A categorical study of spectral dualities

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A CATEGORICAL STUDY OF SPECTRAL DUALITIES

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Résumé

La construction spectrale subsume plusieurs notions mathématiques d'importance, telles que le spectre d'un anneau commutatif en géométrie algébrique ou la dualité de Stone. Entremêlant des aspects catégoriques, topologiques et logiques, elle repose sur la notion de géométrie, une condition reliant une théorie essentiellement algébrique, une extension géométrique axiomatisant une classe d'objets dits locaux, et un système de factorisation, encodant des situations de nature géométrique dans une catégorie d'objets algébriques. Le spectre associé à une géométrie est alors une construction permettant de déployer cette géométrie cachée d'une façon universelle. Plusieurs approches plus ou moins complètes ont été proposées en parallèle pour la construction spectrale, utilisant des formalismes très différents, les uns issus de la théorie catégorique des modèles, d'autres purement catégoriques, d'autres reposant sur le langage des topos. Cette thèse se propose d'unifier ces différents traitements en une approche synthétique combinant les aspects complémentaires de cette construction. Nous discutons par ailleurs en épilogue quelques éléments pour une version 2-catégorique de cette construction, permettant de retrouver les dualités syntaxe-sémantique de la logique du premier ordre comme des constructions spectrales.

Mots-clés : Spectre, géométries, admissibilité, foncteur stable, multi-adjoint, dualité spectrale, adjonction spectrale, objet local, flèche étale, flèche locale, topos modellé.

Abstract

The spectral construction subsumes several prominent mathematical notions, as the spectrum of a commutative ring in algebraic geometry, or also Stone duality. Interweaving categorical, topological and logical aspects, it relies on the notion of geometry. The latter is a condition relating the data of an essentially algebraic theory, a choice of a geometric extension coding for a class of local objects, and a factorization system, encoding a spatial behaviour inside a category of algebraic objects. Then the purpose of the spectrum is to deploy this hidden geometry in a universal way. Several approaches to the spectral construction have been proposed, with unclear links and sometime incomplete treatment, using very different formalisms from category theory, categorical model theory or topos theory. The present thesis unifies those treatments into a synthetic approach combining all the complementary aspects of this construction. We also discuss in an epilogue some elements for a 2-categorical version of this construction in the optics of recovering the syntax-semantics adjunction of first order logics as instances of spectral constructions.

Keywords: Spectrum, geometry, admissibility, stable functor, right multi-adjoint, spectral duality, spectral adjunction, local object, local map, etale map, modelled topos.

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Contents

Introduction en français

Cette thèse se propose de fournir un traitement unifié et synthétique de la théorie des spectres, entrelaçant ses aspects catégoriques, logiques et topologiques.

Les mathématiques nous ont livré plusieurs exemples de dualités spectrales, de la construction classique du spectre de Zariski pour un anneau commutatif, notion de base de la géométrie algébrique, aux diverses dualités à la Stone correspondant aux différentes variétés d'algèbres propositionnelles. Toutes partagent un double intérêt : celui, dans un sens, de déployer des informations de nature géométrique cachées dans des structures algébriques, et inversement, de fournir des " axiomatisations" d'espaces dans certaines classes en leur associant des objets de nature algébrique encodant leur structure géométrique.

Ce procédé repose sur un choix de "données locales" dans le versant algébrique, lesquelles se comporteront comme des points dans le versant géométrique. Ces données locales sont généralement une classe d'objets disposant d'une structure algébrique plus simple, où certaines propriétés deviennent plus aisées à tester, et pouvant être utilisés pour sonder des objets plus complexes. De la sorte des problèmes algébriques difficiles portant sur des objets compliqués peuvent se réduire à des questions plus faciles sur les données locales qui leurs sont attachées, ce qui revient, dans le versant géométrique, à prouver certaines propriétés spatiales en les testant localement en chaque point.

La notion de spectre étudiée ici s'est révélée utile dans au moins deux domaines :

- la construction spectrale sous-tendant la géométrie algébrique, le *spectre de Zariski* d'un anneau commutatif, est indéniablement une contribution majeure autant à la théorie des anneaux qu'à la géométrie en général et pour plusieurs raisons. Dans un sens, elle a aidé à formaliser la façon dont les variétés algébriques pouvaient être paramétrisées par équations polynomiales et a rendu possible l'utilisation de méthodes géométriques dans l'étude de la structure géométrique de certaines classes d'espaces ; dans l'autre sens, elle a permis de convertir des problèmes de la théorie des anneaux, dont la formulation purement algébrique demeurait absconse, en des problèmes géométriques plus clairs. Ici, c'était la structure plus simple des *anneaux locaux* qui a été exploitée. Par ailleurs, cette construction des spectres d'anneaux pouvant être équipés de leur *faisceau structural* tout comme de faisceaux de modules, des techniques géométriques ont pu être appliquées pour obtenir des résultats sur les catégories de modules sur un anneau donné. Enfin elle a permis d'obtenir des preuves constructivistes de résultats classiques de la théorie des anneaux.

- les dualités à la Stone, bien que souvent présentées comme des cas de dualités concrètes, sont également de nature spectrale encore plus patente du fait des liens profonds unissant les treillis à la topologie. Mais leurs contributions majeures se trouvent du côté de la théorie des preuves et de la théorie des modèles, dans la mesure où les différents fragments de la logique propositionnelle correspondent à des catégories de structures ordonnées chacune éligible à sa propre dualité de type Stone. Ici le spectre d'une structure ordonnée peut être vu comme l'espace des modèles de la théorie propositionnelle correspondante, dont les propositions sont représentées par des ouverts, et les propriétés logiques se concrétisent en propriétés topologiques. Ceci a permis de formaliser l'emploi d'intuitions topologiques pour résoudre des problèmes logiques plus abstraits, par exemple des résultats de complétude ou d'interpolation. De plus, bien que cet aspect soit généralement éludé dans leur présentation habituelle, ces dualités ont également une notion de faisceau structural, et d'ailleurs, les techniques de représentation en faisceaux se sont avérées particulièrement efficaces dans ce cadre pour sonder ces structures ordonnées de nature souvent opaque ou obtenir des versions constructivistes de résultats classiques.

La construction spectrale est un procédé général, de nature catégorique, englobant ces exemples disparates. Cette construction a été théorisée de plusieurs façons très différentes, exploitant chacune des techniques propres adaptées à décrire l'un ou l'autre aspect de ce problème, mais ces techniques ayant été développées séparément sans que leurs relations n'aient jamais été abordées, on manquait en quelque sorte d'une vision globale. Cette thèse se propose donc de clarifier ces différentes techniques et de les unifier en une même construction.

Géométries

Donnons à présent un aperçu synthétique de notre sujet. Les données dont nous partirons pour construire une notion de spectre sont organisées en la notion de *géométrie*, consistant en les éléments suivants:

- Une catégorie d'objets ambiants – cette catégorie sera toujours supposée localement finiment présentable, de sorte que ses objets pourront être vus comme les modèles ensemblistes d'une théorie à limites finies. Par exemple, les objets ambiants de la géométrie de Zariski seront les anneaux commutatifs.

- Des données de factorisations : celles-ci seront matérialisées dans la catégorie des objets ambiants par un système de factorisation généré à gauche – cette dernière condition permettant de transporter ce système de factorisation dans les catégories de modèles de la théorie derrière les objets ambiants dans n'importe quel topos, ou de le présenter sous la forme plus invariante d'une classe étale. Les flèches à gauche et à droite de ce système, qui seront qualifiées respectivement d'*étales* et de *locales*, jouent des rôles complémentaires intervenant aux différents niveaux de la construction. D'un point de vue logique, les flèches étales de présentation finie encodent des opérations, dont les termes de sortie sont reflétés par les flèches locales. Par exemple, la catégorie des anneaux locaux est équipée du système de factorisation (localisations, morphismes conservatifs), les localisations créant de nouveaux éléments inversibles et les morphismes conservatifs reflétant ces derniers. D'un point de vue géométrique, les flèches étales sont duales des inclusions généralisées – parmi lesquelles, celles de présentation finie formeront une base de topologie – tandis que les flèches locales seront utilisées pour encoder des informations algébriques résiduelles.

- Une catégorie d'objets distingués, dit locaux, modèles d'une extension géométrique de la théorie des objets ambiants ; ceux-ci joueront le rôle de points dans la géométrie, ou plus exactement, des espaces focaux – c'est-à-dire présentant un plus petit point pour l'ordre de spécialisation. Un intérêt spécial sera consacré aux *formes locales* d'un objet ambiant, c'est-à-dire aux flèches étales sortant de cet objet vers un objet local : elles seront révélées comme étant duales des inclusions de composantes focales en un point. Les objets locaux et les flèches locales sont reliés par une condition dite de "glissement", et forment conjointement une sous-catégorie de la catégorie des objets ambiants avec des propriétés particulières.

La notion de géométrie axiomatise la relation qui doit exister entre ces différents ingrédients pour que cette interprétation géométrique fasse sens et pouvoir lui associer une notion de spectre. Cette condition peut se résumer d'une façon très simple. Les objets locaux, en tant que modèles d'une extension géométrique de la théorie ambiante, peuvent être caractérisés par une condition d'injectivité relativement à une famille de cônes, duaux des familles couvrantes encodant cette extension géométrique dans le site syntactique de la théorie ambiante. Il suffit alors d'imposer que ces cônes soient constitués de flèches étales de présentation finie pour s'assurer de la bonne relation entre les flèches étales, les flèches locales et les objets locaux.

Lorsque cette condition est satisfaite, on obtient la condition dite d'*admissibilité*, assertant que l'objet intermédiaire dans factorisation étale-locale d'une flèche pointant vers un objet local est lui-même local. Spatialement, ceci dit simplement que l'image d'un espace focal dans un autre espace topologique doit être une composante focale. Catégoriquement, cette condition est encapsulée par la notion de foncteur *stable*, ou de façon équivalente, par celle de foncteur *multi-adjoint à droite*. Cette dernière notion correspond à une situation d'adjonction partielle où un foncteur, bien que ne disposant pas globalement d'un adjoint à gauche, en admet un pour sa restriction à la tranche au-dessus de chaque objet. Cette condition se matérialise également par la présence sous chaque objet d'un cône d'*unités locales* assumant conjointement la propriété universelle de l'unité d'une adjonction. Dans le cas d'une géométrie, nous verrons que les formes locales sous un objet donné correspondent exactement aux unités locales exhibant l'inclusion des objets locaux et flèches locales

comme un exemple de *multi-réflexion*. Du point de vue de la théorie des modèles, cette situation peut être vue comme un défaut de construction libre. Si toute extension limite-finie de la théorie ambiante admet une construction libre permettant de choisir de façon unique sous chaque objet un modèle de cette extension – cette construction libre définissant un adjoint à gauche à l'inclusion des modèles de cette extension parmi les objets ambiants – il n'est pas possible d'en faire de même pour une extension géométrique arbitraire en ce qu'il n'existe pas de meilleurs choix d'objet libre. Cependant, en présence d'un système de factorisation satisfaisant la condition d'admissibilité relativement aux modèles de cette extension, les unités locales peuvent être vues comme un choix de meilleurs candidats formant conjointement l'objet libre. De plus, cette situation se reproduit dans la catégorie des modèles de la théorie ambiante dans n'importe quel topos, produisant une multi-réflexion des objets et flèches locales en chaque topos.

Le spectre

Le spectre permet de concrétiser l'intuition spatiale encodée dans une géométrie. A tout modèle de la théorie ambiante, tant ensembliste qu'à valeur dans un topos arbitraire, c'est-à-dire, à tout *topos modelé*, on peut associer un espace construit à partir des données qui lui sont associées dans la géométrie, son spectre.
D'un point de vue catégorique, le spectre prend la forme d'un topos équipé d'un objet local distingué classifiant les formes locales sous notre modèle de la théorie ambiante, et ce d'une façon universelle – c'est-à-dire d'une façon encodant les formes locales de toute ses images inverses possibles le long des morphismes géométriques.

Spatialement, le spectre peut être construit de deux façons, soit à partir des données de factorisations sous la forme d'un site, soit directement à partir de ses points comme un espace topologique. Dans la première méthode, on considère le site constitué de toutes les flèches étales de présentation finie sous ce modèle, équipé d'une topologie construite à partir de la topologie syntactique – plus d'éventuelles données gardant la trace de la topologie présentant le topos sous-jacent. Les formes locales sous ce modèle constituent les points de son spectre ; l'intuition que les objets locaux sont des espaces focaux se manifeste à traverse le fait que leur spectre est toujours un topos local, tandis que le spectre des flèches étales est toujours un morphisme géométrique étale, confirmant leur rôle d'inclusions généralisées.

A un niveau plus global, la construction du spectre définit un adjoint a gauche à l'inclusion de la 2-catégorie des topos localement modelés dans la catégorie ambiante des topos modelés. En un sens, ce procédé permet donc de corriger la situation de départ, où l'on se trouvait avec une multi-adjonction en chaque topos, en une véritable adjonction globale. Ceci est rendu possible en s'échappant du topos dans lequel vit un objet ambiant pour un topos fait sur mesure pour indexer ses formes locales : celui-ci est alors équipé d'un faisceau structural rassemblant ces formes locales en un unique objet vivant au-dessus du spectre et se comportant comme un objet libre universel.

Plan de la thèse

Cette thèse se décompose en deux parties plus un épilogue : la première partie traite de la notion de géométrie et décrit la façon dont elle encode un comportement spatial en des termes algébriques, et la seconde partie traite de la construction du spectre associé à une géométrie, dont il déploie le contenu d'une façon universelle.

Le premier chapitre est un préambule consacré aux techniques d'orthogonalités et de factorisations qui seront ensuite utilisées tout au long de la thèse. Après avoir rappelé les généralités de la théorie des structures d'orthogonalité et des systèmes de factorisation, nous rappellerons une forme particulière de l'*argument du petit objet* adaptée aux catégories localement finiment présentables et la notion associée de système de factorisation généré à gauche. Dans ces système la classe à gauche est générée par colimites filtrantes à partir d'une classe petite de flèches de présentation finies que nous utiliserons ensuite pour construire les sites spectraux. Nous discutons ensuite la notion de *classe étale* dans un topos et donnons une preuve détaillée d'un théorème de génération utilisé en [33].

Le chapitre 2 se consacre à la théorie des multi-adjoints à droite et les notions parentes de foncteur localement adjoint à droite et stable, dont les relations exactes sont étudiées entres autres aspects techniques. Nous rappelons ensuite l'approche des foncteurs multi-adjoints à droite par la complétion produit libre, laquelle constitue une première façon – quoique discrète et non géométrique – de corriger une multi-adjonction en une adjonction. Nous finissons ce chapitre par les aspects d'orthogonalité et de factorisation implicitement présents dans toute situation de multi-adjonction.

Le chapitre 3, le plus important de cette première partie, décrit la notion de géométrie et la façon dont la logique et la topologie y interagissent via la sémantique, entremêlant les points de vue complémentaires de la théorie des modèles et de la théorie des topos. En particulier nous décrivons le rôle des flèches étales et locales, de même que celui des objets locaux, et leur généralisation à tout topos. Nous montrons aussi que, dans une géométrie, la catégorie objets locaux et flèches locales possède un plongement multi-réflectif et accessible dans la catégorie des objets ambiants et comment ceci se généralise au-dessus de tout topos. Nous décrivons aussi les transformations de géométries et décrivons la 2-catégorie qu'elles forment.

Le chapitre 4 continue la comparaison entre les multi-adjoints à droite et les géométries. Après avoir rappelé la théorie de Diers sur les catégories multi-présentables et leurs relations avec les multi-adjonctions, nous décrivons un procédé, inverse de celui étudié au chapitre précédent, permettant de convertir un contexte de Diers à base de multi-adjoint à droite en une géométrie, et établissons l'existence d'une 2-adjonction entre les contextes de Diers et les géométries.

La seconde partie détaille les différentes façons de construire le spectre associé à une géométrie. Le chapitre 5 traite d'une façon abstraite la construction du spectre comme un objet classifiant les formes locales sous un objet donné. Deux méthodes sont comparées. Nous décrivons d'abord avec le plus grand détail la construction, suggérée par Cole, via une séquence de bilimites finies, en nous attachant à décrire tout les aspects 2-catégoriques impliqués dans cette construction. En particulier nous y distinguons deux étapes, l'une traitant seulement des données de factorisation, la seconde intégrant la spécification des objets locaux. Dans un second temps nous traitons la méthode de Dubuc, partiellement similaire, mais prenant en fait assez tôt un cheminement différent en construisant le spectre comme le topos étale relativement à une certaine classe étale dans le classifiant des flèches envoyées par un objet donné vers les objets locaux. Dans chaque cas, nous procédons dans la (2-catégorie opposée de la) tranche oplax au-dessus du topos classifiant la théorie ambiante et celle au-dessus du topos classifiant les objets locaux, nous appuyant sur la représentation des modèles via des morphismes géométriques.

Le chapitre 6 décrit les bicatégories des topos modelés et localement modelés ; nous consacrons en particulier un certain effort à décrire explicitement les bilimites et bicolimites de topos modelés, et établissons que les topos localement modelés héritent de ces bilimites : cependant ce dernier résultat repose sur une stratégie abstraite à base de considérations pseudomonadiques, objet de la seconde partie de ce chapitre.

Le chapitre 7, de loin le plus long et le plus dense de cette thèse, décrit en détail les propriétés géométriques du spectre et la construction de son site spectral. Bien que la définition originale de ce site provienne de [19], la plupart des méthodes de cette section sont de notre cru et reposent sur une analyse en profondeur du spectre. Nous donnons d'abord la description détaillée du spectre d'un modèle ensembliste et quelques résultats généraux tels que la localité du spectre des objets locaux, le caractère étale et net du spectre des flèches étales de présentation finies, et celui totalement connecté des flèches locales. Nous discutons la forme simplifiée que revêt l'adjonction spectrale restreinte aux modèles ensemblistes, que nous établissons par des considérations sur les propriétés universelles du faisceau structural et d'une certaine *flèche étale générique* classifiant les données d'admissibilités de façon universelle à image inverse près. La construction du site spectral pour les topos modelés arbitraires est plus compliquée et doit prendre en compte la structure topologique du topos sous-jacent. Afin de comprendre exactement cette relation, nous rappelons et améliorons la notion de [86] de site fibré et de topos fibré, introduisant une variante permettant de prendre en compte une topologie sur la catégorie de base, et construisant un certain site dont le topos de faisceaux coïncide avec la catégorie des sections continues d'une fibration associée. Nous appliquons ce résultat au cas des topos modelés, en montrant que leur spectre est un exemple de topos de sections continues associé à une fibration spectrale. De façon générale nous accorderons un certain intérêt aux aspects fibrationnels en jeu. La fonctorialité de la construction du site spec-

tral sera aussi abordée, et bien que cet aspect ait été jusque-là négligé par nos sources, il s'avère hautement non trivial à établir. Finalement nous décrivons le site spectral des topos localement modelés, que nous montrons être aussi local sur leur topos de base ; nous utilisons ce résultat pour produire une preuve alternative de l'adjonction spectrale. Nous discutons également la façon dont cette construction circule le long des transformations de géométries.

Le chapitre 8 est consacré à la construction à la Diers du spectre par une méthode beaucoup plus concrète, à partir d'un multi-adjoint à droite. Après avoir rappelé la construction originale de Diers pour un objet ambiant ensembliste, nous proposons une généralisation de l'adjonction spectrale au niveau des espaces modelés. Nous finissons ce chapitre par une tentative d'axiomatisation de la notion de dualité spectrale, que nous prouvons être suffisante pour reconstruire un multi-adjoint à droite par un procédé inverse à la construction du spectre.

Nous finissons cette partie par une abondante liste d'exemples. Si ceux issus de la théorie des anneaux sont généralement bien connus, nous nous attachons aussi à donner la version spectrale, moins standard, des dualités de Stone, et introduisons aussi deux exemples "quasi-spectraux" pour les frames et les topoi.

Finalement, un épilogue introduit quelques éléments pour une version 2-catégorique des notions de géométrie et de spectre, motivée par une approche topologique des dualités syntaxe-sémantique telles que Gabriel-Ulmer et leur correspondance avec les dualités à la Stone pour les fragments de la logique propositionnelle leur correspondant. En particulier, un examen de la localisation de Grothendieck-Verdier d'un topos en un modèle permet de déduire l'existence d'un système de factorisation (focalisations, terminalement connectés) sur la 2-catégorie des petites catégories lex, dont nous discutons le rôle sémantique. Ce système de factorisation devrait pouvoir être exploité ultérieurement pour produire des exemples de 2-géométries correspondant aux différentes doctrines du premier ordre, objet d'un travail en cours de catégorification des résultats de la présente thèse.

Introduction

This thesis is aimed at providing a unified and synthetic account of the theory of spectra, interweaving its categorical, logical and topological aspects.

Various instances of spectral dualities are known across mathematics, from the classical construction of Zariski spectrum of a commutative ring which is at the very basis of algebraic geometry, to the diverse Stone-like spectra for the different kind of propositional algebras. In each case, their purpose is, in one direction, to deploy geometric information hidden in algebraic structures, and in the converse direction, to provide "axiomatizations" of spaces of certain kind by constructing simpler, algebraic objects determining their topological structure.

This process involves a choice of "local data" on the algebraic side, that will behave in a point-like manner on the geometric side: those local data are defines in particular a class of objects with a simpler algebraic structure, where makes certain properties become easier to test. Then one can reduce difficult algebraic problems about complex objects to more manageable questions about the local data attached to them; on the geometric side this will amount to proving some dual spatial properties by testing them locally, at the level of points.

There are mainly two distinct domains where spectrum as studied here has proven useful:

- the spectral construction underlying algebraic geometry, the *Zariksi spectrum* of commutative rings, was a major contribution for both ring theory and geometry itself in numerous ways; in one sense it formalized the manner algebraic varieties were parametrized by polynomial equations, and enabled algebraic methods to investigate the geometric structure of various classes of spaces; on the other hand, it allowed to convert problems of ring theory whose algebraic formulation was difficult into easier geometric questions. It exploited the fact that *local rings* had a simpler structure than arbitrary commutative rings. This construction also equipped spectrum of rings with their *structure sheaf* or also sheaves of modules and exploited geometric techniques to prove algebraic properties of the categories of modules for a given ring. Finally, it was also used to prove choice-free versions of classical results about rings.

- Stone-like dualities, although often presented from the point of view of concrete dualities, also are spectral in nature, in an even more evident way because of the deep connection between topology and lattice theory. But their major contributions were found on the side of proof theory and model theory, since the different fragments of propositional logic correspond to categories of ordered structures eligible to one or another Stone-like duality. Think for instance of the celebrated duality between boolean algebras (which are propositional theories in classical logic) and boolean spaces, Esakia duality for Heyting algebras (propositional theories of intuitionistic logic), Stone duality for distributive lattices (propositional theories of coherent logic), or also examples of dualities for residuated lattices which are used to study of substructural logics. Here the spectrum of an ordered structure is to be seen as spaces of models of the corresponding theory, propositions are turned into compact open sets, and logical properties into topological ones. This allowed to use topological intuition to solve abstract problems, for instance prove fragments of logics to enjoy interpolation properties, or completeness results. Moreover, though it was not always part of their traditional formulation, those dualities also have sheaf theoretic content, and sheaf representations techniques applied in the context of those spectral dualities have proven to be powerful tools to probe the intricate nature of ordered structures or reprove constructive versions of classical results.

The spectral construction provides a unified, systematic approach to these different methods. Several parallel attempts have been done, whose relations were however unclear for they are very

divergent in their strategy. The purpose of this thesis is both to clarify and develop each of those techniques, as well as unify them into a single method entangling all the different aspects they previously focused on separately.

Our contribution

Before giving an overlook of the content of the construction itself, we should directly list the main contributions of this thesis. The reader should be aware that the principal results (the different versions of the spectral adjunctions) were already stated in our different sources. This redundancy is somewhat characteristic of this topic: in fact, these different sources themselves *re-proved* each time the same core result with different methods, and for some of them ignoring that their results already existed - because the most ancient sources remained very elusive or were lost for decades. While each of those "rediscoveries" of the spectral construction had its own interest, some remained very elliptical, dispensing themselves with proving rigorously important results, while certain aspects of the construction were hitherto never investigated.

We hence chose to devote this thesis to understanding the exact relations between those previous methods, clarifying what they had let ambiguous, giving rigorous proofs to results that were lacking one, interpreting what was really going on and generalizing their results. In fact, this project involved proving numerous intermediate results forming altogether an in-depth investigation of the theory. Sometimes we also reproduced or reformulated already existing but hardly available or elusive proofs, for the sake of completeness. Our hope is that this work may be used by people knowing part of the spectral construction for them to complete their understanding of it or to check facts they may need, as well as an introduction to the spectral construction where newcomers will find, among the different formulations examined here, the one which fits their purposes at best.

The core results this thesis revolves around are the different versions of the spectral adjunction, corresponding each to a different chapter in Part II:

- Cole adjunction of theorem 5.2.1.7, which builds on the method suggested by [17], together with an intermediate version theorem 5.1.2.8 for factorization geometries. Here we give very careful and detailed proofs, which were only sketched in [17], and describe the universal property of the *generic etale map* under an object at proposition 5.1.2.2. We also describe the functoriality of the process and the treatment of transformations of geometries.

- Dubuc adjunction at theorem 5.3.3.5; while we add nothing new to this version which is clearly treated in [33], we give an overview of the strategy in order to connect it with the previous version and interpret it; in particular, this examination reveals the surprising double nature of the spectrum as both an inverter and a coinverter at 5.3.4.7.

- The adjunction from site theoretic approach, theorem 7.5.2.1 together with its set-valued restriction theorem 7.1.6.2; while such a construction already exists in [19], [67] and [4], we follow a totally different proof relying on an in-depth analysis of the geometric structure of the spectrum in section 7.5.1, and a careful examination of the universal properties of the structure sheaf and the generic etale map and local form throughout section 7.1.5.

- Diers adjunction theorem 8.1.5.5 is already treated with enough details in [31]; however, being only available in French and without any diagram, we see as important to give this beautiful proof to make it more easily accessible to everyone; moreover we then provide a generalization of it at theorem 8.2.4.3, which is brand new and requires numerous intermediate results. This generalization is the point-set version of the previous topos-theoretic ones, and we connect it with a well-known characterization of right multi-adjoints through free product completion at proposition 8.2.4.5.

Those different versions of the spectral adjunctions involve different sets of data (geometries, admissibility, Dubuc contexts, Diers contexts...). Part I of this work spends time on the relations between those different axiomatizations. The relations between *geometries* - which are more akin to [19] and [67] methods - and *admissibility structures* of [17] and [48] - are very natural as they are in some sense two steps of a same method; this is carefully examined throughout section 3.3. On the other hand, the relation between those and Diers axiomatization through right multi-adjoints are more subtle and will be one of the main topics of Part I, with key results at theorem 3.3.1.12, theorem 3.3.3.6, corollary 4.2.2.5, culminating in a "Coste-Diers 2-adjunction" at theorem 4.2.5.1.

This is also an occasion to prove accessibility and closure properties of categories of local objects relative to connected limits at lemma 3.3.1.10, proposition 3.3.1.13, proposition 3.3.3.5 and proposition 3.3.3.7. These relations justify to spend some time on Diers theory of *multipresentable* categories at the beginning of Chapter 4; we take here the occasion to provide a few new results as multi-version of the characterization of accessible right adjoints at theorem 4.1.2.1, and slightly generalized versions of results of [25] at proposition 4.1.2.4 with new proof.

Though this was somewhat folklore, we prove once for all the equivalence between the notions of stable functor and local right adjoint at ; we also prove a hitherto unnoticed Beck-Chevalley property of slice-wise adjunctions of local right adjoints at theorem 2.1.1.5. We also give analogs of the limit and colimit closures properties of multireflective categories at theorem 2.1.3.15.

Although it is mostly a technical part aimed at recalling preliminary factorization techniques, there are quite a few novelties in Chapter I; in fact, most of the small object argument was done in a clear manner in [4], but we think important to carefully characterize the etale generator throughout section 1.1.2; this closely related to the concrete description of the generator of finitely presented objects of coslices of locally finitely presentable categories we provided at [83]. We also prove some new colimit decomposition for those etale generators at theorem 1.1.4.3. We also give the very long and technical proof of proposition 1.2.2.2, which was left as an exercice in [33].

To complete the list of our contributions, let us get back to Part II. Chapter 6 provides new fibrational approaches to the bicategories of (locally) modelled topoi; we describe bilimits and bicolimits of modelled topoi, and prove that locally modelled topoi inherit bilimits at theorem 6.3.2.3. This generalizes a known, yet poorly explained fact, that limits of locally ringed spaces do exist (in the geometric convention), while local rings only have connected ones. Our strategy relies on a result on closure properties of categories of pseudo-algebras for a pseudomonad under bilimits, which we would have expected to be already present in the litterature; for we failed to find such an exact instance of this statement, we proved it at theorem 6.3.1.6.

We also provided a lot of geometric characterizations of the spectrum of the different ingredients of a geometry, as tidyness for finitely presented etale maps (proposition 7.1.2.5), terminal connectedness for local maps (proposition 7.1.3.10), localness for local objects (proposition 7.1.3.6 and its generalization corollary 7.5.1.4)... We also describe the spectrum of an arbitrary modelled topos from a fibrational point of view, proving it to be the topos of *continuous sections* of the *direct fibration* of a *fibered topos*. This exploits a generalization given at theorem 7.2.3.7 of a result of [86] about topos of sections of a fibered topos, this time allowing to consider a topology on the base category. More generally, we believe that Chapter 7 is the most important of this thesis, and is aimed at giving an in-depth description of the spatial properties of the spectrum.

We also provide a very careful examination of the functoriality of the spectrum, which was neglected until now. While this does not rise major complications in the abstract version (see 5.2.1.5), the site-theoretic aspects of this are considerably more complicated, yet meaningfully related to the admissible factorization, and are treated in section 7.4.

We also examine each time the *2-functoriality* of the spectral construction relative to transformations of geometries.

At the end of Chapter 8, we define a 2-category of *spectral dualities* axiomatizing the point-set version of the spectral construction, and prove at theorem 8.3.4.1 that such data are sufficient to reconstruct right multi-adjoints. We hope to complete this in a topos-theoretic version in a future work.

All the examples we consider are known in a way or another from litterature, although some of them are not usually thought of as spectral: for instance, Stone-like situations are seldom referred as such - except [10]. The presentation of Jipsen-Moshier as a geometry - and its link with Gabriel-Ulmer - are a novelty; also the "geometry-like" situation for frames and the mysterious "terminal geometry" for Grothendieck topoi first introduced at remark 7.1.3.11.

Finally our epilogue, while its overall philosophy is somewhat "in the air", proves a new factorization result at theorem 10.2.2.12 in the 2-category of lex categories, identifying two new classes of lex functors, *focalizations* and *terminally connected* ones, which we prove to be somewhat in-

volved in Gabriel-Ulmer duality. This is the first step of a future categorification of the spectral construction targeting first order syntax-semantics dualities.

Geometries

Let us synthesize the presentation we chose of the topic. The context from which we start are the following data:

- A category of ambient objects - we will always suppose this category to be locally finitely presentable, hence those objects to be models of a finite limit theory. For instance those are commutative rings in Zariski geometry.

- Factorization data: those will be generally materialized by a left generated factorization system on the category of models of this finite limit theory - though the hypothesis of left generation makes it in fact more universal and allows to interpret it in the categories of models in arbitrary topoi, or to present it in the more invariant form of an etale class. The left and right class of this factorization system, which will be called respectively etale and local maps, play complementary roles which can be seen at different levels. At the logical level, one could say that (basic) left maps encode operations whose outputs are reflected by local maps. For instance, the category of rings is equipped with the factorization system (localizations, conservative morphisms) where localizations create new invertible elements while conservative morphisms reflect them. At a geometric level, one can see etale maps as generalized inclusions, amongst which basic ones will be distinguished, from which one can generate a topology, while local maps will be used to encode residual algebraic information.

