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Manuscrit

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par

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de l'Institut de Mathématiques de Toulouse

A hamiltonian approach to special Kähler and Sasaki metrics.

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Chapter 1

Presentation

1.1 List of Publications

[ACL] V. Apostolov, D. Calderbank, E. Legendre. *Weighted K-stability of polarized varieties and extremality of Sasaki manifolds*, to appear in Advances in Mathematics.

[L5] E. Legendre, *Localizing the Donaldson–Futaki invariant*, Intern. Journal of Mathematics, Vol. 32, No. 08.

[BHLTF3] C. Boyer, H.Huang, E.Legendre, C. W. Tønnesen-Friedman, *Some open problems in Sasaki geometry*, Australian-German Workshop on Differential Geometry in the Large, Cambridge University Press. 2020

[ACGL1] V. Apostolov, D. Calderbank, P. Gauduchon, E. Legendre. *Contact toric manifolds of arbitrairy codimension*, Int.Math.Res.Not. 2020 (10), 2020, 2436–2467.

[ACGL2] V. Apostolov, D. Calderbank, P. Gauduchon, E. Legendre. *Levi-Kähler reduction of CR structures, products of spheres, and toric geometry*, Math. Res. Lett., 27 (2020), 1565–1629.

[L4] E. Legendre, *A note on extremal toric almost Kähler metrics*, Moduli of K-stable varieties, volume 31 of Springer INdAM Serie, 53-74 Springer, (2019).

[LR] E. Legendre, Y. Rollin *Hamiltonian stationary Lagrangian fibrations*, J. of Symplectic Geometry, 3, Volume 17 (2019) 753–791.

[BHL] C. Boyer, H.Huang, E.Legendre, *An application of the Duistermaat–Heckman Theorem in Sasaki geometry*, Geometry & Topology, Volume 22, Number 7 (2018), 4205–4234.

[LSD] E. Legendre, R. Sena-Dias *Toric aspects of the first eigenvalue*, The Journal of Geometric Analysis, 2018, 28 (3), 2395–2421.

[BHLTF2] C. Boyer, H.Huang, E.Legendre, C. W. Tønnesen-Friedman, *Reducibility in Sasakian geometry*, Trans. AMS., 2018, 370, 6825–6869.

[BHLTF1] C. Boyer, H.Huang, E.Legendre, C. W. Tønnesen-Friedman *The Einstein–Hilbert functional and the Sasaki-Futaki invariant*, Int.Math.Res.Not., 2017 (7), 1942–1974.

[L3] E. Legendre, *Toric Kähler–Einstein metrics and convex compact polytopes*, Journal of Geometric Analysis, 2016, 26 (1), 399–427.

[GLSD] V. Guillemin, E. Legendre, R. Sena-Dias, *Simple spectrum and Rayleigh quotients*, Geometric and Spectral Analysis. ed. P. Albin, D. Jakobson, and F. Rochon. Contemp. Math. AMS. Rhode Island. (2014).

[LTF] E. Legendre, C. Tønnesen-Friedman, *Toric Generalized Kähler–Ricci Solitons with Hamiltonian 2–form*, Math. Z. (2013) **274** 1177–1209.

[L2] E. Legendre, *Existence and non uniqueness of constant scalar curvature toric Sasaki metrics*, Compositio Mathematica, **147** (2011), 1613–1634.

[L1] E. Legendre, *Toric geometry of quadrilaterals*, Journal of Symplectic Geometry **9** (2011), 343–385.

[L0] E. Legendre, *Extrema of low eigenvalues for the mixed problem on the disk*, Canadian Journal of Mathematics **62** (2010), 808–826.

The content of the articles [L0], [L1], [L2] and [LTF] will not be presented in this dissertation as they only contain materials from my PhD or even before. Moreover [GLSD] is a review and does not contain any new results which are worth presenting here.

The aricles [LR] and [BHLTF2] gather new results but the topics are not central to my current research interests. Therefore, I will not discuss these results in this memoir.

1.1.0.1 List of pre-publications

[DL] R. Dervan, E. Legendre, *Valuative stability of polarised varieties* arXiv:math.DG/2010.04023. Submitted.

[dBL] M. de Borbon, E. Legendre, *Toric Sasaki-Einstein metrics with conical singularities*, arXiv:math.DG /2005.03502. Submitted.

It is a rule of the University of Toulouse's doctoral school, that unpublished results should not appear in the core of a dissertation such as the present one. The main results of [DL] and [dBL] are discussed in Chapter 5 dedicated to current projects and perspectives.

1.2 Scientific context and summary

A central problem in Kähler geometry, proposed by Calabi [57] in the 80's, is whether or not a canonical Kähler metric can be found in a given cohomology class of a compact Kähler manifold. Calabi suggested looking for *extremal Kähler metrics* and characterized in [58] these special structures as follows. A Kähler manifold (M, J, ω, g) is *extremal* if and only if its scalar curvature Scal_g is a *Killing potential*, meaning that its gradient $\nabla^g \mathrm{Scal}_g$ is real holomorphic, i.e. $\mathcal{L}_{\nabla^g \mathrm{Scal}_g} J = 0$ or equivalently its hamiltonian vector field $X_{\mathrm{Scal}_g} = -J\nabla^g \mathrm{Scal}_g$ is *Killing*, i.e., lies in the Lie algebra of the isometry group of (M, g). Constant scalar curvature (cscK for short) and the more famous Kähler-Einstein metrics are particular examples of such metrics.

At that time, Calabi already highlighted some obstructions to the existence of an extremal Kähler metric in relation with the symmetries of the underlying complex manifold. More precisely, the Calabi structure theorem [58] implies that, in that case, X_{Scal_g} must lie in a maximal torus in the center of the isometry group $G = \mathrm{Isom}(M, g)$ which itself must be maximal among all compact connected subgroups of $\mathrm{Aut}(M, J)$. This jusifies working on a maximal compact torus $\mathbb{T} \subset \mathrm{Aut}(M, J)$ chosen in advance.

The existence of an extremal Kähler metric in a given Kähler class, is now conjecturally equivalent to a certain notion of K-stability through an extension of the Yau-Tian-Donaldsons's (YTD) conjecture [207, 197, 82], introduced in [189, 193] in the polarised case and in [78] for general Kähler class. This conjecture, its ramification and its extension have generated tremendous efforts from the community and has led to many interesting developments during the last decades.

The Sasaki version of the Calabi problem has interest on its own and also plays a role in the Calabi extremal Kähler metrics problem and YTD conjec-

ture for polarized Kähler manifolds. Indeed, a prototypical example of Sasaki manifolds is a circle bundle over a polarised Kähler manifold, with a connection associated to the metric. A striking illustration of the interest of such structures in the Kähler setting can be found in the work of Donaldson and Sun [90, 91] who have used the fundamental result of Martelli–Sparks–Yau [168] about the classical transversal Futaki invariant of Sasaki–Einstein typed manifolds to study the topology of moduli spaces of K-stable Fano varieties. Note that this application relies on the identification of the later space with Fano Kähler–Einstein varieties, via the YTD conjecture for Fano varieties, for which there are now few proofs, notably [62, 33, 209].

In this dissertation, I account for a large part of the research I did after my Ph.D. around the notion of Calabi extremal metrics in Kähler and Sasaki geometry. I organized the core of the manuscript into three chapters, each including an introduction and presenting the technics and the main ideas behind my contributions as well as relation to the work of others and some perspectives. Here below is a very short summary of these chapters.

In **Chapter** 2, are explained my contributions to the study of the Futaki invariant (relative to a fixed compact torus $\mathbb{T}$) and mainly its Sasaki version obstructing the existence of a compatible Sasaki metric with constant scalar curvature (cscS). We have given in [BHLTF1] a variational version of this invariant by introducing a homogeneous Einstein–Hilbert functional EH, defined on the Reeb cone $\mathfrak{t}_+ \subset \mathrm{Lie}\mathbb{T}$, which is a finite dimensional strictly convex open cone. This extends Martelli–Sparks–Yau main Theorem [168] from Calabi-Yau cone to general Kähler cones. I also present our argument from [BHL], proving that the Einstein–Hilbert functional is proper and rational on $\mathfrak{t}_+$, using the Atiyah–Bott–Berligne–Vergne equivariant localisation formula. This technic is also exploited in the final section of this Chapter to study the Donaldson–Futaki invariant of a smooth compact Kähler test configuration over a Kähler manifold as I have done in [L5]. It is proven there that this invariant coincides with the classical Futaki invariant of the central fiber when this one has at most orbifold singularities, extending a result known in the polarised case [82] to the transcendental case. Chapter 2 contains also a brief recap on Sasaki geometry.

The first part of **Chapter** 3 is devoted to two processes, that we have called Levi and Levi-Kähler reduction and introduced respectively in [ACGL1] and in [ACGL2]. The Levi reduction is the contact/symplectic counter part while the Levi-Kähler reduction incorporates a CR/complex structure. This is similar to the relation between symplectic versus Kähler reduction and indeed, Levi-Kähler reduction sometimes coincide with a Kähler reduction but, in general, is different. An advantage of the Levi-Kähler reduction on the general Kähler reduction is that the Sasaki deformation technics can be more naturally integrated in the process and the curvature of the quotient seems more sensitive to the Chern curvature of the CR structure. We have studied only the toric compact case of these reductions and obtained a partial classification in the contact/symplectic version which extends the Delzant/Lerman classification of compact toric contact/symplectic manifolds. Computing the curvature of the Levi–Kähler reduction of product of odd dimensional spheres, we have observed in [ACGL2] that a type of *weighted extremal metrics* (sometimes called

weighted cscK metrics) appear naturally in this context. This takes us to the second part of **Chapter** 3 where is presented a correspondence between extremal Sasaki metrics with Reeb vector field $\xi \in \mathfrak{t}_+$ and a specific type of weighted extremal metrics on the Kähler quotient by a circle action commuting with the compact torus induced by ξ. This correspondence allowed us to use directly Lahdili Theorem [142] on general weighted extremal Kähler metrics to prove some properness of the Sasaki version of the Mabuchi functional. We also have proposed a *global Futaki invariant* which coincides with Collins–Székelyhidi [68] Donaldson–Futaki invariant of the (reduced) central fiber of smooth Sasaki test configurations. Combining these results, we get that the Donaldson–Futaki invariant of any smooth Sasaki test configuration with reduced central fiber over a cscS manifold is strictly positive, unless it is a product type. All these results are established in [ACL] and reported in **Chapter** 3.

Chapter 4 gathers my results in toric Kähler geometry. The symplectic approach in this field [1, 82, 115] has lead to establish an explicit dictionary between toric extremal Kähler metrics on a fixed toric symplectic manifold $(M, \omega, \mathbb{T})$ and convex solutions of a (scalar) PDE problem on a convex compact polytopes $\overline{P} \subset (\mathrm{Lie}\mathbb{T})^* =: \mathfrak{t}^*$. The convex solution must satisfy some boundary condition encoded by a *labelling* ℓ or equivalently a measure on the boundary of $\overline{P}$. We end up with an "abstract" boundary value PDE problem that I call the *Calabi problem on polytopes*.

I give the details of the correspondence above in §4.1, with a precise definition of the Calabi problem on polytopes along with my contribution from [L3], where building on [86], I proved that for each simple compact polytope there exists a Kähler–Einstein type solution of the Calabi problem, unique up to a dilatation. Actually the labelling is unique, up to a dilatation, and the result follows then an abstract toric version of the Wang–Zhu Theorem [202] on existence of toric Kähler–Ricci solitons and Kähler–Einstein metrics. Along come an abstract toric version of Zhu's Theorem [211] and Yau's solution of the Calabi conjecture [206]. Geometric applications includes a classification of compact toric Kähler–Einstein orbifolds up to a torus invariant symplectomorphism in terms of integral polytopes as well as the finding of families of singular Kähler with conical singularities. I include also the result I have obtained in [L4] that the convex affine geometry point of view, combined with the resolution by Chen–Cheng [61] of an analytical version of the Yau–Tian–Donaldson conjecture implies that the existence of an extremal toric *almost*–Kähler metrics on a compact symplectic toric manifold is sufficient to ensure the existence of an extremal toric Kähler metric.

Finally, the last section of **Chapter** 4 is of a slightly different flavour as we leave the extremal problem of Calabi to consider bounds on the first non-zero eigenvalue λ_1 of compact toric Kähler manifolds. For this again, we have used in [LSD] the symplectic approach and the characterization of toric Kähler–Einstein metrics by their first invariant eigenspace to provide a sharp bound on λ_1, characterizing the Fubini–Study metric.

As a guiding principle, in this dissertation, the demonstration are not presented in details. Only ideas, inspirations and technics are discussed, unless I want to emphasize on some technical detail for which I believe a potential extension could hold and be of some interest. In **Chapter** 5 are gathered my

main current research projects, including unpublished (but submitted) works like [DL] and [dBL].

Chapter 2 contains a brief recap on Sasaki geometry and there is an Appendix A on Calabi's structure Theorem and the extremal vector field.

1.3 Summary of notation

Throughout N and M denote connected manifolds, N is in general a compact manifold which may be equipped with a contact structure and M is a $2n$-dimensional manifold. The pair (M, J) refers to a manifold M and $J \in \mathrm{End}(TM)$ an integrable almost complex structure corresponding to a complex structure on M. Sometimes, when no confusion is possible, M denotes a complex manifold. The operator J extends naturally to tensors and the twisted differential on forms is $d^c = J^{-1} \circ d \circ J$. In particular, for $f \in C^\infty(M)$, $d^c f = -df \circ J$.

The pair (M, ω) denotes a symplectic manifold and the convention used in this memoir is that a vector field X on M is hamiltonian if there exists a function f such that

$$-df = \omega(X, \cdot).$$

Given $f \in C^\infty(M)$, X_f is the unique hamiltonian vector field satisfying the latter equation.

The action of a Lie group G on a manifold M (or N) is a Lie group morphism $\psi : G \to \mathrm{diffeo}(M)$ (or $G \to \mathrm{diffeo}(N)$). We denote the Lie algebra $\mathfrak{g} := T_e G$ where $e \in G$ in the neutral element and for $a \in \mathfrak{g}$, the associated vector field on M (or N) is

$$\underline{a} := d_e \psi(a).$$

Essentially only torus action are considered in this text and we denote $\mathbb{T}$ a compact (connected) torus with Lie algebra $\mathfrak{t} = T_e \mathbb{T}$. Typically, we assume the action is effective, that is ψ is injective. Given a space, say $\mathcal{A}$, on which an action of $\mathbb{T}$ is specified, $\mathcal{A}^{\mathbb{T}}$ denotes the subspace of $\mathbb{T}$–invariant elements of $\mathcal{A}$.

Chapter 2

Various approaches to the Futaki invariant

Given a compact Kähler manifold (M, J, ω, g), via the *momentum map picture* drawn by Fujiki and Donaldson [100, 82], the (hermitian) scalar curvature is the momentum map of the action of $\mathrm{Ham}(M, \omega)$ on the space of ω–compatible (almost) complex structures. Restricting to $\mathbb{T}$-invariant ω–compatible complex structures $\mathcal{C}(\omega)^{\mathbb{T}}$, it means that

$$\begin{aligned} \mu \colon \mathcal{C}(\omega)^{\mathbb{T}} &\to (C^{\infty}_{\omega,0}(M)^{\mathbb{T}})^{*} \\ J &\mapsto \left(f \mapsto \int_M f \mathrm{Scal}_{J,\omega} \omega^n \right) \end{aligned} \tag{2.1}$$

is a momentum map with respect to a formal symplectic structure on $\mathcal{C}(\omega)^{\mathbb{T}}$, when identifying the formal Lie algebra of $\mathrm{Ham}(M, \omega)$ with the space $C^{\infty}_{\omega,0}(M)^{\mathbb{T}}$ of functions integrating to 0 against ω^n. Now $\mathbb{T}$ acts trivially on $\mathcal{C}(\omega)^{\mathbb{T}}$ thus $\int_M f \mathrm{Scal}_{J,\omega} \omega^n$ does not depend on $J \in \mathcal{C}(\omega)^{\mathbb{T}}$. This is a symplectic version of the Futaki invariant [82, 154] which coincides with the restriction of the classical Futaki character [109] to the (complex) Lie algebra of real holomorphic vector fields $\mathfrak{t} \oplus J\mathfrak{t}$, where $\mathfrak{t} = \mathrm{Lie}\ \mathbb{T}$.

In sum, the symplectic relative (to a torus $\mathbb{T}$) version of the Futaki invariant is the L^2-projection of Scal_g on the space of Killing potentials identified with the space of affine linear functions on the moment polytope integrating to 0 against the Duistermaat-Heckman measure. If there exists a ω-compatible cscK metric, this invariant must be zero.

The main results I want to discuss in this Chapter are related to the Sasaki/contact version of the Futaki invariant. This version of the invariant is an obstruction to the existence of a *transversal Kähler structure* of constant scalar curvature. A transversal Kähler structure consists in a Kähler structure on local leaf space of a foliation, in Sasaki geometry, the foliation is induced by a nowhere vanishing vector field $\xi \in \mathfrak{t}$ called the *Reeb vector field*. The transversal Futaki invariant thus depends on that vector field ξ sitting in a convex polyhedral cone $\mathfrak{t}_+ \subset \mathfrak{t}$ called the *Reeb cone*.

Understanding how the transversal Futaki invariant $\mathbf{F}_\xi$ varies when ξ runs within the Reeb cone could eventually provide tools to study moduli space of polarised cscK manifolds [81] similarly to the case of K-stable Fano varieties [90, 91].

With my collaborators C. Boyer, H. Huang and C. Tønnesen-Friedman we have proved in [BHLTF1] that the zeros of the Sasaki-Futaki invariant correspond essentially to the critical points of the *Einstein–Hilbert functional* see §2.1.3.1. Thus motivated, we have studied the local behaviour of that functional around its critical points. I give an overview of our conclusions in §2.1.3. Note that these results extend the famous Theorem of Martelli–Sparks–Yau [168] on the volume functional defined on a subset of the Reeb cone of a Calabi-Yau cone, see Theorem 2.24.

Using equivariant localisation formulas and Morse–Bott theory we managed to prove in a subsequent paper [BHL], that the Einstein–Hilbert functional is rational (in the variable of $\xi \in \mathfrak{t}_+$) and proper. Thus, the rays of Reeb vector fields with vanishing transversal Futaki invariant are in a compact algebraic subspace of $\mathfrak{t}_+/\mathbb{R}_+$. I discuss this result in §2.1.4 together with a corollary of our computations implying that given a $\mathbb{T}$–invariant scalar flat conical Kähler manifold (with compact smooth cross section) the total transversal scalar curvature, seen as a function $\mathbf{S} : \mathfrak{t}_+ \to \mathbb{R}$, is non-negative.

The Futaki invariant is involved in the finer obstruction to the existence of cscK metric provided by K-stability. In essence, the Donaldson-Futaki invariant of a test configuration is the Futaki invariant of the central fiber. This has a precise meaning when the central fiber is normal and can be verified easily for polarised test configurations, using equivariant cohomology [82]. In §2.2, is briefly presented another way, using equivariant localisation, to relate the intersection product formulation of the Donaldson-Futaki invariant of a compact Kähler test configuration to the Futaki invariant of the central fiber when this one is smooth or has orbifold singularities. These hypothesis ensure that we find the classical Futaki invariant of the central fiber. The interest of doing such a tedious computation is questionable but I include the result in this memoir because it has lead me to consider Kähler tests configurations from the hamiltonian perspective and some of my current projects are natural follow up.

Moreover, on a personal level, studying these test configurations and equivariant cohomology motivated me to learn more on the algebraic side of the YTD conjecture which eventually lead me to study Fujita and Chi Li works [104, 160, 161]. They provide alternative approaches to K-stability in terms of "dreamy" divisors and/or valuations for Fano varieties, which turned to be very useful practical tools to detect K-stability or study the moduli space of K-stable Fano varieties. With R. Dervan, we have extended Fujita's "valuative stability" by introducing a numerical invariant generalizing his β–invariant. We have proved that K-stability with respect to test configurations with irreducible, reduced central fiber is equivalent to this valuative stability. Our results are explained in details in the prepublication [DL] and briefly presented in §5.1,[1]

[1] It is a rule of the doctoral school in University of Toulouse, that unpublished results should not appear in the core of a dissertation such the present one.

Chapter 5.

Sasaki geometry is not as classical as Kähler geometry and the notation might be confusing at first. I have included a background section 2.1.1 on Sasaki geometry where notations and basic facts are explained. The complex cone point of view is treated separately in Section 2.1.2 and contains some original material from my joint work with Apostolov and Calderbank [ACL], where we have clarified some results contained in the literature and brought a new point of view on radial potentials (Kähler potentials on the cone).

2.1 The Einstein–Hilbert functional and cscS

2.1.1 Preliminaries on Sasaki geometry

A Kähler structure is a triple (ω, g, J) where $J \in \mathrm{End}(TM)$ is an integrable almost complex structure, ω a symplectic form et g a riemannian metric such that $\omega = g(J\cdot, \cdot)$ and we need to add a compatibility relation, eg. $g = g(J\cdot, J\cdot)$. Thus any pair of these structures (if compatible) determines the third and therefore the Kähler structure.

Likewise a Sasaki structure is determined by two compatible objects, in this dissertation we will essentially only consider combinations among

(i) a contact distribution $\mathcal{D}$ + a transversal holomorphic structure (ξ, J^ξ);

(ii) a CR structure $(\mathcal{D}, J)$ + a Reeb vector field ξ;

(iii) a CR structure $(\mathcal{D}, J)$ + a contact form η.

However, relations between these objects are not as straightforward as the ones between the structures involved in Kähler geometry. For example, the contact form determines the contact distribution and thus the Reeb vector field but not the CR structure nor the transversal holomorphic one, while a transversal holomorphic structure makes no sense without a fixed Reeb vector field. The notation above in $(i), (ii), (iii)$ are the typical ones used in this text and this section contains a brief recap on basic properties and relations between them.

2.1.1.1 Sasaki structures

Let $(N, \mathcal{D})$ be a compact connected *contact* $(2m+1)$-manifold[2] which means that $\mathcal{D}$ is a codimension one distribution whose Levi form defined by

$$L_{\mathcal{D}}(X, Y) = -q_{\mathcal{D}}([X, Y]) \tag{2.2}$$

where $q_{\mathcal{D}} \colon TN \to TN/\mathcal{D}$ is the quotient map, is surjective. A *contact vector field* is a vector field X on N such that $\mathcal{L}_X(\Gamma^\infty(\mathcal{D})) \subset \Gamma^\infty(\mathcal{D})$. We denote by $\mathfrak{con}(N, \mathcal{D})$ the infinite dimensional Lie algebra of contact vector fields of $(N, \mathcal{D})$.

Classically, the dual of the (real) line bundle $TN/\mathcal{D} \to N$ is identified with the annihilator $\mathcal{D}^0 \subset T^*N$ of the contact distribution $\mathcal{D}$ and a nowhere vanishing section[3] $\eta \in \Gamma(\mathcal{D}^0)$ is a *contact* 1*–form*. Classical ODE theory tells us that to

[2] of codimension 1, in this chapter there is mention of higher co-dimension contact manifold.

[3] Such section exists if and only if, by definition, $\mathcal{D}$ is co-oriented, which will be assumed (and needed) for our purpose.

contact 1–form η is associated a unique contact vector field $\xi = \xi_\eta \in \mathfrak{con}(N, \mathcal{D})$, so that $\eta(\xi) \equiv 1$. This contact vector field ξ is the so-called *Reeb vector field* of η and is uniquely determined by the conditions

$$\eta(\xi) = 1, \quad \mathcal{L}_\xi \eta = 0. \tag{2.3}$$

Conversely, any contact vector field $\xi \in \mathfrak{con}(N, \mathcal{D})$ such that $q_{\mathcal{D}}(\xi)$ is nowhere vanishing determines a unique contact 1–form $\eta \in \Gamma(\mathcal{D}^0)$ satisfying (2.3).

Definition/Remark 2.1. Writing $X = f\xi + Z$ where $Z \in \Gamma(\mathcal{D})$ and thus $f = \eta(X)$, we get that $X \in \mathfrak{con}(N, \mathcal{D})$ if and only if $d\eta(X, \cdot) = df(\xi)\eta - df$ (indeed, $\mathcal{L}_X \eta$ must be colinear to η). As $d\eta$ is non-degenerated on $\mathcal{D}$ (by definition of a contact distribution), the latter equation determines X uniquely from the pair (f, η) and we get a linear isomorphism between $\mathfrak{con}(N, \mathcal{D})$ and $\Gamma^\infty(TN/\mathcal{D})$. Given a fixed contact form η the *η-contact vector field of f* is the unique $X_f \in \mathfrak{con}(N, \mathcal{D})$, such that $f = \eta(X_f)$.

Now suppose $J \in \mathrm{End}(\mathcal{D})$ is a CR structure on $(N, \mathcal{D})$, i.e., a (fiberwise) complex structure on $\mathcal{D}$ such that $\mathcal{D}^{(1,0)} := \{X - iJX |\ X \in \mathcal{D}\}$ is closed under Lie bracket in $TN \otimes \mathbb{C}$; then we denote by

$$\mathfrak{cr}(N, \mathcal{D}, J) := \{X \in \mathfrak{con}(N, \mathcal{D}) |\ \mathcal{L}_X J = 0\}$$

the Lie subalgebra of $\mathfrak{con}(N, \mathcal{D})$, whose elements correspond to CR vector fields X on $(N, \mathcal{D}, J)$. If moreover $(\mathcal{D}, J)$ is strictly pseudoconvex that is $\mathcal{D}^0$, equivalently $TN/\mathcal{D}$, has a canonical orientation : the *positive sections* $\eta \in \Gamma(\mathcal{D}^0)$ are those for which $\eta \circ L_{\mathcal{D}}(\cdot, J\cdot)$ is positive definite. Note that $\eta \circ L_{\mathcal{D}}(\cdot, J\cdot) = (d\eta)|_{\mathcal{D}}(\cdot, J\cdot)$ since $\ker \eta = \mathcal{D}$ and that η is positive if and only $f\eta$ is positive for any smooth positive function f. We let $\mathfrak{con}_+(N, \mathcal{D})$ denote the space of Reeb vector fields of $(N, \mathcal{D})$ for which the associated contact 1–form η (2.3) is positive.

We then have the following fundamental definitions (see e.g. [48]).

Definition 2.2. Let $(N, \mathcal{D}, J)$ be a strictly pseudoconvex CR manifold. Then the *full Sasaki–Reeb cone* of $(N, \mathcal{D}, J)$ is $\mathfrak{cr}_+(N, \mathcal{D}, J) := \mathfrak{cr}(N, \mathcal{D}, J) \cap \mathfrak{con}_+(N, \mathcal{D})$. If $\mathfrak{cr}_+(N, \mathcal{D}, J)$ is nonempty then $(N, \mathcal{D}, J)$ is said to be of *Sasaki type*, an element $\xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ is called *Sasaki–Reeb vector field* or a *Sasaki structure* on $(N, \mathcal{D}, J)$, and $(N, \mathcal{D}, J, \xi)$ is called a *Sasaki manifold*. We say ξ is *quasiregular* if the flow of ξ generates an $\mathbb{S}^1$ action on N and *regular* if, moreover, this action is free. If ξ is not quasiregular, it is *irregular*.

Remark 2.3. A Sasaki manifold $(N, \mathcal{D}, J, \xi)$ is equivalently completely determined by (D, J) and the contact 1-form $\eta \in \Gamma(\mathcal{D}^0)$ defined by (2.3) so we may as well denote $(N, \mathcal{D}, J, \xi)$ by $(N, \mathcal{D}, J, \eta)$ if there is no possible confusion. Observe also that with a Sasaki manifold comes a Riemannian metric, the *Sasaki metric*

$$g := \eta^2 + \frac{1}{2} d\eta(\cdot, J\cdot) \tag{2.4}$$

on N, for which the Reeb vector field is unitary.

Example 2.4. Let (M, J, g, ω) be a Kähler manifold such that $[\omega/2\pi]$ is an integral de Rham class. Then there is a principal $\mathbb{S}^1$-bundle $\pi : N \to M$ with a connection 1-form η satisfying $d\eta = \pi^*\omega$. Thus $(N, \mathcal{D}, J, \xi)$ is a Sasaki manifold,

where $\mathcal{D} = \ker \eta \subset TN$, J is the pullback of the complex structure on TM to $\mathcal{D} \cong \pi^* TM$ and ξ is the generator of the $\mathbb{S}^1$ action (with $\eta(\xi) = 1$, so $\eta = \eta^{\xi}_{\mathcal{D}}$).

Conversely, if $\xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ is (quasi)regular Sasaki–Reeb vector field on $(N, \mathcal{D}, J)$, then N is a principal $\mathbb{S}^1$-bundle (or orbibundle) $\pi : N \to M$ over a Kähler manifold (or orbifold) M. Irrespective of regularity, this correspondence between Kähler geometry and Sasaki geometry holds locally: any point of a Sasaki manifold $(N, \mathcal{D}, J, \xi)$ has a neighbourhood in which the leaf space M of the flow of ξ is a manifold and has a Kähler structure (g, J, ω) induced, using the local identification of $\mathcal{D}$ and $\pi^* TM$, by the *transversal Kähler structure* (g_ξ, J, ω_ξ) on $\mathcal{D}$, where $\omega_\xi := dq^{\xi}_{\mathcal{D}}|_{\mathcal{D}}$ and $g_\xi := \omega_\xi(\cdot, J\cdot)$. Indeed g_ξ, J, and ω_ξ are all ξ-basic, so they all descend to M, and we refer to (M, g, J, ω) as a *Sasaki–Reeb quotient* of $(N, \mathcal{D}, J, \xi)$.

Definition/Remark 2.5. Given any pair of Reeb vector field $\xi_o, \xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$ the respective associated contact 1–forms $\eta_o, \eta \in \Gamma(\mathcal{D}^0)$ are related $\eta = f\eta_o$ where $f = \eta(\xi_o) > 0$. The deformation

$$\mathfrak{cr}_+(N, \mathcal{D}, J) \ni \xi \mapsto \left(\mathcal{D}, J, \frac{1}{\eta_o(\xi)}\eta_o\right)$$

is called a *Deformation of type I* in [47]. This includes *transversal homothety* $0 < \lambda \mapsto \left(\mathcal{D}, J, \lambda^{-1}\eta_o\right) = \left(\mathcal{D}, J, \lambda\xi_o\right)$.

2.1.1.2 Transversal holomorphic structures

Let N be a connected $(2m+1)$-manifold, let ξ be a nowhere zero vector field, and let

$$\beta_\xi : TN \to \mathcal{D}_\xi := TN/\mathrm{Span}_{\mathbb{R}}\xi$$

be the quotient of TN by the span of ξ. Thus $\mathcal{D}_\xi$ is everywhere locally isomorphic to the pullback of the tangent bundle of local quotients of N by ξ.

Definition 2.6. A *transversal holomorphic structure* on (N, ξ) is a complex structure J^ξ on $\mathcal{D}_\xi$ which is everywhere locally the pullback of a complex structure (i.e., an integrable almost complex structure) on a local quotient of N by ξ.

Any Sasaki structure $(\mathcal{D}, J, \xi)$ on N induces a transversal holomorphic structure on (N, ξ): since $\mathcal{D}$ is transverse to ξ, $\beta_\xi|_{\mathcal{D}}$ is a bundle isomorphism $\mathcal{D} \to \mathcal{D}_\xi$, and the complex structure on $\mathcal{D}_\xi$ induced by J is ξ-invariant and integrable because J is.

Definition 2.7. [48] A transversal holomorphic structure J^ξ on (N, ξ) has *Sasaki type* if N admits a Sasaki structure $(\mathcal{D}, J, \xi)$ which is *compatible* with (ξ, J^ξ), i.e.,

$$\beta_\xi|_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}_\xi \text{ intertwines } J \text{ and } J^\xi.$$

Such compatible Sasaki structures on (N, ξ, J^ξ) are completely determined by their contact forms or, equivalently, by the corresponding contact distributions: we let $\mathcal{S}(\xi, J^\xi) \subset \Omega^1(N)$ be the subspace of contact forms of Sasaki structures compatible with (ξ, J^ξ), and write $\eta^{\xi}_{\mathcal{D}}$ for the unique element of $\mathcal{S}(\xi, J^\xi)$ with $\ker \eta^{\xi}_{\mathcal{D}} = \mathcal{D}$.

As is well-known, the geometry of local quotients of N may be described using the basic de Rham complex

$$\Omega^\bullet_\xi(N) = \{\alpha \in \Omega^\bullet(N) : \imath_\xi \alpha = 0 = \mathcal{L}_\xi \alpha\},$$

with differential d_ξ given by restriction of d (which preserves basic forms). Note that $\Omega^0_\xi(N) = C^\infty_N(\mathbb{R})^\xi$ and we also denote the closed and exact forms in $\Omega^\bullet_\xi(N)$ by $\Omega^\bullet_{\xi,\mathrm{cl}}(N)$ and $\Omega^\bullet_{\xi,\mathrm{ex}}(N)$ respectively.

If (N, ξ, J^ξ) is compact of Sasaki type, then $\Omega^\bullet_\xi(N)$ has a Hodge decomposition with respect to the induced transversal metric of any $\eta \in \mathcal{S}(\xi, J^\xi)$, and the $d_\xi d^c_\xi$-lemma and transversal Kähler identities are satisfied [98], with the following consequences, cf. [47, 48].

Lemma 2.8. *A 1-form γ on N is basic with J^ξ-invariant exterior derivative if and only if $\gamma = d^c_\xi \varphi + \alpha$ for a basic function $\varphi \in \Omega^0_\xi(N)$ and a closed basic 1-form $\alpha \in \Omega^1_{\xi,\mathrm{cl}}(N)$; further, α is uniquely determined by γ, as is φ up to an additive constant.*

In particular $\mathcal{S}(\xi, J^\xi)$ is an open subset of an affine space with translation group $\Omega^0_\xi(N)/\mathbb{R} \times \Omega^1_{\xi,\mathrm{cl}}(N)$, where $\Omega^0_\xi(N)/\mathbb{R}$ denotes the quotient of $\Omega^0_\xi(N)$ by constants; furthermore this open subset is $\Omega^1_{\xi,\mathrm{cl}}(N)$-invariant and convex.

Definition/Remark 2.9. By Lemma 2.8 every two contact 1–forms $\eta, \eta_0 \in \mathcal{S}(\xi, J^\xi)$ are related by

$$\eta = \eta_0 + d^c_\xi \varphi + \alpha$$

where $\alpha \in \Omega^1_{\xi,\mathrm{cl}}(N)$ and $\varphi \in \Omega^0_\xi(N)$. Such a deformation within $\mathcal{S}(\xi, J^\xi)$, that is preserving the transversal holomorphic structure, is called a Deformation of type II in [47]. Given $\eta_0 \in \mathcal{S}(\xi, J^\xi)$ we denote

$$\hat{\Xi}(\xi, J^\xi, \eta_0) := \{\varphi \in C^\infty(N)^\xi \,|\, \eta_0 + d^c_\xi \varphi \in \mathcal{S}(\xi, J^\xi)\} \tag{2.5}$$

the space of η_0-Sasaki potentials and the corresponding slice for the action of $\Omega^1_{\xi,\mathrm{cl}}(N)$ on $\mathcal{S}(\xi, J^\xi)$ that we denote

$$\Xi(\xi, J^\xi, \eta_0) := \{\eta_0 + d^c_\xi \varphi \,|\, \varphi \in \hat{\Xi}(\xi, J^\xi, \eta_0)\} \tag{2.6}$$

Not all of the deformations of type II are relevant for our purpose. Indeed, since $d\eta = d\eta_0 + d_\xi d^c_\xi \varphi$, the quotient $\mathcal{S}(\xi, J^\xi)/\Omega^1_{\xi,\mathrm{cl}}(N)$ may be viewed as the *transversal Kähler class* of (N, ξ, J^ξ). When ξ is quasi-regular, denoting (M, J, ω_0) the Sasaki-Kähler orbifold quotient, all the η_0–Sasaki potentials correspond to Kähler potentials with respect to ω_0.

2.1.1.3 The Sasaki-Reeb cone

Definition 2.10. The diffeomorphism group $\mathrm{Diff}(N)$ acts naturally on pairs (ξ, J^ξ), and the *automorphism group* $\mathrm{Aut}(N, \xi, J^\xi)$ is the stabiliser of (ξ, J^ξ); its formal Lie algebra $\mathfrak{aut}(N, \xi, J^\xi)$ consists of ξ-invariant vector fields Y with $\mathcal{L}_{\beta_\xi Y} J^\xi = 0$.

The automorphism group is infinite dimensional: its formal Lie algebra contains an infinite dimensional abelian ideal $\mathfrak{o}(N, \xi)$ of ξ-invariant vector fields in the span of ξ; however, the quotient $\mathfrak{aut}(N, \xi, J^\xi)/\mathfrak{o}(N, \xi)$ may be identified

with the space of holomorphic sections of $\mathcal{D}_\xi$ and hence is a finite dimensional complex Lie algebra if N is compact [48].

Suppose N is compact with transversal holomorphic structure (ξ, J^ξ); then the automorphism group of any compatible Sasaki structure is compact. Hence if (ξ, J^ξ) has Sasaki type, there is a torus $\mathbb{T} \subset \mathrm{Aut}(N, \xi, J^\xi)$ whose Lie algebra $\mathfrak{t}$ contains ξ, and we now fix such a $\mathbb{T}$. The considerations of the previous subsection apply equally to the space $\Omega^\bullet_\xi(N)^\mathbb{T}$ of $\mathbb{T}$-invariant basic forms, and the space $\mathcal{S}(\xi, J^\xi)^\mathbb{T}$ of $\mathbb{T}$-invariant elements of $\mathcal{S}(\xi, J^\xi)$. This space is acted upon by $\mathbb{T}$-equivariant automorphisms, i.e., by the centralizer $\mathrm{Aut}(N, \xi, J^\xi)^\mathbb{T}$ of $\mathbb{T}$ in $\mathrm{Aut}(N, \xi, J^\xi)$.

By definition, any $\zeta \in \mathfrak{t}$ is a CR vector field for the CR structure induced by any $\eta \in \mathcal{S}(\xi, J^\xi)^\mathbb{T}$. Furthermore, using the equation in Definition/Remark 2.1 to compare the signs of $\eta(\zeta)$ for various contact 1-forms $\eta \in \mathcal{S}(\xi, J^\xi)^\mathbb{T}$ we get that the Sasaki–Reeb cones all agree. Precisely, it means the following.

Lemma 2.11. *Suppose that N is compact and $\zeta \in \mathfrak{t}$ is a Sasaki–Reeb vector field with respect to the CR structure induced by some $\eta_0 \in \mathcal{S}(\xi, J^\xi)^\mathbb{T}$. Then ζ is a Sasaki–Reeb vector field with respect to the CR structure induced by any $\eta \in \mathcal{S}(\xi, J^\xi)^\mathbb{T}$.*

Definition 2.12. The *Reeb cone* of $(\xi, J^\xi, \mathbb{T})$ is the cone [168]

$$\mathfrak{t}^\xi_+ := \{\zeta \in \mathfrak{t} |\ \eta(\zeta) > 0\} \tag{2.7}$$

in $\mathfrak{t}$ determined by some (and hence any) $\eta \in \mathcal{S}(\xi, J^\xi)^\mathbb{T}$.

Using Remark 2.5, $\mathfrak{cr}(N, \mathcal{D}, J) \subset \mathfrak{con}(\mathcal{D}, J)$ and Lemma 2.11, we get that

$$\mathfrak{t}_+ := \mathfrak{t}^\xi_+ = \mathfrak{t} \cap \mathfrak{cr}_+(N, \mathcal{D}, J) = \mathfrak{t}^\zeta_+$$

for any pair $\xi, \zeta \in \mathfrak{t} \cap \mathfrak{cr}_+(N, \mathcal{D}, J)$.

Remark 2.13. Alternatively, for $\eta \in \mathcal{S}(\xi, J^\xi)$, the map

$$\begin{array}{rccc} \mu_\eta : & N & \longrightarrow & \mathfrak{t}^* \\ & x & \mapsto & (a \mapsto \eta(X_a)), \end{array}$$

is a contact momentum map in the sense of Lerman [155], who proved that its image

$$P_\xi := \mu_\eta(N)$$

is a convex compact polytope in the hyperplane $\{x \in \mathfrak{t}^* \,|\, \langle \xi, x\rangle = 1\}$. Thus, the Reeb cone is naturally identified with the dual of the cone over P_ξ, or equivalently,

$$\mathfrak{t}_+ = \{f \in \mathrm{Aff}(P_\xi, \mathbb{R}) \,|\, f > 0\}$$

as highlighted in [168, 148]. Consequently, the Reeb cone $\mathfrak{t}_+$ of $(\xi, J^\xi, \mathbb{T})$, equivalently the reduced Sasaki-Reeb of $(\mathcal{D}, J, \mathbb{T})$, is a strictly convex polyhedral cone.

2.1.2 Sasaki structures via complex cones

2.1.2.1 Polarised complex cone and radial potentials

A conical Kähler structure on a smooth complex manifold (Y^{n+1}, J) is a compatible Kähler form $\hat{\omega}$ and a real holomorphic vector field ξ on Y (i.e $\mathcal{L}_\xi J = 0$) such that the *radial vector field* $-J\xi$ induces a proper and free action of $\mathbb{R}^+$ and $\mathcal{L}_{-J\xi}\hat{\omega} = 2\hat{\omega}$. In that case $(Y, J, \xi, \hat{\omega})$ is called a Kähler cone (or a conical Kähler manifold).

