - c(x)\eta = f_4, \quad (2.1.11)$$

with the boundary conditions

$$\varphi(0) = \varphi(L) = 0, \quad \text{and} \quad \psi(0) = \psi(L) = 0. \quad (2.1.12)$$

Let  $(h, g) \in H_0^1(0, L) \times H_0^1(0, L)$ . Multiplying Equations (2.1.9) and (2.1.11) by  $\bar{h}$  and  $\bar{g}$  respectively, integrating over  $(0, L)$ , we obtain

$$\int_0^L a\varphi_x \bar{h}_x dx + \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x)\eta \bar{h} dx + \int_0^L c(x)\xi \bar{h} dx = \int_0^L f_2 \bar{h} dx, \quad (2.1.13)$$

$$\int_0^L \psi_x \bar{g}_x dx + \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x)\xi \bar{g} dx - \int_0^L c(x)\eta \bar{g} dx = \int_0^L f_4 \bar{g} dx. \quad (2.1.14)$$

Inserting Equations (2.1.8) and (2.1.10) into (2.1.13) and (2.1.14), we get

$$\int_0^L a\varphi_x \bar{h}_x dx = \int_0^L f_2 \bar{h} dx + \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x)f_1 \bar{h} dx + \int_0^L c(x)f_3 \bar{h} dx, \quad (2.1.15)$$

$$\int_0^L \psi_x \bar{g}_x dx = \int_0^L f_4 \bar{g} dx + \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x)f_3 \bar{g} dx - \int_0^L c(x)f_1 \bar{g} dx. \quad (2.1.16)$$

Adding Equations (2.1.15) and (2.1.16), we obtain

$$a((\varphi, \psi), (h, g)) = L(h, g), \quad \forall (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L), \quad (2.1.17)$$

where

$$a((\varphi, \psi), (h, g)) = a \int_0^L \varphi_x \bar{h}_x dx + \int_0^L \psi_x \bar{g}_x dx \quad (2.1.18)$$

and

$$L(h, g) = \int_0^L f_2 \bar{h} dx + \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x)f_1 \bar{h} dx + \int_0^L c(x)f_3 \bar{h} dx + \int_0^L f_4 \bar{g} dx + \int_0^L (\mathbb{1}_{(\alpha,\beta)})(x)f_3 \bar{g} dx - \int_0^L c(x)f_1 \bar{g} dx. \quad (2.1.19)$$

Thanks to (2.1.18), (2.1.19), we have that  $a$  is a bilinear continuous coercive form on  $(H_0^1(0, L) \times H_0^1(0, L))^2$ , and  $L$  is a linear continuous form on  $H_0^1(0, L) \times H_0^1(0, L)$ . Then, using Lax-Milgram Theorem, we deduce that there exists  $(\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L)$  unique solution of the variational problem (2.1.17). Applying the classical elliptic regularity we deduce that  $\Phi = (\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  is the unique solution of (2.1.7). The proof is thus complete.  $\square$

From Proposition 2.1.1, the operator  $\mathcal{A}_a$  is m-dissipative on  $\mathcal{H}_a$  and consequently, generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_a})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8. Then, the solution of the evolution Equation (2.1.6) admits the following representation

$$\Phi(t) = e^{t\mathcal{A}_a}\Phi_0, \quad t \geq 0,$$

which leads to the well-posedness of (2.1.6). Hence, we have the following result.

**Theorem 2.1.2.** *Let  $\Phi_0 \in \mathcal{H}_a$  then, problem (2.1.6) admits a unique weak solution  $\Phi$  that satisfies*

$$\Phi(t) \in C^0(\mathbb{R}^+, \mathcal{H}_a).$$

*Moreover, if  $\Phi_0 \in D(\mathcal{A}_a)$  then, problem (2.1.6) admits a unique strong solution  $\Phi$  that satisfies*

$$\Phi(t) \in C^1(\mathbb{R}^+, \mathcal{H}_a) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_a)).$$

## 2.1.2 Strong Stability

**Theorem 2.1.3.** *The  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_a})_{t \geq 0}$  is strongly stable in  $\mathcal{H}_a$ ; i.e. for all  $U_0 \in \mathcal{H}_a$ , the solution of (2.1.6) satisfies*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_a}\Phi\|_{\mathcal{H}_a} = 0.$$

**Lemma 2.1.4.** *For  $\lambda \in \mathbb{R}$ , we have  $i\lambda I - \mathcal{A}_a$  is injective i.e.*

$$\ker(i\lambda I - \mathcal{A}_a) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

*Proof.* From Proposition 2.1.1, we have  $0 \in \rho(\mathcal{A}_a)$ . We still need to show the result for  $\lambda \in \mathbb{R}^*$ . Suppose that there exists a real number  $\lambda \neq 0$  and  $\Phi = (\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$ , such that

$$\mathcal{A}_a\Phi = i\lambda\Phi.$$

Equivalently, we have

$$\eta = i\lambda\varphi, \tag{2.1.20}$$

$$a\varphi_{xx} - (\mathbb{1}_{(\alpha, \beta)})(x)\eta - c(x)\xi = i\lambda\eta, \tag{2.1.21}$$

$$\xi = i\lambda\psi, \tag{2.1.22}$$

$$\psi_{xx} - (\mathbb{1}_{(\alpha, \beta)})(x)\xi + c(x)\eta = i\lambda\xi. \tag{2.1.23}$$

Next, a straightforward computation gives

$$0 = \operatorname{Re}(\langle i\lambda\Phi, \Phi \rangle_{\mathcal{H}_a}) = \operatorname{Re}(\langle \mathcal{A}_a\Phi, \Phi \rangle_{\mathcal{H}_a}) = - \int_0^L (\mathbb{1}_{(\alpha, \beta)})(x) |\eta|^2 dx - \int_0^L (\mathbb{1}_{(\alpha, \beta)})(x) |\xi|^2 dx,$$



consequently, we deduce that

$$\eta = \xi = 0 \quad \text{on} \quad (\alpha, \beta), \quad (2.1.24)$$

then

$$\varphi = \psi = 0 \quad \text{on} \quad (\alpha, \beta). \quad (2.1.25)$$

Substituting Equations (2.1.20), (2.1.22) in Equations (2.1.21), (2.1.23), we get

$$\lambda^2 \varphi + a\varphi_{xx} - i\lambda(\mathbb{1}_{(\alpha, \beta)})(x)\varphi - i\lambda c(x)\psi = 0, \quad \text{in} \quad (0, L) \quad (2.1.26)$$

$$\lambda^2 \psi + \psi_{xx} - i\lambda(\mathbb{1}_{(\alpha, \beta)})(x)\psi + i\lambda c(x)\varphi = 0, \quad \text{in} \quad (0, L) \quad (2.1.27)$$

with the boundary conditions

$$\varphi(0) = \varphi(L) = \psi(0) = \psi(L) = 0. \quad (2.1.28)$$

Our goal is to prove that  $\varphi = \psi = 0$  on  $(0, L)$ . For simplicity, we divide the proof into three steps.

**Step 1.** The aim of this step is to show that  $\varphi = \psi = 0$  on  $(0, \alpha)$ . We have that  $c(x) = 0$  on  $(0, \alpha)$ , then we obtain the following system

$$\begin{cases} \lambda^2 \varphi + a\varphi_{xx} = 0 & \text{over} \quad (0, \alpha), \\ \lambda^2 \psi + \psi_{xx} = 0 & \text{over} \quad (0, \alpha). \end{cases} \quad (2.1.29)$$

Since  $(\varphi, \psi) \in C^1([0, L])$  and the fact that  $\varphi = \psi = 0$  on  $(\alpha, \beta)$ , we get

$$\varphi(\alpha) = \varphi_x(\alpha) = \psi(\alpha) = \psi_x(\alpha) = 0. \quad (2.1.30)$$

It is easy to see that System (2.1.29) admits only a trivial solution on  $(0, \alpha)$  under the boundary condition (2.1.30).

**Step 2.** The aim of this step is to show that  $\varphi = \psi = 0$  on  $(\beta, \gamma)$ . Using the Equation (2.1.25) and the fact that  $(\varphi, \psi) \in C^1([0, L])$ , we obtain the boundary conditions

$$\varphi(\beta) = \varphi_x(\beta) = \psi(\beta) = \psi_x(\beta) = 0. \quad (2.1.31)$$

Using the fact that  $c(x) = c_0$  then, system (2.1.26)-(2.1.27) becomes

$$\begin{cases} \lambda^2 \varphi + a\varphi_{xx} - i\lambda c_0 \psi = 0 & \text{over} \quad (\beta, \gamma) \\ \lambda^2 \psi + \psi_{xx} + i\lambda c_0 \varphi = 0 & \text{over} \quad (\beta, \gamma). \end{cases} \quad (2.1.32)$$

Combining the Equations of system (2.1.32), we get

$$a\varphi_{xxxx} + (a+1)\lambda^2 \varphi_{xx} + \lambda^2(\lambda^2 - c_0^2)\varphi = 0. \quad (2.1.33)$$

The characteristic equation of system (2.1.33) is

$$P(r) := ar^4 + (a+1)\lambda^2 r^2 + \lambda^2(\lambda^2 - c_0^2).$$

Setting

$$P_0(m) := am^2 + (a+1)\lambda^2 m + \lambda^2(\lambda^2 - c_0^2).$$

The polynomial  $P_0$  has two distinct real roots  $m_1$  and  $m_2$  given by:

$$m_1 = \frac{-\lambda^2(a+1) - \sqrt{\lambda^4(a-1)^2 + 4ac_0^2\lambda^2}}{2a} \quad \text{and} \quad m_2 = \frac{-\lambda^2(a+1) + \sqrt{\lambda^4(a-1)^2 + 4ac_0^2\lambda^2}}{2a}.$$

It is clear that  $m_1 < 0$  and the sign of  $m_2$  depends on the value of  $\lambda$  with respect to  $c_0$ . We distinguish the following three cases:  $\lambda^2 < c_0^2$ ,  $\lambda^2 = c_0^2$  and  $\lambda^2 > c_0^2$ .

**Case 1.** If  $\lambda^2 < c_0^2$ , then  $m_2 > 0$ . Setting

$$r_1 = \sqrt{-m_1} \quad \text{and} \quad r_2 = \sqrt{m_2}.$$

Then  $P$  has four simple roots  $ir_1$ ,  $-ir_1$ ,  $r_2$  and  $-r_2$ , and hence the general solution of system (2.1.32), is given by

$$\begin{cases} \varphi(x) &= c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 \cosh(r_2 x) + c_4 \sinh(r_2 x), \\ \psi(x) &= \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} (c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} (c_3 \cosh(r_2 x) + c_4 \sinh(r_2 x)), \end{cases}$$

where  $c_j \in \mathbb{C}$ ,  $j = 1, \dots, 4$ . In this case, the boundary condition in Equation (2.1.31), can be expressed by

$$M_1 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

where

$$M_1 = \begin{pmatrix} \sin(r_1 \beta) & \cos(r_1 \beta) & \cosh(r_2 \beta) & \sinh(r_2 \beta) \\ r_1 \cos(r_1 \beta) & -r_1 \sin(r_1 \beta) & r_2 \sinh(r_2 \beta) & r_2 \cosh(r_2 \beta) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \sin(r_1 \beta) & \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \cos(r_1 \beta) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} \cosh(r_2 \beta) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} \sinh(r_2 \beta) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \cos(r_1 \beta) & -\frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \sin(r_1 \beta) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} r_2 \sinh(r_2 \beta) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} r_2 \cosh(r_2 \beta) \end{pmatrix}.$$

The determinant of  $M_1$  is given by

$$\det(M_1) = \frac{r_1 r_2 a^2 (r_1^2 + r_2^2)^2}{\lambda^2 c_0^2}.$$

System (2.1.32) with the boundary conditions (2.1.31), admits only a trivial solution  $\varphi = \psi = 0$  if and only if  $\det(M_1) \neq 0$ , i.e.  $M_1$  is invertible. Since,  $r_1^2 + r_2^2 = m_2 - m_1 \neq 0$ , then  $\det(M_1) \neq 0$ . Consequently, if  $\lambda^2 < c_0^2$ , we obtain  $\varphi = \psi = 0$  on  $(\beta, \gamma)$ .

**Case 2.** If  $\lambda^2 = c_0^2$ , then  $m_2 = 0$ . Setting

$$r_1 = \sqrt{-m_1} = \sqrt{\frac{(a+1)c_0^2}{a}}.$$

Then  $P$  has two simple roots  $ir_1$ ,  $-ir_1$  and 0 is a double root. Hence the general solution of System (2.1.32) is given by

$$\begin{cases} \varphi(x) &= c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 x + c_4, \\ \psi(x) &= \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} (c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \frac{\lambda}{ic_0} (c_3 x + c_4), \end{cases}$$

where  $c_j \in \mathbb{C}$ , for  $j = 1, \dots, 4$ . Also, the boundary condition in Equation (2.1.31), can be expressed by

$$M_2 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

where

$$M_2 = \begin{pmatrix} \sin(r_1 \beta) & \cos(r_1 \beta) & \beta & 1 \\ r_1 \cos(r_1 \beta) & -r_1 \sin(r_1 \beta) & 1 & 0 \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \sin(r_1 \beta) & \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \cos(r_1 \beta) & \frac{\lambda \beta}{ic_0} & \frac{\lambda}{ic_0} \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \cos(r_1 \beta) & -\frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \sin(r_1 \beta) & \frac{\lambda}{ic_0} & 0 \end{pmatrix}.$$

The determinant of  $M_2$  is given by

$$\det(M_2) = \frac{-a^2 r_1^5}{\lambda^2 c_0^2}.$$

Since  $r_1 = \sqrt{-m_1} \neq 0$ , then  $\det(M_2) \neq 0$ . Thus, System (2.1.26), (2.1.27) with the boundary conditions (2.1.31), admits only a trivial solution  $\varphi = \psi = 0$  on  $(\beta, \gamma)$ .

**Case 3.** If  $\lambda^2 > c_0^2$ , then  $m_2 < 0$ . Setting

$$r_1 = \sqrt{-m_1} \quad \text{and} \quad r_2 = \sqrt{-m_2}.$$

Then  $P$  has four simple roots  $ir_1$ ,  $-ir_1$ ,  $ir_2$  and  $-ir_2$ , and hence the general solution

## 2 Stabilization of coupled wave equations with two interior viscous damping – 2.1

### Well posedness and strong stability

of System (2.1.32) is given by

$$\begin{cases} \varphi(x) &= c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 \sin(r_2 x) + c_4 \cos(r_2 x), \\ \psi(x) &= \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} (c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} (c_3 \sin(r_2 x) + c_4 \cos(r_2 x)), \end{cases}$$

where  $c_j \in \mathbb{C}$ , for  $j = 1, \dots, 4$ . Also, the boundary condition in Equation (2.1.31), can be expressed by

$$M_3 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

where

$$M_3 = \begin{pmatrix} \sin(r_1 \beta) & \cos(r_1 \beta) & \sin(r_2 \beta) & \cos(r_2 \beta) \\ r_1 \cos(r_1 \beta) & -r_1 \sin(r_1 \beta) & r_2 \cos(r_2 \beta) & -r_2 \sin(r_2 \beta) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \sin(r_1 \beta) & \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \cos(r_1 \beta) & \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} \sin(r_2 \beta) & \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} \cos(r_2 \beta) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \cos(r_1 \beta) & -\frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \sin(r_1 \beta) & \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} r_2 \cos(r_2 \beta) & -\frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} r_2 \sin(r_2 \beta) \end{pmatrix}.$$

The determinant of  $M_3$  is given by

$$\det(M_3) = -\frac{r_1 r_2 a^2 (r_1^2 - r_2^2)^2}{\lambda c_0^2}.$$

Since  $r_1^2 - r_2^2 = m_2 - m_1 \neq 0$ , then  $\det(M_3) \neq 0$ . Thus, System (2.1.32) with the boundary condition (2.1.31), admits only a trivial solution  $\varphi = \psi = 0$  on  $(\beta, \gamma)$ . Consequently, we obtain  $\Phi = 0$  on  $(\beta, \gamma)$ .

**Step 3.** The aim of this step is to show that  $\varphi = \psi = 0$  on  $(\gamma, L)$ . From Equations (2.1.32) and the fact that  $c(x) = 0$  on  $(\gamma, L)$ , we obtain the following system

$$\begin{cases} \lambda^2 \varphi + a\varphi_{xx} &= 0 & \text{over } (\gamma, L) \\ \lambda^2 \psi + \psi_{xx} &= 0 & \text{over } (\gamma, L). \end{cases} \quad (2.1.34)$$

Since  $(\varphi, \psi) \in C^1([0, L])$  and the fact that  $\varphi = \psi = 0$  on  $(\beta, \gamma)$ , we get

$$\varphi(\gamma) = \varphi_x(\gamma) = \psi(\gamma) = \psi_x(\gamma) = 0. \quad (2.1.35)$$

Finally, it is easy to see that System (2.1.34) admits only a trivial solution on  $(\gamma, L)$  under the boundary condition (2.1.35).

Consequently, we proved that  $\Phi = 0$  on  $(0, L)$ . The proof is thus complete.  $\square$

**Proof of Theorem 2.1.3.** Using Lemma 2.1.4, we have that  $\mathcal{A}_a$  has non pure imaginary eigenvalues. Since  $D(\mathcal{A}_a)$  is compactly embeded in  $\mathcal{H}_a$ ,  $\mathcal{A}_a$  is maximal dissipative,

$\mathcal{A}_a$  has no pure imaginary eigenvalues, and  $\mathcal{A}_a$  has a compact resolvent. Then, using Arendt-Batty Theorem 1.3.3, the  $C_0$ -semigroup  $(e^{t\mathcal{A}_a})_{t \geq 0}$  is strongly stable.

## 2.2 Exponential Stability

The main result of this section is to show the exponential stability of System (2.1.1).

**Theorem 2.2.1.** *The  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_a})_{t \geq 0}$  is exponentially stable, i.e. there exists constants  $M \geq 1$  and  $\tau > 0$  independent of  $\Phi_0$  such that*

$$\|e^{t\mathcal{A}_a}\Phi_0\|_{\mathcal{H}_a} \leq M e^{-\tau t} \|\Phi_0\|_{\mathcal{H}_a}, \quad \forall \quad t \geq 0.$$

According to Huang [53] and Prüss [85] result 1.3.10, we have to check if the following conditions hold:

$$i\mathbb{R} \subseteq \rho(\mathcal{A}_a) \tag{H1}$$

and

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A}_a)^{-1}\|_{\mathcal{L}(\mathcal{H}_a)} = O(1). \tag{H2}$$

By using the same argument of Lemma 2.1.4, the operator  $\mathcal{A}_a$  has no pure imaginary eigenvalues. Then, condition (H1) holds. We will prove condition (H2) using an argument of contradiction. Indeed, suppose there exists

$$\{(\lambda_n, \Phi_n = (\varphi_n, \eta_n, \psi_n, \xi_n))\}_{n \geq 1} \subset \mathbb{R}_+^* \times D(\mathcal{A}_a)$$

such that

$$\lambda_n \rightarrow +\infty \quad \text{and} \quad \|\Phi_n\|_{\mathcal{H}_a} = 1 \tag{2.2.1}$$

and there exists a sequence  $F_n = (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}) \in \mathcal{H}_a$  such that

$$(i\lambda_n I - \mathcal{A}_a)\Phi_n = F_n \rightarrow 0 \quad \text{in} \quad \mathcal{H}_a. \tag{2.2.2}$$

Detailing (2.2.2), we get the following system

$$i\lambda\varphi_n - \eta_n = f_{1,n} \quad \text{in} \quad H_0^1(0, L), \tag{2.2.3}$$

$$i\lambda\eta_n - a(\varphi_n)_{xx} + (\mathbb{1}_{(\alpha, \beta)})(x)\eta_n + c(x)\xi_n = f_{2,n} \quad \text{in} \quad L^2(0, L), \tag{2.2.4}$$

$$i\lambda\psi_n - \xi_n = f_{3,n} \quad \text{in} \quad H_0^1(0, L), \tag{2.2.5}$$

$$i\lambda\xi_n - (\psi_n)_{xx} + (\mathbb{1}_{(\alpha, \beta)})(x)\xi_n - c(x)\eta_n = f_{4,n} \quad \text{in} \quad L^2(0, L). \tag{2.2.6}$$

In what follows, we will check the condition (H2) by finding a contradiction with (2.2.1) such as  $\|\Phi_n\|_{\mathcal{H}_a} = o(1)$ . For clarity, we divide the proof into several lemmas. From now on, for simplicity, we drop the index  $n$ .

**Lemma 2.2.2.** *The solution  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  of Equations (2.2.3)-(2.2.6) satisfies the*

following asymptotic behavior estimations

$$\int_{\alpha}^{\beta} |\eta|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha}^{\beta} |\xi|^2 dx = o(1).$$

*Proof.* Taking the inner product of (2.2.2) with  $\Phi$  in  $\mathcal{H}_a$ , then using the fact that  $\Phi$  is uniformly bounded in  $\mathcal{H}_a$ , we get

$$\int_{\alpha}^{\beta} |\eta|^2 dx + \int_{\alpha}^{\beta} |\xi|^2 dx = -\operatorname{Re} \langle \mathcal{A}_a \Phi, \Phi \rangle_{\mathcal{H}_a} = \operatorname{Re} \langle (i\lambda I - \mathcal{A}_a) \Phi, \Phi \rangle = o(1).$$

Thus, the proof of the Lemma is complete.  $\square$

Substituting  $\eta$  and  $\xi$  by  $i\lambda\varphi - f_1$  and  $i\lambda\psi - f_3$  respectively in (2.2.4) and (2.2.6), we get the following system

$$\lambda^2 \varphi + a\varphi_{xx} - i\lambda(\mathbb{1}_{(\alpha,\beta)})(x)\varphi - i\lambda c(x)\psi = -i\lambda f_1 - (\mathbb{1}_{(\alpha,\beta)})(x)f_1 - f_2 - c(x)f_3, \quad (2.2.7)$$

$$\lambda^2 \psi + \psi_{xx} - i\lambda(\mathbb{1}_{(\alpha,\beta)})(x)\psi + i\lambda c(x)\varphi = -i\lambda f_3 - (\mathbb{1}_{(\alpha,\beta)})(x)f_3 - f_4 + c(x)f_1. \quad (2.2.8)$$

**Lemma 2.2.3.** Let  $0 < \delta < \frac{\beta-\alpha}{2}$ . The solution  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  of Equations (2.2.3)-(2.2.6) satisfies the following asymptotic behavior estimations

$$\int_{\alpha+\delta}^{\beta-\delta} |\varphi_x|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha+\delta}^{\beta-\delta} |\psi_x|^2 dx = o(1).$$

*Proof.* First, we define the cut-off function  $\theta \in C^1(0, L)$  by

$$0 \leq \theta \leq 1, \quad \theta = 1 \text{ on } (\alpha + \delta, \beta - \delta) \quad \text{and} \quad \theta = 0 \text{ on } (0, \alpha) \cup (\beta, L). \quad (2.2.9)$$

Multiplying Equations (2.2.7) and (2.2.8) by  $\theta\bar{\varphi}$  and  $\theta\bar{\psi}$  respectively, integrating over  $(0, L)$  and using the fact that  $\lambda\varphi$  and  $\lambda\psi$  are uniformly bounded in  $L^2(0, L)$  and  $\|F\| \rightarrow 0$  in  $\mathcal{H}_a$  and taking the real part, we get

$$\int_0^L \theta |\lambda\varphi|^2 dx - a \int_0^L \theta |\varphi_x|^2 dx - a \int_0^L \theta' \bar{\varphi} \varphi_x dx - \operatorname{Re} \left( i\lambda c_0 \int_{\alpha}^{\beta} \theta \psi \bar{\varphi} dx \right) = o(1) \quad (2.2.10)$$

and

$$\int_0^L \theta |\lambda\psi|^2 dx - \int_0^L \theta |\psi_x|^2 dx - \int_0^L \theta' \bar{\psi} \psi_x dx + \operatorname{Re} \left( i\lambda c_0 \int_{\alpha}^{\beta} \theta \varphi \bar{\psi} dx \right) = o(1). \quad (2.2.11)$$

Using the fact that  $\lambda\varphi$  and  $\lambda\psi$  are uniformly bounded in  $L^2(0, L)$ , in particular in  $L^2(\alpha, \beta)$ , and the definition of  $\theta$ , we get

$$\operatorname{Re} \left( i\lambda c_0 \int_{\alpha}^{\beta} \theta \psi \bar{\varphi} dx \right) = o(1) \quad \text{and} \quad \operatorname{Re} \left( i\lambda c_0 \int_{\alpha}^{\beta} \theta \varphi \bar{\psi} dx \right) = o(1). \quad (2.2.12)$$

On the other hand, using the fact that  $\lambda\varphi$ ,  $\lambda\psi$ ,  $\varphi_x$  and  $\psi_x$  are uniformly bounded in  $L^2(0, L)$ ,

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we get

$$a \int_0^L \theta' \bar{\varphi} \varphi_x dx = o(1) \quad \text{and} \quad \int_0^L \theta' \bar{\psi} \psi_x dx = o(1). \quad (2.2.13)$$

Furthermore, using Lemma 2.2.2, Equations (2.2.3), (2.2.5) and the definition of the function  $\theta$  in Equation (2.2.9), we get

$$\int_0^L \theta |\lambda \varphi|^2 dx = o(1) \quad \text{and} \quad \int_0^L \theta |\lambda \psi|^2 dx = o(1). \quad (2.2.14)$$

Inserting Equations (2.2.12)-(2.2.14) in Equations (2.2.10) and (2.2.11), we get the desired results. Thus, the proof of this Lemma is complete.  $\square$

From Lemma 2.2.2 and Lemma 2.2.3, we get  $\|\Phi\|_{\mathcal{H}_a} = o(1)$  on  $(\alpha + \delta, \beta - \delta)$ . In order to complete the proof, we need to show that  $\|\Phi\|_{\mathcal{H}_a} = o(1)$  on  $(\alpha + \delta, \beta - \delta)^c$ .

**Lemma 2.2.4.** *Let  $h \in C^1(0, L)$ . The solution  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  of Equations (2.2.3)-(2.2.6) satisfies the following asymptotic behavior estimation*

$$\begin{aligned} & \int_0^L h' (|\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2) dx - \operatorname{Re} \left( \left[ ah|\varphi_x|^2 \right]_0^L \right) - \operatorname{Re} \left( \left[ h|\psi_x|^2 \right]_0^L \right) \\ & + 2 \operatorname{Re} \left( \int_0^L c(x) h \xi \bar{\varphi}_x dx \right) - 2 \operatorname{Re} \left( \int_0^L c(x) h \eta \bar{\psi}_x dx \right) = 2 \int_0^L h \bar{\varphi}_x f_2 dx \\ & + 2 \int_0^L h \eta (\bar{f}_1)_x dx + 2 \int_0^L h \bar{\psi}_x f_4 dx + 2 \int_0^L h \xi (\bar{f}_3)_x dx + o(1). \end{aligned} \quad (2.2.15)$$

*Proof.* Multiplying Equations (2.2.4) and (2.2.6) by  $2h\bar{\varphi}_x$  and  $2h\bar{\psi}_x$  respectively, integrate over  $(0, L)$  and using the fact that  $\varphi_x, \psi_x$  are uniformly bounded in  $L^2(0, L)$  and  $\|F\|_{\mathcal{H}_a} \rightarrow 0$  and Lemma 2.2.2, we get

$$2 \int_0^L i \lambda h \eta \bar{\varphi}_x dx - 2a \int_0^L h \varphi_{xx} \bar{\varphi}_x dx + 2 \int_0^L c(x) h \xi \bar{\varphi}_x dx = 2 \int_0^L h \bar{\varphi}_x f_2 dx + o(1) \quad (2.2.16)$$

$$2 \int_0^L i \lambda h \xi \bar{\psi}_x dx - 2 \int_0^L h \psi_{xx} \bar{\psi}_x dx - 2 \int_0^L c(x) h \eta \bar{\psi}_x dx = 2 \int_0^L h \bar{\psi}_x f_4 dx + o(1). \quad (2.2.17)$$

From Equations (2.2.3) and (2.2.5), we have

$$-i \lambda \bar{\varphi}_x = \bar{\eta}_x + (\bar{f}_1)_x \quad \text{and} \quad -i \lambda \bar{\psi}_x = \bar{\xi}_x + (\bar{f}_3)_x.$$

Inserting the above equations in Equations (2.2.16) and (2.2.17) and by taking the real part, we obtain

$$-\int_0^L h |\eta|_x^2 dx - a \int_0^L h |\varphi_x|_x^2 dx + 2 \operatorname{Re} \left( \int_0^L c(x) h \xi \bar{\varphi}_x dx \right) = 2 \int_0^L h \bar{\varphi}_x f_2 dx + 2 \int_0^L h \eta (\bar{f}_1)_x dx + o(1), \quad (2.2.18)$$

$$-\int_0^L h |\xi|_x^2 dx - \int_0^L h |\psi_x|_x^2 dx - 2 \operatorname{Re} \left( \int_0^L c(x) h \eta \bar{\psi}_x dx \right) = 2 \int_0^L h \bar{\psi}_x f_4 dx + 2 \int_0^L h \xi (\bar{f}_3)_x dx + o(1). \quad (2.2.19)$$

Using by parts integration in Equations (2.2.18) and (2.2.19), we get the desired results.  $\square$

**Lemma 2.2.5.** *Let  $0 < \delta < \frac{\beta - \alpha}{2}$ . The solution  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  of Equations (2.2.3)-(2.2.6)*

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satisfies the following asymptotic behavior estimation

$$\int_0^{\alpha+\delta} \left( |\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2 \right) dx = o(1).$$

*Proof.* Define the cut-off function  $\tilde{\theta}$  in  $C^1([0, L])$  by

$$0 \leq \tilde{\theta} \leq 1, \quad \tilde{\theta} = 1 \text{ on } (0, \alpha + \delta), \quad \tilde{\theta} = 0 \text{ on } (\beta - \delta, L). \quad (2.2.20)$$

Take  $h = x\tilde{\theta}(x)$  in Equation (2.2.15), we get

$$\begin{aligned} \int_0^L h' \left( |\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2 \right) dx + 2c_0 \operatorname{Re} \left( \int_{\alpha}^{\beta-\delta} x\tilde{\theta}\xi\overline{\varphi}_x dx \right) \\ - 2c_0 \operatorname{Re} \left( \int_{\alpha}^{\beta-\delta} x\tilde{\theta}\eta\overline{\psi}_x dx \right) = o(1). \end{aligned} \quad (2.2.21)$$

Using Lemma (2.2.2) and that  $\varphi_x$  and  $\psi_x$  are uniformly bounded in  $L^2(0, L)$  and in particular in  $L^2(\alpha, \beta - \delta)$ , we get

$$2c_0 \operatorname{Re} \left( \int_{\alpha}^{\beta-\delta} x\tilde{\theta}\xi\overline{\varphi}_x dx \right) = o(1) \quad \text{and} \quad 2c_0 \operatorname{Re} \left( \int_{\alpha}^{\beta-\delta} x\tilde{\theta}\eta\overline{\psi}_x dx \right) = o(1).$$

Inserting the above equations in Equation (2.2.21), and using Lemmas (2.2.2), (2.2.3) and the definition of the function  $\tilde{\theta}$ , we get the desired result.  $\square$

From the preceded results of Lemmas 2.2.2, 2.2.3 and 2.2.5, we deduce that

$$\|\Phi\|_{\mathcal{H}_a} = o(1) \quad \text{on} \quad (0, \beta - \delta).$$

Now, our goal is to prove that  $\|\Phi\|_{\mathcal{H}_a} = o(1)$  on  $(\beta - \delta, L)$ . For this aims, let  $g \in C^1([\beta - \delta, \gamma])$  such that

$$g(\gamma) = -g(\beta - \delta) = 1, \quad \max_{x \in [\beta, \gamma]} |g(x)| = c_g \quad \text{and} \quad \max_{x \in [\beta, \gamma]} |g'(x)| = c_{g'}$$

where  $c_g$  and  $c_{g'}$  are strictly positive constant numbers.

**Remark 2.2.6.** It is easy to see the existence of  $g(x)$ . For example, we can take  $g(x) = \cos\left(\frac{(\gamma - x)\pi}{\gamma - \beta + \delta}\right)$  to get  $g(\gamma) = -g(\beta - \delta) = 1$ ,  $g \in C^1([\beta - \delta, \gamma])$ ,  $|g(x)| \leq 1$  and  $|g'(x)| \leq \frac{\pi}{\gamma - \beta + \delta}$ .

**Lemma 2.2.7.** Let  $0 < \delta < \frac{\beta - \alpha}{2}$ . The solution  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  of Equations (2.2.3)-(2.2.6) satisfies the following asymptotic behavior estimation

$$|\eta(\gamma)|^2 = O(1), \quad |\eta(\beta - \delta)|^2 = O(1), \quad |\xi(\gamma)|^2 = O(1) \quad \text{and} \quad |\xi(\beta - \delta)|^2 = O(1).$$

*Proof.* From Equations (2.2.4) and (2.2.6), we have

$$i\lambda\varphi_x - \eta_x = (f_1)_x \quad \text{and} \quad i\lambda\psi_x - \xi_x = (f_3)_x. \quad (2.2.22)$$



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Multiplying the first equation and the second equation of (2.2.22) respectively by  $2g(x)\bar{\eta}$  and  $2g(x)\bar{\xi}$ , integrating over  $(\beta - \delta, \gamma)$  and using the fact that  $\|F\|_{\mathcal{H}_a} \rightarrow 0$  and  $\eta$  and  $\xi$  are uniformly bounded in  $L^2(0, L)$  in particular in  $L^2(\beta - \delta, \gamma)$ , we get

$$\operatorname{Re} \left( 2i\lambda \int_{\beta-\delta}^{\gamma} g\varphi_x \bar{\eta} dx \right) - \int_{\beta-\delta}^{\gamma} g(x) (|\eta|^2)_x dx = o(1), \quad (2.2.23)$$

$$\operatorname{Re} \left( 2i\lambda \int_{\beta-\delta}^{\gamma} g\psi_x \bar{\xi} dx \right) - \int_{\beta-\delta}^{\gamma} g(x) (|\xi|^2)_x dx = o(1). \quad (2.2.24)$$

Using integration by parts in Equations (2.2.23) and (2.2.24), we get

$$\int_{\beta-\delta}^{\gamma} g'(x) |\eta|^2 dx + \operatorname{Re} \left( 2i\lambda \int_{\beta-\delta}^{\gamma} g\varphi_x \bar{\eta} dx \right) = |\eta(\gamma)|^2 + |\eta(\beta - \delta)|^2 + o(1), \quad (2.2.25)$$

$$\int_{\beta-\delta}^{\gamma} g'(x) |\xi|^2 dx + \operatorname{Re} \left( 2i\lambda \int_{\beta-\delta}^{\gamma} g\psi_x \bar{\xi} dx \right) = |\xi(\gamma)|^2 + |\xi(\beta - \delta)|^2 + o(1). \quad (2.2.26)$$

Multiplying Equations (2.2.4) and (2.2.6) by  $2g(x)\bar{\varphi}_x$  and  $2g(x)\bar{\psi}_x$  respectively, integrating over  $(\beta - \delta, \gamma)$ , using the fact  $\|F\|_{\mathcal{H}_a} \rightarrow 0$ ,  $\varphi_x$  and  $\psi_x$  are uniformly bounded in  $L^2(0, L)$  and Lemma 2.2.2 and taking the real part, we get

$$\operatorname{Re} \left( 2i\lambda \int_{\beta-\delta}^{\gamma} g(x) \eta \bar{\varphi}_x dx \right) - a \int_{\beta-\delta}^{\gamma} g(x) (|\varphi_x|^2)_x dx + 2 \operatorname{Re} \left( c_0 \int_{\beta-\delta}^{\gamma} g(x) \xi \bar{\varphi}_x dx \right) = o(1),$$

$$\operatorname{Re} \left( 2i\lambda \int_{\beta-\delta}^{\gamma} g(x) \xi \bar{\psi}_x dx \right) - \int_{\beta-\delta}^{\gamma} g(x) (|\psi_x|^2)_x dx - 2 \operatorname{Re} \left( c_0 \int_{\beta-\delta}^{\gamma} g(x) \eta \bar{\psi}_x dx \right) = o(1).$$

Using integration by parts in the second terms of the above Equations, we obtain

$$\begin{aligned} \operatorname{Re} \left( 2i\lambda \int_{\beta-\delta}^{\gamma} g(x) \eta \bar{\varphi}_x dx \right) + a \int_{\beta-\delta}^{\gamma} g'(x) |\varphi_x|^2 dx + 2 \operatorname{Re} \left( c_0 \int_{\beta-\delta}^{\gamma} g(x) \xi \bar{\varphi}_x dx \right) \\ = a |\varphi_x(\gamma)|^2 + a |\varphi_x(\beta - \delta)|^2 + o(1) \end{aligned} \quad (2.2.27)$$

and

$$\begin{aligned} \operatorname{Re} \left( 2i \int_{\beta-\delta}^{\gamma} g(x) \xi \bar{\psi}_x dx \right) + \int_{\beta-\delta}^{\gamma} g'(x) |\psi_x|^2 dx - 2 \operatorname{Re} \left( c_0 \int_{\beta-\delta}^{\gamma} g(x) \eta \bar{\psi}_x dx \right) \\ = |\psi_x(\gamma)|^2 + |\psi_x(\beta - \delta)|^2 + o(1). \end{aligned} \quad (2.2.28)$$

Adding Equations (2.2.25)-(2.2.28), we get

$$\begin{aligned} M(\gamma, \beta - \delta) + N(\gamma, \beta - \delta) &= \int_{\beta-\delta}^{\gamma} g'(x) (|\eta|^2 + a |\varphi_x|^2 + |\xi|^2 + |\psi_x|^2) dx \\ &\quad + 2 \operatorname{Re} \left( c_0 \int_{\beta-\delta}^{\gamma} g(x) \xi \bar{\varphi}_x dx \right) - 2 \operatorname{Re} \left( c_0 \int_{\beta-\delta}^{\gamma} g(x) \eta \bar{\psi}_x dx \right) + o(1) \end{aligned} \quad (2.2.29)$$

where

$$\begin{aligned} M(\gamma, \beta - \delta) &= |\eta(\gamma)|^2 + |\eta(\beta - \delta)|^2 + a |\varphi_x(\gamma)|^2 + a |\varphi_x(\beta - \delta)|^2, \\ N(\gamma, \beta - \delta) &= |\xi(\gamma)|^2 + |\xi(\beta - \delta)|^2 + |\psi_x(\gamma)|^2 + |\psi_x(\beta - \delta)|^2. \end{aligned}$$

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From Equation (2.2.29), we get

$$\begin{aligned} M(\gamma, \beta - \delta) + N(\gamma, \beta - \delta) &\leq c_{g'} \int_{\beta - \delta}^{\gamma} \left( |\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2 \right) dx \\ &\quad + c_0 c_g \|\xi\|_{L^2(0,L)} \|\varphi_x\|_{L^2(0,L)} + c_0 c_g \|\eta\|_{L^2(0,L)} \|\psi_x\|_{L^2(0,L)} + o(1). \end{aligned}$$

Using the fact that  $\|\Phi\|$  is uniformly bounded in  $\mathcal{H}_a$ , we obtain the desired result. The proof of this Lemma has been completed.  $\square$

**Lemma 2.2.8.** *Let  $0 < \delta < \frac{\beta - \alpha}{2}$ . The solution  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  of Equations (2.2.3)-(2.2.6) satisfies the following asymptotic behavior estimation*

$$\int_{\beta - \delta}^L \left( |\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2 \right) dx = o(1).$$

*Proof.* Define the cut-off function  $\hat{\theta}$  in  $C^1([0, L])$  by

$$0 \leq \hat{\theta} \leq 1, \quad \hat{\theta} = 1 \text{ on } (\beta - \delta, L), \quad \text{and} \quad \hat{\theta} = 0 \text{ on } (0, \alpha + \delta). \quad (2.2.30)$$

Take  $h = (x - L)\hat{\theta}$  in Equation (2.2.15), using Lemmas (2.2.2), (2.2.3) and the definition of the function  $\hat{\theta}$ , we get

$$\begin{aligned} &\int_{\beta - \delta}^L \left( |\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2 \right) dx + 2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} (x - L) \hat{\theta} \xi \bar{\varphi}_x dx \right) \\ &\quad - 2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} (x - L) \hat{\theta} \eta \bar{\psi}_x dx \right) = o(1). \end{aligned} \quad (2.2.31)$$

Using the fact that  $\xi = i\lambda\psi - f_3$  and  $\eta = i\lambda\varphi - f_1$  in the second and third term of Equation (2.2.31) and that  $\varphi_x, \psi_x$  are uniformly bounded in  $L^2(0, L)$  and the fact that  $\|F\|_{\mathcal{H}_a} \rightarrow 0$ , we get

$$\begin{aligned} &2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} (x - L) \hat{\theta} \xi \bar{\varphi}_x dx \right) - 2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} (x - L) \hat{\theta} \eta \bar{\psi}_x dx \right) = 2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} i\lambda(x - L) \hat{\theta} \psi \bar{\varphi}_x dx \right) \\ &\quad - 2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} i\lambda(x - L) \hat{\theta} \varphi \bar{\psi}_x dx \right) + o(1). \end{aligned}$$

Using integration by parts in the first term of the right hand side of the above equation and the fact that  $\lambda\varphi$  and  $\lambda\psi$  are uniformly bounded in  $L^2(\Omega)$ , we obtain

$$\begin{aligned} &2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} (x - L) \hat{\theta} \xi \bar{\varphi}_x dx \right) - 2c_0 \operatorname{Re} \left( \int_{\beta - \delta}^{\gamma} (x - L) \hat{\theta} \eta \bar{\psi}_x dx \right) = \\ &\quad 2c_0 \operatorname{Re} \left( \left[ i\lambda(x - L) \psi \bar{\varphi} \right]_{\beta - \delta}^{\gamma} \right) + o(1). \end{aligned} \quad (2.2.32)$$

Inserting Equation (2.2.32) in Equation (2.2.31), we obtain

$$\int_{\beta - \delta}^L \left( |\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2 \right) dx = A(\gamma) + B(\beta - \delta) + o(1), \quad (2.2.33)$$

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where

$$\begin{aligned} A(\gamma) &= 2c_0 \operatorname{Re} \left( i\lambda(L - \gamma)\psi(\gamma)\overline{\varphi}(\gamma) \right), \\ B(\beta - \delta) &= 2c_0 \operatorname{Re} \left( i\lambda(\beta - \delta - L)\psi(\beta - \delta)\overline{\varphi}(\beta - \delta) \right). \end{aligned}$$

On the other hand, from Equations (2.2.3) and (2.2.5), we have

$$|\lambda\varphi(s)| \leq |\eta(s)| + |f_1(s)| \quad \text{and} \quad |\lambda\psi(s)| \leq |\xi(s)| + |f_3(s)| \quad \text{for } s \in \{\beta - \delta, \gamma\}. \quad (2.2.34)$$

Using the fact that  $|f_1(s)| \leq s \int_0^s |(f_1)_x|^2 dx \leq sa^{-1} \|F\|_{\mathcal{H}_a}^2$  and  $|f_3(s)| \leq s \int_0^s |(f_3)_x|^2 dx \leq s \|F\|_{\mathcal{H}_a}^2$  for all  $s \in \{\beta - \delta, \gamma\}$ , and using Lemma 2.2.7 in Equation (2.2.34), we obtain

$$|\lambda\varphi(s)| = O(1) \quad \text{and} \quad |\lambda\psi(s)| = O(1), \quad \text{for } s \in \{\beta - \delta, \gamma\}.$$

It follows that

$$A(\gamma) + B(\beta - \delta) = o(1). \quad (2.2.35)$$

Using Equation (2.2.34) in Equation (2.2.33), we obtain

$$\int_{\beta - \delta}^L \left( |\eta|^2 + a|\varphi_x|^2 + |\xi|^2 + |\psi_x|^2 \right) dx = o(1).$$

Thus, the proof has been completed.  $\square$

**Proof of Theorem 2.2.1** Using Lemmas 2.2.2, 2.2.3, 2.2.5 and 2.2.8, we get  $\|\Phi\|_{\mathcal{H}_a} = o(1)$  on  $[0, L]$ , which contradicts Equation (2.2.1). Therefore, (H2) holds, by Huang [53] and Prüss [85] we deduce the exponential stability of the System (2.1.1).

# 3 Stability results of an elastic/viscoelastic transmission problem of locally coupled waves with non smooth coefficients

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## 3.1 Introduction

### 3.1.1 Motivation and aims

There are several mathematical models representing physical damping. The most often encountered type of damping in vibration studies are linear viscous damping and Kelvin-Voigt damping which are special cases of proportional damping. Viscous damping usually models external friction forces such as air resistance acting on the vibrating structures and is thus called "external damping", while Kelvin-Voigt damping originates from the internal friction of the material of the vibrating structures and thus called "internal damping". In 1988, F. Huang in [54] considered a wave equation with globally distributed Kelvin-Voigt damping, i.e. the damping coefficient is strictly positive on the entire spatial domain. He proved that the corresponding semigroup is not only exponentially stable, but also is analytic (see Definition 1.3.1, Theorem 1.3.5 and Theorem 1.3.11 below). Thus, Kelvin-Voigt damping is stronger than the viscous damping when globally distributed. Indeed, it was proved that

### 3 Stability results of an elastic/viscoelastic transmission problem of locally coupled waves with non smooth coefficients – 3.1 Introduction

the semigroup corresponding to the system of wave equations with global viscous damping is exponentially stable but not analytic (see [36] for the one dimensional system and [22] for the higher dimensional system). However, the exponential stability of a wave equation is still true even if the viscous damping is localized, via a smooth or a non smooth damping coefficient, in a suitable subdomain satisfying some geometric conditions (see [22]). Nevertheless, when viscoelastic damping is distributed locally, the situation is more delicate and such comparison between viscous and viscoelastic damping is not valid anymore. Indeed, the stabilization of the wave equation with local Kelvin-Voigt damping is greatly influenced by the smoothness of the damping coefficient and the region where the damping is localized (near or faraway from the boundary) even in the one-dimensional case. So, the stabilization of systems (simple or coupled) with local Kelvin-Voigt damping has attracted the attention of many authors (see the Literature below for the history of this kind of damping). From a mathematical point of view, it is important to study the stability of a system coupling a locally damped wave equation with a conservative one. Moreover, the study of this kind of systems is also motivated by several physical considerations and occurs in many applications in engineering and mechanics. In this direction, recently in 2019, Hassine and Souayah in [50], studied the stabilization of a system of global coupled wave equations with one localized Kelvin-Voigt damping. The system is described by

$$\left\{ \begin{array}{ll} u_{tt} - (u_x + b(x)u_{tx})_x + v_t = 0, & (x, t) \in (-1, 1) \times \mathbb{R}^+, \\ v_{tt} - cv_{xx} - u_t = 0, & (x, t) \in (-1, 1) \times \mathbb{R}^+, \\ u(0, t) = v(0, t) = 0, u(1, t) = v(1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (-1, 1), \\ v(x, 0) = v_0(x), v_t(x, 0) = v_1(x), & x \in (-1, 1), \end{array} \right. \quad (3.1.1)$$

where  $c > 0$ , and  $b \in L^\infty(-1, 1)$  is a non-negative function. They assumed that the damping coefficient is given by  $b(x) = d\mathbb{1}_{[0,1]}(x)$ , where  $d$  is a strictly positive constant. The Kelvin-Voigt damping  $(b(x)u_{tx})_x$  is applied at the first equation and the second equation is indirectly damped through the coupling between the two equations. Under the two conditions that the Kelvin-Voigt damping is localized near the boundary and the two waves are globally coupled, they obtained a polynomial energy decay rate of type  $t^{-\frac{1}{6}}$ . Then the stabilization of System (3.1.1) in the case where the Kelvin-Voigt damping is localized in an arbitrary subinterval of  $(-1, +1)$  and the two waves are locally coupled has been left as an open problem. In addition, we believe that the energy decay rate obtained in [50] can be improved. So, we are interested in studying this open problem.

The main aim of this chapter is to study the stabilization of a system of localized coupled wave equations with only one Kelvin-Voigt damping localized via non-smooth coefficient in a subinterval of the domain. The system is described by

$$u_{tt} - (au_x + b(x)u_{tx})_x + c(x)y_t = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (3.1.2)$$

$$y_{tt} - y_{xx} - c(x)u_t = 0, \quad (x, t) \in (0, L) \times \mathbb{R}^+, \quad (3.1.3)$$

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with fully Dirichlet boundary conditions,

$$u(0, t) = u(L, t) = y(0, t) = y(L, t) = 0, \quad \forall t \in \mathbb{R}^+, \quad (3.1.4)$$

where

$$b(x) = \begin{cases} b_0 & \text{if } x \in (\alpha_1, \alpha_3) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad c(x) = \begin{cases} c_0 & \text{if } x \in (\alpha_2, \alpha_4) \\ 0 & \text{otherwise} \end{cases} \quad (3.1.5)$$

and  $a > 0, b_0 > 0$  and  $c_0 \in \mathbb{R}^*$ , and where we consider  $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L$  (see Figure 3.1). This system is considered with the following initial data

$$u(\cdot, 0) = u_0(\cdot), \quad u_t(\cdot, 0) = u_1(\cdot), \quad y(\cdot, 0) = y_0(\cdot) \quad \text{and} \quad y_t(\cdot, 0) = y_1(\cdot). \quad (3.1.6)$$

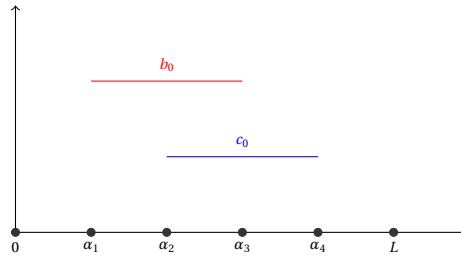


Figure 3.1 – The localization of the damping and coupling coefficients

#### 3.1.2 Literature

The wave is created when a vibrating source disturbs the medium. In order to restrain those vibrations, several dampings can be added such as Kelvin-Voigt damping which is originated from the extension or compression of the vibrating particles. This damping is a viscoelastic structure having properties of both elasticity and viscosity. In the recent years, many researchers showed interest in problems involving this kind of damping (local or global) where different types of stability have been showed. In particular, in the one dimensional case, it was proved that the smoothness of the damping coefficient affects critically the studying of the stability and regularity of the solution of the system. Indeed, in the one dimensional case we can consider the following system

$$\begin{cases} u_{tt} - (u_x + b_1(x)u_{tx})_x = 0, & -1 \leq x \leq 1, t > 0, \\ u(1, t) = u(-1, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & -1 \leq x \leq 1, \end{cases} \quad (3.1.7)$$

with  $b_1 \in L^\infty(-1, 1)$  and

$$b_1(x) = \begin{cases} 0 & \text{if } x \in (0, 1), \\ a_1(x) & \text{if } x \in (-1, 0), \end{cases} \quad (3.1.8)$$

where the function  $a_1(x)$  is non-negative. The case of local Kelvin-Voigt damping was first studied in 1998 [66, 70], it was proved that the semigroup loses exponential stability and

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smooth property when the damping is local and  $a_1 = 1$  or  $b_1(\cdot)$  is the characteristic function of any subinterval of the domain. This surprising result initiated the study of an elastic system with local Kelvin-Voigt damping. In 2002, K. Liu and Z. Liu proved that system (3.1.7) is exponentially stable if  $b_1'(\cdot) \in C^{0,1}([-1, 1])$  (see [67]). Later, in [95], the smoothness on  $b_1$  was weakened to  $b_1(\cdot) \in C^1([-1, 1])$  and a condition on  $a_1$  was taken. In 2004, Renardy's results [87] hinted that the solution of the system (3.1.7) may be exponentially stable under smoother conditions on the damping coefficient. This result was confirmed by K. Liu, Z. Liu and Q. Zhang in [74]. On the other hand, Liu and Rao in 2005 (see [71]) proved that the semigroup corresponding to system (3.1.7) is polynomially stable of order almost 2 if  $a_1(\cdot) \in C(0, 1)$  and  $a_1(x) \geq a_1 \geq 0$  on  $(0, 1)$ . The optimality of this order was later proved in [7]. In 2014, Alves and al., in [8], considered the transmission problem of a material composed of three components; one of them is a Kelvin-Voigt viscoelastic material, the second is an elastic material (no dissipation) and the third is an elastic material inserted with a frictional damping mechanism. They proved that the rate of decay depends on the position of each component. When the viscoelastic component is not in the middle of the material, they proved exponential stability of the solution. However, when the viscoelastic part is in the middle of the material, the solution decays polynomially as  $t^{-2}$ . In 2016, under the assumption that the damping coefficient has a singularity at the interface of the damped and undamped regions and behaves like  $x^\alpha$  near the interface, it was proven by Liu and Zhang [72] that the semigroup corresponding to the system is polynomially or exponentially stable and the decay rate depends on the parameter  $\alpha \in (0, 1]$ . In [12], Ammari and al. generalized the cases of single elastic string with local Kelvin-Voigt damping (in [14, 67]). They studied the stability of a tree of elastic strings with local Kelvin-Voigt damping on some of the edges. They proved exponential/polynomial stability of the system under the compatibility condition of displacement and strain and the continuity condition of damping coefficients at the vertices of the tree.

In [48], Hassine considered the longitudinal and transversal vibrations of the transmission Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam. He proved that the semigroup associated with the equation for the transversal motion of the beam is exponentially stable, although the semigroup associated with the equation for the longitudinal motion of the beam is polynomially stable of type  $t^{-2}$ . In [47], Hassine considered a beam and a wave equation coupled on an elastic beam through transmission conditions with locally distributed Kelvin-Voigt damping that acts through one of the two equations only. He proved a polynomial energy decay rate of type  $t^{-2}$  for both cases where the dissipation acts through the beam equation and through the wave equation. In 2016, Oquendo and Sanz studied the wave equation with internal coupled terms where the Kelvin-Voigt damping is global in one equation and the second equation is conservative. They showed that the semigroup loses speed and decays with the rate  $t^{-\frac{1}{4}}$  and they proved that this decay rate is optimal (see [84]).

#### 3.1.3 Description of the chapter

This chapter is organized as follows: In Subsection 3.2.1, we reformulate the system (3.1.2)-(3.1.6) into an evolution system and we prove the well-posedness of our system by semigroup approach. In Subsection 3.2.2, using a general criteria of Arendt and Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. In Section 3.3, we prove that the system lacks exponential stability using two different approaches. The first case is by taking the damping and the coupling terms to be globally defined, i.e  $b(x) = b_0 > 0$  and

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$c(x) = c_0 \in \mathbb{R}^*$  and we prove the lack of exponential stability using Borichev-Tomilov results. The second case is by taking only the damping term to be localized and we use the method which was developed by Littman and Markus. In Section 3.4, we look for a polynomial decay rate by applying a frequency domain approach combined with a multiplier method based on the exponential stability of the system in chapter 2, where we establish a polynomial energy decay for smooth solution of type  $t^{-1}$ .

## 3.2 Well-Posedness and Strong Stability

In this section, we study the strong stability of System (3.1.2)-(3.1.6). First, using a semigroup approach, we establish well-posedness result of our system.

### 3.2.1 Well-Posedness

Firstly, we reformulate System (3.1.2)-(3.1.6) into an evolution problem in an appropriate Hilbert state space.

The energy of System (3.1.2)-(3.1.6) is given by

$$E(t) = \frac{1}{2} \int_0^L (|u_t|^2 + a|u_x|^2 + |y_t|^2 + |y_x|^2) dx.$$

Let  $(u, u_t, y, y_t)$  be a regular solution of (3.1.2)-(3.1.6). Multiplying (3.1.2), (3.1.3) by  $u_t, y_t$ , respectively, then using the boundary conditions (3.1.4), we get

$$E'(t) = - \int_0^L b(x) |u_{tx}|^2 dx,$$

using the definition of the function  $b(x)$ , we get  $E'(t) \leq 0$ . Thus, System (3.1.2)-(3.1.6) is dissipative in the sense that its energy is a non-increasing function with respect to the time variable  $t$ . Let us define the energy space  $\mathcal{H}$  by

$$\mathcal{H} = (H_0^1(0, L) \times L^2(0, L))^2.$$

The energy space  $\mathcal{H}$  is equipped with the inner product defined by

$$\langle U, U_1 \rangle_{\mathcal{H}} = \int_0^L v \bar{v}_1 dx + a \int_0^L u_x (\bar{u}_1)_x dx + \int_0^L z \bar{z}_1 dx + \int_0^L y_x (\bar{y}_1)_x dx,$$

for all  $U = (u, v, y, z)$  and  $U_1 = (u_1, v_1, y_1, z_1)$  in  $\mathcal{H}$ . We use  $\|U\|_{\mathcal{H}}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{H} \rightarrow \mathcal{H}$  by

$$D(\mathcal{A}) = \left\{ \begin{array}{l} U = (u, v, y, z) \in \mathcal{H}; y \in H^2(0, L) \cap H_0^1(0, L) \\ v, z \in H_0^1(0, L), (au_x + b(x)v_x)_x \in L^2(0, L) \end{array} \right\}$$

and for all  $U = (u, v, y, z) \in D(\mathcal{A})$ ,

$$\mathcal{A}(u, v, y, z) = (v, (au_x + b(x)v_x)_x - c(x)z, z, y_{xx} + c(x)v)^{\top}.$$



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If  $U = (u, u_t, y, y_t)$  is the state of System (3.1.2)-(3.1.6), then this system is transformed into the first order evolution equation on the Hilbert space  $\mathcal{H}$  given by

$$U_t = \mathcal{A}U, \quad U(0) = U_0, \quad (3.2.1)$$

where  $U_0 = (u_0, u_1, y_0, y_1)$ .

**Proposition 3.2.1.** *The unbounded linear operator  $\mathcal{A}$  is  $m$ -dissipative in the energy space  $\mathcal{H}$ .*

*Proof.* For all  $U = (u, v, y, z) \in D(\mathcal{A})$ , we have

$$\operatorname{Re}(\langle \mathcal{A}U, U \rangle_{\mathcal{H}}) = - \int_0^L b(x)|v_x|^2 dx = - \int_{\alpha_1}^{\alpha_3} b_0|v_x|^2 dx \leq 0,$$

which implies that  $\mathcal{A}$  is dissipative. Here  $\operatorname{Re}$  is used to denote the real part of a complex number. Now, let  $F = (f_1, f_2, f_3, f_4)$ , we prove the existence of  $U = (u, v, y, z) \in D(\mathcal{A})$ , solution of the equation

$$-\mathcal{A}U = F. \quad (3.2.2)$$

Equivalently, one must consider the system given by

$$-v = f_1, \quad (3.2.3)$$

$$-(au_x + b(x)v_x)_x + c(x)z = f_2, \quad (3.2.4)$$

$$-z = f_3, \quad (3.2.5)$$

$$-y_{xx} - c(x)v = f_4, \quad (3.2.6)$$

with the boundary conditions

$$u(0) = u(L) = 0, \quad \text{and} \quad y(0) = y(L) = 0. \quad (3.2.7)$$

Let  $(\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L)$ . Multiplying Equations (3.2.4) and (3.2.6) by  $\bar{\varphi}$  and  $\bar{\psi}$  respectively, integrating over  $(0, L)$ , we obtain

$$\int_0^L (au_x + b(x)v_x)\bar{\varphi}_x dx + \int_0^L c(x)z\bar{\varphi} dx = \int_0^L f_2\bar{\varphi} dx, \quad (3.2.8)$$

$$\int_0^L y_x\bar{\psi}_x dx - \int_0^L c(x)v\bar{\psi} dx = \int_0^L f_4\bar{\psi} dx. \quad (3.2.9)$$

Inserting Equations (3.2.3) and (3.2.5) into (3.2.8) and (3.2.9), we get

$$\int_0^L au_x\bar{\varphi}_x dx = \int_0^L f_2\bar{\varphi} dx + \int_0^L b(x)(f_1)_x\bar{\varphi}_x dx + \int_0^L c(x)f_3\bar{\varphi} dx, \quad (3.2.10)$$

$$\int_0^L y_x\bar{\psi}_x dx = \int_0^L f_4\bar{\psi} dx - \int_0^L c(x)f_1\bar{\psi} dx. \quad (3.2.11)$$

Adding Equations (3.2.10) and (3.2.11), we obtain

$$a((u, y), (\varphi, \psi)) = L(\varphi, \psi), \quad \forall (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L), \quad (3.2.12)$$

where

$$a((u, y), (\varphi, \psi)) = a \int_0^L u_x\bar{\varphi}_x dx + \int_0^L y_x\bar{\psi}_x dx \quad (3.2.13)$$

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and

$$L(\varphi, \psi) = \int_0^L f_2 \bar{\varphi} dx + \int_0^L b(x)(f_1)_x \bar{\varphi}_x dx + \int_0^L c(x) f_3 \bar{\varphi} dx + \int_0^L f_4 \bar{\psi} dx - \int_0^L c(x) f_1 \bar{\psi} dx. \quad (3.2.14)$$

Thanks to (3.2.13), (3.2.14), we have that  $a$  is a bilinear continuous coercive form on  $(H_0^1(0, L) \times H_0^1(0, L))^2$ , and  $L$  is a linear continuous form on  $H_0^1(0, L) \times H_0^1(0, L)$ . Then, using Lax-Milgram theorem, we deduce that there exists  $(u, y) \in H_0^1(0, L) \times H_0^1(0, L)$  unique solution of the variational problem (3.2.12). Applying the classical elliptic regularity we deduce that  $U = (u, v, y, z) \in D(\mathcal{A})$  is the unique solution of (3.2.2). The proof is thus complete.  $\square$

From Proposition 3.2.1, the operator  $\mathcal{A}$  is  $m$ -dissipative on  $\mathcal{H}$  and consequently, generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  following Lumer-Phillips Theorem (see Theorem 1.2.8). Then the solution of the evolution Equation (3.2.1) admits the following representation

$$U(t) = e^{t\mathcal{A}} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (3.2.1). Hence, we have the following result.

**Theorem 3.2.2.** *Let  $U_0 \in \mathcal{H}$  then, problem (3.2.1) admits a unique weak solution  $U$  that satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}).$$

*Moreover, if  $U_0 \in D(\mathcal{A})$  then, problem (3.2.1) admits a unique strong solution  $U$  that satisfies*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

#### 3.2.2 Strong Stability

This part is devoted for the proof of the strong stability of the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$ . To obtain strong stability of the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  we use the theorem of Arendt and Batty in [17] (see Theorem 1.3.3).