- A category of distinguished local objects, models of a geometric extension of the theory of ambient objects, will play the role of points of a geometry, or more exactly, of focal spaces - that are spaces with a minimal point. A special interest will be given to local forms of an ambient object, that are those etale maps toward a local object, which behave as inclusion of focal component at a point. Local objects and local maps will enjoy special relations, as a so called gliding property, and will form altogether a non full subcategory of the category of ambient objects with interesting properties.

The notion of geometry axiomatizes the relation between those ingredients in a way ensuring that their behavior really encodes geometrical information and is eligible to a spectral construction. It happens that this condition is actually quite simple. From topos theoretic consideration, we know that local objects, as models of a geometric extension of the theory of ambient objects, are characterized by a cone-injectivity condition relative to the cones dual to the covering families encoding the geometric extension into the syntactic site. Then one just needs to require those cones to be made of finitely presented etale maps to ensure the correct relation between local objects, local maps and etale maps.

Once this latter condition is met, one achieves the condition of *admissibility*, ensuring that the middle object in the etale-local factorization of any arrow toward a local object is itself a local object. Spatially, this condition says that the image of a focal space in a space should be a focal component. Categorically, this condition is encapsulated in the notion of stable functor, or also local right adjoint. The latter are situations of partial adjunction where a functor, while lacking a global left adjoint, has slice-wise local adjoints; this condition can also be visualized by the existence under each object of a cone of local units jointly assuming the role of the unit in a usual situation of adjunction. In the case of a geometry, we will see that local forms under a given object exactly are the local units exhibiting the inclusion of local objects and local maps as an instance of right multi-adjoint. From a logical point of view, this situation can also be seen as failure of free construction: in the case of a finite limit extension of the ambient theory, one could have chosen in a universal way a free local object under a fixed ambient object - this is materialized in particular by the existence of left adjoint to morphisms of locally finitely presentable categories. In the case of an arbitrary extension, this is not anymore the case and there is no way to choose a unique free object. However, in the presence of a factorization system satisfying admissibility relative to those local objects, the universal property of the free object is in some sense jointly assumed by the cone of local forms. Moreover this situation of multi-adjunction generalizes in arbitrary topos, providing topos-wise multireflections of local objects and local maps between into models of the ambient theory.

Spectrum

The purpose of the spectrum is then to deploy the spatial intuition encoded in a geometry. For each model of the theory of ambient objects, either set valued, or even more generally for each arbitrary modelled topos - that is, a topos equipped with a distinguished model - we can construct a space encoding its behavior from the point of view of the underlying geometry.

Categorically, we can see this spectrum as a topos together with a distinguished local object classifying local forms under our model, and moreover, in an universal way - that is in a way that takes into account all the possible existing local forms up to inverse images along geometric morphisms.

Spatially, the spectrum can be constructed in two ways, either as a site, either as a topological space. In the first way, one construct a site made of all the basic etale maps under a model, equipped with a topology induced from the syntactic topology encoding the theory of local objects - plus eventual data remembering the topology presenting the topos it lived in. From this point of view, local forms under an objects are to be seen as point of the spectrum of this object, and the idea that local object are like focal spaces realizes into their spectrum being a local topos, while the role of etale maps as generalized inclusions appears through the etale geometric morphisms they induce between spectra. In the second, more concrete way, one directly takes the local unit and equip them with a topology of etale maps to form a point-set spectrum.

At a more global level, the construction of the spectrum defines a left adjoint to the inclusion of a 2-category of locally modelled topoi into an ambient category of modelled topoi. In some sense, this process is a way to turn the situation of topos-wise multi-adjunctions into a global adjunction, which was made possible by escaping from the base topos in which the ambient object lived for a topos able to index its cone of local forms as a single object: this is the purpose of the structural sheaf of the spectrum of a model, gathering its local forms into a local object living in the spectrum and playing the role of a universal free object.

Examples

We will describe various examples in Chapter 9; to convince the reader of the diversity of spectral constructions, we found relevant to include here an overview of those examples:

- *Zariksi geometry* for commutative rings, where local data are local rings together with conservative maps and etale maps are ring localizations;
- *Pierce geometry* for commutative rings, where local data are connected rings with connected maps and etale maps are localizations at idempotents;
- *Integral domain geometry* for commutative rings, where local data are integral domains with monomorphisms and etale maps are regular epimorphisms;
- *Etale geometry* for commutative rings where local data are strictly henselian local rings with henselian maps and etale maps are etale morphisms of rings;
- *Castiglioni-Menni-Zuluaga-Botero geometry* for commutative rigs and its restriction for integral commutative rigs, where local data are (integral) really local rigs with conservative morphisms and etale maps are localizations;
- *Zariski geometry* for distributive lattices, where local data are 1-local distributive lattices with 1-conservative morphisms and etale maps are 1-minimal quotients;
- *co-Zariski geometry* for distributive lattices, where local data are 0-local distributive lattices with 0-conservative morphisms and etale maps are 0-minimal quotients;
- *Esakia geometry* for Heyting algebras, where local data are 0-local Heyting algebras with monomorphisms and etale maps quotients;
- *Boolean geometry* for Boolean algebras, where local data consist of the sole two-elements lattice and etale maps are epimorphisms;
- *Jipsen-Moshier geometry* for $\wedge$-semilattices, where local maps are 1-conservative morphisms (without specification of local objects) and etale maps are 1-minimal quotients;

– *co-Jipsen-Moshier geometry* for $\vee$-semilattices, where local maps are 0-conservative morphisms (without specification of local objects) and etale maps are 0-minimal quotients;

– *Dubuc-Poveda geometry* for MV-algebras, where local data are MV-chains with monomorphisms and etale maps are quotients.

We should also make mention of the following two "geometry-like" situations, which actually do not admit a presentation as a geometry but as "Diers-like" contexts for non-accessible (2-)categories:

– *Isbell geometry* for frames, where local data are local frames with 1-conservative frames homomorphisms and etale maps are 1-minimal quotients;

– and finally the *terminal geometry* for the opposite bicategory of Grothendieck topoi, where local data are (up to formal duality) local topoi together with terminally connected geometric morphisms and etale maps are pro-etale geometric morphisms.

Those examples of geometries are related by morphisms of geometries, forming the forllowing figure where we informally gathered them:

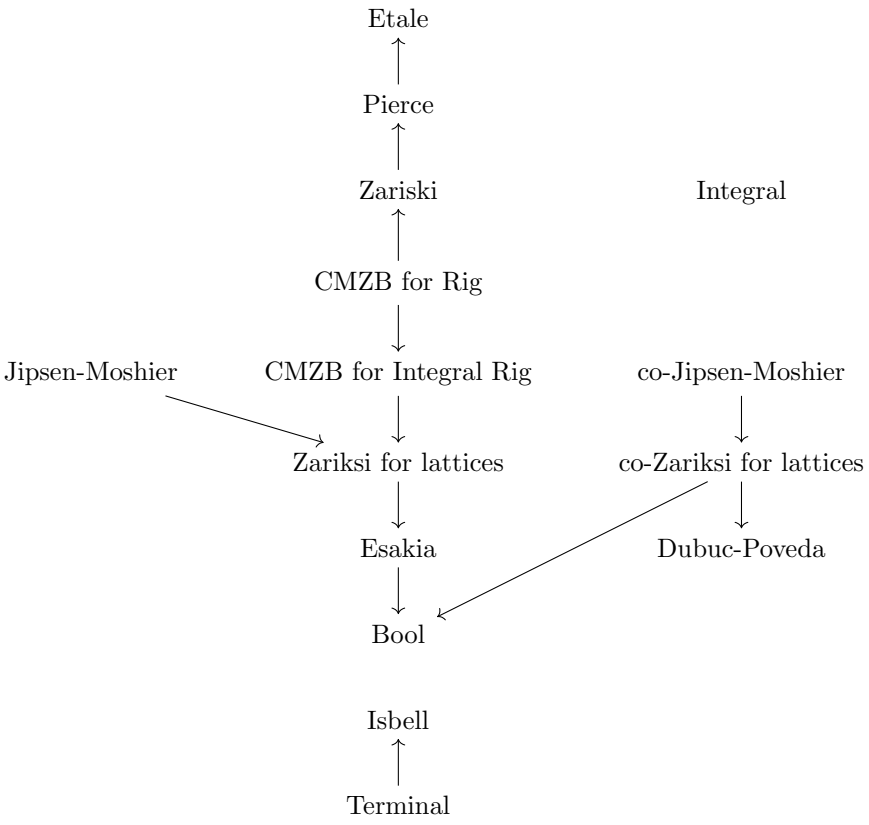


Figure 1: An atlas of geometries

A more formal diagram for the corresponding Diers contexts and the relations between them will be given at fig. 9.1. If here the "terminal geometry" appears to be initial, the motivation for the name "terminal" is its role as a terminal Diers context - beside the role of terminally connected morphisms.

Sources

The main sources for this work are the following, each proposing an independent approach whose relation with the other ones will be carefully examined:

– Cole's paper [17] is the first attempt to a general construction. This paper, which remained elusive and mostly known by reputation for decades, contained several seminal ideas, in particular pointing out the importance of the factorization data as well as the notion of

admissibility and the possibility to process by a sequence of finite bilimit in the bicategory of Grothendieck topoi to construct the spectrum as a classifier. However the construction itself was only sketched, while its technical aspects deserve further precision. Beside the construction of the spectrum, this paper contained beautiful remarks about topos theoretic approach to factorization systems, the behavior of the (pre)sheaf 2-functor on the 2-category of sites, and the ambivalence of some 2-limits in the 2-category of small lex categories.

– Coste's paper [19] emphasized the logical aspects of the spectral construction, interpreting from the point of view of model theory the admissibility situation isolated by Cole and the syntactic content of the different elements of admissibility; it also provided a first explicit construction of the spectrum as a site and explained the relation between subcanonicity and sheaf representation.

– Diers' paper [31] is the most divergent, and seemingly lesser known, yet provides a very elegant approach to admissibility through the notion of right multi-adjoint. However as we shall see the exact relations between Diers context and Cole admissibility structures are subtle, as in fact Diers' work extends out of the world of sketchable categories, where topos theoretic methods and categorical model theory cease to apply. This paper also provided a practical, point-set construction of the spectrum, as well as numerous surprising examples.

– Dubuc's paper [33] revisited the theory of spectrum from a new point of view, substituting the factorization data (which are dependent on the semantics) by etale classes, which are more universal and allow alternative topos theoretic treatment; though it shares with Cole method the idea of constructing the spectrum as a classifying object through universal constructions in the bicategory of Grothendieck topoi, the sequence diverges in order to make use at some crucial step of the construction of the etale topos associated to an etale class; this is related to the paradigm of "gros topos" versus "petit topos".

– Anel's paper [4] emphasizes the topological behavior hidden in factorization data, to this end it developed a particularly suited approach to the small object argument in locally presentable categories; as well as the site description of the spectrum from a site of etale maps under an object, also in the spirit of the gros and petit topos philosophy. It also provides a very explicate account of all the different possible geometries for rings and their relations.

– Lurie's paper [67] proposes a theory of spectra for $(\infty, 1)$-categories through $(\infty, 1)$-topos theoretic methods. In fact most of his method innocently restricts to ordinary 1-categories, where some results even simplifies. It seems that most of the result of this paper were rediscovered by Lurie him-self without help of [30], [17] or [19], for which it contains also fairly alternative proof of most results. While this paper was also important in our understanding of the theory, we follow mostly different strategy, though coinciding sometime with its. We also chose to follow some terminology from it, as the term *geometry* we particularly affectionate for its eloquence.

Besides those main sources, we should also acknowledge a few important additional works in the development of the theory of spectrum:

– The origin of a systematic construction of the spectrum lies in Hakim's thesis [42]; though it restricted its scope to notion of spectra for rings, several key ideas were introduced, as the notion of (locally) ringed topos, the construction of the classifier of local rings through the Zariski site, and the role of the spectrum as a left adjoint to the inclusion of locally modelled topoi into modelled topoi.

– In Johnstone's first book [48][Chapter 6] is found a first synthetic axiomatization of admissibility structure inspired by Cole; in particular the importance of the stability of local maps is pointed out; some steps of construction of the spectrum are also sketched.

– Taylor's paper [93] is related to Diers' approach, and though it drops the spectrum in favor of an alternative construction called the trace, it discusses an interesting formalism in term of stable functors.

– In SGA IV, an exercice nicknamed "médaille en chocolat" [86][Exercise 4.10.6] concerned the construction of a site from maps of a class with suitable properties over a given object of a site, which is related to the gros-petit topos idea and is very close to some of the approach above - though it predates all the works here above mentioned.

Plan of the thesis

This thesis is divided into two parts plus an epilogue: a first part on the notion of *geometry*, that is, the categorical situation encoding spatialness through algebraic terms, and a second part on the construction of the *spectrum* as an object deploying this spatial content in an universal way.

The first chapter of this thesis concerns factorization data. After recalling elements of the general theory of orthogonality structures and factorization systems, it recalls a specific form of the small object argument suited for finitely presentable categories and the notion of left generated factorization system, which will be the one we need for our purpose: those whose left class is generated under filtered colimits from a small class of finitely presented left maps, which will be used to construct the spectral sites under a fixed object. We also discuss at the end of this chapter the theory of etale class and give in particular the proof of a generation theorem from [33].

Chapter 2 is devoted to the theory of right multi-adjoint and the close notions of local right adjoint and stable functor, whose exact relations are investigated as well as several technical aspects. We also recall the approach of right multi-adjoints through free product completion, as it constitutes a first, discrete way to correct a multi-adjunction into a global adjunction. We end with the orthogonality and factorization aspects involved in situations of right multi-adjunctions.

Chapter 3, the most important of the first part, describes the notion of geometry from model theoretic and topos-theoretic approaches. In particular we discuss how its logical and topological aspects interact through semantics, detailing the role of etale and local maps as well as local objects and their generalization in arbitrary topoi. We also explain how in a geometry the category of local objects and local maps has an accessible, multireflective embedding into the category of ambient objects and how this generalizes topos-wisely. We also introduce transformations of geometries and describe the 2-category of geometries.

Chapter 4 pursues the comparison between multi-right adjoints and geometries. After recalling Diers theory of multipresentable categories and its relation with multi-adjunction, it describes a process through which a Diers context can be turned into a geometry, and introduces a 2-adjunction between the 2-category of Diers contexts and geometries.

In the second part we detail the different ways to construct the spectrum associated to a geometry.

Chapter 5 is about the abstract way to construct the spectrum as a classifier of local forms under an object. Two methods are provided. We first give a very explicate and careful account of Cole method through a sequence of finite bilimits, detailing the 2-categorical aspects involved in this construction. In particular we choose to split this process in two phases, one relative to factorization data only, and a second phase including the specifications of local objects. Then we turn to Dubuc's method which bears apparent similarities, yet follows a different strategy as it construct the spectrum as the etale topos relative to a certain etale class in a classifier of morphisms toward local objects under a fixed modelled topos. In each case, we proceed in the (opposite 2-category of the) oplax slices over the classifying topos of the theory of ambient objects and the theory of local objects, emphasizing the representation of models as geometric morphisms.

Chapter 6 describes the bicategories of modelled and locally modelled topoi; in particular it gives a special interest in the computation of limits and colimits of modelled topoi, as well as the inheritance of limits by locally modelled topoi: this latter result requires some pseudomonadic discussions which are the topic of the second section of this chapter.

Chapter 7 is by far the longest of this thesis and describes in detail the geometric property of the spectrum and in particular the underlying spectral site. Though the very definition of the spectral site is from [19], most of the methods of this section are ours and involve in-depth description of the spectrum. We first give the construction of the spectrum of set-valued models and give several results, such as localness of the spectrum of local objects, tidyness and etaleness of the spectrum of basic etale maps. A first version of the spectral adjunction for set-valued model is obtained from the universal properties of a certain generic local map gathering universally all admissibility data under a given object *up to inverse image*. The construction of the spectral site for an arbitrary modelled topos is more involved as it requires to remember also the information of

the underlying topos. To properly understand this relation, we recall and develop [86][VI] notion of fibered site and fibered topos, introducing a notion of fibered site over a category itself endowed with a Grothendieck topology, and constructing a certain site whose topos of sheaves is the topos of continuous sections of the associated fibration. We apply this result to the spectral site of a modelled topos, showing the spectrum to be the topos of continuous section of a spectral fibration. More generally, the fibrational aspects involved are discussed in detail. Functoriality of the construction of the spectrum is also proved - and though it was left implicit in all our sources, this latter aspect is particularly non trivial from the site theoretic approach. Finally we discuss the spectral site of locally modelled topos, proving the spectrum of local objects to be local over its base; we use this result to provide an alternative proof of the spectral adjunction. We also discuss this construction circulates across transformations of geometries.

Chapter 8 is devoted to Diers point-set construction of the spectrum from a right multi-adjoint. After recalling the original Diers construction for set-based ambient objects, we propose a generalized adjunction for "modelled spaces". We end this chapter by an axiomatization of spectral dualities, and prove them to define by a process inverse to the spectral construction a situation of right multi-adjunction.

We end this part with a list of various examples. Examples from ring theory are well known, but Stone like dualities can also be presented from a spectral approach.

Finally, an epilogue is devoted to introducing the 2-categorical analog of geometries and spectra, which is motivated by a topological view of the syntax-semantics adjunction, as Gabriel-Ulmer duality for instance, and their comparison with the corresponding propositional Stone-like dualities which are spectral. In particular, from the Grothendieck-Verdier localization of a topos at a model, we deduce a 2-factorization system (focalization, terminally connected) on the 2-category of small lex categories, and discuss its relation with semantics. This factorization system will provide the factorization data for 2-geometries corresponding to different first-order doctrines, which is the object of ongoing work categorifying the theory described in this thesis.

Prerequisite and conventions

We suppose the reader to be familiar with category theory, topos theory and categorical model theory, though we will recall some notions when a specific approach is emphasized. Throughout this thesis, some recurrent notation or convention will be used.

We will denote as $\mathcal{S}$ the category of sets, **Cat** the 2-category of small categories, **Lex** the 2-category of categories with finite limits and left exact functors, **GTop** the category of Grothendieck topoi and geometric morphisms.

We will also use the following code of letters to systematically refer to the components of a geometry:

- $\mathcal{B}$ will generally denote a locally finitely presentable category, whose generic objects will be denoted in general as variants of B and arbitrary morphisms as f, while its finitely presented objects will generally be denoted as K and morphisms between them as k.

- $\mathbb{T}$ will generally refer to a finite-limit theory; $\mathbb{T}$ and $\mathcal{B}$ will most generally correspond to each other through Gabriel-Ulmer duality and be used alternatively depending on whether we must access the syntactic or the semantic level; their relation and the use of the associated notations $\mathcal{C}_{\mathbb{T}}$, $\mathcal{S}[\mathbb{T}]$, $\mathbb{T}[\mathcal{S}]$ is detailed in the first section of chapter 3 as well as the role of the Grothendieck pretopology J and the associated $\mathbb{T}_J$.

- $\mathcal{V}$ will generally denote a saturated class (see definition 1.1.1.7) in a locally finitely presentable category $\mathcal{B}$ and $\mathcal{V}_B$ the etale generator (see definition 1.1.2.1) under a given object B; in general objects of the etale generator $\mathcal{V}_B$ will be denoted as n;

- $\mathcal{V}$ being generally fixed, the associated factorization system will usually be denoted $(\mathbf{Et}, \mathbf{Loc})$ for etale-local factorization; the associated etale class (see definition 1.2.1.2) will be denoted $\mathscr{H}$;

– arbitrary etale arrows will generally be denoted with l, while local arrows will be denoted with u and subsequent letters; the latter will also denote morphisms in the domain of a stable functor;

– set-valued local objects and objects in the domain of a stable functor will be denoted A

– local unit of an arrow f of a local right adjoint over an object A will usually be denoted as η_f^A, but when seen as candidate, they will be notated as n or n_x or even x when seen as local forms to emphasize their point-like nature;

– Grothendieck topoi will be denoted with calligraphic letters $\mathcal{E}$, $\mathcal{F}$... $\mathbb{T}$-models in Grothendieck topoi other than $\mathcal{S}$ will be denoted as F and local ones as E.

Let also give a word on our notations for the two possible Yoneda embeddings for a small category $\mathcal{C}$. Following a tradition which seems to have originated in [66], we will denote as よ : $\mathcal{C} \hookrightarrow [\mathcal{C}^{\mathrm{op}}, \mathcal{S}]$ the covariant Yoneda embedding into the category of contravariant functors into $\mathcal{S}$, where よ is the Japanese Hiragana for "yo". Dually, we chose to denote as ヨ : $\mathcal{C}^{\mathrm{op}} \hookrightarrow [\mathcal{C}, \mathcal{S}]$ the contravariant Yoneda embedding into the category of covariant functors into $\mathcal{S}$, where ヨ is the Japanese Katakana for "yo"; though we feared at first the use of this latter symbol could be misleading for its fortuitous proximity with the existential symbol $\exists$, we believe this notation to be based for two reasons. First, the existence in the Japanese writing system of two distinct and somewhat "dual" syllabaries has something providential for our need of two dual symbols with the same name. Moreover, the very proximity with the existential happens actually to be meaningful if we recall the way the contravariant Yoneda embedding is used in categorical model theory: indeed, recall that if $\mathbb{T}$ is a finite-limit theory, then for any formula in context $\{\overline{x}, \phi\}$ in the syntactic category $\mathcal{C}_{\mathbb{T}}$, the corepresentable $\exists_{\{\overline{x},\phi\}}$ represent witnesses of the formula $\phi(x)$ in $\mathbb{T}$-models in the sense that an arrow $a : \exists_{\{\overline{x},\phi\}} \to B$ into a set-valued $\mathbb{T}$-model B, especially in display style

$$\exists_{\{\overline{x},\phi\}} \xrightarrow{a} B$$

could hence be read as "there exists a witness a of the formula ϕ in B", for it encodes the name of such a witness.

Part I

Geometry

Chapter 1

Factorization data

This chapter is devoted to the *factorization* data involved in the construction of geometries. This should be seen as a technical preamble gathering all the orthogonality and factorization aspects that will be ubiquitous in the remainder of the thesis.

As we shall see, those aspects do not exclusively manifest as a factorization system: depending on the level at which we approach a geometry, those data manifest themselves as:

- a *saturated class* in a generator of finitely presented objects: this is a class of finitely presented maps encapsulating closure properties needed to generate a left class in a factorization system;
- in the syntactic site associated to the underlying theory, the dual class of maps codes for a class of provably functional formulas, we can see as definable function symbols;
- the saturated class generates an *etale class* in the classifying topos of the theory of ambient objects, which also restricts to an etale class in the classifier of the theory of local objects;
- in the locally finitely presented category of ambient objects, this generates a left class of an orthogonality structure, which happens to be a factorization system thanks to a specific kind of small object argument;
- the right class of this orthogonality structure provides the morphisms we need between local objects of the geometry to achieve admissibility.

In the first section, we recall what we need about orthogonality and factorization. In particular we give a presentation of Anel small object argument at proposition 1.1.3.2. This involves a theory of *saturated classes* which we detail here. We also give a pseudocolimit lemma decomposing the *etale generator* at theorem 1.1.4.3.

The second section, motivated by the construction of the spectrum of [33] as described in chapter 5, focuses on etale classes as a substitute to factorization system, living inside the classifying topos of a theory rather than in its category of models. We will also devote efforts to prove in particular an important generation theorem proposition 1.2.2.2 whose statement was present in [33], yet left unproved - as its proof reveals indeed to be very long because of the numerous conditions involved in the definition of an etale class.

1.1 Left-generated factorization systems

1.1.1 Factorization systems and saturated classes

In this part we recall basic facts about factorization systems. We also give some elements of Anel's presentation of the *small object argument* as presented in [4], which is a process to construct a factorization system. In this case, we need for our purpose a *left-generated* factorization system, where the left maps can be constructed as filtered colimits of left maps of finite presentation.

First, recall that for two maps $l : A \to B$, $r : C \to D$ in a category $\mathcal{C}$, we say that l is *left orthogonal* to r - or equivalently that r is *right orthogonal* to l - if for any square as below

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle r} \\ B & \xrightarrow[g]{} & D \end{array}$$

there exists a unique $d : B \to C$ such that $f = dl$ and $g = rd$. This condition can be encoded by saying that the following square of homset is a pullback:

$$\begin{array}{ccc} \mathcal{C}[B,C] & \xrightarrow{\mathcal{C}[l,C]} & \mathcal{C}[A,C] \\ {\scriptstyle \mathcal{C}[B,r]}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathcal{C}[A,r]} \\ \mathcal{C}[B,D] & \xrightarrow[\mathcal{C}[l,D]]{} & \mathcal{C}[A,D] \end{array}$$

For a class of map $\mathcal{L}$ we denote as $\mathcal{L}^\perp$ the class of maps that are right orthogonal to all maps in $\mathcal{L}$; respectively, for a class of maps $\mathcal{R}$ we denote as ${}^\perp\mathcal{R}$ the class of maps that are left orthogonal to all maps in $\mathcal{R}$. Observe that both of the operations $(-)^\perp$ and ${}^\perp(-)$ are order reversing for the inclusion of classes and form a Galois connexion on the poset of classes of maps of $\mathcal{C}$. An *orthogonality structure* in a category $\mathcal{C}$ is the data of two classes of maps $(\mathcal{L}, \mathcal{R})$, where maps of $\mathcal{L}$ and $\mathcal{R}$ are called respectively *left maps* and *right maps*, such that $\mathcal{R} = \mathcal{L}^\perp$ and $\mathcal{L} = {}^\perp\mathcal{R}$. In particular observe that isomorphisms are always both right and left orthogonal to any map, and that in an orthogonality structure $\mathcal{L} \cap \mathcal{R}$ is exactly the class of isomorphisms.

Now we should list some useful properties of left and right classes in an orthogonality structure. A first important property puts constrains on the factorizations of right maps through left maps (resp., of right maps through left maps):

Lemma 1.1.1.1. *If $(\mathcal{L}, \mathcal{R})$ is an orthogonality structure in $\mathcal{C}$ and we have a factorization as below with r in $\mathcal{R}$ and l in $\mathcal{L}$*

$$\begin{array}{ccc} A & \xrightarrow{r} & B \\ {\scriptstyle l}\downarrow & \nearrow_{f} & \\ C & & \end{array}$$

Then l is a split monomorphism, and f factorizes through r. Dually, for any factorization as below

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & {\scriptstyle l}\searrow & \downarrow{\scriptstyle r} \\ & & C \end{array}$$

then r is a split epimorphism and f factorizes through l.

Proof. Those statements just come from the diagonalization of the squares

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle l}\downarrow & \nearrow_{d} & \downarrow{\scriptstyle r} \\ C & \xrightarrow[f]{} & B \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle l}\downarrow & \nearrow_{d} & \downarrow{\scriptstyle r} \\ C & = & C \end{array}$$

□

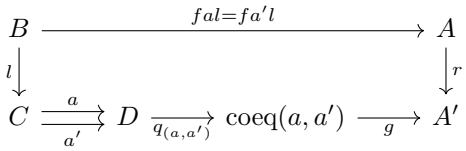
We also have the following property in strong orthogonality structures:

Lemma 1.1.1.2. *Let be $(\mathcal{L}, \mathcal{R})$ an orthogonality structure in a category $\mathcal{C}$ with coequalizers: then if a parallel pair $a, a' : C \rightrightarrows D$ in $\mathcal{C}$ is equalized by a morphism $l : B \to C$ in $\mathcal{L}$, then its coequalizer $q_{a,a'} : D \to \mathrm{coeq}(a, a')$ is in $\mathcal{L}$.*

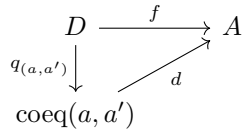
Proof. Let be a square as below with $r \in \mathcal{R}$

$$\begin{array}{ccc} D & \xrightarrow{f} & A \\ {\scriptstyle q_{(a,a')}}\downarrow & & \downarrow{\scriptstyle r} \\ \mathrm{coeq}(a,a') & \xrightarrow[g]{} & A' \end{array}$$

For $al = a'l$ we have $fal = fa'l$, while $gq_{(a,a')}a = gq_{(a,a')}a'$, so that we have a commutative square



and both fa, fa' provide diagonalizations of this square: but such a diagonalization must be unique, so that $fa = fa'$. Hence there exists a unique d factorizing f through the coequalizer as below

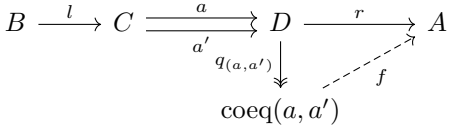


Moreover, we have $udq_{(a,a')} = uf = gq_{(a,a')}$, but as a coequalizer, $q_{(a,a')}$ is an epimorphism: thence necessarily $ud = g$ so that d is the desired lift of the square above. Uniqueness of such a lift proceeds from the uniqueness of the solution in the universal property of the coequalizer. □

Remark 1.1.1.3. Of course we have the dual statement saying that the equalizer of a parallel pair coequalized in $\mathcal{R}$ must be in $\mathcal{R}$.

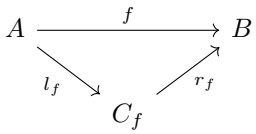
Corollary 1.1.1.4. *Any two arrows which are simultaneously equalized by a morphism in $\mathcal{L}$ and coequalized by a morphism in $\mathcal{R}$ must be equal.*

Proof. Suppose $a, a' : C \rightrightarrows D$ are equalized by some $l : B \to C$ in $\mathcal{L}$ and coequalized by some $r : D \to A$ in $\mathcal{R}$. Then the coequalizer $q_{(a,a')} : D \to \mathrm{coeq}(a, a')$ both is in $\mathcal{L}$ and factorizes r as below



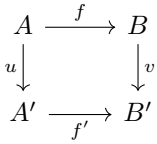
Then from lemma 1.1.1.1 we know $q_{(a,a')}$ to be a split monomorphism; but a coequalizer being always an epimorphism, this forces that actually $q_{(a,a')}$ is an isomorphism, so that $a = a'$. □

Now recall that a *factorization system* on a category $\mathcal{C}$ is the data of an orthogonality structure $(\mathcal{L}, \mathcal{R})$ such that any arrow f in $\mathcal{C}$ admits a factorization

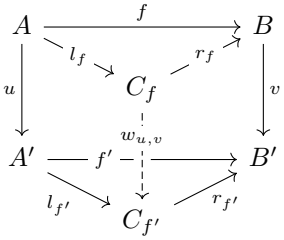


with l_f in $\mathcal{L}$ and r_f in $\mathcal{R}$, which is moreover unique up to a unique isomorphism, that is such that for any other such decomposition $f = rl : A \to C \to B$, there is a unique isomorphism $\alpha : C \simeq C_f$ such that $l = \alpha l_f$ and $r_f = r\alpha$. By orthogonality, we see that any such factorization is terminal amongst those with a left map on the left, and initial amongst those with a right map on the right.

Now a factorization system is said to be *functorial* if for any square as below



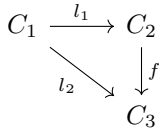
the factorizations are related through a unique map $w_{u,v}$ as below



In particular if u is a left map, resp. if v is a right map, then $w_{u,v}$ is so.

Proposition 1.1.1.5. *If $\mathcal{C}$ is a category endowed with a factorization system $(\mathcal{L}, \mathcal{R})$, then the left and right classes enjoy the following properties :*

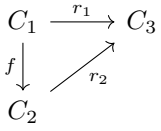
- *$\mathcal{L}$ is closed under composition*
- *$\mathcal{L}$ contains all isomorphisms*
- *$\mathcal{L}$ is right-cancellative: for any triangle*



with l_1, l_2 in $\mathcal{L}$, then f also is in $\mathcal{L}$

- *$\mathcal{L}$ is closed under colimits in $\mathcal{C}^2$*
- *$\mathcal{L}$ is closed under pushout along arbitrary maps*

- *$\mathcal{R}$ is closed under composition*
- *$\mathcal{R}$ contains all isomorphisms*
- *$\mathcal{R}$ is left-cancellative: for any triangle*



with r_1, r_2 in $\mathcal{R}$, then f also is in $\mathcal{R}$

- *$\mathcal{R}$ is closed under limits in $\mathcal{C}^2$*
- *$\mathcal{R}$ is closed under pullback along arbitrary maps*

Proposition 1.1.1.6. *If $(\mathcal{L}, \mathcal{R})$ is a factorization system in $\mathcal{C}$, then f is in $\mathcal{L}$ (resp. $\mathcal{R}$) if and only if r_f (resp. l_f) is invertible.*

Now we come to the main notion of this section. In an factorization system in an abitrary category, both the left and right classes may be large, so in particular it may not be possible to compute explicitly the factorization of a map as a colimit or a limit ranging over its factorizations through a left or a right map. More generally, when starting from an orthogonality structure, one may ask whether it is possible to construct a factorization system from those orthogonality data. This is the topic of the *small object argument*, a version of which in the context of locally finitely presentable categories. For the remaining of this chapter, we fix a locally finitely presentable category $\mathcal{B}$ and denote as $\mathcal{B}_\omega$ its full subcategory of finitely presented objects.

