Local theory and long time behaviour of solutions of nonlinear Schrödinger equations

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In memory of my late grandfather

Contents

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Publications related to this thesis

The work presented in this thesis is based on the following articles:

1 Tin Van Phan, On the Cauchy problem for a derivative nonlinear Schrödinger equation with nonvanishing boundary conditions, *to appear in Evolution Equations & Control Theory*, 2021.

2 Tin Van Phan, On the derivative nonlinear Schrödinger equation on the half line with Robin boundary condition, *to appear in Journal of Mathematical Physics*, 2021.

3 Tin Van Phan, Construction of the multi-soliton, multi kink-soliton of the derivative nonlinear Schrödinger equations, *to appear in Nonlinear Analysis*, 2021, arXiv:2102.00744, submitted.

4 Tin Van Phan, Construction of the multi-soliton for a generalized derivative nonlinear Schrödinger equation, 2021, arXiv:2105.10173, submitted.

5 Tin Van Phan, Instability of algebraic standing waves for nonlinear schrödinger equations with triple power nonlinearities, 2021, arXiv:2110.07899, submitted.

Chapter 1

Introduction

1.1 General theory of classical nonlinear Schrödinger equations

In this section, we would like to introduce the general theory on classical nonlinear Schrödinger equations. We consider the following power type nonlinear Schrödinger equation:

$$\begin{cases} iu_t + \Delta u + \lambda |u|^\alpha u = 0, \\ u(0) = \varphi. \end{cases} \tag{1.1}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $\lambda = \pm 1$ and $0 < \alpha < \frac{4}{N-2}$ ($0 < \alpha < \infty$ if $N = 1, 2$). The equation (1.1) is called focusing if $\lambda = 1$ and defocusing if $\lambda = -1$.

Let I be a open subset of $\mathbb{R}$ with $0 \in I$. We observe that $u \in L^\infty(I, H^1(\mathbb{R}^N))$ is a solution of (1.1) if and only if u satisfies the following integral equation (see [16, Proposition 3.1.3]):

$$u(t) = S(t)\varphi + i\lambda \int_0^t S(t-s)|u|^\alpha u(s)\, ds, \tag{1.2}$$

where $S(t)$ is the Schrödinger group.

It is well known that (1.2) is locally well posed on $H^1(\mathbb{R}^N)$. More precisely, for any $\varphi \in H^1(\mathbb{R}^N)$, there exists a unique maximal solution $u \in C(I, H^1(\mathbb{R}^N))$ of (1.2). This solution u satisfies a blow up alternative and depends continuously on the initial data (see Section 1.3.1 for details). Finally, u satisfies the following conservation laws:

$$M(u(t)) := \frac{1}{2}\|u(t)\|_{L^2}^2 = \frac{1}{2}\|\varphi\|_{L^2}^2, \tag{1.3}$$

$$E(u(t)) := \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 - \frac{\lambda}{\alpha+2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2} = E(\varphi), \tag{1.4}$$

$$P(u(t)) := \mathcal{Im} \int_{\mathbb{R}^N} u(t,x)\nabla\overline{u}(t,x)\, dx = P(\varphi). \tag{1.5}$$

1.1.1 Strichartz estimates

Strichartz estimates are an important tool to study the local well posedness of dispersive equations. In this section, we introduce the Strichartz estimates for the Schrödinger group.

The following well known result is the fundamental estimate for Schrödinger group.

Proposition 1.1. *If $p \in [2,\infty]$ and $t \neq 0$, then $S(t)$ maps $L^{p'}(\mathbb{R}^N)$ continuously to $L^p(\mathbb{R}^N)$ and*

$$\|S(t)\varphi\|_{L^p} \leqslant (4\pi|t|)^{-N\left(\frac{1}{2}-\frac{1}{p}\right)}\|\varphi\|_{L^{p'}}, \quad \text{for all } \varphi \in L^{p'}(\mathbb{R}^N).$$

Before stating the Strichartz estimates, we need the following definition.

Definition 1.2 (Admissible pairs)**.** We say that a pair $(q,r) \in [2,\infty] \times [2,\infty]$ is admissible if

$$\frac{2}{q} = N\left(\frac{1}{2}-\frac{1}{r}\right), \quad (q,r,N) \neq (2,\infty,2).$$

We say that the pair is a *strictly admissible pair* if $(q,r) \neq \left(2,\frac{2N}{N-2}\right)$. The point $\left(2,\frac{2N}{N-2}\right)$ is called *endpoint*.

Theorem 1.3 (Strichartz estimates)**.** *For any admissible pairs (q_1,r_1), (q_2,r_2) there exist $C>0$ such that the following holds:*

- *Homogeneous estimate. For any $\varphi \in L^2(\mathbb{R}^N)$ we have*

$$\|S(t)\varphi\|_{L^{q_1}_t L^{r_1}_x} \leqslant C\|\varphi\|_{L^2}.$$

- *In homogeneous estimate. For $F \in L^{q_2'}_t L^{r_2'}_x(\mathbb{R}\times\mathbb{R}^N)$, we have*

$$\left\|\int_0^t S(t-s)F(s)\,ds\right\|_{L^{q_1}_t L^{r_1}_x} \leqslant C\|F\|_{L^{q_2'}_t L^{r_2'}_x}.$$

Using Strichartz estimates, one can prove the local well posedness of (1.2) in $H^1(\mathbb{R}^N)$ (see e.g [16, Theorem 4.4.1]).

1.1.2 Abstract local theory

In this section, we would like to introduce the general method to establish the local theory for evolution equations. For a deeper discussion of the local well posedness, we refer to [17]. Let X be a Banach space and A be a linear operator in X with $D(A)$ the domain of A. We assume that A is the generator of a bounded continuous group $(S(t))_{t\in\mathbb{R}}$ in X. We consider the following Cauchy problem:

$$\begin{cases} u_t = Au + f(u), \\ u(t=0) = u_0, \end{cases} \tag{1.6}$$

where $u : \mathbb{R} \to X$. We see that $S(t)u_0$ is the unique solution of (1.6) in the case $f \equiv 0$. In Duhamel form, (1.6) is rewritten as follows:

$$u(t) = S(t)u_0 + \int_0^t S(t-s)f(u(s))\,ds. \tag{1.7}$$

Formally, under smoothness and boundedness conditions on f and u, a function u solves (1.6) if only if u solves (1.7) (see [17, Lemma 4.1.1], [17, Proposition 4.1.6], [17, Corollary 4.1.7], [17, Corollary 4.1.8], [17, Proposition 4.1.9]). Thus, we reduce the study of the local theory of (1.6) to the study of the local theory of (1.7). Local well posedness of (1.7) is usually established by using contraction mapping theorem.

In our case, we are interested in the Schrödinger equations i.e $A = i\Delta$. The definition of a strong and weak solution to nonlinear Schrödinger equations is given in [16, Definition 3.1.1]. The definition of locally well posed is given in [16, Definition 3.1.5]. We would like to recall and give some comments on this. We say that the problem (1.7) is locally well-posed in $H^1(\mathbb{R}^N)$ if the following properties hold:

(1) Let $u_0 \in H^1(\mathbb{R}^N)$. Then there exists a unique solution in $H^1(\mathbb{R}^N)$ for the problem (1.7). Moreover, the solution is defined on a maximal interval $(T_{\min}, T_{\max})$, with $T_{\max}$ and $T_{\min}$ depending on u_0. In some cases, it is useful to prove the existence of blow up solutions.

(2) There is the blowup alternative: If $T_{\max} < \infty$ then $\lim_{t\to T_{\max}} \|u(t)\|_{H^1} = \infty$. A similar statement holds for $T_{\min}$. This blowup alternative is useful to prove the existence of global solutions. Indeed, if we can show that $\|u(t)\|_{H^1}$ is bounded when t is close to $T_{\max}$ then $T_{\max} = \infty$.

(3) The solution depends continuously on the initial value i.e if $u_{n0} \to u_0$ in $H^1(\mathbb{R}^N)$ and if $I \subset (T_{\min}, T_{\max})$ is a closed interval, then the corresponding solution u_n with initial data u_{n0} is defined on I for n large enough and satisfy $\|u_n - u\|_{L^\infty(I,H^1)} \to 0$. This property is useful to verify the conservation laws in $H^1(\mathbb{R}^N)$ of (1.7). Indeed, the conservation laws are obtained for an smooth enough and decaying solution of (1.6). We know that under some conditions of f, a solution of (1.6) also solves (1.7). By an approximation argument and using the continuous dependence of the solution on the initial value, we may show rigorously the conservation law for a solution of (1.7).

1.1.3 Global well posedness and blow up

Consider the equation (1.1). As in the previous section, (1.1) is locally well posed on $H^1(\mathbb{R}^N)$ in the energy sub-critical case i.e $0 < \alpha < \frac{4}{N-2}$. Moreover, the conservation laws are satisfied. Let $u \in C((T_{\min}, T_{\max}), H^1(\mathbb{R}^N))$ be the maximal solution of (1.1) corresponding with the initial data $u(0) = u_0$. In this section, we present the well known results of global well posedness ($T_{\max} = \infty$ and $T_{\min} = -\infty$) and blow up of this solution ($T_{\max} < \infty$ or $T_{\min} > -\infty$).

Global well posedness

In the case $\lambda < 0$, using the conservation of mass and energy we may prove that the H^1-norm of u is uniformly bounded in time. This implies that the solution exists globally in time. In the case $\lambda > 0$, the situation is more complex. If $0 < \alpha < \frac{4}{N}$, or $\alpha = \frac{4}{N}$ and $\|u_0\|_{L^2}$ small enough then the solution is global. We may expect the existence of blow up solutions for $\alpha \geqslant \frac{4}{N}$. Thus, in the focusing case, $\alpha = \frac{4}{N}$ is a threshold between global existence an blow up. These are the most complex cases to study the long time dynamic of (1.1). In this section, we focus on introducing the well known results on global existence of solutions of (1.1).

First, in the case of small initial data, the solution is global. More precisely, we have the following result.

Theorem 1.4. *Let $0 \leqslant \alpha < \frac{4}{N-2}$. There exists a number $a > 0$ such that if $\|u_0\|_{H^1} < a$ then the associated solution u of (1.1) is global.*

This theorem is proved by using Sobolev-embedding theorem and a boostrap argument.

In the case α sufficiently large, the solution is global for oscillating data. We have the following result.

Theorem 1.5. *Assume $\frac{4}{N-2} > \alpha > \alpha_0 = \frac{2-N+\sqrt{N^2+12N+4}}{2N}$ and $a = \frac{2\alpha(\alpha+2)}{4-\alpha(N-2)}$. Let $u_0 \in H^1(\mathbb{R}^N)$ be such that $|\cdot|u_0(\cdot) \in L^2(\mathbb{R}^N)$. Given $b \in \mathbb{R}$, set*

$$u_{0b}(x) = e^{\frac{ib|x|^2}{4}} u_0(x),$$

and let u_b be the maximal solution of (1.1) with the initial data u_{0b}. There exists a number b_0 such that if $b \geqslant b_0$ then u_b is global. Moreover, $u_b \in L^a(\mathbb{R}, L^{\alpha+2}(\mathbb{R}^N)) \cap L^\gamma(\mathbb{R}, W^{1,\rho}(\mathbb{R}^N))$ for every admissible pair (γ, ρ).

Moreover, if α is given as in Theorem 1.5 and the space time norm of initial data is small in some space, we also obtain the global existence of solution.

Theorem 1.6. *Let $u_0 \in H^1(\mathbb{R}^N)$, α_0 be as in Theorem 1.5 and u be the associated solution of (1.1) with initial data u_0. There exists a number ε_0 such that if u_0 satisfies*

$$\sup_{t\in\mathbb{R}} t^{\frac{4-(N-2)\alpha}{2\alpha(\alpha+2)}} \|S(t)u_0\|_{L^{\alpha+2}} < \varepsilon_0$$

then u is global and satisfies

$$ess\sup_{t\in\mathbb{R}} t^{\frac{4-(N-2)\alpha}{2\alpha(\alpha+2)}} \|u(t)\|_{L^{\alpha+2}} < \infty.$$

For the proof of the above theorems, we refer to [16] and references therein.

Blow up of solution

As said in the previous section, the existence of blow solution for (1.1) only occurs in the focusing case for $\alpha \geqslant \frac{4}{N}$. These assumptions are made throughout this section.

In [45], the author used the following functional

$$f(t) = \int_{\mathbb{R}^N} |x|^2 |u(t,x)|^2\, dx.$$

Assume that the initial data belongs to weighted space

$$\Sigma = \{\varphi \in H^1(\mathbb{R}^N) : |\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^N)\}.$$

Then the associated solution u of (1.1) satisfies $u \in C((T_{\min}, T_{\max}), \Sigma)$. Thus, the function f is well defined. Moreover, $f \in C^2(T_{\min}, T_{\max})$ and we have the following virial identity

$$f''(t) = 16E(u_0) - \frac{4(N\alpha - 4)}{\alpha + 2}\|u(t)\|_{L^{\alpha+2}}^{\alpha+2},$$

where $E(u_0)$ is the energy. Since $N\alpha \geqslant 4$, if we assume $E(u_0) < 0$ then $f''(t) < \delta < 0$ for some constant δ independent in time. This implies that the time of existence of the solution is finite in both directions. More precisely, we have the following result.

Theorem 1.7 (Glassey [45]). *Let $u_0 \in \Sigma$ be such that $E(u_0) < 0$. Then the corresponding solution of* (1.1) *blows up in finite time.*

In the radial setting, the condition $x|u_0(x)| \in L^2(\mathbb{R}^N)$ can be removed. We have the following result.

Theorem 1.8 (Ogawa-Tsutsumi [96]). *Let $N \geqslant 2$ and*

$$\frac{4}{N} \leqslant \alpha < \frac{4}{N-2} \quad (2 \leqslant \alpha \leqslant 4 \text{ if } N = 2).$$

If $u_0 \in H^1(\mathbb{R}^N)$ is such that $E(u_0) < 0$ and u_0 is radial, then the corresponding solution of (1.1) *blows up in finite time in both directions.*

In the case $N = 1$, $\alpha = 4$, Ogawa-Tsutsumi [97] proved that any solution with negative energy blows up in finite time. More precisely, we have the following result.

Theorem 1.9 (Ogawa-Tsutsumi [97]). *Let $N = 1$, $\alpha = 4$, $u_0 \in H^1(\mathbb{R})$ be such that $E(u_0) < 0$. Then the corresponding solution of* (1.1) *blows up in finite time.*

In the mass critical case $\alpha = \frac{4}{N}$, the existence and uniqueness of blow up solution with critical mass was obtained in [88]. More precisely, we have the following result.

Theorem 1.10 (Merle [88]). *Let $u_0 \in H^1(\mathbb{R}^N)$ be such that the associated solution of* (1.1) *blows up in finite time $T > 0$. Moreover, assume that $\|u_0\|_{L^2} = \|Q\|_{L^2}$, where Q is the unique radial positive solution of the elliptic equation*

$$\Delta u + |u|^{\frac{4}{N}} u = u.$$

There exist $\theta \in \mathbb{R}$, $\omega > 0$, $x_0 \in \mathbb{R}^N$, $x_1 \in \mathbb{R}^N$ such that

$$u_0 = \left(\frac{\omega}{T}\right)^{\frac{N}{2}} e^{i\theta - i|x-x_1|/4T + i\omega^2/T} Q\left(\omega\left(\frac{x - x_1}{T} - x_0\right)\right),$$

and for $t < T$,

$$u(t,x) = \left(\frac{\omega}{T-t}\right)^{\frac{N}{2}} e^{i\theta + i|x-x_1|^2/4(-T+t) - i\omega^2/(-T+t)} Q\left(\frac{\omega}{T-t}((x - x_1) - (T-t)x_0)\right).$$

In the critical case $\alpha = \frac{4}{N}$, if the initial data has larger mass than the mass of ground state profile then the situation is more complex. In [90, 89, 91], the authors proved that in the focusing mass critical case, if the initial data u_0 has the mass near the mass of ground state profile Q, and u_0 has negative energy and zero momentum then the associated solution blows up in finite time. We have the following result.

Theorem 1.11 (Merle-Raphael [90, 89, 91]). *Let $N = 1$ or $N \geqslant 2$ with a spectral assumption. Then there exists a number $a > 0$ and a constant $C > 0$ such that the following is true. Let $u_0 \in H^1(\mathbb{R}^N)$ be such that*

$$0 < \|u_0\|_{L^2}^2 - \|Q\|_{L^2}^2 < a, \quad E(u_0) < \frac{1}{2}\left(\frac{|\mathcal{I}m(\int_{\mathbb{R}^N} \nabla u_0 \overline{u_0}\, dx)|}{\|u_0\|_{L^2}}\right)^2.$$

Let $u(t)$ be the corresponding solution of (1.1)*. Then u blows up in finite time $T > 0$ and for t close to T:*

$$\|\nabla u(t)\|_{L^2}^2 \leqslant C_1 \left(\frac{ln|ln(T-t)|}{T-t}\right)^{\frac{1}{2}},$$

and

$$\|\nabla u(t)\|_{L^2}^2 \geqslant C_2 \left(\frac{ln|ln(T-t)|}{T-t}\right)^{\frac{1}{2}},$$

for some constant $C_1, C_2 > 0$.

1.1.4 Standing waves and stability theory

The equation (1.1) is invariant by Galilean transform. More precisely, if u solves (1.1) then for any $v \in \mathbb{R}^N$, the following function solves (1.1):

$$e^{i\left(\frac{v}{2}(x-vt)+\frac{|v|^2 t}{4}\right)} u(t, x - vt).$$

The equation (1.1) admits a special type of solution called *solitary waves*. A solitary wave of (1.1) is a solution of the form $e^{i\omega t}\varphi(x - vt)$, where $\varphi \in H^1(\mathbb{R}^N)$. In the case $v = 0$, this solution is called *standing wave*.

In the defocusing case $\lambda = -1$, there is no standing wave of (1.1). In the focusing case $\lambda = 1$, there is no standing wave in the case $\omega \leqslant 0$. Throughout this section, we only consider the focusing case i.e $\lambda = 1$. Assume $\omega > 0$, the standing waves of (1.1) are of the form $e^{i\omega t}\varphi(x)$, where φ solves:

$$\begin{cases} -\Delta\varphi + \omega\varphi - |\varphi|^\alpha \varphi = 0 \\ \varphi \in H^1(\mathbb{R}^N) \setminus \{0\} \end{cases} \tag{1.8}$$

The function φ is called *ground state* if it solves the following variational problem

$$\inf\{S_\omega(v); v \text{ is a solution of } (1.8)\},$$

where S_ω is defined by

$$S_\omega(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{\omega}{2}\|u\|_{L^2}^2 - \frac{1}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}.$$

Existence of a radial positive ground state φ can be shown by using variational techniques (see [71] and the references therein). Moreover, the set of ground state is the following
$$\mathcal{G} = \{e^{i\theta}\varphi(\cdot - y); \theta \in \mathbb{R}, y \in \mathbb{R}^N\}.$$

It turns out that in some cases the solution is close to the orbit of the standing wave if the initial data is enough close to the standing wave profile. Before stating the main results, we need the following definition.

Definition 1.12. Let φ be a solution of (1.8). The standing wave $e^{i\omega t}\varphi(x)$ is said to be *orbitally stable* in $H^1(\mathbb{R}^N)$ if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^N)$ satisfies $\|u_0 - \varphi\|_{H^1} < \delta$ then the maximal solution $u(t)$ of (1.1) with $u(0) = u_0$ exists for all $t \in \mathbb{R}$ and

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}, y\in\mathbb{R}^N} \|u(t) - e^{i\theta}\varphi(\cdot - y)\|_{H^1} < \varepsilon.$$

Otherwise, the standing wave is said to be unstable.

In addition, If there exists a sequence $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^N)$ as $n \to \infty$ such that the associated solution u_n of (1.1) with initial data φ_n blows up in finite time for all n, then the standing wave is said to be strongly unstable or *unstable by blow up in finite time*. The strongly instability of standing waves implies its instability.

We have the following result.

Theorem 1.13. *Let φ be a ground state of* (1.8)*. If $0 < \alpha < \frac{4}{N}$ then the standing wave $e^{i\omega t}\varphi(x)$ is orbitally stable.*

There are many methods to prove the stability of standing waves. One of them is the variational method introduced by Cazenave and Lions [15, 18]. This method relies on the following compactness result.

Proposition 1.14. *Let $0 < \alpha < \frac{4}{N}$. Fix $\rho > 0$. Consider the following minimization problem*
$$d_\rho := \inf\{E(v) : v \in H^1(\mathbb{R}^N), \|v\|_{L^2}^2 = \rho\}, \tag{1.9}$$
where E is the functional energy of (1.1) *in the focusing case. Let $v_n \in H^1(\mathbb{R}^N)$ satisfy the following condition:*

$$E(v_n) \to d_\rho, \quad \text{and} \quad \|v_n\|_{L^2}^2 \to \rho.$$

Then there exist a sequence $(y_n) \in \mathbb{R}^N$ and a function $v \in H^1(\mathbb{R}^N)$ such that up to a subsequence we have

$$v_n(\cdot - y_n) \to v \text{ strongly in } H^1(\mathbb{R}^N).$$

In particular, $E(v) = d_\rho$ and $\|v\|_{L^2}^2 = \rho$.

By using Proposition 1.14, Cazenave and Lions [18] proved Theorem 1.13. See also Le Coz [71].

The method of Cazenave and Lions relies on the variational characterization and the uniqueness of ground state under phase shift and translation. In the general case, for standing wave which is not a ground state this method may be not applicable. However, in [49, 50], Grillakis-Shatah-Strauss introduced a famous theory which can treat for larger class of *bound state*. This theory especially treat to the evolution equation with Hamilton structure.

Let E and M be the functional of the energy and the mass of (1.1). Let φ_ω be a solution of (1.8), where the subscript is to exhibit the dependence of solution with the parameter ω. Set $H_\omega = E''(\varphi_\omega) - \omega M''(\varphi_\omega)$ and $d(\omega) = E(\varphi_\omega) - \omega M(\varphi_\omega)$. It turns out that the stability of bound state depends on the convexity or concavity of function $d : \omega \mapsto d(\omega)$. Before stating the main result, we need the following important assumption.

Assumption A1. Assume that H_ω has exactly one simple negative eigenvalue and

$$\ker(H_\omega) = \left\{ i\varphi_\omega, \frac{\partial}{\partial_1}\varphi_\omega, ..., \frac{\partial}{\partial_N}\varphi_\omega \right\},$$

and the rest of its spectrum is positive and bounded away from zero.

The main result is the following.

Theorem 1.15 (Grillakis-Shatah-Strauss [49] Theorem 2, Theorem 4.7)**.** *Under Assumption A1, the bound state $e^{i\omega t}\varphi_\omega$ is orbitally stable if and only if the function $d(\cdot)$ is strictly convex in a neighborhood of ω. If the function d is strictly concave then the bound state $e^{i\omega t}\varphi_\omega$ is orbitally unstable.*

The main ingredient in the proof the stability of the about theorem is a coercivity property of operator H_ω. Consider the case φ_ω is a ground state. Assumption A1 is verified by the work in [71, Lemma 4.14-Lemma 4.19]. The condition $d''(\omega) > 0$ is equivalent to $\alpha < \frac{4}{N}$. Thus, using Theorem 1.15 we obtain the conclusion of Theorem 1.13.

It turn out that the stability of ground states depend on the nonlinear exponent α. Indeed, in the case $\alpha > \frac{4}{N}$, $d''(\omega) < 0$ then using Theorem 1.15 we obtain that the ground state is unstable. Moreover, in this case and the case $\alpha = \frac{4}{N}$, ground states are strongly unstable. More precisely, we have the following result.

Theorem 1.16. *Let φ be a ground state of* (1.8). *If $\alpha \geqslant \frac{4}{N}$ then the standing wave $e^{i\omega t}\varphi(x)$ is unstable by blow up in finite time.*

For the proof of the above theorem, we refer the reader to [71, Theorem 5.3].

1.2 Gross-Pitaevskii equation

In this section, we would like to introduce the following equation:

$$iu_t + \Delta u + u(1 - |u|^2) = 0, \tag{1.10}$$

where $u : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$ satisfies the nonvanishing boundary condition $|u| \to 1$ as $|x| \to \infty$. The equation (1.10) is called Gross-Pitaevskii equation. Its energy is given by

$$E(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{4}\int_{\mathbb{R}^N} (|u|^2 - 1)^2 \, dx,$$

which is defined on the energy space

$$\mathcal{E} = \left\{ u \in H^1_{\mathrm{loc}}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N), |u|^2 - 1 \in L^2(\mathbb{R}^N) \right\}.$$

Consider the *Madelung transform*

$$u = \sqrt{\rho} e^{i\theta}, \qquad \text{for } u \neq 0.$$

The *hydrodynamical variables* $(\rho, v = 2\nabla\theta)$ satisfy the *hydrodynamical system*

$$\begin{cases} \rho_t + \operatorname{div}(\rho v) = 0, \\ v_t + v \cdot \nabla v + 2\nabla\rho = 2\nabla\left(\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right). \end{cases} \tag{1.11}$$

1.2.1 Cauchy problem

First, we recall the definition of Zhidkov spaces which were introduced in [120]:

$$X^k(\mathbb{R}^N) = \left\{ u \in L^\infty(\mathbb{R}^N), \partial^\alpha u \in L^2(\mathbb{R}^N),\ 1 \leqslant |\alpha| \leqslant k \right\}, \tag{1.12}$$

equipped with the natural norm

$$\|u\|_{X^k} = \|u\|_{L^\infty} + \Sigma_{1\leqslant|\alpha|\leqslant k} \|\partial^\alpha u\|_{L^2}.$$

The global well-posedness of (1.10) in one dimension in the energy space $\mathcal{E}$ was proved in [119, 120]. In higher dimensions, the situation is more complex.

As shown in [42], the space $\mathcal{E} \subset X^1(\mathbb{R}^N) + H^1(\mathbb{R}^N)$ is a complete metric space with the following distance metric:

$$d_E(u, \tilde{u}) = \|u - \tilde{u}\|_{X^1 + H^1} + \||u|^2 - |\tilde{u}|^2\|_{L^2}.$$

In [42], the author established the local and global theory of (1.10) in the energy space $\mathcal{E}$.

Theorem 1.17 (Gérard [42])**.** *Let $N = 2, 3$. For each $u_0 \in \mathcal{E}$, there exists a unique solution $u \in C(\mathbb{R}, \mathcal{E})$ of (1.10) with the initial data $u(0) = u_0$. Moreover, u satisfies the following properties:*

- ***Regularity:*** *If $\Delta u_0 \in L^2(\mathbb{R}^N)$ then $\Delta u \in C(\mathbb{R}, L^2(\mathbb{R}^N))$.*
- ***Conservation energy:*** *for all $t \in \mathbb{R}$, we have $E(u(t)) = E(u_0)$.*
- *For each $R > 0$, $T > 0$, there exists $C > 0$ such that for each $u_0, \tilde{u}_0 \in \mathcal{E}$ such that $E(u_0) \leqslant R$ and $E(\tilde{u}_0) \leqslant R$, the corresponding solutions u, $\tilde{u}$ satisfy*

$$\sup_{|t|\leqslant T} d_E(u(t), \tilde{u}(t)) \leqslant C d_E(u_0, \tilde{u}_0).$$

In dimension $N = 4$, in [42], Gérard proved that (1.10) is globally well-posed in the case of small energy of the initial data. The proof uses the contraction mapping theorem.

Theorem 1.18 (Gérard [42]). *Let $N = 4$. There exists $\delta > 0$ such that, for every $u_0 \in \mathcal{E}$ such that $E(u_0) \leqslant \delta$, there exists a unique solution of* (1.10) $u \in C(\mathbb{R}, \mathcal{E})$ *with $\nabla u \in L^2_{loc}(\mathbb{R}, L^4(\mathbb{R}^4))$ and $u(0) = u_0$. Moreover, the energy is conserved and the solution satisfies the regularity property and Lipschitz continuity stated in Theorem 1.17.*

In [69], the authors improved the result of [42] in the case $N = 4$ for arbitrary large energy of the initial data.

Theorem 1.19 (Killip-Oh-Pocovnicu-Visan [69]). *Let $N = 4$ and $u_0 \in \mathcal{E}$. There exists a unique solution $u \in C(\mathbb{R}, \mathcal{E})$ of* (1.10) *with the initial data $u(t = 0) = u_0$.*

We also mention the work of Gallo [41], in which the author proves the local theory on energy space $\mathcal{E}$ for general nonlinearity.

1.2.2 Travelling waves

Travelling waves of (1.10) are special solutions of the form (up to a space rotation)

$$u(t, x) = U_c(x_1 - ct, ..., x_N), \tag{1.13}$$

for a speed $c \in \mathbb{R}$ and the profile U_c solves the equation

$$-ic\partial_1 U_c + \Delta U_c + U_c(1 - |U_c|^2) = 0. \tag{1.14}$$

In dimension $N = 1$, travelling waves for (1.10) are uniquely (up to translation and phase shift) given by

$$U_c(x) = \sqrt{\frac{2 - c^2}{2}} \tanh\left(\frac{\sqrt{2 - c^2}}{2} x\right) + i\frac{c}{\sqrt{2}},$$

for $|c| < \sqrt{2}$. In this case, travelling waves are called *dark solitons*. In the case of higher dimensions, the situation is more complex.

In the case $N \geqslant 2$, the situation is more complex. In many cases, the travelling waves are constant functions. We have the following result.

Theorem 1.20 (Gravejat [46, 47], Bethuel-Saut [12]). *Consider* (1.10) *and a travelling wave profile U_c solving* (1.14). *Assume $c = 0$ for $N \geqslant 2$ or $c > \sqrt{2}$ for $N \geqslant 2$ or $c = \sqrt{2}$ for $N = 2$. Then U_c is a constant function.*

In [81], Maris developed the above result for general cases of (1.10).

Non-existence of non-constant travelling waves also holds in the case of high dimensions with small energy. More precisely, we have the following result.

Theorem 1.21 (Bethuel-Gravejat-Saut [8], de Laire [33]). *Let $N \geqslant 3$. For* (1.10), *there exists a number $\varepsilon > 0$ such that a travelling wave profile U_c with energy*

$$E(U) \leqslant \varepsilon,$$

is constant.

We consider the following minimization problem

$$d_\rho = \inf\{E(u), u \in W(\mathbb{R}^N), p(u) = \rho\}, \tag{1.15}$$

where $\rho \in \mathbb{C}$ and $W(\mathbb{R}^N) = \{1\} + V(\mathbb{R}^N)$ with $V(\mathbb{R}^N)$ is defined by

$$V(\mathbb{R}^N) = \{v : \mathbb{R}^N \to \mathbb{C},\ \text{s.t. } (\nabla v, \mathcal{R}e(v)) \in L^2(\mathbb{R}^N)^2, \mathcal{I}m(v) \in L^4(\mathbb{R}^N),\ \text{and } \nabla\mathcal{R}e(v) \in L^{\frac{4}{3}}(\mathbb{R}^N)\},$$

and p is the first component of momentum function defined by

$$p(u) = \frac{1}{2}\int_{\mathbb{R}^N} \langle i\partial_1 u, u-1\rangle \, dx,$$

where $\langle f, g\rangle = \mathcal{R}ef\mathcal{R}eg + \mathcal{I}mf\mathcal{I}mg$. We have the following result.

Theorem 1.22 (Bethuel-Gravejat-Saut [8]). *The following holds:*

(i) For $N = 2$ and $\rho > 0$, there exists a minimizing travelling wave U_ρ for (1.15).

(ii) For $N = 3$, there exists $\rho_* > 0$ such that there exists a minimizing travelling wave U_ρ for (1.15) *if and only if $\rho \geqslant \rho_*$.*

For the general nonlinearity case, see Chiron-Maris[20].

The uniqueness of solutions to the minimization problem was proved in the case of large momentum. More precisely, we have the following result.

Theorem 1.23 (Chiron-Pacherie [21, 22]). *Let $N = 2$. There exists a number $\rho_0 > 0$ such that, for each $\rho \geqslant \rho_0$, there exists a unique (up to phase shift and translation) minimizer U_ρ of* (1.15)*. Moreover, they form a smooth branch of travelling waves.*

In the case $0 < c < \sqrt{2}$ (Subsonic travelling waves), the existence of non constant travelling wave is proved in dimensions $N \geqslant 3$.

Theorem 1.24 (Maris [82], Bellazzini-Ruiz [4]). *Let $N \geqslant 3$. There exists a non constant travelling wave U_c of* (1.10) *for each $0 < c < \sqrt{2}$.*

1.2.3 Orbital and asymptotic stability of travelling waves

In dimension $N = 1$

Before presenting the well known results, we introduce the following distance in the energy space $\mathcal{E}$

$$d_c(\varphi_1, \varphi_2)^2 = \int_{\mathbb{R}} |\varphi_1' - \varphi_2'|^2 + (1 - |U_c|^2)|\varphi_2 - \varphi_1|^2 + \left||\varphi_1|^2 - |\varphi_2|^2\right|^2 dx.$$

We have the following stability result.

Theorem 1.25 (Bethuel-Gravejat-Saut [7], Bethuel-Gravejat-Saut-Smets [9]). *Let c be such that $c^2 < 2$. There exist $\delta_c > 0$ and $K_c > 0$ such that, for each $u_0 \in \mathcal{E}$ satisfying the condition*

$$\delta := d_c(u_0, U_c) < \delta_c,$$

then the corresponding solution u of (1.10) *is such that there exist two functions $a \in C^1(\mathbb{R},\mathbb{R})$ and $\theta \in C^1(\mathbb{R},\mathbb{R})$ with*

$$\sup_{t\in\mathbb{R}} |a'(t) - c| < K_c\delta,$$

such that the following holds:

$$\sup_{t\in\mathbb{R}} d_c\left(e^{-i\theta(t)}u(\cdot + a(t), t), U_c\right) < K_c\delta.$$

The asymptotic stability of dark solitons is as follows.

Theorem 1.26 (Bethuel-Gravejat-Smets [11], Gravejat-Smets [48], Cuccagna-Jenkins [30])**.** *Let $c \in (-\sqrt{2}, \sqrt{2})$. There exists $\delta_c > 0$ such that for each $u_0 \in \mathcal{E}$ satisfies*

$$d_c(u_0, U_c) < \delta_c,$$

then there exist a number $c_\infty \in (-\sqrt{2}, \sqrt{2})$ and two functions $a \in C^1(\mathbb{R},\mathbb{R})$ and $\theta \in C^1(\mathbb{R},\mathbb{R})$ with

$$a'(t) \to c_\infty, \qquad \text{and } \theta'(t) \to 0,$$

as $t \to \infty$ such that the correspoding solution u of (1.10) *satisfies*

$$e^{-i\theta(t)}u(\cdot + a(t), t) \to U_{c_\infty} \qquad \text{locally uniformly on } \mathbb{R}.$$

In [78], Lin used the abstract theory of Grillakis-Shatah-Strauss [49, 50] to prove the stability and instability of travelling waves in the case of general nonlinearity. More precisely, we have the following result.

Theorem 1.27 (Lin [78])**.** *Consider the following equation:*

$$iu_t + u_{xx} + f(|u|^2)u = 0, \tag{1.16}$$

where $f \in C^2(\mathbb{R}^+)$ and $f(\rho_0) = 0$ for some $\rho_0 > 0$ and satisfies other conditions. Let $c \geqslant 0$ be small enough. Then there exists a travelling wave $U_c(x-ct) = a_c e^{i\theta_c}(x-ct)$ solution to (1.16)*. This solution is stable when $\frac{dP_c}{dc} < 0$ and unstable when $\frac{dP_c}{dc} > 0$, where*

$$P_c = \mathcal{I}m \int_{\mathbb{R}} \overline{U_c}' U_c \left(1 - \frac{\rho_0}{|U_c|^2}\right) dx.$$

Here the stability means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if the initial data $u_0 = a_0 e^{i\theta_0}$ satisfies

$$\int_{s\in\mathbb{R}} (\|a_0^2(\cdot + s) - a_c^2\|_{H^1} + \|\theta_0'(\cdot + s) - \theta_c'\|_{L^2}) < \delta,$$

then

$$\inf_{s\in\mathbb{R}} (\|a(t)^2(\cdot + s) - a_c^2\|_{H^1} + \|\theta(t)'(\cdot + s) - \theta_c'\|_{L^2}) < \varepsilon,$$

for $t \in \mathbb{R}^+$. Here $u(t) = a(t)e^{i\theta(t)}$ is the solution of (1.16) *with $a(0) = a_0$, $\theta(0) = \theta_0$. Instability means that the travelling wave is not stable.*

In higher dimensions

In the case $N = 2, 3$, we equip the energy set $\mathcal{E}$ the following metric distance

$$d(f, g) = \|f - g\|_{L^2(B(0,1))} + \|\nabla f - \nabla g\|_{L^2} + \||f|^2 - |g|^2\|_{L^2}.$$

We have the following result.

Theorem 1.28 (Chiron-Maris [20])**.** *Let $\mathcal{M}_\rho$ be the set of minimizing travelling waves U_ρ with scalar momentum ρ. Fix $U_\rho \in \mathcal{M}_\rho$. For all $\varepsilon > 0$, there exists $\delta > 0$ such that for each $u_0 \in \mathcal{E}$ such that*

$$d(u_0, U_\rho) < \delta,$$

then the corresponding solution u of (1.10) *satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{U\in\mathcal{M}_\rho} d(u(\cdot, t), U) < \varepsilon.$$

The proof of the above theorem used the variational problem of minimizing the energy with fixed momentum.

1.3 The derivative nonlinear Schrödinger equations

This thesis is devoted to the study of Schrödinger-type equations, especially derivative nonlinear Schrödinger equations i.e the equations of the following form:

$$\begin{cases} iu_t + u_{xx} + i\lambda|u|^2u_x + i\mu u^2\overline{u_x} + b|u|^4u = 0, \\ u(0) = u_0, \end{cases} \tag{1.17}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$, $b \in \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}$.

The equation (1.17) is invariant under the scaling transformation:

$$u_\kappa(t, x) := \kappa^{\frac{1}{2}} u(\kappa^2 t, \kappa x).$$

Moreover, in the case $b = 0$, the equation (1.17) has a complete integral structure. We may use inverse scattering techniques to study the long time behaviour of this equation. In [1], by using this techniques, Bahouri and Perelman proved the global existence of solution in $H^{\frac{1}{2}}(\mathbb{R})$. This was an open problem in long time.

Let u be a $H^1(\mathbb{R})$ solution of (1.17). We consider the Gauge transform

$$v(t, x) = u(t, x) \exp\left(ia \int_{-\infty}^{x} |u(t, y)|^2 \, dy\right). \tag{1.18}$$

It is easy to check that v is a $H^1(\mathbb{R})$ solution of the following equation

$$iv_t + v_{xx} + ic_1|v|^2v_x + ic_2v^2\overline{v}_x + c_3|v|^4v = 0, \tag{1.19}$$

where c_1, c_2, c_3 are the constants which depend on a, λ, μ, b. The dynamics of solutions of (1.17) is equivalent to the dynamics of solutions of (1.19). For each of choice of the value of a, we have another equation equivalent to (1.17). In some cases, if we choose a suitable value of a then studying the long time dynamics of solutions of (1.19) is easier than for (1.17). This is one of advantage of this transform. Specially, let u be a solution of the Chen-Liu-Lee equation [19]:

$$iu_t + u_{xx} + i|u|^2 u_x = 0. \tag{1.20}$$

Let v be the Gauge transform of u given by (1.18) with $a = \frac{-1}{2}$. Then v is a solution of the Kaup-Newell equation [68]:

$$iv_t + v_{xx} + i(|v|^2 v)_x = 0. \tag{1.21}$$

Let w be the Gauge transform of u given by (1.18) with $a = \frac{1}{2}$. Then w is a solution of the Gerdzhikov-Ivanov equation [44]:

$$iw_t + w_{xx} - iw^2 \overline{w}_x + \frac{1}{2}|w|^4 w = 0. \tag{1.22}$$

Moreover, (1.17) has some conservation laws in the energy space:

$$\|u(t)\|_{L^2}^2 = \|u_0\|_{L^2}^2, \tag{1.23}$$
$$E(u(t)) = E(u_0), \tag{1.24}$$
$$P(u(t)) = P(u_0), \tag{1.25}$$

where

$$E(\varphi) = \|\varphi_x\|_{L^2}^2 - \frac{\lambda+\mu}{2}\mathcal{I}m \left\langle |\varphi|^2\varphi, \varphi_x \right\rangle - \frac{(\lambda+\mu)\mu}{6}\|\varphi\|_{L^6}^6 - \frac{b}{6}\int_{\mathbb{R}} |\varphi|^6 \, dx,$$

and

$$P(u(t)) = \mathcal{I}m \int_{\mathbb{R}} u_x \overline{u} \, dx + \frac{\mu}{2}\int_{\mathbb{R}} |u|^4 \, dx,$$

1.3.1 Local theory

In this section, we present some well-known results for the local theory of (1.17), some method used and our main goal on establishing local well-posedness of this kind equation.

Local theory of (1.17) has attracted a lot of interests in several years (see e.g [24, 25, 56, 58, 59, 107, 108, 111, 112] and references therein). The main difficulty is the appearance of the derivative term. We cannot use the classical contraction method for this type of nonlinear Schrödinger equations. Some methods were used to overcome this difficulty. In [24, 25, 107, 108], the authors used the Fourier restriction method to established local well-posedness and global well-posedness results for (1.17). By using this method, we can directly use the contraction mapping theorem for the Duhamel form of equation (1.17) to obtain existence results. Another approach was used in [56, 111, 112] where the authors used an approximation argument to prove the existence of solutions. Another method was used in [58, 59, 100], where the authors used a Gauge transform to obtain a system of two equations without derivative nonlinearities from the original equation (1.17). More precisely, we set

$$\varphi(t,x) = \exp\left(i\frac{\lambda}{2}\int_{-\infty}^{x}|u(t,y)|^2\,dy\right)u(t,x),$$

$$\psi(t,x) = \exp\left(i\frac{\lambda}{2}\int_{-\infty}^{x}|u(t,y)|^2\,dy\right)\left(u_x(t,x)+i\frac{\mu}{2}|u|^2u(t,x)\right).$$

We observe that if u solves (1.17) then (φ,ψ) solves the following system

$$\begin{cases} i\varphi_t+\varphi_{xx} = i(\lambda-\mu)\varphi^2\overline{\psi}-b|\varphi|^4\varphi, \\ i\psi_t+\psi_{xx} = -i(\lambda-\mu)\psi^2\overline{\varphi}-\left(\frac{(\lambda-2\mu)\mu}{4}\right)(3|\varphi|^4\psi+2\varphi^3\overline{\varphi}\overline{\psi}) \\ \qquad\qquad -3b|\varphi|^4\psi-2b|\varphi|^2\varphi^2\overline{\psi}. \end{cases} \tag{1.26}$$

By definition, the functions φ and ψ satisfy the following relation

$$\psi = \varphi_x - i\left(\frac{\lambda-\mu}{2}\right)|\varphi|^2\varphi. \tag{1.27}$$

The Cauchy problem of the system (1.26) is established by classical arguments. The main difficulty in this method is to prove that the relation (1.27) is conserved under the flow of the system (1.26). When we prove this relation, the existence of solutions of (1.17) is implied by the existence of solutions of the system (1.26). The uniqueness and continuous dependence on initial data of solutions of (1.17) is obtained by the corresponding properties of solutions of the system.

Recently, the inverse scattering transform (IST) was used to proved global well posed result in the case $b=0$ of (1.17). In [66], Jenkins-Liu-Perry-Sulem proved that for any initial data $u_0\in H^{2,2}(\mathbb{R})=\{u\in H^2(\mathbb{R}),|\cdot|^2u(\cdot)\in L^2(\mathbb{R})\}$ then the associated solution of (1.20) is global existence in $H^{2,2}(\mathbb{R})$. Moreover, in [101], Pelinovsky and Shimabukuro proved the global existence result of solutions of (1.21) in the space $H^2(\mathbb{R})\cap H^{1,1}(\mathbb{R})$. Finally, in [1], Bahouri and Perelman proved that the equation (1.21) is globally well posed in $H^{\frac{1}{2}}(\mathbb{R})$. Moreover, the authors proved that for any initial data in $H^{\frac{1}{2}}(\mathbb{R})$, the associated solution is uniformly bounded in time. This solves an open problem in long time.

1.3.2 Stability theory

In this section, we introduce the well known results on stability and instability of solitons of the equation (1.17).

Solitons

In the case $\lambda\neq 0$ or $\mu\neq 0$, (1.17) has no Galilean invariance as in the case of simple power nonlinearity. Thus, the family of solitary waves has two parameters (frequency and speed) which make the studying of stability and instability is more difficult than the usual cases. Consider (1.17) in the case $\lambda=1$ and $\mu=0$. The solitons of (1.17) are solutions of the form

$$R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct),$$

where $\phi_{\omega,c} \in H^1(\mathbb{R})$. It is clear that $\phi_{\omega,c}$ solves

$$-\phi'' + \omega\phi + ic\phi' - i|\phi|^2\phi' - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}.$$

As in [55], we use the gauge transformation

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x) \exp\left(\frac{ic}{2} - \frac{i}{4}\int_{-\infty}^{x} |\Phi_{\omega,c}(y)|^2\, dy\right).$$

Let $\gamma = 1 + \frac{16}{3}b$. The positive radial profile $\Phi_{\omega,c}$ obtained as follows: if $\gamma > 0$,

$$\Phi^2_{\omega,c}(x) = \begin{cases} \frac{2(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x)-c} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \frac{4c}{(cx)^2+\gamma} & \text{if } c = 2\sqrt{\omega}, \end{cases} \tag{1.28}$$

if $\gamma \leqslant 0$,

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)}\cosh(\sqrt{4\omega - c^2}x) - c} \quad \text{if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}, \tag{1.29}$$

where $s_* = \sqrt{\frac{-\gamma}{1-\gamma}}$.

On stability/instability of solitons

As we know, (1.17) has no Galilean invariance. We know that (1.17) has a two parameter family of solitary waves. In [23], in the case $b = 0$, $\mu = 0$, Colin and Ohta proved that the solitons are orbitally stable in the whole range of parameters values by using variational methods. In this case, in [70], Kwon and Wu showed that the algebraic soliton $u_{\omega,2\sqrt{\omega}}$ is orbitally stable (up to scaling symmetry).

In the case $\mu = 0$, $\lambda = 1$ there are many works on the stability/instability of solitons of (1.17). In [99], in the case $b > 0$, Ohta proved there exists $k = k(b) \in (0, 1)$ such that the solitons $u_{\omega,c}$ of (1.17) is stable if $-2\sqrt{\omega} < c < 2k\sqrt{\omega}$ and unstable if $2k\sqrt{\omega} < c < 2\sqrt{\omega}$. The stability/instability of solitons in the case $c = 2k\sqrt{\omega}$ is an open problem. In [55], Hayashi showed a relation between the stability/instability of solitons and the positivity of momentum of the solitons. More precisely, if the momentum is positive then the solitons are stable and if the momentum is negative then the solitons are unstable. Moreover, the author proved that in the case $b < 0$, the momentum of solitons is positive, hence solitons are orbitally stable. Specially, in [95], Ning-Ohta-Wu showed that the algebraic soliton $u_{\omega,2\sqrt{\omega}}$ is unstable in the case $b > 0$ sufficient small.

1.3.3 Multi-solitons theory

In this section, we present the multi-solitons theory.

A multi-soliton of a dispersive equation is a solution which behaves at large time like a finite or infinite sum of solitons. Usually, in the Cauchy problem theory, when the mass of the initial data is small, the solution exists globally in time. The existence of multi-solitons shows that there also exists a global solution with arbitrary large mass. The main motivation of multi-solitons theory comes from the conjecture called *soliton resolution conjecture*. This conjecture states that all global solutions of a dispersive equation behave at large time like a sum of a radiative term and solitons. Thus, multi-solitons theory gives us more information about the long time behaviour of solutions.

In classical nonlinear Schrödinger equations, the existence of multi-solitons was showed in [26, 28, 35, 72, 73, 84, 118]. For focusing energy-critical nonlinear Schrödinger equation, Jendrej [63] proved existence of pure two-bubles in space dimension $N \geqslant 7$. The main ingredient in [63] is a uniformly bounded of a sequence of solutions and by taking a weak limit to obtain the desired solution. This argument goes back to the works Martel [83], Merle [87], Bellazzini-Ghimenti-Le Coz [3]. A similar argument was used to obtain the existence of two buble solutions for energy critical equations in dimension $N = 6$, see Jendrej [64]. For the energy-critical focusing wave equation with spatial dimension $N = 5$, [65] proved existence of multi-bubble solutions which blows up in infinite time at any K given points, $K \geqslant 2$. For Klein-Gordon equations, see the works [27, 29]. The stability of multi-solitons was shown for generalized Korteweg-de Vries equations and L^2-subcritical nonlinear Schrödinger equations in [85, 86]. In [74], Le Coz and Wu proved a stability result of multi-solitons of (1.17) in the case $b = 0$. In this thesis, we prove the existence of multi solitons of (1.17) for any value of b. First, we recall the definition of multi solitons.

Let $K \in \mathbb{N}^*$ and $(\theta_j, \omega_j, c_j)_{j=1,K}$ be given parameters such that $\omega_j > \frac{c_j^2}{4}$, $c_j \neq c_k$ for $j \neq k$. Let R_{ω_j,c_j} be the soliton associated with the parameters ω_j, c_j for each j. A multi-soliton profile is defined by

$$R(t,x) = \sum_{j=1}^{K} e^{i\theta_j} R_{\omega_j,c_j}(t,x). \tag{1.30}$$

Definition 1.29. A solution u of (1.17) is called a multi-soliton if it behaves like a multi-soliton profile at large time, i.e:

$$\|u(t) - R(t)\|_{H^1} \to 0 \text{ as } t \to \infty.$$

In the next part, we consider the following equation

$$iu_t + u_{xx} + iu^2\overline{u}_x + b|u|^4 u = 0. \tag{1.31}$$

Let $R_{\omega,c}(t,x)$ be a solution of (1.31) of form $e^{i\omega t}\phi_{\omega,c}(x - ct)$. Let $\Phi_{\omega,c}$ be the associated function defined by

$$\Phi_{\omega,c} = \exp\left(-i\frac{c}{2}x + \frac{i}{4}\int_{\infty}^{x} |\phi_{\omega,c}(y)|^2\, dy\right)\phi_{\omega,c}. \tag{1.32}$$

We note that the profile $\Phi_{\omega,c}$ is well defined when $\Phi_{\omega,c}$ restricted on $\mathbb{R}^-$ belongs to $L^2(\mathbb{R}^-)$. Thus, $\Phi_{\omega,c}$ does not need to belong to $L^2(\mathbb{R})$. In this thesis, we prove the existence of multi kink solitons of (1.31). Our motivation comes from the works [73, 72] for classical nonlinear Schrödinger equations. Before stating the next result, we need the definition of multi kink solitons of (1.31).

Definition 1.30. The *half kink solution* $R_{\omega,c}$ of (1.31) is a solution of (1.31) of the type $e^{i\omega t}\phi_{\omega,c}(x-ct)$ where $\phi_{\omega,c}$ is such that the associated function $\Phi_{\omega,c}$ defined in (1.32) verifies

$$\begin{cases} -\Phi'' + \left(\omega - \frac{c^2}{4}\right)\Phi - \frac{c}{2}\Phi^3 + \frac{3}{16}\gamma\Phi^5 = 0, \\ \lim_{x\to\infty}\Phi(x) = 0, \\ \lim_{x\to-\infty}\Phi(x) > 0, \\ \Phi \text{ is a real valued function.} \end{cases} \tag{1.33}$$

We have the following definition of multi-kink-soliton.

Definition 1.31. Let $K \in \mathbb{N}^*$ and R_{ω_0,c_0} be a half kink solution of (1.31). Let $(\theta_j, c_j, \omega_j)_{j=0,..,K}$ be given parameters. The multi-kink-soliton profile is defined by

$$V = \sum_{j=0}^{K} e^{i\theta_j} R_{\omega_j,c_j}. \tag{1.34}$$

A multi-kink-soliton of (1.31) is a solution u of (1.31) such that

$$\|u - V\|_{H^1} \to 0, \qquad \text{as } t \to \infty.$$

In the next part, we consider the following general derivative nonlinear Schrödinger equation:

$$\begin{cases} iu_t + u_{xx} + i|u|^{2\sigma}u_x = 0, \\ u(0) = u_0. \end{cases} \tag{1.35}$$

The local existence and global existence of solutions of (1.35) were studied in many works (see e.g [56, 103] and reference therein).
The equation (1.35) has a two parameters family of solitary waves defined as follows

$$R_{\omega,c} = \varphi_{\omega,c}(x-ct)\exp i\left(\omega t + \frac{c}{2}(x-ct) - \frac{1}{2\sigma+2}\int_{-\infty}^{x-ct}\varphi_{\omega,c}^{2\sigma}(\eta)\,d\eta\right)$$

where $\omega \geqslant \frac{c^2}{4}$ and

$$\varphi_{\omega,c}(y)^{2\sigma} = \frac{(\sigma+1)(4\omega - c^2)}{2\sqrt{\omega}(\cosh(\sigma\sqrt{4\omega-c^2}y) - \frac{c}{2\sqrt{\omega}})}.$$

In [80], Liu-Simpson-Sulem showed that in the case $\sigma \geqslant 2$, the solitons of (1.35) are orbitally unstable; in the case $0 < \sigma < 1$ they are orbitally stable. In the case $\sigma \in (1,2)$, the situation is more complex. The authors proved that there exists $z_0 \in (-1,1)$ such that if $c < 2z_0\sqrt{\omega}$ then the soliton is orbitally stable and if $c > 2z_0\sqrt{\omega}$ the soliton is orbitally unstable. In [52], in the case $1 < \sigma < 2$, the authors proved that the soliton is unstable in the critical frequency case i.e $c = 2z_0\sqrt{\omega}$.

The multi-soliton profile and multi-solitons of (1.35) are defined similarly as the ones of (1.17). The stability of multi-solitons of (1.35) was obtained in [110] in the case $1 < \sigma < 2$.

1.4 Our main results

In this section, we present the main results of this thesis.

1.4.1 Local theory

All well known results on the Cauchy problem of (1.17) are established on the usual Sobolev spaces $H^s(\mathbb{R})$. To our knowledge, there is no result for a local theory of (1.17) under nonvanishing boundary conditions. One of our goals in this thesis is to study the Cauchy problem of (1.17) under nonvanishing boundary conditions. Our main results are the following.

Theorem 1.32. *Let $X^k(\mathbb{R})$ be the Zhidkov space defined in* (1.12). *Consider the following special case of* (1.17)

$$iu_t + u_{xx} = -iu^2\overline{u}_x. \tag{1.36}$$

The equation (1.36) *is locally well-posed in $X^4(\mathbb{R})$ and $\phi+H^2(\mathbb{R})$ for any $\phi \in X^4(\mathbb{R})$. Moreover, if $\|\phi_x\|_{L^2}$ and $\|u_0-\phi\|_{H^1}$ are small enough then there exist $T>0$ and unique solution $u \in \phi+C([-T,T],H^1(\mathbb{R}))\cap L^4([-T,T],W^{1,\infty})$ of* (1.36). *Moreover, all non-vanishing stationary solutions of* (1.36) *in $X^1(\mathbb{R})$ are constant functions or functions of form $e^{i\theta}\sqrt{k}$, where*

$$k(x) = 2\sqrt{B} + \frac{-1}{\sqrt{\frac{5}{72B}\cosh(2\sqrt{B}(x-x_0)) + \frac{5}{12\sqrt{B}}}}, \quad \theta = \theta_0 - \int_x^\infty \left(\frac{B}{k(y)} - \frac{k(y)}{4}\right) dy,$$

for some constants $\theta_0, x_0 \in \mathbb{R}$, $B>0$.

To study the Cauchy problem of (1.36), we use the idea in Hayashi-Ozawa [58, 59, 100]. Set

$$v = u_x + \frac{i}{2}|u|^2u. \tag{1.37}$$

If u solves (1.36) then (u,v) solves the following system

$$\begin{cases} Lu &= -iu^2\overline{v} + \frac{1}{2}|u|^4u, \\ Lv &= i\overline{u}v^2 + \frac{3}{2}|u|^4v + u^2|u|^2\overline{v}, \end{cases} \tag{1.38}$$

where $L = i\partial_t + \partial_{xx}$ is the Schrödinger operator. We establish the local well posedness of solutions of (1.38) in spaces $X^4(\mathbb{R})$, $\phi+H^2(\mathbb{R})$ and $\phi+H^1(\mathbb{R})$ with restrictions on ϕ. Moreover, we prove that the relation (1.37) is conserved. The existence of solutions of (1.36) in $X^4(\mathbb{R})$ is obtained by the following argument. Let $u_0 \in X^4$. Set $v_0 = u_{0x} + \frac{i}{2}|u_0|^2u_0$. Let (u,v) be the corresponding solution of (1.38) with the initial data (u_0,v_0). We may prove that $v = u_x + \frac{i}{2}|u|^2u$ in the time interval of existence of solutions. Thus, u solves

$$Lu = -iu^2\left(\overline{u}_x - \frac{i}{2}|u|^2\overline{u}\right) + \frac{1}{2}|u|^4u = -iu^2\overline{u}_x.$$

This implies the existence in $X^4(\mathbb{R})$ of solutions of (1.36). The uniqueness and other properties of $X^4(\mathbb{R})$ solutions of (1.36) follow from the corresponding properties of the associated solutions of (1.38). The proof of local well posedness of (1.36) is completed. Similarly, we have the local well posedness of (1.36) in the cases $\phi + H^2(\mathbb{R})$ and $\phi + H^1(\mathbb{R})$.

To prove the uniqueness of stationary solution of (1.36) in $X^1(\mathbb{R})$, we use the suitable changes of variables as in [94]. More precisely, let ϕ be a nonvanishing stationary solution of (1.36) in $X^1(\mathbb{R})$. Then we may write ϕ as

$$\phi(x) = \sqrt{k(x)}e^{i\theta(x)},$$

where $k, \theta \in C^2(\mathbb{R})$ and $k > 0$. We prove that θ and k satisfy the following

$$\theta_x = \frac{B}{k} - \frac{k}{4}, \tag{1.39}$$

$$0 = \frac{k_{xx}}{2} - \frac{5}{12}k^3 + 3Bk - 2a, \tag{1.40}$$

for some $B > 0$, $a \in \mathbb{R}$. Since the relation (1.39), we may obtain the formulation of θ by the formulation of R. Moreover, k satisfies $k - 2\sqrt{B} \in H^3(\mathbb{R})$. Combining to (1.40), we have $a = \frac{4B\sqrt{B}}{3}$. Setting $h = k - 2\sqrt{B}$, we have $h \in H^3(\mathbb{R})$ and

$$0 = h_{xx} - \frac{5}{6}h^3 - 5\sqrt{B}h^2 - 4Bh.$$

By a classical argument, we may obtain the explicit formulation of h and thus the explicit formulation of k. This completes the proof of Theorem 1.32. For detail discussion, we refer reader to Chap 2.

1.4.2 Stability theory

In this section, we consider (1.17) on the half line with Robin boundary condition at 0:

$$\begin{cases} iv_t + v_{xx} = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2\overline{v_x} - \frac{3}{16}|v|^4 v, & \forall x \in \mathbb{R}^+, \\ v(0,x) = v_0(x), \\ v_x(t,0) = \alpha v(t,0), & \forall t \in \mathbb{R}, \end{cases} \tag{1.41}$$

where $\alpha \in \mathbb{R}$ is a given constant.

The equation (1.41) has a standing wave of the form $e^{i\omega t}\varphi_\omega(x)$, where $\omega > \alpha^2$ and

$$\varphi_\omega(x) = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right). \tag{1.42}$$

The linear part of (1.41) can be written as follows

$$iv_t + \tilde{H}_\alpha v = 0, \quad v(0) = v_0,$$

where $\tilde{H}_\alpha$ is the self adjoint operator which is defined by

$$\tilde{H}_\alpha : D(\tilde{H}_\alpha) \subset L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+),$$
$$\tilde{H}_\alpha v = v_{xx}, \quad D(\tilde{H}_\alpha) = \left\{v \in H^2(\mathbb{R}^+) : v_x(0) = \alpha v(0)\right\}.$$

The equation (1.41) in Duhamel form is the following

$$v(t)=e^{i\tilde{H}_\alpha t}v_0-i\int_0^t e^{i\tilde{H}_\alpha(t-s)}g(v(s))\,ds, \tag{1.43}$$

where

$$g(v)=\frac{i}{2}|v|^2v_x-\frac{i}{2}v^2\overline{v}_x-\frac{3}{16}|v|^4v.$$

It turns out that the self adjoint operator $\tilde{H}_\alpha$ has a relation with the following delta potential Schrödinger operator on the whole line

$$\begin{aligned}
&H_\gamma: D(H_\gamma)\subset L^2(\mathbb{R})\to L^2(\mathbb{R}),\\
&H_\gamma u=u_{xx},\quad D(H_\gamma)=\left\{u\in H^2(\mathbb{R}\setminus 0)\cap H^1(\mathbb{R}), u_x(0^+)-u_x(0^-)=\gamma u(0)\right\}.
\end{aligned}$$

More precisely, the operator $\tilde{H}_\alpha$ can be seen as the restriction of the operator $H_{2\alpha}$ on even functions and we have

$$e^{i\tilde{H}_\alpha t}\tilde{\varphi}=\left(e^{iH_{2\alpha}t}\varphi\right)|_{\mathbb{R}^+}, \tag{1.44}$$

where $\tilde{\varphi}\in D(\tilde{H}_\alpha)$ and φ is the even function on $\mathbb{R}$ whose restriction on $\mathbb{R}^+$ is $\tilde{\varphi}$. It is well known that the operator $e^{iH_{2\alpha}t}$ is bounded on $H^1(\mathbb{R})$ (see e.g [61]). It implies that the operator $e^{i\tilde{H}_\alpha t}$ is bounded on $H^1(\mathbb{R}^+)$. We assume that (1.41) is locally well posed on $H^1(\mathbb{R}^+)$. By formal calculation, we show that (1.41) has two conservation laws: conservation of the mass and the energy. In this thesis, we use these tools to study the dynamics of (1.41). Our main results are the following.