**Theorem 3.2.3.** *The  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable in  $\mathcal{H}$ ; i.e. for all  $U_0 \in \mathcal{H}$ , the solution of (3.2.1) satisfies*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 3.2.3, since the condition  $(u, v, y, z) \in D(\mathcal{A})$  implies only  $u \in H_0^1(0, L)$ . Therefore, the embedding from  $D(\mathcal{A})$  into  $\mathcal{H}$  is not compact and the resolvent  $(-\mathcal{A})^{-1}$  of the operator  $\mathcal{A}$  is not compact in general. Then according to Theorem 1.3.3, we need to prove that the operator  $\mathcal{A}$  has no pure imaginary eigenvalues and  $\sigma(\mathcal{A}) \cap i\mathbb{R}$  contains only a countable number of continuous spectrum of  $\mathcal{A}$ . The argument for Theorem 3.2.3 relies on the subsequent lemmas.

**Lemma 3.2.4.** *For  $\lambda \in \mathbb{R}$ , we have  $i\lambda I - \mathcal{A}$  is injective i.e.*

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

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*Proof.* From Proposition 3.2.1, we have  $0 \in \rho(\mathcal{A})$ . We still need to show the result for  $\lambda \in \mathbb{R}^*$ . Suppose that there exists a real number  $\lambda \neq 0$  and  $U = (u, v, y, z) \in D(\mathcal{A})$ , such that

$$\mathcal{A}U = i\lambda U.$$

Equivalently, we have

$$v = i\lambda u, \quad (3.2.15)$$

$$(au_x + b(x)v_x)_x - c(x)z = i\lambda v, \quad (3.2.16)$$

$$z = i\lambda y, \quad (3.2.17)$$

$$y_{xx} + c(x)v = i\lambda z. \quad (3.2.18)$$

Next, a straightforward computation gives

$$0 = \operatorname{Re} \langle i\lambda U, U \rangle_{\mathcal{H}} = \operatorname{Re} \langle \mathcal{A}U, U \rangle_{\mathcal{H}} = - \int_0^L b(x)|v_x|^2 dx = - \int_{\alpha_1}^{\alpha_3} b_0|v_x|^2 dx,$$

consequently, we deduce that

$$b(x)v_x = 0 \quad \text{in } (0, L) \quad \text{and} \quad v_x = 0 \quad \text{in } (\alpha_1, \alpha_3). \quad (3.2.19)$$

It follows, from Equation (3.2.15), that

$$u_x = 0 \quad \text{in } (\alpha_1, \alpha_3). \quad (3.2.20)$$

Using Equations (3.2.16), (3.2.17), (3.2.19), (3.2.20) and the definition of  $c(x)$ , we obtain

$$y_x = 0 \quad \text{in } (\alpha_2, \alpha_3). \quad (3.2.21)$$

Substituting Equations (3.2.15), (3.2.17) in Equations (3.2.16), (3.2.18), and using Equation (3.2.19) and the definition of  $b(x)$  in (3.1.5), we get

$$\lambda^2 u + au_{xx} - i\lambda c(x)y = 0, \quad \text{in } (0, L) \quad (3.2.22)$$

$$\lambda^2 y + y_{xx} + i\lambda c(x)u = 0, \quad \text{in } (0, L) \quad (3.2.23)$$

with the boundary conditions

$$u(0) = u(L) = y(0) = y(L) = 0. \quad (3.2.24)$$

Our goal is to prove that  $u = y = 0$  on  $(0, L)$ . For simplicity, we divide the proof into three steps.

**Step 1.** The aim of this step is to show that  $u = y = 0$  on  $(0, \alpha_3)$ . So, using Equation (3.2.20), we have

$$u_x = 0 \quad \text{in } (\alpha_1, \alpha_2).$$

Using the above equation and Equation (3.2.22) and the fact that  $c(x) = 0$  on  $(\alpha_1, \alpha_2)$ , we obtain

$$u = 0 \quad \text{in } (\alpha_1, \alpha_2). \quad (3.2.25)$$

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In fact, system (3.2.22)-(3.2.24) admits a unique solution  $(u, y) \in C^1([0, L])$ , then

$$u(\alpha_1) = u_x(\alpha_1) = 0. \quad (3.2.26)$$

Then, from Equations (3.2.22) and (3.2.26) and the fact that  $c(x) = 0$  on  $(0, \alpha_1)$ , we get

$$u = 0 \quad \text{in} \quad (0, \alpha_1). \quad (3.2.27)$$

Using Equations (3.2.20) and (3.2.25) and the fact that  $u \in C^1([0, L])$ , we get

$$u = 0 \quad \text{in} \quad (\alpha_1, \alpha_3). \quad (3.2.28)$$

Now, using Equations (3.2.20), (3.2.21) and the fact that  $c(x) = c_0$  on  $(\alpha_2, \alpha_3)$  in Equations (3.2.22), (3.2.23), we obtain

$$u = \frac{ic_0}{\lambda} y \quad \text{in} \quad (\alpha_2, \alpha_3). \quad (3.2.29)$$

Using Equation (3.2.28) in Equation (3.2.29), we obtain

$$u = y = 0 \quad \text{in} \quad (\alpha_2, \alpha_3). \quad (3.2.30)$$

Since  $y \in C^1([0, L])$ , then

$$y(\alpha_2) = y_x(\alpha_2) = 0. \quad (3.2.31)$$

So, from Equations (3.2.23) and (3.2.31) and the fact that  $c(x) = 0$  on  $(\alpha_1, \alpha_2)$ , we obtain

$$y = 0 \quad \text{in} \quad (\alpha_1, \alpha_2). \quad (3.2.32)$$

Using the same argument over  $(0, \alpha_1)$ , we get

$$y = 0 \quad \text{in} \quad (0, \alpha_1). \quad (3.2.33)$$

Hence, from Equations (3.2.25), (3.2.27), (3.2.28), (3.2.30), (3.2.32) and (3.2.33), we obtain  $u = y = 0$  on  $(0, \alpha_3)$ . Consequently, we obtain

$$U = 0 \quad \text{in} \quad (0, \alpha_3).$$

**Step 2.** The aim of this step is to show that  $u = y = 0$  on  $(\alpha_3, \alpha_4)$ . Using Equation (3.2.30), and the fact that  $(u, y) \in C^1([0, L])$ , we obtain the boundary conditions

$$u(\alpha_3) = u_x(\alpha_3) = y(\alpha_3) = y_x(\alpha_3) = 0. \quad (3.2.34)$$

Combining Equations (3.2.22), (3.2.23), and the fact that  $c(x) = c_0$  on  $(\alpha_3, \alpha_4)$ , we get

$$au_{xxxx} + (a+1)\lambda^2 u_{xx} + \lambda^2 (\lambda^2 - c_0^2) u = 0. \quad (3.2.35)$$

The characteristic equation of system (3.2.35) is

$$P(r) := ar^4 + (a+1)\lambda^2 r^2 + \lambda^2 (\lambda^2 - c_0^2).$$

Setting

$$P_0(m) := am^2 + (a+1)\lambda^2 m + \lambda^2 (\lambda^2 - c_0^2).$$

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The polynomial  $P_0$  has two distinct real roots  $m_1$  and  $m_2$  given by:

$$m_1 = \frac{-\lambda^2(a+1) - \sqrt{\lambda^4(a-1)^2 + 4ac_0^2\lambda^2}}{2a} \quad \text{and} \quad m_2 = \frac{-\lambda^2(a+1) + \sqrt{\lambda^4(a-1)^2 + 4ac_0^2\lambda^2}}{2a}.$$

It is clear that  $m_1 < 0$  and the sign of  $m_2$  depends on the value of  $\lambda$  with respect to  $c_0$ . We distinguish the following three cases:  $\lambda^2 < c_0^2$ ,  $\lambda^2 = c_0^2$  and  $\lambda^2 > c_0^2$ .

**Case 1.** If  $\lambda^2 < c_0^2$ , then  $m_2 > 0$ . Setting

$$r_1 = \sqrt{-m_1} \quad \text{and} \quad r_2 = \sqrt{m_2}.$$

Then  $P$  has four simple roots  $ir_1$ ,  $-ir_1$ ,  $r_2$  and  $-r_2$ , and hence the general solution of system (3.2.22), (3.2.23), is given by

$$\begin{cases} u(x) &= c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 \cosh(r_2 x) + c_4 \sinh(r_2 x), \\ y(x) &= \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} (c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} (c_3 \cosh(r_2 x) + c_4 \sinh(r_2 x)), \end{cases}$$

where  $c_j \in \mathbb{C}$ ,  $j = 1, \dots, 4$ . In this case, the boundary condition in Equation (3.2.34), can be expressed by

$$M_1 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

where

$$M_1 = \begin{pmatrix} \sin(r_1 \alpha_3) & \cos(r_1 \alpha_3) & \cosh(r_2 \alpha_3) & \sinh(r_2 \alpha_3) \\ r_1 \cos(r_1 \alpha_3) & -r_1 \sin(r_1 \alpha_3) & r_2 \sinh(r_2 \alpha_3) & r_2 \cosh(r_2 \alpha_3) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \sin(r_1 \alpha_3) & \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \cos(r_1 \alpha_3) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} \cosh(r_2 \alpha_3) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} \sinh(r_2 \alpha_3) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \cos(r_1 \alpha_3) & -\frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \sin(r_1 \alpha_3) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} r_2 \sinh(r_2 \alpha_3) & \frac{(\lambda^2 + ar_2^2)}{i\lambda c_0} r_2 \cosh(r_2 \alpha_3) \end{pmatrix}.$$

The determinant of  $M_1$  is given by

$$\det(M_1) = \frac{r_1 r_2 a^2 (r_1^2 + r_2^2)^2}{\lambda^2 c_0^2}.$$

System (3.2.22), (3.2.23) with the boundary conditions (3.2.34), admits only a trivial solution  $u = y = 0$  if and only if  $\det(M_1) \neq 0$ , i.e.  $M_1$  is invertible. Since,  $r_1^2 + r_2^2 = m_2 - m_1 \neq 0$ , then  $\det(M_1) \neq 0$ . Consequently, if  $\lambda^2 < c_0^2$ , we obtain  $u = y = 0$  on  $(\alpha_3, \alpha_4)$ .

**Case 2.** If  $\lambda^2 = c_0^2$ , then  $m_2 = 0$ . Setting

$$r_1 = \sqrt{-m_1} = \sqrt{\frac{(a+1)c_0^2}{a}}.$$

Then  $P$  has two simple roots  $ir_1$ ,  $-ir_1$  and 0 is a double root. Hence the general solution of

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System (3.2.22), (3.2.23) is given by

$$\begin{cases} u(x) &= c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 x + c_4, \\ y(x) &= \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} (c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \frac{\lambda}{ic_0} (c_3 x + c_4), \end{cases}$$

where  $c_j \in \mathbb{C}$ , for  $j = 1, \dots, 4$ . Also, the boundary condition in Equation (3.2.34), can be expressed by

$$M_2 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

where

$$M_2 = \begin{pmatrix} \sin(r_1 \alpha_3) & \cos(r_1 \alpha_3) & \alpha_3 & 1 \\ r_1 \cos(r_1 \alpha_3) & -r_1 \sin(r_1 \alpha_3) & 1 & 0 \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \sin(r_1 \alpha_3) & \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \cos(r_1 \alpha_3) & \frac{\lambda \alpha_3}{ic_0} & \frac{\lambda}{ic_0} \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \cos(r_1 \alpha_3) & -\frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \sin(r_1 \alpha_3) & \frac{\lambda}{ic_0} & 0 \end{pmatrix}.$$

The determinant of  $M_2$  is given by

$$\det(M_2) = \frac{-a^2 r_1^5}{\lambda^2 c_0^2}.$$

Since  $r_1 = \sqrt{-m_1} \neq 0$ , then  $\det(M_2) \neq 0$ . Thus, System (3.2.22), (3.2.23) with the boundary conditions (3.2.34), admits only a trivial solution  $u = y = 0$  on  $(\alpha_3, \alpha_4)$ .

**Case 3.** If  $\lambda^2 > c_0^2$ , then  $m_2 < 0$ . Setting

$$r_1 = \sqrt{-m_1} \quad \text{and} \quad r_2 = \sqrt{-m_2}.$$

Then  $P$  has four simple roots  $ir_1, -ir_1, ir_2$  and  $-ir_2$ , and hence the general solution of System (3.2.22), (3.2.23) is given by

$$\begin{cases} u(x) &= c_1 \sin(r_1 x) + c_2 \cos(r_1 x) + c_3 \sin(r_2 x) + c_4 \cos(r_2 x), \\ y(x) &= \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} (c_1 \sin(r_1 x) + c_2 \cos(r_1 x)) + \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} (c_3 \sin(r_2 x) + c_4 \cos(r_2 x)), \end{cases}$$

where  $c_j \in \mathbb{C}$ , for  $j = 1, \dots, 4$ . Also, the boundary condition in Equation (3.2.34), can be expressed by

$$M_3 \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0,$$

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where

$$M_3 = \begin{pmatrix} \sin(r_1 \alpha_3) & \cos(r_1 \alpha_3) & \sin(r_2 \alpha_3) & \cos(r_2 \alpha_3) \\ r_1 \cos(r_1 \alpha_3) & -r_1 \sin(r_1 \alpha_3) & r_2 \cos(r_2 \alpha_3) & -r_2 \sin(r_2 \alpha_3) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \sin(r_1 \alpha_3) & \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} \cos(r_1 \alpha_3) & \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} \sin(r_2 \alpha_3) & \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} \cos(r_2 \alpha_3) \\ \frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \cos(r_1 \alpha_3) & -\frac{(\lambda^2 - ar_1^2)}{i\lambda c_0} r_1 \sin(r_1 \alpha_3) & \frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} r_2 \cos(r_2 \alpha_3) & -\frac{(\lambda^2 - ar_2^2)}{i\lambda c_0} r_2 \sin(r_2 \alpha_3) \end{pmatrix}.$$

The determinant of  $M_3$  is given by

$$\det(M_3) = -\frac{r_1 r_2 a^2 (r_1^2 - r_2^2)^2}{\lambda c_0^2}.$$

Since  $r_1^2 - r_2^2 = m_2 - m_1 \neq 0$ , then  $\det(M_3) \neq 0$ . Thus, System (3.2.22)-(3.2.23) with the boundary condition (3.2.34), admits only a trivial solution  $u = y = 0$  on  $(\alpha_3, \alpha_4)$ . Consequently, we obtain  $U = 0$  on  $(\alpha_3, \alpha_4)$ .

**Step 3.** The aim of this step is to show that  $u = y = 0$  on  $(\alpha_4, L)$ . From Equations (3.2.22), (3.2.23) and the fact that  $c(x) = 0$  on  $(\alpha_4, L)$ , we obtain the following system

$$\begin{cases} \lambda^2 u + au_{xx} = 0 & \text{over } (\alpha_4, L) \\ \lambda^2 y + y_{xx} = 0 & \text{over } (\alpha_4, L). \end{cases} \quad (3.2.36)$$

Since  $(u, y) \in C^1([0, L])$  and the fact that  $u = y = 0$  on  $(\alpha_3, \alpha_4)$ , we get

$$u(\alpha_4) = u_x(\alpha_4) = y(\alpha_4) = y_x(\alpha_4) = 0. \quad (3.2.37)$$

Finally, it is easy to see that System (3.2.36) admits only a trivial solution on  $(\alpha_4, L)$  under the boundary condition (3.2.37).

Consequently, we proved that  $U = 0$  on  $(0, L)$ . The proof is thus complete.  $\square$

**Lemma 3.2.5.** For all  $\lambda \in \mathbb{R}$ , we have

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}.$$

*Proof.* From Proposition 3.2.1, we have  $0 \in \rho(\mathcal{A})$ . We still need to show the result for  $\lambda \in \mathbb{R}^*$ . Set  $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ , we look for  $U = (u, v, y, z) \in D(\mathcal{A})$  solution of

$$(i\lambda I - \mathcal{A})U = F. \quad (3.2.38)$$

Equivalently, we have

$$v = i\lambda u - f_1, \quad (3.2.39)$$

$$i\lambda v - (au_x + b(x)v_x)_x + c(x)z = f_2, \quad (3.2.40)$$

$$z = i\lambda y - f_3, \quad (3.2.41)$$

$$i\lambda z - y_{xx} - c(x)v = f_4. \quad (3.2.42)$$

Let  $(\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L)$ , multiplying Equations (3.2.40) and (3.2.42) by  $\bar{\varphi}$  and  $\bar{\psi}$  respec-

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tively and integrate over  $(0, L)$ , we obtain

$$\int_0^L i\lambda v \bar{\varphi} dx + \int_0^L a u_x \bar{\varphi}_x dx + \int_0^L b(x) v_x \bar{\varphi}_x dx + \int_0^L c(x) z \bar{\varphi} dx = \int_0^L f_2 \bar{\varphi} dx, \quad (3.2.43)$$

$$\int_0^L i\lambda z \bar{\psi} dx + \int_0^L y_x \bar{\psi}_x dx - \int_0^L c(x) v \bar{\psi} dx = \int_0^L f_4 \bar{\psi} dx. \quad (3.2.44)$$

Substituting  $v$  and  $z$  by  $i\lambda u - f_1$  and  $i\lambda y - f_3$  respectively in Equations (3.2.43)-(3.2.44) and taking the sum, we obtain

$$a((u, y), (\varphi, \psi)) = L(\varphi, \psi), \quad \forall (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L), \quad (3.2.45)$$

where

$$a((u, y), (\varphi, \psi)) = a_1((u, y), (\varphi, \psi)) + a_2((u, y), (\varphi, \psi))$$

with

$$\begin{cases} a_1((u, y), (\varphi, \psi)) = \int_0^L (a u_x \bar{\varphi}_x + y_x \bar{\psi}_x) dx + i\lambda \int_0^L b(x) u_x \bar{\varphi}_x dx, \\ a_2((u, y), (\varphi, \psi)) = -\lambda^2 \int_0^L (u \bar{\varphi} + y \bar{\psi}) dx + i\lambda \int_0^L c(x) (y \bar{\varphi} - u \bar{\psi}) dx, \end{cases}$$

and

$$L(\varphi, \psi) = \int_0^L (f_2 + c(x) f_3 + i\lambda f_1) \bar{\varphi} dx + \int_0^L (f_4 - c(x) f_1 + i\lambda f_3) \bar{\psi} dx + \int_0^L b(x) (f_1)_x \bar{\varphi}_x dx.$$

Let  $V = H_0^1(0, L) \times H_0^1(0, L)$  and  $V' = H^{-1}(0, L) \times H^{-1}(0, L)$  the dual space of  $V$ . Let us consider the following operators,

$$\left\{ \begin{array}{lcl} A: & V & \rightarrow V' \\ & (u, y) & \rightarrow A(u, y) \end{array} \right\} \quad \left\{ \begin{array}{lcl} A_1: & V & \rightarrow V' \\ & (u, y) & \rightarrow A_1(u, y) \end{array} \right\} \quad \left\{ \begin{array}{lcl} A_2: & V & \rightarrow V' \\ & (u, y) & \rightarrow A_2(u, y) \end{array} \right\}$$

such that

$$\left\{ \begin{array}{l} (A(u, y))(\varphi, \psi) = a((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L), \\ (A_1(u, y))(\varphi, \psi) = a_1((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L), \\ (A_2(u, y))(\varphi, \psi) = a_2((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L). \end{array} \right. \quad (3.2.46)$$

Our goal is to prove that  $A$  is an isomorphism operator. For this aim, we divide the proof into three steps.

**Step 1.** In this step, we prove that the operator  $A_1$  is an isomorphism operator. For this goal, following the second equation of (3.2.46) we can easily verify that  $a_1$  is a bilinear continuous coercive form on  $H_0^1(0, L) \times H_0^1(0, L)$ . Then, by Lax-Milgram Lemma, the operator  $A_1$  is an isomorphism.

**Step 2.** In this step, we prove that the operator  $A_2$  is compact. According to the third equation of (3.2.46), we have

$$|a_2((u, y), (\varphi, \psi))| \leq C \| (u, y) \|_{L^2(0, L)} \| (\varphi, \psi) \|_{L^2(0, L)}.$$



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Finally, using the compactness embedding from  $H_0^1(0, L)$  to  $L^2(0, L)$  and the continuous embedding from  $L^2(0, L)$  into  $H^{-1}(0, L)$  we deduce that  $A_2$  is compact.

From steps 1 and 2, we get that the operator  $A = A_1 + A_2$  is a Fredholm operator of index zero. Consequently, by Fredholm alternative, to prove that operator  $A$  is an isomorphism it is enough to prove that  $A$  is injective, i.e.  $\ker\{A\} = \{0\}$ .

**Step 3.** In this step, we prove that  $\ker\{A\} = \{0\}$ . For this aim, let  $(\tilde{u}, \tilde{y}) \in \ker\{A\}$ , i.e.

$$a((\tilde{u}, \tilde{y}), (\varphi, \psi)) = 0, \quad \forall (\varphi, \psi) \in H_0^1(0, L) \times H_0^1(0, L).$$

Equivalently, we have

$$\begin{aligned} -\lambda^2 \int_0^L (\tilde{u}\bar{\varphi} + \tilde{y}\bar{\psi}) dx + i\lambda \int_0^L c(x) (\tilde{y}\bar{\varphi} - \tilde{u}\bar{\psi}) dx + \int_0^L (a\tilde{u}_x\bar{\varphi}_x + \tilde{y}_x\bar{\psi}_x) dx \\ + i\lambda \int_0^L b(x) \tilde{u}_x\bar{\varphi}_x dx = 0. \end{aligned} \quad (3.2.47)$$

Taking  $\varphi = \tilde{u}$  and  $\psi = \tilde{y}$  in equation (3.2.47), we get

$$-\lambda^2 \int_0^L |\tilde{u}|^2 dx - \lambda^2 \int_0^L |\tilde{y}|^2 dx + a \int_0^L |\tilde{u}_x|^2 dx + \int_0^L |\tilde{y}_x|^2 dx - 2\lambda \operatorname{Im} \left( \int_0^L c(x) \tilde{y}\bar{\tilde{u}} dx \right) + i\lambda \int_0^L b(x) |\tilde{u}_x|^2 dx = 0.$$

Taking the imaginary part of the above equality, we get

$$0 = \int_0^L b(x) |\tilde{u}_x|^2 dx,$$

we get,

$$\tilde{u}_x = 0, \quad \text{in } (\alpha_1, \alpha_3). \quad (3.2.48)$$

Then, we find that

$$\begin{cases} -\lambda^2 \tilde{u} - a\tilde{u}_{xx} + i\lambda c(x) \tilde{y} = 0, & \text{in } (0, L) \\ -\lambda^2 \tilde{y} - a\tilde{y}_{xx} - i\lambda c(x) \tilde{u} = 0, & \text{in } (0, L) \\ \tilde{u}_x = \tilde{y}_x = 0, & \text{in } (\alpha_2, \alpha_3) \end{cases}$$

Therefore, the vector  $\tilde{U}$  defined by

$$\tilde{U} = (\tilde{u}, i\lambda\tilde{u}, \tilde{y}, i\lambda\tilde{y})$$

belongs to  $D(\mathcal{A})$  and we have

$$i\lambda\tilde{U} - \mathcal{A}\tilde{U} = 0.$$

Hence,  $\tilde{U} \in \ker(i\lambda I - \mathcal{A})$ , then by Lemma 3.2.4, we get  $\tilde{U} = 0$ , this implies that  $\tilde{u} = \tilde{y} = 0$ . Consequently,  $\ker\{A\} = \{0\}$ .

Therefore, from step 3 and Fredholm alternative, we get that the operator  $A$  is an isomorphism. It is easy to see that the operator  $L$  is continuous from  $V$  to  $L^2(0, L) \times L^2(0, L)$ . Consequently, Equation (3.2.45) admits a unique solution  $(u, y) \in H_0^1(L) \times H_0^1(0, L)$ . Thus, using  $v = i\lambda u - f_1$ ,

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$z = i\lambda y - f_3$  and using the classical regularity arguments, we conclude that Equation (3.2.38) admits a unique solution  $U \in D(\mathcal{A})$ . The proof is thus complete.  $\square$

**Proof of Theorem 3.2.3.** Using Lemma 3.2.4, we have that  $\mathcal{A}$  has non pure imaginary eigenvalues. According to Lemmas 3.2.4, 3.2.5 and with the help of the closed graph theorem of Banach, we deduce that  $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ . Thus, we get the conclusion by applying Theorem 1.3.3 of Arendt Batty. The proof of the theorem is thus complete.

**Remark 3.2.6.** For the case when  $\text{supp}(b) \cap \text{supp}(c) = \emptyset$  it remains as an open problem.

## 3.3 Lack of the exponential Stability

In this section, our goal is to show that system (3.1.2)-(3.1.6) is not exponentially stable.

### 3.3.1 Lack of exponential stability with global Kelvin-Voigt damping.

In this part, assume that

$$b(x) = b_0 > 0 \quad \text{and} \quad c(x) = c_0 \in \mathbb{R}^*, \quad \forall x \in (0, L). \quad (3.3.1)$$

We introduce the following theorem.

**Theorem 3.3.1.** Under hypothesis (3.3.1), for  $\varepsilon > 0$  small enough, we cannot expect the energy decay rate  $\frac{1}{t^{\frac{2}{2-\varepsilon}}}$  for all initial data  $U_0 \in D(\mathcal{A})$  and for all  $t > 0$ .

*Proof.* Following Theorem 1.3.5 of Huang and Prüss it is sufficient to show the existence of a real sequences  $(\lambda_n)_n$  with  $\lambda_n \rightarrow +\infty$ ,  $(U_n)_n \in D(\mathcal{A})$ , and  $(F_n)_n \subset \mathcal{H}$  such that  $(i\lambda_n I - \mathcal{A})U_n = F_n$  is bounded in  $\mathcal{H}$  and  $\lambda_n^{-2+\varepsilon} \|U_n\| \rightarrow +\infty$ . For this aim, take

$$F_n = \left(0, 0, 0, \sin\left(\frac{n\pi x}{L}\right)\right), U_n = \left(A_n \sin\left(\frac{n\pi x}{L}\right), i\lambda_n A_n \sin\left(\frac{n\pi x}{L}\right), B_n \sin\left(\frac{n\pi x}{L}\right), i\lambda_n B_n \sin\left(\frac{n\pi x}{L}\right)\right),$$

where

$$\lambda_n = \frac{n\pi}{L}, \quad A_n = \frac{iL}{c_0 n\pi}, \quad B_n = -\frac{inb_0\pi}{c_0^2 L} - \frac{a-1}{c_0^2}.$$

Clearly that  $U_n \in D(\mathcal{A})$ , and  $F_n$  is bounded in  $\mathcal{H}$ . Let us show that  $(i\lambda_n I - \mathcal{A})U_n = F_n$ . Detailing  $(i\lambda_n I - \mathcal{A})U_n$ , we get

$$(i\lambda_n I - \mathcal{A})U_n = \left(0, D_{1,n} \sin\left(\frac{n\pi x}{L}\right), 0, D_{2,n} \sin\left(\frac{n\pi x}{L}\right)\right),$$

where

$$D_{1,n} = \frac{-(L^2 \lambda_n^2 - a n^2 \pi^2 - i \pi^2 b_0 \lambda_n n^2) A_n}{L^2} + i B_n c_0 \lambda_n, \quad \text{and} \quad D_{2,n} = -i A_n c_0 \lambda_n + \frac{B_n (\pi^2 n^2 - L^2 \lambda_n^2)}{L^2}.$$

Inserting  $\lambda_n, A_n, B_n$  in  $D_{1,n}$  and  $D_{2,n}$ , we get  $D_{1,n} = 0$  and  $D_{2,n} = 1$ . Hence we obtain

$$(i\lambda_n I - \mathcal{A})U_n = \left(0, 0, 0, \sin\left(\frac{n\pi x}{L}\right)\right) = F_n.$$

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Now, we have

$$\|U_n\|_{\mathcal{H}}^2 \geq \int_0^L \left| i\lambda_n B_n \sin\left(\frac{n\pi x}{L}\right) \right|^2 dx = \frac{L\lambda_n^2}{2} |B_n|^2 \sim \lambda_n^4.$$

Therefore, for  $\varepsilon > 0$  small enough, we have

$$\lambda_n^{-2+\varepsilon} \|U_n\|_{\mathcal{H}} \sim \lambda_n^\varepsilon \rightarrow +\infty.$$

Then, we cannot expect the energy decay rate  $\frac{1}{t^{\frac{2}{2-\varepsilon}}}$ . □

#### 3.3.2 Lack of exponential stability with Local Kelvin-Voigt damping.

In this part, under the equal speed wave propagation condition (i.e.  $a = 1$ ), we use the classical method developed by Littman and Markus in [64] (see also [38]), to show that system (3.1.2)-(3.1.6) with local Kelvin-Voigt damping and global coupling is not exponentially stable. For this aim, assume that

$$a = 1, \quad b(x) = \begin{cases} 0 & \text{if } 0 < x \leq \frac{1}{2}, \\ 1 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}, \quad \text{and } c(x) = c \in \mathbb{R}^*. \quad (3.3.2)$$

Our main result in this part is following theorem.

**Theorem 3.3.2.** *Under condition (3.3.2). The semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  generated by the operator  $\mathcal{A}$  is not exponentially stable in the energy space  $\mathcal{H}$ .*

According to Theorem 2.1.6 and Lemma 2.1.11 in [38], one has that

$$s(\mathcal{A}_1) \leq \omega_0(\mathcal{A}_1).$$

By the previous results, one clearly has that  $s(\mathcal{A}) \leq 0$  and the theorem would follow if equality holds in the previous inequality. It therefore amounts to show the existence of a sequence of eigenvalues of  $\mathcal{A}$  whose real parts tend to zero.

Since  $\mathcal{A}$  is dissipative, we fix  $\alpha_0 > 0$  small enough and we study the asymptotic behavior of the eigenvalues  $\lambda$  of  $\mathcal{A}$  in the strip

$$S = \{\lambda \in \mathbb{C} : -\alpha_0 \leq \operatorname{Re}(\lambda) \leq 0\}.$$

First, we determine the characteristic equation satisfied by the eigenvalues of  $\mathcal{A}$ . For this aim, let  $\lambda \in \mathbb{C}^*$  be an eigenvalue of  $\mathcal{A}$  and let  $U = (u, \lambda u, y, \lambda y) \in D(\mathcal{A})$  be an associated eigenvector. Then, the eigenvalue problem is given by

$$\lambda^2 u - (1 + b(x)\lambda) u_{xx} + c\lambda y = 0, \quad x \in (0, 1), \quad (3.3.3)$$

$$\lambda^2 y - y_{xx} - c\lambda u = 0, \quad x \in (0, 1), \quad (3.3.4)$$

with the boundary conditions

$$u(0) = u(1) = y(0) = y(1) = 0.$$

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We define

$$\begin{cases} u^-(x) := u(x), & y^-(x) := y(x) & x \in (0, \frac{1}{2}), \\ u^+(x) := u(x), & y^+(x) := y(x) & x \in [\frac{1}{2}, 1]. \end{cases}$$

Then, system (3.3.3)-(3.3.4) becomes

$$\lambda^2 u^- - u_{xx}^- + c\lambda y^- = 0, \quad x \in (0, 1/2), \quad (3.3.5)$$

$$\lambda^2 y^- - y_{xx}^- - c\lambda u^- = 0, \quad x \in (0, 1/2), \quad (3.3.6)$$

$$\lambda^2 u^+ - (1 + \lambda)u_{xx}^+ + c\lambda y^+ = 0, \quad x \in [1/2, 1), \quad (3.3.7)$$

$$\lambda^2 y^+ - y_{xx}^+ - c\lambda u^+ = 0, \quad x \in [1/2, 1), \quad (3.3.8)$$

with the boundary conditions

$$u^-(0) = y^-(0) = 0, \quad (3.3.9)$$

$$u^+(1) = y^+(1) = 0, \quad (3.3.10)$$

and the continuity conditions

$$u^-(1/2) = u^+(1/2), \quad (3.3.11)$$

$$u_x^-(1/2) = (1 + \lambda)u_x^+(1/2), \quad (3.3.12)$$

$$y^-(1/2) = y^+(1/2), \quad (3.3.13)$$

$$y_x^-(1/2) = y_x^+(1/2). \quad (3.3.14)$$

Here and below, in order to handle, in the case where  $z$  is a non zero non-real number, we denote by  $\sqrt{z}$  the square root of  $z$ ; i.e., the unique complex number whose square is equal to  $z$ , that is defined by

$$\sqrt{z} = \sqrt{\frac{|z| + \operatorname{Re}(z)}{2}} + i \operatorname{sign}(\operatorname{Im}(z)) \sqrt{\frac{|z| - \operatorname{Re}(z)}{2}}.$$

Our aim is to study the asymptotic behavior of the largest eigenvalues  $\lambda$  of  $\mathcal{A}$  in  $S$ . By taking  $\lambda$  large enough, the general solution of system (3.3.5)-(3.3.6) with boundary condition (3.3.9) is given by

$$\begin{cases} u^-(x) &= d_1 \frac{\lambda^2 - r_1^2}{c\lambda} \sinh(r_1 x) + d_2 \frac{\lambda^2 - r_2^2}{c\lambda} \sinh(r_2 x), \\ y^-(x) &= d_1 \sinh(r_1 x) + d_2 \sinh(r_2 x), \end{cases}$$

and the general solution of system (3.3.5)-(3.3.6) with boundary condition (3.3.10) is given by

$$\begin{cases} u^+(x) &= -D_1 \frac{\lambda^2 - s_1^2}{c\lambda} \sinh(s_1(1-x)) - D_2 \frac{\lambda^2 - s_2^2}{c\lambda} \sinh(s_2(1-x)), \\ y^+(x) &= -D_1 \sinh(s_1(1-x)) - D_2 \sinh(s_2(1-x)), \end{cases}$$

where  $d_1, d_2, D_1, D_2 \in \mathbb{C}$ ,

$$r_1 = \lambda \sqrt{1 + \frac{ic}{\lambda}}, \quad r_2 = \lambda \sqrt{1 - \frac{ic}{\lambda}} \quad (3.3.15)$$

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and

$$s_1 = \lambda \sqrt{\frac{1 + \frac{2}{\lambda} + \sqrt{1 - \frac{4c^2}{\lambda^3} - \frac{4c^2}{\lambda^4}}}{2(1 + \frac{1}{\lambda})}}, \quad s_2 = \sqrt{\lambda} \sqrt{\frac{\lambda + 2 - \lambda \sqrt{1 - \frac{4c^2}{\lambda^3} - \frac{4c^2}{\lambda^4}}}{2(1 + \frac{1}{\lambda})}}. \quad (3.3.16)$$

The boundary conditions in (3.3.11)-(3.3.14), can be expressed by  $M(d_1 \ d_2 \ D_1 \ D_2)^\top = 0$ , where

$$M = \begin{pmatrix} \sinh(\frac{r_1}{2}) & \sinh(\frac{r_2}{2}) & \sinh(\frac{s_1}{2}) & \sinh(\frac{s_2}{2}) \\ r_1 \cosh(\frac{r_1}{2}) & r_2 \cosh(\frac{r_2}{2}) & -s_1 \cosh(\frac{s_1}{2}) & -s_2 \cosh(\frac{s_2}{2}) \\ r_1^2 \sinh(\frac{r_1}{2}) & r_2^2 \sinh(\frac{r_2}{2}) & s_1^2 \sinh(\frac{s_1}{2}) & s_2^2 \sinh(\frac{s_2}{2}) \\ r_1^3 \cosh(\frac{r_1}{2}) & r_2^3 \cosh(\frac{r_2}{2}) & -s_1(s_1^2 - \lambda(\lambda^2 - s_1^2)) \cosh(\frac{s_1}{2}) & -s_2(s_2^2 - \lambda(\lambda^2 - s_2^2)) \cosh(\frac{s_2}{2}) \end{pmatrix}.$$

System (3.3.5)-(3.3.14) admits a non trivial solution if and only if  $\det(M) = 0$ . Using Gaussian elimination,  $\det(M) = 0$  is equivalent to  $\det(M_1) = 0$ , where  $M_1$  is given by

$$M_1 = \begin{pmatrix} \sinh(\frac{r_1}{2}) & \sinh(\frac{r_2}{2}) & \sinh(\frac{s_1}{2}) & 1 - e^{-s_2} \\ r_1 \cosh(\frac{r_1}{2}) & r_2 \cosh(\frac{r_2}{2}) & -s_1 \cosh(\frac{s_1}{2}) & -s_2(1 + e^{-s_2}) \\ r_1^2 \sinh(\frac{r_1}{2}) & r_2^2 \sinh(\frac{r_2}{2}) & s_1^2 \sinh(\frac{s_1}{2}) & s_2^2(1 - e^{-s_2}) \\ r_1^3 \cosh(\frac{r_1}{2}) & r_2^3 \cosh(\frac{r_2}{2}) & -s_1(s_1^2 - \lambda(\lambda^2 - s_1^2)) \cosh(\frac{s_1}{2}) & -s_2(s_2^2 - \lambda(\lambda^2 - s_2^2))(1 + e^{-s_2}) \end{pmatrix}.$$

Then, we get

$$\det(M_1) = F_1 + F_2 e^{-s_2}, \quad (3.3.17)$$

where

$$\begin{aligned} F_1 = & -s_1 s_2 (r_1^2 - r_2^2) (s_1^2 - s_2^2) (\lambda + 1) \sinh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) \\ & + r_1 s_2 (r_2^2 - s_1^2) ((\lambda^2 - s_2^2) \lambda + r_1^2 - s_2^2) \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\ & - r_2 s_2 (r_1^2 - s_1^2) ((\lambda^2 - s_2^2) \lambda + r_2^2 - s_2^2) \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\ & - r_1 r_2 (r_1^2 - r_2^2) (s_1^2 - s_2^2) \cosh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\ & + r_2 s_1 (r_1^2 - s_2^2) ((\lambda^2 - s_1^2) \lambda + r_2^2 - s_1^2) \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) \\ & - r_1 s_1 (r_2^2 - s_2^2) ((\lambda^2 - s_1^2) \lambda + r_1^2 - s_1^2) \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) \end{aligned}$$

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and

$$\begin{aligned}
F_2 = & -s_1 s_2 (r_1^2 - r_2^2) (s_1^2 - s_2^2) (\lambda + 1) \sinh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) \\
& + r_1 s_2 (r_2^2 - s_1^2) ((\lambda^2 - s_2^2) \lambda + r_1^2 - s_2^2) \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\
& - r_2 s_2 (r_1^2 - s_1^2) ((\lambda^2 - s_2^2) \lambda + r_2^2 - s_2^2) \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\
& + r_1 r_2 (r_1^2 - r_2^2) (s_1^2 - s_2^2) \cosh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\
& - r_2 s_1 (r_1^2 - s_2^2) ((\lambda^2 - s_1^2) \lambda + r_2^2 - s_1^2) \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) \\
& + r_1 s_1 (r_2^2 - s_2^2) ((\lambda^2 - s_1^2) \lambda + r_1^2 - s_1^2) \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right).
\end{aligned}$$

**Lemma 3.3.3.** Let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $\mathcal{A}$ . Then, we have  $\operatorname{Re}(\lambda)$  is bounded.

*Proof.* Multiplying equations (3.3.5)-(3.3.8) by  $u^-$ ,  $y^-$ ,  $u^+$ ,  $y^+$  respectively, then using the boundary conditions, we get

$$\| \lambda u^- \|^2 + \| u_x^- \|^2 + \| \lambda y^- \|^2 + \| y_x^- \|^2 + \| \lambda u^+ \|^2 + (1 + \operatorname{Re}(\lambda)) \| u_x^+ \|^2 + \| \lambda y^+ \|^2 + \| y_x^+ \|^2 = 0. \quad (3.3.18)$$

Since the operator  $\mathcal{A}$  is dissipative then the real part of  $\lambda$  is negative. It is easy to see that  $u_x^+ \neq 0$ , hence using the fact that  $\|U\|_{\mathcal{H}} = 1$  in (3.3.18), we get that  $\operatorname{Re}(\lambda)$  is bounded below. Therefore, there exists  $\alpha > 0$ , such that

$$-\alpha \leq \operatorname{Re}(\lambda) < 0.$$

□

**Proposition 3.3.4.** Assume that the condition (3.3.2) holds. Then there exists  $n_0 \in \mathbb{N}$  sufficiently large and two sequences  $(\lambda_{1,n})_{|n| \geq n_0}$  and  $(\lambda_{2,n})_{|n| \geq n_0}$  of simple root of  $\det(M_1)$  satisfying the following asymptotic behavior:

**Case 1.** If  $\sin\left(\frac{c}{4}\right) \neq 0$ , then

$$\lambda_{1,n} = 2n\pi i + i\pi - \frac{2 \sin^2\left(\frac{c}{4}\right)(1 - i \operatorname{sign}(n))}{(3 + \cos\left(\frac{c}{2}\right)) \sqrt{|n|\pi}} + O\left(\frac{1}{n}\right) \quad (3.3.19)$$

and

$$\lambda_{2,n} = 2n\pi i + i \arccos\left(\cos^2\left(\frac{c}{4}\right)\right) - \frac{\gamma}{\sqrt{|n|\pi}} + i \frac{\operatorname{sign}(n)\gamma}{\sqrt{|n|\pi}} + O\left(\frac{1}{n}\right), \quad (3.3.20)$$

where

$$\gamma = \frac{\left(\cos\left(\frac{c}{2}\right) \sin\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right) + \sin\left(\frac{3 \arccos(\cos^2(\frac{c}{4}))}{2}\right)\right)}{4 \sqrt{1 - \cos^4\left(\frac{c}{4}\right)} \cos\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right)}.$$

**Case 2.** If  $\sin\left(\frac{c}{4}\right) = 0$ , then

$$\lambda_{1,n} = 2n\pi i + i\pi + \frac{i c^2}{32\pi n} - \frac{(4 + i\pi)c^2}{64\pi^2 n^2} + O\left(\frac{1}{|n|^{\frac{5}{2}}}\right) \quad (3.3.21)$$

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and

$$\lambda_{2,n} = 2n\pi i + O\left(\frac{1}{n}\right). \quad (3.3.22)$$

The proof of Proposition 3.3.4, is divided into two lemmas.

**Lemma 3.3.5.** Assume that condition (3.3.2) holds. Let  $\lambda$  be largest eigenvalue of  $\mathcal{A}$ , then  $\lambda$  is large root of the following asymptotic behavior estimate

$$F(\lambda) := f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1/2}} + \frac{f_2(\lambda)}{8\lambda} + \frac{f_3(\lambda)}{8\lambda^{3/2}} + \frac{f_4(\lambda)}{128\lambda^2} + O(\lambda^{-5/2}), \quad (3.3.23)$$

where

$$\begin{cases} f_0(\lambda) &= \cosh\left(\frac{3\lambda}{2}\right) - \cosh\left(\frac{\lambda}{2}\right) \cos\left(\frac{c}{2}\right), \\ f_1(\lambda) &= \sinh\left(\frac{3\lambda}{2}\right) + \sinh\left(\frac{\lambda}{2}\right) \cos\left(\frac{c}{2}\right), \\ f_2(\lambda) &= c^2 \sinh\left(\frac{3\lambda}{2}\right) - 4 \cosh\left(\frac{3\lambda}{2}\right) + 4 \left( \cosh\left(\frac{\lambda}{2}\right) \cos\left(\frac{c}{2}\right) + c \sinh\left(\frac{\lambda}{2}\right) \sin\left(\frac{c}{2}\right) \right), \\ f_3(\lambda) &= -8 \sinh\left(\frac{3\lambda}{2}\right) + c^2 \cosh\left(\frac{3\lambda}{2}\right) - 12c \cosh\left(\frac{\lambda}{2}\right) \sin\left(\frac{c}{2}\right) - 8 \sinh\left(\frac{\lambda}{2}\right) \cos\left(\frac{c}{2}\right), \\ f_4(\lambda) &= -40c^2 \sinh\left(\frac{3\lambda}{2}\right) + (c^4 + 72c^2 + 48) \cosh\left(\frac{3\lambda}{2}\right) + 32c \left( c \cos\left(\frac{c}{2}\right) + 7 \sin\left(\frac{c}{2}\right) \right) \sinh\left(\frac{\lambda}{2}\right) \\ &\quad - (8c^2 + 8c^3 \sin\left(\frac{c}{2}\right) + 16(4c^2 + 3) \cosh\left(\frac{c}{2}\right)) \cosh\left(\frac{\lambda}{2}\right). \end{cases} \quad (3.3.24)$$

*Proof.* Let  $\lambda$  be a large eigenvalue of  $\mathcal{A}$ , then  $\lambda$  is root of  $\det(M_1)$ . In this Lemma, we give an asymptotic development of the function  $\det(M_1)$  for large  $\lambda$ . First, using the asymptotic expansion in (3.3.15)-(3.3.16), we get

$$\begin{cases} r_1 = \lambda + \frac{ic}{2} + \frac{c^2}{8\lambda} - \frac{ic^3}{16\lambda^2} + O(\lambda^{-3}), \quad r_2 = \lambda - \frac{ic}{2} + \frac{c^2}{8\lambda} + \frac{ic^3}{16\lambda^2} + O(\lambda^{-3}), \\ s_1 = \lambda - \frac{c^2}{2\lambda} + O(\lambda^{-5}), \quad s_2 = \sqrt{\lambda} - \frac{1}{2\sqrt{\lambda}} + \frac{4c^2+3}{8\lambda^{3/2}} + O(\lambda^{-3/2}). \end{cases} \quad (3.3.25)$$

From (3.3.25), we get

$$\begin{cases} 2ic\lambda s_1 s_2 (s_1^2 - s_2^2)(\lambda + 1) = ic\lambda^{11/2} \left( 2 - \frac{1}{\lambda} + \frac{3+4c^2}{4\lambda^2} + O(\lambda^{-3}) \right), \\ r_1 s_2 (r_2^2 - s_1^2) ((\lambda^2 - s_2^2)\lambda + r_1^2 - s_2^2) = -ic\lambda^{11/2} \left( 1 - \frac{1-ic}{2\lambda} + \frac{5c^2+3+14ic}{8\lambda^2} + O(\lambda^{-3}) \right), \\ r_2 s_2 (r_1^2 - s_1^2) ((\lambda^2 - s_2^2)\lambda + r_2^2 - s_2^2) = ic\lambda^{11/2} \left( 1 - \frac{1+ic}{2\lambda} + \frac{5c^2+3-14ic}{8\lambda^2} + O(\lambda^{-3}) \right), \\ 2ic\lambda r_1 r_2 (s_1^2 - s_2^2) = ic\lambda^{11/2} \left( \frac{2}{\sqrt{\lambda}} - \frac{2}{\lambda^{3/2}} + O(\lambda^{-5/2}) \right), \\ r_2 s_1 (r_1^2 - s_2^2) ((\lambda^2 - s_1^2)\lambda + r_2^2 - s_1^2) = -ic\lambda^{11/2} \left( \frac{1}{\sqrt{\lambda}} - \frac{2-3ic}{2\lambda^{3/2}} + O(\lambda^{-5/2}) \right), \\ r_1 s_1 (r_2^2 - s_2^2) ((\lambda^2 - s_1^2)\lambda + r_1^2 - s_1^2) = ic\lambda^{11/2} \left( \frac{1}{\sqrt{\lambda}} - \frac{2+3ic}{2\lambda^{3/2}} + O(\lambda^{-5/2}) \right). \end{cases} \quad (3.3.26)$$

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From equation (3.3.26) and using the fact that  $\text{Re}(\lambda)$  is bounded, we get

$$\begin{aligned} \frac{F_1}{ic\lambda^{11/2}} = & - \left[ \left( 2 - \frac{1}{\lambda} + \frac{4c^2+3}{4\lambda^2} \right) \sinh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) \right. \\ & + \left( 1 - \frac{1}{2\lambda} + \frac{5c^2+3}{8\lambda^2} \right) \left( \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) + \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \right) \sinh\left(\frac{s_1}{2}\right) \\ & + \left( \frac{ic}{2\lambda} + \frac{7ic}{4\lambda^2} \right) \left( \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) - \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \right) \sinh\left(\frac{s_1}{2}\right) \\ & + \left( \frac{2}{\sqrt{\lambda}} - \frac{2}{\lambda^{3/2}} \right) \cosh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\ & + \left( \frac{1}{\sqrt{\lambda}} - \frac{1}{\lambda^{3/2}} \right) \left( \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) + \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \right) \cosh\left(\frac{s_1}{2}\right) \\ & \left. + \left( \frac{3ic}{2\lambda^{3/2}} \right) \left( \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) - \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \right) \cosh\left(\frac{s_1}{2}\right) + O(\lambda^{-5/2}) \right]. \end{aligned} \quad (3.3.27)$$

From equation (3.3.26) and using the fact that  $\text{Re}(\lambda)$  is bounded, we get

$$\begin{aligned} F_2 = & -ic\lambda^{11/2} \left[ 2 \sinh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) \right. \\ & \left. + \left( \cosh\left(\frac{r_1}{2}\right) \sinh\left(\frac{r_2}{2}\right) + \sinh\left(\frac{r_1}{2}\right) \cosh\left(\frac{r_2}{2}\right) \right) \sinh\left(\frac{s_1}{2}\right) + O(\lambda^{-1/2}) \right]. \end{aligned} \quad (3.3.28)$$

Since the real part of  $\sqrt{\lambda}$  is positive, then

$$\lim_{|\lambda| \rightarrow \infty} \lambda^{-5/2} e^{-\sqrt{\lambda}} = 0,$$

hence

$$e^{-\sqrt{\lambda}} = o(\lambda^{-5/2}), \quad (3.3.29)$$

then,

$$F_2 e^{-s_2} = -ic\lambda^{11/2} (o(\lambda^{-5/2})). \quad (3.3.30)$$

Inserting (3.3.27) and (3.3.30) in (3.3.17), we get

$$\det(M_1) = -ic\lambda^{11/2} F(\lambda),$$

where,

$$\begin{aligned} F(\lambda) = & \left( 1 - \frac{1}{2\lambda} + \frac{4c^2+3}{8\lambda^2} \right) \left( \cosh\left(\frac{r_1+r_2}{2}\right) - \cosh\left(\frac{r_1-r_2}{2}\right) \right) \cosh\left(\frac{s_1}{2}\right) \\ & + \left( 1 - \frac{1}{2\lambda} + \frac{5c^2+3}{8\lambda^2} \right) \sinh\left(\frac{r_1+r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) - \left( \frac{ic}{2\lambda} + \frac{7ic}{4\lambda^2} \right) \sinh\left(\frac{r_1-r_2}{2}\right) \sinh\left(\frac{s_1}{2}\right) \\ & + \left( \frac{1}{\sqrt{\lambda}} - \frac{1}{\lambda^{3/2}} \right) \left( \cosh\left(\frac{r_1+r_2}{2}\right) + \cosh\left(\frac{r_1-r_2}{2}\right) \right) \sinh\left(\frac{s_1}{2}\right) \\ & + \left( \frac{1}{\sqrt{\lambda}} - \frac{1}{\lambda^{3/2}} \right) \sinh\left(\frac{r_1+r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) + \left( \frac{3ic}{2\lambda^{3/2}} \right) \sinh\left(\frac{r_1-r_2}{2}\right) \cosh\left(\frac{s_1}{2}\right) + O(\lambda^{-5/2}). \end{aligned} \quad (3.3.31)$$

Therefore, system (3.3.9)-(3.3.14) admits a non trivial solution if and only if  $\det(M_1) = 0$ , if and



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only if the eigenvalues of  $\mathcal{A}$  are roots of the function  $F$ . Next, from (3.3.25) and the fact that real  $\lambda$  is bounded, we get

$$\left\{ \begin{array}{l} \cosh\left(\frac{r_1+r_2}{2}\right) = \cosh(\lambda) + \frac{c^2 \sinh(\lambda)}{8\lambda} + \frac{c^4 \cosh(\lambda)}{128\lambda^2} + O(\lambda^{-3}), \\ \cosh\left(\frac{r_1-r_2}{2}\right) = \cos\left(\frac{c}{2}\right) + \frac{c^3 \sin\left(\frac{c}{2}\right)}{16\lambda^2} + O(\lambda^{-3}), \\ \sinh\left(\frac{r_1+r_2}{2}\right) = \sinh(\lambda) + \frac{c^2 \cosh(\lambda)}{8\lambda} + \frac{c^4 \sinh(\lambda)}{128\lambda^2} + O(\lambda^{-3}), \\ \sinh\left(\frac{r_1-r_2}{2}\right) = i \sin\left(\frac{c}{2}\right) - i \frac{c^3 \cos\left(\frac{c}{2}\right)}{16\lambda^2} + O(\lambda^{-3}), \\ \sinh\left(\frac{s_1}{2}\right) = \sinh\left(\frac{\lambda}{2}\right) - \frac{c^2 \cosh\left(\frac{\lambda}{2}\right)}{4\lambda^2} + O(\lambda^{-4}), \\ \cosh\left(\frac{s_1}{2}\right) = \cosh\left(\frac{\lambda}{2}\right) - \frac{c^2 \sinh\left(\frac{\lambda}{2}\right)}{4\lambda^2} + O(\lambda^{-4}). \end{array} \right. \quad (3.3.32)$$

Inserting (3.3.32) in (3.3.31), we get (3.3.23).  $\square$

**Lemma 3.3.6.** *Under condition (3.3.2), there exists  $n_0 \in \mathbb{N}$  sufficiently large and two sequences  $(\lambda_{1,n})_{|n| \geq n_0}$  and  $(\lambda_{2,n})_{|n| \geq n_0}$  of simple roots of  $F$  satisfying the following asymptotic behavior*

$$\lambda_{1,n} = 2in\pi + i\pi + \epsilon_{1,n} \quad \text{where} \quad \lim_{|n| \rightarrow +\infty} \epsilon_{1,n} = 0 \quad (3.3.33)$$

and

$$\lambda_{2,n} = 2n\pi i + i \arccos\left(\cos^2\left(\frac{c}{4}\right)\right) + \epsilon_{2,n} \quad \text{where} \quad \lim_{|n| \rightarrow +\infty} \epsilon_{2,n} = 0. \quad (3.3.34)$$

*Proof.* First, we look at the roots of  $f_0$ . From (3.3.24), we deduce that  $f_0$  can be written as

$$f_0(\lambda) = 2 \cosh\left(\frac{\lambda}{2}\right) \left( \cosh(\lambda) - \cos^2\left(\frac{c}{4}\right) \right). \quad (3.3.35)$$

Then, the roots of  $f_0$  are given by

$$\left\{ \begin{array}{ll} \mu_{1,n} = 2n\pi i + i\pi, & n \in \mathbb{Z}, \\ \mu_{2,n} = 2n\pi i + i \arccos\left(\cos^2\left(\frac{c}{4}\right)\right), & n \in \mathbb{Z}. \end{array} \right.$$

Now, with the help of Rouché's Theorem, we will show that the roots of  $F$  are close to  $f_0$ . Let us start with the first family  $\mu_{1,n}$ . Let  $B_n = B((2n+1)\pi i, r_n)$  be the ball of centrum  $(2n+1)\pi i$  and radius  $r_n = |n|^{-\frac{1}{4}}$  and  $\lambda \in \partial B_n$ ; i.e.  $\lambda_n = 2n\pi i + i\pi + r_n e^{i\theta}$ ,  $\theta \in [0, 2\pi[$ . Then

$$\cosh\left(\frac{\lambda}{2}\right) = \frac{i(-1)^n r_n e^{i\theta}}{2} + O(r_n^2), \quad \text{and} \quad \cosh(\lambda) = -1 + O(r_n^2). \quad (3.3.36)$$

Inserting (3.3.36) in (3.3.35), we get

$$f_0(\lambda) = -i(-1)^n r_n e^{i\theta} \left( 1 + \cos^2\left(\frac{c}{4}\right) + O(r_n^3) \right).$$

It follows that there exists a positive constant  $C$  such that

$$\forall \lambda \in \partial B_n, \quad |f_0(\lambda)| \geq C r_n = C |n|^{-\frac{1}{4}}.$$

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On the other hand, from (3.3.23), we deduce that

$$|F(\lambda) - f_0(\lambda)| = O\left(\frac{1}{\sqrt{\lambda}}\right) = O\left(\frac{1}{\sqrt{|n|}}\right).$$

It follows that, for  $|n|$  large enough

$$\forall \lambda \in \partial B_n, \quad |F(\lambda) - f_0(\lambda)| < |f_0(\lambda)|.$$

Hence, with the help of Rouché's Theorem, there exists  $n_0 \in \mathbb{N}^*$  large enough, such that  $\forall |n| \geq n_0$ , the first branch of roots of  $F$  denoted by  $\lambda_{1,n}$  are close to  $\mu_{1,n}$ , that is

$$\lambda_{1,n} = \mu_{1,n} + i\pi + \epsilon_{1,n} \quad \text{where} \quad \lim_{|n| \rightarrow +\infty} \epsilon_{1,n} = 0. \quad (3.3.37)$$

Passing to the second family  $\mu_{2,n}$ . Let  $\tilde{B}_n = B(\mu_{2,n}, r_n)$  be the ball of centrum  $\mu_{2,n}$  and radius

$$r_n := \begin{cases} \frac{1}{|n|^{\frac{1}{8}}} & \text{if } \sin\left(\frac{c}{4}\right) = 0, \\ \frac{1}{|n|^{\frac{1}{4}}} & \text{if } \sin\left(\frac{c}{4}\right) \neq 0, \end{cases}$$

such that  $\lambda \in \partial \tilde{B}_n$ ; i.e.  $\lambda_n = \mu_{2,n} + r_n e^{i\theta}$ ,  $\theta \in [0, 2\pi[$ . Then,

$$\cosh(\lambda) - \cos^2\left(\frac{c}{4}\right) = \cosh\left(2n\pi i + i \arccos\left(\cos^2\left(\frac{c}{4}\right) + r_n e^{i\theta}\right)\right) - \cos^2\left(\frac{c}{4}\right).$$

It follow that,

$$\cosh(\lambda) - \cos^2\left(\frac{c}{4}\right) = i r_n \sqrt{1 - \cos^4\left(\frac{c}{4}\right)} e^{i\theta} + \frac{r_n^2 \cos^2\left(\frac{c}{4}\right) e^{2i\theta}}{2} + O(r_n^3), \quad (3.3.38)$$

and

$$\cosh\left(\frac{\lambda}{2}\right) = (-1)^n \cos\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right) + \frac{i r_n e^{i\theta} (-1)^n}{2} \sin\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right) + O(r_n^2). \quad (3.3.39)$$

Inserting (3.3.38) and (3.3.39) in (3.3.35), we get

$$f_0(\lambda) = R_1 e^{i\theta} r_n + R_2 e^{2i\theta} r_n^2 + O(r_n^3), \quad (3.3.40)$$

where

$$\begin{cases} R_1 = i(-1)^n \sqrt{1 - \cos^4\left(\frac{c}{4}\right)} \cos\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right), \\ R_2 = -(-1)^n \sqrt{1 - \cos^4\left(\frac{c}{4}\right)} \sin\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right) + (-1)^n \cos^2\left(\frac{c}{4}\right) \cos\left(\frac{\arccos(\cos^2(\frac{c}{4}))}{2}\right). \end{cases} \quad (3.3.41)$$

We distinguish two cases:

**Case 1.** If  $\sin\left(\frac{c}{4}\right) = 0$ , then

$$R_1 = 0 \quad \text{and} \quad R_2 = (-1)^n \neq 0.$$

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It follows that there exists a positive constant  $C$  such that

$$\forall \lambda \in \partial \tilde{B}_n, \quad |f_0(\lambda)| \geq C r_n^2 = C|n|^{-\frac{1}{4}}.$$

**Case 2.** If  $\sin\left(\frac{\epsilon}{4}\right) \neq 0$ , then  $R_1 \neq 0$ . It follows that, there exists a positive constant  $C$  such that

$$\forall \lambda \in \partial \tilde{B}_n, \quad |f_0(\lambda)| \geq C r_n = C|n|^{-\frac{1}{4}}.$$

On the other hand, from (3.3.23), we deduce that

$$|F(\lambda) - f_0(\lambda)| = O\left(\frac{1}{\sqrt{\lambda}}\right) = O\left(\frac{1}{\sqrt{|n|}}\right).$$

In both cases, for  $|n|$  large enough, we have

$$\forall \lambda \in \partial \tilde{B}_n, \quad |F(\lambda) - f_0(\lambda)| < |f_0(\lambda)|.$$

Hence, with the help of Rouché's Theorem, there exists  $n_0 \in \mathbb{N}^*$  large enough, such that  $\forall |n| \geq n_0$ , the second branch of roots of  $F$ , denoted by  $\lambda_{2,n}$  are close to  $\mu_{2,n}$  that is defined in equation (3.3.34). The proof is thus complete.  $\square$

We are now in position to conclude the proof of Proposition 3.3.4.