When working in a locally finitely presentable category equipped with an orthogonality structure, as arbitrary maps are constructed as filtered colimits of finitely presented maps which form a small class, it is possible to consider the left maps that are finitely presented - that is, whose domain and codomain are finitely presented: they form a small class, and we will see that one can construct a factorization of arbitrary maps as a filtered colimit of its factorizations through finitely presented left maps. This leads us to the following auxiliary notion:

Definition 1.1.1.7. A *saturated class* is a set $\mathcal{V} \subseteq \mathcal{B}_\omega^2$ of finitely presented maps such that:

- $\mathcal{V}$ contains isomorphisms and is stable by composition,
- $\mathcal{V}$ is right-cancellative
- $\mathcal{V}$ is closed under finite colimits in $\mathcal{B}^2$
- $\mathcal{V}$ is closed under pushouts along arbitrary maps between finitely presented objects

Remark 1.1.1.8. A saturated class is always small, as lying in the essentially small generator $\mathcal{B}_\omega^2$.

Remark 1.1.1.9. Observe that saturated classes are closed under retracts in the arrow category since they are supposed to be closed under finite colimits.

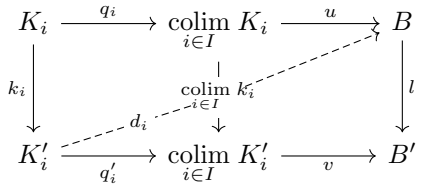
Remark 1.1.1.10. We can see $\mathcal{V}$ as a *full and faithful* subcategory of $\mathcal{B}_\omega^2$. Beware hence that arrows of $\mathcal{V}$ are squares

$$\begin{array}{ccc} K_1 & \xrightarrow{k} & K_2 \\ {\scriptstyle n_1}\downarrow & & \downarrow{\scriptstyle n_2} \\ K_1' & \xrightarrow[k']{} & K_2' \end{array}$$

where however k and k' are *arbitrary* finitely presented arrows, and are not supposed to be in $\mathcal{V}$. This is what it takes for $\mathcal{V}$ to be closed under finite colimits in $\mathcal{B}_\omega^2$, as the colimit inclusions $(q_i, q_i') : n_i \to \mathrm{colim}_{i \in I} n_i$ should not be supposed to be in $\mathcal{V}$.

Proposition 1.1.1.11. *Any set of finitely presented maps $V \subseteq \mathcal{B}^2_\omega$ can be completed into a saturated class $\mathcal{V}$ such that $V^\perp = \mathcal{V}^\perp$.*

Proof. It is clear that $\mathcal{V}^\perp \subseteq V^\perp$ for $V \subseteq \mathcal{V}$. Moreover it is also easy to see that taking the closure by composition and iso does not modify the right class. Stability of the right class after closing the left class under finite colimit is a special case of stability of left classes under colimits, but let us give the detailed proof: let be a map l in $V^\perp$. If $(l_i : K_i \to K'_i)_{i\in I}$ is a diagram in $\mathcal{B}^2$, then in the following diagram we have for each $i \in I$ a lifting



and then by universal property of the colimit, this induces a unique map $\langle d_i\rangle_{i\in I}$ with $\langle d_i\rangle_{i\in I} q'_i = d_i$. We prove this map is a filler. First, universal properties of colimits give us the following sequence of equalities:

$$u = \langle uq_i\rangle_{i\in I} = \langle d_i q'_i l_i\rangle_{i\in I} = \langle d_i\rangle_{i\in I}\langle q'_i l_i\rangle_{i\in I} = \langle d_i\rangle_{i\in I}\operatorname*{colim}_{i\in I} l_i$$

so the upper triangle commutes; similarly we have

$$v = \langle vq'_i\rangle_{i\in I} = \langle ld_i\rangle_{i\in I} = l\langle d_i\rangle_{i\in I}$$

so that the lower triangle also commutes. Finally, the stability of the right class after closing V under pushouts along arbitrary maps is an easy consequence of the universal property of the pushout. □

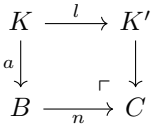
In the remainder of this section, we describe how one can construct a factorization system from a choice of saturated class in a locally finitely presentable category. In particular, as any orthogonality structure induces automatically a saturated class, this will provide a way to turn an orthogonality structure into a factorization system, though as we shall see the left and right classes may be modified in such a process.

1.1.2 The etale generator under a fixed object

When constructing a left-generated factorization system from an orthogonality structure $(\mathcal{L}, \mathcal{R})$ in a locally finitely presentable category $\mathcal{B}$, one will have to consider arbitrary maps whose neither domain nor codomain are supposed to be finitely presented; to this end we need to define at each object B a small class of left maps under it generating other left maps under it. Then one can construct factorization of maps with domain B as a colimit ranging over intermediate factorizations in this small class. The maps in this class will not be finitely presented in the ambient locally finitely presented category $\mathcal{B}$ itself, but in the coslice $B \downarrow \mathcal{B}$.

This subsection is devoted to the construction of such a generator at a fixed object B. Its strategy is inspired from [4] and is the following: from a saturated class $\mathcal{V}$ consisting of *"basic" left maps* in $\mathcal{B}$, we define a generator of left maps (which will be called *etale generator* as in the following left maps will be called etale for our purposes) as the pushouts of basic left maps under it; we prove that such maps are finitely presented in the coslice under B and that this class is to be closed under finite colimits - and in particular retracts - so that it can be seen as the generator of a locally finitely presentable category of maps under B: these maps will be all the left maps under B in our factorization system.

Definition 1.1.2.1. For any object B in $\mathcal{B}$, define the *etale generator at B* as full subcategory $\mathcal{V}_B$ of $B \downarrow \mathcal{B}$ consisting of morphisms $n : B \to C$ such that there exists some $l : K \to K'$ in $\mathcal{V}$ and $a : K \to B$ exhibiting n as the pushout



Remark 1.1.2.2. In particular the etale generator $\mathcal{V}_K$ of a finitely presented object K consists of maps in $\mathcal{V}$ as $\mathcal{V}$ is closed under pushouts and a pushout of a finitely presented map is a finitely presented map as its codomain still is finitely presented. Hence $\mathcal{V}_K$ is a subcategory of $\mathcal{V}$. Moreover, the problem of persistence of maps out of $\mathcal{V}$ in the squares forming the morphisms in $\mathcal{V}$ vanishes in the etale generators of the form $\mathcal{V}_K$ by right cancellation. Hence $\mathcal{V}_K$ is a full subcategory of $K \downarrow \mathcal{B}_\omega$, and yet for any n_1, n_2 and in $\mathcal{V}_K$, any morphism $n_1 \to n_2$ is in $\mathcal{V}$ by right cancellation applied to the triangle

$$\begin{array}{ccc} K & \xrightarrow{n_1} & K_1 \\ & \searrow^{n_2} & \downarrow{\scriptstyle n} \\ & & K_2 \end{array}$$

Remark 1.1.2.3. We are going to use maps of $\mathcal{V}_B$ to constructs lefts maps under B through filtered colimits. Recall to this end that for a small category $\mathcal{C}$, $\mathrm{Ind}(\mathcal{C})$ is the *free inductive completion of* $\mathcal{C}$, which is the subcategory of $\widehat{\mathcal{C}}$ consisting of filtered colimits of representables. It is known that any locally finitely presentable category $\mathcal{B}$ is the inductive completion of its generator of finitely presented objects (see 3.1.1.3), and the same is true for the category of arrows which satisfies $\mathcal{B}^2 \simeq \mathrm{Ind}(\mathcal{B}^2_\omega)$. Here we are going to generate in $\mathcal{B}^2$ full subcategories $\mathrm{Ind}(\mathcal{V})$ and $\mathrm{Ind}(\mathcal{V}_B)$ for each object B, and prove they are the categories of left maps and left maps under a fixed object for an associated left generated factorization system.

Lemma 1.1.2.4. *For each finitely presented K in $\mathcal{B}$, $\mathcal{V}_K$ is closed under finite colimit in the coslice $K \downarrow \mathcal{B}$. In particular, for a finite colimit in $\mathcal{V}_K$, the colimit inclusions are in $\mathcal{V}$.*

Proof. We prove that $\mathcal{V}_K$ has finite coproducts and coequalizers. A coproduct in $\mathcal{V}_K$ is in fact a pushout of the underlying maps

$$\begin{array}{ccc} K & \xrightarrow{n_1} & K_1 \\ {\scriptstyle n_2}\downarrow & \searrow^{n_1+n_2} & \downarrow{\scriptstyle n_{1*}n_2} \\ K_2 & \xrightarrow[n_{2*}n_1]{} & K_1 +_K K_2 \end{array}$$

Then by closure of $\mathcal{V}$ under pushouts, both $n_{2*}n_1$ and $n_{1*}n_2$ are in $\mathcal{V}$, as well as the composite $n_1 + n_2$. The same argument, iterated as needed, ensures closures under finite coproducts.

For coequalizers, consider a parallel pair in $\mathcal{V}_K$

$$\begin{array}{ccc} K & \xrightarrow{n_1} & K_1 \\ & \searrow^{n_2} & {\scriptstyle n}\downarrow\downarrow{\scriptstyle n'} \\ & & K_2 \end{array}$$

Then seeing this diagram of $\mathcal{V}$-maps in $\mathcal{B}^2$, we can compute the coequalizer in $\mathcal{B}^2$ as follows

$$\begin{array}{ccccc} K & \equiv & K & = & K \\ {\scriptstyle n_1}\downarrow & & {\scriptstyle n_2}\downarrow & & \downarrow{\scriptstyle n_{(n,n')}} \\ K_1 & \underset{n'}{\overset{n}{\rightrightarrows}} & K_2 & \xrightarrow[q_{(n,n')}]{} & \mathrm{coeq}(n,n') \end{array}$$

where in particular the bottom line is the coequalizer in $\mathcal{B}$ of n, n'. Then, for $\mathcal{V}$ is supposed to be closed under finite colimits in $\mathcal{B}^2$, we have $n_{(n,n')}$ in $\mathcal{V}$: but this is also the coequalizer of (n, n') in $\mathcal{V}_K$. Moreover by right cancellation, the canonical arrow $q_{(n,n')}$ also is in $\mathcal{V}$.

More generally, for any finite diagram in $\mathcal{V}_K$, we know that the colimit inclusions have to be in $\mathcal{V}$: in each i in a diagram I in $\mathcal{V}$ we have a factorization

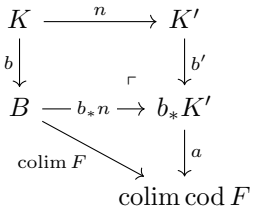
$$\begin{array}{ccccc} & & \mathrm{cod}(\overline{F}(i)) & & \\ & \overset{\overline{F}(i)}{\nearrow} & & \overset{\overline{q}_i}{\searrow} & \\ K & & \xrightarrow[\mathrm{colim}\,\overline{F}]{} & & \mathrm{cod}(\mathrm{colim}\,\overline{F}) \end{array}$$

so right cancellation ensures that q_i is in $\mathcal{V}$. □

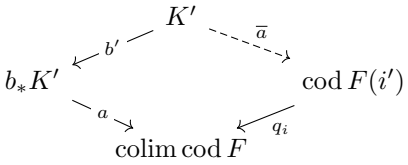
Remark 1.1.2.5. In practice, saturated classes will be constructed by restricting a left class of an orthogonality structure to finitely presented arrows. Hence the argument about coequalizers will be directly proved by applying lemma 1.1.1.2.

Lemma 1.1.2.6. *Objects of $\mathcal{V}_B$ are finitely presented in $B \downarrow \mathcal{B}$*

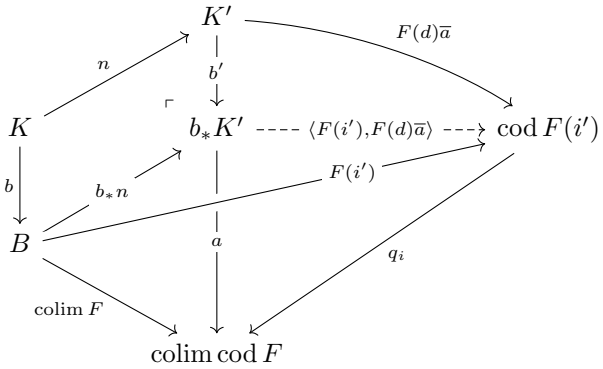
Proof. Let be $F : I \to B \downarrow \mathcal{B}$ a filtered diagram; then form what was said about filtered colimits in the coslice, we have $\mathrm{cod}(\mathrm{colim}\, F) \simeq \mathrm{colim}\, \mathrm{cod}\, F$. Then for any situation as below



the composite arrow $an_*b : K' \to \mathrm{colim}\, \mathrm{cod}\, F$ lifts through some $q_i : \mathrm{cod}\, F(i) \to \mathrm{colim}\, \mathrm{cod}\, F(i)$ as



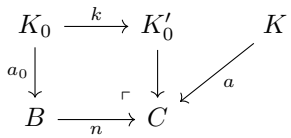
However, it is still not clear that the induced parallel lifts $F(i)b : K \to F(i)$ and $\overline{a}n : K \to F(i)$ commute: but from K is finitely presented and I is filtered, we know they are equalized by some $F(d)$ for some morphism $d : i \to i'$ in I, and moreover, $F(i') = F(d)F(i)$; then the universal property of the pushout induces a universal map



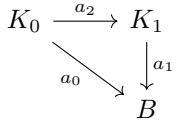
Similarly, we use finite presentedness of K' to prove that any two lifts of a have to be equalized by a further refinement. This proves the pushout map b_*n to be finitely presented in $B \downarrow \mathcal{B}$ □

Before being able to justify the appellation of generator, we need to recall the following useful technical lemma from [4][sub-lemma 12], which is important for our theorem 1.1.4.3 and is also involved in the proof of proposition 1.1.3.2 and theorem 7.1.2.7:

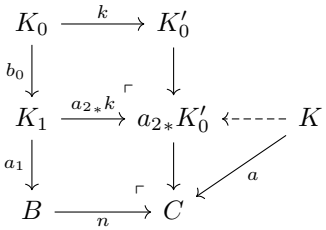
Lemma 1.1.2.7. *Let be a diagram as below*



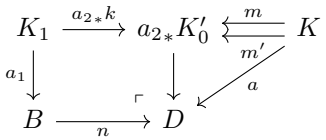
with K, K_0, K'_0 finitely presented and k in $\mathcal{V}$: then there exists a factorization with K_1 finitely presented



such that a factorizes through the following pushout



Remark 1.1.2.8. Observe that from the properties of finitely presented objects, we can say moreover that for any two parallel lifts of the same a

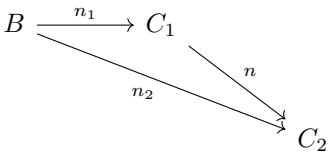


there exists a further factorization $a_1 = a_2 b_1$ with $a_2 : K_2 \to B$ such that m and m' are equalized by the intermediate arrow

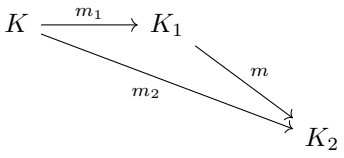
$$(a_{2*}k)b_1 m = (a_{2*}k)b_1 m'$$

This lemma is also crucial to the following fullness-like property of the etale generator in the coslice, which allows to exhibit any arrow between object of the etale generator as a pushout square of finitely presented etale map:

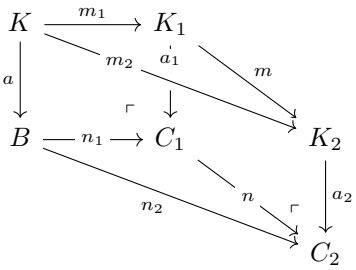
Lemma 1.1.2.9. *For any B in $\mathcal{B}$, and any triangle*



with n_1, n_2 in $\mathcal{V}_B$, there exists a triangle

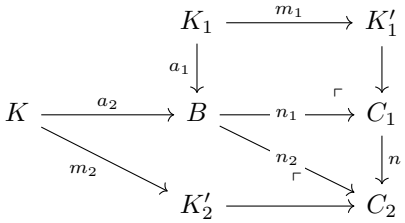


in $\mathcal{V}$ such that all squares below are pushouts

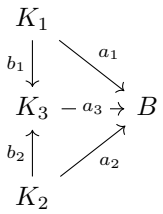


(so that in particular n, as exhibited as $a_{1*}n$, in $\mathcal{V}_{C_1}$).

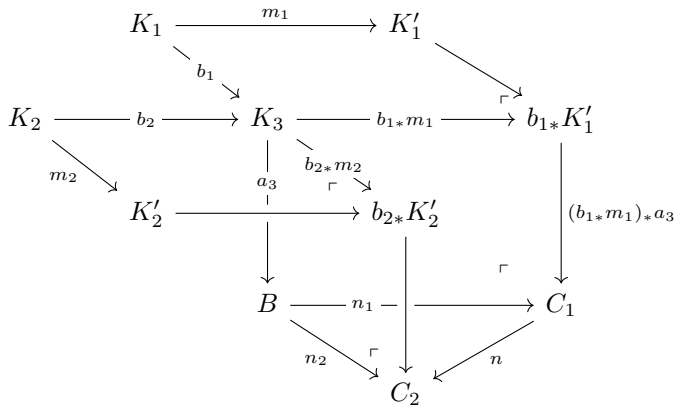
Proof. As n_1 and n_2 are supposed in $\mathcal{V}_B$, they are induced from pushouts as below



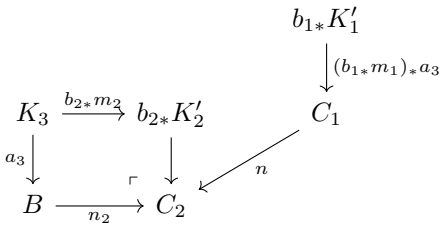
Now by filteredness of $\mathcal{B}_\omega \downarrow B$ there exists $a_3 : K_3 \to B$ and a factorization



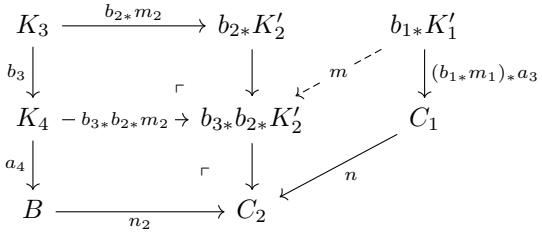
and the pushouts themselves factorize as below



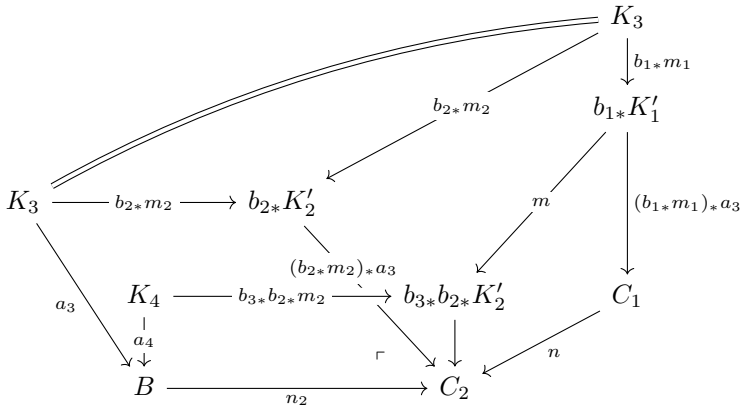
Now apply lemma 1.1.2.7 to the following situation (where b_{1*} still is finitely presented as K_3 is)



to exhibit a further factorization



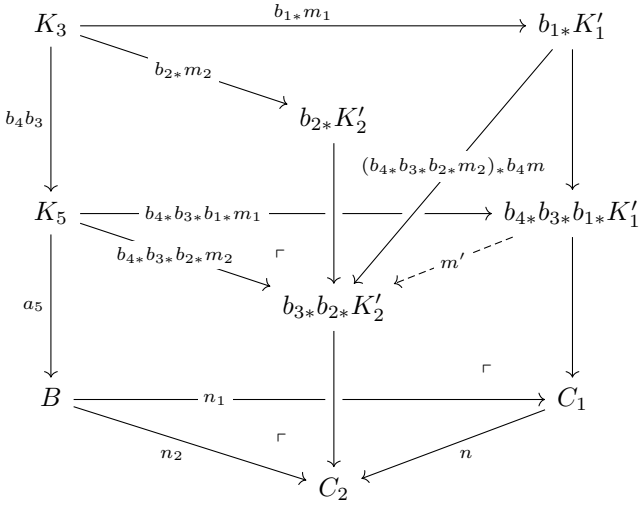
However, we cannot infer at this step that m commutes with the other part of the diagram. Indeed, one cannot infer that $mb_{1*}m_1$ and $(b_{2*}m_2)_*b_{3*}b_{2*}m_2$ commute together. However they are equalized by $n_{2*}a_4$, which provides two parallel lifts of the same situation



so there exists a further factorization $a_4 = a_5 b_4$ with $a_5 : K_5 \to B$ such that

$$\begin{aligned}(b_{4*}b_{3*}b_{2*}m_2)_* b_4 m b_{1*} m_1 &= (b_{4*}b_{3*}b_{2*}m_2)_* b_4 (b_{2*}m_2)_* b_{3*}b_{2*}m_2 \\ &= b_{4*}b_{3*}b_{2*}m_2 b_4 b_3\end{aligned}$$

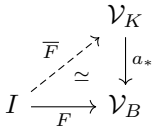
And now by the universal property of the pushout we have an arrow m' as in the diagram below



Combining stability under pushouts and right cancellation of maps in $\mathcal{V}$, we know m' to be in $\mathcal{V}$; moreover, by right cancellation of pushout squares, the right, bottom square is also a pushout, so that n is exhibited as a pushout of m' along the canonical inclusion $(b_{3*}(b_{1*}m_1))_* a_3$. □

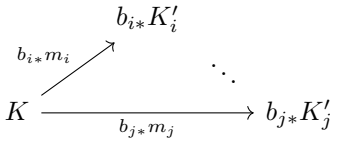
In particular, the following says we can lift any finite diagram in the etale generator of B into a diagram of the same shape made of finitely presented etale arrows, from which it can be induced by pushout:

Lemma 1.1.2.10. *For any finite diagram $F : I \to \mathcal{V}_B$, there is some $a : K \to B$ and some lifts*

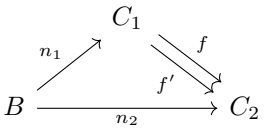


where a_* is the pushout functor. In particular the transition morphisms of F are obtained as pushouts of the corresponding transition morphisms of $\overline{F}$.

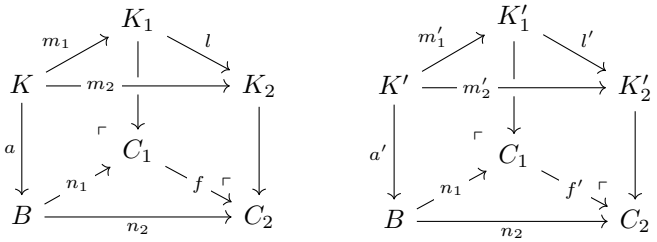
Proof. We saw that one can lift morphisms. Here we prove that one can lift finite discrete diagrams and parallel pairs. Let be a discrete set $(n_i : B \to C_i)_{i \in I}$ with I finite, with $n_i = a_{i*}m_i$ and $m_i : K_i \to K_i'$. Then, for $\mathcal{B}_\omega \downarrow B$ is filtered, there exists some $a : K \to B$ and for each $i \in I$ an arrow $b_i : K_i \to K$ such that $a_i = a b_i$; then in $\mathcal{V}_K$ one gets the following diagram over $(n_i)_{i \in I}$



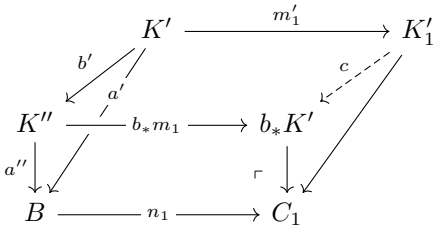
Now consider a parallel pair



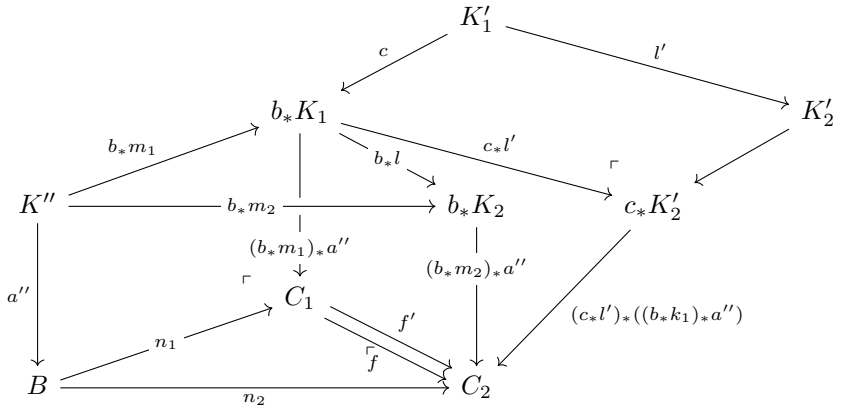
Then from lemma 1.1.2.9 there are respectively two lifts



Now one can find a common refinement $a'' : K'' \to B$, $b : K \to K''$ and $b' : K' \to K''$ of a, a'. Moreover, by applying upstream lemma 1.1.2.7 we can chose this common refinement to be such that there exists a factorization



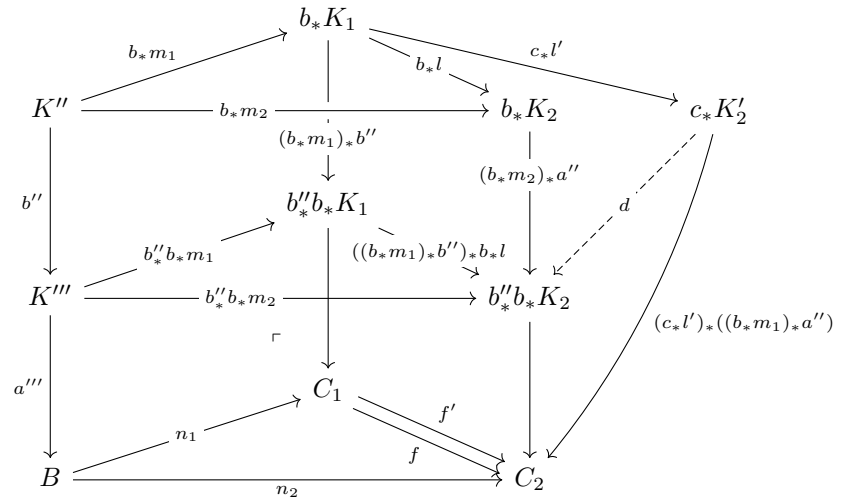
Now we can push the arrow l' along c to get a diagram as below



where, by cancellation of pushouts, we have that

$$f' = ((b_* m_1)_* a'')_* (c_* l')$$

Finally, again by lemma 1.1.2.7, we can get a last factorization $a''' : K''' \to B$ and $b'' : K'' \to K'''$ and a factorization as below



But now, the pair $(dc_* l', b''_* b_* m_2)$ induces a unique arrow

$$b''_* b_* K_1 \xrightarrow{\langle dc_* l', b''_* b_* m_2 \rangle} b''_* b K_2$$

which moreover satisfies $\langle dc_* l', b''_* b_* m_2 \rangle b''_* b_* m_1 = b''_* b_* m_2$, so that by cancellation of pushouts together with the pushout expression of n_1, n_2, we have

$$f' = ((b''_* b_* m_1)_* a''')_* \langle dc_* l', b''_* b_* m_2 \rangle$$

Hence the parallel pair in $\mathcal{V}_{m'''}$

$$b''_* b_* m_1 \underset{((b_* k_1)_* b'')_* b_* l}{\overset{\langle dc_* l', b''_* b_* m_2 \rangle}{\rightrightarrows}} b''_* b_* m_2$$

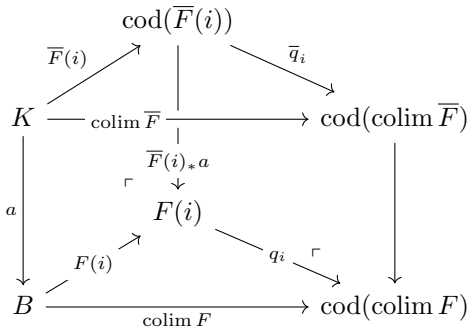
is a lift of the parallel pair f, f' as desired. □

Proposition 1.1.2.11. *The etale generator $\mathcal{V}_B$ is closed under finite colimits in the cocomma $B \downarrow \mathcal{B}$. Hence the category* $\mathrm{Ind}(\mathcal{V}_B)$ *is locally finitely presented.*

Proof. This comes as a consequence of the previous results. For any finite diagram $F : I \to \mathcal{V}_B$, choose a lift $\overline{F} : I \to \mathcal{V}_K$ as provided by lemma 1.1.2.10. As K is finitely presented, $\mathcal{V}_K$ is a full subcategory of $\mathcal{V}$, and by lemma 1.1.2.4, it is closed under finite colimits in $B \downarrow \mathcal{B}$ and colimit inclusions are in $\mathcal{V}$. Then by commutation of pushouts with colimits, we have

$$a_* \mathrm{colim}\, \overline{F} \simeq \mathrm{colim} a_* \overline{F} \simeq \mathrm{colim}\, F$$

Hence for each i in I we have a diagram as below



where the front square is a pushout, exhibiting colim F as an object of $\mathcal{V}_B$. Moreover, the colimit inclusions q_i are obtained as the pushouts

$$q_i = (\overline{F}(i)_* a)_* (\overline{q}_i)$$

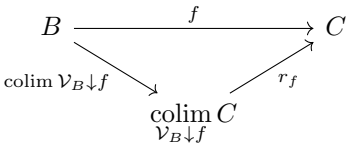
All of this suffices to prove that $\mathcal{V}_B$ is finitely cocomplete and closed under finite colimits in $B \downarrow \mathcal{B}$. Hence its free inductive cocompletion is a locally finitely presentable category. □

Remark 1.1.2.12. In particular, let us emphasize that $\mathcal{V}_B$ has to be closed under retracts, for it is closed under finite colimits. This can also be seen directly, by lifting the idempotent associated to a retract and its splitting, with the latter step enabled by lifting of coequalizers; however we choose to cut this tiresome proof, a close version of which can be found in [83][Lemma 2.10]. The use of splitting of idempotents in this proof explains the discrepancy of our result with [67][Warning 2.2.5], where we are warned that in the context of $(\infty, 1)$-categories, the analogs of $\mathcal{V}_B$ are not anymore closed under retracts. This is due to the specific fact that, unlike in 1-categories, splittings of idempotents in $(\infty, 1)$-categories cease to be constructible by mean of finite (co)limits, so that, in this context, there is no analog to our argument involving expression of the splitting as a coequalizer.