Theorem 1.33. *Let $\alpha>0$ and $v_0\in\Sigma=\left\{v\in D(\tilde{H}_\alpha), xv\in L^2(\mathbb{R}^+)\right\}$. If the energy of v_0 be negative, then the associated $H^1(\mathbb{R}^+)$ solution of (1.36) blows up in finite time.*
Let $\omega>\alpha^2$ and $e^{i\omega t}\varphi_\omega$ be the standing wave of (1.36). If $\alpha<0$ then the standing wave is orbitally stable. If $\alpha\geqslant 0$ then the standing wave is unstable by blow up.

To prove the existence of blow up solutions, we use a similar arguments as in Glassey [45]. Define

$$\begin{aligned}
u(t,x)&=v(t,x)\exp\left(-\frac{i}{4}\int_x^\infty |v(t,y)|^2\right),\\
I(t)&=\int_{\mathbb{R}^+}x^2|v(t)|^2\,dx=\int_{\mathbb{R}^+}x^2|u(t)|^2\,dx,\\
J(t)&=\mathcal{Im}\int_{\mathbb{R}^+}xu_x\overline{u}\,dx.
\end{aligned}$$

By a direct calculation, we have

$$\begin{aligned}
\partial_t I(t)&=4J(t)-\int_{\mathbb{R}^+}x|u(t)|^4\,dx\leqslant 4J(t),\\
\partial_t J(t)&=2\int_{\mathbb{R}^+}|u_x|^2-\mathcal{Im}\int_{\mathbb{R}^+}|u|^2u_x\overline{u}\,dx+\alpha|u(t,0)|^2=4E(v)-\alpha|v(t,0)|^2\\
&\leqslant 4E(v)=4E(v_0).
\end{aligned}$$

Here, we use the condition $\alpha \geqslant 0$. Thus, it is easy to show that

$$I(t) \leqslant I(0) + 4J(0)t + 8E(v_0)t^2.$$

This implies that the time of existence must be finite.

Let $e^{i\omega t}\varphi_\omega$ be the standing wave of (1.41) defined in (1.42). To prove the stability of standing waves in the case $\alpha < 0$, we use variational techniques. Set

$$S_\omega(v) = \frac{1}{2}\left(\|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2\right) - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R}^+)}, \tag{1.45}$$

$$K_\omega(v) = \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2 - \frac{3}{16}\|v\|^6_{L^6(\mathbb{R}^+)}, \tag{1.46}$$

$$N(v) = 3S_\omega(v) - \frac{3}{2}K_\omega(v), \tag{1.47}$$

$$d(\omega) = \inf\{S_\omega(v) | v \in H^1(\mathbb{R}^+) \setminus 0, K_\omega(v) = 0\}. \tag{1.48}$$

First, we prove the following compactness result: If $(v_n) \subset H^1(\mathbb{R}^+)$ satisfy

$$S_\omega(v_n) \to d(\omega),$$
$$K_\omega(v_n) \to 0,$$

then there exists a constant $\theta_0 \in \mathbb{R}$ such that $v_n \to e^{i\theta_0}\varphi_\omega$, where φ_ω is the standing wave profile. Next, we prove that under the assumption $\alpha < 0$, if $(v_n) \subset H^1(\mathbb{R}^+)$ satisfies

$$\|v_n - \varphi_\omega\|_{H^1(\mathbb{R}^+)} \to 0 \text{ as } n \to \infty,$$

then the corresponding solution $v(t)$ of (1.41) satisfies $N(v_n(t)) \to 3d(\omega)$ as $n \to \infty$. Combining the above tools, we conclude the stability of standing waves in the case $\alpha < 0$ by a contradiction argument.

To prove the instability by blow up of the standing waves in the case $\alpha \geqslant 0$, we may use a similar argument as in [71]. The case $\alpha = 0$ is not difficult. We consider the case $\alpha > 0$. Define

$$\mathcal{V} = \{v \in H^1(\mathbb{R}^+) : K_\omega(v) < 0, S_\omega(v) < d(\omega), P(v) < 0\},$$

where $P(v) = \frac{\partial}{\partial \lambda}S(v_\lambda)|_{\lambda=1} = \|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{16}\|v\|^6_{L^6(\mathbb{R}^+)} + \frac{\alpha}{2}|v(0)|^2$. We prove that $\mathcal{V}$ is invariant under flow of (1.41). Moreover, we prove that if $\varphi \in \mathcal{V}$ is such that $|\cdot|\varphi(\cdot) \in L^2(\mathbb{R}^+)$ then the corresponding solution v of (1.41) blows up in finite time on $H^1(\mathbb{R}^+)$. Thus, to conclude the instability by blow up of standing waves, we only need to construct a sequence $\varphi_n \to \varphi$ in $H^1(\mathbb{R}^+)$ such that $\varphi_n \in \mathcal{V}$ and $|\cdot|\varphi_n(\cdot) \in L^2(\mathbb{R}^+)$ for each n. This sequence can be obtained from a scaling of φ_ω. Thus, we complete the proof of Theorem 1.33. For more details, we refer the reader to Chapter 3.

1.4.3 Multi solitons theory

In [72, 73], Le Coz-Li-Tsai proved the existence and uniqueness of multi solitons for classical nonlinear Schrödinger equations by using fixed point arguments around the desired profile. We cannot directly apply this argument to obtain similar results for derivative monlinear Schrödinger equations, because of the presence of derivatives in the nonlinearities. In this thesis, we improve the method of Le Coz-Li-Tsai [72, 73] to obtain the existence of multi- solitons solutions of derivative nonlinear Schrödinger equations under an implicit condition on the parameters. Our first main result is the following.

Theorem 1.34. *Considering* (1.17), *we assume that* $\lambda = 1$, $\mu = 0$. *Let* $(\theta_j, c_j, \omega_j)$ *be sequence of parameters such that* $-2\sqrt{\omega_j} < c_j < 2\sqrt{\omega_j}$ *if* $\gamma > 0$ *and* $-2\sqrt{\omega_j} < c_j < -2s_*\sqrt{\omega_j}$ *if* $\gamma \leqslant 0$, *where* $\gamma = 1 + \frac{16}{3}b$ *and* $s_* = \sqrt{\frac{-\gamma}{1-\gamma}}$. *Let* R *be the multi-soliton profile defined in* (1.30). *Then there exists a certain positive constant* C_* *such that if the parameters* (ω_j, c_j) *satisfy*

$$C_* \left((1 + \|R_x\|_{L_t^\infty L_x^\infty})(1 + \|R\|_{L_t^\infty L_x^\infty}) + \|R\|^4_{L_t^\infty L_x^\infty} \right) \leqslant v_* := \inf_{j \neq k} h_j |c_j - c_k|, \quad (1.49)$$

where $h_j = \sqrt{4\omega_j - c_j^2}$, *then there exist* $T_0 > 0$ *depending on* $\omega_1, ..., \omega_K, c_1, ..., c_K$ *and a solution* u *of* (1.17) *on* $[T_0, \infty)$ *such that*

$$\|u - R\|_{H^1} \leqslant Ce^{-\frac{v_*}{16}t}, \quad \forall t \geqslant T_0,$$

where $\lambda = \frac{v_*}{16}$ *and* C *is a positive constant depending on the parameters* $\omega_1, ..., \omega_K, c_1, ..., c_K$.

We note that the condition (1.49) ensures that $c_i \neq c_j$ for $i \neq j$. Thus, the solitons are separated at large time.

Let us sketch the proof of the above theorem. Define

$$\varphi(t, x) = \exp\left(\frac{i}{2} \int_{-\infty} |u(t, y)|^2 \, dy \right) u(t, x),$$
$$\psi = \varphi_x - \frac{i}{2} |\varphi|^2 \varphi.$$

We observe that if u solves (1.17) then (φ, ψ) solves a system of the form

$$\begin{cases} L\varphi = P(\varphi, \psi), \\ L\psi = Q(\varphi, \psi), \\ \psi = \partial_x \varphi - \frac{i}{2}(|\varphi|^2 \varphi), \end{cases} \quad (1.50)$$

where $L = i\partial_t + \partial_{xx}$ and P, Q are polynomials of variables φ, ψ and their conjugates. Let R be the multi-soliton profile and $q = u - R$. Then R solves

$$LR + i|R|^2 R_x + b|R|^4 R = e^{-\lambda t} v(t, x),$$

where $\lambda = \frac{v_*}{16}$ and $\|v\|_{L_t^\infty H_x^2}$ is bounded. Define

$$h(t, x) = \exp\left(\frac{i}{2} \int_{-\infty}^{x} |R(t, y)|^2 \, dy \right) R(t, x),$$
$$k = h_x - \frac{i}{2} |h|^2 h.$$

We prove that (h, k) solves

$$Lh = P(h, k) + e^{-\lambda t} m(t, x),$$
$$Lk = Q(h, k) + e^{-\lambda t} n(t, x),$$

where m, n satisfy $\|m\|_{L_t^\infty H_x^1} + \|n\|_{L_t^\infty H_x^1}$ bounded. Let $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. Then

$$\tilde{\psi} = \tilde{\varphi}_x - \frac{i}{2}(|\tilde{\varphi} + h|^2(\tilde{\varphi} + h) - |h|^2 h), \tag{1.51}$$

and $(\tilde{\varphi}, \tilde{\psi})$ solves

$$\begin{cases} L\tilde{\varphi} = P(\tilde{\varphi}, \tilde{\psi}) - P(h, k) - e^{-\lambda t} m(t, x), \\ L\tilde{\psi} = Q(\tilde{\varphi}, \tilde{\psi}) - Q(h, k) - e^{-\lambda t} n(t, x). \end{cases} \tag{1.52}$$

We construct the solution of (1.52) by similar arguments as in [73, Proposition 3.1]. The relation (1.51) is proved by using the exponential decay in time of solutions of (1.52) and the assumption (1.49). This implies that the profile (φ, ψ) solves (1.50). Then, by setting

$$u(t, x) = \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |\varphi(t, y)|^2 \, dy\right) \varphi(t, x).$$

we obtain a solution u of (1.17) which satisfies the desired property.

Next, consider the equation (1.31). In [72, 73], Le Coz-Li-Tsai have successfully proved the existence of multi-kink-soliton solutions of classical nonlinear Schrödinger equations. In this thesis, we use a similar method as in the proof of Theorem 1.34 to prove the existence of multi-kink-soliton solutions for the equation (1.31). Our result is the following.

Theorem 1.35. *Consider* (1.31). *We assume that* $b < \frac{5}{16}$. *Let* $(\theta_j, \omega_j, c_j)_{j=0,\dots,K}$ *be parameters such that* $2\sqrt{\gamma} < c_0 < 2\sqrt{\omega_0}$, $2\sqrt{\omega_j} < c_j > 2s_*\sqrt{\omega_j}$, *where* $\gamma = \frac{5}{3} - \frac{16}{3}b$ *and* $s_* = \sqrt{\frac{\gamma}{1+\gamma}}$. *Let* V *be given as in* (1.34). *There exists a certain positive constant* C_* *such that if the parameters* (ω_j, c_j) *satisfy*

$$C_*\left(\left(1 + \|V_x\|_{L_t^\infty L_x^\infty}\right)\left(1 + \|V\|_{L_t^\infty L_x^\infty}\right) + \|V\|_{L_t^\infty L_x^\infty}^4\right) \leqslant v_* := \min\left(\inf_{j\neq k} h_j|c_j - c_k|, \inf_{j\neq 0}|c_j - c_0|\right),$$

where $h_j = \sqrt{4\omega_j - c_j^2}$, *then there exist a solution* u *to* (1.31) *such that*

$$\|u - V\|_{H^1} \leqslant Ce^{-\lambda t}. \quad \forall t \geqslant T_0,$$

where $\lambda = \frac{v_*}{16}$ *and* C, T_0 *are positive constants depending on the parameters* $\omega_0, \dots, \omega_K, c_0, \dots, c_K$.

For details, we refer to Chapter 4.

Consider the equation (1.35). The stability of multi solitons of (1.35) has been studied in [74] for $\sigma = 1$ and in [110] for $\sigma \in (1, 2)$. In this thesis, we give the proof of existence of multi solitons in the cases $\sigma = 1$ or $\sigma = 2$ or $\sigma \geqslant \frac{5}{2}$. We have the following result.

Theorem 1.36. *Let $\sigma \geqslant \frac{5}{2}$ or $\sigma = 1$ or $\sigma = 2$. Let $(\theta_j, \omega_j, c_j)$ be parameters such that $\theta_j \in \mathbb{R}$ and $\omega_j > \frac{c_j^2}{4}$ and R be the multi-soliton profile of (1.35) defined similarly to the one of the equation (1.17). There exists a certain positive constant C_* such that if the parameters (ω_j, c_j) satisfy*

$$C_* \left((1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|R_x\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \right) \leqslant v_* := \inf_{j \neq k} h_j |c_j - c_k|,$$

where $h_j = \sqrt{4\omega_j - c_j^2}$, then there exist a solution u to (1.31) such that

$$\|u - R\|_{H^1} \leqslant Ce^{-\lambda t}. \quad \forall t \geqslant T_0,$$

where $\lambda = \frac{v_*}{16}$ and C, T_0 are positive constants depending on the parameters $\omega_0, ..., \omega_K, c_0, ..., c_K$.

Our method is similar to the one in the case of equation (1.17). In the proof of Theorem 1.36, we use the following inequality

$$(a+b)^{2(\sigma-2)} - a^{2(\sigma-2)} \lesssim b^{2(\sigma-2)} + ba^{2(\sigma-2)-1}, \quad \forall a, b \geqslant 0.$$

The condition $\sigma \geqslant \frac{5}{2}$ ensures that the order of b on the right hand sight of the above inequality is larger than 1. This is an important point to close fixed point argument. For details, we refer to Chapter 5.

1.4.4 Instability of algebraic standing waves

This section is to present our work on the triple power nonlinear Schrödinger equation:

$$iu_t + \Delta u + a_1|u|u + a_2|u|^2u + a_3|u|^3u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \tag{1.53}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $n \in \{1, 2, 3\}$.

In [79], Liu-Tsai-Zwiers studied a 1D triple power nonlinear Schrödinger equation. In particular, the authors presented a picture which shows regions of existence, stability and instability of standing waves with positive frequency. In [36], the authors proved the instability of standing waves with zero frequency i.e algebraic standing waves. Our goal is to study the existence and stability of algebraic standing waves of triple power nonlinear Schrödinger equations. Before stating our main result, we recall the definition of algebraic standing waves for (1.53).

A standing wave of (1.53) is a solution of form $e^{i\omega t}\phi_\omega(x)$. In this thesis, we are interested in the case of frequency equals to zero i.e $\omega = 0$. The profile ϕ_0, which we prefer to denote by ϕ, solves the following equation:

$$\Delta\phi + a_1|\phi|\phi + a_2|\phi|^2\phi + a_3|\phi|^3\phi = 0. \tag{1.54}$$

The equation (1.54) admits a unique radial positive solution. This solution is algebraically decaying in space. Moreover, it is a minimizer of a variational problem. Our main goal is to prove orbital instability of this solution. We have the following result.

Theorem 1.37. *Let ϕ be the radial positive solution of* (1.54) *and $a_1 = -1$, $a_3 = 1$ and a_2 small when $a_2 > 0$. The algebraic standing wave ϕ of* (1.53) *is orbitally unstable in $H^1(\mathbb{R})$.*

The above theorem is a direct consequence of the following result.

Proposition 1.38. *Assume the assumptions of Theorem 1.37 and*

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0, \ \textit{where } v^\lambda(x) = \lambda^{\frac{N}{2}} v(\lambda x).$$

Then the algebraic standing wave ϕ is unstable.

Define

$$\mathcal{N}_\varepsilon = \left\{ v \in H^1(\mathbb{R}^N) : \inf_{(\theta,y)\in\mathbb{R}\times\mathbb{R}^N} \|v - e^{i\theta}\phi(\cdot - y)\|_{H^1} < \varepsilon \right\}.$$

Let $u_0 \in \mathcal{N}_\varepsilon$ and $u(t)$ be the corresponding solution of (1.53). We define the exit time from the tube $\mathcal{N}_\varepsilon$ by

$$T_\varepsilon^\pm(u_0) := \inf\{t > 0 : u(\pm t) \notin \mathcal{N}_\varepsilon\}.$$

Set $I_\varepsilon := (-T_\varepsilon^-(u_0), T_\varepsilon^+(u_0))$. Then, I_ε is the maximal interval for which the solution stays in $\mathcal{N}_\varepsilon$. Thus, to prove the instability of ϕ, we show that there exists $\varepsilon > 0$ such that there exists a sequence (ϕ_n) satisfying $\|\phi_n - \phi\|_{H^1} \to 0$ as $n \to \infty$ and $|I_\varepsilon(\phi_n)| < \infty$ for all n. The conclusion of Theorem 1.37 is proved by the following result.

Proposition 1.39. *There exists $\varepsilon > 0$ such that for all $u_0 \in \mathcal{N}_\varepsilon$ such that $P(u_0) < 0$, $S(u_0) < \mu$ and $|\cdot|u_0(\cdot) \in L^2(\mathbb{R})$, we have $|I_\varepsilon(u_0)| < \infty$, where*

$$\begin{aligned}
S(v) &= \frac{1}{2}\|\nabla v\|_{L^2}^2 + \frac{1}{3}\|v^3\|_{L^3} - \frac{a_2}{4}\|v\|_{L^4}^4 - \frac{1}{5}\|v\|_{L^5}^5, \\
P(v) &= \partial_\lambda S(v^\lambda)|_{\lambda=1} = \|\nabla v\|_{L^2}^2 + \frac{N}{6}\|v\|_{L^3}^3 - \frac{Na_2}{4}\|v\|_{L^4}^4 - \frac{3N}{10}\|v\|_{L^5}^5 \\
K(v) &= \|\nabla v\|_{L^2}^2 + \|v\|_{L^3}^3 - a_2\|v\|_{L^4}^4 - \|v\|_{L^5}^5 \\
\mu &= \inf\left\{ S(v) : v \in \dot{H}^3(\mathbb{R}^N) \cap L^3(\mathbb{R}^N) \setminus \{0\}, K(v) = 0 \right\}.
\end{aligned}$$

The existence of the desired sequence follows by using a suitable scaling of ϕ. For details, we refer to Chapter 6.

Chapter 2

On the Cauchy problem for a derivative nonlinear Schrödinger equation with nonvanishing boundary conditions

2.1 Introduction

In this chapter, we are interested in the Cauchy problem for the following derivative nonlinear Schödinger equation with nonvanishing boundary conditions:

$$\begin{cases} i\partial_t u + \partial^2 u = -iu^2 \partial \overline{u}, \\ u(0) = u_0, \end{cases} \tag{2.1}$$

where $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$, $\partial = \partial_x$ denotes derivative in space and ∂_t denotes derivative in time.

Our attention was drawn to this equation by the work of Hayashi and Ozawa [58] concerning the more general nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial^2 u = i\lambda |u|^2 \partial u + i\mu u^2 \partial \overline{u} + f(u), \\ u(0) = u_0. \end{cases} \tag{2.2}$$

When $\lambda = 0$, $\mu = -1$, $f \equiv 0$, then (2.2) reduces to (2.1). This type of equation is usually refered to as *derivative nonlinear Schrödinger equations*. It may appear in various areas of physics, e.g. in Plasma Physics for the propagation of Alfvén waves [93, 106].

Under Dirichlet boundary conditions in space, the Cauchy problem for (2.1) has been solved in [58]: local well-posedness holds in $H^1(\mathbb{R})$, i.e. for any $u_0 \in H^1(\mathbb{R})$ there exists a unique solution $u \in C(I, H^1(\mathbb{R}))$ of (2.1) on a maximal interval of time I. Moreover, we have continuous dependence with respect to the initial data, blow-up at the ends of the time interval of existence I if I is bounded and conservation of energy, mass and momentum.

The main difficulty is the appearance of the derivative term $-iu^2\overline{u_x}$. We cannot use the classical contraction method for this type of nonlinear Schrödinger equations.

In [58] Hayashi and Ozawa use the Gauge transform to establish the equivalence of the local well-posedness between the equation (2.2) and a system of equations without derivative terms. By studying the Cauchy problem for this system, they obtain the associated results for (2.2). In [56], Hayashi and Ozawa construct a sequence of solutions of approximated equations and prove that this sequence is converging to a solution of (2.2), obtaining this way the local well-posedness of (2.2). The approximation method has also been used by Tsutsumi and Fukuda in [111, 112]. The difference between [56] and [111, 112] lies in the way of constructing the approximate equation. In [56], the authors use approximation on the non-linear term, whereas in [111, 112] the authors use approximation on the linear operator.

To our knowledge, the Cauchy problem for (2.1) has not been studied under non-zero boundary conditions, and our goal in this paper is to initiate this study. Note that non-zero boundary conditions on the whole space are much rarely considered in the literature around nonlinear dispersive equations than Dirichlet boundary conditions. In the case of the nonlinear Schrödinger equation with power-type nonlinearity, we refer to the works of Gérard [42, 43] for local well-posedness in the energy space and to the works of Gallo [40] and Zhidkov [120] for local well-posedness in Zhidkov spaces (see Section 2.2.1 for the definition of Zhidkov spaces) and Gallo [41] for local well-posedness in $u_0 + H^1(\mathbb{R})$. In this paper, using the method of Hayashi and Ozawa as in [58] on the Zhidkov-space $X^k(\mathbb{R})$, $(k \geqslant 4)$ and in the space $\phi + H^k(\mathbb{R})$ $(k = 1, 2)$ for ϕ in a Zhidkov space, we obtain the existence, uniqueness and continuous dependence on the initial data of solutions of (2.1) in these spaces. Using the transform

$$v = \partial u + \frac{i}{2}|u|^2 u, \tag{2.3}$$

we see that if u is a solution of (2.1) then (u, v) is a solution of a system of two equations without derivative terms. It is easy to obtain the local wellposedness of this system on Zhidkov spaces. The main difficulty is how to obtain a solution of (2.1) from a solution of the system. Actually, we must prove that the relation (2.3) is conserved in time. The main difference in our setting with the setting in [59] is that we work on Zhidkov spaces instead of the space of localized functions $H^1(\mathbb{R})$. Our first main result is the following.

Theorem 2.1. *Let $u_0 \in X^4(\mathbb{R})$. Then there exists a unique maximal solution of (2.1) $u \in C((T_{min}, T_{max}), X^4(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^2(\mathbb{R}))$. Moreover, u satisfies the two following properties.*

- Blow-up alternative. *If $T_{max} < \infty$ (resp. $T_{min} > -\infty$) then*

$$\lim_{t \to T_{max}\,(resp.\ T_{min})} \|u(t)\|_{X^2} = \infty.$$

- Continuity with respect to the initial data. *If $u_0^n \in X^4(\mathbb{R})$ is such that $u_0^n \to u_0$ in $X^4(\mathbb{R})$ then for any subinterval $[T_1, T_2] \subset (T_{min}, T_{max})$ the associated solutions of equation* (2.1) (u^n) *satisfy*

$$\lim_{n \to \infty} \|u^n - u\|_{L^\infty([T_1,T_2],X^4)} = 0.$$

To obtain the local wellposedness on $\phi + H^k(\mathbb{R})$ for ϕ in Zhidkov spaces $X^l(\mathbb{R})$. First, we use the transform $v = \partial u + \frac{i}{2}|u|^2 u$. We see that if $u \in \phi + H^k(\mathbb{R})$ then $v \in \frac{i}{2}|\phi|^2\phi + H^{k-1}(\mathbb{R})$. This motivates us to define $\tilde{u} = u - \phi$ and $\tilde{v} = v - \frac{i}{2}|\phi|^2\phi$. We have

$$\tilde{v} = \partial\tilde{u} + \frac{i}{2}(|\tilde{u} + \phi|^2(\tilde{u} + \phi) - |\phi|^2\phi) + \partial\phi. \tag{2.4}$$

We see that if u is a solution of (2.1) then $(\tilde{u}, \tilde{v})$ is a solution of a system of two equations without the derivative terms. For technical reasons, we will need some regularity on ϕ. With a solution of the system in hand, we want to obtain a solution of (2.1). In practice, we need to prove that the relation (2.4) is conserved in time. Our main second result is the following.

Theorem 2.2. *Let $\phi \in X^4(\mathbb{R})$ and $u_0 \in \phi + H^2(\mathbb{R})$. Then the problem (2.1) has a unique maximal solution $u \in C((T_{min}, T_{max}), \phi + H^2(\mathbb{R}))$ which is differentiable as a function of $C((T_{min}, T_{max}), \phi + L^2(\mathbb{R}))$ and such that $u_t \in C((T_{min}, T_{max}), L^2(\mathbb{R}))$. Moreover u satisfies the following properties.*
(1) *Blow-up alternative: If $T_{max} < \infty$ (resp. $T_{min} > -\infty$ then*

$$\lim_{t \to T_{max} (resp.\ T_{min})} (\|u(t) - \phi\|_{H^2(\mathbb{R})}) = \infty.$$

(2) *Continuous dependence on initial data: If $(u_0^n) \subset \phi + H^2(\mathbb{R})$ is such that $\|u_0^n - u_0\|_{H^2} \to 0$ as $n \to \infty$ then for all $[T_1, T_2] \subset (T_{min}, T_{max})$ the associated solutions (u^n) of (2.1) satisfy*

$$\lim_{n\to\infty} \|u^n - u\|_{L^\infty([T_1,T_2],H^2)} = 0.$$

In the less regular space $\phi + H^1(\mathbb{R})$, we obtain the local well posedness under a smallness condition on the initial data. Our third main result is the following.

Theorem 2.3. *Let $\phi \in X^4(\mathbb{R})$ such that $\|\partial\phi\|_{L^2}$ is small enough, $u_0 \in \phi + H^1(\mathbb{R})$ such that $\|u_0 - \phi\|_{H^1(\mathbb{R})}$ is small enough. There exist $T > 0$ and a unique solution u of (2.1) such that*

$$u - \phi \in C([-T, T], H^1(\mathbb{R})) \cap L^4([-T, T], W^{1,\infty}(\mathbb{R})).$$

In the proof of Theorem 2.3, the main difference with the case $\phi + H^2(\mathbb{R})$ is that we use Strichartz estimates to prove the contractivity of a map on $L^\infty([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$. In the case of a general nonlinear term (as in (2.2)), our method is not working. The main reason is that we do not have a proper transform to give a system without derivative terms. Moreover, our method is not working if the initial data lies on $X^1(\mathbb{R})$. It is because when we study the system of equations, we would have to study it on $L^\infty(\mathbb{R})$, but we know that the Schrödinger group is not bounded from $L^\infty(\mathbb{R})$ to $L^\infty(\mathbb{R})$. Thus, the local wellposedness on less regular spaces is a difficult problem for nonlinear derivative Schrödinger equations.

To prove the conservation laws of (2.1), we need to use a localizing function, which is necessary for integrals to be well defined. Indeed, to obtain the conservation of the energy, using (2.1), at least formally, we have

$$\partial_t(|\partial u|^2) = \partial_x(F(u)) + \partial_t(G(u)),$$

for functions F and G which will be defined later. The important thing is that when u is not in $H^1(\mathbb{R})$, there are some terms in $G(u)$ which do not belong to $L^1(\mathbb{R})$, hence, it is impossible to integrate the two sides as in the usual case. However, we can use a localizing function to deal with this problem. Similarly, we use the localizing function to prove the conservation of the mass and the momentum. The localizing function χ is defined as follows

$$\chi \in C^1(\mathbb{R}) \text{ and even }, \quad \operatorname{supp}\chi \subset [-2,2], \quad \text{ and } \chi = 1 \text{ on } [-1,1]. \tag{2.5}$$

For all $a \in \mathbb{R}$ and $R > 0$, we define

$$\chi_{a,R}(x) = \chi\left(\frac{x-a}{R}\right) = \chi\left(\frac{|x-a|}{R}\right). \tag{2.6}$$

To prove the conservation of mass, we use the similar notations as in [34, section 7]

$$m^+(u) = \inf_{a\in\mathbb{R}} \limsup_{R\to\infty} \int_{\mathbb{R}} (|u|^2 - q_0^2)\chi_{a,R}\, dx, \quad m^-(u) = \sup_{a\in\mathbb{R}} \liminf_{R\to\infty} \int_{\mathbb{R}} (|u|^2 - q_0^2)\chi_{a,R}\, dx.$$

If u is such that $m^+(u) = m^-(u)$ we define *generalized mass* as

$$m(u) \equiv m^+(u) = m^-(u).$$

Especially, for $a = 0$ we define

$$\chi_R(x) = \chi\left(\frac{x}{R}\right). \tag{2.7}$$

Our fourth main result is the following.

Theorem 2.4. *Let $q_0 \in \mathbb{R}$ be a constant and $u_0 \in q_0 + H^2(\mathbb{R})$ and $u \in C((T_{min}, T_{max}), q_0 + H^2(\mathbb{R}))$ be the associated solution of* (2.1) *given by Theorem 2.2. Then, we have*

$$\begin{aligned} E(u) &:= \int_{\mathbb{R}} |\partial u|^2\, dx + \frac{1}{2}\mathcal{I}m \int_{\mathbb{R}} (|u|^2\overline{u} - q_0^3)\partial u\, dx \\ &\quad + \frac{1}{6}\int_{\mathbb{R}} (|u|^2 - |q_0|^2)^2(|u|^2 + 2|q_0|^2)\, dx = E(u_0), \end{aligned} \tag{2.8}$$

$$P(u) := \frac{1}{2}\mathcal{I}m \int_{\mathbb{R}} (u - q_0)\partial\overline{u}\, dx - \int_{\mathbb{R}} \frac{1}{4}(|u|^2 - |q_0|^2)^2\, dx = P(u_0), \tag{2.9}$$

for all $t \in (T_{min}, T_{max})$. Moreover, u satisfies $m^+(u(t)) = m^+(u_0)$ (respectively $m^-(u(t)) = m^-(u_0)$). In particular, if u_0 has finite generalized mass then the generalized mass is conserved by the flow, that is $m(u(t)) = m(u_0)$.

Remark 2.5. When $q_0 = 0$, we recover the classical conservation of mass, energy and momentum as usually defined.

In the classical Schrödinger equation, there are special solutions which are called *standing waves*. There are many works on standing waves (see e.g [71], [16] and the references therein). In [120], Zhidkov shows that there are two types of bounded solitary waves possessing limits as $x \to \pm\infty$. These are monotone solutions and solutions which have precisely one extreme point. They are called *kinks* and *soliton-like solutions*, respectively. In [120], Zhidkov studied the stability of kinks of classical Schrödinger equations. In [10], the authors have studied the stability of kinks in the energy space. To our knowledge, all these solitary waves are in Zhidkov spaces i.e the Zhidkov space is largest space we know to find special solutions. We want to investigate stationary solutions of (2.1) in Zhidkov spaces. Before stating the next main result, we need the following definition:

Definition 2.6. The stationary solutions of (2.1) are functions $\phi \in X^2(\mathbb{R})$ satisfying

$$\phi_{xx} + i\phi^2\overline{\phi}_x = 0. \tag{2.10}$$

In [94], the authors proved the existence of periodic traveling waves of a derivative nonlinear Schrödinger equation using a skillful changes of variables. In this paper, we use a similar changes of variables as in [94] to prove the existence and uniqueness of stationary solution of (2.80) on $X^2(\mathbb{R})$. Our fifth main result is the following.

Theorem 2.7. *Let ϕ be a stationary solution of* (2.1) *(see Definition 2.6). The followings is true:*

(1) If ϕ is not a constant function and satisfies

$$\inf_{x\in\mathbb{R}} |\phi(x)| > 0$$

then ϕ is of the form $e^{i\theta}\sqrt{k}$ where

$$k(x) = 2\sqrt{B} + \frac{-1}{\sqrt{\frac{5}{72B}}\cosh(2\sqrt{B}(x-x_0)) + \frac{5}{12\sqrt{B}}}, \qquad \theta = \theta_0 - \int_x^\infty \left(\frac{B}{k(y)} - \frac{k(y)}{4}\right) dy,$$

for some constants $\theta_0, x_0 \in \mathbb{R}$, $B > 0$.

(2) If ϕ is a stationary solution of (2.1) *such that $\phi(\infty) = 0$ then $\phi \equiv 0$ on $\mathbb{R}$.*

Remark 2.8. We have classified stationary solutions of (2.1) for the functions which are vanishing at infinity, and for the functions which are not vanishing on $\overline{\mathbb{R}}$. One question still unanswered is the class of stationary solutions of (2.1) vanishing at a point in $\mathbb{R}$.

This paper is organized as follows. In Section 2.2, we give the proof of local well posedness of solution of (2.1) on Zhidkov spaces. In Section 2.3, we prove the local well posedness on $\phi + H^2(\mathbb{R})$ and $\phi + H^1(\mathbb{R})$, for $\phi \in X^4(\mathbb{R})$ a given function. In Section 2.4, we give the proof of conservation laws when the initial data is in $q_0 + H^2(\mathbb{R})$, for a given constant $q_0 \in \mathbb{R}$. Finally, in Section 2.5, we have some results on stationary solutions of (2.1) on Zhidkov spaces.

Notation. *In this paper, we will use in the following notation L for the linear part of the Schrödinger equation, that is*

$$L = i\partial_t + \partial^2.$$

Moreover, C denotes various positive constants and $C(R)$ denotes the constant depending on R.

2.2 Local existence in Zhidkov spaces

In this section, we give the proof of Theorem 2.1.

2.2.1 Preliminaries on Zhidkov spaces

Before presenting our main results, we give some preliminaries. We start by recalling the definition of Zhidkov spaces, which were introduced by Peter Zhidkov in his pioneering works on Schrödinger equations with non-zero boundary conditions (see [120] and the references therein).

Definition 2.9. Let $k \in \mathbb{N}$, $k \geqslant 1$. The *Zhidkov space* $X^k(\mathbb{R})$ is defined by

$$X^k(\mathbb{R}) = \{u \in L^\infty(\mathbb{R}) : \partial u \in H^{k-1}(\mathbb{R})\}.$$

It is a Banach space when endowed with the norm

$$\|\cdot\|_{X^k} = \|\cdot\|_{L^\infty} + \sum_{\alpha=1}^{k} \|\partial^\alpha \cdot\|_{L^2}.$$

It was proved by Gallo [40, Theorem 3.1 and Theorem 3.2] that the Schrödinger operator defines a group on Zhidkov spaces. More precisely, we have the following result.

Proposition 2.10. *Let $k \geqslant 1$ and $u_0 \in X^k(\mathbb{R})$. For $t \in \mathbb{R}$ and $x \in \mathbb{R}$, the quantity*

$$S(t)u_0(x) := \begin{cases} e^{-i\pi/4}\pi^{-1/2} \lim\limits_{\varepsilon\to 0} \displaystyle\int_{\mathbb{R}} e^{(i-\varepsilon)z^2} u_0(x + 2\sqrt{t}z)dz \ \text{if } t \geqslant 0, \\ e^{i\pi/4}\pi^{-1/2} \lim\limits_{\varepsilon\to 0} \displaystyle\int_{\mathbb{R}} e^{(-i-\varepsilon)z^2} u_0(x + 2\sqrt{-t}z)dz \ \text{if } t \leqslant 0. \end{cases} \tag{2.11}$$

is well-defined and S defines a strongly continuous group on $X^k(\mathbb{R})$. For all $u_0 \in X^k(\mathbb{R})$ and $t \in \mathbb{R}$ we have

$$\|S(t)u_0\|_{X^k} \leqslant C(k)(1 + |t|^{1/4})\|u_0\|_{X^k}.$$

The generator of the group $(S(t))|_{t\in\mathbb{R}}$ on $X^k(\mathbb{R})$ is $i\partial^2$ and its domain is $X^{k+2}(\mathbb{R})$.

Remark 2.11. Since, for all $\phi \in X^k(\mathbb{R})$, we have $\phi + H^k(\mathbb{R}) \subset X^k(\mathbb{R})$, the uniqueness of solution in $X^k(\mathbb{R})$ implies the uniqueness of solution in $\phi + H^k(\mathbb{R})$, and the existence of solution in $\phi + H^k(\mathbb{R})$ implies the existence of solution in $X^k(\mathbb{R})$.

2.2.2 From the equation to the system

The equation (2.1) contains a spatial derivative of u in the nonlinear part, which makes it difficult to work with. In the following proposition, we indicate how to eliminate the derivative in the nonlinearity by introducing an auxiliary function and converting the equation into a system.

Proposition 2.12. *Let $k \geqslant 2$. Given $u \in X^k(\mathbb{R})$, we define v by*

$$v = \partial u + \frac{i}{2}|u|^2 u. \tag{2.12}$$

Hence, $v \in X^{k-1}(\mathbb{R})$. Furthermore, if u satisfies the equation (2.1), then the couple (u, v) verifies the system

$$\begin{cases} Lu = P_1(u,v), \\ Lv = P_2(u,v), \end{cases} \tag{2.13}$$

where P_1 and P_2 are given by

$$\begin{aligned} P_1(u,v) &= -iu^2\overline{v} + \frac{1}{2}|u|^4 u, \\ P_2(u,v) &= i\overline{u}v^2 + \frac{3}{2}|u|^4 v + u^2|u|^2\overline{v}. \end{aligned} \tag{2.14}$$

Proof. Let u be a solution of (2.1) and v be defined by (2.12). Then we have

$$Lu = -iu^2\partial\overline{u} = -iu^2\left(\overline{v} + \frac{i}{2}(|u|^2\overline{u})\right) = -iu^2\overline{v} + \frac{1}{2}|u|^4 u,$$

which gives us the first equation in (2.13).

On the other hand, since L and ∂ commute and u solves (2.1), we have

$$Lv = \partial(Lu) + \frac{i}{2}L(|u|^2u) = \partial(-iu^2\partial\overline{u}) + \frac{i}{2}L(|u|^2u) = -i(u^2\partial^2\overline{u} + 2u|\partial u|^2) + \frac{i}{2}L(|u|^2u). \tag{2.15}$$

Using

$$L(uv) = L(u)v + uL(v) + 2\partial u\partial v, \quad L(\overline{u}) = -\overline{Lu} + 2\partial^2\overline{u}, \tag{2.16}$$

we have

$$\begin{aligned} L(|u|^2u) = L(u^2\overline{u}) &= L(u^2)\overline{u} + u^2L(\overline{u}) + 2\partial(u^2)\partial\overline{u} \\ &= \big(2L(u)u + 2(\partial u)^2)\big)\,\overline{u} + u^2(-\overline{Lu} + 2\partial^2\overline{u}) + 4u|\partial u|^2 \\ &= 2L(u)|u|^2 + 2\overline{u}(\partial u)^2 + 2u^2\partial^2\overline{u} - u^2\overline{Lu} + 4u|\partial u|^2. \end{aligned} \tag{2.17}$$

We now recall that u verifies (2.1) to obtain

$$\frac{i}{2}L(|u|^2u) = u^2\partial\overline{u}|u|^2 + i\overline{u}(\partial u)^2 + iu^2\partial^2\overline{u} + \frac{1}{2}\partial u|u|^4 + 2iu|\partial u|^2. \tag{2.18}$$

Subsituting in (2.15), we get

$$\begin{aligned} Lv &= -i(u^2\partial^2\overline{u} + 2u|\partial u|^2) + u^2\partial\overline{u}|u|^2 + i\overline{u}(\partial u)^2 + iu^2\partial^2\overline{u} + \frac{1}{2}\partial u|u|^4 + 2iu|\partial u|^2, \\ &= u^2\partial\overline{u}|u|^2 + i\overline{u}(\partial u)^2 + \frac{1}{2}\partial u|u|^4. \end{aligned}$$

Observe here that the second order derivatives of u have vanished and only first order derivatives remain. Therefore, using the expression of v given in (2.12) to subsitute ∂u, we obtain by direct calculations

$$Lv = i\overline{u}v^2 + \frac{3}{2}|u|^4v + u^2|u|^2\overline{v},$$

which gives us the second equation in (2.13). □

2.2.3 Resolution of the system

We now establish the local well-posedness of the system (2.13) in Zhidkov spaces.

Proposition 2.13. *Let $k \geqslant 3$, and $(u_0, v_0) \in X^k(\mathbb{R}) \times X^k(\mathbb{R})$. There exist $T_{min} < 0$, $T_{max} > 0$ and a unique maximal solution (u, v) of system (2.13) such that $(u, v) \in C((T_{min}, T_{max}), X^k(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^{k-2}(\mathbb{R}))$. Furthermore the following properties are satisfied.*

- Blow-up alternative. *If $T_{max} < \infty$ (resp. $T_{min} > -\infty$ then*

$$\lim_{t \to T_{max} (resp.\ T_{min})} (\|u(t)\|_{X^1} + \|v(t)\|_{X^1}) = \infty.$$

- Continuity with respect to the initial data. *If $(u_0^n, v_0^n) \in X^k \times X^k$ is such that*

$$\|u_0^n - u_0\|_{X^k} + \|v_0^n - v_0\|_{X^k} \to 0$$

then for any subinterval $[T_1, T_2] \subset (T_{min}, T_{max})$ the associated solution (u^n, v^n) of (2.13) satisfies

$$\lim_{n \to \infty} \left(\|u^n - u\|_{L^\infty([T_1,T_2],X^k)} + \|v^n - v\|_{L^\infty([T_1,T_2],X^k)}\right) = 0.$$

Proof. Consider the operator $A : D(A) \subset X^{k-2}(\mathbb{R}) \to X^{k-2}(\mathbb{R})$ defined by $A = i\partial^2$ with domain $D(A) = X^k(\mathbb{R})$. From Proposition 2.10 we know that the operator A is the generator of the Schrödinger group $S(t)$ on $X^{k-2}(\mathbb{R})$. From classical arguments (see [17, Lemma 4.1.1 and Corollary 4.1.8]) the couple $(u, v) \in C((T_{min}, T_{max}), X^k(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^{k-2}(\mathbb{R}))$ solves (2.13) if and only if the couple $(u, v) \in C((T_{min}, T_{max}), X^k(\mathbb{R}))$ solves

$$\begin{cases} (u, v) = S(t)(u_0, v_0) - i \int_0^t S(t - s)P(u, v)(s)ds, \\ u(0) = u_0 \in X^k(\mathbb{R}), v(0) = v_0 \in X^k(\mathbb{R}), \end{cases} \tag{2.19}$$

where $S(t)(u, v) := (S(t)u, S(t)v)$, $P(u, v) = (P_1(u, v), P_2(u, v))$ and P_1 and P_2 are defined in (2.14). Consider P as a map from $X^k(\mathbb{R}) \times X^k(\mathbb{R})$ into $X^k(\mathbb{R}) \times X^k(\mathbb{R})$. Since P_1 and P_2 are polynomial in u and v, the map P is Lipschitz continuous on bounded sets of $X^k(\mathbb{R}) \times X^k(\mathbb{R})$. Since (see [17, Theorem 4.3.4 and Theorem 4.3.7]), there exists unique maximal solution $(u, v) \in C((T_{min}, T_{max}), X^k(\mathbb{R}) \times X^k(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^{k-2}(\mathbb{R}) \times X^{k-2}(\mathbb{R}))$ of system (2.13). Moreover, (u, v) satisfy blow-up alternative continuous dependence on initial data in $X^k(\mathbb{R}) \times X^k(\mathbb{R})$. It remains to prove the blow-up alternative in $X^1(\mathbb{R}) \times X^1(\mathbb{R})$. We use the similar arguments as in [120, Proof of Theorem 1.2.4]. For each $1 \leqslant s \leqslant k - 1$, since the map P is Lipschitz continuous on bounded sets of $X^s(\mathbb{R}) \times X^s(\mathbb{R})$, there exists T_{smin} and T_{smax} such that (u, v) is the maximal $X^s(\mathbb{R}) \times X^s(\mathbb{R})$ solution of system (2.19) on (T_{smin}, T_{smax}) and (u, v) satisfy:

$$\lim_{t \to T_{smax} (\text{resp. } T_{smin})} (\|u(t)\|_{X^s} + \|v(t)\|_{X^s}) = \infty.$$

It is sufficient to prove that $T_{1max} = T_{max}$ and $T_{1min} = T_{min}$. We have

$$T_{1max} \geqslant T_{2max} \geqslant .. \geqslant T_{(k-1)max} \geqslant T_{max}.$$

We first prove $T_{1max} = T_{2max}$. Assume $T_{1max} > T_{2max}$. For $t \in [0, T_{2max}]$, since (2.19) we have

$$\|u\|_{X^2} + \|v\|_{X^2} \leqslant \|u_0\|_{X^2} + \|v_0\|_{X^2} + \max_{t\in[0,T_{2max}]} (\|u\|_{X^1} + \|v\|_{X^1} + 1)^4 \int_0^t (\|u(s)\|_{X^2} + \|v(s)\|_{X^2})\, ds.$$

By Gronwall's inequality in integral form we obtain

$$\sup_{t\in[0,T_{2max}]} (\|u\|_{X^2} + \|v\|_{X^2}) < \infty.$$

This contradicts to blow-up alternative of (u, v) in $X^2(\mathbb{R}) \times X^2(\mathbb{R})$. Thus, $T_{1max} = T_{2max}$. By apply many times this arguments we obtain $T_{1max} = T_{max}$ and by similar arguments we have $T_{1min} = T_{min}$. This completes the proof of Proposition 2.13. □

2.2.4 Preservation of the differential identity

The following proposition establishes the link from (2.13) to (2.1) by showing preservation along the time evolution of the differential identity

$$v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0.$$

Proposition 2.14. *Let $u_0, v_0 \in X^3(\mathbb{R})$ be such that*

$$v_0 = \partial u_0 + \frac{i}{2} u_0 |u_0|^2.$$

Then the associated solution $(u, v) \in C((-T_{min}, T^{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R}))$ obtained in Proposition 2.13 satisfies for all $t \in (-T_{min}, T^{max})$ the differential identity

$$v = \partial u + \frac{i}{2}|u|^2 u.$$

Proof. Given $(u, v) \in C((-T_{min}, T^{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R}))$ the solution of (2.13) obtained in Proposition 2.13, we define

$$w = \partial u + \frac{i}{2}|u|^2 u.$$

Our goal will be to show that $w = v$. We first have

$$\begin{aligned} Lu &= -iu^2\overline{v} + \frac{1}{2}|u|^4 u \\ &= -iu^2(\overline{v} - \overline{w}) - iu^2\overline{w} + \frac{1}{2}|u|^4 u \\ &= -iu^2(\overline{v} - \overline{w}) - iu^2\partial\overline{u}. \end{aligned}$$

Applying L to w and using (2.17) and the expression previously obtained for Lu, we get

$$\begin{aligned}
Lw &= \partial(Lu) + \frac{i}{2}L(|u|^2u) \\
&= \partial(Lu) + \frac{i}{2}\left(2Lu|u|^2 + 2\overline{u}(\partial u)^2 + 2u^2\partial^2\overline{u} - u^2\overline{Lu} + 4u|\partial u|^2\right) \\
&= \partial(-iu^2(\overline{v}-\overline{w}) - iu^2\partial\overline{u}) \\
&\quad + \frac{i}{2}\left(2(-iu^2\partial\overline{u})|u|^2 + 2\overline{u}(\partial u)^2 - u^2\overline{(-iu^2\partial\overline{u})} + 2u^2\partial^2\overline{u} + 4u|\partial u|^2\right) \\
&\quad + \frac{i}{2}\left[2(-iu^2(\overline{v}-\overline{w}))|u|^2 - u^2\overline{(-iu^2(\overline{v}-\overline{w}))}\right] \\
&= \left(-i\partial(u^2(\overline{v}-\overline{w})) + u^2|u|^2(\overline{v}-\overline{w}) + \frac{1}{2}|u|^4(v-w)\right) \\
&\quad + \left(-i\partial(u^2\partial\overline{u}) + u^2\partial\overline{u}|u|^2 + i\overline{u}(\partial u)^2 + \frac{1}{2}|u|^4\partial u + iu^2\partial^2\overline{u} + 2iu|\partial u|^2\right) \\
&=: I_1 + I_2.
\end{aligned}$$

As in the proof of Proposition 2.12, we obtain

$$I_2 = i\overline{u}w^2 + \frac{3}{2}|u|^4w + |u|^2u^2\overline{w}.$$

Furthermore

$$\begin{aligned}
I_1 &= \partial(-iu^2(\overline{v}-\overline{w})) + u^2|u|^2(\overline{v}-\overline{w}) + \frac{1}{2}|u|^4(v-w) \\
&= -iu^2\partial(\overline{v}-\overline{w}) - 2iu\partial u(\overline{v}-\overline{w}) + u^2|u|^2(\overline{v}-\overline{w}) + \frac{1}{2}|u|^4(v-w).
\end{aligned}$$

It follows that

$$\begin{aligned}
Lw - Lv &= I_1 + (I_2 - Lv) && (2.20) \\
&= I_1 + i\overline{u}(w-v)(w+v) + \frac{3}{2}|u|^4(w-v) + |u|^2u^2(\overline{w}-\overline{v}) && (2.21) \\
&= (w-v)A_1 + (\overline{w}-\overline{v})A_2 - iu^2\partial(\overline{v}-\overline{w}), && (2.22)
\end{aligned}$$

where A_1 and A_2 are polynomials of degree at most 4 in u, ∂u, v, ∂v and their complex conjugates. Hence,

$$(Lw - Lv)(\overline{w}-\overline{v}) = |w-v|^2A_1 + (\overline{w}-\overline{v})^2A_2 - iu^2\frac{\partial(\overline{v}-\overline{w})^2}{2} := K, \tag{2.23}$$

where K is a polynomial of degree at most 6 in u, v, w, ∂u, ∂v, ∂w and their complex conjugates. Remembering that $L = i\partial_t + \partial^2$, and taking imaginary part in the two sides of (2.23) we obtain

$$\frac{1}{2}\partial_t|w-v|^2 + \mathcal{Im}(\partial\left((\partial w - \partial v)(\overline{w}-\overline{v})\right)) = \mathcal{Im}(K). \tag{2.24}$$

Let $\chi : \mathbb{R} \to \mathbb{R}$ be a cut-off function such that

$\chi \in C^1(\mathbb{R}), \quad \mathrm{supp}(\chi) \subset [-2,2], \quad \chi \equiv 1 \,\mathrm{on}\, (-1,1), \quad 0 \leqslant \chi \leqslant 1, \quad |\chi'(x)|^2 \lesssim \chi(x) \,\mathrm{for\ all}\, x \in \mathbb{R}.$

For each $n \in \mathbb{N}$, define

$$\chi_n(x) = \chi\left(\frac{x}{n}\right).$$

Multiplying both sides of (2.24) by χ_n and integrating in space we obtain

$$\frac{1}{2}\partial_t \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \int_{\mathbb{R}} \mathcal{Im}\left(\partial\left((\partial w - \partial v)(\overline{w}-\overline{v})\right)\right)\chi_n dx = \int_{\mathbb{R}} \mathcal{Im}(K)\chi_n dx. \quad (2.25)$$

For the right hand side, we have

$$\int_{\mathbb{R}} \mathcal{Im}(K)\chi_n dx = \mathcal{Im}\int_{\mathbb{R}} |w-v|^2 A_1\chi_n dx + \mathcal{Im}\int_{\mathbb{R}} (\overline{w}-\overline{v})^2 A_2 \chi_n dx - \mathcal{Im}\int_{\mathbb{R}} iu^2 \frac{\partial((\overline{v}-\overline{w})^2)}{2}\chi_n dx,$$

and therefore

$$\left|\int_{\mathbb{R}} \mathcal{Im}(K)\chi_n dx\right| \leqslant \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 \left(\|A_1\|_{L^\infty} + \|A_2\|_{L^\infty}\right) + \frac{1}{2}\left|\int_{\mathbb{R}} u^2 \partial((\overline{v}-\overline{w})^2)\chi_n dx\right|.$$

We now fix some arbitrary interval $[-T_1, T_2]$ such that $0 \in [-T_1, T_2] \subset (-T_{min}, T^{max})$ in which we will be working from now on, and we set

$$R = \|u\|_{L^\infty([T_1,T_2],X^3)} + \|v\|_{L^\infty([T_1,T_2],X^3)}.$$

From the fact that A_1 and A_2 are polynomials in $u, \partial u$, v, ∂v of degree at most 4, for all $t \in [T_1, T_2]$ we have

$$\|A_1\|_{L^\infty} + \|A_2\|_{L^\infty} \leqslant C(R).$$

It follows that

$$\left|\int_{\mathbb{R}} \mathcal{Im}(K)\chi_n dx\right| \leqslant \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 C(R) + \frac{1}{2}\left|\int_{\mathbb{R}} (\overline{v}-\overline{w})^2 \left(\partial(u^2)\chi_n + u^2\partial\chi_n)dx\right)\right|.$$

By definition of χ we have

$$\begin{aligned} \left|\partial(u^2)\chi_n\right| &\leqslant C(R)\chi_n, \\ \left|u^2\partial\chi_n\right| &\leqslant |u^2|\frac{1}{n}\left|\chi'\left(\frac{\cdot}{n}\right)\right| \leqslant \frac{1}{n}C(R)\sqrt{\chi\left(\frac{\cdot}{n}\right)} \leqslant C(R)\frac{1}{n}\sqrt{\chi_n(.)}. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\int_{\mathbb{R}} \mathcal{Im}(K)\chi_n dx\right| &\leqslant \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 C(R) + \frac{C(R)}{n}\left|\int_{\mathbb{R}} (\overline{v}-\overline{w})^2\sqrt{\chi_n}dx\right| \\ &\leqslant C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)^2}{n}\int_{\mathbb{R}} |v-w|\sqrt{\chi_n}dx \\ &\leqslant C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)^2}{n}\int_{-2n}^{2n} |v-w|\sqrt{\chi_n}dx \\ &\leqslant C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)^2}{n}\left(\int_{-2n}^{2n} (|v-w|\sqrt{\chi_n})^2 dx\right)^{\frac{1}{2}}\left(\int_{-2n}^{2n} dx\right)^{\frac{1}{2}} \\ &\leqslant C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{2C(R)^2}{\sqrt{n}}\|(w-v)\sqrt{\chi_n}\|_{L^2}. \end{aligned} \quad (2.26)$$

In addition, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} \mathcal{I}m(\partial\left((\partial w-\partial v)(\overline{w}-\overline{v})\right)\chi_n)dx\right| &= \left|\int_{\mathbb{R}} \mathcal{I}m(((\partial w-\partial v)(\overline{w}-\overline{v}))\,\chi_n')dx\right| \\ &= \left|\int_{\mathbb{R}} \mathcal{I}m\left((\partial w-\partial v)(\overline{w}-\overline{v})\frac{1}{n}\chi'\left(\frac{x}{n}\right)\right)dx\right| \\ &\leqslant \int_{\mathbb{R}} |\partial w-\partial v||w-v|\frac{1}{n}\sqrt{\chi_n}dx \\ &\leqslant \frac{1}{n}\|\partial w-\partial v\|_{L^2}\|(w-v)\sqrt{\chi_n}\|_{L^2} \\ &\leqslant \frac{C(R)}{n}\|(w-v)\sqrt{\chi_n}\|_{L^2}. \end{aligned} \tag{2.27}$$

From (2.25), (2.26), (2.27) we obtain that

$$\partial_t\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 \leqslant C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)}{\sqrt{n}}\|(w-v)\sqrt{\chi_n}\|_{L^2} \tag{2.28}$$

$$\leqslant C(R)\|(w-v)\sqrt{\chi_n}\|_{L^2}^2 + \frac{C(R)}{\sqrt{n}} \tag{2.29}$$

where we have used the Cauchy inequality $|x| \leqslant \frac{|x|^2+1}{2}$. Define the function g : $[-T_1, T_2]$ by

$$g = \|(w-v)\sqrt{\chi_n}\|_{L^2}^2.$$

Then by definition of w we have $g(t=0)=0$. Furthermore, from (2.29) we have

$$\partial_t g \leqslant C(R)g + \frac{C(R)}{\sqrt{n}}.$$

By Gronwall inequality for all $t \in [-T_1, T_2]$ we have

$$g \leqslant \frac{C(R)}{\sqrt{n}}\exp(C(R)(T_2+T_1)) \leqslant \frac{C(R)}{\sqrt{n}}. \tag{2.30}$$

Assume by contradiction that there exist t and x such that

$$w(t,x) \neq v(t,x).$$

By continuity of v and w, there exists $\varepsilon > 0$ such that (for $n > |x|$) we have

$$g(t) = \|(w-v)\sqrt{\chi_n}\|_{L^2}^2 > \varepsilon.$$

Since $\varepsilon > 0$ is independant of n, we obtain a contradiction with (2.30) when n is large enough. Therefore for all t and x, we have

$$v(t,x) = w(t,x),$$

which concludes the proof. $\square$

2.2.5 From the system to the equation

With Proposition 2.14 in hand, we give the proof of Theorem 2.1.

Proof of Theorem 2.1. We start by defining v_0 by

$$v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0 \in X^3(\mathbb{R}).$$

From Proposition 2.13 there exists a unique maximal solution $(u, v) \in C((T_{min}, T_{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^1(\mathbb{R}) \times X^1(\mathbb{R}))$ of the system (2.13) associated with (u_0, v_0). From Proposition 2.14, for all $t \in (T_{min}, T_{max})$ we have

$$v = \partial u + \frac{i}{2}|u|^2 u. \tag{2.31}$$

It follows that

$$Lu = -iu^2\overline{v} + \frac{1}{2}|u|^4 u = -iu^2 \partial \overline{u},$$

and therefore u is a solution of (2.1) on (T_{min}, T_{max}). Furthermore

$$u \in C((T_{min}, T_{max}), X^3(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^1(\mathbb{R})).$$

To obtain the desired regularity on u, we observe that, since v has the same regularity as u, and verifies (2.31), we have

$$\partial u = v - \frac{i}{2}|u|^2 u \in C((T_{min}, T_{max}), X^3(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^1(\mathbb{R}))$$

This implies that

$$u \in C((T_{min}, T_{max}), X^4(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^2(\mathbb{R})).$$

This proves the existence part of the result. Uniqueness is a direct consequence from Proposition 2.12 and Proposition 2.13.

To prove the blow-up alternative, assume that $T_{max} < \infty$. Then from Proposition 2.13 we have

$$\lim_{t \to T_{max}} (\|u(t)\|_{X^1(\mathbb{R})} + \|v(t)\|_{X^1(\mathbb{R})}) = \infty$$

On the other hand, since (2.31) we obtain

$$\lim_{t \to T_{max}} (\|u(t)\|_{X^1(\mathbb{R})} + \|\partial u(t)\|_{X^1(\mathbb{R})}) = \infty.$$

It follows that

$$\lim_{t \to T_{max}} \|u(t)\|_{X^2(\mathbb{R})} = \infty.$$

Finally, we establish the continuity with respect to the initial data. Take a subinterval $[T_1, T_2] \subset (T_{min}, T_{max})$, and a sequence $(u_0^n) \in X^4(\mathbb{R})$ such that $u_0^n \to u_0$ in X^4. Let u_n be the solution of (2.1) associated with u_0^n and define v_n by

$$v_n = \partial u_n + \frac{i}{2}|u_n|^2 u_n. \tag{2.32}$$

By Proposition 2.13 the couple (u_n, v_n) is the unique maximal solution of system (2.13) in

$$C((T_{min}, T_{max}), X^3(\mathbb{R}) \times X^3(\mathbb{R})) \cap C^1((T_{min}, T_{max}), X^1(\mathbb{R}) \times X^1(\mathbb{R})).$$

Moreover, we have

$$\lim_{n\to+\infty} \left(\|u_n - u\|_{L^\infty([T_1,T_2],X^3)} + \|v_n - v\|_{L^\infty([T_1,T_2],X^3)} \right) = 0 \tag{2.33}$$

Since v and v_n verify the differential identity (2.32), we have

$$\partial(u_n - u) = (v_n - v) - \frac{i}{2}\left(|u_n|^2 u_n - |u|^2 u\right).$$

Therefore we have

$$\lim_{n\to+\infty} \|u_n - u\|_{L^\infty([T_1,T_2],X^4)} = 0,$$

which completes the proof. □

2.3 Results on the space $\phi + H^{k-2}(\mathbb{R})$ for $\phi \in X^k(\mathbb{R})$

In this section, we give the proof of Theorem 2.2 and Theorem 2.3.