**Proof of Proposition 3.3.4.** The proof is divided into two steps.

**Calculation of  $\epsilon_{1,n}$ .** From (3.3.37), we have

$$\left\{ \begin{array}{l} \cosh\left(\frac{3\lambda_{1,n}}{2}\right) = -i(-1)^n \sinh\left(\frac{3\epsilon_{1,n}}{2}\right), \quad \sinh\left(\frac{3\lambda_{1,n}}{2}\right) = -i(-1)^n \cosh\left(\frac{3\epsilon_{1,n}}{2}\right), \\ \cosh\left(\frac{\lambda_{1,n}}{2}\right) = i(-1)^n \sinh\left(\frac{\epsilon_{1,n}}{2}\right), \quad \sinh\left(\frac{\lambda_{1,n}}{2}\right) = i(-1)^n \cosh\left(\frac{\epsilon_{1,n}}{2}\right), \\ \frac{1}{\lambda_{1,n}} = -\frac{i}{2\pi n} + \frac{i}{4\pi n^2} + O(\epsilon_{1,n} n^{-2}) + O(n^{-3}), \quad \frac{1}{\lambda_{1,n}^2} = -\frac{1}{4\pi^2 n^2} + O(n^{-3}) \\ \frac{1}{\sqrt{\lambda_{1,n}}} = \frac{1 - i \operatorname{sign}(n)}{2\sqrt{\pi}|n|} + \frac{i - \operatorname{sign}(n)}{8\sqrt{\pi}|n|^3} + O(\epsilon_{1,n}|n|^{-3/2}) + O(|n|^{-5/2}), \\ \frac{1}{\sqrt{\lambda_{1,n}^3}} = \frac{-1 - i \operatorname{sign}(n)}{4\sqrt{\pi^3}|n|^3} + O(|n|^{-5/2}), \quad \frac{1}{\sqrt{\lambda_{1,n}^5}} = O(|n|^{-5/2}). \end{array} \right. \quad (3.3.42)$$

On the other hand, since  $\lim_{|n| \rightarrow +\infty} \epsilon_{1,n} = 0$ , we have the asymptotic expansion

$$\left\{ \begin{array}{l} \sinh\left(\frac{3\epsilon_{1,n}}{2}\right) = \frac{3\epsilon_{1,n}}{2} + O(\epsilon_{1,n}^3), \quad \cosh\left(\frac{3\epsilon_{1,n}}{2}\right) = 1 + \frac{9\epsilon_{1,n}^2}{8} + O(\epsilon_{1,n}^4), \\ \sinh\left(\frac{\epsilon_{1,n}}{2}\right) = \frac{\epsilon_{1,n}}{2} + O(\epsilon_{1,n}^3), \quad \cosh\left(\frac{\epsilon_{1,n}}{2}\right) = 1 + \frac{\epsilon_{1,n}^2}{8} + O(\epsilon_{1,n}^4). \end{array} \right. \quad (3.3.43)$$

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Inserting (3.3.43) in (3.3.42), we get

$$\left\{ \begin{array}{l} \cosh\left(\frac{3\lambda_{1,n}}{2}\right) = -\frac{3i(-1)^n \epsilon_{1,n}}{2} + O(\epsilon_{1,n}^3), \sinh\left(\frac{3\lambda_{1,n}}{2}\right) = -i(-1)^n - \frac{9i(-1)^n \epsilon_{1,n}}{8} + O(\epsilon_{1,n}^4), \\ \cosh\left(\frac{\lambda_{1,n}}{2}\right) = \frac{i(-1)^n \epsilon_{1,n}}{2} + O(\epsilon_{1,n}^3), \sinh\left(\frac{\lambda_{1,n}}{2}\right) = i(-1)^n + \frac{i(-1)^n \epsilon_{1,n}}{8} + O(\epsilon_{1,n}^4), \\ \frac{1}{\lambda_{1,n}} = -\frac{i}{2\pi n} + \frac{i}{4\pi n^2} + O(\epsilon_{1,n} n^{-2}) + O(n^{-3}), \frac{1}{\lambda_{1,n}^2} = -\frac{1}{4\pi^2 n^2} + O(n^{-3}) \\ \frac{1}{\sqrt{\lambda_{1,n}}} = \frac{1-i\operatorname{sign}(n)}{2\sqrt{\pi|n|}} + \frac{i-\operatorname{sign}(n)}{8\sqrt{\pi|n|^3}} + O(\epsilon_{1,n}|n|^{-3/2}) + O(|n|^{-5/2}), \\ \frac{1}{\sqrt{\lambda_{1,n}^3}} = \frac{-1-i\operatorname{sign}(n)}{4\sqrt{\pi^3|n|^3}} + O(|n|^{-5/2}), \frac{1}{\sqrt{\lambda_{1,n}^5}} = O(|n|^{-5/2}). \end{array} \right. \quad (3.3.44)$$

Inserting (3.3.44) in (3.3.23), we get

$$\begin{aligned} & \frac{\epsilon_{1,n}}{2} \left(3 + \cos\left(\frac{c}{2}\right)\right) \left(1 + \frac{i}{4\pi n}\right) + \frac{(1-i\operatorname{sign}(n))(1-\cos(\frac{c}{2}))}{2\sqrt{\pi|n|}} + \frac{i c (4\sin(\frac{c}{2}) - c)}{16\pi n} \\ & - \frac{(2+i\pi)(1+i\operatorname{sign}(n))(1-\cos(\frac{c}{2}))}{8\sqrt{\pi^3|n|^3}} + \frac{4c(7-2i\pi)\sin(\frac{c}{2}) + c^2(2i\pi+5+4\cos(\frac{c}{2}))}{64\pi^2 n^2} \\ & + O(|n|^{-5/2}) + O(\epsilon_{1,n}|n|^{-3/2}) + O(\epsilon_{1,n}^2|n|^{-1/2}) + O(\epsilon_{1,n}^3) = 0. \end{aligned} \quad (3.3.45)$$

We distinguish two cases.

**Case 1.** If  $\sin(\frac{c}{4}) \neq 0$ , then  $1 - \cos(\frac{c}{2}) = 2\sin^2(\frac{c}{4}) \neq 0$ , then from (3.3.45), we get

$$\frac{\epsilon_{1,n}}{2} \left(3 + \cos\left(\frac{c}{2}\right)\right) + \frac{\sin^2(\frac{c}{4})(1-i\operatorname{sign}(n))}{\sqrt{|n|\pi}} + O(\epsilon_{1,n}^3) + O(|n|^{-1/2}\epsilon_{1,n}^2) + O(n^{-1}) = 0,$$

hence, we get

$$\epsilon_{1,n} = -\frac{2\sin^2(\frac{c}{4})(1-i\operatorname{sign}(n))}{(2+\cos(\frac{c}{2}))} + O(n^{-1}). \quad (3.3.46)$$

Inserting (3.3.46) in (3.3.37), we get (3.3.21).

**Case 2.** If  $\sin(\frac{c}{4}) = 0$ ,

$$1 - \cos\left(\frac{c}{2}\right) = 2\sin^2\left(\frac{c}{4}\right) = 0, \quad \sin\left(\frac{c}{2}\right) = 2\sin\left(\frac{c}{4}\right)\cos\left(\frac{c}{4}\right) = 0,$$

then, from (3.3.45), we get

$$\begin{aligned} & 2\epsilon_{1,n} \left(1 + \frac{i}{4\pi n}\right) - \frac{i c^2}{16\pi n} + \frac{c^2(2i\pi+9)}{64\pi^2 n^2} + O(|n|^{-5/2}) + O(\epsilon_{1,n}|n|^{-3/2}) \\ & + O(\epsilon_{1,n}^2|n|^{-1/2}) + O(\epsilon_{1,n}^3) = 0. \end{aligned} \quad (3.3.47)$$

By a straightforward calculation in equation (3.3.47), we get

$$\epsilon_{1,n} = \frac{i c^2}{32\pi n} - \frac{(4+i\pi)c^2}{64\pi^2 n^2} + O(|n|^{-5/2}). \quad (3.3.48)$$

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Inserting (3.3.48) in (3.3.37), we get (3.3.20).

**Calculation of  $\epsilon_{2,n}$ .** From (3.3.34), we have

$$\frac{1}{\sqrt{\lambda_{2,n}}} = \frac{1 - i \operatorname{sign}(n)}{2\sqrt{|n|\pi}} + O(|n|^{-3/2}) \quad \text{and} \quad \frac{1}{\lambda_{2,n}} = O(n^{-1}). \quad (3.3.49)$$

Inserting (3.3.34) and (3.3.49) in (3.3.23), we get

$$\begin{aligned} & \cosh\left(\frac{\lambda_{2,n}}{2}\right) \left( \cosh(\lambda_{2,n}) - \cos^2\left(\frac{\epsilon}{4}\right) \right) \\ & + \frac{(1 - i \operatorname{sign}(n)) \left( \sinh\left(\frac{3\lambda_{2,n}}{2}\right) + \sinh\left(\frac{\lambda_{2,n}}{2}\right) \cos\left(\frac{\epsilon}{2}\right) \right)}{4\sqrt{|n|\pi}} + O(n^{-1}) = 0. \end{aligned} \quad (3.3.50)$$

On the other hand, we have

$$\begin{aligned} \cosh(\lambda_{2,n}) - \cos^2\left(\frac{\epsilon}{4}\right) &= \cosh(2n\pi i + i \arccos(\cos^2(\frac{\epsilon}{4}))) + \epsilon_{2,n} - \cos^2\left(\frac{\epsilon}{4}\right) \\ &= \cos^2\left(\frac{\epsilon}{4}\right) \cosh(\epsilon_{2,n}) + i \sqrt{1 - \cos^4\left(\frac{\epsilon}{4}\right)} \sinh(\epsilon_{2,n}) - \cos^2\left(\frac{\epsilon}{4}\right) \\ &= i \epsilon_{2,n} \sqrt{1 - \cos^4\left(\frac{\epsilon}{4}\right)} + O(\epsilon_{2,n}^2), \end{aligned} \quad (3.3.51)$$

and

$$\begin{cases} \cosh\left(\frac{\lambda_{2,n}}{2}\right) &= (-1)^n \cos\left(\frac{\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) + O(\epsilon_{2,n}), \\ \sinh\left(\frac{\lambda_{2,n}}{2}\right) &= i(-1)^n \sin\left(\frac{\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) + O(\epsilon_{2,n}), \\ \sinh\left(\frac{3\lambda_{2,n}}{2}\right) &= i(-1)^n \sin\left(\frac{3\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) + O(\epsilon_{2,n}). \end{cases} \quad (3.3.52)$$

Inserting (3.3.51) and (3.3.52) in (3.3.50), we get

$$\begin{aligned} & \epsilon_{2,n} \cos\left(\frac{\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) \sqrt{1 - \cos^4\left(\frac{\epsilon}{4}\right)} + O\left(\frac{\epsilon_{2,n}}{n}\right) + O\left(\frac{1}{n}\right) \\ & + \frac{(1 - i \operatorname{sign}(n)) \left( \sin\left(3\frac{\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) + \cos\left(\frac{\epsilon}{2}\right) \sin\left(\frac{\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) \right)}{4\sqrt{|n|\pi}} = 0. \end{aligned} \quad (3.3.53)$$

We distinguish two cases.

**Case 1.** If  $\sin\left(\frac{\epsilon}{4}\right) \neq 0$ , then from (3.3.53), we get

$$\epsilon_{2,n} = - \frac{\left( \cos\left(\frac{\epsilon}{2}\right) \sin\left(\frac{\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) + \sin\left(\frac{3\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) \right) (1 - i \operatorname{sign}(n))}{4\sqrt{1 - \cos^4\left(\frac{\epsilon}{4}\right)} \cos\left(\frac{\arccos(\cos^2(\frac{\epsilon}{4}))}{2}\right) \sqrt{\pi|n|}} + O(n^{-1}). \quad (3.3.54)$$

Inserting (3.3.54) in (3.3.34), we get (3.3.20).

**Case 2.** If  $\sin\left(\frac{\epsilon}{4}\right) = 0$ , we get

$$\epsilon_{2,n} = O(n^{-1}). \quad (3.3.55)$$

Inserting (3.3.55) in (3.3.34), we get (3.3.22). Thus, the proof is complete.

**Proof of Theorem 3.3.2.** From Proposition 3.3.4, the operator  $\mathcal{A}$  has two branches of eigenvalues such that the real parts tending to zero. Then the energy corresponding to the first and second branch of eigenvalues is not exponentially decaying. Then the total energy of the wave

equations with local Kelvin-Voigt damping with global coupling are not exponentially stable in the equal speed case.

### 3.4 Polynomial Stability

From Section 3.3, System (3.1.2)-(3.1.6) is not uniformly (exponentially) stable, so we look for a polynomial decay rate. As the condition  $i\mathbb{R} \subset \rho(\mathcal{A})$  is already checked in Lemma 3.2.4, following Theorem 1.3.10, it remains to prove that condition (1.3.1) holds. This is made with the help of a specific multiplier and by using the exponential decay of the system considered in chapter 2. Our main result in this section is the following theorem.

**Theorem 3.4.1.** There exists a constant  $C > 0$  independent of  $U_0$ , such that the energy of system (3.1.2)-(3.1.6) satisfies the following estimation:

$$E(t) \leq \frac{C}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}). \quad (3.4.1)$$

According to Theorem 1.3.10, by taking  $\ell = 2$ , the polynomial energy decay (3.4.1) holds if the following conditions

$$i\mathbb{R} \subset \rho(A), \quad (H1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^2} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (H2)$$

are satisfied. Condition (H1) is already proved in Lemma 3.2.4. We will prove condition (H2) using an argument of contradiction. For this purpose, suppose that (H2) is false, then there exists  $\{(\lambda_n, U_n = (u_n, v_n, y_n, z_n))\}_{n \geq 1} \subset \mathbb{R} \times D(\mathcal{A})$  and

$$\lambda_n \rightarrow +\infty, \quad \|U_n\|_{\mathcal{H}} = 1, \quad (3.4.2)$$

such that

$$\lambda_n^2 (i\lambda_n U_n - \mathcal{A} U_n) = (f_{1,n}, g_{1,n}, f_{2,n}, g_{2,n}) := F_n \rightarrow 0 \text{ in } \mathcal{H}. \quad (3.4.3)$$

For simplicity, we drop the index  $n$ . Detailing Equation (3.4.3), we obtain

$$i\lambda u - v = \lambda^{-2} f_1 \text{ in } H_0^1(0, L), \quad (3.4.4)$$

$$i\lambda v - (au_x + b(x)v_x)_x + c(x)z = \lambda^{-2} g_1 \text{ in } L^2(0, L), \quad (3.4.5)$$

$$i\lambda y - z = \lambda^{-2} f_2 \text{ in } H_0^1(0, L), \quad (3.4.6)$$

$$i\lambda z - y_{xx} - c(x)v = \lambda^{-2} g_2 \text{ in } L^2(0, L). \quad (3.4.7)$$

Here we will check the condition (H2) by finding a contradiction with (3.4.2) such as  $\|U\|_{\mathcal{H}} = o(1)$ . For clarity, we divide the proof into several lemmas. By taking the inner product of (3.4.3) with  $U$  in  $\mathcal{H}$ , we remark that

$$\int_0^L b(x) |v_x|^2 dx = -\operatorname{Re}(\langle \mathcal{A} U, U \rangle_{\mathcal{H}}) = \operatorname{Re}(\langle (i\lambda I - \mathcal{A}) U, U \rangle_{\mathcal{H}}) = o(\lambda^{-2}).$$

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Then,

$$\int_{\alpha_1}^{\alpha_3} |v_x|^2 dx = o(\lambda^{-2}). \quad (3.4.8)$$

**Remark 3.4.2.** Since  $v$  and  $z$  are uniformly bounded in  $L^2(0, L)$ , then from equations (3.4.4) and (3.4.6), the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (3.4.4)-(3.4.7) satisfies the following asymptotic behavior estimation

$$\|u\| = O(\lambda^{-1}), \quad (3.4.9)$$

$$\|y\| = O(\lambda^{-1}). \quad (3.4.10)$$

Using equation (3.4.4), and equation (3.4.8) we get

$$\int_{\alpha_1}^{\alpha_3} |u_x|^2 dx = o(\lambda^{-4}). \quad (3.4.11)$$

**Lemma 3.4.3.** Let  $\varepsilon < \frac{\alpha_3 - \alpha_1}{4}$ , the solution  $(u, v, y, z) \in D(\mathcal{A})$  of the system (3.4.4)-(3.4.7) satisfies the following estimation

$$\int_{\alpha_1 + \varepsilon}^{\alpha_3 - \varepsilon} |v|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha_1 + \varepsilon}^{\alpha_3 - \varepsilon} |\lambda u|^2 dx = o(1). \quad (3.4.12)$$

*Proof.* We define the function  $\rho \in C_0^\infty(0, L)$  by

$$\rho(x) = \begin{cases} 1 & \text{if } x \in (\alpha_1 + \varepsilon, \alpha_3 - \varepsilon), \\ 0 & \text{if } x \in (0, \alpha_1) \cup (\alpha_3, L), \\ 0 \leq \rho \leq 1 & \text{elsewhere.} \end{cases} \quad (3.4.13)$$

Multiply equation (3.4.5) by  $\frac{1}{\lambda} \rho \bar{v}$ , integrate over  $(0, L)$ , using the fact that  $\|g_1\|_{L^2(0, L)} = o(1)$  and  $v$  is uniformly bounded in  $L^2(\Omega)$ , we get

$$\int_0^L i \rho |v|^2 dx + \frac{1}{\lambda} \int_0^L (a u_x + b(x) v_x) (\rho' \bar{v} + \rho \bar{v}_x) dx + \frac{1}{\lambda} \int_0^L c(x) z \rho \bar{v} dx = o(\lambda^{-3}). \quad (3.4.14)$$

Using Equation (3.4.8), Remark 3.4.2 and the fact that  $v$  and  $z$  are uniformly bounded in  $L^2(\Omega)$ , we get

$$\frac{1}{\lambda} \int_0^L (a u_x + b(x) v_x) (\rho' \bar{v} + \rho \bar{v}_x) dx = o(\lambda^{-2}) \quad \text{and} \quad \frac{1}{\lambda} \int_0^L c(x) z \rho \bar{v} dx = o(1). \quad (3.4.15)$$

Inserting Equation (3.4.15) in Equation (3.4.14), we obtain

$$\int_0^L i \rho |v|^2 dx = o(1). \quad (3.4.16)$$

Hence, we obtain the first estimation in Equation (3.4.12). Now, multiplying Equation (3.4.4) by  $\lambda \rho \bar{u}$  integrating over  $(0, L)$  and using the fact that  $\|f_1\|_{H_0^1(\Omega)} = o(1)$  and Remark 3.4.2, we get

$$\int_0^L i \rho |\lambda u|^2 dx - \int_0^L \rho \lambda v \bar{u} dx = o(\lambda^{-2}).$$

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Using Equation (3.4.16), we get

$$\int_0^L i\rho |\lambda u|^2 dx = o(1).$$

Then, we obtain the desired second estimation in Equation (3.4.12).  $\square$

Inserting equations (3.4.4) and (3.4.6) respectively in equations (3.4.5) and (3.4.7), we get

$$\lambda^2 u + (au_x + b(x)v_x)_x - i\lambda c(x)y = F_1, \quad (3.4.17)$$

$$\lambda^2 y + y_{xx} + i\lambda c(x)u = F_2, \quad (3.4.18)$$

where

$$F_1 = -\lambda^{-2}g_1 - i\lambda^{-1}f_1 - c(x)\lambda^{-2}f_2 \quad \text{and} \quad F_2 = -\lambda^{-2}g_2 - i\lambda^{-1}f_2 + c(x)\lambda^{-2}f_1. \quad (3.4.19)$$

**Lemma 3.4.4.** Let  $\varepsilon < \frac{\alpha_3 - \alpha_1}{4}$ , the solution  $(u, v, y, z) \in D(\mathcal{A})$  of the system (3.4.4)-(3.4.7) satisfies the following estimation

$$\int_{\alpha_2}^{\alpha_3 - 2\varepsilon} |\lambda y|^2 dx = o(1) \quad \text{and} \quad \int_{\alpha_2}^{\alpha_3 - 2\varepsilon} |z|^2 dx = o(1). \quad (3.4.20)$$

*Proof.* We define the function  $\zeta \in C_0^\infty(0, L)$  by

$$\zeta(x) = \begin{cases} 1 & \text{if } x \in (\alpha_1 + 2\varepsilon, \alpha_3 - 2\varepsilon), \\ 0 & \text{if } x \in (0, \alpha_1 + \varepsilon) \cup (\alpha_3 - \varepsilon, L), \\ 0 \leq \zeta \leq 1 & \text{elsewhere.} \end{cases} \quad (3.4.21)$$

Multiply equations (3.4.17) by  $\lambda\zeta\bar{y}$  and (3.4.18) by  $\lambda\zeta\bar{u}$  respectively, integrate over  $(0, L)$ , using Remark 3.4.2 and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, g_1, f_2, g_2)\|_{\mathcal{H}} = o(1)$ , we get

$$\int_0^L \lambda^3 \zeta u \bar{y} dx - \int_0^L \lambda (au_x + b(x)v_x) (\zeta' \bar{y} + \zeta \bar{y}_x) dx - i \int_0^L c(x) \zeta(x) |\lambda y|^2 dx = o(\lambda^{-1}) \quad (3.4.22)$$

and

$$\int_0^L \lambda^3 \zeta y \bar{u} dx - \int_0^L \lambda y_x \zeta' \bar{u}_x dx - \int_0^L \lambda y_x \zeta \bar{u}_{xx} dx + i \int_0^L c(x) \zeta(x) |\lambda u|^2 dx = o(\lambda^{-1}). \quad (3.4.23)$$

Using Remark 3.4.2, Lemma 3.4.3 and the fact that  $y_x$  is uniformly bounded in  $L^2(0, L)$ , we get

$$\int_0^L \lambda (au_x + b(x)v_x) (\zeta' \bar{y} + \zeta \bar{y}_x) dx = o(1), \quad - \int_0^L \lambda y_x \zeta' \bar{u}_x dx = o(1) \quad \text{and} \quad \int_0^L \lambda y_x \zeta \bar{u}_{xx} dx = o(1). \quad (3.4.24)$$

Using Lemma 3.4.3, we have that

$$\int_0^L c(x) \zeta |\lambda u|^2 dx = o(1). \quad (3.4.25)$$

Inserting Equations (3.4.24) and (3.4.25) in Equations (3.4.22) and (3.4.23), and summing the result by taking the imaginary part, and using the definition of the functions  $c$  and  $\zeta$ , we get



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the first estimation of Equation (3.4.20).

Now, multiplying equation (3.4.6) by  $\bar{z}$ , integrating over  $(\alpha_2, \alpha_3 - 2\varepsilon)$  and using the fact that  $\|f_2\|_{H_0^1(0,L)} = o(1)$  and  $z$  is uniformly bounded in  $L^2(0, L)$ , in particular in  $L^2(\alpha_2, \alpha_3 - 2\varepsilon)$ , we get

$$\int_{\alpha_2}^{\alpha_3-2\varepsilon} i\lambda y \bar{z} dx - \int_{\alpha_2}^{\alpha_3-2\varepsilon} |z|^2 dx = o(\lambda^{-2}).$$

Then, using the first estimation of Equation (3.4.20), we get the second desired estimation of Equation (3.4.20).  $\square$

Now, we will construct a new multiplier satisfying some ordinary differential systems.

**Lemma 3.4.5.** Let  $0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < L$  and suppose that  $\varepsilon < \frac{\alpha_3 - \alpha_1}{4}$ , and  $c(x)$  the function defined in Equation (3.1.5). Then, for any  $\lambda \in \mathbb{R}$ , the solution  $(\varphi, \psi) \in ((H^2(0, L) \cap H_0^1(0, L))^2)$  of system

$$\begin{cases} \lambda^2 \varphi + a\varphi_{xx} - i\lambda(\mathbb{1}_{(\alpha_2, \alpha_3-2\varepsilon)})(x)\varphi - i\lambda c(x)\psi = u, & x \in (0, L) \\ \lambda^2 \psi + \psi_{xx} - i\lambda(\mathbb{1}_{(\alpha_2, \alpha_3-2\varepsilon)})(x)\psi + i\lambda c(x)\varphi = y, & x \in (0, L) \\ \varphi(0) = \varphi(L) = 0, \\ \psi(0) = \psi(L) = 0, \end{cases} \quad (3.4.26)$$

satisfies the following estimation

$$\|\lambda\varphi\|_{L^2(0,L)}^2 + \|\varphi_x\|_{L^2(0,L)}^2 + \|\lambda\psi\|_{L^2(0,L)}^2 + \|\psi_x\|_{L^2(0,L)}^2 \leq M \left( \|u\|_{L^2(0,L)}^2 + \|y\|_{L^2(0,L)}^2 \right). \quad (3.4.27)$$

*Proof.* Following Theorem 2.2.1, the exponential stability of System (2.1.1) proved in Chapter 2, and by taking  $\alpha = \alpha_2$ ,  $\beta = \alpha_3 - 2\varepsilon$  and  $\gamma = \alpha_4$ , implies that the resolvent of the operator  $\mathcal{A}_a$  defined by (2.1.4)-(2.1.5) is uniformly bounded on the imaginary axis i.e. there exists  $M > 0$  such that

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A}_a)^{-1}\|_{\mathcal{L}(\mathcal{H}_a)} \leq M < +\infty \quad (3.4.28)$$

where  $\mathcal{H}_a = (H_0^1(0, L) \times L^2(0, L))^2$ . Now, since  $(u, y) \in H_0^1(0, L) \times H_0^1(0, L)$ , then  $(0, -u, 0, -y)$  belongs to  $\mathcal{H}_a$ , and from (3.4.28), there exists  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_a)$  such that  $(i\lambda I - \mathcal{A}_a)(\varphi, \eta, \psi, \xi) = (0, -u, 0, -y)^\top$  i.e.

$$i\lambda\varphi - \eta = 0, \quad (3.4.29)$$

$$i\lambda\eta - a\varphi_{xx} + (\mathbb{1}_{(\alpha_2, \alpha_3-2\varepsilon)})(x)\eta + c(x)\xi = -u, \quad (3.4.30)$$

$$i\lambda\psi - \xi = 0, \quad (3.4.31)$$

$$i\lambda\xi - \psi_{xx} + (\mathbb{1}_{(\alpha_2, \alpha_3-2\varepsilon)})(x)\xi - c(x)\eta = -y, \quad (3.4.32)$$

such that

$$\|(\varphi, \eta, \psi, \xi)\|_{\mathcal{H}_a} \leq M \left( \|u\|_{L^2(0,L)} + \|y\|_{L^2(0,L)} \right). \quad (3.4.33)$$

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From equations (3.4.29)-(3.4.33), we deduce that  $(\varphi, \psi)$  is a solution of (3.4.26) and we have

$$\|\lambda\varphi\|_{L^2(0,L)}^2 + \|\varphi_x\|_{L^2(0,L)}^2 + \|\lambda\psi\|_{L^2(0,L)}^2 + \|\psi_x\|_{L^2(0,L)}^2 \leq M \left( \|u\|_{L^2(0,L)}^2 + \|y\|_{L^2(0,L)}^2 \right).$$

Then, we get our desired result.  $\square$

**Lemma 3.4.6.** Let  $\varepsilon < \frac{\alpha_3 - \alpha_1}{4}$ . Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (3.4.4)-(3.4.7) satisfies the following asymptotic behavior estimation

$$\int_0^L |\lambda u|^2 dx = o(1), \quad (3.4.34)$$

and

$$\int_0^L |\lambda y|^2 dx = o(1). \quad (3.4.35)$$

*Proof.* The proof of this Lemma is divided into two steps.

#### Step 1.

Multiplying equation (3.4.17) by  $\lambda^2 \bar{\varphi}$ , integrating over  $(0, L)$ , and using Equation (3.4.27) and the facts that  $u$  is uniformly bounded in  $L^2(0, L)$  and  $\|F\|_{\mathcal{H}} = \|(f_1, g_1, f_2, g_2)\|_{\mathcal{H}} = o(1)$ , we get

$$\int_0^L (\lambda^2 \bar{\varphi} + a \bar{\varphi}_{xx}) \lambda^2 u dx - \int_0^L \lambda^2 b(x) v_x \bar{\varphi}_x dx - \int_0^L i \lambda^3 c(x) y \bar{\varphi} dx = o(\lambda^{-1}). \quad (3.4.36)$$

Using Equations (3.4.8) and (3.4.27), we get

$$\int_0^L \lambda^2 b(x) v_x \bar{\varphi}_x dx = o(1). \quad (3.4.37)$$

Combining Equations (3.4.36) and (3.4.37), we obtain

$$\int_0^L (\lambda^2 \bar{\varphi} + a \bar{\varphi}_{xx}) \lambda^2 u dx - \int_0^L i \lambda^3 c(x) y \bar{\varphi} dx = o(1). \quad (3.4.38)$$

From System (3.4.26), we have

$$\lambda^2 \bar{\varphi} + a \bar{\varphi}_{xx} = -i \lambda (\mathbb{1}_{(\alpha_2, \alpha_3 - 2\varepsilon)})(x) \bar{\varphi} - i \lambda c(x) \bar{\psi} + \bar{u}. \quad (3.4.39)$$

Substituting (3.4.39) in (3.4.38), we get

$$\int_0^L |\lambda u|^2 dx - \int_0^L i \lambda^3 (\mathbb{1}_{(\alpha_2, \alpha_3 - 2\varepsilon)})(x) u \bar{\varphi} dx - \int_0^L i \lambda^3 c(x) \bar{\psi} u dx - \int_0^L i \lambda^3 c(x) y \bar{\varphi} dx = o(1). \quad (3.4.40)$$

Using Remark 3.4.2, Lemma 3.4.3 and Equation (3.4.27), we obtain

$$\int_0^L i \lambda^3 (\mathbb{1}_{(\alpha_2, \alpha_3 - 2\varepsilon)})(x) u \bar{\varphi} dx = o(1). \quad (3.4.41)$$

Inserting Equation (3.4.41) in Equation (3.4.40), we get

$$\int_0^L |\lambda u|^2 dx - \int_0^L i \lambda^3 c(x) \bar{\psi} u dx - \int_0^L i \lambda^3 c(x) y \bar{\varphi} dx = o(1). \quad (3.4.42)$$

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#### Step 2.

Multiplying equation (3.4.18) by  $\lambda^2 \bar{\psi}$ , integrating over  $(0, L)$ , and using Equation (3.4.27) and the facts that  $y$  is uniformly bounded in  $L^2(0, L)$  and  $\|F\|_{\mathcal{H}} = \|(f_1, g_1, f_2, g_2)\|_{\mathcal{H}} = o(1)$ , we get

$$\int_0^L (\lambda^2 \bar{\psi} + \bar{\psi}_{xx}) \lambda^2 y dx + \int_0^L i \lambda c(x) u \bar{\psi} dx = o(\lambda^{-1}). \quad (3.4.43)$$

From System (3.4.26), we have

$$\lambda^2 \bar{\psi} + a \bar{\psi}_{xx} = -i \lambda (\mathbb{1}_{(\alpha_2, \alpha_3 - 2\varepsilon)})(x) \bar{\psi} + i \lambda c(x) \bar{\varphi} + \bar{y}. \quad (3.4.44)$$

Substituting (3.4.44) in (3.4.43), we get

$$\int_0^L |\lambda y|^2 dx - \int_0^L i \lambda^3 (\mathbb{1}_{(\alpha_2, \alpha_3 - 2\varepsilon)})(x) y \bar{\psi} dx + \int_0^L i \lambda^3 c(x) \bar{\varphi} y dx + \int_0^L i \lambda^3 c(x) u \bar{\psi} dx = o(\lambda^{-1}). \quad (3.4.45)$$

Using Remark 3.4.2, Lemma 3.4.4 and Equation (3.4.27), we obtain

$$\int_0^L i \lambda^3 (\mathbb{1}_{(\alpha_2, \alpha_3 - 2\varepsilon)})(x) y \bar{\psi} dx = o(1). \quad (3.4.46)$$

Inserting Equation (3.4.46) in Equation (3.4.45), we get

$$\int_0^L |\lambda y|^2 dx + \int_0^L i \lambda^3 c(x) \bar{\varphi} y dx + \int_0^L i \lambda^3 c(x) u \bar{\psi} dx = o(1). \quad (3.4.47)$$

Finally, summing up equations (3.4.42) and (3.4.47) we get

$$\int_0^L |\lambda u|^2 dx = o(1) \quad \text{and} \quad \int_0^L |\lambda y|^2 dx = o(1).$$

Hence,

$$\int_0^L |v|^2 dx = o(1) \quad \text{and} \quad \int_0^L |z|^2 dx = o(1). \quad (3.4.48)$$

Then, the proof has been completed.  $\square$

**Lemma 3.4.7.** The solution  $(u, v, y, z) \in D(\mathcal{A})$  of the (3.4.4)-(3.4.7) satisfies the following asymptotic behavior estimations

$$\int_0^L |u_x|^2 dx = o(1) \quad \text{and} \quad \int_0^L |y_x|^2 dx = o(1). \quad (3.4.49)$$

*Proof.* Multiplying (3.4.17) by  $\bar{u}$ , integrating over  $(0, L)$ , using the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, g_1, f_2, g_2)\|_{\mathcal{H}} = o(1)$  and  $u$  is uniformly bounded in  $L^2(0, L)$ , we get

$$\int_0^L |\lambda u|^2 dx - \int_0^L a |u_x|^2 dx - \int_0^L b(x) v_x \bar{u}_x dx - \int_0^L i \lambda c(x) y \bar{u} dx = o(\lambda^{-2}). \quad (3.4.50)$$

Using equations (3.4.8) and (3.4.34), we get

$$\int_0^L |u_x|^2 dx = o(1).$$

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Similarly, multiply (3.4.18) by  $\bar{y}$  and integrate, we get

$$\int_0^L |y_x|^2 dx = o(1).$$

The proof has been completed. □

**Proof of Theorem 3.4.1..** Consequently, from the results of Lemmas 3.4.6 and 3.4.7, we obtain

$$\int_0^L (|v|^2 + |z|^2 + a|u_x|^2 + |y_x|^2) dx = o(1).$$

Hence  $\|U\|_{\mathcal{H}} = o(1)$ , which contradicts (3.4.2). Consequently, condition (H2) holds. This implies, from Theorem 1.3.10, the energy decay estimation (3.4.1). The proof is thus complete.

## 3.5 Conclusion

In this chapter, we have studied the stabilization of a system of locally coupled wave equations with only one internal localized Kelvin-Voigt damping via non-smooth coefficients. We proved the strong stability of the system using Arendt-Batty criteria. Lack of exponential stability results has been proved in both cases: The case of global Kelvin-Voigt damping and the case of localized Kelvin-Voigt damping, taking into consideration that the coupling is global. In addition, if both coupling and damping are localized internally via non-smooth coefficients, we established a polynomial energy decay rate of type  $t^{-1}$ . We can conjecture that the energy decay rate  $t^{-1}$  is optimal. However, if the intersection between the supports of the domains of the damping and the coupling coefficients is empty, the nature of the decay rate of the system will be unknown. This question is still an open problem.

# 4 Energy Decay of some boundary coupled systems involving wave\ Euler-Bernoulli beam with one locally singular fractional Kelvin-Voigt damping

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## 4.1 Introduction

### 4.1.1 Literature

In recent years, many researches showed interest in studying the stability and controllability of certain system. The wave equation with different kinds of damping was studied extensively. The wave is created when a vibrating source disturbs the medium. In order to restrain those vibrations, several dampings can be added such as Kelvin-Voigt damping. Many researchers were interested in problems involving this kind of damping (local or global) where different types of stability have been showed. We refer to [8, 12, 22, 36, 54, 66, 70, 72, 95] and the rich

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references therein.

The beam, or flexural member, is frequently encountered in structures and machines, and its elementary stress analysis constitutes one of the most interesting facts of mechanics of materials. For beams, there was an extensive studying, since 80's, on the stabilization of the beam equations (see [37, 58] for the one dimensional system, and [23] for n-dimensional system). Also, the studies considered the linear and nonlinear boundary feedback acting through shear forces and moments [59–61] and the case control by moment has been studied in [56].

The studying of the beam equation with different types of damping was extensively considered. In 1998, the author in [66] considered the longitudinal and transversal vibrations of the Euler-Bernoulli beam with Kelvin-Voigt damping distributed locally on any subinterval of the region occupied by the beam. It was shown that when the viscoelastic damping is distributed only on a subinterval in the interior of the domain, the exponential stability holds for the transversal but not for the longitudinal motion. In [86], they considered a transmission problem for the longitudinal displacement of a Euler-Bernoulli beam, where one small part of the beam is made of a viscoelastic material with Kelvin-Voigt constitutive relation and they proved that the semigroup associated to the system is exponentially stable.

Another type of damping which was studied extensively in the past few years is the fractional damping. It is widely applied in the domain of science. The fractional-order type is not only important from the theoretical point of view but also for applications. They naturally arise in physical, chemical, biological, and ecological phenomena see for example [77], and the rich references therein. They are used to describe memory and hereditary properties of various materials and processes. For example, in viscoelasticity, due to the nature of the material microstructure, both elastic solid and viscous fluid-like response qualities are involved. Using Boltzmann's assumption, we end up with a stress strain relationship defined by a time convolution. The viscoelastic response occurs in a variety of materials, such as soils, concrete, rubber, cartilage, biological tissue, glasses, and polymers (see [19, 20, 76, 93]). Fractional computing in modeling can improve the capturing of the complex dynamics of natural systems, and controls of fractional order type can improve performance not achievable before using controls of integer-order type. For example, systems in many quantum mechanics, nuclear physics and biological phenomena such as fluid flow are indeed fractional (see for example [76, 83, 93]).

Fractional calculus includes various extensions of the usual definition of derivative from integer to real order, including the Hadamard, Erdelyi-Kober, Riemann-Liouville, Riesz, Weyl, Grünwald-Letnikov, Jumarie and the Caputo representation. A thorough analysis of fractional dynamical systems is necessary to achieve an appropriate definition of the fractional derivative. For example, the Riemann-Liouville definition entails physically unacceptable initial conditions (fractional order initial conditions); conversely, for the Caputo representation, which is introduced by Michele Caputo [34] in 1967, the initial conditions are expressed in terms of integer-order derivatives having direct physical significance; this definition is mainly used to include memory effects. Recently, in [33] a new definition of the fractional derivative was presented without a singular kernel; this derivative possesses very interesting properties, for instance the possibility to describe fluctuations and structures with different scales. The case of wave equation with boundary fractional damping have been treated in [78, 79] where they proved the strong stability and the lack of uniform stabilization. However, the case of the plate

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equation or the beam equation with boundary fractional damping was treated in [1] where they showed that the energy is polynomially stable. In [4], they considered a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain. They established a polynomial energy decay rate for smooth solutions, under some geometric conditions. Ammari et al., in [9], studied the stabilization for a class of evolution systems with fractional-damping. They proved the polynomial stability of the system.

Over the past few years, the coupled systems received a vast attention due to their potential applications. The coupled systems have many applications in the modeling and control of engineering, such as: aircraft, satellite antennas and road traffic(see [26] for example). Most of the work in the coupled system considers the stability of the system with various coupling, damping locations, and damping types. Many researches studied coupling systems with a Kelvin-Voigt damping such as wave-wave system, heat-wave system, Timoshinko (see [94, 97]). In 2012, Tebou in [92] considered the Euler-Bernoulli equation coupled with a wave equation in a bounded domain. The frictional damping is distributed everywhere in the domain and acts through one of the equations only. For the case where the dissipation acts through the Euler-Bernoulli equation he showed that the system is not exponentially stable and that the energy decays polynomially was proved. For the case where the damping acts through the wave equation polynomial stability was proved.

Benaissa et al., in [6], considered the large time behavior of one dimensional coupled wave equations with fractional control applied at the coupled point. They showed an optimal decay result.

In [47], Hassine considered a beam and a wave equations coupled on an elastic beam through transmission conditions where the locally distributed damping acts through one of the two equations only. The systems are described as follows

$$\left\{ \begin{array}{ll} u_{tt} - (u_x + D_a u_{xt})_x = 0, & \text{in } \Omega_1, \\ y_{tt} + y_{xxxx} = 0, & \text{in } \Omega_2, \\ u(\ell, t) = y(\ell, t), & t > 0, \\ y_x(\ell, t) = 0, & t > 0, \\ u_x(\ell, t) + y_{xxx}(\ell, t) = 0, & t > 0, \\ u(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, \ell), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (\ell, L) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ll} y_{tt} + (y_{xx} + D_b y_{xxt})_{xx} = 0, & \text{in } \Omega_1, \\ u_{tt} - u_{xx} = 0, & \text{in } \Omega_2, \\ u(\ell, t) = y(\ell, t), & t > 0, \\ y_x(\ell, t) = 0, & t > 0, \\ u_x(\ell, t) + y_{xxx}(\ell, t) = 0, & t > 0, \\ u(0, t) = y(L, t) = y_x(L, t) = 0, & t > 0, \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, \ell), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (\ell, L) \end{array} \right. \quad (4.1.1)$$

where  $\Omega_1 = (0, \ell) \times \mathbb{R}^+$ ,  $\Omega_2 = (\ell, L) \times \mathbb{R}^+$ ,  $D_a = a(x)\chi_{(e,f)}$  and  $D_b = b(x)\chi_{(e,f)}$  with  $0 < e < f < \ell < L$  and  $a(x), b(x) \geq c_0 > 0$  in  $(e, f)$ . The author proved that for both cases when the dissipation acts through the wave equation or through the beam equation, the energy of this coupled system decays polynomially with a decay rate of type  $t^{-2}$ .

The case of a Euler-Bernoulli beam and a wave equations coupled via the interface by transmission conditions was considered by Hassine in [49], where he supposed that the beam equation is stabilized by a localized distributed feedback. He reached that sufficiently smooth solutions decay logarithmically at infinity even the feedback dissipation affects an arbitrarily small open subset of the interior.

In [42], the authors studied the stabilization system of a coupled wave and a Euler-Bernoulli plate equation where only one equation is supposed to be damped with a frictional damping in the multidimensional case. Under some assumption about the damping and the coupling terms, they showed that sufficiently smooth solutions of the system decay logarithmically at

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infinity without any geometric conditions on the effective damping domain.

In [15], Ammari and Nicaise, considered the stabilization problem for coupling the damped wave equation with a damped Kirchhoff plate equation. They proved an exponential stability result under some geometric condition. In 2018, the authors considered in [63], a system of 1-d coupled string-beam. They obtained two kinds of energy decay rates of the string-beam system with different locations of the frictional damping. On one hand, if the frictional damping is only actuated in the beam part, the system lacks exponential decay. Specifically, the optimal polynomial decay rate  $t^{-1}$  is obtained under smooth initial conditions. On the other hand, if the frictional damping is only effective in the string part, the exponential decay of energy is presented. In 2020, the authors in [45], considered a system of two-dimensional coupled wave-plate with local frictional damping in a bounded domain. The frictional damping is only distributed in the part of the plate's or wave's domain, and the other is stabilized by the transmission through the interface of the plate's and wave's domains. They showed that the energy of the system decays polynomially under some geometric condition when the frictional damping only acts on the part of the plate, and the energy of the system is exponentially stable when the frictional damping acts only on the other part of the wave.

In 2018, Guo and Ren in [44], studied the stabilization for a hyperbolic-hyperbolic coupled system consisting of Euler-Bernoulli beam and wave equations, where the structural damping of the wave equation is taken into account. The coupling is actuated through boundary weak connection. The system is described as follows

$$\left\{ \begin{array}{ll} w_{tt} + w_{xxxx} = 0, & (x, t) \in (0, 1) \times \mathbb{R}^+, \\ u_{tt} - u_{xx} - su_{xt} = 0, & (x, t) \in (0, 1) \times \mathbb{R}^+, \\ w(1, t) = w_{xx}(1, t) = w(0, t) = 0, & t > 0, \\ w_{xx}(0, t) = ru_t(0, t), u(1, t) = 0, & x \in (0, 1), \\ su_{xt}(0, t) + u_x(0, t) = -rw_{xt}(0, t), & x \in (0, 1), \\ w(x, 0) = w_0(x), w_t(x, 0) = w_1(x), & x \in (0, 1), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, 1), \end{array} \right. \quad (4.1.2)$$

where  $(w_0, w_1, u_0, u_1)$  is the initial state and  $r \neq 0, s > 0$  are constants. They concluded the Riesz basis property and the exponential stability of the system.

In [11], the authors considered a stabilization problem for a coupled string-beam system. They proved some decay results of the energy of the system. Moreover, they proved, for the same model but with two control functions, independently of the length of the beam that the energy decay with a polynomial rate for all regular initial data. In [13], the authors considered a stabilization problem for a string-beams network and proved an exponential decay result. In [16], the author considered a boundary stabilization problem for the transmission Bernoulli-Euler plate equation and proved a uniform exponential energy decay under natural conditions. In [88], a coupled system of wave-plate type with thermal effect was studied and exponentially stability was proved. In [40], the authors considered the transmission problem for a coupled system of undamped and structurally damped plate equations in two sufficiently smooth



and bounded subdomains. They showed, independently of the size of the damped part, that the damping is strong enough to produce uniform exponential decay of the energy of the coupled system. In 2019, Liu and Han [46], considered a system of coupled plate equations where indirect structural or Kelvin-Voigt damping is imposed, i.e., only one equation is directly damped by one of these two damping. They showed that the associated semigroup of the system with indirect structural damping is analytic and exponentially stable. However, with the much stronger indirect Kelvin-Voigt damping, they proved that the semigroup is even not differentiable and that the exponential stability is still maintained.

### 4.1.2 Physical interpretation of the models

In the first model (EBB)- $W_{FKV}$ , we investigate the stability of coupled Euler-Bernoulli beam and wave equations. The coupling is via boundary connections with localized non-regular fractional Kelvin-Voigt damping, where the damping acts through the wave equation only (see Figure 4.1). The system that describes this model is as follows

$$\left\{ \begin{array}{ll} u_{tt} - (au_x + d(x)\partial_t^{\alpha,\eta} u_x)_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ y_{tt} + by_{xxxx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ u(L, t) = y(-L, t) = y_x(-L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) + by_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, L), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (-L, 0). \end{array} \right. \quad ((\text{EBB})-W_{FKV})$$

The coefficients  $a, b$  are strictly positive constant numbers,  $\alpha \in (0, 1)$  and  $\eta \geq 0$ . We suppose that there exists  $0 < l_0 < l_1 < L$  and a strictly positive constant  $d_0$ , such that

$$d(x) = \begin{cases} d_0, & x \in (l_0, l_1), \\ 0, & x \in (0, l_0) \cup (l_1, L). \end{cases} \quad (4.1.3)$$

The Caputo's fractional derivative  $\partial_t^{\alpha,\eta}$  of order  $\alpha \in (0, 1)$  with respect to time variable  $t$  is defined by

$$[D^{\alpha,\eta}\omega](t) = \partial_t^{\alpha,\eta}\omega(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \frac{d\omega}{ds}(s) ds, \quad (4.1.4)$$

where  $\Gamma$  denotes the Gamma function.

4 Energy Decay of some boundary coupled systems involving wave\ Euler-Bernoulli beam with one locally singular fractional Kelvin-Voigt damping – 4.1 Introduction

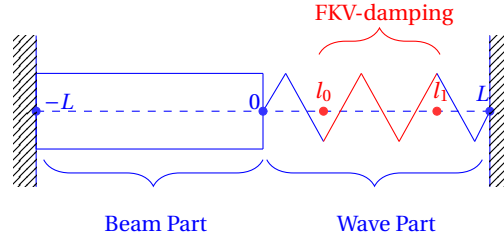


Figure 4.1 – (EBB)- $W_{FKV}$  Model

In the second model  $W-W_{FKV}$ , we investigate the stability of two wave equations coupled through boundary connection with localized non-regular fractional Kelvin-Voigt damping acting through one wave equation only (see Figure 4.2). The system that describes this model is as follows

$$\left\{ \begin{array}{ll} u_{tt} - (au_x + d(x)\partial_t^{\alpha,\eta} u_x)_x = 0, & (x, t) \in (0, L) \times (0, \infty), \\ y_{tt} - by_{xx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ u(L, t) = y(-L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) = by_x(0, t), & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (0, L), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (-L, 0). \end{array} \right. \quad (W-W_{FKV})$$

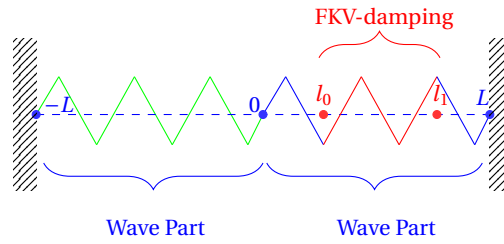


Figure 4.2 –  $W-W_{FKV}$  Model

In the third model  $W-(EBB)_{FKV}$ , we consider a system of coupled Euler-Bernouli beam and wave equations. These two equations are coupled through boundary connection. In this case the localized non-smooth fractional Kelvin-Voigt damping acts only on the Euler-Bernoulli

#### 4 Energy Decay of some boundary coupled systems involving wave\ Euler-Bernoulli beam with one locally singular fractional Kelvin-Voigt damping – 4.1 Introduction

beam (see Figure 4.3). The system that represents this model is as follows

$$\left\{ \begin{array}{ll} u_{tt} - au_{xx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha, \eta} y_{xx})_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\ u(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) + by_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (-L, 0), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (0, L). \end{array} \right. \quad (\text{W}-(\text{EBB})_{FKV})$$

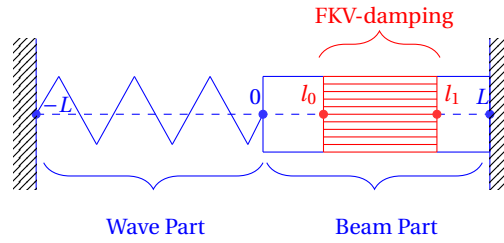


Figure 4.3 – W-(EBB)<sub>FKV</sub> Model

In the fourth model ((EBB)<sub>FKV</sub>), we study a system of Euler-Bernoulli beam with a non-regular localized fractional Kelvin-Voigt damping (see Figure 4.4). The system is as follows

$$\left\{ \begin{array}{ll} y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha, \eta} y_{xx})_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\ y(0, t) = y_x(0, t) = y_{xx}(L, t) = y_{xxx}(L, t) = 0, & t \in (0, \infty), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (0, L). \end{array} \right. \quad ((\text{EBB})_{FKV})$$

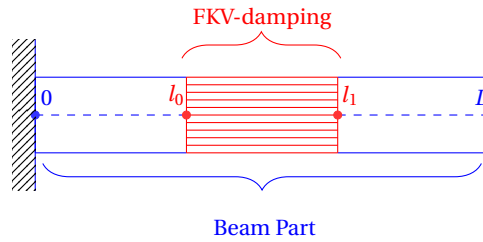


Figure 4.4 – (EBB)<sub>FKV</sub> Model

In the last model (EBB)-(EBB)<sub>FKV</sub>, we consider a system of two Euler-Bernoulli beam equations coupled through boundary connection. The localized non-smooth fractional Kelvin-Voigt

#### 4 Energy Decay of some boundary coupled systems involving wave\ Euler-Bernoulli beam with one locally singular fractional Kelvin-Voigt damping – 4.1 Introduction

damping acts only on one of the two equations (see Figure 4.5). The system that represents this model is as follows

$$\left\{ \begin{array}{ll} u_{tt} + au_{xxxx} = 0, & (x, t) \in (-L, 0) \times (0, \infty), \\ y_{tt} + (by_{xx} + d(x)\partial_t^{\alpha, \eta} y_{xx})_{xx} = 0, & (x, t) \in (0, L) \times (0, \infty), \\ u(-L, t) = u_x(-L, t) = 0, & t \in (0, \infty), \\ y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_{xxx}(0, t) - by_{xxx}(0, t) = 0, & t \in (0, \infty), \\ u_{xx}(0, t) = y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in (-L, 0), \\ y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), & x \in (0, L). \end{array} \right. \quad ((\text{EBB})-(\text{EBB})_{FKV})$$

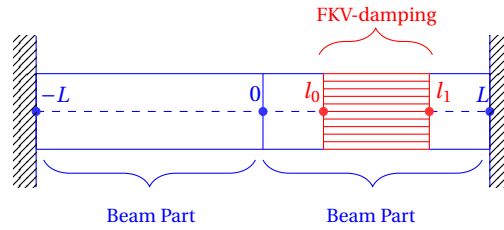


Figure 4.5 –  $(\text{EBB})-(\text{EBB})_{FKV}$

We give the physical meaning of the following variables.

$$\begin{aligned} y &= \text{vertical displacement}, & u_x &= \text{the stress of the wave.} \\ y_x &= \text{rotation}, \\ y_{xx} &= \text{Bending moment}, \\ y_{xxx} &= \text{Shear Force}, \\ y_{xxxx} &= \text{Loading.} \end{aligned} \quad (4.1.5)$$

The way the beam supported is translated into conditions on the function  $y$  and its derivatives. These conditions are collectively referred to as boundary conditions. They are meaningful in physics and engineering. The boundary conditions in the model  $(\text{EBB})_{FKV}$  signifies the following

- $y(0, t) = 0$ : This signifies that the beam is pinned to its support, which means that the beam cannot experience any deflection at  $x = 0$ .
- $y_x(0, t) = 0$ : It signifies that the rotation at the pinned support is zero.
- $y_{xx}(L, t) = 0$ : It means that there is no bending moment at the free end of the beam.
- $y_{xxx}(L, t) = 0$ : This boundary conditions gives the assumption that there is no shearing force acting at the free end of the beam.

This kind of models, supported at one end with the other end free, described in the above four

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conditions can be referred to as a cantilever beam. A good example on the cantilever beam is a balcony, it is supported at one end only, the rest of the beam extends over the open space. Other examples are a cantilever roof in a bus shelter, car park or railway station. We give some of the advantages and disadvantages of the cantilever beam.

Advantages:

- Cantilever beams do not require support on the opposite side.
- The negative bending moment created in cantilever beams helps to counteract the positive bending moments created.
- Cantilever beams can be easily constructed.
- These beam enables erection with little disturbance in navigation.

Disadvantages:

- Cantilever beams are subjected to large deflections.
- Cantilever beams are subjected to larger moments.
- A strong fixed support or a backspan is necessary to keep the structure stable.

In a cantilever beam, the bending moment at the free end always vanishes. In fact, if we connect the beam with a wave (see  $((\text{EBB})-\text{W}_{FKV})$ ,  $(\text{W}-(\text{EBB})_{FKV})$ ) at the free end, the bending moment will be zero and this induces a shear force on the end of the beam. Consequently, the fourth boundary condition above is no longer valid, and it is replaced by  $au_x(0, t) + by_{xxx}(0, t) = 0$ . This condition signifies that the shear force of the beam and the stress force of the wave are such that one cancels the other.

### 4.1.3 Description of the chapter

In this chapter, we investigate the stability results of five models of systems with a non-smooth localized fractional Kelvin-Voigt damping where the coupling is made via boundary connection. In the first model  $((\text{EBB})-\text{W}_{FKV})$  we consider the coupled Euler-Bernoulli beam and wave equation with the damping acts on the wave equation only. In Subsection 4.2.1, we reformulate  $((\text{EBB})-\text{W}_{FKV})$  into an augmented model and we prove the well-posedness of the system by using semigroup approach. Moreover, using a general criteria of Arendt and Batty, we show the strong stability of our system in the absence of the compactness of the resolvent. In section 4.2.2, using the semigroup theory of linear operators and a result obtained by Borichev and Tomilov we show that the energy of the System  $((\text{EBB})-\text{W}_{FKV})$  has a polynomial decay rate of type  $t^{\frac{-4}{2-\alpha}}$ . In the second model  $(\text{W}-\text{W}_{FKV})$ , we consider two wave equations coupled through boundary connection with a non-smooth localized fractional Kelvin-Voigt damping acting only on one of the two equations. We establish a polynomial energy decay rate of type  $t^{\frac{-4}{2-\alpha}}$ . In the third model  $(\text{W}-(\text{EBB})_{FKV})$ , we consider Euler-Bernoulli beam and wave equations coupled through boundary connection with the damping to act through the Euler-Bernoulli beam equation only. For this model, we show that the energy of the System  $(\text{W}-(\text{EBB})_{FKV})$  has a polynomial decay rate of type  $t^{\frac{-2}{3-\alpha}}$ . For the model  $((\text{EBB})_{FKV})$  we consider the Euler-Bernoulli beam with a non-smooth localized fractional Kelvin-Voigt damping. We prove that the energy of the system decays polynomially with a decay rate  $t^{\frac{-2}{1-\alpha}}$ . Finally, for the fifth model  $((\text{EBB})-(\text{EBB})_{FKV})$ , we study the polynomial stability of two Euler-Bernoulli beam equations coupled through boundary connection with damping acting only on one of the two equation. We establish a polynomial energy decay rate of type  $t^{\frac{-2}{3-\alpha}}$ . The table below (Table 4.1) summarizes the decay rate of the energy for the five models. Also, it gives the decay rate of the same five models but with Kelvin-Voigt damping (as  $\alpha \rightarrow 1$ ).

Model	Decay Rate	$\alpha \rightarrow 1$
(EBB)- $W_{FKV}$	$t^{\frac{-4}{2-\alpha}}$	$t^{-4}$
W- $W_{FKV}$	$t^{\frac{-4}{2-\alpha}}$	$t^{-4}$
W-(EBB) $_{FKV}$	$t^{\frac{-2}{3-\alpha}}$	$t^{-1}$
(EBB) $_{FKV}$	$t^{\frac{-2}{1-\alpha}}$	Exponential
(EBB)-(EBB) $_{FKV}$	$t^{\frac{-2}{3-\alpha}}$	$t^{-1}$

Table 4.1 – Decay Results

Some significant implications on the energy decay rate are given below:

- From the decay rate of the models ((EBB)- $W_{FKV}$ ) and (W-(EBB) $_{FKV}$ ) we can deduce that if we want to choose the place where the fractional Kelvin-Voigt damping acts it is better to choose the damping on the wave. Since the energy of this model decays faster compared with that of (W-(EBB) $_{FKV}$ ) model.
- For the model ((EBB)- $W_{FKV}$ ), if we replace the condition of the null bending moment ( $y_{xx}(0) = 0$ ) at the connecting boundary by taking the rotation to be zero ( $y_x(0) = 0$ ), we get the same decay rate. So, this result improves the work in [47] where they reached energy decay rate of type  $t^{-2}$ , however in our work we proved an energy decay rate of type  $t^{-4}$  (as  $\alpha \rightarrow 1$ ) (see Section 4.2).
- For the model ((EBB)- $W_{FKV}$ ), we established an energy decay rate of type  $t^{-4}$  (as  $\alpha \rightarrow 1$ ). By comparing this energy decay rate with that in [94], where the authors considered two wave equations coupled through velocity with localized non smooth Kelvin-Voigt and they established an energy decay rate of type  $t^{-1}$ . We can see that, by comparing the energy decay rate of these two systems that it is better, when considering Kelvin-Voigt damping, to consider the coupling through the boundary connection rather through the velocity.

**Remark 4.1.1.** We note that in the upcoming sections, the letters used to denote the variables are independent from each other in each section.

## 4.2 (EBB)- $W_{FKV}$ Model

In this section, we consider the ((EBB)- $W_{FKV}$ ) model, where we study the stability of the system of a Euler-Bernoulli and wave equations coupled through boundary connection with a localized fractional Kelvin-Voigt damping acting on the wave equation only.

### 4.2.1 Well-Posedness and Strong Stability

In this subsection, we study the strong stability of the system ((EBB)- $W_{FKV}$ ) in the absence of the compactness of the resolvent. First, we will study the existence, uniqueness and regularity of the solution of the system.

#### 4.2.1.1 Augmented model and Well-Posedness.

In this part, using a semigroup approach, we establish the well-posedness for the system ((EBB)- $W_{FKV}$ ). First, we recall theorem 2 stated in [3, 79].

**Theorem 4.2.1.** *Let  $\alpha \in (0, 1)$ ,  $\eta \geq 0$  and  $\mu(\xi) = |\xi|^{\frac{2\alpha-1}{2}}$  be the function defined almost everywhere on  $\mathbb{R}$ . The relation between the 'input'  $V$  and the 'output'  $O$  of the following system*

$$\partial_t \omega(x, \xi, t) + (\xi^2 + \eta) \omega(x, \xi, t) - V(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times (0, \infty), \quad (4.2.1)$$

$$\omega(x, \xi, 0) = 0, \quad (x, \xi) \in (0, L) \times \mathbb{R}, \quad (4.2.2)$$

$$O(x, t) - \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi = 0, \quad (x, t) \in (0, L) \times (0, \infty), \quad (4.2.3)$$

is given by

$$O = I^{1-\alpha, \eta} V, \quad (4.2.4)$$

where

$$[I^{\alpha, \eta} V](x, t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\eta(t-s)} V(s) ds \quad \text{and} \quad \kappa(\alpha) = \frac{\sin(\alpha\pi)}{\pi}.$$

In the above theorem, taking the input  $V(x, t) = \sqrt{d(x)} u_{xt}(x, t)$ , then using Equation (4.1.4), we get that the output  $O$  is given by

$$O(x, t) = \sqrt{d(x)} I^{1-\alpha, \eta} u_{xt}(x, t) = \frac{\sqrt{d(x)}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \partial_s u_x(x, s) ds = \sqrt{d(x)} \partial_t^{\alpha, \eta} u_x(x, t).$$

Therefore, by taking the input  $V(x, t) = \sqrt{d(x)} u_{xt}(x, t)$  in Theorem 4.2.1 and using the above equation, we get

$$\begin{aligned} \partial_t \omega(x, \xi, t) + (\xi^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} u_{xt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} &= 0, \quad (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \\ \omega(x, \xi, 0) &= 0, \quad (x, \xi) \in (0, L) \times \mathbb{R}, \\ \sqrt{d(x)} \partial_t^{\alpha, \eta} u_x(x, t) - \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi &= 0, \quad (x, t) \in (0, L) \times \mathbb{R}_*^+. \end{aligned} \quad (4.2.5)$$

From system (4.2.5), we deduce that system ((EBB)- $W_{FKV}$ ) can be recast into the following augmented model

$$\begin{cases} u_{tt} - \left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ y_{tt} + by_{xxx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} u_{xt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{cases} \quad (4.2.6)$$

with the following transmission and boundary conditions

$$\begin{cases} u(L, t) = y(-L, t) = y_x(-L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) + by_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{cases} \quad (4.2.7)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad \omega(x, \xi, 0) = 0, \quad x \in (0, L), \xi \in \mathbb{R}, \\ y(x, 0) &= y_0(x), \quad y_t(x, 0) = y_1(x), \quad x \in (-L, 0). \end{aligned} \quad (4.2.8)$$

The energy of the system (4.2.6)-(4.2.8) is given by

$$E_1(t) = \frac{1}{2} \int_0^L (|u_t|^2 + a|u_x|^2) dx + \frac{1}{2} \int_{-L}^0 (|y_t|^2 + b|y_{xx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

**Lemma 4.2.2.** *Let  $U = (u, u_t, y, y_t, \omega)$  be a regular solution of the System (4.2.6)-(4.2.8). Then, the energy  $E_1(t)$  satisfies the following estimation*

$$\frac{d}{dt} E_1(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (4.2.9)$$

*Proof.* First, multiplying the first and the second equations of (4.2.6) by  $\bar{u}_t$  and  $\bar{y}_t$  respectively, integrating over  $(0, L)$  and  $(-L, 0)$  respectively, using integration by parts with (4.2.7) and taking the real part  $\text{Re}$ , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^L (|u_t|^2 + a|u_x|^2) dx + \frac{1}{2} \frac{d}{dt} \int_{-L}^0 (|y_t|^2 + b|y_{xx}|^2) dx \\ & + \text{Re} \left( \kappa(\alpha) \int_0^L \sqrt{d(x)} \bar{u}_{tx} \left( \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right) dx \right) = 0. \end{aligned} \quad (4.2.10)$$

Now, multiplying the third equation of (4.2.6) by  $\kappa(\alpha) \bar{\omega}$ , integrating over  $(0, L) \times \mathbb{R}$ , then taking the real part, we get

$$\begin{aligned} & \frac{\kappa(\alpha)}{2} \frac{d}{dt} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx \\ & = \text{Re} \left( \kappa(\alpha) \int_0^L \sqrt{d(x)} u_{xt} \left( \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi, t) d\xi \right) dx \right). \end{aligned} \quad (4.2.11)$$

Finally, by adding (4.2.10) and (4.2.11) we obtain (4.2.9). The proof is thus complete.  $\square$

Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and therefore  $\frac{d}{dt} E_1(t) \leq 0$ . Thus, system (4.2.6)-(4.2.8) is dissipative in the sense that its energy is a non-increasing function with respect to time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_1$  by

$$\mathcal{H}_1 = \{(u, v, y, z, \omega) \in H_R^1(0, L) \times L^2(0, L) \times H_L^2(-L, 0) \times L^2(-L, 0) \times W; \ u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_R^1(0, L) = \{u \in H^1(0, L); u(L) = 0\}, \\ H_L^2(-L, 0) = \{y \in H^2(-L, 0); y(-L) = y_x(-L) = 0\}. \end{cases} \quad (4.2.12)$$

We note that the space  $\mathcal{H}_1$  is a closed subspace of  $H_R^1(0, L) \times L^2(0, L) \times H_L^2(-L, 0) \times L^2(-L, 0) \times W$ .