Example 2.14. Let (M, J, L) a polarised compact manifold (or variety) so that $\pi : L \to M$ is ample and there exists a compatible hermitian metric on L so that its curvature determines a Kähler form ω_h on M. Now consider the total space of the dual line bundle with the zero section removed $Y := L^{-1}\backslash M$ and the smooth function $r_h : Y \to \mathbb{R}_{>0}$ the restriction of the norm of the hermitian metric induced (via duality) by h on L^{-1}. Then $(Y, \frac{1}{4}ddr_h^2)$ is a Kähler cone with radial action given by dilatation along the fibers and Reeb vector field ξ induced by the isometric S^1–action on the fibers. It is the prototypical example of conical Kähler manifold.

Given a Kähler cone $(Y, J, \xi, \hat{\omega})$, as $-J\xi$ generates a free proper action of $\mathbb{R}^+$, we let $N^\xi = Y/\exp(-tJ\xi)$ be the quotient manifold with quotient map $\pi_\xi : Y \to N^\xi$.

Then, if N^ξ is compact (which will be assumed from now) it is easy to show, see e.g. [168] that there is a unique positive function $r : Y \to \mathbb{R}^+$ satisfying

$$\mathcal{L}_\xi r = 0, \quad \mathcal{L}_{-J\xi} r = r, \quad \hat{\omega} = \frac{1}{4} dd^c(r^2). \tag{2.8}$$

Therefore, a conical Kähler structure on (Y^{n+1}, J) is encoded in a pair (ξ, r) where ξ is a real holomorphic vector field and $r \in C^\infty(Y, \mathbb{R}^+)$ satisfies (2.8). This leads to the following.

Definition 2.15. On a complex manifold (Y, J) with a holomorphic vector field ξ, we let

$$\mathcal{R}_\xi(Y, J) := \{r \in C^\infty_Y(\mathbb{R}^+) : \mathcal{L}_\xi r = 0,\ \mathcal{L}_{-J\xi} r = r,\ dd^c r > 0\} \tag{2.9}$$

be the space of (radial) *cone potentials* (ξ-invariant plurisubharmonic functions of homogeneity 1 with respect to $-J\xi$). If (Y, J, ξ) admits a surjective cone potential $r : Y \to \mathbb{R}^+$, then ξ is called a *polarisation* of (Y, J) and (Y, J, ξ) is called a *polarised complex cone*.

Typically in the literature [47, 111, 168] the correspondence between Sasaki and conical Kähler structures is explained as follows. Given a Kähler cone $(Y, \hat{J}, \xi, \hat{\omega})$ the level set (or the *link*) of the associated radial potential $r \in \mathcal{R}_\xi(Y, \hat{J})$, say $N_r := r^{-1}(1)$, inherits of a Sasaki structure

$$(\mathcal{D} := TN \cap \hat{J}TN, J := \hat{J}|_{\mathcal{D}}, \xi) \tag{2.10}$$

with contact 1–form $\eta_r := d^c \log r$. Conversely, given a Sasaki manifold $(N, \mathcal{D}, J, \xi)$ we consider the product $Y = \mathbb{R}^+ \times N$ with $\mathbb{R}^+$–action given by dilatation on the right summand. Then, we extend the endomorphism $J \in \mathrm{End}(\mathcal{D})$ to an endomorphism $\hat{J} \in \mathrm{End}(TY)$ sending $-\xi$ to the generator of that $\mathbb{R}^+$-action.

The fact that ξ is a unitary Killing vector field and that $(\mathcal{D}, J)$ is CR imply that $(Y, \hat{J}, r^2 g + dr^2)$ is a Kähler cone (where r is the projection on the right summand of Y).

However, in this last construction it is not clear that given two Sasaki structures with commuting Reeb vector fields on the same CR manifold, the resulting complex cones are biholomorphic [4]. Also, to compare the Sasaki manifolds obtained as level sets of a radial potentials in $\mathcal{R}_\xi(Y, \hat{J})$ we need to use a diffeomorphism between them which is not always easy to handle. It is often more convenient to work on a single fixed CR manifold. A good candidate for this is the transversally holomorphic quotient $(N^\xi, \mathcal{D}_\xi, J^\xi)$ of (Y, J) by the holomorphic $\mathbb{R}^+$-action induced by the radial vector field $-\hat{J}\xi$.

More precisely, we have the following Lemma in which we denote, for $r \in C^\infty(Y, \mathbb{R}^+)$, $N_r := r^{-1}(1)$ and $\tilde{\eta}_r := d^c \log r$.

Lemma 2.16. *[ACL] Let (Y, J) be a complex manifold with a holomorphic vector field ξ and a function $r \in C^\infty(Y, \mathbb{R}^+)$ such that $\mathcal{L}_\xi r = 0$ and $\mathcal{L}_{-J\xi} r = r$. Then $\tilde{\eta}_r$ is the pullback $\pi_\xi^* \eta_r$ of a 1-form η_r on N^ξ with $\eta_r(\xi) = 1$ and $\mathcal{L}_\xi \eta_r = 0$. Further, the following are equivalent:*

1. *r is plurisubharmonic, i.e., is in $\mathcal{R}_\xi(Y, J)$;*
2. *$\eta_r \in \mathcal{S}(\xi, J^\xi)$.*

Moreover, if $r \in \mathcal{R}_\xi(Y, J)$, $\pi_\xi|_{N_r}$ is a CR isomorphism from N_r (with the hypersurface CR structure) to N^ξ (with the CR structure induced by J^ξ and $\mathcal{D}_r = \ker \eta_r$, i.e., by $\eta_r \in \mathcal{S}(\xi, J^\xi)$).

A useful outcome is that the space of radial potentials gives a base-point free representation of Sasaki/Kähler potentials as highlighted in [ACL].

Proposition 2.17. *[168, 111, ACL] Let (Y, J, ξ) be a polarised complex cone. For any $r, r_0 \in \mathcal{R}_\xi(Y, J)$ we have $\eta_r \in \Xi(N^\xi, \xi, J^\xi, \eta_{r_0})$ and $\eta_r = \eta_{r_0}$ if and only if $r = e^a r_0$ with $a \in \mathbb{R}$ constant.*

Moreover, fixing $r_0 \in \mathcal{R}_\xi(Y, J)$ the map

$$r \in \mathcal{R}_\xi(Y, J) \mapsto \varphi \in C^\infty_N(\mathbb{R}) \quad \text{with} \quad \pi_\xi^* \varphi = \log r - \log r_0 \tag{2.11}$$

is a bijection from $\mathcal{R}_\xi(Y, J)$ to the space of η_{r_0}–Sasaki potentials (cf. (2.5)) on N^ξ and

$$r \mapsto N_r$$

is a bijection from $\mathcal{R}_\xi(Y, J)$ to the set of strictly pseudoconvex (images of) ξ-invariant sections of π_ξ.

In sum, a smooth polarised complex cone (Y, J, ξ) corresponds to a transversal holomorphic manifold (N^ξ, ξ, J^ξ) of Sasaki type with a given *marking* $\Xi \subset \mathcal{S}(N^\xi, \xi, J^\xi)$ that is a slice of the form (2.6). Note that as in §2.1.1.3, we may fix a compact torus $\mathbb{T} \subset \mathrm{Aut}(N^\xi, \xi, J^\xi)$, with $\xi \in \mathfrak{t} = \mathrm{Lie}(\mathbb{T})$, which preserves some compatible Sasaki structure $\eta_0 \in \Xi$. Thus ξ, J^ξ, and $\Xi \cong \Xi(\xi, J^\xi, \eta_0)$ are $\mathbb{T}$-invariant, hence there is an induced holomorphic action of $\mathbb{T}$ on (Y, J, ξ). We denote by $\Xi^{\mathbb{T}}$ the space of $\mathbb{T}$-invariant elements of Ξ and by $\mathcal{R}_\xi(Y, J)^{\mathbb{T}}$ the space of $\mathbb{T}$-invariant cone potentials in $\mathcal{R}_\xi(Y, J)$. The bijections above preserve the $\mathbb{T}$-invariance.

[4] This is true and a corollary of Lemma 2.19.

2.1.2.2 Polarised complex cone and ample line bundles

Let (Y, J, ξ) be polarised complex cone and denote $(N^\xi, \xi, J^\xi, \hat{\Xi})$ the associated marked transversal holomorphic manifold (assumed compact) see §2.1.2.1. We fix a compact torus $\mathbb{T} \subset \mathrm{Aut}(N, \xi, J^\xi)$, with $\xi \in \mathfrak{t} = \mathrm{Lie}(\mathbb{T})$ and assume ξ is quasiregular ($\Lambda \subset \mathfrak{t}$ is the lattice of circle subgroups of $\mathbb{T}$, then the set of quasi regular Reeb vector fields $(\mathbb{R}^+ \otimes_{\mathbb{Z}_+} \Lambda) \cap \mathfrak{t}_+$ is dense in $\mathfrak{t}_+$).

Then, similarly to the regular case (Example 2.4), Y can be identified with the space of nonzero vectors in the dual of an orbi-ample orbiline bundle L over the compact Kähler orbifold (M, J_M, ω_0) which is the Sasaki–Reeb quotient of N by the circle action generated by ξ. Thus Y has a natural one point compactification $\widehat{Y} = Y \cup \{0\}$ which the singular space obtained by blowing down the zero section in the total space of L^*. Since radial potentials correspond to norms of hermitian metrics on L, for any $r \in \mathcal{R}_\xi(Y, J)$, the apex 0 of the cone $\widehat{Y}$ is characterized as the limit of points $p \in Y$ with $r(p) \to 0$. Let

$$\mathcal{H} \cong \bigoplus_{k \in \mathbb{N}} H^0(M, L^k) \tag{2.12}$$

be the space of continuous complex-valued functions of $\widehat{Y}$ which are holomorphic on $Y \cong (L^*)^\times$ and polynomial on each fiber of $(L^*)^\times$ over M. Here $H^0(M, L^k)$ is the space of holomorphic sections s of L^k, which define fiberwise polynomial functions f_s on L^* by $f_s(p) = \langle s, p^k \rangle$. Now $\mathbb{T}_\mathbb{C}$ acts on $\mathcal{H}$ and for $\alpha \in \mathfrak{t}^*$ we let

$$\mathcal{H}_\alpha := \{f \in \mathcal{H} |\ \forall\, \zeta \in \mathfrak{t},\ \mathcal{L}_{-J\zeta} f = \alpha(\zeta) f\}$$

be the α-weight space and

$$\Gamma := \{\alpha \in \mathfrak{t}^* |\ \mathcal{H}_\alpha \neq 0\}$$

be the set of (integral) weights of the action. Thus there is a weight space decomposition

$$\mathcal{H} \cong \bigoplus_{\alpha \in \Gamma} \mathcal{H}_\alpha, \tag{2.13}$$

where the degree k component of (2.12) is the direct sum of the weight spaces $\mathcal{H}_\alpha$ with $\alpha(\xi) = k$. A key fact is that (by ampleness of L) $\mathcal{H}$ separates points of $\widehat{Y}$: in particular for any $p \in \widehat{Y}$ there exists $f \in \mathcal{H}$ with $f(0) = 0$ and $f(p) \neq 0$.

Remark 2.18. In fact orbifold versions of the Kodaira embedding Theorem (see [180]) embed $\widehat{Y}$ as an affine variety in $\mathbb{C}^N$ which is a cone with the singular apex at the origin (see also [200]). The functions in $\mathcal{H}$ are then the regular functions on $\widehat{Y}$, which separate points by definition.

An important idea in [53, 68] is to define the Reeb cone in $\mathfrak{t}$ as dual to the cone generated by Γ in $\mathfrak{t}^*$, i.e., to set

$$\mathfrak{t}_+^\Gamma = \{a \in \mathfrak{t} |\ \forall \alpha \in \Gamma \setminus \{0\},\ \alpha(a) > 0\}$$

and alternatively, as intrinsically defined on the complex cone $(Y, J, \mathbb{T})$ as

$$\mathfrak{t}_+^{Pol} := \{a \in \mathfrak{t} |\ a \text{ is a polarisation of } (Y, J)\}. \tag{2.14}$$

Actually, following the discussion of §2.1.2.1 we can immediately infer that

$$\mathfrak{t}^{\xi}_{+} = \{\zeta \in \mathfrak{t} |\ \mathcal{L}_{-J\zeta} r > 0\} = \{\zeta \in \mathfrak{t} |\ J\zeta \text{ is transverse to } N_r := r^{-1}(1)\}. \quad (2.15)$$

Consequently, (Y, J) has a proper holomorphic action of a complex torus $\mathbb{T}_{\mathbb{C}}$ whose real part coincides with $\mathbb{T}$. It is great that all these definitions coincide.

Lemma 2.19. *[68, 53, ACL] If ξ is quasi regular then $\mathfrak{t}_+ = \mathfrak{t}^{\xi}_{+} = \mathfrak{t}^{Pol}_{+} = \mathfrak{t}^{\Gamma}_{+}$.*

Proof. The equality $\mathfrak{t}_+ = \mathfrak{t}^{\xi}_{+}$ was discussed in 2.1.1. The inclusion $\mathfrak{t}_+ \subset \mathfrak{t}^{Pol}_{+}$ is a straightforward application of the $\mathbb{T}$–invariant version of Proposition 2.17, indeed for $r \in \mathcal{R}_{\xi}(Y, J)$, N_r is a $\mathbb{T}$–invariant section of π_{ζ} for any $\zeta \in \mathfrak{t}_+$. The argument in [68] that $\mathfrak{t}^{Pol}_{+} \subset \mathfrak{t}^{\Gamma}_{+}$ uses that $\mathcal{H}$ separates points and that $f \in \mathcal{H}$ must decreases when approaching the apex. In [ACL] we proved that $\mathfrak{t}^{\Gamma}_{+} \subset \mathfrak{t}^{\xi}_{+}$ as follows.

Let $\zeta \in \mathfrak{t}^{\Gamma}_{+}$ and $r \in \mathcal{R}_{\xi}(Y, J)$. By Remark 2.13, it suffices to show that ζ is in the dual of the momentum cone of the contact momentum map of η_r on N^{ξ} (Definition 2.12). For this, we first let $p_o \in N_r$ have maximal stabiliser with respect to the $\mathbb{T}$-action, i.e., $\mathfrak{t} = \mathfrak{stab}_{\mathbb{T}}(p_o) \oplus \mathrm{Span}\{\xi\}$. Thus $\{\underline{a}_{p_o} | a \in \mathfrak{t}\} = \mathrm{Span}\{\xi_{p_o}\}$ and $\zeta_{p_o} - \eta_r(\zeta)_{\pi_{\xi}(p_o)}\xi_{p_o} \in \mathfrak{stab}_{\mathbb{T}}(p_o)$. Since $\mathcal{H}$ separates points, there exists $f \in \mathcal{H}_{\alpha}$ with $\alpha \in \Gamma \setminus \{0\}$ and $f(p_o) \neq 0$. Let $\Phi_t^{-J\zeta}$ be the flow of $-J\zeta$ and observe that

$$f\left(\Phi_t^{-J\zeta}(p_o)\right) = f\left(\Phi_t^{-\eta_r(\zeta)_{\pi_{\xi}(p_o)} J\xi}(p_o)\right)$$

so that

$$\alpha(\zeta) f(p_o) = \eta_r(\zeta)_{\pi_{\xi}(p_o)} \alpha(\xi) f(p_o).$$

Since $\alpha(\xi) > 0$ and $\alpha(\zeta) > 0$ by assumption, we have $\eta_r(\zeta)_{\pi_{\xi}(p_o)} > 0$. We conclude that $\eta(\zeta) > 0$ at points $\xi = \mu_{\eta_r}(p_o)$ on the edges of the momentum cone. Since the momentum cone is a strictly convex polyhedral cone, the result follows. □

Note that $\mathfrak{t}^{Pol}_{+} = \mathfrak{t}^{\xi}_{+} = \mathfrak{t}^{\zeta}_{+}$ for two commuting Reeb vector fields ξ, ζ on $(N, \mathcal{D}, J)$ implies that their associated polarised complex cone are biholomorphic.

2.1.2.3 Curvatures of a Sasaki manifold and the (contact) Futaki invariant

To end the background section we recall, for future references, the various curvatures associated to a Sasaki manifold and the Futaki invariant. There are various conventions used in the literature for the cone metric, the radial action and even the Sasaki metric. We use the one in Remark 2.3 with the $1/2$ in front of the transversal part which is the convention used for example by Futaki–Ono–Wang [111].

Let $(N, \mathcal{D}, J, \xi)$ be a compact connected Sasaki $(2n+1)$ manifold, we denote by η the associated contact 1–form and g the riemannian metric see Remark 2.3. The transversal Kähler geometry of refers to the geometry of $(\mathcal{D}, J, g_{|_{\mathcal{D}}})$. More precisely, N is foliated by the Reeb flow. So there are local

submersions $\pi_\alpha : U_\alpha \to V_\alpha$, quotient by the flow of ξ, where U_α and V_α are open subsets of N and $\mathbb{C}^n$ respectively, such that $(\pi_\alpha^* i)|_{\mathcal{D}} = J$. In particular, $d\pi_\alpha : (\mathcal{D}_{|_{U_\alpha}}, J) \to (TV_\alpha, i)$ is an isomorphism and the Sasaki metric is sent to a Kähler structure on V_α with a connection ∇^T_α and curvatures R^T_α, Ric^T_α, ρ^T_α, Scal^T_α... Since, $\pi^*_\alpha \nabla^T_\alpha$ and $\pi^*_\beta \nabla^T_\beta$ coincide on $U_\alpha \cap U_\beta$, these objects patch together to define global objects on N, the *transversal* connection and curvatures[5] ∇^T, R^T, Ric^T, ρ^T, Scal^T... See [47, 111] for more details. These tensors are basic, notably the transversal Ricci form $\rho^T \in \Omega^{1,1}(N, \mathrm{cl})$ lies in the basic first Chern class $2\pi c_1^B(\xi)$ but also, as a closed 2-form, it lies in $c_1(\mathcal{D})$, see [198] and [111, Proposition 4.3].

The relations between transversal curvatures and the curvatures Ric^g, Scal_g of the Sasaki riemannian metric g of $(N, \mathcal{D}, J, \xi)$ are well-known [47, §7.3]. In particular, the formulas[6] in [47, Theorem 7.3.12] imply that for any Sasaki structure

$$\mathrm{Ric}^g(\xi, \xi) = 2n \qquad \text{and} \qquad \mathrm{Scal}_g = \mathrm{Scal}^T - 2n. \tag{2.16}$$

More generally, for any *horizontal vector fields* $X, Y \in \Gamma(\mathcal{D})$, we have

$$\mathrm{Ric}^T(X, Y) = \mathrm{Ric}^g(X, Y) + 2g(X, Y). \tag{2.17}$$

Moreover, as recalled in §2.1.2.1, to $(N, \mathcal{D}, J, \xi)$ is associated with a Kähler cone, say $(Y, \hat{J}, \hat{\omega}, \hat{g})$ where $(Y, \hat{g})$ is a riemannian cone over (N, g). Computing the Ricci and scalar curvatures of $\hat{g}$ in terms are those of g is an easy exercise and we get, on N,

$$\mathrm{Ric}^{\hat{g}} = \mathrm{Ric}^g - 2ng \tag{2.18}$$

while $\mathrm{Ric}^{\hat{g}}(\xi, \cdot) = 0 = \mathrm{Ric}^{\hat{g}}(\hat{J}\xi, \cdot)$. Thus, on N, $\mathrm{Scal}_{\hat{g}} = \mathrm{Scal}_g - 2n(2n+1)$.

Therefore, if a Sasaki metric g is Einstein (we then say it is a *Sasaki-Einstein metric*) then $\mathrm{Scal}_g = 2n(2n+1)$. That is the Einstein constant on (N, g) is $2n$. In that case, $(Y, \hat{g})$ is Ricci-flat, $\mathrm{Ric}^T = 2(n+1)g|_{\mathcal{D}}$ and $\mathrm{Scal}^T = 4n(n+1)$. By definition a Sasaki structure $(\mathcal{D}, J, \xi)$ with riemannian–Sasaki metric g is *η–Einstein* if

$$\mathrm{Ric}^g = \mu\eta \otimes \eta + \lambda g$$

for two constants $\mu, \lambda \in \mathbb{R}$. Note that (when $n \geq 2$), these constants must satisfies $\mu + \lambda = 2n$.

Remark 2.20. The transversal scalar curvature Scal^T is homogeneous of order 1 under transversal homotheties $0 < \lambda \mapsto (\mathcal{D}, J, \lambda\xi)$, see Definition/Remark 2.5. In particular, this deformation preserves the cscS property but not Sasaki-Eintein one. Thus, cscS come in rays and if a Sasaki structure $(\mathcal{D}, J, \xi)$ has a positive transversal scalar curvature then there exists a unique colinear Reeb vector field $\xi' \in \mathbb{R}_{>0}\xi$ such that the associated Kähler cone of $(\mathcal{D}, J, \xi')$ is Scalar-flat.

[5] To be very clear we would need to specify all the structures on which depend these curvatures, namely $(\mathcal{D}, J, \xi)$. Such notation would be heavy, so unless there is a possible confusion, we avoid specifying the Sasaki structure in the notation.

[6] The transversal Ricci curvature is insensitive to the convention and thus we can use some computations done in [47] even if they don't use the same convention.

One can define the relative *contact* Futaki invariant as follows, for $a \in \mathfrak{t}$

$$\mathbf{F}_\xi(a) := \frac{1}{n!}\int_N \eta(\underline{a})(\mathrm{Scal}^T - \overline{c}_\xi)\eta \wedge (d\eta)^n \tag{2.19}$$

where $\overline{c}_\xi := \mathbf{S}(\xi)/\mathbf{V}(\xi)$ and $\eta \in \mathcal{S}(N, \mathcal{D}_\xi, J^\xi)^{\mathbb{T}}$ is any compatible CR structure.

Remark 2.21. The expression $\mathbf{F}_\xi$ is not exactly the restriction to some complexification of $\mathfrak{t}$ the classical Sasaki-Futaki invariant Fut_ξ defined in [48, 111] by

$$\mathrm{Fut}_\xi(X) := \int_N (X.\psi)\, \eta \wedge (d\eta)^n$$

for a ξ-basic function ψ satisfying $\Delta_\xi \psi = \mathrm{Scal}^T - \overline{c}_\xi$. Indeed, one can check easily that $\mathrm{Fut}_\xi(\underline{a}) = 0$ for all $a \in \mathfrak{t}$ and the relevant data is then $\mathrm{Fut}(\Phi^\xi(\underline{a}))$ where $\Phi^\xi \in \mathrm{End}(TN)$ extends J with $\Phi^\xi \xi = 0$. Now, one can prove easily using an integration by parts (see the proof of [BHLTF1, Lemma 3.1]) that $\mathbf{F}_\xi = \mathrm{Fut}_\xi \circ \Phi^\xi$.

Theorem 2.22. *[Boyer–Galicki–Simanca [48], Futaki–Ono–Wang [111]] The expression* (2.19) *only depends on $\xi \in \mathfrak{t}_+$ and $a \in \mathfrak{t}$ and does not depends on the chosen CR structure $\eta \in \mathcal{S}(N, \mathcal{D}_\xi, J^\xi)^{\mathbb{T}}$. In particular, if there exists a cscS structure in $\eta \in \mathcal{S}(N, \mathcal{D}_\xi, J^\xi)^{\mathbb{T}}$ then $\mathbf{F}_\xi(a) = 0$ for all $a \in \mathfrak{t}$.*

2.1.3 The Einstein–Hilbert functional and the transversal Futaki invariant

2.1.3.1 The Einstein–Hilbert functional, definition

Let $(N, \mathcal{D}, J, \xi)$ be a compact connected Sasaki $(2n+1)$ manifold. The volume function and the total transversal scalar curvature are

$$\begin{aligned} \mathbf{V}(\mathcal{D}, J, \xi) &:= \frac{1}{n!}\int_N \eta \wedge (d\eta)^n \\ \mathbf{S}(\mathcal{D}, J, \xi) &:= \frac{2}{(n-1)!}\int_N \rho^T \wedge \eta \wedge (d\eta)^{n-1}, \end{aligned} \tag{2.20}$$

by convention[7]. Alternatively, $\mathbf{S}(\xi) = \frac{1}{n!}\int_N \mathrm{Scal}^T \eta \wedge (d\eta)^n$, see [111, (15)].

Note that adding a closed basic 1–form to η in the r.h.s. of the expressions (2.20) does not change the value since there is no non-trivial basic top degree form. Moreover, Futaki–Ono–Wang [111], see also [48], have proved that the Volume and total transversal scalar curvature are constant along the transversal variation $\eta \mapsto \eta + d^c_\xi \phi$. Therefore, the values $\mathbf{V}(\mathcal{D}, J, \xi)$ and $\mathbf{S}(\mathcal{D}, J, \xi)$ do not depend on the specific CR structure $(\mathcal{D}, J)$ chosen inside $\mathcal{S}(\xi, J^\xi)$. So, we drop $(\mathcal{D}, J)$ from the notation and consider these functionals as defined on the Reeb cone

$$\mathbf{V}, \mathbf{S} : \mathfrak{t}_+ \longrightarrow \mathbb{R}.$$

Remark 2.23. Whenever ξ is a quasi regular Reeb vector field (recall that it means it induces a locally free circle action $\mathbb{S}^1_\xi \subset \mathbb{T}$) any $\mathbb{T}$–invariant Sasaki structure $\eta \in \mathcal{S}(N, J^\xi)^{\mathbb{T}}$ determines a Kähler structure ω_ξ on the quotient $M_\xi :=$

[7] The form $\eta \wedge (d\eta)^n/n!$ is 2^{-n} times the volume of the riemannian Sasaki metric of $(\mathcal{D}, J, \xi)$.

$N/\mathbb{S}^1_\xi$, see example 2.4. Actually, N is a circle bundle, η a connection 1–form on the $\mathbb{S}^1$–bundle $\pi_\xi : N \to M_\xi$ and $\pi_\xi^*\omega_\xi = \frac{1}{2}d\eta$. Thus $\mathbf{V}(\xi) = 2^{n+1}2\pi\frac{1}{n!}\int_{M_\xi}\omega_\xi^n$ and $\mathbf{S}(\xi) = 2^{n-1}\frac{2\pi}{(n-1)!}\int_{M_\xi}\rho_\xi \wedge \omega_\xi^{n-1}$.

Clearly for any $\lambda > 0$, $\mathbf{V}(\lambda\xi) = \lambda^{-n-1}\mathbf{V}(\xi)$ and $\mathbf{S}(\lambda\xi) = \lambda^{-n}\mathbf{V}(\xi)$ so the *Einstein-Hilbert functional*, which is defined in this setting as

$$\begin{array}{rccc} \mathrm{EH}: & \mathfrak{t}_+ & \longrightarrow & \mathbb{R} \\ & \xi & \mapsto & \frac{\mathbf{S}(\xi)^{n+1}}{\mathbf{V}(\xi)^n}, \end{array}$$

is homogeneous.

2.1.3.2 First variation of the Einstein–Hilbert functional

Using formulas (2.16) and the fact that the Reeb vector field is unitary, any *Sasaki-Einstein* metric (i.e Sasaki and Einstein metric) must satisfy $\mathrm{Scal}^T = 4n(n+1)$ and $\mathrm{Scal} = 2n(2n+1)$. Therefore, the Reeb vector field of a Sasaki–Einstein structure lies in the set

$$\Sigma := \{\xi \in \mathfrak{t}_+ \mid \mathbf{S}(\xi) = 4n(n+1)\mathbf{V}(\xi)\}.$$

Moreover the condition that ρ^T is a positive multiple of $d\eta$ implies that $c_1^B(\xi) > 0$ and $c_1(\mathcal{D}) = 0$ which, in turns, imply that Σ is a convex set, a transversal polytope of the Reeb cone $\mathfrak{t}_+$, see [168].

Theorem 2.24 (Martelly–Sparks–Yau [167, 168])**.** *The Volume functional restricted on Σ is convex and its unique critical point (if it exists) is the unique Reeb vector field in Σ whose transversal Futaki invariant vanishes.*

The general case, that is when $c_1(\mathcal{D}) \neq 0$, turns out to be more complicated. In this case, the set Σ is not convex and there are examples of Reeb cone admitting multiples rays of cscS metrics [148, 54]. The first examples of such Reeb cones are toric and were found by a careful examination of the Einstein–Hilbert functional on some explicit Reeb cones. With my collaborators, we have extended the basic fact used there to non-necessarily toric Sasaki manifolds. Note that, in what follows, the transversal Futaki invariant is understood in its reduced sense, that is restricted to the complexification of the compact torus $\mathbb{T} \subset \mathrm{Aut}(N, \xi, J^\xi)$ we have fixed. Also, it is more relevant to consider the Futaki invariant on a maximal torus $\mathbb{T} \subset \mathrm{Aut}(N, \xi, J^\xi)$. This is what we assume for the rest of this section.

Theorem 2.25 (Boyer–Huang–L.–Tønnesen-Friedman [BHLTF1])**.** *The set of critical points of the Einstein–Hilbert functional is the union of the zeros of the transversal Futaki invariant and of the total transversal scalar curvature.*

In particular, if a Reeb vector field admits a compatible cscS metric then it is a critical point of the Einstein–Hilbert functional.

Remark 2.26. It is not surprising that the Einstein–Hilbert functional obstructs cscS metrics, it is already the case on a conformal class of riemannian metrics.

The second statement of Theorem 2.25 follows from the first and Theorem 2.22 from [48, 111] recalled above, that the vanishing of the transversal Futaki invariant is a necessary condition for a cscS metric to exist in $\mathcal{S}(\xi, J^\xi)$. The first statement of Theorem 2.25 can be interpreted as an extension of a result of Martelli–Sparks–Yau [167, 168] and is a direct consequence of the next lemma.

Lemma 2.27. *[BHLTF1] The derivative of* EH *at* $\xi \in \mathfrak{t}_+$ *in the direction* $a \in \mathfrak{t}$ *is, up to a factor the transversal Futaki invariant, that is*

$$d_\xi \mathrm{EH}(a) = \frac{n(n+1)\mathbf{S}^n_\xi}{\mathbf{V}^n_\xi}\mathbf{F}_\xi(a).$$

If $\mathbf{S}_\xi = 0$ *then* $d\mathbf{S}_\xi = n\mathbf{F}_\xi(a)$.

Remark 2.28. The toric case is very special as the volume $\mathbf{V}(\xi)$ (respectively the total transversal scalar curvature $\mathbf{S}(\xi)$) coincide with the volume (respectively the area of the boundary) of the transversal polytope $P_\xi := \{\xi \in \mathcal{C} \,|\, \langle \xi, x\rangle = 1\}$ with respect to the measure induced by the lattice of circle subgroups in $\mathfrak{t}$ and the inward primitive normal vectors, see [82, 148] and Chapter 4, so the proof in this case boils to basic calculus on rational functions.

Proof of Lemma 2.27. This is a direct computation. We denote $\Phi \in \mathrm{End}(TN)$, the extension of $J \in \mathrm{End}(\mathcal{D})$ satisfying $\Phi(\xi) = 0$. Given a path ξ_t in $\mathfrak{t}^+_k$ such that $\xi_0 = \xi$, we denote

$$\eta_t = \frac{\eta}{\eta(\xi_t)},\ \Phi_t = \Phi - \Phi(\xi_t)\otimes\eta_t \ \text{ and}$$

$$g_t = \frac{1}{2}d\eta_t(\Phi_t(\cdot),\cdot) + \eta_t\otimes\eta_t \tag{2.21}$$

the associated riemannian metric and Scal_{g_t} its scalar curvature and dv_g its volume form. Since $(\mathcal{D}, J, \xi_t)$ is sasakian $\mathrm{Scal}_{g_t} - 2n = \mathrm{Scal}^T_{\xi_t}$. Therefore, using the computation in [36, p.63], we have

$$\int_N \dot{\mathrm{Scal}}^T_{\xi_t} dv_g = \int_N \dot{\mathrm{Scal}}_g dv_g = -\int_N g(\dot{g}, \mathrm{Ric}_g) dv_g \tag{2.22}$$

where $g := g_0$. Now, denoting $\dot{\phi}_t := \eta(\xi_t)$ here, using the fact that $\eta(\xi_0) = 1$ (thus $d(\eta(\xi_0)) = 0$) and $\dot{\eta}_0 = -\dot{\phi}\eta$ we get that

$$\dot{g} = -\dot{\phi}g - \dot{\phi}\eta\otimes\eta + b \tag{2.23}$$

where b is symmetric tensor. The only properties we will need about b is that if $u, v \in \mathcal{D}_p$ at $p \in N$, we have

$$b(u,v) = \left(\frac{d}{ds}g_s(u,v)\right)_{s=0} + \dot{\phi}g(u,v) = \left(\frac{d}{2ds}(\eta(\xi_s)^{-1})d\eta(u, J(v))\right)_{s=0} + \dot{\phi}g(u,v) = 0.$$

Moreover, by definition, for any $u, v \in T_pN$, we have $2b(u,v) = -(d\dot{\phi}\wedge\eta)(u, \Phi v) + d\eta(u, \dot{\Phi}v)$, thus $\Phi\xi = 0$ and $d\eta(\xi, \cdot) = 0$ lead to $b(\xi, \xi) = 0$.

Inserting (2.23) in (2.22) we get

$$\begin{aligned}\int_N \dot{\mathrm{Scal}}^T_\xi dv_g &= \int_N (\dot\phi(\mathrm{Scal}_g + g(\eta\otimes\eta, \mathrm{Ric}_g)) - g(b, \mathrm{Ric}_g))dv_g \\ &= \int_N \dot\phi(\mathrm{Scal}_g + 2n) - g(b, \mathrm{Ric}_g)dv_g \\ &= \int_N \dot\phi \mathrm{Scal}^T_g dv_g\end{aligned} \tag{2.24}$$

the second line comes from the identity $\mathrm{Ric}_g(\xi,\xi) = 2n$, recalled above. For the last line is used that $g(b, \mathrm{Ric}_g) = 0$ which is due to the fact that b is symmetric and that at each point $p \in N$, $b(\xi,\xi) = 0$, $b(u,v) = 0$ if $u, v \in D_p$, while $\mathrm{Ric}(u,\xi) = 0$ at p whenever $u \in D_p$.

Furthermore, $dv_{g_t} = \frac{1}{n!}\eta_t \wedge (d\eta_t)^n$, thus

$$\left(\frac{d}{dt}dv_t\right)_{t=0} = -(n+1)\eta(a)dv_{g_0} = -(n+1)\dot\phi dv_g. \tag{2.25}$$

Putting the variational formulas (2.25) and (2.24) together we get

$$\begin{aligned}\frac{d}{ds}\left(\mathbf{S}_{\xi_s}\right)_{s=0} &= \int_N (\dot{\mathrm{Scal}}_g dv_g + \mathrm{Scal}^T_g d\dot v_g) \\ &= \int_N (\dot\phi \mathrm{Scal}^T_g dv_g - (n+1)\dot\phi \mathrm{Scal}^T_g dv_g) \\ &= -n\int_N \dot\phi \mathrm{Scal}^T_g dv_g.\end{aligned} \tag{2.26}$$

Hence, using (2.26) and (2.24) we have

$$\begin{aligned}\left(\frac{d}{ds}\mathrm{EH}(\xi_s)\right)_{s=0} &= \frac{\mathbf{S}^n_\xi}{\mathbf{V}^{n+1}_\xi}\left((n+1)\mathbf{V}_\xi \dot{\mathbf{S}}_\xi - n\mathbf{S}_\xi \dot{\mathbf{V}}_\xi\right) \\ &= -n(n+1)\frac{\mathbf{S}^n_\xi}{\mathbf{V}^n_\xi}\left(\int_N \dot\phi\left(\mathrm{Scal}^T_g - \frac{\mathbf{S}_\xi}{\mathbf{V}_\xi}\right)dv_g\right).\end{aligned} \tag{2.27}$$

□

2.1.3.3 Second variation of the Einstein–Hilbert functional

Since EH is homogeneous, it cannot be convex in the usual sense but one can wonder if it is transversally convex. We also know from [148, 54] that global transversal convexity generally fails in the cscS case. However, in all the known examples the critical set of EH consists in a finite number of isolated rays. It is an open question to know if it is always the case [BHLTF1, BHLTF3] and could have some application in the study of moduli space of cscK metrics [91, 81]. As a first approach, we have investigated the local transversal convexity/concavity properties of the functional EH and computed its second variation.

To simplify the expression, we denote $dv_\eta = \eta \wedge (d\eta)^n$ (although this is not exactly the riemannian volume form of g) and use the standard notation for inner product on the space $L^2(N)$ of square integrable real valued functions on N, namely

$$\langle f, h\rangle := \int_N f h dv_\eta \quad \text{and} \quad \|f\|^2 := \int_N f^2 dv_\eta.$$

For a vector field $X \in \Gamma(TN)$ or 1-form $\beta \in \Gamma(T^*N)$, we denote

$$\|X\|^2 := \int_N g(X,X)dv_\eta \quad \text{and} \quad \|\beta\|^2 := \int_N g(\beta,\beta)dv_\eta.$$

Moreover we define the normalized transversal scalar curvature as

$$\overset{\circ}{\mathrm{Scal}}{}^T = \mathrm{Scal}^T - \frac{\mathbf{S}}{\mathbf{V}}$$

which integrates to zero against dv_η.

Lemma 2.29. *[BHLTF1] For $\xi \in \mathfrak{t}_+$ and each $a \in T_\xi \mathfrak{t}_+ = \mathfrak{t}$ and variation $\xi_t = \xi + ta$, we have the following second variation.*

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{EH}(\xi_t)_{t=0} &= n(n+1)(2n+1)\frac{\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\|d(\eta(a))\|^2 - n(n+1)\frac{\mathbf{S}_\xi^{n+1}}{\mathbf{V}_\xi^{n+1}}\|\eta(a)\|^2 \\ &\quad + n(n+1)^2\frac{\mathbf{S}_\xi^n}{\mathbf{V}_\xi^n}\int_N \eta(a)^2 \overset{\circ}{\mathrm{Scal}}{}^T dv_g \\ &\quad + n(n+1)\frac{\mathbf{S}_\xi^{n-1}}{\mathbf{V}_\xi^{n+2}}\left(n\mathbf{V}\langle \overset{\circ}{\mathrm{Scal}}{}^T, \eta(a)\rangle - \mathbf{S}\langle \eta(a), 1\rangle\right)^2.\end{aligned}$$

Again the formula in the last lemma was obtained using direct calculations similar to the ones in the proof of Lemma 2.27. We have used this computation to get the following partial transversal convexity results.

Lemma 2.30. *Let $(N^{2n+1}, \mathcal{D}, J, \xi)$ be a $\mathbb{T}$–invariant cscS compact manifold of non-zero transversal scalar curvature Scal^T where $\mathbb{T} \subset CR(\mathcal{D}, J)$ is a maximal compact torus.*

a) If the transverse scalar curvature is negative, the Einstein–Hilbert functional EH *is transversally convex if n is even and transversally concave if n is odd.*

b) If the transverse scalar curvature is positive and the first non-zero eigenvalue of the Laplacian, restricted to the space of $\mathbb{T}$–invariant functions is bounded below by $\frac{\mathrm{Scal}^T}{2n+1}$, then EH *is convex near ξ.*

c) If $(N^{2n+1}, \mathcal{D}, J, \xi)$ is η–Einstein, then EH *is convex near ξ.*

In these cases the ray of cscS metrics of ξ is isolated in the Sasaki-Reeb cone.

The points a) and b) above are obtained straightforwardly from the second variation formula of Lemma 2.29. The point c) is deduced from b) using the Sasaki version of Matsushima's Theorem [170] which implies that smallest non-zero eigenvalue λ, of a Sasaki η–Einstein of (then positive and constant) transversal scalar curvature $\mathrm{Scal}^T > 0$, is $\lambda = \mathrm{Scal}^T/2n$ which is greater than $\mathrm{Scal}^T/(2n+1)$.

The *general joint construction* refers to the closest notion of "products" in the Sasaki category and consists in a Sasaki type manifold obtained as a circle quotient of a product of Sasaki manifolds. In the case where one factor is $\mathbb{S}^3$ and the other is a general Sasaki manifold N, it is called a $\mathbb{S}^3$*-joint of* N. Boyer and

Tønnesen-Friedman have developed in [54] a formalism allowing them to give a precise description of a 2-dimensional subcone, called the **w**-cone, of the Reeb cone of the quotient. They have produced this way rich families of examples. Notably, in our joint work with them and Hongnian Huang, their expertise was used to study the Einstein–Hilbert functional on that two dimensional cone.

1) [BHLTF1, Example 5.7] exhibits a (positive) cscS metric for which $\mathrm{EH}''(\xi) < 0$. In particular, the bound $\lambda_1 > \frac{\mathrm{Scal}_g^T}{2n+1}$ does not hold for this variation.

2) [BHLTF1, Example 5.8] an example where the **w**-cone is the whole Reeb cone and there are two critical rays of EH which are not in the zero set of the transversal Futaki invariant (but, of course the total scalar curvature vanishes on these rays and they do not admit cscS representative). There is a third critical point of EH corresponding to a (negative) cscS ray.

3) [BHLTF1, Example 5.9] gives an example of cscS metrics with zero transverse scalar curvature for which EH is locally transversally convex even though it vanishes up to order six at the given point.