2.3.1 The local well posedness on $\phi + H^2(\mathbb{R})$

From the equation to the system

Define

$$v = \partial u + \frac{i}{2}|u|^2 u. \tag{2.34}$$

Since Proposition 2.12, if u solves (2.1) then (u, v) solves the following system:

$$\begin{cases} Lu = -iu^2\overline{v} + \frac{1}{2}|u|^4 u, \\ Lv = i\overline{u}v^2 + \frac{3}{2}|u|^4 v + u^2|u|^2\overline{v}, \\ u(0) = u_0, \\ v(0) = v_0 := \partial u_0 + \frac{i}{2}|u_0|^2 u_0. \end{cases} \tag{2.35}$$

Let $\phi \in X^4(\mathbb{R})$. Define $\tilde{u} = u - \phi$, $\tilde{v} = v - \frac{i}{2}|\phi|^2\phi$. We have if u solves (2.1) then $(\tilde{u}, \tilde{v})$ solves:

$$\begin{cases} L\tilde{u} = Q_1(\tilde{u}, \tilde{v}, \phi), \\ L\tilde{v} = Q_2(\tilde{u}, \tilde{v}, \phi), \\ \tilde{u}(0) = \tilde{u}_0 := u_0 - \phi, \\ \tilde{v}(0) = \tilde{v}_0 := v_0 - \frac{i}{2}|\phi|^2\phi, \end{cases} \tag{2.36}$$

where

$$Q_1(\tilde{u}, \tilde{v}, \phi) = -i(\tilde{u} + \phi)^2 \left(\overline{\tilde{v}} - \frac{i}{2}|\phi|^2\overline{\phi}\right) + \frac{1}{2}|\tilde{u} + \phi|^4(\tilde{u} + \phi) - L(\phi), \tag{2.37}$$

$$Q_2(\tilde{u}, \tilde{v}, \phi) = i(\overline{\tilde{u}} + \overline{\phi}) \left(\tilde{v} + \frac{i}{2}|\phi|^2\phi\right)^2 + \frac{3}{2}|\tilde{u} + \phi|^4 \left(\tilde{v} + \frac{i}{2}|\phi|^2\phi\right) \tag{2.38}$$

$$+ (\tilde{u} + \phi)^2|\tilde{u} + \phi|^2 \left(\overline{\tilde{v}} - \frac{i}{2}|\phi|^2\overline{\phi}\right) - \frac{i}{2}L(|\phi|^2\phi). \tag{2.39}$$

Resolution of the system

Let $k \geqslant 1$. We note that if $\phi \in X^{k+2}$ then $Q_1 : (\tilde{u}, \tilde{v}) \to Q_1(\tilde{u}, \tilde{v}, \phi)$ and $Q_2 : (\tilde{u}, \tilde{v}) \to Q_2(\tilde{u}, \tilde{v}, \phi)$ defined as in (2.37) and (2.39) are Lipschitz continuous on bounded set of $H^k(\mathbb{R}) \times H^k(\mathbb{R})$. By similar arguments to the one used for the proof of Proposition 2.13, we obtain the following local well-posedness result:

Proposition 2.15. *Let $k \geqslant 1$, $\phi \in X^{k+2}$, $\tilde{u}_0, \tilde{v}_0 \in H^k(\mathbb{R})$. There exist $T_{min} < 0$, $T_{max} > 0$ and a unique maximal solution $(\tilde{u}, \tilde{v})$ of the system* (2.36) *such that $\tilde{u}, \tilde{v} \in C((T_{min}, T_{max}), H^k(\mathbb{R})) \cap C^1((T_{min}, T_{max}), H^{k-2}(\mathbb{R}))$. Furthermore the following properties are satisfied.*

- Blow-up alternative. *If $T_{max} < \infty$ (resp. $T_{min} > -\infty$ then*

$$\lim_{t \to T_{max} (resp.\ T_{min})} (\|\tilde{u}\|_{H^k} + \|\tilde{v}\|_{H^k}) = \infty.$$

- Continuity with respect to the initial data. *If $\tilde{u}_0^n, \tilde{v}_0^n \in H^k(\mathbb{R})$ are such that*

$$\|\tilde{u}_0^n - \tilde{u}_0\|_{H^k} + \|\tilde{v}_0^n - \tilde{v}_0\|_{H^k} \to 0$$

then for any subinterval $[T_1, T_2] \subset (T_{min}, T_{max})$ the associated solution $(\tilde{u}^n, \tilde{v}^n)$ of (2.36) *satisfies*

$$\lim_{n \to +\infty} \left(\|\tilde{u}^n - \tilde{u}\|_{L^\infty([T_1,T_2],H^k)} + \|\tilde{v}^n - \tilde{v}\|_{L^\infty([T_1,T_2],H^k)}\right) = 0.$$

Preservation of a differential identity

Let $(\tilde{u}_0, \tilde{v}_0)$ be defined as in section 2.3.1. By an elementary calculation, we have

$$\tilde{v}_0 = \partial\tilde{u}_0 + \frac{i}{2}(|\tilde{u}_0 + \phi|^2(\tilde{u}_0 + \phi) - |\phi|^2\phi) + \partial\phi. \tag{2.40}$$

We have the following results:

Proposition 2.16. *Let $\phi \in X^4(\mathbb{R})$ and $\tilde{u}_0, \tilde{v}_0 \in H^2(\mathbb{R})$ satisfy* (2.40). *Then the associated solution $(\tilde{u}, \tilde{v})$ obtained in Proposition 2.15 also satisfy* (2.40) *for all $t \in (T_{min}, T_{max})$.*

Proof. We define

$$\tilde{w} = \partial \tilde{u} + \frac{i}{2}(|\tilde{u}+\phi|^2(\tilde{u}+\phi) - |\phi|^2\phi) + \partial\phi. \tag{2.41}$$

Set $u = \tilde{u} + \phi$, $v = \tilde{v} + \frac{i}{2}|\phi|^2\phi$, $w = \tilde{w} + \frac{i}{2}|\phi|^2\phi$. We have

$$w = \partial u + \frac{i}{2}|u|^2 u. \tag{2.42}$$

Since $(\tilde{u}, \tilde{v})$ is a solution of (2.36), we have (u, v) is a solution of (2.35). We have

$$Lu = -iu^2(\overline{v} - \overline{w}) + H,$$

where H defined by

$$H = -iu^2\overline{w} + \frac{1}{2}|u|^4 u.$$

By using (2.17) and the previously expression obtained for Lu, we get

$$\begin{aligned}
Lw &= \partial(Lu) + \frac{i}{2}L(|u|^2u) \\
&= \partial(Lu) + \frac{i}{2}\left(2L(u)|u|^2 + 2\overline{u}(\partial u)^2 + 2u^2\partial^2\overline{u} - u^2\overline{L(u)} + 4u|\partial u|^2\right) \\
&= \partial\left(-iu^2(\overline{v} - \overline{w})\right) + \partial H \\
&\quad + i\left(H|u|^2 - iu^2|u|^2(\overline{v} - \overline{w}) + \overline{u}(\partial u)^2 + u^2\partial^2\overline{u} - \frac{1}{2}u^2\left(i\overline{u}^2(v-w) + \overline{H}\right) + 2u|\partial u|^2\right) \\
&= -i\partial\left(u^2(\overline{v} - \overline{w})\right) + u^2|u|^2(\overline{v} - \overline{w}) + \frac{1}{2}|u|^4(v-w) + K,
\end{aligned}$$

where K is defined by

$$K = \partial H + iH|u|^2 + i\overline{u}(\partial u)^2 + iu^2\partial^2\overline{u} - \frac{i}{2}u^2\overline{H} + 2iu|\partial u|^2.$$

Using (2.42) to replace the term ∂u in K and remark that the role of w is the same the one of v as in Proposition 2.12, we have

$$K = i\overline{u}w^2 + \frac{3}{2}|u|^4 w + u^2|u|^2\overline{w}.$$

Thus,

$$\begin{aligned}
Lw - Lv &= -i\partial\left(u^2(\overline{v} - \overline{w})\right) + u^2|u|^2(\overline{v} - \overline{w}) + \frac{1}{2}|u|^4(v-w) + (K - L(v)) \\
&= -i\partial\left(u^2(\overline{v} - \overline{w})\right) + u^2|u|^2(\overline{v} - \overline{w}) + \frac{1}{2}|u|^4(v-w) \\
&\quad + i\overline{u}(w^2 - v^2) + \frac{3}{2}|u|^4(w-v) + u^2|u|^2(\overline{w} - \overline{v}) \\
&= -iu^2\partial(\overline{v} - \overline{w}) + A(v-w) + B(\overline{v} - \overline{w}),
\end{aligned}$$

where

$$A := -|u|^4 - i\overline{u}(v+w),$$
$$B := -2iu\partial u = -2iu\left(w - \frac{i}{2}|u|^2 u\right) = -2iuw - |u|^2u^2.$$

This implies that

$$L(\tilde{w}-\tilde{v}) = -i(\tilde{u}+\phi)^2\partial(\overline{\tilde{v}}-\overline{\tilde{w}}) + A(\tilde{v}-\tilde{w}) + B(\overline{\tilde{v}}-\overline{\tilde{w}}). \tag{2.43}$$

Multiplying both sides of (2.43) by $\overline{\tilde{w}}-\overline{\tilde{v}}$, taking the imaginary part, and integrating over space with integration by part for the first term of right hand side of (2.43), we obtain

$$\frac{d}{dt}\|\tilde{w}-\tilde{v}\|_{L^2}^2 \lesssim (\|\tilde{u}+\phi\|_{L^\infty}\|\partial\tilde{u}+\partial\phi\|_{L^\infty} + \|A\|_{L^\infty} + \|B\|_{L^\infty})\|\tilde{w}-\tilde{v}\|_{L^2}^2.$$

By Grönwall's inequality we obtain

$$\|\tilde{w}-\tilde{v}\|_{L^2}^2 \leqslant \|\tilde{w}(0)-\tilde{v}(0)\|_{L^2}^2 \times \exp(C\int_0^t (\|\tilde{u}+\phi\|_{L^\infty}\|\partial\tilde{u}+\partial\phi\|_{L^\infty}+\|A\|_{L^\infty}+\|B\|_{L^\infty})\, ds).$$

Using the fact that $\tilde{w}(0)=\tilde{v}(0)$, we obtain $\tilde{w}=\tilde{v}$, for all t. This implies that

$$\tilde{v} = \partial\tilde{u} + \frac{i}{2}(|\tilde{u}+\phi|^2(\tilde{u}+\phi) - |\phi|^2\phi) + \partial\phi.$$

This completes the proof of Proposition 2.16. □

From the system to the equation

Now, we finish the proof of Theorem 2.2.

Proof of Theorem 2.2. Let $\phi \in X^4(\mathbb{R})$ and $u_0 \in \phi + H^2(\mathbb{R})$. We define $v_0 \in X^1(\mathbb{R})$, $\tilde{u}_0 \in H^2(\mathbb{R})$ and $\tilde{v}_0 \in H^1(\mathbb{R})$ in the following way:

$$v_0 = \partial u_0 + \frac{i}{2}u_0|u_0|^2, \quad \tilde{u}_0 = u_0 - \phi, \quad \text{ and } \tilde{v}_0 = v_0 - \frac{i}{2}|\phi|^2\phi.$$

We have

$$\tilde{v}_0 = \partial\tilde{u}_0 + \frac{i}{2}(|\tilde{u}_0+\phi|^2(\tilde{u}_0+\phi) - |\phi|^2\phi) + \partial\phi.$$

From Proposition 2.15 there exists a unique maximal solution $(\tilde{u},\tilde{v}) \in C((T_{min},T_{max}),H^1(\mathbb{R})) \cap C^1((T_{min},T_{max}),H^{-1}(\mathbb{R}))$ of (2.36). Let $\tilde{u}_0^n \in H^3(\mathbb{R})$ be such that

$$\|\tilde{u}_0^n - \tilde{u}_0\|_{H^2(\mathbb{R})} \to 0$$

as $n \to \infty$. Define $\tilde{v}_0^n \in H^2(\mathbb{R})$ by

$$\tilde{v}_0^n = \partial\tilde{u}_0^n + \frac{i}{2}(|\tilde{u}_0^n+\phi|^2(\tilde{u}_0^n+\phi) - |\phi|^2\phi) + \partial\phi.$$

From Proposition 2.15, there exists a unique solution maximal solution.

$$\tilde{u}^n, \tilde{v}^n \in C((T^n_{min}, T^n_{max}), H^2(\mathbb{R})) \cap C^1((T^n_{min}, T^n_{max}), L^2(\mathbb{R}))$$

of the system (2.36). Let $[T_1, T_2] \subset (T_{min}, T_{max})$ be any closed interval. From [17, proposition 4.3.7], for $n \geqslant N_0$ large enough, we have $[T_1, T_2] \subset (T^n_{min}, T^n_{max})$. By Proposition 2.16, for $n \geqslant N_0$, $t \in [T_1, T_2]$, we have

$$\tilde{v}^n = \partial \tilde{u}^n + \frac{i}{2}(|\tilde{u}^n + \phi|^2(\tilde{u}^n + \phi) - |\phi|^2\phi) + \partial\phi.$$

By Proposition 2.15, we have

$$\lim_{n\to\infty} \sup_{t\in[T_1,T_2]} (\|\tilde{u}^n(t) - \tilde{u}(t)\|_{H^1(\mathbb{R})} + \|\tilde{v}^n(t) - \tilde{v}(t)\|_{H^1(\mathbb{R})}) \to 0.$$

We obtain that for all $t \in [T_1, T_2]$, and then for all $t \in (T_{min}, T_{max})$:

$$\tilde{v} = \partial \tilde{u} + \frac{i}{2}(|\tilde{u} + \phi|^2(\tilde{u} + \phi) - |\phi|^2\phi) + \partial\phi.$$

This follows that

$$\partial\tilde{u} \in C((T_{min}, T_{max}), H^1(\mathbb{R})) \cap C^1((T_{min}, T_{max}), H^{-1}(\mathbb{R})).$$

Hence we have

$$\tilde{u} \in C((T_{min}, T_{max}), H^2(\mathbb{R})) \cap C^1((T_{min}, T_{max}), L^2(\mathbb{R})).$$

Define $u = \phi + \tilde{u}$ and define v by

$$v = \tilde{v} + \frac{i}{2}|\phi|^2\phi = \partial u + \frac{i}{2}|u|^2 u.$$

Since $(\tilde{u}, \tilde{v})$ solves (2.36), we have (u, v) solves (2.35). Therefore, $u \in \phi + C((T_{min}, T_{max}), H^2(\mathbb{R})) \cap C^1((T_{min}, T_{max}), L^2(\mathbb{R}))$ solves:

$$Lu = -iu^2\overline{v} + \frac{1}{2}|u|^4 u = -iu^2\partial\overline{u}.$$

This establishes the existence of a solution to (2.1). To prove uniqueness, assume that $U \in \phi + C((T_{min}, T_{max}), H^2(\mathbb{R})) \cap C^1((T_{min}, T_{max}), L^2(\mathbb{R}))$ is another solution of (2.1). Set $V = \partial U + \frac{i}{2}|U|^2 U$, and $\tilde{U} = U - \phi$, $\tilde{V} = V - \frac{i}{2}|\phi|^2\phi$. Thus, $(\tilde{U}, \tilde{V}) \in C((T_{min}, T_{max}), H^1(\mathbb{R})) \cap C^1((T_{min}, T_{max}), H^{-1}(\mathbb{R}))$ is a solution of (2.36). By the uniqueness statement in Proposition 2.15, we obtain $\tilde{U} = \tilde{u}$. Hence, $u = U$, which proves uniqueness. The blow-up alternative and continuity with respect to the initial data are proved using similar arguments as in the proof of Theorem 2.1. This completes the proof of Theorem 2.2. □

2.3.2 The local well posedness on $\phi + H^1(\mathbb{R})$

In this section, we give the proof of Theorem 2.3, using the method of Hayashi and Ozawa [59]. As in Section 2.3.1, we work with the system (2.36).

Resolution of the system

Since we are working in the less regular space $\phi+H^1(\mathbb{R})$, we cannot use Proposition 2.15. Instead, we establish the following result using Strichartz estimate.

Proposition 2.17. *Consider the system (2.36). Let $\phi \in X^2(\mathbb{R})$, $\tilde{u}_0, \tilde{v}_0 \in L^2(\mathbb{R})$. There exists $R > 0$ such that if $\|\tilde{u}_0\|_{L^2} + \|\tilde{v}_0\|_{L^2} < R$ then there exist $T > 0$ and a unique solution $(\tilde{u}, \tilde{v})$ of the system (2.36) verifying*

$$\tilde{u}, \tilde{v} \in C([-T,T], L^2(\mathbb{R})) \cap L^4([-T,T], L^\infty(\mathbb{R})).$$

Moreover, we have the following continuous dependence on initial data property: If $(\widetilde{u_0^n}, \widetilde{v_0^n}) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ is a sequence such that $\|\widetilde{u_0^n} - \tilde{u}_0\|_2 + \|\widetilde{v_0^n} - \tilde{v}_0\|_2 \to 0$ then for n large enough we have $\|\widetilde{u_0^n}\|_2 + \|\widetilde{v_0^n}\|_2 < R$ and the associated solutions $(\tilde{u}^n, \tilde{v}^n)$ satisfy:

$$\|\tilde{u}^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|\tilde{v}^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} \to 0,$$

where we have used the following notation:

$$L^\infty L^2 = L^\infty([-T,T], L^2(\mathbb{R})), \quad L^4 L^\infty = L^4([-T,T], L^\infty(\mathbb{R}))$$

and the norm on $L^\infty L^2 \cap L^4 L^\infty$ is defined, as usual for the intersection of two Banach spaces, as the sum of the norms on each space.

Proof. Let Q_1, Q_2 be defined as in system (2.36). By direct computations, we have

$$Q_1(\tilde{u},\tilde{v},\phi) = -i(\tilde{u}+\phi)^2\overline{\tilde{v}} - \frac{1}{2}|\phi|^2\overline{\phi}(\tilde{u}^2 + 2\tilde{u}\phi) + \frac{1}{2}(|\tilde{u}+\phi|^4 - |\phi|^4)\tilde{u} - \partial^2\phi, \quad (2.44)$$

$$\begin{aligned} Q_2(\tilde{u},\tilde{v},\phi) = {}& i\overline{\tilde{u}}\left(\tilde{v} + \frac{i}{2}|\phi|^2\phi\right)^2 + i\overline{\phi}\left[\left(\tilde{v} + \frac{i}{2}|\phi|^2\phi\right)^2 - \left(\frac{i}{2}|\phi|^2\phi\right)^2\right] + \frac{3}{2}|\tilde{u}+\phi|^4\tilde{v} \\ &+ \frac{3}{4}i|\phi|^2\phi(|\tilde{u}+\phi|^4 - |\phi|^4) + \overline{\tilde{v}}(\tilde{u}+\phi)|\tilde{u}+\phi|^2 \\ &- \frac{i}{2}|\phi|^2\overline{\phi}((\tilde{u}+\phi)^2|\tilde{u}+\phi|^2 - |\phi|^2\phi^2) - \frac{i}{2}\partial^2(|\phi|^2\phi). \end{aligned} \quad (2.45)$$

Thus,

$$\begin{aligned} |Q_1(\tilde{u},\tilde{v},\phi)| &\lesssim |\tilde{v}|(|\tilde{u}|^2 + |\phi|^2) + |\phi|^3|\tilde{u}|^2 + |\phi|^4|\tilde{u}| + (|\tilde{u}|^5 + |\phi|^4|\tilde{u}|) + |\phi|(|\tilde{u}|^4 + |\tilde{u}||\phi|^3) + |\partial^2\phi| \\ &\lesssim |\tilde{v}||\tilde{u}|^2 + |\tilde{v}||\phi|^2 + |\tilde{u}|^5 + |\tilde{u}||\phi|^4 + |\partial^2\phi|, \\ |Q_2(\tilde{u},\tilde{v},\phi)| &\lesssim |\tilde{u}|(|\tilde{v}|^2 + |\phi|^6) + |\phi|(|\tilde{v}|^2 + |\tilde{v}||\phi|^3) + |\tilde{v}|(|\tilde{u}|^4 + |\phi|^4) \\ &\quad + |\phi|^3(|\tilde{u}|^4 + |\tilde{u}||\phi|^3) + |\tilde{v}|(|\tilde{u}|^3 + |\phi|^3) + |\phi|^3(|\tilde{u}|^4 + |\phi|^3|\tilde{u}|) + |\partial^2(|\phi|^2\phi)| \\ &\lesssim |\tilde{u}||\tilde{v}|^2 + |\tilde{u}||\phi|^6 + |\phi||\tilde{v}|^2 + |\phi|^4|\tilde{v}| + |\tilde{u}|^4|\tilde{v}| + |\phi|^3|\tilde{u}|^4 \\ &\quad + |\tilde{u}|^3|\tilde{v}| + |\phi|^3|\tilde{v}| + |\partial^2(|\phi|^2\phi)|. \end{aligned}$$

Consider the following problem

$$(\tilde{u},\tilde{v}) = S(t)(\tilde{u}_0,\tilde{v}_0) - i\int_0^t S(t-s)Q(\tilde{u},\tilde{v},\phi)\,ds \quad (2.46)$$

where $Q = (Q_1, Q_2)$. Let

$$\Phi(\tilde{u}, \tilde{v}) = S(t)(\tilde{u}_0, \tilde{v}_0) - i \int_0^t S(t-s) Q \, ds.$$

Assume that $\|\tilde{u}_0\|_{L^2(\mathbb{R})} + \|\tilde{v}_0\|_{L^2(\mathbb{R})} \leqslant \frac{R}{4}$ for $R > 0$ small enough. For $T > 0$ we define the space $X_{T,R}$ by

$$X_{T,R} = \left\{ (\tilde{u}, \tilde{v}) \in (C([-T,T], L^2(\mathbb{R})) \cap L^4([-T,T], L^\infty(\mathbb{R})))^2 : \|(\tilde{u}, \tilde{v})\|_{(L^\infty L^2 \cap L^4 L^\infty)^2} \leqslant R \right\}.$$

We are going to prove that for R, T small enough the map Φ is a contraction from $X_{T,R}$ to itself.

We first prove that for R, T small enough, Φ maps $X_{T,R}$ into $X_{T,R}$. Let $(\tilde{u}, \tilde{v}) \in X_{T,R}$. By Strichartz estimates we have

$$\begin{aligned} \|\Phi(\tilde{u}, \tilde{v})\|_{(L^\infty L^2 \cap L^4 L^\infty)^2} &\lesssim \|(\tilde{u}_0, \tilde{v}_0)\|_{L^2 \times L^2} + \|Q\|_{L^1 L^2 \times L^1 L^2}, \\ &\lesssim \frac{R}{4} + (\|Q_1\|_{L^1 L^2} + \|Q_2\|_{L^1 L^2}). \end{aligned}$$

We have

$$\begin{aligned} \|Q_1\|_{L^1 L^2} &\lesssim \||\tilde{u}|^2 \tilde{v}\|_{L^1 L^2} + \||\tilde{v}||\phi|^2\|_{L^1 L^2} + \||\tilde{u}|^5\|_{L^1 L^2} + \|\partial^2 \phi\|_{L^1 L^2} \\ &\lesssim \|\tilde{v}\|_{L^2 L^2} \|\tilde{u}\|^2_{L^4 L^\infty} + \|\tilde{v}\|_{L^2 L^2} \||\phi|\|^2_{L^4 L^\infty} + \|\tilde{u}\|^4_{L^4 L^\infty} \|\tilde{u}\|_{L^\infty L^2} + \|\partial^2 \phi\|_{L^1 L^2} \\ &\lesssim (2T)^{\frac{1}{2}} \|\tilde{v}\|_{L^\infty L^2} \|\tilde{u}\|^2_{L^4 L^\infty} + (2T)^{\frac{1}{2}} \|\tilde{v}\|_{L^\infty L^2} \|\phi\|_{L^\infty} (2T)^{\frac{1}{4}} \\ &+ \|\tilde{u}\|^4_{L^4 L^\infty} \|\tilde{u}\|_{L^\infty L^2} + \|\partial^2 \phi\|_{L^2} (2T) \\ &\lesssim (2T)^{\frac{1}{2}} R^3 + (2T)^{\frac{3}{4}} \|\phi\|_{L^\infty} R + R^5 + (2T) \|\phi\|_{X^2} < \frac{R}{4}. \end{aligned}$$

for T, R small enough. Similarly, we also have

$$\|\tilde{Q}_2\|_{L^1 L^2} < \frac{R}{4}$$

for T, R small enough. Therefore, for T, R small enough, we have

$$\|\Phi(\tilde{u}, \tilde{v})\|_{(L^\infty L^2 \cap L^4 L^\infty)^2} < \frac{3R}{4} < R.$$

Hence, Φ maps from $X_{T,R}$ into itself.

We now show that for T, R small enough, the map Φ is a contraction from $X_{T,R}$ to itself.

Indeed, let $(u_1, v_1), (u_2, v_2) \in X_{T,R}$. By Strichartz estimates we have

$$\begin{aligned} &\|\Phi(u_1, v_1) - \Phi(u_2, v_2)\|_{(L^\infty L^2 \cap L^4 L^\infty)^2} \\ &= \left\| \int_0^t S(t-s) \left(Q(u_1, v_1) - Q(u_2, v_2) \right) \, ds \right\|_{(L^\infty L^2 \cap L^4 L^\infty)^2}, \\ &\lesssim \|Q_1(u_1, v_1) - Q_1(u_2, v_2)\|_{L^1 L^2} + \|Q_2(u_1, v_1) - Q_2(u_2, v_2)\|_{L^1 L^2}. \end{aligned}$$

Using the same kind of arguments as before we obtain that Φ is a contraction on $X_{T,R}$. Therefore, using the Banach fixed-point theorem, there exist $T > 0$ and a unique solution $(\tilde{u},\tilde{v}) \in C([-T,T],L^2(\mathbb{R})) \cap L^4([-T,T],L^\infty(\mathbb{R}))$ of the problem (2.46). As above, we see that if $h,k \in C([-T,T],L^2(\mathbb{R})) \cap L^4([-T,T],L^\infty(\mathbb{R}))$ then $Q_1(h,k,\phi), Q_2(h,k,\phi) \in L^1([-T,T],L^2(\mathbb{R}))$. By [17, Proposition 4.1.9], $(\tilde{u},\tilde{v}) \in C([-T,T],L^2(\mathbb{R})) \cap L^4([-T,T],L^\infty(\mathbb{R}))$ solves (2.46) if only if $(\tilde{u},\tilde{v})$ solves (2.36). Thus, we proved the existence of a solution of (2.36). The uniqueness of solution of (2.36) is obtained by the uniqueness of solution of (2.46).

It is remains to prove the continuous dependence on initial data. Assume that $(u_0^n, v_0^n) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ is such that

$$\|u_0^n - \tilde{u}_0\|_{L^2(\mathbb{R})} + \|v_0^n - \tilde{v}_0\|_{L^2(\mathbb{R})} \to 0,$$

as $n \to \infty$. In particular, for n large enough, we have

$$\|u_0^n\|_{L^2(\mathbb{R})} + \|v_0^n\|_{L^2(\mathbb{R})} < R.$$

There exists a unique maximal solution (u^n, v^n) of system (2.36), and we may assume that for n large enough, (u^n,v^n) is defined on $[-T,T]$. Assume that T small enough such that

$$\|\tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|\tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} + \sup_n (\|u^n\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n\|_{L^\infty L^2 \cap L^4 L^\infty}) \leqslant 2R. \tag{2.47}$$

We have $(\tilde{u},\tilde{v})$ is a solution of the following system

$$(\tilde{u},\tilde{v}) = S(t)(\tilde{u}_0,\tilde{v}_0) - i\int_0^t S(t-s)(Q_1(\tilde{u},\tilde{v},\phi), Q_2(\tilde{u},\tilde{v},\phi)).$$

Similarly, (u^n,v^n) are solutions of the following system

$$(u^n,v^n) = S(t)(u_0^n,v_0^n) - i\int_0^t S(t-s)(Q_1(u^n,v^n,\phi), Q_2(u^n,v^n,\phi)).$$

Hence,

$$\begin{aligned}
&(u^n - u, v^n - v) \\
&= S(t)(u_0^n - \tilde{u}_0, v_0^n - \tilde{v}_0) - i\int_0^t S(t-s)(Q_1(\tilde{u},\tilde{v},\phi) - Q_1(u^n,v^n,\phi), Q_2(\tilde{u},\tilde{v},\phi) - Q_2(u^n,v^n,\phi)).
\end{aligned}$$

Using Strichartz estimates and (2.47), for all $t \in [-T,T]$ and R, T small enough, we have

$$\begin{aligned}
&\|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} \\
&\lesssim \|u_0^n - \tilde{u}_0\|_{L^2} + \|v_0^n - \tilde{v}_0\|_{L^2} \\
&+ \|Q_1(\tilde{u},\tilde{v},\phi) - Q_1(u^n,v^n,\phi)\|_{L^1L^2} + \|Q_2(\tilde{u},\tilde{v},\phi) - Q_2(u^n,v^n,\phi))\|_{L^1L^2} \\
&\lesssim \|u_0^n - \tilde{u}_0\|_{L^2} + \|v_0^n - \tilde{v}_0\|_{L^2} \\
&+ R(\|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty}).
\end{aligned}$$

For $R < \frac{1}{2}$ small enough, we have

$$\frac{1}{2}(\|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty}) \leqslant \|\tilde{u}_0 - u_0^n\|_{L^2(\mathbb{R})} + \|\tilde{v}_0 - v_0^n\|_{L^2(\mathbb{R})}.$$

Letting $n \to +\infty$ we obtain the desired result. $\square$

From the system to the equation

Now, we finish the proof of Theorem 2.3.

Proof of Theorem 2.3. Let $\phi \in X^4(\mathbb{R})$ be such that $\|\partial\phi\|_{L^2}$ is small enough. Let $u_0 \in \phi + H^1(\mathbb{R})$ be such that $\|u_0 - \phi\|_{H^1}$ is small enough. Set $v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0$, $\tilde{u}_0 = u_0 - \phi$ and $\tilde{v}_0 = v_0 - \frac{i}{2}|\phi|^2\phi$. We have

$$\tilde{v}_0 = \partial\tilde{u}_0 + \frac{i}{2}(|\tilde{u}_0 + \phi|^2(\tilde{u}_0 + \phi) - |\phi|^2\phi) + \partial\phi.$$

Furthermore, $\tilde{u}_0 \in H^1(\mathbb{R})$, $\tilde{v}_0 \in L^2(\mathbb{R})$ satisfy:

$$\|\tilde{u}_0\|_{L^2(\mathbb{R})} + \|\tilde{v}_0\|_{L^2(\mathbb{R})} \lesssim \|\tilde{u}_0\|_{H^1(\mathbb{R})} + \|\partial\phi\|_{L^2},$$

which is small enough by the assumption. By Proposition 2.17, there exist $T > 0$ and a unique solution $(\tilde{u}, \tilde{v}) \in C([-T, T], L^2(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$ of the system (2.36). Let $u_0^n \in H^3(\mathbb{R})$ satisfy $\|u_0^n - \tilde{u}_0\|_{H^1(\mathbb{R})} \to 0$ as $n \to +\infty$. Set

$$v_0^n = \partial u_0^n + \frac{i}{2}(|u_0^n + \phi|^2(u_0^n + \phi) - |\phi|^2) + \partial\phi.$$

Let (u^n, v^n) be the $H^2(\mathbb{R})$ solution of the system (2.36) obtained by Proposition 2.15 with data (u_0^n, v_0^n). By Proposition 2.16 we have

$$v^n = \partial u^n + \frac{i}{2}(|u^n + \phi|^2(u^n + \phi) - |\phi|^2\phi) + \partial\phi. \tag{2.48}$$

Furthermore,

$$\|u_0^n - \tilde{u}_0\|_{L^2(\mathbb{R})} + \|v_0^n - \tilde{v}_0\|_{L^2(\mathbb{R})} \to 0.$$

From the continuous dependence on the initial data obtained in Proposition 2.17, $(u^n, v^n), (\tilde{u}, \tilde{v})$ are solutions of the system (2.36) on $[-T, T]$ for n large enough, and

$$\|u^n - \tilde{u}\|_{L^\infty L^2 \cap L^4 L^\infty} + \|v^n - \tilde{v}\|_{L^\infty L^2 \cap L^4 L^\infty} \to 0$$

as $n \to \infty$. Letting $n \to \infty$ on the two sides of (2.48), we obtain for all $t \in [-T, T]$

$$\tilde{v} = \partial\tilde{u} + \frac{i}{2}(|\tilde{u} + \phi|^2(\tilde{u} + \phi) - |\phi|^2\phi) + \partial\phi, \tag{2.49}$$

which makes sense in $H^{-1}(\mathbb{R})$. From (2.49) we see that $\partial\tilde{u} \in C([-T, T], L^2(\mathbb{R}))$ and (2.49) makes sense in $L^2(\mathbb{R})$. Then $\tilde{u} \in C([-T, T], H^1(\mathbb{R})) \cap L^4([-T, T], L^\infty(\mathbb{R}))$. By the Sobolev embedding of $H^1(\mathbb{R})$ in $L^\infty(\mathbb{R})$ we obtain that

$$\begin{aligned}
\||\tilde{u} + \phi|^2(\tilde{u} + \phi) - |\phi|^2\phi\|_{L^4L^\infty} &\lesssim \||\tilde{u}|^3\|_{L^4L^\infty} + \||\tilde{u}||\phi|^2\|_{L^4L^\infty} \\
&< \|\tilde{u}\|_{L^4L^\infty}\|\tilde{u}\|_{L^\infty L^\infty} + \|\tilde{u}\|_{L^4L^\infty}\|\phi\|^2_{L^\infty L^\infty} < \infty.
\end{aligned}$$

Hence, $|\tilde{u}+\phi|^2(\tilde{u}+\phi) - |\phi|^2\phi \in L^4L^\infty$. From (2.49) we obtain that $\partial\tilde{u} \in L^4L^\infty$ which implies $\tilde{u} \in L^4([-T, T], W^{1,\infty}(\mathbb{R}))$. Set $u = \tilde{u} + \phi$, $v = \tilde{v} + \frac{i}{2}|\phi|^2\phi$, then $u - \phi \in$

$C([-T,T],H^1(\mathbb{R}))\cap L^4([-T,T],W^{1,\infty}(\mathbb{R}))$ and $v-\frac{i}{2}|\phi|^2\phi\in C([-T,T],L^2(\mathbb{R}))\cap L^4([-T,T],L^\infty(\mathbb{R}))$. Moreover,

$$v=\partial u+\frac{i}{2}|u|^2u.$$

Since (u,v) solves (2.35), we have

$$Lu=-iu^2\overline{v}+\frac{1}{2}|u|^4u=-iu^2\partial\overline{u}.$$

The existence of a solution of the equation (2.1) follows. To prove the uniqueness property, assume that $U\in C([-T,T],\phi+H^1(\mathbb{R}))\cap L^4([-T,T],\phi+W^{1,\infty}(\mathbb{R}))$ is another solution of the equation (2.1). Set $V=\partial U+\frac{i}{2}|U|^2U$ and $\tilde{U}=U-\phi$, $\tilde{V}=V-\frac{i}{2}|\phi|^2\phi$. Hence $\tilde{U}\in C([-T,T],H^1(\mathbb{R}))\cap L^4([-T,T],W^{1,\infty}(\mathbb{R}))$ and $\tilde{V}\in C([-T,T],L^2(\mathbb{R}))\cap L^4([-T,T],L^\infty(\mathbb{R}))$. Moreover, $(\tilde{U},\tilde{V})$ is a solution of the system (2.36). By the uniqueness of solutions of (2.36), we obtain that $\tilde{U}=\tilde{u}$. Hence, $u=U$, which completes the proof. $\square$

2.4 Conservation of the mass, the energy and the momentum

In this section, we prove Theorem 2.4. Let $q_0\in\mathbb{R}$ and $u\in q_0+H^2(\mathbb{R})$ be a solution of (2.1). Let χ and χ_R be the functions defined as in (2.5) and (2.7). We have

$$\|\chi_R'\|_{L^2(\mathbb{R})}=\left(\int_{\mathbb{R}}\frac{1}{R^2}\left(\chi'\left(\frac{x}{R}\right)\right)^2dx\right)^{\frac{1}{2}}=\frac{1}{R^{\frac{1}{2}}}\|\chi'\|_{L^2(\mathbb{R})}\to 0\,\text{as}\,R\to\infty. \tag{2.50}$$

Similarly, for each $a\in\mathbb{R}$, we have

$$\|\chi_{a,R}'\|_{L^2(\mathbb{R})}\to 0\,\text{as}\,R\to\infty. \tag{2.51}$$

By the continuous dependence on initial data property of solution, we can assume that $u_0\in q_0+H^3(\mathbb{R})$, so that

$$u\in C((T_{min},T_{max}),q_0+H^3(\mathbb{R})).$$

It is enough to prove conservation of generalized mass, conservation of energy (2.8) and conservation of momentum (2.9) for any closed interval $[T_0,T_1]\in(T_{min},T_{max})$. Let $T_0<0$, $T_1>0$ be such that $[T_0,T_1]\subset(T_{min},T_{max})$. Let $M>0$ be defined by

$$M=\sup_{t\in[T_0,T_1]}\|u-q_0\|_{H^3(\mathbb{R})}.$$

2.4.1 Conservation of mass

Multiplying both sides of (2.1) by $\overline{u}$ and taking imaginary part to obtain

$$\mathcal{R}e(u_t\overline{u})+\mathcal{I}m(\partial^2u\overline{u})+\mathcal{R}e(|u|^2u\partial\overline{u})=0.$$

This implies that

$$\begin{aligned}
0 &= \frac{1}{2}\partial_t(|u|^2) + \partial(\mathcal{Im}(\partial u\overline{u})) + \frac{1}{4}\partial(|u|^4)\\
&= \frac{1}{2}\partial_t(|u|^2 - q_0^2) + \partial(\mathcal{Im}(\partial u\overline{u})) + \frac{1}{4}\partial(|u|^4 - q_0^4).
\end{aligned}$$

By multiplying both sides by χ_R, integrating on space, and integrating by part we have

$$\begin{aligned}
0 &= \partial_t \int_{\mathbb{R}} \frac{1}{2}(|u|^2 - q_0^2)\chi_R dx - \int_{\mathbb{R}} \mathcal{Im}(\partial u\overline{u})\chi_R' - \int_{\mathbb{R}} \frac{(|u|^4 - q_0^4)}{4}\chi_R' dx\\
&= \partial_t \int_{\mathbb{R}} \frac{1}{2}(|u|^2 - q_0^2)\chi_R dx - \int_{\mathbb{R}} \left(\mathcal{Im}(\partial u\overline{u}) + \frac{1}{4}(|u|^4 - q_0^4)\right)\chi_R' dx. \qquad (2.52)
\end{aligned}$$

Denote the second term of (2.52) by K, using (2.50), we have

$$|K| \leqslant \|\mathcal{Im}(\partial u\overline{u}) + \frac{1}{4}(|u|^4 - q_0^4)\|_{L^2}\|\chi_R'\|_{L^2} \lesssim C(M)\frac{1}{R^{\frac{1}{2}}} \to 0 \text{ as } R \to \infty.$$

Thus, by integrating from 0 to t and taking R to infinity we obtain

$$\lim_{R\to\infty}\left(\int_{\mathbb{R}} \frac{1}{2}(|u|^2 - q_0^2)\chi_R dx - \int_{\mathbb{R}} \frac{1}{2}(|u_0|^2 - q_0^2)\chi_R dx\right) = 0. \qquad (2.53)$$

Similarly, for each $a \in \mathbb{R}$, we have

$$\lim_{R\to\infty}\left(\int_{\mathbb{R}} \frac{1}{2}(|u|^2 - q_0^2)\chi_{a,R} dx - \int_{\mathbb{R}} \frac{1}{2}(|u_0|^2 - q_0^2)\chi_{a,R} dx\right) = 0. \qquad (2.54)$$

as $R \to \infty$. This implies that $m^+(u(t))$ and $m^-(u(t))$ are conserved in time. In particular, if $m^+(u_0) = m^-(u_0) = m(u_0)$ then $m^+(u(t)) = m^-(u(t)) = m(u(t)) = m(u_0)$. This completes the proof of conservation of mass.

2.4.2 Conservation of energy

Now, we prove the conservation of the energy. Since u solves (2.1), after an elementary calculation, we have

$$\begin{aligned}
\partial_t(|\partial u|^2) &= \partial\left(2\mathcal{Re}(\partial u\partial_t\overline{u}) + \mathcal{Re}(u^2(\partial\overline{u})^2) - |\partial u|^2|u|^2 - |u|^4\mathcal{Im}(\overline{u}\partial u)\right)\\
&\quad + |u|^4\partial\mathcal{Im}(\overline{u}\partial u) + 2\mathcal{Im}(|u|^2\partial\overline{u}u_t). \qquad (2.55)
\end{aligned}$$

Recall that we have

$$\partial\mathcal{Im}(\partial u\overline{u}) = -\frac{1}{2}\partial_t(|u|^2) - \frac{1}{4}\partial(|u|^4). \qquad (2.56)$$

Furthermore,

$$\partial_t\mathcal{Im}(|u|^2u\partial\overline{u}) = 4\mathcal{Im}(u_t|u|^2\partial\overline{u}) + \partial\mathcal{Im}(|u|^2u\partial_t\overline{u}).$$

Thus,

$$2\mathcal{Im}(|u|^2u_t\partial\overline{u}) = \frac{1}{2}\left(\partial_t\mathcal{Im}(|u|^2u\partial\overline{u}) - \partial\mathcal{Im}(|u|^2u\partial_t\overline{u})\right). \tag{2.57}$$

Combining (2.55), (2.56) and (2.57) we obtain

$$\begin{aligned}\partial_t(|\partial u|^2) &= \partial\left(2\mathcal{Re}(\partial u\partial_t\overline{u}) + \mathcal{Re}(u^2(\partial\overline{u})^2) - |u|^2|\partial u|^2 - |u|^4\mathcal{Im}(\partial u\overline{u}) - \frac{1}{2}\mathcal{Im}(|u|^2u\partial_t\overline{u})\right)\\ &\quad + \frac{1}{2}\partial_t\mathcal{Im}(|u|^2u\partial\overline{u}) - \frac{1}{8}\partial(|u|^8) - \frac{1}{6}\partial_t(|u|^6).\end{aligned}$$

Hence,

$$\begin{aligned}&\partial_t\left(|\partial u|^2 - \frac{1}{2}\mathcal{Im}((|u|^2u - q_0^3)\partial\overline{u}) + \frac{1}{6}(|u|^6 - q_0^6)\right)\\ &= \partial\left(2\mathcal{Re}(\partial u\partial_t\overline{u}) + \mathcal{Re}(u^2(\partial\overline{u})^2) - |u|^2|\partial u|^2 - |u|^4\mathcal{Im}(\partial u\overline{u}) - \frac{1}{2}\mathcal{Im}(|u|^2u\partial_t\overline{u}) - \frac{1}{8}(|u|^8 - q_0^8)\right)\\ &\quad + \frac{1}{2}q_0^3\mathcal{Im}\partial_t\partial\overline{(u - q_0)}.\end{aligned}$$

Multiplying both sides by χ_R, integrating in space and integrating by part we obtain

$$\begin{aligned}&\partial_t\int_{\mathbb{R}}\left(|\partial u|^2 - \frac{1}{2}\mathcal{Im}((|u|^2u - q_0^3)\partial\overline{u}) + \frac{1}{6}(|u|^6 - q_0^6)\right)\chi_R\,dx\\ &= -\int_{\mathbb{R}}\chi_R'\left(2\mathcal{Re}(\partial u\partial_t\overline{u}) + \mathcal{Re}(u^2(\partial\overline{u})^2) - |u|^2|\partial u|^2 - |u|^4\mathcal{Im}(\partial u\overline{u}) - \frac{1}{2}\mathcal{Im}(|u|^2u\partial_t\overline{u})\right.\\ &\quad\left. - \frac{1}{8}(|u|^8 - q_0^8)\right)\,dx - \frac{q_0^3}{2}\mathcal{Im}\partial_t\int_{\mathbb{R}}\overline{(u - q_0)}\chi_R'\,dx.\end{aligned}$$

Integrating from 0 to t we obtain

$$\int_{\mathbb{R}}\left(|\partial u|^2 - \frac{1}{2}\mathcal{Im}((|u|^2u - q_0^3)\partial\overline{u}) + \frac{1}{6}(|u|^6 - q_0^6)\right)\chi_R\,dx \tag{2.58}$$

$$-\int_{\mathbb{R}}\left(|\partial u_0|^2 - \frac{1}{2}\mathcal{Im}((|u_0|^2u_0 - q_0^3)\partial\overline{u_0}) + \frac{1}{6}(|u_0|^6 - q_0^6)\right)\chi_R\,dx \tag{2.59}$$

$$\begin{aligned}&= \int_0^t\int_{\mathbb{R}}\chi_R'\left(2\mathcal{Re}(\partial u\partial_t\overline{u}) + \mathcal{Re}(u^2(\partial\overline{u})^2) - |u|^2|\partial u|^2 - |u|^4\mathcal{Im}(\partial u\overline{u})\right.\\ &\quad\left. - \frac{1}{2}\mathcal{Im}(|u|^2u\partial_t\overline{u}) - \frac{1}{8}(|u|^8 - q_0^8)\right)\,dx\,ds\end{aligned} \tag{2.60}$$

$$- \frac{q_0^3}{2}\left(\mathcal{Im}\int_{\mathbb{R}}\overline{(u - q_0)}\partial\chi_R\,dx - \mathcal{Im}\int_{\mathbb{R}}\overline{(u_0 - q_0)}\chi_R'\,dx\right). \tag{2.61}$$

Denoting the term (2.60) by A_R, using (2.50), we have

$$|A_R| \leqslant \|\chi_R'\|_{L^2}\|2\mathcal{Re}(\partial u\partial_t\overline{u}) + \mathcal{Re}(u^2(\partial\overline{u})^2) - |u|^2|\partial u|^2 - |u|^4\mathcal{Im}(\partial u\overline{u}) \tag{2.62}$$

$$- \frac{1}{2}\mathcal{Im}(|u|^2u\partial_t\overline{u}) - \frac{1}{8}(|u|^8 - q_0^8)\|_{L^2} \tag{2.63}$$

$$\lesssim C(M)\|\chi_R'\|_{L^2} \to 0 \text{ as } R \to \infty. \tag{2.64}$$

Moreover, using (2.50) again, we have

$$\left|\mathcal{I}m\int_{\mathbb{R}}\overline{(u-q_0)}\chi_R'\,dx\right| \leqslant \|u-q_0\|_{L^2}\|\chi_R'\|_{L^2} \lesssim C(M)\|\chi_R'\|_{L^2} \to 0 \text{ as } R\to\infty. \tag{2.65}$$

$$\left|\mathcal{I}m\int_{\mathbb{R}}\overline{(u_0-q_0)}\chi_R'\,dx\right| \leqslant \|u_0-q_0\|_{L^2}\|\chi_R'\|_{L^2} \lesssim C(M)\|\chi_R'\|_{L^2} \to 0 \text{ as } R\to\infty. \tag{2.66}$$

To deal with the term (2.58), we need to divide it into two terms. First, using $u\in q_0+H^3(\mathbb{R})$, as $R\to\infty$, we have

$$\int_{\mathbb{R}}\left(|\partial u|^2-\frac{1}{2}\mathcal{I}m((|u|^2u-q_0^3)\partial\overline{u})\right)\chi_R\,dx \to \int_{\mathbb{R}}\left(|\partial u|^2-\frac{1}{2}\mathcal{I}m((|u|^2u-q_0^3)\partial\overline{u})\right)dx. \tag{2.67}$$

Second, by easy calculations, we have

$$\begin{aligned}
&\frac{1}{6}\int_{\mathbb{R}}(|u|^6-q_0^6)\chi_R-\frac{1}{6}\int_{\mathbb{R}}(|u_0|^6-q_0^6)\chi_R\,dx & (2.68)\\
&=\frac{1}{6}\int_{\mathbb{R}}\left[(|u|^2-q_0^2)(|u|^4+q_0^2|u|^2-2q_0^4)+3q_0^4(|u|^2-q_0^2)\right]\chi_R\,dx & (2.69)\\
&\quad-\frac{1}{6}\int_{\mathbb{R}}\left[(|u_0|^2-q_0^2)(|u_0|^4+q_0^2|u_0|^2-2q_0^4)+3q_0^4(|u_0|^2-q_0^2)\right]\chi_R\,dx &\\
&=\frac{1}{6}\int_{\mathbb{R}}(|u|^2-q_0^2)^2(|u|^2+2q_0^2)\chi_R\,dx-\frac{1}{6}\int_{\mathbb{R}}(|u_0|^2-q_0^2)^2(|u_0|^2+2q_0^2)\chi_R\,dx & (2.70)\\
&+\frac{q_0^4}{2}\int_{\mathbb{R}}(|u|^2-q_0^2)\chi_R\,dx-\frac{q_0^4}{2}\int_{\mathbb{R}}(|u_0|^2-q_0^2)\chi_R\,dx. & (2.71)
\end{aligned}$$

Denote the term (2.70) by B_R, we have

$$\begin{aligned}
B_R \to &\frac{1}{6}\int_{\mathbb{R}}(|u|^2-q_0^2)(|u|^4+q_0^2|u|^2-2q_0^4)\,dx\\
&-\frac{1}{6}\int_{\mathbb{R}}(|u_0|^2-q_0^2)(|u_0|^4+q_0^2|u_0|^2-2q_0^4)\,dx
\end{aligned} \tag{2.72}$$

as $R\to+\infty$. The term (2.71) converges to 0 as $R\to\infty$ by (2.53). Finally, we have

$$\begin{aligned}
&\lim_{R\to\infty}\left(\frac{1}{6}\int_{\mathbb{R}}(|u|^6-q_0^6)\chi_R\,dx-\frac{1}{6}\int_{\mathbb{R}}(|u_0|^6-q_0^6)\chi_R\,dx\right)\\
&=\frac{1}{6}\int_{\mathbb{R}}(|u|^2-q_0^2)^2(|u|^2+2q_0^2)\,dx-\frac{1}{6}\int_{\mathbb{R}}(|u_0|^2-q_0^2)^2(|u_0|^2+2q_0^2)\,dx.
\end{aligned} \tag{2.73}$$

Combining (2.73) and (2.67) we have

$$\begin{aligned}
&\lim_{R\to\infty}(\text{ the term (2.58)} - \text{ the term (2.59)})\\
&=\int_{\mathbb{R}}|\partial u|^2-\frac{1}{2}\mathcal{I}m(|u|^2u-q_0^3)\partial\overline{u})\,dx+\frac{1}{6}\int_{\mathbb{R}}(|u|^2-q_0^2)^2(|u|^2+2q_0^2)\,dx\\
&-\int_{\mathbb{R}}|\partial u_0|^2-\frac{1}{2}\mathcal{I}m(|u_0|^2u-q_0^3)\partial\overline{u_0})\,dx-\frac{1}{6}\int_{\mathbb{R}}(|u_0|^2-q_0^2)^2(|u_0|^2+2q_0^2)\,dx
\end{aligned} \tag{2.74}$$

Combining (2.58)-(2.66), (2.74), we have

$$\int_{\mathbb{R}} |\partial u|^2 - \frac{1}{2}\mathcal{Im}(|u|^2 u - q_0^3)\partial \overline{u})\, dx + \frac{1}{6}\int_{\mathbb{R}} (|u|^2 - q_0^2)^2(|u|^2 + 2q_0^2)\, dx$$
$$= \int_{\mathbb{R}} |\partial u_0|^2 - \frac{1}{2}\mathcal{Im}(|u_0|^2 u_0 - q_0^3)\partial \overline{u_0})\, dx + \frac{1}{6}\int_{\mathbb{R}} (|u_0|^2 - q_0^2)^2(|u_0|^2 + 2q_0^2)\, dx.$$

This implies (2.8).

2.4.3 Conservation of momentum

Now, we prove (2.9). Multiplying both sides of (2.1) by $-\partial\overline{u}$ and taking real part we obtain

$$\begin{aligned} 0 &= -\mathcal{Re}(iu_t\partial\overline{u} + \partial^2 u\partial\overline{u} + iu^2(\partial\overline{u})^2) \\ &= \mathcal{Im}(u_t\partial\overline{u}) + \mathcal{Im}(u^2(\partial\overline{u})^2) - \frac{1}{2}\partial(|\partial u|^2). \end{aligned} \tag{2.75}$$

Moreover, by an elementary calculation, we have

$$\partial_t \mathcal{Im}(u\partial\overline{u}) = 2\mathcal{Im}(u_t\partial\overline{u}) + \partial\mathcal{Im}(u\partial_t\overline{u}).$$

Replacing $\mathcal{Im}(u_t\partial\overline{u}) = \frac{1}{2}\left(\partial_t\mathcal{Im}(u\partial\overline{u}) - \partial\mathcal{Im}(u\partial_t\overline{u})\right)$ in (2.75), we obtain that

$$\begin{aligned} 0 &= \left(\frac{1}{2}\partial_t\mathcal{Im}(u\partial\overline{u}) - \frac{1}{2}\partial\mathcal{Im}(u\partial_t\overline{u})\right) + 2\mathcal{Re}(u\partial\overline{u})\mathcal{Im}(u\partial\overline{u}) - \frac{1}{2}\partial(|\partial u|^2) \\ &= \partial_t\left[\frac{1}{2}\mathcal{Im}(u\partial\overline{u}) - \frac{1}{4}(|u|^4 - q_0^4)\right] + \partial\left[\mathcal{Im}(|u|^2 u\partial\overline{u}) - \frac{1}{2}|\partial u|^2 - \frac{1}{6}(|u|^6 - q_0^6)\right]. \end{aligned}$$

Multiply both sides by χ_R, integrating on space and integrating by part, we have

$$\begin{aligned} 0 &= \partial_t\int_{\mathbb{R}}\left[\frac{1}{2}\mathcal{Im}(u\partial\overline{u}) - \frac{1}{4}(|u|^4 - q_0^4)\right]\chi_R dx - \int_{\mathbb{R}}\left[\mathcal{Im}(|u|^2 u\partial\overline{u}) - \frac{1}{2}|\partial u|^2 - \frac{1}{6}(|u|^6 - q_0^6)\right]\chi_R' dx \\ &= \partial_t\int_{\mathbb{R}}\left[\frac{1}{2}\mathcal{Im}(u\partial\overline{u}) - \frac{1}{4}(|u|^2 - q_0^2)^2 - \frac{1}{2}q_0^2(|u|^2 - q_0^2)\right]\chi_R dx \\ &\quad - \int_{\mathbb{R}}\left[\mathcal{Im}(|u|^2 u\partial\overline{u}) - \frac{1}{2}|\partial u|^2 - \frac{1}{6}(|u|^6 - q_0^6)\right]\chi_R' dx. \end{aligned} \tag{2.76}$$

Denoting the second term of (2.76) by D_R, we have

$$|D_R| \leqslant \|\mathcal{Im}(|u|^2 u\partial\overline{u}) - \frac{1}{2}|\partial u|^2 - \frac{1}{6}(|u|^6 - q_0^6)\|_{L^2}\|\chi_R'\|_{L^2} \lesssim C(M)\|\chi_R'\|_{L^2} \to 0 \tag{2.77}$$

as $R \to \infty$. Integrating from 0 to t the two sides of (2.76) and taking R to infinity, using (2.77) and (2.53), we have

$$\int_{\mathbb{R}}\left[\frac{1}{2}\mathcal{Im}(u\partial\overline{u}) - \frac{1}{4}(|u|^2 - q_0^2)^2\right] dx = \int_{\mathbb{R}}\left[\frac{1}{2}\mathcal{Im}(u_0\partial\overline{u_0}) - \frac{1}{4}(|u_0|^2 - q_0^2)^2\right] dx. \tag{2.78}$$

We thus obtain the conservation of momentum, which completes the proof of Theorem 2.4.

2.5 Stationary solutions

In this section, we give the proof of Theorem 2.7. To convenience for readers, we first introduce a fundamental lemma which is a classical version of the Cauchy-Lipschitz theorem:

Lemma 2.18. *Let $C_1, C_2 \in \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. There exists a unique real valued C^2 local solution of following equation*

$$\begin{cases} u_{xx} = f(u), \\ u(0) = C_1, \\ u_x(0) = C_2. \end{cases} \tag{2.79}$$

Remark 2.19. Let $C_1, C_2 \in \mathbb{C}$ and f be considered as C^1 function from $\mathbb{R}^2$ to $\mathbb{R}^2$. By using Picard's uniqueness and existence theorem for system equations, we obtain the existence and uniqueness of complex valued solution for (2.79). However, the Lemma 2.18 is sufficient for our analysis in this paper.

Now, we give the proof of Theorem 2.7. We use the similar of variable changing as in [94, Proof of Proposition 1.1].