The energy space  $\mathcal{H}_1$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_1} &= \int_0^L v \overline{v_1} dx + a \int_0^L u_x (\overline{u_1})_x dx + \int_{-L}^0 z \overline{z_1} dx + b \int_{-L}^0 y_{xx} (\overline{y_1})_{xx} dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1(x, \xi)} d\xi dx, \end{aligned}$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_1$ . We use  $\|U\|_{\mathcal{H}_1}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_1 : D(\mathcal{A}_1) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_1$  by

$$D(\mathcal{A}_1) = \left\{ U = (u, v, y, z, \omega) \in \mathcal{H}_1; \begin{aligned} &(v, z) \in H_R^1(0, L) \times H_L^2(-L, 0), \quad y \in H^4(-L, 0), \\ &\left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \in L^2(0, L), \\ &- (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ &au_x(0) + by_{xxx}(0) = 0, \quad y_{xx}(0) = 0, \text{ and } v(0) = z(0) \end{aligned} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$ ,

$$\mathcal{A}_1(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ \left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \\ z \\ -by_{xxxx} \\ - (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

**Remark 4.2.3.** The condition  $|\xi| \omega(x, \xi) \in W$  is imposed to insure the existence of the integral

$$\int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx \text{ in (4.2.9) and } \sqrt{d(x)} \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \in L^2(0, L).$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (4.2.6)-(4.2.8), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_1$  given by

$$U_t = \mathcal{A}_1 U, \quad U(0) = U_0, \quad (4.2.13)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ .

**Lemma 4.2.4.** Let  $\alpha \in (0, 1)$ ,  $\eta \geq 0$ , then the following integrals

$$\begin{aligned} \mathcal{I}_1(\eta, \alpha) &= \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1 + \xi^2 + \eta} d\xi, \quad \mathcal{I}_2(\eta, \alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{(1 + \xi^2 + \eta)^2} d\xi \\ \text{and } \mathcal{I}_3(\eta, \alpha) &= \int_0^{+\infty} \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} d\xi \end{aligned} \quad (4.2.14)$$

are well defined.

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*Proof.* First,  $I_1(\eta, \alpha)$  can be written as

$$I_1(\eta, \alpha) = 2 \frac{\kappa(\alpha)}{1+\eta} \int_0^{+\infty} \frac{\xi^{2\alpha-1}}{1 + \frac{\xi^2}{1+\eta}} d\xi. \quad (4.2.15)$$

Thus equation (4.2.15) can be simplified by defining a new variable  $y = 1 + \frac{\xi^2}{1+\eta}$ . Substituting  $\xi$  by  $(y-1)^{\frac{1}{2}}(1+\eta)^{\frac{1}{2}}$  in equation (4.2.15), we get

$$I_1(\eta, \alpha) = \frac{\kappa(\alpha)}{(1+\eta)^{1-\alpha}} \int_1^{+\infty} \frac{1}{y(y-1)^{1-\alpha}} dy.$$

Using the fact that  $\alpha \in (0, 1)$ , it's easy to see that  $y^{-1}(y-1)^{\alpha-1} \in L^1(1, +\infty)$ , therefore  $I_1(\eta, \alpha)$  is well defined. Now, for  $I_2(\eta, \alpha)$ , using  $\eta \geq 0$  and  $\alpha \in (0, 1)$ , we get

$$I_2(\eta, \alpha) < \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{1 + \xi^2 + \eta} d\xi = \frac{I_1(\eta, \alpha)}{\kappa(\alpha)} < +\infty.$$

Then,  $I_2(\eta, \alpha)$  is well-defined.

Now, for the integral  $I_3(\eta, \alpha)$ , since

$$\frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} \underset{0}{\sim} \frac{\xi^{2\alpha+1}}{(1 + \eta)^2} \quad \text{and} \quad \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} \underset{+\infty}{\sim} \frac{1}{\xi^{3-2\alpha}},$$

and the fact that  $\alpha \in (0, 1)$ , we get  $I_3(\eta, \alpha)$  is well-defined.

The proof is thus complete. □

**Proposition 4.2.5.** *The unbounded linear operator  $\mathcal{A}_1$  is  $m$ -dissipative in the energy space  $\mathcal{H}_1$ .*

*Proof.* For all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$ , one has

$$\operatorname{Re} \left( \langle \mathcal{A}_1 U, U \rangle_{\mathcal{H}_1} \right) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx \leq 0,$$

which implies that  $\mathcal{A}_1$  is dissipative. Now, let  $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}_1$ , we prove the existence of  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$ , solution of the equation

$$(I - \mathcal{A}_1)U = F.$$

Equivalently, one must consider the system given by

$$u - v = f_1, \quad (4.2.16)$$

$$v - \left( au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x = f_2, \quad (4.2.17)$$

$$y - z = f_3, \quad (4.2.18)$$

$$z + by_{xxxx} = f_4, \quad (4.2.19)$$

$$(1 + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}v_x |\xi|^{\frac{2\alpha-1}{2}} = f_5(x, \xi). \quad (4.2.20)$$

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Using Equations (4.2.16), (4.2.20) and the fact that  $\eta \geq 0$ , we get

$$\omega(x, \xi) = \frac{f_5(x, \xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{d(x)}u_x|\xi|^{\frac{2\alpha-1}{2}}}{1 + \xi^2 + \eta} - \frac{\sqrt{d(x)}(f_1)_x|\xi|^{\frac{2\alpha-1}{2}}}{1 + \xi^2 + \eta}.$$

Inserting the above equation and (4.2.16) in (4.2.17) and (4.2.18) in (4.2.19), we get

$$u - (au_x + d(x)I_1(\eta, \alpha)u_x + J)_x = F_1, \quad (4.2.21)$$

$$y + by_{xxx} = F_2 \quad (4.2.22)$$

where  $I_1(\eta, \alpha)$  is defined in Equation (4.2.14),  $F_1 = f_1 + f_2$ ,  $F_2 = f_3 + f_4$ , and

$$J = d(x)I_1(\eta, \alpha)(f_1)_x - \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi)}{1 + \xi^2 + \eta} d\xi,$$

and with the following boundary conditions

$$u(L) = y(-L) = y_x(-L) = y_{xx}(0) = 0, au_x(0) + by_{xxx}(0) = 0, \text{ and } u(0) = y(0). \quad (4.2.23)$$

Now, we define

$$V = \{(\varphi, \psi) \in H_R^1(0, L) \times H_L^2(-L, 0); \varphi(0) = \psi(0)\}.$$

The space  $V$  is equipped with the following inner product

$$\langle (\varphi, \psi), (\varphi_1, \psi_1) \rangle_V = a \int_0^L \varphi_x \overline{\varphi_{1x}} dx + b \int_{-L}^0 \psi_{xx} \overline{(\psi_1)_{xx}} dx.$$

Let  $(\varphi, \psi) \in V$ . Multiplying equations (4.2.21) and (4.2.22) by  $\overline{\varphi}$  and  $\overline{\psi}$ , and integrating respectively on  $(0, L)$  and  $(-L, 0)$ , then using by parts integration, we get

$$a((u, y), (\varphi, \psi)) = L(\varphi, \psi) \quad \forall (\varphi, \psi) \in V, \quad (4.2.24)$$

where

$$a((u, y), (\varphi, \psi)) = \int_0^L u \overline{\varphi} dx + a \int_0^L u_x \overline{\varphi_x} dx + \int_{-L}^0 y \overline{\psi} dx + b \int_{-L}^0 y_{xx} \overline{\psi_{xx}} dx + I_1(\eta, \alpha) \int_0^L d(x) u_x \overline{\varphi_x} dx$$

and

$$L(\varphi, \psi) = \int_0^L F_1 \overline{\varphi} dx + I_1(\eta, \alpha) \int_0^L d(x) (f_1)_x \overline{\varphi_x} dx - \kappa(\alpha) \int_0^L \sqrt{d(x)} \overline{\varphi_x} \left( \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi)}{1 + \xi^2 + \eta} d\xi \right) dx + \int_{-L}^0 F_2 \overline{\psi} dx.$$

Using the fact that  $I_1(\eta, \alpha) > 0$ , we get that  $a$  is a bilinear, continuous, and coercive form on  $V \times V$ . Next, by using Cauchy-Schwartz inequality and the definition of  $d(x)$ , we get

$$\left| \int_0^L \sqrt{d(x)} \overline{\varphi_x} \left( \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi)}{1 + \xi^2 + \eta} d\xi \right) dx \right| \leq \sqrt{\frac{d_0}{\kappa(\alpha)}} \sqrt{I_2(\eta, \alpha)} \|\varphi_x\|_{L^2(I_0, I_1)} \|f_5\|_W, \quad (4.2.25)$$

where  $I_2(\eta, \alpha)$  is defined in Equation (4.2.14). Hence,  $L$  is a linear continuous form on  $V$ .

Then, using Lax-Milgram theorem, we deduce that there exists unique  $(u, y) \in V$  solution of the variational problem (4.2.24). Applying the classical elliptic regularity, we deduce that  $y \in H^4(-L, 0)$ , and

$$\left( au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_x \in L^2(0, L).$$

Defining

$$\begin{cases} v := u - f_1, & z := y - f_3 \\ \omega(x, \xi) = \frac{f_5(x, \xi)}{1 + \xi^2 + \eta} + \frac{\sqrt{d(x)}u_x |\xi|^{\frac{2\alpha-1}{2}}}{1 + \xi^2 + \eta} - \frac{\sqrt{d(x)}(f_1)_x |\xi|^{\frac{2\alpha-1}{2}}}{1 + \xi^2 + \eta}. \end{cases} \quad (4.2.26)$$

It is easy to see that  $(v, z) \in H_R^1(0, L) \times H_L^2(-L, 0)$  and  $v(0) = z(0)$ .

In order to complete the existence of  $U \in D(\mathcal{A}_1)$ , we need to prove  $\omega(x, \xi)$  and  $|\xi|\omega(x, \xi) \in W$ . From equation (4.2.26), we obtain

$$\int_0^L \int_{\mathbb{R}} |\omega(x, \xi)|^2 dx \leq 3 \int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{(1 + \xi^2 + \eta)^2} d\xi dx + 3d_0 \mathcal{I}_2(\eta, \alpha) \int_{l_0}^{l_1} (|u_x|^2 + |(f_1)_x|^2) dx.$$

Using Lemma 4.2.4, the fact that  $(u, f_1) \in H_R^1(0, L) \times H_R^1(0, L)$ , we obtain

$$\mathcal{I}_2(\eta, \alpha) \int_{l_0}^{l_1} (|u_x|^2 + |(f_1)_x|^2) dx < \infty.$$

On the other hand, using the fact that  $f_5 \in W$ , we get

$$\int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{(1 + \xi^2 + \eta)^2} d\xi dx \leq \frac{1}{(1 + \eta)^2} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < +\infty.$$

It follows that  $\omega(x, \xi) \in W$ . Next, using equation (4.2.26), we get

$$\int_0^L \int_{\mathbb{R}} |\xi \omega(x, \xi)|^2 d\xi dx \leq 3 \int_0^L \int_{\mathbb{R}} \frac{\xi^2 |f_5(x, \xi)|^2}{(1 + \xi^2 + \eta)^2} d\xi dx + 6d_0 \mathcal{I}_3(\eta, \alpha) \left( \int_{l_0}^{l_1} (|u_x|^2 + |(f_1)_x|^2) dx \right),$$

where  $\mathcal{I}_3(\eta, \alpha) = \int_0^{+\infty} \frac{\xi^{2\alpha+1}}{(1 + \xi^2 + \eta)^2} d\xi$ . Using Lemma 4.2.4 we get that  $\mathcal{I}_3(\eta, \alpha)$  is well-defined.

Now, using the fact that  $f_5(x, \xi) \in W$  and

$$\max_{\xi \in \mathbb{R}} \frac{\xi^2}{(1 + \xi^2 + \eta)^2} = \frac{1}{4(1 + \eta)} < \frac{1}{4},$$

we get

$$\int_0^L \int_{\mathbb{R}} \frac{\xi^2 |f_5(x, \xi)|^2}{(1 + \xi^2 + \eta)^2} d\xi dx \leq \max_{\xi \in \mathbb{R}} \frac{\xi^2}{(1 + \xi^2 + \eta)^2} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < \frac{1}{4} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < +\infty.$$

It follows that  $|\xi|\omega \in W$ . Finally, since  $\omega, f_5 \in W$ , we get

$$-(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}} = \omega(x, \xi) - f_5(x, \xi) \in W.$$

Therefore, there exists  $U := (u, v, y, z, \omega) \in D(\mathcal{A}_1)$  solution of  $(I - \mathcal{A}_1)U = F$ . The proof is thus complete.  $\square$

From proposition 4.2.5, the operator  $\mathcal{A}_1$  is m-dissipative on  $\mathcal{H}_1$ , consequently it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_1})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8. Then the solution of the evolution Equation (4.2.13) admits the following representation

$$U(t) = e^{t\mathcal{A}_1} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (4.2.13). Hence, we have the following result.

**Theorem 4.2.6.** *Let  $U_0 \in \mathcal{H}_1$ , then problem (4.2.13) admits a unique weak solution  $U$  that satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}_1).$$

*Moreover, if  $U_0 \in D(\mathcal{A}_1)$ , then problem (4.2.13) admits a unique strong solution  $U$  that satisfies*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}_1) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_1)).$$

#### 4.2.1.2 Strong Stability

This part is devoted to study the strong stability of the system. It is easy to see that the resolvent of  $\mathcal{A}_1$  is not compact. Here, we will use Theorem 1.3.3 to obtain the strong stability of the  $C_0$ -semigroup  $(e^{t\mathcal{A}_1})_{t \geq 0}$ . Our main result in this part is the following theorem.

**Theorem 4.2.7.** *Assume that  $\eta \geq 0$ , then the  $C_0$ -semigroup of contractions  $e^{t\mathcal{A}_1}$  is strongly stable on  $\mathcal{H}_1$  in the sense that*

$$\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_1} U_0\|_{\mathcal{H}_1} = 0, \quad \forall U_0 \in \mathcal{H}_1.$$

In order to prove Theorem 4.2.7 we need to prove that the operator  $\mathcal{A}_1$  has no pure imaginary eigenvalues and  $\sigma(\mathcal{A}_1) \cap i\mathbb{R}$  is countable, where  $\sigma(\mathcal{A}_1)$  denotes the spectrum of  $\mathcal{A}_1$ . For clarity, we divide the proof into several lemmas.

**Lemma 4.2.8.** *Let  $\alpha \in (0, 1)$ ,  $\eta \geq 0$ ,  $\lambda \in \mathbb{R}$  and  $f_5 \in W$ . For  $(\eta > 0$  and  $\lambda \in \mathbb{R})$  or  $(\eta = 0$  and  $\lambda \in \mathbb{R}^*)$ , we have*

$$I_4(\lambda, \eta, \alpha) = i\lambda\kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{i\lambda + \xi^2 + \eta} d\xi < \infty, \quad I_5(\lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{i\lambda + \xi^2 + \eta} d\xi < \infty,$$

and

$$I_6(x, \lambda, \eta, \alpha) := \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi)}{i\lambda + \xi^2 + \eta} d\xi \in L^2(0, L)$$

*Proof.* The integrals  $I_4$  and  $I_5$  can be written in the following form

$$I_4(\lambda, \eta, \alpha) = \lambda^2 I_7(\lambda, \eta, \alpha) + i\lambda I_8(\lambda, \eta, \alpha), \text{ and } I_5(\lambda, \eta, \alpha) = -i\lambda I_7(\lambda, \eta, \alpha) + I_8(\lambda, \eta, \alpha)$$

where

$$I_7(\lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi, \text{ and } I_8(\lambda, \eta, \alpha) = \kappa(\alpha) \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1} (\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi.$$

So, we need to prove that  $I_7(\lambda, \eta, \alpha)$  and  $I_8(\lambda, \eta, \alpha)$  are well defined.

First, we have

$$I_7(\lambda, \eta, \alpha) = 2\kappa(\alpha) \int_0^{+\infty} \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi = 2\kappa(\alpha) \int_0^1 \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi + 2\kappa(\alpha) \int_1^{+\infty} \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi.$$

Hence in the both cases where  $(\eta > 0 \text{ and } \lambda \in \mathbb{R})$  or  $(\eta = 0 \text{ and } \lambda \in \mathbb{R}^*)$ , we have

$$\frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} \underset{0}{\sim} \frac{\xi^{2\alpha-1}}{\lambda^2 + \eta^2} \quad \text{and} \quad \frac{\xi^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} \underset{+\infty}{\sim} \frac{1}{\xi^{5-2\alpha}}.$$

Since  $0 < \alpha < 1$  then  $I_7(\lambda, \eta, \alpha)$  is well-defined. Now, we have

$$I_8(\lambda, \eta, \alpha) = 2\kappa(\alpha) \int_0^{+\infty} \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi = 2\kappa(\alpha) \int_0^1 \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi + 2\kappa(\alpha) \int_1^{+\infty} \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} d\xi.$$

Similar to  $I_7$ , in both cases when  $(\eta > 0 \text{ and } \lambda \in \mathbb{R})$  or  $(\eta = 0 \text{ and } \lambda \in \mathbb{R}^*)$ , we have

$$\frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} \underset{0}{\sim} \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + \eta^2} \quad \text{and} \quad \frac{\xi^{2\alpha-1}(\xi^2 + \eta)}{\lambda^2 + (\xi^2 + \eta)^2} \underset{+\infty}{\sim} \frac{1}{\xi^{3-2\alpha}}.$$

Since  $0 < \alpha < 1$ , then  $I_8(\lambda, \eta, \alpha)$  is well-defined. For  $I_6$ , using Cauchy-Schwarz inequality and the fact that  $f_5 \in W$  and that  $I_7 < \infty$ , we get

$$\begin{aligned} \int_0^L |I_6(x, \lambda, \eta, \alpha)|^2 dx &= \kappa(\alpha)^2 \int_0^L \left| \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} f_5(x, \xi)}{i\lambda + \xi^2 + \eta} d\xi \right|^2 dx \\ &\leq \kappa(\alpha)^2 \left( \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi \right) \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < +\infty. \end{aligned}$$

The proof is thus complete. □

**Lemma 4.2.9.** *Let  $\alpha \in (0, 1)$ ,  $\eta \geq 0$ ,  $\lambda \in \mathbb{R}$ . For  $(\eta > 0 \text{ and } \lambda \in \mathbb{R})$  or  $(\eta = 0 \text{ and } \lambda \in \mathbb{R}^*)$ , we have*

$$I_9(\lambda, \eta, \alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} d\xi \quad \text{and} \quad I_{10}(\lambda, \eta, \alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha+1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi$$

*are well-defined.*

*Proof.* We have

$$I_9(\lambda, \eta, \alpha) = 2 \int_0^1 \frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} d\xi + 2 \int_1^{+\infty} \frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} d\xi$$

Hence in the both cases where  $(\eta > 0 \text{ and } \lambda \in \mathbb{R})$  or  $(\eta = 0 \text{ and } \lambda \in \mathbb{R}^*)$ , we have

$$\frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} \underset{0}{\sim} \frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + \eta^2}} \quad \text{and} \quad \frac{\xi^{2\alpha-1}}{\sqrt{\lambda^2 + (\xi^2 + \eta)^2}} \underset{+\infty}{\sim} \frac{1}{\xi^{3-2\alpha}}.$$

Since  $0 < \alpha < 1$  then  $I_9(\lambda, \eta, \alpha)$  is well-defined. Using the same argument to show that  $I_{10}(\lambda, \eta, \alpha)$  is well-defined. The proof is thus complete. □

**Lemma 4.2.10.** Assume that  $\eta \geq 0$ . Then, for all  $\lambda \in \mathbb{R}$ , we have  $i\lambda I - \mathcal{A}_1$  is injective, i.e.

$$\ker(i\lambda I - \mathcal{A}_1) = \{0\}.$$

*Proof.* Let  $\lambda \in \mathbb{R}$ , such that  $i\lambda$  be an eigenvalue of the operator  $\mathcal{A}_1$  and  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$  a corresponding eigenvector. Therefore, we have

$$\mathcal{A}_1 U = i\lambda U. \quad (4.2.27)$$

Equivalently, we have

$$v = i\lambda u, \quad (4.2.28)$$

$$\left( au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x = i\lambda v, \quad (4.2.29)$$

$$z = i\lambda y, \quad (4.2.30)$$

$$-by_{xxxx} = i\lambda z, \quad (4.2.31)$$

$$(i\lambda + |\xi|^2 + \eta) \omega(x, \xi) = \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}}. \quad (4.2.32)$$

with the boundary conditions

$$\begin{cases} u(L) = y(-L) = y_x(-L) = 0, \\ y_{xx}(0) = 0, \end{cases} \quad (4.2.33)$$

and with the continuity transmission conditions

$$\begin{cases} u(0) = y(0), \\ au_x(0) = -by_{xxx}(0). \end{cases} \quad (4.2.34)$$

A straightforward calculation gives

$$0 = \operatorname{Re}(\langle i\lambda U, U \rangle_{\mathcal{H}_1}) = \operatorname{Re}(\langle \mathcal{A}_1 U, U \rangle_{\mathcal{H}_1}) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx.$$

Consequently, we deduce that

$$\omega(x, \xi) = 0 \text{ a.e. in } (0, L) \times \mathbb{R}. \quad (4.2.35)$$

Inserting Equation (4.2.35) into (4.2.32) and using the definition of  $d(x)$ , we get

$$v_x = 0 \text{ in } (l_0, l_1). \quad (4.2.36)$$

It follows, from Equation (4.2.28), that

$$\lambda u_x = 0 \text{ in } (l_0, l_1). \quad (4.2.37)$$

Here we will distinguish two cases.

**Case 1.** If  $\lambda = 0$ :

From (4.2.28) and (4.2.30) we get

$$v = z = 0 \quad \text{on} \quad (0, L).$$

Using Equations (4.2.29), (4.2.31) and (4.2.35) we get

$$u_{xx} = y_{xxxx} = 0.$$

Using the boundary conditions in (4.2.33) we can write  $u$  and  $y$  as

$$u = c_1(x - L) \quad \text{and} \quad y = c_3 \left( \frac{x^3}{6} - \frac{L^2}{2}x - \frac{L^3}{3} \right)$$

where  $c_1, c_3$  are constant numbers to be determined. Now, using conditions in (4.2.34) we get

$$\begin{cases} c_1 = \frac{L^2}{2}c_3, \\ ac_1 = -bc_3. \end{cases} \quad (4.2.38)$$

Then,  $c_3 \left( a \frac{L^2}{2} + b \right) = 0$ . Since  $a, b > 0$ , we deduce that  $c_1 = c_3 = 0$ . Then we get  $u = y = 0$ . Hence,  $U = 0$ . In this case the proof is complete.

**Case 2.** If  $\lambda \neq 0$ :

From Equation (4.2.37), we get

$$u_x = 0 \quad \text{in} \quad (l_0, l_1). \quad (4.2.39)$$

Using Equations (4.2.35) and (4.2.39) in (4.2.29), and using Equation (4.2.28) we get

$$u = 0 \quad \text{in} \quad (l_0, l_1). \quad (4.2.40)$$

Substituting equations (4.2.28) and (4.2.30) into Equations (4.2.29) and (4.2.31) and using Equation (4.2.35), we get

$$\lambda^2 u + au_{xx} = 0, \quad \text{over} \quad (0, L), \quad (4.2.41)$$

$$\lambda^2 y - by_{xxxx} = 0, \quad \text{over} \quad (-L, 0), \quad (4.2.42)$$

From Equations (4.2.40) and (4.2.41) and using the unique continuation theorem (see [62]) we get

$$u = 0 \quad \text{in} \quad (0, L). \quad (4.2.43)$$

From Equations (4.2.42), (4.2.33)-(4.2.34), and using (4.2.43) we get the following system

$$\begin{cases} \lambda^2 y - by_{xxxx} = 0, \quad \text{over} \quad (-L, 0) \\ y(0) = y_{xx}(0) = y_{xxx}(0) = 0, \\ y(-L) = y_x(-L) = 0. \end{cases} \quad (4.2.44)$$

It's easy to see that  $y = 0$  is the unique solution of (4.2.44). Hence  $U = 0$ . The proof is thus completed.  $\square$



**Lemma 4.2.11.** Assume that  $\eta = 0$ . Then, the operator  $-\mathcal{A}_1$  is not invertible and consequently  $0 \in \sigma(\mathcal{A}_1)$ .

*Proof.* Let  $F = \left(x^2 - L^2, 0, \frac{-x^4}{3L^2} - \frac{4}{3}Lx - L^2, 0, 0\right) \in \mathcal{H}_1$  and assume that there exists  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$  such that  $-\mathcal{A}_1 U = F$ . It follows that

$$v_x = 2x \text{ in } (0, L) \quad \text{and} \quad \xi^2 \omega(x, \xi) - 2x \sqrt{d(x)} |\xi|^{\frac{2\alpha-1}{2}} = 0.$$

From the above equation, we deduce that  $\omega(x, \xi) = 2|\xi|^{\frac{2\alpha-5}{2}} x \sqrt{d(x)} \notin W$ , therefore the assumption of the existence of  $U$  is false and consequently the operator  $-\mathcal{A}_1$  is not invertible. The proof is thus complete.  $\square$

**Lemma 4.2.12.** If  $(\eta > 0 \text{ and } \lambda \in \mathbb{R})$  or  $(\eta = 0 \text{ and } \lambda \in \mathbb{R}^*)$ , then  $i\lambda I - \mathcal{A}_1$  is surjective.

*Proof.* Let  $F = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}_1$ , we look for  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$  solution of

$$(i\lambda I - \mathcal{A}_1)U = F. \quad (4.2.45)$$

Equivalently, we have

$$i\lambda u - v = f_1, \quad (4.2.46)$$

$$i\lambda v - \left(au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi\right)_x = f_2, \quad (4.2.47)$$

$$i\lambda y - z = f_3, \quad (4.2.48)$$

$$i\lambda z + by_{xxxx} = f_4, \quad (4.2.49)$$

$$(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}v_x |\xi|^{\frac{2\alpha-1}{2}} = f_5(x, \xi). \quad (4.2.50)$$

Using Equations (4.2.46) and (4.2.50) and the fact that  $\eta \geq 0$  we get

$$\omega(x, \xi) = \frac{f_5(x, \xi)}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{d(x)}i\lambda u_x |\xi|^{\frac{2\alpha-1}{2}}}{i\lambda + \xi^2 + \eta} - \frac{\sqrt{d(x)}(f_1)_x |\xi|^{\frac{2\alpha-1}{2}}}{i\lambda + \xi^2 + \eta}. \quad (4.2.51)$$

Substituting  $v$  and  $z$  in (4.2.46) and (4.2.48) into Equations (4.2.47) and (4.2.49), and using (4.2.51) we get

$$\lambda^2 u + (au_x + d(x)I_4(\lambda, \eta, \alpha)u_x - g(x, \lambda, \eta, \alpha))_x = f, \quad (4.2.52)$$

$$\lambda^2 y - by_{xxxx} = F, \quad (4.2.53)$$

such that

$$\begin{cases} f = -(f_2 + i\lambda f_1) \in L^2(0, L), \\ g(x, \lambda, \eta, \alpha) = I_5(\lambda, \eta, \alpha)d(x)(f_1)_x - \sqrt{d(x)}I_6(x, \lambda, \eta, \alpha), \\ F = -(f_4 + i\lambda f_3) \in L^2(-L, 0), \end{cases}$$

and  $I_4(\lambda, \eta, \alpha)$ ,  $I_5(\lambda, \eta, \alpha)$  and  $I_6(\lambda, \eta, \alpha)$  are defined in Lemma 4.2.8.

Now, we distinguish two cases:

**Case 1:**  $\eta > 0$  and  $\lambda = 0$ , then system (4.2.52)-(4.2.53) becomes

$$\begin{aligned} (au_x - g(x, 0, \eta, \alpha))_x &= -f_2, \\ by_{xxxx} &= f_4. \end{aligned}$$

By applying Lax-Milgram theorem, and using Lemma 4.2.8 it is easy to see that the above system has a unique strong solution  $(u, y) \in V$ .

**Case 2:**  $\eta \geq 0$  and  $\lambda \in \mathbb{R}^*$ . Then system (4.2.52)-(4.2.53) becomes

$$\lambda^2 u + (au_x + d(x)I_4(\lambda, \eta, \alpha)u_x)_x = G, \quad (4.2.54)$$

$$\lambda^2 y - by_{xxxx} = F, \quad (4.2.55)$$

such that

$$G = f + g_x(x, \lambda, \eta, \alpha).$$

We first define the linear unbounded operator  $\mathcal{L} : \mathbb{H} := H_R^1(0, L) \times H_L^2(-L, 0) \longrightarrow \mathbb{H}'$  where  $\mathbb{H}'$  is the dual space of  $\mathbb{H}$  by

$$\mathcal{L}U = \begin{pmatrix} -(au_x + d(x)I_4(\lambda, \eta, \alpha)u_x)_x \\ by_{xxxx} \end{pmatrix}, \quad \forall U \in \mathbb{H}.$$

Thanks to Lax-Milgram Theorem, it is easy to see that  $\mathcal{L}$  is isomorphism. The system (4.2.54)-(4.2.55) is equivalent to

$$(\lambda^2 \mathcal{L}^{-1} - I)U = \mathcal{L}^{-1}\mathcal{F}, \quad \text{where } U = (u, y)^\top \text{ and } \mathcal{F} = (G, F)^\top. \quad (4.2.56)$$

Since the operator  $\mathcal{L}^{-1}$  is isomorphism and  $I$  is a compact operator from  $\mathbb{H}$  to  $\mathbb{H}'$ . Then,  $\mathcal{L}^{-1}$  is compact operator from  $\mathbb{H}$  to  $\mathbb{H}$ . Consequently, by Fredholm's alternative, proving the existence of  $U$  solution of (4.2.56) reduces to proving  $\ker(\lambda^2 \mathcal{L}^{-1} - I) = \{0\}$ . Indeed, if  $(\tilde{u}, \tilde{y}) \in \ker(\lambda^2 \mathcal{L}^{-1} - I)$ , then  $\lambda^2(\tilde{u}, \tilde{y}) - \mathcal{L}(\tilde{u}, \tilde{y}) = 0$ . It follows that,

$$\lambda^2 \tilde{u} + (a\tilde{u}_x + d(x)I_4(\lambda, \eta, \alpha)\tilde{u}_x)_x = 0, \quad (4.2.57)$$

$$\lambda^2 \tilde{y} - b\tilde{y}_{xxxx} = 0, \quad (4.2.58)$$

$$\tilde{u}(L) = \tilde{y}(-L) = \tilde{y}_x(-L) = \tilde{y}_{xx}(0) = 0, \quad (4.2.59)$$

$$a\tilde{u}_x(0) + b\tilde{y}_{xxx}(0) = 0, \tilde{u}(0) = \tilde{y}(0). \quad (4.2.60)$$

Multiplying (4.2.57) and (4.2.58) by  $\bar{\tilde{u}}$  and  $\bar{\tilde{y}}$  respectively, integrating over  $(0, L)$  and  $(-L, 0)$  respectively and taking the sum, then using by parts integration, the boundary conditions (4.2.59)-(4.2.60), and taking the imaginary part we get

$$d_0 \operatorname{Im}(I_4(\lambda, \alpha, \eta)) \int_{l_0}^{l_1} |\tilde{u}_x|^2 dx = 0.$$

From Lemma 4.2.8 we have  $\operatorname{Im}(I_4(\lambda, \alpha, \eta)) = \lambda I_8(\lambda, \eta, \alpha) \neq 0$ , we get  $\tilde{u}_x = 0$  in  $(l_0, l_1)$ .

Then, system (4.2.57)-(4.2.59) becomes

$$\lambda^2 \tilde{u} + a \tilde{u}_{xx} = 0 \quad \text{over } (0, L), \quad (4.2.61)$$

$$\lambda^2 \tilde{y} - b \tilde{y}_{xxxx} = 0 \quad \text{over } (-L, 0), \quad (4.2.62)$$

$$\tilde{u}_x = 0 \quad \text{over } (l_0, l_1). \quad (4.2.63)$$

It is now easy to see that if  $(\tilde{u}, \tilde{y})$  is a solution of system (4.2.61)-(4.2.63), then the vector  $\tilde{U}$  defined by  $\tilde{U} := (\tilde{u}, i\lambda \tilde{u}, \tilde{y}, i\lambda \tilde{y}, 0)$  belongs to  $D(\mathcal{A}_1)$ , and  $i\lambda \tilde{U} - \mathcal{A}_1 \tilde{U} = 0$ .

Therefore,  $\tilde{U} \in \ker(i\lambda I - \mathcal{A}_1)$ , then by using Lemma 4.2.10, we get  $\tilde{U} = 0$ . This implies that system (4.2.56) admits a unique solution due to Fredholm's alternative, hence (4.2.56) admits a unique solution in  $V$ . Thus, we define  $v := i\lambda u - f_1$ ,  $z := i\lambda y - f_3$  and

$$\omega(x, \xi) = \frac{f_5(x, \xi)}{i\lambda + \xi^2 + \eta} + \frac{\sqrt{d(x)} i\lambda u_x |\xi|^{\frac{2\alpha-1}{2}}}{i\lambda + \xi^2 + \eta} - \frac{\sqrt{d(x)} (f_1)_x |\xi|^{\frac{2\alpha-1}{2}}}{i\lambda + \xi^2 + \eta}. \quad (4.2.64)$$

Since  $F \in \mathcal{H}_1$ , it is easy to see that  $v \in H_R^1(0, L)$ ,  $z \in H_L^2(-L, 0)$ ,  $v(0) = z(0)$  and

$$\left( au_x + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_x \in L^2(0, L).$$

It is left to prove that  $\omega$  and  $|\xi|\omega \in W$  (for the both cases). From equation (4.2.64), we get

$$\int_0^L \int_{\mathbb{R}} |\omega(x, \xi)|^2 d\xi dx \leq 3 \int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx + 3d_0 \left( \int_{l_0}^{l_1} (|\lambda u_x|^2 + |(f_1)_x|^2) dx \right) I_7(\lambda, \eta, \alpha).$$

Using the fact that  $f_5 \in W$  and  $(\eta > 0$  and  $\lambda \in \mathbb{R})$  or  $(\eta = 0$  and  $\lambda \in \mathbb{R}^*)$ , we obtain

$$\int_0^L \int_{\mathbb{R}} \frac{|f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi \leq \frac{1}{\lambda^2 + \eta^2} \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < +\infty.$$

Using Lemma 4.2.8, it follows that  $\omega \in W$ . Next, using equation (4.2.64), we get

$$\int_0^L \int_{\mathbb{R}} |\xi \omega|^2 d\xi \leq 3 \int_0^L \int_{\mathbb{R}} \frac{\xi^2 |f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx + 3d_0 \int_0^L \left( |\lambda u_x|^2 + |(f_1)_x|^2 \right) I_{10}(\lambda, \eta, \alpha),$$

where  $I_{10}(\lambda, \eta, \alpha) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha+1}}{\lambda^2 + (\xi^2 + \eta)^2} d\xi < +\infty$  by using Lemma 4.2.9. Now, using the fact that  $f_5 \in W$  and

$$\max_{\xi \in \mathbb{R}} \frac{\xi^2}{\lambda^2 + (\xi^2 + \eta)^2} = \frac{\sqrt{\eta^2 + \lambda^2}}{\lambda^2 + \left( \sqrt{\eta^2 + \lambda^2} + \eta \right)^2} = C(\lambda, \eta),$$

we get

$$\int_0^L \int_{\mathbb{R}} \frac{\xi^2 |f_5(x, \xi)|^2}{\lambda^2 + (\xi^2 + \eta)^2} d\xi dx \leq \int_0^L \max_{\xi \in \mathbb{R}} \frac{\xi^2}{\lambda^2 + (\xi^2 + \eta)^2} \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi = C(\lambda, \eta) \int_0^L \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi dx < +\infty.$$

It follows that  $|\xi|\omega \in W$ . Finally, since  $\omega \in W$ , we get

$$-(\xi^2 + \eta)\omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}} = i\lambda \omega(x, \xi) - f_5(x, \xi) \in W.$$

Thus, we obtain  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_1)$  solution of  $(i\lambda I - \mathcal{A}_1)U = F$ . The proof is thus complete.  $\square$

**Proof of Theorem 4.2.7.** First, using Lemma 4.2.10, we directly deduce that  $\mathcal{A}_1$  has no pure imaginary eigenvalues. Next, using Lemmas 4.2.11, 4.2.12 and with the help of the closed graph theorem of Banach, we deduce that  $\sigma(\mathcal{A}_1) \cap i\mathbb{R} = \{\emptyset\}$  if  $\eta > 0$  and  $\sigma(\mathcal{A}_1) \cap i\mathbb{R} = \{0\}$  if  $\eta = 0$ . Thus, we get the conclusion by Applying Theorem 1.3.3 of Arendt Batty.

## 4.2.2 Polynomial Stability in the case $\eta > 0$

In this section, we study the polynomial stability of the system (4.2.6)-(4.2.8) in the case  $\eta > 0$ . For this purpose, we will use a frequency domain approach method, namely we will use Theorem 1.3.10. Our main result in this section is the following theorem.

**Theorem 4.2.13.** Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_1})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_1 > 0$  such that for every  $U_0 \in D(\mathcal{A}_1)$ , we have

$$E_1(t) \leq \frac{C_1}{t^{\frac{4}{2-\alpha}}} \|U_0\|_{D(\mathcal{A}_1)}^2, \quad t > 0, \forall U_0 \in D(\mathcal{A}_1). \quad (4.2.65)$$

According to Theorem 1.3.10, by taking  $\ell = 1 - \frac{\alpha}{2}$ , the polynomial energy decay (4.2.65) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_1), \quad (H_1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{1-\frac{\alpha}{2}}} \|(i\lambda I - \mathcal{A}_1)^{-1}\|_{\mathcal{L}(\mathcal{H}_1)} < \infty, \quad (H_2)$$

are satisfied. Since Condition (H<sub>1</sub>) is already proved in Lemma 4.2.10. We will prove condition (H<sub>2</sub>) by an argument of contradiction. For this purpose, suppose that (H<sub>2</sub>) is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))^T)\} \subset \mathbb{R}^* \times D(\mathcal{A}_1)$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_1} = \|(u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))\|_{\mathcal{H}_1} = 1, \quad (4.2.66)$$

such that

$$(\lambda_n)^{1-\frac{\alpha}{2}} (i\lambda_n I - \mathcal{A}_1) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}(\cdot, \xi))^T \rightarrow 0 \quad \text{in } \mathcal{H}_1. \quad (4.2.67)$$

For simplicity, we drop the index  $n$ . Equivalently, from (4.2.67), we have

$$i\lambda u - v = \frac{f_1}{\lambda^{1-\frac{\alpha}{2}}} \quad \text{in } H_R^1(0, L), \quad (4.2.68)$$

$$i\lambda v - (S_d)_x = \frac{f_2}{\lambda^{1-\frac{\alpha}{2}}} \quad \text{in } L^2(0, L), \quad (4.2.69)$$

$$i\lambda y - z = \frac{f_3}{\lambda^{1-\frac{\alpha}{2}}} \quad \text{in } H_L^2(-L, 0), \quad (4.2.70)$$

$$i\lambda z + by_{xxxx} = \frac{f_4}{\lambda^{1-\frac{\alpha}{2}}} \quad \text{in } L^2(-L, 0), \quad (4.2.71)$$

$$(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}v_x|\xi|^{\frac{2\alpha-1}{2}} = \frac{f_5(x, \xi)}{\lambda^{1-\frac{\alpha}{2}}} \quad \text{in } W, \quad (4.2.72)$$

where  $S_d = au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi$ .

Here we will check the condition (H<sub>2</sub>) by finding a contradiction with (4.2.66) by showing  $\|U\|_{\mathcal{H}_1} = o(1)$ . For clarity, we divide the proof into several Lemmas.

**Lemma 4.2.14.** *Let  $\alpha \in (0, 1)$ ,  $\eta > 0$  and  $\lambda \in \mathbb{R}$ , then*

$$\begin{cases} I_{11}(\lambda, \eta, \alpha) = \int_{\mathbb{R}} \frac{|\xi|^{\alpha+\frac{1}{2}}}{(|\lambda| + \xi^2 + \eta)^2} d\xi = c_1 (|\lambda| + \eta)^{\frac{\alpha}{2} - \frac{5}{4}}, \\ I_{12}(\lambda, \eta) = \left( \int_{\mathbb{R}} \frac{1}{(|\lambda| + \xi^2 + \eta)^2} d\xi \right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}} \frac{1}{(|\lambda| + \eta)^{\frac{3}{4}}}, \\ I_{13}(\lambda, \eta) = \left( \int_{\mathbb{R}} \frac{\xi^2}{(|\lambda| + \xi^2 + \eta)^4} d\xi \right)^{\frac{1}{2}} = \frac{\sqrt{\pi}}{4} \frac{1}{(|\lambda| + \eta)^{\frac{5}{4}}} \end{cases}$$

where  $c_1 = \int_1^\infty \frac{(y-1)^{\frac{\alpha}{2} - \frac{1}{4}}}{y^2} dy$ .

*Proof.*  $I_{11}$  can be written as

$$I_{11}(\lambda, \eta, \alpha) = \frac{2}{(\lambda + \eta)^2} \int_0^\infty \frac{\xi^{\alpha+\frac{1}{2}}}{\left(1 + \frac{\xi^2}{|\lambda| + \eta}\right)^2} d\xi. \quad (4.2.73)$$

Thus, equation (4.2.73) may be simplified by defining a new variable  $y = 1 + \frac{\xi^2}{\lambda + \eta}$ . Substituting  $\xi$  by  $(y-1)^{\frac{1}{2}} (\lambda + \eta)^{\frac{1}{2}}$  in equation (4.2.73), we get

$$I_{11}(\lambda, \eta, \alpha) = (\lambda + \eta)^{\frac{\alpha}{2} - \frac{5}{4}} \int_1^\infty \frac{(y-1)^{\frac{\alpha}{2} - \frac{1}{4}}}{y^2} dy.$$

Using the fact that  $\alpha \in ]0, 1[$ , it is easy to see that  $y^{-2} (y-1)^{\frac{\alpha}{2} - \frac{1}{4}} \in L^1(1, +\infty)$ . Hence, the last integral in the above equation is well defined. Now,  $I_{12}(\lambda, \eta)$  can be written as

$$(I_{12}(\lambda, \eta))^2 = \frac{2}{(\lambda + \eta)^2} \int_0^\infty \frac{1}{\left(1 + \left(\frac{\xi}{\sqrt{\lambda + \eta}}\right)^2\right)^2} d\xi = \frac{2}{(\lambda + \eta)^{\frac{3}{2}}} \int_0^\infty \frac{1}{(1 + s^2)^2} ds = \frac{2}{(\lambda + \eta)^{\frac{3}{2}}} \times \frac{\pi}{4} = \frac{\pi}{2(\lambda + \eta)^{\frac{3}{2}}},$$

Therefore,  $I_{12}(\lambda, \eta) = \sqrt{\frac{\pi}{2}} \frac{1}{(\lambda + \eta)^{\frac{3}{4}}}$ . Finally,  $I_{13}(\lambda, \eta)$  can be written as

$$(I_{13}(\lambda, \eta))^2 = \frac{2}{(\lambda + \eta)^4} \int_0^\infty \frac{\xi^2}{\left(1 + \left(\frac{\xi}{\sqrt{\lambda + \eta}}\right)^2\right)^4} d\xi = \frac{2}{(\lambda + \eta)^{\frac{5}{2}}} \int_0^\infty \frac{s^2}{(1 + s^2)^4} ds = \frac{2}{(\lambda + \eta)^{\frac{5}{2}}} \times \frac{\pi}{32}.$$

Then  $I_{13}(\lambda, \eta) = \frac{\sqrt{\pi}}{4} \frac{1}{(\lambda + \eta)^{\frac{5}{4}}}$ . The proof has been completed.  $\square$

**Lemma 4.2.15.** Assume that  $\eta > 0$ . Then, the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_1)$  of system (4.2.68)–(4.2.72) satisfies the following asymptotic behavior estimations

$$\int_0^L \int_{\mathbb{R}} (|\xi|^2 + \eta) |\omega(x, \xi)|^2 d\xi dx = o\left(\lambda^{-1+\frac{\alpha}{2}}\right), \int_{l_0}^{l_1} |v_x|^2 dx = o\left(\lambda^{-\frac{\alpha}{2}}\right) \text{ and } \int_{l_0}^{l_1} |u_x|^2 dx = o\left(\lambda^{-2-\frac{\alpha}{2}}\right). \quad (4.2.74)$$

*Proof.* For clarity, we divide the proof into several steps.

**Step 1.** Taking the inner product of  $F$  with  $U$  in  $\mathcal{H}_1$ , then using (4.2.66) and the fact that  $U$  is uniformly bounded in  $\mathcal{H}_1$ , we get

$$\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx = -\operatorname{Re}(\langle \mathcal{A}_1 U, U \rangle_{\mathcal{H}_1}) = \operatorname{Re}(\langle (i\lambda I - \mathcal{A}_1)U, U \rangle_{\mathcal{H}_1}) = o\left(\lambda^{-1+\frac{\alpha}{2}}\right).$$

**Step 2.** Our aim here is to prove the second estimation in (4.2.74).

From (4.2.72), we get

$$\sqrt{d(x)} |\xi|^{\frac{2\alpha-1}{2}} |v_x| \leq (|\lambda| + \xi^2 + \eta) |\omega(x, \xi)| + |\lambda|^{-1+\frac{\alpha}{2}} |f_5(x, \xi)|.$$

Multiplying the above inequality by  $(|\lambda| + \xi^2 + \eta)^{-2} |\xi|$ , integrate over  $\mathbb{R}$ , we get

$$\sqrt{d(x)} I_{11}(\lambda, \eta, \alpha) |v_x| \leq I_{12}(\lambda, \eta) \left( \int_{\mathbb{R}} |\xi \omega(x, \xi)|^2 d\xi \right)^{\frac{1}{2}} + |\lambda|^{-1+\frac{\alpha}{2}} I_{13}(\lambda, \eta) \left( \int_{\mathbb{R}} |f_5(x, \xi)|^2 d\xi \right)^{\frac{1}{2}}, \quad (4.2.75)$$

where  $I_{11}(\lambda, \eta, \alpha)$ ,  $I_{12}(\lambda, \eta)$  and  $I_{13}(\lambda, \eta)$  are defined in Lemma 4.2.14. Using Young's inequality and the definition of the function  $d(x)$  in (4.2.75), we get

$$\int_{l_0}^{l_1} |v_x|^2 dx \leq 2 \frac{I_{12}^2}{I_{11}^2} \frac{o(1)}{|\lambda|^{1-\frac{\alpha}{2}}} + 2 \frac{I_{13}^2}{I_{11}^2} \frac{o(1)}{|\lambda|^{2-\alpha}}.$$

It follows from Lemma 4.2.14 that

$$\int_{l_0}^{l_1} |v_x|^2 dx \leq \frac{1}{c_1 (|\lambda| + \eta)^{\alpha-1}} \frac{o(1)}{|\lambda|^{1-\frac{\alpha}{2}}} + \frac{\sqrt{\pi}}{4} \frac{1}{c_1 (|\lambda| + \eta)^{\alpha}} \frac{o(1)}{|\lambda|^{2-\alpha}}. \quad (4.2.76)$$

Since  $\alpha \in (0, 1)$ , we have  $\min(\frac{\alpha}{2}, 2) = \frac{\alpha}{2}$ , hence from the above equation, we get the second desired estimation in (4.2.74).

**Step 3.** From Equation (4.2.68) we have

$$i\lambda u_x = v_x - \lambda^{-1+\frac{\alpha}{2}} (f_1)_x.$$

It follows that

$$\|\lambda u_x\|_{L^2(l_0, l_1)} \leq \|v_x\|_{L^2(l_0, l_1)} + |\lambda|^{-1+\frac{\alpha}{2}} \|(f_1)_x\|_{L^2(l_0, l_1)} \leq \frac{o(1)}{\lambda^{\frac{\alpha}{4}}} + \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}.$$

Since  $\alpha \in (0, 1)$ , we have  $\min(1 + \frac{\alpha}{4}, 2 - \frac{\alpha}{2}) = 1 + \frac{\alpha}{4}$ , hence from the above equation, we get

$$\int_{l_0}^{l_1} |u_x|^2 dx = \frac{o(1)}{\lambda^{2+\frac{\alpha}{2}}}.$$

The proof is thus completed.  $\square$

**Lemma 4.2.16.** *Let  $0 < \alpha < 1$  and  $\eta > 0$ . Then, the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_1)$  of system (4.2.68)-(4.2.72) satisfies the following asymptotic behavior*

$$\int_{l_0}^{l_1} |S_d|^2 dx = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}. \quad (4.2.77)$$

*Proof.* Using the fact that  $|P + Q|^2 \leq 2P^2 + 2Q^2$ , we obtain

$$\begin{aligned} \int_{l_0}^{l_1} |S_d|^2 dx &= \int_{l_0}^{l_1} \left| au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right|^2 dx \\ &\leq 2a^2 \int_{l_0}^{l_1} |u_x|^2 dx + 2d_0\kappa(\alpha)^2 \int_{l_0}^{l_1} \left( \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} \sqrt{\xi^2 + \eta}}{\sqrt{\xi^2 + \eta}} \omega(x, \xi) d\xi \right)^2 dx \\ &\leq 2a^2 \int_{l_0}^{l_1} |u_x|^2 dx + c_2 \int_{l_0}^{l_1} \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx \end{aligned}$$

where  $c_2 = d_0\kappa(\alpha)^2 \mathcal{I}_{14}(\alpha, \eta)$  and  $\mathcal{I}_{14}(\alpha, \eta) = \int_{\mathbb{R}} \frac{|\xi|^{2\alpha-1}}{|\xi|^2 + \eta} d\xi$ . We have

$$\frac{|\xi|^{2\alpha-1}}{|\xi|^2 + \eta} \underset{0}{\sim} \frac{|\xi|^{2\alpha-1}}{\eta} \quad \text{and} \quad \frac{|\xi|^{2\alpha-1}}{|\xi|^2 + \eta} \underset{+\infty}{\sim} \frac{1}{|\xi|^{3-2\alpha}}.$$

Since  $0 < \alpha < 1$  and  $\eta > 0$ , then  $\mathcal{I}_{14}(\alpha, \eta)$  is well defined. Using the first and the third estimations in (4.2.74), we get our desired result.  $\square$

**Lemma 4.2.17.** *Assume that  $\eta > 0$ . Let  $g \in C^1([l_0, l_1])$  such that*

$$g(l_1) = -g(l_0) = 1, \quad \max_{x \in (l_0, l_1)} |g(x)| = m_g \quad \text{and} \quad \max_{x \in (l_0, l_1)} |g'(x)| = m_{g'}$$

where  $m_g$  and  $m_{g'}$  are strictly positive constant numbers. Then, the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_1)$  of system (4.2.68)-(4.2.72) satisfies the following asymptotic behavior

$$|v(l_1)|^2 + |v(l_0)|^2 \leq \left( \frac{\lambda^{1-\frac{\alpha}{2}}}{2} + 2m_{g'} \right) \int_{\ell_0}^{l_1} |v|^2 dx + \frac{o(1)}{\lambda} \quad (4.2.78)$$

and

$$|S_d(l_1)|^2 + |S_d(l_0)|^2 \leq \frac{\lambda^{1+\frac{\alpha}{2}}}{2} \int_{l_0}^{l_1} |v|^2 dx + o(1). \quad (4.2.79)$$

*Proof.* First we will prove Equation (4.2.78). From Equation (4.2.68), we have

$$v_x = i\lambda u_x - \lambda^{-1+\frac{\alpha}{2}} (f_1)_x. \quad (4.2.80)$$

Multiply Equation (4.2.80) by  $2g\bar{v}$  and integrate over  $(l_0, l_1)$ , we get

$$|v(l_1)|^2 + |v(l_0)|^2 = \int_{l_0}^{l_1} g' |v|^2 dx + \operatorname{Re} \left( 2i\lambda \int_{l_0}^{l_1} u_x g \bar{v} dx \right) - \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} (f_1)_x g \bar{v} dx \right). \quad (4.2.81)$$

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Then,

$$|v(l_1)|^2 + |v(l_0)|^2 \leq m_{g'} \int_{l_0}^{l_1} |v|^2 dx + 2\lambda m_g \int_{l_0}^{l_1} |u_x| |\bar{v}| dx + 2m_g \lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} |(f_1)_x| |\bar{v}| dx. \quad (4.2.82)$$

Using Young's inequality we have

$$\begin{cases} 2\lambda m_g |u_x| |\bar{v}| \leq \frac{\lambda^{1-\frac{\alpha}{2}}}{2} |v|^2 + 2\lambda^{1+\frac{\alpha}{2}} m_g^2 |u_x|^2, \\ 2m_g \lambda^{-1+\frac{\alpha}{2}} |(f_1)_x| |\bar{v}| \leq m_{g'} |v|^2 + \frac{m_g^2}{m_{g'}} \lambda^{-2+\alpha} |(f_1)_x|^2. \end{cases} \quad (4.2.83)$$

Using Equation (4.2.83), then Equation (4.2.82) becomes

$$\begin{aligned} |v(l_1)|^2 + |v(l_0)|^2 &\leq \left( \frac{\lambda^{1-\frac{\alpha}{2}}}{2} + 2m_{g'} \right) \int_{l_0}^{l_1} |v|^2 dx + 2\lambda^{1+\frac{\alpha}{2}} m_g^2 \int_{l_0}^{l_1} |u_x|^2 dx \\ &\quad + \frac{m_g^2}{m_{g'}} \lambda^{-2+\alpha} \int_{l_0}^{l_1} |(f_1)_x|^2 dx. \end{aligned} \quad (4.2.84)$$

Using the third estimation in Equation (4.2.74) and the fact that  $\|(f_1)_x\|_{L^2(l_0, l_1)} = o(1)$ , we obtain

$$|v(l_1)|^2 + |v(l_0)|^2 \leq \left( \frac{\lambda^{1-\frac{\alpha}{2}}}{2} + 2m_{g'} \right) \int_{l_0}^{l_1} |v|^2 dx + \frac{o(1)}{\lambda} + \frac{o(1)}{\lambda^{2-\alpha}}. \quad (4.2.85)$$

Since  $\alpha \in (0, 1)$ , hence

$$|v(l_1)|^2 + |v(l_0)|^2 \leq \left( \frac{\lambda^{1-\frac{\alpha}{2}}}{2} + 2m_{g'} \right) \int_{l_0}^{l_1} |v|^2 dx + \frac{o(1)}{\lambda}. \quad (4.2.86)$$

Now, we will prove (4.2.79). For this aim, multiply Equation (4.2.69) by  $-2g\bar{S}_d$  and integrate over  $(l_0, l_1)$ , we get

$$|S_d(l_1)|^2 + |S_d(l_0)|^2 = \int_{l_0}^{l_1} g' |S_d|^2 dx + \operatorname{Re} \left( 2i\lambda \int_{l_0}^{l_1} v g \bar{S}_d dx \right) - \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} f_2 g \bar{S}_d dx \right). \quad (4.2.87)$$

Then,

$$|S_d(l_1)|^2 + |S_d(l_0)|^2 \leq m_{g'} \int_{l_0}^{l_1} |S_d|^2 dx + 2\lambda m_g \int_{l_0}^{l_1} |v| |S_d| dx + 2m_g \lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} |f_2| |S_d| dx. \quad (4.2.88)$$

Using Young's inequality and Equation (4.2.77) we obtain

$$2\lambda m_g |v| |\bar{S}_d| \leq \frac{\lambda^{1+\frac{\alpha}{2}}}{2} |v|^2 + 2m_g^2 \lambda^{1-\frac{\alpha}{2}} |S_d|^2 \leq \frac{\lambda^{1+\frac{\alpha}{2}}}{2} |v|^2 + o(1). \quad (4.2.89)$$

Using Cauchy-Schwarz inequality, Equation (4.2.77) and the fact that  $\|f_2\|_{L^2(l_0, l_1)} = o(1)$ , we obtain

$$2m_g \lambda^{-1+\frac{\alpha}{2}} \int_{l_0}^{l_1} |f_2| |S_d| dx \leq \frac{1}{\lambda^{1-\frac{\alpha}{2}}} \|f_2\|_{L^2(l_0, l_1)} \|S_d\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda^{\frac{3}{2}-\frac{3\alpha}{4}}}. \quad (4.2.90)$$



Using Equations (4.2.77), (4.2.89) and (4.2.90) in (4.2.88), and using the fact that  $\alpha \in (0, 1)$  we get

$$|S_d(l_1)|^2 + |S_d(l_0)|^2 \leq \frac{\lambda^{1+\frac{\alpha}{2}}}{2} \int_{l_0}^{l_1} |v|^2 dx + o(1). \quad (4.2.91)$$

□

**Lemma 4.2.18.** *Let  $0 < \alpha < 1$  and  $\eta > 0$ . Then, the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_1)$  of system (4.2.68)–(4.2.72) satisfies the following asymptotic behavior*

$$\int_{l_0}^{l_1} |v|^2 dx = \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}. \quad (4.2.92)$$

*Proof.* Multiply Equation (4.2.69) by  $-i\lambda^{-1}\bar{v}$  and integrate over  $(l_0, l_1)$ , we get

$$\int_{l_0}^{l_1} |v|^2 dx = \operatorname{Re} \left( i\lambda^{-1} \int_{l_0}^{l_1} S_d \bar{v}_x dx \right) - [i\lambda^{-1} S_d \bar{v}]_{l_0}^{l_1} + \operatorname{Re} \left( i\lambda^{-2+\frac{\alpha}{2}} \int_{l_0}^{l_1} f_2 \bar{v} dx \right). \quad (4.2.93)$$

Estimation of the term  $\operatorname{Re} \left( i\lambda^{-1} \int_{l_0}^{l_1} S_d \bar{v}_x dx \right)$ . Using Cauchy-Schwarz inequality, the second estimation in (4.2.74) and the estimation in (4.2.77), we get

$$\left| \operatorname{Re} \left( i\lambda^{-1} \int_{l_0}^{l_1} S_d \bar{v}_x dx \right) \right| \leq \frac{1}{\lambda} \left( \int_{l_0}^{l_1} |S_d|^2 dx \right)^{\frac{1}{2}} \left( \int_{l_0}^{l_1} |v_x|^2 dx \right)^{\frac{1}{2}} = \frac{o(1)}{\lambda^{\frac{3}{2}}}. \quad (4.2.94)$$

Estimation of the term  $\operatorname{Re} \left( i\lambda^{-2+\frac{\alpha}{2}} \int_{l_0}^{l_1} f_2 \bar{v} dx \right)$ . Using Cauchy-Schwarz inequality,  $v$  is uniformly bounded in  $L^2(l_0, l_1)$  and  $\|f_2\|_{L^2(l_0, l_1)} = o(1)$ , we get

$$\left| \operatorname{Re} \left( i\lambda^{-2+\frac{\alpha}{2}} \int_{l_0}^{l_1} f_2 \bar{v} dx \right) \right| \leq \lambda^{-2+\frac{\alpha}{2}} \left( \int_{l_0}^{l_1} |f_2|^2 dx \right)^{\frac{1}{2}} \left( \int_{l_0}^{l_1} |v|^2 dx \right)^{\frac{1}{2}} = \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}}. \quad (4.2.95)$$

Inserting Equations (4.2.94) and (4.2.95) in (4.2.93), using the fact that  $\min(\frac{3}{2}, 2 - \frac{\alpha}{2}) = \frac{3}{2}$ , and using Young's inequality on the second term of (4.2.93) we get

$$\int_{l_0}^{l_1} |v|^2 dx \leq \frac{\lambda^{-1+\frac{\alpha}{2}}}{2} [|v(l_1)|^2 + |v(l_0)|^2] + \frac{\lambda^{-1-\frac{\alpha}{2}}}{2} [|S_d(l_1)|^2 + |S_d(l_0)|^2] + \frac{o(1)}{\lambda^{\frac{3}{2}}}. \quad (4.2.96)$$

Now, inserting (4.2.78) and (4.2.79) in (4.2.96), we get

$$\int_{l_0}^{l_1} |v|^2 dx \leq \left( \frac{1}{2} + m_{g'} \lambda^{-1+\frac{\alpha}{2}} \right) \int_{l_0}^{l_1} |v|^2 dx + \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}} + \frac{o(1)}{\lambda^{2-\frac{\alpha}{2}}}. \quad (4.2.97)$$

Since  $\alpha \in (0, 1)$  then  $\min(1 + \frac{\alpha}{2}, 2 - \frac{\alpha}{2}) = 1 + \frac{\alpha}{2}$ . Then Equation (4.2.97) becomes

$$\left( \frac{1}{2} - m_{g'} \lambda^{-1+\frac{\alpha}{2}} \right) \int_{l_0}^{l_1} |v|^2 dx \leq \frac{o(1)}{\lambda^{1+\frac{\alpha}{2}}}. \quad (4.2.98)$$

Using the fact that  $|\lambda| \rightarrow +\infty$ , we can take  $\lambda \geq 4^{\frac{1}{2-\alpha}} m_{g'}^{\frac{1}{2-\alpha}}$ , and we get our desired result. The proof is thus complete. □

**Lemma 4.2.19.** Assume that  $\eta > 0$ . Let  $h \in C^1([0, L])$  and  $\varphi \in C^2([-L, 0])$ , then the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_1)$  of system (4.2.68)-(4.2.72) satisfies the following estimation

$$\begin{aligned} & \int_0^L h'(|v|^2 + a^{-1}|S_d|^2) dx + \int_{-L}^0 \varphi'(|z|^2 + 3b|y_{xx}|^2) dx + 2b \int_{-L}^0 y_{xx} \varphi'' \bar{y}_x dx \\ & + b\varphi(-L)|y_{xx}(-L)|^2 + h(0)|v(0)|^2 - ah(L)|u_x(L)|^2 + ah(0)|u_x(0)|^2 \\ & + \operatorname{Re}(2by_{xxx}(0)\varphi(0)\bar{y}_x(0)) - \varphi(0)|z(0)|^2 = o(1). \end{aligned} \quad (4.2.99)$$

*Proof.* The proof is divided into several steps.