Part b) in Lemma 2.30 can be strengthened using the variational formulas for the transversal scalar curvature found by Boyer–Galicki–Simanca [48, Proposition 7.4]. Actually, for a transversal variation

$$t \mapsto \left(\mathcal{D}, J, \eta_t := \frac{\eta}{\eta(\xi_t)}\right)$$

given by a path $t \mapsto \xi_t \in \mathfrak{t}_+$, the computation is greatly simplified with Apostolov–Calderbank formula [10]. This formula reads, when denoting $f_t = \eta(\xi_t)$,

$$\mathrm{Scal}_{\eta_t}^T = f_t \mathrm{Scal}_\eta^T - 2(n+1)\Delta_B^\eta f_t - (n+1)(n+2) f_t^{-1} |df_t|_\eta^2. \tag{2.28}$$

Here, the CR-structure $(\mathcal{D}, J)$ is fixed and each Sasaki structure in the path is determined by its contact forms $\eta_t := f_t^{-1}\eta$. The basic Laplacian Δ_B^η, the pointwise norm of 1–forms $|\cdot|_\eta$ and the transversal scalar curvature appearing on the right hand side of the equation 2.28 are all computed with respect to the fixed Sasaki structure $(\mathcal{D}, J, \eta)$. Now, assume ξ_0 is the Reeb vector field of η, that is $f_0 \equiv 1$, and consider the variation of $\mathrm{Scal}_{\eta_t}^T$ at $t = 0$. Observing that $df_0 \equiv 0$, the variation of the third part of 2.28 vanishes at $t = 0$ and we are left with

$$\left(\frac{d}{dt}\mathrm{Scal}_{\eta_t}^T\right)_{t=0} = \dot{f}\mathrm{Scal}_\eta^T - 2(n+1)\Delta_B^\eta \dot{f}. \tag{2.29}$$

Note also that $\dot{f} = \left(\frac{d}{dt}\eta(\xi_t)\right)_{t=0}$ is a $\mathbb{T}$–Killing potential, i.e, a Killing potential with contact vector field lying in the Lie algebra of $\mathbb{T}$. From this observation and considering the projection of the transversal curvature on the space of Killing potential as a map from $\mathfrak{t}_+$ to $\mathfrak{t}$, we get the following condition.

Lemma 2.31. *[BHLTF3] Assume that $(N, \mathcal{D}, J, \eta)$ is a $\mathbb{T}$–invariant (compact connected) cscS manifold of transversal scalar curvature $S \neq 0$. If none of the non trivial $\mathbb{T}$–Killing potential is an eigenfunction of the basic Laplacian Δ_B^η for the eigenvalue $S/2(n+1)$ then the ray of $(N, \mathcal{D}, J, \eta)$ is isolated in the set of rays of Reeb vector fields with vanishing Futaki invariant in the Reeb cone $\mathfrak{t}_+$ of $\mathbb{T}$.*

2.1.4 Properness and rationality of the Einstein–Hilbert functional

With my collaborators Charles Boyer (University of New Mexico, United States) and Hongnian Huang (University of New Mexico, United States), developing on an idea of [168] we proved in [BHL] that the functionals $\mathbf{V}, \mathbf{S}, \mathrm{EH} : \mathfrak{t}_+ \to \mathbb{R}$ are proper and rationals. A part of these results, mainly on $\mathbf{V}$, can be obtained by a careful study of the Hilbert serie of some line bundle [168, 68]. Our main result can be stated as follows (or could be equally stated in terms of a transversal holomorphic compact manifold of Sasaki type (N, ξ, J^ξ) thanks to the invariance of the functionals involved).

Theorem 2.32 (Boyer–Huang–L. [BHL])**.** *Let $(N, \mathcal{D}, J)$ be a CR manifold Sasaki type, $\mathbb{T} \subset \mathrm{Aut}(N, \mathcal{D}, J)$ be a compact torus denote $\mathfrak{t}_+$ the Sasaki-Reeb cone of in $\mathfrak{t} := Lie\mathbb{T}$. Then the Volume, total transversal curvature and Einstein-Hilbert functionals $\mathbf{V}, \mathbf{S}, \mathrm{EH} : \mathfrak{t}_+ \to \mathbb{R}$ are rational functions in the variable $\xi \in \mathfrak{t}_+ \subset \mathfrak{t}$. Moreover if*

$$\lim_{s \to +\infty} \xi_s \in \partial \mathfrak{t}_+ \backslash \{0\}$$

then $\mathbf{V}(\xi_s) \to +\infty$, $\mathbf{S}(\xi_s) \to +\infty$ *and* $\mathrm{EH}(\xi_s) \to +\infty$ *when* $s \to +\infty$.

In the process of proving this Theorem we obtained a formula relating the total transverse scalar curvature of two commuting Reeb vector fields see (2.33) which as by product gives the following.

Corollary 2.33. *Let $(Y, J, \hat{\omega}, \xi)$ be a scalar-flat Kähler cone, for any compact torus $\mathbb{T} \subset \mathrm{Aut}(Y, J, \hat{\omega})^\xi$ the total scalar curvature is strictly positive on the Reeb cone in $\mathbb{T}$.*

Corollary 2.34. *Within a Sasaki cone there is at most one ray of vanishing transverse scalar curvature. Moreover, if there is one Sasaki structure with non negative transverse scalar curvature then the total transverse scalar curvature is non-negative on the whole Sasaki cone.*

The approach we took falls mainly into differential geometry, actually hamiltonian technics and uses the Atiyah–Bott–Berline–Vergne localisation formula, in the form given by Berline and Vergne [31]. We recall first briefly this theory (only for torus actions) in order to introduce notation, and then discuss the proof of Theorem 2.32.

2.1.4.1 Localisation in equivariant cohomology

Some classic references for this theory are [28, 31, 117].

Let M be a smooth manifold with an effective and smooth action of a compact torus $\mathbb{T}$, that is $\nu : \mathbb{T} \hookrightarrow \mathrm{diffeo}(M)$. We will mostly work with the induced infinitesimal action of $\mathfrak{t} := \mathrm{Lie}\, T$. For $a \in \mathfrak{t}$, we denote the induced vector field on M by $\underline{a} = \nu_*(a)$.

An equivariant form is a polynomial map $\psi : \mathfrak{t} \to \Omega^*(M)^{\mathbb{T}}$ where $\Omega^*(M)^{\mathbb{T}}$ is the graded complex of T-invariant forms. We denote $\Omega^*_{\mathbb{T}}(M)$ the set of equivariant forms over M. There is an appropriate notion of degree for these forms so that the equivariant differential $d_{\mathfrak{t}} : \Omega^*_{\mathbb{T}}(M) \longrightarrow \Omega^*_{\mathbb{T}}(M)$ which is defined by

$$a \mapsto (d_{\mathfrak{t}}\psi)_a := (d - \underline{a} \lrcorner)\psi_a$$

increases the degree of 1. An equivariant form $\psi \in \Omega^*_{\mathbb{T}}(M)$ is *equivariantly closed* if $d_{\mathfrak{t}}\psi \equiv 0$.

It is known that $d^2_{\mathfrak{t}} = 0$ and the equivariant cohomology can be defined as the cohomology of the associated chain complex:

$$H^*_{\mathbb{T}}(M) := \frac{\ker d_{\mathfrak{t}}}{\text{im } d_{\mathfrak{t}}}.$$

Remark 2.35. There is another definition of equivariant cohomology introduced by Atiyah and Bott [28] and valid for non connected compact Lie group G, which is the cohomology of the quotient $M_G := EG \times_G M$ where $EG \to BG$ is the universal principal G–bundle. In the case of connected compact Lie group both definitions coincide.

Let $\mathbf{Z} := \text{Fix}_M T$ be the fixed points set of T in M. A classical compact group action argument see eg. [117] implies that $\mathbf{Z}$ consists in a disjoint union of smooth submanifolds $\mathbf{Z} = \sqcup Z$ of even codimension. Given a connected component, say $Z \subset \mathbf{Z}$ of codimension $2m_Z$, the normal bundle $p : E^M_Z \longrightarrow Z$ bears a complex structure induced by the action [117] and inherits of a $\mathbb{T}$–equivariant bundle structure. The crucial point in this theory is that the equivariant Euler class, say $\bar{\mathbf{e}}(E^M_Z)$, of that bundle E^M_Z is invertible in $H^*_{\mathbb{T}}(Z)$.

Recall that the equivariant Euler (respectively Todd, Thom, Chern...) class of a rank r G–equivariant bundle $E \to M$ is the Euler (respectively Todd, Thom, Chern...) class of the corresponding rank r bundle $E_G \to M_G$. As a bundle over Z, there is an equivariant splitting into irreducible summands $E^M_Z = \oplus_j E^Z_j$ where $\mathbb{T}$ acts with weight $\mathbf{w}^Z_j \in \mathfrak{t}^*$ on the component E^Z_j. The condition for $\bar{\mathbf{e}}(E^M_Z)$ being invertible in $H^*_{\mathbb{T}}(Z)$ translate in our case as

$$\det \mathbf{w}^Z(a) := \Pi_j \langle \mathbf{w}^Z_j, a\rangle^{\text{rank}E^Z_j}$$

being non-zero for generic $a \in \mathfrak{t}$, which is the case if $\underline{a}$ vanishes only on the fixed points set of $\mathbb{T}$. When the action is effective, the set of generic elements of $\mathfrak{t}$ is open and dense.

Following the conventions of [168] and [94], we have

$$\bar{\mathbf{e}}(E^M_Z)_a = (2\pi)^{-2m} \det \mathbf{w}^Z(a) \prod_j \det\left(I + \frac{i\Omega_j}{\langle \mathbf{w}^Z_j, a\rangle}\right)$$

With these notations the *equivariant Localization Formula* reads.

Theorem 2.36. *[Atiyah–Bott [28], Berline–Vergne [30]] Let $\psi \in \Omega^*_{\mathbb{T}}(M)$ be an equivariently closed form with M compact. For any generic $a \in \mathfrak{t}$ we have*

$$\int_M \psi_a = \sum_Z \int_Z \frac{\iota^*_Z \psi_a}{\bar{\mathbf{e}}_a(E^M_Z)}. \tag{2.30}$$

$\bar{\mathbf{e}}(E^M_Z)^{-1}$ is the inverse of the equivariant Euler class of E^M_Z in $H^*_{\mathbb{T}}(Z)$.

The Equivariant Localization Formula has been extended to orbifolds [171] (but we need to multiply the Euler class of every normal E^M_Z bundle by the order of the isotropy group of Z) and the approach of Berline–Vergne [30] extends straightforwardly for non compact manifold M as soon as for each $a \in \mathfrak{t}$, ψ_a is compactly supported or more generally integrable on M.

2.1.4.2 Localisation on the total space of the polarised complex cone

Sketch of proof of Theorem 2.32. Getting back to the context of Theorem 2.32, $(N, \mathcal{D}, J)$ is a CR manifold Sasaki type, $\mathbb{T} \subset \mathrm{Aut}(N, \mathcal{D}, J)$ a compact torus with Sasaki-Reeb cone $\mathfrak{t}_+$ lying in $\mathfrak{t} := \mathrm{Lie}\mathbb{T}$. We pick a quasi-regular Reeb vector field $\xi_o \in \mathfrak{t}_+$ (the set of such is dense since $\mathfrak{t}_+$ is open and $\mathbb{T}$ is a compact torus) and consider the associated Sasaki structure $(\mathcal{D}, J, \xi_o)$ with contact form η_o see (2.3). We will also use the radial potential $r \in \mathcal{R}_\xi(Y, J)^{\mathbb{T}}$ associated to the Kähler structure induced on the associated complex polarised cone (Y, J, ξ) as well as Sasaki-Kähler orbifold quotient (M, J, ω_o) with an ample orbiline bundle $L \to M$ see §2.1.2.2 associated to $(\mathcal{D}, J, \xi_o)$.

We work on the total space $X = L^{-1}$, denote $\pi : X \to M$ the bundle map. Note that the square of the radial potential $r^2 : X \to \mathbb{R}$ is smooth, it is the square of a fiberwise distance to the zero section. Also, while dr and $d^c r$ are not individually defined on X, the product $dr \wedge d^c r$ is (as a combination of $dd^c r^2$ and $r^2 \pi^* \rho_{\omega_o}$) and its pull back to the zero section vanishes. In [BHL], we modified the 2–form $dd^c r^2$ to apply the idea of [168], we were working on a bundle constructed from a reduction but it can be done on X.

Given $\zeta \in \mathfrak{t}_+$, there is a unique associated contact form η^ζ on $(N, \mathcal{D}, J)$ that we extend as a $-J\xi$-invariant 1-form on Y and $\mu^\zeta : Y \to \mathfrak{t}^*$ as $\langle \mu^\zeta, a \rangle = \eta^\zeta(\underline{a})$. We considered the 2-form $\Omega_\zeta := \frac{1}{2} d(r^2 \eta^\zeta)$ and checked that

$$\Omega_\zeta - \frac{r^2 \mu^\zeta}{2} \tag{2.31}$$

is $\mathbb{T}$-equivariantly closed form on Y, extends continuously on X and vanishes when pulled back to the zero section. In particular, $r^2/2$ is a Ω_ζ–hamiltonian for ζ. Then, an elementary integration rule gives

$$\mathbf{V}(\zeta) = \frac{1}{(n+1)!} \int_Y e^{-r^2/2} \Omega_\zeta^{n+1} = \int_X \exp\left(\Omega_\zeta - \frac{r^2}{2}\right).$$

The r.h.s integrant is the exponential of the equivariantly closed form (2.31) evaluated at ζ. The exponential of an equivariantly closed form being closed itself we can use the Equivariant Localisation Formula (the orbifold version) to get

$$\mathbf{V}(\zeta) = \sum_Z \frac{1}{d_Z} \int_Z \frac{\iota_Z^* \exp\left(\Omega_\zeta - \frac{r^2}{2}\right)}{\bar{\mathbf{e}}_a(E_Z^X)} = \sum_Z \int_Z \frac{1}{\bar{\mathbf{e}}_\zeta(E_Z^X)} \tag{2.32}$$

because $Z \subset M = \{r = 0\}$. Here and below d_Z is the order of the isotropy group of $Z \subset X$.

To get a similar formula for $\mathbf{S}$ we had to work a little harder. The expression of the transversal Ricci form ρ_ζ^T of $(\mathcal{D}, J, \zeta)$ in terms of ρ_ξ^T is not so straightforward and ρ_ζ^T is not ξ–basic. However, since the difference $\rho_\zeta^T - \rho_\xi^T$ is exact we were able[8] to prove that

$$\mathbf{S}(\zeta) = \frac{1}{n!} \int_N \frac{\mathrm{Scal}_\xi^T}{\eta_o(\zeta)^n} \eta_o \wedge (d\eta_o)^n + \frac{n+1}{2} \int_N \frac{|d\eta_o(\zeta)|_{g_o}^2}{\eta_o(\zeta)^{n+2}} \eta_o \wedge (d\eta_o)^n \tag{2.33}$$

[8] That was before [10] gives an exact formula between Scal_ξ^T and Scal_ζ^T

where η_o is the contact form of $(N, \mathcal{D}, J, \xi)$ and g_o its riemannian metric.

With formula (2.33) in hands, $\mathbf{S}(\xi)$ coincides with the intersection product of an integrable equivariantly closed form (evaluated at ζ) on the cone. Namely,

$$\mathbf{S}(\zeta) = \int_Y (\rho_o^T - \frac{1}{2}\Delta^{g_o}\eta_o(\zeta)) \exp\left(\Omega_\zeta - \frac{r^2}{2}\right) \tag{2.34}$$

where ρ_o^T is the transversal Ricci form of $(N, \mathcal{D}, J, \xi)$ thus can be identified with the pull back of the Ricci form of ω_o to X. Note that $\Delta^{g_o}\eta_o(\zeta)$ is also ξ-basic, well-defined on M and X. Moreover, on a fixed component $Z \subset \mathbf{Z}$, where the weights (with multiplicity) of the $\mathbb{T}$ action on the normal bundle of Z in M, are $\mathbf{w}_1, \dots, \mathbf{w}_{n_Z}$, we have $\Delta^{g_o}\eta_o(\zeta) = -2\sum_{j=1}^{n_j}\langle \mathbf{w}_j, \zeta\rangle$. Thus, using $\iota_M^*\Omega_\zeta = 0$, we get

$$\mathbf{S}(\zeta) = \sum_Z \frac{1}{d_Z}\int_Z \frac{\iota_Z^*(\rho_o^T) + \sum_{j=1}^{n_j}\langle \mathbf{w}_j, \zeta\rangle}{\bar{\mathbf{e}}_\zeta(E_Z^X)} \tag{2.35}$$

On each summand $Z \subset \mathbf{Z}$, using the decomposition $E_Z^X = E_Z^M \oplus E_0^Z$ and the notation of §2.2, with $\mathbf{w}_0 \in \mathfrak{t}^*$ the weight on E_0^Z and n_Z the codimension of Z in M, we can rewrite the formulas above as

$$\mathbf{V}(\zeta) = \sum_Z \frac{1}{d_Z}\int_Z \frac{(2\pi)^{n_Z+1}}{\det \mathbf{w}^Z(\zeta)} \frac{1}{\bar{\mathbf{e}}(E_Z^M)} \sum_{m\geq 0}\left(\frac{c_1(L)}{\langle \mathbf{w}_0, \zeta\rangle}\right)^m. \tag{2.36}$$

Here this uses $E_0^Z \simeq \iota_Z^* L^{-1}$ and the formula $(1-x)^{-1} = \sum_{m\geq 0} x^m$. We get a similar formula for $\mathbf{S}$, namely

$$\mathbf{S}(\zeta) = \sum_Z \frac{1}{d_Z}\int_Z \frac{(2\pi)^{n_Z+1}}{\det \mathbf{w}^Z(\zeta)} \frac{c_1(M) + \sum_{j=1}^{n_j}\langle \mathbf{w}_j, \zeta\rangle}{\bar{\mathbf{e}}(E_Z^M)} \sum_{m\geq 0}\left(\frac{c_1(L)}{\langle \mathbf{w}_0, \zeta\rangle}\right)^m. \tag{2.37}$$

Now, to prove the properness claim of Theorem 2.32, let $\zeta \to a \in \partial\mathfrak{t}_+$ away from the apex $0 \in \mathfrak{t}_+$. Write $\zeta_s := (1-s)\xi - sa$ for $s \in [0,1]$, by convexity $\zeta_s \in \mathfrak{t}_+$ when $s < 1$ and there exists $p \in N$ such that $\eta_o(\underline{a})_p = \eta_o(\zeta_1)_p = 0$. The crucial input is that the function $\eta_o(\zeta_s)$ is a Morse-Bott function on N (and on M) of even index (for any $s \in \mathbb{R}$). By an argument of Guillemin–Sternberg [116], $\eta_o(\zeta_s)$ has a unique minimum in the sense that there is a unique connected component $\tilde{Z} \subset \mathrm{crit}(\eta_o(\zeta_s)) \subset N$ such that $\tilde{Z}$ is a local minimum, which is then a global minimum. Picking s close enough to 1, there is no other part of N on which $\eta_o(\zeta_s)$ tends to 0 faster than it does on $\tilde{Z}$. The argument holds on M as well, let $Z = \pi_\xi(\tilde{Z})$ where $\pi_\xi : N \to M$ is the quotient by $\mathbb{S}^1_\xi$. The leading terms in (2.35) and (2.32) are the integrals over that same minimal locus Z.

We can modify ζ_s slightly so that $[\zeta_s] = sa \in \mathfrak{t}/\mathbb{R}\xi$ is generic. Then

$$\langle \mathbf{w}_j^Z, \zeta\rangle = s\langle \mathbf{w}_j^Z, a\rangle > 0$$

for $j = 1, \dots, n_Z$ because Z is a minimum.[9] On the other hand[10]

$$\langle \mathbf{w}_0^Z, \zeta_s\rangle = \eta(\zeta_s)_p = (1-s)$$

[9] We use the localisation formula of [168], so we need to stick with their convention.

[10] Here the weights are computed with respect to the convention of [168], in [BHL], the reduction we took made it clear. Another way to get convinced that this weight has to be positive with their convention is that this is also a minimum in the fiber direction.

tends to 0. Thus the leading term will be the one with the highest power of $(\langle \mathbf{w}_0^Z, \zeta_s\rangle)^{-1}$. For the volume it is

$$\frac{1}{d_Z}\int_Z \frac{(2\pi)^{n_Z+1}}{\det \mathbf{w}^Z(\zeta)}\left(\frac{c_1(L)}{\langle \mathbf{w}_0,\zeta\rangle}\right)^{n-n_Z}. \tag{2.38}$$

and the leading term of the total scalar curvature is

$$\frac{1}{d_Z}\int_Z \frac{(2\pi)^{n_Z+1}\sum_{j=1}^{n_j}\langle \mathbf{w}_j,\zeta\rangle}{\det \mathbf{w}^Z(\zeta)}\left(\frac{c_1(L)}{\langle \mathbf{w}_0,\zeta\rangle}\right)^{n-n_Z} \tag{2.39}$$

All the factors are positive and comparing the corresponding powers in the expression of EH we get Theorem 2.32.

□

2.2 Localisation on test configurations

2.2.1 The Donaldson–Futaki invariant as an intersection of equivariant forms

Following [78, 185], we call a (regular) *test configuration* for a Kähler manifold (M,ω) a following set of data

1. A smooth compact Kähler manifold $(\mathcal{M},\Omega)$;
2. $\mathbb{C}^*$–action $\nu:\mathbb{C}^*\hookrightarrow \mathrm{Aut}(\mathcal{M})$ such that $[\nu(t)^*\Omega]=[\Omega]$;
3. a $\mathbb{C}^*$–equivariant surjective map $\pi:\mathcal{M}\longrightarrow\mathbb{P}^1$ for the standard action on $\mathbb{P}^1:=\mathbb{C}\cup\{+\infty\}$;
4. a $\mathbb{C}^*$–equivariant biholomorphism

$$\psi:\mathcal{M}^*:=\mathcal{M}\backslash\pi^{-1}(0)\xrightarrow{\sim} M\times\mathbb{P}^1\backslash\{0\}$$

for the trivial action on M times the restriction of the standard action on $\mathbb{P}^1$ and $\mathrm{pr}_2(\psi(x))=\pi(x)$.

Moreover, for $t\in\mathbb{P}^1$, we denote $M_t:=\pi^{-1}(t)$ and the inclusion $\iota_t:M_t\hookrightarrow\mathcal{M}$. We have, for $t\neq 0$, $M_t\overset{\psi_t}{\simeq} M$, that is $\psi_t=\mathrm{pr}_1\circ\psi\circ\iota_t$. Finally we denote $\Omega_t=\iota_t^*\Omega$ and assume that

$$[\Omega_t]=[\psi_t^*\omega]. \tag{2.40}$$

In this context, the *Donaldson–Futaki* invariant of $(\mathcal{M},\Omega)$, when $\dim_{\mathbb{C}} M = n$, can be defined to be:

$$\mathbf{DF}(\mathcal{M},\Omega):=\frac{n}{n+1}\overline{c}_{[\omega]}\int_{\mathcal{M}}\Omega^{n+1}-\int_{\mathcal{M}}(\rho-\pi^*\rho^{FS})\wedge\Omega^n \tag{2.41}$$

where $\overline{c}_{[\omega]}=\int_M\rho^\omega\wedge\omega^{n-1}/\int_M\omega^n$, ρ^ω is the Ricci curvature of ω and ρ^Ω is any representative of $2\pi c_1(M)$.

The quantity (2.41) coincides with the invariant introduced by Donaldson in [82] when the test configuration is the compactification of a polarised test configuration as shown in Odaka [175] and Wang [201], see also [82, p.315] and Li–Xu [164, Proposition 6].

Remark 2.37. In the Kähler setting [78, 185] the test configurations over $(M,[\omega])$ considered are more generally (compact) Kähler analytical spaces. Then the Donadson-Futaki invariant as (2.41) is defined for a smooth resolution over it. It is proved then that the definition is independent of the resolution and that to test K-semistability (i.e that $\mathbf{DF}(\mathcal{M},[\Omega])\geq 0$ for all test configurations over $(M,[\omega])$) and even uniform K-stability (see eg.[78]) it is enough to consider smooth test configurations, see [78, Proposition 2.23]. However, on the resolution, the pull back of Ω is not positive everywhere and thus does not necessarily belong to a Kähler class. Note that (2.41) still make sense for more general $(1,1)$-forms Ω on M as soon as, as above, ρ is defined as any representative of $2\pi c_1(M)$.

Clearly, the quantity $\mathbf{DF}(\mathcal{M},\Omega)$ does not depend on the representative $\Omega\in[\Omega]\in H^{1,1}(\mathcal{M},\mathbb{R})$ and we assume, without loss of generality, Ω is $\mathbb{S}^1$-invariant Kähler for the standard inclusion $\mathbb{S}^1\subset\mathbb{C}^*$.

We denote $V:=\nu_*(\partial_\theta)$ the real holomorphic vector field on $\mathcal{M}$ induced from the generator of $\mathbb{S}^1$ via the action. By assumption this is also a Killing vector field with zeros and thus it is a Hamiltonian vector field on $(\mathcal{M},\Omega)$. We pick a Hamiltonian function $\mu:\mathcal{M}\to\mathbb{R}$ that is

$$-d\mu=\Omega(V,\cdot).$$

We get two $\mathbb{S}^1$-equivariantly closed forms[11]

$$\begin{aligned}\mathcal{A}_\Omega&:=(\Omega-\mu)\\ \mathcal{B}_\Omega&:=\frac{n\overline{c}}{n+1}(\Omega-\mu)-\left(\rho^\Omega-\frac{1}{2}\Delta^\Omega\mu\right)+(\pi^*\omega_{FS}-\pi^*\mu_{FS})\end{aligned}\tag{2.42}$$

where μ_{FS} is a Hamiltonian for the standard $\mathbb{S}^1$ action on $(\mathbb{P}^1,\omega_{FS})$. Indeed, by Bochner formula we have

$$-\frac{d}{2}\Delta^\Omega\mu=\rho(V,\cdot).$$

Observe that the equivariant classes $[\mathcal{A}_\Omega]$, $[\mathcal{B}_\Omega]\in H^2_{\mathbb{S}^1}(\mathcal{M})$ are independent of the chosen representative in $[\Omega]\in H^{(1,1)}(\mathcal{M},\mathbb{R})$. Indeed, $\mathcal{A}_{\Omega+dd^c\varphi}=\mathcal{A}_\Omega+d_{\mathbb{S}^1}d^c\varphi$ because $\mu+d^c\varphi(\underline{a})$ is a momentum map for $\Omega+dd^c\varphi$.

Remark 2.38. When Ω is degenerated on subvarieties of $\mathcal{M}$ which itself is smooth (see Remark 2.37), the Ricci form ρ^Ω and the Laplacian do not make sense. However, we can pick any representative $\rho\in 2\pi c_1(\mathcal{M})$, by the Calabi-Yau Theorem [206], there exists a metric $\widetilde{\Omega}$ on $\mathcal{M}$ such that $\rho^{\widetilde{\Omega}}=\rho$ and thus $\rho(V,\cdot)=-\frac{d}{2}\Delta^{\widetilde{\Omega}}\mu$ is exact. Then the equivariant form

$$\mathcal{B}_\Omega=\frac{n\overline{c}}{n+1}(\Omega-\mu)-\left(\rho-\frac{1}{2}\Delta^{\widetilde{\Omega}}\mu\right)+(\pi^*\omega_{FS}-\pi^*\mu_{FS})$$

is well-defined and, up to an addition by a equivariantly exact form, does not depend on the representative $\Omega\in[\Omega]\in H^{(1,1)}(\mathcal{M},\mathbb{R})$ and $\rho\in 2\pi c_1(\mathcal{M})$.

Since the integration picks up only the $2(n+1)$–degree terms we have the following.

Lemma 2.39. *[L5]* $\mathbf{DF}(\mathcal{M},\Omega)=[\mathcal{A}_\Omega]^n\cup[\mathcal{B}_\Omega]([\mathcal{M}])$

A similar formula was found independently by Inoue [129].

[11] The notion is recalled in §2.1.4.1.

2.2.2 Localisation of the Donaldson–Futaki invariant

Since $\pi : \mathcal{M} \to \mathbb{P}^1$ is equivariant the fixed points set of the $\mathbb{S}^1$ action on $\mathcal{M}$, $\mathrm{Fix}_{\mathbb{S}^1}\mathcal{M}$, lies in $M_0 \sqcup M_\infty$. Following a classical argument of Riemannian geometry, $\mathrm{Fix}_{\mathbb{S}^1}\mathcal{M}$ is a disjoint union of smooth submanifolds, $\mathrm{Fix}_{\mathbb{S}^1}\mathcal{M} = M_\infty \sqcup (\sqcup_l Z_l)$. One of the component is M_∞ because of the condition (4) defining a test configuration. The remaining components lie in M_0. We denote then

$$\mathrm{Fix}_{\mathbb{S}^1}\mathcal{M} = (\sqcup Z) \sqcup M_\infty$$

where Z denotes a generic component of $\mathrm{Fix}_{\mathbb{S}^1}\mathcal{M}$ in M_0.

Applying Theorem 2.36 on the expression found in Lemma 2.39 above, we obtain the next proposition where the following notation is used: given any subset $S \subset \mathcal{M}$ we denote the inclusion $\iota_S : S \to \mathcal{M}$ and, for any form/function/tensor.. α on $\mathcal{M}$ the pull-back to S is denoted $\alpha_S := \iota_S^* \alpha$.

Proposition 2.40. *[L5] Let $(\mathcal{M}, [\Omega], \nu, \pi, \psi)$ be a regular (compact) test configuration over $(M^n, [\omega])$. We pick $\Omega \in [\Omega]$ a $\mathbb{S}^1$–invariant Kähler metric. Let $V = \nu_*(\partial\theta)$ be the vector field induced by the underlying $\mathbb{S}^1$–action, $\mu : \mathcal{M} \to \mathbb{R}$ be any hamiltonian function for V. The Donaldson–Futaki invariant of $(\mathcal{M}, [\Omega], \nu, \pi, \psi)$ is*

$$\begin{aligned}\frac{\mathbf{DF}(\mathcal{M},\Omega)}{n!} =& \sum_Z \int_Z \frac{n\bar{c}_{[\omega]}(\Omega_Z - \mu_Z)^{n+1}}{(n+1)!\bar{\mathbf{e}}(E_Z^{\mathcal{M}})(V)} \\ &- \sum_Z \int_Z \frac{(\rho_Z^\Omega + \sum_{i=0}^{n-n_Z}\langle \mathbf{w}_i, \partial\theta\rangle) \wedge (\Omega_Z - \mu_Z)^n}{n!\bar{\mathbf{e}}(E_Z^{\mathcal{M}})(V)} \\ &+ \sum_Z \int_Z \frac{(\Omega_Z - \mu_Z)^n}{n!\bar{\mathbf{e}}(E_Z^{\mathcal{M}})(V)}.\end{aligned} \tag{2.43}$$

where Z runs into the set of connected components of the fixed points set of the $\mathbb{S}^1$–action lying in the central fiber $M_0 = \pi^{-1}(0)$. The equivariant vector bundle $N_Z^{\mathcal{M}}$ over Z is the normal bundle of Z in $\mathcal{M}$ and $\mathbf{w}_0^Z, \ldots, \mathbf{w}_{n-n_Z}^Z \in (Lie\ \mathbb{S}^1)^*$ are the weights (with multiplicity) of the induced action of $\mathbb{S}^1$ on the normal bundle $E_Z^{\mathcal{M}}$.

The proof of the last proposition boils down to show that the contribution of M_∞ in the equivariant localisation of $\mathcal{A}_\Omega^n \cup \mathcal{B}_\Omega$ vanishes identically. Moreover, $\sum_{i=0}^{n-n_Z}\langle \mathbf{w}_i, \partial\theta\rangle$ is the value of $-\frac{1}{2}\Delta^\Omega\mu$ at any point of Z. In particular, (2.43) holds as well when Ω is degenerated and M smooth, and ρ^Ω is replaced in $\mathcal{B}_\Omega$ by any representative of $2\pi c_1(\mathcal{M})$, see Remark 2.38.

Comment 2.41. It is not hard to show that there exists a value $r \in \mathbb{R}$ such that $\mu^{-1}(r) \subset \mathcal{M}\backslash(M_0 \sqcup M_\infty)$. Moreover $\mu^{-1}(r)/\mathbb{S}^1 \simeq M$, actually, $\mathcal{M}\backslash(M_0 \sqcup M_\infty)/\mathbb{C}^* \simeq M$ as complex manifolds. Therefore the Kirwan map from $\mathbb{S}^1$-equivariant cohomology on $\mathcal{M}$ to de Rham cohomology of M, denoted $\kappa : H^*_{\mathbb{S}^1}(\mathcal{M}) \to H^*_{dR}(\mathcal{M})$, is sujective [139]. Following an idea of Kalkman [135] we can localise on $\mathcal{M}_- := \{p \in \mathcal{M} \,|\, \mu(p) > r\}$, any (closed) $2n$–form α on M. That is there exists an equivariantly closed form $\tilde{\alpha}$ on $\mathcal{M}$ such that

$$\int_M \alpha = \sum_Z \int_Z \frac{\tilde{\alpha}_Z}{\bar{\mathbf{e}}(E_Z^{\mathcal{M}})(V)}.$$

Thus, theoritically, we could compare the classical functionals (Mabuchi, Aubin, ...) on the space of Kähler potentials with the value of the Donaldson–Futaki invariant given by (2.43). However, given such α, finding a representative for $\tilde{\alpha} \in \kappa^{-1}([\alpha])$ is not straightforward.

Using Proposition 2.40 and [97, Proposition 6] it is clear that **DF** is an equivariant class on the central fiber M_0. In the case where the latter is smooth, orbifold type or more generally irreducible we should find back the classical Futaki invariant of the induced $\mathbb{S}^1$-action on M_0. Donaldson gave in [82] a very simple proof of this fact in the polarised case when the central fiber is smooth using the (equivariant) Hirzebruch–Riemann–Roch formula and the definition of **DF** as a coefficient of the expansion of the normalized weight of the induced action on $H^0(\mathcal{L}^k)$. As an application of Proposition 2.40, I showed that it was indeed the case when the central fiber inherits of an orbifold structure from the inclusion in $\mathcal{M}$.

Theorem 2.42 (Donaldson [82], L. [L5])**.** *Let $(\mathcal{M}^{n+1}, [\Omega])$ be a regular (compact) test configuration over $(M^n, [\omega])$ with $\pi : \mathcal{M} \to \mathbb{P}^1$ and the $\mathbb{C}^*$–action $\nu : \mathbb{C}^* \hookrightarrow Aut(\mathcal{M})$. Let $V = \nu_*(\partial\theta)$ be the vector field induced by the underlying S^1–action and $\mu : \mathcal{M} \to \mathbb{R}$ be any Hamiltonian function for V. Assume that the central fiber $M_0 := \pi^{-1}(0)$ inherits a Kähler orbifold structure from the inclusion $\iota_0 : M_0 \hookrightarrow \mathcal{M}^{n+1}$. Then*

$$\frac{\mathbf{DF}(\mathcal{M}, [\Omega])}{n!} = -\pi \mathrm{Fut}_{(M_0, [\Omega_0])}(J_0 V) \tag{2.44}$$

where $\Omega_0 = \iota_{M_0}^* \Omega$ is the pull-back on M_0.

Test configurations with smooth central fiber but with π not being a submersion are known to exist see [23, 196]. Examples with π being a submersion also fulfill the hypothesis of the Theorem, such examples include the so-called "product tests configuration".

The proof of Theorem 2.42 is very direct when the central fiber is smooth. Indeed, in that case the normal bundle $E_0 := T\mathcal{M}/TM_0$ provides a splitting $E_Z^{\mathcal{M}} = \iota_Z^* E_0 \oplus E_Z^{M_0}$ and we can easily clear out the expression of the Futaki invariant $-\pi Fut_{(M_0,[\Omega_0])}(J_0 V)$ localized on $\mathrm{Fix}_{\mathbb{S}^1}(M_0) = \mathrm{Fix}_{\mathbb{S}^1}(\mathcal{M}) \cap M_0$. In the orbifold case, we need to work a little bit more. Each connected component Z of the S^1–fixed point set is a smooth Kähler submanifold of $(\mathcal{M}, \Omega)$. Since connected (smooth) compact metric manifold cannot be isometric to a metric orbifold with more than one type of isotropy group, see [143, Lemma 2.1], there exists a single orbifold uniformising chart $(\widetilde{U}_Z, \Gamma_Z)$ for each component Z. That is $\widetilde{U}_Z \subset \mathbb{C}^n$ is open and $\Gamma_Z \subset \mathrm{Gl}(\mathbb{C}^n)$ is a finite subgroup, and there is a Γ_Z–invariant surjective holomorphic map $\mathbf{q}_Z : \widetilde{U}_Z \to U_Z \simeq \widetilde{U}_Z/\Gamma_Z$ where U_Z is a neighbourhood of Z in M_0. Also $Z \simeq \mathbf{q}_Z^{-1}(Z)$ is the only fixed points of Γ_Z. By [157], there is a group extension $\Gamma_Z \hookrightarrow H_Z \to \mathbb{S}^1$ and H_Z acts on $\widetilde{U}_Z$ covering the action of $\mathbb{S}^1$ on U_Z. The connected component of the identity of H_Z is a circle and the normal bundle $E_Z^{\widetilde{U}_Z}$ is then a S^1–equivariant bundle over Z with weights $\tilde{\mathbf{w}}_1, \ldots, \tilde{\mathbf{w}}_{\ell_Z} \in (\mathrm{Lie}S^1)^* \simeq \mathbb{R}$ (the weights might not lie in the (same) weights lattice of our original representation of S^1 but it is not a problem in terms of localisation formula [171]).

With this notation prove Theorem 2.42 it is sufficient to show that these three following conditions hold on each connected component Z of the fixed point set:

a) $\iota_Z^*[\rho^\Omega] = \iota_Z^*[\rho^{\Omega_0}] \in H^2_{dR}(Z)$;
b) $\sum_{i=0}^{\ell_Z} \langle \mathbf{w}_i^Z, \partial\theta \rangle = 1 + \sum_{i=1}^{\ell_Z} \langle \tilde{\mathbf{w}}_i^Z, \partial\theta \rangle$;
c) $\bar{\mathbf{e}}(E_Z^{\mathcal{M}}) = d_Z \bar{\mathbf{e}}(E_Z^{M_0})/2\pi$;

here d_Z is the order of the isotropy group of a generic point in Z. The relation a) follows directly from the irreducibility of the central fiber. I proved the second equality b) using the characterisation of each side as the value at $z \in Z$ of the Laplacian of a Hamiltonian function of the $\mathbb{S}^1$ on $\widetilde{U}_Z$ and on $\mathcal{M}$. For the last relation I compared directly the equivariant Thom forms of $E_Z^{\mathcal{M}}$ and $E_Z^{M_0}$.

Chapter 3

Transversal Kähler structures, Levi reductions and weighted extremal Kähler metrics

Typically the Kähler reduction of an extremal Kähler metric is not extremal itself [55, 108]. In particular the so-called Guillemin (Kähler) metric, which exists on any compact toric orbifold and is obtained by the Kähler reduction of the flat complex space, is extremal only on weighted projective spaces [1, 115]. However, the Kähler reduction of a scalar-flat Kähler cone, by the $\mathbb{S}^1$-action induced by the Reeb vector field (in the quasi-regular case) is cscK and more generally, the Kähler reduction of a compatible rigid extremal Kähler toric fibrations by the torus is cscK [13]. These are are very special cases and to get extremal Kähler metric involved more generally one needs to include some other kind of special metrics, see eg [22].

With these facts in minds, with my collaborators [ACGL1, ACGL2], we have proposed a slight modification of the Kähler reduction process incorporating sasakian deformation technics. We have called this construction *Levi-Kähler reduction*. We have worked essentially only in a toric setting, extending the notion of Sasaki toric manifold to higher co-dimension CR structure and developed the theory of toric contact manifold of arbitrary co-dimension. I explain our results below §3.1.

Our initial hope in introducing Levi-Kähler reduction was to build new extremal Kähler metrics. To be honest, in that respect, it was not a big success (although see Proposition 3.19 below) but we have found some unexpected *weighted extremal Kähler metrics* along the way, see Theorem 3.18. At that time, these special Kähler metrics were studied in relation with Einstein–Maxwell structures and conformally Einstein manifolds [20]. Since then Lahdili has developed a very general theory of *weighted cscK metrics* see [142], including the type of weighted extremal metric we had found in [ACGL2], those of [20], the μ-cscK metric introduced by Inoue in relation with Kähler–Ricci solitons [127] and re-

cently, in [19], the v–soliton of Berman–Nyström [35].

Starting from our discovery (i.e, Theorem 3.18 below or [ACGL2, Theorem 6]), my collaborators V. Apostolov and D. Calderbank exhibited in [10] the exact relation between extremal Sasaki structure and weighted extremal Kähler metrics in the Sasaki setting.

Precisely, with the notation of 2.1.1, they showed that given two commuting Reeb vector fields $\xi, \xi_o \in \mathfrak{t}_+$ on a Sasaki-CR manifold $(N, \mathcal{D}, J)$, with ξ_o quasi-regular and Sasaki-Kähler quotient (M, J, ω_o), the transversal scalar curvature of $(N, \mathcal{D}, J, \xi)$ is some $([\xi], \kappa)$–weighted scalar curvature of (M, J, ω_o) where the weights $([\xi], \kappa)$ are determined by $(N, \mathcal{D}, J, \xi)$ as explained in §3.2.

In a subsequent paper [ACL], with V.Apostolov and D.Calderbank, we have extended the relation they had found to

Proposition 3.1. *[ACL] There is an extremal Sasaki metric in the class $\mathcal{S}(J^\xi, \xi)$ if and only there is a $([\xi], \kappa)$–weighted Kähler metric in $(M, [\omega_o])$.*

Using this we were able to prove that when ξ_o is regular (thus M is smooth and we can use Darvas–Rubinstein theory [73]), the existence of an extremal Sasaki metric in the class $\mathcal{S}(J^\xi, \xi)$ implies (relative) $([\xi], \kappa)$ weighted K-stability of $(M, J, [\omega_o])$ with respect to a maximal torus and smooth equivariant test configurations with reduced central fiber. We also compared weighted K-stability to the K-stability of the corresponding polarised affine cone (introduced by Collins–Székelyhidi in [68]), and proved that they agree on the class of test configurations we consider. In order to do that, we have expressed the Collins–Székelyhidi numerical invariant as an intersection product formula on the total space of the Sasaki test configuration.