Proof of Theorem 2.7. Let ϕ be a nonconstant solution of (2.10) such that $m = \inf_{x\in\mathbb{R}} |\phi(x)| > 0$. From (2.10), we have $\phi \in X^3(\mathbb{R})$. Using the assumptions on ϕ we can write ϕ as

$$\phi(x) = R(x)e^{i\theta(x)}$$

where $R > 0$ and $R, \theta \in \mathcal{C}^2(\mathbb{R})$ are real-valued functions. We have

$$\begin{aligned} \phi_x &= e^{i\theta}(R_x + i\theta_x R), \\ \phi_{xx} &= e^{i\theta}(R_{xx} + 2iR_x\theta_x + iR\theta_{xx} - R\theta_x^2). \end{aligned}$$

Hence, since ϕ satisfies (2.10) we obtain

$$0 = (R_{xx} - R\theta_x^2 + R^3\theta_x) + i(2R_x\theta_x + R\theta_{xx} + R^2R_x).$$

This is equivalent to

$$0 = R_{xx} - R\theta_x^2 + R^3\theta_x, \tag{2.80}$$

$$0 = 2R_x\theta_x + R\theta_{xx} + R^2R_x. \tag{2.81}$$

The equation (2.81) is equivalent to

$$0 = \partial_x\left(R^2\theta_x + \frac{1}{4}R^4\right).$$

Hence there exists $B \in \mathbb{R}$ such that

$$B = R^2\theta_x + \frac{1}{4}R^4. \tag{2.82}$$

This implies

$$\theta_x = \frac{B}{R^2} - \frac{R^2}{4}. \tag{2.83}$$

Substituting the above equality in (2.80) we obtain

$$\begin{aligned} 0 &= R_{xx} - R\left(\frac{B}{R^2} - \frac{R^2}{4}\right)^2 + R^3\left(\frac{B}{R^2} - \frac{R^2}{4}\right) \\ &= R_{xx} - \frac{B^2}{R^3} - \frac{5R^5}{16} + \frac{3BR}{2}. \end{aligned} \tag{2.84}$$

We prove that the set $V = \{x \in \mathbb{R} : R_x(x) \neq 0\}$ is dense in $\mathbb{R}$. Indeed, assume there exists $x \in \mathbb{R} \setminus \overline{V}$. Thus, there exists ε such that $B(x, \varepsilon) \in \mathbb{R} \setminus \overline{V}$. It implies that for all $y \in B(x, \varepsilon)$, we have $R_x(y) = 0$ so $R \equiv C_0$ on $B(x, \varepsilon)$ for some constant C_0. Let $x_0 \in B(x, \varepsilon)$ then $R(x_0) = C_0$ and $R_x(x_0) = 0$. By Lemma 2.18, $R \equiv C_0$. By (2.83), θ_x is constant. Thus, $\phi(x)$ is of form $Ce^{i\alpha x}$, for some constants $C, \alpha \in \mathbb{R}$. If $\alpha = 0$, ϕ is a constant and if $\alpha \neq 0$ ϕ is not in $X^1(\mathbb{R})$, which contradicts the assumption of ϕ. From (2.84), we have

$$\begin{aligned} 0 &= R_x\left(R_{xx} - \frac{B^2}{R^3} - \frac{5R^5}{16} + \frac{3BR}{2}\right) \\ &= \frac{d}{dx}\left[\frac{1}{2}R_x^2 + \frac{B^2}{2R^2} - \frac{5}{96}R^6 + \frac{3B}{4}R^2\right]. \end{aligned}$$

Hence there exists $a \in \mathbb{R}$ such that

$$a = \frac{1}{2}R_x^2 + \frac{B^2}{2R^2} - \frac{5}{96}R^6 + \frac{3B}{4}R^2.$$

This is equivalent to

$$\begin{aligned} 0 &= R_x^2 R^2 + B^2 - \frac{5}{48}R^8 + \frac{3B}{2}R^4 - 2aR^2 \\ &= \frac{1}{4}[(R^2)_x]^2 + B^2 - \frac{5}{48}R^8 + \frac{3B}{2}R^4 - 2aR^2. \end{aligned}$$

Set $k = R^2$. We have

$$0 = \frac{1}{4}k_x^2 + B^2 - \frac{5}{48}k^4 + \frac{3B}{2}k^2 - 2ak. \tag{2.85}$$

Differentiating the two sides of (2.85) we have

$$0 = k_x\left(\frac{k_{xx}}{2} - \frac{5}{12}k^3 + 3Bk - 2a\right)$$

On the other hand, since $k_x = 2R_xR \neq 0$ for a.e x in $\mathbb{R}$, we obtain the following equation for a.e x in $\mathbb{R}$, hence, by continuity of k, it is true for all x in $\mathbb{R}$:

$$0 = \frac{k_{xx}}{2} - \frac{5}{12}k^3 + 3Bk - 2a. \tag{2.86}$$

Now, using Lemma 2.20 we have $k - 2\sqrt{B} \in H^3(\mathbb{R})$. Combining with (2.86) we obtain $a = \frac{4B\sqrt{B}}{3}$. Set $h = k - 2\sqrt{B}$. Then from (2.86) $h \in H^3(\mathbb{R})$ solves

$$\begin{cases} 0 = h_{xx} - \frac{5}{6}h^3 - 5\sqrt{B}h^2 - 4Bh, \\ h > -2\sqrt{B}, \end{cases} \tag{2.87}$$

Since $h \in H^3(\mathbb{R})$, there exists $x_0 \in \mathbb{R}$ such that $h_x(x_0) = 0$. Indeed, if h_x does not change sign on $\mathbb{R}$ then $|h(-\infty)| > 0$ or $|h(\infty)| > 0$. This contradicts to $h \in H^3(\mathbb{R})$. Multiplying both sides of (2.87) by h_x we obtain

$$0 = \frac{1}{2}\partial_x(h_x^2) - \frac{5}{24}\partial_x(h^4) - \frac{5\sqrt{B}}{3}\partial_x(h^3) - 2B\partial_x(h^2).$$

Since $h \in H^3(\mathbb{R})$ we have $h(\infty) = h_x(\infty) = 0$ and hence,

$$\frac{1}{2}(h_x)^2 = \frac{5}{24}h^4 + \frac{5\sqrt{B}}{3}h^3 + 2Bh^2. \tag{2.88}$$

Using $h_x(x_0) = 0$, since (2.88), we have $h(x_0) = 0$ or $h(x_0) = \frac{4}{5}(-5 \pm \sqrt{10})\sqrt{B}$. If $h(x_0) = 0$ then by using Lemma 2.18, we have $h \equiv 0$, this is a contradiction. Since $h > -2\sqrt{B}$, we obtain $h(x_0) = \frac{4}{5}(-5 + \sqrt{10})\sqrt{B}$. Define $v(x) = h(x + x_0)$. We have

$$\begin{cases} 0 = v_{xx} - \frac{5}{6}v^3 - 5\sqrt{B}v^2 - 4Bv, \\ v(0) = \frac{4}{5}(-5 + \sqrt{10})\sqrt{B}, \\ v_x(0) = 0. \end{cases} \tag{2.89}$$

Using Lemma 2.18, there exists a unique solution v of (2.89). Moreover, we can check that the following function is a solution of (2.89):

$$v(x) = \frac{-1}{\sqrt{\frac{5}{72B}\cosh(2\sqrt{B}x) + \frac{5}{12\sqrt{B}}}}.$$

Hence,

$$h(x) = \frac{-1}{\sqrt{\frac{5}{72B}\cosh(2\sqrt{B}(x - x_0)) + \frac{5}{12\sqrt{B}}}}$$

This implies

$$k = 2\sqrt{B} + h = 2\sqrt{B} + \frac{-1}{\sqrt{\frac{5}{72B}\cosh(2\sqrt{B}(x - x_0)) + \frac{5}{12\sqrt{B}}}}.$$

Furthermore, using $\theta_x = \frac{B}{k} - \frac{k}{4}$, there exists $\theta_0 \in \mathbb{R}$ such that

$$\theta(x) = \theta_0 - \int_x^\infty \left(\frac{B}{k} - \frac{k}{4}\right) dy.$$

Now, assume that ϕ is a solution of (2.10) such that $\phi(\infty) = 0$. We prove $\phi \equiv 0$ on $\mathbb{R}$. Multiplying both sides of (2.10) by $\overline{\phi}$ then taking the imaginary part we obtain

$$\partial_x \mathcal{Im}(\phi_x\overline{\phi}) + \frac{1}{4}\partial_x(|\phi|^4) = 0$$

On the other hand, $\phi(\infty)=\phi_x(\infty)=0$ then on $\mathbb{R}$ we have

$$\mathcal{Im}(\phi_x\overline{\phi})+\frac{1}{4}|\phi|^4=0. \tag{2.90}$$

If there exists y_0 such that $\phi_x(y_0)=0$ then from (2.90) we have $\phi(y_0)=0$. By the uniqueness of Cauchy problem we obtain $\phi\equiv 0$ on $\mathbb{R}$. Otherwise, ϕ_x does not vanish on $\mathbb{R}$. From now on, we will consider this case. Multiplying both sides of (2.10) by $\overline{\phi_x}$ then taking the real part, we have

$$\begin{aligned}
0 &= \mathcal{Re}(\phi_{xx}\overline{\phi_x})-\mathcal{Im}(\phi^2\overline{\phi_x}^2)\\
&= \frac{1}{2}\frac{d}{dx}|\phi_x|^2-2\mathcal{Re}(\phi\overline{\phi_x})\mathcal{Im}(\phi\overline{\phi_x})\\
&= \frac{1}{2}\frac{d}{dx}|\phi_x|^2-\partial_x(|\phi|^2)\frac{1}{4}|\phi|^4\\
&= \frac{d}{dx}\left(\frac{1}{2}|\phi_x|^2-\frac{1}{12}|\phi|^6\right).
\end{aligned}$$

This implies that

$$|\phi_x|^2-\frac{1}{6}|\phi|^6=0.$$

Hence, since ϕ_x is non vanishing, ϕ is also non vanishing on $\mathbb{R}$. We can write $\phi=\rho e^{i\theta}$ for $\rho>0$, $\rho,\theta\in\mathcal{C}^2(\mathbb{R})$. Similar to (2.80) we have

$$0=-\rho\theta_x^2+\rho_{xx}+\rho^3\theta_x. \tag{2.91}$$

Replacing $\phi=\rho e^{i\theta}$ in (2.90) we have

$$0=\rho^2\theta_x+\frac{1}{4}\rho^4.$$

Then $\theta_x=\frac{-1}{4}\rho^2$, replacing this equality in (2.91) we obtain

$$0=\rho_{xx}-\frac{5}{16}\rho^5.$$

Multiplying both sides of the above equality by ρ_x we obtain

$$0=\rho_{xx}\rho_x-\frac{5}{16}\rho^5\rho_x=\frac{d}{dx}\left(\frac{1}{2}\rho_x^2-\frac{5}{96}\rho^6\right).$$

Hence,

$$0=\rho_x^2-\frac{5}{48}\rho^6.$$

Moreover, ϕ is non vanishing on $\mathbb{R}$ then $\rho>0$ and then ρ_x is not change sign on $\mathbb{R}$. If $\rho_x>0$ then since $\rho(\infty)=0$ we have $\rho<0$ on $\mathbb{R}$, a contradiction. Hence, $\rho_x<0$ and $\rho_x=-\sqrt{\frac{5}{48}}\rho^3$. From this we easily check that

$$\rho^2(x)=\frac{1}{\rho(0)^2+\sqrt{5/12}\,x},$$

which implies the contradiction, for the right hand side is not a continuous function on $\mathbb{R}$. This completes the proof. □

Lemma 2.20. *Let $B > 0$ be the constant given as the above. The following is true:*

$$k - 2\sqrt{B} \in L^2(\mathbb{R}), \quad k \in X^3(\mathbb{R}).$$

Proof. Using $\phi \in L^\infty(\mathbb{R})$ we obtain $k \in L^\infty(\mathbb{R})$. On the other hand, since $\phi \in X^3(\mathbb{R})$, we have $\phi_x \in L^2(\mathbb{R})$, $\phi_{xx} \in L^2(\mathbb{R})$ and it easy to see that

$$|\phi_x|^2 = \frac{k_x^2}{4k} + k\theta_x^2 \in L^1(\mathbb{R}),$$
$$|\phi_{xx}|^2 = \left|\frac{k_x\theta_x}{\sqrt{k}} + \theta_{xx}\sqrt{k}\right|^2 + \left|\frac{k_{xx}}{2\sqrt{k}} - \sqrt{k}\theta_x^2 - \frac{k_x^2}{4k\sqrt{k}}\right|^2 \in L^1(\mathbb{R}).$$

This implies

$$\frac{k_x}{2\sqrt{k}} \in L^2(\mathbb{R}) \text{ and } \sqrt{k}\theta_x \in L^2(\mathbb{R})$$
$$\frac{k_x\theta_x}{\sqrt{k}} + \theta_{xx}\sqrt{k} \in L^2(\mathbb{R}) \text{ and } \frac{k_{xx}}{2\sqrt{k}} - \sqrt{k}\theta_x^2 - \frac{k_x^2}{4k\sqrt{k}} \in L^2(\mathbb{R}).$$

Using $\sqrt{m} < k < \|k\|_{L^\infty}$, $\theta_x = \frac{4B-k^2}{4k} \in L^\infty(\mathbb{R})$, $k_x = 2RR_x \in L^\infty($ indeed $|\phi_x|^2 = |R_x|^2 + |R\theta_x|^2 \in L^\infty(\mathbb{R}))$ we have

$$k_x \in L^2 \text{ and } \theta_x \in L^2,$$
$$\theta_{xx} \in L^2 \text{ and } k_{xx} \in L^2.$$

By using $\theta_x = \frac{4B-k^2}{4k} \in L^2(\mathbb{R})$, we have $4B - k^2 \in L^2(\mathbb{R})$. Thus, $B \geqslant 0$ and $2\sqrt{B} - k \in L^2(\mathbb{R})$. If $B = 0$ then $k \in L^2(\mathbb{R})$, hence, $R \in L^2(\mathbb{R})$. Which contradicts to the assumption $m > 0$. Thus, $B > 0$. It remains to prove that $k_{xxx} \in L^2(\mathbb{R})$. Indeed, from $\phi_{xxx} \in L^2(\mathbb{R})$ we have

$$|\phi_{xxx}|^2 = |\theta_{xxx}\sqrt{k} + \mathcal{M}|^2 + \left|\frac{k_{xxx}}{2\sqrt{k}} + \mathcal{N}\right|^2 \in L^1(\mathbb{R}) \tag{2.92}$$

where $\mathcal{M}, \mathcal{N}$ are functions of $\theta, \theta_x, \theta_{xx}, k, k_x, k_{xx}$. We can easily check that $\mathcal{M}, \mathcal{N} \in L^2(\mathbb{R})$. Hence, from (2.92) and the facts that $\theta_x \in H^1(\mathbb{R})$, $k \in X^2(\mathbb{R})$, k bounded from below we obtain $\theta_{xxx}, k_{xxx} \in L^2(\mathbb{R})$. This implies the desired results. □

From now on, we will denote ϕ_B is the stationary solution of (2.10) given by Theorem 2.7 with $\theta_0 = 0$. We have

$$\phi_B = e^{i\theta}\sqrt{k}\ , \tag{2.93}$$
$$k(x) = 2\sqrt{B} + \frac{-1}{\sqrt{\frac{5}{72B}\cosh(2\sqrt{B}x) + \frac{5}{12\sqrt{B}}}}\ , \tag{2.94}$$
$$\theta(x) = -\int_x^\infty \frac{B}{k(y)} - \frac{k(y)}{4}\ dy. \tag{2.95}$$

We have the following asymptotic properties for ϕ_B at ∞.

Proposition 2.21. *Let $B > 0$ and ϕ_B be kink solution of* (2.1)*. Then for $x > 0$, we have*

$$|\phi_B - \sqrt{2\sqrt{B}}| \lesssim e^{-\sqrt{B}x}.$$

As consequence ϕ_B converges to $\sqrt{2\sqrt{B}}$ as x tends to ∞ and there exists limit of ϕ_B as x tends to $-\infty$.

Proof. Since (2.94) we have

$$|k - 2\sqrt{B}| \lesssim e^{-2\sqrt{B}x}.$$

Hence, for all $x \in \mathbb{R}$ we have

$$\begin{aligned} |\phi_B(x) - \sqrt{2\sqrt{B}}| &\lesssim |e^{i\theta(x)}\sqrt{k(x)} - \sqrt{k(x)}| + |\sqrt{k(x)} - \sqrt{2\sqrt{B}}| && (2.96)\\ &\lesssim \|k\|_{L^\infty}^{\frac{1}{2}}|e^{i\theta(x)} - 1| + e^{-\sqrt{B}x} && (2.97) \end{aligned}$$

Moreover, for $x > 0$, we have

$$\begin{aligned} |e^{i\theta(x)} - 1| \leqslant |\theta(x)| &\leqslant \int_x^\infty \left|\frac{B}{k} - \frac{k}{4}\right| dx \\ &\leqslant \int_x^\infty \left|\frac{B}{k} - \frac{\sqrt{B}}{2}\right| + \left|\frac{\sqrt{B}}{2} - \frac{k}{4}\right| dx \\ &\lesssim \int_x^\infty \left|k - 2\sqrt{B}\right| dx \lesssim \int_x^\infty e^{-2\sqrt{B}x}\, dx \lesssim e^{-2\sqrt{B}x}. \end{aligned}$$

Combining with (2.97) we obtain

$$|\phi_B(x) - \sqrt{2\sqrt{B}}| \lesssim e^{-\sqrt{B}x}.$$

As consequence ϕ_B converges to $\sqrt{2\sqrt{B}}$ as x tends to ∞. Since (2.94), we have $|k - 2\sqrt{B}| \in L^1(\mathbb{R})$ and $k > \left(2 - \frac{1}{\frac{5}{12}+\sqrt{\frac{5}{72}}}\right)\sqrt{B}$. Thus, $\frac{B}{k} - \frac{k}{4} = \frac{4B-k^2}{4k} \in L^1(\mathbb{R})$. Hence since (2.95) we have

$$\lim_{x\to-\infty} \theta(x) = -\int_{-\infty}^\infty \left(\frac{B}{k(y)} - \frac{k(y)}{4}\right) dy.$$

Hence,

$$\lim_{x\to-\infty} \phi_B(x) = \exp\left(-i\int_{-\infty}^\infty \left(\frac{B}{k(y)} - \frac{k(y)}{4}\right) dy\right)\sqrt{2\sqrt{B}}.$$

This completes the proof. □

Chapter 3

On the derivative nonlinear Schrödinger equation on the half line with Robin boundary condition

3.1 Introduction

In this chapter, we consider the derivative nonlinear Schrödinger equation on $[0,+\infty)$ with Robin boundary condition at 0:

$$\begin{cases} iv_t + v_{xx} = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2\overline{v_x} - \frac{3}{16}|v|^4 v \quad \text{for } x \in \mathbb{R}_+, \\ v(0) = \varphi, \\ \partial_x v(t,0) = \alpha v(t,0) \quad \forall t \in \mathbb{R}, \end{cases} \tag{3.1}$$

where $\alpha \in \mathbb{R}$ is a given constant.

The linear parts of (3.1) can be rewritten in the following forms:

$$\begin{cases} iv_t + \widetilde{H}_\alpha v = 0 \text{ for } x \in \mathbb{R}_+, \\ v(0) = \varphi, \end{cases} \tag{3.2}$$

where $\widetilde{H}_\alpha$ are self-adjoint operators defined by

$$\begin{aligned} &\widetilde{H}_\alpha : D(\widetilde{H}_\alpha) \subset L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), \\ &\widetilde{H}_\alpha u = u_{xx}, \quad D(\widetilde{H}_\alpha) = \left\{ u \in H^2(\mathbb{R}_+) : u_x(0^+) = \alpha u(0^+)) \right\}. \end{aligned}$$

We call $e^{i\widetilde{H}_\alpha t} : \mathbb{R} \to \mathcal{L}(L^2(\mathbb{R}^+))$ is group defining the solution of (3.2).

The derivative nonlinear Schrödinger equation was originally introduced in Plasma Physics as a simplified model for Alfvén wave propagation. Since then, it has attracted a lot of attention from the mathematical community (see e.g [24, 25, 57, 58, 60, 68, 108, 109]).

Consider the equation (3.1), and set

$$u(t,x) = \exp\left(\frac{3i}{4}\int_\infty^x |v(t,y)|^2\,dy\right) v(t,x).$$

Using the Gauge transformation, we see that u solves

$$iu_t + u_{xx} = i\partial_x(|u|^2 u), \quad t \in \mathbb{R},\ x \in (0,\infty), \tag{3.3}$$

under a boundary condition $\partial_x u(t,0) = \alpha u(t,0) + \frac{3i}{4}|u(t,0)|^2 u(t,0)$. In all line case, there are many papers to deal with Cauchy problem of (3.3) (see e.g [59, 111, 112]). In [59], the authors establish the local well posedness in $H^1(\mathbb{R})$ by using a Gauge transform. Indeed, since u solves (3.3) on $\mathbb{R}$, by setting

$$\begin{aligned} h(t,x) &= \exp\left(-i\int_{-\infty}^{x} |u(t,y)|^2\,dy\right) u(t,x),\\ k &= h_x + \frac{i}{2}|h|^2 h, \end{aligned} \tag{3.4}$$

we have h, k solve

$$\begin{cases} ih_t + h_{xx} = -ih^2\overline{k},\\ ik_t + k_{xx} = ik^2\overline{h}. \end{cases} \tag{3.5}$$

By classical arguments, we can prove that there exists a unique solution $h, k \in C([0,T], L^2(\mathbb{R})) \cap L^4([0,T], L^\infty(\mathbb{R}))$ given $h_0, k_0 \in L^2(\mathbb{R})$ are satisfy (3.4). To obtain the existence solution of (3.1), the authors prove that the relation (3.4) satisfies for all $t \in [0,T]$. Thus, if we set

$$u(t,x) = \exp\left(i\int_{-\infty}^{x} |h(t,y)|^2\,dy\right) h(t,x),$$

then $u \in C([0,T], H^1(\mathbb{R}))$ solves (3.1). In [1], the authors have proved the global well posedness of (3.3) given initial data in $H^{\frac{1}{2}}(\mathbb{R})$. In half line case, [116] Wu prove existence of blow up solution of (3.3) under Dirichlet boundary condition, given initial data in $\Sigma := \{u_0 \in H^2(\mathbb{R}^+), xu_0 \in L^2(\mathbb{R}^+)\}$. In this paper, we give a proof of existence of blow up solution of (3.1) under Robin boundary condition.

To study equation (3.1), we start by the definition of solution on $H^1(\mathbb{R}^+)$. Since (3.1) contains a Robin boundary condition, the notion of solution in $H^1(\mathbb{R}^+)$ is not completely clear. We use the following definition. Let I be an open interval of $\mathbb{R}$. We say that v is a $H^1(\mathbb{R}^+)$ solution of the problem (3.1) on I if $v \in C(I, H^1(\mathbb{R}^+))$ satisfies the following equation

$$v(t) = e^{i\widetilde{H}_\alpha t}\varphi - i\int_0^t e^{i\widetilde{H}_\alpha (t-s)} g(v(s))\,ds, \tag{3.6}$$

where g is the function defined by

$$g(v) = \frac{i}{2}|v|^2 v_x - \frac{i}{2}v^2\overline{v}_x - \frac{3}{16}|v|^4 v.$$

Let $v \in C(I, D(\widetilde{H}_\alpha))$ be classical solution of (3.1). At least formally, we have

$$\frac{1}{2}\partial_t(|v|^2) = -\partial_x \mathcal{I}m(v_x\overline{v}).$$

Therefore, using the Robin boundary condition we have

$$\begin{aligned}\partial_t\left(\frac{1}{2}\int_0^\infty |v|^2\,dx\right) &= -\mathcal{Im}(v_x\overline{v})(\infty)+\mathcal{Im}(v_x\overline{v})(0)\\ &= \mathcal{Im}(v_x\overline{v})(0)\\ &= \alpha\mathcal{Im}(|v(0)|^2)\\ &= 0.\end{aligned}$$

This implies the conservation of the mass. By an elementary calculation, we have

$$\partial_t\left(|v_x|^2-\frac{1}{16}|v|^6\right)=\partial_x\left(2\mathcal{Re}(v_x\overline{v}_t)-\frac{1}{2}|v|^2|v_x|^2+\frac{1}{2}v^2\overline{v}_x^2\right).$$

Hence, integrating the two sides in space, we obtain

$$\partial_t\left(\int_{\mathbb{R}_+}|v_x|^2\,dx-\frac{1}{16}|v|^6\,dx\right)=-2\mathcal{Re}(v_x(0)\overline{v}_t(0))+\frac{1}{2}|v(0)|^2|v_x(0)|^2-\frac{1}{2}v(0)^2\overline{v_x(0)}^2$$

Using the Robin boundary condition for v, we obtain

$$\partial_t\left(\int_{\mathbb{R}_+}|v_x|^2\,dx-\frac{1}{16}|v|^6\,dx\right)=-2\alpha\mathcal{Re}(v(0)\overline{v_t(0)})=-\alpha\partial_t(|v(0)|^2).$$

This implies the conservation of the energy.

In this paper, we will need the following assumption.

Assumption A. We assume that for all $\varphi\in H^1(\mathbb{R}^+)$ there exist a solution $v\in C(I,H^1(\mathbb{R}^+))$ of (3.1) for some interval $I\subset\mathbb{R}$. Moreover, v satisfies the following conservation law:

$$\begin{aligned}M(v) &:= \frac{1}{2}\|v\|^2_{H^1(\mathbb{R}^+)}=M(\varphi),\\ E(v) &:= \frac{1}{2}\|v_x\|^2_{L^2(\mathbb{R}^+)}-\frac{1}{32}\|v\|_{L^6(\mathbb{R}^+)}+\frac{\alpha}{2}|v(0)|^2.\end{aligned}$$

The existence of blowing up solutions for classical nonlinear Schrödinger equations was considered by Glassey [45] in 1977. He introduced a concavity argument based on the second derivative in time of $\|xu(t)\|^2_{L^2}$ to show the existence of blowing up solutions. In this paper, we are also interested in studying the existence of blowing-up solutions of (3.1). In the limit case $\alpha=+\infty$, which is formally equivalent to Dirichlet boundary condition if we write $v(0)=\frac{1}{\alpha}v'(0)=0$. In [116], Wu proved the blow up in finite time of solutions of (3.3) with Dirichlet boundary condition and some conditions on the initial data. Using the method of Wu [116] we obtain the existence of blowing up solutions in the case $\alpha\geqslant 0$, under a weighted space condition for the initial data and negativity of the energy. Our first main result is the following.

Theorem 3.1. *We assume that Assumption A holds. Let $\alpha\geqslant 0$ and $\varphi\in\Sigma$ where*

$$\Sigma=\left\{u\in D(\widetilde{H}_\alpha),xu\in L^2(R_+))\right\}$$

such that $E(\varphi)<0$. Then the solution v of (3.1) blows-up in finite time i.e $T_{min}>-\infty$ and $T_{max}<+\infty$.

Remark 3.2. In (3.1), if we consider nonlinear term $i|v|^2v_x$ instead of $\frac{i}{2}|v|^2v_x - \frac{i}{2}v^2\overline{v_x} - \frac{3}{16}|v|^4v$ then there is no conservation of energy of solution. Indeed, set

$$u(t,x) = v(t,x)\exp\left(-\frac{i}{4}\int_{\infty}^{x}|v(t,y)|^2\,dy\right).$$

If v solves

$$\begin{cases} iv_t + v_{xx} = i|v|^2v_x, \\ \partial_x v(t,0) = \alpha v(t,0) \end{cases}$$

then u solves

$$\begin{cases} iu_t + u_{xx} = \frac{i}{2}|u|^2u_x - \frac{i}{2}u^2\overline{u_x} - \frac{3}{16}|u|^4u, \\ \partial_x u(t,0) = \alpha u(t,0) - \frac{i}{4}|u(t,0)|^2u(t,0). \end{cases} \tag{3.7}$$

By an elementary calculation, since u solves (3.7), we have

$$\partial_t\left(|u_x|^2 - \frac{1}{16}|u|^6\right) = \partial_x\left(2\mathcal{Re}(u_x\overline{u_t}) - \frac{1}{2}|u|^2|u_x|^2 + \frac{1}{2}u^2\overline{u_x}^2\right).$$

Integrating the two sides in space, we obtain

$$\partial_t\left(\int_{\mathbb{R}^+}|u_x|^2 - \frac{1}{16}|u|^6\,dx\right) = -2\mathcal{Re}(u_x(0)\overline{u_t}(0)) + \frac{1}{2}|u(0)|^2|u_x(0)|^2 - \frac{1}{2}u(0)^2\overline{u_x(0)}^2.$$

Using the boundary condition of u, we obtain

$$\begin{aligned} \partial_t\left(\int_{\mathbb{R}^+}|u_x|^2 - \frac{1}{16}|u|^6\,dx\right) &= -2\alpha\mathcal{Re}(u(0)\overline{u_t(0)}) - \frac{1}{2}\mathcal{Im}(u(0)|u(0)|^2\overline{u_t(0)}) \\ &\quad + \frac{1}{2}|u(0)|^4\left(\alpha^2 + \frac{1}{16}|u(0)|^4 - \left(\alpha + \frac{i}{4}|u(0)|^2\right)^2\right) \\ &= -\alpha\partial_t(|u(0)|^2) + A, \end{aligned}$$

where $A = -\frac{1}{2}\mathcal{Im}(u(0)|u(0)|^2\overline{u_t(0)}) + \frac{1}{2}|u(0)|^4\left(\alpha^2 + \frac{1}{16}|u(0)|^4 - \left(\alpha + \frac{i}{4}|u(0)|^2\right)^2\right)$. Moreover, we can not write A in form $\partial_t B(u(0))$, for some function $B : \mathbb{C} \to \mathbb{C}$. Then, there is no conservation of energy of u and hence, there is no conservation of energy of v.

The stability of standing waves for classical nonlinear Schrödinger equations was originally studied by Cazenave and Lions [18] with variational and compactness arguments. A second approach, based on spectral arguments, was introduced by Weinstein [114, 115] and then considerably generalized by Grillakis, Shatah and Strauss [49, 50] (see also [31], [32]). In our work, we use the variational techniques to study the stability of standing waves. First, we define

$$\begin{aligned} S_\omega(v) &:= \frac{1}{2}\left[\|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2\right] - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R}^+)}, \\ K_\omega(v) &:= \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2 - \frac{3}{16}\|v\|^6_{L^6(\mathbb{R}^+)}. \end{aligned}$$

We are interested in the following variational problem:

$$d(\omega) := \inf\left\{S_\omega(v) \mid K_\omega(v) = 0, v \in H^1(\mathbb{R}^+) \setminus \{0\}\right\}. \tag{3.8}$$

We have the following result.

Proposition 3.3. *Let $\omega, \alpha \in \mathbb{R}$ such that $\omega > \alpha^2$. All minimizers of* (3.8) *are of form $e^{i\theta}\varphi$, where $\theta \in \mathbb{R}$ and φ is given by*

$$\varphi = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}x + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right).$$

We give the definition of stability and instability by blow up in $H^1(\mathbb{R}^+)$. Let $w(t,x) = e^{i\omega t}\varphi(x)$ be a standing wave solution of (3.1).

(1) The standing wave w is called *orbitally stable* in $H^1(\mathbb{R}^+)$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R}^+)$ satisfies

$$\|v_0 - \varphi\|_{H^1(\mathbb{R}^+)} \leqslant \delta,$$

then the associated solution v of (3.1) satisfies

$$\sup_{t\in\mathbb{R}} \inf_{\theta\in\mathbb{R}} \|v(t) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} < \varepsilon.$$

Otherwise, w said to be *unstable.*

(2) The standing wave w is called *unstable by blow up* if there exists a sequence (φ_n) such that $\lim_{n\to\infty} \|\varphi_n - \varphi\|_{H^1(\mathbb{R}^+)} = 0$ and the associated solution v_n of (3.1) blows up in finite time for all n.

Our second main result is the following.

Theorem 3.4. *Let $\alpha, \omega \in \mathbb{R}$ be such that $\omega > \alpha^2$. The standing wave $e^{i\omega t}\varphi$, where φ is the profile as in Proposition 3.3, solution of* (3.1)*, satisfies the following properties.*

(1) If $\alpha < 0$ then the standing wave is orbitally stable in $H^1(\mathbb{R}^+)$.

(2) If $\alpha > 0$ then the standing wave is strongly unstable.

Remark 3.5. To our knowledge, the conservation law play an important role to study the stability of standing waves. However, the existence of conservation of energy is not always true (see remark 3.2). Our work can only extend for the models with nonlinear terms provide the conservation law of solution.

This paper is organized as follows. First, under the assumption of local well posedness in $H^1(\mathbb{R}^+)$, we prove the existence of blowing up solutions using a virial argument Theorem 3.1. In section 3.2.1, we give the proof of Theorem 3.1. Second, in the case $\alpha < 0$, using similar arguments as in [23], we prove the orbital stability of standing waves of (3.1). In the case $\alpha > 0$, using similar arguments as in [71], we prove the instability by blow up of standing waves. The proof of Theorem 3.4 is obtained in Section 3.2.2.

3.2 Proof of the main results

We consider the equation (3.1) and assume that Assumption A holds.

3.2.1 The existence of a blow-up solutions

In this section, we give the proof of Theorem 3.1 using a virial argument (see e.g [45] or [116] for similar arguments). Let $\alpha \geqslant 0$. Let v be a solution of (3.1). To prove the existence of blowing up solutions we use similar arguments as in [116]. Set

$$I(t) = \int_0^\infty x^2 |v(t)|^2 \, dx.$$

Let

$$u(t,x) = v(t,x) \exp\left(-\frac{i}{4} \int_x^{+\infty} |v|^2 \, dy \right) \tag{3.9}$$

be a Gauge transform in $H^1(\mathbb{R}_+)$. Then the problem (3.1) is equivalent with

$$\begin{cases} iu_t + u_{xx} = i|u|^2 u_x, \\ u_x(0) = \alpha u(0) + \frac{i}{4}|u(0)|^2 u(0). \end{cases} \tag{3.10}$$

The equation (3.10) has a simpler nonlinear form, but we pay this simplification with a nonlinear boundary condition. Observe that

$$I(t) = \int_0^\infty x^2 |u(t)|^2 \, dx = \int_0^\infty x^2 |v(t)|^2 \, dx.$$

By a direct calculation, we get

$$\partial_t I(t) = 2\mathcal{Re} \int_0^\infty x^2 \overline{u(t,x)} \partial_t u(t,x) \, dx = 2\mathcal{Re} \int_0^\infty x^2 \overline{u} (iu_{xx} + |u|^2 u_x) \, dx \tag{3.11}$$

$$= 2\mathcal{Im} \int_0^\infty 2x \overline{u} u_x \, dx - \frac{1}{2} \int_0^\infty 2x |u|^4 \, dx \tag{3.12}$$

$$= 4\mathcal{Im} \int_0^\infty x u_x \overline{u} \, dx - \int_0^\infty x |u|^4 \, dx. \tag{3.13}$$

Define

$$J(t) = \mathcal{Im} \int_0^\infty x u_x \overline{u} \, dx.$$

We have

$$\begin{aligned}
\partial_t J(t) &= \int_0^\infty x u_x \overline{u}_t \, dx + \int_0^\infty x \overline{u} u_{xt} \, dx \\
&= -\mathcal{Im} \int_0^\infty x u_t \overline{u}_x \, dx - \mathcal{Im} \int_0^\infty (x\overline{u})_x u_t \, dx \\
&= -2\mathcal{Im} \int_0^\infty x u_t \overline{u}_x \, dx - \mathcal{Im} \int_0^\infty u_t \overline{u} \, dx \\
&= -2\mathcal{Im} \int_0^\infty x \overline{u}_x (i u_{xx} + |u|^2 u_x) \, dx - \mathcal{Im} \int_0^\infty \overline{u} (i u_{xx} + |u|^2 u_x) \, dx \\
&= -2\mathcal{Re} \int_0^\infty x \overline{u}_x u_{xx} \, dx - \mathcal{Re} \int_0^\infty \overline{u} u_{xx} \, dx - \mathcal{Im} \int_0^\infty |u|^2 u_x \overline{u} \, dx \\
&= - \int_0^\infty x \partial_x |u_x|^2 \, dx - \mathcal{Re}(\overline{u} u_x)(+\infty) + \mathcal{Re}(\overline{u} u_x)(0) + \mathcal{Re} \int_0^\infty \overline{u_x} u_x \, dx - \mathcal{Im} \int_0^\infty |u|^2 u_x \overline{u} \, dx \\
&= \int_0^\infty |u_x|^2 \, dx + \mathcal{Re}(\overline{u}(0) u_x(0)) + \int_0^\infty |u_x|^2 \, dx - \mathcal{Im} \int_0^\infty |u|^2 u_x \overline{u} \, dx \\
&= 2 \int_0^\infty |u_x|^2 \, dx - \mathcal{Im} \int_0^\infty |u|^2 u_x \overline{u} \, dx + \mathcal{Re}(\overline{u}(0) u_x(0)).
\end{aligned}$$

Using the Robin boundary condition we have

$$\partial_t J(t) = 2 \int_0^\infty |u_x|^2 \, dx - \mathcal{Im} \int_0^\infty |u|^2 u_x \overline{u} \, dx + \alpha |u(0)|^2.$$

Moreover using the expression of v in term of u given in (3.9), we get

$$\begin{aligned}
\partial_t J(t) &= 2 \int_0^\infty |v_x|^2 \, dx - \frac{1}{8} \int_0^\infty |v|^6 \, dx + \alpha |v(0)|^2 \\
&= 4E(v) - \alpha |v(0)|^2 \leqslant 4E(v) = 4E(\varphi).
\end{aligned}$$

By integrating the two sides of the above inequality in time we have

$$J(t) \leqslant J(0) + 4E(\varphi) t. \tag{3.14}$$

Integrating the two sides of (3.11) in time we have

$$\begin{aligned}
I(t) &= I(0) + 4 \int_0^t J(s) \, ds - \int_0^t \int_0^\infty x |u(s,x)|^4 \, dx \, ds \\
&\leqslant I(0) + 4 \int_0^t J(s) \, ds.
\end{aligned}$$

Using (3.14) we have

$$\begin{aligned}
I(t) &\leqslant I(0) + 4 \int_0^t (J(0) + 4E(\varphi) s) \, ds \\
&\leqslant I(0) + 4J(0) t + 8E(\varphi) t^2.
\end{aligned}$$

From the assumption $E(\varphi) < 0$, there exists a finite time $T_* > 0$ such that $I(T_*) = 0$,

$$I(t) > 0 \text{ for } 0 < t < T_*.$$

Note that

$$\int_0^\infty |\varphi(x)|^2\,dx = \int_0^\infty |v(t,x)|^2\,dx = -2\mathcal{Re}\int_0^\infty xv(t,x)\overline{v_x}(t,x)\,dx$$
$$\leqslant 2\|xv\|_{L^2_x(\mathbb{R}_+)}\|v_x\|_{L^2_x(\mathbb{R}_+)} = 2\sqrt{I(t)}\|v_x\|_{L^2_x(\mathbb{R}_+)}.$$

Then there exists a constant $C = C(\varphi) > 0$ such that

$$\|v_x\|_{L^2_x(\mathbb{R}_+)} \geqslant \frac{C}{2\sqrt{I(t)}} \to +\infty \text{ as } t \to T_*.$$

Then the solution v blows up in finite time in $H^1(\mathbb{R}^+)$. This complete the proof of Theorem 3.1.

3.2.2 Stability and instability of standing waves

In this section, we give the proof of Theorem 3.4 and Proposition 3.3. First, we find the form of the standing waves of (3.1).

Standing waves

Let $v = e^{i\omega t}\varphi$ be a solution of (3.1). Then φ solves

$$\begin{cases} 0 = \varphi_{xx} - \omega\varphi + \frac{1}{2}\mathcal{Im}(\varphi_x\overline{\varphi})\varphi + \frac{3}{16}|\varphi|^4\varphi, & \text{for } x > 0 \\ \varphi_x(0) = \alpha\varphi(0), \\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \tag{3.15}$$

Set

$$A := \omega - \frac{1}{2}\mathcal{Im}(\varphi_x\overline{\varphi}) - \frac{3}{16}|\varphi|^4$$

By writing $\varphi = f + ig$ for f and g real valued functions, for $x > 0$, we have

$$f_{xx} = Af,$$
$$g_{xx} = Ag.$$

Thus,

$$\partial_x(f_xg - g_xf) = f_{xx}g - g_{xx}f = 0 \text{ when } x \neq 0.$$

Hence, by using $f, g \in H^2(\mathbb{R}^+)$, we have

$$f_x(x)g(x) - g_x(x)f(x) = 0 \text{ when } x \neq 0.$$

Then, for all $x \neq 0$, we have

$$\mathcal{Im}(\varphi_x(x)\overline{\varphi(x)}) = g_x(x)f(x) - f_x(x)g(x) = 0,$$

hence, (3.15) is equivalent to

$$\begin{cases} 0 = \varphi_{xx} - \omega\varphi + \frac{3}{16}|\varphi|^4\varphi, & \text{for } x > 0 \\ \varphi_x(0) = \alpha\varphi(0), \\ \varphi \in H^2(\mathbb{R}^+). \end{cases} \tag{3.16}$$

We have the following description of the profile φ.

Proposition 3.6. *Let $\omega > \alpha^2$. There exists a unique (up to phase shift) solution φ of* (3.16)*, which is of the form*

$$\varphi = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}x + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right), \tag{3.17}$$

for all $x > 0$.

Proof. Let w be the even function defined by

$$w(x) = \begin{cases} \varphi(x) \text{ if } x \geqslant 0, \\ \varphi(-x) \text{ if } x \leqslant 0. \end{cases}$$

Then w solves

$$\begin{cases} 0 = -w_{xx} + \omega w - \frac{3}{16}|w|^4 w, \text{ for } x \neq 0, \\ w_x(0^+) - w_x(0^-) = 2\alpha w(0), \\ w \in H^2(\mathbb{R}) \setminus \{0\} \cap H^1(\mathbb{R}). \end{cases} \tag{3.18}$$

Using the results of Fukuizumi and Jeanjean [38], we obtain that

$$w(x) = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right)$$

up to phase shift provided $\omega > \alpha^2$. Hence, for $x > 0$ we have

$$\varphi(x) = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right)$$

up to phase shift. This implies the desired result. $\square$

The variational problems

In this section, we give the proof of Proposition 3.3. First, we introduce another variational problem:

$$\widetilde{d}(\omega) := \inf\left\{\widetilde{S}_\omega(v) \mid v \text{ even}, \widetilde{K}_\omega(v) = 0, v \in H^1(\mathbb{R}) \setminus \{0\}\right\}, \tag{3.19}$$

where $\widetilde{S}_\omega$, $\widetilde{K}_\omega$ are defined for all $v \in H^1(\mathbb{R})$ by

$$\begin{aligned} \widetilde{S}_\omega(v) &:= \frac{1}{2}\left[\|v_x\|^2_{L^2(\mathbb{R})} + \omega\|v\|^2_{L^2(\mathbb{R})} + 2\alpha|v(0)|^2\right] - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R})}, \\ \widetilde{K}_\omega(v) &:= \|v_x\|^2_{L^2(\mathbb{R})} + \omega\|v\|^2_{L^2(\mathbb{R})} + 2\alpha|v(0)|^2 - \frac{3}{16}\|v\|^6_{L^6(\mathbb{R})}. \end{aligned}$$

The functional $\widetilde{K}_\omega$ is called Nehari functional. The following result has proved in [38, 39].

Proposition 3.7. *Let $\omega > \alpha^2$ and φ satisfies*

$$\begin{cases} -\varphi_{xx} + 2\alpha\delta\varphi + \omega\varphi - \frac{3}{16}|\varphi|^4\varphi = 0, \\ \varphi \in H^1(\mathbb{R}) \setminus \{0\}. \end{cases} \tag{3.20}$$

Then, there exists a unique positive solution φ of (3.20)*. This solution is the unique positive minimizer of* (3.19)*. Furthermore, we have an explicit formula for φ*

$$\varphi(x) = 2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right).$$

We have the following relation between the variational problems.

Proposition 3.8. *Let $\omega > \alpha^2$. We have*

$$d(\omega) = \frac{1}{2}\widetilde{d}(\omega).$$

Proof. Assume v is a minimizer of (3.8), define the $H^1(\mathbb{R})$ function w by

$$w(x) = \begin{cases} v(x) \text{ if } x > 0, \\ v(-x) \text{ if } x < 0. \end{cases}$$

The function $w \in H^1(\mathbb{R}) \setminus \{0\}$ verifies

$$\widetilde{S}_\omega(w) = 2S_\omega(v) = 2d(\omega),$$
$$\widetilde{K}_\omega(w) = 2K_\omega(v) = 0.$$

This implies that

$$\widetilde{d}(\omega) \leqslant \widetilde{S}_\omega(w) = 2d(\omega). \tag{3.21}$$

Now, assume v is a minimizer of (3.19). Let w be the restriction of v on $\mathbb{R}^+$, then,

$$K_\omega(w) = \frac{1}{2}\widetilde{K}_\omega(v) = 0.$$

Hence, we obtain

$$\widetilde{d}(\omega) = \widetilde{S}_\omega(v) = 2S_\omega(w) \geqslant 2d(\omega). \tag{3.22}$$

Combining (3.21) and (3.22) we have

$$\widetilde{d}(\omega) = 2d(\omega).$$

This implies the desired result. □

Proof of Proposition 3.3. Let v be a minimizer of (3.8). Define $w(x) \in H^1(\mathbb{R})$ by

$$w(x) = \begin{cases} v(x) \text{ if } x > 0, \\ v(-x) \text{ if } x < 0. \end{cases}$$

Then, w is an even function. Moreover, w satisfies

$$\widetilde{K}_\omega(w) = 2K_\omega(v) = 0,$$
$$\widetilde{S}_\omega(w) = 2S_\omega(v) = 2d(\omega) = \widetilde{d}(\omega).$$

Hence, w is a minimizer of (3.19). From Propositions 3.7, 3.8, w is of the form $e^{i\theta}\varphi$, where $\theta \in \mathbb{R}$ is a constant and φ is of the form

$$2\sqrt[4]{\omega}\,\mathrm{sech}^{\frac{1}{2}}\left(2\sqrt{\omega}|x| + \tanh^{-1}\left(\frac{-\alpha}{\sqrt{\omega}}\right)\right).$$

Hence, $v = w|_{\mathbb{R}^+}$ satisfies

$$v(x) = e^{i\theta}\varphi(x),$$

for $x > 0$. This completes the proof of Proposition 3.3. □

Stability and instability of standing waves

In this section, we give the proof of Theorem 3.4. We use the notations $\widetilde{S}_\omega$ and $\widetilde{K}_\omega$ as in Section 3.2.2. First, we define

$$N(v) := \frac{3}{16}\|v\|^6_{L^6(\mathbb{R}^+)}, \tag{3.23}$$

$$L(v) := \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2. \tag{3.24}$$

We can rewrite S_ω, K_ω as follows

$$\begin{aligned} S_\omega &= \frac{1}{2}L - \frac{1}{6}N, \\ K_\omega &= L - N. \end{aligned}$$

We have the following classical properties of the above functions.

Lemma 3.9. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\omega > \alpha^2$. The following assertions hold.*

(1) There exists a constant $C > 0$ such that

$$L(v) \geqslant C\|v\|^2_{H^1(\mathbb{R}^+)} \quad \forall v \in H^1(\mathbb{R}^+).$$

(2) We have $d(\omega) > 0$.

(3) If $v \in H^1(\mathbb{R}^+)$ satisfies $K_\omega(v) < 0$ then $L(v) > 3d(\omega)$.

Proof. We have

$$\begin{aligned} |v(0)|^2 &= -\int_0^\infty \partial_x(|v(x)|^2)\,dx = -2\mathcal{R}e\int_0^\infty v(x)\overline{v}_x(x)\,dx \\ &\leqslant 2\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)}. \end{aligned}$$

Hence,

$$\begin{aligned} L(v) &= \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \alpha|v(0)|^2 \\ &\geqslant \|v_x\|^2_{L^2(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} - 2|\alpha|\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geqslant C\|v\|^2_{H^1(\mathbb{R}^+)} + (1-C)\|v_x\|^2_{L^2(\mathbb{R}^+)} + (\omega - C)\|v\|^2_{L^2(\mathbb{R}^+)} - 2|\alpha|\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)} \\ &\geqslant C\|v\|^2_{H^1(\mathbb{R}^+)} + (2\sqrt{(1-C)(\omega-C)} - 2|\alpha|)\|v\|_{L^2(\mathbb{R}^+)}\|v_x\|_{L^2(\mathbb{R}^+)}. \end{aligned}$$

From the assumption $\omega > \alpha^2$, we can choose $C \in (0,1)$ such that

$$2\sqrt{(1-C)(\omega-C)} - 2|\alpha| > 0.$$

This implies (1). Now, we prove (2). Let v be an element of $H^1(\mathbb{R}^+)$ satisfying $K_\omega(v) = 0$. We have

$$C\|v\|^2_{H^1(\mathbb{R}^+)} \leqslant L(v) = N(v) \leqslant C_1\|v\|^6_{H^1(\mathbb{R}^+)}.$$

Then,

$$\|v\|^2_{H^1(\mathbb{R}^+)} \geqslant \sqrt[4]{\frac{C}{C_1}}.$$

From the fact that, for v satisfying $K_\omega(v) = 0$, we have $S_\omega(v) = S_\omega(v) - \frac{1}{6}K_\omega(v) = \frac{1}{3}L(v)$, this implies that

$$d(\omega) = \frac{1}{3}\inf\left\{L(v) : v \in H^1(\mathbb{R}^+), K_\omega(v) = 0\right\} \geqslant \frac{C}{3}\sqrt[4]{\frac{C}{C_1}} > 0.$$

Finally, we prove (3). Let $v \in H^1(\mathbb{R}^+)$ satisfying $K_\omega(v) < 0$. Then, there exists $\lambda_1 \in (0,1)$ such that $K_\omega(\lambda_1 v) = \lambda_1^2 L(v) - \lambda_1^6 N(v) = 0$. Since $v \neq 0$, we have $3d(\omega) \leqslant L(\lambda_1 v) = \lambda_1^2 L(v) < L(v)$. □

Define

$$\tilde{N}(v) := \frac{3}{16}\|v\|^6_{L^6}, \tag{3.25}$$

$$\tilde{L}(v) := \|v_x\|^2_{L^2} + \omega\|v\|^2_{L^2} + 2\alpha|v(0)|^2. \tag{3.26}$$

We can rewrite S_ω, K_ω as follows

$$\tilde{S}_\omega = \frac{1}{2}\tilde{L} - \frac{1}{6}\tilde{N},$$

$$\tilde{K}_\omega = \tilde{L} - \tilde{N}.$$

As consequence of the previous lemma, we have the following result.

Lemma 3.10. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\omega > \alpha^2$. The following assertions hold.*

(1) There exists a constant $C > 0$ such that

$$\tilde{L}(v) \geqslant C\|v\|^2_{H^1} \quad \forall v \in H^1(\mathbb{R}).$$

(2) We have $\tilde{d}(\omega) > 0$.

(3) If $v \in H^1$ satisfies $\tilde{K}_\omega(v) < 0$ then $\tilde{L}(v) > 3\tilde{d}(\omega)$.

We introduce the following properties.

Lemma 3.11 (Brezis-Lieb [14])**.** *Let $2 \leqslant p < \infty$ and (f_n) be a bounded sequence in $L^p(\mathbb{R})$. Assume that $f_n \to f$ a.e in $\mathbb{R}$. Then we have*

$$\|f_n\|^p_{L^p} - \|f_n - f\|^p_{L^p} - \|f\|^p_{L^p} \to 0.$$

Lemma 3.12. *The following minimization problem is equivalent to the problem* (3.19) *i.e same minimum and the minimizers:*

$$d := \inf\left\{\frac{1}{16}\|u\|^6_{L^6} : u \text{ even }, u \in H^1(\mathbb{R}) \setminus \{0\}, \tilde{K}_\omega(u) \leqslant 0\right\}. \tag{3.27}$$

Proof. We see that the minimizer problem (3.19) is equivalent to following problem:

$$\inf\left\{\frac{1}{16}\|u\|_{L^6}^6 : u \text{ even } u\in H^1(\mathbb{R})\setminus\{0\}, \tilde{K}_\omega(u)=0\right\}. \tag{3.28}$$

Let v be a minimizer of (3.19) then $\tilde{K}_\omega(v)\leqslant 0$, hence, $\tilde{d}(\omega)=\frac{1}{16}\|v\|_{L^6}^6\geqslant d$. Now, let v be a minimizer of (3.27). We prove that $\tilde{K}_\omega(v)=0$. Indeed, assuming $\tilde{K}_\omega(v)<0$, we have

$$\tilde{K}_\omega(\lambda v)=\lambda^2\left(\|v_x\|_{L^2}^2+\omega\|v\|_{L^2}^2+2\alpha|v(0)|^2-\frac{3\lambda^4}{16}\|v\|_{L^6}^6\right)\leqslant 0,$$

as $0<\lambda$ is small enough. Thus, by continuity, there exists a $\lambda_0\in(0,1)$ such that $\tilde{K}_\omega(\lambda_0 v)=0$. We have $d<\tilde{d}(\omega)\leqslant\frac{1}{16}\|\lambda_0 v\|_{L^6}^6<\frac{1}{16}\|v\|_{L^6}^6=d$. Which is a contradiction. It implies that $\tilde{K}_\omega(v)=0$ and v is a minimizer of (3.28), hence v is a minimizer of (3.19). This completes the proof. $\square$

Now, using the similar arguments in [39, Proof of Proposition 2], we have the following result.

Proposition 3.13. *Let $(\omega,\alpha)\in\mathbb{R}^2$ be such that $\alpha<0$, $\omega>\alpha^2$ and $(w_n)\subset H^1(\mathbb{R})$ be a even sequence satisfying the following properties*

$$\begin{aligned}\widetilde{S}_\omega(w_n)&\to\tilde{d}(\omega),\\ \widetilde{K}_\omega(w_n)&\to 0.\end{aligned}$$

as $n\to\infty$. Then, there exists a minimizer w of (3.19) *such that $w_n\to w$ strongly in $H^1(\mathbb{R})$ up to subsequence.*

Proof. In what follows, we shall often extract subsequence without mentioning this fact explicitly. We divide the proof into two steps.

Step 1. Weakly convergence to a nonvanishing function of minimizer sequence We have

$$\frac{1}{3}\tilde{L}(w_n)=\tilde{S}_\omega(w_n)-\frac{1}{6}\tilde{K}_\omega(w_n)\to\tilde{d}(\omega),$$

as $n\to\infty$. Then, (w_n) is bounded in $H^1(\mathbb{R})$ and there exists $w\in H^1(\mathbb{R})$ even such that $w_n\rightharpoonup w$ in $H^1(\mathbb{R})$ up to subsequence. We prove $w\neq 0$. Assume that $w\equiv 0$. Define, for $u\in H^1(\mathbb{R})$,

$$\begin{aligned}S_\omega^0(u)&=\frac{1}{2}\|u_x\|_{L^2}^2+\frac{\omega}{2}\|u\|_{L^2}^2-\frac{1}{32}\|u\|_{L^6}^6,\\ K_\omega^0(u)&=\|u_x\|_{L^2}^2+\omega\|u\|_{L^2}^2-\frac{3}{16}\|u\|_{L^6}^6.\end{aligned}$$

Let ψ_ω be minimizer of following problem

$$\begin{aligned}d^0(\omega)&=\inf\left\{S_\omega^0(u) : u \text{ even }, u\in H^1(\mathbb{R})\setminus\{0\}, K_\omega^0(u)=0\right\}\\ &=\inf\left\{\frac{1}{16}\|u\|_{L^6}^6 : u \text{ even }, u\in H^1(\mathbb{R})\setminus\{0\}, K_\omega^0(u)\leqslant 0\right\}.\end{aligned}$$

We have $K_\omega^0(w_n) = \tilde{K}_\omega(w_n) - 2\alpha|w_n(0)|^2 \to 0$, as $n \to \infty$. Since, $\alpha < 0$. we have $\tilde{K}_\omega(\psi_\omega) < 0$ and hence we obtain

$$\tilde{d}(\omega) < \frac{1}{16}\|\psi_\omega\|_{L^6}^6 = d^0(\omega) \tag{3.29}$$

We set

$$\lambda_n = \left(\frac{\|\partial_x w_n\|_{L^2}^2 + \omega\|w_n\|_{L^2}^2}{\frac{3}{16}\|w_n\|_{L^6}^6}\right)^{\frac{1}{4}}.$$

We here remark that $0 < \tilde{d}(\omega) = \lim_{n\to\infty} \frac{1}{16}\|w_n\|_{L^6}^6$. It follows that

$$\lambda_n^4 - 1 = \frac{K_\omega^0(w_n)}{\frac{3}{16}\|w_n\|_{L^6}^6} \to 0,$$

as $n \to \infty$. We see that $K_\omega^0(\lambda_n w_n) = 0$ and $\lambda_n w_n \neq 0$. By the definition of $d^0(\omega)$, we have

$$d^0(\omega) \leqslant \frac{1}{16}\|\lambda_n w_n\|_{L^6}^6 \to \tilde{d}(\omega) \text{ as } n \to \infty.$$

This contradicts to (3.29). Thus, $w \neq 0$.

Step 2. Conclude the proof Using Lemma 3.11 we have

$$\tilde{K}_\omega(w_n) - \tilde{K}_\omega(w_n - w) - \tilde{K}_\omega(w) \to 0, \tag{3.30}$$

$$\tilde{L}(w_n) - \tilde{L}(w_n - w) - \tilde{L}(w) \to 0. \tag{3.31}$$

Now, we prove $\tilde{K}_\omega(w) \leqslant 0$ by contradiction. Suppose that $\tilde{K}_\omega(w) > 0$. By the assumption $\tilde{K}_\omega(w_n) \to 0$ and (3.30), we have

$$\tilde{K}_\omega(w_n - w) \to -\tilde{K}_\omega(w) < 0.$$

Thus, $\tilde{K}_\omega(w_n - w) < 0$ for n large enough. By Lemma 3.10 (3), we have $\tilde{L}(w_n - w) \geqslant 3\tilde{d}(\omega)$. Since $\tilde{L}(w_n) \to 3\tilde{d}(\omega)$, by (3.31), we have

$$\tilde{L}(w) = \lim_{n\to\infty}(\tilde{L}(w_n) - \tilde{L}(w_n - w)) \leqslant 0.$$

Moreover, $w \neq 0$ and by Lemma 3.10 (1), we have $\tilde{L}(w) > 0$. This is a contradiction. Hence, $\tilde{K}_\omega(w) < 0$. By Lemma 3.10 (2), (3) and weakly lower semicontinuity of $\tilde{L}$, we have

$$3\tilde{d}(\omega) \leqslant \tilde{L}(w) \leqslant \lim_{n\to\infty}\inf \tilde{L}(w_n) = 3\tilde{d}(\omega).$$

Thus, $\tilde{L}(w) = 3\tilde{d}(\omega)$. Combining with (3.31), we have $\tilde{L}(w_n - w) \to 0$, as $n \to \infty$. By Lemma 3.10 (1), we have $w_n \to w$ strongly in $H^1(\mathbb{R})$. Hence, w is a minimizer of (3.19). This completes the proof. $\square$

To prove the stability statement (1) for $\alpha < 0$ in Theorem 3.4, we will use similar arguments as in the work of Colin and Ohta [23]. We need the following property.

Lemma 3.14. *Let $\alpha < 0$, $\omega > \alpha^2$. If a sequence $(v_n) \subset H^1(\mathbb{R}^+)$ satisfies*

$$S_\omega(v_n) \to d(\omega), \tag{3.32}$$
$$K_\omega(v_n) \to 0, \tag{3.33}$$

then there exist a constant $\theta_0 \in \mathbb{R}$ such that $v_n \to e^{i\theta_0}\varphi$, up to subsequence, where φ is defined as in Proposition 3.3.

Proof. Define the sequence $(w_n) \subset H^1(\mathbb{R})$ as follows,

$$w_n(x) = \begin{cases} v_n(x) \text{ for } x > 0, \\ v_n(-x) \text{ for } x < 0. \end{cases}$$

We can check that

$$\begin{aligned} \widetilde{S}_\omega(w_n) &= 2S_\omega(v_n) \to 2d(\omega) = \tilde{d}(\omega), \\ \widetilde{K}_\omega(w_n) &= 2K_\omega(v_n) \to 0, \end{aligned}$$

as $n \to \infty$. Using Proposition 3.13, there exists a minimizer w_0 of (3.19) such that $w_n \to w_0$ strongly in $H^1(\mathbb{R})$, up to subsequence. For convenience, we assume that $w_n \to w_0$ strongly in $H^1(\mathbb{R})$. By Proposition 3.7, there exists a constant $\theta_0 \in \mathbb{R}$ such that

$$w_0 = e^{i\theta_0}\tilde{\varphi},$$

where $\tilde{\varphi}$ is defined by

$$\tilde{\varphi}(x) = \begin{cases} \varphi(x) \text{ for } x > 0, \\ \varphi(-x) \text{ for } x < 0. \end{cases} \tag{3.34}$$

Hence, the sequence (v_n) is the restriction of the sequence (w_n) on $\mathbb{R}^+$, and satisfies

$$v_n \to e^{i\theta_0}\varphi, \text{ strongly in } H^1(\mathbb{R}^+),$$

up to subsequence. This completes the proof. $\square$

Define

$$\begin{aligned} \mathcal{A}_\omega^+ &= \left\{ v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) > 0 \right\}, \\ \mathcal{A}_\omega^- &= \left\{ v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), K_\omega(v) < 0 \right\}, \\ \mathcal{B}_\omega^+ &= \left\{ v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) < 3d(\omega) \right\}, \\ \mathcal{B}_\omega^- &= \left\{ v \in H^1(\mathbb{R}^+) \setminus \{0\} : S_\omega(v) < d(\omega), N(v) > 3d(\omega) \right\}. \end{aligned}$$

We have the following result.

Lemma 3.15. *Let $\omega, \alpha \in \mathbb{R}^2$ such that $\alpha < 0$ and $\omega > \alpha^2$.*

(1) The sets $\mathcal{A}_\omega^+$ and $\mathcal{A}_\omega^-$ are invariant under the flow of (3.1).

(2) $\mathcal{A}_\omega^+ = \mathcal{B}_\omega^+$ and $\mathcal{A}_\omega^- = \mathcal{B}_\omega^-$.