**Step 1.** Multiplying Equation (4.2.69) by  $2a^{-1}h\bar{S}_d$  and integrating over  $(0, L)$ , we get

$$\begin{aligned} & \operatorname{Re}\left(2a^{-1}i\lambda \int_0^L v h \bar{S}_d dx\right) + a^{-1} \int_0^L h'|S|^2 dx - a^{-1}h(L)|S_d(L)|^2 \\ & + a^{-1}h(0)|S_d(0)|^2 = \operatorname{Re}\left(2a^{-1}\lambda^{-1+\frac{\alpha}{2}} \int_0^L f_2 h \bar{S}_d dx\right). \end{aligned} \quad (4.2.100)$$

From Equation (4.2.68) we have

$$i\lambda \bar{u}_x = -\bar{v}_x - \lambda^{-1+\frac{\alpha}{2}}(\bar{f}_1)_x.$$

Then

$$i\lambda a^{-1}\bar{S}_d = -\bar{v}_x - \lambda^{-1+\frac{\alpha}{2}}(\bar{f}_1)_x + i\lambda a^{-1}\sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi. \quad (4.2.101)$$

Then, the first term of (4.2.100) becomes

$$\begin{aligned} \operatorname{Re}\left(2a^{-1}i\lambda \int_0^L v h \bar{S}_d dx\right) &= \int_0^L h'|v|^2 dx - h(L)|v(L)|^2 + h(0)|v(0)|^2 - \operatorname{Re}\left(2\lambda^{-1+\frac{\alpha}{2}} \int_0^L v h (\bar{f}_1)_x dx\right) \\ &+ \operatorname{Re}\left(2i\lambda a^{-1}\kappa(\alpha) \int_0^L h v \sqrt{d(x)} \left(\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi\right) dx\right). \end{aligned} \quad (4.2.102)$$

Inserting Equation (4.2.102) into (4.2.100), and using the fact that  $v$  and  $S_d$  are uniformly bounded in  $L^2(0, L)$  and  $\|f_2\|_{L^2(0, L)} = o(1)$  and  $\|f_1\|_{H_L^1(0, L)} = o(1)$ , we obtain

$$\begin{aligned} & \int_0^L h'(|v|^2 + a^{-1}|S_d|^2) dx + h(0)|v(0)|^2 - h(L)|u_x(L)|^2 + h(0)a|u_x(0)|^2 \\ & + \operatorname{Re}\left(2i\lambda a^{-1}\kappa(\alpha) \int_0^L h v \sqrt{d(x)} \left(\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi\right) dx\right) = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}. \end{aligned} \quad (4.2.103)$$

Estimation of the term  $\operatorname{Re}\left(2i\lambda a^{-1}\kappa(\alpha) \int_0^L h v \sqrt{d(x)} \left(\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi\right) dx\right)$ . Using the definition of  $d(x)$  and Cauchy-Schwarz inequality, the fact that  $0 < \alpha < 1$ ,  $\eta > 0$ , and using the first estimation in (4.2.74) and Equation (4.2.92), we obtain

$$\left|\operatorname{Re}\left(2i\lambda a^{-1}\kappa(\alpha) \int_0^L h v \sqrt{d(x)} \left(\int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \bar{\omega}(x, \xi) d\xi\right) dx\right)\right| = o(1). \quad (4.2.104)$$

Inserting Equation (4.2.104) in Equation (4.2.103), and using the fact that  $\alpha \in (0, 1)$  we obtain

$$\int_0^L h'(|v|^2 + a^{-1}|S_d|^2) dx + h(0)|v(0)|^2 - ah(L)|u_x(L)|^2 + ah(0)|u_x(0)|^2 = o(1). \quad (4.2.105)$$

**Step 2.** Multiplying Equation (4.2.71) by  $2\varphi\bar{y}_x$  and integrating over  $(-L, 0)$ , we get

$$\begin{aligned} & \operatorname{Re} \left( 2i\lambda \int_{-L}^0 z\varphi\bar{y}_x dx \right) - 2b \int_{-L}^0 y_{xxx}\varphi'\bar{y}_x dx - 2b \int_{-L}^0 y_{xxx}\varphi\bar{y}_{xx} dx \\ & + \operatorname{Re} \left( [2by_{xxx}\varphi'\bar{y}_x]_{-L}^0 \right) = \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 f_4\varphi\bar{y}_x dx \right). \end{aligned} \quad (4.2.106)$$

Integrating by parts the second and third terms of the above equation we get

$$\begin{aligned} & \operatorname{Re} \left( 2i\lambda \int_{-L}^0 z\varphi\bar{y}_x dx \right) + 2b \int_{-L}^0 \varphi'|y_{xx}|^2 dx + 2b \int_{-L}^0 y_{xx}\varphi''\bar{y}_x dx \\ & - [2by_{xx}\varphi'\bar{y}_x]_{-L}^0 + b \int_{-L}^0 \varphi'|y_{xx}|^2 dx - [b\varphi|y_{xx}|^2]_{-L}^0 \\ & + \operatorname{Re} (2by_{xxx}(0)\varphi(0)\bar{y}_x(0)) = \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 f_4\varphi\bar{y}_x dx \right). \end{aligned} \quad (4.2.107)$$

From Equation (4.2.70), we have

$$i\lambda\bar{y}_x = -\bar{z}_x - \lambda^{-1+\frac{\alpha}{2}}(\bar{f}_3)_x. \quad (4.2.108)$$

By inserting Equation (4.2.108) into the first term of (4.2.107), we get

$$\begin{aligned} \operatorname{Re} \left( 2i\lambda \int_{-L}^0 \varphi z\bar{y}_x dx \right) &= \operatorname{Re} \left( -2 \int_{-L}^0 \varphi z\bar{z}_x dx - 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 (\bar{f}_3)_x \varphi z dx \right) \\ &= \int_{-L}^0 \varphi'|z|^2 dx - \varphi(0)|z(0)|^2 - \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 (\bar{f}_3)_x \varphi z dx \right). \end{aligned} \quad (4.2.109)$$

Inserting Equation (4.2.109) into (4.2.107), and using the boundary conditions we get

$$\begin{aligned} & \int_{-L}^0 \varphi'(|z|^2 + 3b|y_{xx}|^2) dx + 2b \int_{-L}^0 \varphi'' y_{xx}\bar{y}_x dx + b\varphi(-L)|y_{xx}(-L)|^2 \\ & + \operatorname{Re} (2b\varphi(0)y_{xxx}(0)\bar{y}_x(0)) - \varphi(0)|z(0)|^2 = \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 \varphi f_4\bar{y}_x dx \right) \\ & + \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 \varphi(\bar{f}_3)_x z dx \right). \end{aligned} \quad (4.2.110)$$

Estimation of the term  $\operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 \varphi f_4\bar{y}_x dx \right)$ . Using Poincaré inequality, Cauchy-Schwarz inequality, the definition of  $\varphi$ , the fact that  $y_{xx}$  is uniformly bounded in  $L^2(-L, 0)$ , and that  $\|f_4\|_{L^2(-L, 0)} = o(1)$ , we get

$$\begin{aligned} \left| \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 \varphi f_4\bar{y}_x dx \right) \right| &\leq \lambda^{-1+\frac{\alpha}{2}} \|y_x\|_{L^2(0, L)} \|f_4\|_{L^2(0, L)} \\ &\leq \lambda^{-1+\frac{\alpha}{2}} c_p \|y_{xx}\|_{L^2(0, L)} \|f_4\|_{L^2(0, L)} = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}. \end{aligned} \quad (4.2.111)$$

Estimation of the term  $\operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 \varphi(\overline{f_3})_x z dx \right)$ . Using Cauchy-Schwarz inequality, the definition of  $\varphi$ , the fact that  $z$  is bounded in  $L^2(-L, 0)$ , and that  $\|(f_3)_x\|_{L^2(-L, 0)} = o(1)$  we get

$$\left| \operatorname{Re} \left( 2\lambda^{-1+\frac{\alpha}{2}} \int_{-L}^0 \varphi(\overline{f_3})_x z dx \right) \right| \leq \lambda^{-1+\frac{\alpha}{2}} \left( \int_0^L |z|^2 dx \right)^{\frac{1}{2}} \left( \int_0^L |(f_3)_x|^2 dx \right)^{\frac{1}{2}} = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}. \quad (4.2.112)$$

Inserting (4.2.111) and (4.2.112) into (4.2.110), we get

$$\begin{aligned} & \int_{-L}^0 \varphi' \left( |z|^2 + 3b|y_{xx}|^2 \right) dx + 2b \int_{-L}^0 \varphi'' y_{xx} \overline{y}_x dx + b\varphi(-L) |y_{xx}(-L)|^2 \\ & + \operatorname{Re} \left( 2b\varphi(0) y_{xxx}(0) \overline{y}_x(0) \right) - \varphi(0) |z(0)|^2 = \frac{o(1)}{\lambda^{1-\frac{\alpha}{2}}}. \end{aligned} \quad (4.2.113)$$

Now, summing Equations (4.2.105) and (4.2.113), we get our desired result.  $\square$

**Lemma 4.2.20.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_1)$  of system (4.2.68)-(4.2.72) satisfies the following estimation

$$\|U\|_{\mathcal{H}_1} = o(1). \quad (4.2.114)$$

*Proof.* The proof of this Lemma is divided into several steps.

**Step 1.** In this step we will prove that  $\|v\|_{L^2(0, L)} = o(1)$  and  $\|u_x\|_{L^2(0, L)} = o(1)$ .

Taking  $h(x) = x\theta_1(x) + (x-L)\theta_2(x)$  and  $\varphi(x) = 0$  in Equation (4.2.99), where  $\theta_1, \theta_2 \in C^1([0, L])$  are defined as follows

$$\theta_1(x) = \begin{cases} 1 & \text{if } x \in [0, l_0], \\ 0 & \text{if } x \in [l_1, L], \\ 0 \leq \theta_1 \leq 1 & \text{elsewhere,} \end{cases} \quad (4.2.115)$$

and

$$\theta_2(x) = \begin{cases} 1 & \text{if } x \in [l_1, L], \\ 0 & \text{if } x \in [0, l_0], \\ 0 \leq \theta_2 \leq 1 & \text{elsewhere,} \end{cases} \quad (4.2.116)$$

we get

$$\int_0^L (\theta_1 + \theta_2) (|v|^2 + a^{-1}|S_d|^2) dx + \int_0^L (x\theta_1' + (x-L)\theta_2') (|v|^2 + a^{-1}|S|^2) dx = o(1). \quad (4.2.117)$$

Using Equations (4.2.77), (4.2.92) and the definition of  $\theta_1$  and  $\theta_2$ , we get

$$\int_0^L (\theta_1 + \theta_2) (|v|^2 + a^{-1}|S_d|^2) dx = o(1). \quad (4.2.118)$$

Hence, we deduce that

$$\int_0^{l_0} |v|^2 dx = o(1) \quad \text{and} \quad a \int_0^{l_0} |u_x|^2 dx = o(1) \quad (4.2.119)$$

and

$$\int_{l_1}^L |v|^2 dx = o(1) \quad \text{and} \quad a \int_{l_1}^L |u_x|^2 dx = o(1). \quad (4.2.120)$$

Using (4.2.119), (4.2.120), (4.2.77) and (4.2.92), we get the desired result of Step 1.

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**Step 2.** Taking  $h(x) = x - L$  and  $\varphi(x) = 0$  in Equation (4.2.99), we get

$$\int_0^L (|v|^2 + a^{-1}|S|^2) dx - L|v(0)|^2 - aL|u_x(0)|^2 = o(1). \quad (4.2.121)$$

By using Step 1, we get

$$|v(0)|^2 = o(1) \quad \text{and} \quad |u_x(0)|^2 = o(1). \quad (4.2.122)$$

**Step 3.** The aim of this step is to prove that  $\|z\|_{L^2(-L,0)} = o(1)$  and  $\|y_{xx}\|_{L^2(-L,0)} = o(1)$ .

Taking  $h(x) = 0$  and  $\varphi(x) = x + L$  in Equation (4.2.99), we get

$$\int_{-L}^0 (|z|^2 + 3b|y_{xx}|^2) dx + \operatorname{Re}(2Lby_{xxx}(0)\bar{y}_x(0)) - L|z(0)|^2 = o(1). \quad (4.2.123)$$

Using (4.2.122) and the transmission conditions, we obtain

$$b|y_{xxx}(0)| = a|u_x(0)| = o(1) \quad \text{and} \quad |z(0)|^2 = |v(0)|^2 = o(1). \quad (4.2.124)$$

Inserting Equation (4.2.124) in Equation (4.2.123) and using the fact that  $|y_x(0)| \leq \sqrt{L}\|y_{xx}\|_{L^2(-L,0)} = O(1)$  we get

$$\int_{-L}^0 |z|^2 dx = o(1) \quad \text{and} \quad b \int_{-L}^0 |y_{xx}|^2 dx = o(1). \quad (4.2.125)$$

Finally, using Equations Step 1. and Equation (4.2.125), we get that  $\|U\|_{\mathcal{H}_1} = o(1)$ .  $\square$

**Proof of Theorem 4.2.13.** From Lemma 4.2.20 we get that  $\|U\|_{\mathcal{H}_1} = o(1)$ , which contradicts (4.2.66). Consequently, condition (H<sub>2</sub>) holds. This implies, from Theorem 1.3.10, the energy decay estimation (4.2.65). The proof is thus complete.

### 4.3 W-W<sub>FKV</sub> Model

In this section, we consider the (W-W<sub>FKV</sub>) model, where we study the stability of a system of two wave equations coupled through boundary connection with a localized fractional Kelvin-Voigt damping acting on one equation only.

By taking the input  $V(x, t) = \sqrt{d(x)}u_{xt}(x, t)$  in Theorem 4.2.1, we get that the output  $O$  is given by

$$O(x, t) = \sqrt{d(x)}I^{1-\alpha,\eta}u_{xt}(x, t) = \frac{\sqrt{d(x)}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \partial_s u_x(x, s) ds = \sqrt{d(x)} \partial_t^{\alpha,\eta} u_x(x, t).$$

Then system (W-W<sub>FKV</sub>) can be recast into the following augmented model

$$\begin{cases} u_{tt} - \left( au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_x = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ y_{tt} - by_{xx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)}u_{xt}(x, t)|\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{cases} \quad (4.3.1)$$

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with the following transmission and boundary conditions

$$\begin{cases} u(L, t) = y(-L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) = by_x(0, t), & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{cases} \quad (4.3.2)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & \omega(x, \xi, 0) &= 0, & x &\in (0, L), \xi \in \mathbb{R}, \\ y(x, 0) &= y_0(x), & y_t(x, 0) &= y_1(x), & & & x &\in (-L, 0). \end{aligned} \quad (4.3.3)$$

The energy of the system (4.3.1)-(4.3.3) is given by

$$E_2(t) = \frac{1}{2} \int_0^L (|u_t|^2 + a|u_x|^2) dx + \frac{1}{2} \int_{-L}^0 (|y_t|^2 + b|y_x|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

Using similar computations as in Lemma 4.2.2, we obtain

$$\frac{d}{dt} E_2(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (4.3.4)$$

Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and therefore  $\frac{d}{dt} E_2(t) \leq 0$ . Thus, system (4.3.1)-(4.3.3) is dissipative in the sense that its energy is a non-increasing function with respect to time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_2$  by

$$\mathcal{H}_2 = \{(u, v, y, z, \omega) \in H_R^1(0, L) \times L^2(0, L) \times H_L^1(-L, 0) \times L^2(-L, 0) \times W; \ u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_R^1(0, L) = \{u \in H^1(0, L); u(L) = 0\}, \\ H_L^1(-L, 0) = \{y \in H^1(-L, 0); y(-L) = 0\}. \end{cases} \quad (4.3.5)$$

The energy space  $\mathcal{H}_2$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_2} &= \int_0^L v \overline{v_1} dx + a \int_0^L u_x (\overline{u_1})_x dx + \int_{-L}^0 z \overline{z_1} dx + b \int_{-L}^0 y_x (\overline{y_1})_x dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1(x, \xi)} d\xi dx, \end{aligned}$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_2$ . We use  $\|U\|_{\mathcal{H}_2}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_2 : D(\mathcal{A}_2) \subset \mathcal{H}_2 \rightarrow \mathcal{H}_2$  by

$$D(\mathcal{A}_2) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega) \in \mathcal{H}_2; (v, z) \in H_R^1(0, L) \times H_L^1(-L, 0), y \in H^2(-L, 0), \\ \left( au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \in L^2(0, L), \\ -(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ au_x(0) = by_x(0), \text{ and } v(0) = z(0) \end{array} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_2)$ ,

$$\mathcal{A}_2(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ \left( au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_x \\ z \\ by_{xx} \\ -(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} v_x |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (4.3.1)-(4.3.3), then the system can be rewritten as an evolution equation on the Hilbert space  $\mathcal{H}_2$  given by

$$U_t = \mathcal{A}_2 U, \quad U(0) = U_0, \quad (4.3.6)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ .

In a similar way to Section 4.2.1, we can see that the unbounded linear operator  $\mathcal{A}_2$  is m-dissipative in the energy space  $\mathcal{H}_2$ . Also, the  $C_0$ -semigroup of contractions  $e^{t\mathcal{A}_2}$  is strongly stable on  $\mathcal{H}_2$  in the sense that  $\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_2} U_0\|_{\mathcal{H}_2} = 0$ .

**Theorem 4.3.1.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_2})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_2 > 0$  such that for every  $U_0 \in D(\mathcal{A}_2)$ , we have*

$$E_2(t) \leq \frac{C_1}{t^{\frac{4}{2-\alpha}}} \|U_0\|_{D(\mathcal{A}_2)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_2). \quad (4.3.7)$$

According to Theorem 1.3.10 of Borichev-Tomilov, by taking  $\ell = 1 - \frac{\alpha}{2}$ , the polynomial energy decay (4.3.7) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_2), \quad (G_1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{1-\frac{\alpha}{2}}} \|(i\lambda I - \mathcal{A}_2)^{-1}\|_{\mathcal{L}(\mathcal{H}_2)} < \infty, \quad (G_2)$$

are satisfied. Since Condition (G<sub>1</sub>) is satisfied (similar way as in Section 4.2.1). We will prove condition (G<sub>2</sub>) by an argument of contradiction. For this purpose, suppose that (G<sub>2</sub>) is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))^\top)\} \subset \mathbb{R}^* \times D(\mathcal{A}_2)$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_2} = \|(u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))\|_{\mathcal{H}_2} = 1, \quad (4.3.8)$$

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such that

$$(\lambda_n)^{1-\frac{\alpha}{2}} (i\lambda_n I - \mathcal{A}_2) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}(\cdot, \xi))^T \rightarrow 0 \text{ in } \mathcal{H}_2. \quad (4.3.9)$$

For simplicity, we drop the index  $n$ . Equivalently, from (4.3.9), we have

$$i\lambda u - v = \frac{f_1}{\lambda^{1-\frac{\alpha}{2}}} \text{ in } H_R^1(0, L), \quad (4.3.10)$$

$$i\lambda v - (S_d)_x = \frac{f_2}{\lambda^{1-\frac{\alpha}{2}}} \text{ in } L^2(0, L), \quad (4.3.11)$$

$$i\lambda y - z = \frac{f_3}{\lambda^{1-\frac{\alpha}{2}}} \text{ in } H_L^1(-L, 0), \quad (4.3.12)$$

$$i\lambda z - by_{xx} = \frac{f_4}{\lambda^{1-\frac{\alpha}{2}}} \text{ in } L^2(-L, 0), \quad (4.3.13)$$

$$(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}v_x|\xi|^{\frac{2\alpha-1}{2}} = \frac{f_5(x, \xi)}{\lambda^{1-\frac{\alpha}{2}}} \text{ in } W, \quad (4.3.14)$$

where  $S_d = au_x + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi$ .

Here we will check the condition  $(G_2)$  by finding a contradiction with (4.3.8) by showing  $\|U\|_{\mathcal{H}_2} = o(1)$ . In order to get this contradiction, we follow similar arguments as in Section 4.2.2. We get the same results as in Lemmas 4.2.14, 4.2.15, 4.2.16, 4.2.17, 4.2.18.

Using the same computations as in Lemma 4.2.19, we get that the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_2)$  of system (4.3.10)-(4.3.14) satisfies the following estimation

$$\begin{aligned} & \int_0^L h'(|v|^2 + a^{-1}|S_d|^2) dx + \int_{-L}^0 \varphi'(|z|^2 + b|y_x|^2) dx + h(0)|v(0)|^2 \\ & - ah(L)|u_x(L)|^2 + ah(0)|u_x(0)|^2 - \varphi(0)|z(0)|^2 - b\varphi(0)|y_x(0)|^2 \\ & + b\varphi(-L)|y_x(-L)|^2 = o(1). \end{aligned} \quad (4.3.15)$$

Next, we proceed in a similar way to Lemma 4.2.20.

**Step 1.** Taking  $h(x) = x\theta_1(x) + (x-L)\theta_2(x)$  and  $\varphi(x) = 0$  in Equation (4.3.15), where  $\theta_1, \theta_2 \in C^1([0, L])$  are defined in Lemma 4.2.20, yields

$$\|v\|_{L^2(0, L)} = o(1) \quad \text{and} \quad \|u_x\|_{L^2(0, L)} = o(1).$$

**Step 2.** Taking  $h(x) = x - L$  and  $\varphi(x) = 0$  in Equation (4.3.15), we obtain

$$\int_0^L (|v|^2 + a^{-1}|S_d|^2) dx - L|v(0)|^2 - aL|u_x(0)|^2 = o(1). \quad (4.3.16)$$

By using Step 1, we get

$$|v(0)|^2 = |u_x(0)|^2 = o(1). \quad (4.3.17)$$

**Step 3.** Taking  $h(x) = 0$  and  $\varphi(x) = x + L$  in Equation (4.3.15), we get

$$\int_{-L}^0 (|z|^2 + b|y_x|^2) dx - L|z(0)|^2 - bL|y_x(0)|^2 = o(1). \quad (4.3.18)$$



Using (4.3.17) and the transmission conditions we have

$$b|y_x(0)| = a|u_x(0)| = o(1) \quad \text{and} \quad |z(0)|^2 = |v(0)|^2 = o(1). \quad (4.3.19)$$

Inserting Equation (4.3.19) in Equation (4.3.18), we get

$$\int_{-L}^0 |z|^2 dx = o(1) \quad \text{and} \quad b \int_{-L}^0 |y_x|^2 dx = o(1). \quad (4.3.20)$$

**Proof of Theorem 4.3.1.** From Step 1. and Step 3. we deduce that  $\|U\|_{\mathcal{H}_2} = o(1)$ , which contradicts (4.3.8). Consequently, condition (G<sub>2</sub>) holds. This implies, from Theorem 1.3.10, the energy decay estimation (4.3.7). The proof is thus complete.

## 4.4 W-(EBB)<sub>FKV</sub> Model

This section is devoted to study the stability of the model (W-(EBB)<sub>FKV</sub>), where we consider the Euler-Bernoulli beam and wave equations coupled through boundary connection. We take the fractional Kelvin-Voigt damping to be a localized internal damping acting on the Euler-Bernoulli beam only.

### 4.4.1 Well-Posedness and Strong Stability

In this subsection, we give the strong stability result of the system (W-(EBB)<sub>FKV</sub>). First, using a semigroup approach, we establish the well-posedness result of the system (W-(EBB)<sub>FKV</sub>). In Theorem 4.2.1, taking the input  $V(x, t) = \sqrt{d(x)}y_{xxt}(x, t)$ , then using (4.1.4), we get the output  $O$  is given by

$$O(x, t) = \sqrt{d(x)}I^{1-\alpha, \eta}y_{xxt}(x, t) = \frac{\sqrt{d(x)}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \partial_s y_{xx}(x, s) ds = \sqrt{d(x)} \partial_t^{\alpha, \eta} y_{xx}(x, t).$$

Therefore, by taking the input  $V(x, t) = \sqrt{d(x)}y_{xxt}(x, t)$  in Theorem 4.2.1 and using the above equation, we get

$$\begin{aligned} \partial_t \omega(x, \xi, t) + (\xi^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} y_{xxt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} &= 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times (0, \infty), \\ \omega(\xi, 0) &= 0, & (x, \xi) \in (0, L) \times \mathbb{R}, \\ \sqrt{d(x)} \partial_t^{\alpha, \eta} y_{xx}(x, t) - \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi &= 0, & (x, t) \in (0, L) \times (0, \infty). \end{aligned} \quad (4.4.1)$$

From system (4.4.1), we deduce that system (W-(EBB)<sub>FKV</sub>) can be recast into the following augmented model

$$\begin{cases} u_{tt} - au_{xx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_*^+, \\ y_{tt} + \left( by_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} y_{xxt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{cases} \quad (4.4.2)$$

with the following transmission and boundary conditions

$$\begin{cases} u(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_x(0, t) + by_{xxx}(0, t) = 0, y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{cases} \quad (4.4.3)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x) & x \in (-L, 0) \\ y(x, 0) &= y_0(x), \quad y_t(x, 0) = y_1(x), \quad \omega(x, \xi, 0) = 0 & x \in (0, L), \xi \in \mathbb{R}. \end{aligned} \quad (4.4.4)$$

The energy of the system (4.4.2)-(4.4.4) is given by

$$E_3(t) = \frac{1}{2} \int_{-L}^0 (|u_t|^2 + a|u_x|^2) dx + \frac{1}{2} \int_0^L (|y_t|^2 + b|y_{xx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

By using similar computations as in Lemma 4.2.2, it is easy to see that the energy  $E_3(t)$  satisfies the following estimation

$$\frac{d}{dt} E_3(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (4.4.5)$$

Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and therefore  $\frac{d}{dt} E_3(t) \leq 0$ . Thus, system (4.4.2)-(4.4.4) is dissipative in the sense that its energy is a non-increasing function with respect to time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_3$  by

$$\mathcal{H}_3 = \{(u, v, y, z, \omega) \in H_L^1(-L, 0) \times L^2(-L, 0) \times H_R^2(0, L) \times L^2(0, L) \times W; \ u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_L^1(-L, 0) = \{u \in H^1(-L, 0); u(-L) = 0\}, \\ H_R^2(0, L) = \{y \in H^2(0, L); y(L) = y_x(L) = 0\}. \end{cases} \quad (4.4.6)$$

The energy space  $\mathcal{H}_3$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_3} &= \int_{-L}^0 v \overline{v_1} dx + a \int_{-L}^0 u_x (\overline{u_1})_x dx + \int_0^L z \overline{z_1} dx + b \int_0^L y_{xx} (\overline{y_1})_{xx} dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1(x, \xi)} d\xi dx, \end{aligned}$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_3$ . We use  $\|U\|_{\mathcal{H}_3}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_3 : D(\mathcal{A}_3) \subset \mathcal{H}_3 \rightarrow \mathcal{H}_3$  by

$$D(\mathcal{A}_3) = \left\{ \begin{array}{l} U = (u, v, y, z, \omega) \in \mathcal{H}_3; (v, z) \in H_L^1(-L, 0) \times H_R^2(0, L), u \in H^2(-L, 0), \\ \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \in L^2(0, L), \\ -(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ au_x(0) + by_{xxx}(0) = 0, y_{xx}(0) = 0, \text{ and } v(0) = z(0) \end{array} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_3)$ ,

$$\mathcal{A}_3(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ au_{xx} \\ z \\ -\left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \\ -(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (4.4.2)-(4.4.4), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_3$  given by

$$U_t = \mathcal{A}_3 U, \quad U(0) = U_0, \quad (4.4.7)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ .

Similar to Proposition 4.2.5, the operator  $\mathcal{A}_3$  is m-dissipative on  $\mathcal{H}_3$ , consequently it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_3})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8. Then, the solution of the evolution Equation (4.4.7) admits the following representation

$$U(t) = e^{t\mathcal{A}_3} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (4.4.7). Hence, we have the following result.

**Theorem 4.4.1.** *Let  $U_0 \in \mathcal{H}_3$ , then problem (4.4.7) admits a unique weak solution  $U$  that satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}_3).$$

*Moreover, if  $U_0 \in D(\mathcal{A}_3)$ , then problem (4.4.7) admits a unique strong solution  $U$  that satisfies*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}_3) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_3)).$$

**Theorem 4.4.2.** *Assume that  $\eta \geq 0$ , then the  $C_0$ -semigroup of contractions  $e^{t\mathcal{A}_3}$  is strongly stable on  $\mathcal{H}_3$  in the sense that  $\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_3} U_0\|_{\mathcal{H}_3} = 0, \forall U_0 \in \mathcal{H}_3$ .*

**Proof of Theorem 4.4.2.** The proof of this theorem follows by proceeding with similar arguments as in Subsection 4.2.1, and using Theorem 1.3.3 of Arendt-Batty.

## 4.4.2 Polynomial Stability in the case $\eta > 0$

The aim of this part is to study the polynomial stability of system (4.4.2)-(4.4.4) in the case  $\eta > 0$ . As the condition  $i\mathbb{R} \subset \rho(\mathcal{A}_3)$  is already checked in the subsection 4.4.1, it remains to prove that condition (1.3.1) holds (see Theorem 1.3.10). This is established by using specific multipliers, some interpolation inequalities and by solving differential equations of order 4. Our main result in this part is the following theorem.

**Theorem 4.4.3.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_3})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_3 > 0$  such that for every  $U_0 \in D(\mathcal{A}_3)$ , we have*

$$E_3(t) \leq \frac{C_2}{t^{\frac{2}{3-\alpha}}} \|U_0\|_{D(\mathcal{A}_3)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_3). \quad (4.4.8)$$

According to Theorem 1.3.10, by taking  $\ell = 3 - \alpha$ , the polynomial energy decay (4.4.8) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_3), \quad (R_1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{3-\alpha}} \|(i\lambda I - \mathcal{A}_3)^{-1}\|_{\mathcal{L}(\mathcal{H}_3)} < \infty, \quad (R_2)$$

are satisfied. Since condition (R<sub>1</sub>) is already checked (see Subsection 4.4.1), we still need to prove condition (R<sub>2</sub>). For this purpose we will use an argument of contradiction. Suppose that (R<sub>2</sub>) is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))^T)\} \subset \mathbb{R}^* \times D(\mathcal{A}_3)$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_3} = \|(u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))\|_{\mathcal{H}_3} = 1, \quad (4.4.9)$$

such that

$$(\lambda_n^{3-\alpha})(i\lambda_n I - \mathcal{A}_3)U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}(\cdot, \xi))^T \rightarrow 0 \quad \text{in } \mathcal{H}_3. \quad (4.4.10)$$

For simplicity, we drop the index  $n$ . Equivalently, from (4.4.10), we have

$$i\lambda u - v = \frac{f_1}{\lambda^{3-\alpha}} \quad \text{in } H_L^1(-L, 0), \quad (4.4.11)$$

$$i\lambda v - au_{xx} = \frac{f_2}{\lambda^{3-\alpha}} \quad \text{in } L^2(-L, 0), \quad (4.4.12)$$

$$i\lambda y - z = \frac{f_3}{\lambda^{3-\alpha}} \quad \text{in } H_R^2(0, L), \quad (4.4.13)$$

$$i\lambda z + S_{xx} = \frac{f_4}{\lambda^{3-\alpha}} \quad \text{in } L^2(0, L), \quad (4.4.14)$$

$$(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}z_{xx}|\xi|^{\frac{2\alpha-1}{2}} = \frac{f_5(x, \xi)}{\lambda^{3-\alpha}} \quad \text{in } W, \quad (4.4.15)$$

where  $S = by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi$ . Here we will check the condition (R<sub>2</sub>) by finding a contradiction with (4.4.9) by showing  $\|U\|_{\mathcal{H}_3} = o(1)$ . For clarity, we divide the proof into several Lemmas.

**Lemma 4.4.4.** *Assume that  $\eta > 0$ . Then, the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-*

(4.4.15) satisfies the following asymptotic behavior estimations

$$\begin{aligned} \int_0^L \int_{\mathbb{R}} (|\xi|^2 + \eta) |\omega(x, \xi)|^2 d\xi dx &= \frac{o(1)}{\lambda^{3-\alpha}}, \quad \int_{l_0}^{l_1} |z_{xx}|^2 dx = \frac{o(1)}{\lambda^2}, \\ \int_{l_0}^{l_1} |y_{xx}|^2 dx &= \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_{l_0}^{l_1} |S|^2 dx = \frac{o(1)}{\lambda^{3-\alpha}}. \end{aligned} \quad (4.4.16)$$

*Proof.* For the clarity we divide the proof into several steps.

**Step 1.** Taking the inner product of  $F$  with  $U$  in  $\mathcal{H}_3$ , then using (4.4.9) and the fact that  $U$  is uniformly bounded in  $\mathcal{H}_3$ , we get

$$\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx = -\operatorname{Re}(\langle \mathcal{A}_3 U, U \rangle_{\mathcal{H}_3}) = \operatorname{Re}(\langle (i\lambda I - \mathcal{A}_3)U, U \rangle_{\mathcal{H}_3}) = o(\lambda^{-3+\alpha}).$$

**Step 2.** Our aim here is to prove the second estimation in (4.4.16).

From (4.4.15), we get

$$\sqrt{d(x)} |\xi|^{\frac{2\alpha-1}{2}} |z_{xx}| \leq (|\lambda| + \xi^2 + \eta) |\omega(x, \xi)| + |\lambda|^{-3+\alpha} |f_5(x, \xi)|.$$

Multiplying the above inequality by  $(|\lambda| + \xi^2 + \eta)^{-2} |\xi|$ , integrating over  $\mathbb{R}$  and proceeding in a similar way as in Lemma 4.2.15 (Subsection 4.2.2), we get the second desired estimation in (4.4.16).

**Step 3.** From Equation (4.4.13) we have that

$$\|\lambda y_{xx}\|_{L^2(l_0, l_1)} \leq \|z_{xx}\|_{L^2(l_0, l_1)} + |\lambda|^{-3+\alpha} \|(f_3)_{xx}\|_{L^2(l_0, l_1)}.$$

Using Step 2, the fact that  $\|f_3\|_{H_R^2(0, L)} = o(1)$ , and that  $\alpha \in (0, 1)$ , we get the third estimation in (4.4.16).

**Step 4.** Using the fact that  $|P + Q|^2 \leq 2P^2 + 2Q^2$ , we obtain

$$\begin{aligned} \int_{l_0}^{l_1} |S|^2 dx &= \int_{l_0}^{l_1} \left| by_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right|^2 dx \\ &\leq 2b^2 \int_{l_0}^{l_1} |y_{xx}|^2 dx + 2d_0 \kappa(\alpha)^2 \int_{l_0}^{l_1} \left( \int_{\mathbb{R}} \frac{|\xi|^{\frac{2\alpha-1}{2}} \sqrt{\xi^2 + \eta}}{\sqrt{\xi^2 + \eta}} \omega(x, \xi) d\xi \right)^2 dx \\ &\leq 2b^2 \int_{l_0}^{l_1} |y_{xx}|^2 dx + c_2 \int_{l_0}^{l_1} \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi)|^2 d\xi dx \end{aligned}$$

where  $c_2 = d_0 \kappa(\alpha)^2 \mathbb{I}_{14}(\alpha, \eta)$  is defined in Lemma 4.2.16. Thus, we get the last estimation in (4.4.16). Hence, the proof is complete.  $\square$

**Lemma 4.4.5.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior

$$\|z\|_{H^2(l_0, l_1)} = \frac{o(1)}{\lambda}, \quad \|y\|_{H^2(l_0, l_1)} = \frac{o(1)}{\lambda^2}, \quad \text{and} \quad \|S_x\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda^{\frac{3-\alpha}{4}}}. \quad (4.4.17)$$

*Proof.* The proof of this Lemma will be divided into two steps.

**Step 1.** Let  $0 < \varepsilon < \frac{l_1 - l_0}{2}$ . We define  $h \in C^\infty([0, L])$ ,  $0 \leq h \leq 1$  on  $[0, L]$ ,  $h = 1$  on  $(l_0 + \varepsilon, l_1 - \varepsilon)$ , and  $h = 0$  on  $(0, l_0) \cup (l_1, L)$ . Also, we define  $\max_{x \in [0, L]} |h'(x)| = m'_h$  and  $\max_{x \in [0, L]} |h''(x)| = m''_h$ , where

$m'_h$  and  $m''_h$  are strictly positive constant numbers. Multiply equation (4.4.14) by  $-i\lambda^{-1}h\bar{z}$  and integrate over  $(l_0, l_1)$ , we get

$$\int_{l_0}^{l_1} h|z|^2 dx = i\lambda^{-1} \int_{l_0}^{l_1} S(h''\bar{z} + 2h'z_x + hz_{xx}) dx - i\lambda^{-4+\alpha} \int_{l_0}^{l_1} hf_4\bar{z} dx. \quad (4.4.18)$$

Using Nirenberg inequality Theorem (see [81]), Equations (4.4.9) and (4.4.16), we obtain

$$\|z_x\|_{L^2(l_0, l_1)} \leq \|z_{xx}\|_{L^2(l_0, l_1)}^{1/2} \|z\|_{L^2(l_0, l_1)}^{1/2} + \|z\|_{L^2(l_0, l_1)} \leq O(1). \quad (4.4.19)$$

Estimation of the term  $\left(i\lambda^{-1} \int_{l_0}^{l_1} h''S\bar{z}\right)$ . Using Cauchy Schwarz inequality, last estimation in (4.4.16) and the fact that  $z$  is uniformly bounded in  $L^2(0, L)$  and that  $0 < \alpha < 1$ , we get

$$\left|i\lambda^{-1} \int_{l_0}^{l_1} h''S\bar{z} dx\right| \leq \frac{m''_h}{\lambda} \int_{l_0}^{l_1} |S||z| dx \leq \frac{o(1)}{\lambda^{\frac{5-\alpha}{2}}}. \quad (4.4.20)$$

Estimation of the term  $\left(i\lambda^{-1} \int_{l_0}^{l_1} hSz_{xx} dx\right)$ . Using Cauchy Schwarz inequality, the second and the last estimations in (4.4.16), we get

$$\left|i\lambda^{-1} \int_{l_0}^{l_1} hS\bar{z}_{xx} dx\right| \leq \frac{m'_h}{\lambda} \left(\int_{l_0}^{l_1} |S|^2 dx\right)^{1/2} \left(\int_{l_0}^{l_1} |z_{xx}|^2 dx\right)^{1/2} = \frac{o(1)}{\lambda^{\frac{7-\alpha}{2}}}. \quad (4.4.21)$$

Estimation of the term  $\left(i\lambda^{-1} \int_{l_0}^{l_1} h'Sz_x dx\right)$ . Cauchy Schwarz inequality, the last estimation in (4.4.16), Equation (4.4.19) and the fact that  $0 < \alpha < 1$ , we get

$$\left|i\lambda^{-1} \int_{l_0}^{l_1} h'Sz_x dx\right| \leq \frac{m'_h}{\lambda} \int_{l_0}^{l_1} |S||z_x| dx \leq \frac{o(1)}{\lambda^{\frac{5-\alpha}{2}}}. \quad (4.4.22)$$

Estimation of the term  $\left(i\lambda^{-4+\alpha} \int_{l_0}^{l_1} hf_4\bar{z} dx\right)$ . Using Cauchy-Schwarz inequality, the fact that  $\|f_4\|_{L^2(0, L)} = o(1)$  and the fact that  $z$  is uniformly bounded in  $L^2(0, L)$ , we get

$$\left|i\lambda^{-4+\alpha} \int_{l_0}^{l_1} hf_4\bar{z} dx\right| = \frac{o(1)}{\lambda^{4-\alpha}}. \quad (4.4.23)$$

Thus, using Equations (4.4.20)-(4.4.23) in (4.4.18), we get

$$\int_{l_0}^{l_1} h|z|^2 dx = \frac{o(1)}{\lambda^{\frac{5-\alpha}{2}}} \quad \text{and} \quad \int_{l_0+\varepsilon}^{l_1-\varepsilon} |z|^2 dx = \frac{o(1)}{\lambda^{\frac{5-\alpha}{2}}}. \quad (4.4.24)$$

**Step 2.** Applying the interpolation theorem involving compact subdomain ([2], Theorem 4.23), we obtain

$$\|z_x\|_{L^2(l_0, l_1)} \leq \|z_{xx}\|_{L^2(l_0, l_1)} + \|z\|_{L^2(l_0+\varepsilon, l_1-\varepsilon)}.$$

Then, by using (4.4.24) and that  $0 < \alpha < 1$ , we get

$$\|z_x\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda}. \quad (4.4.25)$$

Also, using the same theorem we get that

$$\|z\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda}. \quad (4.4.26)$$

Thus, using the first estimation in (4.4.16), (4.4.25) and (4.4.26), we obtain the first estimation in Lemma 4.4.5.

Now, from Equation (4.4.13) we have  $i\lambda y - z = \lambda^{-3+\alpha} f_3$ , then

$$\|y\|_{H^2(l_0, l_1)} \leq \frac{1}{\lambda} \|z\|_{H^2(l_0, l_1)} + \frac{1}{\lambda^{4-\alpha}} \|f_3\|_{H^2(l_0, l_1)},$$

using the fact that  $\alpha \in (0, 1)$ ,  $\|f_3\|_{H^2(l_0, l_1)} = o(1)$ , and the first estimation in Lemma 4.4.5, we obtain the second estimation of Lemma 4.4.5. Using Nirenberg inequality Theorem (see [81]), Lemma 4.4.4, (4.4.14) and the first estimation in (4.4.26), we get

$$\|S_x\|_{L^2(l_0, l_1)} \leq \|S_{xx}\|_{L^2(l_0, l_1)}^{\frac{1}{2}} \|S\|_{L^2(l_0, l_1)}^{\frac{1}{2}} + \|S\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda^{\frac{3-\alpha}{4}}}. \quad (4.4.27)$$

Thus, the proof is completed.  $\square$

**Lemma 4.4.6.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior

$$|y(l_0)| = \frac{o(1)}{\lambda^2}, |y_x(l_0)| = \frac{o(1)}{\lambda^2}, |y(l_1)| = \frac{o(1)}{\lambda^2}, \text{ and } |y_x(l_1)| = \frac{o(1)}{\lambda^2}. \quad (4.4.28)$$

Moreover,

$$\frac{1}{\lambda^{\frac{1}{2}}} |y_{xxx}(l_0^-)| = \frac{o(1)}{\lambda^{\frac{5-\alpha}{4}}} \quad \text{and} \quad \frac{1}{\lambda^{\frac{1}{2}}} |y_{xxx}(l_1^+)| = \frac{o(1)}{\lambda^{\frac{5-\alpha}{4}}}. \quad (4.4.29)$$

*Proof.* For the proof of (4.4.28). Since  $y, z \in H^2(0, L)$ , Sobolev embedding theorem implies that  $y, z \in C^1[0, L]$ . Then, using the second estimation in Lemma 4.4.5 we get (4.4.28).

Define

$$J(z)(x) = \frac{1}{\lambda^{3-\alpha}} \int_x^L \int_s^L z(\tau) d\tau ds \quad (4.4.30)$$

and

$$X = \frac{1}{i\lambda} [-S + J(f_4)]. \quad (4.4.31)$$

From (4.4.14) and (4.4.31), we get

$$X_{xx} = z. \quad (4.4.32)$$

From (4.4.31), Lemma 4.4.4 and using the fact that  $\|f_4\|_{L^2(0, L)} = o(1)$ , we have that

$$\|\lambda X\|_{L^2(l_0, l_1)} \leq \|S\|_{L^2(l_0, l_1)} + \|J(f_4)\|_{L^2(l_0, l_1)} \leq \frac{o(1)}{\lambda^{\frac{3-\alpha}{2}}}. \quad (4.4.33)$$

It follows that,

$$\|X\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda^{\frac{5-\alpha}{2}}}. \quad (4.4.34)$$

Using Equation (4.4.32) and Lemma 4.4.5, we get

$$\|X_{xx}\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda}, \|X_{xxx}\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda}, \text{ and } \|X_{xxxx}\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda}. \quad (4.4.35)$$

Now, using the interpolation inequality theorem ( see in [81]), and  $\alpha \in (0, 1)$ , we get

$$\|X_x\|_{L^2(l_0, l_1)} \leq \|X_{xx}\|_{L^2(l_0, l_1)}^{\frac{1}{2}} \|X\|_{L^2(l_0, l_1)}^{\frac{1}{2}} + \|X\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda^{\frac{7-\alpha}{4}}}. \quad (4.4.36)$$

By using Equations (4.4.34)-(4.4.36), we get

$$\|X\|_{H^4(l_0, l_1)} = \frac{o(1)}{\lambda}. \quad (4.4.37)$$

From the interpolation inequality (see Theorem 4.17 in [2]), we have

$$\|\lambda^{1/2} X\|_{H^2(l_0, l_1)} \lesssim \|\lambda^{1/2} X\|_{H^4(l_0, l_1)}^{\frac{1}{2}} \cdot \|\lambda^{1/2} X\|_{L^2(l_0, l_1)}^{\frac{1}{2}} = \frac{o(1)}{\lambda^{\frac{5-\alpha}{4}}}. \quad (4.4.38)$$

We note that  $S = by_{xx}$  on  $(0, l_0) \cup (l_1, L)$ . Also, we have that  $y \in H^4(0, l_0)$  and  $y \in H^4(l_1, L)$ .

From (4.4.31), we have

$$by_{xxx}(l_0^-) = (J(f_4) - i\lambda X)_x(l_0) \quad \text{and} \quad by_{xxx}(l_1^+) = (J(f_4) - i\lambda X)_x(l_1). \quad (4.4.39)$$

Dividing Equation (4.4.39) by  $\lambda^{\frac{1}{2}}$ , and using Equation (4.4.38) and the fact that  $\|f_4\|_{L^2(0, L)} = o(1)$ , we get Equation (4.4.29). Thus, the proof of the Lemma is complete.  $\square$

**Lemma 4.4.7.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior for every  $h \in C^2([0, L])$  and  $h(0) = h(L) = 0$ , and for every  $g \in C^1[-L, 0]$

$$\begin{aligned} & \int_0^L h' |z|^2 dx + 2b \int_0^L h' |y_{xx}|^2 dx + b \int_0^{l_0} h' |y_{xx}|^2 dx \\ & + b \int_{l_1}^L h' |y_{xx}|^2 dx + \operatorname{Re} \left( 2 \int_0^L h'' S \bar{y}_x dx \right) - \operatorname{Re} \left( 2 \int_{l_0}^{l_1} h S_x \bar{y}_{xx} dx \right) \\ & - bh(l_0) |y_{xx}(l_0^-)|^2 + bh(l_1) |y_{xx}(l_1^+)|^2 = \frac{o(1)}{\lambda^{3-\alpha}} \end{aligned} \quad (4.4.40)$$

and

$$\int_{-L}^0 g' (|v|^2 + a|u_x|^2 dx) - g(0) |v(0)|^2 - ag(0) |u_x(0)|^2 + ag(-L) |u_x(-L)|^2 = \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.41)$$

*Proof.* Multiply Equation (4.4.14) by  $2h\bar{y}_x$  and integrate over  $(0, L)$ , we get

$$\operatorname{Re} \left( 2i\lambda \int_0^L z h \bar{y}_x dx \right) + \operatorname{Re} \left( 2 \int_0^L h S_{xx} \bar{y}_x dx \right) = \operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^L f_4 h \bar{y}_x dx \right). \quad (4.4.42)$$

From Equation (4.4.13) we have

$$i\lambda \bar{y}_x = -\bar{v}_x - \lambda^{-3+\alpha} (\bar{f}_3)_x.$$

Then

$$\operatorname{Re} \left( 2i\lambda \int_0^L z h \bar{y}_x dx \right) = \int_0^L h' |z|^2 dx - \operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^L h z (\bar{f}_3)_x dx \right). \quad (4.4.43)$$

Estimation of the term  $\operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^L h z (\bar{f}_3)_x dx \right)$ . Using Cauchy-Schwarz inequality, the



definition of  $h$ , the fact that  $\|f_3\|_{H^2(0,L)} = o(1)$ , and that  $z$  is uniformly bounded in  $L^2(0,L)$ , we get

$$\left| \operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^L h z (\overline{f_3})_x dx \right) \right| \lesssim \frac{1}{|\lambda|^{3-\alpha}} \left( \int_0^L |z|^2 dx \right)^{1/2} \left( \int_0^L |(f_3)_x|^2 dx \right)^{1/2} = \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.44)$$

Then, using Equation (4.4.44) in Equation (4.4.43) we get

$$\operatorname{Re} \left( 2i\lambda \int_0^L z h \overline{y_x} dx \right) = \int_0^L h' |z|^2 dx + \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.45)$$

Estimation of the term  $\operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^L f_4 h \overline{y_x} dx \right)$ . Using Cauchy-Schwarz inequality, the definition of  $h$ , the fact that  $\|f_4\|_{L^2(0,L)} = o(1)$ , and that  $\|y_x\|_{L^2(0,L)} \lesssim \|y_{xx}\|_{L^2(0,L)} = O(1)$ , we get

$$\operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^L f_4 h \overline{y_x} dx \right) \lesssim \frac{1}{\lambda^{3-\alpha}} \left( \int_0^L |f_4|^2 dx \right)^{1/2} \left( \int_0^L |y_{xx}|^2 dx \right)^{1/2} \leq \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.46)$$

For the second term of Equation (4.4.42), integrating by parts we get

$$\operatorname{Re} \left( 2 \int_0^L h S_{xx} \overline{y_x} dx \right) = \operatorname{Re} \left( 2 \int_0^L h'' S \overline{y_x} dx \right) + \operatorname{Re} \left( 2 \int_0^L h' S \overline{y_{xx}} dx \right) - \operatorname{Re} \left( 2 \int_0^L h S_x \overline{y_{xx}} dx \right). \quad (4.4.47)$$

For the term  $\operatorname{Re} \left( 2 \int_0^L h' S \overline{y_{xx}} dx \right)$ , we have

$$\operatorname{Re} \left( 2 \int_0^L h' S \overline{y_{xx}} dx \right) = 2b \int_0^L h' |y_{xx}|^2 dx + \operatorname{Re} \left( 2 \int_0^L h' \overline{y_{xx}} \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi dx \right). \quad (4.4.48)$$

Estimation of the term  $\operatorname{Re} \left( 2 \int_0^L h' \overline{y_{xx}} \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi dx \right)$ . Using Cauchy-Schwarz inequality, the definition of the functions  $d(x)$  and  $h$ , and using Lemma 4.4.4, we get

$$\left| \operatorname{Re} \left( 2 \int_0^L h' \overline{y_{xx}} \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi dx \right) \right| = \frac{o(1)}{\lambda^{\frac{7-\alpha}{2}}}. \quad (4.4.49)$$

This yields that,

$$\operatorname{Re} \left( 2 \int_0^L h' S \overline{y_{xx}} dx \right) = 2b \int_0^L h' |y_{xx}|^2 dx + \frac{o(1)}{\lambda^{\frac{7-\alpha}{2}}}. \quad (4.4.50)$$

For the term  $-\operatorname{Re} \left( 2 \int_0^L h S_x \overline{y_{xx}} dx \right)$ . We have

$$-\operatorname{Re} \left( 2 \int_0^L h S_x \overline{y_{xx}} dx \right) = -\operatorname{Re} \left( 2b \int_0^{l_0} h y_{xxx} \overline{y_{xx}} dx + 2b \int_{l_1}^L h y_{xxx} \overline{y_{xx}} dx \right) - \operatorname{Re} \left( 2 \int_{l_0}^{l_1} h S_x \overline{y_{xx}} dx \right). \quad (4.4.51)$$

Integrating the above Equation by parts, we get

$$\begin{aligned} -\operatorname{Re}\left(2\int_0^L hS_x\bar{y}_{xx}dx\right) &= b\int_0^{l_0} h'|y_{xx}|^2dx + b\int_{l_1}^L h'|y_{xx}|^2dx - bh(l_0)|y_{xx}(l_0)|^2 \\ &+ bh(l_1)|y_{xx}(l_1)|^2 - \operatorname{Re}\left(2\int_{l_0}^{l_1} hS_x\bar{y}_{xx}dx\right). \end{aligned} \quad (4.4.52)$$

Then, substituting Equation (4.4.50) and (4.4.52) in Equation (4.4.47), we get

$$\begin{aligned} \operatorname{Re}\left(2\int_0^L hS_{xx}\bar{y}_x dx\right) &= 2b\int_0^L h'|y_{xx}|^2dx + b\int_0^{l_0} h'|y_{xx}|^2dx + b\int_{l_1}^L h'|y_{xx}|^2dx \\ &- bh(l_0)|y_{xx}(l_0)|^2 + bh(l_1)|y_{xx}(l_1)|^2 - \operatorname{Re}\left(2\int_{l_0}^{l_1} hS_x\bar{y}_{xx}dx\right) \\ &+ \operatorname{Re}\left(2\int_0^L h''S\bar{y}_x dx\right) + \frac{o(1)}{\lambda^{3-\alpha}}. \end{aligned} \quad (4.4.53)$$

Therefore, substituting Equations (4.4.45), (4.4.46), and (4.4.53) into (4.4.42), we get our desired result.

Now, we will prove Equation (4.4.41). For this aim, multiply Equation (4.4.12) by  $2g\bar{u}_x$  and integrate over  $(-L, 0)$ , we get

$$\operatorname{Re}\left(2i\lambda\int_{-L}^0 \nu g\bar{u}_x dx\right) - \operatorname{Re}\left(2a\int_{-L}^0 gu_{xx}\bar{u}_x dx\right) = \operatorname{Re}\left(2\lambda^{-3+\alpha}\int_{-L}^0 f_2 g\bar{u}_x dx\right). \quad (4.4.54)$$

We have  $i\lambda\bar{u}_x = -\bar{v}_x - \lambda^{-3+\alpha}(\bar{f}_1)_x$ . Then,

$$\operatorname{Re}\left(2i\lambda\int_{-L}^0 \nu g\bar{u}_x dx\right) = \int_{-L}^0 g'|v|^2 dx - g(0)|v(0)|^2 - \operatorname{Re}\left(2\lambda^{-3+\alpha}\int_{-L}^0 \nu g(\bar{f}_1)_x dx\right). \quad (4.4.55)$$

Estimation of the term  $\operatorname{Re}\left(2\lambda^{-3+\alpha}\int_{-L}^0 \nu g(\bar{f}_1)_x dx\right)$ . Using Cauchy-schwarz inequality,  $\|f_1\|_{H_L^2(-L,0)} = o(1)$ , and the fact that  $\nu$  is bounded in  $L^2(0, L)$ , we get

$$\left|\operatorname{Re}\left(2\lambda^{-3+\alpha}\int_{-L}^0 \nu g(\bar{f}_1)_x dx\right)\right| \leq \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.56)$$

Then, using the above equation, (4.4.55) becomes

$$\operatorname{Re}\left(2i\lambda\int_{-L}^0 \nu g\bar{u}_x dx\right) = \int_{-L}^0 g'|v|^2 dx - g(0)|v(0)|^2 + \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.57)$$

Integrating the second term of Equation (4.4.54), we get

$$-\operatorname{Re}\left(2a\int_{-L}^0 gu_{xx}\bar{u}_x dx\right) = a\int_{-L}^0 g'|u_x|^2 dx - g(0)|u_x(0)|^2 + ag(-L)|u_x(-L)|^2. \quad (4.4.58)$$

Estimation of the term  $\operatorname{Re}\left(2\lambda^{-3+\alpha}\int_{-L}^0 f_2 g\bar{u}_x dx\right)$ . Using Cauchy-Schwarz inequality,  $\|f_2\|_{L^2(-L,0)} =$

$o(1)$ , and the fact that  $u_x$  is bounded in  $L^2(-L, 0)$ , we get

$$\left| \operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_{-L}^0 f_2 g \overline{u_x} dx \right) \right| = \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.59)$$

Thus, summing Equations (4.4.57), (4.4.58) and (4.4.59) we obtain our desired result.  $\square$

**Lemma 4.4.8.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior

$$|\lambda y(0)| = O(1), \quad |y_{xx}(l_0^-)| = O(1) \quad \text{and} \quad |y_{xx}(l_1^+)| = O(1). \quad (4.4.60)$$

*Proof.* Take  $g(x) = x + L$  in Equation (4.4.41), we get

$$\int_{-L}^0 (|v|^2 + a|u_x|^2 dx) - L|v(0)|^2 - aL|u_x(0)|^2 = \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.61)$$

From the above Equation and using the fact that  $v$  and  $u_x$  are uniformly bounded in  $L^2(-L, 0)$ , we get

$$|v(0)| = O(1) \quad \text{and} \quad |u_x(0)| = O(1). \quad (4.4.62)$$

From Equation (4.4.13) we have

$$i\lambda u = v + \lambda^{-3+\alpha} f_1.$$

Then, using (4.4.62) and the fact that  $\|f_1\|_{L^2(-L,0)} = o(1)$ , we get

$$|\lambda u(0)| \leq |v(0)| + \lambda^{-3+\alpha} |f_1(0)| \leq |v(0)| + \lambda^{-3+\alpha} \sqrt{L} \|f_1\|_{L^2(-L,0)} = O(1). \quad (4.4.63)$$

From the continuity condition ( $u(0) = y(0)$ ), we deduce that  $|\lambda y(0)| = O(1)$ . In order to prove the second term in the Equation (4.4.60) we proceed as follows. From Equation (4.4.13) we have

$$z = i\lambda y - \lambda^{-3+\alpha} f_3.$$

Substituting the above Equation into Equation (4.4.14), we get

$$-\lambda^2 y + b y_{xxxx} = F \quad \text{on} \quad (0, l_0) \cup (l_1, L), \quad (4.4.64)$$

where  $F = \lambda^{-3+\alpha} (f_4 + i\lambda f_3)$ .

Multiply Equation (4.4.64) by  $2\zeta \overline{y_x}$ , where  $\zeta \in C^2[0, l_0]$ ,  $\zeta(0) = \zeta'(l_0) = 0$  and  $\zeta(l_0) = 1$ , and integrate over  $(0, l_0)$ , we get

$$-\operatorname{Re} \left( 2\lambda^2 \int_0^{l_0} \zeta y \overline{y_x} dx \right) + \operatorname{Re} \left( 2b \int_0^{l_0} \zeta y_{xxxx} \overline{y_x} dx \right) = \operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^{l_0} (f_4 + i\lambda f_3) \zeta \overline{y_x} dx \right). \quad (4.4.65)$$

Estimation of the term  $\operatorname{Re} \left( 2\lambda^2 \int_0^{l_0} \zeta y \overline{y_x} dx \right)$ . Integrating by parts and using Equation (4.4.28), we get

$$-\operatorname{Re} \left( 2\lambda^2 \int_0^{l_0} \zeta y \overline{y_x} dx \right) = \int_0^{l_0} \zeta' |\lambda y|^2 dx + \frac{o(1)}{\lambda^2}. \quad (4.4.66)$$

Integrating by parts the second term in Equation (4.4.65), and using Lemma 4.4.6, we get

$$\begin{aligned} \operatorname{Re} \left( 2b \int_0^{l_0} \zeta y_{xxx} \bar{y}_x dx \right) &= -\operatorname{Re} \left( 2b \int_0^{l_0} \zeta' y_{xx} \bar{y}_x dx \right) - \operatorname{Re} \left( 2b \int_0^{l_0} \zeta y_{xxx} \bar{y}_{xx} dx \right) \\ &\quad + \operatorname{Re} (2b y_{xxx}(l_0^-) \bar{y}_x(l_0)) \\ &= \operatorname{Re} \left( 2b \int_0^{l_0} \zeta'' y_{xx} \bar{y}_x dx \right) + 3b \int_0^{l_0} \zeta' |y_{xx}|^2 dx \\ &\quad - b \zeta(l_0) |y_{xx}(l_0)|^2 + \frac{o(1)}{\lambda^{\frac{11-\alpha}{4}}}. \end{aligned} \quad (4.4.67)$$

Integrating by parts the last term of Equation (4.4.65) and using (4.4.28) and the fact that  $|f_3(l_0)| \lesssim \|f_3\|_{H_R^2(0,L)} = o(1)$ , we get

$$\begin{aligned} \operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^{l_0} (f_4 + i\lambda f_3) \zeta \bar{y}_x dx \right) &= \operatorname{Re} \left( 2\lambda^{-3+\alpha} \int_0^{l_0} f_4 \zeta \bar{y}_x dx \right) \\ &\quad - \operatorname{Re} \left( 2i\lambda^{-3+\alpha} \int_0^{l_0} f_3 \zeta' \lambda \bar{y} dx \right) - \operatorname{Re} \left( 2i\lambda^{-3+\alpha} \int_0^{l_0} (f_3)_x \zeta \lambda \bar{y} dx \right) + \frac{o(1)}{\lambda^{4-\alpha}}. \end{aligned} \quad (4.4.68)$$

Substituting Equations (4.4.66), (4.4.67), and (4.4.68) in Equation (4.4.65), and using the fact that  $\|U\|_{\mathcal{H}_3} = 1$ ,  $\|y_x\|_{L^2(0,L)} \leq c_p \|y_{xx}\|_{L^2(0,L)} = O(1)$  and  $\|f_3\|_{H_R^2(0,L)} = o(1)$ , we obtain our desired term. For the last term in Equation (4.4.60), we proceed in a similar way as above and thus the proof of the Lemma is complete.  $\square$

**Lemma 4.4.9.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior

$$|y_{xx}(l_0^-)| = \frac{o(1)}{\lambda} \quad \text{and} \quad |y_{xx}(l_1^+)| = \frac{o(1)}{\lambda}. \quad (4.4.69)$$

*Proof.* Equation (4.4.64) can be written as

$$\left( \frac{\partial}{\partial x} - i\mu \right) \left( \frac{\partial}{\partial x} + i\mu \right) \left( \frac{\partial^2}{\partial x^2} - \mu^2 \right) y = \frac{1}{b} F, \text{ on } (0, l_0) \cup (l_1, L). \quad (4.4.70)$$

where  $\mu = \sqrt{\frac{\lambda}{\sqrt{b}}}$ .