1.1.3 Small object argument

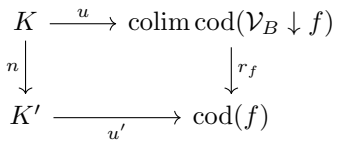
1.1.3.1. Now we recall the construction of the factorization system as done in [4] and also [19]. We saw that the etale generator at any object is closed under finite colimits. Then for any $f : B \to C$, the category $\mathcal{V}_B \downarrow f$ of finitely presented etale maps under B above f is filtered. Moreover, recall that the codomain functor $B \downarrow \mathcal{B}$ preserves filtered colimits. Now we can construct the factorization of any arrow f in $\mathcal{B}$:

Proposition 1.1.3.2. *An arbitrary arrow $f : B \to C$ in $\mathcal{B}$ admits a factorization*

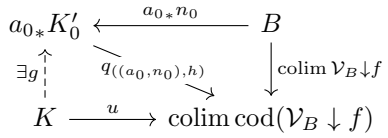


with l_f is in $\mathrm{Ind}(\mathcal{V}_B)$ *and r_f is in $\mathcal{V}^{\perp}$.*

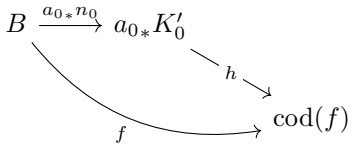
Proof. Let $n : K \to K'$ a map in $\mathcal{V}$ and a square



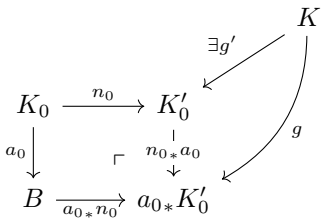
then, as we have $\mathrm{cod}(\operatorname{colim} \mathcal{V}_B \downarrow f) \simeq \operatorname{colim} \mathrm{cod}(\mathcal{V}_B \downarrow f)$ which is a finitely filtered colimit and K is finitely presented, we have a factorization



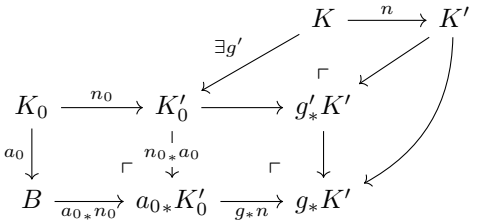
for some $((a_0, n_0), h)$ in $\mathcal{V}_B \downarrow f$, that is for (a_0, n_0) in $\mathcal{V}_c$ such that f factorizes through $a_{0*}k_0$ as



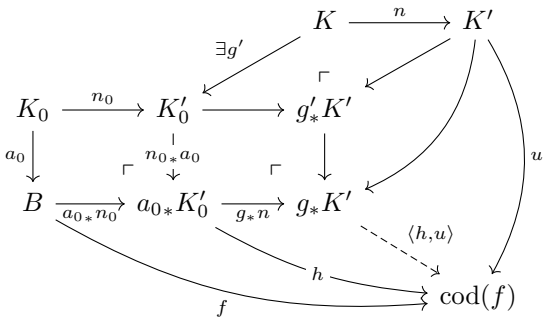
Then from lemma 1.1.2.7, we could have chosen (a_0, k_0) such that g factorizes through $n_{0*}a_0$ as



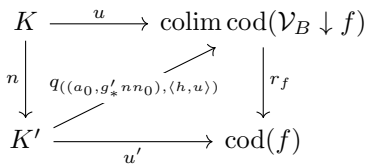
so that the composite of pushouts



provides a factorization of f by pushout property: from one hand we supposed that there was a factorization of f through $a_{0*}n_0$ by some h, but then this jointly provide with u a universal map



But now observe that the composite of pushouts $g_*na_{0*}n_0 : B \to g_*K$ coincides with the pushout of g'_*nn_0 along a_0 as $g_*n = (n_{0*}a_0)_*g'_*n$ by left cancellation of pushout, and the later is in $\mathcal{V}$ by closure under pushout along arbitrary maps. In other words, we have a factorization though $g'_*nn_{\rangle}$. But then the data of $((a_0, g'_*nn_0), \langle h, u\rangle)$ define an object of $\mathcal{V}_B \downarrow f$, and the corresponding inclusion into the colimit provides the unique lifting

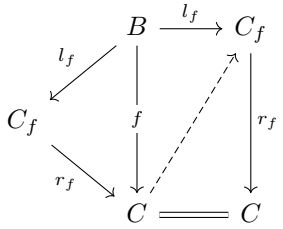


□

We have hence proved that $(\mathrm{Ind}(\mathcal{V}), \mathcal{V}^{\perp})$ is an factorization system on $\mathcal{C}$. We must now check this is an orthogonal factorization system, that is, that the left and right class are mutually orthogonal. This is the purpose of this statement:

Proposition 1.1.3.3. *The factorization above is orthogonal, that is,* $\mathrm{Ind}(\mathcal{V}) = {}^{\perp}(\mathcal{V}^{\perp})$.

Proof. We saw that left classes are closed under (filtered) colimits. For $l \in {}^{\perp}(\mathcal{V}^{\perp})$ with factorization $f = r_f l_f$, f is left orthogonal to its own right part, thence a unique filler in the diagram below



But now right cancellation of left maps, together with left cancellation of right maps, enforces that this filler is an isomorphism, which forces r_f to be iso, so that f is in $\mathrm{Ind}(\mathcal{V}_B)$. □

Definition 1.1.3.4. A factorization system is *left-generated* if it is of the form $(\mathrm{Ind}(\mathcal{V}), \mathcal{V}^{\perp})$. In particular, the factorization of an arrow is obtained as a filtered colimit as in proposition 1.1.3.2.

Remark 1.1.3.5. Saturated classes and factorization systems on a locally finitely presentable category $\mathcal{B}$ form posets $\mathbf{Sat}_{\mathcal{B}}$ and $\mathbf{Fact}_{\mathcal{B}}$ with inclusion as wide subcategory of $\mathcal{B}_{\omega}$ for $\mathbf{Sat}_{\mathcal{B}}$ and inclusion of the left class for $\mathbf{Fact}_{\mathcal{B}}$. Saturated class and factorization systems in locally finitely presentable categories are related through an adjunction:

$$\begin{array}{ccc} \mathbf{Sat}_{\mathcal{B}} & \overset{\perp}{\rightleftarrows} & \mathbf{Fact}_{\mathcal{B}} \\ \mathcal{V} & \longmapsto & (\mathrm{Ind}(\mathcal{V}), \mathcal{V}^{\perp}) \\ \mathcal{L} \cap \mathcal{B}_{\omega}^{2} & \longmapsfrom & (\mathcal{L}, \mathcal{R}) \end{array}$$

where $\mathbf{Sat}_{\mathcal{B}} \hookrightarrow \mathbf{Fact}_{\mathcal{B}}$ is a mono because for any saturated class $\mathcal{V} = \mathrm{Ind}(\mathcal{V})$. On the converse, it is not true that a factorization system is generated from a saturated class, so that in general we only have

$$(\mathrm{Ind}(\mathcal{L} \cap \mathcal{B}_{\omega}^{2}), (\mathcal{L} \cap \mathcal{B}_{\omega}^{2})^{\perp}) \leq (\mathcal{L}, \mathcal{R})$$

so that in particular $\mathcal{R} \subseteq (\mathcal{L} \cap \mathcal{B}_{\omega}^{2})^{\perp}$. However in the following we are going to restrict our attention to the factorizations systems having this property; the general case will however be met again when dealing with Diers contexts.

1.1.3.6. Left-generated factorizations systems in locally finitely presentable categories are in some sense factorization systems that can be axiomatized by a finite limit theory. Indeed, from the axioms of a saturated class, the category $\mathcal{V}$ is closed under finite colimits. But as a consequence its inductive completion $\mathrm{Ind}(\mathcal{V})$ is a locally finitely presentable category. In fact the full inclusion

$$\mathcal{V} \hookrightarrow \mathcal{B}_{\omega}^{2}$$

preserves finite colimits; but for $\mathcal{B}^{2} \simeq \mathrm{Ind}(\mathcal{B}_{\omega}^{2})$ we have a locally finitely presentable functor

$$\mathcal{B} \to \mathrm{Ind}(\mathcal{V})$$

sending each arrow f to its left part l_f, with full and faithful left adjoint. Hence the category of left maps in a left-generated system is a locally finitely presentable category; the corresponding finite limit theory has the category $\mathcal{V}^{\mathrm{op}}$ as syntactic site, and $\widehat{\mathcal{V}^{\mathrm{op}}}$ as classifying topos.

1.1.4 A pseudocolimit formula

1.1.4.1. From what preceeds, we know $\mathrm{Ind}(\mathcal{V}_B)$ to be a locally finitely presented category, which will be exhibited later to be the left class of a factorization system. However before recalling the construction of a factorization system from the data above, let us give the following result, which relates the canonical cone of the codomain of an arbitrary etale arrow and the etale generators through a pseudocolimit construction. First recall an arbitrary arrow $l : B \to C$ in $\mathrm{Ind}(\mathcal{V}_B)$ defines

a left exact functors $\text{ɟ}_l : \mathcal{V}_B^{op} \to \mathcal{S}$. Then the category of elements $\int \text{ɟ}_l$ is cofiltered. Moreover, recall that, as a standard fact, we have an equivalence of categories

$$\int \text{ɟ}_l \simeq \mathcal{V}_B \downarrow l$$

for the category of elements has pairs (n, a) with n in $\mathcal{V}_B$ and $a \in \text{ɟ}_l(n) = \text{Ind}(\mathcal{V}_B)[n, l]$ is an object of $\mathcal{V}_B \downarrow l$, while an arrow $(n_1, a_1) \to (n_2, a_2)$ consists in some $m : n_1 \to n_2$ in $\mathcal{V}_B$ such that $a_2 m = a_1$. Moreover, observe that we have a pseudofunctor

$$\mathcal{V}_{\text{cod}(-)} : \int \text{ɟ}_l^{\text{op}} \to \mathcal{C}at$$

which sends any $(n, a) \in \int \text{ɟ}_l$ to $\mathcal{V}_{\text{cod}(n)}$ and any $m : (n_1, a_1) \to (n_2, a_2)$ to the pushout functor

$$\mathcal{V}_{\text{cod}(n_1)} \xrightarrow{m_*} \mathcal{V}_{\text{cod}(n_2)}$$

associating to a map $n : \text{cod}(n_1) \to D$ in the etale generator $\mathcal{V}_{\text{cod}(n_1)}$ the pushout map $m_* n : \text{cod}(n_2) \to m_* D$. The latter is indeed in $\mathcal{V}_{\text{cod}(n_2)}$, as if n was induced from a map $k : K \to K'$ in $\mathcal{V}$ along some $a : K \to \text{cod}(n_1)$, then in the following diagram

$$\begin{array}{ccccc} K & \xrightarrow{a} & \text{cod}(n_1) & \xrightarrow{m} & \text{cod}(n_2) \\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle n} & & \downarrow{\scriptstyle m_* n} \\ K' & \longrightarrow & D & \longrightarrow & m_* D \end{array}$$

the outer square is also a pushout.

1.1.4.2. Recall that the Grothendieck construction of a pseudofunctor is its oplax colimit, and that the pseudolimit is constructed by localizing the oplax colimit at the opcartesian morphisms. For generalities about pseudocolimits of categories, see [86][Proposition 6.5] and also [23].

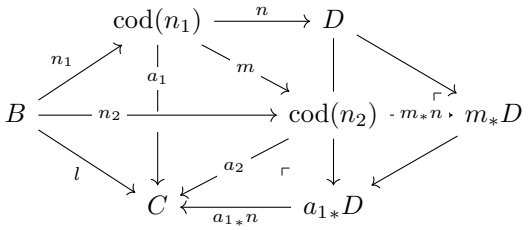
Theorem 1.1.4.3. *For any $l : B \to C$ in $\text{Ind}(\mathcal{V}_B)$, we have a pseudocolimit*

$$\mathcal{V}_C \simeq \underset{(n,a) \in \int \text{ɟ}_l}{\text{pscolim}} \mathcal{V}_{\text{cod}(n)}$$

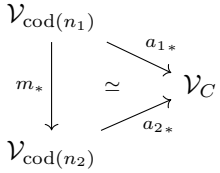
Proof. $\text{Ind}(\mathcal{V}_B)$ is a locally finitely presentable category where $l \simeq \text{colim}\, \mathcal{V}_B \downarrow l$; this colimit being filtered, it is preserved by the codomain functor, so that $C \simeq \text{colim}_{(n,a) \in \int \text{ɟ}_l} \text{cod}(n)$. From this we are going to produce a pseudococone

$$(\mathcal{V}_{\text{cod}(n)} \xrightarrow{a_*} \mathcal{V}_C)_{(n,a) \in \int \text{ɟ}_l}$$

where the a_* are the pushout functors; the pseudocommutativity of this diagram can be seen as follows: for a morphism $m : (n_1, a_1) \to (n_2, a_2)$, that is such that $n_2 = m n_1$ and $a_2 m = a_1$, and for an object $n : \text{cod}(n_1) \to D$ in $\mathcal{V}_{\text{cod}(n_1)}$ we have by composition of pushouts the following diagram



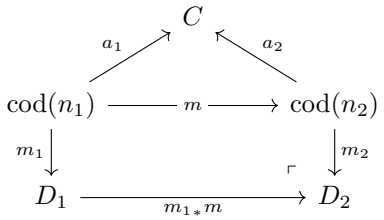
where the canonical isomorphism $a_1 * n \simeq a_{2*} m_* n$ provides the value at n of the natural isomorphism ensuring the pseudocommutativity of the triangle



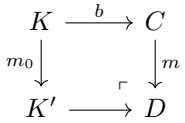
Hence the universal property of the pseudocolimit returns a functor

$$\operatorname*{colim}_{(n,a)\in\int \not\perp_l} \mathcal{V}_{\mathrm{cod}(n)} \xrightarrow{\langle a_* \rangle_{(n,a)\in\int \not\perp_l}} \mathcal{V}_C$$

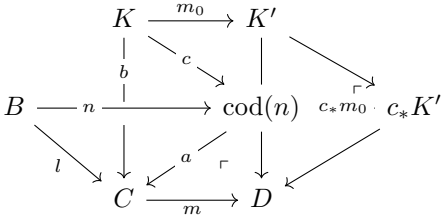
where the pseudocolimit $\mathrm{colim}_{(n,a)\in\int \not\perp_l} \mathcal{V}_{\mathrm{cod}(n)}$ is equivalent to the localization of the oplax colimit oplaxcolim $_{(n,a)\in\int \not\perp_l} \mathcal{V}_{\mathrm{cod}(n)}$ at the class of opcartesian morphisms which are the morphisms $(m, 1) : ((n_1, a_1), m_1) \to ((n_2, a_2), m_2)$ coding for pushout squares exhibiting m_2 as $m_2 = m_* m_1$ as below



We are going to prove this functor to be both essentially surjective and full and faithful. Let be an object m in $\mathcal{V}_C$ is induced as a pushout from some finitely presented etale map m_0 in $\mathcal{V}$



As K is finitely presentable, it factorizes through some a for $(n, a) \in \int \not\perp_l$ as $b = ac$, and then we can exhibit m as arising as the pushout of $c_* m_0$ along a by right cancellation of pushouts

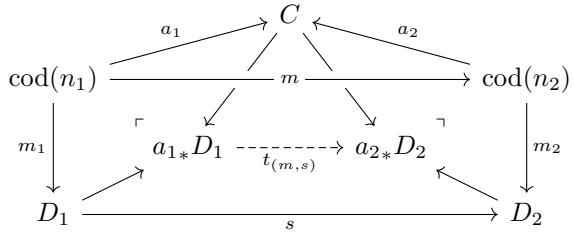


That is, $m = b_* m_0 = a_* c_* m_0$; moreover, any such two lifts of b are identified in the canonical cone of C, hence so are the induced pushouts of m_0. Hence the functor $\langle a_* \rangle_{(n,a)\in\int \not\perp_l}$ is essentially surjective on objects.

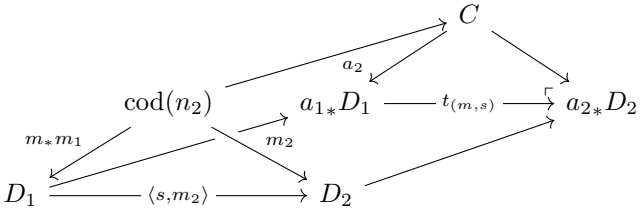
Now we want for any $((n_1, a_1), m_1),\ ((n_2, a_2), m_2)$ an isomorphism

$$\operatorname*{colim}_{(n,a)\in\int \not\perp_l} \mathcal{V}_{\mathrm{cod}(n)}[((n_1, a_1), m_1),\ ((n_2, a_2), m_2)] \simeq \mathcal{V}_C[a_{1*} m_1, a_{2*} m_2]$$

In one direction, to any $(m, s) : ((n_1, a_1), m_1) \to ((n_2, a_2), m_2)$ we can associate the unique map $t_{(m,s)}$ induced as below from the universal property of the pushout

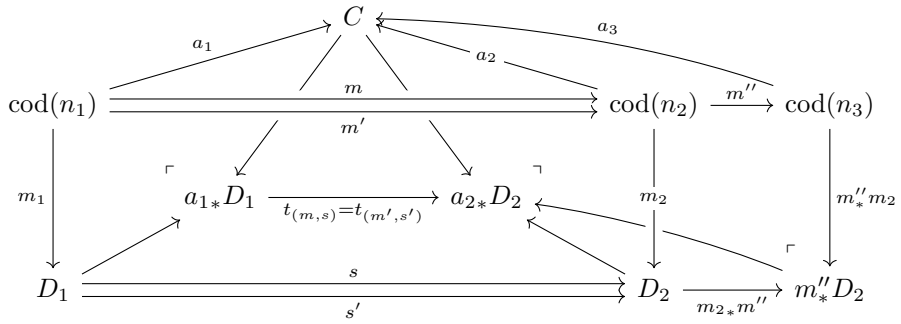


Observe that this map is also obtained as the pushout $t_{(m,s)} = ((m_* m_2)_* a_2)_* \langle s, m_2 \rangle$ as below



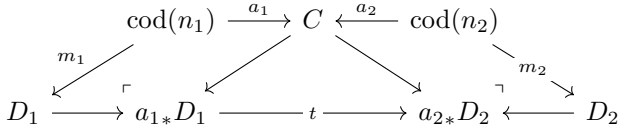
where right cancellation of pushouts makes the bottom square a pushout and $\langle s, m_2 \rangle$ is the map uniquely induced by s.

Suppose that we have $(m, s), (m', s')$ such that $t_{(m,s)} = t_{(m',s')}$. Then for $\int \luparrow_l$ is filtered, there is some (n_3, a_3) in $\int \luparrow_l$ such that $m''m = m''m'$. Then we have that $m_{2*}m''sm_1 = m''_* m_s m''m = m''_* m_s m''m' = m_{2*}m''s'm_1$ as seen in the following diagram

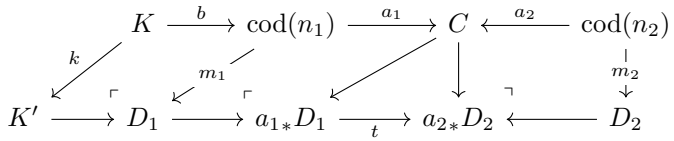


and therefore (m, s) and (m', s') are equalized in oplaxcolim $_{(n,a)\in\int \luparrow_l} \mathcal{V}_{\mathrm{cod}(n)}$ by the opcartesian morphism $(m'', m_{2*}m'')$. But the latter becomes invertible in the pseudocolimit, where the morphisms (m, s) and (m', s') are hence identified. Hence the faithfulness.

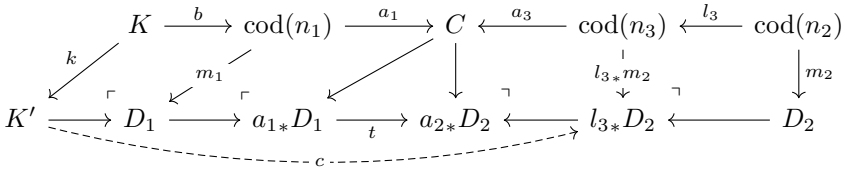
Now in a a situation as below



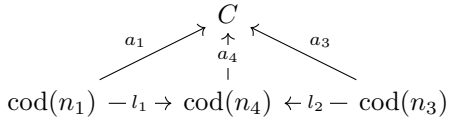
if m_1 is induced through pushouts from map $k : K \to K'$ in $\mathcal{V}$ as below, we get a composite arrow



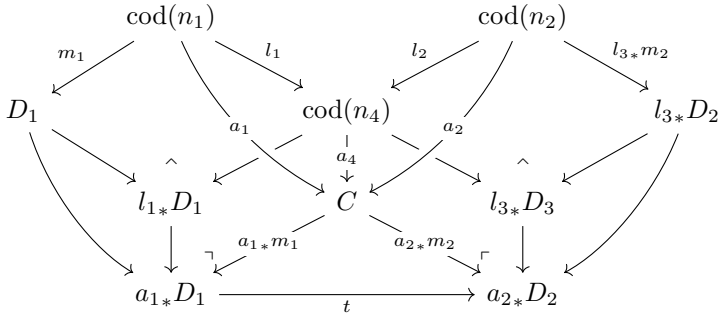
Now by lemma 1.1.2.7 there exists (a_3, n_3) in $\int \luparrow_l$ and $l_3 : (n_2, a_2) \to (n_3, a_3)$ such that we have a factorization



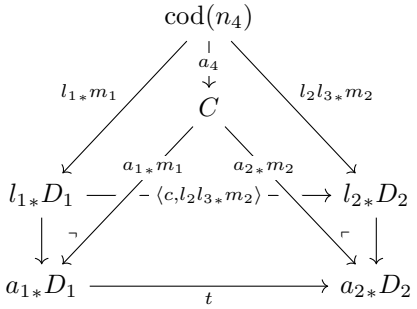
Then again by filteredness of $\int \luparrow_l$, there exists (n_3, a_3) and a span



along which we can consider pushouts



But now for $l_{1*}D_1 = l_1 b_* K'$, we have a universal map $\langle c,\, l_2 l_{3*} m_2 \rangle : l_{1*}D_1 \to l_{3*}D_3$, and moreover in the diagram below



the bottom square is a forced to be a pushout by right cancellation. This exhibits t as the induced maps $t_{(1_{n_4}, \langle c,\, l_2 l_{3*} m_2 \rangle)}$, and to conclude, observe that $((n_1, a_1), m_1)$ and $((n_4, a_4), l_{1*} m_1)$ are related by an opcartesian morphism, while on the other hand $((n_2, a_2), m_2)$ and $((n_4, a_4), l_2 l_{3*} m_2)$ are also related by an opcartesian morphism, and are then identified in the pseudocolimit. Hence the pair $(1_{n_4}, \langle c,\, l_2 l_{3*} m_2 \rangle)$ can be seen as an antecedent of t in the homset above. Thence the fullness.

Therefore the functor $\mathrm{colim}_{(n,a) \in \int \nleftarrow_l} \mathcal{V}_{\mathrm{cod}(n)} \to \mathcal{V}_C$, being at the same time essentially surjective, full and faithful, defines an equivalence of category as desired. □

Remark 1.1.4.4. Observe that the condition of right cancellation of the maps in $\mathcal{V}$ seems of no use in the result above, which could apply actually to any class of maps that is closed under pushout along arbitrary map. This theorem will be useful to get an expression of the spectral site of an arbitrary etale arrow in theorem 7.1.2.7.

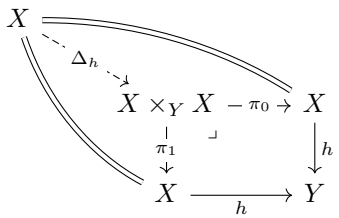
1.2 Etale classes

In [33], the notion of factorization system is substituted with the notion of *etale class*, which is defined inside a Grothendieck topos. This method allows not to be dependent on semantical data, as the etale class exists independently either of the eventual set-valued models of the ambient theory or the site presentation. This becomes necessary for the more general method developed in [33] where one starts from an arbitrary topos that may not classify a finite limit theory. However as we shall see, etale classes and left-generated factorization systems are related, for a saturated class generating a left factorization system also generates an etale class, while an etale class restricts on a saturated class for a choice of lex site of presentation.

In this section, we recall general facts about etale classes, and describe the relation with saturated class. Beware that saturated classes are "left-like" for they are defined as maps amongst finitely presented points, while the axioms of etale classes are "right-like", though they are actually on "the same side": this is because for a given theory $\mathbb{T}$, a saturated class in $\mathbb{T}[\mathcal{S}]$ lives in $\mathbb{T}[\mathcal{S}]_\omega \simeq \mathcal{C}_{\mathbb{T}}^{\mathrm{op}}$, and the associated etale class will live in the classifier $\mathcal{S}[\mathbb{T}]$, and $\mathcal{C}_{\mathbb{T}} \hookrightarrow \mathcal{S}[\mathbb{T}]$, so that the left-like properties in $\mathbb{T}[\mathcal{S}]$ correspond to right-like properties in $\mathcal{S}[\mathbb{T}]$. In other words, a saturated class and its associated left class live between points of a topos, while the corresponding etale class lives amongst objects of the topos. However etale classes require also more axioms - that are neither specifically left or right-like - which precise their interaction with epimorphisms. For more on the theory of etale classes, see [33] and [54].

1.2.1 Properties of etale classes

1.2.1.1. In the following we fix a Grothendieck topos $\mathcal{E}$. Moreover, recall in a category $\mathcal{C}$ with pullbacks, the *diagonal* of a morphism $h : X \to Y$ is the map $\Delta_h : X \to X \times_Y X$ induced as follows



Moreover, for any h, the diagonal Δ_h is always a monomorphism.

Definition 1.2.1.2. An *etale class* in $\mathcal{E}$ is a class of maps $\mathscr{H} \subseteq \mathcal{E}^2$ such that:

- $\mathscr{H}$ is closed under composition
- $\mathscr{H}$ contains isomorphisms
- $\mathscr{H}$ is closed under diagonals: if $h \in \mathscr{H}$, then $\Delta_h \in \mathscr{H}$
- $\mathscr{H}$ is closed under pullback along arbitrary maps
- $\mathscr{H}$ *descends* along epimorphisms: if $e : Z \twoheadrightarrow Y$ is an epimorphism and $h : X \to Y$ is such that $e^*h \in \mathscr{H}$, then $h \in \mathscr{H}$
- if in the triangle below $e : Z \to X$ is an epimorphism and both g and Δ_h are in $\mathscr{H}$

$$\begin{array}{ccc} Z & \xrightarrow{\quad e \quad} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array}$$

  then h itself is in $\mathscr{H}$
- $\mathscr{H}$ is closed under coproducts in $\mathcal{E}^2$: if $(h_i : X_i \to Y_i)_{i\in I}$ is a family of maps in $\mathscr{H}$, the coproduct maps $\coprod_{i\in I} h_i : \coprod_{i\in I} X_i \to \coprod_{i\in I} Y_i$ is also in $\mathscr{H}$
- for any I, the map $\coprod_{i\in I} 1 \to 1$ is in $\mathscr{H}$.

Remark 1.2.1.3. The two last axioms entail in particular that for a family $(h_i : X_i \to X)_{i\in I}$ of maps in $\mathscr{H}$, the universal arrow $\langle h_i\rangle_{i\in I} : \coprod_{i\in I} X_i \to X$ is also in $\mathscr{H}$. As a consequence of a property below, the inclusions $q_i : X_i \to \coprod_{i\in I} X_i$ will also be forced to be in $\mathscr{H}$.

Proposition 1.2.1.4. *If $\mathcal{H}$ is an etale class, then for any X in E and any diagram of maps $(h_i : H_i \to X)_{i\in I}$ in $\mathscr{H}$, the universal map induced from the colimit $\langle h\rangle_{i\in I} : \mathrm{colim}_{i\in I} H_i \to X$ is in $\mathscr{H}$*

Proof. By the properties of etale maps, we can see in the diagram below

$$\begin{array}{ccccc} H_i & \xrightarrow{q_{(H,h)}} & \coprod_{i\in I} H_i & \twoheadrightarrow & \operatorname{colim}_{i\in I} H_i \\ & {\scriptstyle h_i}\searrow & \downarrow{\scriptstyle \langle h_i\rangle_{i\in I}} & \swarrow{\scriptstyle \langle h_i\rangle_{i\in I}} & \\ & & X & & \end{array}$$

that the map induced from the coproduct is etale by the remark above, as each h_i is, and hence, so is the map induced from the colimit as the colimit is obtained from the coproduct through an epimorphism. □

Proposition 1.2.1.5. *Let $\mathscr{H}$ be an etale class; then in a triangle as follows*

$$\begin{array}{ccc} Z & \xrightarrow{\quad f \quad} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array}$$

we have g and Δ_h in $\mathscr{H}$, then f is also in $\mathscr{H}$. In particular, any etale class is left-cancellative.

For a complete proof, see [33][Proposition 2.2(a)]. The left cancellativity follows from $g \in \mathscr{H}$ implies that $\Delta_g \in \mathscr{H}$. Moreover, the following says that an etale class has also some partial right-cancellation along epimorphisms that are also etale (this is [33][Proposition 2.5]):

Proposition 1.2.1.6. *Let $\mathscr{H}$ be an etale class; then in a triangle as follows*

$$\begin{array}{ccc} Z & \xrightarrow{\quad e \quad} & X \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle h} & \\ & Y & \end{array}$$

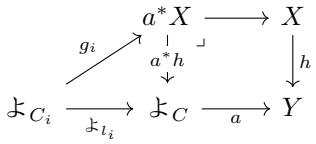
if both e and g are in $\mathscr{H}$ and moreover e is an epimorphism, then h is in $\mathscr{H}$.

1.2.2 The etale class generated from a saturated class

1.2.2.1. From the nature of the axioms given at definition 1.2.1.2, it appears that any class of maps in a Grothendieck topos can be completed into an etale class: for a class of maps $\mathscr{H}_0$ in $\mathcal{E}$, there exists a smallest etale class $\overline{\mathscr{H}_0}$ such that $\mathscr{H}_0 \subseteq \overline{\mathscr{H}_0}$. Conversely, for $\mathscr{H}$ an etale class in a Grothendieck topos and $(\mathcal{C}, J)$ a small site of presentation for $\mathcal{E}$, we denote as $\mathscr{H} \cap \mathcal{C}^2$ the set of arrows $\{f \in \mathcal{C}^2 \mid \text{よ}_f \in \mathscr{H}\}$. In particular, for a convenient site of presentation, it is possible to characterise the etale class generated from a class of map living in the site:

Proposition 1.2.2.2. *Let be $(\mathcal{C}, J)$ a small lex site of presentation for $\mathcal{E}$ and Λ a class of maps in $\mathcal{C}$ such that J is generated in Λ. Denote $\mathscr{H}$ the etale class generated by the maps of the form よ_f for $f \in \Lambda$. Then a map $h : X \to Y$ is in $\mathscr{H}$ if and only the following two conditions holds:*

- *for each $a : \text{よ}_C \to Y$ with $C \in \mathcal{C}$, the pullback projection a^*h is in $\mathscr{H}$, and for each of those pullbacks, there exists some family of maps $(l_i : C_i \to C)_{i \in I}$ in Λ and for each $i \in I$ a lift $g_i : \text{よ}_{C_i} \to X$ such that we have a factorization*



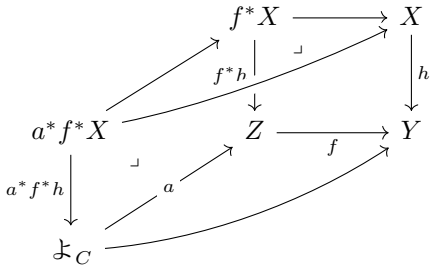
  and $\langle g_i \rangle_{i \in I} : \coprod_{i \in I} \text{よ}_{C_i} \to a^*X$ is an epimorphism;

- *the diagonal $\Delta_h : X \to X \times_{\text{よ}_C} X$ satisfies the same condition.*

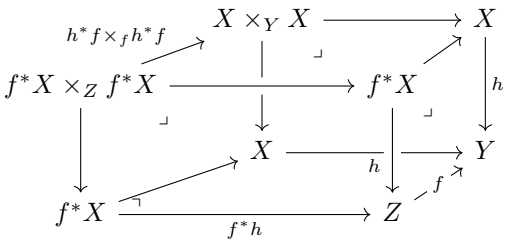
Remark 1.2.2.3. The first condition says in some sense that a map is locally in $\mathscr{H}$, while the second ensures the closure under diagonals.