There is an increasing interest in weighted cscK metrics see eg [162] and recently, building on the proof of [ACL, Theorem 1] (see Theorem 3.32 below), Apostolov–Jubert[1]–Lahdili have proved more generally that compact weighted extremal Kähler manifold has coercive weighted Mabuchi energy with respect to a maximal complex torus in the reduced group of complex automorphisms. They also have showed that this coercivity property implies weighted K-polystability on any non-product equivariant smooth Kähler test configuration with reduced central fiber extending our result. This extends and unifies many results in the literature.

3.1 Levi–Kähler reduction

Underlying a Sasaki manifold there is an odd dimensional manifold N^{2n+1} whose tangent splits

$$TN = \mathcal{D} \oplus \mathbb{R}\xi \tag{3.1}$$

where $\xi \in \Gamma(TN)$ is a nowhere vanishing vector field, the Reeb vector field, and $\mathcal{D} \subset TN$ is a distribution equipped with a complex (or CR) structure $J \in \mathrm{End}(\mathcal{D})$. In view of example 2.4, the interest of Sasaki geometry for Kähler geometers lies in the transversal Kähler geometry of such structure, which refers

[1] Simon Jubert is a PhD student that I co-supervise with Vestislav Apostolov. Simon is working on some class of weighted toric extremal Kähler metrics.

to the Kähler structure induced on the local leaves space of Reeb foliation. As discussed above, it might be wise to vary the splitting (3.1) in the search of interesting (extremal,...) transversal Kähler structures.

Our thesis in [ACGL1, ACGL2] is that these ideas need not be limited to 1-dimensional foliations. Indeed, any Sasaki manifold has an underlying codimension one CR structure, whereas CR manifolds arise naturally in arbitrary codimension [40]. This prompted us to introduce transverse "Reeb foliations" $\mathcal{F}$ on arbitrary CR manifolds $(N^{2n+k}, \mathcal{D}, J)$, $\operatorname{rank}(\mathcal{D}) = 2n$. Such objects are particular examples of polarised CR manifold in the sense of Meersseman [172]. Our focus is on the *transversal Kähler geometry* of $(\mathcal{D}, J)$, i.e., the Kähler structures induced on the space of (local) leaves of the Reeb foliation $\mathcal{F}$ by *contact forms*. Whereas in codimension one, the exterior derivative of the contact form equips the horizontal distribution $\mathcal{D}$ with a nondegenerate 2-form (which, together with the complex structure J on $\mathcal{D}$, defines the horizontal Kähler structure on $\mathcal{D}$), in higher codimension, the non-integrability of $\mathcal{D}$ is measured by a 2-form on $\mathcal{D}$ with values in $TN/\mathcal{D}$, called the *Levi form* $Lv_{\mathcal{D}}$, namely

$$L_{\mathcal{D}}(X, Y) = -q_{\mathcal{D}}([X, Y]) \tag{3.2}$$

where $q_{\mathcal{D}} \colon TN \to TN/\mathcal{D}$ is the quotient map.

In order to construct a Kähler metric on the leaf space of the Reeb foliation, we consider a section $\eta \in \Gamma(\mathcal{D}^0)$ where the annihilator $\mathcal{D}^0 \subset T^*N$ is identified with the dual of the bundle $TN/\mathcal{D}$ over N. Then[2] $\langle \eta, L_{\mathcal{D}} \rangle = d\eta|_{\mathcal{D}}$ and η is *positive* if $h^{\eta}_{\mathcal{D}} := \langle \eta, L_{\mathcal{D}} \rangle(\cdot, J\cdot)$ is positive definite on $\mathcal{D}$. In that case, $(\mathcal{D}, J, h^{\eta}_{\mathcal{D}})$ would define local Kähler structures on the leaf space of the Reeb foliation if η and J are invariant along the foliation.

For this to work, $(\mathcal{D}, J)$ needs to be pseudo-convex and, thus, $\mathcal{D}$ has to be *contact* which is a strong condition §3.1.1.1. Unlike the classical codimension 1 case, contact structures of co-dimension higher than one have local invariants, cf Remark 3.4, and the theory becomes quickly very rich. In our works [ACGL1, ACGL2], we focused on *toric* contact manifolds and toric CR submanifolds of $\mathbb{C}^d$, in particular product of odd dimensional spheres.

3.1.1 Toric contact manifolds of arbitrary codimension

3.1.1.1 Symplectization

In this section, $(N^{2n+k}, \mathcal{D})$ denotes a compact connected manifold N of dimension $2n + k$, together with a distribution seen as a subbundle $\mathcal{D} \subset TN$ of codimension $k \geq 0$.

For $k = 1$, such a pair $(N, \mathcal{D})$ is contact (in the classical sense) if the Liouville symplectic form Ω of T^*N is non-degenerated on the annihilator $\mathcal{D}^0 \backslash \{0\} \subset T^*N$, equivalently a local contact form exists around each point. For higher k, we could simply ask that the annihilator $\mathcal{D}^0 \backslash 0$-section of $\mathcal{D}$ in T^*N is a symplectic submanifold of (T^*N, Ω). This definition is used in subriemannian geometry [173] and includes classical codimension 1 contact submanifolds. However, in higher codimension this would be too restrictive for our purpose. Even product of codimension 1 contact manifolds would not be contact in general. So we define it as "a local contact form exists around each point" as follows.

[2] $\ker \eta \supset \mathcal{D}$, thus $d\eta(X, Y) = -\eta([X, Y])$ for sections $X, Y \in \Gamma(\mathcal{D})$

Definition 3.2. $(N, \mathcal{D})$ is a *contact manifold* if the *nondegeneracy locus* $U_{\mathcal{D}} := \{\eta \in \mathcal{D}^0 \cong (TN/\mathcal{D})^* |\ \eta \circ L_{\mathcal{D}} \text{ is nondegenerate}\}$ meets every fiber of $\mathcal{D}^0$. A (local) section of $U_{\mathcal{D}}$ is called a (local) *contact form* on N.

Example 3.3. If $(N_i, \mathcal{D}_i)$ are contact manifolds, with codimensions k_i, for $i \in \{1, \dots m\}$, then so is $(\prod_{i=1}^m N_i, \mathcal{D}_1 \oplus \dots \oplus \mathcal{D}_n)$, with codimension $k = k_1 + \dots + k_m$ and $U_{\mathcal{D}} = \prod_{i=1}^k U_{\mathcal{D}_i}$. In particular, the product of $m = k$ codimension one contact spheres $\mathbb{S}^{2n_1+1} \times \dots \times \mathbb{S}^{2n_k+1}$ is a codimension k contact manifold.

Remark 3.4. If $U_{\mathcal{D}} \cap \mathcal{D}^0_z$ is nonempty then, since nondegeneracy is an open condition, $\mathcal{D}^0_z$ has a basis $\eta_1, \dots, \eta_k$ in $U_{\mathcal{D}}$ and $U_{\mathcal{D}} \cap \mathcal{D}^0_z$ is the complement of the set where $(\sum_{i=1}^k t_i \eta_i) \circ L_{\mathcal{D}}$ degenerates (that is if $\mathrm{rank}\mathcal{D} = n$, $(\sum_{i=1}^k t_i d\eta_i)^n = 0$ on $\mathcal{D}_z$), which is the cone over a projective hypersurface $V_{\mathcal{D},z} \subset \mathbb{RP}(\mathcal{D}^0_z)$ of degree m. In [ACGL1], we called $V_{\mathcal{D}} \subset \mathbb{RP}(\mathcal{D}^0)$ the *degeneracy variety* of $\mathcal{D}$. The degeneracy variety is a local contact invariant of a contact manifold $(N, \mathcal{D})$. For example $\mathbb{S}^1 \times \mathbb{S}^5$ and $\mathbb{S}^3 \times \mathbb{S}^3$ cannot be even locally contactomorphic because the degeneracy variety of the latter has two points in each fiber, whereas the former has only one (with multiplicity two).

Example 3.5. *(The middle step of some symplectic reductions)* Let $(\widetilde{M}^{2d}, \widetilde{\omega})$ be a symplectic manifold and $\mathbb{G} \subset \mathrm{Ham}(\widetilde{M}, \widetilde{\omega})$ be a k–dimensional compact torus with momentum map $\mu : \widetilde{M} \to \mathfrak{g}^*$. Given a regular value of the moment map $\alpha \in \mathfrak{g}^*$, we consider the submanifold $N := \mu^{-1}(\alpha)$ in $\widetilde{M}$ and denote the inclusion $\iota_\alpha : N \hookrightarrow \widetilde{M}$. The orbits of $\mathbb{G}$ are tangent to N and the induced action of $\mathbb{G}$ on N is locally free. At each point $z \in N$, the tangent space of the orbit $\mathfrak{g}_z := T_z(\mathbb{G} \cdot z)$ has dimension k and $\mathcal{D}_z := \mathfrak{g}_z^{\perp_\omega} \cap T_z N$ has dimension $2n := 2(d-k)$.

In that generality, the Levi form $L_{\mathcal{D}}$ might be degenerated, that is $(N, \mathcal{D})$ is not necessarily contact (eg. $M \times \mathbb{G}$ with $\mathcal{D} = TM$ where M is a manifold). However, whenever $\iota_\alpha^* \widetilde{\omega} = d\eta$ with $\ker(\eta) \supset \mathcal{D}$ then $(N, \mathcal{D})$ is contact. Examples of such $(\widetilde{M}, \widetilde{\omega})$ include open subsets of linear symplectic spaces, products of k symplectic cones or some symplectic submanifolds of T^*N with the Liouville symplectic form [ACGL1].

Contactomorphisms are naturally defined in the category of contact manifolds and we denote by $\mathfrak{con}(N, \mathcal{D})$ the Lie algebra of *infinitesimal contactomorphisms* of $(N, \mathcal{D})$. To recall the notation: an effective action of a torus $\mathbb{T}$ on N corresponds to an embedding $\mathbb{T} \overset{\imath}{\hookrightarrow} \mathrm{Diffeo}(M)$. It is a *contact action* if the linear injection $\imath_* : \mathfrak{t} \hookrightarrow \Gamma(TN)$, where $\mathfrak{t} := \mathrm{Lie}(\mathbb{T})$ embeds $\mathfrak{t}$ in $\mathfrak{con}(N, \mathcal{D})$. Given $\xi \in \mathfrak{t}$ we denote $\underline{\xi} = \imath_*(\xi) \in \Gamma(TN)$ the vector field on N induced by the local action $t \mapsto \exp(t\xi) \subset \mathbb{T}$.

Proposition 3.6. *[ACGL1] Let $(N, \mathcal{D})$ be a contact manifold of codimension k. Then $U_{\mathcal{D}}$ is the open subset of $\mathcal{D}^0$ over which the pullback, say $\omega_{\mathcal{D}}$, of the canonical symplectic form on T^*N to $\mathcal{D}^0$ is nondegenerate. Thus $(U_{\mathcal{D}}, \omega_{\mathcal{D}})$ is an (open) exact symplectic manifolds. Moreover any local contact vector field on $(N, \mathcal{D})$ has a canonical lift to a local hamiltonian vector field on $(U_{\mathcal{D}}, \omega_{\mathcal{D}})$. In particular, if a compact torus $\mathbb{T}$ induces a contact action on $(N, \mathcal{D})$ then the action of $\mathbb{T}$ canonically lifts to a hamiltonian action on $(U_{\mathcal{D}}, \omega_{\mathcal{D}})$ and has a natural homogeneous momentum map $\mu : U_{\mathcal{D}} \to \mathfrak{t}^*$, the restriction of the homogeneous momentum map of the Liouville symplectic structure.*

Let $\mathbb{T}$ be a $(n+k)$–dimensional torus acting effectively on $(N, \mathcal{D})$. In this case, $(N, \mathcal{D}, \mathbb{T})$ is a *toric contact manifold* if moreover the action is contact. A toric contact manifold $(N, \mathcal{D}, \mathbb{T})$ is of *Reeb type* if there exists a k–dimensional linear subalgebra $\mathfrak{g} \subset \mathfrak{t}$ satisfying the following Reeb condition: at every point $z \in N$,

$$T_z N = \mathcal{D}_z \oplus \{\underline{\xi}_z \,|\, \xi \in \mathfrak{g}\}. \tag{3.3}$$

The last summand $\underline{\mathfrak{g}}_z := \{\underline{\xi}_z \,|\, \xi \in \mathfrak{g}\}$ is called the *Reeb distribution* and plays the role of the Reeb vector field. We might call $\mathfrak{g}$ itself a *Reeb subspace* if no confusion is possible.

Remark 3.7. The Reeb type condition is not automatic even on codimension 1 toric contact manifolds. It corresponds in that case to the existence of a global $\mathbb{T}$–invariant contact 1-form, thus needed for a compatible Sasaki metric to exist see eg [46].

Lerman developed the theory for codimension 1 toric contact manifolds [155, 156]. In that case, $U_{\mathcal{D}} = \mathcal{D}^0 \backslash 0$-section, assuming $\mathcal{D}$ co-oriented (or Reeb type), $U_{\mathcal{D}}$ has two components and picking one of them, say $\mathcal{D}^0_+$, we get a symplectic cone $(\mathcal{D}^0_+, \omega_{\mathcal{D}})$ as above. Lerman proved that the image of the induced homogeneous momentum map $\mu : \mathcal{D}^0_+ \to \mathfrak{t}^*$ is a *Delzant polyhedral cone*[3] without its apex, which means that $C := \{x \in \mathfrak{t}^* \backslash \{0\} \,|\, \langle x, \ell_s \rangle \geq 0,\ s = 1, \ldots, d\}$ where d is the number of facets of C and $\ell = \{\ell_1, \ldots, \ell_d\}$ is a set of primitive vectors in the lattice of circle subgroups of $\mathbb{T}$ satisfying the condition of Remark 4.1. The Reeb type condition is equivalent to the strict convexity of the moment cone C. Also, picking a $\mathbb{T}$–invariant contact form $\eta \in \Gamma(\mathcal{D}^0_+)$ and applying the hamiltonian technics of Guillemin–Sternberg [116], he showed that the image of $\mu \circ \eta$ is a convex compact simple polytope, transverse to the moment cone and lying inside the hyperplane $\xi^{-1}(1)$ where $\xi \in \mathfrak{t}_+$ is the Reeb vector field associated to η (2.3).

In higher codimension $U_{\mathcal{D}}$ might have more than two components (cf Remark 3.4) but assuming there is a global $\mathbb{T}$–invariant contact form $\eta \in \Gamma(\mathcal{D}^0)$ its image lies in a single component, say $U \subset U_{\mathcal{D}}$, which we call a *Reeb component*. Noticing that in the partially ordered set of closed $\mathbb{T}$ orbit strata in N, the analogue of a facet is a closed orbit stratum stabilized by a circle (only), we get that the collection of these "facets" is in bijection with a set of primitive vectors $\ell = \{\ell_1, \ldots, \ell_s\}$ of the lattice of circle subgroups Λ of $\mathbb{T}$. The set ℓ does not generates $\mathfrak{t}$ if and only if $(N, \mathcal{D})$ is a trivial $\mathbb{H}$-principal bundle over a toric contact manifold where $\mathbb{H} \subset \mathbb{T}$ is a non-trivial subtorus. Extending Lerman arguments [156] and slices theorem for contact manifold [ACGL1, §2.2] we proved that this set ℓ can be chosen uniquely so that the following holds.

Proposition 3.8. *[ACGL1] Let $(N^{2n+k}, \mathcal{D}, \mathbb{T})$ be a codimension k toric contact manifold of Reeb type with a global $\mathbb{T}$–invariant contact form $\eta \in \Gamma(U)$ where U is connected component of $U_{\mathcal{D}}$. Let d be the number of orbit strata in N having circle stabilizer. Then the momentum image $\mu(U)$ lies in*

$$C := \{x \in \mathfrak{t}^* \backslash \{0\} \,|\, \langle x, \ell_s \rangle \geq 0,\ s = 1 \ldots, d\}$$

[3] Lerman called these cones *good*.

where $\ell = \{\ell_1, \dots, \ell_d\}$ *is a set of primitive vectors in the lattice of circle subgroups* Λ *of* $\mathbb{T}$*. Moreover,* (C, ℓ) *satisfies a Delzant condition with respect to* $\mu(U)$*, namely if* $p \in \mu(U)$ *then* $\mathrm{Span}_{\mathbb{Z}}\{\ell_s \,|\, \ell_s(p) = 0\} = \Lambda \cap \mathrm{Span}_{\mathbb{R}}\{\ell_s \,|\, \ell_s(p) = 0\}$*. Finally,* $\mu(\eta(N))$ *is convex compact polytope with* d *facets contained in a* n*–dimensional affine subspace* $\mathcal{A}$ *of* $\mathfrak{t}^*$ *and* $\mu(\eta(N)) = \{x \in \mathcal{A} \,|\, \ell_s(x) \geq 0,\ s = 1 \dots, d\}$*.*

Actually this result in [ACGL1] is stated in terms of a Levi pair which is defined below. To get the version above we need to pick a Reeb subspace compatible with η and show that η is the associated contact form of a Levi pair.

3.1.1.2 Labelled polytope associated to a Levi pair

Given a Reeb subspace $\mathfrak{g} \subset \mathfrak{t}$, we denote $\eta_{\mathfrak{g}} \colon TN \to \mathfrak{g}$ the connection 1-form of $\mathfrak{g}$ (i.e $\ker \eta_{\mathfrak{g}} = \mathcal{D}$ and $\eta_{\mathfrak{g}}(\underline{\xi}) = \xi$).

Definition 3.9. We say $(\mathfrak{g}, \lambda)$ is a *Levi pair* on $(N, \mathcal{D})$ if $\mathfrak{g} \subset \mathfrak{t}$ induces a Reeb distribution and $\lambda \in \mathfrak{g}^*$ defines a global contact form as

$$\eta^{\lambda}_{\mathfrak{g}} := \lambda \circ \eta_{\mathfrak{g}} \in \Gamma(U_{\mathcal{D}})$$

(i.e. equivalently $d\eta^{\lambda}_{\mathfrak{g}}|_{\mathcal{D}}$ is a nondegenerated 2-form on $\mathcal{D}$).

Any Levi pair $(\mathfrak{g}, \lambda)$ is associated a unique simple convex labelled polytope $(P_{\mathfrak{g},\lambda}, \ell^{\mathfrak{g},\lambda})$ via Proposition 3.8. More precisely in this case we can say that $P = P_{\mathfrak{g},\lambda}$ lies in an affine copy of $(\mathfrak{t}/\mathfrak{g})^*$ inside $\mathfrak{t}^*$. Indeed, for any $\xi \in \mathfrak{g}$, $\mu \circ \eta^{\lambda}_{\mathfrak{g}}(\underline{\xi}) = \langle \lambda, \xi \rangle$ and thus P is a subset of the annihilator of $\jmath(\ker \lambda)$ in $\mathfrak{t}^*$, where $\jmath : \mathfrak{g} \to \mathfrak{t}$ is the inclusion. This annihilator is naturally identified to $(\mathfrak{t}/(\jmath(\ker \lambda)))^*$ and fits in the dual sequence of

$$0 \to \mathfrak{g}/(\ker \lambda) \longrightarrow \mathfrak{t}/(\ker \lambda) \longrightarrow \mathfrak{t}/\mathfrak{g} \longrightarrow 0$$

which identifies $(\mathfrak{t}/\mathfrak{g})^*$ to $(\jmath^*)^{-1}(\lambda)$. Then we showed $\mu(\eta^{\lambda}_{\mathfrak{g}}(N)) = C \cap (\jmath^*)^{-1}(\lambda) =: \overline{P}_{\mathfrak{g},\lambda}$ that is $\overline{P}_{\mathfrak{g},\lambda} = \{x \in (\mathfrak{t}/\mathfrak{g})^* \,|\, \ell_s(x) \geq 0, s = 1, \dots, d\}$ is a simple compact polytope labelled by ℓ.

Conversely, if $\mu(U)$ meets transversally a n–dimensional affine subspace $\mathcal{A}$ in a compact polytope $\overline{P} = \mathcal{A} \cap \mu(U)$ then $T_x\mathcal{A} = \mathfrak{g}^0$ for a k–dimensional subspace $\jmath : \mathfrak{g} \hookrightarrow \mathfrak{t}$ and $\mathcal{A} = (\jmath^*)^{-1}(\lambda)$ for $\lambda \in \mathfrak{g}^*$. The local properties of the momentum map $\mu : U \to \mathfrak{t}^*$ imply that $\mathfrak{g}$ is Reeb.

Corollary 3.10. *Given a (compact, connected) toric contact manifold* $(N, \mathcal{D}, \mathbb{T})$ *of codimension* k *and Reeb type with a Reeb component* $U \subset U_{\mathcal{D}}$*, to a Levi pair* $(\mathfrak{g}, \lambda)$ *with associated Reeb component* U *is associated a unique simple labelled polytope in* $(\mathfrak{t}/\mathfrak{g})^*$ *obtained as a transversal polytope to* $\mu(U)$ *where* $\mu : U \to \mathfrak{t}^*$ *is the natural momentum map of the symplectization. Conversely, any* n*–dimensional labelled polytope transversal to* $\mu(U)$ *determines uniquely a Levi pair on* $(N, \mathcal{D}, \mathbb{T})$ *with associated Reeb component* U*.*

The crucial ingredient to prove the convexity of $P_{\mathfrak{g},\lambda}$ is that for any levi pair $(\mathfrak{g}, \lambda)$ on $(N, \mathcal{D})$ and $a \in \mathfrak{t}$ the function $\mu(\eta^{\lambda}_{\mathfrak{g}}(\underline{a})) \in C^{\infty}(N)$ is Morse-Bott of even index. This allows us to use Atiyah's argument [27]. This convexity result has been proved in a slightly more general situation by Ishida [130].

3.1.1.3 The grassmannian image

The labelled polytope associated to a toric contact manifold with a Levi pair does not seem to determine the underlying toric contact manifold uniquely up to a contactomorphism see [ACGL1, Example 4]. More information is given by the *grassmanian momentum image* defined for a compact toric contact manifold $(N, \mathcal{D})$ as the set $\Xi := \{\mu_z(\mathcal{D}^0_z) \,|\, z \in N\} \subset \mathrm{Gr}_k(\mathfrak{t}^*)$ which is well-defined because the homogeneous momentum map of the Liouville symplectic form is linear along the fibers. However, for Ξ to have nice properties of momentum image we need to assume that $(N, \mathcal{D})$ has Reeb type, as we do in the rest of this section.

Using the theory developed in [137] for non-compact symplectic manifold like $(U_{\mathcal{D}}, \omega_{\mathcal{D}})$ we obtained that $N/\mathbb{T}$ and Ξ are diffeomorphic manifolds with corners. The orbit strata of N corresponding to circle stabilizers are send to the facets of Ξ lying in subgrassmanians of codimension 1. The existence of a Levi pair $(\mathfrak{g}, \lambda)$, actually the choice of a Reeb component of $U_{\mathcal{D}}$, provides a labelling ℓ satisfying the Delzant condition above. Also, the Reeb subspace $\mathfrak{g} \subset \mathfrak{t}$, implies that Ξ lies in an affine subspace of $\mathrm{Gr}_k(\mathfrak{t}^*)$ corresponding to the k–planes transverse to the annihilator of $\mathfrak{g}$ inside $\mathfrak{t}^*$. Moreover, using $\lambda \in \mathfrak{g}^*$ this manifold with corners Ξ must be sent bijectively by an affine map to a convex simple polytope $\overline{P}_{\mathfrak{g},\lambda}$. This last condition encodes the needed convexity and compactness properties. Summing up these conditions we get that the grassmannian image Ξ of a compact connected toric contact manifold of Reeb type $(N, \mathcal{D}, \mathbb{T})$ is *Delzant and polyhedral of Reeb type in* $\mathrm{Gr}_k(\mathfrak{t}^*)$.

Theorem 3.11 (Apostolov–Calderbank–Gauduchon–L. [ACGL1])**.** *The grassmannian image Ξ of a codimension k compact connected toric contact manifold of Reeb type $(N, \mathcal{D}, \mathbb{T})$ is compact polyhedral manifold with corners in $\mathrm{Gr}_k(\mathfrak{t}^*)$, and any Reeb component $U \subset U_{\mathcal{D}}$ induces a labelling ℓ such that (Ξ, ℓ) is Delzant of Reeb type. Conversely, any Delzant labelled polyhedral manifold with corners (Ξ, ℓ) of Reeb type in $\mathrm{Gr}_k(\mathfrak{t}^*)$, is the grassmannian image of a codimension k compact connected toric contact manifold with a Reeb component whose associated labelled manifold with corners is (Ξ, ℓ).*

For the last claim, we used a variant of Delzant construction to obtain the symplectization, actually a Reeb component $(U, \omega_{\mathcal{D}})$, of the toric contact manifold as a symplectic reduction of $(V \times \mathbb{T} \times \mathbb{C}^d, \Omega - \omega_{std})$ where $V \subset \mathfrak{t}^*$ is open, $V \times \mathbb{T} \subset T^*\mathbb{T}$ is endowed with its Liouville symplectic form Ω.

We would have liked to prove that toric contact manifolds are classified by their grassmannian image, with a suitable labelling. This is true locally by the local models we obtained in [ACGL1, §2.2] using symplectic slices. However, as in toric symplectic geometry [157] and toric contact geometry in codimension one [156], there remains a local-to-global question governed by the cohomology of a sheaf. Thus we have the following result.

Theorem 3.12 (Apostolov–Calderbank–Gauduchon–L. [ACGL1])**.** *The grassmannian image Ξ of a compact toric contact manifold of Reeb type $(N, \mathcal{D}, \mathbb{T})$ is Delzant and polyhedral of Reeb type in $\mathrm{Gr}_k(\mathfrak{t}^*)$, and there is a sheaf $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ on Ξ such that toric contact manifolds of Reeb type with grassmannian image Ξ are parametrized up to isomorphism by $H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D}))$.*

In contrast to the symplectic and codimension one cases, $\mathfrak{con}^{\mathbb{T}}(\mathcal{D})$ is the sheaf of solutions of a linear partial differential equation (for infinitesimal contactomorphisms), which is overdetermined and typically not involutive. We

show that $H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D}))$ vanishes in special cases, such as when $(N, \mathcal{D}, \mathbf{K})$ is a product of codimension one toric contact manifolds. However, it remains an open question whether it vanishes in general.

Proposition 3.13. *[ACGL1] If N is a product of codimension one contact manifolds of Reeb type, then $H^1(\Xi, \mathfrak{con}^{\mathbb{T}}(\mathcal{D})) = 0$.*

3.1.2 Levi-Kähler reduction

Here we incorporate a CR structure in the picture drawn in the last subsection. Namely, we assume $(N, \mathcal{D}, \mathbb{T})$ is a codimension k (compact connected) toric contact manifold of Reeb type and, in addition, there is a $\mathbb{T}$–invariant CR structure materialized by an endomorphism $J \in \mathrm{End}(\mathcal{D})$ such that $J^2 = -\mathrm{Id}$ and $\mathcal{D}^{1,0} \subset \mathcal{D} \otimes \mathbb{C}$ is integrable (closed under the Lie bracket). It makes the Levi form J–invariant (or of "(1,1)" type) in the sense that $-2L_{\mathcal{D}}(X, Y) = \mathbf{q}_{\mathcal{D}}([X, Y] + [J\cdot, J\cdot])$. Therefore,

$$U_{\mathcal{D}}^{+} := \{\eta \in \mathcal{D}^0 |\ \eta \circ L_{\mathcal{D}}(X, Y) \text{ is positive definite}\}$$

is an open and closed submanifold of $U_{\mathcal{D}}$. Of course, we are interested in the cases where this set is non-empty and, thus, coincide with a Reeb component from the last subsection.

A *Levi-Kähler pair* is a Levi pair (cf Definition 3.9) $(\mathfrak{g}, \lambda)$ such that $\eta_{\mathfrak{g}}^{\lambda} = \lambda \circ \eta_{\mathfrak{g}} \in \Gamma(U_{\mathcal{D}}^{+})$. Such Levi-Kähler pair defines a metric $h_{\mathfrak{g},\lambda} := d\eta_{\mathfrak{g}}^{\lambda}|_{\mathcal{D}}(\cdot, J\cdot)$ on the complex bundle $(\mathcal{D}, J)$. Local quotient by the foliation induced by the Reeb foliation takes that structure to a local Kähler structure (this uses that the structure is $\mathfrak{g}$–invariantand that $d\eta_{\mathfrak{g}}^{\lambda}$ is $\mathfrak{g}$-basic). Thus, a Levi-Kähler pair defines transversal Kähler structures in the sense discussed in this memoir.

Note that if $\mathfrak{g}$ coincides with the Lie algebra of a subtorus $\mathbb{G} \subset \mathbb{T}$ then any Levi-Kähler pair $(\mathfrak{g}, \lambda)$ defines a Kähler structure, say $(\omega_{\mathfrak{g},\lambda}, J)$, on the orbifold quotient $M = N/\mathbb{G}$.

We used these Levi Kähler pairs in [ACGL2] to construct compatible Kähler structures on compact toric complex manifolds by revisiting the Delzant construction [77] as follows.

Given a n–dimensional simple labelled polytope (P, ℓ) with $P \subset \mathbb{R}^n$ and defining affine linear functions $\ell = \{\ell_1, \dots, \ell_d\} \subset \mathrm{Aff}(\mathbb{R}^n, \mathbb{R})$ (cf 4.1.1 for the definition of a labelled polytope). Denote $\mathbb{T}^d = \mathbb{S}^1 \times \cdots \mathbb{S}^1$ and consider its standard hamiltonian action on $(\mathbb{C}^d, \omega_{std} := \sum_j \frac{i}{2\pi} dz_j \wedge d\bar{z}_j)$ with momentum map

$$\begin{array}{rccc} \mu_{std}: & \mathbb{C}^d & \longrightarrow & \mathbb{R}^d \\ & z & \mapsto & \frac{1}{2}(|z_1|^2, \dots, |z_d|^2). \end{array}$$

The Delzant construction, see [77, 157], associates to (P, ℓ) the following objects

- a $(k = d - n)$–dimensional subalgebra $\jmath_{\mathfrak{g}} : \mathfrak{g}_P \hookrightarrow \mathbb{R}^d$ defined by

$$\mathfrak{g}_P := \ker\left(\mathbb{R}^d \ni x \mapsto \sum_{s=1}^{d} x_s d\ell_s \in \mathbb{R}^n\right);$$

- an element in the dual of that subalgebra $\lambda_P := \jmath^*_{\mathfrak{g}_P}(\ell_1(0), \dots, \ell_d(0)) \in \mathfrak{g}_P^*$
- a level set $N_P := \mu_{\mathfrak{g}}^{-1}(\lambda)$ of the $\mathfrak{g}_P$–momentum map $\mu_{\mathfrak{g}} := \jmath^* \circ \mu_{std} : \mathbb{C}^d \to \mathfrak{g}^*$ and thus a toric CR submanifold of $\mathbb{C}^d$

  $$(N_P, \mathcal{D}_P, J)$$

  with $\mathcal{D}_P = TN_P \cap JTN_P$ where J denotes the integrable almost complex structure of $\mathbb{C}^d$.

A key observation [ACGL2] is that a part of the proof of Delzant only use the *combinatorics* of P, by which we refer to the partially ordered set of closed facets of P. Using this and the type of equations defining the submanifolds in Delzant's proof we get the following.

Lemma 3.14. *[ACGL2] The pair $(\mathfrak{g}_P, \lambda_P)$ is Levi-Kähler on $(N_Q, \mathcal{D}_Q, J)$ for any labelled polytope $(Q, \tilde{\ell})$ provided that Q has the same combinatorics than P up to a renumbering of the facets. Moreover, the labelled polytope associated to $(N_Q, \mathcal{D}_Q, J, \mathfrak{g}_P, \lambda_P)$ in §3.1.1.2 is (P, ℓ) (does not depend on Q). In particular, if $\mathfrak{g} = Lie(\mathbb{G})$ for a subtorus $\mathbb{G}$ of $\mathbb{T}^d$ the Levi quotient $(N_Q/\mathbb{G}, \omega_{\mathfrak{g}_P, \lambda_P})$ is $\mathbb{T}/\mathbb{G}$–equivariantly symplectomorphic to the Delzant-Lerman-Tolman toric symplectic orbifold associated to (P, ℓ).*

Note that even on $(N_P/\mathbb{G}, J)$ the Levi-Kähler structure $(J, \omega_{\mathfrak{g}_P, \lambda_P})$ is not necessarily isometric to the so-called Guillemin metric [115] which is obtained as a standard Kähler reduction of $(\mathbb{C}^d, \omega_{std})$ for the group $\mathbb{G}$ (when (P, ℓ) is rational) at the level λ_P. Indeed, there is no particular reason for ω_{std} to restrict as $d\eta^{\lambda_P}_{\mathfrak{g}_P}$ on $\mathcal{D}_P$ and in general it does not. However, the complex structure underlying the Guillemin Kähler structure is the one obtained from the CR structure $(N_P, \mathcal{D}_P, J)$ by identifying $\mathcal{D}$ to $T(N_P/\mathbb{G})$. Thus, there is a canonical identification between the complex orbifold underlying Guillemin "metric" and the one underlying the Levi-Kähler reduction. Thus, given a rational labelled polytope (P, ℓ) and associated Delzant-Lerman-Tolman toric symplectic orbifold (M, ω), there is a distinguished ω–compatible Kähler structure $h_{(P,\ell)}$ obtained by a Levi-Kähler reduction (here we denote $h_{(P,\ell)}$ the Kähler triple $(g_{(P,\ell)}, \omega, J_{(P,\ell)})$ where $g_{(P,\ell)}$ is a riemannian metric $g_{(P,\ell)} = -\omega J_{(P,\ell)}$).
Our goal in this project was to put special metrics on compact toric symplectic orbifold. Another insight that leads us is that a Levi–Kähler reduction of a CR manifold $(N, \mathcal{D}, J)$ can be expected to have nice curvature properties if $(N, \mathcal{D}, J)$ does. The simplest examples, in codimension one, are round CR $(2m+1)$-spheres, which are the toric CR submanifolds associated to m-simplices, and are circle orbibundles over complex weighted projective spaces. The Levi–Kähler quotients in this case give rise to the natural *Bochner flat* Kähler metrics on weighted complex projective spaces, studied in [56, 71, 203], which are also *extremal* in the sense of Calabi [57].

Recall here, the *Bochner part* of of the curvature tensor of a Kähler metric is its pointwise orthogonal projection onto the $U(m)$-submodule of (formal) Kähler curvature tensors with vanishing Ricci trace. In our language, Webster [203] showed that in codimension one the Bochner tensor of a Levi–Kähler quotient M of CR manifold N pulls back to the Chern–Moser tensor of N, which is a local CR invariant [66] and vanishes when the N is locally CR diffeomorphic to

a standard CR sphere $(\mathbb{S}^{2n+1}, \mathcal{D}, J)$. In particular, every Levi–Kähler quotient (M^{2n}, g, J) of $\mathbb{S}^{2n+1}$ is Bochner-flat.

Similarly, we showed in [ACGL2] that the Levi–Kähler quotients of *products* of CR spheres $N = \mathbb{S}^{2n_1+1} \times \cdots \mathbb{S}^{2n_k+1}$ provide a natural extension of the Bochner flat Kähler metrics in higher codimension. The product structure on N induces distributions on M and we show that the curvature of M has vanishing Bochner component on each such distribution, simply because CR spheres have vanishing Chern–Moser tensor.

Theorem 3.15 (Apostolov–Calderbank–Gauduchon–L. [ACGL2])**.** *Let (M, ω) be a compact $2n$ dimensional symplectic toric orbifold with Delzant labelled polytope (P, ℓ). Then P has the combinatorics of a product of simplices if and only if (M, ω) can be obtained as a Levi–Kähler reduction of a product of odd dimensional CR spheres. In particular, such an orbifold admits a distinguished compatible Kähler metric $h_{(P,\ell)}$ for which the tangent bundle admits a decomposition into orthogonal distributions and such that the curvature of $h_{(P,\ell)}$ has vanishing Bochner component on each such distribution.*

For 3-dimensional CR manifolds, the vanishing of the Chern–Moser tensor is automatic and we compute instead the scalar curvature of a Levi–Kähler quotient of a product of 3-spheres and observe that, when the polytope is projectively equivalent to a cube, the Levi–Kähler quotient metric can be characterized as being extremal in a weighted sense that was introduced (in a special case) in [20]. Here, recall that a polytope P having the combinatorics of a cube is *projectively equivalent to a cube* if the intersections of pairs of opposite facets lie in a hyperplane $H \subset \mathfrak{t}^*$. Thus there is a unique, up to scale, affine linear function $w \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ which is positive on P. We showed by direct computation that for the scalar curvature of the Kähler metric $(g = g_{(P,\ell)}, \omega)$ corresponding to the Levi-Kähler reduction of Lemma 3.14 the following function is affine-linear on P

$$\mathrm{Scal}_{g,w} := w^2 \mathrm{Scal}_g - 2(n+1) w \Delta^g w - (n+2)(n+1) g^{-1}(dw, dw). \qquad (3.4)$$

Apostolov and Maschler have introduced similar weighted curvature (3.4) in relation with Maxwell–Einstein metric [20]. Given a toric symplectic orbifold $(M, \omega, \mathbb{T})$ with momentum image $\overline{P}$ and $(w, p) \in \mathrm{Aff}(P, \mathbb{R}^+) \times \mathbb{N}$, they have defined in the case $p = 2n$ the following operator on ω–compatible Kähler metrics

$$\mathrm{Scal}_{g,w,p} := w^2 s\mathrm{Scal}_{g,w} - 2(p-1) w \Delta_J w - p(p-1) g_J^{-1}(dw, dw). \qquad (3.5)$$

Solutions g of $\mathrm{Scal}_{g,w,p} \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ enjoy the basic property a "canonical metric" should have:

Proposition 3.16. *[20, ACGL2] Let $(M, \omega, \mathbb{T})$ be a compact toric symplectic orbifold with labelled Delzant polytope (P, ℓ) in $\mathbb{R}^n$ and w a positive affine-linear function on Δ. Then,*

(a) *There exists at most one (up to equivariant isometry) ω compatible toric metric g on $(M, \omega, \mathbb{T})$, for which $\mathrm{Scal}_{g,w,p}$ is an affine-linear function.*
(b) *The affine-linear function in* (a) *is uniquely determined by (P, ℓ, w, p).*

Definition 3.17. The (unique) compatible toric metric satisfying the condition (a) of Proposition 3.16 is called the (w, p)-*extremal* metric of $(M, \omega, \mathbb{T})$.

Thus, our finding that (3.4) is affine-linear on a projective cube $P \subset \mathbb{R}^n$ was actually an occurrence of weighted $(w, n+2)$-extremal metric which is therefore unique in a given class.

Theorem 3.18 (Apostolov–Calderbank–Gauduchon–L. [ACGL2])**.** *Suppose (P, ℓ) is a labelled projective cube in $\mathbb{R}^n$, corresponding to a compact toric orbifold $(M, \omega, \mathbb{T})$ and let h_ℓ be the Levi–Kähler quotient metric defined by 3.14. Then h_ℓ is the $(w, m+2)$-extremal metric of (P, ℓ), where w is the unique up to scale affine-linear function on $\mathbb{R}^m$, vanishing on the hyperplane containing the intersections of opposite facets of P and positive on P.*

An important source of toric Kähler orbifolds M whose Delzant polytope has the combinatorics of a product of simplices—and to which Theorem A applies—is obtained from the generalized Calabi construction (see [14]), where both the base and the fiber are toric orbifolds with Delzant polytopes having the combinatorics of product of simplices. This includes the complex Hirzebruch surfaces, holomorphic projective bundles over a projective space, the Bott manifolds, and, inductively, rigid toric fibrations where the base and the fiber are one of the aforementioned smooth complex manifolds. We show that in this special setting, the Kähler metric corresponding to the Levi–Kahler quotient of the product of spheres associated to M is obtained from the generalized Calabi construction, where the metrics on the base and on the fiber are themselves Levi–Kahler quotients of product of spheres. The framework given by Calabi ansatz is very convenient to compute scalar curvature and we were able to produce new cscK metrics.

Proposition 3.19. *There exists a countable family of compact constant scalar curvature Kähler 6-orbifolds obtained as Levi–Kähler reductions of $\mathbb{S}^5 \times \mathbb{S}^3$.*

3.2 Weighted cscK and Sasaki geometry

3.2.1 Extremal Sasaki metrics and weighted cscK

Let $(N^{2n+1}, \mathcal{D}_\xi, J^\xi)$ be a compact transversally holomorphic manifold of Sasaki type. In analogy with the Calabi extremal metrics [57], Boyer–Galicki–Simanca introduced in [48] extremal Sasaki structures as critical points of the functional

$$\mathcal{S}(N, \mathcal{D}_\xi, J^\xi) \ni \eta \mapsto \int_N \mathrm{Scal}(g_\eta)^2 \eta \wedge (d\eta)^n.$$

Here, $\mathrm{Scal}(g_\eta)$ is the scalar curvature of the Sasaki-riemannian metric associated to η through (2.4) but of course the critical points are the same if we replace it with the transversal scalar curvature since they differ by a constant and that the total scalar curvature is constant on $\mathcal{S}(N, \mathcal{D}_\xi, J^\xi)$. They proved in the same paper that the condition of being extremal was characterized by the contact vector field associated to the scalar curvature as follows. Recall that given a contact manifold $(N, \mathcal{D})$ with a global contact form η and associated Reeb vector field ξ, any function $f \in C^\infty(N)$ determines a unique η–contact vector field $X_f \in \mathfrak{con}(N, \mathcal{D})$ as

$$X_f = f\xi + Z_f$$

where $Z_f \in \Gamma(\mathcal{D})$ is the unique section satisfying $d\eta(Z_f, \cdot) = \xi.f - df$, see Definition/Remark 2.1.