On the interval  $(0, l_0)$ :

Let  $Y_{l_0}^1 = \left( \frac{\partial}{\partial x} + i\mu \right) \left( \frac{\partial^2}{\partial x^2} - \mu^2 \right) y$ . Solving on the interval  $(0, l_0)$  the following Equation

$$\left( \frac{\partial}{\partial x} - i\mu \right) Y_{l_0}^1 = \frac{1}{b} F$$

we get

$$Y_{l_0}^1 = K_1 e^{i\mu(x-l_0)} + \frac{1}{b} \int_{l_0}^x e^{i\mu(x-z)} F(z) dz, \quad (4.4.71)$$

where

$$K_1 = y_{xxx}(l_0^-) - \mu^2 y_x(l_0) + i\mu y_{xx}(l_0^-) - i\mu^3 y(l_0).$$

Let  $Y_{l_0}^2 = \left( \frac{\partial^2}{\partial x^2} - \mu^2 \right) y$ . We will solve the following differential equation

$$\left( \frac{\partial}{\partial x} + i\mu \right) Y_{l_0}^2 = Y_{l_0}^1. \quad (4.4.72)$$

By using the solution  $Y_{l_0}^1$ , we obtain the solution of the differential equation (4.4.72)

$$Y_{l_0}^2 = K_2 e^{-i\mu(x-l_0)} + \frac{K_1}{\mu} \sin((x-l_0)\mu) + \frac{1}{b\mu} \int_{l_0}^x \int_{l_0}^s e^{i\mu(2s-x-z)} F(z) dz ds. \quad (4.4.73)$$

Integrating by parts the last term of the above Equation, we get

$$\frac{1}{b\mu} \int_{l_0}^x \int_{l_0}^s e^{i\mu(2s-x-z)} F(z) dz ds = \frac{1}{b\mu} \int_{l_0}^x \sin((x-z)\mu) F(z) dz.$$

Inserting the above Equation in (4.4.73), we get

$$Y_{l_0}^2 = K_2 e^{-i\mu(x-l_0)} + \frac{K_1}{\mu} \sin((x-l_0)\mu) + \frac{1}{b\mu} \int_{l_0}^x \sin((x-z)\mu) F(z) dz \quad (4.4.74)$$

where

$$K_2 = y_{xx}(l_0^-) - \mu^2 y(l_0).$$

Let  $Y_{l_0}^3 = \left( \frac{\partial}{\partial x} - \mu \right) y$ . The solution of the following differential equation

$$\left( \frac{\partial}{\partial x} + \mu \right) Y_{l_0}^3 = Y_{l_0}^2 \quad (4.4.75)$$

is given by

$$\begin{aligned} Y_{l_0}^3 &= K_3 e^{-\mu(x-l_0)} + \frac{K_2}{\mu(1-i)} [e^{-i\mu(x-l_0)} - e^{-\mu(x-l_0)}] \\ &+ \frac{1}{b\mu} \int_{l_0}^x \int_{l_0}^s e^{\mu(s-x)} \sin((s-z)\mu) F(z) dz ds \\ &- \frac{K_1}{2\mu^2} [\cos((x-l_0)\mu) - \sin((x-l_0)\mu) - e^{-\mu(x-l_0)}] \end{aligned} \quad (4.4.76)$$

where

$$K_3 = y_x(l_0) - \mu y(l_0).$$

Take  $x = 0$  in (4.4.76) and multiply the equation by  $\mu e^{-\mu l_0}$ , we get

$$\begin{aligned} \sqrt{\frac{5}{4}} |y_{xx}(l_0^-)| &\leq |\mu y_x(0)| e^{-\mu l_0} + |\mu^2 y(0)| e^{-\mu l_0} + |y_x(l_0)| + |\mu y(l_0)| \\ &+ \frac{1}{2\mu} |y_{xxx}(l_0)| [2e^{-\mu l_0} + 1] + \frac{1}{2} |\mu^2 y(l_0)| (2e^{-\mu l_0} + 1) + \frac{1}{2} |\mu y_x(l_0)| (2e^{-\mu l_0} + 1) \\ &+ \frac{1}{\sqrt{2}} |\mu^2 y(l_0)| (e^{-\mu l_0} + 1) + \left(1 + \frac{1}{\sqrt{2}}\right) |y_{xx}(l_0^-)| e^{-\mu l_0} \\ &+ \left| \frac{1}{b} \int_0^{l_0} \int_{l_0}^s e^{(s-l_0)\mu} \sin((s-z)\mu) F(z) dz ds \right|. \end{aligned} \quad (4.4.77)$$

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For the integral in the above Equation, we have

$$\begin{aligned} & \frac{1}{b} \int_0^{l_0} \int_{l_0}^s e^{\mu(s-l_0)} \sin((s-z)\mu) F(z) dz ds = \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_{l_0}^s e^{\mu(s-l_0)} \sin((s-z)\mu) f_4(z) dz ds \\ & + \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_{l_0}^s e^{\mu(s-l_0)} \sin((s-z)\mu) i\lambda f_3(z) dz ds. \end{aligned} \quad (4.4.78)$$

Estimation of the first integral in the right side of Equation (4.4.78), using the fact that  $\|f_4\|_{L^2(0,L)} = o(1)$ , we get

$$\left| \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_{l_0}^s e^{\mu(s-l_0)} \sin((s-z)\mu) f_4(z) dz ds \right| \leq \frac{l_0^{3/2}}{\lambda^{3-\alpha}} \int_0^{l_0} |f_4(z)|^2 dz = \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.79)$$

Estimation of the second term in the second side of Equation (4.4.78). Integrating by parts, using the fact that  $|f_3(0)| \lesssim \|(f_3)_x\|_{L^2(0,L)} = o(1)$ ,  $|f_3(l_0)| \lesssim \|(f_3)_x\|_{L^2(0,L)} = o(1)$  and  $\|f_3\|_{L^2(0,L)} = o(1)$ , we get

$$\begin{aligned} & \left| \frac{1}{b\lambda^{3-\alpha}} \int_0^{l_0} \int_{l_0}^s e^{\mu(s-l_0)} \sin((s-z)\mu) i\lambda f_3(z) dz ds \right| \\ &= \left| \frac{i\lambda e^{-\mu l_0}}{b\lambda^{3-\alpha}} \int_0^{l_0} \left( \int_0^z e^{\mu s} \sin((s-z)\mu) ds \right) f_3(z) dz \right| \\ &= \left| \frac{i\lambda e^{-\mu l_0}}{2b\mu\lambda^{3-\alpha}} \int_0^{l_0} (\cos(\mu z) + \sin(\mu z) - e^{\mu z}) f_3(z) dz \right| \\ &\leq \frac{\lambda}{2be^{\mu l_0}\lambda^{3-\alpha}\mu} \left[ 2 \int_0^{l_0} |f_3(z)| dz + \left| \int_0^{l_0} e^{\mu z} f_3(z) dz \right| \right] \\ &\leq \frac{\lambda}{2be^{\mu l_0}\lambda^{3-\alpha}\mu} \left[ 2 \int_0^{l_0} |f_3(z)| dz + \frac{1}{\mu} (|f_3(0)| + |f_3(l_0)|) e^{\mu l_0} + \frac{e^{\mu l_0}}{\mu} \int_0^{l_0} |f_3'(z)| dz \right] = \frac{o(1)}{\lambda^{3-\alpha}}. \end{aligned}$$

Hence,

$$\left| \frac{1}{b} \int_0^{l_0} \int_{l_0}^s e^{\mu(s-l_0)} \sin((s-z)\mu) F(z) dz ds \right| = \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.80)$$

Since  $e^{\sqrt{\zeta}} \geq \frac{1}{5}\zeta^2$ , and taking  $\zeta = \frac{\lambda l_0^2}{\sqrt{b}} \forall \lambda \in \mathbb{R}_+^*$ , and using Lemmas 4.4.6 and 4.4.8, and Equation (4.4.80) in Equation (4.4.77), we get that

$$|y_{xx}(l_0^-)| = \frac{o(1)}{\lambda}.$$

On the interval  $(l_1, L)$ :

Proceeding with a similar computation as on  $(0, l_0)$ , we get

$$Y_{l_1}^2 = \mathbf{K}_2 e^{-i\mu(x-l_1)} + \frac{\mathbf{K}_1}{\mu} \sin((x-l_1)\mu) + \frac{1}{b\mu} \int_{l_1}^x \sin((x-z)\mu) F(z) dz \quad (4.4.81)$$

where

$$\mathbf{K}_1 = y_{xxx}(l_1^+) - \mu^2 y_x(l_1) + i\mu y_{xx}(l_1^+) - i\mu^3 y(l_1)$$

and

$$\mathbf{K}_2 = y_{xx}(l_1^+) - \mu^2 y(l_1).$$

We will solve the following differential Equation

$$\left( \frac{\partial}{\partial x} - \mu \right) Y_{l_1}^3 = Y_{l_1}^2 \quad (4.4.82)$$

where  $Y_{l_1}^3 = \left( \frac{\partial}{\partial x} + \mu \right) y$ . The solution of (4.4.82) is

$$\begin{aligned} Y_{l_1}^3 = & \mathbf{K}_3 e^{\mu(x-l_1)} - \frac{\mathbf{K}_1}{2\mu^2} \left[ \cos((x-l_1)\mu) + \sin((x-l_1)\mu) - e^{\mu(x-l_1)} \right] \\ & - \frac{\mathbf{K}_2}{1+i} \left[ e^{-i\mu(x-l_1)} - e^{\mu(x-l_1)} \right] + \frac{1}{b\mu} \int_{l_1}^x \int_{l_1}^s e^{\mu(x-s)} \sin((s-z)\mu) F(z) dz ds \end{aligned} \quad (4.4.83)$$

where

$$\mathbf{K}_3 = y_x(l_1) + \mu y(l_1).$$

Taking  $x = L$  in Equation (4.4.83) and multiplying by  $\mu e^{-\mu(L-l_1)}$ , we get

$$\begin{aligned} \frac{1}{\sqrt{2}} |y_{xx}(l_1^+)| \leq & |\mu y_x(l_1)| + |\mu^2 y(l_1)| + \frac{e^{-\mu(L-l_1)}}{\sqrt{2}} |y_{xx}(l_1^+)| + \frac{1}{\sqrt{2}} |\mu^2 y_x(l_1)| \left( e^{-\mu(L-l_1)} + 1 \right) \\ & + \frac{1}{\mu} |y_{xxx}(l_1^+)| \left( 2e^{-\mu(L-l_1)} + 1 \right) + |\mu y_x(l_1)| \left( 2e^{-\mu(L-l_1)} + 1 \right) + 2 |y_{xx}(l_1^+)| e^{-\mu(L-l_1)} \\ & + |\mu^2 y(l_1)| \left( 2e^{-\mu(L-l_1)} + 1 \right) + \left| \frac{1}{b} \int_{l_1}^L \int_{l_1}^s e^{-\mu(s-l_1)} \sin((s-z)\mu) F(z) dz ds \right|. \end{aligned} \quad (4.4.84)$$

Using the same computation in Equation (4.4.78), we get

$$\frac{1}{b} \int_{l_1}^x \int_{l_1}^s e^{-\mu(s-l_1)} \sin((s-z)\mu) F(z) dz ds = \frac{o(1)}{\lambda^{3-\alpha}}. \quad (4.4.85)$$

Since  $e^{\sqrt{\zeta}} \geq \frac{1}{5} \zeta^2$  and taking  $\zeta = \lambda(L-l_1)^2$ , and using Lemmas 4.4.6, 4.4.8, and Equation (4.4.85) in (4.4.84), we get

$$|y_{xx}(l_1^+)| = \frac{o(1)}{\lambda}.$$

Thus the proof of this Lemma is complete.  $\square$

**Lemma 4.4.10.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior

$$\int_0^L |z|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad b \int_0^L |y_{xx}|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.4.86)$$

*Proof.* Taking  $h(x) = x\theta_1(x)$  in Equation (4.4.40), where  $\theta_1 \in C^1([0, L])$  is defined as follows

$$\theta_1(x) = \begin{cases} 1 & \text{if } x \in [0, l_0], \\ 0 & \text{if } x \in [l_1, L], \\ 0 \leq \theta_1 \leq 1 & \text{elsewhere.} \end{cases} \quad (4.4.87)$$

Estimation of the term  $\operatorname{Re} \left( 2 \int_0^L h'' S \bar{y}_x dx \right)$ . Using Cauchy-Schwarz inequality, the definition of  $h$ , Lemmas 4.4.4 and 4.4.5, we get

$$\left| \operatorname{Re} \left( 2 \int_0^L h'' S \bar{y}_x dx \right) \right| = \frac{o(1)}{\lambda^{\frac{7-\alpha}{2}}}. \quad (4.4.88)$$

Estimation of the term  $\operatorname{Re} \left( 2 \int_{l_0}^{l_1} h S_x \bar{y}_{xx} dx \right)$ . Using Cauchy-Schwarz inequality, the definition of  $h$ , Lemmas 4.4.4 and 4.4.5, we get

$$\left| \operatorname{Re} \left( 2 \int_{l_0}^{l_1} h S_x \bar{y}_{xx} dx \right) \right| \leq \left( \int_{l_0}^{l_1} |S_x|^2 dx \right)^{1/2} \left( \int_{l_0}^{l_1} |y_{xx}|^2 dx \right)^{1/2} = \frac{o(1)}{\lambda^{\frac{11-\alpha}{4}}}. \quad (4.4.89)$$

Using Lemma 4.4.9, Equations (4.4.88) and (4.4.89) in (4.4.40), we get

$$\int_0^L h' |z|^2 dx + 2b \int_0^L h' |y_{xx}|^2 dx + b \int_0^{l_0} h' |y_{xx}|^2 dx + b \int_{l_1}^L h' |y_{xx}|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.4.90)$$

By using the definition of  $h$ , we obtain

$$\int_0^{l_0} |z|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad b \int_0^{l_0} |y_{xx}|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.4.91)$$

Now, taking  $h(x) = x\theta_2(x)$  in Equation (4.4.40), where  $\theta_2 \in C^1([0, L])$  is defined as follows

$$\theta_2(x) = \begin{cases} 1 & \text{if } x \in [l_1, L], \\ 0 & \text{if } x \in [0, l_0], \\ 0 \leq \theta_2 \leq 1 & \text{elsewhere.} \end{cases} \quad (4.4.92)$$

Proceeding in a similar way as above we get

$$\int_{l_1}^L |z|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad b \int_{l_1}^L |y_{xx}|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.4.93)$$

Therefore, using the third estimation of (4.4.16), Lemma 4.4.5 and combining Equations (4.4.91), and (4.4.93) we get our desired result.  $\square$

**Lemma 4.4.11.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior

$$|y_{xxx}(0)| = o(1). \quad (4.4.94)$$

*Proof.* From the interpolation inequality Theorem (see [81]), and using the fact that  $y \in H^4(0, l_0)$ ,  $\|f_4\|_{L^2(0, l_0)} = o(1)$ , Equation (4.4.14) and Lemma 4.4.10, we get

$$\begin{aligned} \|y_{xxx}\|_{L^2(0, l_0)} &\lesssim \|y_{xxxx}\|_{L^2(0, l_0)}^{1/2} \cdot \|y_{xx}\|_{L^2(0, l_0)}^{1/2} + \|y_{xx}\|_{L^2(0, l_0)} \\ &\leq \left[ \lambda^{\frac{1}{2}} \|z\|_{L^2(0, l_0)}^{\frac{1}{2}} + \frac{1}{\lambda^{\frac{3-\alpha}{2}}} \|f_4\|_{L^2(0, l_0)}^{\frac{1}{2}} \right] \|y_{xx}\|_{L^2(0, l_0)}^{\frac{1}{2}} + \|y_{xx}\|_{L^2(0, l_0)} = \frac{o(1)}{\lambda^{\frac{1}{2}}}. \end{aligned} \quad (4.4.95)$$



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Using Equation (4.4.14) and the definition of  $S$  on  $(0, l_0)$  and Lemma 4.4.10, we get

$$\|y_{xxxx}\|_{L^2(0, l_0)} = o(1). \quad (4.4.96)$$

Then, from (4.4.95) and (4.4.96), we get

$$\|y_{xxx}\|_{H^1(0, l_0)} = o(1).$$

Since  $H^1(0, l_0) \subset C([0, l_0])$ , then we get the desired result. Thus, the proof is complete.  $\square$

**Lemma 4.4.12.** Assume that  $\eta > 0$ . The solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_3)$  of system (4.4.11)-(4.4.15) satisfies the following asymptotic behavior

$$\int_{-L}^0 (|v|^2 + a|u_x|^2) dx = o(1). \quad (4.4.97)$$

*Proof.* Using the transmission condition and Lemma 4.4.11, we have

$$|u_x(0)| = |y_{xxx}(0)| = o(1). \quad (4.4.98)$$

Now, let  $q \in C^2([0, l_0])$  such that  $q(l_0) = q_x(l_0) = 0$ ,  $q(0) = 1$ . Multiply Equation (4.4.64) by  $q\bar{y}_x$  and integrate over  $(0, l_0)$ , and using the fact that  $\|f_3\|_{H_R^2(0, L)} = o(1)$ ,  $\|f_4\|_{L^2(0, L)} = o(1)$ , and  $\|y_x\|_{L^2(0, l_0)} \leq \|y_x\|_{L^2(0, L)} \lesssim \|y_{xx}\|_{L^2(0, L)} = O(1)$ , we get

$$\begin{aligned} & \frac{1}{2} \int_0^{l_0} q' (|\lambda y|^2 + 3|y_{xx}|^2) dx + b \int_0^{l_0} q'' y_{xx} \bar{y}_x dx + \frac{1}{2} |\lambda y(0)|^2 \\ & - \operatorname{Re} (b y_{xxx}(0) \bar{y}_x(0)) = \frac{o(1)}{\lambda^{2-\alpha}}. \end{aligned} \quad (4.4.99)$$

By using Lemmas 4.4.10, 4.4.11 and the fact that  $|y_x(0)| \lesssim \|y_{xx}\|_{L^2(0, L)} = O(1)$  and  $\|y_x\|_{L^2(0, l_0)} \leq \|y_x\|_{L^2(0, L)} \lesssim \|y_{xx}\|_{L^2(0, L)} = O(1)$  in Equation (4.4.99), we obtain

$$|\lambda y(0)|^2 = o(1). \quad (4.4.100)$$

From the transmission conditions we have

$$|\lambda u(0)| = |\lambda y(0)| = o(1).$$

It follows from the above Equation and Equation (4.4.11) that

$$|v(0)| = o(1). \quad (4.4.101)$$

Thus, by taking  $g(x) = x + L$  in Equation (4.4.41), and using Equations (4.4.98) and (4.4.101) we get our desired result.  $\square$

**Proof of Theorem 4.2.13.** From Lemmas 4.4.10 and 4.4.12, we get that  $\|U\|_{\mathcal{H}_3} = o(1)$ , which contradicts (4.4.9). Consequently, condition (R<sub>2</sub>) holds. This implies, from Theorem 1.3.10, the energy decay estimation (4.4.8). The proof is thus complete.

**Remark 4.4.13.** • The result in [47] can be improved. Indeed, in [47], Fathi considered a Euler-Bernoulli beam and wave equations coupled through transmission conditions. The

#### 4 Energy Decay of some boundary coupled systems involving wave\ Euler-Bernoulli beam with one locally singular fractional Kelvin-Voigt damping – 4.5 (EBB)<sub>FKV</sub> Model

damping is locally distributed and acts on the wave equation, and the rotation vanishes at the connecting point ( $y_x(\ell) = 0$ ). The system is given in the left side of Equation (4.1.1). He proved the polynomial stability with energy decay rate of type  $t^{-2}$ . By using similar computations as in Section 4.2 by taking  $\alpha = 1$ , and by solving the ordinary differential equations in Section 4.4 we can reach that  $|y_{xx}(\ell)| = o(1)$ . Thus, with the same technique of the proof of Section 4.2, we can reach that energy of the system (the left system in Equation (4.1.1)) of the mentioned paper satisfies the decay rate  $t^{-4}$ .

• In [47], when the damping acts on the beam equation the energy decay rate reached was  $t^{-2}$  (the left system in Equation (4.1.1)), when taking the condition  $y_x(\ell) = 0$  at the connecting point. However, in this paper, we proved in section 4.4 the polynomial energy decay rate of type  $t^{-1}$ , when taking  $y_{xx}(0) = 0$  at the connecting point. From this comparison, we see that the boundary conditions play a critical role in the energy decay rate for the system (4.4.2).

### 4.5 (EBB)<sub>FKV</sub> Model

In this section, we consider the Euler-Bernoulli beam with localized fractional Kelvin-Voigt damping. We study the polynomial stability of the system ((EBB)<sub>FKV</sub>). In Theorem 4.2.1, taking the input  $V(x, t) = \sqrt{d(x)}y_{xxt}(x, t)$ , then using (4.1.4), we get the output  $O$  is given by

$$O(x, t) = \sqrt{d(x)}I^{1-\alpha, \eta}y_{xxt}(x, t) = \frac{\sqrt{d(x)}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \partial_s y_{xx}(x, s) ds = \sqrt{d(x)} \partial_t^{\alpha, \eta} y_{xx}(x, t).$$

Then, we deduce that system ((EBB)<sub>FKV</sub>) can be recast into the following augmented model

$$\begin{cases} y_{tt} + \left( b y_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)} y_{xxt}(x, t) |\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{cases} \quad (4.5.1)$$

with the boundary conditions

$$y(0, t) = y_x(0, t) = 0, y_{xx}(L, t) = 0, y_{xxx}(L, t) = 0, \quad t \in (0, \infty), \quad (4.5.2)$$

and with the following initial conditions

$$y(x, 0) = y_0(x), y_t(x, 0) = y_1(x), \omega(x, \xi, 0) = 0, x \in (0, L), \xi \in \mathbb{R}. \quad (4.5.3)$$

The energy of the system (4.5.1)-(4.5.3) is given by

$$E_4(t) = \frac{1}{2} \int_0^L (|y_t|^2 + b|y_{xx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

**Lemma 4.5.1.** *Let  $U = (y, y_t, \omega)$  be a regular solution of the System (4.5.1)-(4.5.3). Then, the energy  $E_4(t)$  satisfies the following estimation*

$$\frac{d}{dt} E_4(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (4.5.4)$$

Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and therefore  $\frac{d}{dt} E_4(t) \leq 0$ . Thus, system (4.5.1)-(4.5.3) is

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dissipative in the sense that its energy is a non-increasing function with respect to time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_4$  by

$$\mathcal{H}_4 = H_L^2(0, L) \times L^2(0, L) \times W,$$

where  $W = L^2((0, L) \times \mathbb{R})$  and  $H_L^2(0, L) = \{y \in H^2(0, L); y(0) = y_x(0) = 0\}$ .

The energy space  $\mathcal{H}_4$  is equipped with the inner product defined by

$$\langle U, U_1 \rangle_{\mathcal{H}_4} = \int_0^L z \overline{z_1} dx + b \int_0^L y_{xx}(\overline{y_1})_{xx} dx + \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1}(x, \xi) d\xi dx,$$

for all  $U = (y, z, \omega)$  and  $U_1 = (y_1, z_1, \omega_1)$  in  $\mathcal{H}_4$ . We use  $\|U\|_{\mathcal{H}_4}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_4 : D(\mathcal{A}_4) \subset \mathcal{H}_4 \rightarrow \mathcal{H}_4$  by

$$D(\mathcal{A}_4) = \left\{ \begin{array}{l} U = (y, z, \omega) \in \mathcal{H}_4; z \in H_L^2(0, L), \\ \left( b y_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} \in L^2(0, L), \\ -(|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ y_{xx}(L) = 0, \text{ and } y_{xxx}(L) = 0. \end{array} \right\},$$

and for all  $U = (y, z, \omega) \in D(\mathcal{A}_4)$ ,

$$\mathcal{A}_4(y, z, \omega)^\top = \left( \begin{array}{l} - \left( b y_{xx} + \sqrt{d(x)} \kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} \\ - (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}} \end{array} \right).$$

If  $U = (y, y_t, \omega)$  is a regular solution of system (4.5.1)-(4.5.3), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_4$  given by

$$U_t = \mathcal{A}_4 U, \quad U(0) = U_0, \quad (4.5.5)$$

where  $U_0 = (y_0, y_1, 0)$ .

We can see that (in a smiliar way as in Section 4.2) the unbounded linear operator  $\mathcal{A}_4$  is m-dissipative in the energy space  $\mathcal{H}_4$ . Also, the  $C_0$ -semigroup of contractions  $e^{t\mathcal{A}_4}$  is strongly stable on  $\mathcal{H}_4$  in the sense that  $\lim_{t \rightarrow +\infty} \|e^{t\mathcal{A}_4} U_0\|_{\mathcal{H}_4} = 0, \forall U_0 \in \mathcal{H}_4$ .

**Theorem 4.5.2.** Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_4})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_4 > 0$  such that for every  $U_0 \in D(\mathcal{A}_4)$ , we have

$$E_4(t) \leq \frac{C_4}{t^{\frac{2}{1-\alpha}}} \|U_0\|_{D(\mathcal{A}_4)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_4). \quad (4.5.6)$$

According to Borichev-Tomilov Theorem 1.3.10, by taking  $\ell = 1 - \alpha$ , the polynomial energy decay (4.5.6) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_4), \quad (\text{P}_1)$$

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and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{1-\alpha}} \|(i\lambda I - \mathcal{A}_4)^{-1}\|_{\mathcal{L}(\mathcal{H}_4)} < \infty, \quad (\text{P}_2)$$

are satisfied. Since condition (P<sub>1</sub>) is satisfied (similar way as in Subsection 4.4.1), we still need to prove condition (P<sub>2</sub>). For this purpose we will use an argument of contradiction. Suppose that (P<sub>2</sub>) is false, then there exists  $\{(\lambda_n, U_n := (y_n, z_n, \omega_n(\cdot, \xi))^T)\} \subset \mathbb{R}^* \times D(\mathcal{A}_4)$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_4} = 1, \quad (4.5.7)$$

such that

$$(\lambda_n^{1-\alpha})(i\lambda_n I - \mathcal{A}_4)U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}(\cdot, \xi))^T \rightarrow 0 \quad \text{in } \mathcal{H}_4. \quad (4.5.8)$$

For simplicity, we drop the index  $n$ . Equivalently, from (4.5.8), we have

$$i\lambda y - z = \frac{f_1}{\lambda^{1-\alpha}} \quad \text{in } H_L^2(0, L), \quad (4.5.9)$$

$$i\lambda z + S_{xx} = \frac{f_2}{\lambda^{1-\alpha}} \quad \text{in } L^2(0, L), \quad (4.5.10)$$

$$(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}z_{xx}|\xi|^{\frac{2\alpha-1}{2}} = \frac{f_3(x, \xi)}{\lambda^{1-\alpha}} \quad \text{in } W, \quad (4.5.11)$$

where  $S = by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi$ . Here we will check the condition (P<sub>2</sub>) by finding a contradiction with (4.5.7) by showing  $\|U\|_{\mathcal{H}_4} = o(1)$ . In order to reach this contradiction, we proceed in a similar way as in Subsection 4.4.2. We give the estimation results directly considering that the proof of these results can follow using similar computations as in Subsection 4.4.2.

Assume that  $\eta > 0$ . Then, the solution  $(y, z, \omega) \in D(\mathcal{A}_4)$  of system (4.5.1)-(4.5.3) satisfies the asymptotic behavior estimations mentioned below.

Similar to Lemma 4.4.4, we obtain

$$\int_0^L \int_{\mathbb{R}} (|\xi|^2 + \eta) |\omega(x, \xi)|^2 d\xi dx = \frac{o(1)}{\lambda^{1-\alpha}}, \quad \int_{l_0}^{l_1} |z_{xx}|^2 dx = o(1) \quad \text{and} \quad \int_{l_0}^{l_1} |y_{xx}|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.5.12)$$

Similar to Lemma 4.4.4, we obtain

$$\int_{l_0}^{l_1} |S|^2 dx = \frac{o(1)}{\lambda^{1-\alpha}}. \quad (4.5.13)$$

Similar to Lemma 4.4.5, we get

$$\|z\|_{H^2(l_0, l_1)} = o(1). \quad (4.5.14)$$

Similar to Lemma 4.4.6, we obtain that the solution of the system (4.5.1)-(4.5.3) satisfies the following estimations

$$|y(l_0)| = \frac{o(1)}{\lambda}, \quad |y_x(l_0)| = \frac{o(1)}{\lambda}, \quad |y(l_1)| = \frac{o(1)}{\lambda}, \quad |y_x(l_1)| = \frac{o(1)}{\lambda} \quad (4.5.15)$$

and

$$\frac{1}{\lambda^{\frac{1}{2}}} |y_{xxx}(l_0^-)| = \frac{o(1)}{\lambda^{\frac{1}{4}-\frac{\alpha}{4}}} \quad \text{and} \quad \frac{1}{\lambda^{\frac{1}{2}}} |y_{xxx}(l_1^+)| = \frac{o(1)}{\lambda^{\frac{1}{4}-\frac{\alpha}{4}}}. \quad (4.5.16)$$

Similar to Lemma 4.4.9, we obtain

$$|y_{xx}(l_0^-)| = o(1) \quad \text{and} \quad |y_{xx}(l_1^+)| = o(1). \quad (4.5.17)$$

Finally, by proceeding in a similar way as in Lemma 4.4.10, we reach our desired result

$$\int_0^L |z|^2 dx = o(1) \quad \text{and} \quad b \int_0^L |y_{xx}|^2 dx = o(1) \quad (4.5.18)$$

**Proof of Theorem 4.5.7.** From the first estimation in Equation (4.5.12) and Equation (4.5.18), we get that  $\|U\|_{\mathcal{H}_4} = o(1)$ , which contradicts (4.5.7). Consequently, condition (P<sub>2</sub>) holds. This implies, from Theorem 1.3.10, the energy decay estimation (4.5.6). The proof is thus complete.

## 4.6 (EBB)-(EBB)<sub>FKV</sub> Model

In this section, we consider a system of two Euler-Bernoulli beam equations coupled via boundary connection with a localized non-regular fractional Kelvin-Voigt damping acting on one of the two equations only. In this part, we study the polynomial stability of the system. In Theorem 4.2.1, taking the input  $V(x, t) = \sqrt{d(x)}y_{xxt}(x, t)$ , then using (4.1.4), we get the output  $O$  is given by

$$O(x, t) = \sqrt{d(x)}I^{1-\alpha, \eta}y_{xxt}(x, t) = \frac{\sqrt{d(x)}}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} \partial_s y_{xx}(x, s) ds = \sqrt{d(x)} \partial_t^{\alpha, \eta} y_{xx}(x, t).$$

Then, we deduce that system ((EBB)-(EBB)<sub>FKV</sub>) can be recast into the following augmented model

$$\begin{cases} u_{tt} + au_{xxxx} = 0, & (x, t) \in (-L, 0) \times \mathbb{R}_*^+, \\ y_{tt} + \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi, t) d\xi \right)_{xx} = 0, & (x, t) \in (0, L) \times \mathbb{R}_*^+, \\ \omega_t(x, \xi, t) + (|\xi|^2 + \eta) \omega(x, \xi, t) - \sqrt{d(x)}y_{xxt}(x, t)|\xi|^{\frac{2\alpha-1}{2}} = 0, & (x, \xi, t) \in (0, L) \times \mathbb{R} \times \mathbb{R}_*^+, \end{cases} \quad (4.6.1)$$

with the following transmission and boundary conditions

$$\begin{cases} u(-L, t) = u_x(-L, t) = y(L, t) = y_x(L, t) = 0, & t \in (0, \infty), \\ au_{xxx}(0, t) - by_{xxx}(0, t) = 0, u_{xx}(0, t) = y_{xx}(0, t) = 0, & t \in (0, \infty), \\ u(0, t) = y(0, t), & t \in (0, \infty), \end{cases} \quad (4.6.2)$$

and with the following initial conditions

$$\begin{aligned} u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), & x \in (-L, 0), \\ y(x, 0) &= y_0(x), \quad y_t(x, 0) = y_1(x), \quad \omega(x, \xi, 0) = 0 & x \in (0, L), \xi \in \mathbb{R}. \end{aligned} \quad (4.6.3)$$

The energy of the system (4.6.1)-(4.6.3) is given by

$$E_5(t) = \frac{1}{2} \int_{-L}^0 (|u_t|^2 + a|u_{xx}|^2) dx + \frac{1}{2} \int_0^L (|y_t|^2 + b|y_{xx}|^2) dx + \frac{\kappa(\alpha)}{2} \int_0^L \int_{\mathbb{R}} |\omega(x, \xi, t)|^2 d\xi dx.$$

By similar computation to Lemma 4.2.2, it is easy to see that the energy  $E_5(t)$  satisfies the following estimation

$$\frac{d}{dt}E_5(t) = -\kappa(\alpha) \int_0^L \int_{\mathbb{R}} (\xi^2 + \eta) |\omega(x, \xi, t)|^2 d\xi dx. \quad (4.6.4)$$

Since  $\alpha \in (0, 1)$ , then  $\kappa(\alpha) > 0$ , and therefore  $\frac{d}{dt}E_5(t) \leq 0$ . Thus, system (4.6.1)-(4.6.3) is dissipative in the sense that its energy is a non-increasing function with respect to time variable  $t$ . Now, we define the following Hilbert energy space  $\mathcal{H}_5$  by

$$\mathcal{H}_5 = \{(u, v, y, z, \omega) \in H_L^2(-L, 0) \times L^2(-L, 0) \times H_R^2(0, L) \times L^2(0, L) \times W; \ u(0) = y(0)\},$$

where  $W = L^2((0, L) \times \mathbb{R})$  and

$$\begin{cases} H_L^2(-L, 0) = \{u \in H^2(-L, 0); u(-L) = u_x(-L) = 0\}, \\ H_R^2(0, L) = \{y \in H^2(0, L); y(L) = y_x(L) = 0\}. \end{cases} \quad (4.6.5)$$

The energy space  $\mathcal{H}_5$  is equipped with the inner product defined by

$$\begin{aligned} \langle U, U_1 \rangle_{\mathcal{H}_5} &= \int_{-L}^0 v \overline{v_1} dx + a \int_{-L}^0 u_{xx} \overline{u_1}_{xx} dx + \int_0^L z \overline{z_1} dx + b \int_0^L y_{xx} \overline{y_1}_{xx} dx \\ &+ \kappa(\alpha) \int_0^L \int_{\mathbb{R}} \omega(x, \xi) \overline{\omega_1}(x, \xi) d\xi dx, \end{aligned}$$

for all  $U = (u, v, y, z, \omega)$  and  $U_1 = (u_1, v_1, y_1, z_1, \omega_1)$  in  $\mathcal{H}_5$ . We use  $\|U\|_{\mathcal{H}_5}$  to denote the corresponding norm. We define the unbounded linear operator  $\mathcal{A}_5 : D(\mathcal{A}_5) \subset \mathcal{H}_5 \rightarrow \mathcal{H}_5$  by

$$D(\mathcal{A}_5) = \left\{ U = (u, v, y, z, \omega) \in \mathcal{H}_5; \begin{aligned} &(v, z) \in H_L^2(-L, 0) \times H_R^2(0, L), \ u \in H^4(-L, 0), \\ &\left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \in L^2(0, L), \\ &- (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}}, \quad |\xi| \omega(x, \xi) \in W, \\ &au_{xxx}(0) - by_{xxx}(0) = 0, \ u_{xx}(0) = y_{xx}(0) = 0, \text{ and } v(0) = z(0) \end{aligned} \right\},$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A}_5)$ ,

$$\mathcal{A}_5(u, v, y, z, \omega)^\top = \begin{pmatrix} v \\ -au_{xxxx} \\ z \\ - \left( by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi \right)_{xx} \\ - (|\xi|^2 + \eta) \omega(x, \xi) + \sqrt{d(x)} z_{xx} |\xi|^{\frac{2\alpha-1}{2}} \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t, \omega)$  is a regular solution of system (4.6.1)-(4.6.3), then the system can be rewritten as evolution equation on the Hilbert space  $\mathcal{H}_5$  given by

$$U_t = \mathcal{A}_5 U, \quad U(0) = U_0, \quad (4.6.6)$$

where  $U_0 = (u_0, u_1, y_0, y_1, 0)$ .

Similar to Proposition 4.2.5, the operator  $\mathcal{A}_5$  is m-dissipative on  $\mathcal{H}_5$ , consequently it generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_5})_{t \geq 0}$  following Lumer-Phillips Theorem 1.2.8. Then the solution of the evolution Equation (4.6.6) admits the following representation

$$U(t) = e^{t\mathcal{A}_5} U_0, \quad t \geq 0,$$

which leads to the well-posedness of (4.6.6). Hence, we have the following result.

**Theorem 4.6.1.** *Let  $U_0 \in \mathcal{H}_5$ , then problem (4.6.6) admits a unique weak solution  $U$  that satisfies*

$$U(t) \in C^0(\mathbb{R}^+, \mathcal{H}_5).$$

*Moreover, if  $U_0 \in D(\mathcal{A}_5)$ , then problem (4.6.6) admits a unique strong solution  $U$  that satisfies*

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}_5) \cap C^0(\mathbb{R}^+, D(\mathcal{A}_5)).$$

**Theorem 4.6.2.** *Assume that  $\eta > 0$ . The  $C_0$ -semigroup  $(e^{t\mathcal{A}_5})_{t \geq 0}$  is polynomially stable; i.e. there exists constant  $C_5 > 0$  such that for every  $U_0 \in D(\mathcal{A}_5)$ , we have*

$$E_5(t) \leq \frac{C_5}{t^{\frac{2}{3-\alpha}}} \|U_0\|_{D(\mathcal{A}_5)}^2, \quad \forall t > 0, \forall U_0 \in D(\mathcal{A}_5). \quad (4.6.7)$$

According to Theorem 1.3.10, by taking  $\ell = 3 - \alpha$ , the polynomial energy decay (4.6.7) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}_5), \quad (Q_1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{3-\alpha}} \|(i\lambda I - \mathcal{A}_5)^{-1}\|_{\mathcal{L}(\mathcal{H}_5)} < \infty, \quad (Q_2)$$

are satisfied. Since condition (Q<sub>1</sub>) is satisfied (similar to Subsection 4.4.1), we still need to prove condition (Q<sub>2</sub>). For this purpose we will use an argument of contradiction. Suppose that (Q<sub>2</sub>) is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))^T)\} \subset \mathbb{R}^* \times D(\mathcal{A}_5)$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}_5} = \|(u_n, v_n, y_n, z_n, \omega_n(\cdot, \xi))\|_{\mathcal{H}_5} = 1, \quad (4.6.8)$$

such that

$$(\lambda_n^{3-\alpha})(i\lambda_n I - \mathcal{A}_5)U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n}, f_{5,n}(\cdot, \xi))^T \rightarrow 0 \quad \text{in } \mathcal{H}_5. \quad (4.6.9)$$

For simplicity, we drop the index  $n$ . Equivalently, from (4.6.9), we have

$$i\lambda u - v = \frac{f_1}{\lambda^{3-\alpha}} \quad \text{in } H_L^2(-L, 0), \quad (4.6.10)$$

$$i\lambda v + au_{xxxx} = \frac{f_2}{\lambda^{3-\alpha}} \quad \text{in } L^2(-L, 0), \quad (4.6.11)$$

$$i\lambda y - z = \frac{f_3}{\lambda^{3-\alpha}} \quad \text{in } H_R^2(0, L), \quad (4.6.12)$$

$$i\lambda z + S_{xx} = \frac{f_4}{\lambda^{3-\alpha}} \quad \text{in } L^2(0, L), \quad (4.6.13)$$

$$(i\lambda + \xi^2 + \eta)\omega(x, \xi) - \sqrt{d(x)}z_{xx}|\xi|^{\frac{2\alpha-1}{2}} = \frac{f_5(x, \xi)}{\lambda^{3-\alpha}} \quad \text{in } W, \quad (4.6.14)$$

where  $S = by_{xx} + \sqrt{d(x)}\kappa(\alpha) \int_{\mathbb{R}} |\xi|^{\frac{2\alpha-1}{2}} \omega(x, \xi) d\xi$ . Here we will check the condition (Q<sub>2</sub>) by finding a contradiction with (4.6.8) by showing  $\|U\|_{\mathcal{H}_5} = o(1)$ . We need to prove several asymptotic behavior estimations for the solution to obtain this contradiction. Here, we give the estimations directly since the proof can be done in a smiliar way as in Subsection 4.4.2. Assume that  $\eta > 0$ . Similar to Lemma 4.4.4, the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_5)$  of system (4.6.10)-(4.6.14) satisfies the following asymptotic behavior estimations

$$\begin{aligned} \int_0^L \int_{\mathbb{R}} (|\xi|^2 + \eta) |\omega(x, \xi)|^2 d\xi dx &= \frac{o(1)}{\lambda^{3-\alpha}}, \quad \int_{l_0}^{l_1} |z_{xx}|^2 dx = \frac{o(1)}{\lambda^2}, \\ \int_{l_0}^{l_1} |y_{xx}|^2 dx &= \frac{o(1)}{\lambda^4} \quad \text{and} \quad \int_{l_0}^{l_1} |S|^2 dx = \frac{o(1)}{\lambda^{3-\alpha}}. \end{aligned} \quad (4.6.15)$$

Similar to Lemma 4.4.5, we have that the solution of the system (4.6.10)-(4.6.14) satisfies

$$\|z\|_{H^2(l_0, l_1)} = \frac{o(1)}{\lambda}, \quad \|y\|_{H^2(l_0, l_1)} = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \|S_x\|_{L^2(l_0, l_1)} = \frac{o(1)}{\lambda^{\frac{3-\alpha}{4}}}. \quad (4.6.16)$$

Similar to Lemma 4.4.6, we get that the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_5)$  of system (4.6.10)-(4.6.14) satisfies the following asymptotic behavior

$$|y(l_0)| = \frac{o(1)}{\lambda^2}, |y_x(l_0)| = \frac{o(1)}{\lambda^2}, |y(l_1)| = \frac{o(1)}{\lambda^2}, \text{ and } |y_x(l_1)| = \frac{o(1)}{\lambda^2}. \quad (4.6.17)$$

Moreover,

$$\frac{1}{\lambda^{\frac{1}{2}}} |y_{xxx}(l_0^-)| = \frac{o(1)}{\lambda^{\frac{5-\alpha}{4}}} \quad \text{and} \quad \frac{1}{\lambda^{\frac{1}{2}}} |y_{xxx}(l_1^+)| = \frac{o(1)}{\lambda^{\frac{5-\alpha}{4}}}. \quad (4.6.18)$$

Similar to Lemma 4.4.7, by multiplying Equation (4.6.13) by  $2h\bar{y}_x$ , where  $h \in C^2([0, L])$  and  $h(0) = h(L) = 0$  and proceeding as Lemma 4.4.7, we obtain

$$\begin{aligned} \int_0^L h' |z|^2 dx + 2b \int_0^L h' |y_{xx}|^2 dx + b \int_0^{l_0} h' |y_{xx}|^2 dx + b \int_{l_1}^L h' |y_{xx}|^2 dx + \text{Re} \left( 2 \int_0^L h'' S \bar{y}_x dx \right) \\ - \text{Re} \left( 2 \int_{l_0}^{l_1} h S_x \bar{y}_{xx} dx \right) - bh(l_0) |y_{xx}(l_0^-)|^2 + bh(l_1) |y_{xx}(l_1^+)|^2 = \frac{o(1)}{\lambda^{3-\alpha}}. \end{aligned} \quad (4.6.19)$$



Similar to Lemma 4.4.8, we get that

$$|y_{xx}(l_0^-)| = O(1) \quad \text{and} \quad |y_{xx}(l_1^+)| = O(1). \quad (4.6.20)$$

Similar to Lemma 4.4.9, we obtain that the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_5)$  of system (4.6.10)-(4.6.14) satisfies the following asymptotic behavior

$$|y_{xx}(l_0^-)| = \frac{o(1)}{\lambda} \quad \text{and} \quad |y_{xx}(l_1^+)| = \frac{o(1)}{\lambda}. \quad (4.6.21)$$

Similar to Lemma 4.4.10, we have that the solution  $(u, v, y, z, \omega) \in D(\mathcal{A}_5)$  of system (4.6.10)-(4.6.14) satisfies the following asymptotic behavior

$$\int_0^L |z|^2 dx = \frac{o(1)}{\lambda^2} \quad \text{and} \quad b \int_0^L |y_{xx}|^2 dx = \frac{o(1)}{\lambda^2}. \quad (4.6.22)$$

Similar to Lemma 4.4.11, we reach that

$$|y_{xxx}(0)| = o(1). \quad (4.6.23)$$

Using the transmission condition  $(au_{xxx}(0) - by_{xxx}(0) = 0)$  and Equation (4.6.23), we obtain

$$|u_{xxx}(0)| = o(1). \quad (4.6.24)$$

Similar the computations in Equations (4.4.99) and using the fact that  $|y_x(0)| \lesssim \|y_{xx}\|_{L^2(0,L)} = O(1)$ , we obtain

$$|\lambda y(0)|^2 = o(1). \quad (4.6.25)$$

By using the transmission conditions and Equation (4.6.25), we get

$$|\lambda u(0)|^2 = |\lambda y(0)|^2 = o(1). \quad (4.6.26)$$

Substituting  $v = i\lambda u - \lambda^{-2+\alpha}$  into Equation (4.6.11), we get

$$-\lambda^2 u + au_{xxx} = \lambda^{-3+\alpha} f_2 + i\lambda^{-2+\alpha} f_1. \quad (4.6.27)$$

Now, multiply Equation (4.6.27) by  $2(x+L)\overline{u_x}$  and integrate over  $(-L, 0)$ , and by using boundary conditions and the fact that  $\|f_1\|_{H_L^2(-L,0)} = o(1)$ ,  $\|f_2\|_{L^2(-L,0)} = o(1)$ ,  $|u_x(0)| \leq \|u_{xx}\|_{L^2(-L,0)} = O(1)$ ,  $\|u_x\|_{L^2(-L,0)} \leq c_p \|u_{xx}\|_{L^2(-L,0)} = O(1)$ , we get

$$\int_{-L}^0 (|\lambda u|^2 + 3|u_{xx}|^2) dx - L|\lambda y(0)|^2 + \text{Re}(2aLu_{xxx}(0)\overline{u_x}(0)) = \frac{o(1)}{\lambda^{2-\alpha}}. \quad (4.6.28)$$

By using equations (4.6.24) and (4.6.26), and the fact that  $|u_x(0)| \lesssim \|u_{xx}\|_{L^2(-L,0)} = O(1)$  in Equation (4.6.28), we obtain

$$\int_{-L}^0 |v|^2 dx = o(1) \quad \text{and} \quad a \int_{-L}^0 |u_{xx}|^2 dx = o(1). \quad (4.6.29)$$

**Proof of Theorem 4.6.8.** From the first estimation in Equation (4.6.15) and Equations (4.6.22) and (4.6.29), we get that  $\|U\|_{\mathcal{H}_5} = o(1)$ , which contradicts (4.6.8). Consequently, condition  $(Q_2)$  holds. This implies, from Theorem 1.3.10, the energy decay estimation (4.6.7). The proof

is thus complete.

## 4.7 Conclusion

We have studied the stabilization of five models of systems. We considered a Euler-Bernoulli beam equation and a wave equation coupled through boundary connection with a localized non-regular fractional Kelvin-Voigt damping that acts through the wave equation only. We proved the strong stability of the system using Arendt-Batty criteria. In addition, we established a polynomial energy decay rate of type  $t^{\frac{-4}{2-\alpha}}$ . Also, we considered two wave equations coupled via boundary connections with localized non-smooth fractional Kelvin-Voigt damping. We showed a polynomial energy decay rate of type  $t^{\frac{-4}{2-\alpha}}$ . Moreover, we studied the system of Euler-Bernoulli beam and wave equations coupled through boundary connections where the dissipation acts through the beam equation. We proved a polynomial energy decay rate of type  $t^{\frac{-2}{3-\alpha}}$ . In addition, we considered the Euler-Bernoulli beam alone with the same localized non-smooth damping. We established a polynomial energy decay rate of type  $t^{\frac{-2}{1-\alpha}}$ . In the last model, we studied the polynomial stability of a system of two Euler-Bernoulli beam equations coupled through boundary conditions with a localized non-regular fractional Kelvin-Voigt damping acting only on one of the two equations. We reached a polynomial energy decay rate of type  $t^{\frac{-2}{3-\alpha}}$ .

# 5 A N-dimensional elastic\viscoelastic transmission problem with Kelvin-Voigt damping and non smooth coefficient at the interface

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## 5.1 Introduction

Let  $\Omega \subset \mathbb{R}^N$  be a bounded open set with Lipschitz boundary  $\Gamma$ . We consider the following two wave equations coupled through velocities with a viscoelastic damping:

$$\begin{cases} u_{tt} - \operatorname{div}(a \nabla u + b(x) \nabla u_t) + c(x) y_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ y_{tt} - \Delta y - c(x) u_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \end{cases} \quad (5.1.1)$$

with the following initial conditions:

$$u(x, 0) = u_0(x), \quad y(x, 0) = y_0(x), \quad u_t(x, 0) = u_1(x), \quad y_t(x, 0) = y_1(x) \quad x \in \Omega, \quad (5.1.2)$$

and the following boundary conditions:

$$u(x, t) = y(x, t) = 0 \quad \text{on } \Gamma \times \mathbb{R}^+. \quad (5.1.3)$$

The functions  $b, c \in L^\infty(\Omega)$  such that  $b : \Omega \rightarrow \mathbb{R}_+$  is the viscoelastic damping coefficient and  $c : \Omega \rightarrow \mathbb{R}^*$  is the coupling function. The constant  $a$  is a strictly positive constant.

The stabilization of the wave equation with localized damping has received a special attention since the seventies (see [22, 39, 41, 91]). The stabilization of a material composed of two

parts: one that is elastic and the other one that is a Kelvin-Voigt type viscoelastic material was studied extensively. This type of material is encountered in real life when one uses patches to suppress vibrations, the modeling aspect of which may be found in [21]. This type of damping was examined in the one-dimensional setting in [66, 67, 71]. Later on, the wave equation with Kelvin-Voigt damping in the multidimensional setting was studied. Let us consider the wave equation with Kelvin-Voigt damping given in the following system

$$\begin{cases} u_{tt} - \operatorname{div}(a \nabla u + b(x) \nabla u_t) = 0, & \text{in } \Omega \times \mathbb{R}_+^*, \\ u(x, t) = 0, & \text{on } \Gamma \times \mathbb{R}_+^*, \\ u(\cdot, 0) = u_0, u_t(\cdot, 0) = u_1, & \text{in } \Omega. \end{cases} \quad (5.1.4)$$

In [54], the author proved that when the Kelvin-Voigt damping  $\operatorname{div}(b(x) \nabla u_t)$  is globally distributed, i.e.  $b(x) \geq b_0 > 0$  for almost all  $x \in \Omega$ , the wave equation generates an analytic semi-group. In [69], the authors considered the wave equation with local visco-elastic damping distributed around the boundary of  $\Omega$ . They proved that the energy of the system decays exponentially to zero as  $t$  goes to infinity for all usual initial data under the assumption that the damping coefficient satisfies:  $b \in C^{1,1}(\Omega)$ ,  $\Delta b \in L^\infty(\Omega)$  and  $|\nabla b(x)|^2 \leq M_0 b(x)$  for almost every  $x$  in  $\Omega$  where  $M_0$  is a positive constant. On the other hand, in [91], the author studied the stabilization of the wave equation with Kelvin-Voigt damping. He established a polynomial energy decay rate of type  $t^{-1}$  provided that the damping region is localized in a neighborhood of a part of the boundary and verifies certain geometric condition. Also, in [80], under the same assumptions on  $b$ , the authors established the exponential stability of the wave equation with local Kelvin-Voigt damping localized around a part of the boundary and an extra boundary with time delay where they added an appropriate geometric condition. Later on, in [10], the wave equation with Kelvin-Voigt damping localized in a subdomain  $\omega$  far away from the boundary without any geometric conditions was considered. The authors established a logarithmic energy decay rate for smooth initial data. In [35], the authors proved an exponential decay of the energy of a wave equation with two types of locally distributed mechanisms; a frictional damping and a Kelvin-Voigt damping where the location of each damping is such that none of them is able to exponentially stabilize the system. Under an appropriate geometric condition, piecewise multiplier geometric condition in short PMGC introduced by K. Liu in [65], on a subset  $\omega$  of  $\Omega$  where the dissipation is effective, they proved that the energy of the system decays polynomially of type  $t^{-1}$  in the absence of regularity of the Kelvin-Voigt damping coefficient  $b$ . In [4], the authors considered a multidimensional wave equation with boundary fractional damping acting on a part of the boundary of the domain and they proved stability results under geometric control condition (GCC in short, see Definition 1.4.1). In [5], the author established a polynomial energy decay rate of type  $t^{-1}$  for smooth initial data under some geometric conditions. Also, they proved a general polynomial energy decay estimate on a bounded domain where the geometric conditions on the localized viscoelastic damping are violated and they applied it on a square domain where the damping is localized in a vertical strip. Recently, in [89], the authors analyzed the long time behavior of the wave equation with local Kelvin-Voigt damping where they showed the logarithmic decay rate for energy of the system without any geometric assumption on the subdomain on which the damping is effective. Further, in [31], the author showed how perturbative approaches and the black box strategy allow to obtain decay rates for Kelvin-Voigt damped wave equations from quite standard resolvent estimates (for example Carleman estimates or geometric control

## 5 A $N$ -dimensional elastic\viscoelastic transmission problem with Kelvin-Voigt damping and non smooth coefficient at the interface – 5.1 Introduction

estimates). Recently, in [30], the authors studied the energy decay rate of the Kelvin-Voigt damped wave equation with piecewise smooth damping on the multi-dimensional domain. Under suitable geometric assumptions on the support of the damping, they obtained an optimal polynomial decay rate. In 2021, in [32], they studied the decay rates for Kelvin-Voigt damped wave equations under a geometric control condition. When the damping coefficient is sufficiently smooth they showed that exponential decay follows from geometric control conditions.

Over the past few years, the coupled systems received a vast attention due to their potential applications. The system of coupled wave equations with only one Kelvin-Voigt damping was considered in [84]. The authors considered the damping and the coupling coefficients to be constants and they established a polynomial energy decay rate of type  $t^{-1/2}$  and an optimality result. In [68], exponential stability for the wave equations with local Kelvin-Voigt damping was considered where the local viscoelastic damping distributed around the boundary of the domain. They showed that the energy of the system goes uniformly and exponentially to zero for all initial data of finite energy. In [96], the author considered the wave equation with Kelvin-Voigt damping in a non empty bounded convex domain  $\Omega$  with partition  $\Omega = \Omega_1 \cap \Omega_2$  where the viscoelastic damping is localized in  $\Omega_1$ , the coupling is through a common interface. Under the condition that the damping coefficient  $b$  is non smooth, she established a polynomial energy decay rate of type  $t^{-1}$  for smooth initial data. Also, in [43], the authors studied the stability of coupled wave equations under Geometric Control Condition (GCC in short) where they considered one viscous damping. Finally, in [51], the authors considered a system of weakly coupled wave equations with one or two locally internal Kelvin-Voigt damping and non-smooth coefficient at the interface. They established some polynomial energy decay estimates under some geometric condition. The stability of wave equations coupled through velocity and with non-smooth coupling and damping coefficients is not considered yet. Also, the study of the coupled wave equations under several geometric condition is not covered. In this work, we consider the coupled system represented in (5.1.1)-(5.1.3) by considering several geometric conditions (H1), (H2), (H3), (H4), and (H5) ( see Section 5.4) where the coupling is made via velocities and with non smooth coupling and damping coefficients. In addition, this work is a generalization of the work in Chapter 3 to a multidimensional case and we study the stability of the system (5.1.1)-(5.1.3) under several geometric control conditions. We establish polynomial stability when there is an intersection between the damping and the coupling regions. Also, when the coupling region is a subset of the damping region and under Geometric Control Condition GCC. Moreover, in the absence of any geometric condition, we study the stability of the system on the 2-dimensional square domain.

The chapter is organized as follows: first, in Section 5.2, we show that the system (5.1.1)-(5.1.3) is well-posed using semi-group approach. Then, using a unique continuation result based on a Carleman estimate and a general criteria of Arendt-Batty, we prove the strong stability of the system in the absence of the compactness of the resolvent and without any geometric condition. In Section 5.3, using a spectral analysis, we prove the non uniform stability of the system in the case where  $b(x) = b \in \mathbb{R}_+$  and  $c(x) = c \in \mathbb{R}^*$ . In Section 5.4, we establish some polynomial energy decay rates under several geometric conditions by using a frequency domain approach combined with a multiplier method. In addition, we establish two polynomial energy decay rates on a square domain where the damping and the coupling are localized in a vertical strip.

## 5.2 Well-Posedness and Strong Stability

### 5.2.1 Well posedness

In this part, using a semigroup approach, we establish the well-posedness result for the system (5.1.1)-(5.1.3).

Let  $(u, u_t, y, y_t)$  be a regular solution of the system (5.1.1)-(5.1.3). The energy of the system is given by

$$E(t) = \frac{1}{2} \int_{\Omega} (|u_t|^2 + |y_t|^2 + a|\nabla u|^2 + |\nabla y|^2) dx. \quad (5.2.1)$$

A straightforward computation gives

$$E'(t) = - \int_{\Omega} b(x) |\nabla u_t|^2 dx \leq 0.$$

Thus, the system (5.1.1)-(5.1.3) is dissipative in the sense that its energy is a non increasing function with respect to the time variable  $t$ . We define the energy Hilbert space  $\mathcal{H}$  by

$$\mathcal{H} = (H_0^1(\Omega) \times L^2(\Omega))^2$$

equipped with the following inner product

$$\langle U, \tilde{U} \rangle = \int_{\Omega} (a \nabla u \cdot \nabla \bar{\tilde{u}} + \nabla y \cdot \nabla \bar{\tilde{y}} + v \bar{\tilde{v}} + z \bar{\tilde{z}}) dx,$$

for all  $U = (u, v, y, z)^{\top} \in \mathcal{H}$  and  $\tilde{U} = (\tilde{u}, \tilde{v}, \tilde{y}, \tilde{z})^{\top} \in \mathcal{H}$ . Finally, we define the unbounded linear operator  $\mathcal{A}$  by

$$D(\mathcal{A}) = \{ U = (u, v, y, z, \omega) \in \mathcal{H} : v, z \in H_0^1(\Omega), \operatorname{div}(a(x) \nabla u + b(x) \nabla v) \in L^2(\Omega), y \in H^2(\Omega) \cap H_0^1(\Omega) \}$$

and for all  $U = (u, v, y, z, \omega) \in D(\mathcal{A})$ ,

$$\mathcal{A}(u, v, y, z)^{\top} = \begin{pmatrix} v \\ \operatorname{div}(a \nabla u + b(x) \nabla v) - c(x) z \\ z \\ \Delta y + c(x) v \end{pmatrix}.$$

If  $U = (u, u_t, y, y_t)^{\top}$  is a regular solution of system (5.1.1)-(5.1.3), then we rewrite this system as the following evolution equation

$$U_t = \mathcal{A}U, \quad U(0) = U_0 \quad (5.2.2)$$

where  $U_0 = (u_0, u_1, y_0, y_1)^{\top}$ .

**Proposition 5.2.1.** *The unbounded linear operator  $\mathcal{A}$  is  $m$ -dissipative in the energy space  $\mathcal{H}$ .*

*Proof.* For all  $U = (u, v, y, z)^{\top} \in D(\mathcal{A})$ , we have

$$\operatorname{Re}(\mathcal{A}U, U)_{\mathcal{H}} = - \int_{\Omega} b(x) |\nabla v|^2 dx \leq 0, \quad (5.2.3)$$

which implies that  $\mathcal{A}$  is dissipative. Now, let  $F = (f_1, f_2, f_3, f_4)^\top \in \mathcal{H}$ , we prove the existence of

$$U = (u, v, y, z)^\top \in D(\mathcal{A})$$

unique solution of the equation

$$-\mathcal{A}U = F. \quad (5.2.4)$$

Equivalently, we have the following system

$$-v = f_1, \quad (5.2.5)$$

$$-\operatorname{div}(a\nabla u + b(x)\nabla v) + c(x)z = f_2, \quad (5.2.6)$$

$$-z = f_3, \quad (5.2.7)$$

$$-\Delta y - c(x)v = f_4. \quad (5.2.8)$$

Inserting (5.2.5), (5.2.7) into (5.2.6) and (5.2.8), we get

$$-\operatorname{div}(a\nabla u - b(x)\nabla f_1) = f_2 + c(x)f_3, \quad (5.2.9)$$

$$-\Delta y = f_4 - c(x)f_1. \quad (5.2.10)$$

Let  $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ . Multiplying (5.2.9) and (5.2.10) by  $\bar{\varphi}$  and  $\bar{\psi}$  respectively, and integrating over  $\Omega$ , we obtain

$$a((u, v), (\varphi, \psi)) = L(\varphi, \psi), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \quad (5.2.11)$$

where

$$a((u, v), (\varphi, \psi)) = \int_{\Omega} (a\nabla u \cdot \nabla \bar{\varphi} + \nabla y \cdot \nabla \bar{\psi}) dx \quad (5.2.12)$$

and

$$L(\varphi, \psi) = \int_{\Omega} (f_2 + c(x)f_3)\bar{\varphi} dx + \int_{\Omega} b(x)\nabla f_1 \cdot \nabla \bar{\varphi} dx + \int_{\Omega} (f_4 - c(x)f_1)\bar{\psi} dx. \quad (5.2.13)$$

Thanks to (5.2.12), (5.2.13), we have that  $a$  is a sesquilinear, continuous and coercive form on  $(H_0^1(\Omega) \times H_0^1(\Omega))^2$ , and  $L$  is a antilinear continuous form on  $H_0^1(\Omega) \times H_0^1(\Omega)$ . Then, using Lax-Milgram theorem, we deduce that there exists  $(u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$  unique solution of the variational problem (5.2.11). By using the classical elliptic regularity, we deduce that (5.2.9)-(5.2.10) admits a unique solution  $(u, y) \in H_0^1(\Omega) \times (H^2(\Omega) \cap H_0^1(\Omega))$  such that  $\operatorname{div}(a\nabla u - b(x)\nabla f_1) \in L^2(\Omega)$ . By taking  $F = (0, 0, 0, 0)^\top$  in (5.2.4) it is easy to see that  $\ker\{\mathcal{A}\} = \{0\}$ . Consequently, we get  $U = (u, -f_1, y, -f_3)^\top \in D(\mathcal{A})$  is a unique solution of (5.2.4). Then,  $\mathcal{A}$  is an isomorphism and since  $\rho(\mathcal{A})$  is open set of  $\mathbb{C}$  (see Theorem 6.7 (Chapter III) in [57]), we easily get  $R(\lambda I - \mathcal{A}) = \mathcal{H}$  for a sufficiently small  $\lambda > 0$ . This, together with the dissipativeness of  $\mathcal{A}$ , imply that  $D(\mathcal{A})$  is dense in  $\mathcal{H}$  and that  $\mathcal{A}$  is m-dissipative in  $\mathcal{H}$  (see Theorem 4.5, 4.6 in [82]). The proof is thus complete.  $\square$

According to Lumer-Phillips Theorem 1.2.8, Proposition 5.2.1 implies that the operator  $\mathcal{A}$  generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}})_{t \geq 0}$  in  $\mathcal{H}$  which gives the well-posedness of (5.2.2). Then, we have the following result

**Theorem 5.2.2.** For any  $U_0 \in \mathcal{H}$ , Problem (5.2.2) admits a unique weak solution  $U$  that satisfies

$$U(t) \in C^0(\mathbb{R}^+; \mathcal{H}).$$

Moreover, if  $U_0 \in D(\mathcal{A})$ , then Problem (5.2.2) admits a unique strong solution  $U$  that satisfies

$$U(t) \in C^1(\mathbb{R}^+, \mathcal{H}) \cap C^0(\mathbb{R}^+, D(\mathcal{A})).$$

## 5.2.2 Strong Stability

This subsection is devoted to study the strong stability of System (5.1.1)-(5.1.3) in the sense that its energy converges to zero when  $t$  goes to infinity for all initial data in  $\mathcal{H}$ . The proof will be done using the unique continuation theorem based on a Carleman estimate and a general criteria of Arendt-Batty (see Theorem 1.3.3). For this aim, we assume that there exists constants  $b_0 > 0$  and  $c_0 > 0$  and two nonempty open sets  $\omega_b \subset \Omega$  and  $\omega_c \subset \Omega$ , such that

$$b(x) \geq b_0 > 0, \quad \forall x \in \omega_b, \quad (5.2.14)$$

$$c(x) \geq c_0 > 0, \quad \forall x \in \omega_c. \quad (5.2.15)$$

In this part, we prove that the energy of the System (5.1.1)-(5.1.3) decays to zero as  $t$  tends to infinity if one of the following assumptions hold:

(A1) Assume that  $\omega_b$  and  $\omega_c$  are non-empty open subsets of  $\Omega$  such that  $\omega_c \subset \omega_b$  and  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$  (see Figures 5.1, 5.2, 5.3).