Proof. (1) Let $u_0 \in \mathcal{A}_\omega^+$ and $u(t)$ the associated solution for (3.1) on (T_{min}, T_{max}). By $u_0 \neq 0$ and the conservation laws, we see that $S_\omega(u(t)) = S_\omega(u_0) < d(\omega)$ for $t \in (T_{min}, T_{max})$. Moreover, by definition of $d(\omega)$ we have $K_\omega(u(t)) \neq 0$ on (T_{min}, T_{max}). Since the function $t \mapsto K_\omega(u(t))$ is continuous, we have $K_\omega(u(t)) > 0$ on (T_{min}, T_{max}). Hence, $\mathcal{A}_\omega^+$ is invariant under flow of (3.1). By the same way, $\mathcal{A}_\omega^-$ is invariant under flow of (3.1).
(2) If $v \in \mathcal{A}_\omega^+$ then by (3.26), (3.25) we have $N(v) = 3S_\omega(v) - 2K_\omega(v) < 3d(\omega)$, which shows $v \in \mathcal{B}_\omega^+$, hence $\mathcal{A}_\omega^+ \subset \mathcal{B}_\omega^+$. Now, let $v \in \mathcal{B}_\omega^+$. We show $K_\omega(v) > 0$ by contradiction. Suppose that $K_\omega(v) \leqslant 0$. Then, by Lemma 3.10 (3), $L(v) \geqslant 3d(\omega)$. Thus, by (3.26) and (3.25), we have

$$S_\omega(v) = \frac{1}{2}L(v) - \frac{1}{6}N(v) \geqslant d(\omega),$$

which contradicts $S_\omega(v) < d(\omega)$. Therefore, we have $K_\omega(v) > 0$, which shows $v \in \mathcal{A}_\omega^+$ and $\mathcal{B}_\omega^+ \subset \mathcal{A}_\omega^+$. Next, if $v \in \mathcal{A}_\omega^-$, then by Lemma 3.10 (3), $L(v) > 3d(\omega)$. Thus, by (3.26) and (3.25), we have $N(v) = L(v) - K_\omega(v) > 3d(\omega)$, which shows $v \in \mathcal{B}_\omega^-$. Thus, $\mathcal{A}_\omega^- \subset \mathcal{B}_\omega^-$. Finally, if $v \in \mathcal{B}_\omega^-$, then by (3.26) and (3.25), we have $2K_\omega(v) = 3S_\omega(v) - N(v) < 3d(\omega) - 3d(\omega) = 0$, which shows $v \in \mathcal{A}_\omega^-$, hence, $\mathcal{B}_\omega^- \subset \mathcal{A}_\omega^-$. This completes the proof. □

From Proposition 3.3, we have

$$d(\omega) = S_\omega(\varphi).$$

Since $\alpha < 0$, we see that

$$d''(\omega) = \partial_\omega \|\varphi\|^2_{L^2(\mathbb{R}^+)} = \frac{1}{2}\partial_\omega \|\tilde{\varphi}\|^2_{L^2(\mathbb{R})} > 0,$$

where $\tilde{\varphi}$ is defined as (3.34) and we know from [39], [38] that

$$\partial_\omega \|\tilde{\varphi}\|^2_{L^2(\mathbb{R})} > 0,$$

for $\alpha < 0$. We define the function $h : (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ by

$$h(\tau) = d(\omega \pm \tau),$$

for $\varepsilon_0 > 0$ sufficiently small such that $h''(\tau) > 0$ and the sign $+$ or $-$ is selected such that $h'(\tau) > 0$ for $\tau \in (-\varepsilon_0, \varepsilon_0)$. Without loss of generality, we can assume

$$h(\tau) = d(\omega + \tau).$$

Lemma 3.16. *Let $(\omega, \alpha) \in \mathbb{R}^2$ such that $\omega > \alpha^2$ and let h be defined as above. Then, for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\delta > 0$ such that if $v_0 \in H^1(\mathbb{R}^+)$ satisfies $\|v_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$, then the solution v of (3.1) with $v(0) = v_0$ satisfies $3h(-\varepsilon) < N(v(t)) < 3h(\varepsilon)$ for all $t \in (T_{min}, T_{max})$.*

Proof. The proof of the above lemma is similar to the one of [23] or [104]. Let $\varepsilon \in (0, \varepsilon_0)$. Since h is increasing, we have $h(-\varepsilon) < h(0) < h(\varepsilon)$. Moreover, by $K_\omega(\varphi) = 0$ and (3.25), (3.26), we see that $3h(0) = 3d(\omega) = 3S_\omega(\varphi) = N(\varphi)$. Thus, if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ then we have $3h(0) = N(u_0) + O(\delta)$ and $3h(-\varepsilon) < N(u_0) < 3h(\varepsilon)$ for sufficiently small $\delta > 0$. Since $h(\pm\varepsilon) = d(\omega \pm \varepsilon)$ and the set $\mathcal{B}_\omega^\pm$ are invariant under the flow of (3.1) by Lemma 3.15, to conclude the proof, we only have to show that there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ then $S_{\omega\pm\varepsilon}(u_0) < h(\pm\varepsilon)$. Assume that $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$. We have

$$\begin{aligned} S_{\omega\pm\varepsilon}(u_0) &= S_{\omega\pm\varepsilon}(\varphi) + O(\delta) \\ &= S_\omega(\varphi) \pm \varepsilon M(\varphi) + O(\delta) \\ &= h(0) \pm \varepsilon h'(0) + O(\delta). \end{aligned}$$

On the other hand, by the Taylor expansion, there exists $\tau_1 = \tau_1(\varepsilon) \in (-\varepsilon_0, \varepsilon_0)$ such that

$$h(\pm\varepsilon) = h(0) \pm \varepsilon h'(0) + \frac{\varepsilon^2}{2} h''(\tau_1).$$

Since $h''(\tau_1) > 0$ by definition of h, we see that there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^+)$ satisfies $\|u_0 - \varphi\|_{H^1(\mathbb{R}^+)} < \delta$ then $S_{\omega\pm\varepsilon}(u_0) < h(\pm\varepsilon)$. This completes the proof. □

Proof of Theorem 3.4 (1). Assume that $e^{i\omega t}\varphi$ is not stable for (3.1). Then, there exists a constant $\varepsilon_1 > 0$, a sequence of solutions (v^n) to (3.1), and a sequence $\{t_n\} \in (0, \infty)$ such that

$$v_n(0) \to \varphi \text{ in } H^1(\mathbb{R}^+), \ \inf_{\theta\in\mathbb{R}} \|v_n(t_n) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} \geqslant \varepsilon_1. \tag{3.35}$$

By using the conservation laws of solutions of (3.1), we have

$$S_\omega(v_n(t_n)) = S_\omega(v_n(0)) \to S_\omega(\varphi) = d(\omega). \tag{3.36}$$

Using Lemma 3.16, we have

$$N(v_n(t_n)) \to 3d(\omega). \tag{3.37}$$

Combined (3.36) and (3.37), we have

$$K_\omega(v_n(t_n)) = 2S_\omega(v_n(t_n)) - \frac{2}{3}N(v_n(t_n)) \to 0.$$

Therefore, using Lemma 3.14, there exists $\theta_0 \in \mathbb{R}$ such that $(v_n(t_n, .))$ has a subsequence (we denote it by the same letter) that converges to $e^{i\theta_0}\varphi$ in $H^1(\mathbb{R}^+)$, where φ is defined as in Proposition 3.3. Hence, we have

$$\inf_{\theta\in\mathbb{R}} \|v_n(t_n) - e^{i\theta}\varphi\|_{H^1(\mathbb{R}^+)} \to 0, \tag{3.38}$$

as $n \to \infty$, this contradicts (3.35). Hence, we obtain the desired result. □

Next, we give the proof of Theorem 3.4 (2). We divide the proof in two cases.

First, let $\alpha = 0$. In this case, we use similar arguments as in Weinstein [113]. We have

$$E_\omega(v) = \frac{1}{2}\|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{32}\|v\|^6_{L^6(\mathbb{R}^+)},$$
$$P(v) = \|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{16}\|v\|^6_{L^6(\mathbb{R}^+)}.$$

Thus, $E(\varphi_\omega) = P(\varphi_\omega) = 0$. Let $\varepsilon > 0$ and $\varphi_{\omega,\varepsilon} = (1+\varepsilon)\varphi_\omega$. We have

$$E(\varphi_{\omega,\varepsilon}) = (1+\varepsilon)^2\frac{1}{2}\|\varphi_\omega\|^2_{L^2(\mathbb{R}^+)} - (1+\varepsilon)^6\frac{1}{32}\|\varphi_\omega\|^6_{L^6(\mathbb{R}^+)} = ((1+\varepsilon)^2-(1+\varepsilon)^6)\frac{1}{2}\|\varphi_\omega\|^2_{L^2(\mathbb{R}^+)} < 0.$$

In the addition, $|x|\varphi_{\omega,\varepsilon}(x) \in L^2(\mathbb{R}^+)$ by exponential decay of φ_ω. Using Theorem 3.1, the solution associated to $\varphi_{\omega,\varepsilon}$ blows up in finite time. As $\varphi_{\omega,\varepsilon} \to \varphi_\omega$ in $H^1(\mathbb{R}^+)$, we obtain the instability by blow-up of standing waves.

Now, let $\alpha > 0$ and $e^{i\omega t}\varphi$ be the standing wave solution of (3.1). We use similar arguments as in [71]. Introduce the scaling

$$v_\lambda(x) = \lambda^{\frac{1}{2}} v(\lambda x).$$

Let S_ω, K_ω be defined as in Proposition 3.3, for convenience, we will remove the index ω. Define

$$P(v) := \frac{\partial}{\partial\lambda}S(v_\lambda)|_{\lambda=1} = \|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{16}\|v\|^6_{L^6(\mathbb{R}^+)} + \frac{\alpha}{2}|v(0)|^2.$$

In the following lemma, we investigate the behaviour of the above functional under scaling.

Lemma 3.17. *Let $v \in H^1(\mathbb{R}^+) \setminus \{0\}$ be such that $v(0) \neq 0$, $P(v) \leqslant 0$. Then there exists $\lambda_0 \in (0,1]$ such that*

(i) $P(v_{\lambda_0}) = 0$,

(ii) $\lambda_0 = 1$ *if only if* $P(v) = 0$,

(iii) $\frac{\partial}{\partial\lambda}S(v_\lambda) = \frac{1}{\lambda}P(v_\lambda)$,

(iv) $\frac{\partial}{\partial\lambda}S(v_\lambda) > 0$ *on* $(0,\lambda_0)$ *and* $\frac{\partial}{\partial\lambda}S(v_\lambda) < 0$ *on* (λ_0,∞),

(v) The function $\lambda \to S(v_\lambda)$ *is concave on* (λ_0,∞).

Proof. A simple calculation leads to

$$P(v_\lambda) = \lambda^2\|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{\lambda^2}{16}\|v\|^6_{L^6(\mathbb{R}^+)} + \frac{\lambda\alpha}{2}|v(0)|^2.$$

Then, for $\lambda > 0$ small enough, we have

$$P(v_\lambda) > 0.$$

By continuity of P, there exists $\lambda_0 \in (0,1]$ such that $P(v_{\lambda_0}) = 0$. Hence (i) is proved. If $\lambda_0 = 1$ then $P(v) = 1$. Conversely, if $P(v) = 0$ then

$$0 = P(v_{\lambda_0}) = \lambda_0^2 P(v) + \frac{\lambda_0 - \lambda_0^2}{2}\alpha|v(0)|^2 = \frac{\lambda_0 - \lambda_0^2}{2}\alpha|v(0)|^2.$$

By the assumption $v(0) \neq 0$, we have $\lambda_0 = 1$, hence (ii) is proved. Item (iii) is obtained by a simple calculation. To obtain (iv), we use (iii). We have

$$\begin{aligned} P(v_\lambda) &= \lambda^2\lambda_0^{-2}P(v_{\lambda_0}) + \left(\frac{\lambda\alpha}{2} - \frac{\lambda^2\lambda_0^{-1}\alpha}{2}\right)|v(0)|^2 \\ &= \frac{\lambda\alpha(\lambda_0 - \lambda)}{2\lambda_0}|v(0)|^2. \end{aligned}$$

Hence, $P(v_\lambda) > 0$ if $\lambda < \lambda_0$ and $P(v_\lambda) < 0$ if $\lambda > \lambda_0$. This proves (iv). Finally, we have

$$\frac{\partial^2}{\partial\lambda^2}S(v_\lambda) = P(v) - \frac{\alpha}{2}|v(0)|^2 < 0.$$

This proves (v). □

In the case of functions such that $v(0) = 0$, we have the following lemma.

Lemma 3.18. *Let $v \in H^1(\mathbb{R}^+) \setminus \{0\}$, $v(0) = 0$ and $P(v) = 0$ then we have*

$$S(v_\lambda) = S(v) \quad \text{for all } \lambda > 0.$$

Proof. The proof is simple, using the fact that

$$\frac{\partial}{\partial\lambda}S(v_\lambda) = \frac{1}{\lambda}P(v_\lambda) = \lambda P(v) = 0.$$

Hence, we obtain the desired result. □

Now, consider the minimization problems

$$d_{\mathcal{M}} := \inf\left\{S(v) : v \in \mathcal{M}\right\}, \tag{3.39}$$

$$m := \inf\left\{S(v), v \in H^1(\mathbb{R}^+) \setminus 0, S'(v) = 0\right\}, \tag{3.40}$$

where

$$\mathcal{M} = \left\{v \in H^1(\mathbb{R}^+) \setminus 0, P(v) = 0, K(v) \leqslant 0\right\}.$$

By classical arguments, we can prove the following property.

Proposition 3.19. *Let m be defined as above. Then, we have*

$$m = \inf\left\{S(v) : v \in H^1(\mathbb{R}^+) \setminus 0, K(v) = 0\right\}.$$

We have the following relation between the minimization problems m and $d_{\mathcal{M}}$.

Lemma 3.20. *Let m and $d_{\mathcal{M}}$ be defined as above. We have*

$$m = d_{\mathcal{M}}.$$

Proof. Let $\mathcal{G}$ be the set of all minimizers of (3.40). If $\varphi \in \mathcal{G}$ then $S'(\varphi) = 0$. By the definition of S, P, K we have $P(\varphi) = 0$ and $K(\varphi) = 0$. Hence, $\varphi \in \mathcal{M}$, this implies $S(\varphi) \geqslant d_{\mathcal{M}}$. Thus, $m \geqslant d_{\mathcal{M}}$.

Conversely, let $v \in \mathcal{M}$. If $K(v) = 0$ then $S(v) \geqslant m$, using Proposition 3.19. Otherwise, $K(v) < 0$. Using the scaling $v_\lambda(x) = \lambda^{\frac{1}{2}} v(\lambda x)$, we have

$$K(v_\lambda) = \lambda^2 \|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{3\lambda^2}{16}\|v\|^6_{L^6(\mathbb{R}^+)} + \omega\|v\|^2_{L^2(\mathbb{R}^+)} + \frac{\alpha\lambda}{2}|v(0)|^2 \to \omega\|v\|^2_{L^2(\mathbb{R}^+)} > 0,$$

as $\lambda \to 0$. Hence, $K(v_\lambda) > 0$ as $\lambda > 0$ is small enough. Thus, there exists $\lambda_1 \in (0,1)$ such that $K(v_{\lambda_1}) = 0$. Using Proposition 3.19, $S(v_{\lambda_1}) \geqslant m$. We consider two cases. First, if $v(0) = 0$ then using Lemma 3.18, we have $S(v) = S(v_{\lambda_1}) \geqslant m$. Second, if $v(0) \neq 0$ then using Lemma 3.17, we have $S(v) \geqslant S(v_{\lambda_1}) \geqslant m$. In any case, $S(v) \geqslant m$. This implies $d_{\mathcal{M}} \geqslant m$, and completes the proof. □

Define

$$\mathcal{V} := \left\{ v \in H^1(\mathbb{R}^+) \setminus \{0\} : K(v) < 0, P(v) < 0, S(v) < m \right\}.$$

We have the following important lemma.

Lemma 3.21. *If $v_0 \in \mathcal{V}$ then the solution v of* (3.1) *associated with v_0 satisfies $v(t) \in \mathcal{V}$ for all t in the time of existence.*

Proof. Since $S(v_0) < 0$, by conservation of the energy and the mass we have

$$S(v) = E(v) + \omega M(v) = E(v_0) + \omega M(v_0) = S(v_0) < m. \tag{3.41}$$

If there exists $t_0 > 0$ such that $K(v(t_0)) \geqslant 0$ then by continuity of K and v, there exists $t_1 \in (0, t_0]$ such that $K(v(t_1)) = 0$. This implies $S(v(t_1)) \geqslant m$, using Proposition 3.19. This contradicts (3.41). Hence, $K(v(t)) < 0$ for all t in the time of existence of v. Now, we prove $P(v(t)) < 0$ for all t in the time of existence of v. Assume that there exists $t_2 > 0$ such that $P(v(t_2)) \geqslant 0$, then, there exists $t_3 \in (0, t_2]$ such that $P(v(t_3)) = 0$. Using the previous lemma, $S(v(t_3)) \geqslant m$, which contradicts (3.41). This completes the proof. □

Using the above lemma, we have the following property of solutions of (3.1) when the initial data lies on $\mathcal{V}$.

Lemma 3.22. *Let $v_0 \in \mathcal{V}$, v be the corresponding solution of* (3.1) *in* (T_{min}, T_{max})*. There exists $\delta > 0$ independent of t such that $P(v(t)) < -\delta$, for all $t \in (T_{min}, T_{max})$.*

Proof. Let $t \in (T_{min}, T_{max})$, $u = v(t)$ and $u_\lambda(x) = \lambda^{\frac{1}{2}} u(\lambda x)$. Using Lemma 3.17, there exists $\lambda_0 \in (0,1)$ such that $P(u_{\lambda_0}) = 0$. If $K(u_{\lambda_0}) \leqslant 0$ then we keep λ_0. Otherwise, $K(u_{\lambda_0}) > 0$, then, there exists $\widetilde{\lambda}_0 \in (\lambda_0, 1)$ such that $K(u_{\widetilde{\lambda}_0}) = 0$. We replace λ_0 by $\widetilde{\lambda}_0$. In any case, we have

$$S(u_{\lambda_0}) \geqslant m. \tag{3.42}$$

By (v) of Lemma 3.17 we have

$$S(u) - S(u_{\lambda_0}) \geqslant (1-\lambda_0)\frac{\partial}{\partial\lambda}S(u_\lambda)|_{\lambda=1} = (1-\lambda_0)P(u).$$

In addition $P(u) < 0$, we obtain

$$S(u) - S(u_{\lambda_0}) \geqslant (1-\lambda_0)P(u) > P(u). \tag{3.43}$$

Combined (3.42) and (3.43), we obtain

$$S(v_0) - m = S(v(t)) - m = S(u) - m \geqslant S(u) - S(u_{\lambda_0}) > P(u) = P(v(t)).$$

Setting

$$-\delta := S(v_0) - m,$$

we obtain the desired result. □

Using the previous lemma, if the initial data lies on $\mathcal{V}$ and satisfies a weight condition then the associated solution blows up in finite time on $H^1(\mathbb{R}^+)$. More precisely, we have the following result.

Proposition 3.23. *Let $\varphi \in \mathcal{V}$ such that $|x|\varphi \in L^2(\mathbb{R}^+)$. Then the corresponding solution v of* (3.1) *blows up in finite time on $H^1(\mathbb{R}^+)$.*

Proof. By Lemma 3.22, there exists $\delta > 0$ such that $P(v(t)) < -\delta$ for $t \in (T_{min}, T_{max})$. Remember that

$$\frac{\partial}{\partial t}\|xv(t)\|^2_{L^2(\mathbb{R}^+)} = J(t) - \int_{\mathbb{R}^+} x|v|^4\,dx, \tag{3.44}$$

where $J(t)$ satisfies

$$\partial_t J(t) = 4\left(2\|v_x\|^2_{L^2(\mathbb{R}^+)} - \frac{1}{8}\|v\|^6_{L^6(\mathbb{R}^+)} + \alpha|v(0)|^2\right) = 8(P(v(t))) < -8\delta.$$

This implies that

$$J(t) = J(0) + 8\int_0^t P(v(s))\,ds < J(0) - 8\delta t.$$

Hence, from (3.44), we have

$$\begin{aligned}
\|xv(t)\|^2_{L^2(\mathbb{R}^+)} &= \|xv(0)\|^2_{L^2(\mathbb{R}^+)} + \int_0^t J(s)\,ds - \int_0^t\int_{\mathbb{R}^+} x|v|^4\,dx\,ds \\
&\leqslant \|xv(0)\|^2_{L^2(\mathbb{R}^+)} + \int_0^t (J(0) - 8\delta s)\,ds \\
&\leqslant \|xv(0)\|^2_{L^2(\mathbb{R}^+)} + J(0)t - 4\delta t^2.
\end{aligned}$$

Thus, for t sufficiently large, there is a contradiction with $\|xv\|_{L^2(\mathbb{R}^+)} \geqslant 0$. Hence, $T_{max} < \infty$ and $T_{min} > -\infty$. By the blow up alternative, we have

$$\lim_{t\to T_{max}} \|v_x\|_{L^2(\mathbb{R}^+)} = \lim_{t\to T_{min}} \|v_x\|_{L^2(\mathbb{R}^+)} = \infty.$$

This completes the proof. □

Proof of Theorem 3.4 (2). Using Proposition 3.23, we need to construct a sequence $(\varphi_n) \subset \mathcal{V}$ such that φ_n converges to φ in $H^1(\mathbb{R}^+)$. Define

$$\varphi_\lambda(x) = \lambda^{\frac{1}{2}} \varphi(\lambda x).$$

We have

$$S(\varphi) = m, \quad P(\varphi) = K(\varphi) = 0, \quad \varphi(0) \neq 0.$$

By (iv) of Lemma 3.17,

$$S(\varphi_\lambda) < m \text{ for all } \lambda > 0.$$

In the addition,

$$P(\varphi_\lambda) < 0 \text{ for all } \lambda > 1.$$

Moreover,

$$\begin{aligned}
\frac{\partial}{\partial \lambda} K(\varphi_\lambda) &= 2\lambda \left(\|\varphi_x\|^2_{L^2(\mathbb{R}^+)} - \frac{3}{16} \|\varphi\|^6_{L^6(\mathbb{R}^+)} \right) + \alpha |\varphi(0)|^2 \\
&= 2\lambda (K(\varphi) - \omega \|\varphi\|^2_{L^2(\mathbb{R}^+)} - \alpha |\varphi(0)|^2) + \alpha |\varphi(0)|^2 \\
&= -2\omega\lambda \|\varphi\|^2_{L^2(\mathbb{R}^+)} - \alpha (2\lambda - 1) |\varphi(0)|^2 \\
&< 0,
\end{aligned}$$

when $\lambda > 1$. Thus, $K(\varphi_\lambda) < K(\varphi) = 0$ when $\lambda > 1$. This implies $\varphi_\lambda \in \mathcal{V}$ when $\lambda > 1$. Let $\lambda_n > 1$ such that $\lambda_n \to 1$ as $n \to \infty$. Define, for $n \in \mathbb{N}^*$

$$\varphi_n = \varphi_{\lambda_n},$$

then, the sequence (φ_n) satisfies the desired property. This completes the proof of Theorem 3.4. $\square$

Chapter 4

Multi-solitons Part 1: Construction of multi-solitons and multi kink-solitons of derivative nonlinear Schrödinger equations

4.1 Introduction

In this chapter, we consider the derivative nonlinear Schrödinger equation:

$$\begin{cases} iu_t + u_{xx} + i\alpha|u|^2u_x + i\mu u^2\overline{u_x} + f(u) = 0, \\ u(0) = u_0. \end{cases} \tag{4.1}$$

where $\alpha, \mu \in \mathbb{R}$, $f : \mathbb{C} \to \mathbb{C}$ is a given function and u is a complex valueed function of $(t, x) \in \mathbb{R} \times \mathbb{R}$.

In [111, 112], Tsutsumi and Fukuda used an approximation argument to prove the existence of solutions of (4.1) in the case $\alpha = -2$, $\mu = -1$. In this case with $f = 0$, Biagioni and Linares [13] proved that the solution map from $H^s(\mathbb{R})$ to $C([-T, T], H^s(\mathbb{R}))$ is not locally uniformly continuous, for $T > 0$ and $s < \frac{1}{2}$. The $H^{\frac{1}{2}}$ solution in this case is global if $\|u_0\|^2_{L^2} < 2\pi$ by the work of Miao-Wu-Xu [92]. Later, Guo and Wu [53] improved this result; that is, $H^{\frac{1}{2}}$ solution is global if $\|u_0\|^2_{L^2} < 4\pi$. The Cauchy problem of (4.1) was also studied as in [107], where gauge transformation and Fourier restriction method are used to obtain local well-posedness in H^s, $s \geqslant 1/2$. In [100], Ozawa studied the Cauchy problem and gave a sufficient condition of global well-posedness for (4.1). The proof was used gauge transformations which reduce the original equations to systems of equations without derivative nonlinearities. In [58, 59], in the case $\alpha = 2\mu$, Hayashi-Ozawa proved the unique global existence of solutions to (4.1) in Sobolev spaces and in the weighted spaces with smallness on the initial data $\|u_0\|^2_{L^2} < \frac{4\pi}{|\alpha|}$. In the case $\alpha = -2$, $\mu = -1$, $f = 0$, Wu [116] improved the global results in [58, 59]. More precisely, the author proved that the solutions exist globally in time under smallness on the initial data $\|u_0\|_{L^2} < \sqrt{2\pi} + \varepsilon_*$, where ε_* is a small positive constant. Later, Wu [117] improved this results for larger bounded on the initial data $\|u_0\|_{L^2} < \sqrt{4\pi}$.

The proof combines a gauge transformation and conservation laws with a sharp Gagliardo-Nirenberg inequality. In [37], by using variational argument, Fukaya-Hayashi-Inui gave results covering the result of Wu [117]. The authors showed that in the case $f = 0$, $\alpha = 1$, $\mu = 0$, the H^1 solutions of (4.1) exist globally in time for the initial satisfies $\|u_0\|_{L^2}^2 < 4\pi$ or $\|u_0\|_{L^2}^2 = 4\pi$ and $P(u_0) < 0$, where P is the momentum functional which is conserved under the flow of (4.1). In [25], Colliander-Keel-Staffilani-Takaoka-Tao proved by the so-called *I-method* the global well posedness in $H^s(\mathbb{R})$, $s > \frac{1}{2}$ of (4.1) if $\|u_0\|_{L^2}^2 < 2\pi$ (see also [24]). In the case $f = 0$ and $\mu = 0$, (4.1) is a completely integrable equation. The complete integrability structure of equation was used to prove global existence of solutions in $H^{2,2}(\mathbb{R})$ by [66] and in $H^s(\mathbb{R})$, $s > \frac{1}{2}$ by [1].

In the case $\mu = 0$ and $f(u) = b|u|^4u$, there were a lot of works on studying stability and instability of solitons of (4.1). The family of solitons of (4.1) has two parameters (ω, c). In the case $b = 0$, Guo and Wu [51] proved that the solitons are orbitally stable when $\omega > \frac{c^2}{4}$ and $c < 0$ by using the abstract theory of Grillakis-Shatah-Strauss [49, 50]. After that, Colin and Ohta [23] improved this result for all $\omega > \frac{c^2}{4}$ using variational techniques. In [99], Ohta proved that for each $b > 0$ there exists a unique $s^* = s^*(b) > 0 \in (0, 1)$ such that the soliton $u_{\omega,c}$ is orbitally stable if $-2\sqrt{\omega} < c < 2s^*\sqrt{\omega}$ and orbitally unstable if $2s^*\sqrt{\omega} < c < 2\sqrt{\omega}$. In the case $b < 0$, the stability result is obtained in [54]. In the case $b = 0$, Kwon-Wu [70] proved a stability result of solitons in the zero mass case. Removing the effect of scaling in the stability result of this work is an open question.

4.1.1 Multi-solitons

First, we focus on studying the following special form of (4.1):

$$iu_t + u_{xx} + i|u|^2u_x + b|u|^4u = 0. \tag{4.2}$$

Our first goal in this paper is to study the long time behaviour of solutions of (4.2). More precisely, we study the multi-solitons theory of (4.2). The existence of multi-solitons is a step towards the proof of the *soliton resolution conjecture*, which states that all global solutions of a dispersive equation behave at large times as a sum of a radiative term and solitons. The theory of multi-soliton has attracted a lot of interest. In [72, 73], Le Coz-Li-Tsai proved existence and uniqueness of finite and infinite soliton and kink-soliton trains of classical nonlinear Schrödinger equations, using fixed point arguments around of the desired profile. Another method was introduced in [84] for the simple power nonlinear Schrödinger equation with L^2-subcritical nonlinearities. The proof was established by two ingredients: uniform estimates and a compactness property. The arguments were later modified to obtain the results for L^2-supercritical equations [28] and for profiles made with excited states [26]. One can also cite the works on the logarithmic Schrödinger equation (logNLS) in the focusing regime in [35]. In [118], the inverse scattering transform method (IST) was used to construct multi-solitons of the one dimensional cubic focusing NLS. We would like also to mention the works on the non-linear Klein-Gordon equation [29] and [27], and on the stability of multi-solitons for generalized Korteweg-de Vries equations and L^2-subcritical nonlinear Schrödinger equations from Martel, Merle and Tsai [85],[86]. In [74], Le Coz-Wu proved a stability result of multi-solitons of (4.2) in the case $b = 0$. Our motivation is to prove the existence of a multi-solitons in a similar sense as in [73, 72]. The method used in [73, 72] cannot apply directly in our case. The reason is the appearance of the derivative nonlinearities. To overcome this difficulty, we use a Gauge transformation to obtain a system of Schrödinger equations without derivative nonlinearities. We may use Strichartz estimates and fixed point argument to construct a suitable solution of this system. This solution satisfies a relation which is proved by using the Grönwall inequality and the condition on the parameters and we obtain a solution of (4.2). This solution satisfies the desired property.

Consider equation (4.2). The soliton of equation (4.2) is a solution of the form $R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x - ct)$, where $\phi_{\omega,c} \in H^1(\mathbb{R})$ solves

$$-\phi_{xx} + \omega\phi + ic\phi_x - i|\phi|^2\phi_x - b|\phi|^4\phi = 0, \quad x \in \mathbb{R}. \tag{4.3}$$

Applying the following gauge transform to $\phi_{\omega,c}$

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x)\exp\left(i\frac{c}{2}x - \frac{i}{4}\int_{-\infty}^{x}|\Phi_{\omega,c}(y)|^2\,dy\right),$$

it is easy to verify that $\Phi_{\omega,c}$ (see e.g [23, Proof of Lemma 2]) satisfies the following equation.

$$-\Phi_{xx} + \left(\omega - \frac{c^2}{4}\right)\Phi + \frac{c}{2}|\Phi|^2\Phi - \frac{3}{16}\gamma|\Phi|^4\Phi = 0, \quad \gamma := 1 + \frac{16}{3}b. \tag{4.4}$$

The positive even solution of (4.4) is explicitly obtained by: if $\gamma > 0$ ($b > \frac{-3}{16}$),

$$\Phi^2_{\omega,c}(x) = \begin{cases} \frac{2(4\omega - c^2)}{\sqrt{c^2+\gamma(4\omega-c^2)}\cosh(\sqrt{4\omega-c^2}x)-c} & \text{if } -2\sqrt{\omega} < c < 2\sqrt{\omega}, \\ \frac{4c}{(cx)^2+\gamma} & \text{if } c = 2\sqrt{\omega}, \end{cases} \tag{4.5}$$

and if $\gamma \leqslant 0$ ($b \leqslant -\frac{3}{16}$),

$$\Phi^2_{\omega,c}(x) = \frac{2(4\omega - c^2)}{\sqrt{c^2 + \gamma(4\omega - c^2)}\cosh(\sqrt{4\omega - c^2}x) - c} \text{ if } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega},$$

where $s_* = s_*(\gamma) = \sqrt{\frac{-\gamma}{1-\gamma}}$. We note that the following condition on the parameters γ and (ω, c) is a necessary and sufficient condition for the existence of non-trivial solutions of (4.2) vanishing at infinity (see [5]):

$$\begin{aligned} &\text{if } \gamma > 0 (\Leftrightarrow b > \frac{-3}{16}) \text{ then } -2\sqrt{\omega} < c \leqslant 2\sqrt{\omega}, \\ &\text{if } \gamma \leqslant 0 (\Leftrightarrow b \leqslant \frac{-3}{16}) \text{ then } -2\sqrt{\omega} < c < -2s_*\sqrt{\omega}. \end{aligned}$$

Let (c_j, ω_j) satisfying for each $1 \leqslant j \leqslant K$ the condition of existence of soliton. For each $j \in \{1, 2, .., K\}$, we set

$$R_j(t,x) = e^{i\theta_j}R_{\omega_j,c_j}(t,x).$$

The profile of a multi-soliton is a sum of the form:

$$R = \sum_{j=1}^{K} R_j. \tag{4.6}$$

A solution of (4.2) is called a multi-soliton if

$$\|u(t) - R(t)\|_{H^1} \leqslant Ce^{-\lambda t},$$

for some $C, \lambda > 0$ and t large enough. For convenience, we set $h_j = \sqrt{4\omega_j - c_j^2}$. We rewrite

$$\Phi_{\omega_j,c_j}(x) = \sqrt{2}h_j \left(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j\right)^{-\frac{1}{2}}. \tag{4.7}$$

As each soliton is in $H^\infty(\mathbb{R})$, we have $R \in H^\infty(\mathbb{R})$. Our first main result is the following.

Theorem 4.1. *Let $K \in \mathbb{N}^*$ and for each $1 \leqslant j \leqslant K$, let $(\theta_j, c_j, \omega_j)$ be a set of parameters such that $\theta_j \in \mathbb{R}$, $c_j \neq c_k$, for $j \neq k$ and c_j such that $-2\sqrt{\omega_j} < c_j < 2\sqrt{\omega_j}$ if $\gamma > 0$ and $-2\sqrt{\omega_j} < c_j < -2s_*\sqrt{\omega_j}$ if $\gamma \leqslant 0$. The multi-soliton profile R is given as in (4.6). Then there exists a certain positive constant C_* such that if the parameters (ω_j, c_j) satisfy*

$$C_* \left((1 + \|R_x\|_{L_t^\infty L_x^\infty})(1 + \|R\|_{L_t^\infty L_x^\infty}) + \|R\|^4_{L_t^\infty L_x^\infty}\right) \leqslant v_* := \inf_{j\neq k} h_j|c_j - c_k|, \tag{4.8}$$

then there exist $T_0 > 0$ depending on $\omega_1, ..., \omega_K, c_1, ..., c_K$ and a solution u of (4.2) on $[T_0, \infty)$ such that

$$\|u - R\|_{H^1} \leqslant Ce^{-\lambda t}, \quad \forall t \geqslant T_0, \tag{4.9}$$

where $\lambda = \frac{v_*}{16}$ and C is a positive constant depending on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$.

We observe that the formula for solitons in the case $\gamma > 0$ and in the case $\gamma \leqslant 0$ is similar. Thus, in the proof of Theorem 4.1, we only consider the case $\gamma > 0$. The case $\gamma \leqslant 0$ is treated by similar arguments.

Remark 4.2. We give an example of parameters satisfying (4.8). Let $d_j < 0$, $h_j \in \mathbb{R}$ for all $j \in \{1, 2, ..., K\}$ such that $d_j \neq d_k$ for all $j \neq k$. Let $(c_j, \omega_j) = \left(Md_j, \frac{1}{4}(h_j^2 + M^2 d_j^2)\right)$. We prove that for M large enough, the condition (4.8) is satisfied. By this choosing, we have $h_j \ll |c_j|$ and $c_j < 0$ for all j. We have

$$\|\Phi_{\omega_j,c_j}\|^2_{L^\infty} \leqslant \frac{2h_j^2}{\sqrt{c_j^2 + \gamma h_j^2} - c_j} \lesssim \frac{h_j^2}{|c_j|}.$$

Moreover,

$$\partial\Phi_{\omega_j,c_j} = \frac{-\sqrt{2}}{2} h_j^2 \sqrt{c_j^2 + \gamma h_j^2}\sinh(h_j x)\left(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j x) - c_j\right)^{-\frac{3}{2}}.$$

Thus, for all j, we obtain

$$
\begin{aligned}
|\partial\Phi_{\omega_j,c_j}| &\lesssim h_j^2\sqrt{c_j^2+\gamma h_j^2}|\sinh(h_jx)|\left(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_jx)-c_j\right)^{-\frac{3}{2}} \\
&\lesssim h_j^2\left(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_jx)-c_j\right)^{-\frac{1}{2}} \\
&\approx h_j|\Phi_{\omega_j,c_j}| \lesssim \frac{h_j^2}{\sqrt{|c_j|}}.
\end{aligned}
$$

In the addition, we have

$$
\begin{aligned}
\|\partial R_j\|_{L^\infty} = \|\partial\phi_{\omega_j,c_j}\|_{L^\infty} &\approx \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \left\|\frac{c_j}{2}\Phi_{\omega_j,c_j} - \Phi^3_{\omega_j,c_j}\right\|_{L^\infty} \\
&\leqslant \|\partial\Phi_{\omega_j,c_j}\|_{L^\infty} + \frac{|c_j|}{2}\|\Phi_{\omega_j,c_j}\|_{L^\infty} + \|\Phi_{\omega_j,c_j}\|^3_{L^\infty} \\
&\lesssim \frac{h_j^2}{\sqrt{|c_j|}} + h_j\sqrt{|c_j|} + \frac{h_j^3}{\sqrt{|c_j|^3}}.
\end{aligned}
$$

Thus, the left hand side of (4.8) is bounded by

$$
C_*\left(\left(1+\sum_{1\leqslant j\leqslant K}\left(\frac{h_j^2}{\sqrt{|c_j|}}+h_j\sqrt{|c_j|}+\frac{h_j^3}{\sqrt{|c_j|^3}}\right)\right)\left(1+\sum_{1\leqslant j\leqslant K}\frac{h_j}{\sqrt{|c_j|}}\right)+\sum_{1\leqslant j\leqslant K}\frac{h_j^4}{c_j^2}\right). \tag{4.10}
$$

By our choosing, (4.10) is order $M^{\frac{1}{2}}$ and the right hand side of (4.8) is order M^1. Thus, (4.8) is satisfied for M large enough.

4.1.2 Multi kink-solitons

Second, we consider another special case of (4.1) as follows

$$
iu_t + u_{xx} + iu^2\overline{u_x} + b|u|^4u = 0. \tag{4.11}
$$

Our goal is to construct multi kink-solitons of (4.11). The motivation comes from [73, 72], where the authors have constructed an infinite multi kink-soliton train for classical nonlinear Schrödinger equations by using fixed point arguments. However, in the case of (4.11), this method can not directly be used due to the appearing of a derivative term. To overcome this difficulty, use a transformation and work on a system of two equations without derivative nonlinearites.

Consider the equation (4.11). First, we would like to define a kink solution of (4.11). Let $R_{\omega,c}$ be a smooth solution of (4.11) of the form:

$$
R_{\omega,c}(t,x) = e^{i\omega t}\phi_{\omega,c}(x-ct), \tag{4.12}
$$

where $\phi_{\omega,c}$ is smooth and solves

$$
-\phi_{xx} + \omega\phi + ic\phi_x - i\phi^2\overline{\phi_x} - b|\phi|^4\phi = 0, \quad x\in\mathbb{R}. \tag{4.13}
$$

If $\phi_{\omega,c}\ |_{\mathbb{R}^+} \in H^1(\mathbb{R}^+)$ then the following Gauge transform is well defined:

$$\Phi_{\omega,c} = \exp\left(-i\frac{c}{2}x + \frac{i}{4}\int_{\infty}^{x} |\phi_{\omega,c}(y)|^2\,dy\right)\phi_{\omega,c}.$$

Since $\phi_{\omega,c}$ solves (4.13), $\Phi_{\omega,c}$ is smooth and solves

$$-\Phi_{xx}+\left(\omega-\frac{c^2}{4}\right)\Phi-\frac{3}{2}\mathcal{I}m(\overline{\Phi}\Phi_x)\Phi-\frac{c}{2}|\Phi|^2\Phi+\frac{3}{16}\gamma|\Phi|^4\Phi=0,\quad \gamma:=\frac{5}{3}-\frac{16}{3}b. \tag{4.14}$$

Since $\Phi_{\omega,c}\ |_{\mathbb{R}^+} \in H^2(\mathbb{R}^+)$, by similar arguments as in [23, Proof of Lemma 2], we can prove that

$$\mathcal{I}m(\overline{\Phi_{\omega,c}}\partial_x\Phi_{\omega,c}) = 0.$$

Thus, $\Phi_{\omega,c}$ solves

$$-\Phi_{xx}+\left(\omega-\frac{c^2}{4}\right)\Phi-\frac{c}{2}|\Phi|^2\Phi+\frac{3}{16}\gamma|\Phi|^4\Phi=0. \tag{4.15}$$

Now, we give the definition of a half-kink of (4.2).

Definition 4.3. The function $R_{\omega,c}$ is called a half-kink solution of (4.2) if $R_{\omega,c}$ is of the form (4.12) and the associated $\Phi_{\omega,c}$ is a real valued function solving (4.15) and satisfying:

$$\begin{cases}\lim_{x\to\pm\infty}\Phi(x)\neq 0,\\ \lim_{x\to\mp\infty}\Phi(x)=0,\end{cases} \tag{4.16}$$

where $\tilde{\omega}=\omega-\frac{c^2}{4}$, $f:\mathbb{R}\to\mathbb{R}$.

For more convenience, we define

$$f(s)=\frac{c}{2}s^3-\frac{3}{16}\gamma s^5.$$

The following result about the existence of a half-kink profile is stated in [72] as follows.

Proposition 4.4. *Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1 function with $f(0)=0$ and define $F(s):=\int_0^s f(t)\,dt$. For $\tilde{\omega}\in\mathbb{R}$, let*

$$\zeta(\tilde{\omega}):=\inf\left\{\zeta>0, F(\zeta)-\frac{1}{2}\tilde{\omega}\zeta^2=0\right\}$$

and assume that there exists $\tilde{\omega}_1\in\mathbb{R}$ such that

$$\zeta(\tilde{\omega}_1)>0,\quad f'(0)-\tilde{\omega}_1<0,\quad f(\zeta(\tilde{\omega}_1))-\tilde{\omega}_1\zeta(\tilde{\omega}_1)=0. \tag{4.17}$$

Then, for $\tilde{\omega}=\tilde{\omega}_1$, there exists a half-kink profile $\Phi\in C^2(\mathbb{R})$ of (4.16) i.e Φ is unique (up to translation), positive and satisfies $\Phi'>0$ on $\mathbb{R}$ and the boundary conditions

$$\lim_{x\to-\infty}\Phi(x)=0,\quad \lim_{x\to\infty}\Phi(x)=\zeta(\tilde{\omega}_1)>0. \tag{4.18}$$

If in addition,

$$f'(\zeta(\tilde{\omega}_1))-\tilde{\omega}_1<0, \tag{4.19}$$

then for any $0<a<\tilde{\omega}_1-\max\{f'(0),f'(\zeta(\tilde{\omega}_1))\}$ there exists $D_a>0$ such that

$$|\Phi'(x)|+|\Phi(x)1_{x<0}|+|(\zeta(\tilde{\omega}_1)-\Phi(x))1_{x>0}|\leqslant D_a e^{-a|x|},\quad \forall x\in\mathbb{R}. \tag{4.20}$$

We have the following remarks.

Remark 4.5.

(1) As in [72, Remark 1.15], using the symmetry $x \to -x$ and Proposition 4.4 implies the existence and uniqueness of half-kink profile Φ satisfying

$$\lim_{x\to-\infty}\Phi(x)=\zeta(\tilde{\omega}_1)>0,\quad \lim_{x\to\infty}\Phi(x)=0.$$

(2) In our case, $f(s)=\frac{c}{2}s^3-\frac{3}{16}\gamma s^5$. We may check that if $\gamma>0$, $c>0$ then there exist $\tilde{\omega}_1=\frac{c^2}{4\gamma}$ and $\zeta(\tilde{\omega}_1)=\sqrt{\frac{2c}{\gamma}}$ satisfying the conditions (4.17), (4.19) and the definition of the function ζ. Thus, using Proposition 4.4, if $\gamma>0$, $c>0$ then there exists a half-kink solution of (4.2) and the constant a in Proposition 4.4 satisfy

$$0<a<\frac{c^2}{4\gamma}.$$

(3) Consider the half-kink profile Φ of Proposition 4.4. Since Φ solves (4.16) and satisfies (4.20), we have

$$|\Phi''(x)|+|\Phi'''(x)|\leqslant D_a e^{-a|x|}.$$

Now, we assume $\gamma>0$. Let $K>0$, $\theta_0,\omega_0,c_0\in\mathbb{R}$ be such that $2\sqrt{\omega_0}>c_0>\sqrt{2\gamma}$. For $1\leqslant j\leqslant K$, let $(\theta_j,\omega_j,c_j)\in\mathbb{R}$ be such that $c_j>c_0$, $c_j\neq c_k$ for $j\neq k$, $2\sqrt{\omega_j}>c_j>2s_*\sqrt{\omega_j}$ for $s_*=\sqrt{\frac{\gamma}{1+\gamma}}$. Set $R_j=e^{i\theta_j}R_{\omega_j,c_j}$, where $R_{\omega_j,c_j}\in H^1(\mathbb{R})$ is the soliton solution of (4.11) with the associated profile defined in (4.5). Let Φ_0 be the half-kink profile given in Remark 4.5 (1) associated with the parameters ω_0,c_0 and R_{ω_0,c_0} be the associated half-kink solution of (4.11). Set $R_0=e^{i\theta}R_{\omega_0,c_0}$. The multi kink-soliton profile of (4.11) is defined as follows:

$$V=R_0+\sum_{j=1}^{K}R_j. \tag{4.21}$$

Our second main result is the following.

Theorem 4.6. *Considering* (4.11)*, we assume that* $b<\frac{5}{16}$ $(\gamma>0)$*. Let* V *be given as in* (4.21)*. There exists a certain positive constant* C_* *such that if the parameters* (ω_j,c_j) *satisfy*

$$C_*\left(\left(1+\|V_x\|_{L_t^\infty L_x^\infty}\right)\left(1+\|V\|_{L_t^\infty L_x^\infty}\right)+\|V\|^4_{L_t^\infty L_x^\infty}\right)\leqslant v_*:=\min\left(\inf_{j\neq k}h_j|c_j-c_k|,\inf_{j\neq 0}|c_j-c_0|\right), \tag{4.22}$$

then there exist a solution u *to* (4.11) *such that*

$$\|u-V\|_{H^1}\leqslant Ce^{-\lambda t}.\quad \forall t\geqslant T_0, \tag{4.23}$$

where $\lambda=\frac{v_*}{16}$ *and* C,T_0 *are positive constants depending on the parameters* $\omega_0,...,\omega_K,c_0,...,c_K$*.*

We have some following discussions about the above theorem.

Remark 4.7.

(1) The condition $c_0^2 > 2\gamma$ in Theorem 4.6 is a technical condition and we can remove this. The constant a in Proposition 4.4 satisfies

$$0 < a < \frac{c_0^2}{4\gamma}.$$

Thus, under the condition $c_0^2 > 2\gamma$, we can choose $a = \frac{1}{2}$. This fact makes the proof easier and we have

$$|\Phi_0'''(x)| + |\Phi_0''(x)| + |\Phi_0'(x)| + |\Phi_0(x)1_{x>0}| + \left| \left(\sqrt{\frac{2c_0}{\gamma}} - \Phi_0(x) \right) 1_{x<0} \right| \lesssim e^{-\frac{1}{2}|x|}. \quad (4.24)$$

(2) By similar arguments as above, we can prove that there exists a half-kink solution of (4.2) which satisfies the definition 4.3. To our knowledge, there are no result about stability or instability of this kind of solution.

(3) Let $\gamma > 0$. We give an example of parameters satisfying the condition (4.22) of Theorem 4.6. As in Remark 4.2, we have

$$\Phi_{\omega_j,c_j} = \sqrt{2}h_j \left(\sqrt{c_j^2 - \gamma h_j^2} \cosh(h_j x) + c_j \right)^{\frac{-1}{2}}, \quad \forall j = 1, ..., K.$$

Hence, choosing $h_j \ll c_j$, for all j, we have

$$\|\Phi_{\omega_j,c_j}\|_{L^\infty}^2 \leqslant \frac{2h_j^2}{\sqrt{c_j^2 - \gamma h_j^2} + c_j} \lesssim \frac{h_j^2}{c_j}.$$

By similar arguments as in Remark 4.2, for all $1 \leqslant j \leqslant K$, we have

$$\|\partial R_j\|_{L^\infty} \lesssim \frac{h_j^2}{\sqrt{c_j}} + h_j\sqrt{c_j} + \frac{h_j^3}{\sqrt{c_j^3}}.$$

Now, we treat to the case $j = 0$. Let Φ_0 be the profile given as in Proposition 4.4 associated to the parameters c_0, ω_0 and R_0 be the associated half-kink solution of (4.2). From (4.20), Remark 4.5 and Remark 4.7 we have

$$\|\Phi_0\|_{L^\infty} \lesssim \sqrt{c_0},$$
$$\|\partial\Phi_0\|_{L^\infty} \lesssim 1,$$

Thus,

$$\|R_0\|_{L^\infty L^\infty} \lesssim \sqrt{c_0},$$
$$\|\partial R_0\|_{L^\infty L^\infty} \lesssim 1 + c_0^{\frac{3}{2}} \lesssim c_0^{\frac{3}{2}}.$$

This implies that for $h_j \ll c_j$ $(j = 1, .., K)$ the left hand side of (4.22) is estimated by:

$$C_* \left(\left(1 + c_0^{\frac{3}{2}} + \sum_{j=1}^{K} \left(\frac{h_j^2}{\sqrt{c_j}} + h_j\sqrt{c_j} + \frac{h_j^3}{\sqrt{c_j^3}} \right) \right) \left(1 + \sqrt{c_0} + \sum_{j=1}^{K} \frac{h_j}{\sqrt{c_j}} \right) \right).$$

Choosing $c_0 \approx 1$, the above expression is estimated by:

$$C_* \left(\left(1 + \sum_{j=1}^{K} \left(\frac{h_j^2}{\sqrt{c_j}} + h_j\sqrt{c_j} + \frac{h_j^3}{\sqrt{c_j^3}} \right) \right) \left(1 + \sum_{j=1}^{K} \frac{h_j}{\sqrt{c_j}} \right) \right). \tag{4.25}$$

Let $h_j, d_j \in \mathbb{R}^+$, $d_j \neq d_k$ for all $j \neq k$, $1 \leqslant j,k \leqslant K$. Set $c_j = Md_j$, $\omega_j = \frac{1}{4}(h_j^2 + M^2 d_j^2)$. We have (4.25) is of order M^0 and the right hand side of (4.22) is of order M^1. Thus, by these choices of parameters, when M is large enough, the condition (4.22) is satisfied.

The proof of Theorem 4.6 uses similar arguments as in the one of Theorem 4.1. To prove Theorem 4.1, our strategy is the following. Let R be the multi-soliton profile. Our aim is to construct a solution of (4.2) which behaves as R at large times. Using the Gauge transform (4.26), we construct a system of equations of (φ, ψ). Let h, k be the profile under the Gauge transform of R. We see that h, k solves the same system as φ, ψ up to exponential decay pertubations. The decay of these terms is showed by using the separation of solitons. Set $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. We see that if u solves (4.2) then $(\tilde{\varphi}, \tilde{\psi})$ solves (4.35). By using the Banach fixed point theorem, we show that there exists a solution of this system which decays exponentially fast at infinity. Using this property and combining with the condition (4.8), we may prove a relation between $\tilde{\varphi}$ and $\tilde{\psi}$. This relation allows us to obtain a solution of (4.2) satisfying the desired property.

This chapter is organized as follows. In the section 4.2, we prove the existence of multi-solitons for the equation (4.2). In the section 4.3, we prove the existence of multi kink-solitons for the equation (4.11). In the section 4.4, we prove some tools which is used in the proofs in the section 4.2 and the section 4.3. More precisely, we prove the exponential decay of the pertubations in the equations of h, k (Lemma 4.11, Lemma 4.14) and the existence of exponential decay solutions of the systems considered in the proofs of the main results in the section 4.2 (Lemma 4.13). Before proving the main results, we recall Strichartz estimates and introduce some notations used in this chapter. We need the following definition of admissible pairs.

Definition 4.8. Let $N \in \mathbb{N}^*$. We say that a pair (q, r) is admissible if

$$\frac{2}{q} = N\left(\frac{1}{2} - \frac{1}{r}\right),$$

and

$$2 \leqslant r \leqslant \frac{2N}{N-2} \quad (2 \leqslant r \leqslant \infty \text{ if } N = 1\, 2 \leqslant r < \infty \text{ if } N = 2).$$

Lemma 4.9. *(Strichartz estimates)(see e.g [16, Theorem 2.3.3]) Let $S(t)$ be the Schrödinger group. The following properties holds:*

(i) There exists a constant C such that for all $\varphi \in L^2(\mathbb{R}^N)$, we have

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R}, L^r)} \leqslant C\|\varphi\|_{L^2},$$

for every admissible pair (q, r).

(ii) Let I be an interval of $\mathbb{R}$ and $t_0 \in \overline{I}$. Let (γ, ρ) be an admissible pair and $f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N))$. Then, for all admissible pair (q, r), the function

$$t \mapsto \Phi_f(t) = \int_{t_0}^{t} S(t-s) f(s)\, ds$$

belong to $L^q(I, L^r(\mathbb{R}^N)) \cap C(I, L^2(\mathbb{R}^N))$. Moreover, there exists a constant C independent of I such that

$$\|\Phi_f\|_{L^q(I,L^r)} \leqslant C \|f\|_{L^{\gamma'}(I,L^{\rho'})}, \quad \textit{for all } f \in L^{\gamma'}(I, L^{\rho'}(\mathbb{R}^N)).$$

Notation.

(1) For $t > 0$, the Strichartz space $S([t,\infty))$ is defined via the norm

$$\|u\|_{S([t,\infty))} = \sup_{(q,r) \ \textit{admissible}} \|u\|_{L^q_\tau L^r_x([t,\infty)\times\mathbb{R})}$$

The dual space is denoted by $N([t,\infty)) = S([t,\infty))^*$.

(2) For $z = (a, b) \in \mathbb{C}^2$ a vector, we denote $|z| = |a| + |b|$.

(3) We denote $a \lesssim b$, for $a, b > 0$, if a is smaller than b up to multiplication by a positive constant. Moreover, we denote $a \approx b$ if a equal to b up to multiplication by a positive constant.

(4) We denote $a \lesssim_k b$ if there exists a constant $C(k)$ depending only on k such that $a \leqslant C(k) b$.
Particularly, we denote $a \lesssim_p b$ if there exists a constant C depending only on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$ such that $a \leqslant Cb$.

(5) Let $f \in C^1(\mathbb{R})$. We use ∂f or f_x to denote the derivative in space of the function f.

(6) Let $f(x, y, z, ..)$ be a function. We denote $|df| = |f_x| + |f_y| + |f_z| + ...$

4.2 Proof of Theorem 4.1

In this section, we give the proof of Theorem 4.1. We divide our proof into three steps.

Step 1. Preliminary analysis

Considering the following transform:

$$\begin{cases} \varphi(t,x) = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |u(t,y)|^2\, dy\right) u(t,x), \\ \psi = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi. \end{cases} \tag{4.26}$$

By similar arguments as in [59] and [100], we see that if u solves (4.2) then (φ, ψ) solves the following system

$$\begin{cases} L\varphi = i\varphi^2\overline{\psi} - b|\varphi|^4\varphi, \\ L\psi = -i\psi^2\overline{\varphi} - 3b|\varphi|^4\psi - 2b|\varphi|^2\varphi^2\overline{\psi}, \\ \varphi\mid_{t=0} = \varphi_0 = \exp\left(\frac{i}{2}\int_{-\infty}^{x} |u_0(y)|^2\, dy\right) u_0, \\ \psi\mid_{t=0} = \psi_0 = \partial\varphi_0 - \frac{i}{2}|\varphi_0|^2\varphi_0, \end{cases} \tag{4.27}$$

where $L = i\partial_t + \partial_{xx}$. Define

$$P(\varphi,\psi) = i\varphi^2\overline{\psi} - b|\varphi|^4\varphi,$$
$$Q(\varphi,\psi) = -i\psi^2\overline{\varphi} - 3b|\varphi|^4\psi - 2b|\varphi|^2\varphi^2\overline{\psi}.$$

Let R be the multi soliton profile given in (4.6). Since R_j solves (4.2), for all j, by an elementary calculation, we have

$$iR_t+R_{xx}+i|R|^2R_x+b|R|^4R = i\left(|R|^2R_x - \sum_{j=1}^K |R_j|^2R_{jx}\right)+b\left(|R|^4R - \sum_{j=1}^K |R_j|^4R_j\right). \tag{4.28}$$

From Lemma 4.11, we have

$$\left\| |R|^2R_x - \sum_{j=1}^K |R_j|^2R_{jx}\right\|_{H^2} + \left\| |R|^4R - \sum_{j=1}^K |R_j|^4R_j\right\|_{H^2} \leqslant e^{-\lambda t}, \tag{4.29}$$

where $\lambda = \frac{1}{16}v_*$. Thus, we rewrite (4.28) as follows

$$iR_t + R_{xx} + i|R|^2R_x + b|R|^4R = e^{-\lambda t}v(t,x), \tag{4.30}$$

where $v(t)\in H^2(\mathbb{R})$ is such that $\|v(t)\|_{H^2}$ is uniformly bounded in t. Define

$$h(t,x) = \exp\left(\frac{i}{2}\int_{-\infty}^x |R|^2\,dy\right)R(t,x), \tag{4.31}$$
$$k = h_x - \frac{i}{2}|h|^2h. \tag{4.32}$$

By an elementary calculation, we have

$$Lh = ih^2\overline{k} - b|h|^4h + e^{-t\lambda}m(t,x) = P(h,k) + e^{-t\lambda}m(t,x),$$
$$Lk = -ik^2\overline{h} - 3b|h|^4k - 2b|h|^2h^2\overline{k} + e^{-t\lambda}n(t,x) = Q(h,k) + e^{-t\lambda}n(t,x),$$

where m,n satisfy

$$m = v\exp\left(\frac{i}{2}\int_{-\infty}^x |R|^2\,dy\right) - h\int_{-\infty}^x \mathcal{Im}(v\overline{R})\,dy, \tag{4.33}$$
$$n = m_x - i|h|^2m + \frac{i}{2}h^2\overline{m}. \tag{4.34}$$

From Lemma 4.12, we have $\|m(t)\|_{H^1} + \|n(t)\|_{H^1}$ uniformly bounded in t. Set $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. Then $\tilde{\varphi}$, $\tilde{\psi}$ solve:

$$\begin{cases} L\tilde{\varphi} = P(\varphi,\psi) - P(h,k) - e^{-t\lambda}m(t,x),\\ L\tilde{\psi} = Q(\varphi,\psi) - Q(h,k) - e^{-t\lambda}n(t,x).\end{cases} \tag{4.35}$$

Set $\eta = (\tilde{\varphi},\tilde{\psi})$, $W = (h,k)$, $H = -e^{-t\lambda}(m,n)$ and $f(\varphi,\psi) = (P(\varphi,\psi),Q(\varphi,\psi))$. We express solutions of (4.35) in the following form:

$$\eta(t) = i\int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\,ds, \tag{4.36}$$

where $S(t)$ is the Schrödinger group. Moreover, by using $\psi = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi$, we have

$$\tilde{\psi} = \partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi}+h|^2(\tilde{\varphi}+h) - |h|^2 h). \tag{4.37}$$

Step 2. Existence a solution of (4.35)

From Lemma 4.13, there exists $T_* \gg 1$ such that for $T_0 \geqslant T_*$ there exists a unique solution η defined on $[T_0,\infty)$ of (4.35) such that

$$e^{t\lambda}(\|\eta\|_{S([t,\infty))\times S([t,\infty))}) + e^{t\lambda}(\|\eta_x\|_{S([t,\infty))\times S([t,\infty))}) \leqslant 1, \quad \forall t \geqslant T_0, \tag{4.38}$$

Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}, \tag{4.39}$$

Step 3. Existence of multi-solitons

Let η be the solution of (4.35) found in step 1. We prove that the solution $\eta = (\tilde{\varphi},\tilde{\psi})$ of (4.35) satisfies the relation (4.37). Set $\varphi = \tilde{\varphi} + h$, $\psi = \tilde{\psi} + k$ and

$$v = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi.$$

Since h solves $Lh = P(h,k) + e^{-t\lambda}m(t,x)$ and $\tilde{\varphi}$ solves $L\tilde{\varphi} = P(\varphi,\psi) - P(h,k) - e^{-t\theta}m(t,x)$, we have $L\varphi = P(\varphi,\psi)$. Similarly, $L\psi = Q(\varphi,\psi)$. We have

$$\begin{cases} L\varphi = P(\varphi,\psi), \\ L\psi = Q(\varphi,\psi). \end{cases}$$

Thus,

$$\begin{aligned} L\psi - Lv &= Q(\varphi,\psi) - \left(\partial L\varphi - \frac{i}{2}L(|\varphi|^2\varphi)\right) \\ &= Q(\varphi,\psi) - \left(\partial L\varphi - \frac{i}{2}(L(\varphi^2)\overline{\varphi} + \varphi^2 L(\overline{\varphi}) + 2\partial(\varphi^2)\partial\overline{\varphi})\right) \\ &= Q(\varphi,\psi) - \left(\partial L\varphi - \frac{i}{2}(2L\varphi|\varphi|^2 + 2(\partial\varphi)^2\overline{\varphi} - \varphi^2\overline{L\varphi} + 2\varphi^2\partial_{xx}\overline{\varphi}) + 4\varphi|\partial\varphi|^2)\right). \end{aligned} \tag{4.40}$$

Moreover,

$$\begin{aligned} L\varphi = P(\varphi,\psi) &= i\varphi^2\overline{\psi} - b|\varphi|^4\varphi \\ &= i\varphi^2\overline{(\psi - v)} + i\varphi^2\overline{v} - b|\varphi|^4\varphi. \end{aligned} \tag{4.41}$$

Combining (4.41) and (4.40) and by an elementary calculation, we obtain

$$
\begin{aligned}
L\psi - Lv &= Q(\varphi,\psi) - \partial(i\varphi^2\overline{(\psi - v)}) - |\varphi|^2\varphi^2\overline{(\psi - v)} - \frac{1}{2}|\varphi|^4(\psi - v) - Q(\varphi, v)\\
&= (Q(\varphi,\psi) - Q(\varphi,v)) - 2i\varphi\partial\varphi\overline{(\psi - v)} - i\varphi^2\partial\overline{(\psi - v)}\\
&\quad - |\varphi|^2\varphi^2\overline{(\psi - v)} - \frac{1}{2}|\varphi|^4(\psi - v)\\
&= -i(\psi^2 - v^2)\overline{\varphi} - 3b|\varphi|^4(\psi - v) - 2b|\varphi|^2\varphi^2\overline{(\psi - v)}\\
&\quad - 2i\varphi\left(v + \frac{i}{2}|\varphi|^2\varphi\right)\overline{(\psi - v)} - i\varphi^2\partial\overline{(\psi - v)}\\
&\quad - |\varphi|^2\varphi^2\overline{(\psi - v)} - \frac{1}{2}|\varphi|^4(\psi - v).
\end{aligned}
\tag{4.42}
$$

Define $\tilde{v} = v - k$. Since $\tilde{\psi} - \tilde{v} = \psi - v$ and (4.42) we have

$$
L\tilde{\psi} - L\tilde{v} = (\tilde{\psi} - \tilde{v})A(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) + \overline{(\tilde{\psi} - \tilde{v})}B(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) - i(\tilde{\varphi} + h)^2\partial\overline{(\tilde{\psi} - \tilde{v})}, \tag{4.43}
$$

where

$$
\begin{aligned}
A &= -i(\tilde{\psi} + \tilde{v} + 2k)(\overline{\tilde{\varphi} + h}) - 3b|\tilde{\varphi} + h|^4 - \frac{1}{2}|\tilde{\varphi} + h|^4,\\
B &= -2b|\tilde{\varphi} + h|^2(\tilde{\varphi} + h)^2 - 2i(\tilde{\varphi} + h)\left(\tilde{v} + k + \frac{i}{2}|\tilde{\varphi} + h|^2(\tilde{\varphi} + h)\right) - |\tilde{\varphi} + h|^2(\tilde{\varphi} + h)^2.
\end{aligned}
$$

We see that A, B are polynomials of degree at most 4 in $(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k)$. Multiplying both sides of (4.43) by $\overline{\tilde{\psi} - \tilde{v}}$ then taking imaginary part and integrating over space using integration by parts, we obtain

$$
\begin{aligned}
\frac{1}{2}\partial_t\|\tilde{\psi} - \tilde{v}\|_{L^2}^2 = \mathcal{I}m\int_{\mathbb{R}} &(\tilde{\psi} - \tilde{v})^2 A(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k) + \overline{(\tilde{\psi} - \tilde{v})}^2 B(\tilde{\psi}, \tilde{v}, \tilde{\varphi}, h, k)\\
&+ \frac{i}{2}\partial(\tilde{\varphi} + h)^2\overline{(\tilde{\psi} - \tilde{v})}^2\, dx.
\end{aligned}
$$

Thus,

$$
\left|\frac{1}{2}\partial_t\|\tilde{\psi} - \tilde{v}\|_{L^2}^2\right| \lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2(\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty}).
$$

By using Grönwall inequality, we obtain

$$
\begin{aligned}
&\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2\\
&\lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde{\varphi} + h)^2\|_{L^\infty}\, ds\right).
\end{aligned}
\tag{4.44}
$$

Combining (4.38), (4.39), using $k = h_x - \frac{i}{2}|h|^2 h$, $\tilde{v} = \partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^2(\tilde{\varphi} + h) - |h|^2 h)$, $|h| = |R|$ and the Sobolev embedding $H^1(\mathbb{R}) \hookrightarrow L^\infty$, we have, for $t \geqslant T_0$:

$$
\begin{aligned}
&\|\tilde{\varphi} + h\|_{L^\infty} \lesssim 1 + \|h\|_{L^\infty},\\
&\qquad \|\tilde{v}\|_{L^\infty} = \left\|\partial\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^2(\tilde{\varphi} + h) - |h|^2 h)\right\|_{L^\infty} \lesssim \|\partial\tilde{\varphi}\|_{L^\infty} + \|\tilde{\varphi}\|_{L^\infty}^3 + \|\tilde{\varphi}\|_{L^\infty}\|h\|_{L^\infty}^2\\
&\qquad\qquad \lesssim 1 + \|\partial\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty}^2.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde\varphi + h)^2\|_{L^\infty})\, ds \\
&\lesssim \int_t^N (\|\tilde\psi\|_{L^\infty} + \|\tilde v\|_{L^\infty} + \|k\|_{L^\infty})\|\tilde\varphi + h\|_{L^\infty} + \|\tilde\varphi + h\|_{L^\infty}^4 + \|\tilde\varphi + h\|_{L^\infty}\|\tilde v + k\|_{L^\infty} \\
&\quad + (\|\tilde\varphi\|_{L^\infty} + \|h\|_{L^\infty})(\|\partial\tilde\varphi\|_{L^\infty} + \|h_x\|_{L^\infty})\, ds \\
&\lesssim \int_t^N (1 + \|\tilde v\|_{L^\infty} + \|k\|_{L^\infty})(1 + \|h\|_{L^\infty}) + 1 + \|h\|_{L^\infty}^4 + (1 + \|h\|_{L^\infty})(\|\tilde v\|_{L^\infty} + \|k\|_{L^\infty}) \\
&+ (1 + \|h\|_{L^\infty})(\|\partial\tilde\varphi\|_{L^\infty} + \|h_x\|_{L^\infty})\, ds \\
&\lesssim \int_t^N 1 + \|h\|_{L^\infty}^4 + \|k\|_{L^\infty}(1 + \|h\|_{L^\infty}) + \|\tilde v\|_{L^\infty}(1 + \|h\|_{L^\infty}) \\
&\quad + (1 + \|h\|_{L^\infty})(\|\partial\tilde\varphi\|_{L^\infty} + \|h_x\|_{L^\infty})\, ds \\
&\lesssim \int_t^N 1 + \|h\|_{L^\infty}^4 + \|k\|_{L^\infty}(1 + \|h\|_{L^\infty}) + \|\partial\tilde\varphi\|_{L^\infty}(1 + \|h\|_{L^\infty}) \\
&\quad + (1 + \|h\|_{L^\infty})(\|\partial\tilde\varphi\|_{L^\infty} + \|k\|_{L^\infty} + \|h\|_{L^\infty}^3)\, ds \\
&\lesssim \int_t^N 1 + \|h\|_{L^\infty}^4 + \|k\|_{L^\infty}(1 + \|h\|_{L^\infty}) + \|\partial\tilde\varphi\|_{L^\infty}(1 + \|h\|_{L^\infty})\, ds \\
&\lesssim (N-t)(1 + \|h\|_{L^\infty L^\infty}^4 + \|k\|_{L^\infty L^\infty}(1 + \|h\|_{L^\infty L^\infty})) \\
&\quad + \|\partial\tilde\varphi\|_{L^4(t,N)L^\infty}(\|1\|_{L^{\frac43}(t,N)} + \|h\|_{L^{\frac43}(t,N)L^\infty}) \\
&\lesssim (N-t)(1 + \|R\|_{L^\infty L^\infty}^4 + (\|h_x\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^3)(1 + \|R\|_{L^\infty L^\infty})) \\
&\quad + (N-t)^{\frac34}(1 + \|R\|_{L^\infty L^\infty}^{\frac43}) \\
&\lesssim (N-t)(1 + \|R\|_{L^\infty L^\infty}^4 + \|R_x\|_{L^\infty L^\infty}(1 + \|R\|_{L^\infty L^\infty})) + (N-t)^{\frac34}(1 + \|R\|_{L^\infty L^\infty}^{\frac43}).
\end{aligned}
$$

Thus, there exists a certain positive constant C_0 such that

$$
\begin{aligned}
&\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde\varphi + h)^2\|_{L^\infty})\, ds \\
&\leqslant C_0 \Big((N-t)(1 + \|R\|_{L^\infty L^\infty}^4 + \|R_x\|_{L^\infty L^\infty}(1 + \|R\|_{L^\infty L^\infty})) + (N-t)^{\frac34}(1 + \|R\|_{L^\infty L^\infty}^{\frac43}) \Big).
\end{aligned}
$$

Let $C_* = 32C_0$. From the assumption (4.8), we have

$$C_0 \left((1 + \|R_x\|_{L^\infty L^\infty})(1 + \|R\|_{L^\infty L^\infty}) + \|R\|_{L^\infty L^\infty}^4 \right) \leqslant \frac{v_*}{32} = \frac{\lambda}{2}.$$

Hence, fix t and let N large enough, we have

$$\int_t^N (\|A\|_{L^\infty} + \|B\|_{L^\infty} + \|\partial(\tilde\varphi + h)^2\|_{L^\infty})\, ds \leqslant (N-t)\lambda.$$

Combining with (4.39) and (4.44), we obtain, for N large enough:

$$\|\tilde\psi(t) - \tilde v(t)\|_{L^2}^2 \lesssim e^{-2\lambda N} e^{(N-t)\lambda} = e^{-\lambda N - t\lambda}.$$

Let $N \to \infty$, we obtain
$$\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 = 0.$$
This implies that $\tilde{\psi} = \tilde{v}$ and we have
$$\psi = v = \partial\varphi - \frac{i}{2}|\varphi|^2\varphi. \tag{4.45}$$
Define $u = \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |\varphi(y)|^2\,dy\right)\varphi$. Combining (4.45) with the fact that (φ, ψ) solves
$$\begin{cases} L\varphi = P(\varphi, \psi), \\ L\psi = Q(\varphi, \psi), \end{cases}$$
we obtain that u solves (4.2). Moreover,
$$\begin{aligned}\|u - R\|_{H^1} &= \left\|\exp\left(-\frac{i}{2}\int_{-\infty}^{x} |\varphi(y)|^2\,dy\right)\varphi - \exp\left(-\frac{i}{2}\int_{-\infty}^{x} |h(y)|^2\,dy\right)h\right\|_{H^1} \\ &\lesssim \|\varphi - h\|_{H^1} = \|\tilde{\varphi}\|_{H^1}\end{aligned}$$
Combining with (4.39), for $t \geqslant T_0$, we have
$$\|u - R\|_{H^1} \leqslant Ce^{-\lambda t},$$
for a constant C depending on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$. This completes the proof of Theorem 4.1.