Definition 3.20. [48] A Sasaki manifold $(N^{2n+1}, \mathcal{D}, J, \eta)$ is *extremal* if the η–contact vector field of the transversal scalar curvature $\mathrm{Scal}^T(\eta)$ is transversally holomorphic, i.e, lies in $\mathfrak{cr}(\mathcal{D}, J)$.

In particular, a cscS metric is extremal because its contact vector field is a constant times the Reeb vector field.

A useful result [48, Theorem 4.8] is that, similarly to the Kähler case, the presence of an extremal Sasaki structure in $\mathcal{S}(N, \mathcal{D}_\xi, J^\xi)$ implies strong consequences on the structure of the finite dimensional Lie algebra of $\mathfrak{aut}(N, \xi, J^\xi)/\mathfrak{o}(N, \xi)$ (see §2.1.1.3). Their result implies in particular that the isometry group $\mathrm{Aut}(N, \ker\eta, J, \xi)$ of an extremal structure η, should it exists, would be compact maximal in $\mathrm{Aut}(N, \xi, J^\xi)$ and the contact vector field of the transversal scalar curvature of the extremal representative would lies in its centre.

From these considerations, we are in a position to define a (relative) *contact Futaki invariant* and *contact extremal vector field*, similar to [154] as follows. As usual, we fix a torus $\mathbb{T}$ but now in $\mathrm{Con}(N, \mathcal{D})$ and with $\xi \in \mathfrak{t} := \mathrm{Lie}\mathbb{T}$. Then, we define the *extremal vector field* ξ_{ext} (which depends on $\mathbb{T}$) as the projection on $\mathfrak{t}$ of $\xi_{\mathrm{Scal}^T(\eta)}$, the η–contact vector field of the transversal scalar curvature $\mathrm{Scal}^T(\eta)$ of the Sasaki structure $(N, \ker\eta, J, \xi)$ associated to $\eta \in \mathcal{S}(N, \mathcal{D}_\xi, J^\xi)$. Namely, $\xi_{ext} \in \mathfrak{t}$ is uniquely defined so that

$$\langle\!\langle a, \xi_{ext} - \xi_{\mathrm{Scal}^T(\eta)} \rangle\!\rangle_\eta := \int_N \eta(a)(\eta(\xi_{ext}) - \eta(\xi_{\mathrm{Scal}^T(\eta)}))\eta \wedge (d\eta)^n = 0 \qquad \forall a \in \mathfrak{t}.$$

The product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_\eta$ depends on η^ξ and we have picked $\mathbb{T} \subset \mathrm{Con}(N, \mathcal{D})$ but the resulting vector field $\xi_{ext} \in \mathfrak{t}$ does not depend on the form $\eta \in \mathcal{S}(\xi, J^\xi)^{\mathbb{T}}$ following standard arguments [48, 110, 111, ACL].

Remark 3.21. The momentum map picture [82, 100], adapted to the Sasaki setting [121, 10], allows us to give a stronger statement and gives a conceptual explanation for this invariant. Indeed, the group $\mathrm{Con}(N, \mathcal{D})^{\mathbb{T}}$ acts on the space of $\mathbb{T}$–invariant CR-structures $\mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}}$, compatible with a fixed orientation of $\mathcal{D}^0$. Fixing $\xi \in \mathfrak{t}_+$ and denoting $\eta^\xi \in \Gamma(\mathcal{D}^0)$ the associated contact form (see (2.3)) the map

$$\begin{aligned} \mu_\xi \colon \mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}} &\to (\mathfrak{con}(N, \mathcal{D})^{\mathbb{T}})^* \\ J &\mapsto \Big(a \mapsto \langle\!\langle \mathrm{Scal}^T(\eta^\xi, J), a \rangle\!\rangle_{\eta^\xi}\Big) \end{aligned} \tag{3.6}$$

is a momentum map for the action of $\mathrm{Con}(N, \mathcal{D})^{\mathbb{T}}$ for a formal symplectic structure on $\mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}}$ [10, 121]. Now, $\mathbb{T}$ acts trivially on $\mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}}$ thus $\langle\!\langle \mathrm{Scal}^T(\eta^\xi, J), a \rangle\!\rangle_{\eta^\xi}$ does not depend on $J \in \mathcal{C}_+(N, \mathcal{D})^{\mathbb{T}}$ for $a \in \mathfrak{t}$.

Definition 3.22. (see e.g. [121, 122]) The (relative) Mabuchi energy is the functional

$$\mathbf{M}^\zeta \colon \hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}} \to \mathbb{R}$$

characterized by

$$\begin{aligned} (d\mathbf{M}^\zeta)_\psi(\dot{\psi}) &= \int_N \dot{\psi}\Big(\mathrm{Scal}^T(\eta_\psi) - \eta_\psi(\zeta_{\mathrm{ext}})\Big)\,\eta_\psi \wedge d\eta_\psi^m, \\ \mathbf{M}^\zeta(0) &= 0, \end{aligned}$$

where for any $\psi \in \hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$, $\eta_\psi := \eta_0^\zeta + d_\zeta^c \psi$ stands for the corresponding contact form in $\Xi(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$ whereas $\mathrm{Scal}^T(\eta_\psi)$ denotes the corresponding transversal scalar curvature.

Clearly if a Sasaki structure in $\Xi(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$ is extremal in the sense of [48] and $\zeta_{ext} \in \mathfrak{t}$, it is a critical point of $\mathbf{M}^\zeta$. W.He and Li [121, 122] have imported Kähler technics to study the modified Mabuchi (the non relative one, thus defined on a bigger subspace of $\hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)$) along geodesics in the space of Sasaki potentials and extended some of the results known in the Kähler case. In particular they proved that there exists an extremal Sasaki metric $\Xi(\zeta, J^\zeta, \eta_0^\zeta)$ if and only if $\mathbf{M}^\zeta$ satisfies some relative properness condition discussed below.

In our work [ACL], our primary goal was to translate the transversal Kähler geometry of one polarisation $\zeta \in \mathfrak{t}_+$ into some "weighted" geometry on the Sasaki-Kähler quotient of another (quasi-regular) polarisation $\xi \in \mathfrak{t}_+$ and ultimately understand the (Sasaki) K-stability of $(N, \mathcal{D}_\zeta, J^\zeta)$ as an appropriate "weighted" stability notion on the Sasaki-Kähler quotient $(M/\mathbb{S}^1_\xi, J, \omega_\xi)$. Of course, for this to work we need to restrict to $\mathbb{T}$–invariant transversal Kähler structures where $\mathfrak{t} := \mathrm{Lie}\mathbb{T}$ contains at least ζ and ξ. Then necessarily any $\mathbb{T}$–invariant structures in $\Xi(N, \mathcal{D}_\zeta, J^\zeta, \eta_0^\zeta)$ exists, in another form, on $(N/\mathbb{S}^1_\xi, J, \omega_\xi)$. Before explaining the results, some basic observations are in order.

Definition 3.23. Given a Sasaki manifold $(N, \mathcal{D}, J, \eta)$, a function $h \in C^\infty(N)$ is a *Killing potential* if $h = \eta(a)$ with $a \in \mathfrak{cr}(\mathcal{D}, J)$. This is well-defined using Definition/Remark 2.1 and since $\mathfrak{cr}(\mathcal{D}, J) \subset \mathfrak{con}(N, \mathcal{D})$. More generally, given $\eta \in \mathcal{S}(N, \mathcal{D}_\xi, J^\xi))$, h is a η-Killing potential if it is a Killing potential for $(\ker \eta, J, \xi)$.

Note that a $\mathbb{T}$–invariant function h is a Killing potential for $(N, \mathcal{D}, J, \zeta)$ if and only if $f\,h$ is a Killing potential for $(N, \mathcal{D}, J, \xi)$ with $f = \eta^\xi_{\mathcal{D}}(\zeta)$ where $\eta^\xi_{\mathcal{D}}$ denotes the unique $\eta^\xi_{\mathcal{D}} \in \mathcal{S}(N, \mathcal{D}_\xi, J^\xi))$ with $\ker \eta^\xi_{\mathcal{D}} = \mathcal{D}$ (see Definition/Remark 2.5). Therefore checking if the function $\mathrm{Scal}^T(J, \eta^\zeta_{\mathcal{D}})$ is Killing for $(N, \mathcal{D}, J, \zeta)$ amounts to verify that

$$f\,\mathrm{Scal}^T(J, \eta^\zeta_{\mathcal{D}}) \text{ is a Killing potential of } (N, \mathcal{D}, J, \xi). \tag{3.7}$$

Remark 3.24. Denote $\varpi_\xi : N \to M := N/\mathbb{S}^1_\xi$ the quotient by the circle group $\mathbb{S}^1_\xi$ generated by $\xi \in \mathfrak{t}$. Thus, $\check{\mathbb{T}} := \mathbb{T}/\mathbb{S}^1_\xi$ acts effectively on M. Any $\eta \in \mathcal{S}(N, \mathcal{D}_\xi, J^\xi)$ determines a Kähler structure (J, ω_η) on M and $\mathfrak{con}(N, \ker \eta)^\xi/\mathrm{Span}_{\mathbb{R}}\xi$ are the hamiltonian vector fields on (M, ω_η). Given $a \in \mathfrak{con}(N, \ker \eta)^\xi$ a hamiltonian function for $[a] \in \mathfrak{con}(N, \ker \eta)^\xi/\mathrm{Span}_{\mathbb{R}}\xi$ is given by $\eta(a)$. In particular, Killing potentials on (M, J, ω_η) pull back to ξ–invariant Killing potentials on $(N, \ker \eta, J, \xi)$.

Apostolov and Calderbank have compared the transversal scalar curvature $(N, \mathcal{D}, J, \zeta)$ and $(N, \mathcal{D}, J, \xi)$ using a Tanaka connection [70] and proved in [10] that

$$\mathrm{Scal}^T(g_\zeta) = f\mathrm{Scal}^T(g_\xi) - 2(n+1)\Delta^{g_\xi} f - (n+2)(n+1)f^{-1}|df|^2_{g_\xi} \tag{3.8}$$

where $f = \eta^\xi_{\mathcal{D}}(\zeta)$, g_ξ (resp. g_ζ) denotes the Sasaki-riemannian metric of $(N, \mathcal{D}, J, \xi)$ (resp. $(N, \mathcal{D}, J, \zeta)$) and $\mathrm{Scal}^T(g_\zeta)$ is the transversal scalar curvature associated with the structure $(N, \mathcal{D}, J, \zeta)$.

Remark 3.25. Note that (M, J^ξ, ω_ξ) coincides with the Levi-Kähler reduction for $\mathfrak{g} = \mathbb{R}\xi$ and $\lambda \in \mathfrak{g}^*$ such that $\lambda(\xi) = 1$. Indeed, $(N, \mathcal{D}, J)$ is a CR manifold of codimension one. Thus, the varying Levi-Kähler pairs $(\mathfrak{g}, \lambda)$ and looking at the transversal Kähler structures is equally achieved by varying the Reeb vector field in the Reeb cone $\mathfrak{t}_+ \subset \mathfrak{t} := \mathrm{Lie}\mathbb{T}$ and thus present a simplification (lower codimension) of the framework studied in [ACGL1, ACGL2] and recalled above §3.1,3.1.2.

With the observations above and Apostolov-Calderbank formula we are lead to study the following weighted scalar curvature on (M, J, ω_o)

$$\mathrm{Scal}_f(\omega_o) := f^2\mathrm{Scal}(\omega_o) - 2(n+1)f\Delta^{\omega_o} f - (n+2)(n+1))|df|^2_{\omega_o} \tag{3.9}$$

where $\mathrm{Scal}(\omega_o)$ is the scalar curvature associated to the Kähler structure (J, ω_o). It prompts to introduce the following notion.

Definition 3.26. Let $f \in C^\infty(M, \mathbb{R}_{>0})$ be a Killing potential on (M, J, ω_o). We say that (J, ω_o) is a f–extremal[4] Kähler structure if $\mathrm{Scal}_f(\omega_o)$ is a Killing potential.

Lemma 3.27. *[10] Let (M, J, ω) be the Kähler orbifold corresponding to the Sasaki–Reeb quotient of $(N, \mathcal{D}, J)$ with respect to a quasiregular $\xi \in \mathfrak{cr}_+(N, \mathcal{D}, J)$, and $f = \eta^\xi_{\mathcal{D}}(K) > 0$ the induced positive Killing potential on (M, J, ω) by $\zeta \in \mathfrak{cr}_+(N, \mathcal{D}, J)^\xi$. Then $(\mathcal{D}, J, \zeta)$ is an extremal Sasaki structure on N if and only if (J, ω) is a f–extremal Kähler structure.*

Of course the notion of weighted extremal metrics make sense on Sasaki manifolds itself withtout asking one of the Reeb vector field to be quasiregular. Then, given a $f \in C^\infty(N, \mathbb{R}_{>0})$ a Killing potential for a Sasaki structure $(N, \mathcal{D}, J, \xi)$, we would say that $(N, \mathcal{D}, J, \xi)$ is *f–extremal* if

$$f^2\mathrm{Scal}^T(g_\xi) - 2(n+1)f\Delta^{g_\xi} f - (n+2)(n+1))|df|^2_{g_\xi}$$

is a Killing potential. By (3.8), it is the case if and only $(N, \mathcal{D}, J, \zeta)$ is a Sasaki extremal structure. This is already interesting in its own but a bit restrictive because the CR structure is fixed. The goal of our work [ACL] was to make this correspondence more flexible by allowing variations within the compatible CR structures of the transversal holomorphic structures $\mathcal{S}(\mathcal{D}_\xi, J^\xi)$ and $\mathcal{S}(\mathcal{D}_\zeta, J^\zeta)$.

3.2.2 A correspondence between radial potentials

A key observation highlighted in [ACL] is that given $\xi, \zeta \in \mathfrak{t}_+$ we have bijection

$$\begin{array}{rccc} \Theta\colon & \mathcal{R}_\xi(Y,J)^{\mathbb{T}} & \to & \mathcal{R}_\zeta(Y,J)^{\mathbb{T}} \\ & r & \mapsto & \tilde{r} \end{array} \tag{3.10}$$

between $\mathbb{T}$-invariant radial potentials with respect to ξ and such potentials with respect to ζ, characterized uniquely by $\Theta(r) = \tilde{r}$ if and only if $N_{\tilde{r}} := \tilde{r}^{-1}(1)$ coincides with $N_r := r^{-1}(1)$.

[4] In order to fit with Lahdili's notion [142], Apostolov-Calderbank call the latter notion a (X_f, κ)–weighted extremal Kähler structure with X_f the ω_o–hamiltonian vector field of f and $\kappa := \int_M f\omega_o^n$.

Indeed, N_r is a pseudo-convex image of a $\mathbb{T}$-invariant section of both $\pi_\xi : Y \to N^\xi = Y/\mathbb{R}^+_\xi$ and $\pi_\zeta : Y \to N^\zeta = Y/\mathbb{R}^+_\zeta$ and the same for and $N_{\tilde{r}}$. Thus, the bijection is a straightforward application of the $\mathbb{T}$-invariant of formulation of the correspondence between links and radial potentials recalled in Proposition 2.17.

Fixing base points $r_0 \in \mathcal{R}_\xi(Y,J)^{\mathbb{T}}$ and $\tilde{r}_0 := \Theta(r_0) \in \mathcal{R}_\zeta(Y,J)^{\mathbb{T}}$, we also have a bijection

$$\begin{aligned} \widehat{\Theta}\colon \hat{\Xi}(N^\xi,\xi,J^\xi,\eta_{r_0})^{\mathbb{T}} &\to \hat{\Xi}(N^\zeta,\zeta,J^\zeta,\eta_{\tilde{r}_0})^{\mathbb{T}} \\ \varphi &\mapsto \psi \quad \text{with} \quad \Theta(e^\varphi r_0) = e^\psi \tilde{r}_0. \end{aligned} \tag{3.11}$$

Pulled-back on $N_{e^\varphi r_0} = N_{e^\psi \tilde{r}_0}$, the contact form $\alpha_{r_0,\varphi} := \widetilde{\eta_{e^\varphi r_0}} \in \mathcal{S}(\xi, J^\xi)$ and $\alpha_{\Theta(r_0),\widehat{\Theta}(\varphi)} := \widetilde{\eta_{e^\psi \tilde{r}_0}} \in \mathcal{S}(\zeta, J^\zeta)$ determine the *same* CR structure, namely the one induced by inclusion in (Y,J), see Lemma 2.16. Then φ and $\widehat{\Theta}(\varphi)$ induce Sasaki structures with the same underlying CR structure. This is a useful property of the map of $\widehat{\Theta}$. Indeed, using this and Apostolov-Calderbank formula (3.8), we get

$$\mathrm{Scal}^T_{(J^\zeta,\alpha_{\Theta(r_0),\widehat{\Theta}(\varphi)})} = f\mathrm{Scal}^T_{(J^\xi,\alpha_{r_0,\varphi})} - 2(n+1)\Delta^{\alpha_{r_0,\varphi}} f - (n+2)(n+1)|df|^2_{\alpha_{r_0,\varphi}}$$

where $f = \alpha_{r_0,\varphi}(\xi)$ and letting φ running into the set of Sasaki potentials we conclude

Lemma 3.28. *Let $(N, \mathcal{D}_\xi, J^\xi, \xi)$ be a Sasaki type transversal holomorphic manifold and let $\mathbb{T} \subset \mathrm{Aut}(N, \mathcal{D}_\xi, J^\xi)$ with $\xi \in \mathfrak{t}_+$. For any $\zeta \in \mathfrak{t}_+$, there is an extremal Sasaki metric in $\mathcal{S}(\mathcal{D}_\zeta, J^\zeta)^{\mathbb{T}}$ if and only if there exists a $\eta(\zeta)$–extremal Sasaki metric in $\mathcal{S}(\mathcal{D}_\xi, J^\xi)^{\mathbb{T}}$ for a (hence any) $\eta \in \mathcal{S}(\mathcal{D}_\xi, J^\xi)^{\mathbb{T}}$.*

When ξ is quasiregular, the latter lemma translates on the Sasaki-Reeb quotient $(M, J, [\omega])$ as follows.

Proposition 3.29. *[ACL] Let $(N, \mathcal{D}_0, J)$ be a compact CR manifold of Sasaki type and $\xi, \zeta \in \mathfrak{cr}_+(N, \mathcal{D}_0, J)$ with $[\xi, \zeta] = 0$. Suppose ξ is quasiregular with Sasaki–Reeb quotient the compact Kähler orbifold (M, J, ω). Denote $\check{\zeta}$ the image of ζ in the automorphisms group of (M, J, ω) and $\kappa = \langle\langle \zeta, \xi \rangle\rangle_{\eta_0^\xi}$. Then there is an extremal Sasaki structure in $\mathcal{S}(\zeta, J^\zeta)$ if and only if there is a $(\check{\zeta}, \kappa)$–weighted cscK metric in the Kähler class $[\omega]$.*

We also have computed explicitly the derivatives of $\widehat{\Theta}$ to show first that it is a smooth map but also to be able to compare various functionals defined respectively on these spaces of potentials, see below. Unsurprisingly, the derivative of $\widehat{\Theta}$ is closely related to the deformation of type I (see Definition/Remark 2.5) through the maps $\Psi^\xi_r := (\pi_\xi)|_{N_r} : N_r \xrightarrow{\sim} N^\xi$, $\Psi^\zeta_{\Theta(r)} := (\pi_\zeta)|_{N_{\Theta(r)}} : N_{\Theta(r)} \xrightarrow{\sim} N^\zeta$ and

$$\Psi_\varphi := \Psi^\zeta_{\Theta(e^\varphi r_0)} \circ (\Psi^\xi_{e^\varphi r_0})^{-1} : N^\xi \xrightarrow{\sim} N^\zeta.$$

Next, we consider the induced action of $\mathbb{T}$ and of its complexification $\mathbb{G} = \mathbb{T}_{\mathbb{C}}$ on the complex cone (Y,J). The important feature of this setting is that $\mathbb{G}$ naturally acts on the spaces $\mathcal{R}_\xi(Y,J)^{\mathbb{T}}$ and $\mathcal{R}_\zeta(Y,J)^{\mathbb{T}}$ of $\mathbb{T}$-invariant cone potentials of (Y,J,ξ) and (Y,J,ζ), respectively. Using the basepoints $r_0^\xi \in \mathcal{R}_\xi(Y,J)^{\mathbb{T}}$ and $r_0^\zeta \in \mathcal{R}_\zeta(Y,J)^{\mathbb{T}}$ corresponding to the contact forms η_0^ξ, η_0^ζ, there is an induced

action of $\mathbb{G}$ on the corresponding spaces $\hat{\Xi}(\xi, J^\xi, \eta_0^\xi)^{\mathbb{T}}$ and $\hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$ of $\mathbb{T}$-invariant Sasaki potentials.

In [ACL], we introduced a twisted version of the (relative) Mabuchi energy

$$\mathbf{M}^\xi_\zeta \colon \hat{\Xi}(\xi, J^\xi, \eta_0^\xi)^{\mathbb{T}} \to \mathbb{R}$$

using the usual characterization (as above) and proved that

(1) $\mathbf{M}^\zeta \circ \widehat{\Theta} = \mathbf{M}^\xi_\zeta$ (*thus we can take that as a definition*)

(2) for $\varphi \in C^\infty(M)$,
$$\mathbf{M}^\xi_\zeta(\varpi_\xi^* \varphi) = \mathbf{M}_{\check{\zeta},\kappa}(\varphi)$$
where $\check{\zeta} \in \mathfrak{t}/\mathbb{R}\xi$ is a Killing vector field on the Sasaki-Kähler quotient (M, J, ω_φ), $\kappa = \langle\!\langle \zeta, \xi \rangle\!\rangle_{\eta_0^\xi}$ and $\mathbf{M}_{\check{\zeta},\kappa}$ is the weighted Mabuchi energy of Lahdilli [142].

(3) $\widehat{\Theta}$ is bi-lipschitz with respect to the d_1-distance of Darvas [72] on $\hat{\Xi}(\xi, J^\xi, \eta_0^\xi)^{\mathbb{T}}$ extended to the Sasaki case by He–Li [122].

(4) Θ and $\widehat{\Theta}$ are $\mathbb{G}$–equivariant.

(5) $\mathbb{G}$ acts by d_1 isometries on both $\hat{\Xi}(\xi, J^\xi, \eta_0^\xi)^{\mathbb{T}}$ and $\hat{\Xi}(\xi, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$
Thus we can define, for any $\psi_0, \psi_1 \in \hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$
$$d_1^{\mathbb{G}}(\psi_0, \psi_1) := \inf_{g_0, g_1 \in \mathbb{G}} \{d_1(g_0 \cdot \psi_0, g_1 \cdot \psi_1)\} = \inf_{g \in \mathbb{G}} \{d_1(\psi_0, g \cdot \psi_1)\}.$$

Definition 3.30. A functional $F : \mathcal{P}^{\mathbb{T}} \to \mathbb{R}$ defined on a metric space $(\mathcal{P}^{\mathbb{T}}, d_1)$ on which $\mathbb{G}$ acts isometrically is $\mathbb{G}$-*proper* if there exist constants $\Lambda > 0$ and C, such that for any $\psi \in \mathcal{P}^{\mathbb{T}}$ we have $F(\psi) \geq \Lambda d_1^{\mathbb{G}}(0, \psi) - C$.

(6) $\mathbf{M}^\zeta$ is $\mathbb{G}$–proper if and only if $\mathbf{M}^\zeta_\xi$ is $\mathbb{G}$–proper. (*note* $\widehat{\Theta}(0) = 0$)

Whenever ξ is regular (thus M is smooth) (2) above allows to use directly the work of Lahdilli [142] to obtain a refinement (in the regular case) of the uniqueness result [199] assuming $\mathbb{T}$ is maximal in $\mathrm{Aut}(N, \mathcal{D}, J)$.

Lemma 3.31. *[ACL] Suppose $(N, \mathcal{D}, J, \mathbb{T}, \xi, \zeta)$ is as above with ξ regular and $\mathbb{T}$ is maximal in $\mathrm{Aut}(N, \mathcal{D}, J)$. Then any two extremal Sasaki structures $\psi, \tilde{\psi} \in \hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$ are isometric under the action of $\mathbb{G}$.*

Combining the properties of $\widehat{\Theta}$ listed above together with [122] saying that $\mathbf{M}^\zeta$ is defined on the d_1-completion of $\hat{\Xi}(\xi, J^\xi, \eta_0^\xi)^{\mathbb{T}}$, one can directly check that all the conditions of the general existence/properness principle of Darvas–Rubinstein [73, Section 3] hold true with

$$(\mathcal{R}, d, F, G) = \Big(\hat{\Xi}(\xi, J^\xi, \eta_0^\xi)^{\mathbb{T}}, d_1, \mathbf{M}^K, \mathbb{G}\Big).$$

From this we conclude the following.

Theorem 3.32 (Apostolov–Calderbank–L. [ACL])**.** *Let $(N, \mathcal{D}, J)$ be a compact CR manifold of Sasaki type, $\mathbb{T} \subset \mathrm{Aut}(N, \mathcal{D}, J)$ a maximal torus, and $\xi, \zeta \in \mathfrak{t}_+$ with ξ regular. If there exists an extremal Sasaki structure η_ψ associated to $\psi \in \hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$, then $\mathbf{M}^\zeta$ is $\mathbb{G}$-proper.*

Via Proposition 3.1, Theorem 3.32 is a particular case of "existence of a weighted cscK implies $\mathbb{G}$–properness of the corresponding weighted Mabuchi energy" proved recently by Apostolov–Jubert–Lahdili [19].

In [121] W. He considers the identity component $G := \mathrm{Aut}_0(N, \zeta, J^\zeta)$ of the group of diffeomorphisms of N preserving ζ and the transversal holomorphic structure J^ζ. This group G naturally acts on the space $\mathcal{S}(\zeta, J^\zeta)$ of compatible contact forms by pullback. There is also an induced action of G on the quotient space $\mathcal{S}(\zeta, J^\zeta)/\Omega^1_{X,\mathrm{cl}}(N)$ and hence, by using the base point η_0^ζ, on the marking $\Xi(\zeta, J^\zeta, \eta_0^\zeta)$. As G does not act on the space of η_0^ζ-relative Sasaki potentials $\hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)$, a slice $\hat{\Xi}_0(\zeta, J^\zeta, \eta_0^\zeta) := (\mathbf{I}^\zeta)^{-1}(0)$ is introduced, where $\mathbf{I}^\zeta : \hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta) \to \mathbb{R}$ is a Sasaki version of the Aubin–Mabuchi functional defined by

$$\begin{aligned} (d\mathbf{I}^\zeta)_\psi(\dot{\psi}) &= \int_N \dot{\psi}\, \eta_\psi \wedge d\eta_\psi^m, \\ \mathbf{I}^\zeta(0) &= 0. \end{aligned} \tag{3.12}$$

Unlike $\mathbf{M}^\zeta$, $\mathbf{I}^\zeta$ is not invariant but is equivariant under additive real constants and thus determines a unique representative $\psi_0 \in \hat{\Xi}_0(\zeta, J^\zeta, \eta_0^\zeta)$ of $\eta_\psi \in \Xi(\zeta, J^\zeta, \eta_0^\zeta)$. This leads to an action of G on the slice $\hat{\Xi}_0(\zeta, J^\zeta, \eta_0^\zeta)$, which we denote by $[g] \cdot \psi_0$, where $[g] \in [G]$, the effective quotient of G. One direction of the statements in [121, Theorem 1] and [122, Theorem 5.1] then yields that if there exists an extremal Sasaki metric in $\mathcal{S}(\zeta, J^\zeta)^{\mathbb{T}}$, the Mabuchi energy $\mathbf{M}^\zeta$ is proper on $\hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$ with respect to $[G]$, i.e., satisfies

$$\mathbf{M}^\zeta(\psi) \geq \Lambda d_1^{[G]}(0, \psi_0) - C, \tag{3.13}$$

where $d_1^{[G]}(0, \psi_0) := \inf_{g \in G}\{d_1(0, [g] \cdot \psi_0)\}$.

Now $\mathbb{G}$ also acts on $\mathcal{S}(\zeta, J^\zeta)^{\mathbb{T}}$, hence on $\hat{\Xi}_0(\zeta, J^\zeta, \eta_0^\zeta)$, and we let $[\mathbb{G}]$ denote the effective quotient. Although $\mathbb{G}$ is not a subgroup of G, $[\mathbb{G}]$ is a subgroup of $[G]$ and the $\mathbb{G}$-orbits of induced Sasaki structures are inside the $[G]$-orbits on $\hat{\Xi}_0(\zeta, J^\zeta, \eta_0^\zeta)$. On the other hand, the action of $\mathbb{G}$ on $\hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$ includes translations by constants, so we conclude that for any $\psi \in \hat{\Xi}(\zeta, J^\zeta, \eta_0^\zeta)^{\mathbb{T}}$ with projection ψ_0 to $\hat{\Xi}_0(\zeta, J^\zeta, \eta_0^\zeta)$ we have

$$d_1^{[\mathbb{G}]}(0, \psi_0) := \inf_{g \in \mathbb{G}}\{d_1(0, [g] \cdot \psi_0)\} \geq d_1^{[G]}(0, \psi_0), \quad d_1^{\mathbb{G}}(0, \psi) = d_1^{\mathbb{G}}(0, \psi_0) \leq d_1^{[\mathbb{G}]}(0, \psi_0).$$

Thus proving the existence of an extremal Sasaki metric implies $[\mathbb{G}]$-properness would be an improvement of both [122, Theorem 5.1] and Theorem 3.32. The missing ingredient is the transitivity result which holds for $\mathbb{G}$ thanks to Lemma 3.31 and for $[G]$ thanks to [199].

3.2.3 Sasaki K-stability and weighted K-stability

3.2.3.1 Sasaki K-stability following Collins and Székelyhidi

In [68], Collins and Székelyhidi proposed a numerical invariant to supply a notion of K-stability they introduced for Sasaki manifolds via their affine cone. The quantity they suggest to study is computed on the central fiber of the test

configuration. In [ACL] we have exhibited a global formula (i.e an expression on the total Sasaki link of the test configuration) for this invariant.

Here $\widehat{Y} \subset \mathbb{C}^N$ is a $\mathbb{T}_{\mathbb{C}}$-homogeneous affine variety, and $\mathbb{T}_{\mathbb{C}} \leq \mathrm{GL}(\mathbb{C}^N)$ is a diagonal torus whose real form is identified with $\mathbb{T}$. When ζ is quasiregular, one can further identify the space $\mathcal{H}$ of holomorphic functions on $Y := \widehat{Y}\backslash\{0\}$ introduced in §2.1.2.2 with the space of regular functions on $\widehat{Y}$ (see Remark 2.4), i.e., $\mathcal{H} = \mathbb{C}[x_1, \ldots, x_N]/I$ for a $\mathbb{T}_{\mathbb{C}}$-homogeneous ideal I. Thus, in the decomposition (2.13), $\Gamma \subset \mathfrak{t}^*$ is the set of weights of the $\mathbb{T}_{\mathbb{C}}$-action on the space of regular functions. Given a polarisation $\zeta \in \mathfrak{t}_+$, the pair $(\widehat{Y}, \zeta)$ is referred to as a *polarised affine cone*.

In this latter setting, Collins and Székelyhidi [68] introduced the $\mathbb{T}$-equivariant index character

$$H(\zeta, t) := \sum_{\alpha \in \Gamma} e^{-t\alpha(\zeta)} \dim \mathcal{H}_\alpha, \quad \mathrm{Re}(t) > 0, \tag{3.14}$$

and prove (cf. [68, Prop. 4.3]) that $F(\zeta, t)$ is a well-defined function in t, which has meromorphic extension to a neighbourhood of the origin, with Laurent expansion

$$H(\zeta, t) = \frac{a_0(\zeta) m!}{t^{m+1}} + \frac{a_1(\zeta)(m-1)!}{t^m} + O(t^{1-m}).$$

According to [68, Theorem 4.10], the coefficients $a_0(\zeta) > 0$ and $a_1(\zeta)$ are smooth functions on $\zeta \in \mathfrak{t}_+$. With these ingredients, the following definition is proposed in [68].

Definition 3.33. Given a $\mathbb{T}_{\mathbb{C}}$-homogeneous affine variety $\widehat{Y}$ polarised by $\zeta \in \mathfrak{t}_+$, and $\nu \in \mathfrak{t}$, the ζ-Donaldson–Futaki invariant is defined to be the quantity

$$\mathbf{DF}_{\widehat{Y},\zeta}(\nu) := \frac{a_0(\zeta)}{m} D_\nu(a_1(\zeta)/a_0(\zeta)) + \frac{a_1(\zeta) D_\nu a_0(\zeta)}{m(m+1) a_0(\zeta)},$$

where D_ν denotes the derivative in the direction of ν on $\mathfrak{t}$.

Following [68], Definition 3.33 can be used to assign a numerical invariant to any $\mathbb{T}$-*equivariant test configuration* associated to the polarised affine cone $(\widehat{Y}, \zeta)$, as follows. Let

$$w = (w_1, \ldots, w_N), \ w_i \in \mathbb{Z}$$

be a set of integers and denote by $\rho_w \colon \mathbb{C}^\times \to GL(\mathbb{C}^N)$ the weighted $\mathbb{C}^\times$-action defined by w. Taking the flat limit across $0 \in \mathbb{C}^\times$ of the $\mathbb{C}^\times$ orbit of $\widehat{Y}$ under the action of ρ_w, we obtain a flat family of affine schemes over $\mathbb{C}$, with central fiber C_0. Besides the $\mathbb{T}$-action on $\widehat{Y}_0$, the central fiber inherits the $\mathbb{C}^\times$-action ρ_w, and we denote by ν_w the generator of the corresponding $\mathbb{S}^1$-action on $\mathbb{C}^k$ whose flow preserves $\widehat{Y}_0$. It is shown in [68, Section 5] that for a fixed $\zeta \in \mathfrak{t}_+$, Definition 3.33 can be applied on the scheme $\widehat{Y}_0$ and gives a necessary condition for the existence of a CSC Sasaki structures (known as K-semistability).

3.2.3.2 The global Futaki invariant of a polarised cone

The notion of a (compactified) polarised test configuration associated to $(M, J, L, \mathbb{T})$ goes back to [82, 196]. In [ACL], we considered an analogous notion for polarised

cones (with compact cross section) simply by considering the dual bundles involved in a standard polarised test configuration. Let's recall the definition of the latter and introduce notation.

Definition 3.34. A $\mathbb{T}$-*equivariant polarised test configuration of exponent* $s \in \mathbb{N}$ associated to $(M, J, L, \mathbb{T})$ is a normal polarised variety $(\mathcal{M}, \mathcal{L})$ endowed an action $\mathbb{T} \leq \mathrm{Aut}(\mathcal{M}, \mathcal{L})$, and

1. a (flat) surjective morphism $\pi\colon \mathcal{M} \to \mathbb{C}P^1$ such that $\check{\mathbb{T}}$ preserves each fiber $M_t := \pi^{-1}(t)$ and, for $t \neq 0$, $(M_t, L_t := \mathcal{L}_{|M_t})$ is $\mathbb{T}$-equivariantly isomorphic, as a polarised variety, to $(M, L, \mathbb{T})$;

2. a $\mathbb{C}^\times$-action ρ on $(\mathcal{M}, \mathcal{L})$ commuting with $\mathbb{T}$ and covering the usual $\mathbb{C}^\times$-action on $\mathbb{C}P^1$;

3. an isomorphism of polarised varieties

$$\lambda\colon \Big(M \times (\mathbb{C}P^1 \setminus \{0\}),\, L^s \otimes \mathcal{O}_{\mathbb{C}P^1}(s)\Big) \cong (\mathcal{M} \setminus M_0,\, \mathcal{L}),$$

which is equivariant with respect to the actions of $\mathbb{T} \times \mathbb{S}^1_0$ on $\big(M \times (\mathbb{C}P^1 \setminus \{0\}),\, L^s \otimes \mathcal{O}_{\mathbb{C}P^1}(s)\big)$ and $\mathbb{T} \times \mathbb{S}^1_\rho$ on $\big(\mathcal{M} \setminus M_0,\, \mathcal{L}\big)$, where $\mathbb{S}^1_0$ stands for the standard $\mathbb{S}^1$-action on $\mathcal{O}(1) \to \mathbb{C}P^1$ and $\mathbb{S}^1_\rho$ is the $\mathbb{S}^1$-action induced by ρ.

The test configuration $(\mathcal{M}, \mathcal{L})$ as above is called *product* if $\big(\mathcal{M} \setminus M_\infty, \pi\big)$ is $\check{\mathbb{T}}$-equivariantly isomorphic (as morphism of complex varieties) to $\big(M \times (\mathbb{C}P^1 \setminus \{\infty\}), \pi_{\mathbb{C}P^1}\big)$.

Assuming that $s = 1$ and $\mathcal{M}$ is smooth for simplicity, we consider the corresponding complex cones $\mathcal{Y} := (\mathcal{L}^*)^\times$ and $Y := (L^*)^\times$ and denote by $\tilde{\zeta}$ and ζ the corresponding (regular) cone polarisations. We thus get the following objects:

(i) a flat surjective morphism $\pi\colon \mathcal{Y} \to \mathbb{C}P^1$ such that $\mathbb{T}$ preserves each fiber $Y_t := \pi^{-1}(t)$ and, for $t \neq 0$, $(Y_t, \tilde{\zeta}_{|Y_t})$ is $\mathbb{T}$-equivariantly isomorphic, as a polarised complex cone, to (Y, ζ);

(ii) a $\mathbb{C}^\times$-action ρ on $\mathcal{Y}$ commuting with $\mathbb{T}$ and covering the usual $\mathbb{C}^\times$-action on $\mathbb{C}P^1$;

(iii) a biholomorphism

$$\lambda\colon Y \times (\mathbb{C}P^1 \setminus \{0\}) \cong \mathcal{Y} \setminus Y_0, \tag{3.15}$$

which is equivariant with respect to the actions of $\mathbb{T} \times \mathbb{S}^1_0$ on $Y \times (\mathbb{C}P^1 \setminus \{0\})$ and $\mathbb{T} \times \mathbb{S}^1_\rho$ on $\mathcal{Y} \setminus Y_0$, where $\mathbb{S}^1_0$ stands for the standard $\mathbb{S}^1$-action on $\mathcal{O}(1) \to \mathbb{C}P^1$.

This prompts the following

Definition 3.35. Let (Y, J, ζ) be a smooth polarised complex cone endowed with a holomorphic action of a compact torus $\mathbb{T}$, and $\zeta \in \mathfrak{t}_+$ a Sasaki–Reeb polarisation. A *smooth* $\mathbb{T}$-*equivariant test configuration of* (Y, J, ζ) is a smooth polarised complex cone $(\mathcal{Y}, \mathcal{J}, \tilde{\zeta})$, endowed with a $\mathbb{T}$-action such that $\tilde{\zeta} \in \tilde{\mathfrak{t}}^+$, satisfying the conditions (i)–(iii) above.

Using the description of the Reeb cone as the dual of the momentum cone see Remark 2.13 (and thus are convex polyhedral cone) we get that the Reeb cone (in $\mathfrak{t}$) of $(\mathcal{Y}, \mathcal{J}, \tilde{\zeta})$ and (Y, J, ζ) coincide and that

$$\zeta = \tilde{\zeta}.$$

Now, from Proposition 2.17 a smooth complex polarised cone say $(\mathcal{Y}, \mathcal{J}, \zeta)$ corresponds to a transversal holomorphic manifold $(\mathcal{N} := \mathcal{Y}/\mathbb{R}^+_{(-\mathcal{J}\zeta)}, \zeta, \mathcal{J}^\zeta)$ together with a marking $\Xi \subset \mathcal{S}(\mathcal{N}, \zeta, \mathcal{J}^K)$ i.e a slice in the space of compatible contact forms induced by radial potentials via Lemma 2.16. Each $r \in \mathcal{R}_\zeta(\mathcal{Y})^{\mathbb{T}}$ determines a form $\eta_r \in \Xi$ such that the Sasaki structure induced on $(\mathcal{N}, \zeta, \mathcal{J}^K)$ is Sasaki isomorphic to the Sasaki manifold

$$(\mathcal{N}_r := r^{-1}(1), \mathscr{D} := T\mathcal{N} \cap \ker(d^c \log r), \mathcal{J}|_{\mathscr{D}}, \zeta)$$

with contact form $\eta_r := d^c \log r$. The gobal invariant we propose below (3.16), does not depend on the choice of $r \in \mathcal{R}_\zeta(\mathcal{Y})^{\mathbb{T}}$ following a standard argument [111, Prop. 4.4].

Definition 3.36. [ACL] Given a smooth $\mathbb{T}$-equivariant test configuration $(\mathcal{Y}, \zeta)$ of the polarised complex cone (Y, ζ) with $\zeta \in \mathfrak{t}_+$ as in Definition 3.35, and a radial potential $r \in \mathcal{R}_\zeta(\mathcal{Y})^{\mathbb{T}}$, we define the *global Futaki invariant* of $(\mathcal{Y}, \zeta)$ as

$$\begin{aligned} \mathbf{GF}_\zeta(\mathcal{Y}) := &- \frac{1}{(n+1)!} \int_{\mathcal{N}_r} \Big(\mathrm{Scal}^T_{\mathcal{J}}(\eta_r) - c_\zeta\Big) \eta_r \wedge (d\eta_r)^{n+1} \\ &+ \frac{2}{n!} \int_{\mathcal{N}_r} (\pi^* \omega_{FS}) \wedge \eta_r \wedge (d\eta_r)^n, \end{aligned} \tag{3.16}$$

where $\mathcal{N}_r := r^{-1}(1)$, $\eta_r := (d^c \log r)|_{T\mathcal{N}_r}$, $\mathrm{Scal}^T_{\mathcal{J}}(\eta_r)$ is the transversal scalar curvature of the Sasaki structure by the inclusion $\mathcal{N}_r \subset \mathcal{Y}$ and η_r, $c_\zeta = \mathbf{S}(\zeta)/\mathbf{V}(\zeta)$, and ω_{FS} is the Fubini–Study metric on $\mathbb{C}P^1$ satisfying $\mathrm{Ric}(\omega_{FS}) = \omega_{\mathrm{FS}}$.