Proof. This is the content of [33][Proposition 5.1.1]; however, as this statement is not proved there, it is worth giving a complete proof. We have to check that the class of maps having the property above is etale. We check that it satisfies the different axioms. In the following of the proof, we denote as $\mathscr{H}_0$ this class.

First, it is clear that this class contains isomorphisms. Now, Before the other axioms, we need stability under pullback. Let be $h : X \to Y$ already in $\mathscr{H}_0$. Then for $f : Z \to Y$, $C \in \mathcal{C}$ and $a : \text{よ}_C \to Z$, cancellation of pullbacks tells us that in the following diagram

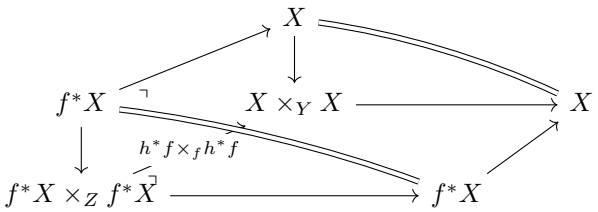


a^*f^*h is in $\mathscr{H}_0$ for h is in $\mathscr{H}_0$. Hence f^*h is locally in $\mathscr{H}_0$. Now we must prove that Δ_{f^*h} is also in $\mathscr{H}_0$. Consider the following diagram:



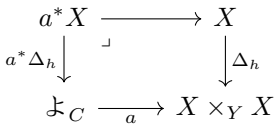
where the map $h^*f \times_f h^*f$ denotes the unique map induced by the property of the pullback. Then

we see that $\Delta_{f^*h} = (h^*f \times_f h^*f)^*\Delta_h$ by cancellation of pullback applied to the following diagram

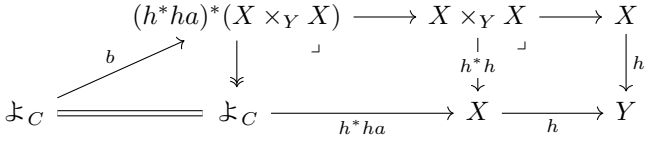


for $(h^*f \times_f h^*f)^* 1_X = 1_{f^*X}$. Hence Δ_{f^*h} is in $\mathscr{H}_0$ for Δ_h is and the previous statement.

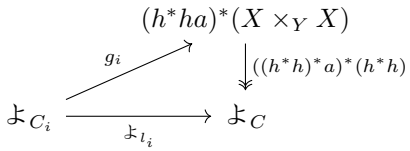
As to diagonals: suppose that $h : X \to Y$ is in $\mathscr{H}_0$; then for any C in $\mathscr{H}$ and any $a : \text{よ}_C \to X \times_Y X$ the following pullback



We have to check this pullback is in $\mathscr{H}_0$. We can use the fact that in the following diagram



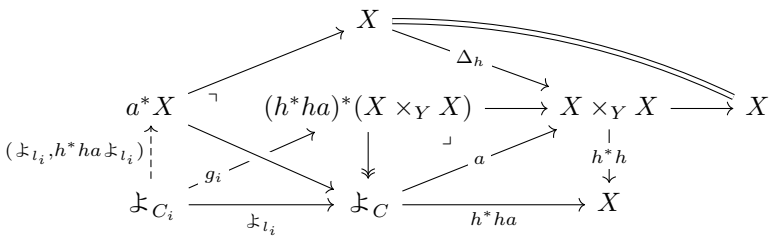
the pullback projection h^*h is in $\mathscr{H}_0$, and so is $((h^*h)^*a)^*(h^*h)$. As it is also in $\mathscr{H}_0$ with a representable codomain, there exists a family $(l_i : C_i \to C)_{i\in I}$ in Λ and $g_i : \text{よ}_{C_i} \to a^*X$ such that we have a factorization



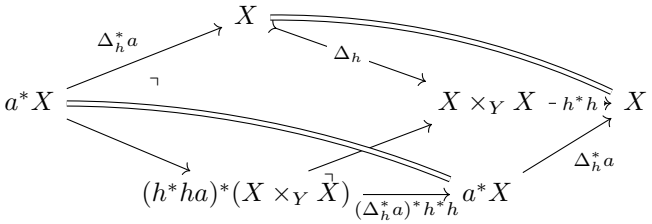
and an epimorphism

$$\coprod_{i\in I} \text{よ}_{C_i} \xrightarrow{\langle g_i\rangle_{i\in I}} (h^*ha)^*(X \times_Y X)$$

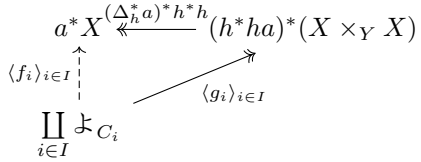
Now observe that each g_i factorizes by the universal property of the pullback trough a map $(\text{よ}_{l_i}, h^*ha\,\text{よ}_{l_i}) : \text{よ}_{C_i} \to a^*X$ as below



while there is also a canonical factorization so that the epimorphism above factorizes as $\langle f_i\rangle_{i\in I}$. Now we can construct an epimorphism $(h^*ha)^*(X \times_Y X) \twoheadrightarrow a^*X$ as the pullback $(\Delta_h^* a)^* h^* h$ as seen below

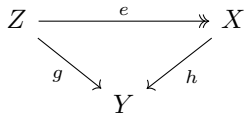


where h^*h is an epimorphism as being a retract of the diagonal. Hence the family $(l_i)_{i\in I}$ together with the composite epimorphism

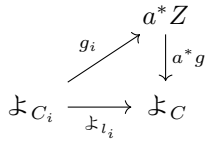


witnesses that $a^*\Delta_h$ is in $\mathscr{H}_0$, and then h is itself shown to be locally in $\mathscr{H}_0$. Finally we know that Δ_{Δ_h} is trivially in $\mathscr{H}$ as it is an isomorphism, for Δ_h is monic.

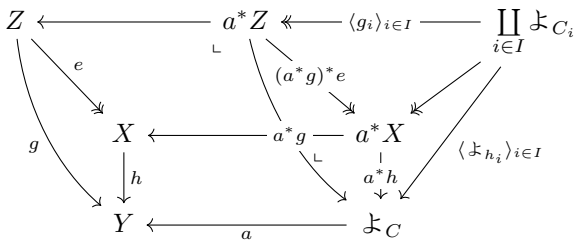
Now we prove cancellation along epimorphisms. Let be



with g and Δ_h in $\mathscr{H}_0$. Then for $C \in \mathcal{C}$ and $a : \text{よ}_C \to Y$, we know from the previous item that a^*g is a map in $\mathscr{H}_0$ with representable codomain, and therefore there exists $(l_i : C_i \to C)_{i\in I}$ in Λ such that for each $i \in I$ one has

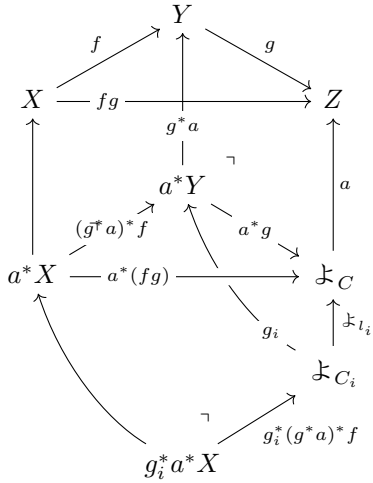


and $\langle g_i \rangle_{i\in I}$ is epic. Then the composite family $(\text{よ}_{l_i})_{i\in I}$ together with the composite $(f^*a)^*eg_i$ for $i \in I$ provide a family as desired as seen in the diagram below

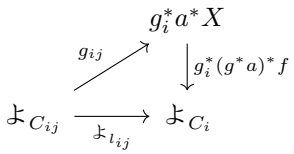


proving a^*h to be locally in $\mathscr{H}_0$, and hence h itself so. Moreover, by hypothesis, Δ_h is already in $\mathscr{H}_0$.

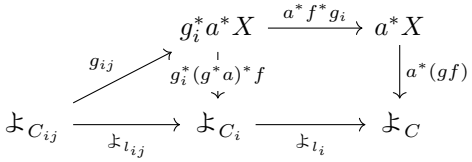
Now we prove stability under composition. Let be $f : X \to Y$ and $g : Y \to Z$ in $\mathscr{H}_0$. We prove that gf is locally in $\mathscr{H}_0$. Let be $a : \text{よ}_C \to Z$: then a^*g is in $\mathscr{H}_0$, so there exists a family $(l_i : C_i \to C)_{i\in I}$ and $(g_i : \text{よ}_{C_i} \to a^*Y$ satisfying the condition of map $\mathscr{H}_0$ with representable codomain; but then for each $i \in I$ form the pullback



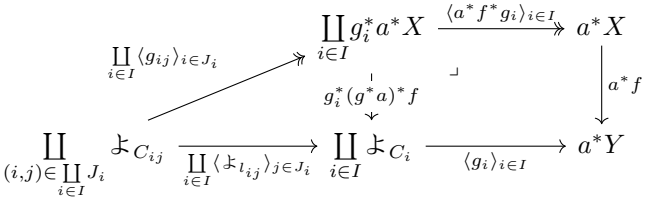
where $g_i^*(g^*a)^*f$ is in $\mathscr{H}_0$ as f is. Hence in each $i \in I$ we have a family $(l_{ij} : C_{ij} \to C_i)_{j \in J_i}$ and $(g_{ij} : \text{ﻉ}_{C_{ij}} \to g_i^* a^* X)_{j \in J_i}$ such that we have a factorization



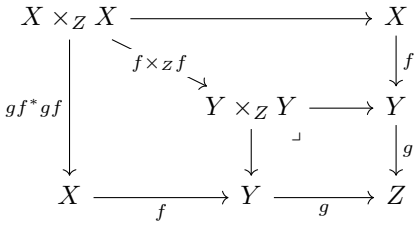
and $\langle g_{ij} \rangle_{j \in J_i} : \coprod_{j \in J_i} \text{ﻉ}_{C_{ij}} \twoheadrightarrow g_i^* a^* X$ is epic. Then the gluing of the composite families $(l_i l_{ij} : C_{ij} \to C)_{(i,j) \in \coprod_{i \in I} J_i}$ and the maps $(g_i g_{ij})_{(i,j) \in \coprod_{i \in I} J_i}$ together exhibits $a^*(gf)$ as being in $\mathscr{H}_0$ for we have in each $(i,j) \in \coprod_{i \in I} J_i$ a factorization



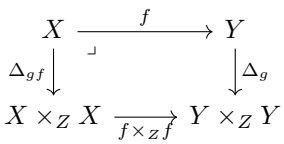
an epimorphism as below



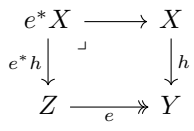
by stability of coproducts in a Grothendieck topos: hence gf is locally in $\mathscr{H}_0$. Now we have to prove that Δ_{gf} also is locally in $\mathscr{H}_0$. To see why, consider the map $f \times_Z f$ induced from the universal property of the pullback in the following diagram



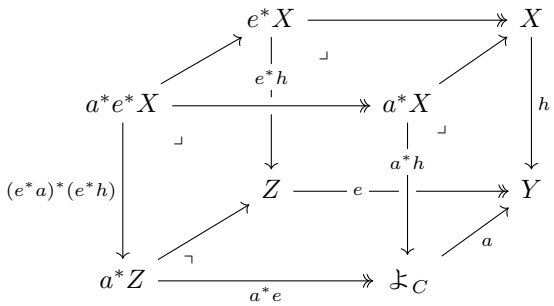
Now this allows us to exhibit Δ_{gf} as the following pullback, where and Δ_g is in $\mathscr{H}_0$.



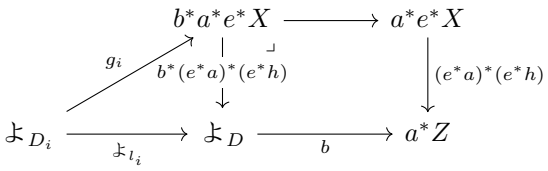
Now we prove descent along epimorphisms. Let be a diagram as below



with e^*h in $\mathscr{H}_0$. Then for any $a : \text{ﻉ}_C \to Y$ form the pullback cube



where all the horizontal arrows are epimorphisms. Then the map $(e^*a)^*(e^*h)$ is in $\mathscr{H}_0$ as e^*h is, as its pullback along all the maps $b : \text{ﻉ}_D \to a^*Z$ which moreover have representable codomain. Hence for each such pair (D, d) in the comma category $\text{ﻉ} \downarrow a^*Z$ we have a family $((l_i : D_i \to D)_{i \in I_{(D,d)}}$ in Λ and $(g_i : \text{ﻉ}_{D_i} \to b^*a^*e^*Z)$ such that



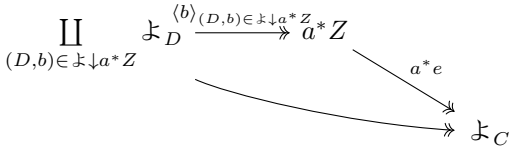
and we have an epimorphism

$$\coprod_{i \in I} \text{ﻉ}_{D_i} \xrightarrow{\langle g_i \rangle_{i \in I}} b^*a^*e^*X$$

But now, as $\mathcal{C}$ is a generator for $\mathcal{E}$, the comma category $\text{ﻉ} \downarrow a^*Z$ indexes a canonical diagram and we have also an epimorphism

$$\coprod_{(D,b) \in \text{ﻉ} \downarrow a^*Z} \text{ﻉ}_D \xrightarrow{\langle b \rangle_{(D,b) \in \text{ﻉ} \downarrow a^*Z}} a^*Z$$

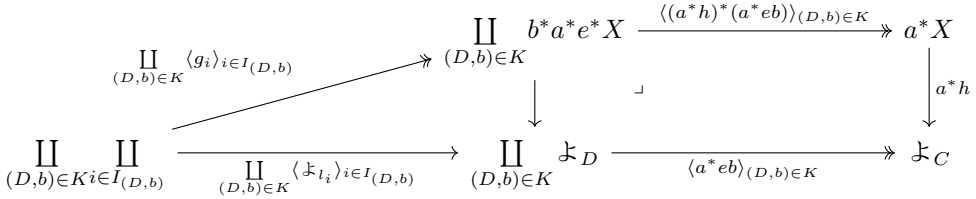
Now if we compose this epimorphism with the epimorphism a^*e we get an epimorphism



Now, by fullness of the Yoneda embedding, each composite $a^*eb : \text{ﻉ}_D \to \text{ﻉ}_C$ comes from some $f_{D,d} : D \to C$ in $\mathcal{C}$; however we cannot infer that it is in Λ. But this later condition means there is a subset $K \subseteq \text{ﻉ} \downarrow a^*Z$ such that the family $(f_{(D,b)})_{(D,b) \in K}$ is in $J(C)$, and then from the hypothesis that J is generated in Λ, we can choose this family to be actually in Λ; then we have an epimorphism in $\mathcal{E}$

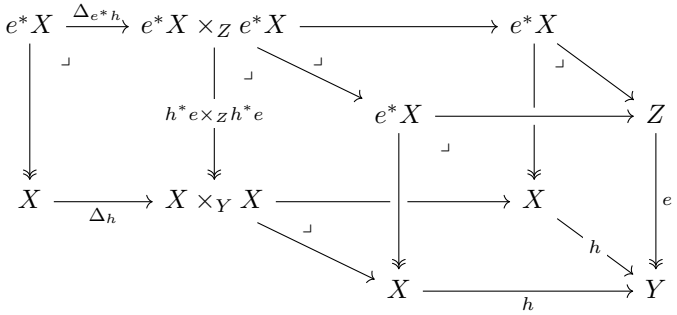
$$\coprod_{(D,b) \in K} \text{ﻉ}_D \xrightarrow{\langle \text{ﻉ}_{f_{(D,b)}} \rangle_{(D,b) \in K}} \text{ﻉ}_C$$

where actually each $\text{ﻉ}_{f_{(D,b)}} = a^*eb$. Hence we have an epimorphism consisting of the upper row of the diagram below



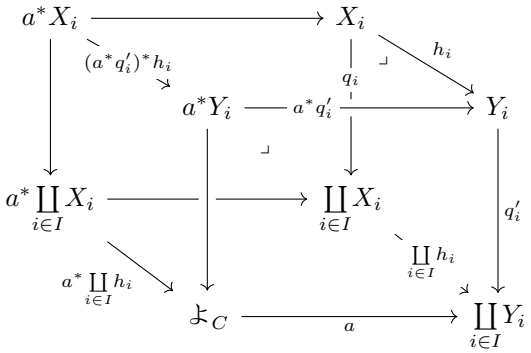
and moreover each composite arrow $f_{(D,b)}l_i$ is in Λ. This proves that a^*h is in $\mathscr{H}_0$, and hence that h is locally in $\mathscr{H}_0$.

Now we have to prove that the diagonal Δ_h also is locally in $\mathscr{H}_0$: but we saw when proving stability under pullback that the diagonal of the pullback map is computed as the pullback



where all the vertical arrows are pullback; but the map $h^*e \times_Z h^*e$ is epic, and Δ_{e^*h} is in $\mathscr{H}_0$ for e^*h is so by hypothesis: but then one has to apply again the previous strategy to show that Δ_h is locally in $\mathscr{H}_0$.

Now for closure under coproducts, let be $(h_i : X_i \to Y_i)_{i\in I}$ a small family in $\mathscr{H}_0$; then for any arrow $a : \text{ɟ}_C \to \coprod_{i\in I} Y_i$ and any $i \in I$ consider the pullback cube



where every square is a pullback (for the right square, this is a consequence of extensivity of $\mathcal{E}$) and the arrow $(a^*q_i')^*h_i$ is in $\mathscr{H}_0$ as each h_i is. Now the strategy is the same as for descent along epimorphisms, which the only difference being that the intermediate epimorphisms we use is actually the isomorphism $\coprod a^*Y_i \simeq \text{ɟ}_C$. Now to see that the diagonal $\Delta_{\coprod_{i\in I} h_i}$ is in $\mathscr{H}_0$, observe that by stability of coproducts we have a pullback

$$\begin{array}{ccc} \coprod_{i\in I} X_i \times_{Y_i} X_i & \longrightarrow & \coprod_{i\in I} X_i \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \coprod_{i\in I} h_i} \\ \coprod_{i\in I} X_i & \xrightarrow[\coprod_{i\in I} h_i]{} & \coprod_{i\in I} Y_i \end{array}$$

and

$$\Delta_{\coprod_{i\in I} X_i} = \coprod_{i\in I} \Delta_{h_i}$$

But as each h_i is in $\mathscr{H}_0$, hence by the first part of this item we know that the diagonal is so.

Finally, for I a set, the canonical I-indexed epimorphism $\coprod_{i\in I} 1 \twoheadrightarrow 1$ is a case of a morphism with representable codomain for $1 = \text{ɟ}_1$ since the Yoneda embedding preserves the terminal object, and the equality together with the canonical inclusion $q_i : 1 \to \coprod_{i\in I} 1$ witness that this map is locally in $\mathscr{H}_0$. The argument for the diagonal is similar to the previous item.

With all of this, we proved that $\mathscr{H}_0$ is an etale class cointaining the maps of the form ɟ_l for $l \in \Lambda$. But now, any etale class that contains those also contains $\mathscr{H}_0$: indeed, any etale class satisfies in particular the closure properties of $\mathscr{H}_0$ for it is stable under pullback and diagonals and has the cancellation property along epimorphisms. Hence $\mathscr{H}_0$ is the etale class generated from Λ, which is $\mathscr{H}$. □

Etale classes are transferable along geometric morphisms:

1.2.2.4. Let be $f : \mathcal{F} \to \mathcal{E}$ a geometric morphism, and $\mathscr{H}$ and $\mathscr{G}$ two etale classes in $\mathcal{F}$ and $\mathcal{E}$ respectively. Then denote as

- $f(\mathscr{H})$ the class in $\mathcal{E}$ of maps $h \in \mathcal{E}^2$ such that $f^*(h)$ is in $\mathscr{H}$;
- $f^{-1}(\mathscr{G})$ the class in $\mathcal{F}$ of maps of the form $f^*(h)$ with h in $\mathscr{G}$.

Then both $f(\mathscr{H})$ and $f^{-1}(\mathscr{G})$ are etale classes. Moreover, one has

$$f^{-1}f(\mathscr{H}) \subseteq \mathscr{H} \qquad \mathscr{G} \subseteq ff^{-1}(\mathscr{G})$$

Chapter 2

Stable functors and Right Multi-adjoints

In this chapter we give a detailed account of the notion of right multi-adjoint and the related notions of local adjoint and stable functor. This notion is at the core of [31] construction of the spectrum, and we shall see that it is also involved in the other way of constructing the spectrum, especially in the next chapter where we construct multireflections froms geometries.

Though the theory of multi-adjoints is known thanks to Diers, the main sources about it are not that easily available, and since this notion will play an enlightening role in this thesis, we find appropriate to devote some efforts to it in a mostly expository chapter, in which we shall nevertheless give some result which seems to be absent elsewhere; further connections with the notion of geometry will be investigated in chapter 4.

We first give a very formal, categorical approach to the key notions and their different characterizations. In particular we provide a Beck-Chevalley result at theorem 2.1.1.5 relating the slice-wise adjunctions in a local right adjoint, and prove once for all that stable functors are the same as local right adjoint at theorem 2.1.4.6. We also discuss inheritance of (co)limits in multireflective subcategories at theorem 2.1.3.15.

In the second section we discuss the characterization of right multi-adjoints through the free product completion; though it will play no technical role in this thesis, we find this result to have an important meaning: it is in some sense the simplest way to correct a multi-adjoint into an adjoint (see proposition 2.2.2.3), which is also the purpose of the spectrum - though the latter does so in the most universal way. We shall see in chapter 8 that the adjunction induced by a right multi-adjoint through the free product completion is in some sense the restriction of Diers spectral adjunction to discrete spaces, see proposition 8.2.4.5.

We also discuss in the last section factorizations and orthogonality aspects involved in a situation of multi-adjunction, which will play in important technical role in chapter 4 and 8.

2.1 Local right adjoints and stable functors

In this first section, we recall our three notions of interest, namely local right adjoints, right multi-adjoints and stable functors. We first give some technical points about the behavior of the local units of the local adjunctions. We also prove that for a local right adjoint, the local adjunctions enjoy automatically a Beck-Chevalley condition, which was seemingly unnoticed until now. Then we turn to different characterizations of local adjointness in term of nerves and initial family, and introduce the stronger notion of right multi-adjoint and recall a variant of Freyd adjoint functor theorem for multi-adjoint. Finally we turn to the notion of stable functor, as studied by Taylor in [93], and also in [96], and we prove equivalence with the notion of local right adjoint.

2.1.1 Local right adjoints and Beck-Chevalley condition

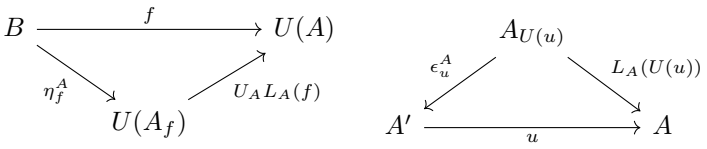
Local right adjoint are functors that, while lacking a global left adjoint, have slice-wise left adjoints at each object of their codomain.

Definition 2.1.1.1. A functor $U : \mathcal{A} \to \mathcal{B}$ is said to be a *local right adjoint* if for each object A of $\mathcal{A}$ the restriction of U to the slice $\mathcal{A}/A$ has a left adjoint

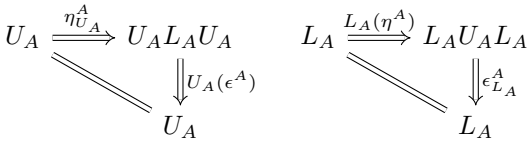
$$\mathcal{A}/A \underset{U/A}{\overset{L_A}{\rightleftarrows}} \mathcal{B}/U(A) \quad (\perp)$$

where we denote A_f the domain of the arrow $L_A(f)$ in $\mathcal{A}/A$. In the following we shall also denote U/A as U_A for concision. The maps η_f^A for $f : B \to U(A)$ are called *local units* under B. We shall refer to the family of local units under a given object as its *cone of local units*.

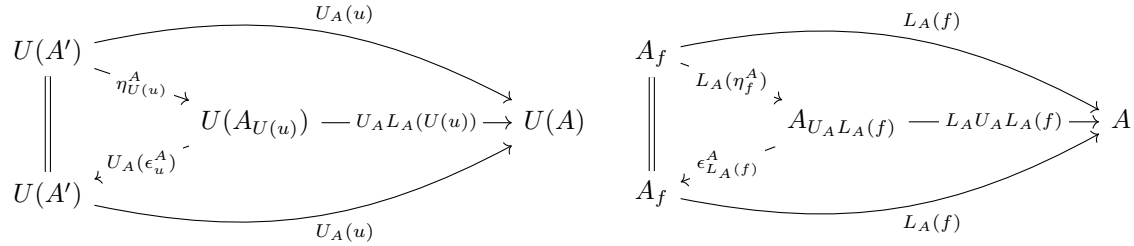
2.1.1.2. The definition of a local right adjoint means that for any arrow $f : B \to U(A)$ in $\mathcal{B} \downarrow U$ and $u : A' \to A$ in $\mathcal{A}/A$ we have triangles in $\mathcal{B}$ and $\mathcal{A}$ respectively



Moreover those natural unit and counit must satisfy the triangle identities



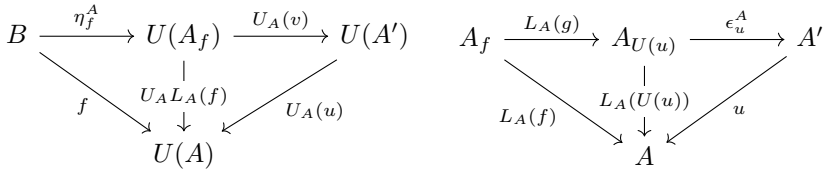
which incarnate respectively as the following retractions



In particular the local adjunction at A

$$\mathcal{A}/A[L_A(f), u] \simeq \mathcal{B}/U(A)[f, U(u)]$$

sends an arrow $v : L_A(f) \to u$, resp. an arrow $g : f \to U(u)$, to the composite triangle on the left, resp. on the right



Remark 2.1.1.3. Beware that, in general, we cannot enforce the counits to be pointwise iso, that is, to require each U_A to be full and faithful. Hence the factorization of a morphism in the range of U may not be trivial. Morally, the factorization through the unit only takes in account the object of $\mathcal{A}$ whose strict image is the codomain, while, even when the domain is in the image of U, the factorization may not remember the precise object in $\mathcal{A}$ it comes from.

Remark 2.1.1.4. For any $u : A_1 \to A_2$ in $\mathcal{A}$, functoriality of U makes the following square commute up to equality

$$\begin{array}{ccc} \mathcal{A}/A_1 & \xrightarrow{U_{A_1}} & \mathcal{B}/U(A_1) \\ {\scriptstyle A/u}\downarrow & = & \downarrow{\scriptstyle \mathcal{B}/U(u)} \\ \mathcal{A}/A_2 & \xrightarrow[U_{A_2}]{} & \mathcal{B}/U(A_2) \end{array}$$

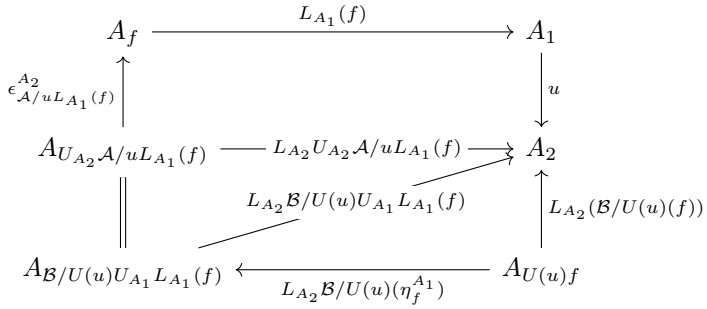
This equality 2-cell possesses a *Beck-Chevalley mate*

$$\begin{array}{ccc} \mathcal{A}/A_1 & \xleftarrow{L_{A_1}} & \mathcal{B}/U(A_1) \\ {\scriptstyle A/u}\downarrow & \overset{\sigma^u}{\Leftarrow} & \downarrow{\scriptstyle \mathcal{B}/U(u)} \\ \mathcal{A}/A_2 & \xleftarrow[L_{A_2}]{} & \mathcal{B}/U(A_2) \end{array}$$

defined as the composite

$$L_{A_2}\mathcal{B}/U(u) \xrightarrow{L_{A_2}\mathcal{B}/U(u)(\eta^{A_1})} L_{A_2}\mathcal{B}/U(u)U_{A_1}L_{A_1} = L_{A_2}U_{A_2}\mathcal{A}/uL_{A_1} \xrightarrow{\epsilon^{A_2}_{\mathcal{A}/uL_{A_1}}} \mathcal{A}/uL_{A_1}$$

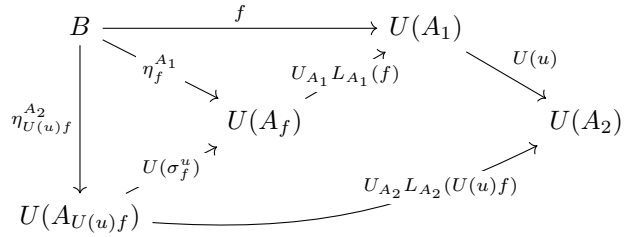
This mate relates in a canonical way the unit of any $f : B \to U(A_1)$ and the unit of the composite $U(u)f : B \to U(A_2)$ as seen in the following diagram



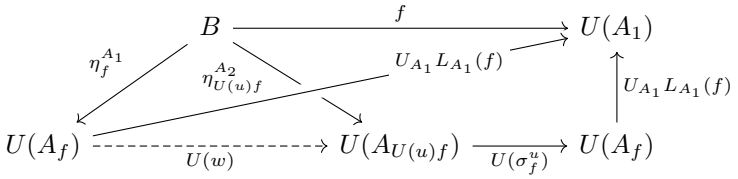
But surprisingly, this mate is automatically an isomorphism because of the universal property of the units, as stated in the following proposition:

Theorem 2.1.1.5. *Let $U : \mathcal{A} \to \mathcal{B}$ be a local right adjoint. Then for any $u : A_1 \to A_2$ in $\mathcal{A}$, we have the Beck-Chevalley condition at u, that is, the canonical transformation σ^u is a point-wise isomorphism.*

Proof. Remark that for each $u : A_1 \to A_2$ and $f : B \to U(A_1)$, the morphism $\sigma^u_f : L_{A_2}(\mathcal{B}/U(u)(f)) \to L_{A_1}(f)$ is in $\mathcal{A}$, and we have a factorization



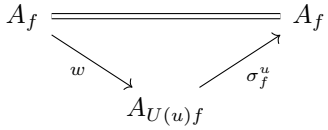
Observe that σ^u_f is the unique arrow in $\mathcal{A}$ provided by the universal property of the unit $\eta^{A_2}_{U(u)f}$ at $\eta^{A_1}_f$ seen as an arrow $U(u)f \to U(u)U_{A_1}L_{A_1}(f)$ in $\mathcal{B}/U(A_2)$. But on the other hand, by the universal property of $\eta^{A_1}_f$ at $\eta^{A_2}_{U(u)f}$ seen as an arrow $f \to U_{A_1}L_{A_1}(f)U(\sigma^u_f)$ in $\mathcal{B}/U(A_1)$, there exists a unique arrow $w : A_f \to A_{U(u)f}$ in $\mathcal{A}$ such that



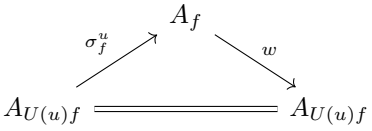
Now we prove that w and σ^u_f are mutual inverses in $\mathcal{A}$. First, as

$$U(\sigma^u_f)\eta^{A_2}_{U(u)f} = \eta^{A_1}_f$$

and $1_{L_{A_1}(f)}$ is the unique map induced by $\eta^{A_1}_f$ seen as an arrow $f \to U_{A_1}L_{A_1}(f)$, then necessarily we have a retraction in $\mathcal{A}$



but again, as now $\eta^{A_2}_{U(u)f} = U(w)\eta^{A_1}_f$ and $1_{L_{A_2}(U(u)f)}$ is the unique map induced by $\eta^{A_2}_{U(u)f}$ as an arrow $U(u)f \to U_{A_2}L_{A_2}(U(u)f)$, we have a retraction in $\mathcal{A}$

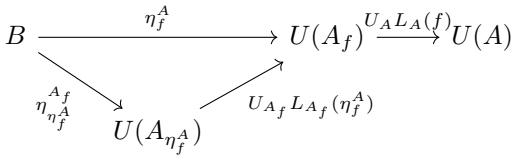


and σ^u_f defines both an iso $A_{U(u)f} \simeq A_f$ in $\mathcal{A}$ and $L_{A_2}\mathcal{A}/u(f) \simeq \mathcal{B}/U(u)L_{A_1}(f)$ which can be shown to be natural. □

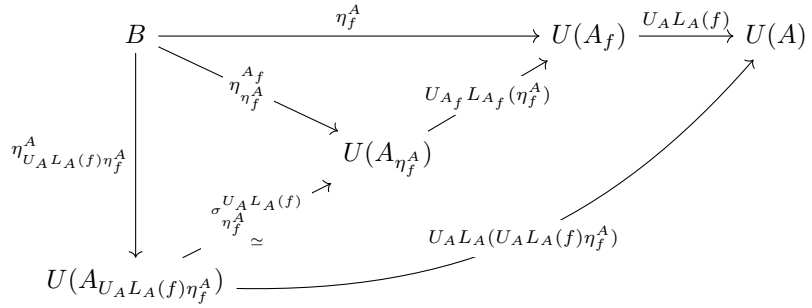
Remark 2.1.1.6. Beck-Chevalley condition says that factorization through local units are not modified by postcomposing with arrows in the range of U: that is, for $f : B \to U(A_1)$ and $u : A_1 \to A_2$, then the Beck chevalley transformation provides an isomorphism $A_f \simeq A_{U(u)f}$. The next corollary exploits this result to establish that local units are their own local unit over their codomain:

Corollary 2.1.1.7. *Take $f : B \to U(A)$: then we have $A_f \simeq A_{\eta^A_f}$ in $\mathcal{A}$ and $\eta^{A_f}_{\eta^A_f} \simeq \eta^A_f$ in $B \downarrow U$.*

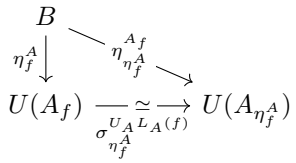
Proof. Consider the following diagram



Then by what precedes we have $\sigma^{U_AL_A(f)}_{\eta^A_f}$ is an iso as seen in the following diagram



But $U_AL_A(f)\eta^A_f = f$, exhibiting an isomorphism



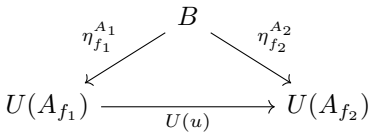
□

2.1.2 Local units as a multi-initial family

We have seen that any local right adjoint $U : \mathcal{A} \to \mathcal{B}$ automatically satisfies the Beck-Chevalley condition at any morphism in the domain category $\mathcal{A}$, and deduced that postcomposition with morphisms in the range of U did not modify the local unit: this suggests some initialness property of the local units, we are going to discuss in this subsection.