4.3 Proof of Theorem 4.6

In this section, we prove Theorem 4.6. We use the similar idea in the proof of Theorem 4.1. However, the argument used in this section cannot apply to (4.2) (see Remark 4.10). We divide our proof into three steps:

Step 1. Preliminary analysis

Set
$$v := u_x + \frac{i}{2}|u|^2 u.$$
By an elementary calculation, we see that if u solves (4.2) then (u, v) solves the following system:
$$\begin{cases} Lu = -iu^2\overline{v} + \left(\frac{1}{2} - b\right)|u|^4 u, \\ Lv = iv^2\overline{u} + \left(\frac{3}{2} - 3b\right)|u|^4 v + (1-2b)|u|^2 u^2 \overline{v}, \\ u\mid_{t=0} = u_0, \\ v\mid_{t=0} = v_0 = \partial u_0 + \frac{i}{2}|u_0|^2 u_0. \end{cases} \tag{4.46}$$
Define
$$P(u, v) = -iu^2\overline{v} + \left(\frac{1}{2} - b\right)|u|^4 u,$$
$$Q(u, v) = iv^2\overline{u} + \left(\frac{3}{2} - 3b\right)|u|^4 v + (1-2b)|u|^2 u^2 \overline{v}.$$

Let V be the multi kink-soliton profile defined in (4.21). Since R_j solves (4.2), for all j, by an elementary calculation, we have

$$iV_t + V_{xx} + iV^2\overline{V_x} + b|V|^4V = i\left(V^2\overline{V_x} - \sum_{j=0}^{K} R_j^2\overline{R_{jx}}\right) + b\left(|V|^4V - \sum_{j=0}^{K}|R_j|^4R_j\right). \tag{4.47}$$

From Lemma 4.14, we have

$$\left\|V^2\overline{V_x} - \sum_{j=0}^{K} R_j^2\overline{R_{jx}}\right\|_{H^2} + \left\||V|^4V - \sum_{j=0}^{K}|R_j|^4R_j\right\|_{H^2} \leqslant e^{-\lambda t}, \tag{4.48}$$

for $\lambda = \frac{1}{16}v_*$. Thus, we rewrite (4.47) as follows

$$iV_t + V_{xx} + iV^2\overline{V_x} + b|V|^4V = e^{-\lambda t}m(t,x), \tag{4.49}$$

where $m(t) \in H^2(\mathbb{R})$ such that $\|m(t)\|_{H^2}$ uniformly bounded in t. Define

$$h = V,$$
$$k = h_x + \frac{i}{2}|h|^2h.$$

By an elementary calculation, h, k satisfy the following system.

$$Lh = -ih^2\overline{k} + \left(\frac{1}{2} - b\right)|h|^4h + e^{-t\lambda}m = P(h,k) + e^{-t\lambda}m,$$
$$Lk = ik^2\overline{h} + \left(\frac{3}{2} - 3b\right)|h|^4k + (1-2b)|h|^2h^2\overline{k} + e^{-t\lambda}n = Q(h,k) + e^{-t\lambda}n.$$

where $n = m_x + i|h|^2m - \frac{i}{2}h^2\overline{m}$ satisfies $\|n(t)\|_{H^1}$ uniformly bounded in t. Let $\tilde{u} = u - h$ and $\tilde{v} = v - k$. Then $(\tilde{u}, \tilde{v})$ solves:

$$\begin{cases} L\tilde{u} = P(u,v) - P(h,k) - e^{-t\lambda}m, \\ L\tilde{v} = Q(u,v) - Q(h,k) - e^{-t\lambda}n. \end{cases} \tag{4.50}$$

Define $\eta = (\tilde{u}, \tilde{v})$, $W = (h,k)$, $H = e^{-t\lambda}(m,n)$ and $f(u,v) = (P(u,v), Q(u,v))$. We find a solution of (4.50) in the Duhamel form

$$\eta = -i\int_t^{\infty} S(t-s)[f(W+\eta) - f(W) + H](s)\,ds. \tag{4.51}$$

Moreover, from $v = u_x + \frac{i}{2}|u|^2u$, we have

$$\tilde{v} = \tilde{u}_x + \frac{i}{2}(|\tilde{u}+h|^2(\tilde{u}+h) - |h|^2h). \tag{4.52}$$

Step 2. Existence a solution of (4.51)

From Lemma 4.13, there exists $T_* \gg 1$ such that for $T_0 \gg T_*$ there exists a unique solution η defined on $[T_0, \infty)$ of (4.51) such that

$$e^{t\lambda}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{t\lambda}\|\eta_x\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0, \tag{4.53}$$

where $\lambda = \frac{v_*}{16}$. Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{u}\|_{H^1} + \|\tilde{v}\|_{H^1} \lesssim e^{-t\lambda}. \tag{4.54}$$

Step 3. Existence of multi kink-solitons

By using similar arguments as in the proof of Theorem 4.1 we can prove that the solution $\eta = (\tilde{\varphi}, \tilde{\psi})$ of (4.51) satisfies the relation (4.52) provided assumption (4.22) is verified. This implies that

$$\tilde{v} = \tilde{u}_x + \frac{i}{2}(|\tilde{u}+h|^2(\tilde{u}+h) - |h|^2 h).$$

Set $u = \tilde{u} + h$, $v = \tilde{v} + k$. We have

$$v = u_x + \frac{i}{2}|u|^2 u. \tag{4.55}$$

Since $(\tilde{u}, \tilde{v})$ solves (4.50), we infer that u, v solve

$$\begin{aligned} Lu &= P(u,v),\\ Lv &= Q(u,v). \end{aligned}$$

Combining with (4.55), we have u solves (4.2). Moreover, for $t \geqslant T_0$, we have

$$\|u - V\|_{H^1} = \|\tilde{u}\|_{H^1} \lesssim e^{-\lambda t}.$$

This completes the proof of Theorem 4.6.

Remark 4.10. We do not have the proof for the construction of multi kink-solitons for (4.2). The reason is that if the profile R in the proof of Theorem 4.1 is not in $H^1(\mathbb{R})$ then the function h defined as in (4.31) is not in $H^1(\mathbb{R})$. Thus, the functions m, n defined as in (4.33) and (4.34) are not in $H^1(\mathbb{R})$ and we can not apply Lemma 4.13 to construct a solution of system (4.35).

4.4 Some technical lemmas

4.4.1 Properties of solitons

In this section, we prove some estimates on the multi-soliton profile used in the proof of Theorem 4.1.

Lemma 4.11. *There exist $T_0 > 0$ and a constant $\lambda > 0$ such that the estimate (4.29) is uniformly true for $t \geqslant T_0$.*

Proof. First, we need some estimates on the soliton profile. We have

$$\begin{aligned} |R_j(x,t)| = |\Phi_{\omega_j,c_j}(x - c_j t)| &= \sqrt{2} h_j \left(\sqrt{c_j^2 + \gamma h_j^2}\cosh(h_j(x - c_j t)) - c_j\right)^{-\frac{1}{2}}\\ &\lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x - c_j t|}. \end{aligned}$$

Moreover,

$$
\begin{aligned}
|\partial R_j(x,t)| &= |\partial \phi_{\omega_j,c_j}(x-c_jt)| \\
&= \frac{-\sqrt{2}}{2} h_j^2 \sqrt{c_j^2+\gamma h_j^2}\,|\sinh(h_j(x-c_jt)|\left(\sqrt{c_j^2+\gamma h_j^2}\cosh(h_j(x-c_jt))-c_j\right)^{-\frac{3}{2}} \\
&\lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|}.
\end{aligned}
$$

By an elementary calculation, we have

$$|\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)| \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|}.$$

For convenience, we set

$$\chi_1 = i|R|^2 R_x - i\sum_{j=1}^{K} |R_j|^2 R_{jx}, \tag{4.56}$$

$$\chi_2 = |R|^4 R - \sum_{j=1}^{K} |R_j|^4 R_j. \tag{4.57}$$

Fix $t>0$. For $x \in \mathbb{R}$, choose $m = m(x) \in \{1,2,...,K\}$ so that

$$|x-c_mt| = \min_j |x-c_jt|.$$

For $j \neq m$, we have

$$|x-c_jt| \geqslant \frac{1}{2}|c_jt-c_mt| = \frac{t}{2}|c_j-c_m|.$$

Thus, we have

$$
\begin{aligned}
&|(R-R_m)(x,t)| + |(\partial R - \partial R_m(x,t))| + |\partial^2 R - \partial^2 R_m| + |\partial^3 R - \partial^3 R_m| \\
&\leqslant \sum_{j\neq m}(|R_j(x,t)| + |\partial R_j(x,t)| + |\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)|) \\
&\lesssim_{h_1,..,h_K,|c_1|,..,|c_K|} \delta_m(x,t) := \sum_{j\neq m} e^{\frac{-h_j}{2}|x-c_jt|}
\end{aligned}
$$

Recall that

$$v_* = \inf_{j\neq k} h_j|c_j-c_k|.$$

We have

$$|(R-R_m)(x,t)|+|(\partial R-\partial R_m(x,t))|+|\partial^2 R-\partial^2 R_m|+|\partial^3 R-\partial^3 R_m| \lesssim \delta_m(x,t) \lesssim e^{\frac{-1}{4}v_*t}.$$

Let f_1, g_1, r_1 and f_2, g_2, r_2 be the polynomials of u, u_x, u_{xx}, u_{xxx} and conjugates satisfying:

$$
\begin{aligned}
i|u|^2u_x &= f_1(u,\overline{u},u_x), \quad & |u|^4u &= f_2(u,\overline{u}), \\
\partial(i|u|^2u_x) &= g_1(u,u_x,u_{xx},\overline{u},..), \quad & \partial(|u|^4u) &= g_2(u,u_x,\overline{u},..), \\
\partial^2(i|u|^2u_x) &= r_1(u,u_x,u_{xx},u_{xxx},\overline{u},..), \quad & \partial^2(|u|^4u) &= r_2(u,u_x,u_{xx},\overline{u},..).
\end{aligned}
$$

Denote

$$A = \sup_{|u|+|u_x|+|u_{xx}|+|u_{xxx}|\leqslant \sum_{j=1}^{K}\|R_j\|_{H^4}} (|df_1| + |df_2| + |dg_1| + |dg_2| + |dr_1| + |dr_2|),$$

We have

$$\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2| \\
&\leqslant |f_1(R, R_x) - f_1(R_m, R_{mx})| + |f_2(R) - f_2(R_m)| + \sum_{j\neq m}(|f_1(R_j, R_{jx})| + |f_2(R_j)|) \\
&\quad + |g_1(R, R_x, R_{xx}, ..) - g_1(R_m, R_{mx}, R_{mxx}, ..)| + |g_2(R, R_x, ..) - g_2(R_m, R_{mx}, ..)| \\
&+ \sum_{j\neq m}(g_1(R_j, R_{jx}, R_{jxx}, ..) + g_2(R_j, R_{jx}), ..) \\
&\quad + |r_1(R, R_x, R_{xx}, R_{xxx}, ..) - r_1(R_m, R_{mx}, R_{mxx}, R_{mxxx}, ..)| \\
&\quad + |r_2(R, R_x, R_{xx}, ..) - r_2(R_m, R_{mx}, R_{mxx}, ..)| \\
&\quad + \sum_{j\neq m}(r_1(R_j, R_{jx}, R_{jxx}, R_{jxxx}, ..) + r_2(R_j, R_{jx}, R_{jxx}, ..)) \\
&\leqslant A(|R - R_m| + |R_x - R_{mx}| + |R_{xx} - R_{mxx}| + |R_{xxx} - R_{mxxx}|) \\
&\quad + \sum_{j\neq m} A(|R_j| + |R_{jx}| + |R_{jxx}| + |R_{jxxx}|) \\
&\leqslant 2A\sum_{j\neq m}(|R_j| + |R_{jx}| + |R_{jxx}| + |R_{jxxx}|) \\
&\lesssim_p \delta_m(t, x).
\end{aligned}$$

In particular,

$$\|\chi_1\|_{W^{2,\infty}} + \|\chi_2\|_{W^{2,\infty}} \lesssim_p e^{-\frac{1}{4}v_* t}.$$

Moreover, we have

$$\begin{aligned}
&\|\chi_1\|_{W^{2,1}} + \|\chi_2\|_{W^{2,1}} \\
&\lesssim \sum_{j=1}^{K}(\||R_j|^2 R_{jx}\|_{L^1} + \|\partial(|R_j|^2 R_{jx})\|_{L^1} + \|\partial^2(|R_j|^2 R_{jx})\|_{L^1} \\
&\quad + \|R_j^5\|_{L^1} + \|\partial(|R_j|^4 R_j)\|_{L^1} + \|\partial^2(|R_j|^4 R_j)\|_{L^1}) \\
&\lesssim \sum_{j=1}^{K}(\|R_j\|_{H^1}^3 + \|R_j\|_{H^2}^3 + \|R_j\|_{H^3}^3 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^1}^5 + \|R_j\|_{H^2}^5) < C < \infty
\end{aligned}$$

By Holder inequality, for $1 < r < \infty$, we have

$$\|\chi_1\|_{W^{2,r}} + \|\chi_2\|_{W^{2,r}} \lesssim_p e^{-(1-\frac{1}{r})\frac{1}{4}v_* t}, \quad \forall r \in (1, \infty).$$

Choosing $r = 2$ we obtain:

$$\|\chi_1\|_{H^2} + \|\chi_2\|_{H^2} \lesssim_p e^{-\frac{v_*}{8}t},$$

Thus, for $t \geqslant T_0$, where T_0 large enough depend on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$, we have

$$\|\chi_1\|_{H^2} + \|\chi_2\|_{H^2} \leqslant e^{-\frac{v_*}{16}t}, \quad \forall t \geqslant T_0.$$

Let $\lambda = \frac{v_*}{16}$, we obtain the desired result. $\square$

4.4.2 Prove the boundedness of v, m, n

Let v, m and n be given as in (4.28), (4.33) and (4.34) respectively. In this section, we prove the uniform in time boundedness in $H^2(\mathbb{R})$ of v and in $H^1(\mathbb{R})$ of m, n. We have the following result.

Lemma 4.12. *There exist $C > 0$ and $T_0 > 0$ such that for all $t > T_0$ the functions v, m, n satisfy*

$$\|v(t)\|_{H^2} + \|m(t)\|_{H^1} + \|n(t)\|_{H^1} \leqslant C,$$

Proof. Let χ_1 and χ_2 be defined as in (4.56) and (4.57) respectively. We have

$$e^{-\lambda t} v = \chi_1 + b\chi_2.$$

By Lemma 4.11, we have $\|v(t)\|_{H^2} \leqslant D$, for some constant $D > 0$. From (4.33), we have

$$\|m\|_{H^2} \lesssim \|v\|_{H^2} + \|h\|_{H^2}\|v\|_{H^2}\|R\|_{H^2} \leqslant C_1,$$

for some constant $C_1 > 0$. From, (4.34), we have

$$\|n\|_{L^2} \lesssim \|m_x\|_{L^2} + \|h\|_{H^1}^2\|m\|_{H^1} \leqslant \|m\|_{H^1}(1 + \|h\|_{H^1}^2) \leqslant C_2,$$

for some constant $C_2 > 0$. Moreover, we have

$$\|n_x\|_{L^2} \lesssim \|m_{xx}\|_{L^2} + \|h\|_{H^1}^2\|m\|_{H^1} \leqslant \|m\|_{H^2}(1 + \|h\|_{H^1}^2) \leqslant C_3,$$

for some constant $C_3 > 0$. Choosing $C = D + C_1 + C_2 + C_3$, we obtain the desired result. ☐

4.4.3 Existence solution of system equation

In this section, we prove the existence of solutions of (4.36). For convenience, we recall the equation:

$$\eta(t) = i\int_t^\infty S(t-s)[f(W+\eta) - f(W) + H](s)\,ds, \tag{4.58}$$

where $\eta = (\tilde{u}, \tilde{v})$ is unknown function, $W = (h, k)$, $H = -e^{-t\lambda}(m, n)$ and $f(u, v) = (P(u, v), Q(u, v))$, where P, Q are defined by

$$P(u, v) = -iu^2\overline{v} + \left(\frac{1}{2} - b\right)|u|^4 u,$$
$$Q(u, v) = iv^2\overline{u} + \left(\frac{3}{2} - 3b\right)|u|^4 v + (1 - 2b)|u|^2 u^2 \overline{v}.$$

The existence of solutions of (4.58) is established in the following lemma.

Lemma 4.13. *Let $H = H(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{C}^2$, $W = W(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{C}^2$ be given vector functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geqslant 0$:*

$$\|W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2 \times L^2} \leqslant C_1 \quad \forall t \geqslant T_0, \tag{4.59}$$
$$\|\partial W(t)\|_{L^2 \times L^2} + \|\partial W(t)\|_{L^\infty \times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2 \times L^2} \leqslant C_2, \quad \forall t \geqslant T_0. \tag{4.60}$$

Consider equation (4.58). There exists a constant λ_* such that if $\lambda \geqslant \lambda_*$ then there exists a unique solution η to (4.58) on $[T_0,\infty) \times \mathbb{R}$ satisfying

$$e^{\lambda t}\|\eta\|_{S([t,\infty)) \times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty)) \times S([t,\infty))} \leqslant 1, \quad \forall t \geqslant T_0.$$

Proof. We use similar arguments as in [72, 73]. We rewrite (4.58) into $\eta = \Phi\eta$. We shall show that, for λ sufficiently large, Φ is a contraction map in the ball

$$B = \left\{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty)) \times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty)) \times S([t,\infty))} \leqslant 1\right\}.$$

Step 1. Proof that Φ maps B into B

Let $t \geqslant T_0$, $\eta = (\eta_1, \eta_2) \in B$, $W = (w_1, w_2)$ and $H = (h_1, h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t,\infty)) \times S([t,\infty))} \lesssim \|f(W+\eta) - f(W)\|_{N([t,\infty)) \times N([t,\infty))} \tag{4.61}$$
$$+ \|H\|_{L^1_\tau L^2_x([t,\infty)) \times L^1_\tau L^2_x([t,\infty))}. \tag{4.62}$$

For (4.62), using (4.59), we have

$$\begin{aligned}\|H\|_{L^1_\tau L^2_x([t,\infty)) \times L^1_\tau L^2_x([t,\infty))} &= \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \int_t^\infty e^{-\lambda\tau}\, d\tau \leqslant \frac{1}{\lambda}e^{-\lambda t}.\end{aligned}$$

For (4.61), we have

$$\begin{aligned}|P(W+\eta) - P(W)| &= |P(w_1+\eta_1, w_2+\eta_2) - P(w_1,w_2)|\\ &\lesssim |(w_1+\eta_1)^2\overline{(w_2+\eta_2)} - w_1^2\overline{w_2}| + ||\eta_1+w_1|^4(\eta_1+w_1) - |w_1|^4 w_1|\\ &\lesssim |\eta_1| + |\eta_2| + |\eta_1|^5\end{aligned}$$

Thus,

$$\begin{aligned}\|P(W+\eta) - P(W)\|_{N([t,\infty))} &\lesssim \|\eta_1\|_{N([t,\infty))} + \|\eta_2\|_{N([t,\infty))} + \|\eta_1^5\|_{N([t,\infty))}\\ &\lesssim \|\eta_1\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_2\|_{L^1_\tau L^2_x(t,\infty)} + \|\eta_1^5\|_{L^1_\tau L^2_x(t,\infty)}\\ &\lesssim \int_t^\infty e^{-\lambda\tau}\, d\tau + \int_t^\infty \|\eta_1(\tau)\|_{L^{10}}^5\, d\tau\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t} + \int_t^\infty \|\eta_1(\tau)\|_{L^2}^{\frac{7}{2}}\|\partial\eta_1(\tau)\|_{L^2}^{\frac{3}{2}}\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t} + \int_t^\infty e^{-(7/2\lambda + 3/2\lambda)\tau}\, d\tau\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t} + \frac{1}{7/2\lambda + 3/2\lambda}e^{-(7/2\lambda+3/2\lambda)t} \lesssim \frac{1}{\lambda}e^{-\lambda t}.\end{aligned}$$

By similar arguments as above, we have

$$\|Q(W+\eta)-Q(W)\|_{N([t,\infty))}\lesssim \frac{1}{\lambda}e^{-\lambda t}.$$

Thus, for λ large enough, we have

$$\|\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))}\leqslant \frac{1}{10}e^{-\lambda t}.$$

It remains to estimate $\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))}$. By Strichartz estimate we have

$$\begin{aligned}\|\partial\Phi\eta\|_{S([t,\infty)\times S([t,\infty)))} &\lesssim \|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))} &(4.63)\\ &\quad+\|\partial H\|_{N([t,\infty))\times N([t,\infty))}. &(4.64)\end{aligned}$$

For (4.64), using (4.60), we have

$$\begin{aligned}\|\partial H\|_{N([t,\infty))\times N([t,\infty))} &\leqslant \|\partial h_1\|_{L^1_\tau L^2_x([t,\infty))}+\|\partial h_2\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau=\frac{1}{\lambda}e^{-\lambda t}. \qquad (4.65)\end{aligned}$$

For (4.63), we have

$$\begin{aligned}&\|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))}\\ &=\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))}+\|\partial(Q(W+\eta)-Q(W))\|_{N([t,\infty))}\end{aligned}$$

Furthermore,

$$\begin{aligned}&|\partial(P(W+\eta)-P(W))|\\ &\lesssim |\partial((w_1+\eta_1)^2\overline{(w_2+\eta_2)}-w_1^2\overline{w_2})|+|\partial(|w_1+\eta_1|^4(w_1+\eta_1)-|w_1|^4w_1)|\\ &\lesssim |\partial\eta|(|\eta|^2+|W|^2)+|\partial W|(|\eta|^2+|W||\eta|)\\ &\quad+|\partial\eta|(|\eta|^4+|W|^4)+|\partial W|(|\eta|^4+|\eta||W|^3).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))}\\ &\lesssim \||\partial\eta|(|\eta|^2+|W|^2)\|_{N([t,\infty))}+\||\partial W|(|\eta|^2+|W||\eta|)\|_{N([t,\infty))} &(4.66)\\ &\quad+\||\partial\eta|(|\eta|^4+|W|^4)\|_{N([t,\infty))}+\||\partial W|(|\eta|^4+|\eta||W|^3)\|_{N([t,\infty))}. &(4.67)\end{aligned}$$

For (4.66), using (4.59) and (4.60) and the assumption $\eta\in B$ we have

$$\begin{aligned}&\||\partial\eta|(|\eta|^2+|W|^2)\|_{N([t,\infty))}+\||\partial W|(|\eta|^2+|W||\eta|)\|_{N([t,\infty))}\\ &\lesssim \||\partial\eta||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))}+\||\partial\eta||W|^2\|_{L^1_\tau L^2_x([t,\infty))}+\||\partial W||\eta|^2\|_{L^1_\tau L^2_x([t,\infty))}\\ &\quad+\||\partial W||W||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \||\partial\eta|\|_{L^2_\tau L^2_x([t,\infty))}\||\eta|\|^2_{L^4_\tau L^\infty}+\||\partial\eta|\|_{L^1_\tau L^2_x([t,\infty))}\||W|\|^2_{L^\infty L^\infty}\\ &\quad+\||\partial W|\|_{L^\infty L^\infty}\||\eta|\|_{L^4_\tau L^\infty_x([t,\infty))}\||\eta|\|_{L^{4/3}_\tau L^2_x([t,\infty))}\\ &\quad+\||W|\|_{L^\infty L^\infty}\||\partial W|\|_{L^\infty L^\infty}\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\ &\lesssim \frac{1}{\lambda}e^{-\lambda t}.\end{aligned}$$

For (4.67), using (4.59) and (4.60) and the assumption $\eta \in B$ we have

$$\begin{aligned}
&\||\partial\eta|(|\eta|^4+|W|^4)\|_{N([t,\infty))}+\||\partial W|(|\eta|^4+|\eta||W|^3)\|_{N([t,\infty))}\\
&\lesssim \||\partial\eta|(|\eta|^4+|W|^4)\|_{L^1_\tau L^2_x([t,\infty))}+\||\partial W|(|\eta|^4+|\eta||W|^3)\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \|\partial\eta\|_{L^\infty_\tau L^2_x([t,\infty))}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))}+\|W\|^4_{L^\infty L^\infty}\|\partial\eta\|_{L^1_\tau L^2_x([t,\infty))}\\
&\quad+\|\partial W\|_{L^\infty L^2}\|\eta\|^4_{L^4_\tau L^\infty_x([t,\infty))}+\|\partial W\|_{L^\infty L^\infty}\||W|\|^3_{L^\infty L^\infty}\|\eta\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \frac{1}{\lambda}e^{-\lambda t}.
\end{aligned}$$

Hence,

$$\|\partial(P(W+\eta)-P(W))\|_{N([t,\infty))}\lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{4.68}$$

By similar arguments, we have

$$\|\partial(Q(W+\eta)-Q(W))\|_{N([t,\infty))}\lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{4.69}$$

Combining (4.68) and (4.69), we obtain

$$\|\partial(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))}\lesssim \frac{1}{\lambda}e^{-\lambda t}. \tag{4.70}$$

Combining (4.65) and (4.70), we obtain

$$\|\partial\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}\lesssim \frac{1}{\lambda}e^{-\lambda t}\leqslant \frac{1}{10}e^{-\lambda t},$$

if $\lambda>0$ is large enough. Thus, for $\lambda>0$ large enough

$$\|\Phi\eta\|_X\leqslant 1. \tag{4.71}$$

This implies that Φ map B onto B.

Step 2. Φ is contraction map on B

By using (4.59) and (4.60) and similar estimates as for the proof of (4.71), we can show that, for any $\eta\in B$, $\kappa\in B$,

$$\|\Phi\eta-\Phi\kappa\|_X\leqslant \frac{1}{2}\|\eta-\kappa\|_X.$$

By Banach fixed point theorem there exists a unique solution on B of (4.58). □

4.4.4 Properties of multi kink-solitons profile

In this section, we prove some estimates on the multi kink-solitons profile used in the proof of Theorem 4.6.

Lemma 4.14. *There exist $T_0>0$ and a constant $\lambda>0$ such that the estimate (4.48) is uniformly true for $t\geqslant T_0$.*

Proof. For convenience, set

$$R = \sum_{j=1}^{K} R_j.$$

By similar arguments in the proof of Lemma 4.11, we have

$$|R_j(x,t)| + |\partial R_j(x,t)| + |\partial^2 R_j(x,t)| + |\partial^3 R_j(x,t)| \lesssim_{h_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|},$$

for all $1 \leqslant j \leqslant K$. Define

$$\chi_1 = iV^2\overline{V_x} - i\sum_{j=0}^{K} R_j^2\overline{R_{jx}},$$

$$\chi_2 = |V|^4 V - \sum_{j=0}^{K} |R_j|^4 R_j.$$

Fix $t > 0$. For $x \in \mathbb{R}$, we choose $m = m(x) \in \mathbb{N}$ such that

$$|x - c_m t| = \min_{j \in \mathbb{N}} |x - c_j t|.$$

If $m \geqslant 1$ then by the assumption $c_0 < c_j$ for $j > 0$ we have $x > c_0 t$. Thus, by the asymptotic behaviour of Φ_0 as in Remark 4.7, we can see R_0 as a soliton. More precise, we have

$$|R_0(t,x)| + |R_0'(t,x)| + |R_0''(t,x)| + |R_0'''(t,x)| \lesssim e^{-\frac{1}{2}|x-c_0t|} \lesssim e^{-\frac{1}{4}v_* t}.$$

Using similar argument as in the proof of Lemma 4.11, we have:

$$|(R-R_m)(x,t)| + |(\partial R - \partial R_m)(x,t)| + |(\partial^2 R - \partial^2 R_m)(x,t)| + |\partial^3 R - \partial^3 R_m| \lesssim e^{-\frac{1}{4}v_* t}.$$

Let f_1, g_1, r_1 and f_2, g_2, r_2 be the polynomials of u, u_x, u_{xx}, u_{xxx} and their conjugates such that for all $u \in H^3(\mathbb{R})$:

$$\begin{aligned} iu^2\overline{u_x} &= f_1(u, \overline{u}, u_x), & |u|^4 u &= f_2(u, \overline{u}), \\ \partial(iu^2\overline{u_x}) &= g_1(u, u_x, u_{xx}, \overline{u}, ..), & \partial(|u|^4 u) &= g_2(u, u_x, \overline{u}, ..), \\ \partial^2(iu^2\overline{u_x}) &= r_1(u, u_x, u_{xx}, u_{xxx}, \overline{u}, ..), & \partial^2(|u|^4 u) &= r_2(u, u_x, u_{xx}, \overline{u}, ..). \end{aligned}$$

Denote

$$A = \sup_{|u|+|u_x|+|u_{xx}|+|u_{xxx}| \leqslant \|R_0\|_{W^{4,\infty}} + \sum_{j=1}^{K} \|R_j\|_{H^4(\mathbb{R})}} (|df_1| + |df_2| + |dg_1| + |dg_2| + |dr_1| + |dr_2|).$$

In the case $m = 1$, we have

$$
\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2| \\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + |f_1(V,..) - f_1(R,..)| + |f_2(V,...) - f_2(R,..)| \\
&\quad + |g_1(V,..) - g_1(R,..)| + |g_2(V,..) - g_2(R,..)| \\
&\quad + |r_1(V,..) - r_1(R,..)| + |r_2(V,..) - r_2(R,..)| + |f_1(R, R_x, \overline{R}) - \sum_{j=1}^{K} f_1(R_j, R_{jx}, \overline{R_j})| \\
&\quad + |f_2(R, \overline{R}) - \sum_{j=0}^{K} f_2(R_j, \overline{R_j})| + |g_1(R, R_x, ..) - \sum_{j=1}^{K} g_1(R_j, R_{jx}, ..)| \\
&\quad + |g_2(R, R_x, ..) - \sum_{j=0}^{K} g_2(R_j, R_{jx}, ..)| + |r_1(R, R_x, ..) - \sum_{j=0}^{K} r_1(R_j, R_{jx}, ..)| \\
&\quad + |r_2(R, R_x, ..) - \sum_{j=0}^{K} r_2(R_j, R_{jx}, ..)| \\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + A|R_0| \\
&\quad + A(|(R - R_m)(x,t)| + |(\partial R - \partial R_m)(x,t)| + |(\partial^2 R - \partial^2 R_m)(x,t)| + |\partial^3 R - \partial^3 R_m|) \\
&\quad + A \sum_{j=1, j\neq m}^{K} (|R_j| + |\partial R_j| + |\partial^2 R_j| + |\partial^3 R_j|) \\
&\lesssim |R_0|^2|R_{0x}| + |R_0|^5 + A|R_0| + A \sum_{j=1, j\neq m}^{K} (|R_j| + |\partial R_j| + |\partial^2 R_j| + |\partial^3 R_j|) \\
&\lesssim_p e^{-\frac{1}{4}v_* t},
\end{aligned}
$$

In the case $m = 0$, we have

$$
\begin{aligned}
&|\chi_1| + |\chi_2| + |\partial\chi_1| + |\partial\chi_2| + |\partial^2\chi_1| + |\partial^2\chi_2| \\
&\lesssim \sum_{v=1,2} (|f_v(V, V_x, ..) - f_v(R_0, \partial R_0, ..)| + |g_v(V, V_x, ..) - g_v(R_0, \partial R_0, ..)| \\
&\quad + |r_v(V, V_x, ..) - r_v(R_0, \partial R_0)|) \\
&\quad + \sum_{j=1,\dots,K; v=1,2} (|f_v(R_j, R_{jx}, ..)| + |g_v(R_j, R_{jx}, ..)| + |r_v(R_j, R_{jx}, ..)|) \\
&\lesssim A|R| + A\sum_{j=1}^{K} (|R_j| + |\partial R_j| + |\partial^2 R_j| + |\partial^3 R_j|) \\
&\lesssim_p e^{-\frac{1}{4}v_* t}.
\end{aligned}
$$

In all case we have

$$
\|\chi_1(t)\|_{W^{2,\infty}} + \|\chi_2(t)\|_{W^{2,\infty}} \lesssim_p e^{-\frac{1}{4}v_* t}. \tag{4.72}
$$

On one hand,

$$\begin{aligned}
&\|\chi_1(t)\|_{W^{2,1}} \\
&\lesssim \sum_{j=0}^{K}(\|R_j^2\overline{R_{jx}}\|_{L^1} + \|\partial(R_j^2\overline{R_{jx}})\|_{L^1} + \|\partial^2(R_j^2\overline{R_{jx}})\|_{L^1}) \\
&\lesssim \sum_{j=1}^{K}\|R_j\|_{H^3}^3 + \|\partial R_0\|_{W^{2,1}} < C < \infty.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\|\chi_2(t)\|_{W^{2,1}} \\
&\lesssim \||V|^4V - |R_0|^4R_0\|_{W^{2,1}} + \sum_{j=1}^{K}\||R_j|^5\|_{W^{2,1}} \\
&\lesssim \left\| |R_0|^4 \sum_{j=1}^{K}|R_j| + \sum_{j=1}^{K}|R_j|^5 \right\|_{W^{2,1}} + \sum_{j=1}^{K}\|R_j\|_{W^{2,1}}^5 \\
&\lesssim \sum_{j=1}^{K}\||R_0|^4|R_j|\|_{W^{2,1}} + \sum_{j=1}^{K}\|R_j\|_{W^{2,1}}^5 \\
&\lesssim \sum_{j=1}^{K}(\|R_j\|_{W^{2,1}}\|R_0\|_{W^{2,\infty}}^4 + \|R_j\|_{H^3}^5) < C < \infty.
\end{aligned}$$

Thus,

$$\|\chi_1(t)\|_{W^{2,1}} + \|\chi_1(t)\|_{W^{2,1}} < \infty. \tag{4.73}$$

From (4.72) and (4.73), using Hölder inequality, we have

$$\|\chi_1(t)\|_{H^2} + \|\chi_2(t)\|_{H^2} \lesssim_p e^{-\frac{1}{8}v_*t}.$$

Let T_0 be large enough, we have

$$\|\chi_1(t)\|_{H^2} + \|\chi_2(t)\|_{H^2} \leqslant e^{-\frac{1}{16}v_*t}, \quad \forall t \geqslant T_0.$$

Setting $\lambda = \frac{1}{16}v_*$, we obtain the desired result. □

Chapter 5

Multi-solitons Part 2:Construction of multi-solitons for a generalized derivative nonlinear Schrödinger equations

5.1 Introduction

In this chapter, we consider the following generalized derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \tag{5.1}$$

where $\sigma \in \mathbb{R}^+$ is a given constant and $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}$.

The local well-posedness and global well- posedness of (5.1) was studied in [56] when the initial data is in the Sobolev space $H_0^1(\Omega)$, where Ω is any unbounded interval of $\mathbb{R}$. In this work, Hayashi-Ozawa used an approximation argument. In [103], Santos proved the local well-posedness for small size initial data in weighted Sobolev spaces. The arguments used in this work follow parabolic regularization approach introduced by Kato [67].

The equation (5.1) has a two parameters family of solitons. The stability of the solitons has attracted the attention of many researchers. In [80], by using the abstract theory of Grillakis-Shatah-Strauss [49, 50], Liu-Simpson-Sulem proved that in the case $\sigma \geqslant 2$, the solitons of (5.1) are orbitally unstable; in the case $0 < \sigma < 1$, they are orbitally stable and in the case $\sigma \in (1,2)$ they are orbitally stable if $c < 2z_0\sqrt{\omega}$ and orbitally unstable if $c > 2z_0\sqrt{\omega}$ for some constant $z_0 \in (0,1)$. In the critical case $c = 2z_0\sqrt{\omega}$, Guo-Ning-Wu [52] proved that solitons are always orbitally unstable. In [110], in the case $\sigma \in (1,2)$, Tang and Xu proved the stability of the sum of two solitary waves in the energy space using perturbation arguments, modulational analysis and an energy argument as in [85, 86]. In this chapter, we show the existence of multi-soliton trains in energy space in the case $\sigma \geqslant \frac{5}{2}$. Before stating the main result, we give some preliminaries on multi-soliton trains of (5.1).

As mentioned in [80], the equation (5.1) admits a two-parameters family of solitary waves solutions given by

$$\psi_{\omega,c}(t,x) = \varphi_{\omega,c}(x-ct)\exp\left(i\left(\omega t + \frac{c}{2}(x-ct) - \frac{1}{2\sigma+2}\int_{-\infty}^{x-ct}\varphi_{\omega,c}^{2\sigma}(\eta)\,d\eta\right)\right), \tag{5.2}$$

where $\omega > \frac{c^2}{4}$ and

$$\varphi_{\omega,c}^{2\sigma}(y) = \frac{(\sigma+1)(4\omega-c^2)}{2\sqrt{\omega}\left(\cosh(\sigma\sqrt{4\omega-c^2}y) - \frac{c}{2\sqrt{\omega}}\right)}. \tag{5.3}$$

The profile $\varphi_{\omega,c}$ is a positive solution of

$$-\partial_y^2\varphi_{\omega,c} + \left(\omega - \frac{c^2}{4}\right)\varphi_{\omega,c} + \frac{c}{2}|\varphi_{\omega,c}|^{2\sigma}\varphi_{\omega,c} - \frac{2\sigma+1}{(2\sigma+2)^2}|\varphi_{\omega,c}|^{4\sigma}\varphi_{\omega,c} = 0. \tag{5.4}$$

Define

$$\phi_{\omega,c}(y) = \varphi_{\omega,c}(y)e^{i\theta_{\omega,c}(y)}, \tag{5.5}$$

where

$$\theta_{\omega,c}(y) = \frac{c}{2}y - \frac{1}{2\sigma+2}\int_{-\infty}^{y}\varphi_{\omega,c}^{2\sigma}(\eta)\,d\eta. \tag{5.6}$$

Clearly, we have

$$\psi_{\omega,c}(x,t) = e^{i\omega t}\phi_{\omega,c}(x-ct). \tag{5.7}$$

and $\phi_{\omega,c}$ solves

$$-\partial_y^2\phi_{\omega,c} + \omega\phi_{\omega,c} + ic\partial_y\phi_{\omega,c} - i|\phi_{\omega,c}|^{2\sigma}\partial_y\phi_{\omega,c} = 0, \quad y \in \mathbb{R}. \tag{5.8}$$

Let $K \in \mathbb{N}$. For each $1 \leqslant j \leqslant K$, let $(\omega_j, c_j, x_j, \theta_j) \in \mathbb{R}^4$ be parameters such that $\omega_j > \frac{c_j^2}{4}$. Define, for each $j = 1, ..., K$

$$R_j(t,x) = e^{i\theta_j}\psi_{\omega_j,c_j}(t, x-x_j)$$

and define the multi-soliton profile by

$$R = \sum_{j=1}^{K} R_j. \tag{5.9}$$

For convenience, define $h_j = \sqrt{4\omega_j - c_j^2}$, for each $j = 1, ..., K$. Our main result is the following.

Theorem 5.1. *Let $\sigma \geqslant \frac{5}{2}$, $K \in \mathbb{N}^*$ and for each $1 \leqslant j \leqslant K$, $(\theta_j, \omega_j, c_j, x_j)$ be a sequence of parameters such that $x_j \in \mathbb{R}$, $\theta_j \in \mathbb{R}$, $c_j \neq c_k$, for $j \neq k$. The multi-soliton profile R is given as in (5.9). There exists a certain positive constant C_* such that if the parameters (ω_j, c_j) satisfy*

$$C_*\left((1+\|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1+\|R\|_{L^\infty H^1}^2)(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1})\right) \leqslant v_* = \inf_{j\neq k} h_j|c_j - c_k|, \tag{5.10}$$

then there exists a solution u of (5.1) *such that*

$$\|u-R\|_{H^1} \leqslant Ce^{-\lambda t}, \quad \forall t \geqslant T_0,$$

for positive constants C, T_0 depending only on the parameters $\omega_1, ..., \omega_K, c_1, ..., c_K$ and $\lambda = \frac{1}{16}v_*$.

We have the following comment about the restriction $\sigma \geqslant \frac{5}{2}$.

Remark 5.2. By Lemma 5.7, the following inequality holds for $\sigma \geqslant 2$:

$$(a+b)^{2(\sigma-2)} - a^{2(\sigma-2)} \lesssim b^{2(\sigma-2)} + ba^{2(\sigma-2)-1}, \qquad \text{for all } a, b > 0. \tag{5.11}$$

The condition $\sigma \geqslant \frac{5}{2}$ ensures that the order of b on the right hand side of (5.11) is larger than 1. This is used in the proof of Lemma 5.9.

The condition (5.10) is an implicit condition on the parameters. Below, we show that for large, negative and enough separated velocities, the condition (5.10) holds.

Remark 5.3. We prove that there exist parameters $(\omega_j, c_j, \theta_j, x_j)$ for $1 \leqslant j \leqslant K$ such at the condition (5.10) is satisfied. Let $M > 0$, $h_j > 0$, $d_j < 0$, for each $1 \leqslant j \leqslant K$. We chose $(c_j, \omega_j) = \left(Md_j, \frac{1}{4}(h_j^2 + M^2 d_j^2)\right)$. We verify that this choice satisfies the condition (5.10) for M large enough. Indeed, we see that $c_j < 0$ and $h_j \ll |c_j|$ for M large enough. We have

$$\begin{aligned}
\varphi_{\omega_j,c_j}^{2\sigma} &\approx \frac{h_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}}\right)} \\
\partial_x \varphi_{\omega_j,c_j} &\approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}}.
\end{aligned}$$

Using $|\sinh(x)| \leqslant |\cosh(x)|$ for all $x \in \mathbb{R}$ we have

$$|\partial_x \varphi_{\omega_j,c_j}| \leqslant \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{1}{(\cosh(\sigma h_j y) - \frac{c_j}{2\sqrt{\omega_j}})^{\frac{1}{2\sigma}}} \lesssim |\varphi_{\omega_j,c_j}|.$$

Thus,

$$\begin{aligned}
\|R_j\|_{L^\infty L^\infty} &= \|\varphi_{\omega_j,c_j}\|_{L^\infty} \lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \ll 1 \\
\|\partial_x R_j\|_{L^\infty L^\infty} &= \|\partial_x \phi_{\omega_j,c_j}\|_{L^\infty L^\infty} \\
&\lesssim \|\partial_x \varphi_{\omega_j,c_j}\|_{L^\infty} + \left\|\frac{c_j}{2}\varphi_{\omega_j,c_j} - \frac{1}{2\sigma+2}\varphi_{\omega_j,c_j}^{2\sigma+1}\right\|_{L^\infty} \\
&\lesssim \|\varphi_{\omega_j,c_j}\|_{L^\infty} + |c_j| \|\varphi_{\omega_j,c_j}\|_{L^\infty} \\
&\lesssim \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}}.
\end{aligned}$$

Hence,

$$\|R\|_{L^\infty L^\infty} \lesssim \sum_j \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \lesssim 1$$

$$\|\partial_x R\|_{L^\infty L^\infty} \lesssim \sum_j \left(\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right).$$

Furthermore,

$$\begin{aligned}
\|R_j\|_{L^\infty H^1}^2 &= \|R_j\|_{L^\infty L^2}^2 + \|\partial_x R_j\|_{L^\infty L^2}^2 = \|\varphi_{\omega_j,c_j}\|_{L^2}^2 + \|\partial_x \varphi_{\omega_j,c_j}\|_{L^2}^2 \\
&\lesssim \|\varphi_{\omega_j,c_j}\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \left\| \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \right\|_{L^2}^2 \lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \|e^{-\frac{h_j}{2}|y|}\|_{L^2}^2 \\
&\approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{\sigma}} \frac{1}{h_j} \lesssim h_j^{\frac{1}{\sigma}} h_j^{-1} = h_j^{\frac{1}{\sigma}-1},
\end{aligned}$$

where we use $h_j \leqslant 2\sqrt{\omega_j}$. Thus,

$$\|R\|_{L^\infty H^1}^2 \lesssim \sum_j h_j^{\frac{1}{\sigma}-1}.$$

The condition (5.10) satisfies if the following estimate holds:

$$\left(1 + \sum_j h_j^{\frac{1}{\sigma}-1}\right) \left(1 + \sum_j \left(\sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} + |c_j| \sqrt[2\sigma]{\frac{h_j^2}{|c_j|}} \right)\right) \ll \inf_{j\neq k} h_j |c_j - c_k|. \tag{5.12}$$

We see that the left hand side of (5.12) is order $M^{1-\frac{1}{2\sigma}}$ and the right hand side of (5.12) is order M^1. Hence, the condition (5.10) satisfies if we choose M large enough.

Remark 5.4. We may replace the condition (5.10) by the following condition

$$(1 + \|R\|_{L^\infty L^\infty}^{2(\sigma-1)})(1 + \|R\|_{L^\infty H^1}^2)(1 + \|\partial_x R\|_{L^\infty L^\infty} + \|R\|_{L^\infty L^\infty}^{2\sigma+1}) \leqslant C v_* = \inf_{j\neq k} h_j |c_j - c_k|, \tag{5.13}$$

where C is a certain positive number. We do not know exactly what this constant is. The condition (5.10) says that we can choose the parameters such that the right hand side of (5.10) is arbitrary larger than the left hand side and hence the condition (5.13) satisfies.

Our strategy of the proof of Theorem 5.1 is as follows. First, we define φ, ψ based on u in such a way that φ and ψ satisfy a system of nonlinear Schrödinger equations without derivatives (see (5.16)). Let R be a multi-soliton profile which satisfies the assumptions of Theorem 5.1. Then R solves (5.1) up to a small perturbation. Let (h, k) be defined in a similar way as (φ, ψ) but replace u by R. We see that (h, k) solves (5.16) up to small perturbations. Setting $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$, we see that if u solves (5.1) then $(\tilde{\varphi}, \tilde{\psi})$ solves a system and a relation between $\tilde{\varphi}$ and $\tilde{\psi}$ holds and vice versa. By using the Banach fixed point theorem, we prove that there exists a solution $(\tilde{\varphi}, \tilde{\psi})$ of this system which exponential decays in time on $H^1(\mathbb{R})$ for t large. Combining with the assumption (5.10), we can prove a relation between $\tilde{\varphi}$ and $\tilde{\psi}$. Thus, we easily obtain the solution u of (5.1) satisfying the desired property.

This chapter is organized as follows. In Section 5.2, we prove the existence of multi-soliton trains for the equation (5.1). In Section 5.3, we prove some technical results which are used in the proof of the main result Theorem 5.1. More precisely, we prove the exponential decay of perturbations in the equations of h, k (Lemma 5.6) and the existence of decaying solutions for the system of equations of $\tilde{\varphi}, \tilde{\psi}$ (Lemma 5.9).

Before proving the main result, we introduce some notation used in this chapter.

Notation.

(1) We denote the Schrödinger operator as follows

$$L = i\partial_t + \partial_x^2.$$

(2) Given a time $t \in \mathbb{R}$, the Strichartz space $S([t,\infty))$ is defined via the norm

$$\|u\|_{S([t,\infty))} = \sup_{(q,r)\ \textit{admissible}} \|u\|_{L_t^q L_x^r([t,\infty)\times\mathbb{R})}.$$

We denote the dual space by $N[t,\infty) = S([t,\infty))^*$. Hence for any (q,r) admissible pair we have

$$\|u\|_{N([t,\infty))} \leqslant \|u\|_{L_t^{q'} L_x^{r'}([t,\infty)\times\mathbb{R})}.$$

(3) For $a, b \in \mathbb{R}^2$, we denote $|(a,b)| = |a| + |b|$.

(4) Let $a, b > 0$. We denote $a \lesssim b$ if a is smaller than b up to multiplication by a positive constant and denote $a \lesssim_c b$ if a is smaller than b up to multiplication by a positive constant depending on c. Moreover, we denote $a \approx b$ if a equals to b up to multiplication by a positive constant.

5.2 Proof of the main result

In this section we give the proof of Theorem 5.1. We use the Banach fixed point theorem and Strichartz estimates. We divide our proof in three steps.