The invariant $\mathbf{GF}_\zeta$ looks like the intersection formulation of the Donaldson–Futaki invariant discussed in 2.2 and, unsurprisingly, it coincides, up to a factor 2π, with the Donaldson–Futaki invariant of the Sasaki Reeb quotient whenever ζ is quasiregular. Indeed, when ζ is induced by a circle action $\mathbb{S}^1_\zeta$ and given $r \in \mathcal{R}_\zeta(\mathcal{Y})^{\mathbb{T}}$, $\mathcal{N}_r$ is a principal $\mathbb{S}^1_\zeta$–bundle over $\mathcal{M}_\zeta := \mathcal{N}_r/\mathbb{S}^1_\zeta$ and η_r a connection 1-form, so the integrals in (3.16) (see that apart from η_r all the forms are ζ-basic) are 2π times the corresponding integrals on $\mathcal{M}_\zeta$. Moreover, $\mathcal{M}_\zeta$ (together with its induced complex structure and polarisation $\mathcal{L}_\zeta$ coming from $\mathcal{Y}_\zeta$) is a polarised (orbifold) test configuration over the compact orbifold $Y_1/\mathbb{C}^*_\zeta$ where $Y_1 = \pi^{-1}(-1) \subset \mathcal{Y}$ is the fiber at 1. Using this, together with the fact that quasi regular vector fields are dense in $\mathfrak{t}_+$ and applying Donaldson's argument [82, p.315], see also [175, 201, 164], to relate the Donaldson-Futaki invariant (asymptotic weight on the central fiber) to an intersection formula of orbiline bundles over the orbifold base $\mathcal{M}_\zeta$. Then Ross–Thomas extension of K-stability theory [180] was already related to the concept of K-stability in [68], which we used to get the following.

Proposition 3.37. *[ACL] Let $(\mathcal{Y}, \zeta)$ be a smooth $\mathbb{T}$-invariant test configuration of polarised complex cones associated to (Y, ζ) with $\zeta \in \mathfrak{t}_+$. Consider the*

corresponding polarised affine cone $(\widehat{\mathcal{Y}}, \zeta)$ *with central fiber the polarised affine variety* $(\widehat{Y}_0, \zeta)$ *and associated Donaldson–Futaki invariant* $\mathbf{DF}_{\widehat{Y}_0,\zeta}(\nu_w)$*. Then*

$$\mathbf{DF}_{\widehat{Y}_0,\zeta}(\nu_w) = \lambda(n)\mathbf{GF}_\zeta(\mathcal{Y}). \tag{3.17}$$

for a dimensional constant $\lambda(n) > 0$*.*

3.2.3.3 Applications to K-polystability

Berman–Darvas–Lu [34] have used properness (relative to the reduced group of automorphisms) of the Mabuchi energy to prove K-polystability of polarised cscK manifolds. Lahdili in [142] modified the argument from [34] (and from [34, 186]) to incorporate the weights he is working with, doing so, he proved that the existence of weighted cscK metric implies that his weighted Donaldson-Futaki invariant $\mathcal{F}_{\xi,\kappa}$ is non-negative on any smooth $\mathbb{T}$–equivariant test configurations with reduced central fiber. Using again the assumption that ξ is regular and Proposition 3.1 to translate all our problem into the framework of weighted cscK structures on a smooth compact Kähler manifold, we were actually able to improve [142, Theorem 2] in our setting. The argument simplifies thanks to the fact that $\mathbb{G}$ acts directly on the potentials we consider (no-normalisation, by Aubin functional, is needed and even the original proof of [34] is simplified as we can take a fixed base point). Note also that compare to Lahdili's framework the weights in our setting come from a natural geometric model and are rational functions. Finally, using integration by parts we were able to relate the weighted Donaldson-Futaki invariant of Lahdili to the global Futaki invariant of the associated polarised cone defined above.

Theorem 3.38. *[Apostolov–Calderbank–L. [ACL]] Let* $(N, \mathcal{D}, J, \xi)$ *be a compact regular Sasaki manifold and* $\mathbb{T}$ *a maximal torus in* $\mathrm{Aut}(N, \mathcal{D}, J)$*. Let* $\zeta \in \mathfrak{t}_+$ *be a Sasaki–Reeb vector field with associated polarised complex cone* (Y, J, ζ)*. If there exists an extremal Sasaki structure in* $\mathcal{S}(\zeta, J^\zeta)$*, then for any smooth polarised* $\mathbb{T}$*–equivariant test configuration* $(\mathcal{Y}, \mathcal{J}, \zeta)$ *over* (Y, J, ζ) *with reduced central fiber and which is not a product we have*

$$\mathbf{GF}_\zeta(\mathcal{Y}, \mathcal{J}) > 0.$$

In particular, $\mathbf{DF}_{\widehat{Y}_0,\zeta}(\nu_w) > 0$*.*

Chapter 4

Extremal metrics in toric geometries

As highlighted in [86], the symplectic point of view (fixing the symplectic form instead of the complex structure) has the advantage to translate more directly the Calabi problem into the formalism of Gauge Theory. In this setting, almost Kähler metrics can be included in the picture naturally and this is on this space that the Donaldson–Fujiki momentum map approach to the YTD conjecture [100, 82] takes its place. In (compact) toric geometry, the correspondence between symplectic versus complex point of views has a simpler form via the affine geometric dictionary established along the years. In particular, from the symplectic side, a $\mathbb{T}$–invariant extremal Kähler metric corresponds to a convex solution of non-linear PDE equation (recalled below) on the moment polytope $\overline{P} \subset \mathfrak{t}^*$. The convex solution must satisfy some boundary condition encoded by a *labelling* ℓ, a set of affine functions over $\mathfrak{t}^*$, say $\ell = \{\ell_1, \dots, \ell_d\}$, so that

$$\overline{P} := \{x \in \mathfrak{t}^* \,|\, \ell_s(x) \geq 0\}$$

uniquely determined by the lattice of circle subgroup $\Lambda \subset \mathfrak{t}$. This correspondence is obtained by combining works from Guillemin [115], Abreu [1, 2], Donaldson [82] (precision about the boundary conditions were given in [83, 12]) building on the Delzant–Lerman–Tolman correspondence for toric symplectic orbifolds [77, 157].

As advertized by Donaldson in [82, 86], the resulting *Calabi problem* on (P, ℓ) makes sense more generally as an abstract 4th order PDE problem. Moreover, ($\mathbb{T}$–relative) K-stability should have an affine-convex/analytical counter part in terms of (P, ℓ) which would obstruct the PDE problem to admit a convex solution. This idea has been exploited and pursued by many Kähler geometers during the last two decades, notably [9, 13, 63, 147, 210] and Donaldson used this theory to prove the YTD conjecture for toric surfaces with vanishing Futaki invariant in [82, 83, 84, 82].

Using important inputs of Zhou–Zhu [210] and Donaldson [82], the recent resolution by Chen–Cheng [61] and He [120] of an analytical version of the YTD conjecture can be translated in terms of the labelled polytope data (P, ℓ) as explained in [L4] and reported in §4.2. In this dissertation, this condition on the

labelled polytope (P, ℓ) *uniform K-stability*[1], a notion (in the toric case) going back to Donaldson seminal paper [82] and recalled in Definition 4.25 below. Uniform K-stability was proven to be necessary when a smooth convex solution of the Calabi problem exists on (P, ℓ) by Chen–Li–Sheng [63]. As I pointed out in [L4], their proof holds as well for toric extremal almost Kähler metric of involutive type. A direct application is that the existence of an extremal toric *almost*–Kähler metrics (of involutive type) on a compact symplectic toric manifold is equivalent to the existence of an extremal toric Kähler metric as predicted by Donaldson [82].

As a guiding principle, all the $\mathbb{T}$–invariant topological/geometric properties of a toric symplectic manifold have a combinatorial counterpart expressed in terms of its labelled polytope (P, ℓ). An example that will be explained below is that a symplectic manifold (M, ω) is monotone if and only if $\exists \lambda > 0$ such that $\lambda[\omega] = c_1(M)$. A toric symplectic manifold is monotone if and only if there exits a point $p \in P$ such that $\ell_1(p) = \cdots = \ell_d(p)$. The latter condition makes sense for any polytope likewise the Futaki invariant is translated by a condition on the barycenters of P and ∂P with measures defined below. In [L3], I proved that given any convex compact polytope there exists a unique labeling such that (P, ℓ) is monotone and has Futaki invariant zero. This result complement observations from [82]. Then inspired by, and building on, [86] I gave an argument to extend the Wang–Zhu Theorem [202] to the abstract setting and ensure the existence of a solution of a Kähler–Einstein equation on *simple* labelled polytopes. Simple polytopes have a natural simplicial fan attached at each vertex and I used these as symplectic charts even in the abstract case when they don't glue together to give a compact toric manifold. This idea was also mentioned by Donaldson in [84]. I discuss these technics, problems and solutions in §4.1 together with geometric applications. In particular, we get existence and uniqueness of a Kähler–Einstein metric with Einstein constant 1, with singularity of conical type along the invariant divisor, on any compact toric manifold. The angle on the singularity can change depending on the component of the divisor and it can be greater, equal or smaller than 2π. These solutions include those found by Berman–Berndtsson [32] using Monge–Ampère technics.

A basic idea used in the results above is that any torus invariant tensor/function on a toric symplectic manifold $(M^n, \omega, \mathbb{T})$ correspond to a tensor/function defined on the moment polytope with appropriate boundary condition. In a joint work with Sena-Dias [LSD], we explored the limit of this idea showing that the first eigenvalue of the Laplacian of any n–toric Kähler manifold is sharply bounded by the one of the Fubini-Study metric Laplacian on $\mathbb{CP}^n$ but the set of invariant first Laplacian eigenvalues of compatible toric Kähler metrics on $(M^n, \omega, \mathbb{T})$ is $]0, +\infty[$.

[1] It is not clear a priori that this notion coincides with other types of uniform K-stability proposed in the literature [126, 8] but it is the only notion of uniform K-stability discussed here, there should be no confusion.

4.1 The Calabi problem on polytopes

4.1.1 The Abreu equation on symplectic potentials and applications

Here $\mathfrak{t}^*$ is a n–dimensional affine space and $P \subset \mathfrak{t}^*$ is a relatively compact open convex polytope which is *simple*(i.e any vertex of P is the intersection of exactly n–*facets* (codimension 1-faces)). The solution we seek must satisfy some boundary condition in terms of a set of inward normal vectors to P encoded in a (minimal) set of affine-linear functions $\ell_1, \dots, \ell_d \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ such that

$$P = \{x \in \mathfrak{t}^* \,|\, \ell_s(x) > 0, s = 1, \dots, d\}.$$

We call such a pair (P, ℓ) a *labelled polytope* and $\ell = \{\ell_1, \dots, \ell_d\}$ a *labelling* of P. A *symplectic potential* of (P, ℓ) is a convex function $u \in C^0(\overline{P}) \cap C^\infty(P)$ whose restriction to any face's interior is smooth and convex and such that

$$u - \frac{1}{2}\sum_{s=1}^{d} \ell_s \log \ell_s \in C^\infty(\overline{P}).$$

We denote $\mathcal{S}(P, \ell)$ the set of symplectic potentials of (P, ℓ). The function $\frac{1}{2}\sum_{s=1}^{d} \ell_s \log \ell_s \in C^0(\overline{P}) \cap C^\infty(P)$ lies in $\mathcal{S}(P, \ell)$ and is called the *Guillemin potential*. As we will recall below in §4.1.4, to each $u \in \mathcal{S}(P, \ell)$ is associated a riemannian metric g_u on $P \times \mathfrak{t} \subset T^*\mathfrak{t}$ which, together with the Liouville structure defines a Kähler structures.

A symplectic potential $u \in \mathcal{S}(P, \ell)$ satisfies the *Abreu equation* if its "scalar curvature" is affine-linear on P, that is

$$S(u) := -\sum_{i,j=1}^{n} u^{ij}_{,i,j} \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R}) \tag{4.1}$$

where u^{ij} is this inverse of the Hessian of u and $(f_{,i,j})$ denotes the Hessian of a function f. Particular cases include the *Abreu cscK equation* which is

$$S(u) \text{ is constant}$$

and the *Kähler–Einstein equation* (with Einstein constant λ) which is the matrix-equation

$$\rho(u) := -\frac{1}{2}\left(\sum_{i}^{n} u^{ik}_{,i,j}\right)_{1 \le j,k \le n} = \lambda \mathrm{Id}_n \tag{4.2}$$

where Id_n is the $n \times n$ identity matrix.

Elementary considerations, see Definition/Proposition4.14, imply that if (4.2) holds then $\lambda > 0$. Also, if $S(u) \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ then $S(u) = A(P, \ell)$ where $A(P, \ell)$ is predetermined by the data (P, ℓ). Moreover, an argument due to Guan [114] can be applied directly here to claim that, up to addition by an affine-linear function, there exists at most one solution $u \in \mathcal{S}(P, \ell)$ of (4.1).

Solutions of the Calabi problem on polytopes include at least three interesting special geometric structures:

1) When (P,ℓ) has *rational type* that is when the vectors $d\ell_1,\dots,d\ell_d \in \mathfrak{t}$ span a lattice $\Lambda \subset \mathfrak{t}$, a solution $u \in \mathcal{S}(P,\ell)$ to the Abreu (resp. KE) equation corresponds to an extremal Kähler metric (resp. KE metric) on the toric symplectic orbifold associated to (P,ℓ,Λ) through Delzant-Lerman-Tolman correspondence [115, 1, 2, 82, 12].

2) When $\ell_1,\dots,\ell_d \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$ span a lattice Λ in $\mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$, a solution $u \in \mathcal{S}(P,\ell)$ to the Abreu (resp. KE) equation corresponds to an extremal Sasaki metric (resp. η–Einstein metric) on the contact orbifold associated to (P,ℓ,Λ) through Lerman correspondence [3, 148, 167] see also 5.2.1.

3) When $(P,\tilde{\ell})$ has rational type with lattice $\Lambda = \mathrm{Span}_{\mathbb{Z}}\{d\tilde{\ell}_1,\dots d\tilde{\ell}_d\}$, a solution $u \in \mathcal{S}(P,\ell)$ to the Abreu (resp. KE) equation for another set of labelling ℓ of P, corresponds to an extremal Kähler metric (resp. KE metric) with edges singularity along the invariant divisor on the toric symplectic orbifold associated to $(P,\tilde{\ell},\Lambda)$ [L3], see4.1.4.

Remark 4.1. To substitute "orbifolds" by "manifolds" in the cases 1 and 3 above one needs to impose the following *Delzant condition:* for each vertex $p \in \overline{P}$ the set of vectors $\{d\ell_s \,|\, \ell_s(p)=0\}$ is a basis of the lattice Λ span by the normal vectors [77]. In the contact Sasaki case (case 2 above) the condition is $\mathrm{Span}_{\mathbb{Z}}\{\ell_s \,|\, \ell_s(p)=0\} = \Lambda \cap \mathrm{Span}_{\mathbb{R}}\{\ell_s \,|\, \ell_s(p)=0\}$ for any vertex p, [148]. In both cases, the Delzant condition imposes that P is simple.

4.1.2 Unicity of Kähler–Einstein symplectic potential on simple polytopes.

With this terminology in place, the main result in [L3] is the following.

Theorem 4.2 (Donaldson [86], L. [L3])**.** *For any (relatively compact simple) labelled polytope P and $\lambda > 0$, there exists a unique labelling ℓ and a solution $u \in \mathcal{S}(P,\ell)$ of the Kähler–Einstein equation* (4.2) *with Einstein constant λ. Moreover, this solution u is unique in $\mathcal{S}(P,\ell)$ up to addition by an affine-linear function.*

This extends the Theorem of Wang–Zhu [202] which ensures the existence of a smooth toric Kähler–Einstein metric on any toric Fano manifolds with vanishing Futaki invariant (the case of Kähler-Ricci soliton is discussed below). Shi and Zhu [183] proved the orbifold version. A proof of the existence part of Theorem 4.2 was proposed in [86] using only the affine geometry side of the picture. The demonstration in [L3] uses the symplectic charts associated to the vertices of P (and the labelling) to extends the Theorem of Wang–Zhu in this abstract context and will be discussed below. First, we explain some consequences of this result.

Combined with the openness result of Donaldson [84] and the linear constraint $A(P,\ell) =$ constant, Theorem 4.2 implies the following corollary where the set of labellings of P is denoted $\mathcal{L}ab(P)$. The latter is an open quadrant in the space $\bigoplus_{i=1}^{d} F_i^0 \simeq \mathbb{R}^d$ where F_i^0 is the annihilator in $\mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$ of the facet $F_i \subset \overline{P} \subset \mathfrak{t}^*$.

Corollary 4.3. *[L3] For any simple, relatively compact, polytope P, there is a non empty open set of labellings $U \subset \mathcal{L}ab(P)$ such that for each $\ell \in U$, $\mathcal{S}(P,\ell)$ contains a solution of the extremal equation* (4.1)*. Moreover U intersect non-trivially a $(d-n)$–dimensional linear subspace so that each solution with labelling in the intersection has constant scalar curvature $S(u)$.*

4.1.3 Applications on existence of toric Kähler–Einstein orbifolds

Checking when the labelling found in Theorem 4.2 is rational leads to the following.

Corollary 4.4. *[L3] Any compact simple polytope $\overline{P}$ whose vertices span a lattice in $\mathfrak{t}^*$, is the moment polytope of a Kähler–Einstein orbifold, unique up to finite orbifold covering and homothety.*

Remark 4.5. 1) The volume of a toric symplectic manifold $(M^n, \omega, \mathbb{T} = \mathfrak{t}/\Lambda)$ is the volume of a polytope with respect to the measure on $\mathfrak{t}^*$ induced by the lattice Λ, [115]. The point of view here is to fix a polytope P but not the lattice. Going through Delzant construction with $(P, a\ell, a\Lambda)$ where a runs in $\mathbb{R}_{>0}$ produces homothetic symplectic manifolds/orbifolds.
2) Observe also that corollary 4.4 does not come in contradiction with the fact that there is only a finite number of toric Fano Gorenstein varieties in each dimension. As discussed in [157, Remark 9.6] the classification of toric symplectic manifolds is not equivalent to the one of toric algebraic varieties, the co-dimension one orbifold singularities "disapear" in algebraic geometry.

There exist smooth irregular toric Sasaki–Einstein metrics see [113]. The labelled polytope (P,ℓ) associated to such object cannot be rational (by unicity of Sasaki–Einstein metric in the Reeb cone [167]). However, the Kähler–Einstein solution u with scalar curvature $2n(2n+2)$ on a lattice polytope P defines a Sasaki–Einstein metric on a contact orbifold associated the cone, labelled by ℓ, over P in $\mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$, [3, 148].

Comment 4.6. It is not stated as such in [L3] (partly because the classification of toric contact orbifold is not developed as much as the symplectic one) but the contact/Sasaki counterpart of Corollary 4.4 should be expressed in terms of the embedding of P into $\mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})^*$ by the evaluation map (i.e for $x \in \mathfrak{t}^*$, $\mathrm{ev}_x \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})^*$). Namely that, any simple relatively compact polytope $P \subset \mathfrak{t}^*$ such that $ev_x(\overline{P})$ is a lattice polytope in $\mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})^*$ is the associated moment polytope of a Sasaki–Einstein orbifold, unique up to covering.

4.1.4 Toric Kähler-Einstein metrics with conical singularity along a divisor

Let $(M^n, \omega, \mathbb{T} = \mathfrak{t}/\Lambda)$ be a toric symplectic manifold with momentum map $\mu : M \to \mathfrak{t}^*$ and moment polytope $\mu(M) = \overline{P}$. The natural labelling $\tilde{\ell}$ here is the one so that the normal inwards vectors $d\tilde{\ell}_1, \ldots, d\tilde{\ell}_d \in \mathfrak{t}$ are primitive in the lattice of circle subgroups $\Lambda \subset \mathfrak{t}$.

We consider $P \times \mathfrak{t} \subset T^*\mathfrak{t}$ with its canonical symplectic Liouville form ω_{can}, that is, if (x, θ) are coordinates on $P \times \mathfrak{t}$ (where $x : \mathfrak{t}^* \to \mathbb{R}^n$ are affine and

dual to θ) $\omega_{\rm can} = \sum_{i=1}^n dx_i \wedge d\theta_i$. The toric symplectic theory [115] implies that there is a $\mathbb{T}$–equivariant symplectomorphism

$$\Phi : \mathring{M} := \mu^{-1}(P) \longrightarrow (P \times \mathbb{T}, \omega_{\rm can})$$

so that the first projection $\mathring{M} \to P$ coincides with μ. Also, each convex function $\tilde{u} \in C^\infty(P)$ defines a $\mathbb{T}$–invariant $\omega_{\rm can}$–compatible Kähler metric

$$g_{\tilde{u}} = \sum_{i,j=1}^n \tilde{u}_{,i,j} dx_i \otimes dx_j + \tilde{u}^{ij} d\theta_i \otimes d\theta_j \tag{4.3}$$

on $(P \times \mathfrak{t}, \omega_{\rm can})$, thus on $(P \times \mathbb{T}, \omega_{\rm can})$. The theory established in [2, 12, 84] states that the pull back $\Phi^* g_{\tilde{u}}$ is the restriction to $\mathring{M}$ of a smooth $\mathbb{T}$–invariant Kähler metric on (M, ω) if and only if $\tilde{u} \in \mathcal{S}(P, \tilde{\ell})$.

Now, for any labelling ℓ of P, a symplectic potential $u \in \mathcal{S}(P, \ell)$ determines a Kähler metric g_u on the manifold $P \times \mathfrak{t} \subset T^*\mathfrak{t}$ compatible with its canonical symplectic form via formula (4.3) and thus on $(P \times \mathbb{T}, \omega_{\rm can})$. However, when $\ell \neq \tilde{\ell}$, $\Phi^* g_u$ is not the restriction of a smooth metric on M. Analysing the behaviour of $\Phi^* g_u$ along the invariant divisor in M, it is not surprising to find conical singularity with angle $2\pi\tilde{\ell}_s/\ell_s =: 2\pi\beta_s$ along the preimage $\mu^{-1}(\mathring{F}_s)$ of the interior of the facet $F_s := \ell_s^{-1}(0) \cap \overline{P} = \tilde{\ell}_s^{-1}(0) \cap \overline{P}$. Precisely, assuming up to a rotation that $d\ell_s = dx_1$ taking polar coordinates $(\frac{1}{2}r^2 := \ell_s, \theta_1)$ transverse to $\mu^{-1}(\mathring{F}_s)$, recycling the computation in [12], we get

$$g_u =_{loc} \frac{1}{\beta_k} \left(dr \otimes dr + \beta_k^2 r^2 d\theta_1 \otimes d\theta_1 \right) + smooth. \tag{4.4}$$

Consequently, we obtain

Proposition 4.7. *[L3] Let $(M, \omega, \mathbb{T} = \mathfrak{t}/\Lambda)$ be a smooth compact symplectic toric manifold associated to the Delzant labelled polytope $(P, \tilde{\ell}, \Lambda)$. For any labelling ℓ of P, the symplectic potentials in $\mathcal{S}(P, \ell)$ define $\mathbb{T}$–invariant, compatible, Kähler metrics on the open dense subset of M where $\mathbb{T}$ acts freely. Each of these metrics has a conical singularity of angle $2\pi\tilde{\ell}_s/\ell_s > 0$ along the pre-image of the interior of the facet $\mathring{F}_s$.*

Denote ℓ^λ the labelling obtained in Theorem 4.2 corresponding to the Einstein constant $\lambda > 0$. As a corollary of Theorem 4.2 we get the following.

Proposition 4.8. *[L3] Let $(M, \omega, \mathbb{T} = \mathfrak{t}/\Lambda)$ be a smooth compact symplectic toric manifold associated to the Delzant labelled polytope $(P, \tilde{\ell}, \Lambda)$.*

For any $\lambda > 0$, there exists a $\mathbb{T}$–invariant Kähler–Einstein metric $g_{KE,\lambda}$ smooth on the open dense subset $\mathring{M}$ where the torus acts freely, compatible with ω and with Einstein constant λ. The type of singularity of $g_{KE,\lambda}$ along the pre-image of the interior of the facet $\mathring{F}_s$ is of edge type with angle $2\pi\tilde{\ell}_s/\ell_s^\lambda > 0$ along the pre-image of the interior of the facet $\mathring{F}_s$. Moreover, any ω–compatible $\mathbb{T}$–invariant Kähler–Einstein metric on $\mathring{M}$ with Einstein constant λ is $\mathbb{T}$-equivariantly isometric to $g_{KE,\lambda}$.

According to Lerman–Tolman [157], any two $\mathbb{T}$–invariant complex structures compatible with ω on M are $\mathbb{T}$–equivariantly biholomorphic. This biholomorphism can be made more explicit by using the Legendre duality, namely given

two symplectic potentials $u, u_o \in \mathcal{S}(P, \tilde{\ell})$, normalized so that both reach their minimum at the same point $p \in P$, the map

$$(du_o \times \mathrm{Id})^{-1} \circ (du \times \mathrm{Id}) : P \times \mathbb{T} \to P \times \mathbb{T}$$

is the restriction of a smooth diffeomorphism of M and send the complex structure associated to g_u to the one associated to g_{u_o} see [84, 8]. Of course, if u and u_o are symplectic potentials with respect to distinct labellings of P the map above does not extend smoothly on M. However, on $\mathring{M}$ it identifies the complex structures associated respectively to g_u and g_{u_o} and allows to express the singularity (4.4) in complex coordinates which turns out to be conical singularity along divisors as studied in [89].

Remark 4.9. Passing from symplectic to complex point of views is direct in (compact) toric geometry. When (P, ℓ) is the labelled polytope of a smooth compact toric symplectic manifold (M, ω) with momentum map $\mu : M \to \overline{P}$, to $u \in \mathcal{S}(P, \sigma)$ is associated a ω–compatible Kähler structure (g_u, J_u) on M. In this context, Guillemin proved in [115] that the map

$$(x, \theta) \mapsto d_x u + \sqrt{-1}\theta$$

provides complex coordinates on $(T_q M, J_u) \simeq \mathbb{C}^n$ where $\mu(q) \in P$ is the minimum of u. Thus it identifies the "big" orbit $(\mathring{M}, J_u) \simeq (\mathbb{C}^*)^n$ through the $\mathbb{T}_{\mathbb{C}}$ orbit of q, i.e. the flows starting at q of the vector fields $\underline{e}_1, \ldots, \underline{e}_n, J_u\underline{e}_1, \ldots, J_u\underline{e}_n$ where $e_1, \ldots, e_n$ is a basis of $\mathfrak{t}$.

In coordinates $(y, \theta) = (d_x u, \theta)$, a Kähler potential of the Kähler form ω is given by the Legendre transform $\psi_u : \mathfrak{t} \to \mathbb{R}$ of u, that is

$$\psi_u(y) = \langle x, y \rangle - u(x) \tag{4.5}$$

for $y = d_x u$.

The constraints $\lambda = 1$ and $\beta_k \leq 1$ (i.e angles smaller than 2π) impose condition on P which can be expressed in terms of the barycenter of P. In [32], Berman and Berndtsson have discovered this condition using Monge-Ampère technics (working in complex coordinates) and have interpreted their result in terms of K-stability of the Fano pairs, confirming the Yau–Tian–Donaldson conjecture in this case. They also have related their result to Chi Li toric formula of the Ricci lower bound [158] which was further studied in [159].

Remark 4.10. Consider the 2–sphere with a smooth Kähler–Einstein metric $(\mathbb{S}^2, g_{KE}, \omega)$ with Einstein constant 1 and let (P, ℓ) be its associated Delzant labelled polytope. For $a > 0$, the Kähler–Einstein symplectic potential in $(P, a\ell)$ defines (via Proposition 4.8) a singular Kähler–Einstein structure $(\mathbb{S}^2, g_{KE,a}, \omega)$. The existence of these metrics are well-known. It is also well-known that there is no $\mathbb{S}^1$–invariant csc metric on sphere, smooth outside the North and South poles, with conical singularity of *distinct* angles at the two poles. Proposition 4.8 extend this fact to any toric symplectic manifold.

Comment 4.11. Let $(M, \omega, \mathbb{T})$ be a smooth compact symplectic toric manifold associated to the Delzant labelled polytope $(P, \tilde{\ell}, \Lambda)$. There is a $(d - n)$–dimensional cone of labellings of P that can be identified with the Kähler cone

of M (remember that all two $\mathbb{T}$–invariant complex structures on M are $\mathbb{T}$–equivariantly biholomorphic). However, the change of coordinates in this case do not preserve the Abreu equation in general and extremal Kähler potentials in other classes are not sent to extremal Kähler potentials on P. I suspect a kind of weighted csck equation in the sense of Lahdilli [142] (with maybe more than one weight function v) should be involved here.

4.1.5 Obstructions to existence of Kähler–Einstein symplectic potentials

The Kähler–Einstein equation (4.2) has a better and more classical expression in terms of the Ricci potential. To express this in terms of (P, ℓ), note that from the duality established in [115], given $u \in \mathcal{S}(P, \ell)$ the only smooth pluriharmonic functions are the affine-linear functions in the variable $du \in \mathfrak{t}$, namely $dd^c f := -d(df \circ J_u) = 0$ iff $f(x) = \langle \alpha, d_x u\rangle + c$ for some $\alpha \in \mathfrak{t}^*, c \in \mathbb{R}$, where the complex structure J_u depends on u. Thus, $u \in \mathcal{S}(P, \ell)$ is a solution of (4.2) with $\lambda > 0$ if and only if there exist $p_o \in \mathfrak{t}^*$ and $c \in \mathbb{R}$ such that

$$-\frac{1}{2}\log(\det u^{ij}(x)) - \lambda(\langle x, d_x u\rangle - u(x)) - \lambda\langle p_o, d_x u\rangle + c = 0. \tag{4.6}$$

Indeed, the condition is that the difference between the Ricci potential of u, which is

$$R_u := -\frac{1}{2}\log\det u^{ij} \tag{4.7}$$

(see [1]) and the Kähler potential (see [115])

$$\psi_u(x) := \langle x, d_x u\rangle - u(x)$$

is a affine-linear function in the variable $d_x u$.

We may assume $c = 0$ in (4.6) by adding c to u. However, the smoothness over $\overline{P}$ of the l.h.s. (4.6) does not depend on the representative $u \in \mathcal{S}(P, \ell)$. A straightforward computation see [153] with the Guillemin potential $\frac{1}{2}\sum_{s=1}^{d} \ell_s \log \ell_s \in \mathcal{S}(P, \ell)$, leads to the condition that

$$\ell_1(p_o) = \cdots = \ell_d(p_o) = \frac{1}{\lambda}.$$

Definition 4.12. A labelled polytope (P, ℓ) is *monotone* if there exists $p_o \in P$ with $\ell_1(p_o) = \cdots = \ell_d(p_o)$.

Remark 4.13. The polytope associated to a toric Fano variety $(M, -K_M)$ is naturally "normalized" so that $p_o = 0$. Indeed, there is canonical way to linearize the torus action on $-K_M$ which fixes the polytope. More generally the complex point of view in toric geometry also needs a point to be chosen in the dense orbit $p \in \mathring{M}$ to identify $T_p M \simeq \mathbb{C}^n$. Thus, in the literature, the convention $p_o = 0$ (obtained by a translation) is often taken which makes (4.6) looks better (and it looks even more better in complex coordinates $y + i\theta = d_x u + i\theta$) but this normalisation might be confusing when the labelled polytopes come in families (eg. toric Sasaki geometry) see eg §5.2.1.

A second obstruction to solve the Kähler–Eintein equation is given by the Futaki invariant. In the toric case, it turns out that the extremal vector field of Futaki–Mabuchi [110] can be encoded in an affine linear function $A_{(P,\ell)}$ which is the unique candidate for the scalar curvature of an extremal toric Kähler metric. This was proved by Donaldson in [82, 84] and this works the same in the abstract setting too, as we recall now.

A labelling ℓ on P determines a measure σ_ℓ on the boundary of P as follows

$$\ell_s \wedge \sigma_\ell = -\mathrm{dv} \qquad \text{on } F_s \tag{4.8}$$

where again $dx = dx_1 \wedge \cdots \wedge dx_n$ is a fixed affine invariant volume form on $\mathfrak{t}^*$.

Definition/Proposition 4.14. [82] Given a labelled polytope (P,ℓ), there exists a unique affine-linear function $A_{(P,\ell)} \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$, called the *extremal affine function* such that

$$\mathcal{L}_{(P,\ell)}(f) := \int_P f A_{(P,\ell)}\,\mathrm{dv} - 2\int_{\partial P} f\sigma_\ell = 0 \tag{4.9}$$

for any affine–linear function $f \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$. Moreover, if there exists $u \in \mathcal{S}(P,\ell)$ solving the Abreu equation (4.1) then $S(u) = A_{(P,\ell)}$. Finally, $A_{(P,\ell)}$ is constant if and only if $\mathrm{bar}(P,\mathrm{dv}) = \mathrm{bar}(\partial P,\sigma_\ell)$. In that case, $A_{(P,\ell)} = 2\int_{\partial P}\sigma_\ell(\int_P A_{(P,\ell)}\,\mathrm{dv})^{-1} > 0$.

The key lemma leading to Theorem 4.2 is the following.

Lemma 4.15. *[L3] For any relatively compact polytope P, there is a n–dimensional cone of labelling $\ell \in \mathcal{L}ab(P)$ such that (P,ℓ) is monotone. This cone contains exactly one ray of labelling ℓ such that $A_{(P,\ell)}$ is constant.*

The existence part of such labelling was proved in [84] when $\dim P = 2$.

In [L3] using a triangulation the rest of the statement of the lemma is proved using the basic observation that if (P,ℓ) is a labelled polytope, $p \in P$ and $\lambda > 0$ then

$$\left\{\frac{\ell_1}{\lambda\ell_1(p)},\ldots,\frac{\ell_1}{\lambda\ell_d(p)}\right\}$$

is a monotone labelling of P. Conversely, any monotone labelling of P is of this type. In [dBL, Lemma 3.10], we observed that if (P,ℓ) is monotone with respect to $q_\ell \in P$ with $\ell_s(q) = c > 0$ then the measure σ_ℓ can be defined explicitly as

$$\sigma_\ell = \frac{1}{c}\sum_{i=1}^n (-1)^{i+1}(x_i - q_{\ell,i})dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

with the classical notations in coordinates. We check easily that σ_ℓ satisfies (4.8) and $d\sigma_\ell = \frac{n}{c}\mathrm{dv}$ from which we end up (see [dBL, Lemma 3.10]) with the equation

$$\left(1+\frac{1}{n}\right)\mathrm{bar}(P,\mathrm{dv}) = \mathrm{bar}(\partial P,\sigma_\ell) + \frac{q_\ell}{n}. \tag{4.10}$$

This provides a simpler proof of the last claim of Lemma 4.15. Indeed, on a monotone polytope (P,ℓ) with respect to $q_\ell \in P$, the condition that the Futaki invariant of (P,ℓ) vanishes, equivalently $A_{(P,\ell)}$ is constant or $\mathrm{bar}(\partial P,\sigma_\ell) = \mathrm{bar}(P,\mathrm{dv})$, see Definition/Proposition 4.14, implies by (4.10) that $q_\ell = \mathrm{bar}(P,\mathrm{dv})$ which in turns fixes the labelling ℓ up to a dilatation.

4.1.6 The Wang–Zhu Theorem on monotone labelled polytopes

Recall that a Kähler–Ricci soliton on a complex manifold (M, J) refers to a Kähler metric ω and a holomorphic vector field Z satisfying $\rho^\omega - \lambda\omega = \mathcal{L}_Z\omega$ where $\lambda \in \mathbb{R}$. In the toric compact context of this chapter we always have $\lambda > 0$ and Z is of the form $Z = JX_a - X_a \in \mathrm{Lie}\mathbb{T}_\mathbb{C} \simeq \mathfrak{t} \oplus J\mathfrak{t}$. The Kähler–Ricci equation translates on symplectic potentials as

$$x \mapsto -\frac{1}{2}\log(\det u^{ij}(x)) - \lambda(\langle x, d_x u\rangle - u(x)) - \lambda\langle p_o, d_x u\rangle \quad \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R}). \tag{4.11}$$

Here again the existence of a solution of this equation imposes (P, ℓ) to be monotone with respect to the point p_o. Also, adding a constant to u if necessary we replace the r.h.s. of (4.11) by $\mathfrak{t} = (\mathfrak{t}^*)^*$. Then, if (4.11) holds, the linear function on the right is the unique $a \in \mathfrak{t}$ such that

$$\int_P e^{-\langle a,x\rangle}(f(x) - f(p_o))\mathrm{dv} = 0 \qquad \forall f \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R}), \tag{4.12}$$

where again $\mathrm{dv} = dx_1 \wedge \cdots \wedge dx_n$ is a fixed affine invariant volume form on $\mathfrak{t}^*$. see [86, 183].

The Theorems of Wang–Zhu [202] and Shi–Zhu [183] in the abstract symplectic point of view translate as follows.

Theorem 4.16. *(Affine-convex version of Wang–Zhu Theorem [202, 183, 86, L3]] Let (P, ℓ) be a simple, relatively compact, monotone labelled polytope with respect to the point p_o. Then there exists $u \in \mathcal{S}(P, \ell)$ solving* (4.11) *and this solution is unique up to addition by an affine-linear function. Moreover, u solves the Kähler–Einstein equation* (4.6) *if and only if $A_{(P,\ell)}$ is constant.*

From the discussion in the last subsection the last claim of the Theorem is straightforward. Indeed, $a = 0$ in (4.12) if and only if $p_o = \mathrm{bar}(P, \mathrm{dv})$ which, by (4.10), holds if and only if $A_{(P,\ell)}$ is constant using Definition/Proposition 4.14.

Wang–Zhu proved their Theorem (which is equivalent to Theorem 4.16 when (P, ℓ) is monotone and Delzant) using a continuity method. They picked a $\mathbb{T}$–invariant Kähler form $\omega_0 \in c_1(M)$ defined a path of Monge–Ampère equations depending on $t \in [0, 1]$, say

$$F_t(\psi) \equiv 0$$

on $\mathbb{T}$–invariant Kähler potentials $\{\psi \in C^\infty(M)^\mathbb{T} \,|\, \omega_0 + dd^c\psi > 0\}$ with appropriate normalisations involving ω_0 and X. A solution of the equation at $t = 1$ was the one they seek, at $t = 0$, the solution ψ_0 solves Zhu's equation [211], namely that $\omega = \omega_0 + dd^c\psi_0$ satisfies

$$\rho^\omega - \mathcal{L}_Z\omega = \rho^{\omega_0}. \tag{4.13}$$

Then Wang and Zhu proved that the set of $t \in [0, 1]$ for which a solution exists is open and closed using the ellipticity of their equation and a priori estimates allowing them to use the now standard bootstrap argument.

In fact, the argument of Wang–Zhu holds for any labelled polytope without any deep modification. When (P,ℓ) is not necessarily rational, following [86], we consider $P\times\mathfrak{t}$ with its symplectic structure (that is, $P\times\mathfrak{t}\subset\mathfrak{t}^*\times\mathfrak{t}\simeq T^*\mathfrak{t}$) and the $\mathfrak{t}$–hamiltonian action by translation on the second factor, the moment map is the projection on the first factor. The invariant Kähler metric g_u, defined via (4.3), for a symplectic potential $u\in\mathcal{S}(P,\nu)$, is simply a $\mathfrak{t}$–invariant Kähler metric on $P\times\mathfrak{t}$ with specific behavior along $\partial P\times\mathfrak{t}$.

As introduced in [96], see also [86, 84], given a simple labelled polytope (P,ℓ), each vertex p of P provides a basis $\{d\ell_s\,|\,\ell_s(p)=0\}$ of $\mathfrak{t}$ and thus determines uniquely an open toric symplectic manifold $(M_p,\omega_p,\mathbb{T}_p)$ with $\mathbb{T}_p=\mathfrak{t}/\mathrm{Span}_{\mathbb{Z}}\{d\ell_s\,|\,\ell_s(p)=0\}$ following the well-known caracterisation of linear hamiltonian torus actions by their weights, see eg [157]. These *toric charts* are the orbifold uniformizing charts when (P,ℓ) is rational. The boundary condition on symplectic potential $u\in\mathcal{S}(P,\ell)$ corresponds exactly to the condition that g_u defines a smooth $\mathbb{T}_p$–invariant Kähler metric on each toric chart $(M_p,\omega_p,\mathbb{T}_p)$.

The potentials ψ appearing in the proof of Wang–Zhu, behave as smooth functions defined on the compact set $\overline{P}$ on which we can consider the $L^{p,q}$ norms and so on. The reference metric (J,ω_0) corresponds to a symplectic potential $u_0\in\mathcal{S}(P,\ell)$ for which the boundary condition, suitably interpreted, allows to apply the (local) computations of Yau [206] and Tian–Zhu [195] on each chart (M_p,ω_p,T_p). Along the way, we show that both Yau's Theorem [206] and Zhu's Theorem [211] hold in this extended setting.