Another consequence of theorem 2.1.1.5 is that local units that are related by an arrow in the range of U must actually be isomorphic as objects under their domain:

Proposition 2.1.2.1. *Take $f_1 : B \to U(A_1)$ and $f_2 : B \to U(A_2)$, such that there exists a morphism u in $\mathcal{A}$ and a triangle*



then u is an isomorphism.

Proof. By theorem 2.1.1.5, we have that

$$\eta^{A_{f_1}}_{\eta^{A_1}_{f_1}} \simeq \eta^{A_1}_{f_1} \qquad\qquad \eta^{A_{f_2}}_{\eta^{A_2}_{f_2}} \simeq \eta^{A_2}_{f_2}$$

But by Beck-Chevalley condition at u, we also have

$$\eta^{A_{f_1}}_{\eta^{A_1}_{f_1}} \simeq \eta^{A_{f_2}}_{U(u)\eta^{A_1}_{f_1}} = \eta^{A_{f_2}}_{\eta^{A_2}_{f_2}}$$

□

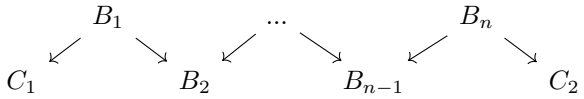
Corollary 2.1.2.2. *If $U : \mathcal{A} \to \mathcal{B}$ is local right adjoint, then for any B, any two $f_1 : B \to U(A_1)$, $f_2 : B \to U(A_2)$ in the same connected component of $B \downarrow U$ factorize through the same unit, that is $\eta_{f_1}^{A_1} \simeq \eta_{f_2}^{A_2}$, $A_{f_1} \simeq A_{f_2}$.*

Corollary 2.1.2.3. *For any $f : B \to U(A)$, η_f^A is initial in $(B \downarrow U) \downarrow f$.*

Hence, the local units under a given object have some initialness property; however, different from global units that are globally initial in the slice under their domain, local units only *jointly* assume this property; such a situation is encoded in the following notion:

Definition 2.1.2.4. Let $\mathcal{C}$ be a category; a *multi-initial family* in $\mathcal{C}$ is a family of objects $(X_i)_{i\in I}$ such that for any C in $\mathcal{C}$ there is a unique $i \in I$ and a unique arrow $X_i \to C$. A X_i for $i \in I$ is a local initial object.

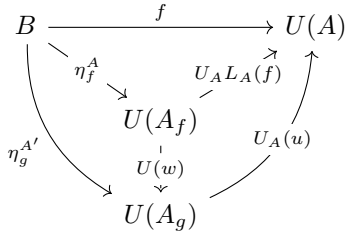
Remark 2.1.2.5. Observe that in this definition, if one has an arrow $f : C_1 \to C_2$, then C_1 and C_2 lie under the same local initial object. More generally, if two objects C_1, C_2 are in the same connected component and $X_i \to C_1$, $X_j \to C_2$ are the initial maps above them, then one must actually have $X_i \simeq X_j$: for there is a zigzag



and by uniqueness of the local initial object over a given object, the local object over B_1 is necessarily the same as the local initial object over B_3 because they both lie over B_2 and so on. Conversely any two objects under a same local initial object are in the same connected component. Hence there is exactly one local initial object by connected component.

Proposition 2.1.2.6. *Let $U : \mathcal{A} \to \mathcal{B}$ be a local right adjoint and B in $\mathcal{B}$: then the comma category $B \downarrow U$ has a multi-initial family.*

Proof. We claim that the (large) class of local units under B is a multi-initial family in $B \downarrow U$. First, for any $f : B \to U(A)$, we have by local adjunction in A an arrow $L_A(f)$ in $B \downarrow U$; but now suppose there is another $g : B \to U(A')$ such that $\eta_g^{A'}$ has a map $g \to f$ in $B \downarrow U$, that is there is a map u in $\mathcal{A}$ such that $f = U(u)\eta_g^{A_1}$; but then by the universal property of the unit there is a unique factorization



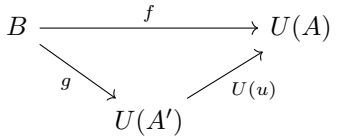
But by proposition 2.1.2.1, this forces $\eta_f^A \simeq \eta_g^{A'}$. □

From their very universal property, local units under a given object have to live isolated from each other, each one in its connected component of the comma $B \downarrow U$.

Remark 2.1.2.7. Observe that without further assumption, this multi-initial family under B might not be small. This is the point of the following notion:

Definition 2.1.2.8. A functor $U : \mathcal{A} \to \mathcal{B}$ is said to be a *right multi-adjoint* if for any B in $\mathcal{B}$ there is a *small* multi-initial family in the comma $B \downarrow U$. A (non necessarily full) subcategory $\mathcal{A} \hookrightarrow \mathcal{B}$ is *multireflective* if the inclusion functor is a right multi-adjoint.

Remark 2.1.2.9. Observe that this definition is indexed by the domain, that is, by the object B in $\mathcal{B}$, while the definition of local right adjoint was indexed by the objects A of $\mathcal{A}$. However it is easy to see that any right multi-adjoint is in particular a local right adjoint: for any $f : B \to U(A)$ define η_f^A to be the unique local initial object over f, and $L_A(f)$ to be the unique map $\eta_f^A \to f$. Then η_f^A has the universal property of the unit as any triangle $f \to U(u)$ in $\mathcal{B}/U(A)$



can be seen as a triangle $g \to f$ in $B \downarrow U$ forcing g and f to be in the same connected component. Therefore η_f^A is also the initial object over g, inducing an unique arrow $\eta_f^A \to g$ in $B \downarrow U$, that is a unique arrow $L_A(f) \to u$ in $\mathcal{A}/A$ as desired.

2.1.3 Connected limits, multi(co)limits and local adjoint functor theorem

We should end this section with a word on closure properties of multireflectives subcategories. Existence of a multi-initial object in the comma is reminiscent of the so-called solution set condition in Freyd Adjoint Functor Theorem. Let us make precise this fact, in order to retrieve an analogous multi-adjoint theorem, in the vein of [93]. First we need a weakening of the notion of initial family

Definition 2.1.3.1. Let $\mathcal{C}$ be a category; a *weakly initial family* is a family $(X_i)_{i \in I}$ such that for any object C, there is some $i \in I$ and some arrow $X_i \to C$.

Remark 2.1.3.2. Observe that in this definition, there is no requirement of uniqueness, nor for the index, nor for the arrow, so there may be several weakly initial objects over an arbitrary one. In the following we are interested in *small* weakly initial families:

Definition 2.1.3.3. A functor $U : \mathcal{A} \to \mathcal{B}$ is said to satisfy the *solution set condition* if each of the comma $B \downarrow U$ admits a small weakly initial family: that is a family $(n_i : B \to U(A_i))_{i \in I_B}$ such that any map from B to U factorizes through some (non necessarily unique) n_i.

Proposition 2.1.3.4. *A functor $U : \mathcal{A} \to \mathcal{B}$ is a right multi-adjoint if and only if it is local right adjoint and satisfies the solution set condition.*

Proof. It is obvious that a right multi-adjoint satisfies the solution set condition as for any object B the small multi-initial family of $B \downarrow U$ is in particular a small weakly initial family, and any right multi-adjoint is trivially local right adjoint. For the converse, if a local right adjoint satisfies the Solution Set Condition, consider the small weakly initial family $(f_i : B \to U(A_i))_{i \in I}$ of $B \downarrow U$ and then take the corresponding units $(n_{f_i} : B \to U(A_{f_i})_{i \in I}$: we claim this is a small multi-initial family. □

Now, as well as right adjointness can be detected by combining the solution set condition and preservation of small limits, we will be able to characterize right multi-adjoints thanks to preservation of a certain class of limits in the presence of the solution set:

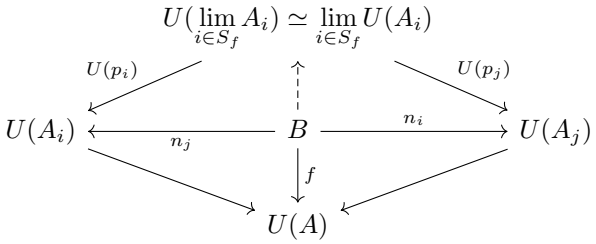
Definition 2.1.3.5. A *connected limit* is a limit indexed by a connected category; a *wide pullback* is a limit of a diagram over a set of arrows with common codomain. In particular any wide-pullback is a connected limit.

Proposition 2.1.3.6. *A category has wide pullbacks if and only if each slice $\mathcal{A}/A$ has products. A functor preserves connected limits if and only if it preserves wide pullbacks.*

In Freyd adjoint functor theorem, assuming completeness and local smallness of the domain category $\mathcal{A}$, one can prove that a functor preserving limits is right adjoint if it satisfies the solution set condition: this is achieved by constructing an initial object in any $B \downarrow U$ as the limit $\lim_{i\in I} U(A_i) \simeq U(\lim_{i\in I} A_i)$. Similarly :

Proposition 2.1.3.7. *Let $\mathcal{A}$ be a category with wide pullbacks; then a functor $U : \mathcal{A} \to \mathcal{B}$ is a right multi-adjoint if and only if it satisfies the solution set condition and preserves wide pullbacks.*

Proof. Indeed, assuming that $\mathcal{A}$ has connected limits and that U preserves them, the solution set condition will enable us to prove that its restriction on any slice $U_A : \mathcal{A}/A \to \mathcal{B}/U(A)$ is right adjoint by constructing an initial object in any of the comma $f \downarrow U_A$ whit $f : B \to U(A) \in \mathcal{B}/U(A)$ (this is nothing but the category of factorizations of f through U) as the following. If $S_B = (n_i : B \to U(A_i))_{i\in I_B}$ is the small weakly initial family in $B \downarrow U$ given by the solution set condition, define $S_f = S_B \downarrow f$ consisting of all the pairs (u, i) with $i \in I_B$ and $f = F(u)n_i$ a factorization of f through n_i: then we can do the wide pullback of the A_i over A and it is preserved by U, so that we have



Now we claim that this unique arrow $B \to U(\lim_{i\in S_f} A_i)$ is the desired initial object in $f \downarrow U_A$, as any factorization of f factorizes itself through some of the n_i in S_f hence through the wide pullback.

For the converse, observe that the small family of local units of a right multi-adjoint produces a small (weakly) initial family, so that it always satisfies the solution set condition. Preservation of wide pullback just comes from the fact that they are ordinary products in the slices, hence preserved by the restriction as it is right adjoint. □

Now we turn to another facet of the local adjunction.

Definition 2.1.3.8. Let $U : \mathcal{A} \to \mathcal{B}$ be a functor; then the *conerve* of U is the functor

$$\mathcal{B}^{\mathrm{op}} \xrightarrow{\mathcal{N}^U} [\mathcal{A}, \mathcal{S}]$$

sending each B in $\mathcal{B}$ to the functor $\mathcal{N}^U_B = \mathcal{B}[B, U(-)] : \mathcal{A} \to \mathcal{S}$.

For any B in $\mathcal{B}$, we have a discrete opfibration

$$\pi_B : \int \mathcal{N}^U_B \to \mathcal{A}$$

whose objects are pairs (A, f) with A in $\mathcal{A}$ and $f : B \to U(A)$, and morphisms $(A_1, f_1) \to (A_2, f_2)$ are $u : A_1 \to A_2$ with $U(u)f_1 = f_2$. There is a general result saying that a functor is a right adjoint if the projection from the category of elements of its co-nerve at each object has a limit; in fact this says that the co-nerve functor is representable, that is, there is an initial object in the category of elements, which is the unit. We will give here the corresponding statement for a right multi-adjoint, which requires first to introduce some multi-version of limits - and colimits which we will use latter on.

Definition 2.1.3.9. Let $F : I \to \mathcal{A}$ be a functor: then a *multilimit* of F is a small family of cones

$$(p^j_i : L_j \to F(i))_{\substack{i\in I \\ j\in J}}$$

such that for any cone $(f_i : X \to F(i))_{i\in I}$ there is a unique $j \in J$ and a unique factorization of the cone $(f_i)_{i\in I}$ through the cone $(p^j_i)_{i\in I}$. A category is (finitely) *multicomplete* if any (finite) diagram has a multi-limits.

A functor $U : \mathcal{A} \to \mathcal{B}$ preserves multi-limits (or also, is *multicontinuous*) if for any multi-limit $(p_i^j : L_j \to F(i))_{i\in I,\, j\in J}$ in $\mathcal{A}$, there is a multi-limit $(q_i^k : M_k \to UF(i))_{i\in I, k\in K}$ in $\mathcal{B}$ and for each $k \in K$ we have

$$M_k \simeq \coprod_{j\in J_k} U(L_j)$$

where J_k is the set of $j \in J$ such that the cone $(U(p_i^j) : U(L_j) \to UF(i))_{i\in I}$ factorizes through M_k.

Before going further, we think relevant to introduce the dual notion of multilimits, for we are going to use them also at the end of this subsection:

Definition 2.1.3.10. Let $F : I \to \mathcal{A}$ be a functor. Then a *multicolimit* of F is a small family of cocones

$$((q_i^j : F(i) \to X_j)_{i\in I})_{j\in J}$$

that is multi-initial the the category of cocones over F: that is, any other cocone $(f_i : F(i) \to A)_{i\in I}$ factorizes uniquely through one of the $(q_i^j : F(i) \to X_j)_{i\in I}$ for a unique $j \in J$. A category is (finitely) *multicocomplete* if any (finite) diagram admits a multi-colimit.

A functor $U : \mathcal{A} \to \mathcal{B}$ is *multicocontinuous* if for any multicolimit $((q_i^j : F(i)_i \to X_j)_{i\in I})_{j\in J}$ in $\mathcal{A}$, then the composite $UF : I \to \mathcal{B}$ has a multicolimit $((s_i^k : UF(i) \to Y_k)_{i\in I})_{k\in K}$ such that for any $k \in K$ we have a coproduct decomposition

$$Y_k \simeq \coprod_{J_k} U(X_j)$$

where $J_k = \{j \in J \mid (U(q_i^j) : UF(i) \to U(X_j))_{i\in I}$ factorizes through $(s_i^k)_{i\in I}\}$.

Remark 2.1.3.11. The universal property of the multicolimits can be encapsulated, for any other X in $\mathcal{B}$, in the following isomorphism

$$\lim_{i\in I} \mathcal{B}[F(i), X] \simeq \coprod_{j\in J} \mathcal{B}[X_j, X]$$

Remark 2.1.3.12. Remark that any (co)limit is in particular a multi-(co)limit with a single cone. In particular (co)completeness implies multi-(co)completeness, and (co)continuity implies multi-(co)continuity. Then in particular corepresentables functors are multicontinuous while representables send multicolimits to multilimits.

The following observation gives the obvious analog of the characterization of right adjoint in terms of the existence of the limit of the projection of the category of elements of the nerve:

Proposition 2.1.3.13. *A functor $U : \mathcal{A} \to \mathcal{B}$ is a local right adjoint if and only if for any B in $\mathcal{B}$, each connected component of $\int \mathcal{N}_B^U$ has an initial object.*

Now from what was said, it appears that a right multi-adjoint is a functor such that the associated conerve in any object is "locally representable". Indeed, any arrow from an object B toward U is determined first by the connected component it lies in, which corresponds to the local unit it factorizes through, and secondly by a choice of map in $\mathcal{A}$. This amounts to the following:

Proposition 2.1.3.14. *For a functor $U : \mathcal{A} \to \mathcal{B}$ the following are equivalent:*

- *U is a right multi-adjoint*
- *for each B, $\int \mathcal{N}_B^U$ has a small set of connected components with an initial object in each of them*
- *for each B, the functor $\pi_B : \int \mathcal{N}_B^U \to \mathcal{A}$ has a multilimit in $\mathcal{A}$ and U preserves it.*
- *for each B, one has*

  $$\mathcal{N}_B^U \simeq \coprod_{i\in I_B} \mathcal{A}[A_i, -]$$

  with I_B the set of connected components of B and $n_i : B \to U(A_i)$ the initial object of the connected component i

The first half of this fact is tautological; for the second part, one can adapt [8][proposition 3.3.2].

We finally end this subsection with a result on multireflective subcategories. Recall that full reflective subcategories inherit limits and colimits from their ambient category in the following way. Limits can be computed directly in the subcategory and are created by the inclusion, which we get by proving that the reflection of a limit computed in the ambient category is a limit in the subcategory, and observing the latter must be preserved by inclusion, so that this reflection was actually an iso. For colimits, one computes the colimit in the ambient category, then takes its reflection. In the context of multireflection, the correct analogs of those statements are in terms of limits and multicolimits.

Theorem 2.1.3.15. *Let $U : \mathcal{A} \hookrightarrow \mathcal{B}$ be a full multireflective subcategory; then:*

- *if $\mathcal{B}$ has colimits, then $\mathcal{A}$ has multicolimits*
- *if $\mathcal{B}$ has limits, then $\mathcal{A}$ has connected limits and they are created by U.*

Proof. For multicolimits, take a functor $F : I \to \mathcal{A}$; then one can compute the colimit $(q_i : UF(i) \to \mathrm{colim}_{i\in I} UF(i))_{i\in I}$ in $\mathcal{B}$. Then consider its small cone of local units $n_x : (\mathrm{colim}_{i\in I} UF(i) \to U(A_x))_{x\in I_{\mathrm{colim}_{i\in I} UF(i)}}$. Then by fullness of U each composite $n_x q_i : UF(i) \to U(A_x)$ comes from a unique map $q_i^x : F(i) \to A_x$ and $(q_i^x : F(i) \to A_x)_{i\in I, x\in I_{\mathrm{colim}_{i\in I} UF(i)}}$ is a multicolimit of F.

Now for connected limits, if we suppose $F : I \to \mathcal{A}$ with I connected, then the category $\lim_{i\in I} UF(i) \downarrow UF$ is connected; then by corollary 2.1.2.2, all the limit projections p_i factorize through a same local unit n_F

$$\begin{array}{ccc} \lim\limits_{i\in I} UF(i) & \xrightarrow{p_i} & UF(i) \\ {\scriptstyle n_F}\downarrow & \nearrow_{U(u_i^F)} & \\ U(A_F) & & \end{array}$$

Then $(U(u_i^F) : A_F \to A_i)_{i\in I}$ is a limiting cone in $\mathcal{A}$: indeed, any cone $(u_i : A \to F(i))_{i\in I}$ in $\mathcal{A}$ is transported by U to a cone in $\mathcal{B}$, were it induces a unique arrow $(U(u_i))_{i\in I} U(A) \to \lim_{i\in I} UF(i)$ and the composite $n_F(U(u_i))_{i\in I} : U(A) \to U(A_F)$ comes uniquely from an arrow $A \to A_F$ as desired. Hence A_F is a limit of the connected diagram F, but as U preserves connected limits, n_F was actually an isomorphism $\lim_{i\in I} UF(i) \simeq U(A_F)$. □

Remark 2.1.3.16. In fact, concerning connected limits, the condition that U is full can be slightly relaxed into the condition of being *relatively full and faithful*, which will be defined later in definition 2.3.1.4.

2.1.4 Stable functors

Now we come to an alternative notion encapsulating the property of being a local right adjoint, but in a way that is more related to factorization systems. This was studied in [93] and [96], and we prove here that this notion coincides with local right adjointness. It relies on an alternative presentation of local units in a pattern which is more akin to factorization systems.

Definition 2.1.4.1. A *candidate* (*diagonally universal morphism toward U* in the terminology of Diers), is a morphism $n : B \to U(A)$ such that for any square of the following form there exists a unique morphism $w : A \to A_1$ such that $U(w)$ diagonalizes uniquely the square and the left triangle already commutes in $\mathcal{A}$

$$\begin{array}{ccc} B & \xrightarrow{f} & U(A_1) \\ {\scriptstyle n}\downarrow & \nearrow_{U(w)} & \downarrow{\scriptstyle U(v)} \\ U(A) & \xrightarrow[U(u)]{} & U(A_2) \end{array} \qquad \begin{array}{ccc} & & A_1 \\ & \nearrow_{\exists! w} & \downarrow{\scriptstyle v} \\ A & \xrightarrow[u]{} & A_2 \end{array}$$

Definition 2.1.4.2. A functor $U : \mathcal{A} \to \mathcal{B}$ is *stable* when any morphism $f : B \to U(A)$ factorizes uniquely through the range of U as

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & U(A) \\ & {\scriptstyle n_f}\searrow \quad \nearrow{\scriptstyle U(u_f)} & \\ & U(A_f) & \end{array}$$

where $n_f : B \to U(A_f)$ is a candidate. We refer to this factorization as *the stable factorization of* f and to n_f as *the candidate of* f.

Remark 2.1.4.3. Then the candidate for f is the initial object in the category of factorizations of f through the range of U. Indeed, for any other factorization through U

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & U(A) \\ & {\scriptstyle g}\searrow \quad \nearrow{\scriptstyle U(u)} & \\ & U(A') & \end{array}$$

one gets a square as below, where n_f produces a unique v such that $U(v)$ is a filler

$$\begin{array}{ccc} B & \xrightarrow{\quad g \quad} & U(A') \\ {\scriptstyle n_f}\downarrow & \nearrow{\scriptstyle U(v)} & \downarrow{\scriptstyle U(u)} \\ U(A_f) & \xrightarrow[U(u_f)]{} & U(A) \end{array}$$

Proposition 2.1.4.4. *For a functor $U : \mathcal{A} \to \mathcal{B}$ and B in $\mathcal{B}$ we have the following:*

- *If a candidate $n_1 : B \to U(A_1)$ admits an arrow $n_2 \to n_1$ from another candidate $n_2 : B \to U(A_2)$ in $B \downarrow U$, then we have $n_1 \simeq n_2$ in $B \downarrow U$ and $A_1 \simeq A_2$ in $\mathcal{A}$.*
- *In particular, any two candidates in a same connected component of $B \downarrow U$ are isomorphic.*
- *If $f : B \to U(A)$ admits a stable factorization, then it is unique up to unique isomorphism.*
- *In particular, when U is stable, the stable factorization of any arrow is unique up to unique isomorphism.*

Proof. The first item is easily shown to imply the other ones. Suppose we have n_1, n_2 and a triangle

$$\begin{array}{ccc} B & \xrightarrow{\quad n_1 \quad} & U(A_1) \\ {\scriptstyle n_2}\downarrow & \nearrow{\scriptstyle U(u)} & \\ U(A_2) & & \end{array}$$

Then we have a unique filler

$$\begin{array}{ccc} B & \xrightarrow{\quad n_2 \quad} & U(A_2) \\ {\scriptstyle n_1}\downarrow & \nearrow{\scriptstyle U(v)} & \downarrow{\scriptstyle U(u)} \\ U(A_1) & \xrightarrow[U(1_{A_1})]{} & U(A_1) \end{array}$$

But now there is a unique filler of the square

$$\begin{array}{ccc} B & \xrightarrow{\quad n_1 \quad} & U(A_1) \\ {\scriptstyle n_2}\downarrow & \nearrow{\scriptstyle U(w)} & \downarrow{\scriptstyle U(v)} \\ U(A_2) & \xrightarrow[U(1_{A_2})]{} & U(A_2) \end{array}$$

so that $u : A_2 \to A_2$ is both a retraction and a section in $A\mathcal{A}$, hence an isomorphism, so that $n_1 \simeq n_2$ in $B \downarrow U$. □

Proposition 2.1.4.5. *For any square as below*

$$\begin{array}{ccc} B_1 & \xrightarrow{\quad f_1 \quad} & U(A_1) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle U(u)} \\ B_2 & \xrightarrow[f_2]{} & U(A_2) \end{array}$$

the stable factorizations of f_1 and f_2 are related by a unique morphism in $\mathcal{A}$ such that

$$\begin{array}{ccccc} B_1 & \xrightarrow{n_{f_1}} & U(A_{f_1}) & \xrightarrow{U(u_{f_1})} & U(A_1) \\ {\scriptstyle f}\downarrow & & \vdots{\scriptstyle U(w_{g,u})} & & \downarrow{\scriptstyle U(u)} \\ B_2 & \xrightarrow[n_{f_2}]{} & U(A_{f_2}) & \xrightarrow[U(u_{f_2})]{} & U(A_2) \end{array}$$

Proof. The desired $w_{g,u}$ is the filler of the square

$$\begin{array}{ccc} B_1 & \xrightarrow{n_{f_1}} & U(A_{f_1}) \\ {\scriptstyle n_{f_2} f}\downarrow & & \downarrow{\scriptstyle U(u u_{f_1})} \\ U(A_{f_2}) & \xrightarrow[U(u_{f_2})]{} & U(A_2) \end{array}$$

□

Theorem 2.1.4.6. *Stable functors and local right adjoints coincide.*

Proof. Let $U : \mathcal{A} \to \mathcal{B}$ be a stable functor. For each A defines the functor L_A returning the left part of the initial factorization of an arrow by its associated candidate:

$$\begin{array}{rccc} L_A : & \mathcal{B}/U(A) & \to & \mathcal{A}/A \\ & B \xrightarrow{f} U(A) & & \\ & {\scriptstyle n_f}\searrow \quad \nearrow{\scriptstyle U(u_f)} & \mapsto & A_f \xrightarrow{u_f} A \\ & U(A_f) & & \end{array}$$

We can easily prove this functor to be left adjoint to U/A, but it is more direct to observe that the family of candidates under B is a multi-initial family in $B \downarrow U$. Hence U is a local right adjoint.

For the converse, suppose U is a local right adjoint. We claim that candidates are arrows $n : B \to U(A)$ such that $L_A(n)$ provides an iso $A_f \simeq A$ in $\mathcal{A}$ such that $\eta^A_n \simeq n$ in $B \downarrow U$. Consider a square

$$\begin{array}{ccc} B & \xrightarrow{g} & U(A_1) \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle U(u)} \\ U(A) & \xrightarrow[U(v)]{} & U(A_2) \end{array}$$

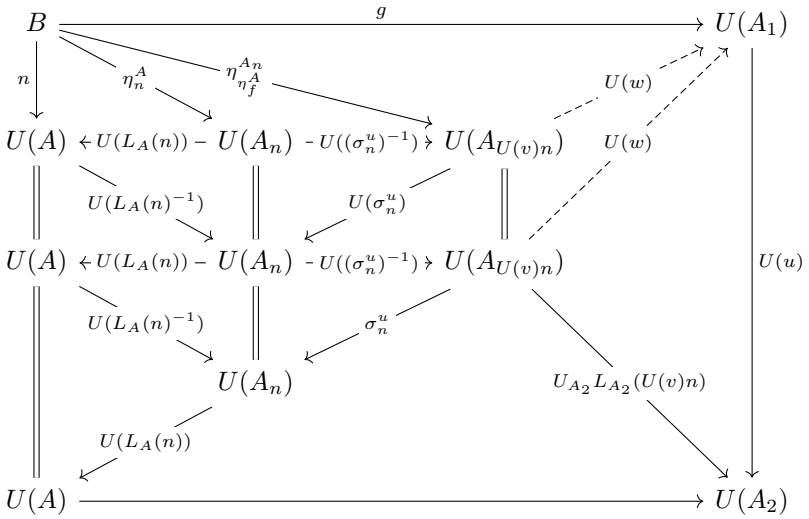
Recall by theorem 2.1.1.5 we also have that composing with $U(v)$ does not modify the unit, as we have an isomorphism $\sigma^u_n : \eta^{A_2}_{U(v)n} \simeq \eta^A_f$. But the triangle

$$\begin{array}{ccc} B & \xrightarrow{g} & U(A_1) \\ & {\scriptstyle U(u)n}\searrow & \downarrow{\scriptstyle U(u)} \\ & & U(A_2) \end{array}$$

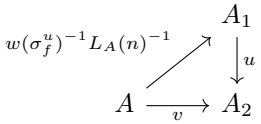
provides us with a unique arrow $w : A_{U(v)n} \to A_1$ such that

$$\begin{array}{ccc} & & A_1 \\ & {\scriptstyle w}\nearrow & \downarrow{\scriptstyle U(u)} \\ A_{U(v)n} & \xrightarrow[L_A(U(u)n)]{} & A_2 \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{g} & U(A_1) \\ {\scriptstyle \eta^{A_2}_{U(v)n}}\downarrow & \nearrow{\scriptstyle U(w)} & \downarrow{\scriptstyle U(u)} \\ U(A_{U(v)n}) & \xrightarrow[U_{A_2} L_{A_2}(U(u)n)]{} & U(A_2) \end{array}$$

Then by inserting the local inverses of $L_A(n)$ and σ^u_n in the square above and using the universal property of $\eta^{A_n}_{\eta^A_n}$ produces a path in the diagram below



where the sequence of back-and-forth along the isomorphisms is used to ensure the commutation of both the upper and lower part of this diagram, exhibiting the composite $U(w)U((\sigma_f^u)^{-1})U(L_A(n)^{-1})$ as a lift of the square above; but for n was a candidate, we know that such a lift comes from a unique factorization in $\mathcal{A}$ as below



In particular, local units are candidates by corollary 2.1.1.7. Hence for any arrow $f : B \to U(A)$, the factorization through the unit as $f = U_A L_A(f)\eta_f^A$ provides a stable factorization through a candidate. □

This achieves to prove that stable functors and local right adjoint can be used indifferently and are two ways of encapsulating the same property.