Step 1. Preliminary analysis. Let $u \in C(I, H^1(\mathbb{R}))$ be a $H^1(\mathbb{R})$ solution of (5.1) on I. Consider the following transform:

$$\varphi(t,x) = \exp(i\Lambda)u(t,x), \tag{5.14}$$

$$\psi = \exp(i\Lambda)\partial_x u = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi, \tag{5.15}$$

where

$$\Lambda = \frac{1}{2}\int_{-\infty}^{x} |u(t,y)|^{2\sigma}\, dy.$$

As in [56, section 4], we have

$$\partial_t \Lambda = -\sigma \mathcal{Im}(|u|^{2(\sigma-1)}\overline{u}\partial_x u) + \sigma \mathcal{Im}\left[\int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u \, dy\right] - \frac{1}{4}|u|^{4\sigma}.$$

Thus, using $|u| = |\varphi|$ and $\mathcal{Im}(\overline{u}\partial_x u) = \mathcal{Im}(\overline{\varphi}\psi)$, we have

$$\begin{aligned}
\partial_t \Lambda &= -\sigma|\varphi|^{2(\sigma-1)}\mathcal{Im}(\overline{\varphi}\psi) + \sigma \int_{-\infty}^{x} \partial_x(|u|^{2(\sigma-1)})\mathcal{Im}(\overline{u}\partial_x u)\, dx - \frac{1}{4}|\varphi|^{4\sigma} \\
&= -\sigma|\varphi|^{2(\sigma-1)}\mathcal{Im}(\overline{\varphi}\psi) + \sigma \int_{-\infty}^{x} \partial_x(|\varphi|^{2(\sigma-1)})\mathcal{Im}(\overline{\varphi}\psi)\, dx - \frac{1}{4}|\varphi|^{4\sigma}.
\end{aligned}$$

Since u solves (5.1), we have

$$\begin{aligned}
L\varphi &= L(\exp(i\Lambda))u + \exp(i\Lambda)Lu + 2\partial_x(\exp(i\Lambda))\partial_x u \\
&= L(\exp(i\Lambda))u + \exp(i\Lambda)(Lu + i|u|^{2\sigma}u) \\
&= L(\exp(i\Lambda))u \\
&= (i\partial_t + \partial_x^2)(\exp(i\Lambda))u, \\
&= \left[-\exp(i\Lambda)\partial_t\Lambda + \partial_x(\exp(i\Lambda)\frac{i}{2}|u|^{2\sigma})\right] u \\
&= -\varphi\partial_t\Lambda + \left[\exp(i\Lambda)\frac{-1}{4}|u|^{2\sigma} + \frac{i}{2}\exp(i\Lambda)\partial_x(|u|^{2\sigma})\right] u \\
&= -\varphi\partial_t\Lambda + \varphi\left[-\frac{1}{4}|\varphi|^{4\sigma} + \frac{i}{2}\partial_x(|\varphi|^{2\sigma})\right] \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{Im}(\overline{\varphi}\psi) - \sigma\varphi\int_{-\infty}^{x}\partial_x(|\varphi|^{2(\sigma-1)})\mathcal{Im}(\overline{\varphi}\psi)\, dx \\
&\quad + \frac{1}{4}|\varphi|^{4\sigma}\varphi - \frac{1}{4}\varphi|\varphi|^{4\sigma} + i\sigma|\varphi|^{2(\sigma-1)}\varphi\mathcal{Re}(\overline{\varphi}\partial_x\varphi) \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{Im}(\overline{\varphi}\psi) + i\mathcal{Re}(\overline{\varphi}\partial_x\varphi)) \\
&\quad - \sigma\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}(\sigma-1)\partial_x(|\varphi|^2)\mathcal{Im}(\overline{\varphi}\psi)\, dx \\
&= \sigma|\varphi|^{2(\sigma-1)}\varphi(\mathcal{Im}(\overline{\varphi}\psi) + i\mathcal{Re}(\overline{\varphi}\psi)) \\
&\quad - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}2\mathcal{Re}(\overline{\varphi}\psi)\mathcal{Im}(\overline{\varphi}\psi)\, dx \\
&= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2)\, dy.
\end{aligned}$$

As in [56, section 4], we have

$$
\begin{aligned}
L\psi &= L(\exp(i\Lambda)\partial_x u)\\
&= \exp(i\Lambda)\left[-\frac{i}{2}\partial_x(|u|^{2\sigma})\partial_x u + \sigma|u|^{2(\sigma-1)}\mathcal{Im}(\overline{u}\partial_x u)\partial_x u\right.\\
&\quad\left. -\sigma\int_{-\infty}^{x}\mathcal{Im}(\partial_x(|u|^{2(\sigma-1)}\overline{u})\partial_x u)\,dy\partial_x u\right]\\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\mathcal{Im}(\overline{\varphi}\psi)\psi - \sigma\int_{-\infty}^{x}\partial_x(|u|^{2(\sigma-1)})\mathcal{Im}(\overline{u}\partial_x u)\,dy\psi\\
&= -\frac{i}{2}\partial_x(|\varphi|^{2\sigma})\psi + \sigma|\varphi|^{2(\sigma-1)}\psi\mathcal{Im}(\overline{\varphi}\psi) - \sigma\psi\int_{-\infty}^{x}\partial_x(|\varphi|^{2(\sigma-1)})\mathcal{Im}(\overline{\varphi}\psi)\,dy\\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{Im}(\overline{\varphi}\psi) - i\mathcal{Re}(\overline{\varphi}\partial_x\varphi))\\
&\quad -\sigma\psi\int_{-\infty}^{x}(\sigma-1)|\varphi|^{2(\sigma-1)}2\mathcal{Re}(\overline{\varphi}\partial\varphi)\mathcal{Im}(\overline{\varphi}\psi)\,dy\\
&= \sigma|\varphi|^{2(\sigma-1)}\psi(\mathcal{Im}(\overline{\varphi}\psi) - i\mathcal{Re}(\overline{\varphi}\psi))\\
&\quad -\sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}2\mathcal{Re}(\overline{\varphi}\psi)\mathcal{Im}(\overline{\varphi}\psi)\mathcal{Im}(\overline{\varphi}\psi)\,dy\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi} - \sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2)\,dy.
\end{aligned}
$$

Thus, if u solves (5.1) then (φ,ψ) solves

$$
\begin{cases}
L\varphi &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2)\,dy,\\
L\psi &= -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi} - \sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2)\,dy.
\end{cases}
\tag{5.16}
$$

For convenience, we define

$$
P(\varphi,\psi) = i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{\psi} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2), \tag{5.17}
$$

$$
Q(\varphi,\psi) = -i\sigma|\varphi|^{2(\sigma-1)}\psi^2\overline{\varphi} - \sigma(\sigma-1)\psi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{Im}(\psi^2\overline{\varphi}^2). \tag{5.18}
$$

Let R be the multi-soliton profile defined in (5.9). Define h,k by

$$
\begin{aligned}
h(t,x) &= \exp\left(\frac{i}{2}\int_{-\infty}^{x}|R(t,x)|^{2\sigma}\,dy\right)R(t,x),\\
k &= \partial_x h - \frac{i}{2}|h|^{2\sigma}h.
\end{aligned}
$$

Since R_j solves (5.1) for each $1 \leqslant j \leqslant K$, we have

$$
LR + i|R|^{2\sigma}R_x = -\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x. \tag{5.19}
$$

By Lemma 5.6 for $t \gg T_0$ large enough we have

$$
\left\| -\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x \right\|_{H^2} \leqslant e^{-\lambda t}. \tag{5.20}
$$

Thus, we rewrite (5.19) as follows:

$$LR + i|R|^{2\sigma}R_x = e^{-\lambda t}v, \tag{5.21}$$

where

$$v = e^{\lambda t}(-\sum_j i|R_j|^{2\sigma}R_{jx} + i|R|^{2\sigma}R_x). \tag{5.22}$$

By an elementary calculation, we have

$$\begin{cases} Lh = i\sigma|h|^{2(\sigma-1)}h^2\overline{k} - \sigma(\sigma-1)h\int_{-\infty}^{x}|h|^{2(\sigma-2)}\mathcal{Im}(k^2\overline{h}^2)\,dy + e^{-\lambda t}m(t,x), \\ Lk = -i\sigma|h|^{2(\sigma-1)}k^2\overline{h} - \sigma(\sigma-1)k\int_{-\infty}^{x}|h|^{2(\sigma-2)}\mathcal{Im}(k^2\overline{h}^2)\,dy + e^{-\lambda t}n(t,x). \end{cases} \tag{5.23}$$

where

$$m = \exp\left(\frac{i}{2}\int_{-\infty}^{x}|R|^{2\sigma}\,dy\right)v - \sigma h\int_{-\infty}^{x}|R|^{2(\sigma-1)}\mathcal{Im}(\overline{R}v)\,dy, \tag{5.24}$$

$$n = \exp\left(\frac{i}{2}\int_{-\infty}^{x}|R|^{2\sigma}\,dy\right)e^{-\lambda t}(\partial_x v - \sigma\partial_x R\int_{-\infty}^{x}|R|^{2(\sigma-1)}\mathcal{Im}(\overline{R}v)\,dy). \tag{5.25}$$

Since v is uniformly bounded in time in $H^2(\mathbb{R})$, we see that m, n are uniformly bounded in time in $H^1(\mathbb{R})$. Let $\tilde{\varphi} = \varphi - h$ and $\tilde{\psi} = \psi - k$. Then $(\tilde{\varphi}, \tilde{\psi})$ solves:

$$\begin{cases} L\tilde{\varphi} = P(\varphi,\psi) - P(h,k) - e^{-\lambda t}m(t,x), \\ L\tilde{\psi} = Q(\varphi,\psi) - Q(h,k) - e^{-\lambda t}n(t,x). \end{cases} \tag{5.26}$$

Set $\eta = (\tilde{\varphi}, \tilde{\psi})$, $W = (h,k)$ and $f(\varphi,\psi) = (P(\varphi,\psi), Q(\varphi,\psi)$ and $H = e^{-\lambda t}(m,n)$. We find a solutions of (5.26) in Duhamel form:

$$\eta(t) = i\int_t^{\infty}[f(W+\eta) - f(W) + H](s)\,ds, \tag{5.27}$$

where $S(t)$ denote the Schrödinger group. Moreover, since $\psi = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$, we have

$$\tilde{\psi} = \partial_x\tilde{\varphi} - \frac{i}{2}(|\tilde{\varphi} + h|^{2\sigma}(\tilde{\varphi} + h) - |h|^{2\sigma}h). \tag{5.28}$$

Step 2. Existence of a solution of the system

From Lemma 5.9, there exists $T_* \gg 1$ such that for $T_0 \geqslant T_*$ there exists a unique solution η of (5.26) defined on $[T_0, T_*)$ such that

$$\|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial_x\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1 \quad \forall t \geqslant T_0. \tag{5.29}$$

Thus, for all $t \geqslant T_0$, we have

$$\|\tilde{\varphi}\|_{H^1} + \|\tilde{\psi}\|_{H^1} \lesssim e^{-\lambda t}. \tag{5.30}$$

Step 3. Existence of a multi-soliton train

We prove that the solution $\eta = (\tilde\varphi, \tilde\psi)$ of (5.26) satisfies the relation (5.28). Set $\varphi = \tilde\varphi + h$, $\psi = \tilde\psi + k$ and $v = \partial_x\varphi - \frac{i}{2}|\varphi|^2\varphi$ and $\tilde v = v - k$. Since $(\tilde\varphi, \tilde\psi)$ solves (5.26) and (h, k) solves (5.23), we have (φ, ψ) solves (5.16). Furthermore,

$$Lv = \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi). \tag{5.31}$$

Moreover,

$$\begin{aligned}
&L(|\varphi|^{2\sigma}\varphi)\\
&= (i\partial_t + \partial_x^2)(\varphi^{\sigma+1}\overline{\varphi}^{\sigma}) = i\partial_t(\varphi^{\sigma+1}\overline{\varphi}^{\sigma}) + \partial_x^2(\varphi^{\sigma+1}\overline{\varphi}^{\sigma})\\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\overline{\varphi}\\
&\quad + \partial_x((\sigma+1)|\varphi|^{2\sigma}\partial_x\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\overline{\varphi})\\
&= i(\sigma+1)|\varphi|^{2\sigma}\partial_t\varphi + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_t\overline{\varphi} + (\sigma+1)\left[\partial_x^2\varphi|\varphi|^{2\sigma} + \partial_x\varphi\partial_x(|\varphi|^{2\sigma})\right]\\
&\quad + \sigma\left[\partial_x^2\overline{\varphi}|\varphi|^{2(\sigma-1)}\varphi^2 + (\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + (\sigma-1)|\varphi|^{2(\sigma-2)}\varphi^3(\partial_x\overline{\varphi})^2\right]\\
&= (\sigma+1)|\varphi|^{2\sigma}(i\partial_t\varphi + \partial_x^2\varphi) + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(i\partial_t\overline{\varphi} + \partial_x^2\overline{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma})\\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\\
&= (\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\overline{\varphi}) + (\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma})\\
&\quad + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3.
\end{aligned}$$

Combining with (5.31) and using (5.16), we have

$$\begin{aligned}
Lv &= \partial_x L\varphi - \frac{i}{2}L(|\varphi|^{2\sigma}\varphi)\\
&= \partial_x L\varphi - \frac{i}{2}\left[(\sigma+1)|\varphi|^{2\sigma}L\varphi + \sigma|\varphi|^{2(\sigma-1)}\varphi^2(-\overline{L\varphi} + 2\partial_x^2\overline{\varphi})\right.\\
&\quad \left.+(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right]\\
&= \partial_x(P(\varphi,\psi) - P(\varphi,v)) + \partial_x P(\varphi,v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi,\psi) - P(\varphi,v))\\
&\quad - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi,v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\psi)} - \overline{P(\varphi,v)})\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi,v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi}\\
&\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\right.\\
&\quad \left.+\sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right]\\
&= \partial_x(P(\varphi,\psi) - P(\varphi,v)) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi,\psi) - P(\varphi,v))\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi,\psi)} - \overline{P(\varphi,v)}) + G(\varphi,v),
\end{aligned}$$

where $G(\varphi, v)$ contains the remaining ingredients and $G(\varphi, v)$ only depends on φ

and v:

$$\begin{aligned}
&G(\varphi, v)\\
&= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma} P(\varphi, v) + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)}\\
&\quad - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\right.\\
&\quad \left.+\sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right]. \qquad (5.32)
\end{aligned}$$

As the calculations of $L\psi$ in the step 1, noting that the role of v is similar to the role of ψ in the process of calculation, we have $G(\varphi, v) = Q(\varphi, v)$ (see Lemma 5.8 for a detailed proof). Hence,

$$\begin{aligned}
L\psi - Lv = Q(\varphi, \psi) - Q(\varphi, v) - \partial_x(P(\varphi, \psi) - P(\varphi, v))&\\
+ \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\left(P(\varphi, \psi) - P(\varphi, v)\right)&\\
- \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \psi)} - \overline{P(\varphi, v)}).&
\end{aligned}$$

Thus,

$$\begin{aligned}
L\tilde{\psi} - L\tilde{v} &= L\psi - Lv\\
&= Q(\varphi, \tilde{\psi}+k) - Q(\varphi, \tilde{v}+k) - \partial_x(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k)\\
&\quad + \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k))\\
&\quad - \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \tilde{\psi}+k)} - \overline{P(\varphi, \tilde{v}+k)}). \qquad (5.33)
\end{aligned}$$

Multiplying both side of (5.33) by $\overline{\tilde{\psi} - \tilde{v}}$, taking imaginary part and integrating over space with integration by parts we obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t\|\tilde{\psi} - \tilde{v}\|_{L^2}^2\\
&= \mathcal{Im}\int_{\mathbb{R}}(Q(\varphi, \tilde{\psi}+k) - Q(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx \qquad (5.34)\\
&\quad - \mathcal{Im}\int_{\mathbb{R}}\partial_x(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx \qquad (5.35)\\
&\quad + (\sigma+1)\mathcal{Im}\int_{\mathbb{R}}\frac{i}{2}|\varphi|^{2\sigma}(P(\varphi, \tilde{\psi}+k) - P(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx \qquad (5.36)\\
&\quad - \sigma\mathcal{Im}\int_{\mathbb{R}}\frac{i}{2}|\varphi|^{2(\sigma-1)}\varphi^2(\overline{P(\varphi, \tilde{\psi}+k)} - \overline{P(\varphi, \tilde{v}+k)})(\overline{\tilde{\psi}} - \overline{\tilde{v}})\,dx. \qquad (5.37)
\end{aligned}$$

We denote by A, B, C, D the terms (5.34), (5.35), (5.36) and (5.37) respectively.

First, we try to estimate A, B, C, D in term of R. We have

$$
\begin{aligned}
|A| &\lesssim \left| \int_{\mathbb{R}} (Q(\varphi, \tilde{\psi}+k) - Q(\varphi, \tilde{v}+k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \overline{\varphi}((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2)(\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \left[(\tilde{\psi}+k) \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}((\tilde{\psi}+k)^2 \overline{\varphi}^2)\, dy \right.\right. \\
&\quad \left.\left. -(\tilde{v}+k) \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}((\tilde{v}+k)^2 \overline{\varphi}^2)\, dy \right] (\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \overline{\varphi}((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2)(\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \left[(\tilde{\psi}-\tilde{v}) \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}((\tilde{\psi}+k)^2 \overline{\varphi}^2)\, dy \right] (\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \left[(\tilde{v}+k) \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}(\overline{\varphi}^2((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2))\, dy \right] (\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty} \\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 \left\| \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}((\tilde{\psi}+k)^2 \overline{\varphi}^2)\, dy \right\|_{L_x^\infty} \\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2} \|\tilde{v}+k\|_{L^2} \left\| \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}(\overline{\varphi}^2((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2))\, dy \right\|_{L_x^\infty} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 \|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L_x^1} \\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2} \|\tilde{v}+k\|_{L^2} \|\varphi^{2(\sigma-1)}((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2)\|_{L^1} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty} + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 \|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L^1} \\
&\quad + \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 \|\tilde{v}+k\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2} \\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2 K_1, \qquad (5.38)
\end{aligned}
$$

where,

$K_1 := \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi}+\tilde{v}+2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)}(\tilde{\psi}+k)^2\|_{L^1} + \|\tilde{v}+k\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}$.

Furthermore,

$$
\begin{aligned}
|B| &\lesssim \left| \int_{\mathbb{R}} \partial_x (|\varphi|^{2(\sigma-1)} \varphi^2 (\overline{\tilde{\psi}} - \overline{\tilde{v}}))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \partial_x \left(\varphi \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}(\overline{\varphi}^2((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2))\, dy \right) (\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\lesssim \left| \int_{\mathbb{R}} \partial_x (|\varphi|^{2(\sigma-1)} \varphi^2)(\overline{\tilde{\psi}} - \overline{\tilde{v}})^2\, dx \right| + \left| |\varphi|^{2(\sigma-1)} \varphi^2 \frac{1}{2} \partial_x ((\tilde{\psi}-\tilde{v})^2)\, dx \right| \qquad (5.39) \\
&\quad + \left| \int_{\mathbb{R}} \partial_x \varphi \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}(\overline{\varphi}^2 (\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\, dy (\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right| \\
&\quad + \left| \int_{\mathbb{R}} \varphi |\varphi|^{2(\sigma-2)} \mathcal{Im}(\overline{\varphi}^2 (\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))(\overline{\tilde{\psi}} - \overline{\tilde{v}})\, dx \right|.
\end{aligned}
$$

By using integration by parts for the second term of (5.39) and using Hölder inequality we have

$$
\begin{aligned}
|B| &\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}\\
&\quad+\|\partial_x\varphi\|_{L^2}\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy\|_{L_x^\infty}\|\tilde{\psi}-\tilde{v}\|_{L^2}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}\\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}\\
&\quad+\|\partial_x\varphi\|_{L^2}\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L_x^1} \qquad (5.40)\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}\\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}\\
&\quad+\|\partial_x\varphi\|_{L^2}\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}\\
&= \|\tilde{\psi}-\tilde{v}\|_{L^2}^2K_2, \qquad (5.41)
\end{aligned}
$$

where

$$K_2:=\|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty}+\|\partial_x\varphi\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}+\|\varphi^{2\sigma-1}(\tilde{\psi}+\tilde{v}+2k)\|_{L^\infty}.$$

Using (5.17), we have

$$
\begin{aligned}
|C| &\lesssim \left|\int_{\mathbb{R}}|\varphi|^{2\sigma}|\varphi|^{2(\sigma-1)}\varphi^2(\overline{\tilde{\psi}}-\overline{\tilde{v}})^2\,dx\right|\\
&\quad+\left|\int_{\mathbb{R}}|\varphi|^{2\sigma}\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2((\tilde{\psi}+k)^2-(\tilde{v}+k)^2))\,dy(\overline{\tilde{\psi}}-\overline{\tilde{v}})\,dx\right|\\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}\\
&\quad+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\|\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(\overline{\varphi}^2(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k))\,dy\|_{L_x^\infty}\\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}-\tilde{v})(\tilde{\psi}+\tilde{v}+2k)\|_{L^1}\\
&\lesssim \|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{4\sigma}\|_{L^\infty}+\|\tilde{\psi}-\tilde{v}\|_{L^2}^2\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}\\
&= \|\tilde{\psi}-\tilde{v}\|_{L^2}^2K_3, \qquad (5.42)
\end{aligned}
$$

where

$$K_3:=\|\varphi^{4\sigma}\|_{L^\infty}+\|\varphi^{2\sigma+1}\|_{L^2}\|\varphi^{2(\sigma-1)}(\tilde{\psi}+\tilde{v}+2k)\|_{L^2}.$$

Now, we give an estimate for D. We have

$$
\begin{aligned}
|D| &\lesssim \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 |\varphi|^{2(\sigma-1)} \overline{\varphi}^2 (\tilde{\psi} - \tilde{v})(\overline{\tilde{\psi}} - \overline{\tilde{v}}) \, dx \right| \\
&\quad + \left| \int_{\mathbb{R}} |\varphi|^{2(\sigma-1)} \varphi^2 \varphi \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}(\overline{\varphi}^2((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2)) \, dy (\overline{\tilde{\psi}} - \overline{\tilde{v}}) \, dx \right| \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} \\
&\quad + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \| \int_{-\infty}^{x} |\varphi|^{2(\sigma-2)} \mathcal{Im}(\overline{\varphi}^2((\tilde{\psi}+k)^2 - (\tilde{v}+k)^2)) \, dy \|_{L_x^\infty} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2} \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} - \tilde{v})(\tilde{\psi} + \tilde{v} + 2k)\|_{L^1} \\
&\lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{4\sigma}\|_{L^\infty} + \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\
&= \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 K_4, \qquad (5.43)
\end{aligned}
$$

where

$$
K_4 := \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2}.
$$

Combining (5.38), (5.41), (5.42) and (5.43), we have

$$
\left| \partial_t \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 \right| \lesssim \|\tilde{\psi} - \tilde{v}\|_{L^2}^2 (K_1 + K_2 + K_3 + K_4).
$$

Using the Grönwall inequality, we have

$$
\begin{aligned}
\|\tilde{\psi}(t) - \tilde{v}(t)\|_{L^2}^2 &\lesssim \|\tilde{\psi}(N) - \tilde{v}(N)\|_{L^2}^2 \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) \, ds \right) \\
&\leqslant e^{-2\lambda N} \exp\left(\int_t^N (K_1 + K_2 + K_3 + K_4) \, ds \right). \qquad (5.44)
\end{aligned}
$$

Now, we try to estimate $K_1 + K_2 + K_3 + K_4$ in term of R. When we have this kind of estimate, we will use the assumption (5.10) to obtain that $\tilde{\psi} = \tilde{v}$. We have

$$
\begin{aligned}
&\int_t^N (K_1 + K_2 + K_3 + K_4) \, ds \\
&= \int_t^N \|\varphi\|_{L^\infty}^{2\sigma-1} \|\tilde{\psi} + \tilde{v} + 2k\|_{L^\infty} + \|\varphi^{2(\sigma-1)}(\tilde{\psi} + k)^2\|_{L^1} \\
&\quad + \|\tilde{v} + k\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \, ds \qquad (5.45) \\
&\quad + \int_t^N \|\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2)\|_{L^\infty} + \|\partial_x \varphi\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \\
&\quad + \|\varphi^{2\sigma-1}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^\infty} \, ds \qquad (5.46) \\
&\quad + \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \, ds \qquad (5.47) \\
&\quad + \int_t^N \|\varphi^{4\sigma}\|_{L^\infty} + \|\varphi^{2\sigma+1}\|_{L^2} \|\varphi^{2(\sigma-1)}(\tilde{\psi} + \tilde{v} + 2k)\|_{L^2} \, ds \qquad (5.48)
\end{aligned}
$$

Using (5.29) and (5.30), we have

$$\|\varphi\|_{L^\infty} \leqslant \|\tilde{\varphi}\|_{L^\infty} + \|h\|_{L^\infty} \lesssim 1 + \|h\|_{L^\infty} \tag{5.49}$$

$$\|\varphi\|_{L^2} \leqslant \|\tilde{\varphi}\|_{L^2} + \|h\|_{L^2} \lesssim 1 + \|h\|_{L^2} \tag{5.50}$$

$$\|\psi\|_{L^\infty} \lesssim 1 \tag{5.51}$$

We denote by Z_1, Z_2, Z_3, Z_4 the terms (5.45), (5.46), (5.47) and (5.48) respectively. Using (5.49), (5.50), (5.51), (5.29) and (5.30), for $N \gg t$, we have

$$\begin{aligned}
|Z_1| &\lesssim \|\varphi\|^3_{L^4(t,N)L^\infty}\|\varphi\|^{2(\sigma-2)}_{L^\infty L^\infty}\|\tilde{\psi}+\tilde{v}+2k\|_{L^4(t,N)L^\infty} \\
&\quad + (N-t)\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}(\|\tilde{\psi}\|_{L^\infty L^2}+\|k\|_{L^\infty L^2})^2 \\
&\quad + \|\tilde{v}+k\|_{L^{\frac{4}{3}}(t,N)L^2}\|\varphi\|_{L^\infty L^2}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^4(t,N)L^\infty}+\|k\|_{L^4(t,N)L^\infty}) \\
&\lesssim (N-t)^{\frac{3}{4}}\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty}(1+\|k\|_{L^\infty L^\infty}(N-t)^{\frac{1}{4}}) \\
&\quad + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|^2_{L^\infty L^2}) \\
&\quad + (N-t)^{\frac{3}{4}}(1+\|k\|_{L^\infty L^2})(1+\|h\|_{L^\infty L^2})(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+(N-t)^{\frac{1}{4}}\|k\|_{L^\infty L^\infty}) \\
&\lesssim (N-t)\|k\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty}) + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|^2_{L^\infty L^2}) \\
&\quad + (N-t)\|k\|_{L^\infty L^\infty}(1+\|k\|_{L^\infty L^2})(1+\|h\|_{L^\infty L^2})(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty}) \\
&:= (N-t)W_1(h,k).
\end{aligned}$$

Similarly, for $N \gg t$, we have

$$\begin{aligned}
|Z_2| &\lesssim \|\partial_x\varphi\varphi^{2\sigma-1}\|_{L^1(t,N)L^\infty} + (N-t)\|\partial_x\varphi\|_{L^\infty(t,N)L^2}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}\|\tilde{\psi}+\tilde{v}+k\|_{L^\infty(t,N)L^2} \\
&\quad + (N-t)^{\frac{3}{4}}\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^4(t,N)L^\infty}+\|k\|_{L^4(t,N)L^\infty}) \\
&\lesssim (N-t)^{\frac{3}{4}}(\|\partial_x\tilde{\varphi}\|_{L^4(t,N)L^\infty}+\|\partial_x h\|_{L^4(t,N)L^\infty})\|\varphi\|^{2\sigma-1}_{L^\infty L^\infty} \\
&\quad + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2}) \\
&\quad + (N-t)^{\frac{3}{4}}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty})(1+(N-t)^{\frac{1}{4}}\|k\|_{L^\infty L^\infty}) \\
&\lesssim (N-t)\|\partial_x h\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty}) + (N-t)(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2}) \\
&\quad + (N-t)\|k\|_{L^\infty L^\infty}(1+\|h\|^{2\sigma-1}_{L^\infty L^\infty}) \\
&:= (N-t)W_2(h,k),
\end{aligned}$$

and

$$\begin{aligned}
&|Z_3| = |Z_4| \\
&\lesssim (N-t)(\|\tilde{\varphi}\|_{L^\infty L^\infty}+\|h\|_{L^\infty L^\infty})^{4\sigma} \\
&\quad + (N-t)\|\varphi\|_{L^\infty L^2}\|\varphi\|^{2\sigma}_{L^\infty L^\infty}\|\varphi\|^{2(\sigma-1)}_{L^\infty L^\infty}(\|\tilde{\psi}+\tilde{v}\|_{L^\infty L^2}+\|k\|_{L^\infty L^2}) \\
&\lesssim (N-t)(1+\|h\|^{4\sigma}_{L^\infty L^\infty}) + (N-t)(1+\|h\|_{L^\infty L^2})(1+\|h\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|k\|_{L^\infty L^2}) \\
&:= (N-t)W_3(h,k).
\end{aligned}$$

Hence, from (5.44), we have

$$\begin{aligned}
\|\tilde{\psi(t)}-\tilde{v(t)}\|^2_{L^2} &\lesssim e^{-2\lambda N}\exp\left(\int_t^N (K_1+K_2+K_3+K_4)\,ds\right) \\
&\lesssim e^{-2\lambda N}\exp((N-t)(W_1(h,k)+W_2(h,k)+W_3(h,k)))
\end{aligned} \tag{5.52}$$

The above estimate is not enough explicit. As said above, we would like to estimate the right hand side of (5.52) in terms of R. Noting that $|h| = |R|$ and $|k| = |\partial_x R|$, we have

$$\begin{aligned}
W_1(h,k) &= \|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|^{2\sigma-1}_{L^\infty L^\infty}) + (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|\partial_x R\|^2_{L^\infty L^2}) \\
&\quad + \|\partial_x R\|_{L^\infty L^\infty}(1+\|\partial_x R\|_{L^\infty L^2})(1+\|R\|_{L^\infty L^2})(1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty}) \, [\|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|_{L^\infty L^\infty}) + (1+\|\partial_x R\|_{L^\infty L^2}) \\
&\quad + \|\partial_x R\|_{L^\infty L^\infty}(1+\|\partial_x R\|_{L^\infty L^2})(1+\|R\|_{L^\infty L^2})] \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\times \\
&\quad \times \left[\|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|_{L^\infty H^1}) + (1+\|R\|^2_{L^\infty H^1}) + \|\partial_x R\|_{L^\infty L^\infty}(1+\|R\|^2_{L^\infty H^1})\right] \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty}).
\end{aligned}$$

Similarly, by noting that $|\partial_x h| \leqslant |k| + |h|^{2\sigma+1}$, we have

$$\begin{aligned}
W_2(h,k) &\lesssim (\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty})(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\quad + (1+\|h\|^{2(\sigma-1)})(1+\|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1+\|h\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty}) \\
&\lesssim (1+\|h\|^{2(\sigma-1)})\times \\
&\quad \times \left[(\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty})(1+\|h\|_{L^\infty L^\infty})\right. \\
&\quad \left. + (1+\|k\|_{L^\infty L^2}) + \|k\|_{L^\infty L^\infty}(1+\|h\|_{L^\infty L^\infty})\right] \\
&\lesssim (1+\|h\|^{2(\sigma-1)})\times \\
&\quad \times \left[(1+\|h\|_{L^\infty L^\infty})(\|k\|_{L^\infty L^\infty} + \|h\|^{2\sigma+1}_{L^\infty L^\infty}) + (1+\|k\|_{L^\infty L^2})\right] \\
&= (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})\times \\
&\quad \times \left[(1+\|R\|_{L^\infty L^\infty})(\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) + (1+\|\partial_x R\|_{L^\infty L^2})\right] \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}),
\end{aligned}$$

and

$$\begin{aligned}
W_3(h,k) &= (1+\|R\|^{4\sigma}_{L^\infty L^\infty}) + (1+\|R\|_{L^\infty L^2})(1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|\partial_x R\|_{L^\infty L^2}) \\
&\lesssim (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty}) \left[(1+\|R\|^2_{L^\infty L^\infty}) + (1+\|R\|_{L^\infty L^2})(1+\|\partial_x R\|_{L^\infty L^2})\right] \\
&\lesssim (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1}).
\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}
&W_1(h,k) + W_2(h,k) + W_3(h,k) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) \\
&\quad + (1+\|R\|^{4\sigma-2}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}) \\
&\quad + (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^{2\sigma}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1}) \\
&\lesssim (1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty} + \|R\|^{2\sigma+1}_{L^\infty L^\infty}).
\end{aligned}$$

Thus, there exists a positive constant C_0 such that

$$\begin{aligned}&W_1(h,k)+W_2(h,k)+W_3(h,k)\\&\leqslant C_0\left((1+\|R\|^{2(\sigma-1)}_{L^\infty L^\infty})(1+\|R\|^2_{L^\infty H^1})(1+\|\partial_x R\|_{L^\infty L^\infty}+\|R\|^{2\sigma+1}_{L^\infty L^\infty})\right).\end{aligned}$$

Let $C_*=16C_0$. Using the assumption (5.10), we have

$$W_1(h,k)+W_2(h,k)+W_3(h,k)\leqslant\frac{v_*}{16}=\lambda,$$

for t large enough. Thus, by (5.52), we have

$$\|\tilde{\psi}(t)-\tilde{v}(t)\|^2_{L^2}\leqslant e^{-2\lambda N+(N-t)\lambda},$$

for t large enough. Letting $N\to\infty$ in the above estimate, we obtain

$$\|\tilde{\psi}(t)-\tilde{v}\|^2_{L^2}=0,$$

for all t large enough. This implies that

$$\tilde{\psi}=\partial_x\varphi-\frac{i}{2}|\varphi|^2\varphi-k, \tag{5.53}$$

and then

$$\psi=\partial_x\varphi-\frac{i}{2}|\varphi|^2\varphi.$$

Moreover, since $(\tilde{\psi},\tilde{\varphi})$ solves (5.26) we have (ψ,φ) solves (5.16). Combining with (5.53), if we set

$$u=\exp\left(-\frac{i}{2}\int_{-\infty}^{x}|\varphi|^{2\sigma}\,dy\right)\varphi$$

then u solves (5.1). Furthermore,

$$\begin{aligned}\|u-R\|_{H^1}&=\left\|\exp\left(-\frac{i}{2}|\varphi|^{2\sigma}\,dy\right)\varphi-\exp\left(\frac{i}{2}|h|^{2\sigma}\,dy\right)h\right\|_{H^1}\\&\lesssim C(\|\varphi\|_{H^1},\|h\|_{H^1})\|\varphi-h\|_{H^1}\lesssim\|\tilde{\varphi}\|_{H^1}\lesssim e^{-\lambda t},\end{aligned}$$

Thus for t large enough, we have

$$\|u-R\|_{H^1}\leqslant Ce^{-\lambda t}, \tag{5.54}$$

for $\lambda=\frac{1}{16}v_*$ and $C=C(\omega_1,...,\omega_K,c_1,...,c_K)$. This completes the proof of Theorem 5.1.

Remark 5.5. In the case $\sigma=1$, the integrals in (5.16) disappear. In the case, $\sigma=2$, the integrals (5.16) reduce into $\int_{-\infty}^{x}\mathcal{Im}(\psi^2\overline{\varphi}^2)\,dy$, we do not need to use the inequality (5.56). Thus, by similar arguments as in the proof of Theorem 5.1 we may prove that there exist multi-solitons solutions of (5.1) when $\sigma=1$ or $\sigma=2$.

5.3 Some technical lemmas

5.3.1 Properties of solitons

In this section, we give the proof of (5.20). We have the following result.

Lemma 5.6. *There exist $C > 0$ and a constant $\lambda > 0$ such that for $t > 0$ large enough, the estimate (5.20) uniformly holds in time.*

Proof. First, we need some estimates on the profile. We have

$$\begin{aligned}
|R_j(t,x)| &= |\psi_{\omega_j,c_j}(t,x)| = |\phi_{\omega_j,c_j}(x-c_jt)| = |\varphi_{\omega_j,c_j}(x-c_jt)| \\
&\approx \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j(x-c_jt)) - \frac{c_j}{2\sqrt{\omega_j}}\right)}\right)^{\frac{1}{2\sigma}} \\
&\lesssim \left(\frac{4\omega_j - c_j^2}{2\sqrt{\omega_j}\left(\cosh(\sigma h_j(x-c_jt)) - \frac{|c_j|}{2\sqrt{\omega_j}}\cosh(\sigma h_j(x-c_jt))\right)}\right)^{\frac{1}{2\sigma}} \\
&\lesssim \left(\frac{4\omega_j - c_j^2}{(2\sqrt{\omega_j} - |c_j|)\cosh(\sigma h_j(x-c_jt))}\right)^{\frac{1}{2\sigma}} \lesssim \left(\frac{2\sqrt{\omega_j} + |c_j|}{\cosh(\sigma h_j(x-c_jt))}\right)^{\frac{1}{2\sigma}} \\
&\lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|x-c_jt|},
\end{aligned}$$

Furthermore,

$$\partial_x\varphi_{\omega_j,c_j}(y) \approx \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{-\sinh(\sigma h_j y)}{\left(\cosh(\sigma h_j y) - \frac{c_j}{\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}}.$$

Thus,

$$\begin{aligned}
|\partial_x\varphi_{\omega_j,c_j}(y)| &\lesssim \left(\frac{h_j^2}{2\sqrt{\omega_j}}\right)^{\frac{1}{2\sigma}} \frac{|\sinh(\sigma h_j y)|}{\left(1 - \frac{|c_j|}{\sqrt{\omega_j}}\right)^{1+\frac{1}{2\sigma}}\cosh(\sigma h_j y)^{1+\frac{1}{2\sigma}}} \\
&\lesssim_{\omega_j,|c_j|} \frac{1}{\cosh(\sigma h_j y)^{\frac{1}{2\sigma}}} \lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|y|},
\end{aligned}$$

Using the above estimates, we have

$$\begin{aligned}
|\partial_x R_j(t,x)| &= |\partial_x\psi_{\omega_j,c_j}(t,x)| = |\partial_x\phi_{\omega_j,c_j}(x-c_jt)| \\
&= |\partial_x\varphi_{\omega_j,c_j}(x-c_jt) + i\varphi_{\omega_j,c_j}(x-c_jt)\partial_x\theta_{\omega_j,c_j}(x-c_jt)| \\
&\lesssim |\partial_x\varphi_{\omega_j,c_j}(x-c_jt)| + |\varphi_{\omega_j,c_j}(x-c_jt)||\partial_x\theta_{\omega_j,c_j}(x-c_jt)| \\
&\lesssim_{\omega_j,|c_j|} |\partial_x\varphi_{\omega_j,c_j}(x-c_jt)| + e^{\frac{-h_j}{2}|x-c_jt|} \\
&\lesssim_{\omega_j,|c_j|} e^{-\frac{h_j}{2}|x-c_jt|}.
\end{aligned}$$

By similar arguments, we have

$$|\partial_x^2 R_j(t,x)| + |\partial_x^3 R_j(t,x)| \lesssim_{\omega_j,|c_j|} e^{\frac{-h_j}{2}|x-c_jt|},$$

For convenience, we set

$$\begin{aligned}
\chi &= -i|R|^{2\sigma}\partial_x R + i\Sigma_j |R_j|^{2\sigma}\partial_x R_j,\\
f(R,\overline{R},\partial_x R) &= i|R|^{2\sigma}\partial_x R,\\
g(R,\overline{R},\partial_x R,\partial_x\overline{R},\partial_x^2 R) &= i\partial_x(|R|^{2\sigma}\partial_x R),\\
r(R,\partial_x R,..,\partial_x^3 R,\partial_x\overline{R},\partial_x^2\overline{R}) &= i\partial_x^2(|R|^{2\sigma}\partial_x R).
\end{aligned}$$

Fix $t>0$, for each $x\in\mathbb{R}$, choose $m=m(x)\in\{1,2,...,K\}$ so that

$$|x-c_mt| = \min_j |x-c_jt|.$$

For $j\neq m$ we have

$$|x-c_jt| \geqslant \frac{1}{2}(|x-c_jt|+|x-c_mt|) \geqslant \frac{1}{2}|c_jt-c_mt| = \frac{t}{2}|c_j-c_m|.$$

Thus, we have

$$\begin{aligned}
&|(R-R_m)(t,x)| + |\partial_x(R-R_m)(t,x)| + |\partial_x^2(R-R_m)(t,x)| + |\partial_x^3(R-R_m)(t,x)|\\
&\leqslant \sum_{j\neq m}(|R_j(t,x)| + |\partial_x R_j(t,x)| + |\partial_x^2 R_j(t,x)| + |\partial_x^3 R_j(t,x)|)\\
&\lesssim_{\omega_1,..,\omega_K,|c_1|,..,|c_K|} \delta_m(t,x) := \sum_{j\neq m} e^{\frac{-h_j}{2}|x-c_jt|}.
\end{aligned}$$

Recall that

$$v_* = \inf_{j\neq k} h_j|c_j-c_k|.$$

We have

$$\begin{aligned}
&|(R-R_m)(t,x)| + |\partial_x(R-R_m)(t,x)| + |\partial_x^2(R-R_m)(t,x)| + |\partial_x^3(R-R_m)(t,x)| \lesssim \delta_m(t,x)\\
&\lesssim e^{-\frac{1}{4}v_*t}.
\end{aligned}$$

We see that f,g,r are polynomials in R, $\partial_x R$, $\partial_x^2 R$, $\partial_x^3 R$, $\partial_x\overline{R}$ and $\partial_x^2\overline{R}$. Denote

$$A = \sup_{|u|+|\partial_x u|+|\partial_x^2 u|+|\partial_x^3 u|\leqslant \sum_j \|R_j\|_{H^4}} (|df|+|dg|+|dr|).$$

We have

$$\begin{aligned}
&|\chi| + |\partial_x\chi| + |\partial_x^2\chi|\\
&\leqslant |f(R,\overline{R},\partial_x R) - f_{R_m,\partial_x\overline{R}_m,R_m}| + |g(R,\overline{R},\partial_x R,..) - g(R_m,\overline{R}_m,\partial_x R_m,..)|\\
&\quad + |r(R,\partial_x R,..,\partial_x^3 R,\overline{R},..) - r(R_m,\partial_x R_m,..,\partial_x^3 R_m,\overline{R}_m,..)|\\
&+ \Sigma_{j\neq m}(f(R_j,\overline{R}_j,\partial_x R_j) + g(R_j,\partial_x R_j,\partial_x^2 R_j,\overline{R}_j,\partial_x\overline{R}_j) + r(R_j,...,\partial_x^3 R_j,\overline{R}_j,...,\partial_x^2\overline{R}_j))\\
&\lesssim A(|R-R_m| + |\partial_x(R-R_m)| + |\partial_x^2(R-R_m)| + |\partial_x^3(R-R_m)|)\\
&\quad + A\Sigma_{j\neq m}(|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|)\\
&\lesssim 2A\Sigma_{j\neq m}(|R_j| + |\partial_x R_j| + |\partial_x^2 R_j| + |\partial_x^3 R_j|)\\
&\lesssim 2A\delta_m(t,x).
\end{aligned}$$

In particular,
$$\|\chi\|_{W^{2,\infty}} \lesssim e^{-\frac{1}{4}v_* t}. \tag{5.55}$$
Moreover,
$$\begin{aligned}\|\chi\|_{W^{2,1}} &\lesssim \Sigma_j(\||R_j|^{2\sigma}\partial_x R_j\|_{L^1} + \|\partial_x(|R_j|^{2\sigma}\partial_x R_j)\|_{L^1} + \|\partial_x^2(|R_j|^{2\sigma}\partial_x R_j)\|_{L^1})\\ &\lesssim \Sigma_j(\|R_j\|_{H^1}^{(}2\sigma+1) + \|R_j\|_{H^2}^{2\sigma+1} + \|R_j\|_{H^3}^{2\sigma+1}) < \infty.\end{aligned}$$
Thus, using Hölder inequality we obtain
$$\|\chi\|_{H^2} \lesssim_{\omega_1,..,\omega_K,|c_1|,..,|c_K|} e^{-\frac{1}{8}v_* t}.$$
It follows that if $t \gg \max\{\omega_1, ..., \omega_K, |c_1|, ..., |c_K|\}$ is large enough then
$$\|\chi\|_{H^2} \leqslant e^{-\frac{1}{16}v_* t}.$$
Setting $\lambda = \frac{1}{16}v_*$, we obtain the desired result. $\square$

5.3.2 Some useful estimates

Lemma 5.7. *Let $x \geqslant 0$. Then there exists $C = C(x)$ such that*
$$(a+b)^x - a^x \leqslant C(x)(b^x + ba^{x-1}). \tag{5.56}$$
for all $a, b \geqslant 0$.

Proof. If $x = 0$ or $x = 1$ or $b = 0$ or $a = 0$ then (5.56) is true for $C(x) = 1$. Consider $a, b > 0$. If $0 < x < 1$ then using $m^x > m$ for $m < 1$ and $0 < x < 1$ we have
$$\left(\frac{a}{a+b}\right)^x + \left(\frac{b}{a+b}\right)^x > \frac{a}{a+b} + \frac{b}{a+b} = 1.$$
Hence,
$$(a+b)^x < a^x + b^x,$$
if we choose $C(x) = 1$ then (5.56) holds. Considering $a, b > 0$ and $x > 1$, we set
$$g(z) = z^x, \quad \forall z \in \mathbb{R}.$$
We have g is class C^1. Thus, there exists $\xi \in (a, a+b)$ such that
$$|(a+b)^x - a^x| = |g(a+b) - g(a)| = |bg'(\xi)| = bx\xi^{x-1} < xb(a+b)^{x-1}.$$
If $x-1 \leqslant 1$ then $(a+b)^{x-1} \leqslant a^{x-1}+b^{x-1}$ and hence we choose $C(x) = x$. If $x-1 > 1$ then by Jensen's inequality for convex function $f(z) = z^{x-1}$ we have
$$\left(\frac{a+b}{2}\right)^{x-1} \leqslant \frac{a^{x-1}+b^{x-1}}{2}.$$
We obtain
$$(a+b)^x - a^x < xb(a+b)^{x-1} \leqslant 2^{x-2}xb(a^{x-1}+b^{x-1}).$$
Choosing $C(x) = 2^{x-2}x$, we obtain the desired result. $\square$

5.3.3 Proof $G(\varphi, v) = Q(\varphi, v)$

Let $G(\varphi, v)$ be defined as in (5.32) and Q be defined as in (5.18). Then we have the following result.

Lemma 5.8. *Let $v = \partial_x\varphi - \frac{i}{2}|\varphi|^2\varphi$. Then the following equality holds:*

$$G(\varphi, v) = Q(\varphi, v).$$

Proof. We have

$$\begin{aligned}
P(\varphi, v) &= i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{v} - \sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy,\\
Q(\varphi, v) &= -i\sigma|\varphi|^{2(\sigma-1)}v^2\overline{\varphi} - \sigma(\sigma-1)v\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
G(\varphi, v) &= \partial_x P(\varphi, v) - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}P(\varphi, v)\\
&\quad + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{P(\varphi, v)} - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi}\\
&\quad - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\right.\\
&\quad \left.+\sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right].
\end{aligned}$$

The term contains $\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy$ in the expression of $G(\varphi, v)$ is the following.

$$\begin{aligned}
&-\sigma(\sigma-1)\partial_x\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
&\quad -\frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(-1)\sigma(\sigma-1)\varphi\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
&\quad +\frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-1)\sigma(\sigma-1)\overline{\varphi}\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\\
&= -\sigma(\sigma-1)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\left(\partial_x\varphi - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}\varphi + \frac{i}{2}\sigma|\varphi|^{2\sigma}\varphi\right)\\
&= -\sigma(\sigma-1)\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy\left(\partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi\right)\\
&= -\sigma(\sigma-1)v\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy,
\end{aligned}$$

which equals to the term contains $\int_{-\infty}^{x}|\varphi|^{2(\sigma-2)}\mathcal{I}m(v^2\overline{\varphi}^2)\,dy$ in the expression of $Q(\varphi, v)$. We only need to check the equality of the remaining terms. The remaining terms of $G(\varphi, v)$ is the following.

$$
\begin{aligned}
& i\sigma\partial_x(|\varphi|^{2(\sigma-1)}\varphi^2\overline{v}) - \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\mathcal{Im}(v^2\overline{\varphi}^2) \\
& - \frac{i}{2}(\sigma+1)|\varphi|^{2\sigma}(i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\overline{v}) \\
& + \frac{i}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(-i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}^2 v) - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} \qquad (5.57) \\
& - \frac{i}{2}\left[(\sigma+1)\partial_x\varphi\partial_x(|\varphi|^{2\sigma}) + \sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi \right. \\
& \left. + \sigma(\sigma-1)(\partial_x\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\right]. \qquad (5.58)
\end{aligned}
$$

Noting that $\partial_x(|\varphi|^2) = 2\mathcal{Re}(v\overline{\varphi})$ and $v = \partial_x\varphi - \frac{i}{2}|\varphi|^{2\sigma}\varphi$, we have

$$
\begin{aligned}
& \text{the term (5.57)} \\
&= i\sigma\partial_x(|\varphi|^{2(\sigma-1)})\varphi^2\overline{v} + i\sigma|\varphi|^{2(\sigma-1)}2\varphi\partial_x\varphi\overline{v} + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\overline{v} \\
&\quad - \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi 2\mathcal{Re}(v\overline{\varphi})\mathcal{Im}(v\overline{\varphi}) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} \\
&\quad + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{Re}(\varphi\overline{v}) - i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x^2\overline{\varphi} \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{Re}(v\overline{\varphi})\varphi^2\overline{v} + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\overline{v} + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x(\overline{v} - \overline{\partial_x\varphi}) \\
&\quad - 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi\mathcal{Re}(v\overline{\varphi})\mathcal{Im}(v\overline{\varphi}) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{Re}(\varphi\overline{v}) \\
&= 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{Re}(v\overline{\varphi})\varphi(i\varphi\overline{v} - \mathcal{Im}(v\overline{\varphi})) + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\overline{v} \\
&\quad + i\sigma|\varphi|^{2(\sigma-1)}\varphi^2\partial_x\left(\frac{i}{2}|\varphi|^{2\sigma}\overline{\varphi}\right) + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{Re}(\varphi\overline{v}) \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{Re}(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} \\
&\quad - \frac{1}{2}\sigma|\varphi|^{2(\sigma-1)}\varphi^2(2\sigma|\varphi|^{2(\sigma-1)}\mathcal{Re}(v\overline{\varphi}) + |\varphi|^{2\sigma}\partial_x\overline{\varphi}) \\
&\quad + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} + \sigma^2|\varphi|^{4\sigma-2}\varphi\mathcal{Re}(\varphi\overline{v}) \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{Re}(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} \\
&\quad - \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\partial_x\overline{\varphi} + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2\overline{v} \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{Re}(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} + \frac{1}{2}\sigma|\varphi|^{4\sigma-2}\varphi^2(\overline{v} - \partial_x\overline{\varphi}) \\
&= 2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{Re}(v\overline{\varphi}))^2 + 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi.
\end{aligned}
$$

Moreover, using $\mathcal{R}e(\partial_x\varphi\overline{\varphi}) = \mathcal{R}e(v\overline{\varphi})$ we have

$$\begin{aligned}
&\text{the term (5.58)}\\
&= \frac{-i}{2}\big[\sigma(\sigma+1)|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi(\partial_x\overline{\varphi}\varphi + \partial_x\varphi\overline{\varphi})\\
&\quad +\sigma(\sigma-1)(\partial\overline{\varphi})^2|\varphi|^{2(\sigma-2)}\varphi^3\big]\\
&= \frac{-i}{2}\big[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2(\partial_x\varphi\overline{\varphi} + \partial_x\overline{\varphi}\varphi)\\
&\quad +2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\big]\\
&= \frac{-i}{2}\big[2\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2\mathcal{R}e(v\overline{\varphi})\\
&\quad +2\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\big]\\
&= -i\big[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\partial_x\overline{\varphi}\varphi^2\mathcal{R}e(v\overline{\varphi})\\
&\quad +\sigma(\sigma+1)|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\big]\\
&= -i\big[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + \sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\mathcal{R}e(v\overline{\varphi})\varphi(\partial_x\overline{\varphi}\varphi + \partial_x\varphi\overline{\varphi})\\
&\quad +2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\big]\\
&= -i\big[\sigma|\partial\varphi|^2|\varphi|^{2(\sigma-1)}\varphi + 2\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}(\mathcal{R}e(v\overline{\varphi}))^2\varphi\big]\\
&= -2i\sigma(\sigma-1)|\varphi|^{2(\sigma-2)}\varphi(\mathcal{R}e(v\overline{\varphi}))^2\\
&\quad - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi}).
\end{aligned}$$

Combining the above expressions we obtain

$$\begin{aligned}
&\text{the remaining term of } G(\varphi, v)\\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\varphi\partial_x\varphi\overline{v} + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi - 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{R}e(v\overline{\varphi})\\
&= 2i\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(\varphi\overline{v} - \mathcal{R}e(v\overline{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&= -2\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi\mathcal{I}m(\varphi\overline{v}) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi - i\sigma|\partial_x\varphi|^2|\varphi|^{2(\sigma-1)}\varphi\\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\overline{v}) + i\partial_x\overline{\varphi}\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(2\mathcal{I}m(\varphi\partial_x\overline{\varphi}) + |\varphi|^{2\sigma+2} + i\mathcal{R}e(\varphi\partial_x\overline{\varphi}) - \mathcal{I}m(\varphi\partial_x\overline{\varphi})) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -\sigma|\varphi|^{2(\sigma-1)}\partial_x\varphi(|\varphi|^{2\sigma+2} + i\overline{\varphi}\partial_x\varphi) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}(\partial_x\varphi)^2 - \sigma|\varphi|^{4\sigma}\partial_x\varphi + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right)^2 - \sigma|\varphi|^{4\sigma}\left(v + \frac{i}{2}|\varphi|^{2\sigma}\varphi\right) + \frac{i}{4}\sigma|\varphi|^{6\sigma}\varphi\\
&= -i\sigma|\varphi|^{2(\sigma-1)}\overline{\varphi}v^2.
\end{aligned}$$

This is exactly the remaining terms of $Q(\varphi, v)$. Thus, $G(\varphi, v) = Q(\varphi, v)$. $\square$

5.3.4 Existence of a solution of the system

In this section, using similar arguments as in [72, 73], we prove the existence of a solution of (5.26). For convenience, we recall the equation:

$$\eta(t) = i\int_t^\infty S(t-s)[f(W+\eta)-f(W)+H](s)\,ds, \tag{5.59}$$

where

$$\begin{aligned} W &= (h,k), \\ H &= e^{-\lambda t}(m,n), \\ f(\varphi,\psi) &= (P(\varphi,\psi), Q(\varphi,\psi)). \end{aligned}$$

We have the following lemma.

Lemma 5.9. *Let $H = H(t,x) : [0,\infty)\times\mathbb{R}\to\mathbb{C}^2$, $W = W(t,x) : [0,\infty)\times\mathbb{R}\to\mathbb{C}^2$ be given vector functions which satisfy for some $C_1 > 0$, $C_2 > 0$, $\lambda > 0$, $T_0 \geqslant 0$:*

$$\|W(t)\|_{L^\infty\times L^\infty} + e^{\lambda t}\|H(t)\|_{L^2\times L^2} \leqslant C_1, \quad \forall t\geqslant T_0, \tag{5.60}$$

$$\|\partial W(t)\|_{L^2\times L^2} + \|\partial W(t)\|_{L^\infty\times L^\infty} + e^{\lambda t}\|\partial H(t)\|_{L^2\times L^2} \leqslant C_2, \quad \forall t\geqslant T_0. \tag{5.61}$$

Consider equation (5.59). There exists a constant λ_* independent of C_2 such that if $\lambda\geqslant\lambda_*$ then there exists a unique solution η of (5.59) on $[T_0,\infty)\times\mathbb{R}$ satisfying

$$e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1, \quad \forall t\geqslant T_0.$$

Proof. We rewrite (5.59) by $\eta = \Phi\eta$. We show that, for λ large enough, Φ is a contraction map in the following ball

$$B = \left\{\eta : \|\eta\|_X := e^{\lambda t}\|\eta\|_{S([t,\infty))\times S([t,\infty))} + e^{\lambda t}\|\partial_x\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant 1\right\}.$$

We will use condition $\lambda\gg 1$ in the proof without specifying it.

Step 1. Proof Φ maps B into B

Let $t\geqslant T_0$, $\eta = (\eta_1,\eta_2)\in B$, $W = (w_1,w_2)$ and $H = (h_1,h_2)$. By Strichartz estimates, we have

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \lesssim \|f(W+\eta)-f(W)\|_{N([t,\infty))\times N([t,\infty))}, \tag{5.62}$$

$$+\|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. \tag{5.63}$$

For (5.63), using (5.60), we have

$$\begin{aligned} \|H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))} &= \|h_1\|_{L^1_\tau L^2_x([t,\infty))} + \|h_2\|_{L^1_\tau L^2_x([t,\infty))} \\ &\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau \leqslant \frac{1}{\lambda}e^{-\lambda t} < \frac{1}{10}e^{-\lambda t}. \end{aligned} \tag{5.64}$$

For (5.62), we have

$$\begin{aligned} &|P(W+\eta)-P(W)| \\ &= |P(w_1+\eta_1, w_2+\eta_2) - P(w_1,w_2)| \\ &\lesssim \left||w_1+\eta_1|^{2\sigma-1)}(w_1+\eta_1)^2\overline{w_2+\eta_2} - |w_1|^{2(\sigma-1)}w_1^2\overline{w_2}\right| \quad (5.65) \\ &+\left|(w_1+\eta_1)\int_{-\infty}^x |w_1+\eta_1|^{2(\sigma-2)}\mathcal{I}m((w_2+\eta_2)^2(\overline{w_1}+\overline{\eta_1})^2)\right. \\ &\left.\quad -w_1\int_{-\infty}^x |w_1|^{2(\sigma-2)}\mathcal{I}m(w_2^2\overline{\eta_1}^2)\right|. \quad (5.66) \end{aligned}$$

Using the assumption $\sigma \geqslant \frac{5}{2}$ and Lemma 5.7 we have

$$\begin{aligned}
&\text{the term (5.65)}\\
&\lesssim \big||w_1+\eta_1|^{2(\sigma-1)} - |w_1|^{2(\sigma-1)}||w_1+\eta_1|^2|w_2+\eta_2|\big|\\
&\quad + \big||w_1|^{2(\sigma-1)}|(w_1+\eta_1)^2 - w_1^2||w_2+\eta_2|\big| + \big||w_1|^{2(\sigma-1)}|w_1|^2|\eta_2|\big|\\
&\lesssim (|\eta_1|^{2(\sigma-1)} + |\eta_1||w_1|^{2(\sigma-1)-1})(|W|+|\eta|)^3\\
&\quad + |w_1|^{2(\sigma-1)}(|w_1||\eta_1| + |\eta_1|^2)|w_2+\eta_2| + |w_1|^{2\sigma}|\eta_2|\\
&\lesssim (|\eta|^{2(\sigma-1)} + |\eta||W|^{2(\sigma-1)-1})(|W|^3+|\eta|^3)\\
&\quad + |W|^{2(\sigma-1)}(|W||\eta| + |\eta|^2)(|W|+|\eta|) + |W|^{2\sigma}|\eta|\\
&\lesssim |\eta|(|\eta|^{2\sigma-3} + |W|^{2\sigma-3})(|\eta|^3 + |W|^3) + |\eta||W|^{2(\sigma-1)}(|W|^2+|\eta|^2) + |W|^{2\sigma}|\eta|\\
&\lesssim |\eta|(|\eta|^{2\sigma} + |W|^{2\sigma}) + |\eta||W|^{2\sigma} + |\eta|^3|W|^{2(\sigma-1)} + |W|^{2\sigma}|\eta|\\
&\lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\text{the term (5.66)}\\
&\lesssim |\eta_1|\int_{-\infty}^{x} |w_1+\eta_1|^{2(\sigma-2)}|w_2+\eta_2|^2|w_1+\eta_1|^2\,dy\\
&\quad + |w_1|\int_{-\infty}^{x} (|w_1+\eta_1|^{2(\sigma-2)} - |w_1|^{2(\sigma-2)})|w_2+\eta_2|^2|w_1+\eta_1|^2\,dy\\
&\quad + |w_1|\int_{-\infty}^{x} |w_1|^{2(\sigma-2)}|\mathcal{Im}((w_2+\eta_2)^2 - w_2^2)(\overline{w_1}+\overline{\eta_1})^2|\,dy\\
&\quad + |w_1|\int_{-\infty}^{x} |w_1|^{2(\sigma-2)}|\mathcal{Im}(w_2^2((\overline{w_1}+\eta_1)^2 - \overline{\eta_1}^2))|\,dy\\
&\lesssim |\eta|\int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\,dy + |W|\int_{-\infty}^{x} (|\eta_1|^{2(\sigma-2)} + |\eta_1||w_1|^{2\sigma-5})(|W|^4+|\eta|^4)\,dy\\
&\quad + |W|\int_{-\infty}^{x} |W|^{2(\sigma-2)}(|\eta_2|^2 + |w_2||\eta_2|)(|W|^2+|\eta|^2)\,dy\\
&\quad + |W|\int_{-\infty}^{x} |W|^{2(\sigma-2)}|w_2|^2(|\eta_1|^2 + |\eta_1||w_1|)\,dy\\
&\lesssim |\eta|\int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\,dy + |W|\int_{-\infty}^{x} |\eta|(|W|^{2\sigma} + |\eta|^{2\sigma})\,dy\\
&+ |W|\int_{-\infty}^{x} |W|^{2(\sigma-2)}|\eta|(|W|^3+|\eta|^3)\,dy + |W|\int_{-\infty}^{x} |W|^{2(\sigma-2)}|W|^2|\eta|(|W|+|\eta|)\,dy\\
&\lesssim |\eta|\int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\,dy + |W|\int_{-\infty}^{x} |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma}\,dy.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&|P(W+\eta) - P(W)|\\
&\lesssim |\eta|^{2\sigma+1} + |\eta||W|^{2\sigma} + |\eta|\int_{-\infty}^{x} |W|^{2\sigma} + |\eta|^{2\sigma}\,dy + |W|\int_{-\infty}^{x} |\eta||W|^{2\sigma-1} + |\eta|^{2\sigma}\,dy.
\end{aligned}$$

Similarly,

$$
\begin{aligned}
&|Q(W+\eta)-Q(W)|\\
&\lesssim |\eta|^{2\sigma+1}+|\eta||W|^{2\sigma}+|\eta|\int_{-\infty}^{x}|W|^{2\sigma}+|\eta|^{2\sigma}\,dy+|W|\int_{-\infty}^{x}|\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy.
\end{aligned}
$$

Hence, using $\sigma \geqslant \frac{5}{2}$, we have:

$$
\begin{aligned}
&\|f(W+\eta)-f(W)\|_{N([t,\infty))\times N([t,\infty))}\\
&\lesssim \|P(W+\eta)-P(W)\|_{L^1_\tau L^2_x([t,\infty))}+\|Q(W+\eta)-Q(W)\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \||\eta|^{2\sigma+1}\|_{L^1_\tau L^2_x([t,\infty))}+\||\eta|\int_{-\infty}^{x}|W|^{2\sigma}+|\eta|^{2\sigma}\,dy\|_{L^1_\tau L^2_x([t,\infty))}\\
&\quad+\||W|\int_{-\infty}^{x}|\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim \||\eta|\|_{L^\infty L^2_x([t,\infty))}\||\eta|\|^4_{L^4_\tau L^\infty_x([t,\infty))}\\
&\quad+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\left\|\int_{-\infty}^{x}|W|^{2\sigma}+|\eta|^{2\sigma}\,dy\right\|_{L^\infty_\tau L^\infty_x([t,\infty))}\\
&\quad+\||W|\|_{L^\infty_\tau L^2_x([t,\infty))}\|\int_{-\infty}^{x}|\eta||W|^{2\sigma-1}+|\eta|^{2\sigma}\,dy\|_{L^1_\tau L^\infty_x([t,\infty))}\\
&\lesssim e^{-5\lambda t}+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\||W|^{2\sigma}+|\eta|^{2\sigma}\|_{L^\infty_\tau L^1_x}\\
&\quad+\|W\|_{L^\infty_t L^2_x}\|\eta\|_{L^1_\tau L^2_x([t,\infty))}\||W|^{2\sigma-1}+|\eta|^{2\sigma-1}\|_{L^\infty_\tau L^2_x([t,\infty))}\\
&\lesssim e^{-5\lambda t}+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}=e^{-5\lambda t}+\int_t^\infty e^{-\lambda\tau}d\tau\\
&\lesssim e^{-5\lambda t}+\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t},
\end{aligned}
$$

Combining with (5.64) and (5.62), (5.63) we obtain

$$
\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}<\frac{1}{5}e^{-\lambda t}. \tag{5.67}
$$

We have

$$
\begin{aligned}
\|\partial_x\Phi\eta\|_{S([t,\infty))\times S([t,\infty))}&\lesssim \|\partial_x(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))} && (5.68)\\
&\quad+\|\partial_x H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}. && (5.69)
\end{aligned}
$$

For (5.69), using (5.61) we have

$$
\|\partial_x H\|_{L^1_\tau L^2_x([t,\infty))\times L^1_\tau L^2_x([t,\infty))}\lesssim \int_t^\infty e^{-\lambda\tau}\,d\tau=\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t}, \tag{5.70}
$$

For (5.68), we have

$$
\begin{aligned}
\|\partial_x(f(W+\eta)-f(W))\|_{N([t,\infty))\times N([t,\infty))}&=\|\partial_x(P(W+\eta)-P(W))\|_{N([t,\infty))}\\
&\quad+\|\partial_x(Q(W+\eta)-Q(W))\|_{N([t,\infty))}.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
&|\partial_x(P(W+\eta)-P(W))| \\
&\lesssim |\partial_x(|w_1+\eta_1|^{2(\sigma-1)}(w_1+\eta_1)^2(\overline{w}_2+\overline{\eta}_2)-|w_1|^{2(\sigma-1)}w_1^2\overline{w}_2)| && (5.71)\\
&\quad+\left|\partial_x(w_1+\eta_1)\int_{-\infty}^{x}|w_1+\eta_1|^{2(\sigma-2)}\mathcal{Im}((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\,dy\right.\\
&\quad\left.-\partial_x w_1\int_{-\infty}^{x}|w_1|^{2(\sigma-2)}\mathcal{Im}(w_2^2\overline{w}_1^2)\,dy\right| && (5.72)\\
&\quad+\big|(w_1+\eta_1)|w_1+\eta_1|^{2(\sigma-2)}\mathcal{Im}((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\\
&\quad-w_1|w_1|^{2(\sigma-2)}\mathcal{Im}(w_2^2\overline{w}_1)\big|. && (5.73)
\end{aligned}
$$

For (5.71), we have

$$
\begin{aligned}
&\text{the term } (5.71)\\
&\lesssim(|\eta|+|\eta|^{2\sigma}+|\partial_x\eta|)(|W|+|W|^{2\sigma}+|\eta|+|\eta|^{2\sigma}+|\partial_x\eta|)
\end{aligned}
$$

Thus,

$$
\|\text{the term } (5.71)\|_{L^1_\tau L^2_x([t,\infty))}\lesssim\||\eta|+|\partial\eta|\|_{L^1_\tau L^2_x}\lesssim\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t}.
$$

For (5.72), using Lemma 5.7, we have

$$
\begin{aligned}
&\|\text{ the term } (5.72)\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim\|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))}\|\int_{-\infty}^{x}|w_1+\eta_1|^{2(\sigma-2)}\mathcal{Im}((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\,dy\|_{L^\infty_t L^\infty_x}\\
&\quad+\|\partial_x w_1\|_{L^\infty_t L^2_x}\times\\
&\quad\times\left\|\int_{-\infty}^{x}(|w_1+\eta_1|^{2(\sigma-2)}\mathcal{Im}((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)-|w_1|^{2(\sigma-2)}\mathcal{Im}(w_2^2\overline{w}_1^2))\,dy\right\|_{L^1{}_\tau L^\infty_x}\\
&\lesssim\|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))}\||w_1+\eta_1|^{2(\sigma-2)}\mathcal{Im}((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)\|_{L^\infty_t L^1_x}\\
&\quad+\||w_1+\eta_1|^{2(\sigma-2)}\mathcal{Im}((w_2+\eta_2)^2(\overline{w}_1+\overline{\eta}_1)^2)-|w_1|^{2(\sigma-2)}\mathcal{Im}(w_2^2\overline{w}_1^2)\|_{L^1_\tau L^1_x}\\
&\lesssim\|\partial\eta_1\|_{L^1_\tau L^2_x([t,\infty))}+\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\leqslant\int_t^\infty e^{-\lambda\tau}\,d\tau\lesssim\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t},
\end{aligned}
$$

For (5.73), using Lemma 5.7, we have

$$
\begin{aligned}
&\|\text{the term } (5.73)\|_{L^1_\tau L^2_x([t,\infty))}\\
&\lesssim\||\eta|\|_{L^1_\tau L^2_x([t,\infty))}\\
&\leqslant\int_t^\infty e^{-\lambda\tau}\,d\tau\lesssim\frac{1}{\lambda}e^{-\lambda t}<\frac{1}{10}e^{-\lambda t},
\end{aligned}
$$

Combining the above estimates, we obtain

$$
\begin{aligned}
&\|\partial_x(P(W+\eta)-P(W))\|_{N([t,\infty))}\\
&\leqslant\|\partial_x(P(W+\eta)-P(W))\|_{L^1_\tau L^2_x([t,\infty))}\leqslant\frac{3}{10}e^{-\lambda t}, && (5.74)
\end{aligned}
$$

Similarly,

$$\|\partial_x(Q(W+\eta)-Q(W))\|_{N([t,\infty))} \leqslant \frac{3}{10}e^{-\lambda t}, \tag{5.75}$$

Combining the estimates (5.68), (5.69), (5.70), (5.74) and (5.75), we have

$$\|\partial_x \Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant \frac{7}{10}e^{-\lambda t}. \tag{5.76}$$

Combining (5.67) with (5.76), we obtain

$$\|\Phi\eta\|_{S([t,\infty))\times S([t,\infty))} + \|\partial_x \Phi\eta\|_{S([t,\infty))\times S([t,\infty))} \leqslant \frac{9}{10}e^{-\lambda t}, \tag{5.77}$$

Thus, for λ large enough

$$\|\Phi\eta\|_X < 1.$$

This implies that Φ maps B into B.
Step 2. Φ is a contraction map on B
By using (5.60), (5.61) and a similar estimate of (5.77), we can show that, for any $\eta \in B$ and $\kappa \in B$ we have

$$\|\Phi\eta - \Phi\kappa\|_X \leqslant \frac{1}{2}\|\eta-\kappa\|_X.$$

for λ large enough. From Banach fixed point theorem, there exists a unique solution in B of (5.59) and thus a solution of (5.26). This completes the proof of Lemma 5.9. $\square$

Chapter 6

Instability of algebraic standing waves for nonlinear Schrödinger equations with triple power nonlinearities

6.1 Introduction

In this chapter, we are interested in the following triple power nonlinear Schrödinger equation:

$$iu_t + \Delta u + a_1|u|u + a_2|u|^2u + a_3|u|^3u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \tag{6.1}$$

where $a_1, a_2, a_3 \in \mathbb{R}$ and $n \in \{1, 2, 3\}$.