To state these last results we need to introduce the equivalent of $(1,1)$–Dolbeaut classes for abstract labelled polytopes. This is also done by Donaldson in [86] we recall the idea. Given $f\in C^\infty(P)$ one can define the class $[f]:=\{\tilde{f}\in C^\infty(P)\,|\,f-\tilde{f}\in C^\infty_{\mathbb{R}}(\overline{P})\}$ where we recall that a function is in $C^\infty(\overline{P})$ if it is the restriction to $\overline{P}$ of a smooth function defined on an open set containing $\overline{P}$. The class $[f]$ is an analogue of the $\mathbb{T}$–invariant $(1,1)$-Dolbeaut cohomology class. Indeed, in the case where (P,ℓ) is rational, thus associated to a toric symplectic orbifold $(M,\omega,\mathbb{T})$ with momentum map $\mu:M\to\overline{P}$, a theorem of Schwarz [182] implies that $C^\infty(M)^{\mathbb{T}}=C^\infty(\overline{P})$. Combining this with the dd^c–lemma and the equivariant identification $\mathring{M}\simeq P\times\mathbb{T}$, we get that every two co-cohomological $\mathbb{T}$–invariant smooth $(1,1)$ forms on M correspond to functions in $C^\infty(P)$ belonging to the same class $[f]$.

For $u\in\mathcal{S}(P,\ell)$, we denote $R_u:=-\frac{1}{2}\log(\det u^{ij}(x))$, this is a potential for the Ricci form of the Kähler metric (g_u,ω). Observe that $R_u+\frac{1}{2}\sum_{s=1}^d\log(\ell_s)\in C^\infty(\overline{P})$.

Theorem 4.17. *(Affine-convex version of Yau's Theorem, [206, 86, L3]) Given $R\in C^\infty(P)$ such that $R-R_{u_o}\in C^\infty(\overline{P})$, there exists $u\in\mathcal{S}(P,\ell)$ such that $R=\frac{1}{2}\log\det(\mathrm{Hess}\,u)_x$ and this solution is unique up to addition by an affine-linear function.*

Theorem 4.18. *(Affine-convex version of Zhu's Theorem, [211, 86, L3]) Given a convex function $R\in C^\infty(P)$ such that $R-R_{u_o}\in C^\infty(\overline{P})$ and $a\in\mathfrak{t}$, there exists $u\in\mathcal{S}(P,\ell)$ such that $R_u-R=a$ and this solution is unique up to addition by an affine-linear function.*

4.2 Toric almost Kähler metrics

An almost Kähler structure on a $2n$-dimensional manifold M^{2n} is a triple (g, J, ω) such that g is a riemannian metric, ω is a symplectic form, and $J \in \Gamma(\mathrm{End}(TM))$ squares to minus the identity, together with the following compatibility relation: $g(J\cdot, J\cdot) = g(\cdot, \cdot) \quad g(J\cdot, \cdot) = \omega(\cdot, \cdot)$. A *toric almost Kähler metric* (g, J) is then an almost Kähler metric on a toric symplectic manifold/orbifold $(M, \omega, \mathbb{T})$ such that (g, J) is compatible with ω and g (equivalently J) is invariant by the torus $\mathbb{T}$. As proved in [82, 154], from the momentum map picture the appropriate extremality condition for an almost Kähler metric has to be expressed in terms of the scalar curvature of the Chern (or Hermitian) connection, which does not agree with the Levi-Civita connection when the complex structure is not integrable.

The main result discussed in this section folllows from the observation that the proof of the Chen–Li–Sheng Theorem [63] works equally for *extremal toric almost Kähler metric of involutive type*. These special almost Kähler metrics were introduced by Lejmi [154] and the definition is recalled below. Combining this observation with Chen–Cheng [61] and He [120] existence Theorem, we get that the existence of an extremal toric Kähler metric is equivalent to the existence of an extremal toric almost Kähler metric of involutive type.

Proposition 4.19. *[L4] Let (M, J, g, ω) be a compact toric Kähler manifold with associated Delzant labelled polytope (P, ℓ), such that (M, ω) admits a compatible extremal toric almost Kähler metrics of involutive type then (P, ℓ) is uniformly K–stable*[2]*. Moreover, there exists a ω–compatible extremal toric* almost *Kähler metrics of involutive type on M if and only if there exists a ω–compatible extremal toric Kähler metric on M.*

As explained below, in the toric setting an almost Kähler structure of involutive type is encoded in a smooth symmetric bilinear form $H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*)$ on which the extremal equation is linear and of order 2. This is very specific to the toric case and then Theorem 4.19 implies that a weaker conjecture than the Yau–Tian–Donaldson holds in the toric case [82, Discussions $(4a)$]. Moreover, combining Theorem 4.19 with Theorem 4.2, we get the following corollary.

Corollary 4.20. *[L4] Given a simple relatively compact polytope $P \subset \mathfrak{t}^*$, the set of labellings $\ell \in \mathcal{L}ab(P)$ such that $\mathcal{S}(P, \ell)$ contains a symplectic potential solving the Abreu equation 4.1 is convex and non-empty.*

Remark 4.21. It is unlikely that in general, for compact Kähler manifold of non-toric type, the existence of an extremal almost Kähler metric (M, J, ω) implies uniform K–stability of (M, J) or the existence of an extremal Kähler metric compatible with ω. However, as pointed out in [138], a certain notion of stability could generalize the conjecture and the theory to almost Kähler metrics.

In [14], for any $k_2, k_1 > 0$ and any toric symplectic form ω on the total space of the projective bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2)) \to \mathbb{P}^1$, they construct explicit examples of almost Kähler metrics compatible with ω. One can check directly

[2] Here uniform K-stability should be understood as defined below, see Remark §4.26.

that these metrics are of involutive type and as an application of Theorem 4.19 we get the following.

Corollary 4.22. *[L4] Each Kähler class of $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2))$ admits a compatible extremal toric Kähler metric.*

Remark 4.23. Theorem 4.19 has been recently extended to a class of weighted toric extremal metric by Jubert [134] and applied to prove that each Kähler class of the projective bundle $\mathbb{P}_\Sigma(\mathcal{O} \oplus \mathcal{O}(k_1) \oplus \mathcal{O}(k_2))$ over an elliptic curve Σ admits an extremal representative.

The rest of this section is dedicated to the proof of Proposition 4.19 which is a combinaison of Theorem 4.27 and Lemma 4.31 below.

4.2.1 Toric almost Kähler metrics in action-angle coordinates

Let $(M, \omega, \mathbb{T})$ be a toric symplectic manifold with a momentum map $x : M \to \mathfrak{t}^*$ with associated labelled polytope (P, ℓ). We use notation layed in §4.1 and fix a set of affine coordinates $x = (x_1, \ldots, x_n)$ on $\mathfrak{t}^*$. In [154], the author proves among other things that $\mathbb{T}$–invariant almost Kähler structures compatible with (M, ω) with involutive g–orthogonal distributions to the orbits (such structure is called of *involutive type*) are parametrized by symmetric bilinear forms

$$H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*) \tag{4.14}$$

satisfying some conditions pointed out in [12] that we now recall.

(i) **Smoothness** H is the restriction on $\overline{P}$ of a smooth $\mathrm{Sym}^2(\mathfrak{t}^*)$–valued function defined on an open neighbourhood of $\overline{P}$.

(ii) **Boundary condition** For any point y in interior of a codimension 1 face $F_s \subset \overline{P}$, denoting $v_s = d\ell_s$ we have

$$H_y(v_s, \cdot) = 0 \tag{4.15}$$

$$dH_y(v_s, v_s) = 2v_s. \tag{4.16}$$

(iii) **Positivity** For any point y in interior $\mathring{F}$ of a face $F \subset \overline{P}$, H is positive definite as $\mathrm{Sym}^2(T_y\mathring{F})$–valued function.

Proposition 4.24. *[Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman [12]] Let (M, ω, T) be a toric symplectic manifold and (g, J) be a compatible T–invariant almost Kähler metric of involutive type compatible with ω. Then the symmetric bilinear form defined for $a, b \in \mathfrak{t}$ and $x \in \overline{P}$ by $H_x(a, b) := g_p(X_a, X_b)$ for any $p \in M$ such that $x(p) = x$, satisfies the conditions* (i), (ii) *and* (iii). *Moreover, for any such symmetric bilinear form $H : \overline{P} \to Sym^2(\mathfrak{t}^*)$ satisfying conditions* (i), (ii) *and* (iii) *there is a unique compatible T–invariant almost Kähler metric (g_H, J_H) of involutive type satisfying $H_{x(p)}(a, b) = g_p^H(X_a, X_b)$ for any $p \in M$. With respect to action angle coordinates $(x, [\theta])$ on $\mathfrak{t}^* \times T \simeq \mathring{M}$, the metric g is given as*

$$g = \sum_{i,j} G_{ij} dx_i \otimes dx_j + H_{ij} d\theta_i \otimes d\theta_j, \tag{4.17}$$

where $G = (G_{ij}) = H^{-1}$.

Thanks to Proposition 4.24 we parametrize the space of compatible toric almost Kähler metrics of involutive type as

$$\mathcal{AK}(P,\ell) := \{H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*) \,|\, H \text{ satisfies conditions (i), (ii) and (iii)}\}.$$

The inverse (u^{ij}) of the Hessian of symplectic potential $u \in \mathcal{S}(P,\ell)$ can be extended as a bilinear form $H^u \in \mathcal{AK}(P,\ell)$. Observe also that for $H_0, H_1 \in \mathcal{AK}(P,\ell)$ we have

$$H_t = tH_1 + (1-t)H_0 \in \mathcal{AK}(P,\ell) \qquad \forall t \in [0,1].$$

The space $\mathcal{AK}(P,\ell)$ is then a *convex* infinite dimensional set of metrics.

As proved in [82, 154], given $H \in \mathcal{AK}(P,\ell)$, the hermitian scalar curvature of (g_H, J_H, ω) in coordinates (x,θ), is

$$S(H) := -\sum_{i,j=1}^{n} \frac{\partial H_{ij}}{\partial x_i \partial x_j}. \tag{4.18}$$

A toric almost Kähler metric (g_H, J_H, ω) is *extremal* if and only if $S(H) \in \mathrm{Aff}(\mathfrak{t}^*, \mathbb{R})$ which in turn happens if and only if

$$S(H) = A_{(P,\ell)} \tag{4.19}$$

where $A_{(P,\ell)}$ is the affine-linear function of (P,ℓ) defined in Definition/Proposition 4.14. Indeed, this follows from the following observation, made in [82]: for $f \in C^2(\overline{P})$, the boundary condition (ii) above implies

$$\int_P f S(H)\,\mathrm{dv} = \int_P \langle H, \mathrm{Hess} f\rangle\,\mathrm{dv} + 2\int_{\partial P} f\,\sigma. \tag{4.20}$$

4.2.2 Uniform K-stability of labelled polytopes

Consider the functional

$$\mathcal{L}_{(P,\ell)}(f) := \int_{\partial P} f\sigma_\ell - \frac{1}{2}\int_P f A_{(P,\ell)} dx$$

which can be defined on various spaces of functions on $\overline{P}$, for example $C^0(\overline{P})$. From Definition/Proposition 4.14 we get that $\mathcal{L}_{(P,\ell)}$ vanishes identically on the space of affine-linear functions.

Following [63, 82, 210], for a fixed point $p_o \in P$, we define the sets

$$\begin{aligned} \mathcal{C}_\infty :=& \{f \in C^0(\overline{P}) \cap C^\infty(P) \,|\, f \text{ convex }\} \\ \mathcal{C}^*_\infty(p_o) :=& \{f \in \mathcal{C}_\infty \,|\, f(p) \geq f(p_o) = 0 \qquad \forall p \in P\} \end{aligned}$$

$$\mathcal{C}^*(p_o) := \left\{ f \in C^0(\overline{P}) \;\middle|\; \begin{array}{r} \exists K > 0, \exists f^{(k)} \in \mathcal{C}_\infty(p_o),\ \displaystyle\int_{\partial P} f^{(k)}\sigma_\ell < K, \\ f^{(k)} \text{ converges locally uniformly to } f \end{array} \right\}.$$

We have $\mathcal{S}(P,\ell) \subset \mathcal{C}_\infty$ for any $\ell \in \mathcal{L}ab(P)$. Note that the only affine-linear function in $\mathcal{C}^*(p_o)$ is the trivial one.

Definition 4.25. A labelled polytope (P,ℓ) is *uniformly K–stable* if there exists $\lambda > 0$ such that

$$\mathcal{L}_{(P,\ell)}(f) \geq \lambda \int_{\partial P} f\sigma_\ell$$

for any $f \in \mathcal{C}^*(p_o)$.

Remark 4.26. Let $\mathcal{T}(P)$ be the set of continuous piecewise linear convex functions on $\overline{P}$, that is $f \in \mathcal{T}(P)$ if there are $f_1,\dots,f_m \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$ such that $f(x) = \max\{f_1(x),\dots,f_m(x)\}$ for $x \in \overline{P}$. Given a lattice $\Lambda \subset \mathfrak{t}$, we define $\mathcal{T}(P,\Lambda) \subset \mathcal{T}(P)$, the set of continuous piecewise linear convex functions on P taking integral values on the dual lattice $\Lambda^* \subset \mathfrak{t}^*$. When (P,η,Λ) is rational Delzant and its vertices lie in the dual lattice $\Lambda^* \subset \mathfrak{t}^*$, the associated symplectic manifold (M,ω) is rational (that is $[\omega] \in H^2_{dR}(M,\mathbb{Q})$) and for any compatible toric complex structure J on M the Kähler manifold $(M,J,k[\omega])$ (for some k big enough) is polarised by a line bundle $L^k \to M$. In this situation, Donaldson presents in [82] a way to associate a test configuration $(\mathcal{X}_f,\mathcal{L}_f)$ over (M,L) to any function $f \in \mathcal{T}(P,\Lambda)$ such that the Donaldson–Futaki invariant of $(\mathcal{X}_f,\mathcal{L}_f)$ coincides, up to a positive multiplicative constant, with $\mathcal{L}_{(P,\ell)}(f)$. These test configurations are called *toric degenerations* in [82] and [210]. The Yau–Tian–Donaldson conjecture predicts that if $A_{P,\ell}$ is a constant and there exists a solution $u \in \mathcal{S}(P,\ell)$ of the Abreu equation (4.1) then

$$\mathcal{L}_{(P,\ell)}(f) \geq 0$$

for any $f \in \mathcal{T}(P,\Lambda)$ with equality if and only f is affine-linear.

Observe that the map $f \mapsto \|f\|_* := \int_{\partial P} f\sigma_\ell$ is a norm on $\mathcal{C}^*(p_o)$. Therefore, Definition 4.25 coincides with the notion of uniform K–stability in the sense of Székelyhidi [189] but with a different norm and adapted to the toric situation. Moreover, this is the notion of stability in Definition 4.25 that Chen–Li–Sheng used in [63] to prove the following when the solution $H = H^u$ is the inverse hessian of some $u \in \mathcal{S}(P,\ell)$.

Theorem 4.27 (Chen–Li–Sheng[63], L.[L4])**.** *Let (P,ℓ) be a labelled polytope and assume there exists a solution $H \in \mathcal{AK}(P,\ell)$ of the equation* (4.19) *then (P,ℓ) is uniformly K–stable.*

The main argument to prove Theorem 4.27 in [63] only uses the fact that the Hessian and inverse Hessian $H^u = (u_{ij})^{-1}$ of the solution $u \in \mathcal{S}(P,\ell)$ are positive definite on the interior of P. One important step for their proof is to show that : *a labelled polytope (P,ℓ) is uniformly K–stable if and only if $\mathcal{L}_{(P,\ell)}(f) \geq 0$ on some closed balls*

$$\mathcal{C}^K_*(p_o) := \{ f \in \mathcal{C}^*(p_o) \mid \|f\|_* \leq K \}$$

for $K >> 0$, see [63, Proposition 4.5]. This is general and does not need any hypothesis on the existence of a solution of the Abreu equation. The key observation is the following, if $H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*)$ satisfies equation (4.19), that is $S(H) = -\sum_{i,j=1}^n H_{ij,ij} = A_{(P,\ell)}$ then the boundary conditions (ii) of §4.2.1, in particular (4.20), implies that

$$\mathcal{L}_{(P,\ell)}(f) = \int_P \langle H, \mathrm{Hess} f \rangle dx \tag{4.21}$$

whenever f is twice differentiable.

Therefore, let H be a solution of equation (4.19), then for any interval $I \subset\subset P$ and sequence of convex functions $f_k \in \mathcal{C}_\infty \subset C^\infty(P)$ converging locally uniformly to $f \in \mathcal{C}^K_*$ then we have, using (4.21) and weak convergence of Monge–Ampère measures, that

$$\mathcal{L}_{(P,\ell)}(f_k) \geq \tau m_I(f) \tag{4.22}$$

where $m_I(f)$ is the Monge–Ampère measure induced by f on I and τ is a positive constant independent of k. This is [63, Lemma 5.1] from which one can derive Theorem 4.19 using the same argument than [63] in the last paragraph of their section 5. That is, if (P,ℓ) is not uniformly K-stable, there is a sequence of convex functions $f^{(k)} \in \mathcal{C}_\infty(p_o)$ of norm 1 such that $\mathcal{L}_{(P,\ell)}(f^{(k)}) \to 0$ and thus

$$\int_P f^{(k)} A_{(P,\ell)} \to 1.$$

Then $f^{(k)}$ locally uniformly converges to a function $f \in \mathcal{C}^*(p_o)$ and we can deduce from (4.22) that $m_I(f) = 0$ for any segment $I \subset \overline{P}$. This implies $f = 0$ but with these conditions we have

$$\int_P f^{(k)} A_{(P,\ell)} \to \int_P f A_{(P,\ell)} = 1$$

as proved in [82] ans from this we conclude Theorem 4.27.

Remark 4.28. In the toric case, hamiltonian 2–form, as introduced [11], are used to build explicit solution $H : \overline{P} \to \mathrm{Sym}^2(\mathfrak{t}^*)$ of equation (4.19) on some simple labelled polytopes (P,ℓ). This is a separation of variables and the equation splits in this case into a set of ODE. In many cases, smooth solution H satisfying the boundary condition (ii) appearing in Proposition 4.24 do exist and the issue is to know if H satisfies the positivity condition (iii). In case it does not satisfy the positive condition (but the boundary condition) the method of the proof above yields that (P,ℓ) is unstable. This was used successfully in [13] to produce unstable Kähler orbifolds.

4.2.3 Existence Theorem of Chen-Cheng and He

Let K be a maximal connected compact subgroup of $\mathrm{Aut}_{red}(M,J)$ with extremal vector field $X_{ext}([\omega],K)$, see appendix A.2. We consider the space of K–invariant Kähler potentials $\mathcal{H}^K_\omega := \{\phi \in \mathcal{C}^\infty(M)^K \mid \omega_\phi := \omega + dd^c\phi > 0\}$ and the *modified Mabuchi K–energy* introduced in [114] as a functional $\mathbf{M}^K : \mathcal{H}^K_\omega \to \mathbb{R}$ satisfying $\mathbf{M}^K(0) = 0$ and

$$d\mathbf{M}^K_\phi(\psi) = \int_M \psi(\mathrm{Scal}_{\omega_\phi} - f_{ext}(\omega))\omega^n_\phi.$$

To get an effective parameterization of Kähler metrics, we need also to fixe a normalisation which, classically is the subset $\mathcal{H}^K_{\omega,0} \subset \mathcal{H}^K_\omega$ given by the level set of the Aubin–Mabuchi functional (see (3.12) in the Sasaki context) introduced in [165], that is $0 \in \mathcal{H}^K_{\omega,0}$ and any smooth path ψ_t belongs to $\mathcal{H}^K_{\omega,0}$ if and only if

$$\int_M \cdot\psi_t\, \omega^n_{\psi_t} = 0.$$

An alternative definition see eg. [112, §4.1], implies that $\mathcal{H}^K_{\omega,0}$ is a slice for the addition by constants and thus $\mathcal{H}^K_{\omega,0}$ parameterized effectively the Kähler class. The complexification G of K in $\mathrm{Aut}_{red}(M,J)$ does not act directly on $\mathcal{H}^K_\omega$ (because ω is not invariant by G). However, given $\psi \in \mathcal{H}^K_{\omega,0}$ and $\gamma \in G$, $\gamma^*(\omega_\phi) = \omega + dd^c([\gamma](\psi))$ for a unique $[\gamma](\psi) \in \mathcal{H}^K_{\omega,0}$. This provides an action of G on $\mathcal{H}^K_{\omega,0}$.

An important ingredient in this theory is the distance $d_{1,G}$ on the quotient $\mathcal{H}^K_\omega/[G]$ introduced by Darvas [72] and corresponding to the L^1–norm on $T_\phi\mathcal{H}^K_{\omega,0}$. That is for $\psi \in T_\phi\mathcal{H}^K_{\omega,0}$, the norm $\int_M |\psi|\omega^n_\phi$ allows to compute the length of curves and then $d_1(\phi_0,\phi_1)$ is the infimum of the length of the curves joining ϕ_0 and ϕ_1. It turns out that G acts isometrically on $(\mathcal{H}^K_{\omega,0}, d_1)$, [73] and

$$d_{1,G}(\phi_0,\phi_1) := \inf_{\gamma\in G} d_1(\phi_0,[\gamma]\phi_1).$$

Theorem 4.29. *[He [120, Theorem 4]] There is a K–invariant extremal Kähler metrics in $(M,J,[\omega])$ if and only if the modified Mabuchi energy $\mathbf{M}^K$ is bounded below on $\mathcal{H}^K_\omega$ and proper with respect to $d_{1,G}$.*

Since the extremal vector field lies in the center of the reduced automorphism group [57] see appendix A.1, thus within a maximal torus $\mathbb{T}$, the Mabuchi energy $\mathbf{M}^{\mathbb{T}}$ restricts to $\mathcal{H}^K_\omega$ as $\mathbf{M}^K$. Moreover, since $d_{1,G} \leq d_{1,\mathbb{T}_\mathbb{C}}$ on $\mathcal{H}^K_\omega$, He's Theorem can be equally stated for $K = \mathbb{T}$ and $G = \mathbb{T}_\mathbb{C}$.

On a toric manifold, following Donaldson [82], one can define the K–energy on the space of symplectic potentials as follow. Let (P,ℓ) be a labelled compact simple polytope with extremal affine function $A_{P,\ell} \in \mathrm{Aff}(\mathfrak{t}^*,\mathbb{R})$ and $u \in \mathcal{S}(P,\ell)$, the modified Mabuchi K–energy is essentially

$$\mathbf{M}_{(P,\ell)}(u) = \mathcal{L}_{(P,\ell)}(u) - \int_P \log\det(u_{ij})\mathrm{dv}. \tag{4.23}$$

Remark 4.30. Indeed, direct calculation shows that the critical points of (4.23) on $\mathcal{S}(P,\nu)$ are the symplectic potentials satisfying (4.1). To identify $\mathbf{M}_{(P,\ell)}$ with the modified Mabuchi energy, we need to fix a Kähler metric (ω_0,J_0), equivalently a symplectic potential $u_0 \in \mathcal{S}(P,\ell)$. Then the ω_0–Kähler potential of the Kähler structure (ω_0,J_u) is the pull back by the momentum map of the function $x \mapsto \phi_{u,u_0}(x) := \psi_u(d_xu) - \psi_{u_0}(d_xu)$ where ψ_u is the Legendre transform of u (4.5). That is ϕ_{u,u_0} extends smoothly on $\overline{P}$ and the $\mathbb{T}$–equivariant diffeomorphism $\Phi : M \to M$ (inducing the diffeomorphism $(du)^{-1} \circ du_0$ on P) send J_0 on J_u and ω_0 on $\omega_0 + dd^{c_u}\phi_{u,u_0}$. Then $\mathbf{M}_{(P,\ell)}(u) = \mathbf{M}^{\mathbb{T}}(\phi_{u,u_0}) + C$ where $C = -\mathbf{M}^{\mathbb{T}}(0)$.

This allows to translate He's Theorem (recalled in Theorem 4.29 above) in terms of (P,ℓ) only and putting together [82, Proposition 5.1.2] and [210, Lemma 2.3] we will get

Lemma 4.31. *[L4] If (P,ℓ) is uniformly K–stable then $\mathbf{M}_{(P,\ell)}$ is bounded below on $\mathcal{C}^*(p_o)$ and is proper with respect to $d_{1,\mathbb{T}_\mathbb{C}}$.*

The first condition is given by

Proposition 4.32. *(Donaldson [82, Proposition 5.1.2]) If (P,ℓ) is uniformly K–stable then $\mathbf{M}_{(P,\ell)}$ is bounded below on $\mathcal{C}^*(p_o)$.*

We will derive the second using the following result.

Lemma 4.33. *[Zhou–Zhu [210, Lemma 2.3]] If (P,ℓ) is uniformly K–stable then there exist real constants $C>0, D$ such that*

$$\mathbf{M}_{(P,\ell)}(u) \geq C\int_P u\mathrm{dv} - D \tag{4.24}$$

for all $u\in\mathcal{C}^*(p_o)$.

To work on the space of Kähler potentials we need to fix a metric $u_0\in\mathcal{S}(P,\ell)$. Then, we need to prove that one can choose $C,D>0$ so that when $u\in\mathcal{C}^*(p_o)$, $C\int_P u\mathrm{dv}-D$ is an upper bound for $d_{1,\mathbb{T}_\mathbb{C}}(0,\tilde{\phi}_{u,u_0})$ with the notation ϕ_{u,u_0} of Remark 4.30, $\tilde{\phi}$ being the translation by a constant of $\phi\in\mathcal{H}^{\mathbb{T}}_{\omega_0}$ lying in the slice $\mathcal{H}^{\mathbb{T}}_{\omega_0,0}$ and u_0. The left hand side is insensitive of the normalisation $\mathbf{M}_{(P,\ell)}(u+f)=\mathbf{M}_{(P,\ell)}(u)$ for $f\in\mathrm{Aff}(P,\mathbb{R})$ but the right hand side is.

The Legendre transform relation (4.5) implies $\dot{u_t}=-\dot{\psi_{u_t}}$ for any path u_t of symplectic potentials and thus, with fixed $u_0\in\mathcal{C}^*_\infty(p_o)\cap\mathcal{S}(P,\ell)$, we get that

$$\phi_{u,u_0}\in\mathcal{H}^{\mathbb{T}}_{\omega_0,0} \;\Leftrightarrow\; \left(d_{p_o}u=0 \text{ and } \int_P u\mathrm{dv}=\int_P u_0\mathrm{dv}\right)$$

as highlighted in [8]. Therefore, for $u\in\mathcal{C}^*_\infty(p_o)\cap\mathcal{S}(P,\ell)$, $\tilde{u}:=u+\frac{1}{\int_P\mathrm{dv}}\int_P(u_0-u)\mathrm{dv}$ is such $\phi_{\tilde{u},u_0}\in\mathcal{H}^{\mathbb{T}}_{\omega_0,0}$ and

$$\begin{aligned} C\int_P u\mathrm{dv}-D &= C\int_P|u|\mathrm{dv}-D\geq C\int_P|u-u_0|\mathrm{dv}-D'\\ &\geq\frac{C}{2}\int_P|\tilde{u}-u_0|\mathrm{dv}-D'\geq\frac{C}{2}d_1(0,\phi_{\tilde{u},u_0})-D'\geq\frac{C}{2}d_{1,\mathbb{T}_\mathbb{C}}(0,\phi_{\tilde{u},u_0})-D'. \end{aligned}$$

Here for $\int_P|\tilde{u}-u_0|\mathrm{dv}\geq C'd_1(0,\phi_{\tilde{u},u_0})$ we use that the length of the path $t\tilde{u}-(t-1)u_0$ is an upper bound of the distance $d_1(\psi_{u_0},\psi_{\tilde{u}})$ (via a change of coordinates). In [L4], I have been careless with the normalisation and the middle step above is taken from [210, 8].

4.3 The first eigenvalue of a toric Kähler manifold

In this section, I present results obtained in collaboration with Rosa Sena-Dias (IST, Portugal) on the first eigenvalue of the Laplacian of compact toric Kähler manifolds, published in [LSD].

4.3.1 The Bourguignon–Li–Yau bound on the first eigenvalue of a toric Kähler manifold

Bounding the first (non-zero)) eigenvalue λ_1 of the Laplacian on some space of riemannian metrics on a compact manifold M is a classical topic in riemannian geometry see eg [29]. A result of interest (from the geometric point of view) is that g is extremal for the map $g\mapsto\lambda_1(g)$ on the space of volume 1 riemannian metrics on M if and only if a $L^2(g)$–orthogonal basis of the λ_1–eigenspace

$f_0, \ldots, f_N$ embeds isometrically (M, g) into a a minimal submanifold of the sphere $\mathbb{S}^N \subset \mathbb{R}^{N+1}$, [99, 174]. By a result of Bourguignon–Li–Yau [45], when (M^{2n}, J) is compact, complex, of Kähler type and polarised by an ample line bundle $\omega \in c_1(L)$, the Kodaira embedding into $\mathbb{P}(H^0(M, L^N))$ provides an explicit upper bound on the first eigenvalue $\lambda_1(g)$ for any J–compatible Kähler metric (g, ω) with $[\omega] \in 2\pi c_1(L)$. With R. Sena-Dias we have translated their result in terms of the labelled polytope data (P, ℓ) in the case where $(M^{2n}, J, [\omega])$ is toric with integral Delzant labelled polytope (P, ℓ) and extended the bound to any rational Delzant labelled polytope by approximation as follows.

Theorem 4.34. *[L.–Sena-Dias [LSD]] Let (M^{2n}, ω) be a toric symplectic manifold endowed with a toric Kähler structure whose Riemannian metric we denote by g. Let $\overline{P} \subset \mathfrak{t}^*$ be its moment polytope. There is an integer, $k_0(P) \geq 1$ such that for any $k \geq k_0(P)$*

$$\lambda_1(g) \leq \frac{2nk(N_k + 1)}{N_k}.$$

where $N_k + 1 = \sharp(\overline{P} \cap \mathbb{Z}^n/k)$. If P is integral (i.e its vertices lie in the lattice of circle subgroups Λ), then we have a finer bound given by

$$\lambda_1(g) \leq \frac{2n(N+1)}{N}, \tag{4.25}$$

where $N + 1 = \sharp(\overline{P} \cap \Lambda)$ is the number of lattice points in P.

The Fubini-Study metric realizes the bound in the above theorem. In fact we show that this is the *only* toric Kähler metric that does saturate this bound in the integral case.

Theorem 4.35. *[LSD] Let (M^{2n}, ω) be an integral toric symplectic manifold endowed with a toric Kähler structure whose Riemannian metric we denote by g. Let $N + 1$ be the number of lattice points in the moment polytope of M. If*

$$\lambda_1(g) = \frac{2n(N+1)}{N},$$

then M is equivariantly symplectomorphic to $\mathbb{CP}^n$ and this symplectomorphism takes g into the Fubini-Study metric on $\mathbb{CP}^n$.

It was previously known (see [29]) that the Fubini-Study metric on $\mathbb{CP}^n$ is determined by the spectrum among all Kähler metrics on $\mathbb{CP}^n$ compatible with the standard complex structure. It was also proved by Tanno (see [194]) that, if a Kähler manifold of real dimension less than 12 has the same spectrum as $\mathbb{CP}^n$ with the Fubini-Study metric, then it is holomorphically isometric to it. A simple consequence of the above theorem is that the spectrum of the Laplacian of a toric Kähler metric on an integral toric manifold determines if the manifold is $\mathbb{CP}^n$ endowed with the Fubini-Study metric.

Corollary 4.36. *[LSD] An integral toric Kähler manifolds which has the same spectrum as $(\mathbb{CP}^n, \omega_{FS})$ is holomorphically isometric to it.*

Sketch of the proof of Theorem 4.35. Recall that the first eigenspace of $(\mathbb{CP}^N, \omega_{FS})$ has a basis given by the functions $[Z] \mapsto \Psi_{ij}(Z) - \frac{\delta_{ij}}{N+1}$ where for $i, j \in \{0, 1, \ldots, N\}$,

$$\Psi_{ij}(Z) = \frac{Z_i \overline{Z}_j}{\sum_{k=0}^N |Z_k|^2}$$

is one component of the $SU(N+1)$ moment map $\Psi : \mathbb{CP}^N \hookrightarrow \mathfrak{su}^*_{N+1}$. Given a polarised Kähler manifold (M,ω), the key step of the proof of the main Theorem in [45], is to show that, given a full (Kodaira) embedding $\Phi : M \hookrightarrow \mathbb{CP}^N$, there exists a unique $B \in SL(N+1,\mathbb{C})$ such that $B^* = B > 0$ and

$$\frac{1}{\int_M \omega^n} \int_M (\Psi_{ij} \circ B \circ \Phi)(p)\omega^n = \frac{\delta_{ij}}{N+1}. \tag{4.26}$$

Said differently, $(B \circ \Phi)^*\omega_{FS}$ is $(\omega^n/n!)$–balanced, (see [85]).

Denote $f^B_{ij} = \Psi_{ij} \circ B \circ \Phi - \frac{\delta_{ij}}{N+1} \in C^\infty(M)$. The Rayleigh principle implies that

$$\lambda_1(M,\omega) \int_M |f^B_{ij}|^2 \frac{\omega^n}{n!} \leq \int_M |\nabla^\omega f^B_{ij}|^2 \frac{\omega^n}{n!} \tag{4.27}$$

with equality if and only f^B_{ij} is an eigenfunction of Δ^ω for the eigenvalue $\lambda_1(M,\omega)$. Bourguignon–Li–Yau continued their proof summing over i,j on the left side of (4.27) and using, on the right side, that on $\mathbb{CP}^N$,

$$\sum_{i,j} d\Psi_{ij} \wedge d^c\overline{\Psi}_{ij} = 2\omega_{FS}. \tag{4.28}$$

In [LSD], we checked that when (M,ω) is toric, B is diagonal and deduced that $(B \circ \Phi)^*\omega_{FS}$ is also toric. Note that the map on M given in coordinates by $(f^B_{00},\dots,f^B_{NN})$ is momentum map for $(B \circ \Phi)^*\omega_{FS}$ but not necessarily for ω so the equality case in (4.27) does not implies straightforwardly that ω is Kähler–Einstein using Proposition 4.37 below.

However, we used the Guillemin [115] explicit embedding for toric polarised Kähler manifold $(M, g_u\omega)$ with $u \in \mathcal{S}(P,\ell)$ and momentum map $\mu : M \to \overline{P}$, that is working in action–angle coordinates (x,θ) on $P \times \mathfrak{t}$, we get a holomorphic full equivariant embedding

$$\Phi_u : M \to \mathbb{CP}^N$$

where the homogenous coordinates are indexed by $p \in \overline{P} \cap \Lambda$ and explicitely given by

$$Z_p = e^{\langle (d_x u + i\theta), p\rangle}.$$

With this embedding we proved that equality case in (4.27) for $\lambda_1(g_u)$ implies that (up to a composition with a diagonal matrix B) $\Phi^*_u\Psi_{00},\dots,\Phi^*_u\Psi_{NN}$ are linear combinaisons of the momentum coordinates μ. This in turns, implies that $N = \dim M + 1$ (thus $M \simeq \mathbb{CP}^n$) and using Proposition 4.37 below we get that (g_u,ω) is Kähler-Einstein. □

The following proposition is a refinement of Matsushima Theorem [170] in the toric case.

Proposition 4.37. [3] *[LSD] Let $(M,\omega,\mathbb{T})$ be a compact symplectic toric orbifold with moment map $\mu : M \to \mathfrak{t}^*$. Then $(M,g,J,\omega,\mathbb{T})$ is a Kähler–Einstein toric orbifold with Einstein constant λ if and only if, up to an additive constant, the moment map satisfies*

$$2\lambda\langle \mu, b\rangle = \Delta^g\langle \mu, b\rangle \qquad \forall b \in \mathfrak{t}. \tag{4.29}$$

[3] It is possible that this result was previously known but we did not find a reference for it in the literature.

In that case, 2λ is the smallest non-zero eigenvalue for the Kähler–Einstein orbifold toric metric.

Matsushima's theorem implies that a necessary condition for a toric Kähler metric to be Kähler-Einstein is that its λ_1 be a multiple eigenvalue with multiplicity at least equal to half the dimension of the manifold. What's more, it follows from the above proposition that one can see if a metric is Kähler- Einstein by simply checking if its moment map coordinates are eigenfunctions for 2λ.

Proof of Proposition 4.37. This is a direct computation. First, from the Kähler version of the Bochner identity (used many times in this memoir) that $\Delta^g \mu$ is a momentum map for the Ricci form $2\rho^g$, thus if $(M, g, J, \omega, \mathbb{T})$ is a Kähler–Einstein $\Delta^g \mu - 2\lambda\mu$ is a constant. Conversely, let $u \in \mathcal{S}(P, \ell)$ and consider the associated Kähler structure (g_u, ω) on M. Expressing the Laplacian (i.e $\Delta^u = -\mathrm{Div}^{g_u} \circ \nabla^{g_u}$) in the action angle coordinates (x, θ) on $P \times \mathfrak{t}$ see (4.3), we get

$$\Delta^u = -\sum_{i,j=1}^{n} \left[G_{ij} \frac{\partial^2}{\partial\theta_i \partial\theta_j} + \frac{\partial}{\partial x_i} \left(H_{ij} \frac{\partial}{\partial x_j} \right) \right], \tag{4.30}$$

where $G = \mathrm{Hess}\, u$ and $H = G^{-1}$ as above. Assuming (4.29), we have

$$\Delta^{g_u} x_i = -\sum_{j=1}^{n} \frac{\partial H_{ij}}{\partial x_j} = 2\lambda x_i$$

for $i = 1, \dots n$. Inserting this in the expression of the Ricci form in action-angle coordinates (4.2), we get

$$\begin{aligned} \rho^{g_u}(\cdot, \cdot) &= \frac{-1}{2} \sum_{i,l,k=1}^{n} H_{li,ik} dx_k \wedge d\theta_l = \frac{1}{2} \sum_{l,k=1}^{n} \frac{\partial}{\partial x_k} (2\lambda x_l) dx_k \wedge d\theta_l \\ &= \lambda \sum_{k=1}^{n} dx_k \wedge d\theta_k = \lambda\omega. \end{aligned} \tag{4.31}$$

. □

Comment 4.38. By replacing "Kähler–Einstein" by "Hermitian-Einstein almost-Kähler" in Proposition 4.37, the statement holds for toric almost Kähler metric of involutive type (g_H, ω), with $H \in \mathcal{AK}(P, \ell)$. Indeed, from [154, Lemma 4.2], also in this case, $\Delta^{g_H} \mu$ is a momentum map for the Hermitian–Ricci form $2\rho^{g_H}$. Conversely, given $H \in \mathcal{AK}(P, \ell)$, the Hermitian–Ricci form $2\rho^{g_H}$ of a toric almost Kähler metric (g_H, ω) expressed in action-angle coordinates (x, θ) is in the form

$$\rho^{g_H}(\cdot, \cdot) = \frac{-1}{2} \sum_{i,l,k=1}^{n} H_{li,ik} dx_k \wedge d\theta_l$$

see [154, Equation (4.5)]. Thus, the calculation (4.31) applies in this case too.

Another interesting question is that of spectrally characterizing either constant scalar-curvature, extremal or Kähler-Einstein toric Kähler metrics. In [93]

the authors prove that the equivariant spectrum determines if a toric Kähler metric has constant scalar curvature. A variation of the argument there would show that the equivariant spectrum also determines if a metric is extremal.

Going back to the first eigenvalue, there are various bounds that one can write down for toric Kähler manifolds using Bourguignon–Li–Yau's bound. It would be interesting to see what the best bound is for a given toric manifold, once we fix the polytope. In particular, one could hope to improve the bound in Theorem 4.34 for special classes of manifolds (monotone, Fano..) or special classes of metrics say extremal toric Kähler metric, or Kähler-Einstein metrics. In [18] the authors prove that a toric Kähler–Einstein manifold whose connected component of automorphism group is a torus is never λ_1–*extremal*, where λ_1–extremal means extremal for the first eigenvalue with respect to local variations in the Kähler metrics space. Hence, in general, we cannot expect a toric Kähler–Einstein metric to saturate fine bounds. Another natural candidate to consider is a balanced metric when it exists [37].

4.3.2 The $\mathbb{T}$–invariant first eigenvalue is unbounded

On a toric manifold endowed with a torus invariant metric one can consider a toric version of λ_1 namely $\lambda_1^{\mathbb{T}}$ defined to be the smallest non-zero invariant eigenvalue of the Laplacian i.e. the smallest eigenvalue of the Laplacian restricted to torus invariant functions. We clearly have $\lambda_1 \leq \lambda_1^{\mathbb{T}}$. In [5], Abreu–Freitas studied $\lambda_1^{\mathbb{T}}$ for the simplest toric manifold, namely S^2 with the usual S^1 action by rotations around an axis. They proved it was unbounded (both above and below) among S^1-invariant metrics. Their argument for the upper bound does extend straightforwardly to higher dimension because the components of the metric in action-angle coordinates are not scalar functions when the (complex) dimension is greater than 1. With R. Sena-Dias (IST, Portugal) we have managed to modify their argument and by approximation over compact subsets of the polytope, we are able to prove the following.

Theorem 4.39. *[L.–Sena-Dias [LSD]] Let $(M, \omega, \mathbb{T})$ be a compact symplectic toric orbifold, let $\mathcal{K}_\omega^{\mathbb{T}}$ be the set of all toric Kähler metrics on $(M, \omega, \mathbb{T})$.*

$$\inf\{\lambda_1(g) \,|\, g \in \mathcal{K}_\omega^{\mathbb{T}}\} = 0.$$

and

$$\sup\{\lambda_1^{\mathbb{T}}(g) \,|\, g \in \mathcal{K}_\omega^{\mathbb{T}}\} = +\infty.$$

Combining Theorem 4.34 and 4.39, we see that there are toric Kähler manifolds for which λ_1 does not coincide with $\lambda_1^{\mathbb{T}}$. For toric Kähler–Einstein metrics, it follows from Matsushima Theorem [170] that $\lambda_1 = \lambda_1^{\mathbb{T}}$ as there are invariant eigenfunctions for λ_1.