(A2) Assume that  $\omega_b$  and  $\omega_c$  are non-empty open subsets of  $\Omega$  such that  $\omega = \omega_b \cap \omega_c$ . Also, assume that  $\omega$  satisfies  $\text{meas}(\overline{\omega} \cap \Gamma) > 0$  (see Figures 5.4, 5.5, 5.6).

(A3) Assume that  $\omega_b$  and  $\omega_c$  are non-empty open subsets of  $\Omega$  such that  $\omega = \omega_b \cap \omega_c$ ,  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$  and  $\omega_c$  not near the boundary (see Figure 5.7).

(A4) Assume that  $\omega_b$  is non-empty open subset of  $\Omega$  and  $c(x) = c_0 \in \mathbb{R}^*$  in  $\Omega$ . Also, assume that  $\omega$  is not near the boundary (see Figure 5.8).

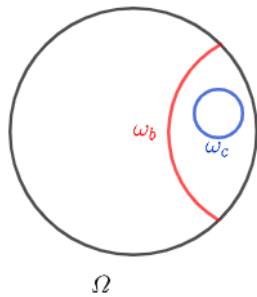


Figure 5.1

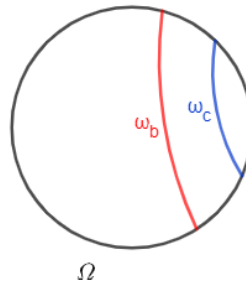


Figure 5.2



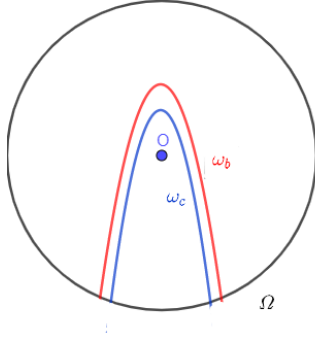


Figure 5.3

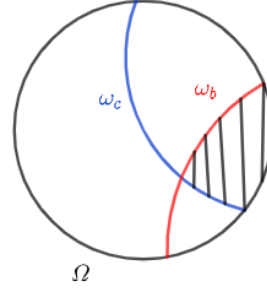


Figure 5.4

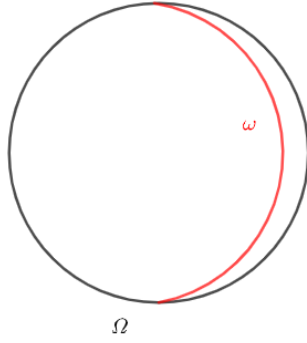


Figure 5.5

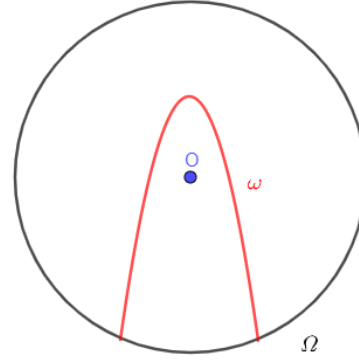


Figure 5.6

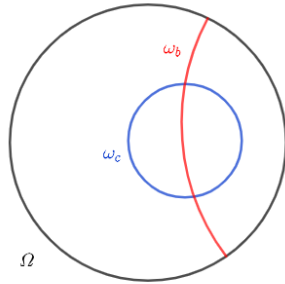


Figure 5.7

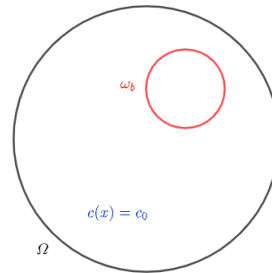


Figure 5.8

Before stating the main theorem of this section, we will give the proof of a local unique continuation result for a coupled system of wave equations.

We define the following elliptic operator  $P$  defined on a product space by

$$\begin{aligned} P : H^2(V) \times H^2(V) &\rightarrow L^2(V) \times L^2(V) \\ (u, y) &\rightarrow (\Delta u, \Delta y) \end{aligned} \quad (5.2.16)$$

and the following function  $g$  defined by

$$\begin{aligned} g : L^2(V) \times L^2(V) &\rightarrow L^2(V) \times L^2(V) \\ (u, y) &\rightarrow \left( \frac{1}{a} (-\lambda^2 u + c(x) i \lambda y), -\lambda^2 y - c(x) i \lambda u \right) \end{aligned} \quad (5.2.17)$$

**Lemma 5.2.3.** (See [52] also [51]) Let  $V$  be a bounded open set in  $\mathbb{R}^N$  and let  $\varphi = e^{\rho\psi}$  with  $\psi \in C^\infty(\mathbb{R}^N, \mathbb{R})$ ;  $|\nabla_x \psi| > 0$  and  $\rho > 0$  large enough. Then, there exist  $\tau_0$  large enough and  $C > 0$  such that

$$\tau^3 \|e^{\tau\varphi} u\|_{L^2(V)}^2 + \tau \|e^{\tau\varphi} \nabla_x u\|_{L^2(V)}^2 \leq C \|e^{\tau\varphi} \Delta u\|_{L^2(V)}^2, \quad \text{for all } u \in H_0^2(V) \text{ and } \tau > \tau_0. \quad (5.2.18)$$

**Proposition 5.2.4.** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^N$  and  $x_0$  be a point in  $\Omega$ . In a neighborhood  $V$  of  $x_0 \in \Omega$ , we take a function  $f$  such that  $\nabla f \neq 0$  in  $\overline{V}$ . Moreover, let  $(u, y) \in H^2(V) \times H^2(V)$  be a solution of  $P(u, y) = g(u, y)$ . If  $u = y = 0$  in  $\{x \in V; f(x) \geq f(x_0)\}$  then  $u = y = 0$  in a neighborhood of  $x_0$ .

*Proof.* We call  $W$  the region  $\{x \in V; f(x) \geq f(x_0)\}$ . We choose  $V'$  and  $V''$  neighborhoods of  $x_0$  such that  $V'' \subseteq V' \subseteq V$ , and we choose a function  $\chi \in C_c^\infty(V')$  such that  $\chi = 1$  in  $V''$ . Set  $\tilde{u} = \chi u$  and  $\tilde{y} = \chi y$ . Then,  $(\tilde{u}, \tilde{y}) \in H_0^2(V) \times H_0^2(V)$ . Let  $\psi = f(x) - c|x - x_0|^2$  and set  $\varphi = e^{\rho\psi}$ . Then, apply the Carleman estimate of Lemma 5.2.3 to  $\tilde{u}$  and  $\tilde{y}$  respectively, then sum the two inequalities we obtain

$$\tau^3 \int_{V'} e^{2\tau\varphi} (|\tilde{u}|^2 + |\tilde{y}|^2) dx + \tau \int_{V'} e^{2\tau\varphi} (|\nabla \tilde{u}|^2 + |\nabla \tilde{y}|^2) dx \leq C \int_{V'} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx.$$

As  $V'' \subseteq V'$  and  $\chi \in C_c^\infty(V')$  such that  $\chi = 1$  in  $V''$ , we get

$$\begin{aligned} \tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx + \tau \int_{V''} e^{2\tau\varphi} (|\nabla u|^2 + |\nabla y|^2) dx &\leq C \int_{V''} e^{2\tau\varphi} (|\Delta u|^2 + |\Delta y|^2) dx \\ + C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \end{aligned} \quad (5.2.19)$$

This implies that,

$$\tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_{V''} e^{2\tau\varphi} (|\Delta u|^2 + |\Delta y|^2) dx + C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx.$$

We have that,  $a\Delta u = -\lambda^2 u + c(x) i \lambda y$  and  $\Delta y = -\lambda^2 u - c(x) i \lambda u$ . Then, there exists  $C_{\lambda, c, a} > 0$  such that

$$(\tau^3 - C_{\lambda, c, a}) \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \quad (5.2.20)$$

Then, there exists  $\tau > 0$  large enough and  $C > 0$  such that

$$\tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_{V' \setminus V''} e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \quad (5.2.21)$$

By using that  $u = y = 0$  in  $W$ , we obtain

$$\tau^3 \int_{V''} e^{2\tau\varphi} (|u|^2 + |y|^2) dx \leq C \int_S e^{2\tau\varphi} (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \quad (5.2.22)$$

where  $S = V' \setminus (V'' \cup W)$ .

For all  $\varepsilon \in \mathbb{R}$ , we set  $V_\varepsilon = \{x \in V; \varphi(x) \leq \varphi(x_0) - \varepsilon\}$  and  $V'_\varepsilon = \{x \in V; \varphi(x) \geq \varphi(x_0) - \frac{\varepsilon}{2}\}$ . There exists  $\varepsilon$  such that  $S \subset V_\varepsilon$ . Then, choose a ball  $B_0$  with center  $x_0$  such that  $B_0 \subset V'' \cap V'_\varepsilon$ . Hence, using (5.2.22), we have

$$\int_{B_0} (|u|^2 + |y|^2) dx \leq \frac{Ce^{-\tau\varphi}}{\tau^3} \int_S (|\Delta \tilde{u}|^2 + |\Delta \tilde{y}|^2) dx. \quad (5.2.23)$$

Letting  $\tau$  tends to infinity, we obtain  $u = y = 0$  in  $B_0$ . Hence, we reached our desired result.  $\square$

**Theorem 5.2.5.** (*Calderón Theorem*) Let  $\Omega$  be a connected open set in  $\mathbb{R}^N$  and let  $\omega \subset \Omega$ , with  $\omega \neq \emptyset$ . If  $(u, y) \in H^2(\Omega) \times H^2(\Omega)$  satisfies  $P(u, y) = g(u, y)$  in  $\Omega$  and  $u = y = 0$  in  $\omega$ , then  $u$  and  $y$  vanishes in  $\Omega$ .

*Proof.* By setting  $F = \text{supp } u \cup \text{supp } y$  and using Proposition 5.2.4 instead of Proposition 4.1 in the proof of the Theorem 4.2 in [62] the result holds.  $\square$

**Theorem 5.2.6.** Assume that either (A1), (A2) (A3) or (A4) holds. Then, the  $C_0$ -semigroup  $(e^{t\mathcal{A}})_{t \geq 0}$  is strongly stable in  $\mathcal{H}$  in the sense that for all  $U_0 \in \mathcal{H}$ , the solution  $U(t) = e^{t\mathcal{A}} U_0$  of (5.2.2) satisfies

$$\lim_{t \rightarrow \infty} \|e^{t\mathcal{A}} U_0\|_{\mathcal{H}} = 0.$$

For the proof of Theorem 5.2.6, the resolvent of  $\mathcal{A}$  is not compact. Then, in order to prove this Theorem we will use a general criteria Arendt-Batty. We need to prove that the operator  $\mathcal{A}$  has no pure imaginary eigenvalues and  $\sigma(\mathcal{A}) \cap i\mathbb{R}$  contains only a countable number of continuous spectrum of  $\mathcal{A}$ . The argument for Theorem 5.2.6 relies on the subsequent lemmas.

**Lemma 5.2.7.** Assume that (A1) holds. Then, we have

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

*Proof.* From Proposition 5.2.1,  $0 \in \rho(\mathcal{A})$ . We still need to show the result for  $\lambda \in \mathbb{R}^*$ . Suppose that there exists a real number  $\lambda \neq 0$  and  $U = (u, v, y, z)^\top \in D(\mathcal{A})$  such that

$$\mathcal{A}U = i\lambda U. \quad (5.2.24)$$

From (5.2.3) and (5.2.24), we have

$$0 = \text{Re}(i\lambda \|U\|_{\mathcal{H}}^2) = \text{Re}(\langle \mathcal{A}U, U \rangle_{\mathcal{H}}) = - \int_{\Omega} b(x) |\nabla v|^2 dx. \quad (5.2.25)$$

Using the condition (5.2.14) and Poincaré's inequality implies that

$$b(x) \nabla v = 0 \text{ in } \Omega \text{ and } v = 0 \text{ in } \omega_b. \quad (5.2.26)$$

Detailing (5.2.24) and using (5.2.26), we get the following system

$$v = i\lambda u \quad \text{in } \Omega, \quad (5.2.27)$$

$$a\Delta u - c(x)z = i\lambda v \quad \text{in } \Omega, \quad (5.2.28)$$

$$z = i\lambda y \quad \text{in } \Omega, \quad (5.2.29)$$

$$\Delta y + c(x)v = i\lambda z \quad \text{in } \Omega. \quad (5.2.30)$$

From (5.2.26), (5.2.27) and (5.2.28) with the assumption (A1), we get

$$z = 0 \text{ on } \omega_c. \quad (5.2.31)$$

Inserting (5.2.27) into (5.2.28), and using (5.2.26) we get

$$\begin{cases} \lambda^2 u + a\Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{in } \omega_b. \end{cases} \quad (5.2.32)$$

Using the unique continuation theorem, we get

$$u = 0 \text{ on } \Omega. \quad (5.2.33)$$

Now, substituting (5.2.29) in (5.2.30), using (5.2.31) and the definition of  $c(x)$ , we get

$$\begin{cases} \lambda^2 y + \Delta y = 0 & \text{in } \Omega, \\ y = 0 & \text{in } \omega_c. \end{cases} \quad (5.2.34)$$

Using the unique continuation theorem, we get

$$y = 0 \text{ on } \Omega. \quad (5.2.35)$$

Therefore,  $U = 0$ . thus the proof is complete.  $\square$

**Lemma 5.2.8.** Assume that either (A2), (A3) or (A4) holds. Then, we have

$$\ker(i\lambda I - \mathcal{A}) = \{0\}, \quad \forall \lambda \in \mathbb{R}.$$

*Proof.* From Proposition 5.2.1,  $0 \in \rho(\mathcal{A})$ . We still need to show the result for  $\lambda \in \mathbb{R}^*$ . Suppose that there exists a real number  $\lambda \neq 0$  and  $U = (u, v, y, z)^\top \in D(\mathcal{A})$  such that

$$\mathcal{A}U = i\lambda U. \quad (5.2.36)$$

From (5.2.3) and (5.2.36), we have

$$0 = \operatorname{Re}(i\lambda \|U\|_{\mathcal{H}}^2) = \operatorname{Re}(\langle \mathcal{A}U, U \rangle_{\mathcal{H}}) = - \int_{\Omega} b(x) |\nabla v|^2 dx. \quad (5.2.37)$$

Condition (5.2.14) implies that

$$\nabla v = 0 \text{ in } \omega_b. \quad (5.2.38)$$

Detailing (5.2.36) and using (5.2.38), we get the following system

$$v = i\lambda u \text{ in } \Omega, \quad (5.2.39)$$

$$a\Delta u - c(x)z = i\lambda v \text{ in } \Omega, \quad (5.2.40)$$

$$z = i\lambda y \text{ in } \Omega, \quad (5.2.41)$$

$$\Delta y + c(x)v = i\lambda z \text{ in } \Omega. \quad (5.2.42)$$

Now, we will distinguish between the following three cases:

**Case 1.** If (A2) holds. Then, by using Poincaré's inequality we get

$$v = 0 \text{ on } \omega. \quad (5.2.43)$$

From (5.2.40), (5.2.43) and using (5.2.41) we get

$$y = 0 \text{ on } \omega. \quad (5.2.44)$$

Inserting (5.2.39) and (5.2.41) into (5.2.40) and (5.2.42) respectively, we get

$$\begin{cases} \lambda^2 u + a\Delta u - i\lambda c(x)y = 0 & \text{in } \Omega, \\ \lambda^2 y + \Delta y + i\lambda c(x)u = 0 & \text{in } \Omega, \\ u = y = 0 & \text{in } \omega. \end{cases} \quad (5.2.45)$$

Then, using Theorem 5.2.5 we get that  $u = y = 0$  in  $\Omega$ . Thus, we deduce that  $U = 0$  in  $\Omega$  and we reached our desired result.

**Case 2.** If (A3) holds. Then, by using Poincaré's inequality we get

$$v = 0 \text{ on } \omega_b. \quad (5.2.46)$$

Proceeding in the same way as in Case 1., we get

$$\begin{cases} \lambda^2 u + a\Delta u - i\lambda c(x)y = 0 & \text{in } \Omega, \\ \lambda^2 y + \Delta y + i\lambda c(x)u = 0 & \text{in } \Omega, \\ u = y = 0 & \text{in } \omega. \end{cases} \quad (5.2.47)$$

Then, using Theorem 5.2.5 we get that  $u = y = 0$  in  $\Omega$ . Thus, we deduce that  $U = 0$  in  $\Omega$  and we reached our desired result.

**Case 3.** Assume that (A4) holds. By differentiating (5.2.40) and using the fact that  $c(x) = c_0$  in  $\Omega$ , we obtain

$$\partial_j y = 0 \text{ in } \omega_b, \quad \forall j = 1 \dots, N.$$

Then, for all  $j = 1, \dots, N$ , we have the following system

$$\begin{cases} \lambda^2 \partial_j u + a\Delta \partial_j u - i\lambda c_0 \partial_j y = 0 & \text{in } \Omega, \\ \lambda^2 \partial_j y + \Delta \partial_j y + i\lambda c_0 \partial_j u = 0 & \text{in } \Omega, \\ \partial_j u = \partial_j y = 0 & \text{in } \omega. \end{cases} \quad (5.2.48)$$

By applying Theorem 5.2.5, we obtain

$$\partial_j u = \partial_j y = 0 \text{ in } \Omega, \quad \forall j = 1 \dots, N.$$

Using the fact that  $u = y = 0$  on  $\Gamma$ , we get  $u = y = 0$  in  $\Omega$ . Consequently,  $U = 0$  in  $\Omega$ . □

**Lemma 5.2.9.** Assume that either (A1), (A2), (A3) or (A4) holds. Then, we have

$$R(i\lambda I - \mathcal{A}) = \mathcal{H}, \quad \text{for all } \lambda \in \mathbb{R}.$$

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*Proof.* From Proposition 5.2.1, we have  $0 \in \rho(\mathcal{A})$ . We still need to show the result for  $\lambda \in \mathbb{R}^*$ . Set  $F = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ , we look for  $U = (u, v, y, z) \in D(\mathcal{A})$  solution of

$$(i\lambda I - \mathcal{A})U = F. \quad (5.2.49)$$

Equivalently, we have

$$v = i\lambda u - f_1, \quad (5.2.50)$$

$$i\lambda v - \operatorname{div}(a\nabla u + b(x)\nabla v) + c(x)z = f_2, \quad (5.2.51)$$

$$z = i\lambda y - f_3, \quad (5.2.52)$$

$$i\lambda z - \Delta y - c(x)v = f_4. \quad (5.2.53)$$

Let  $(\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$ , multiplying Equations (5.2.51) and (5.2.53) by  $\bar{\varphi}$  and  $\bar{\psi}$  respectively and integrating over  $\Omega$ , we obtain

$$\int_{\Omega} i\lambda v \bar{\varphi} dx + \int_{\Omega} a\nabla u \nabla \bar{\varphi} dx + \int_{\Omega} b(x)\nabla v \nabla \bar{\varphi} dx + \int_{\Omega} c(x)z \bar{\varphi} dx = \int_{\Omega} f_2 \bar{\varphi} dx, \quad (5.2.54)$$

$$\int_{\Omega} i\lambda z \bar{\psi} dx + \int_{\Omega} \nabla y \nabla \bar{\psi} dx - \int_{\Omega} c(x)v \bar{\psi} dx = \int_{\Omega} f_4 \bar{\psi} dx. \quad (5.2.55)$$

Substituting  $v$  and  $z$  in (5.2.50) and (5.2.52) into (5.2.54) and (5.2.55) and taking the sum, we obtain

$$a((u, y), (\varphi, \psi)) = L(\varphi, \psi), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \quad (5.2.56)$$

where

$$a((u, y), (\varphi, \psi)) = a_1((u, y), (\varphi, \psi)) + a_2((u, y), (\varphi, \psi))$$

with

$$\begin{cases} a_1((u, y), (\varphi, \psi)) = \int_{\Omega} (a\nabla u \nabla \bar{\varphi} + \nabla y \nabla \bar{\psi}) dx + i\lambda \int_{\Omega} b(x)\nabla u \nabla \bar{\varphi} dx, \\ a_2((u, y), (\varphi, \psi)) = -\lambda^2 \int_{\Omega} (u\bar{\varphi} + y\bar{\psi}) dx + i\lambda \int_{\Omega} c(x)(y\bar{\varphi} - u\bar{\psi}) dx, \end{cases}$$

and

$$L(\varphi, \psi) = \int_{\Omega} (f_2 + c(x)f_3 + i\lambda f_1) \bar{\varphi} dx + \int_{\Omega} (f_4 - c(x)f_1 + i\lambda f_3) \bar{\psi} dx + \int_{\Omega} b(x)\nabla f_1 \bar{\varphi}_x dx.$$

Let  $V = H_0^1(\Omega) \times H_0^1(\Omega)$  and  $V' = H^{-1}(\Omega) \times H^{-1}(\Omega)$  the dual space of  $V$ . Let us consider the following operators,

$$\left\{ \begin{array}{lcl} A: & V & \rightarrow V' \\ & (u, y) & \rightarrow A(u, y) \end{array} \right\} \quad \left\{ \begin{array}{lcl} A_1: & V & \rightarrow V' \\ & (u, y) & \rightarrow A_1(u, y) \end{array} \right\} \quad \left\{ \begin{array}{lcl} A_2: & V & \rightarrow V' \\ & (u, y) & \rightarrow A_2(u, y) \end{array} \right\}$$

such that

$$\left\{ \begin{array}{l} (A(u, y))(\varphi, \psi) = a((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \\ (A_1(u, y))(\varphi, \psi) = a_1((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega), \\ (A_2(u, y))(\varphi, \psi) = a_2((u, y), (\varphi, \psi)), \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega). \end{array} \right. \quad (5.2.57)$$

Our goal is to prove that  $A$  is an isomorphism operator. For this aim, we divide the proof into

three steps.

**Step 1.** In this step, we prove that the operator  $A_1$  is an isomorphism operator. For this aim, following the second equation of (5.2.57) we can easily verify that  $a_1$  is a bilinear continuous coercive form on  $H_0^1(\Omega) \times H_0^1(\Omega)$ . Then, by Lax-Milgram Lemma, the operator  $A_1$  is an isomorphism.

**Step 2.** In this step, we prove that the operator  $A_2$  is compact. According to the third equation of (5.2.57), we have

$$|a_2((u, y), (\varphi, \psi))| \leq C \|(u, y)\|_{L^2(\Omega)} \|(\varphi, \psi)\|_{L^2(\Omega)}.$$

Finally, using the compactness embedding from  $H_0^1(\Omega)$  to  $L^2(\Omega)$  and the continuous embedding from  $L^2(\Omega)$  into  $H^{-1}(\Omega)$  we deduce that  $A_2$  is compact.

From steps 1 and 2, we get that the operator  $A = A_1 + A_2$  is a Fredholm operator of index zero. Consequently, by Fredholm alternative, to prove that operator  $A$  is an isomorphism it is enough to prove that  $A$  is injective, i.e.  $\ker\{A\} = \{0\}$ .

**Step 3.** In this step, we prove that  $\ker\{A\} = \{0\}$ . For this aim, let  $(\tilde{u}, \tilde{y}) \in \ker\{A\}$ , i.e.

$$a((\tilde{u}, \tilde{y}), (\varphi, \psi)) = 0, \quad \forall (\varphi, \psi) \in H_0^1(\Omega) \times H_0^1(\Omega).$$

Equivalently, we have

$$\begin{aligned} & -\lambda^2 \int_{\Omega} (\tilde{u}\bar{\varphi} + \tilde{y}\bar{\psi}) dx + i\lambda \int_{\Omega} c(x) (\tilde{y}\bar{\varphi} - \tilde{u}\bar{\psi}) dx + \int_{\Omega} (a\nabla\tilde{u}\nabla\bar{\varphi} + \nabla\tilde{y}\nabla\bar{\psi}) dx \\ & + i\lambda \int_{\Omega} b(x)\nabla\tilde{u}\nabla\bar{\varphi} dx = 0. \end{aligned} \quad (5.2.58)$$

Taking  $\varphi = \tilde{u}$  and  $\psi = \tilde{y}$  in equation (5.2.58), we get

$$\begin{aligned} & -\lambda^2 \int_{\Omega} |\tilde{u}|^2 dx - \lambda^2 \int_{\Omega} |\tilde{y}|^2 dx + a \int_{\Omega} |\nabla\tilde{u}|^2 dx + \int_0^L |\nabla\tilde{y}|^2 dx - 2\lambda \operatorname{Im} \left( \int_{\Omega} c(x) \tilde{y}\bar{\tilde{u}} dx \right) \\ & + i\lambda \int_{\Omega} b(x) |\nabla\tilde{u}|^2 dx = 0. \end{aligned}$$

Taking the imaginary part of the above equality, we get

$$\int_{\Omega} b(x) |\nabla\tilde{u}|^2 dx = 0,$$

we get,

$$\nabla\tilde{u} = 0, \quad \text{in } \omega_b. \quad (5.2.59)$$

Then, we find that

$$\begin{cases} -\lambda^2 \tilde{u} - a\Delta\tilde{u} + i\lambda c(x) \tilde{y} & = 0, & \text{in } \Omega \\ -\lambda^2 \tilde{y} - \Delta\tilde{y} - i\lambda c(x) \tilde{u} & = 0, & \text{in } \Omega \\ \tilde{u} = \tilde{y} & = 0, & \text{in } \omega \end{cases}$$

Now, it is easy to see that the vector  $\tilde{U}$  defined by  $\tilde{U} = (\tilde{u}, i\lambda\tilde{u}, \tilde{y}, i\lambda\tilde{y})$  belongs to  $D(\mathcal{A})$  and

we have  $i\lambda\tilde{U} - \mathcal{A}\tilde{U} = 0$ . Therefore,  $\tilde{U} \in \ker(i\lambda I - \mathcal{A})$ , then by Lemmas 5.2.7 and 5.2.8, we get  $\tilde{U} = 0$ , this implies that  $\tilde{u} = \tilde{y} = 0$ . Consequently,  $\ker\{A\} = \{0\}$ . Thus, from step 3 and Fredholm alternative, we get that the operator  $A$  is an isomorphism. It is easy to see that the operator  $L$  is continuous from  $V$  to  $L^2(\Omega) \times L^2(\Omega)$ . Consequently, Equation (5.2.56) admits a unique solution  $(u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$ . Thus, using  $v = i\lambda u - f_1$ ,  $z = i\lambda y - f_3$  and using the classical regularity arguments, we conclude that Equation (5.2.49) admits a unique solution  $U \in D(\mathcal{A})$ . The proof is thus complete.  $\square$

**Proof of Theorem 5.2.6.** Using Lemma 5.2.7 and 5.2.8, we have that  $\mathcal{A}$  has non pure imaginary eigenvalues. According to Lemmas 5.2.7, 5.2.8, 5.2.9 and with the help of the closed graph theorem of Banach, we deduce that  $\sigma(\mathcal{A}) \cap i\mathbb{R} = \emptyset$ . Thus, we get the conclusion by applying Arend-Batty Theorem 1.3.3. The proof of the theorem is thus complete.

## 5.3 Non Uniform Stability

In this section, our aim is to prove the non-uniform stability of the system (5.1.1)-(5.1.3). For this aim, we assume that

$$b(x) = b \in \mathbb{R}_+ \quad \text{and} \quad c(x) = c \in \mathbb{R}^*. \quad (5.3.1)$$

Our main result in this section is the following theorem.

**Theorem 5.3.1.** *Under condition (5.3.1). The energy of the system (5.1.1)-(5.1.3) does not decay uniformly in the energy space  $\mathcal{H}$ .*

For the proof of Theorem 5.3.1, we aim to study the asymptotic behavior of the eigenvalues of the operator  $\mathcal{A}$  near the imaginary axis. First, we will determine the characteristic equation satisfied by the eigenvalues of  $\mathcal{A}$ . So, let  $\lambda \in \mathbb{C}$  be an eigenvalue of  $\mathcal{A}$  and let  $U = (u, v, y, z)^\top \in D(\mathcal{A})$  be an associated eigenvector, i.e,

$$\mathcal{A}U = \lambda U,$$

Equivalently,

$$v = \lambda u, \quad (5.3.2)$$

$$\operatorname{div}(a\nabla u + b\nabla v) - cz = \lambda v, \quad (5.3.3)$$

$$z = \lambda y, \quad (5.3.4)$$

$$\operatorname{div}(\nabla y) + cv = \lambda z. \quad (5.3.5)$$

Inserting (5.3.2) and (5.3.4) into (5.3.3) and (5.3.5) respectively, we get

$$\lambda^2 u - (a + \lambda b)\Delta u + c\lambda y = 0, \quad (5.3.6)$$

$$\lambda^2 y - \Delta y - c\lambda u = 0. \quad (5.3.7)$$

From (5.3.7), we have

$$u = \frac{1}{\lambda c} [\Delta y - \lambda^2 y]. \quad (5.3.8)$$



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Substitute (5.3.8) in (5.3.6), we get

$$\begin{cases} (a + \lambda b)\Delta^2 y - [(1 + a)\lambda^2 + b\lambda^3] \Delta y + \lambda^2 (\lambda^2 + \alpha^2) y = 0, & \text{in } \Omega \\ y = \Delta y = 0 & \text{on } \Gamma. \end{cases} \quad (5.3.9)$$

Now, let  $(\mu_k, \varphi_k)$  be, respectively, the sequence of the eigenvalues and the eigenvectors of the Laplace operator with fully Dirichlet boundary conditions in  $\Omega$ , i.e,

$$\begin{cases} -\Delta \varphi_k = \mu_k^2 \varphi_k & \text{in } \Omega, \\ \varphi_k = 0 & \text{on } \Gamma. \end{cases} \quad (5.3.10)$$

Then by taking  $y = \varphi_k$  in (5.3.9), we deduce the following characteristic equation

$$P(\lambda) = \lambda^4 + b\mu_k^2 \lambda^3 + [(1 + a)\mu_k^2 + c^2] \lambda^2 + b\mu_k^4 \lambda + a\mu_k^4 = 0. \quad (5.3.11)$$

**Proposition 5.3.2.** *There exists  $k_0 \in \mathbb{N}^*$  sufficiently large and two sequences  $(\lambda_{1,k})_{|k| \geq k_0}$  and  $(\lambda_{2,k})_{|k| \geq k_0}$  of simple roots of  $P$  satisfying the following asymptotic behavior*

$$\lambda_{1,k} = i\mu_k - \frac{c^2}{2b\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right) \quad (5.3.12)$$

and

$$\lambda_{2,k} = -i\mu_k - \frac{c^2}{2b\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right). \quad (5.3.13)$$

*Proof.* Set  $\xi = \frac{\lambda}{\mu_k}$  and  $\zeta_k = \frac{1}{\mu_k}$  in (5.3.11), we obtain

$$h(\xi) = b\xi^3 + b\xi + a\zeta_k + c^2 \xi^2 \zeta_k^3 + (1 + a) \xi^2 \zeta_k + \xi^4 \zeta_k = 0. \quad (5.3.14)$$

Now, in order to find the eigenvalues of the operator  $\mathcal{A}$  we need to give the roots of  $h$ . For this aim, we will proceed in the following two steps.

**Step 1.** Let

$$f(\xi) = b(\xi^3 + \xi) \quad \text{and} \quad f_1(\xi) = a\zeta_k + c^2 \xi^2 \zeta_k^3 + (1 + a) \xi^2 \zeta_k + \xi^4 \zeta_k.$$

We look for  $r_k$  sufficiently small such that

$$|f| > |h - f| = |f_1| \quad \text{on } \partial D,$$

where  $D = \{\xi \in \mathcal{C}; |\xi - i| \leq r_k\}$ .

Let  $\xi \in \partial D(i, r_k)$ , then  $\xi = i + r_k e^{i\theta}$  with  $0 \leq \theta \leq 2\pi$ . We have

$$f(\xi) = b(\xi^3 + \xi) = b\xi r_k (2ie^{i\theta} + r_k e^{2i\theta}).$$

But, if  $r_k \leq \frac{1}{2}$  then

$$|\xi| \geq |1 - r_k| \geq \frac{1}{2}$$

and

$$\left| 2ie^{i\theta} + r_k e^{2i\theta} \right| \geq \left| 2ie^{i\theta} \right| - r_k \geq \frac{3}{2}.$$

This implies that

$$|f| = |b(\xi^3 + \xi)| \geq \frac{3br_k}{4}, \quad \text{if } r_k \leq \frac{1}{2}.$$

On the other hand, since  $\xi$  is bounded in  $D$  and  $\xi_k \rightarrow 0$  we have,

$$|f_1(\xi)| \leq c\xi_k, \quad \text{for some constant } c > 0.$$

So, it is enough to choose  $r_k = \frac{4c}{3b}\xi_k$ .

Similarly, we can find  $r_k$  sufficiently small such that

$$|f| > |h - f| = |f_1| \quad \text{on } \partial D' = \partial \{\xi \in \mathbb{C}; |\xi + i| \leq r_k\}.$$

**Step 2.** By using Step 1. and Rouché's Theorem, there exists  $k_0$  large enough such that for all  $|k| \geq k_0$  the roots of the polynomial  $h$  are close to the roots of the polynomial  $f(\xi) = b(\xi^3 + \xi)$ . Then,

$$\xi_k^+ = i + \varepsilon_k^+ \quad \text{and} \quad \xi_k^- = -i + \varepsilon_k^-, \quad \text{with} \quad \lim_{|k| \rightarrow \infty} \varepsilon_k^\pm = 0. \quad (5.3.15)$$

Inserting Equation (5.3.15) in Equation (5.3.14) and using the fact that  $\lambda_k^\pm = \mu_k \xi_k^\pm$ , we get

$$\varepsilon_k^\pm = o\left(\frac{1}{\mu_k}\right) \quad \text{and} \quad \lambda_k^\pm = \pm i\mu_k + \tilde{\varepsilon}_k, \quad \text{where} \quad \lim_{|k| \rightarrow +\infty} \tilde{\varepsilon}_k = 0. \quad (5.3.16)$$

Multiplying Equation (5.3.11) by  $\frac{1}{\mu_k^4}$ , we get

$$\frac{1}{\mu_k^4} \lambda^4 + \frac{b}{\mu_k^2} \lambda^3 + \frac{(1+a)}{\mu_k^2} \lambda^2 + \frac{c^2}{\mu_k^4} \lambda^2 + b\lambda + a = 0. \quad (5.3.17)$$

Inserting Equation (5.3.16) in Equation (5.3.17), we get

$$\tilde{\varepsilon}_k = -\frac{c^2}{2b\mu_k^2} + o\left(\frac{1}{\mu_k^3}\right). \quad (5.3.18)$$

The proof is thus complete.  $\square$

**Proof of Theorem 5.3.1.** From Proposition 5.3.2 the large eigenvalues in (5.3.12)-(5.3.13) approach the imaginary axis and therefore the system (5.1.1)-(5.1.3) is not uniformly stable in the energy space  $\mathcal{H}$ .

## 5.4 Polynomial Stability

In this section, we will study the polynomial energy decay rate of the system (5.1.1)-(5.1.3). In order to study the energy decay rate of the system, we consider the following geometric assumptions on  $\omega_b$ ,  $\omega_c$  and  $\omega = \omega_b \cap \omega_c \neq \emptyset$ :

(H1) The open subset  $\omega$  verifies the GCC (see Figure 5.6 and Figure 5.9).

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(H2) Assume that  $\text{meas}(\overline{\omega_c} \cap \Gamma) > 0$  and  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$ . Also, assume that  $\omega_c \subset \omega_b$  and  $\omega_c$  satisfies the GCC (see Figure 5.3).

(H3) Assume that  $\omega_b \subset \Omega$ ,  $\overline{\omega_c} \subset \omega_b$  such that  $\Omega$  is a non-convex open set and  $\omega_c$  satisfies GCC (see Figure 5.10).

(H4) Assume that  $\Omega = (0, L) \times (0, L)$ ,  $\omega_c \subset \omega_b$  such that  $\omega_b = \{(x, y) \in \mathbb{R}^2; \varepsilon_1 < x < \varepsilon_4 \text{ and } 0 < y < L\}$ ,  $\omega_c = \{(x, y) \in \mathbb{R}^2; \varepsilon_2 < x < \varepsilon_3 \text{ and } 0 < y < L\}$  for  $0 < \varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \varepsilon_4 < L$  (see Figure 5.11).

(H5) Assume that  $\Omega = (0, L) \times (0, L)$ ,  $\omega_c \subset \omega_b$  such that  $\omega_b = \{(x, y) \in \mathbb{R}^2; 0 < x < \varepsilon_2 \text{ and } 0 < y < L\}$  and  $\omega_c = \{(x, y) \in \mathbb{R}^2; 0 < x < \varepsilon_1 \text{ and } 0 < y < L\}$  for  $0 < \varepsilon_1 < \varepsilon_2 < L$  (see Figure 5.12).

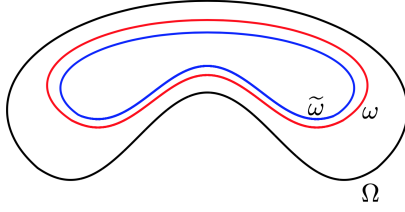


Figure 5.9

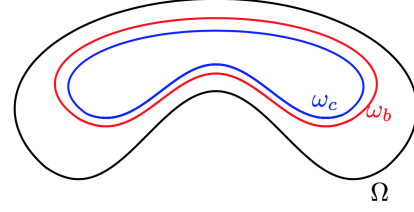


Figure 5.10

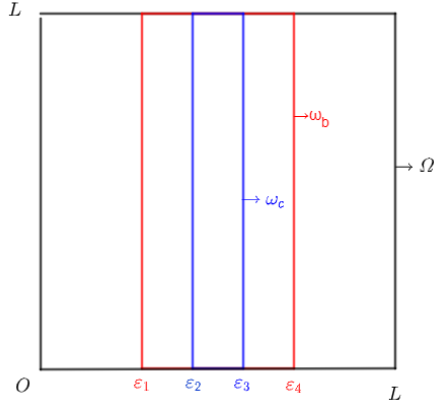


Figure 5.11

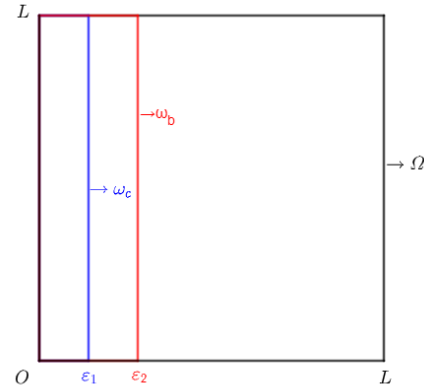


Figure 5.12

**Remark 5.4.1. (About Geometric Conditions and Smoothness of the boundary.)**

1. If the  $\Gamma$ -condition applies, then it is enough to give a Lipschitz boundary conditions to  $\Gamma$ .
2. The GCC is an optimal condition, then we need more regularity to the  $\Gamma$ -condition, thus we need to take  $\Gamma$  of class  $C^3$ .
3. In (H1), if  $\Omega$  is a convex domain. Then, the condition  $\omega$  satisfies the GCC means that  $\text{meas}(\overline{\omega} \cap \Gamma) > 0$  (i.e  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$  and  $\text{meas}(\overline{\omega_c} \cap \Gamma) > 0$ ).
4. In (H1), if  $\Omega$  is a non convex domain. When  $\omega$  satisfies the GCC, we study the case when  $\text{meas}(\overline{\omega_b} \cap \Gamma) > 0$  and without the condition  $\text{meas}(\overline{\omega_c} \cap \Gamma) > 0$ . For the case when both  $\omega_b$  and  $\omega_c$  are not near the boundary, we didn't study it since the strong stability remains an open problem in this case.
5. In (H4) and (H5),  $\omega_b$  and  $\omega_c$  does not satisfy any geometric condition.

One of the main tools to prove the polynomial stability of (5.1.1)-(5.1.3) such that the assumption (H1) holds and such that  $c \in W^{1,\infty}(\Omega)$  is to use the exponential energy decay of the coupled wave equations via velocities with two viscous dampings. We consider the following system

$$\begin{cases} \varphi_{tt} - a\Delta\varphi + d(x)\varphi_t + c(x)\psi_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \psi_{tt} - \Delta\psi + d(x)\psi_t - c(x)\varphi_t = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ \varphi(x) = \psi(x) = 0 & \text{on } \Gamma \times \mathbb{R}^+, \\ (\varphi(x,0), \psi(x,0)) = (\varphi_0, \psi_0) \quad \text{and} \quad (\varphi_t(x,0), \psi_t(x,0)) = (\varphi_1, \psi_1) & \text{in } \Omega. \end{cases} \quad (5.4.1)$$

where  $d \in W^{1,\infty}(\Omega)$  such that

$$d(x) \geq d_0 > 0 \quad \text{on } \omega_\epsilon \subset \omega \subset \Omega.$$

The energy of System (5.4.1) is given by

$$E_{aux}(t) = \frac{1}{2} \int_{\Omega} \left( |\varphi_t|^2 + a|\nabla\varphi|^2 + |\psi_t|^2 + |\nabla\psi|^2 \right) dx$$

and by a straightforward calculation, we have

$$\frac{d}{dt} E_{aux}(t) = - \int_{\Omega} d(x) |\varphi_t|^2 dx - \int_{\Omega} d(x) |\psi_t|^2 dx \leq 0.$$

Thus, System (5.4.1) is dissipative in the sense that its energy is a non-increasing function with respect to the time variable  $t$ . The auxiliary energy Hilbert space of Problem (5.4.1) is given by

$$\mathcal{H}_{aux} = \left( H_0^1(\Omega) \times L^2(\Omega) \right)^2.$$

We denote by  $\eta = \varphi_t$  and  $\xi = \psi_t$ . The auxiliary energy space  $\mathcal{H}_{aux}$  is endowed with the following norm

$$\|\Phi\|_{\mathcal{H}_{aux}}^2 = \|\eta\|^2 + a\|\nabla\varphi\|^2 + \|\xi\|^2 + \|\nabla\psi\|^2,$$

where  $\|\cdot\|$  denotes the norm of  $L^2(\Omega)$  and  $\Phi = (\varphi, \psi, \eta, \xi)$  in  $\mathcal{H}_{aux}$ . We define the unbounded linear operator  $\mathcal{A}_{aux}$  by

$$D(\mathcal{A}_{aux}) = \left( (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega) \right)^2, \quad (5.4.2)$$

and

$$\mathcal{A}_{aux}(\varphi, \eta, \psi, \xi)^\top = \begin{pmatrix} \eta \\ a\Delta\varphi - d(x)\eta - c(x)\xi \\ \xi \\ \Delta\psi - d(x)\xi + c(x)\eta \end{pmatrix}.$$

If  $\Phi = (\varphi, \psi, \eta, \xi)$  is the state of System (5.4.1), then this system is transformed into a first order evolution equation on the auxiliary Hilbert space  $\mathcal{H}_{aux}$  given by

$$\Phi_t = \mathcal{A}_{aux}\Phi, \quad \Phi(0) = \Phi_0,$$

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where  $\Phi_0 = (\varphi_0, \eta_0, \psi_0, \xi_0)$ . It is easy to see that  $\mathcal{A}_{aux}$  is m-dissipative and generates a  $C_0$ -semigroup of contractions  $(e^{t\mathcal{A}_{aux}})_{t \geq 0}$ .

**Remark 5.4.2.** From [43], we know that when  $\omega$  satisfies the GCC condition and under the equality speed condition we have that the system of two wave equations coupled through velocity with one viscous damping is exponentially stable (see Theorem 3.1 in [43]). Taking this result into consideration and the fact that our system is considered with two viscous dampings and that  $d, c \in W^{1,\infty}(\Omega)$ , by proceeding with a similar proof with  $a \in \mathbb{R}_+^*$  as in Theorem 3.1 in [43], we can reach that the system (5.4.1) decays exponentially such that there exists  $M \geq 1$  and  $\theta > 0$  such that for all initial data  $U_0 \in \mathcal{H}$ , the energy of the system (5.4.1) satisfies the following estimation

$$E_{aux}(t) \leq M e^{-\theta t} E(0), \quad \forall t > 0.$$

Now, we will state the main theorems in this section.

**Theorem 5.4.3.** Assume that the boundary  $\Gamma$  is of class  $C^3$ . Also, assume that assumption (H1) holds and that  $c \in W^{1,\infty}(\Omega)$ . Then, for all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $C > 0$  independent of  $U_0$ , such that the energy of the system (5.1.1)-(5.1.3) satisfies the following estimation

$$E(t, U) \leq \frac{C}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (5.4.3)$$

According to Theorem 1.3.10 of Borichev and Tomilov, by taking  $\ell = 2$ , the polynomial energy decay (5.4.3) holds if the following conditions

$$i\mathbb{R} \subset \rho(\mathcal{A}), \quad (C1)$$

and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^2} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (C2)$$

are satisfied. Since Condition (C1) is already proved in Lemmas 5.2.7 and 5.2.8. We will prove condition (C2) by an argument of contradiction. For this purpose, suppose that (C2) is false, then there exists

$\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n)^\top)\} \subset \mathbb{R}^* \times D(\mathcal{A})$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1, \quad (5.4.4)$$

such that

$$(\lambda_n)^2 (i\lambda_n I - \mathcal{A}) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^\top \rightarrow 0 \text{ in } \mathcal{H}. \quad (5.4.5)$$

For simplicity, we drop the index  $n$ . Equivalently, from (5.4.5), we have

$$i\lambda u - v = \frac{f_1}{\lambda^2} \text{ in } H_0^1(\Omega), \quad (5.4.6)$$

$$i\lambda v - \operatorname{div}(a\nabla u + b(x)\nabla v) + c(x)z = \frac{f_2}{\lambda^2} \text{ in } L^2(\Omega), \quad (5.4.7)$$

$$i\lambda y - z = \frac{f_3}{\lambda^2} \text{ in } H_0^1(\Omega), \quad (5.4.8)$$

$$i\lambda z - \operatorname{div}(\nabla y) - c(x)v = \frac{f_4}{\lambda^2} \text{ in } L^2(\Omega). \quad (5.4.9)$$

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Here we will check the condition (C2) by finding a contradiction with (5.4.4) such as  $\|U\|_{\mathcal{H}} = o(1)$ . From Equations (5.4.4), (5.4.6) and (5.4.8) we obtain

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda} \quad \text{and} \quad \|y\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}. \quad (5.4.10)$$

For clarity, we will divide the proof into several lemmas.

**Lemma 5.4.4.** *Assume that the assumption (H1) holds. Then, we have that the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.6)-(5.4.9) satisfies the following estimations*

$$\|\nabla v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda}, \quad \|v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda}, \quad \|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2} \quad \text{and} \quad \|\nabla u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \quad (5.4.11)$$

*Proof.* Taking the inner product of (5.4.5) with  $U$  in  $\mathcal{H}$ , we get

$$\int_{\Omega} b(x) |\nabla v|^2 dx = -\operatorname{Re}(\langle \mathcal{A}U, U \rangle_{\mathcal{H}}) = \operatorname{Re}(\langle (i\lambda I - \mathcal{A})U, U \rangle_{\mathcal{H}}) = \frac{o(1)}{\lambda^2}. \quad (5.4.12)$$

Then,

$$\int_{\omega_b} |\nabla v|^2 dx = \frac{o(1)}{\lambda^2}. \quad (5.4.13)$$

By using Poincaré inequality and Equation (5.4.13), we get the second estimation in (5.4.11). From Equation (5.4.6) and the second estimation in (5.4.11), we obtain

$$\|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}.$$

By using (5.4.6) and the first estimation in (5.4.11), we get the last estimation.  $\square$

Inserting Equations (5.4.6) and (5.4.8) into (5.4.7) and (5.4.9), we get

$$\lambda^2 u + \operatorname{div}(a \nabla u + b(x) \nabla v) - i \lambda c(x) y = -\frac{f_2}{\lambda^2} - c(x) \frac{f_3}{\lambda^2} - \frac{i f_1}{\lambda}, \quad (5.4.14)$$

$$\lambda^2 y + \Delta y + i \lambda c(x) u = -\frac{f_4}{\lambda^2} + c(x) \frac{f_1}{\lambda^2} - \frac{i f_3}{\lambda}. \quad (5.4.15)$$

**Lemma 5.4.5.** *Assume that the assumption (H1) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.6)-(5.4.9) satisfies the following estimation*

$$\int_{\omega_{\varepsilon}} |\lambda y|^2 dx = o(1) \quad (5.4.16)$$

where  $\omega_{\varepsilon} \subset \omega$  such that  $\omega_{\varepsilon}$  satisfies the GCC condition.

*Proof.* First, we define the function  $\zeta \in C_c^{\infty}(\mathbb{R}^N)$  such that

$$\zeta(x) = \begin{cases} 1 & \text{if } x \in \omega_{\varepsilon}, \\ 0 & \text{if } x \in \Omega \setminus \omega, \\ 0 \leq \zeta \leq 1 & \text{elsewhere,} \end{cases} \quad (5.4.17)$$

such that  $\omega_{\varepsilon} \subset \omega$  satisfies the GCC condition. Multiply Equation (5.4.14) by  $\lambda \zeta \bar{y}$  and integrate over  $\Omega$  and using Green's formula, and using Equation (5.4.10) and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we get

$$\lambda^3 \int_{\Omega} u \bar{\zeta} \bar{y} dx - \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\bar{y} \nabla \bar{\zeta} + \bar{\zeta} \nabla \bar{y}) dx - i \int_{\Omega} c(x) \bar{\zeta} |\lambda y|^2 dx = \frac{o(1)}{\lambda}. \quad (5.4.18)$$

Estimation of the first term in (5.4.18). Using Cauchy-Schwarz inequality, (5.4.11) and (5.4.10) we get

$$\left| \lambda^3 \int_{\Omega} u \bar{\zeta} \bar{y} dx \right| \leq \lambda^3 \|u\|_{L^2(\omega)} \cdot \|y\|_{L^2(\omega)} = o(1). \quad (5.4.19)$$

Estimation of the second term in (5.4.18). Using Cauchy-Schwarz inequality, (5.4.10), (5.4.11), the fact that  $\text{supp } \zeta \subset \omega$ , and that  $\|U\|_{\mathcal{H}} = 1$ , we obtain the following estimations

$$\left| \lambda \int_{\Omega} a \nabla u \cdot \nabla \bar{\zeta} \bar{y} dx \right| \leq \lambda \|\nabla u\|_{L^2(\omega)} \cdot \|y\|_{L^2(\omega)} = \frac{o(1)}{\lambda^2}, \quad (5.4.20)$$

$$\left| \lambda \int_{\Omega} a \nabla u \bar{\zeta} \nabla \bar{y} dx \right| \leq \lambda \|\nabla u\|_{L^2(\omega)} \cdot \|\nabla y\|_{L^2(\omega)} = \frac{o(1)}{\lambda}, \quad (5.4.21)$$

$$\left| \lambda \int_{\Omega} b(x) \nabla v \bar{\zeta} \nabla \bar{y} dx \right| \leq \lambda \|\nabla v\|_{L^2(\omega)} \cdot \|\nabla y\|_{L^2(\omega)} = o(1), \quad (5.4.22)$$

$$\left| \lambda \int_{\Omega} b(x) \nabla v \cdot \nabla \bar{\zeta} \bar{y} dx \right| \leq \lambda \|\nabla v\|_{L^2(\omega)} \cdot \|y\|_{L^2(\omega)} = \frac{o(1)}{\lambda}. \quad (5.4.23)$$

Inserting Equations (5.4.19)-(5.4.23) in Equation (5.4.18), we get that

$$i \int_{\Omega} c(x) \bar{\zeta} |\lambda y|^2 dx = o(1). \quad (5.4.24)$$

Using the definition of the function  $c$  and  $\zeta$ , we obtain our desired result.  $\square$

**Lemma 5.4.6.** *For any  $\lambda \in \mathbb{R}$ , the solution  $(\varphi, \psi) \in ((H^2(\Omega) \cap H_0^1(\Omega))^2)$  of the system*

$$\begin{cases} \lambda^2 \varphi + a \Delta \varphi - i \lambda d(x) \varphi - i \lambda c(x) \psi = u, & \text{in } \Omega \\ \lambda^2 \psi + \Delta \psi - i \lambda d(x) \psi + i \lambda c(x) \varphi = y, & \text{in } \Omega \\ \varphi = \psi = 0, & \text{on } \Gamma \end{cases} \quad (5.4.25)$$

*satisfies the following estimation*

$$\|\lambda \varphi\|_{L^2(\Omega)} + \|\nabla \varphi\|_{L^2(\Omega)} + \|\lambda \psi\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} \leq M (\|u\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}). \quad (5.4.26)$$

*Proof.* Using Remark 5.4.2, then the resolvent set of the associated operator  $\mathcal{A}_{aux}$  contains  $i\mathbb{R}$  and  $(i\lambda I - \mathcal{A}_{aux})^{-1}$  is uniformly bounded on the imaginary axis. Consequently, there exists  $M > 0$  such that

$$\sup_{\lambda \in \mathbb{R}} \|(i\lambda I - \mathcal{A}_{aux})^{-1}\|_{\mathcal{L}(\mathcal{H}_{aux})} \leq M < +\infty. \quad (5.4.27)$$

Now, since  $(u, y) \in H_0^1(\Omega) \times H_0^1(\Omega)$ , then  $(0, -u, 0, -y)$  belongs to  $\mathcal{H}_{aux}$ , and from (5.4.27), there exists  $(\varphi, \eta, \psi, \xi) \in D(\mathcal{A}_{aux})$  such that  $(i\lambda I - \mathcal{A}_{aux})(\varphi, \eta, \psi, \xi) = (0, -u, 0, -y)^{\top}$  i.e.

$$i\lambda\varphi - \eta = 0, \quad (5.4.28)$$

$$i\lambda\eta - a\Delta\varphi + d(x)\eta + c(x)\xi = -u, \quad (5.4.29)$$

$$i\lambda\psi - \xi = 0, \quad (5.4.30)$$

$$i\lambda\xi - \Delta\psi + d(x)\xi - c(x)\eta = -y, \quad (5.4.31)$$

such that

$$\|(\varphi, \eta, \psi, \xi)\|_{\mathcal{H}_a} \leq M(\|u\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}). \quad (5.4.32)$$

From equations (5.4.28)-(5.4.32), we deduce that  $(\varphi, \psi)$  is a solution of (5.4.25) and we have

$$\|\lambda\varphi\|_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} + \|\lambda\psi\|_{L^2(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} \leq M(\|u\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}). \quad (5.4.33)$$

Thus, we get our desired result.  $\square$

**Lemma 5.4.7.** Assume that the assumption (H1) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.6)-(5.4.9) satisfies the following estimations

$$\int_{\Omega} |\lambda u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 dx = o(1). \quad (5.4.34)$$

*Proof.* For clarity, we will divide the proof of this Lemma into two steps.

**Step 1.** Multiply (5.4.14) by  $\lambda^2 \bar{\varphi}$  and integrate over  $\Omega$ , and using Green's formula, Equation (5.4.26), and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we obtain

$$\int_{\Omega} (\lambda^2 \bar{\varphi} + a\Delta \bar{\varphi}) \lambda^2 u dx - \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \bar{\varphi} dx - \int_{\Omega} i\lambda^3 c(x) y \bar{\varphi} dx = \frac{o(1)}{\lambda}. \quad (5.4.35)$$

From Equations (5.4.11) and (5.4.26), we obtain

$$\left| \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \bar{\varphi} dx \right| = o(1). \quad (5.4.36)$$

By using Equation (5.4.36) in (5.4.35), we get

$$\int_{\Omega} (\lambda^2 \bar{\varphi} + a\Delta \bar{\varphi}) \lambda^2 u dx - \int_{\Omega} i\lambda^3 c(x) y \bar{\varphi} dx = o(1). \quad (5.4.37)$$

Now, from System (5.4.25), we have that

$$\lambda^2 \bar{\varphi} + a\Delta \bar{\varphi} = -i\lambda d(x) \bar{\varphi} - i\lambda c(x) \bar{\psi} + \bar{u}. \quad (5.4.38)$$

Inserting Equation (5.4.38) into (5.4.37), we obtain

$$\int_{\Omega} |\lambda u|^2 dx - i\lambda^3 \int_{\Omega} d(x) u \bar{\varphi} dx - i\lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx - \int_{\Omega} i\lambda^3 c(x) y \bar{\varphi} dx = o(1). \quad (5.4.39)$$

By using (5.4.11) and (5.4.26), we get

$$\left| i\lambda^3 \int_{\Omega} d(x) u \bar{\varphi} dx \right| \leq \lambda^3 \|u\|_{L^2(\omega_\varepsilon)} \cdot \|\varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda}. \quad (5.4.40)$$



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Now, inserting Equation (5.4.40) into (5.4.39), we get

$$\int_{\Omega} |\lambda u|^2 dx - i\lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx - i\lambda^3 \int_{\Omega} c(x) y \bar{\varphi} dx = o(1). \quad (5.4.41)$$

**Step 2.**

Multiply Equation (5.4.15) by  $\lambda^2 \bar{\psi}$ , integrate over  $\Omega$ , using Green's formula, and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we obtain

$$\int_{\Omega} (\lambda^2 \bar{\psi} + \Delta \bar{\psi}) \lambda^2 y dx + i\lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx = \frac{o(1)}{\lambda}. \quad (5.4.42)$$

From System (5.4.25), we have

$$\lambda^2 \bar{\psi} + \Delta \bar{\psi} = -i\lambda d(x) \bar{\psi} + i\lambda c(x) \bar{\varphi} + \bar{y} \quad (5.4.43)$$

Inserting (5.4.43) into (5.4.42), we get

$$\int_{\Omega} |\lambda y|^2 dx - i\lambda^3 \int_{\Omega} d(x) y \bar{\psi} dx + i\lambda^3 \int_{\Omega} c(x) y \bar{\varphi} dx + i\lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx = \frac{o(1)}{\lambda}. \quad (5.4.44)$$

Using Cauchy-Schwarz inequality, Lemma 5.4.5, and Equation (5.4.26)

$$\left| i\lambda^3 \int_{\Omega} d(x) y \bar{\psi} dx \right| = o(1). \quad (5.4.45)$$

Inserting (5.4.45) into (5.4.44), we get

$$\int_{\Omega} |\lambda y|^2 dx + i\lambda^3 \int_{\Omega} c(x) y \bar{\varphi} dx + i\lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx = o(1). \quad (5.4.46)$$

Adding Equations (5.4.41) and (5.4.46), we get

$$\int_{\Omega} |\lambda u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 dx = o(1). \quad (5.4.47)$$

Thus, the proof of the Lemma is completed.  $\square$

**Lemma 5.4.8.** Assume that the assumption (H1) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.6)-(5.4.9) satisfies the following estimations

$$\int_{\Omega} |\nabla u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1). \quad (5.4.48)$$

*Proof.* Multiply Equation (5.4.14) by  $\bar{u}$ , integrating over  $\Omega$ , Green's formula, Equation (5.4.10) and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we obtain

$$\int_{\Omega} |\lambda u|^2 dx - a \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} b(x) \nabla v \cdot \nabla \bar{u} dx - \int_{\Omega} i\lambda c(x) y \bar{u} dx = \frac{o(1)}{\lambda^2} \quad (5.4.49)$$

Using Equation (5.4.11) and Lemma 5.4.7, we obtain

$$\int_{\Omega} |\nabla u|^2 dx = o(1). \quad (5.4.50)$$

Multiplying Equation (5.4.15) by  $\bar{y}$  and proceeding in a similar way as above, we get

$$\int_{\Omega} |\nabla y|^2 dx = o(1). \quad (5.4.51)$$

□

**Proof of Theorem 5.4.3.** Consequently, from the results of Lemmas 5.4.7 and 5.4.8, we obtain

$$\int_{\Omega} (|v|^2 + |z|^2 + a|\nabla u|^2 + |\nabla y|^2) dx = o(1).$$

Hence  $\|U\|_{\mathcal{H}} = o(1)$ , which contradicts (5.4.4). Consequently, condition (C2) holds. This implies that the energy decay estimation (5.4.3). The proof is thus complete.

**Remark 5.4.9.** In the case when  $d, c \in L^\infty(\Omega)$  such that they are discontinuous functions, we didn't find any result in the literature on the stability of the system (5.4.1). But, we can conjecture that the system (5.4.1) is exponentially stable. Further, we have that the system (5.4.1) with  $d, c$  are discontinuous functions is exponentially stable in the dimension 1 (see Chapter 2).

One of the main tools to prove the polynomial stability of the system (5.1.1)-(5.1.3) when one of the assumptions (H2), (H3), (H4) or (H5) holds is to use the exponential or polynomial decay of the wave equation with viscous damping. We consider the following system

$$\begin{cases} \varphi_{tt} - \Delta \varphi + \mathbb{1}_{\omega_c}(x) \varphi_t = 0 & \text{in } \Omega \times (0, +\infty) \\ \varphi = 0 & \text{in } \Gamma \times (0, +\infty) \\ \varphi(\cdot, 0) = \varphi_0, \quad \varphi_t(\cdot, 0) = \varphi_1. \end{cases} \quad (5.4.52)$$

**Remark 5.4.10. (About System (5.4.52))**

1. If (H2) or (H3) holds, system (5.4.52) is exponentially stable (see [29] and Lemma 3.8 in [5]).
2. If (H4) holds, the energy of the wave equation (5.4.52) with local viscous damping decays polynomially as  $t^{-1}$  for smooth initial data (see Example 3 in [71]).
3. If (H5) holds, the energy of the wave equation (5.4.52) with local viscous damping decays polynomially as  $t^{-\frac{4}{3}}$  for smooth initial data (see [90]).