The standing waves of (6.1) are solutions of the form $u_\omega(t,x) = e^{i\omega t}\phi_\omega(x)$, where ϕ_ω solves:

$$-\omega\phi_\omega + \Delta\phi_\omega + a_1|\phi_\omega|\phi_\omega + a_2|\phi_\omega|^2\phi_\omega + a_3|\phi_\omega|^3\phi_\omega = 0. \tag{6.2}$$

In [79], the authors study existence and stability of standing waves of (6.1) in one dimension. Existence of standing waves is obtained by ODE arguments. By studying the properties of the nonlinearity, the authors give domains of parameters for existence and nonexistence of standing waves. Stability results are obtained by studying the sign of an integral found by Iliev and Kirchev [62], based on the criteria of stability of Grillakis, Shatah and Strauss [49, 50, 105].

In the special case $\omega = 0$, the profile ϕ_0, which for convenience we denote by ϕ, satisfies:

$$\Delta\phi + a_1|\phi|\phi + a_2|\phi|^2\phi + a_3|\phi|^3\phi = 0. \tag{6.3}$$

The equation (6.3) can be rewritten as $S'(\phi) = 0$ where S is defined by

$$S(v) := \frac{1}{2}\|\nabla v\|_{L^2}^2 - \frac{a_1}{3}\|v\|_{L^3}^3 - \frac{a_2}{4}\|v\|_{L^4}^4 - \frac{a_3}{5}\|v\|_{L^5}^5. \tag{6.4}$$

Define

$$X := \dot{H}^1(\mathbb{R}^n) \cap L^3(\mathbb{R}^n), \qquad \text{and } \|u\|_X := \|\nabla u\|_{L^2} + \|u\|_{L^3}, \tag{6.5}$$

$$d := \inf\{S(v) : v \in X \setminus \{0\}, S'(v) = 0\}. \tag{6.6}$$

The algebraic standing waves are standing waves with algebraic decay. In this paper, we are only interested in a special kind of algebraic standing waves which are minimizers of the problem (6.6). Throughout this paper, for convenience, we define an algebraic standing wave as a solution of (6.3) solving problem (6.6). Thus, the function ϕ is an algebraic standing wave of (6.1) if $\phi \in \mathcal{G}$, where $\mathcal{G}$ is defined by

$$\mathcal{G} := \{v \in X \setminus \{0\} : S'(v) = 0, S(v) = d\}. \tag{6.7}$$

The instability of algebraic standing waves was studied in [36] for double power nonlinearities. Using similar arguments as in [36], we study existence and instability of algebraic standing waves for the nonlinear Schrödinger equation with triple power nonlinearities (6.1).

First, we study the existence of algebraic standing waves of (6.1). As in [79], we will use the abbreviation D: defocusing when $a_i < 0$ and F: focusing when $a_i > 0$. In Section 6.2, we prove the following result.

Proposition 6.1. *Let $n = 1$. The equation* (6.3) *has an unique even positive solution ϕ in the space $H^1(\mathbb{R})$ in the following cases: DFF, DDF, DFD and $a_1 = a_3 = -1$, $a_2 > \frac{8}{\sqrt{15}}$. Moreover, all solutions of* (6.3) *are of the form $e^{i\theta}\phi(x - x_0)$ for some $\theta, x_0 \in \mathbb{R}$. They are all algebraic standing waves of* (6.1).

In high dimensions, the situation is more complex than in the one dimension. The solutions of (6.3) are very diverse. It is not easy to describe all such solutions as in the dimension one. Thus, classifying the algebraic standing waves of (6.1) is not easy problem. It turns out that a radial positive solutions of (6.3) is also an algebraic standing wave of (6.1). To study the positive radial solutions of (6.3), we prove the following result in Section 6.2.

Proposition 6.2. *Let $n = 2, 3$ and DDF or DFF. Then there exists a unique radial positive solution of* (6.3).

Before stating the next results, we need some definitions. Firstly, we define the *Nehari functional* as follows:

$$K(v) := \langle S'(v), v \rangle = \|\nabla v\|_{L^2}^2 - a_1\|v\|_{L^3}^3 - a_2\|v\|_{L^4}^4 - a_3\|v\|_{L^5}^5. \tag{6.8}$$

The *rescaled function* is defined by:

$$v^\lambda(x) := \lambda^{\frac{n}{2}} v(\lambda x). \tag{6.9}$$

The following is *Pohozhaev functional*:

$$P(v) := \partial_\lambda S(v^\lambda)|_{\lambda=1} = \|\nabla v\|_{L^2}^2 - \frac{na_1}{6}\|v\|_{L^3}^3 - \frac{na_2}{4}\|v\|_{L^4}^4 - \frac{3na_3}{10}\|v\|_{L^5}^5. \tag{6.10}$$

The *Nehari manifold* is defined by:

$$\mathcal{K} := \{v \in X \setminus \{0\} : K(v) = 0\}.$$

Moreover, we consider the following minimization problem:

$$\mu := \inf\{S(v) : v \in \mathcal{K}\}. \tag{6.11}$$

The following is the set of minimizers of problem (6.11):

$$\mathcal{M} := \{v \in \mathcal{K} : S(v) = \mu\}. \tag{6.12}$$

Finally, we define a specific set which uses in our proof:

$$\mathcal{B} := \left\{v \in H^1(\mathbb{R}^n) : S(v) < \mu, P(v) < 0\right\}. \tag{6.13}$$

It turns out that the solution of (6.3) given by Proposition 6.2 satisfies a variational characterization and each algebraic standing wave of (6.1) is up to phase shift and translation of this special solution. More precise, in Section 6.3, we prove the following result.

Proposition 6.3. *Let $n = 1, 2, 3$ and DDF or DFF. Then the radial positive solution ϕ of* (6.3) *given by Proposition 6.1 and Proposition 6.2 satisfies*

$$S(\phi) = \mu.$$

where S and μ are defined as in (6.4), (6.11) *respectively. Moreover, all algebraic standing waves of equation* (6.1) *are of the form*

$$e^{i\theta_0}\phi(\cdot - x_0),$$

for some $\theta \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$.

Remark 6.4. (1) In case DFD, we only obtain the result on existence of algebraic standing waves when $n = 1$ (see Proposition 6.1). The variational characterization of algebraic standing waves and stability or instability of these solutions are open problems, even in dimension one.

(2) By using similar arguments as in [36, Proof of Proposition 3.5], we prove that the algebraic standing waves in higher dimensions ($n = 2, 3$) are also in $H^1(\mathbb{R}^n)$.

(3) By scaling invariance of (6.1), we may assume $|a_1| = |a_3| = 1$ without loss of generality. This assumption will be made throughout the rest of this paper.

Before stating the main result, we define the orbital stability and orbital instability of standing waves.

Definition 6.5. Let $u_\omega(t,x) = e^{i\omega t}\phi_\omega(x)$ be a standing wave solution of (6.1). We say that this solution is orbitally stable if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for each $u_0 \in H^1(\mathbb{R}^n)$ such that $\|u_0 - \varphi_\omega\|_{H^1} < \delta$ then the associated solution u of (6.1) is global and satisfies

$$\inf_{\theta\in\mathbb{R},\ y\in\mathbb{R}^n} \|u(t) - e^{i\theta}\varphi_\omega(\cdot - y)\|_{H^1} < \varepsilon.$$

Otherwise, u_ω is orbitally unstable.

Our main result is the following.

Theorem 6.6. *Let $n = 1, 2, 3$. Assume that the parameters of* (6.1) *satisfy DDF or DFF when $n = 2, 3$ or DFF and $a_2 < \frac{32}{15\sqrt{6}}$ when $n = 1$. Then the algebraic standing wave ϕ given as in Proposition 6.1 and Proposition 6.3 is orbitally unstable in $H^1(\mathbb{R}^n)$.*

The rest of this paper is organized as follows. In Section 6.2, we find the region of parameters a_1, a_2, a_3 in which there exist solutions of the elliptic equation (6.3). Specially, in one dimension, all solution of (6.3) are algebraic standing waves. In Section 6.3, we establish the variational characterization of solutions given in Section 6.2. The existence of algebraic standing waves in high dimensions is also proved in section 6.3. In Section 6.4, we prove instability of algebraic standing waves.

6.2 Existence of solution of the elliptic equation

First, we find the region of parameters a_1, a_2, a_3 in which there exist solutions of (6.3).

6.2.1 In dimension one

Let $n = 1$. To study the existence of algebraic standing waves, we use the following lemma (see [5], [79, Proposition 2.1])

Lemma 6.7. *Let g be a locally Lipschitz continuous function with $g(0) = 0$ and let $G(t) = \int_0^t g(s)\,ds$. A necessary and sufficient condition for the existence of a solution ϕ of the problem*

$$\begin{cases} \phi \in C^2(\mathbb{R}), \quad \lim_{x\to\pm\infty} \phi(x) = 0, \quad \phi(0) > 0, \\ \phi_{xx} + g(\phi) = 0, \end{cases} \tag{6.14}$$

is that $c = \inf\{t > 0 : G(t) = 0\}$ exists, $c > 0$, $g(c) > 0$.

Using Lemma 6.7, we have the following result.

Lemma 6.8. *Let $g(u) = a_1u^2 + a_2u^3 + a_3u^4$ be such that g satisfies the assumptions of Lemma 6.7 for some $a_1, a_2, a_3 \in \mathbb{R}$. Then there exists a positive solution ϕ of* (6.14)*. Moreover, all complex valued solutions of* (6.14) *are of form:*

$$e^{i\theta_0}\phi(x - x_0),$$

for some $\theta_0, x_0 \in \mathbb{R}$.

Proof. By Lemma 6.7, there exists a real valued solution ϕ of (6.14). We have

$$\phi_{xx} + a_1\phi^2 + a_2\phi^3 + a_3\phi^4 = 0. \tag{6.15}$$

Since $\lim_{x\to\pm\infty} \phi(x) = 0$, there exists x_0 such that $\phi_x(x_0) = 0$. Multiplying two sides of (6.15) by ϕ_x and noting that $\lim_{x\to\infty} \phi(x) = 0$ we obtain

$$\frac{1}{2}\phi_x^2 + \frac{a_1}{3}\phi^3 + \frac{a_2}{4}\phi^4 + \frac{a_3}{5}\phi^5 = 0. \tag{6.16}$$

We see that ϕ is not vanishing on $\mathbb{R}$. Indeed, if $\phi(x_1) = 0$ for some $x_1 \in \mathbb{R}$ then $\phi_x(x_1) = 0$ by (6.16). Thus, $\phi \equiv 0$ by uniqueness of solutions of (6.16) which is a contradiction. Then, we can assume that $\phi > 0$.

The value $\phi(x_0)$ is a positive solution of $G(u) = \frac{a_1}{3}u^3 + \frac{a_2}{4}u^4 + \frac{a_3}{5}u^5 = 0$. Since g satisfies the condition in Lemma 6.7, it follows that $G(u) = 0$ has a first positive solution c such that $g(c) > 0$. If $\phi(x_0) \neq c$ then G has another positive zero $d > c$ such that $d = \phi(x_0)$. By continuity of ϕ, there exists $x_1 > x_0$ such that $\phi(x_1) = c$ and by (6.16) $\phi_x(x_1) = 0$. This conclusion implies that every positive solution of (6.15) has a critical point such that the value of solution at this point equals to c.

Let u be a complex valued solution of (6.14). We prove that $u = e^{i\theta_0}\phi(x-x_0)$, for some $\theta_0, x_0 \in \mathbb{R}$. We use similar arguments as in [16, Theorem 8.1.4]. Multiplying the equation by $\overline{u}_x$ and taking real part, we obtain:

$$\frac{d}{dx}\left(\frac{1}{2}|u_x|^2 + \frac{a_1}{3}|u|^3 + \frac{a_2}{4}|u|^4 + \frac{a_3}{5}|u|^5\right) = 0.$$

Thus,

$$\frac{1}{2}|u_x|^2 + \frac{a_1}{3}|u|^3 + \frac{a_2}{4}|u|^4 + \frac{a_3}{5}|u|^5 = K.$$

Using $\lim_{x\to\pm\infty} u(x) = 0$ we have $K = 0$. In particular, $|u| > 0$. Indeed, if u vanishes then u_x vanish at the same point, hence, $u \equiv 0$. Therefore, we may write $u = \rho e^{i\theta}$, where $\rho > 0$ and $\rho, \theta \in C^2(\mathbb{R})$. Substituting $u = \rho e^{i\theta}$ in (6.14) we have $2\rho_x\theta_x + \rho\theta_{xx} = 0$ which implies there exists $\tilde{K} \in \mathbb{R}$ such that $\rho^2\theta_x = \tilde{K}$ and so $\theta_x = \frac{\tilde{K}}{\rho^2}$. Moreover, since $|u_x|$ is bounded, it follows that $\rho^2\theta_x^2$ is bounded. Thus, $\frac{\tilde{K}^2}{\rho^2}$ is bounded. Since $\rho(x) \to 0$ as $x \to \infty$, we have $\tilde{K} = 0$. Thus, since $\rho > 0$ we have $\theta \equiv \theta_0$ for some $\theta_0 \in \mathbb{R}$. Thus $u = e^{i\theta_0}\rho$. Since ρ is a positive solution of (6.15), there exists $x_2 \in \mathbb{R}$ such that $\rho(x_2) = c$ and $\rho_x(x_2) = 0$. Thus, by uniqueness of solution of (6.15), there exists $x_3 \in \mathbb{R}$ such that $\rho(x) = \phi(x - x_3)$ and $u = e^{i\theta_0}\phi(x - x_3)$. This implies the desired result. □

Moreover, we have the following result.

Lemma 6.9. *Let g and ϕ be as in Lemma 6.8. Then $\phi \in H^1(\mathbb{R})$.*

Proof. Firstly, since g satisfies the assumption of Lemma 6.7, we have $a_1 < 0$ (see the arguments in the proof of Proposition 6.1). As in the proof of Lemma 6.8, up to a translation, we may assume that $\phi_x(0) = 0$ and let $c = \phi(0)$. Then ϕ is an even function of x. Furthermore, ϕ satisfies

$$\frac{1}{2}\phi_x^2 + G(\phi) = 0. \tag{6.17}$$

Moreover, $\phi_{xx}(0) = -g(\phi(0)) = -g(c) < 0$. Therefore, there exists $a > 0$ such that $\phi_x < 0$ on $(0, a)$. We claim that $a = \infty$. Otherwise, there would exists $b > 0$ such that $\phi_x < 0$ on $(0, b)$ and $\phi_x(b) = 0$. Thus, $\phi(b) < c$ is a positive zero of G. This is a contradiction since c is the first positive solution of G. Hence, $\phi_x < 0$ on $(0, \infty)$. Thus, there exists $0 \leqslant l < c$ such that $\lim_{x\to\infty} \phi(x) = l$. In particular, there exists $x_m \to \infty$ such that $\phi_x(x_m) \to 0$ as $m \to \infty$. Passing to the limit in (6.17) we have

$G(l) = 0$ and hence $l = 0$ by definition of c. Therefore ϕ decreases to 0, as $x \to \infty$. Thus, from (6.17), for $|x|$ large enough, we have

$$\phi_x^2 \approx -\frac{a_1}{3}\phi^3.$$

Then

$$-\phi_x \approx c\phi^{\frac{3}{2}}, \text{ for some } c > 0.$$

Thus, for $|x|$ large enough, we have

$$0 \geqslant \phi_x + c\phi^{\frac{3}{2}}.$$

It follows that $\phi \leqslant \frac{1}{(cx+d)^2}$ for some $c, d > 0$. Hence $\phi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, especially $\phi \in L^2(\mathbb{R})$. Combining this and (6.17), we obtain that $\phi_x \in L^2(\mathbb{R})$. Thus, $\phi \in H^1(\mathbb{R})$, this completes the proof of Lemma 6.9. □

Now, we comeback to the proof of Proposition 6.1.

Proof of Proposition 6.1. A solution of (6.3) in the space X satisfies

$$u_{xx} + g(u) = 0, \quad u \in C^2(\mathbb{R}), \quad \text{and } \lim_{x\to\pm\infty} u(x) = 0, \tag{6.18}$$

From Lemma 6.7, the necessary condition for existence of solutions of (6.18) is $a_1 < 0$. Indeed, let c is the first positive root of $G(u)$ then $G'(c) = g(c) > 0$. Thus, G do not change sign on $(0, c)$ and is increasing in a neighborhood of c. It follows that $G < 0$ on $(0, c)$ and hence $a_1 < 0$.
To conclude the existence of solution of (6.18), we consider the three cases DDF, DFF, DFD. In the case DDD we have $G < 0$ on $(0, \infty)$, therefore there is no solution of (6.18).
In the case DDF (i.e $a_1 = -1$, $a_2 < 0$, $a_3 = 1$), we have

$$g(s) = -s^2 + a_2 s^3 + s^4,$$
$$G(s) = -\frac{1}{3}s^3 + \frac{a_2}{4}s^4 + \frac{1}{5}s^5.$$

Thus ,

$$c = \frac{-\frac{a_2}{4} + \sqrt{\frac{a_2^2}{16} + \frac{4}{15}}}{\frac{2}{5}},$$

and $g(c) = c^2(c^2 + a_2 c - 1)$. It easy to check that c is larger than the largest root of $x^2 + a_2 x - 1$. Thus, $g(c) > 0$. It follows that in case DDF, there exists a solution of (6.18).
By similar arguments, in the case DFF, (6.18) has a solution. In the case DFD, (6.18) has a solution if and only if $a_2 > \frac{8}{\sqrt{15}}$.
Let ϕ be a solution of (6.18). From Lemma 6.8 all solution of (6.18) are of the form $e^{i\theta}\phi(x - x_0)$, and belong to $H^1(\mathbb{R})$ by Lemma 6.9. Thus, they are all algebraic standing waves of (6.1). This completes the proof of Proposition 6.1. □

6.2.2 In higher dimensions

In this section, we prove existence and uniqueness of a radial positive solution of (6.3) when $a_1 = -1$, $a_3 = 1$ and $n = 2, 3$. The existence result is a consequence of the following theorem.

Theorem 6.10 ([6],Theorem 1.1)**.** *Let g be a locally Lipschitz continuous function from $\mathbb{R}^+$ to $\mathbb{R}$ with $g(0) = 0$, satisfying*

(1) $\alpha = \inf\{\zeta > 0, g(\zeta) \geqslant 0\}$ *exists, and* $\alpha > 0$.

(2) *There exists a number* $\zeta > 0$ *such that* $G(\zeta) > 0$*, where*

$$G(t) = \int_0^t g(s)\, ds.$$

Define $\zeta_0 = \inf\{\zeta > 0, G(\zeta) > 0\}$*. Then,* ζ_0 *exists, and* $\zeta_0 > \alpha$.

(3) $\lim_{s\downarrow\alpha} \frac{g(s)}{s-\alpha} > 0$.

(4) $g(s) > 0$ *for* $s \in (\alpha, \zeta_0]$*. Let* $\beta = \inf\{\zeta > \zeta_0, g(\zeta) = 0\}$*. Then,* $\zeta_0 < \beta \leqslant \infty$.

(5) *If* $\beta = \infty$ *then* $\frac{g(s)}{s^l} = 0$*, with* $l < \frac{n+2}{n-2}$*, (If* $n = 2$*, we may choose for* l *just any finite real number).*

Then there exists a number $\zeta \in (\zeta_0, \beta)$ *such that the solution* $u \in C^2(\mathbb{R}^+)$ *of the Initial Value problem*

$$\begin{cases} -u'' - \frac{n-1}{r}u' = g(u), \text{ for } r > 0, \\ u(0) = \zeta, \;\; u'(0) = 0 \end{cases}$$

has the properties: $u > 0$ *on* $\mathbb{R}^+$*,* $u' < 0$ *on* $\mathbb{R}^+$ *and*

$$\lim_{r\to\infty} u(r) = 0.$$

In our case, we have

$$g(s) = -s^2 + a_2 s^3 + s^4, \tag{6.19}$$

$$G(s) = \frac{-1}{3}s^3 + \frac{a_2}{4}s^4 + \frac{1}{5}s^5. \tag{6.20}$$

It is easy to check that the function g and G satisfy the conditions of Theorem 6.10 when $n = 2, 3$ with $\alpha = \frac{-a_2+\sqrt{a_2^2+4}}{2}$ (the positive zero of g), $\zeta_0 = \frac{-a_2+\sqrt{a_2^2+\frac{64}{15}}}{\frac{8}{5}}$ (the positive zero of G), $\beta = \infty$ and $4 < l < 5$ when $n = 3$ and $l > 4$ when $n = 2$. Thus, in high dimensions ($n = 2, 3$), there exists a decreasing radial positive solution of (6.3).

The uniqueness of a radial positive solution is obtained by following result.

Theorem 6.11 ([102],Theorem 1)**.** *Let us consider, for* $n \geqslant 2$*, the following equation*

$$\Delta u + g(u) = 0, \tag{6.21}$$

where g *satisfies the following conditions:*

(a) g is continuous on $[0,\infty)$ and $g(0)=0$,

(b) g is a C^1-function on $(0,\infty)$,

(c) There exists $a>0$ such that $g(a)=0$ and

$$\begin{aligned} &g(u)<0 \text{ for } 0<u<a,\\ &g(u)>0 \text{ for } u>a. \end{aligned}$$

(d) $\frac{d}{du}\left[\frac{G(u)}{g(u)}\right] \geqslant \frac{n-2}{2n}$, for $u>0, u\neq a$, where $G(s)=\int_0^s f(\tau)d\tau$.

Then (6.21) *admits at most one radial positive solution.*

The function g given in (6.19) satisfies conditions (a), (b), (c) of Theorem 6.11 for a the positive root of g. When $n=2,3$, the condition (d) is satisfied if only if

$$\frac{d}{ds}\left[\frac{\frac{1}{5}s^3+\frac{a_2}{4}s^2-\frac{1}{3}s}{s^2+a_2s-1}\right] \geqslant \frac{n-2}{2n}, \text{ for } s>0, s\neq a. \tag{6.22}$$

We prove that (6.22) holds. We only need to show that

$$\frac{d}{ds}\left[\frac{\frac{1}{5}s^3+\frac{a_2}{4}s^2-\frac{1}{3}s}{s^2+a_2s-1}\right] \geqslant \frac{1}{6}, \text{ for } s\neq a.$$

This is equivalent to

$$\frac{1}{5}s^4+\frac{2a_2}{5}s^3+\left(\frac{a_2^2}{2}+\frac{2}{5}\right)-a_2s+1\geqslant 0,$$

which is true for all $s>0$, $a_2\in\mathbb{R}$ by the fact that

$$\frac{1}{5}s^4+\frac{2a_2}{5}s^3+\left(\frac{a_2^2}{2}+\frac{2}{5}\right)-a_2s+1=\frac{1}{5}(s^2+a_2s)^2+\frac{3}{10}\left(a_2-\frac{5}{3}\right)^2+\frac{2}{5}s^2+\frac{1}{6}>0.$$

Thus, there exists a unique radial positive solution of (6.3) by Theorem 6.11. This completes the proof of Proposition 6.2.

6.3 Variational characterization

Let $n=1,2,3$. In this section, we prove Proposition 6.3. By the assumption of Proposition 6.3, we may pick $a_1=-1$ and $a_3=1$. We recall that S,K,P are defined in (6.4), (6.8) and (6.10).

Let $\mathcal{M}$ and $\mathcal{K}$ be defined as (6.12) and (6.8). First, as in [36], we prove that $\mathcal{M}$ is not empty. We set

$$J(v)=\frac{1}{4}\|\nabla v\|_{L^2}^2+\frac{1}{12}\|v\|_{L^3}^3+\frac{1}{20}\|v\|_{L^5}^5,$$

which is well defined on X. The functional S is rewritten as

$$S(v) = \frac{1}{2}K(v) - \frac{1}{6}\|v\|_{L^3}^3 + \frac{a_2}{4}\|v\|_{L^4}^4 + \frac{3}{10}\|v\|_{L^5}^5,$$
$$S(v) = \frac{1}{4}K(v) + J(v).$$

We can rewrite μ as

$$\mu = \inf\{J(v) : v \in \mathcal{K}\}. \tag{6.23}$$

Lemma 6.12. *Let $v \in H^1(\mathbb{R}^n)$. If $K(v) < 0$ then $\mu < J(v)$. In particular,*

$$\mu = \inf\{J(v) : v \in X \setminus \{0\}, K(v) \leqslant 0\}. \tag{6.24}$$

Proof. Since $K(v) < 0$ and $K(\lambda v) > 0$ if $\lambda > 0$ small enough, there exists $\lambda_1 \in (0,1)$ such that $K(\lambda_1 v) = 0$. Therefore, by (6.23) and since the function $\lambda \mapsto J(\lambda v)$ on $(0,\infty)$ is increasing, we have

$$\mu \leqslant J(\lambda_1 v) < J(v).$$

This completes the proof. □

Lemma 6.13. *The following is true:*

$$\mu > 0.$$

Proof. Let $v \in \mathcal{K}$. By using the Gagliardo-Nirenberg inequalities, for some $\theta \in (0,5)$ and $\tilde{\theta} \in (0,4)$, we have

$$\|v\|_{L^5}^5 \lesssim \|\nabla v\|_{L^2}^{\theta}\|v\|_{L^3}^{5-\theta} \leqslant C_1\|\nabla v\|_{L^2}^5 + C_2\|v\|_{L^3}^5,$$
$$\|v\|_{L^4}^4 \lesssim \|\nabla v\|_{L^2}^{\tilde{\theta}}\|v\|_{L^3}^{4-\tilde{\theta}} \leqslant C_3\|\nabla v\|_{L^2}^4 + C_4\|v\|_{L^3}^4,$$

we have

$$0 = K(v) \geqslant (1 - C_1\|\nabla v\|_{L^2}^3 - |a_2|C_3\|\nabla v\|_{L^2}^2)\|\nabla v\|_{L^2}^2 + (1 - C_2\|v\|_{L^3}^2 - |a_2|C_4\|v\|_{L^3})\|v\|_{L^3}^3,$$

It follows that $1 \leqslant C_1\|\nabla v\|_{L^2}^3 + |a_2|C_3\|\nabla v\|_{L^2}^2 \leqslant C\|\nabla v\|_{L^2}^3 + \frac{1}{2}$ or $1 \leqslant C_2\|v\|_{L^3}^2 + |a_2|C_4\|v\|_{L^3} \leqslant \tilde{C}\|v\|_{L^3}^2 + \frac{1}{2}$, for some $C, \tilde{C} > 0$. Hence, $\|\nabla v\|_{L^2}$ or $\|v\|_{L^3}^3$ bounded below by some constant. In two cases, $J(v)$ is bounded below by some constant. Combining with (6.23) we have the conclusion. □

We need the following results.

Lemma 6.14 ([2, 76]). *Let $p \geqslant 1$. Let (f_n) be a bounded sequence in $\dot{H}^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$. Assume that there exists $q \in (p, 2^*-1)$ such that* $\limsup_{n\to\infty}\|f_n\|_{L^{q+1}} > 0$. *Then there exist $(y_n) \subset \mathbb{R}^n$ and $f \in \dot{H}^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n) \setminus \{0\}$ such that $(f_n(\cdot - y_n))$ has a subsequence that converges to f weakly in $\dot{H}^1(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$.*

Lemma 6.15 ([14]). *Let $1 \leqslant r < \infty$. Let (f_n) be a bounded sequence in $L^r(\mathbb{R}^n)$ and $f_n \to f$ a.e in $\mathbb{R}^n$ as $n \to \infty$. Then*

$$\|f_n\|_{L^r}^r - \|f_n - f\|_{L^r}^r - \|f\|_{L^r}^r \to 0,$$

as $n \to \infty$.

Now, we comeback to prove the set $\mathcal{M}$ is not empty.

Lemma 6.16. *If $(v_n) \in X$ is a minimizing sequence for μ, that is,*

$$K(v_n) \to 0, \quad S(v_n) \to \mu,$$

then there exist $(y_n) \subset \mathbb{R}^n$, a subsequence (v_{n_j}), and $v_0 \in X \setminus \{0\}$ such that $v_{n_j}(\cdot - y_{n_j}) \to v_0$ in X. In particular, $v_0 \in \mathcal{M}$.

Proof. Since $K(v_n) \to 0$ and $S(v_n) \to \mu$, we have

$$J(v_n) \to \mu, \tag{6.25}$$

$$\frac{-1}{6}\|v\|_{L^3}^3 + \frac{a_2}{4}\|v\|_{L^4}^4 + \frac{3}{10}\|v\|_{L^5}^5 \to \mu. \tag{6.26}$$

From (6.25), we infer that (v_n) is bounded in X. Also, since $\mu > 0$ by Lemma 6.13 and the Gagliardo-Nirenberg inequality $\|v\|_{L^5}^5 \lesssim \|\nabla v\|_{L^2}^5 + \|v\|_{L^4}^5$, we have $\limsup_{n\to\infty}\|v_n\|_{L^4} > 0$. Then, by Lemma 6.14 there exist $(y_n) \subset \mathbb{R}^n$ and $v_0 \in X\setminus\{0\}$ and a subsequence of $(v_n(\cdot - y_n))$, which we still denote by the same notation, such that $v_n(\cdot - y_n) \rightharpoonup v_0$ weakly in X. we put $w_n := v_n(\cdot - y_n)$.

We can assume that $w_n \to v_0$ a.e in $\mathbb{R}^n$ and we prove that $w_n \to v_0$ strongly in X. By Lemma 6.15, we have

$$J(w_n) - J(w_n - v_0) \to J(v_0), \tag{6.27}$$

$$K(w_n) - K(w_n - v_0) \to K(v_0). \tag{6.28}$$

Since $J(v_0) > 0$ by $v_0 \neq 0$, it follows from (6.27) and (6.25) that

$$\lim_{n\to\infty} J(w_n - v_0) = \lim_{n\to\infty} J(w_n) - J(v_0) < \lim_{n\to\infty} J(w_n) = \mu.$$

From this and (6.24) we have $K(w_n - v_0) > 0$ for n large. Thus, since $K(v_n) \to 0$ and (6.28) we obtain $K(v_n) \leqslant 0$. By (6.24) and weak lower semicontinuity of the norms, we have

$$\mu \leqslant J(v_0) \leqslant \lim_{n\to\infty} J(w_n) = \mu.$$

Combining with (6.27) imply that $J(w_n - v_0) \to 0$ thus, $w_n \to v_0$ strongly in X. This completes the proof. $\square$

Proof of Proposition 6.3. Firstly, we prove the variational characterization of ϕ as follows

$$S(\phi) = \mu.$$

This means that ϕ is a minimizer of (6.11). From Lemma 6.16, we have $\mathcal{M} \neq \emptyset$. Let $\varphi \in \mathcal{M}$. We divide the proof of this to three steps.
Step 1. There exists $\theta \in \mathbb{R}$ such that $e^{i\theta}\varphi$ is a positive function.
We use similar arguments as in [36, Lemma 2.10]. Put $v := |\mathcal{R}e\varphi|$, $w := |\mathcal{I}m\varphi|$ and $\psi := v + iw$. By a phase modulation, we may assume that $v \neq 0$.

Since $|\psi| = |\varphi|$ and $|\nabla\psi| = |\nabla\varphi|$, we have $K(\psi) = K(\varphi)$ and $S(\psi) = S(\varphi)$. Thus, $\psi \in \mathcal{M}$. Then, there exists $\gamma \in \mathbb{R}$ such that

$$S'(\psi) = \gamma K'(\psi).$$

Hence,

$$\gamma \langle K'(\psi), \psi\rangle = \langle S'(\psi), \psi\rangle = K(\psi) = 0. \tag{6.29}$$

Moreover, using $K(\psi) = 0$ we have

$$\begin{aligned}
\langle K'(\psi), \psi\rangle &= \partial_\lambda K(\lambda\psi)|_{\lambda=1} \\
&= \partial_\lambda K(\lambda\psi)|_{\lambda=1} - 4K(\psi) \\
&= (2\|\nabla\psi\|_{L^2}^2 + 3\|\psi\|_{L^3}^3 - 4a_2\|\psi\|_{L^4}^4 - 5\|\psi\|_{L^5}^5) \\
&\quad - 4(\|\nabla\psi\|_{L^2}^2 + \|\psi\|_{L^3}^3 - a_2\|\psi\|_{L^4}^4 - \|\psi\|_{L^5}^5) \\
&= -2\|\nabla\psi\|_{L^2}^2 - \|\psi\|_{L^3}^3 - \|\psi\|_{L^5}^5 < 0.
\end{aligned}$$

Combining with (6.29), we deduce $\gamma = 0$. Thus, $S'(\psi) = 0$. Hence, v solves the following equation

$$(-\Delta + |\varphi| - a_2|\varphi|^2 - |\varphi|^3)v = 0.$$

Since v is nonnegative and not identically equal to zero, using [77, Theorem 9.10], we infer that v is positive function. Furthermore, since $K(|\psi|) \leqslant K(\psi)$ and $S(|\psi|) \leqslant S(\psi)$, it follows from Lemma 6.12 we have $K(|\psi|) = K(\psi)$ and $S(|\psi|) = S(\psi)$. Then, $\|\nabla|\psi|\|_{L^2} = \|\nabla\psi\|_{L^2}$. By [77, Theorem 7.8], there exists a constant c such that $w = cv$ for some $c \geqslant 0$.

Since v is continuous and positive, $\mathcal{R}e\varphi$ and $\mathcal{I}m\varphi$ do not change sign. Then, there exist constants $\lambda = \pm 1$ and $\eta \in \mathbb{R}$ such that $\mathcal{R}e\varphi = \lambda v$ and $\mathcal{I}m\varphi = \eta v$. Taking $\theta \in \mathbb{R}$ such that $e^{-i\theta} = \frac{\lambda + i\eta}{|\lambda + i\eta|}$, we have $e^{i\theta}\varphi = e^{i\theta}(\lambda + i\eta)v = |\lambda + i\eta|v$. This completes the step 1.

Step 2. Radial symmetry of minimizer.

Since [75, Theorem 1], there exists $y \in \mathbb{R}^n$ such that $e^{i\theta}\varphi(\cdot - y)$ is a radial and decreasing function.

Step 3. Conclusion.

Since ϕ and $e^{i\theta}\varphi(\cdot - y)$ are positive radial solutions of (6.3), using Proposition 6.2, we obtain

$$\phi = e^{i\theta}\varphi(\cdot - y),$$

Thus, $S(\phi) = S(\varphi) = \mu$, $\phi \in \mathcal{M}$ and each element of $\mathcal{M}$ is of form $e^{i\theta}\phi(\cdot - x_0)$ for some $\theta, x_0 \in \mathbb{R}$.

It remains to classify all algebraic standing waves of (6.1). We only need to prove that $\mathcal{G} = \mathcal{M} \neq \emptyset$, where $\mathcal{G}$ and $\mathcal{M}$ are defined in (6.7) and (6.12), respectively. We use similar arguments as in [36, Proof of Theorem 2.1]. We divide the proof of this in two steps.

Step 1. $\mathcal{M} \subset \mathcal{G}$.

Let $\psi \in \mathcal{M}$. Then, $S'(\psi) = 0$. Now, we show that $\psi \in \mathcal{G}$. Let $v \in X \setminus \{0\}$ such that $S'(v) = 0$. From $K(v) = \langle S'(v), v\rangle = 0$ and by definition of $\mathcal{M}$, we have $S(\psi) \leqslant S(v)$. Thus, $\psi \in \mathcal{G}$ and $\mathcal{M} \subset \mathcal{G}$.

Step 2. $\mathcal{G} \subset \mathcal{M}$ and conclusion.

Let $\psi \in \mathcal{G}$. Then $K(\psi) = \langle S'(\psi), \psi\rangle = 0$. As the above, $\phi \in \mathcal{M}$. As in step 1, $\phi \in \mathcal{G}$. Therefore, $S(\psi) = S(\phi) = \mu$, which implies $\psi \in \mathcal{M}$. Thus $\mathcal{G} \subset \mathcal{M}$, which completes the proof of Proposition 6.3. □

It turns out that the algebraic standing waves of (6.1) in high dimensions ($n = 2, 3$) belongs to $H^1(\mathbb{R}^n)$. To prove this, we need the following lemma (see [36, Lemma 3.4]).

Lemma 6.17. *Let $\varphi \in C^1([0,\infty))$ be a positive function. If there exist $\rho, A > 0$such that*

$$\varphi'(r) + A\varphi(r)^{1+\rho} \leqslant 0, \text{ for all } r > 0,$$

then

$$\varphi(r) \leqslant \left(\frac{1}{\rho A r}\right)^{\frac{1}{\rho}}.$$

Proof of Remark 6.4(2). We use similar arguments as in [36, Proof of Proposition 3.5]. Firstly, we denote $\phi(r)$ as function of ϕ respect to variable $r = |x|$. Since ϕ is positive decreasing radial function, we have

$$\|\phi\|_{L^3}^3 \geqslant \int_{|x|\leqslant R} |\phi|^3 \, dx \geqslant |\mathcal{B}(R)||\phi(R)|^3 = CR^n|\phi(R)|^3,$$

for all $R > 0$. Hence,

$$\phi(x) \leqslant |x|^{-\frac{n}{3}} \|\phi\|_{L^3}, \text{ for all } x \in \mathbb{R}.$$

For $r > r_0$ large enough, we have

$$|a_2|\phi^3 + \phi^4 \leqslant \frac{1}{2}\phi^2,$$

Since ϕ solves (6.3) and is decreasing as a function of r, this implies

$$\phi''(r) \geqslant \phi''(r) + \frac{n-1}{r}\phi'(r) = \phi^2 - a_2\phi^3 - \phi^4 \geqslant \frac{1}{2}\phi^2, \text{ for } r > r_0.$$

Multiplying the two sides by ϕ' and integrating it on $[r, \infty)$, we get

$$\phi'(r)^2 \geqslant \frac{1}{3}\phi^3, \text{ for } r \geqslant r_0.$$

Since $\phi' < 0$ we obtain that

$$\phi'(r) + \sqrt{\frac{1}{3}}\phi^{\frac{3}{2}} \leqslant 0, \text{ for } r \geqslant r_0.$$

By Lemma 6.17, we deduce that

$$\phi(r) \leqslant Cr^{-2}, \text{ for } r \geqslant r_0.$$

Thus, $\phi \in L^2(\mathbb{R}^n)$, for $n = 1, 2, 3$. From the proof of Proposition 6.3, we have $\phi \in \mathcal{M}$. Hence, $|\nabla\phi| \in L^2(\mathbb{R}^n)$ and $\phi \in H^1(\mathbb{R}^n)$. This completes the proof. □

6.4 Instability of algebraic standing waves

Let $n = 1, 2, 3$. In this section, we prove Theorem 6.6. Throughout this section, we consider the case DDF or DFF and a_2 small. Then we may pick $a_1 = -1$ and $a_3 = 1$. First, we prove the following result by using similar arguments as in [98] (see also [36, Proof of Proposition 5.1]).

Proposition 6.18. *Assume that*

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0, \text{ where } v^\lambda(x) := \lambda^{\frac{n}{2}} v(\lambda x). \tag{6.30}$$

Then the algebraic standing wave ϕ is unstable.

We define a tube around the standing wave by

$$\mathcal{N}_\varepsilon := \left\{ v \in H^1(\mathbb{R}^n) : \inf_{(\theta,y)\in\mathbb{R}\times\mathbb{R}^n} \|v - e^{i\theta}\phi(\cdot - y)\|_{H^1} < \varepsilon \right\}.$$

Lemma 6.19. *Assume* (6.30) *holds. Then there exist $\varepsilon_1, \delta_1 \in (0,1)$ such that: For any $v \in \mathcal{N}_{\varepsilon_1}$ there exists $\Lambda(v) \in (1-\delta_1, 1+\delta_1)$ such that*

$$\mu \leqslant S(v) + (\Lambda(v) - 1)P(v).$$

Proof. First, we recall that S, K and P are defined as in (6.4), (6.8) and (6.10), respectively.

Since $\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0$, by the continuity of the function

$$(\lambda, v) \mapsto \partial_\lambda^2 S(v^\lambda),$$

there exist $\varepsilon_1, \delta_1 \in (0,1)$ such that $\partial_\lambda^2 S(v^\lambda) < 0$ for any $\lambda \in (1-\delta_1, 1+\delta_1)$ and $v \in \mathcal{N}_{\varepsilon_1}$. Moreover, by the definition of P we have

$$S(v^\lambda) \leqslant S(v) + (\lambda - 1)P(v), \tag{6.31}$$

for $\lambda \in (1-\delta_1, 1+\delta_1)$ and $v \in \mathcal{N}_{\varepsilon_1}$.
Moreover, consider the map:

$$(\lambda, v) \mapsto K(v^\lambda) = \lambda^2 \|\nabla v\|_{L^2}^2 + \lambda^{\frac{n}{2}} \|v\|_{L^3}^3 - a_2 \lambda^n \|v\|_{L^4}^4 - \lambda^{\frac{3n}{2}} \|v\|_{L^5}^5.$$

Note that $K(\phi) = 0$ and

$$\partial_\lambda K(\phi^\lambda)|_{\lambda=1} = 2\|\nabla\phi\|_{L^2}^2 + \frac{n}{2}\|\phi\|_{L^3}^3 - na_2\|\phi\|_{L^4}^4 - \frac{3n}{2}\|\phi\|_{L^5}^5.$$

Thus,

$$\begin{aligned}
\partial_\lambda K(\phi^\lambda)|_{\lambda=1} &= \partial_\lambda K(\phi^\lambda)|_{\lambda=1} - 5P(\phi) \\
&= -3\|\nabla\phi\|_{L^2}^2 - \frac{n}{3}\|\phi\|_{L^3}^3 + \frac{na_2}{4}\|\phi\|_{L^4}^4.
\end{aligned}$$

Thus, in the case $a_2 < 0$, we have $\partial_\lambda K(\phi^\lambda)|_{\lambda=1} < 0$. In the case $a_2 \geqslant 0$, using $P(\phi) = 0$, we have

$$\begin{aligned}\frac{na_2}{4}\|\phi\|_{L^4}^4 &= \|\nabla\phi\|_{L^2}^2 + \frac{n}{6}\|\phi\|_{L^3}^3 - \frac{3n}{10}\|\phi\|_{L^5}^5 \\ &\leqslant 3\|\nabla\phi\|_{L^2}^2 + \frac{n}{3}\|\phi\|_{L^3}^3,\end{aligned}$$

hence we also have $\partial_\lambda K(\phi^\lambda)|_{\lambda=1} < 0$. In all cases, by the implicit function theorem, taking ε_1 and δ_1 small enough, for any $v \in \mathcal{N}_{\varepsilon_1}$ there exists $\Lambda(v) \in (1-\delta_1, 1+\delta_1)$ such that $\Lambda(\phi) = 1$ and $K(v^{\Lambda(v)}) = 0$. Therefore, by definition of μ as in (6.11) we obtain:

$$\mu \leqslant S(v^{\Lambda(v)}) \leqslant S(v) + (\Lambda(v) - 1)P(v).$$

This completes the proof. $\square$

Let $u_0 \in \mathcal{N}_\varepsilon$ and $u(t)$ be the associated solution of (6.1). We define the exit time from the tube $\mathcal{N}_\varepsilon$ by

$$T_\varepsilon^\pm(u_0) := \inf\{t > 0 : u(\pm t) \notin \mathcal{N}_\varepsilon\}.$$

We set $I_\varepsilon(u_0) := (-T_\varepsilon^-(u_0), T_\varepsilon^+(u_0))$.

Lemma 6.20. *Assume* (6.30) *holds and let ε_1 be given by Lemma 6.19. Then for any $u_0 \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1}$, where $\mathcal{B}$ is defined as in* (6.13)*, there exists $m = m(u_0) > 0$ such that $P(u(t)) \leqslant -m$ for all $t \in I_{\varepsilon_1}(u_0)$.*

Proof. For $t \in I_{\varepsilon_1}(u_0)$, since $u(t) \in \mathcal{N}_{\varepsilon_1}$, it follows from Lemma 6.19 that

$$\mu - S(u_0) = \mu - S(u(t)) \leqslant -(1 - \Lambda(u(t)))P(u(t)).$$

In particular, since $\mu > S(u_0)$ by $u_0 \in \mathcal{B}$, we have $P(u(t)) \neq 0$. By continuity of the flow and $P(u_0) < 0$ we obtain

$$P(u(t)) < 0, \quad 1 - \Lambda(u(t)) > 0.$$

Therefore, we obtain

$$-P(u(t)) \geqslant \frac{\mu - S(u_0)}{1 - \Lambda(u(t))} \geqslant \frac{\mu - S(u_0)}{\delta_1} =: m(u_0) > 0.$$

This completes the proof. $\square$

Lemma 6.21. *Assume* (6.30) *holds. Then $|I_{\varepsilon_1}| < \infty$ for all $u_0 \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1} \cap \Sigma$, where*

$$\Sigma = \left\{v \in H^1(\mathbb{R}) : xv \in L^2(\mathbb{R})\right\}. \tag{6.32}$$

Proof. Let $u(t)$ be associated solution of $u_0 \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1} \cap \Sigma$. By the virial identity and Lemma 6.20 we have

$$\frac{d^2}{dt^2}\|xu(t)\|_{L^2}^2 = 8P(u(t)) \leqslant -8m(u_0)$$

for all $t \in I_{\varepsilon_1}(u_0)$, which implies $|I_{\varepsilon_1}(u_0)| < \infty$. This completes the proof. $\square$

Let χ be a smooth cut-off function such that

$$\chi(r) := \begin{cases} 1 & \text{if } 0 \leqslant r \leqslant 1, \\ 0 & \text{if } r \geqslant 2. \end{cases}$$

and for $R > 0$ define $\chi_R(x) = \chi\left(\frac{|x|}{R}\right)$.

The following is similar as in [36, Lemma 4.5].

Lemma 6.22. *There exists a function $R : (1, \infty) \to (0, \infty)$ such that $\chi_{R(\lambda)}\phi^\lambda \in \mathcal{B} \cap \Sigma \cap \mathcal{N}_{\varepsilon_1}$ for all $\lambda > 1$ close to 1, and that $\chi_{R(\lambda)}\phi^\lambda \to \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$.*

Proof. We divide the proof in three steps.
Step 1: Prove $\phi^\lambda \to \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$.
We have

$$\begin{aligned}
&\|\phi^\lambda - \phi\|_{\dot{H}^1} + \|\phi^\lambda - \phi\|_{L^2} \\
&\leqslant \|\lambda^{\frac{n}{2}}\phi(\lambda\cdot) - \phi(\lambda\cdot)\|_{\dot{H}^1} + \|\phi(\lambda\cdot) - \phi(\cdot)\|_{\dot{H}^1} + \|\lambda^{\frac{n}{2}}\phi(\lambda\cdot) - \phi(\lambda\cdot)\|_{L^2} + \|\phi(\lambda\cdot) - \phi(\cdot)\|_{L^2} \\
&= (\lambda^{\frac{n}{2}} - 1)(\lambda^{1-\frac{n}{2}}\|\phi\|_{\dot{H}^1} + \lambda^{\frac{-n}{2}}\|\phi\|_{L^2}) \qquad (6.33) \\
&\quad + \|\phi(\lambda\cdot) - \phi(\cdot)\|_{\dot{H}^1} + \|\phi(\lambda\cdot) - \phi(\cdot)\|_{L^2}. \qquad (6.34)
\end{aligned}$$

The term (6.33) converges to zero as $\lambda \to 1$. To prove the term (6.34) converges to zero as $\lambda \to 1$, we prove for all $\phi \in L^p$, $1 < p < \infty$, then the following holds

$$\|\phi(\lambda x) - \phi(x)\|_{L^p} \to 0, \text{ as } \lambda \to 1.$$

Indeed, we only need to consider ϕ is a integrable step function, by density of step function in $L^p(\mathbb{R}^n)$. It is sufficient to consider $\phi = \mathbb{1}_{\mathcal{A}}$, for some measurable set $\mathcal{A}$. We have $\phi(\lambda x) = \mathbb{1}_{\frac{1}{\lambda}\mathcal{A}}$ and

$$\begin{aligned}
\|\phi(\lambda x) - \phi(x)\|_{L^p}^p &= \|\mathbb{1}_{\frac{1}{\lambda}\mathcal{A}} - \mathbb{1}_{\mathcal{A}}\|_{L^p}^p \\
&= \mu(\{\lambda x \in \mathcal{A}, x \notin \mathcal{A}\} \cup \{x \in \mathcal{A}, \lambda x \notin \mathcal{A}\}) \\
&\leqslant \mu(\mathcal{A}) + \mu\left(\frac{1}{\lambda}\mathcal{A}\right) - 2\mu\left(\mathcal{A} \cap \frac{1}{\lambda}\mathcal{A}\right),
\end{aligned}$$

this converges to zero when λ converges to 1. Thus, if we consider $\nabla\phi$ as a vector function then the term (6.34) converges to zero as λ converges to 1.
Step 2: $\chi_{R(\lambda)}\phi^\lambda \to \phi$ as $\lambda \to 1$ for some function R.
Choosing $R : (1, \infty) \to (0, \infty)$ such that $R(\lambda) \to \infty$ as $\lambda \to 1$. Thus, for all $v \in H^1(\mathbb{R}^n)$, we have

$$\chi_{R(\lambda)}v \to v, \text{ as } \lambda \to 1$$

and $\chi_{R(\lambda)}\phi^\lambda \to \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$, since step 1.
Step 3: Conclusion.
We claim that $\phi^\lambda \in \mathcal{B}$ for $\lambda > 1$ close to 1. Since $\partial_\lambda S(\phi^\lambda)|_{\lambda=1} = 0$ and $\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0$, there exists $\lambda_1 > 1$ such that $\partial_\lambda S(\phi^\lambda) < 0$ and $S(\phi^\lambda) < \mu$ for $\lambda \in (1, \lambda_1)$. We see that $P(\phi^\lambda) = \lambda\partial_\lambda S(\phi^\lambda) < 0$ for $\lambda \in (1, \lambda_1)$. Moreover, taking λ_1 close to 1, we get $\phi^\lambda \in \mathcal{N}_{\varepsilon_1}$ for all $\lambda \in (1, \lambda_1)$. Since $\chi_{R(\lambda)}$ has compact support and $\|\chi_{R(\lambda)}\phi^\lambda - \phi^\lambda\|_{H^1} \to 0$ as $\lambda \to 1$, we have $\chi_{R(\lambda)}\phi^\lambda \in \mathcal{B} \cap \mathcal{N}_{\varepsilon_1} \cap \Sigma$ for λ close to 1. This completes the proof. $\square$

Proof of Proposition 6.18. By Lemma 6.22, there exists $R : (1,\infty) \to (0,\infty)$ such that $\chi_{R(\lambda)}\phi^\lambda \to \phi$ in $H^1(\mathbb{R}^n)$ as $\lambda \downarrow 1$. Moreover, $\chi_{R(\lambda)}\phi^\lambda \in \mathcal{B} \cap \Sigma \cap \mathcal{N}_{\varepsilon_1}$ for $\lambda > 1$ close to 1. Thus, by Lemma 6.21, $|I_{\varepsilon_1}(\chi_{R(\lambda)}\phi^\lambda)| < \infty$ for $\lambda > 1$ close to 1 and since $\chi_{R(\lambda)}\phi^\lambda \to \phi$ as $\lambda \to 1$ in $H^1(\mathbb{R}^n)$ we have ϕ is unstable. This completes the proof. $\square$

Proof of Theorem 6.6. Using Proposition 6.18, we only need to check the condition (6.30). We have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\nabla\phi\|_{L^2}^2 + \frac{n(n-2)}{12}\|\phi\|_{L^3}^3 - \frac{n(n-1)a_2}{4}\|\phi\|_{L^4}^4 - \frac{3n(3n-2)}{20}\|\phi\|_{L^5}^5.$$

We divide into three cases.
Case $n = 1$:
In this case, we have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\phi'\|_{L^2}^2 - \frac{1}{12}\|\phi\|_{L^3}^3 - \frac{3}{20}\|\phi\|_{L^5}^5.$$

In the case DDF, using $K(\phi) = 0$ and $P(\phi) = 0$ we have

$$0 = P(\phi) - \frac{1}{4}K(\phi) = \frac{3}{4}\|\phi'\|_{L^2}^2 - \frac{1}{12}\|\phi\|_{L^3}^3 - \frac{1}{20}\|\phi\|_{L^5}^5.$$

Thus,

$$\|\phi'\|_{L^2}^2 = \frac{1}{9}\|\phi\|_{L^3}^3 + \frac{1}{15}\|\phi\|_{L^5}^5.$$

It follows that

$$\begin{aligned}
\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} &= \frac{1}{36}\|\phi\|_{L^3}^3 - \frac{1}{12}\|\phi\|_{L^5}^5 \\
&= \frac{1}{36}\|\phi\|_{L^3}^3 - \frac{1}{12}\frac{10}{3}\left(\|\phi'\|_{L^2}^2 + \frac{1}{6}\|\phi\|_{L^3}^3 - \frac{a_2}{4}\|\phi\|_{L^4}^4 - P(\phi)\right) \\
&= -\frac{5}{18}\|\phi'\|_{L^2}^2 - \frac{1}{54}\|\phi\|_{L^3}^3 + \frac{5a_2}{72}\|\phi\|_{L^4}^4. \qquad (6.35)
\end{aligned}$$

Thus,

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} < 0.$$

This implies the instability of algebraic standing waves in the case DDF.
In the case DFF, using (6.35) and the fact that $a\|\phi\|_{L^3}^3 + b\|\phi\|_{L^5}^5 \geqslant 2\sqrt{ab}\|\phi\|_{L^4}^4$ for all $a, b > 0$ we have

$$\begin{aligned}
\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} &= -\frac{5}{18}\left(\frac{1}{9}\|\phi\|_{L^3}^3 + \frac{1}{15}\|\phi\|_{L^5}^5\right) - \frac{1}{54}\|\phi\|_{L^3}^3 + \frac{5a_2}{72}\|\phi\|_{L^4}^4 \\
&= -\frac{4}{81}\|\phi\|_{L^3}^3 - \frac{1}{54}\|\phi\|_{L^5}^5 + \frac{5a_2}{72}\|\phi\|_{L^4}^4 \\
&\leqslant -\frac{4}{27\sqrt{6}}\|\phi\|_{L^4}^4 + \frac{5a_2}{72}\|\phi\|_{L^4}^4 < 0,
\end{aligned}$$

since we have assumed $a_2 < \frac{32}{15\sqrt{6}}$. Thus, in the case DFF and $a_2 < \frac{32}{15\sqrt{6}}$ we obtain the instability of algebraic standing waves.

Case $n=2$:
In this case, we have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\nabla\phi\|_{L^2}^2 - \frac{a_2}{2}\|\phi\|_{L^4}^4 - \frac{6}{5}\|\phi\|_{L^5}^5. \tag{6.36}$$

Moreover,

$$0 = P(\phi) = \|\nabla\phi\|_{L^2}^2 + \frac{1}{3}\|\phi\|_{L^3}^3 - \frac{a_2}{2}\|\phi\|_{L^4}^4 - \frac{3}{5}\|\phi\|_{L^5}^5.$$

Replacing $\frac{a_2}{2}\|\phi\|_{L^4}^4 = \|\nabla\phi\|_{L^2}^2 + \frac{1}{3}\|\phi\|_{L^3}^3 - \frac{3}{5}\|\phi\|_{L^5}^5$ in (6.36), we obtain

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = -\frac{1}{3}\|\phi\|_{L^3}^3 - \frac{3}{5}\|\phi\|_{L^5}^5 < 0.$$

The instability of algebraic standing waves in the case $n=2$ follows.
Case $n=3$:
In this case, we have

$$\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} = \|\nabla\phi\|_{L^2}^2 + \frac{1}{4}\|\phi\|_{L^3}^3 - \frac{3a_2}{2}\|\phi\|_{L^4}^4 - \frac{63}{20}\|\phi\|_{L^5}^5. \tag{6.37}$$

Moreover,

$$0 = P(\phi) = \|\nabla\phi\|_{L^2}^2 + \frac{1}{2}\|\phi\|_{L^3}^3 - \frac{3a_2}{4}\|\phi\|_{L^4}^4 - \frac{9}{10}\|\phi\|_{L^5}^5.$$

Hence,

$$\begin{aligned}\partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} &= \partial_\lambda^2 S(\phi^\lambda)|_{\lambda=1} - 2P(\phi) \\ &= -\|\nabla\phi\|_{L^2}^2 - \frac{3}{4}\|\phi\|_{L^3}^3 - \frac{27}{20}\|\phi\|_{L^5}^5 < 0.\end{aligned}$$

The instability of algebraic standing waves in case $n=3$ follows. This completes the proof of Theorem 6.6. $\square$

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