Chapter 5

Projects and perspectives

5.1 Valuative approach to K-stability

This section contains an overview of a joint work with Ruadhaí Dervan (Cambridge, UK), the main results of this project are presented in the prepublication [DL]. The tools used here fall into a more classical theory with standard terminology and notation. Therefore, there is no background section and I refer to [144] for definitions and basic properties. In this section, and the next, X and Y denote normal varieties, as it is the common notation in algebraic geometry.

Since the resolution the YTD conjecture on Fano varieties by Chen-Donaldson-Sun [62], the algebraic side of the picture has progressed massively via the introduction of new invariants and corresponding notions of stability. Notably, Fujita and Li's reinterpretation of K-stability in terms of valuations [104, 161] are major contributions as well as the very recent breakthrough by K. Zhang [208, 209]. These ideas, together with significant input from birational geometry, have led to an almost-complete understanding of K-stability of Fano varieties. Indeed these invariants are new tools to study the moduli spaces of K-stable Fano varieties and they were used successfully to this end, for example in [6, 38, 67, 105].

Fano varieties (and their complex cone) are very special and the alternative stability notions proposed above are sometimes not even defined for general polarised variety while K-stability is. This motivates to try to adapt some of these points of view to general polarised varieties in hope to get a better grasp on K-stability. With Ruadhaí Dervan, we proposed a notion of valuative stability and associated invariant valid for a normal prolarized variety (X, L) with $\mathbb{Q}$–Cartier canonical divisor K_X. We proved that, when $L = -K_X$, our notion coicides with Fujita's valuative stability [104], and moreover, that this condition *valuative stability* is equivalent to K-stability with respect to test configurations with *integral* (reduced and irreducible) central fibre. I discuss the main result and its proof in §5.1.1.

We do *not* expect that valuative stability (as we have defined it) is equivalent to K-stability in general and we show that an equivariant version of this statement fails in the toric case §5.1.1.1). More precisely, an example in [82] shows that K-unstable toric surfaces might have a vanishing Futaki invariant. This last condition can be understood as K-stability with respect to equivariant

test configurations with integral central fibre while the destabilizing test configuration exhibited in [82] is equivariant with a reducible central fiber (with 2 components). This situation is known to be impossible for toric Fano manifolds/varieties using Wang-Zhu's Theorem [202]. Thus, our result highlights the major role of Li–Xu Theorem [164] in the theory of K-stable Fano varieties. Their theorem implies that for K-stability of *Fano* varieties $(X, -K_X)$, it *is* equivalent to check K-stability with respect to test configurations with integral central fibre. This result is not expected to be true in general and its demonstration crucially relies on the fact that Fano varieties are Mori dream spaces.

Nevertheless, test configurations with smooth, hence reduced and irreducible, central fibre play an important role in many analytic works concerning the existence of constant scalar curvature Kähler metrics [65, 191], and hence one should expect Theorem 5.4 to be a useful tool producing concrete obstruction to K-stability of polarised varieties.

5.1.1 The β–invariant, main result and idea of the proof.

Let X and Y be normal projective varieties, and let $\pi : Y \to X$ be a surjective birational morphism, with X and Y of dimension n. A prime divisor $F \subset Y$ for such Y is called a prime divisor *over* X.

We view F as defining a divisorial valuation on X in the sense that it determines vector subspace $H^0(X, kL - xF) \subset H^0(X, kL)$ for any line bundle L over X, defined via the canonical identifications

$$H^0(X, kL - xF) := H^0(Y, k\pi^*L - xF) \subset H^0(Y, k\pi^*L) \cong H^0(X, kL).$$

Alternatively, a section of $H^0(X, kL - xF)$ is a section of $H^0(X, kL)$ vanishing up to order x along $\pi(F)$.

The *volume* of $L - xF$ is the volume of the associated valuation, that is

$$\mathrm{Vol}(L - xF) = \limsup_{k\to\infty} \frac{\dim H^0(X, kL - kxF)}{k^n/n!}$$

which is known to be a continuously differentiable function on the big cone [43], vanishing identically outside this cone. The value $x \in \mathbb{R}$ where $L - xF$ leaves the big cone is finite and known as the pseudoeffective threshold, that is $\tau(F) = \sup\{x \,|\, \mathrm{Vol}(L - xF) > 0\}$. Therefore, the function

$$x \mapsto \mathrm{Vol}'(L - xF).K_X = \frac{d}{dt}\mathrm{Vol}(L - xF + tK_X)\Big|_{t=0}$$

is compactly supported and integrable against dx on $\mathbb{R}$.

Definition 5.1. Let (X, L) be a normal polarised variety with $\mathbb{Q}$–Cartier canonical divisor K_X. Given F, a prime divisor over X, we associate

$$\beta_L(F) = A_X(F)\mathrm{Vol}(L) + n\overline{c}_L \int_0^\infty \mathrm{Vol}(L - xF)dx + \int_0^\infty \mathrm{Vol}'(L - xF).K_X dx$$

where

$$\overline{c}_L = \overline{c}(X, L) = \frac{-K_X.L^{n-1}}{L^n}$$

is a topological constant and $A_X(F)$ is the log discrepancy of F in X.

When $L = -K_X$, integrating by parts the third term of β_{-K_X} turns it into a multiple of the second factor and we recover the original β–invariant of Fujita. To define β_L we need to be able to define the log discrepancy and this where we ask K_X to be $\mathbb{Q}$-Cartier, then for Y as above and satisfying the same conditions we have $A_X(F) = \mathrm{ord}_F(K_Y - \pi^* K_X) + 1$.

Remark 5.2. In essence, the log discrepancy measures the singularity type of a variety near the center of F. Its appearance in the β–invariant and the equivalence between K-stability and Fujita's valuative stability for Fano varieties [104] are involved in the characterization of the kind of singularities K-stable Fano varieties might bear [107]. The log discrepancy appearing in β_L for a general ample bundle L could eventually lead to similar results. However, to translate this invariant on the polarised complex cone $\widehat{Y}$ associated to L (see §2.1.2.2) and eventually find back Chi Li's notion of stability, it might be a problem because $\widehat{Y}$ is not $\mathbb{Q}$–Gorenstein for general ample L.

To state the result we need to introduce a finite generation condition.

Definition 5.3. [104, Definition 1.3] We say that F is *dreamy* if for some (equivalently any) $r \in \mathbf{Z}_{>0}$ the $\mathbf{Z}_{\geq 0}^{\otimes 2}$-graded $\mathbb{C}$-algebra

$$\bigoplus_{j,k \in \mathbf{Z}_{\geq 0}} H^0(X, krL - jF)$$

is finitely generated.

We then say that (X, L) is *valuatively stable* if $\beta(F) > 0$ for all dreamy divisor F over X.

Theorem 5.4. *[Dervan–L.[DL]] A polarised variety is valuatively stable if and only it is K-stable with respect to test configurations with reduced and irreducible central fibre.*

We also have proved analogous results for K-semistability and uniform K-stability with respect to norms that we have introduced following Fujita's corresponding notions.

Remark 5.5. An example of a non-dreamy prime divisor F over $(\mathbb{P}^2, -K_{\mathbb{P}^2})$ has been produced by Fujita [106, Example 3.8]. However, he proved also that for a Fano variety, valuative stability with respect to dreamy divisorial valuation is equivalent to valuative stability on any divisorial valuations. For a general polarised variety it is not known.

Sketch of the proof of Theorem 5.4. Our proof is modelled on that of Fujita [103, 104] and most of his arguments apply without much change to our situation. I briefly describe the steps in order to discuss details and possible generalisations.
1) *Correspondence between dreamy divisors and integral test configuration.* Given a dreamy divisor F over X, we get a linearly bounded decreasing finitely generated $\mathbb{Z}$–filtration defined by the subspaces

$$\mathcal{F}^j H^0(X, kL) = \begin{cases} H^0(X, kL - \lfloor j \rfloor F) & \text{if } j \geq 0 \\ V_k & \text{otherwise} \end{cases} \tag{5.1}$$

The correspondence between test configurations and decreasing linearly bounded $\mathbb{Z}$–filtrations is now well understood [204, 193, 44] and the relative Proj over $\mathbb{C}$ of the filtration (5.1) gives us a test configuration $(\mathcal{X}, \mathcal{L})$ over (X, L). The homogenous coordinates ring of the central fibre is given by the Rees algebra of the filtration and in our case, each piece is characterized by their *exact* order of vanishing on the centre Z in X of the valuation F. Thus, to check that the central fibre is integral we need to show that this algebra is an integral domain. This is true here because the order of vanishing along Z is additive under tensor product of sections.

Conversely, let $(\mathcal{X}, \mathcal{L})$ be an integral test configuration. We fix a resolution of indeterminacy as follows.

$$\begin{array}{ccc} \mathcal{Y} & & \\ {\scriptstyle q}\downarrow & \searrow^{p} & \\ X \times \mathbb{P}^1 & \dashrightarrow & \mathcal{X} \end{array} \tag{5.2}$$

Thus $(\mathcal{Y}, p^*\mathcal{L})$ is a test configuration over (X, L) with central fibre $\mathcal{Y}_0$. Let $\hat{\mathcal{X}}_0 \subset \mathcal{Y}_0$ be the strict transform of $\mathcal{X}_0$ in $\mathcal{Y}$. By [44, Section 4.2], each irreducible component $E \subset \mathcal{Y}_0$ defines a divisorial valuation on X, say v_E, via pull back from X to $\mathcal{Y}$. Moreover, denoting $D = p^*\mathcal{L} - q^*L$, each of these valuations defines a filtration

$$F_E^\lambda H^0(X, kL) = \{f \in H^0(X, kL) \,|\, v_E(f) \geq \lambda - kb_E^{-1}\mathrm{ord}_E(D)\}$$

where $b_E = \mathrm{ord}_E(\mathcal{Y}_0)$. From [44, Lemma 5.17], the filtration corresponding to the test configuration $(\mathcal{Y}, p^*\mathcal{L})$ is defined by the intersections

$$\bigcap_{\mathrm{irred.}E\subset\mathcal{Y}_0} F_E^\lambda H^0(X, kL).$$

By the projection formula, this intersection coincides with $F^\lambda_{\hat{\mathcal{X}}_0} H^0(X, kL)$ which in turn implies that $v_{\hat{\mathcal{X}}_0}$ is a dreamy divisorial valuation.

2) *The numerical invariants match.* To conclude the proof we need to show $\beta_L(F) = \mathbf{DF}(\mathcal{X}, \mathcal{L})$ with the notations above. The way to compute the Donaldson with the filtration data is known [193] and following Fujita's demonstration [104] we get that

$$\mathbf{DF}(\mathcal{X}, \mathcal{L}) = \frac{n!}{L^n}\left(-2f_n + n\bar{c}(X, L)f_{n+1}\right)$$

where for $k >> 0$, $\sum_{j=0}^{k\tau(F)} \dim F^j\mathcal{H}_k = f_{n+1}k^{n+1} + f_nk^n + O(k^{n-1})$ and $\mathcal{H}_k = H^0(X, kL)$. In general $L - xF$ is not ample on the whole interval $(0, \tau(F))$ so we cannot apply an asymptotic Riemann-Roch formula these coefficients into intersections products. However, like Fujita did, we can use a scaled MMP [136, Theorem 4.2] which ensures that there is a partition of that interval

$$0 = \tau_0 < \tau_1 < \cdots < \tau_m = \tau_L(F),$$

and normal projective varieties $Y_1, \ldots, Y_m$ with birational contractions

$$\phi_j : Y \dashrightarrow Y_j$$

where $(\phi_j)_*(L - xF)$ is ample for all $x \in (\tau_{j-1}, \tau_j)$ and for all $x \in [\tau_{j-1}, \tau_j]$, $(\phi_j)_*(L - xF)$ is semiample and $H^0(Y, k(L - xF)) = H^0(Y_j, (\phi_j)_*(kL - kxF))$.

Then applying Fujita's asymptotic Riemann-Roch formula [104, Proposition 4.1] we have

$$f_{n+1} = \sum_{j=1}^{m} \frac{1}{n!} \int_{\tau_{j-1}}^{\tau_j} (L_j - xF_j)^n dx,$$
$$f_n = -\sum_{j=1}^{m} \frac{1}{2(n-1)!} \int_{\tau_{j-1}}^{\tau_j} ((L_j - xF_j)^{n-1}.(K_{Y_j} + F_j) dx.$$

Moreover for $x \in [\tau_{j-1}, \tau_j]$, we have an equality $\mathrm{Vol}(L - xF) = (L_j - xF_j)^n$ which would help. We also have from [104, Claim 5.6] that

$$K_{Y_j} - (\phi_j)_* \pi^* K_X = (A_X(F) - 1)F_j.$$

For Fujita, the two previous formulas were enough to express f_n and f_{n+1} in terms of $L = -K_X$ and F on X only. However, when $L \neq -K_X$ there is one term left, involving $L^{n-1}.K_{Y_j}$. So we needed to relate the derivative of the volume on X to the one on Y_j. Actually, it comes from [136, Remark 2.4 (i)] and from the differentiability of the volume [43] that in the situation described here: for any line bundle E on Y and any $x \in [\tau_{j-1}, \tau_j]$, there is an equality

$$(L_j - xF_j)^{n-1}.((\phi_j)_* E) = \frac{1}{n} \mathrm{Vol}'(L - xF).E.$$

This allows to replace all the terms depending on j in the formulas for f_n and f_{n+1} above by constant (in the variable j) like $A_X(F)$, $\mathrm{Vol}(L - xF)$, $\mathrm{Vol}'(L - xF).(K_X)$ and we ended up with $\beta_L(F)$. □

Comment 5.6. Valuative stability, as we have defined it in [DL], tests one divisorial valuation at a time following Fujita's approach. Theorem 5.4 confirms that it indeed corresponds to a subcase of K-stability. On the other hand, from Boucksom et al. [44, §4, §5] an ample test configuration $(\mathcal{X}, \mathcal{L})$ is associated to a *set* of divisorial valuations: the $v_{E_1}, \ldots, v_{E_N}$ with $E_1, \ldots, E_N$ being the irreducible components of the central fibre of the test configuration (actually, the resolution dominating $X \times \mathbb{P}^1$). We could start with a set of divisorial valuations with the dreaminess property replaced by the intersection of the filtrations $H^0(kL - xE_i)$ being finitely generated. However, it would be unlikely that we would be able to express $\mathbf{DF}(\mathcal{X}, \mathcal{L})$ as a function on $\beta_L(E_1), \ldots, \beta_L(E_N)$, apart from very special case. Maybe it is possible to get some relations between $\mathbf{DF}(\mathcal{X}, \mathcal{L})$ and the values $\beta_L(E_1), \ldots, \beta_L(E_N)$ but even in the toric case (see below) it is not clear.

5.1.1.1 The toric valuative stability

Here we assume that (X, L) is a normal polarised toric variety and F is toric prime divisor *over* X, which means that there exists a normal toric variety Y and $\mathbb{T}_C$-equivarient birational morphism $\pi : Y \to X$ with $F \subset Y$, a prime divisor which is fixed by a subgroup $\mathbb{C}^* \subset \mathbb{T}_C$. We assume furthermore that K_X and K_Y are $\mathbb{Q}$-Cartier (for example this is ensured if they are $\mathbb{Q}$-factorial which corresponds to the fact that they have at worst orbifold singularities).

In this situation, the prime divisor F corresponds uniquely to a ray ρ_F of the fan of Y, then the divisorial valuation F only depends on that ray and we can assume that Y is obtained by a subdivision. We denote by u_F the primitive vector in $\rho_F \cap N$, where N is the lattice of circle subgroups.

Now, following the standard theory of toric algebraic geometry (see eg. [69]), to L is associated a convex polytope $\overline{P} \subset \mathfrak{t}^*$ where, following the standard notation of that field, $\mathfrak{t} = N_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} N$. Observe that in the context of toric algebraic geometry, the polytope associated to L is integral and the outward normal of a facet corresponds to the primitive vector in a ray of the fan of X. In particular, the lattice determines a volume form dv on $N_{\mathbb{R}}$ and the measure on the boundary σ introduced in [82], see (4.8), is well-defined. We denote bar_{P_L} (respectively $\mathrm{bar}_{\partial P_L}$) the barycentre of P (respectively ∂P) with respect to their respective measures. Using essentially the co-area formula we proved in [DL] that

$$\frac{\beta_L(F)}{\mathrm{Vol}_\Sigma(\partial P)} = (A_X(F) - 1)\frac{\mathrm{Vol}_M P}{\mathrm{Vol}_\Sigma(\partial P)} + \langle \mathrm{bar}_P - \mathrm{bar}_{\partial P}, u_F \rangle. \tag{5.3}$$

Remark 5.7. Whenever $L = -K_X$, we have (see (4.10) where $q = 0$)

$$\left(1 + \frac{1}{n}\right) \mathrm{bar}_P = \mathrm{bar}_{\partial P}$$

and we recover Fujita's formula [101, Theorem 6.1] from (5.3).

Note that from [69, Lemma 11.4.10] we get that $A_X(F) \geq 1$ with equality if and only if u_F lies in a ray of the fan of X. This implies that if the Futaki invariant of (X, L) is zero (that is $\mathrm{bar}_P = \mathrm{bar}_{\partial P}$) then

$$\beta_L(F) \geq 0$$

for any toric prime divisor F over X. Conversely, assume $\beta_L(F) \geq 0$ for any toric prime divisor F over X. Because X compact, there are some positive real numbers $t_\rho > 0$ satisfy $\sum_{\rho \in \Sigma(1)} t_\rho u_\rho = 0$ where $\Sigma(1)$ denotes the rays in the fan of X. By linearity of the Futaki invariant and by (5.3), we have

$$0 \leq \sum_{\rho \in \Sigma(1)} t_\rho \beta_L(D_\rho) = \mathrm{Vol}_\Sigma(\partial P_L) \sum_{\rho \in \Sigma(1)} t_\rho \langle \mathrm{bar}_P - \mathrm{bar}_{\partial P}, u_\rho \rangle = 0.$$

Thus, $\beta_L(D_\rho) = 0$ for any $\rho \in \Sigma(1)$.

Theorem 5.8. *[Dervan–L.[DL]] The Futaki invariant of (X, L) vanishes identically (on the torus) if and only if*

$$\beta_L(F) \geq 0$$

for any toric prime divisor F over X.

This result is not surprising given Theorem 5.4 and indicates that this notion of *toric* valuative stability is too strong.

Comment 5.9. Many questions remain open as for example : does valuative stability with respect to dreamy divisorial valuation is equivalent to valuative stability with respect to any divisorial valuation ? Can we define similarly valuative stability with respect to other type of valuations (quasimonomial..) ? See also Comment 5.6.

5.2 Various projects in toric Kähler geometry

5.2.1 Sasaki-Einstein metrics with conical singularity along a divisor.

Let (Y,ω) be a smooth toric Kähler cone of complex dimension $n+1$ with smooth compact cross section. In what follows, $\widehat{\mathbb{T}} = \widehat{\mathfrak{t}}/\Lambda$ denotes a compact torus of dimension $n+1$ acting efficiently, holomorphically and in a Hamiltonian fashion on (Y,ω). We denote the unique homogeneous of order 2 momentum map $\mu : Y \to \widehat{\mathfrak{t}}^*$. The associated moment cone is the strictly convex cone

$$C := \mu(Y) = \{x \in \widehat{\mathfrak{t}}^* \setminus \{0\} \ \ | \langle \ell_s, x\rangle \geq 0 \ \text{ for } s = 1,\dots,d\},$$

where ℓ_s are inward normal vectors, primitive in Λ, defining the facets $\widehat{F}_s = \{\ell_s^{-1}(0)\} \cap C$. The Reeb cone is the interior of the dual cone of C, where

$$\mathfrak{t}_+ = \{q \in \mathbb{R}^{n+1} \ \ | \langle x, q\rangle > 0 \ \text{ for all } x \in C\}$$

Before stating the next result, two simple observations are in order. Each $\xi \in \widehat{\mathfrak{t}}_+$, determines an abstract labelled polytope in the sense of (4.1.1), denoted (P_ξ, ℓ), defined as

$$\overline{P}_\xi := \mathring{C} \cap (H_\xi := \{x \in \widehat{\mathfrak{t}}^* \,|\, \langle x, \xi\rangle = 1/2\})$$

and $\ell = \{\ell_1,\dots,\ell_d\} \in \mathrm{Aff}(H_\xi, \mathbb{R})$ (note that the restriction of a linear function on $\widehat{\mathfrak{t}}^*$ on any affine hyperplane is affine-linear).

Moreover, the linear map $L : \widehat{\mathfrak{t}}^* \to \mathbb{R}^d$

$$x \mapsto (\langle \ell_1, x\rangle, \dots, \langle \ell_d, x\rangle) \tag{5.4}$$

injects the moment cone into the positive quadrant and we call the image of C by L the *angles' cone*, that is

$$\mathcal{B} := \{\beta = (\beta_1,\dots,\beta_d) \in \mathbb{R}^d_{>0}\} \cap \mathrm{Image}(L).$$

Note that Y denotes the cone *without* its apex. In particular, Y as well as its invariant divisors $D_s := \mu^{-1}(\widehat{F}_s) \subset X$ are smooth submanifolds. We fix (Y,ω) as a symplectic cone manifold and consider toric Kähler cone metrics on X smooth on $\mu^{-1}(\mathring{C})$ and with conical singularity of angle $2\pi\beta_s$ along D_s. With Martin De Borbon (King's College, UK), applying Theorem 4.2 above and in particular Proposition 4.8, we have obtained the following classification.

Theorem 5.10. *[de Borbon–L. [dBL]] There is a $(n+1)$-dimensional family of $\widehat{\mathbb{T}}$–invariant compatible Ricci-flat cone metrics on (Y,ω) smooth on $\mu^{-1}(\mathring{C})$ and with cone singularities along its invariant divisors. The family of metrics can be parameterized in the following two equivalent ways.*

(1) Fixing the Reeb vector field. *For every $\xi \in \mathfrak{t}_+$, there is a unique $\beta \in \mathcal{B}$ such that Y has a $\mathbb{T}$–invariant Ricci-flat cone metric with cone angles $2\pi\beta_s$ along D_s and Reeb vector equal to ξ.*

(2) Fixing the cone angles. *For every β in $\mathcal{B}$ there is a unique ξ in $\mathfrak{t}_+$ such that Y has a $\mathbb{T}$–invariant Calabi-Yau cone metric with cone angles $2\pi\beta_s$ along D_s and Reeb vector equal to ξ.*

In either case, the toric Ricci-flat cone metric with prescribed Reeb vector or cone angles is unique up to isometry.

The remaining of this section is a sketch of the proof of that last Theorem.

The toric symplectic point of view on toric kähler geometry, developed in compact case by [1, 2, 12, 82, 86, 115] has been extended to the Kähler cone case by [167], see also [3, 148]. In particular, we have suitable action–angle coordinates $(x, [\theta])$ on the open dense subset $\mathring{Y}$ of Y where $\widehat{\mathbb{T}}$ acts freely, providing a $\widehat{\mathbb{T}}$–equivariant symplectomorphism $(\mathring{Y}, \omega) \simeq (\mathring{C} \times \widehat{\mathbb{T}}, \sum_{i=1}^n dx_i \wedge d\theta_i)$. Any convex smooth function $\widehat{u} \in C^\infty(\mathring{C})$ defines a ω–compatible Kähler metric as

$$g_{\widehat{u}} := \widehat{u}_{ij} dx_i \otimes dx_j + \widehat{u}^{ij} d\theta_i \otimes d\theta_j. \tag{5.5}$$

Martelli, Sparks and Yau gave the conditions on $\widehat{u}$ for the metric $g_{\widehat{u}}$ to be the restriction to $\mathring{Y}$ of a smooth $\mathbb{T}$–invariant Kähler cone metric on (Y, ω), [167]. A key observation is that their conditions translate exactly as

1. the Hessian of $\widehat{u}$ is homogeneous of order -1 with respect to the $\mathbb{R}^+$–dilatation on $\widehat{\mathfrak{t}}^*$, thus $\xi_j = \sum_{i=0}^n \widehat{u}_{ij} x_i$ are constant and $\xi = \sum_{i=0}^n \xi_i \frac{\partial}{\partial \theta_i} \in \widehat{\mathfrak{t}}$ is the Reeb vector field;
2. the restriction of $\widehat{u}$ to the polytope $P_\xi := \mathring{C} \cap \{x \in \widehat{\mathfrak{t}}^* \,|\, \langle x, \xi \rangle = 1/2\}$ lies in $\mathcal{S}(P_\xi, \ell)$.

In particular, every symplectic potential $u \in \mathcal{S}(P_\xi, \ell)$ defines a unique Kähler cone metric on Y, with potential $\widehat{u}$ with Reeb vector field ξ. The correspondence can be made explicit, see [3] by taking

$$\widehat{u} = 2\mu u\left(\frac{\mu}{2\langle \mu, \xi \rangle}\right) + \frac{\mu}{2\langle \mu, \xi \rangle} \log \langle \mu, \xi \rangle.$$

Actually, Martelli, Sparks and Yau proved that in addition to the natural homogeneity and convexity properties, $\widehat{u} \in C^\infty(\mathring{C}) \cap C^0(C)$ defines a smooth ω-compatible Kähler cone structure $g_{\widehat{u}}$ on Y, restricting to (5.5) on $\mathring{Y}$, if and only if

$$\widehat{u} - \sum_s (\ell_s \log \ell_s) \in C^\infty(C).$$

Therefore, the local analysis [L3, §6.3] (recalled in §4.1.4 above, holds here as well and we get the following.

Proposition 5.11. *[dBL] Let $\beta_1, \dots, \beta_d \in \mathbb{R}_{>0}$ and $\widehat{u} \in C^0(C) \cap C^\infty(\mathring{C})$. The tensor $g_{\widehat{u}}$ of (5.5) is the restriction to $\mathring{Y}$ of a toric Kähler cone metric with conical singularities of angles $2\pi\beta_s$ along the divisors D_s if:*

(i) the restriction of $\widehat{u}$ to the interior of C, and to the interior of any of its faces of positive dimension, is smooth and strictly convex and such that

$$\widehat{u} - \frac{1}{2} \sum_{s=0}^d \beta_s^{-1} \ell_s \log \ell_s \in C^\infty(C); \tag{5.6}$$

(ii) the Hessian of $\widehat{u}$ is homogeneous of order -1 with respect to the $\mathbb{R}^+$–dilatation on $\widehat{\mathfrak{t}}^*$.

In that case the Reeb vector field $\xi = (\xi_0, \dots, \xi_n) \in \widehat{\mathfrak{t}}_+$ *is given in coordinates as* $\xi_j = \sum_{i=0}^n \widehat{u}_{ij} x_i$.

Said differently, $g_{\widehat{u}}$ of (5.5) is the restriction to $\mathring{Y}$ of a toric Kähler cone metric with conical singularities of angles $2\pi\beta_s$ along the divisors D_s if, in addition to the natural homogeneity property, its restriction to P_ξ lies in $\mathcal{S}(P_\xi, \beta_1^{-1}\ell_1, \dots, \beta_d^{-1}\ell_d)$. Combined with the relation between Ricci curvature of the cone metric and the transversal Ricci curvature, see §2.1.2.3, the set of angles $\beta \in \mathcal{B}$ whose existence is claimed in part (1) of Theorem 5.10 is the unique one such that $(P_\xi, \beta_1^{-1}\ell_1, \dots, \beta_d^{-1}\ell_d)$ is monotone (see Definition 4.12) with vanishing Futaki invariant and $A_{(P_\xi, \beta_1^{-1}\ell_1, \dots, \beta_d^{-1}\ell_d)} = 4n(n+1)$ (see Definition/Proposition 4.14). Such labelling exists and is unique by Lemma 4.15. Moreover, by Theorem 4.16 there exists a unique Kähler–Einstein symplectic potential $u \in \mathcal{S}(P_\xi, \beta_1^{-1}\ell_1, \dots, \beta_d^{-1}\ell_d)$ up to an addition by an affine-linear function. The associated symplectic potential on the cone $\widehat{u}$ satisfies the conditions of Proposition 5.11 and a Kähler–Ricci-flat metric on $\mathring{Y}$.

To explain the part (2) of Theorem 5.10, note that any $\beta \in \mathcal{B}$ is the image of a unique point $p_\beta \in \mathring{C}$ via the map (5.4) and we can define the set Ξ_β of Reeb vector fields whose transversal polytope P_ξ contains p_β, that is

$$\Xi_\beta := \{\xi \in \widehat{\mathfrak{t}}_+ \,|\, p_\beta \in P_\xi\},$$

which is a cross section of $\widehat{\mathfrak{t}}_+$ and a convex open polytope. Then, extending the main result of [167], we prove the following where dv_ξ is a Lebesgue volume form on H_ξ.

Theorem 5.12 (Martelli–Sparks–Yau [167],de Borbon–L.[dBL])**.** *The volume functional*

$$\mathbf{V} : \Xi_\beta \to \mathbb{R}$$

is strictly convex and its unique minimum $\xi_\beta \in \Xi_\beta$ *is the only Reeb vector field in* Ξ_β *satisfying* $p_\beta = bar(P_\xi, \mathrm{dv}_\xi)$*. Moreover,* ξ_β *is the only Reeb vector field in* $\widehat{\mathfrak{t}}_+$ *admitting a compatible toric Calabi-Yau cone metric with cone angles* $2\pi\beta_s$ *along the toric divisors* D_s*.*

The proof of this statement in [dBL] is not, in essence, different than the one of the (toric) convexity result of Martelli–Sparks–Yau [167], although the last claim uses the unicity of Theorem 4.16. However, the formalism of abstract labelled polytope reveals the simplicity and generality of the argument, si I present the proof here.

Proof. Consider the labelling $\tilde{\ell}$ on P_ξ such that $\tilde{\ell}_s = \beta_s^{-1}\ell_s$. By definition of p_β we have

$$\tilde{\ell}_1(p_\beta) = \cdots = \tilde{\ell}_d(p_\beta) = 1.$$

Thus the labelled polytope $(P_\xi, \tilde{\ell})$ is monotone in the sense of Definition 4.12. The key observation is that the associated measure (4.8) introduced by Donaldson [82] can be given by

$$\sigma = \sum_{i=1}^n (-1)^{i+1}(x_i - p_{\beta,i}) dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

with classical notations in affine coordinates $(x_1, \dots, x_n)$ on the hyperplane containing P_ξ. Note that $d\sigma = n\mathrm{dv}$ and thus,

$$\int_{\partial P_\xi} \sigma = n \int_{P_\xi} \mathrm{dv}.$$

Then integration by parts leads to (4.10), that we recall

$$\left(1 + \frac{1}{n}\right) \mathrm{bar}(P_\xi, \mathrm{dv}) = \mathrm{bar}(\partial P_\xi, \sigma) + \frac{p_\beta}{n}. \tag{5.7}$$

This implies that $p_\beta = \mathrm{bar}(P_\xi, \mathrm{dv})$ if and only if $\mathrm{bar}(P_\xi, \mathrm{dv}) = \mathrm{bar}(\partial P_\xi, \sigma)$, which in turns is equivalent to the vanishing of the Futaki invariant of $(P_\xi, \tilde{\ell})$ that is $A_{(P_\xi, \tilde{\ell})}$ is constant see Definition/Proposition 4.14.

The remaining step is to establish the link with the volume functional $\mathbf{V} : \Xi_\beta \to \mathbb{R}$. Note that for any $\xi \in \Xi_\beta$ by definition $T_\xi \Xi_\beta$ is the annihilator of p_β in $\mathfrak{t}$. Thus, from and easy computation $d_\xi \mathbf{V}(a) = -(n+1) \int_{P_\xi} \langle x, a \rangle \mathrm{dv}$ we get that ξ is a critical point of $\mathbf{V}$ in Ξ_β if and only if $p_\beta = \mathrm{bar}(P_\xi, \mathrm{dv})$. Also for any variation $\xi + ta$ in Ξ_β, we have

$$\left(\frac{d^2}{dt^2} \mathbf{V}(\xi + ta)\right)_{t=0} = (n+1)(n+2) \int_{P_\xi} \langle x, a \rangle^2 \mathrm{dv}.$$

Finally, the localisation argument in the toric case can be done directly on the polytope and is equivalent to an integration by parts. From this we get that $\mathbf{V}$ is proper which concludes the proof. □

5.2.2 Almost Kähler metrics of constant scalar curvature on $\mathbb{CP}^2$ blown-up at three points

Unsurprisingly, there are few known explicit examples of extremal Kähler metrics on compact manifolds and orbifolds. Actually, essentially all the explicit extremal Kähler metrics are obtained by the Calabi ansatz [57] and its generalisation using Hamiltonian 2–forms [11, 12, 13]. This includes the extremal metrics on weighted projective spaces of Bryant [56]. The interest of such explicit description is highlighted by the famous example of Apostolov–Calderbank–Gauduchon–Tønnesen-Friedman [13] providing a counter-example of a preliminary version of the YTD conjecture.

As explained in [92], a better knowledge of some Kähler–Einstein metrics could also be useful for physicists which lead the authors to propose various methods to approximate numerically the Kähler–Einstein metric on the (Fano) third del Pezzo surface. Let (M, J) denote this complex surface, that is (M, J) is the toric complex manifold obtained by blowing up $\mathbb{CP}^2$ at its 3 fixed point. It is known that (M, J) is a toric Fano manifold with vanishing Futaki invariant, thus by Wang–Zhu Theorem [202], it admits a compatible Kähler–Einstein metric ω_{KE}. Actually, in this simpler case the fact that (M, J) admits a Kähler-Einstein metric was known before and proved by Mabuchi [166].

In this line of idea, I think it would be interesting to describe, more explicitly, the toric extremal *almost* Kähler metrics on the symplectic manifold (M, ω)

where M is the smooth manifold underlying the third del Pezzo surface and ω is a $\mathbb{T}$–invariant symplectic form in the same cohomology class than ω_{KE}, that is $2\pi c_1(M)$. From [154, Lemma 4.2] and because there exists one Kähler–Einstein metric on M, there exists an infinite family of toric extremal almost Kähler metrics of involutive type on (M,ω). All these metrics are csc since the affine extremal function only depends on the boundary condition of the associated labelled polytope, see §4.1.5. Denoting (P,ℓ) the labelled polytope of (M,ω) and working in action–angle coordinates (x,θ), recall that a ω–compatible toric almost Kähler metric (g_H, J_H) is encoded in a matrix valued function $H \in \mathcal{AK}(P,\ell)$ see §4.2.1. The equation we obtain is then

$$\sum_{i,j=1}^{2} \frac{\partial^2}{\partial x_j \partial x_i} H_{ij} = -4.$$

If we could find an explicit (or nearly explicit) description of the family, say $\mathcal{AK}_C \subset \mathcal{AK}(P,\ell)$, of toric csc almost Kähler metrics on (M,ω), the next natural step would be to study the behaviour of the functional $F : \mathcal{AK}_C \to \mathbb{R}$, introduced in [82], and defined by

$$F(H) := -\int_P \log(\det H)\mathrm{dv}.$$

Donaldson proved in [82, Discussions $(4a)$] that, on the space of extremal almost Kähler metrics of involutive type, the critical point of that functional (if it exists) is integrable.

An alternative approach would be to consider only the Hermitian-Einstein almost Kähler metrics of involutive type on (M,ω), say $\mathcal{AK}_{HE} \subset \mathcal{AK}(P,\ell)$. This also consists in an infinite dimensional family by Lejmi's Lemma [154, Lemma 4.2]. Moreover, using the almost Kähler version of Proposition 4.37 from [LSD], see Comment 4.38, these elements $H \in \mathcal{AK}_{HE}$ are caracterized by the following degree one differential equations

$$\sum_{j=1}^{2} \frac{\partial}{\partial x_j} H_{ij} = -2x_i \qquad\qquad i = 1,2. \tag{5.8}$$

Relaxing the condition $H \in \mathcal{AK}(P,\ell)$ by allowing non-positive definite matrix valued functions that do not satisfies boundary condition (4.15) and (4.16), we get easily many solutions of (5.8). Also, note that a smooth solution H satisfying the boundary condition (4.15) and (4.16) satisfies automatically some positivity condition because (P,ℓ) is uniformly K-stable, see Remark 4.28.

Appendices

Appendix A

Basics on extremal Kähler metrics

For the reader's convenience, below are gathered classical results of Calabi [58] and Futaki–Mabuchi in [110] following the explanation given in Gauduchon's notes [112, §2.4, §4.13].

A.1 Calabi structure Theorem

Let (M, J, ω, g) be a compact Kähler manifold. A common description of the reduced automorphism group $\mathrm{Aut}_{red}(M, J) \subset \mathrm{Aut}(M, J)$ is that it is the connected Lie group of real holomorphic vector fields, with Lie algebra $\mathfrak{h}_{red} = \{X \in \mathrm{Lie}(\mathrm{Aut}(M, J)) \,|\, \exists p \in M, X_p = 0\}$. Actually, denoting $\mathfrak{harm}_g$ the space of harmonic 1–forms, $\mathfrak{h}_{red}$ coincides with the kernel of the map

$$\mathrm{Lie}(\mathrm{Aut}(M, J)) \ni X \mapsto (\alpha \mapsto \alpha(X)) \in \mathfrak{harm}_g^*.$$

This explains many properties of $\mathfrak{h}_{red}$. For example, $\mathrm{Lie}(\mathrm{Aut}(M, J))/\mathfrak{h}_{red} \simeq H^1(M, \mathbb{R})$.

Now, assume that (M, J, ω, g) is a compact *extremal* Kähler manifold in the sense of Calabi. Thus, by definition, $X := X_{\mathrm{Scal}_\omega} = -J\nabla^g \mathrm{Scal}^g$ is a Killing vector field, it is also hamiltonian and thus $X \in \mathfrak{h} := \mathrm{Lie}(\mathrm{Aut}(M, J)))$. In this situation, Calabi proved the following:

- An L^2–orthogonal decomposition holds

  $$\mathfrak{h} = \mathfrak{h}^{(0)} \oplus \left(\bigoplus_{\lambda > 0} \mathfrak{h}^{(\lambda)} \right)$$

  where, for $\lambda \geq 0$, $\mathfrak{h}^{(\lambda)} := \{Z \in \mathfrak{h} \,|\, \mathcal{L}_X Z = \lambda J Z\}$. Thus, $\mathfrak{h}^{(0)}$ is the centralizer of X.
- $\mathfrak{h}^{(0)}$ is reductive and decomposes as

  $$\mathfrak{h}^{(0)} = \mathfrak{a} \oplus \mathfrak{i}_{ham} \oplus J\mathfrak{i}_{ham}$$

  where $\mathfrak{a}$ is abelian (and the Lie algebra of a compact complex torus), and $\mathfrak{i}_{ham}$ is the space of hamiltonian Killing vector fields.

- For $\lambda > 0$, $\mathfrak{h}^{(\lambda)} \subset \mathfrak{h}_{red}$ and $\mathfrak{h}_{red} = \mathfrak{i}_{ham} \oplus J\mathfrak{i}_{ham} \oplus \left(\bigoplus_{\lambda>0} \mathfrak{h}^{(\lambda)}\right)$.
- The subspace of Killing vector fields is $\mathfrak{a} \oplus \mathfrak{i}_{ham}$.

From this, Calabi deduced that the connected component of the isometry group of a compact extremal Kähler manifold is maximal among all compact, connected Lie subgroups of $\mathrm{Aut}(M, J)$. More precisely for $\lambda > 0$, the flow of a non trivial element of $\mathfrak{h}^{(\lambda)}$ cannot be contained in a compact group.

A.2 The extremal vector field of a Kähler class

Fixing a maximal compact connected subgroup $K \subset \mathrm{Aut}_{red}(M, J)$, we consider the space of K–invariant metrics $\mathcal{M}^K_\Omega$, with Kähler class $\Omega = [\omega] \in H^{1,1}(M, \mathbb{R})$. By Calabi structure Theorem, an extremal Kähler metric, if any, belongs to $\mathcal{M}^{hKh^{-1}}_\Omega$ for some $h \in \mathrm{Aut}_{red}(M, J)$. Moreover, $\mathcal{M}^{hKh^{-1}}_\Omega$ contains an extremal metric if and only if $\mathcal{M}^K_\Omega$ does.

Observe that all the metrics in $\mathcal{M}^K_\Omega$ have the same space of hamiltonian Killing vector fields, namely $\mathfrak{k} = \mathrm{Lie}K$. Given $\omega \in \mathcal{M}^K_\Omega$ the space of Killing potentials $\mathcal{K}(M, \omega)$ is the subspace of smooth function $f \in C^\infty(M)$ such that the hamiltonian $X_f \in \mathfrak{k}$ and we denote the L^2–projection

$$\pi^K_\omega : C^\infty(M) \to \mathcal{K}(M, \omega).$$

Then we can define the *extremal vector field* of Ω and K as the hamiltonian vector field of the projection of the scalar curvature

$$X_{ext}(\Omega, K) := X_{\pi^K_\omega(\mathrm{Scal}_\omega)}.$$

Futaki and Mabuchi proved that $X_{ext}(\Omega, K)$ only depends on Ω and K (and not on the particular metric $\omega \in \mathcal{M}^K_\Omega$). Furthemore $X_{ext}(\Omega, K)$ lies in the center of K and thus, if $\mathbb{T} \subset K$ is a maximal torus, then

$$X_{ext}(\Omega, \mathbb{T}) = X_{ext}(\Omega, K).$$

For any $\omega \in \mathcal{M}^K_\Omega$, there exists a unique Killing potential $f_{ext}(\omega) \in \mathcal{K}(M, \omega)$ such that $X_{f_{ext}(\omega)} = X_{ext}(\Omega, K)$ and

$$\int_M f_{ext}\omega^n = \int_M \mathrm{Scal}_\omega \omega^n.$$

Thus $\omega \in \mathcal{M}^K_\Omega$, is extremal if and only if

$$\mathrm{Scal}_\omega = f_{ext}(\omega).$$

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