**Theorem 5.4.11.** Assume that the boundary  $\Gamma$  is of class  $C^3$ . Also, assume that assumption (H2) or (H3) holds. Also, assume that the energy of the system (5.4.52) is exponentially stable. Then, for all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $C_2 > 0$  independent of  $U_0$ , the energy of the system (5.1.1)-(5.1.3) satisfies the following estimation

$$E(t, U) \leq \frac{C_2}{t} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (5.4.53)$$

Following Theorem 1.3.10 of Borichev and Tomilov, the polynomial energy decay (5.4.53) holds if (C1) and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^2} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (C3)$$

holds. Since Condition (C1) is already proved (see Lemmas 5.2.7 and 5.2.8). We will prove condition (C3) by an argument of contradiction. For this purpose, suppose that (C3) is false, then there exists

$\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n)^\top)\} \subset \mathbb{R}^* \times D(\mathcal{A})$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1, \quad (5.4.54)$$

such that

$$\lambda_n^2 (i\lambda_n I - \mathcal{A}) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^\top \rightarrow 0 \text{ in } \mathcal{H}. \quad (5.4.55)$$

For simplicity, we drop the index  $n$ . Equivalently, from (5.4.55), we have

$$i\lambda u - v = \lambda^{-2} f_1 \text{ in } H_0^1(\Omega), \quad (5.4.56)$$

$$i\lambda v - \operatorname{div}(a\nabla u + b(x)\nabla v) + c(x)z = \lambda^{-2} f_2 \text{ in } L^2(\Omega), \quad (5.4.57)$$

$$i\lambda y - z = \lambda^{-2} f_3 \text{ in } H_0^1(\Omega), \quad (5.4.58)$$

$$i\lambda z - \operatorname{div}(\nabla y) - c(x)v = \lambda^{-2} f_4 \text{ in } L^2(\Omega). \quad (5.4.59)$$

Here we will check the condition (C3) by finding a contradiction with (5.4.54) such as  $\|U\|_{\mathcal{H}} = o(1)$ . From Equations (5.4.54), (5.4.56) and (5.4.58) we obtain

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda} \quad \text{and} \quad \|y\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}. \quad (5.4.60)$$

**Lemma 5.4.12.** *Assume that the assumption (H2) or (H3) holds. We have that the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.56)-(5.4.59) satisfies the following estimations*

$$\|\nabla v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda} \quad \text{and} \quad \|\nabla u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \quad (5.4.61)$$

The proof of this Lemma is similar to that of Lemma 5.4.4.

**Lemma 5.4.13.** *Under the assumption (H2). We have that the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.56)-(5.4.59) satisfies the following estimations*

$$\|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \quad (5.4.62)$$

*Proof.* By using Poincaré inequality and Equation (5.4.61), we get

$$\|v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda}. \quad (5.4.63)$$

From Equation (5.4.56) and (5.4.63), we obtain

$$\|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^2}. \quad (5.4.64)$$

□

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Inserting Equations (5.4.56) and (5.4.58) into (5.4.57) and (5.4.59), we get

$$\lambda^2 u + \operatorname{div}(a \nabla u + b(x) \nabla v) - i \lambda c(x) y = -\frac{f_2}{\lambda^2} - c(x) \frac{f_3}{\lambda^2} - \frac{i f_1}{\lambda}, \quad (5.4.65)$$

$$\lambda^2 y + \Delta y + i \lambda c(x) u = -\frac{f_4}{\lambda^2} + c(x) \frac{f_1}{\lambda^2} - \frac{i f_3}{\lambda}. \quad (5.4.66)$$

**Lemma 5.4.14.** Assume that assumption (H3) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.56)-(5.4.59) satisfies the following estimation

$$\int_{\tilde{\omega}_b} |\lambda u|^2 dx = o(1). \quad (5.4.67)$$

such that  $\omega_c \subset \tilde{\omega}_b \subset \omega_b$ .

*Proof.* Let a non-empty open subset  $\tilde{\omega}_b$  such that  $\omega_c \subset \tilde{\omega}_b \subset \omega_b$ . Then, we define the function  $h_1 \in C_c^\infty(\mathbb{R}^N)$  such that

$$h_1(x) = \begin{cases} 1 & \text{if } x \in \tilde{\omega}_b, \\ 0 & \text{if } x \in \Omega \setminus \omega_b, \\ 0 \leq h_1 \leq 1 & \text{elsewhere.} \end{cases} \quad (5.4.68)$$

Multiply (5.4.65) by  $h_1 \bar{u}$  and integrate over  $\Omega$ , we get

$$\int_{\Omega} h_1 |\lambda u|^2 dx - \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (h_1 \nabla \bar{u} + \nabla h_1 \bar{u}) dx - i \lambda \int_{\Omega} h_1 c(x) y \bar{u} dx = \frac{o(1)}{\lambda^2}. \quad (5.4.69)$$

Using (5.4.60) and (5.4.61), we have

$$\left| \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (h_1 \nabla \bar{u} + \nabla h_1 \bar{u}) dx \right| = \frac{o(1)}{\lambda^2} \quad (5.4.70)$$

and

$$\left| i \lambda \int_{\Omega} h_1 c(x) y \bar{u} dx \right| = \frac{O(1)}{\lambda}. \quad (5.4.71)$$

Thus, by using Equations (5.4.70) and (5.4.71) in (5.4.69), we obtain

$$\int_{\Omega} h_1 |\lambda u|^2 dx = \frac{O(1)}{\lambda}. \quad (5.4.72)$$

Thus, we reach our desired result.  $\square$

**Lemma 5.4.15.** Assume that assumption (H2) or (H3) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.56)-(5.4.59) satisfies the following estimation

$$\int_{\omega_c} |\lambda y|^2 dx = o(1). \quad (5.4.73)$$

*Proof.* **Case1.** Assume that assumption (H2) holds, we define the function  $\rho \in C_c^\infty(\mathbb{R}^N)$  such that

$$\rho(x) = \begin{cases} 1 & \text{if } x \in \omega_c, \\ 0 & \text{if } x \in \Omega \setminus \omega_b, \\ 0 \leq \rho \leq 1 & \text{elsewhere.} \end{cases} \quad (5.4.74)$$

Now, multiplying (5.4.65) by  $\lambda \rho \bar{y}$ , integrating over  $\Omega$  and using Green's formula, (5.4.60) and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we get

$$\lambda^3 \int_{\Omega} u \rho \bar{y} dx - \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \bar{y} + \rho \nabla \bar{y}) dx - i \int_{\Omega} c(x) \rho |\lambda y|^2 dx = \frac{o(1)}{\lambda}. \quad (5.4.75)$$

Using (5.4.60), (5.4.61) and Cauchy-Schwarz we obtain

$$\left| \lambda^3 \int_{\Omega} u \rho \bar{y} dx \right| \leq \lambda^3 \|u\|_{L^2(\omega_b)} \cdot \|y\|_{L^2(\omega_b)} = o(1) \quad (5.4.76)$$

and

$$\left| \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \bar{y} + \rho \nabla \bar{y}) dx \right| = o(1). \quad (5.4.77)$$

Thus, using Equations (5.4.76) and (5.4.77) in (5.4.75) we obtain our desired result for the first case.

**Case 2.** Assume that assumption (H3) holds. Define the function  $h_2 \in C_c^\infty(\mathbb{R}^N)$  such that

$$h_2(x) = \begin{cases} 1 & \text{if } x \in \omega_c, \\ 0 & \text{if } x \in \Omega \setminus \tilde{\omega}_b, \\ 0 \leq h_2 \leq 1 & \text{elsewhere.} \end{cases} \quad (5.4.78)$$

Multiply (5.4.65) by  $\lambda h_2 \bar{y}$  and integrate over  $\Omega$ , and using Green's formula, (5.4.60) and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we get

$$\lambda^3 \int_{\Omega} u h_2 \bar{y} dx - \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla h_2 \bar{y} + h_2 \nabla \bar{y}) dx - i \int_{\Omega} c(x) h_2 |\lambda y|^2 dx = \frac{o(1)}{\lambda}. \quad (5.4.79)$$

Using (5.4.60), (5.4.61) and Cauchy-Schwarz, we obtain

$$\left| \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla h_2 \bar{y} + h_2 \nabla \bar{y}) dx \right| = o(1). \quad (5.4.80)$$

By using Equation (5.4.80) in (5.4.79), we obtain

$$\lambda^3 \int_{\Omega} u h_2 \bar{y} dx - i \int_{\Omega} c(x) h_2 |\lambda y|^2 dx = o(1). \quad (5.4.81)$$

Now, multiply (5.4.66) by  $\lambda h_2 \bar{u}$  and integrate over  $\Omega$ , and using Green's formula, (5.4.60) and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we get

$$\lambda^3 \int_{\Omega} y h_2 \bar{u} dx - \lambda \int_{\Omega} \nabla y \cdot (\nabla h_2 \bar{u} + h_2 \nabla \bar{u}) dx + i c_0 \int_{\omega_c} |\lambda u|^2 dx = \frac{o(1)}{\lambda}. \quad (5.4.82)$$

Using (5.4.54), (5.4.60) and Cauchy-Schwarz, we obtain

$$\left| \lambda \int_{\Omega} \nabla y \cdot (\nabla h_2 \bar{u} + h_2 \nabla \bar{u}) dx \right| = o(1). \quad (5.4.83)$$

Inserting (5.4.83) into (5.4.82), we get

$$\lambda^3 \int_{\Omega} y h_2 \bar{u} dx + i c_0 \int_{\omega_c} |\lambda u|^2 dx = o(1). \quad (5.4.84)$$

Summing Equations (5.4.81) and (5.4.84), and taking the imaginary part and using Equation (5.4.67), we obtain our desired result.  $\square$

**Lemma 5.4.16.** *Assume that assumption (H2) or (H3) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.56)-(5.4.59) satisfies the following estimations*

$$\int_{\Omega} |\lambda u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 dx = o(1). \quad (5.4.85)$$

*Proof.* Let  $\varphi, \psi \in H^2(\Omega) \cap H_0^1(\Omega)$  be the solution of the following system

$$\begin{cases} \lambda^2 \varphi + a \Delta \varphi - i \lambda \mathbb{1}_{\omega_c}(x) \varphi = u, & \text{in } \Omega \\ \lambda^2 \psi + \Delta \psi - i \lambda \mathbb{1}_{\omega_c}(x) \psi = y, & \text{in } \Omega \\ \varphi = \psi = 0, & \text{on } \Gamma \end{cases} \quad (5.4.86)$$

where  $(u, v, y, z)$  is the solution of (5.4.56)-(5.4.59). Since either (H2) or (H3) holds, then system (5.4.52) is exponentially stable. Thus, there exists  $M > 0$  such that system (5.4.86) satisfies the following estimation

$$\|\lambda \varphi\|_{L^2(\Omega)} + \|\nabla \varphi\|_{L^2(\Omega)} + \|\lambda \psi\|_{L^2(\Omega)} + \|\nabla \psi\|_{L^2(\Omega)} \leq M (\|u\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}). \quad (5.4.87)$$

**Case 1.** Under the assumption (H3). Multiply (5.4.65) by  $\lambda^2 \bar{\varphi}$  and integrate over  $\Omega$ , and using Green's formula, Equation (5.4.87), and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we obtain

$$\int_{\Omega} (\lambda^2 \bar{\varphi} + a \Delta \bar{\varphi}) \lambda^2 u dx - \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \bar{\varphi} dx - \int_{\Omega} i \lambda^3 c(x) y \bar{\varphi} dx = \frac{o(1)}{\lambda}. \quad (5.4.88)$$

From Equation (5.4.61) and (5.4.87), we obtain

$$\left| \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \bar{\varphi} dx \right| = o(1). \quad (5.4.89)$$

Now, using System (5.4.86) and Equation (5.4.89) in (5.4.88), we get

$$\int_{\Omega} |\lambda u|^2 dx - i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) u \bar{\varphi} dx - i \lambda^3 \int_{\Omega} c(x) y \bar{\varphi} dx = o(1). \quad (5.4.90)$$

By using (5.4.62), (5.4.87) and the fact that  $\omega_c \subset \omega_b$

$$\left| i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) u \bar{\varphi} dx \right| \leq \lambda^3 \|u\|_{L^2(\omega_c)} \cdot \|\varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda}. \quad (5.4.91)$$

Using (5.4.73) and (5.4.87), we get that

$$\left| i \lambda^3 \int_{\Omega} c(x) y \bar{\varphi} dx \right| = o(1). \quad (5.4.92)$$

Now, inserting Equations (5.4.91) and (5.4.92) into (5.4.90), we get

$$\int_{\Omega} |\lambda u|^2 dx = o(1). \quad (5.4.93)$$

Multiply (5.4.66) by  $\lambda^2 \bar{\psi}$  and integrate over  $\Omega$ , and using Green's formula, Equation (5.4.87), and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we obtain

$$\int_{\Omega} (\lambda^2 \bar{\psi} + \Delta \bar{\psi}) \lambda^2 y dx + i \lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx = \frac{o(1)}{\lambda}. \quad (5.4.94)$$

By using System (5.4.86) in (5.4.94), we get

$$\int_{\Omega} |\lambda y|^2 dx - i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) y \bar{\psi} dx + i \lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx = o(1). \quad (5.4.95)$$

Using (5.4.62), (5.4.73), and (5.4.87) and the fact that  $\omega_c \subset \omega_b$ , we get

$$\left| i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) y \bar{\psi} dx \right| = o(1). \quad (5.4.96)$$

and

$$\left| i \lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx \right| = \frac{o(1)}{\lambda}. \quad (5.4.97)$$

Inserting (5.4.96) and (5.4.97) into (5.4.95), we obtain

$$\int_{\Omega} |\lambda y|^2 dx = o(1). \quad (5.4.98)$$

**Case 2.** Under the assumption (H3). We proceed in the same way as in Case 1., the only change is that we have the following two estimations instead of (5.4.91) and (5.4.97), by using (5.4.67) and (5.4.87) we get

$$\left| i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) u \bar{\varphi} dx \right| = o(1) \quad (5.4.99)$$

and

$$\left| i \lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx \right| = o(1). \quad (5.4.100)$$

Thus, we reach our desired result.  $\square$

**Lemma 5.4.17.** Assume that either the assumption (H2) or (H3) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.56)-(5.4.59) satisfies the following estimations

$$\int_{\Omega} |\nabla u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1). \quad (5.4.101)$$

*Proof.* The proof is similar to the proof of Lemma 5.4.8.  $\square$

**Proof of Theorem 5.4.11.** Consequently, from the results of Lemmas 5.4.16, and 5.4.17, we obtain that  $\|U\|_{\mathcal{H}} = o(1)$ , which contradicts (5.4.54). Consequently, condition (C3) holds. This implies, from Theorem 1.3.10, the energy decay estimation (5.4.53). The proof is thus complete.

**Theorem 5.4.18.** *Assume that assumption (H4) or (H5) holds. Then, for all initial data  $U_0 \in D(\mathcal{A})$ , there exists a constant  $C_3 > 0$  independent of  $U_0$ , the energy of the system (5.1.1)-(5.1.3) satisfies the following estimation*

$$E(t, U) \leq \frac{C_3}{t^{\frac{2}{2+4\beta}}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0. \quad (5.4.102)$$

where

$$\beta = \begin{cases} 2 & \text{if (H4) holds} \\ \frac{3}{2} & \text{if (H5) holds.} \end{cases} \quad (5.4.103)$$

Following Theorem 1.3.10 of Borichev and Tomilov, the polynomial energy decay (5.4.102) holds if (C1) and

$$\limsup_{\lambda \in \mathbb{R}, |\lambda| \rightarrow \infty} \frac{1}{|\lambda|^{2+4\beta}} \|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad (C4)$$

are satisfied. Since Condition (C1) is already satisfied (see Lemmas 5.2.7 and 5.2.8). We will prove condition (C4) by an argument of contradiction. For this purpose, suppose that (C4) is false, then there exists  $\{(\lambda_n, U_n := (u_n, v_n, y_n, z_n)^\top)\} \subset \mathbb{R}^* \times D(\mathcal{A})$  with

$$|\lambda_n| \rightarrow +\infty \quad \text{and} \quad \|U_n\|_{\mathcal{H}} = \|(u_n, v_n, y_n, z_n)\|_{\mathcal{H}} = 1, \quad (5.4.104)$$

such that

$$\lambda_n^{2+4\beta} (i\lambda_n I - \mathcal{A}) U_n = F_n := (f_{1,n}, f_{2,n}, f_{3,n}, f_{4,n})^\top \rightarrow 0 \quad \text{in } \mathcal{H}. \quad (5.4.105)$$

For simplicity, we drop the index  $n$ . Equivalently, from (5.4.105), we have

$$i\lambda u - v = \frac{f_1}{\lambda^{2+4\beta}} \quad \text{in } H_0^1(\Omega), \quad (5.4.106)$$

$$i\lambda v - \operatorname{div}(a\nabla u + b(x)\nabla v) + c(x)z = \frac{f_2}{\lambda^{2+4\beta}} \quad \text{in } L^2(\Omega), \quad (5.4.107)$$

$$i\lambda y - z = \frac{f_3}{\lambda^{2+4\beta}} \quad \text{in } H_0^1(\Omega), \quad (5.4.108)$$

$$i\lambda z - \operatorname{div}(\nabla y) - c(x)v = \frac{f_4}{\lambda^{2+4\beta}} \quad \text{in } L^2(\Omega). \quad (5.4.109)$$

Here we will check the condition (C4) by finding a contradiction with (5.4.104) such as  $\|U\|_{\mathcal{H}} = o(1)$ . From Equations (5.4.104), (5.4.106) and (5.4.108), we obtain

$$\|u\|_{L^2(\Omega)} = \frac{O(1)}{\lambda} \quad \text{and} \quad \|y\|_{L^2(\Omega)} = \frac{O(1)}{\lambda}. \quad (5.4.110)$$

**Lemma 5.4.19.** *Under the assumptions (H4) or (H5). We have that the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.106)-(5.4.109) satisfies the following estimations*

$$\|\nabla v\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{1+2\beta}} \quad \|\nabla u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{2+2\beta}} \quad \text{and} \quad \|u\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{2+2\beta}}. \quad (5.4.111)$$

The proof of this Lemma is similar to that of Lemma 5.4.4.



Inserting Equations (5.4.106) and (5.4.108) into (5.4.107) and (5.4.109), we get

$$\lambda^2 u + \operatorname{div}(a \nabla u + b(x) \nabla v) - i \lambda c(x) y = -\frac{f_2}{\lambda^{2+4\beta}} - c(x) \frac{f_3}{\lambda^{2+4\beta}} - \frac{i f_1}{\lambda^{1+4\beta}}, \quad (5.4.112)$$

$$\lambda^2 y + \Delta y + i \lambda c(x) u = -\frac{f_4}{\lambda^{2+4\beta}} + c(x) \frac{f_1}{\lambda^{2+4\beta}} - \frac{i f_3}{\lambda^{1+4\beta}}. \quad (5.4.113)$$

**Lemma 5.4.20.** *Assume that assumption (H4) or (H5) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.106)-(5.4.109) satisfies the following estimation*

$$\int_{\omega_c} |\lambda y|^2 dx = \frac{o(1)}{\lambda^{2\beta}}. \quad (5.4.114)$$

*Proof.* Assume that either assumption (H4) or (H5) holds. Define the function  $\rho \in C_c^\infty(\mathbb{R}^N)$  such that

$$\rho(x) = \begin{cases} 1 & \text{if } x \in \omega_c, \\ 0 & \text{if } x \in \Omega \setminus \omega_b, \\ 0 \leq \rho \leq 1 & \text{elsewhere.} \end{cases} \quad (5.4.115)$$

Now, multiply (5.4.112) by  $\lambda \rho \bar{y}$ , integrate over  $\Omega$  and using Green's formula, (5.4.110) and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we get

$$\lambda^3 \int_{\Omega} u \rho \bar{y} dx - \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \bar{y} + \rho \nabla \bar{y}) dx - i \int_{\Omega} c(x) \rho |\lambda y|^2 dx = \frac{o(1)}{\lambda^{1+4\beta}}. \quad (5.4.116)$$

Using (5.4.110), (5.4.111) and Cauchy-Schwarz we obtain

$$\left| \lambda^3 \int_{\Omega} u \rho \bar{y} dx \right| \leq \lambda^3 \|u\|_{L^2(\omega_b)} \cdot \|y\|_{L^2(\omega_b)} = \frac{o(1)}{\lambda^{2\beta}} \quad (5.4.117)$$

and

$$\left| \lambda \int_{\Omega} (a \nabla u + b(x) \nabla v) \cdot (\nabla \rho \bar{y} + \rho \nabla \bar{y}) dx \right| = \frac{o(1)}{\lambda^{2\beta}}. \quad (5.4.118)$$

Thus, using Equation (5.4.117) and (5.4.118) in (5.4.116) we obtain our desired result.  $\square$

**Lemma 5.4.21.** *Assume that the assumption (H4) or (H5) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.106)-(5.4.109) satisfies the following estimations*

$$\int_{\Omega} |\lambda u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\lambda y|^2 dx = o(1). \quad (5.4.119)$$

*Proof.* Let  $\varphi, \psi \in H^2(\Omega) \cap H_0^1(\Omega)$  be the solution of the following system

$$\begin{cases} \lambda^2 \varphi + a \Delta \varphi - i \lambda \mathbb{1}_{\omega_c}(x) \varphi = u, & \text{in } \Omega \\ \lambda^2 \psi + \Delta \psi - i \lambda \mathbb{1}_{\omega_c}(x) \psi = y, & \text{in } \Omega \\ \varphi = \psi = 0, & \text{on } \Gamma \end{cases} \quad (5.4.120)$$

where  $(u, v, y, z)$  is the solution of (5.4.106)-(5.4.109). We suppose that the energy of the System (5.4.52) satisfies the following estimate

$$E(t, U) \leq \frac{C}{t^{\frac{2}{\beta}}} \|U_0\|_{D(\mathcal{A})}^2, \quad \forall t > 0.$$

5 A  $N$ -dimensional elastic\viscoelastic transmission problem with Kelvin-Voigt damping and non smooth coefficient at the interface – 5.4 Polynomial Stability

When assumption (H4) holds, we have that System (5.4.52) is polynomially stable with an energy decay rate  $t^{-1}$ , i.e  $\beta = 2$ . However, when assumption (H5) holds then we have that System (5.4.52) is polynomially stable with an energy decay rate  $t^{-\frac{4}{3}}$ , i.e  $\beta = \frac{3}{2}$ . Thus, there exists  $M > 0$  such that system (5.4.120) satisfies the following estimation

$$\|\lambda\varphi\|_{L^2(\Omega)} + \|\nabla\varphi\|_{L^2(\Omega)} + \|\lambda\psi\|_{L^2(\Omega)} + \|\nabla\psi\|_{L^2(\Omega)} \leq M|\lambda|^\beta (\|u\|_{L^2(\Omega)} + \|y\|_{L^2(\Omega)}). \quad (5.4.121)$$

Assuming that (H4) or (H5) holds. Multiply (5.4.112) by  $\lambda^2\bar{\varphi}$  and integrate over  $\Omega$ , and using Green's formula, Equation (5.4.87), and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we obtain

$$\int_{\Omega} (\lambda^2\bar{\varphi} + a\Delta\bar{\varphi}) \lambda^2 u dx - \lambda^2 \int_{\Omega} b(x) \nabla v \cdot \nabla \bar{\varphi} dx - \int_{\Omega} i \lambda^3 c(x) y \bar{\varphi} dx = \frac{o(1)}{\lambda^{1+3\beta}}. \quad (5.4.122)$$

From Equation (5.4.111) and (5.4.121), we obtain

$$\left| \lambda^2 \int_{\Omega} b(x) \nabla v \nabla \bar{\varphi} dx \right| = \frac{o(1)}{\lambda^\beta}. \quad (5.4.123)$$

Now, using System (5.4.120) and Equation (5.4.123) in (5.4.122), we get

$$\int_{\Omega} |\lambda u|^2 dx - i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) u \bar{\varphi} dx - i \lambda^3 \int_{\Omega} c(x) y \bar{\varphi} dx = \frac{o(1)}{\lambda^\beta}. \quad (5.4.124)$$

By using (5.4.111), (5.4.121) and the fact that  $\omega_c \subset \omega_b$

$$\left| i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) u \bar{\varphi} dx \right| \leq \lambda^3 \|u\|_{L^2(\omega_c)} \cdot \|\varphi\|_{L^2(\Omega)} = \frac{o(1)}{\lambda^{1+\beta}}. \quad (5.4.125)$$

Using (5.4.114) and (5.4.121), we get that

$$\left| i \lambda^3 \int_{\Omega} c(x) y \bar{\varphi} dx \right| = o(1). \quad (5.4.126)$$

Now, inserting Equation (5.4.125) and (5.4.126) into (5.4.124), we get

$$\int_{\Omega} |\lambda u|^2 dx = o(1). \quad (5.4.127)$$

Multiply (5.4.113) by  $\lambda^2\bar{\psi}$  and integrate over  $\Omega$ , and using Green's formula, Equation (5.4.121), and the fact that  $\|F\|_{\mathcal{H}} = \|(f_1, f_2, f_3, f_4)\|_{\mathcal{H}} = o(1)$ , we obtain

$$\int_{\Omega} (\lambda^2\bar{\psi} + \Delta\bar{\psi}) \lambda^2 y dx + i \lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx = \frac{o(1)}{\lambda^{1+3\beta}}. \quad (5.4.128)$$

By using System (5.4.120) in (5.4.128) we get

$$\int_{\Omega} |\lambda y|^2 dx - i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) y \bar{\psi} dx + i \lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx = \frac{o(1)}{\lambda^{1+3\beta}}. \quad (5.4.129)$$

Using (5.4.111), (5.4.114), and (5.4.121) and the fact that  $\omega_c \subset \omega_b$ , we get

$$\left| i \lambda^3 \int_{\Omega} \mathbb{1}_{\omega_c}(x) y \bar{\psi} dx \right| = o(1). \quad (5.4.130)$$

and

$$\left| i\lambda^3 \int_{\Omega} c(x) u \bar{\psi} dx \right| = \frac{o(1)}{\lambda^{1+\beta}}. \quad (5.4.131)$$

Inserting (5.4.130) and (5.4.131) into (5.4.129), we obtain

$$\int_{\Omega} |\lambda y|^2 dx = o(1). \quad (5.4.132)$$

□

**Lemma 5.4.22.** *Assume that either the assumption (H4) or (H5) holds. Then, the solution  $(u, v, y, z) \in D(\mathcal{A})$  of (5.4.106)-(5.4.109) satisfies the following estimations*

$$\int_{\Omega} |\nabla u|^2 dx = o(1) \quad \text{and} \quad \int_{\Omega} |\nabla y|^2 dx = o(1). \quad (5.4.133)$$

*Proof.* The proof is similar to the proof of Lemma 5.4.8. □

**Proof of Theorem 5.4.18.** Consequently, from the results of Lemma 5.4.21, 5.4.22, we obtain that  $\|U\|_{\mathcal{H}} = o(1)$ , which contradicts (5.4.104). Consequently, condition (C4) holds. This implies, from Theorem 1.3.10, the energy decay estimation (5.4.102). The proof is thus complete.

# Bibliography

- [1] Zineb Achouri, Nour Eddine Amroun, and Abbes Benaissa. “The Euler–Bernoulli beam equation with boundary dissipation of fractional derivative type”. In: *Mathematical Methods in the Applied Sciences* 40.11 (), pp. 3837–3854. DOI: <https://doi.org/10.1002/mma.4267>. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/mma.4267>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/mma.4267> (cit. on p. 98).
- [2] R. A. Adams. *Sobolev spaces* / Robert A. Adams. English. Academic Press New York, 1975, xviii, 268 p. : ISBN: 0120441500 (cit. on pp. 137, 139).
- [3] Mohammad Akil, Yacine Chitour, Mouhammad Ghader, et al. “Stability and exact controllability of a Timoshenko system with only one fractional damping on the boundary”. In: *Asymptotic Analysis* 119 (Oct. 2019), pp. 1–60. DOI: [10.3233/ASY-191574](https://doi.org/10.3233/ASY-191574) (cit. on pp. 20, 106).
- [4] Mohammad Akil and Ali Wehbe. “Stabilization of multidimensional wave equation with locally boundary fractional dissipation law under geometric conditions”. In: *Mathematical Control & Related Fields* 8 (Jan. 2018), pp. 1–20. DOI: [10.3934/mcrf.2019005](https://doi.org/10.3934/mcrf.2019005) (cit. on pp. 98, 159).
- [5] Nahla Noun Ali Wehbe Rayan Nasser. “Stability of N-D transmission problem in viscoelasticity with localized Kelvin–Voigt damping under different types of geometric conditions”. In: *Mathematical Control & Related Fields* 0 (2020), pp. - (cit. on pp. 47, 159, 181).
- [6] Hayet Allouni, Mohamed Kesri, and Benaissa Abbes. “On the asymptotic behaviour of two coupled strings through a fractional joint damper”. In: *Rendiconti del Circolo Matematico di Palermo Series 2* 69 (Aug. 2020), pp. 613–640. DOI: [10.1007/s12215-019-00423-2](https://doi.org/10.1007/s12215-019-00423-2) (cit. on p. 98).
- [7] Margareth Alves, Jaime Rivera, Mauricio Sepúlveda, et al. “The asymptotic behaviour of the linear transmission problem in viscoelasticity”. In: *Mathematische Nachrichten* 287 (Apr. 2014). DOI: [10.1002/mana.201200319](https://doi.org/10.1002/mana.201200319) (cit. on p. 66).
- [8] Margareth Alves, Jaime Rivera, Mauricio Sepúlveda, et al. “The Lack of Exponential Stability in Certain Transmission Problems with Localized Kelvin–Voigt Dissipation”. In: *SIAM Journal on Applied Mathematics* 74 (Mar. 2014). DOI: [10.1137/130923233](https://doi.org/10.1137/130923233) (cit. on pp. 66, 96).

- [9] Kaïs Ammari, Hassine Fathi, and Luc Robbiano. “Fractional-feedback stabilization for a class of evolution systems”. In: *Journal of Differential Equations* 268.10 (2020), pp. 5751–5791. ISSN: 0022-0396. DOI: <https://doi.org/10.1016/j.jde.2019.11.022>. URL: <http://www.sciencedirect.com/science/article/pii/S002203961930556X> (cit. on p. 98).
- [10] Kaïs Ammari, Fathi Hassine, and Luc Robbiano. “Stabilization for the Wave Equation with Singular Kelvin–Voigt Damping”. In: *Archive for Rational Mechanics and Analysis* (Nov. 2019). ISSN: 1432-0673. DOI: [10.1007/s00205-019-01476-4](https://doi.org/10.1007/s00205-019-01476-4). URL: <http://dx.doi.org/10.1007/s00205-019-01476-4> (cit. on p. 159).
- [11] Kaïs Ammari, Mohamed Jellouli, and Michel Mehrenberger. “Feedback stabilization of a coupled string-beam system”. In: *NHM* 4 (Mar. 2009), pp. 19–34. DOI: [10.3934/nhm.2009.4.19](https://doi.org/10.3934/nhm.2009.4.19) (cit. on p. 99).
- [12] Kaïs Ammari, Zhuangyi Liu, and Shel Farhat. “Stability of the wave equations on a tree with local Kelvin–Voigt damping”. In: *Semigroup Forum* (Mar. 2018). DOI: [10.1007/s00233-019-10064-7](https://doi.org/10.1007/s00233-019-10064-7) (cit. on pp. 66, 96).
- [13] Kaïs Ammari and Michel Mehrenberger. “Study of the nodal feedback stabilization of a string-beams network”. In: *JAMC J Appl Math Comput* 36 (Aug. 2011). DOI: [10.1007/s12190-010-0412-9](https://doi.org/10.1007/s12190-010-0412-9) (cit. on p. 99).
- [14] Kaïs Ammari, Denis Mercier, Virginie Régnier, et al. “Spectral analysis and stabilization of a chain of serially Euler–Bernoulli beams and strings”. In: *Communications on Pure and Applied Analysis* 11 (May 2010). DOI: [10.3934/cpaa.2012.11.785](https://doi.org/10.3934/cpaa.2012.11.785) (cit. on p. 66).
- [15] Kaïs Ammari and Serge Nicaise. “Stabilization of a transmission wave/plate equation”. In: *Journal of Differential Equations* 249.3 (2010), pp. 707–727. ISSN: 0022-0396. DOI: <https://doi.org/10.1016/j.jde.2010.03.007>. URL: <http://www.sciencedirect.com/science/article/pii/S0022039610000926> (cit. on p. 99).
- [16] Kaïs Ammari and Georgi Vodev. “Boundary stabilization of the transmission problem for the Bernoulli–Euler plate equation”. In: *Cubo* 5 (Jan. 2009) (cit. on p. 99).
- [17] W. Arendt and C. J. K. Batty. “Tauberian theorems and stability of one-parameter semigroups”. In: *Trans. Amer. Math. Soc.* 306.2 (1988), pp. 837–852. ISSN: 0002-9947. DOI: [10.2307/2000826](https://doi.org/10.2307/2000826). URL: <http://dx.doi.org/10.2307/2000826> (cit. on pp. 17, 23, 36, 40, 44, 45, 69).
- [18] Wolfgang Arendt, Charles Batty, Matthias Hieber, et al. *Vector-Valued Laplace Transforms and Cauchy Problems*. Vol. 96. Jan. 2001. ISBN: 978-3-0348-0086-0. DOI: [10.1007/978-3-0348-0087-7](https://doi.org/10.1007/978-3-0348-0087-7) (cit. on p. 46).
- [19] R. L. Bagley and P. J. Torvik. “A Theoretical Basis for the Application of Fractional Calculus to Viscoelasticity”. In: *Journal of Rheology* 27.3 (1983), pp. 201–210. DOI: [10.1122/1.549724](https://doi.org/10.1122/1.549724). eprint: <https://doi.org/10.1122/1.549724>. URL: <https://doi.org/10.1122/1.549724> (cit. on p. 97).

- [20] Ronald L. Bagley and Peter J. Torvik. “Fractional calculus - A different approach to the analysis of viscoelastically damped structures”. In: *AIAA Journal* 21.5 (1983), pp. 741–748. DOI: [10.2514/3.8142](https://doi.org/10.2514/3.8142). eprint: <https://doi.org/10.2514/3.8142>. URL: <https://doi.org/10.2514/3.8142> (cit. on p. 97).
- [21] H. Banks, R. Smith, and Y. Wang. “The modeling of piezoceramic patch interactions with shells, plates, and beams”. In: *Quarterly of Applied Mathematics* 53 (1995), pp. 353–381 (cit. on p. 159).
- [22] Claude. Bardos, Gilles. Lebeau, and Jeffrey. Rauch. “Sharp Sufficient Conditions for the Observation, Control, and Stabilization of Waves from the Boundary”. In: *SIAM Journal on Control and Optimization* 30.5 (1992), pp. 1024–1065. DOI: [10.1137/0330055](https://doi.org/10.1137/0330055). URL: <https://doi.org/10.1137/0330055> (cit. on pp. 64, 96, 158).
- [23] J. Bartolomeo and R. Triggiani. “Uniform Energy Decay Rates for Euler–Bernoulli Equations with Feedback Operators in the Dirichlet/Neumann Boundary Conditions”. In: *SIAM Journal on Mathematical Analysis* 22.1 (1991), pp. 46–71. DOI: [10.1137/0522004](https://doi.org/10.1137/0522004). eprint: <https://doi.org/10.1137/0522004>. URL: <https://doi.org/10.1137/0522004> (cit. on p. 97).
- [24] Charles J. K. Batty and Thomas Duyckaerts. “Non-uniform stability for bounded semi-groups on Banach spaces”. In: *J. Evol. Equ.* 8.4 (2008), pp. 765–780. ISSN: 1424-3199. DOI: [10.1007/s00028-008-0424-1](https://doi.org/10.1007/s00028-008-0424-1). URL: <http://dx.doi.org/10.1007/s00028-008-0424-1> (cit. on pp. 40, 44, 46).
- [25] Stuart Bennett. *A history of control engineering, 1800-1930*. London: Peregrinus on behalf of the Institution of Electrical Engineers, 1986. ISBN: 978-0863410475 (cit. on p. 13).
- [26] Saroj K. Biswas and N. U. Ahmed. “Optimal control of large space structures governed by a coupled system of ordinary and partial differential equations”. In: *Math. Control. Signals Syst.* 2.1 (1989), pp. 1–18. DOI: [10.1007/BF02551358](https://doi.org/10.1007/BF02551358). URL: <https://doi.org/10.1007/BF02551358> (cit. on p. 98).
- [27] Alexander Borichev and Yuri Tomilov. “Optimal polynomial decay of functions and operator semigroups”. In: *Math. Ann.* 347.2 (2010), pp. 455–478. ISSN: 0025-5831. DOI: [10.1007/s00208-009-0439-0](https://doi.org/10.1007/s00208-009-0439-0). URL: <http://dx.doi.org/10.1007/s00208-009-0439-0> (cit. on pp. 19, 23, 40, 44, 46).
- [28] Haim Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York, 2010. DOI: [10.1007/978-0-387-70914-7](https://doi.org/10.1007/978-0-387-70914-7). URL: <https://doi.org/10.1007/978-0-387-70914-7> (cit. on pp. 40, 41, 43, 44).
- [29] N. Burq. “Contrôlabilité exacte des ondes dans des ouverts peu réguliers”. In: *Asymptotic Analysis* 14 (1997), pp. 157–191 (cit. on p. 181).
- [30] N. Burq and Chenmin Sun. “DECAY FOR THE KELVIN-VOIGT DAMPED WAVE EQUATION: PIECEWISE SMOOTH DAMPING”. In: *arXiv: Analysis of PDEs* (2020) (cit. on p. 160).

- [31] Nicolas Burq. “Decays for Kelvin–Voigt Damped Wave Equations I: The Black Box Perturbative Method”. In: *SIAM Journal on Control and Optimization* 58.4 (2020), pp. 1893–1905. DOI: [10.1137/19M1259080](https://doi.org/10.1137/19M1259080). eprint: <https://doi.org/10.1137/19M1259080>. URL: <https://doi.org/10.1137/19M1259080> (cit. on p. 159).
- [32] Nicolas Burq and Chenmin Sun. “Decays rates for Kelvin-Voigt damped wave equations II: the geometric control condition”. In: (Oct. 2020) (cit. on p. 160).
- [33] M. Caputo and Mauro Fabrizio. “A new definition of fractional derivative without singular Kernel”. In: *Prog Fract Differ Appl* 1 (Apr. 2015), pp. 73–85. DOI: [10.12785/pfda/010201](https://doi.org/10.12785/pfda/010201) (cit. on p. 97).
- [34] Michele Caputo. “Linear Models of Dissipation whose Q is almost Frequency Independent-II”. In: *Geophysical Journal* 13.5 (Nov. 1967), pp. 529–539. DOI: [10.1111/j.1365-246X.1967.tb02303.x](https://doi.org/10.1111/j.1365-246X.1967.tb02303.x) (cit. on p. 97).
- [35] Marcelo Cavalcanti, Valeria Domingos Cavalcanti, and Louis Tebou. “Stabilization of the wave equation with localized compensating frictional and kelvin-voigt dissipating mechanisms”. In: *Electronic Journal of Differential Equations* 2017 (Jan. 2017), pp. 1–18 (cit. on p. 159).
- [36] G. Chen, S. A. Fulling, F. J. Narcowich, et al. “Exponential Decay of Energy of Evolution Equations with Locally Distributed Damping”. In: *SIAM Journal on Applied Mathematics* 51.1 (1991), pp. 266–301. DOI: [10.1137/0151015](https://doi.org/10.1137/0151015). URL: <https://doi.org/10.1137/0151015> (cit. on pp. 64, 96).
- [37] Goong Chen, Michel Delfour, A. Krall, et al. “Modeling, Stabilization and Control of Serially Connected Beams”. In: *Siam Journal on Control and Optimization - SIAM J CONTR OPTIMIZAT* 25 (May 1987). DOI: [10.1137/0325029](https://doi.org/10.1137/0325029) (cit. on p. 97).
- [38] Ruth F. Curtain and Hans Zwart. *An introduction to infinite-dimensional linear systems theory*. Vol. 21. Texts in Applied Mathematics. Springer-Verlag, New York, 1995, pp. xviii+698. ISBN: 0-387-94475-3. DOI: [10.1007/978-1-4612-4224-6](https://doi.org/10.1007/978-1-4612-4224-6). URL: <http://dx.doi.org/10.1007/978-1-4612-4224-6> (cit. on pp. 40, 44, 46, 78).
- [39] C.M. Dafermos. “Asymptotic Behavior of Solutions of Evolution Equations”. In: *Nonlinear Evolution Equations*. Ed. by Michael G. Crandall. Academic Press, 1978, pp. 103–123. ISBN: 978-0-12-195250-1. DOI: <https://doi.org/10.1016/B978-0-12-195250-1.50010-2>. URL: <https://www.sciencedirect.com/science/article/pii/B9780121952501500102> (cit. on p. 158).
- [40] Robert Denk and Felix Kammerlander. “Exponential stability for a coupled system of damped–undamped plate equations”. In: *IMA Journal of Applied Mathematics* 83.2 (Feb. 2018), pp. 302–322. ISSN: 0272-4960. DOI: [10.1093/imamat/hxy002](https://doi.org/10.1093/imamat/hxy002). eprint: <https://academic.oup.com/imamat/article-pdf/83/2/302/25194104/hxy002.pdf>. URL: <https://doi.org/10.1093/imamat/hxy002> (cit. on p. 99).



- [41] Zuazua Enrike. “Exponential Decay for The Semilinear Wave Equation with Locally Distributed Damping”. In: *Communications in Partial Differential Equations* 15.2 (1990), pp. 205–235. DOI: [10.1080/03605309908820684](https://doi.org/10.1080/03605309908820684). eprint: <https://doi.org/10.1080/03605309908820684>. URL: <https://doi.org/10.1080/03605309908820684> (cit. on p. 158).
- [42] Xiaoyu Fu and Qi Lu. *Stabilization of the weakly coupled wave-plate system with one internal damping*. 2017. arXiv: [1801.00232 \[math.OA\]](https://arxiv.org/abs/1801.00232) (cit. on p. 98).
- [43] Stéphane Gerbi, Chiraz Kassem, Amina Mortada, et al. “Exact Controllability and Stabilization of Locally Coupled Wave Equations: Theoretical Results”. In: *Zeitschrift für Analysis und ihre Anwendungen* 40 (Jan. 2021), pp. 67–96. DOI: [10.4171/ZAA/1673](https://doi.org/10.4171/ZAA/1673) (cit. on pp. 160, 176).
- [44] Bao-Zhu Guo and Han-Jing Ren. “Stability and regularity transmission for coupled beam and wave equations through boundary weak connections”. In: *ESAIM Control Optimisation and Calculus of Variations* 26 (Sept. 2020), 29 pp. DOI: [10.1051/cocv/2019056](https://doi.org/10.1051/cocv/2019056) (cit. on p. 99).
- [45] Ya-Ping Guo, Jun-Min Wang, and Dong-Xia Zhao. “Energy decay estimates for a two-dimensional coupled wave-plate system with localized frictional damping”. In: *ZAMM - Journal of Applied Mathematics and Mechanics / Zeitschrift für Angewandte Mathematik und Mechanik* 100.2 (2020), e201900030. DOI: <https://doi.org/10.1002/zamm.201900030>. eprint: <https://onlinelibrary.wiley.com/doi/pdf/10.1002/zamm.201900030>. URL: <https://onlinelibrary.wiley.com/doi/abs/10.1002/zamm.201900030> (cit. on p. 99).
- [46] Han, Zhong-Jie and Liu, Zhuangyi. “Regularity and stability of coupled plate equations with indirect structural or Kelvin-Voigt damping”. In: *ESAIM: COCV* 25 (2019), p. 51. DOI: [10.1051/cocv/2018060](https://doi.org/10.1051/cocv/2018060). URL: <https://doi.org/10.1051/cocv/2018060> (cit. on p. 100).
- [47] Fathi Hassine. “Energy decay estimates of elastic transmission wave/beam systems with a local Kelvin-Voigt damping”. In: *International Journal of Control* (Dec. 2015), pp. 1–29. DOI: [10.1080/00207179.2015.1135509](https://doi.org/10.1080/00207179.2015.1135509) (cit. on pp. 20, 66, 98, 105, 148, 149).
- [48] Fathi Hassine. “Stability of elastic transmission systems with a local Kelvin-Voigt damping”. In: *European Journal of Control* 23 (Mar. 2015). DOI: [10.1016/j.ejcon.2015.03.001](https://doi.org/10.1016/j.ejcon.2015.03.001) (cit. on p. 66).
- [49] Fathi Hassine. “Asymptotic behavior of the transmission Euler-Bernoulli plate and wave equation with a localized Kelvin-Voigt damping”. In: *Discrete and Continuous Dynamical Systems - Series B* 21 (June 2016), pp. 1757–1774. DOI: [10.3934/dcdsb.2016021](https://doi.org/10.3934/dcdsb.2016021) (cit. on p. 98).
- [50] Fathi Hassine and Nadia Souayah. “Stability for coupled waves with locally disturbed Kelvin-Voigt damping”. In: *Semigroup Forum* 102 (2019), pp. 134–159 (cit. on pp. 16, 64).



- [51] Alaa Hayek, Serge Nicaise, Zaynab Salloum, et al. “A transmission problem of a system of weakly coupled wave equations with Kelvin–Voigt dampings and non-smooth coefficient at the interface”. In: *SeMA Journal* 77.3 (June 2020), pp. 305–338. DOI: [10.1007/s40324-020-00218-x](https://doi.org/10.1007/s40324-020-00218-x). URL: <https://doi.org/10.1007/s40324-020-00218-x> (cit. on pp. 160, 165).
- [52] Lars Hörmander. *Linear Partial Differential Operators*. Springer Berlin Heidelberg, 1969. DOI: [10.1007/978-3-662-30722-9](https://doi.org/10.1007/978-3-662-30722-9). URL: <https://doi.org/10.1007/978-3-662-30722-9> (cit. on p. 165).
- [53] Fa Lun Huang. “Characteristic conditions for exponential stability of linear dynamical systems in Hilbert spaces”. In: *Ann. Differential Equations* 1.1 (1985), pp. 43–56. ISSN: 1002-0942 (cit. on pp. 40, 44, 45, 56, 62).
- [54] Falun. Huang. “On the Mathematical Model for Linear Elastic Systems with Analytic Damping”. In: *SIAM Journal on Control and Optimization* 26.3 (1988), pp. 714–724. DOI: [10.1137/0326041](https://doi.org/10.1137/0326041). URL: <https://doi.org/10.1137/0326041> (cit. on pp. 63, 96, 159).
- [55] Vasile Ion Istratescu. *Inner Product Structures*. Springer Netherlands, 1987. DOI: [10.1007/978-94-009-3713-0](https://doi.org/10.1007/978-94-009-3713-0). URL: <https://doi.org/10.1007/978-94-009-3713-0> (cit. on pp. 40, 42).
- [56] Guangcao Ji and Irena Lasiecka. “Nonlinear Boundary Feedback Stabilization for a Semilinear Kirchhoff Plate with Dissipation Acting Only via Moments-Limiting Behavior”. In: *Journal of Mathematical Analysis and Applications* 229.2 (1999), pp. 452–479. ISSN: 0022-247X. DOI: <https://doi.org/10.1006/jmaa.1998.6170>. URL: <http://www.sciencedirect.com/science/article/pii/S0022247X98961704> (cit. on p. 97).
- [57] Tosio Kato. *Perturbation Theory for Linear Operators*. Springer Berlin Heidelberg, 1995. DOI: [10.1007/978-3-642-66282-9](https://doi.org/10.1007/978-3-642-66282-9). URL: <https://doi.org/10.1007/978-3-642-66282-9> (cit. on pp. 40, 42, 162).
- [58] Jong Uhn Kim and Yuriko Renardy. “Boundary Control of the Timoshenko Beam”. In: *SIAM Journal on Control and Optimization* 25.6 (1987), pp. 1417–1429. DOI: [10.1137/0325078](https://doi.org/10.1137/0325078). eprint: <https://doi.org/10.1137/0325078>. URL: <https://doi.org/10.1137/0325078> (cit. on p. 97).
- [59] John E. Lagnese. “Uniform boundary stabilization of homogeneous isotropic plates”. In: (1987). Ed. by Franz Kappel, Karl Kunisch, and Wilhelm Schappacher, pp. 204–215 (cit. on p. 97).
- [60] I Lasiecka. “Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary”. In: *Journal of Differential Equations* 79.2 (1989), pp. 340–381. ISSN: 0022-0396. DOI: [https://doi.org/10.1016/0022-0396\(89\)90107-1](https://doi.org/10.1016/0022-0396(89)90107-1). URL: <http://www.sciencedirect.com/science/article/pii/0022039689901071> (cit. on p. 97).

- [61] I. Lasiecka. “Asymptotic behavior of solutions to plate equations with nonlinear dissipation occurring through shear forces and bending moments”. In: *Applied Mathematics and Optimization* 21 (1990), pp. 167–189 (cit. on p. 97).
- [62] Le Rousseau, Jérôme and Lebeau, Gilles. “On Carleman estimates for elliptic and parabolic operators. Applications to unique continuation and control of parabolic equations”. In: *ESAIM: COCV* 18.3 (2012), pp. 712–747. DOI: [10.1051/cocv/2011168](https://doi.org/10.1051/cocv/2011168). URL: <https://doi.org/10.1051/cocv/2011168> (cit. on pp. 115, 166).
- [63] Yan-Fang Li, Zhong-Jie Han, and Gen-Qi Xu. “Explicit decay rate for coupled string-beam system with localized frictional damping”. In: *Applied Mathematics Letters* 78 (2018), pp. 51–58. ISSN: 0893-9659. DOI: <https://doi.org/10.1016/j.aml.2017.11.003>. URL: <http://www.sciencedirect.com/science/article/pii/S0893965917303294> (cit. on p. 99).
- [64] W. Littman and L. Markus. “Stabilization of a hybrid system of elasticity by feedback boundary damping”. In: *Annali di Matematica Pura ed Applicata* 152.1 (Dec. 1988), pp. 281–330. DOI: [10.1007/bf01766154](https://doi.org/10.1007/bf01766154). URL: <https://doi.org/10.1007/bf01766154> (cit. on p. 78).
- [65] Kangsheng Liu. “Locally Distributed Control and Damping for the Conservative Systems”. In: *SIAM Journal on Control and Optimization* 35.5 (1997), pp. 1574–1590. DOI: [10.1137/S0363012995284928](https://doi.org/10.1137/S0363012995284928). eprint: <https://doi.org/10.1137/S0363012995284928>. URL: <https://doi.org/10.1137/S0363012995284928> (cit. on p. 159).
- [66] Kangsheng Liu and Zhuangyi Liu. “Exponential Decay of Energy of the Euler–Bernoulli Beam with Locally Distributed Kelvin–Voigt Damping”. In: *SIAM Journal on Control and Optimization* 36 (May 1998), pp. 1086–1098. DOI: [10.1137/S0363012996310703](https://doi.org/10.1137/S0363012996310703) (cit. on pp. 65, 96, 97, 159).
- [67] Kangsheng Liu and Zhuangyi Liu. “Exponential decay of energy of vibrating strings with local viscoelasticity”. In: *Zeitschrift Fur Angewandte Mathematik Und Physik - ZAMP* 53 (Mar. 2002), pp. 265–280. DOI: [10.1007/s00033-002-8155-6](https://doi.org/10.1007/s00033-002-8155-6) (cit. on pp. 66, 159).
- [68] Kangsheng Liu and Bopeng Rao. “Stabilité exponentielle des équations des ondes avec amortissement local de Kelvin–Voigt”. In: *Comptes Rendus Mathématique* 339.11 (2004), pp. 769–774. ISSN: 1631-073X. DOI: <https://doi.org/10.1016/j.crma.2004.09.029>. URL: <https://www.sciencedirect.com/science/article/pii/S1631073X04004856> (cit. on p. 160).
- [69] Kangsheng Liu and Bopeng Rao. “Exponential stability for the wave equations with local Kelvin–Voigt damping”. In: *Zeitschrift für angewandte Mathematik und Physik* 57 (May 2006), pp. 419–432. DOI: [10.1007/s00033-005-0029-2](https://doi.org/10.1007/s00033-005-0029-2) (cit. on p. 159).

- [70] Kangsheng. Liu, Shuping. Chen, and Zhuangyi. Liu. “Spectrum and Stability for Elastic Systems with Global or Local Kelvin–Voigt Damping”. In: *SIAM Journal on Applied Mathematics* 59.2 (1998), pp. 651–668. DOI: [10.1137/S0036139996292015](https://doi.org/10.1137/S0036139996292015). eprint: <https://doi.org/10.1137/S0036139996292015>. URL: <https://doi.org/10.1137/S0036139996292015> (cit. on pp. 65, 96).
- [71] Zhuangyi Liu and Bopeng Rao. “Characterization of polynomial decay rate for the solution of linear evolution equation”. In: *Z. Angew. Math. Phys.* 56.4 (2005), pp. 630–644. ISSN: 0044-2275. DOI: [10.1007/s00033-004-3073-4](https://doi.org/10.1007/s00033-004-3073-4). URL: <http://dx.doi.org/10.1007/s00033-004-3073-4> (cit. on pp. 40, 44, 46, 66, 159, 181).
- [72] Zhuangyi Liu and Qiong Zhang. “Stability of a String with Local Kelvin–Voigt Damping and Nonsmooth Coefficient at Interface”. In: *SIAM Journal on Control and Optimization* 54 (Jan. 2016), pp. 1859–1871. DOI: [10.1137/15M1049385](https://doi.org/10.1137/15M1049385) (cit. on pp. 66, 96).
- [73] Zhuangyi Liu and Songmu Zheng. *Semigroups associated with dissipative systems*. Vol. 398. Chapman & Hall/CRC Research Notes in Mathematics. Chapman & Hall/CRC, Boca Raton, FL, 1999, pp. x+206. ISBN: 0-8493-0615-9 (cit. on pp. 15, 23, 29, 33).
- [74] Liu, Kangsheng, Liu, Zhuangyi, and Zhang, Qiong. “Eventual differentiability of a string with local Kelvin–Voigt damping”. In: *ESAIM: COCV* 23.2 (2017), pp. 443–454. DOI: [10.1051/cocv/2015055](https://doi.org/10.1051/cocv/2015055). URL: <https://doi.org/10.1051/cocv/2015055> (cit. on p. 66).
- [75] Zheng-hua Luo, Bao-Zhu Guo, Omer Morgul, et al. *Stability and Stabilization of Infinite Dimensional Systems with Applications*. 1st. Berlin, Heidelberg: Springer-Verlag, 1999. ISBN: 1852331240 (cit. on pp. 40, 44, 46).
- [76] M. Mainardi and E. Bonetti. “The application of real-order derivatives in linear viscoelasticity”. In: (1988). Ed. by Hanswalter Giesekus and Mark Francis Hibberd, pp. 64–67 (cit. on p. 97).
- [77] Denis Matignon and Christophe Prieur. “Asymptotic stability of Webster–Lokshin equation”. In: *Mathematical Control and Related Fields* 4 (2014), p. 481. ISSN: 2156-8472. DOI: [10.3934/mcrf.2014.4.481](https://doi.org/10.3934/mcrf.2014.4.481). URL: <http://aims sciences.org/article/id/fd4eead6-2403-4cdb-a863-8d64828c8791> (cit. on p. 97).
- [78] B. Mbodje and G. Montseny. “Boundary fractional derivative control of the wave equation”. In: *IEEE Transactions on Automatic Control* 40.2 (1995), pp. 378–382. DOI: [10.1109/9.341815](https://doi.org/10.1109/9.341815) (cit. on p. 97).
- [79] Brahim Mbodje. “Wave energy decay under fractional derivative controls”. In: *IMA Journal of Mathematical Control and Information* 23.2 (June 2006), pp. 237–257. ISSN: 0265-0754. DOI: [10.1093/imamci/dni056](https://doi.org/10.1093/imamci/dni056). eprint: <https://academic.oup.com/imamci/article-pdf/23/2/237/1829221/dni056>.

- pdf. URL: <https://doi.org/10.1093/imamci/dni056> (cit. on pp. 20, 97, 106).
- [80] Serge Nicaise and Cristina Pignotti. “Stability of the wave equation with localized Kelvin-Voigt damping and boundary delay feedback”. In: *Discrete and Continuous Dynamical Systems - Series S* 9 (Apr. 2016), pp. 791–813. DOI: [10.3934/dcdss.2016029](https://doi.org/10.3934/dcdss.2016029) (cit. on p. 159).
- [81] Louis Nirenberg. “An extended interpolation inequality”. en. In: *Annali della Scuola Normale Superiore di Pisa - Classe di Scienze* Ser. 3, 20.4 (1966), pp. 733–737. URL: [www.numdam.org/item/ASNSP\\_1966\\_3\\_20\\_4\\_733\\_0/](http://www.numdam.org/item/ASNSP_1966_3_20_4_733_0/) (cit. on pp. 137–139, 147).
- [82] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Vol. 44. Applied Mathematical Sciences. Springer-Verlag, New York, 1983, pp. viii+279. ISBN: 0-387-90845-5. DOI: [10.1007/978-1-4612-5561-1](https://doi.org/10.1007/978-1-4612-5561-1). URL: <http://dx.doi.org/10.1007/978-1-4612-5561-1> (cit. on pp. 15, 23, 29, 33, 43, 44, 162).
- [83] Igor Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Mathematics in Science and Engineering. London: Academic Press, 1999. URL: <https://cds.cern.ch/record/395913> (cit. on p. 97).
- [84] Higidio Portillo Oquendo and Patricia Sanz. “Optimal decay for coupled waves with Kelvin-Voigt damping”. In: *Applied Mathematics Letters* 67 (Dec. 2016). DOI: [10.1016/j.aml.2016.11.010](https://doi.org/10.1016/j.aml.2016.11.010) (cit. on pp. 66, 160).
- [85] Jan Prüß. “On the spectrum of  $C_0$ -semigroups”. In: *Trans. Amer. Math. Soc.* 284.2 (1984), pp. 847–857. ISSN: 0002-9947. DOI: [10.2307/1999112](https://doi.org/10.2307/1999112). URL: <http://dx.doi.org/10.2307/1999112> (cit. on pp. 40, 44, 45, 56, 62).
- [86] Carlos Raposo, Waldemar Bastos, and J. Avila. “A Transmission Problem for Euler-Bernoulli beam with Kelvin-Voigt Damping”. In: *Applied Mathematics and Information Sciences* 5 (Jan. 2011), pp. 17–28 (cit. on p. 97).
- [87] M. Renardy. “On localized Kelvin-Voigt damping”. In: *Zamm-zeitschrift Fur Angewandte Mathematik Und Mechanik* 84 (Apr. 2004), pp. 280–283. DOI: [10.1002/zamm.200310100](https://doi.org/10.1002/zamm.200310100) (cit. on p. 66).
- [88] Jaime Rivera and Mauro Santos. “Analytic property of a coupled system of wave-plate type with thermal effect”. In: *Differential and Integral Equations* 24 (Sept. 2011) (cit. on p. 99).
- [89] Luc Robbiano and Qiong Zhang. “Logarithmic Decay of Wave Equation with Kelvin-Voigt Damping”. In: *Mathematics* 8.5 (2020). ISSN: 2227-7390. DOI: [10.3390/math8050715](https://doi.org/10.3390/math8050715). URL: <https://www.mdpi.com/2227-7390/8/5/715> (cit. on p. 159).

- [90] Reinhard Stahn. “Optimal decay rate for the wave equation on a square with constant damping on a strip”. In: *Zeitschrift für angewandte Mathematik und Physik* 68.2 (Feb. 2017). DOI: [10.1007/s00033-017-0781-0](https://doi.org/10.1007/s00033-017-0781-0). URL: <https://doi.org/10.1007/s00033-017-0781-0> (cit. on p. 181).
- [91] Louis Tebou. “A constructive method for the stabilization of the wave equation with localized Kelvin–Voigt damping”. In: *Comptes Rendus Mathématique* 350 (June 2012), pp. 603–608. DOI: [10.1016/j.crma.2012.06.005](https://doi.org/10.1016/j.crma.2012.06.005) (cit. on pp. 158, 159).
- [92] Louis Tebou. “Energy decay estimates for some weakly coupled Euler-Bernoulli and wave equations with indirect damping mechanisms”. In: *Mathematical Control & Related Fields* 1 (Mar. 2012). DOI: [10.3934/mcrf.2012.2.45](https://doi.org/10.3934/mcrf.2012.2.45) (cit. on p. 98).
- [93] P. J. Torvik and R. L. Bagley. “On the Appearance of the Fractional Derivative in the Behavior of Real Materials”. In: *Journal of Applied Mechanics* 51.2 (June 1984), pp. 294–298. ISSN: 0021-8936. DOI: [10.1115/1.3167615](https://doi.org/10.1115/1.3167615). eprint: [https://asmedigitalcollection.asme.org/appliedmechanics/article-pdf/51/2/294/5457297/294\\_1.pdf](https://asmedigitalcollection.asme.org/appliedmechanics/article-pdf/51/2/294/5457297/294_1.pdf). URL: <https://doi.org/10.1115/1.3167615> (cit. on p. 97).
- [94] Ali Wehbe, Ibtissam Issa, and Mohammad Akil. “Stability Results of an Elastic/Viscoelastic Transmission Problem of Locally Coupled Waves with Non Smooth Coefficients”. In: *Acta Applicandae Mathematicae* 171.1 (Feb. 2021). DOI: [10.1007/s10440-021-00384-8](https://doi.org/10.1007/s10440-021-00384-8). URL: <https://doi.org/10.1007/s10440-021-00384-8> (cit. on pp. 3, 34, 98, 105).
- [95] Qiong Zhang. “Exponential stability of an elastic string with local Kelvin–Voigt damping”. In: *Zeitschrift Fur Angewandte Mathematik Und Physik - ZAMP* 61 (Dec. 2010), pp. 1009–1015. DOI: [10.1007/s00033-010-0064-5](https://doi.org/10.1007/s00033-010-0064-5) (cit. on pp. 66, 96).
- [96] Qiong Zhang. “Polynomial decay of an elastic/viscoelastic waves interaction system”. In: *Zeitschrift für angewandte Mathematik und Physik* 69.4 (June 2018). DOI: [10.1007/s00033-018-0981-2](https://doi.org/10.1007/s00033-018-0981-2). URL: <https://doi.org/10.1007/s00033-018-0981-2> (cit. on p. 160).
- [97] Xu Zhang and Enrique Zuazua. “Long-Time Behavior of a Coupled Heat-Wave System Arising in Fluid-Structure Interaction”. In: *Archive for Rational Mechanics and Analysis* 184 (Apr. 2007), pp. 49–120. DOI: [10.1007/s00205-006-0020-x](https://doi.org/10.1007/s00205-006-0020-x) (cit. on p. 98).