However in the following, and especially in the second paper, we shall give more interest to right multi-adjoint for the smallness condition allows us to manipulate local units without size issue.

2.2 Right multi-adjoints through free product completion

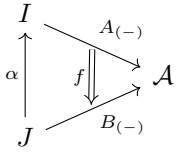
In this section, we give the characterization of right multi-adjoints though the free product completion, following loosely [25] and [94]. In the second part of this work, we are going to show how the notion of spectrum is a way to turn a local right adjoint into a right adjoint, the spectrum functor being the desired left adjoint. But this construction, motivated by geometric and duality theoretic conceptions, proceeds by extracting as much as possible topological and geometric information from a situation of local adjunction: in some way, it exploits the defect of universality on the algebraic side in order to produce a richer structure on the geometric side. In this section, we recall another way to turn a situation of multi-adjunction into an honest adjunction, which is purely algebraic and purer in some sense, but also devoid of any geometric content for this very reason. The relation between those two approaches will be studied in more detail in the second part of this work, where this approach through free product completion will appear as the "discrete version" of the spectral adjunction.

The main intuition of this part is that, for a right multi-adjoint $U : \mathcal{A} \to \mathcal{B}$, the cone of local units under a given object B defines a family of objects in $\mathcal{A}$ given by the codomains of those local units. Hence, at the level of families of objects, the multiversality of the construction can be fixed and U will induce an honest adjunction between categories of families of objects of $\mathcal{A}$ and $\mathcal{B}$. The good notion of "category of families" here is the one provided by the free product completion, the beginning of this part is devoted to.

2.2.1 Free product completion

Definition 2.2.1.1. For a category $\mathcal{A}$, the *free product completion* of $\mathcal{A}$ is the category $\Pi\mathcal{A}$ whose

- objects are functors $A_{(-)} : I \to \mathcal{A}$ (also denoted $(A_i)_{i\in I}$) with I a set,
- and arrows $(A_i)_{i\in I} \to (B_j)_{j\in J}$ consist of the data of an application $\alpha : J \to I$ and a natural transformation



that is, a J-indexed family $(f_j : A_{\alpha(j)} \to B_j)_{j\in J}$.

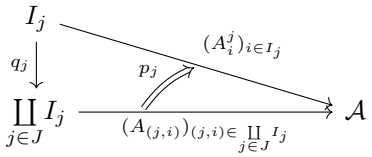
Proposition 2.2.1.2. *We have the following properties of the free product completion, for a given category $\mathcal{A}$:*

- *$\Pi\mathcal{A}$ has small products;*
- *There is a codense full embedding $\iota_{\mathcal{A}} : \mathcal{A} \hookrightarrow \Pi\mathcal{A}$ whose essential image is the subcategory $(\Pi\mathcal{A})_{coco}$ of co-connected objects;*
- *Moreover, the embedding $\mathcal{A} \to \Pi\mathcal{A}$ has a right adjoint if and only if $\mathcal{A}$ already had products;*
- *We have a full embedding $\Pi\mathcal{A} \hookrightarrow [\mathcal{A}, \mathcal{S}]^{\mathrm{op}}$ whose essential image consists of all small products of representable;*
- *For any category $\mathcal{B}$ with small products, we have an equivalence of categories*

$$[\mathcal{A}, \mathcal{B}] \simeq [\Pi\mathcal{A}, \mathcal{B}]_{\Pi}$$

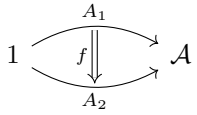
(where $[\Pi\mathcal{A}, \mathcal{B}]_{\Pi}$ is the category of functors preserving small products) sending any $F : \mathcal{A} \to \mathcal{B}$ on its right Kan extension $\mathrm{ran}_{\iota_{\mathcal{A}}} F$ *and any $G : \Pi\mathcal{A} \to \mathcal{B}$ on its restriction $G\iota_{\mathcal{A}}$.*

Proof. For the first item: the product in $\Pi\mathcal{A}$ of a family of families $((A_i^j)_{i\in I_j})_{j\in J}$ has as indexing set the disjoint union $\coprod_{j\in J} I_j$ and whose member of index (j,i) is the object $A_{(j,i)} = A_j^i$; the projections are given for each $j \in J$ as the transformation



where p_i is the pointwise equality $A_{(j,i)} = A_i^j$.

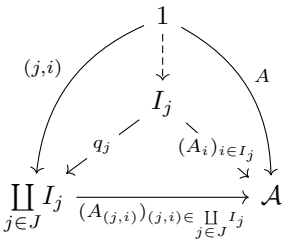
For the second item, the embedding sends an object A of $\mathcal{A}$ to the one element family $A : 1 \to \mathcal{A}$ and a morphism $f : A_1 \to A_2$ to the natural transformation



Now we prove objects in the image of this embedding are coconnected, which says that for a family of families $((A_i^j)_{i\in I_j})_{j\in J}$, we have

$$\Pi\mathcal{A}\Big[\prod_{j\in J}(A_i^j)_{i\in I_j}, \iota_{\mathcal{A}}(A)\Big] \simeq \coprod_{j\in J} \Pi\mathcal{A}\big[(A_i^j)_{i\in I}, \iota_{\mathcal{A}}(A)\big]$$

Indeed, any arrow $\Pi_{j\in J}(A_i^j)_{i\in I_j} \to \iota_{\mathcal{A}}(A)$ defines an arrow $1 \to \coprod_{j\in J} I_j$ pointing at some pair (j,i) with $j \in J$ and $i \in I_j$, and a natural transformation

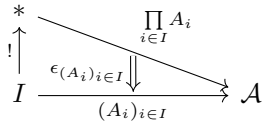


which is nothing but an arrow $f : A_{(j,i)} \to A$. But in such a case, as 1 is a connected object in $\mathcal{S}$, this arrow $(j,i) : 1 \to \coprod_{j\in J} I_j$ factorizes through I_j for some $j \in J$, pointing the corresponding $i \in I_j$, and the natural transformation f factorizes through the componentwise identity $p^j : A_{(i,j)} = A_i^j$ so we have an arrow $(A_i^j)_{i\in I_j} \to \iota_{\mathcal{A}}(A)$ as desired. Conversely, any coconnected object is of the form $\iota_{\mathcal{A}}(A)$: indeed a family $(A_i)_{i\in I} : I \to \mathcal{A}$ is nothing but the product in $\Pi\mathcal{A}$ of the family $(\iota_{\mathcal{A}}(A_i))_{i\in I}$ as the set I decomposes as the coproduct $\coprod_I 1$ in $\mathcal{S}$; and any family should be indexed by a connected set to be a coconnected object in $\Pi\mathcal{A}$, but 1 is the only connected set. This also suffices to prove that any object is a product of objects in the range of ι_A.

Now suppose that $\mathcal{A}$ has products. Then for any family $(A_i)_{i\in I}$ in $\Pi\mathcal{A}$ one can compute the product in $\mathcal{A}$, $\prod_{i\in I} A_i$. Now for an object A in $\mathcal{A}$, we have

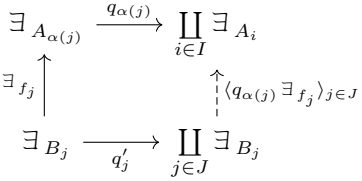
$$\Pi\mathcal{A}[\iota_{\mathcal{A}}(A), (A_i)_{i\in I}] \simeq \mathcal{A}[A, \prod_{i\in I} A_i]$$

sending a family of arrows $(A \to A_i)_{i\in I}$ to the universal map $A \to \prod_{i\in I}$. The unit of this adjunction is iso as $\iota_{\mathcal{A}}$ is full and faithful, while the counit is the transformation



where $\epsilon_{(A_i)_{i\in I}}$ as the projection $p_i : \prod_{i\in I} A_i \to A_i$ has component in i. For the converse, it is easy to see that any right adjoint of the embedding $\iota_{\mathcal{A}}$ sends a family on an object in $\mathcal{A}$ with the universal property of the product.

Recall now that we denote as $\exists : \mathcal{A}^{\mathrm{op}} \hookrightarrow [\mathcal{A}, \mathcal{S}]$ the contravariant Yoneda embedding; then it factorizes through the embedding $\Pi\mathcal{A} \hookrightarrow [\mathcal{A}, \mathcal{S}]^{\mathrm{op}}$ sending a family $(A_i)_{i\in I}$ to the coproduct $\coprod_{i\in I} \exists_{A_i}$ and an arrow $(\alpha, (f_j)_{j\in J}) : (A_i)_{i\in I} \to (B_j)_{j\in J}$ to the opposite of the induced map between the corresponding coproducts in $[\mathcal{A}, \mathcal{S}]$ as depicted below



Finally, for a functor $U : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ having products; we claim that the right Kan extensions of U is pointwise and can be computed as

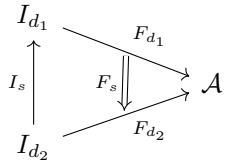
$$\mathrm{ran}_{\iota_{\mathcal{A}}} U(A_i)_{i\in I} = \prod_{i\in I} U(A_i)$$

Indeed for any $(A_i)_{i\in I}$ the comma category $(A_i)_{i\in I} \downarrow \iota_{\mathcal{A}}$ has a small initial I-indexed subcategory consisting of the objects $(i, 1_{A_i})$ for $i \in I$, and this subcategory is discrete. Hence calculating the pointwise right Kan extension reduces to calculating the product above. Moreover, as $\iota_{\mathcal{A}}$ is full and faithful, restricting back this Kan extension along $\iota_{\mathcal{A}}$ gives back U, in fact up to equality in this context. □

Proposition 2.2.1.3. *The embedding $\mathcal{A} \hookrightarrow \Pi\mathcal{A}$ creates connected limits in $\mathcal{A}$. Moreover, $\Pi\mathcal{A}$ is complete if and only if $\mathcal{A}$ has connected limits.*

Proof. Let D be a connected category and $F : D \to \mathcal{A}$; we prove that the singleton $\iota_{\mathcal{A}}(\lim F)$ is the limit of $\iota_{\mathcal{A}}F$ in $\Pi\mathcal{A}$. Consider a cone $(\alpha_d, (f_d)_{d\in D} : (A_i)_{i\in I}\iota_{\mathcal{A}}F$ in $\Pi\mathcal{A}$ consisting for each $d \in D$ of an arrow $f_d : A_{\alpha_d} \to F(d)$ where $\alpha_d : 1 \to I$ points to some index; but as D is connected and I is a set, necessarily the α_d are all equal to the same index α, so that we actually have a cone $(f_d : A_\alpha \to F(d))_{d\in D}$ in $\mathcal{A}$, inducing a unique arrow $f : A_\alpha \to \lim F$ in $\mathcal{A}$. This defines a unique arrow $(\alpha, f) : (A_i)_{i\in I} \to \iota_{\mathcal{A}}(\lim F)$. By what precedes, it is also clear that any connected cone that $\iota_{\mathcal{A}}$ sends to a limiting cone was already limiting.

Now, recall that a category is complete if and only if it has connected limits and products. But $\Pi\mathcal{A}$ always has products, so we have to show that $\Pi\mathcal{A}$ has connected limits if and only if $\mathcal{A}$ does. Let D be a connected category, and $F : D \to \Pi\mathcal{A}$ a functor, with $F_d : I_d \to \mathcal{A}$ its component in d and with transition morphism



for each $s : d_1 \to d_2$ in D. Then F defines an oplax cocone $(F_d : I_d \to \mathcal{A})_{d\in D}$ in **Cat**, defining uniquely a functor

$$\int I \xrightarrow{\langle F_d\rangle_{d\in D}} \mathcal{A}$$

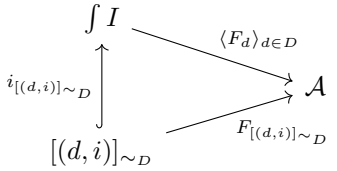
where $\int I$ is the category of elements of the functor $I : D^{\mathrm{op}} \to \mathcal{S}$ returning the indexing set $I_d : I_{d_2} \to I_{d_1}$ and the transition map I_s for $s : d_1 \to d_2$: it is indeed well known that the category of elements is the oplax colimit in **Cat**, and we see here the I_d as discrete categories. Now, as D was small and each I_d was a set, the category $\int I$ has a small set $\pi_0(\int I)$ of connected components. In this context, one can describe the connected components as follows. In set, the colimit of the diagram I is the quotient

$$\operatorname*{colim}_{d\in D} I_d \simeq \coprod_{d\in D} I_d/\sim_D$$

where $(d, i) \sim_D (d', i')$ if there is a zigzag in D relating i and i': this exactly amounts to say that (d, i) and (d', i') are in the same connected component of $\int I$, so we also have that the connected components of $\int I$ are exactly equivalence classes $[(d, i)]_{\sim_D}$ and

$$\coprod_{d\in D} I_d/\sim_D \simeq \pi_0(\int I)$$

Now, if we restrict the induced functor $\langle F_d\rangle_{d\in D}$ along the inclusion of a connected component



we can compute the limit $\lim F_{[(d,i)]_{\sim_D}}$ in $\mathcal{A}$, and this limit is preserved by the inclusion functor $\iota_{\mathcal{A}}$. So the desired limit of F in $\Pi\mathcal{A}$ is the family

$$\pi_0(\int I) \to \mathcal{A}$$

sending the connected component $[(d, i)]_{\sim_D}$ to the connected limit $\lim F_{[(d,i)]_{\sim_D}}$, and this actually coincides with the product in $\Pi\mathcal{A}$ of the family $(\lim F_{[(d,i)]_{\sim_D}} : 1 \to \mathcal{A})_{[(d,i)]_{\sim_D}\in\pi_0(\int I)}$. □

2.2.2 Right multi-adjoints have left adjoint in the free product completion

Now we see that one can extend canonically a functor between two categories into a product-preserving functor between their free product completion as stated below; we shall see that this extension becomes in fact a global right adjoint, for the family of units form altogether an ordinary arrow in the category of families.

Proposition 2.2.2.1. *Any functor $U : \mathcal{A} \to \mathcal{B}$ extends uniquely into a functor ΠU, called its* free product extension*, making the square below to commute up to equality*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathcal{B} \\ {\scriptstyle \iota_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \iota_{\mathcal{B}}} \\ \Pi\mathcal{A} & \xrightarrow[\Pi U]{} & \Pi\mathcal{B} \end{array}$$

Proof. The functor ΠU just is the right Kan extension $\mathrm{ran}_{\iota_{\mathcal{A}}} \iota_{\mathcal{B}} U$, and is defined by sending a family $(A_i)_{i\in I}$ to $(U(A_i))_{i\in I}$. □

The following proposition is tautological:

Proposition 2.2.2.2. *$\mathcal{A}$ has a multi-initial family if and only if $\Pi\mathcal{A}$ has an initial object.*

The following proposition is the core idea of [25][part 4], though we present here a quite different proof.

Proposition 2.2.2.3. *For a functor $U : \mathcal{A} \to \mathcal{B}$, the following are equivalent:*

1. *U is a right multi-adjoint*
2. *U has a relative left adjoint along $\iota_{\mathcal{A}}$*
3. *its free product extension $\Pi U : \Pi\mathcal{A} \to \Pi\mathcal{B}$ is a right adjoint*

Proof. Suppose that U is a right multi-adjoint, with I_B the set of local units $\eta_x : B \to U(A_x)$ and $\pi_B : I_B \to \mathcal{A}$ its projection sending x to A_x. Define a functor $L : \mathcal{B} \to \Pi\mathcal{A}$ sending an object B to the family $\pi_B : I_B \to \mathcal{A}$, and an arrow $f : B_1 \to B_2$ to the transformation

$$\begin{array}{ccc} I_{B_1} & & \\ {\scriptstyle I_f}\uparrow & \overset{\pi_{B_1}}{\searrow}\ L_f\Downarrow & \mathcal{A} \\ I_{B_2} & \underset{\pi_{b_2}}{\nearrow} & \end{array}$$

where I_f sends x to the index of the unit $\eta^{A_x}_{n_x f} : B \to U(A_{n_x f}) = n_{I_f(x)}$ and L_f has component $L_{A_x}(n_x f) : A_{n_x f} \to A_x$ as provided in each $x \in I_{B_2}$ by the factorization

$$\begin{array}{ccc} B_1 & \xrightarrow{f} & B_2 \\ {\scriptstyle \eta^{A_x}_{n_x f}}\downarrow & & \downarrow{\scriptstyle n_x} \\ U(A_{n_x f}) & \xrightarrow[U_{A_x} L_{A_x}(n_x f)]{} & U(A_x) \end{array}$$

Observe that the local units of B define in particular a morphism of families

$$\begin{array}{ccc} 1 & \overset{B}{\to} & \mathcal{B} \\ {\scriptstyle !}\uparrow & \overset{n}{\Rightarrow} & \uparrow{\scriptstyle U} \\ I_B & \xrightarrow[\pi_B]{} & \mathcal{A} \end{array}$$

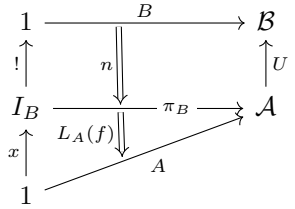
corresponding to the family $(n_x : B \to U(A_x))_{x\in I_B}$. Now it is easy to see that this functor is a relative left adjoint to U along $\iota_{\mathcal{A}}$, that is, that for any B in $\mathcal{B}$ and A in $\mathcal{A}$ we have

$$\Pi\mathcal{A}[L(B), \iota_{\mathcal{A}}(A)] \simeq \mathcal{B}[B, U(A)]$$

Indeed, any arrow $f : B \to U(A)$ factorizes through a unique $n_x : B \to U(A_x)$, where x is the index of the unit η^A_f, while $L_A(f) : A_x \to A$ provides a morphism in $\Pi\mathcal{A}$

$$\begin{array}{ccc} I_B & & \\ {\scriptstyle x}\uparrow & \overset{\pi_B}{\searrow}\ L_A(f)\Downarrow & \mathcal{A} \\ 1 & \underset{A}{\nearrow} & \end{array}$$

while any arrow $(x, u) : L(B) \to \iota_{\mathcal{A}}(A)$ can be pasted with the family of units

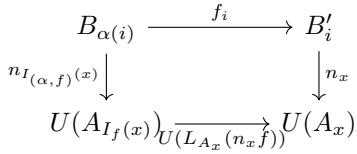


to return an arrow $B \to U(A)$, and one has to use the universal properties of the local units to see that those processes are mutual inverses.

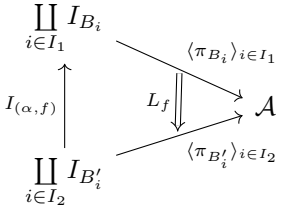
This functor L extends to $\Pi\mathcal{B}$ as follows: for a family $(B_i)_{i\in I}$, define $L(B_i)_{i\in I}$ as the family

$$\coprod_{i\in I} I_{B_i} \xrightarrow{\langle \pi_{B_i}\rangle_{i\in I}} \mathcal{A}$$

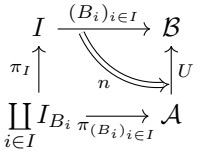
sending (i, x) with $i \in I$ and $x \in I_{B_i}$ to the associated A_x; for an arrow $(\alpha, f = (f_i)_{i\in I}) : (B_i)_{i\in I_1} \to (B'_i)_{i\in I_2}$, that is a family $(f_i : B_{\alpha(i)} \to B'_i)_{i\in I_2}$, each pair $(i, x) \in \coprod_{i\in I_2} I_{B'_i}$ defines uniquely some $I_{(\alpha,f)}(x) \in I_{B_{\alpha(i)}}$ indexing the unit through which factorizes the composite $n_x f_i : B_{\alpha(i)} \to U(A_x)$, that is such that



Define then $I_{(\alpha,f)} : \coprod_{i\in I_2} I_{B'_i} \to \coprod_{i\in I} I_{B_i}$ as sending (i, x) to this $I_{(\alpha,f)}(x)$: the desired morphism $L(\alpha, f)$ is



where L_f denotes the family $(L_{A_x}(n_x f) : A_{I_{(\alpha,f)}} \to A_x)_{(i,x)\in\coprod_{i\in I_2} I_{B'_i}}$. In particular we have a morphism



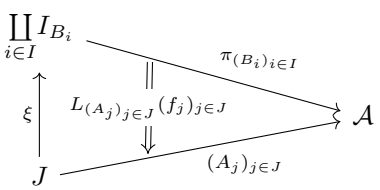
where $\pi_I : \coprod_{i\in I} I_{B_i} \to I$ is the projection sending (i, x) on $i \in I$, $\pi_{(B_i)_{i\in I}} : \coprod_{i\in I} I_{B_i} \to \mathcal{A}$ sends (i, x) on A_x, and n has as components

$$(n_{(i,x)} = n_x : B_i \to U(A_x))_{(i,x)\in\coprod_{i\in I} I_{B_i}}$$

Then for any $(A_j)_{j\in J}$ in $\Pi\mathcal{A}$ and $(B_i)_{i\in I}$ in $\Pi\mathcal{B}$ we have an isomorphism

$$\Pi\mathcal{A}\big[L(B_i)_{i\in I}, (A_j)_{j\in J}\big] \simeq \Pi\mathcal{B}\big[(B_i)_{i\in I}, (U(A_j))_{j\in J}\big]$$

Indeed a morphism of family $(\alpha, f) : (B_i)_{i\in I} \to (U(A_j))_{j\in J}$, that is a family $(f_j : B_{\alpha(j)} \to U(A_j))_{j\in J}$, defines an application $\xi : J \to \coprod_{i\in I} I_{B_i}$ sending i to the index of the local unit $n_{\xi(i)} = \eta^{A_i}_{f_i} : B_{\alpha(i)} \to A_{\xi(i)}$, and a morphism of families



where $L_{(A_j)_{j\in J}}(f_j)_{j\in J}$ consists of all right part in $\mathcal{A}$ of their factorization

$$(L_{A_j}(f_j) : A_{\xi(j)} \to A_j)_{j\in J}$$

For the converse, use the same argument as for the proof of the first implication by pasting.

Now we prove that if U is such that ΠU is right adjoint to a functor L, then U is multi-adjoint. Observe that with this hypothesis, we have in particular for any B in $\mathcal{B}$ a unit $\eta_{(*,B)} : (*, B) \to \Pi UL(*, B)$. So if I_B denote the indexing set of $\Pi UL(*, B)$ and A_i is the object in $\mathcal{A}$ corresponding to the ith index of I_B in this family, then we have a cone $(\eta_i : B \to U(A_i))_{i\in I_B}$. Using the unit property of $\Pi UL(*, B)$, this can be shown to be a cone of local units: for any A in $\mathcal{A}$ and $f : B \to U(A)$, the unit property produces a factorization in $\Pi\mathcal{B}$ inducing a map $i_f : * \to I_B$ pointing at the index of the local unit η^A_f, together with a transformation $L_A(f)$ returning the image of f along the local left adjoint at A. □

2.3 Factorization aspects

As suggested by the definition of candidates in the notion of stability, orthogonality structures are hidden in the notion of local adjunction. In the same vein, one could ask whether the stable factorization of arrows toward U can be generalized to any arrow, that is, if the orthogonality structure provided by the candidates on the left and the morphisms in the range of U on the right can be completed into a factorization system. In the context studied in [31], this is possible through a small object argument in the context of locally finitely presentable category. This step is essential in general in the construction of spectra, and also takes place in the topos-theoretic approach of [19], though it is mainly left implicit. The reference for this is [4], we mostly follow there modulo some adaptations, and in combination with elements from [31].

2.3.1 Diagonally universal morphisms

Here we list the properties of the left-like maps induced by a local right adjoint U, and how they are related to the local units; however in this section it is more practical to speak of stable functor for they are directly defined through some kind of factorization condition of arrows into the range of U, we want to generalize to arbitrary arrows in this section.

Definition 2.3.1.1. Let $U : \mathcal{A} \to \mathcal{B}$ a local right adjoint. A morphism $n : B \to C$ is said to be *diagonally universal* if it is left orthogonal to morphisms in the range of U, that is, if for any morphism $u : A_1 \to A_2$ in $\mathcal{A}$ and any square as below, there exists a unique filler $d : C \to U(A_1)$ making both the upper and lower triangles to commute

$$\begin{array}{ccc} B & \xrightarrow{f} & U(A_1) \\ {\scriptstyle n}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle U(u)} \\ C & \xrightarrow[g]{} & U(A_2) \end{array}$$

As a left class in an orthogonality structure, diagonally universal morphisms enjoy the following properties, which are standard and then do not need proofs here.

Proposition 2.3.1.2. *We have the following:*

- *diagonally universal morphisms are stable under composition and contain isomorphisms*
- *if $n : B \to C$ is diagonally universal and $m : C \to D$ is such that mn is diagonally universal, then m is also diagonally universal. In particular, diagonally universal morphisms are stable under retracts.*
- *diagonally universal morphisms are stable under pushout along arbitrary morphisms*
- *diagonally universal morphisms are stable under colimits and retracts in the arrow category $\mathcal{B}^2$*

Remark 2.3.1.3. Beware that without further assumption, a diagonally universal morphism with codomain in the range of U, that is of the form $n : B \to U(A)$, needs not be a candidate, as the filler needs not arise from a morphism in $\mathcal{A}$. The condition ensuring this is the following:

Definition 2.3.1.4. A functor $U : \mathcal{A} \to \mathcal{B}$ is said to be *relatively full and faithful* if for any triangle as below

$$\begin{array}{ccc} U(A_1) & \xrightarrow{\quad f \quad} & U(A_2) \\ & {\scriptstyle U(u_1)} \searrow \quad \swarrow {\scriptstyle U(u_2)} & \\ & U(A) & \end{array}$$

then f comes uniquely from some $u : A_1 \to A_2$ such that $U(u) = f$.

Proposition 2.3.1.5. *For any $u : A_1 \to A_2$, $U(u)$ is diagonally universal if and only if $U(u)$ is an isomorphism. If moreover U is relatively full and faithful, then $U(u)$ is diagonally universal if and only if u is an isomorphism.*

Proof. The unique filler of the square

$$\begin{array}{ccc} U(A_1) & = & U(A_1) \\ {\scriptstyle U(u)}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle U(u)} \\ U(A_2) & = & U(A_2) \end{array}$$

is both a right and left inverse to $U(u)$; and if moreover U is relatively full and faithful, it comes from a unique morphism $d = U(v)$ which is both a section of u from the lower triangle, but it is also a retraction because there is a unique morphism in $\mathcal{A}$ lifting $U(u)$ in the upper triangle, and this must be u. □

Proposition 2.3.1.6. *A morphism $n : B \to U(A)$ is diagonally universal if and only if $U_A L_A(n)$ is an isomorphism.*

Proof. The unique lifter d in the following square

$$\begin{array}{ccc} B & \xrightarrow{\eta^A_n} & U(A_n) \\ {\scriptstyle n}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle U_A L_A(n)} \\ U(A) & = & U(A) \end{array}$$

is a section of $U_A L_A(n)$. This provides also a filler of the square

$$\begin{array}{ccc} B & \xrightarrow{\eta^A_n} & U(A_n) \\ {\scriptstyle \eta^A_n}\downarrow & \overset{dU_A L_A(n)}{\nearrow} & \downarrow{\scriptstyle U_A L_A(n)} \\ U(A_n) & \xrightarrow[U_A L_A(n)]{} & U(A) \end{array}$$

But $1_{U(A)} = U(1_A)$ is the only filler of the square because η^A_n is a candidate. So d is also a retraction of $U_A L_A$, which is hence an isomorphism. Conversely, if $U_A L_A(n)$ has an inverse, then one can use the candidate property at η^A_f to get a filler in any square with a morphism in the range of U on the right. □

Remark 2.3.1.7. Beware that U needs not be conservative, so that the inverse of $U_A L_A(n)$ needs not come from a morphism in $\mathcal{A}$ making $L_A(n)$ an isomorphism itself, so that the remark above does not say that $n \simeq \eta^A_n$ in $B \downarrow U$; in particular $n : B \to U(A)$ may be diagonally universal without being a candidate. However, in the case where U is relatively full and faithful, the filler we had above must come from a unique morphism $d = U(v)$ which must satisfy the same commutations, hence providing an inverse of $L_A(n)$: thence the following corollary.

Corollary 2.3.1.8. *Suppose that U is relatively full and faithful; then for a morphism $n : B \to U(A)$ with codomain in the range of U, the following are equivalent:*

- *n is diagonally universal*
- *n is a candidate*
- *$L_A(n)$ is an isomorphism*

2.3.1.9. Denote as **Diag** the class of diagonally universal morphisms; this is the left orthogonal ${}^{\perp}U(\mathcal{A}^2)$. Hence we end with an orthogonality structure $(\mathbf{Diag}, \mathbf{Diag}^{\perp})$ where $\mathbf{Diag}^{\perp}$ is the double-orthogonal $({}^{\perp}U(\mathcal{A}^2))^{\perp}$. Arrows in $\mathbf{Diag}^{\perp}$ lie now out of the essential image of U and may have arbitrary domain and codomain. However, we have the following fullness property of the essential image of U relatively to $\mathbf{Diag}^{\perp}$:

Proposition 2.3.1.10. *Let $u : U(A_1) \to U(A_2)$ be an arrow in $\mathbf{Diag}^{\perp}$. Then $u \simeq U(L_{A_2}(u))$ in $\mathcal{B}/U(A_2)$ and $\eta_u^{A_2}$ is an isomorphism. In particular u is an arrow in the essential image of U.*

Proof. Indeed, u is right orthogonal to the local unit in its stable factorization, so there exists a unique w as below

$$\begin{array}{ccc} U(A_1) & = & U(A_1) \\ {\scriptstyle \eta_u^{A_2}}\downarrow & \overset{w}{\nearrow} & \downarrow{\scriptstyle u} \\ U(A_u) & \xrightarrow[U(L_{A_2}(u))]{} & U(A_2) \end{array}$$

which is both diagonally universal by left cancellation, and in $\mathbf{Diag}^{\perp}$ by right cancellation, and is hence an isomorphism. In particular $\eta_u^{A_2}$ is an iso, being section of an iso. □

2.3.2 Diagonal axiomatizability

Now, we explain how, in a suitable context, the stable factorization of morphisms towards U extends to a factorization system in $\mathcal{B}$, where the diagonally universal morphisms form the left class. To do so, we are going to adapt [4] version of the small object argument in the context of locally presentable categories.

2.3.2.1. In the following we suppose that $\mathcal{B}$ is a locally finitely presentable category and $U : \mathcal{A} \to \mathcal{B}$ is a local right adjoint. Then denote $\mathcal{D}$ the class of diagonally universal morphisms between finitely presented objects. This coincides with the intersection of the class of diagonally universal morphisms and the class of finitely presented morphisms, that is, $\mathcal{D} = \mathbf{Diag} \cap \mathcal{B}^2_{\omega}$. We are going to use $\mathcal{D}$ to left-generate a factorization system, which will enjoy some degree of accessibility.

Proposition 2.3.2.2. *The class $\mathcal{D}$ has the following properties:*

- *$\mathcal{D}$ is closed under composition and contains isomorphisms between finitely presented objects*
- *$\mathcal{D}$ is left cancellative*
- *$\mathcal{D}$ is closed under pushouts along arbitrary finitely presented morphisms*
- *$\mathcal{D}$ is closed under finite colimit in the arrow category $\mathcal{B}^2$*
- *Any filtered colimit in $\mathcal{B}^2$ of morphisms in $\mathcal{D}$ is diagonally universal.*

Proof. The two first propositions are obvious. The third is an easy consequence of the universal property of the pushout. The fourth comes from the fact that $\mathcal{B}^2_{\omega}$ is itself closed under finite colimits in $\mathcal{B}^2$ because $\mathcal{B}_{\omega}$ is so in $\mathcal{B}$ and colimits in the arrow category are sent to colimit by domain and codomain functors, while **Diag** is also closed under colimit as a left class in an orthogonality structure. This last argument also proves the last item. □

2.3.2.3. Now we invoke proposition 1.1.3.2 to construct a factorization system $(\mathrm{Ind}(\mathcal{D}), \mathcal{D}^{\perp})$. In our context, we can use the small class $\mathcal{D}$ of finitely presented diagonally universal morphisms as the desired saturated class through which we can generate a factorization system in $\mathcal{B}$. In particular we know that the factorization above is orthogonal, that is, $\mathrm{Ind}(\mathcal{D}) = {}^{\perp}(\mathcal{D}^{\perp})$.

Recall that $\mathcal{B}^2$ also is locally finitely presentable, with $\mathcal{B}^2_{\omega}$ as generator of finitely presented objects. We have now a functor preserving finite colimits

$$\mathcal{D} \overset{\iota_{\mathcal{D}}}{\hookrightarrow} \mathcal{B}^2_{\omega}$$

which extends into pair of adjoint functors

$$\mathcal{B}^2 \underset{\iota_{\mathcal{D}*}}{\overset{\iota_{\mathcal{D}}^*}{\rightleftarrows}} \mathrm{Ind}(\mathcal{D}) \quad (\perp)$$

where $\mathrm{Ind}(\mathcal{D})$ itself is locally finitely presentable. This gives rise to an idempotent comonad, one could see as returning the left part of a factorization system. In fact, the adjunction $\iota_{\mathcal{D}}^* \dashv \iota_{\mathcal{D}*}$ defines a morphism of locally finitely presentable categories, for the left adjoint $\iota_{\mathcal{D}}^*$ restricts to finitely presented objects, as morphisms in $\mathcal{D}$ are in particular finitely presented in $\mathcal{B}^2$.

It is well known that a left adjoint between locally finitely presented categories sends finitely presented object to finitely presented objects if and only if its right adjoint preserves filtered colimits. In our case, this says that the functor ι_{D*}, which returns the left part of the factorization, preserves filtered colimits. This means that for any filtered diagram $f_{(-)} : I \to \mathcal{B}^2$ of arrows $f_i : B_i \to B_i'$ in $\mathcal{B}^2$, the left part of the filtered colimit is the filtered colimit of the left parts

$$l_{\operatorname*{colim}_{i\in I} f_i} \simeq \operatorname*{colim}_{i\in I} l_{f_i}$$

In particular, if we apply this to the canonical diagram of an arrow $f \simeq \operatorname{colim} \mathcal{B}^2_\omega \downarrow f$, we have

$$l_f \simeq \operatorname*{colim}_{\mathcal{B}^2_\omega \downarrow f} l_k$$

where the colimit is indexed by all the (k, a, a') in $\mathcal{B}^2_\omega \downarrow f$. Hence this proves that arrows of the form l_k for $k \in \mathcal{K}$ form a generator in $\mathrm{Ind}(\mathcal{D})$.

Moreover, objects in $\mathrm{Ind}(\mathcal{D})$ are then models of a finite limit theory, which motivates the following definition:

Definition 2.3.2.4. For a local right adjoint U, a diagonally universal morphism is said to be *axiomatisable* if it lies in $\mathrm{Ind}(\mathcal{D})$.

Remark 2.3.2.5. Hence we have for any local right adjoint U a factorization system $(\mathrm{Ind}(\mathcal{D}), \mathcal{D}^\perp)$. Observe however that in general one cannot force arbitrary diagonally universal morphisms to be decomposable as a filtered colimit of morphisms in $\mathcal{D}$. That is, we only have $\mathrm{Ind}(\mathcal{D}) \subseteq \mathbf{Diag}$ (and hence $\mathbf{Diag}^\perp \subseteq \mathrm{Ind}(\mathcal{D})^\perp$) in the general case. For this reason, with the inclusion of right class a morphism between orthogonality structures, the factorization system $(\mathrm{Ind}(\mathcal{D}), \mathrm{Ind}(\mathcal{D})^\perp)$ is the free left generated factorization system associated to the orthogonality structure $(\mathbf{Diag}, \mathbf{Diag}^\perp)$.

Observe that in the general case, the local units under a given object may not be obtained as filtered colimits of finitely presented diagonally universal morphisms under them. If one successively take the stable factorization and the $(\mathrm{Ind}(\mathcal{D}), \mathcal{D}^\perp)$-factorization

$$\begin{array}{ccc} B & \xrightarrow{f} & U(A) \\ {\scriptstyle \operatorname*{colim}_{\mathcal{D}_B\downarrow\eta_f^A} n}\downarrow & \searrow^{\eta_f^A} & \uparrow{\scriptstyle U(L_A(f))} \\ \operatorname*{colim}_{\mathcal{D}_B\downarrow\eta_f^A} C & \xrightarrow[u_{\eta_f^A}]{} & U(A_f) \end{array}$$

then $u_f = U(L_A(f))u_{\eta_f^A}$ is in $\mathrm{Ind}(\mathcal{D})^\perp$ by uniqueness of the factorization because $\mathbf{Diag}^\perp \subseteq \mathrm{Ind}(\mathcal{D})^\perp$, so that the right part of η_f^A is also the right part of f, but this only says that the functor

$$\mathcal{D}_B \downarrow \eta_f^A \to \mathcal{D}_B \downarrow f$$

is cofinal since it induces the same colimit, without ensuring equality between the local unit and the left part. Remark also that $u_{\eta_f^A}$ is in $\mathbf{Diag} \cap \mathrm{Ind}(\mathcal{D})^\perp$, which however does not force it however to be an isomorphism.

Moreover, this situation cannot even be improved in the case of a right multi-adjoint, where the local units under a given object form a small set. This is why, in all generality, we have to impose explicitly the condition that local units be filtered colimits of finitely presented diagonally universal morphisms above them amongst conditions isolated by Diers to enable the construction of a spectra from a right multi-adjoint.

Definition 2.3.2.6. For a local right adjoint functor U we will say that:

– U is *diagonally axiomatisable* if we have $\mathbf{Diag} = \mathrm{Ind}(\mathcal{D})$, that is, if any diagonally universal morphism is axiomatisable;

– U satisfies *Diers condition* if for any B and any $f : B \to U(A)$, the local unit η^A_f is in $\mathrm{Ind}(\mathcal{D})$, that is, if at least local units are axiomatisable.

Remark 2.3.2.7. Diers condition implies in particular that the stable factorization of an arrow f through its local unit coincides with the $(\mathbf{Et}_U, \mathbf{Loc}_U)$-factorization of f. This is a strictly weaker condition than being diagonally axiomatisable as it does not require any diagonally universal morphism to be axiomatisable.

In [31] were isolated conditions enabling the construction of a point-set notion of spectrum. We sum up those conditions under the following notion:

Definition 2.3.2.8. Define a *Diers context* as the data of

– a locally finitely presentable category $\mathcal{B}$

– a right multi-adjoint functor $U : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$

– satisfying Diers condition: any local unit is the filtered colimit of all the diagonally universal morphisms of finite presentation above it.

In some situations we can produce local right adjoints with the desired property if we start from a left generated factorization system and a class of objects enjoying an adequate "gliding" condition:

Definition 2.3.2.9. Take a functor $U : \mathcal{A} \to \mathcal{B}$ and $\mathcal{R}$ a class of maps in $\mathcal{B}$. We say that U *lifts $\mathcal{R}$ maps* if for any A in $\mathcal{A}$ and any $r : B \to U(A)$, there exists $u : A_0 \to A$ and an isomorphism $\alpha : U(A_0) \simeq B$ such that $r\alpha = U(u)$.

2.3.2.10. Let $\mathcal{V}$ be a saturated class in a locally finitely presentable category $\mathcal{B}$ and $(\mathcal{L}, \mathcal{R})$ the associated left generated system, with $\mathcal{L} = \mathrm{Ind}(\mathcal{V})$ and $\mathcal{R} = \mathcal{V}^\perp$. Now suppose that $U_0 : \mathcal{A}_0 \to \mathcal{B}$ is a functor lifting $\mathcal{R}$-maps. Then define $\iota_0 : \mathcal{A} \hookrightarrow \mathcal{A}_0$ as the wide subcategory whose arrows are those whose image under U_0 is in $\mathcal{R}$ and U as the restriction $U = U_0\iota_0$.

Proposition 2.3.2.11. *Suppose that U_0 is relatively full and faithful; then the induced functor $U : \mathcal{A} \to \mathcal{B}$ is stable and diagonally axiomatisable.*

Proof. For any $f : B \to U(A)$, consider the axiomatisable factorization

$$\begin{array}{ccc} B & \xrightarrow{\quad f \quad} & U(A) \\ & {\scriptstyle l_f}\searrow \quad \nearrow{\scriptstyle r_f} & \\ & C_f & \end{array}$$

where l_f is obtained as the filtered colimit $l_f = \mathrm{colim}\mathcal{D}_B \downarrow f$ in $B \downarrow \mathcal{B}$. For U_0 lifts along $\mathcal{R}$-maps, there exists $u_f : A_f \to A$ in $\mathcal{A}$ sent by U to r_f, and moreover, this morphism is essentially unique in $\mathcal{A}$ as U_0 is relatively full and faithful. But then l_f is the local unit of U, or equivalently, is a candidate for U for it is diagonally universal with its image in the range of U by corollary 2.3.1.8. □

There is also a converse property:

Proposition 2.3.2.12. *Let $U : \mathcal{A} \to \mathcal{B}$ a diagonally axiomatizable right multi-adjoint: then U lifts its local maps.*

Proof. For any local map of the form $r : C \to U(A)$, any precomposition with a diagonally universal morphism with codomain C as below

$$B \xrightarrow{\ l\ } C \xrightarrow{\ r\ } U(A)$$

must also coincide up to iso with its own factorization, for in the square below

$$\begin{array}{ccc} B & \xrightarrow{\eta^A_{rl}} & U(A_{rl}) \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle UL_A(rl)} \\ C & \xrightarrow[r]{} & U(A) \end{array}$$

we have both a diagonal $C \to U(A_{rl})$ induced from the fact that l is diagonally universal, and a diagonal $U(A_{rl}) \to C$ from the fact that η^A_{rl} is diagonally universal and r is a local map, hence is in $\mathbf{Diag}^\perp$ by diagonal axiomatizability. It is easy to see that those maps are mutual inverse. Applying the result before with $l = 1_C$, we get an isomorphism $\eta^A_r : C \simeq U(A_r)$, and we have $r = UL_A(r)\eta^A_r$. □

2.3.3 Stable functor from a factorization system

A typical example of non full multireflection is the following as pointed out in [93]:

Proposition 2.3.3.1. *For any unique factorization system $(\mathcal{L},\mathcal{R})$, the inclusion $\mathcal{R} \hookrightarrow \mathcal{B}$ is a local right adjoint. If moreover $(\mathcal{L},\mathcal{R})$ is left-generated, then $\mathcal{R} \hookrightarrow \mathcal{B}$ is right multi-adjoint.*

Conversely any stable functor (resp. right multi-adjoint) which is surjective on objects and faithful is of this form.

Proof. If we have a factorization system, any morphism $f : B \to \iota_{\mathcal{R}}(A)$ factorizes uniquely as $\iota_{\mathcal{R}}(r)l$ and l is orthogonal to any morphism in $\mathcal{R}$ so we can see it as the desired candidate.

If $U : \mathcal{A} \to \mathcal{B}$ is stable, faithful and surjective on objects, any object is some $U(A)$ and any $f : B \to U(A)$ factorizes through a candidate which is left orthogonal to the morphisms in the range of U, hence the class of candidates constitutes the left part and the morphisms in $U(A^2)$ the right part. □

The following terminology was suggested by Anel, and was also identified in [31][part 4] amongst condition to produce a spectral construction:

Definition 2.3.3.2. Let $\mathcal{R}$ be a class of maps in a category and $\mathcal{A}$ a class of objects. We say that $\mathcal{A}$ has the *glidding property relatively to* $\mathcal{R}$ if for any arrow $l : B \to A$ in $\mathcal{R}$ with A an object of $\mathcal{A}$, then B must also be in $\mathcal{A}$.

Theorem 2.3.3.3. *Let $\mathcal{B}$ be a category equipped with a factorization system $(\mathcal{L},\mathcal{R})$ and $\mathcal{A}$ be a class of objects of $\mathcal{B}$ with the gliding condition relative to $\mathcal{R}$. Then the inclusion $\mathcal{A}^{\mathcal{R}} \hookrightarrow \mathcal{B}$ of $\mathcal{A}$ objects equipped with arrows of $\mathcal{R}$ between them defines a relatively full and faithful stable functor.*

Proof. This follows from the previous proposition: for any B in $\mathcal{B}$ and any arrow $f : B \to A$ with A an object in $\mathcal{A}$, as there exists a unique factorization $f = u_f n_f$ with n_f in $\mathcal{L}$ and u_f in $\mathcal{R}$, then A_f is also an object in $\mathcal{A}$; this factorization is initial amongst those through a morphism in $\mathcal{R}$ on the right. Moreover n_f is a morphism in $\mathcal{L}$ with an objects in $\mathcal{A}$ as codomain, and such arrows are exactly the candidates for the inclusion as for any square with A_0, A_1, A_2 objects in $\mathcal{A}$, we have the diagonalization

$$\begin{array}{ccc} B & \longrightarrow & A_1 \\ {\scriptstyle n}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle u} \\ A_0 & \xrightarrow[u']{} & A_2 \end{array}$$

But recall that $\mathcal{R}$ is left-cancellative as any right class, so that u must itself be in $\mathcal{R}$. Hence in the factorization above n_f is a candidate and the inclusion is stable; left-cancellativity of $\mathcal{R}$ also enforces that this inclusion is relatively full and faithful. □

Chapter 3

Geometries and their avatars

This chapter is devoted to the notion of *geometry*, which is the seed a spectrum grows up from. This notion encapsulates in an algebraic manner the spatial content of the spectral construction, which is just a way to deploy it in an universal way.

In fact there are several, closely related (yet not exactly equivalent) notions that could be qualified as geometries, isolated separately by the few different sources: they are listed and compared in the second section of this chapter. we shall in particular emphasize here the version that originates in [17] "admissibility", though the exact definition we shall use is from [19].

In section 3, we describe the interplay between the logical, algebraic, factorization and topological phenomenology encoded in a geometry. In particular we describe the different roles of the factorization data and the local data, emphasizing successively how they can be interpreted from a topological intuition thanks to the notion of focal spaces, or how they provide an instance of multi-adjunction as defined in the previous chapter.

For this latter aspect, it is enlightening to think of geometries as situations of multi-adjunction: we shall see later how the spectrum can be used to "fix" those multi-adjunctions into global adjunctions. The first section of this chapter gives the ambient philosophy under which this approach makes sense. We prove also how geometries induce accessible multireflection at theorem 3.3.1.12, and how this generalizes to categories of models in arbitrary Grothendieck topoi at theorem 3.3.3.6 - which encapsulate the universality of the admissible factorization.

Finally a last section will be devoted to the notion of *transformation of geometries* and to a global 2-categories of geometries, relatively to which the construction of the spectrum will prove to be 2-functorial.

3.1 The Yoga of Gabriel-Ulmer and the problem of the free object

This first section contains very standard facts about finite limit theories and locally finitely presentable categories. Its purpose is to recall all the equivalent ways to describe a finite limit theory, and how one can toggle between the logical, functorial and toposic ways of seeing models. We emphasize the way the data of a finite limit theory, its syntactic category, its classifying topos and its class of set valued models are mutually determined by giving as much as possible of the involved equivalence of categories, for we shall need to have in mind those translations to make more intuitive the rest of this work.

3.1.1 Finite limit theory and finitely presentable categories

While all the well known examples of the spectral construction are defined relatively to algebraic categories, which are the object of interest of universal algebra and are axiomatized by Lawvere theories, those latter are not the most suited notion to proceed for our purpose. Indeed, when working with topoi as we are going to do, it is more natural to consider *locally finitely presentable categories* and their syntactic counterparts, the *finite limit theories*. The reason is that each Grothendieck topos admits a *standard presentation*, exhibiting it as a category of sheaves over a small site with finite limits, and that the extensions theorem as Diaconescu are formulated relatively to finite limits. In the following, we call functors preserving finite limits *lex functors* and

denote as **Lex** the 2-category of small categories with finite limits, lex functors between them, and natural transformations between lex functors; for concision we also call small categories with finite limits *lex categories*. For a lex category $\mathcal{C}$ and a Grothendieck topos $\mathcal{E}$ we denote as $\mathbf{Lex}[\mathcal{C},\mathcal{E}]$ the category of lex functors and natural transformations between them. We also denote as $\mathcal{S}$ the topos of sets.

3.1.1.1. In the following we fix a first order language $\mathcal{L}$. For a theory $\mathbb{T}$ in $\mathcal{L}$, a sequent is said to be $\mathbb{T}$-provable if it can be deduced from the axioms of $\mathbb{T}$ and the structural rules of first order logic, and we denote as $\vdash_{\mathbb{T}}$ the relation of $\mathbb{T}$-provability. The *relation of* $\mathbb{T}$*-equivalence* is the equivalence relation $\dashv\vdash_{\mathbb{T}}$ on formulas in $\mathcal{L}$ defined as

$$\phi(\overline{x}) \dashv\vdash_{\mathbb{T}} \psi(\overline{x}) \text{ iff } \phi(\overline{x}) \vdash_{\mathbb{T}} \psi(\overline{x}),\ \psi(\overline{x}) \vdash_{\mathbb{T}} \phi(\overline{x})$$

Its classes are denoted as $[\phi(\overline{x})]_{\mathbb{T}}$.

We recall that a *finite limit theory* in $\mathcal{L}$ is a first order theory in $\mathcal{L}$ whose sequents consist of *cartesian formulas*, that are formulas built from atomic formulas with finite conjunctions and existentials with proof of uniqueness. A $\mathbb{T}$-*provably functional formula* in $\mathcal{L}$ is a formula $\theta[\overline{x},\overline{y}]$ such that the following sequents are $\mathbb{T}$-provable:

- $\theta(\overline{x},\overline{y}) \vdash_{\mathbb{T}} \psi(\overline{y})$
- $\phi(\overline{x}) \vdash_{\mathbb{T}} \exists\overline{y}\theta(\overline{x},\overline{y})$
- $\theta(\overline{x},\overline{y}) \wedge \theta(\overline{x},\overline{y}') \vdash_{\mathbb{T}} \overline{y} = \overline{y}'$

A provably functional formula $\theta[\overline{x},\overline{y}]$ should be thought as a statement of equality exhibiting $\overline{y}$ as a function of $\overline{x}$. The first condition says that the image of this function should satisfy the codomain formula; the second says that this function is defined over witnesses of the domain formula, and the third is the condition of functionality, ensuring that an object has a unique image through this function.

3.1.1.2. To any first order theory $\mathbb{T}$ we can associate a category $\mathcal{C}_{\mathbb{T}}$, the *syntactic category of* $\mathbb{T}$, whose

- objects are formulas in context $\{\overline{x},\phi\}$ in the language of $\mathbb{T}$
- morphisms $\{\overline{x},\phi\} \to \{\overline{y},\psi\}$ are equivalence classes $[\theta(\overline{x},\overline{y})]_{\mathbb{T}}$ for the relation of $\mathbb{T}$-provability of functional formulas.

If $\mathbb{T}$ is cartesian then $\mathcal{C}_{\mathbb{T}}$ is a small category with finite limits. Conversely, any lex category $\mathcal{C}$ is the $\mathcal{C}_{\mathbb{T}}$ of some cartesian theory $\mathbb{T}$. However in the following we shall not need to explicitly give a syntactic presentation of the finite limit theory associated to a lex category, so we do not recall the process here.

A *set-valued model* of $\mathbb{T}$ is a lex functor $F : \mathcal{C}_{\mathbb{T}} \to \mathcal{S}$. The intuition behind this definition is that such a functor associates to any formula in context its set of *witnesses* in the corresponding model

$$F(\{\overline{x},\phi\}) = \{\overline{a} \in F \mid\ F \models \phi(\overline{a})\}$$

and to any morphism of the syntactic category, the application

$$F(\{\overline{x},\phi\}) \xrightarrow{F([\theta(\overline{x},\overline{y})]_{\mathbb{T}})} F(\{\overline{y},\psi\})$$

associating to each $\overline{a}$ in $F(\{\overline{x},\phi\})$ the unique $\overline{b}$ in $F(\{\overline{y},\psi\})$ such that $F \models \theta[\overline{a},\overline{b}]$.
A morphism of $\mathcal{L}$-structures between $\mathbb{T}$-models F, G is a natural transformation $F \Rightarrow G$ in $\mathbf{Lex}[\mathcal{C}_{\mathbb{T}},\mathcal{S}]$. Set-valued $\mathbb{T}$-models and morphisms of $\mathcal{L}$-structures between them form a category $\mathbf{Lex}[\mathcal{C}_{\mathbb{T}},\mathcal{S}]$ which we denote as $\mathbb{T}[\mathcal{S}]$.

3.1.1.3. Now we turn to the general properties of the categories of set-valued models of finite limit theories. A *locally finitely presentable category* $\mathcal{B}$ is a category with:

- filtered colimits
- small limits

– a small generator $\mathcal{B}_\omega$ of finitely presented objects such that any object B is the filtered colimit of $\mathcal{B}_\omega \downarrow B$

In particular any locally finitely presentable category also has small colimits. Moreover, the full subcategory $\mathcal{B}_\omega$ of finitely presented objects is closed in $\mathcal{B}$ under finite colimits, and the inclusion $\mathcal{B}_\omega \hookrightarrow \mathcal{B}$ exhibits $\mathcal{B}$ as the inductive completion

$$\mathcal{B} \simeq \mathrm{Ind}(\mathcal{B}_\omega)$$

where $\mathrm{Ind}(\mathcal{B}_\omega)$ is the full subcategory of $[\mathcal{B}_\omega^{\mathrm{op}}, \mathcal{S}]$ consisting of functors that are filtered limits of representables.

A *morphism of locally finitely presentable categories* is a functor $F : \mathcal{B}_1 \to \mathcal{B}_2$ preserving filtered colimits and small limits. It can be shown by variation of Freyd adjoint functor theorem that any such functor has a left adjoint F^* sending finitely presented objects of $\mathcal{B}_2$ to finitely presented objects of $\mathcal{B}_1$. We denote as $\mathcal{LFP}$ the 2-category of locally finitely presentable categories, with natural modifications between locally finitely presentable categories as 2-cells.

3.1.1.4. Locally finitely presentable categories are exactly the categories of models of Cartesian theories; this is the content of Gabriel-Ulmer duality:

$$\begin{array}{rcl} \mathbf{Lex}^{\mathrm{op}} & \simeq & \mathbf{LFP} \\ \mathcal{C} = \mathcal{C}_{\mathbb{T}} & \mapsto & \mathbf{Lex}[\mathcal{C}, \mathcal{S}] \\ \mathcal{B}_\omega^{\mathrm{op}} & \leftarrow\!\shortmid & \mathcal{B} \end{array}$$

where the left to right process associates to any locally finitely presentable category the lex category made of the opposite category of the small subcategory of finitely presented objects. Objects of the small generator of finitely presented objects in $\mathbf{Lex}[\mathcal{C}_{\mathbb{T}}, \mathcal{S}]$ are the corepresentable functors, which we denote as $\exists_{\{\overline{x},\phi\}} = \mathcal{C}_{\mathbb{T}}[\{\overline{x}, \phi\}, -]$. Moreover, for any object B in $\mathcal{B}$, the restriction to $\mathcal{B}_\omega$ of the corresponding representable functor $\lrcorner_B = \mathcal{B}[-, B]$ defines a lex functor

$$\mathcal{B}_\omega^{\mathrm{op}} \xrightarrow{\lrcorner_B} \mathcal{S}$$

This defines an equivalence of category $\mathcal{B} \simeq \mathbf{Lex}[\mathcal{B}_\omega^{\mathrm{op}}, \mathcal{S}]$.

3.1.1.5. For a language $\mathcal{L}$ and any string of variables $\overline{x}$, we can construct the *free* $\mathcal{L}$-structure $\langle \overline{x} \rangle_{\mathcal{L}}$ consisting of all terms in $\mathcal{L}$ over the variables $\overline{x}$. For a finite limit theory $\mathbb{T}$, this can be completed into the free model of $\mathbb{T}$ by forcing the equality of terms that are $\mathbb{T}$-provable in $\mathbb{T}$, and we denote it as $\langle \overline{x} \rangle_{\mathbb{T}}$.

For a formula in context $\{\overline{x}, \phi\}$ in $\mathcal{C}_{\mathbb{T}}$, the corresponding finitely presented set-valued models of $\mathbb{T}$ can be seen as the quotient $\langle \overline{x} \rangle_{\mathbb{T}} / \phi(\overline{x})$. We denote this finitely presented model as K_ϕ, and seen as an object of $\mathbf{Lex}[\mathcal{C}_{\mathbb{T}}, \mathcal{S}]$, K_ϕ coincides with the corepresentable functor $\exists_{\{\overline{x},\phi\}}$.

Because any lex category is the syntactic category of a finite limit theory, Gabriel-Ulmer duality says that any locally finitely presentable $\mathcal{B}$ category is the category of models of a finite limit theory $\mathbb{T}$ such that $\mathbb{T}[\mathcal{S}] \simeq \mathbf{Lex}[\mathcal{C}_{\mathbb{T}}, \mathcal{S}]$. Then it can be shown that any finitely presented model K in $\mathcal{B}$ is uniquely determined by a presentation formula $\{\overline{x}, \phi\}$ with ϕ a formula in the language of $\mathbb{T}$, exhibiting K as $\langle \overline{x} \rangle_{\mathbb{T}} / \phi(\overline{x})$. Similarly, any map of finite presentation $f : K \to K'$ such that $K = \langle \overline{x} \rangle_{\mathbb{T}} / \phi(\overline{x})$ and $K' = \langle \overline{y} \rangle_{\mathbb{T}} / \psi(\overline{y})$ can be presented by a formula $\theta_f(\overline{y}, \overline{x})$ encoding the expression of the image by f of the generators of K in terms of the generators of K': that is, if $\overline{x} = x_1, ..., x_n$ one has $f(x_i) = \tau_i[\overline{y}]$ for $i = 1, ..., n$, then θ_f can be chosen as the formula

$$\theta_f(\overline{y}, \overline{x}) \Leftrightarrow \bigwedge_{i=1,...,n} x_i = \tau_i[\overline{y}]$$

This correspondence is part of an equivalence of categories

$$\mathcal{C}_{\mathbb{T}} \simeq \mathcal{B}_\omega^{\mathrm{op}}$$

In the following we shall denote $K_\phi = \langle \overline{x} \rangle / \phi$ the finitely presented model presented by the formula ϕ and f_θ the finitely presented arrow presented by θ.

Because K_ϕ corresponds to the representable functor $\text{ﻉ}^*_{\{\overline{x},\phi\}}$ in $\mathcal{B} \simeq \mathbf{Lex}[\mathcal{C},\mathcal{S}]$, the Yoneda lemma tells us that any arrow $K_\phi \to B$ in $\mathcal{B}$ is the name of some witness in $\mathcal{B}$ of the formula $\phi(\overline{x})$:

$$\mathcal{B}[K_\phi, B] \simeq B(\{\overline{x},\phi\}) = \{\overline{a} \in B \mid \ B \models \phi(\overline{a})\}$$

In the following, if $\overline{a}$ is a witness in B of some $\phi(\overline{x})$, then we denote as $\ulcorner a \urcorner : K_\phi \to B$ the corresponding arrow in $\mathcal{B}$.

3.1.1.6. As well as we considered set-valued models of a finite limit theory $\mathbb{T}$, we can also consider models in arbitrary Grothendieck topoi: those are the lex functors

$$\mathbb{T}[\mathcal{E}] \simeq \mathbf{Lex}[\mathcal{C}_\mathbb{T}, \mathcal{E}]$$

with natural transformations between them. In the case of finite limit theories, a $\mathbb{T}$-model in a sheaf topos $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}, J)$ is the same as a sheaf of $\mathbb{T}$-models over $(\mathcal{C}, J)$. Indeed, if $F : \mathcal{C}_\mathbb{T} \to \mathcal{E}$ is a $\mathbb{T}$-model, then not only for each $\{\overline{x},\phi\}$ we have a sheaf $F(\{\overline{x},\phi\})$ over $(\mathcal{C}, J)$, sending an object c to a set $F(\{\overline{x},\phi\})(c)$, but for any c in $\mathcal{C}$, we have functor

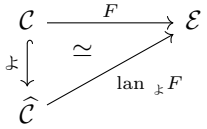
$$\mathcal{C}_\mathbb{T} \overset{F(-)(c)}{\longrightarrow} \mathcal{S}$$

sending any $\{\overline{x},\phi\}$ to $F(\{\overline{x},\phi\})(c)$, and this functor is lex: indeed evaluation at an object preserves (finite) limits, because limits are computed pointwisely in categories of functors. Hence $F(-)(c)$ is a set-valued model of $\mathbb{T}$.

3.1.1.7. Now we turn to the classifying topos for $\mathbb{T}$. For a category $\mathcal{C}$, we denote as $\widehat{\mathcal{C}}$ the presheaf topos $[\mathcal{C}^{\text{op}}, \mathcal{S}]$. For any Grothendieck topos $\mathcal{E}$, we have an equivalence of categories

$$\mathbf{Geom}[\mathcal{E}, \widehat{\mathcal{C}}] \simeq \mathbf{Lex}[\mathcal{C}, \mathcal{E}]$$

sending any lex functor $F : \mathcal{C} \to \mathcal{E}$ to its *Yoneda extension*, that is, the left Kan extension along the Yoneda embedding



where the canonical transformation $F \Rightarrow \text{lan}_{\text{ﻉ}} F \text{ﻉ}$ is invertible for ﻉ being full and faithful. Moreover the functor $\text{lan}_{\text{ﻉ}} F$ is left adjoint to the functor $F_* : \mathcal{E} \to \widehat{\mathcal{C}}$. Moreover $\text{lan}_{\text{ﻉ}} F$ is still lex, hence it defines a geometric morphism $\text{lan}_{\text{ﻉ}} F \dashv F_*$. In the following we shall denote $F^* = \text{lan}_{\text{ﻉ}} F$. The functoriality of this process is ensured by the universal property of the left Kan extension. Conversely any geometric morphism $F^* \dashv F_*$ induces a lex functor $F^* \text{ﻉ} : \mathcal{C} \to \mathcal{E}$.

Hence in particular for any finite limit theory $\mathbb{T}$, we have an equivalence of categories $\mathbb{T}[\mathcal{E}] \simeq \mathbf{Geom}[\mathcal{E}, \widehat{\mathcal{C}_\mathbb{T}}]$. For this reason, we call the presheaf topos $\widehat{\mathcal{C}_\mathbb{T}}$ the *classifying topos of* $\mathbb{T}$ and denote it $\mathcal{S}[\mathbb{T}]$. In particular the set-valued models of $\mathbb{T}$ are the points of the classifying topos $\mathcal{S}[\mathbb{T}]$, that is

$$\mathbb{T}[\mathcal{S}] \simeq \text{Ind}(\mathcal{C}_\mathbb{T}^{\text{op}}) \simeq \mathbf{Lex}[\mathcal{C}_\mathbb{T}, \mathcal{S}] \simeq \mathbf{Geom}[\mathcal{S}, \widehat{\mathcal{C}_\mathbb{T}}]$$

If one starts with a locally finitely presentable category $\mathcal{B}$, this gives the following equivalence of categories

$$\mathcal{B} \simeq \text{Ind}(\mathcal{B}_\omega) \simeq \mathbf{Lex}[\mathcal{B}_\omega^{\text{op}}, \mathcal{S}] \simeq \mathbf{Geom}[\mathcal{S}, \widehat{\mathcal{B}_\omega^{\text{op}}}]$$

3.1.1.8. We also have the following fact for finite limit theories: if a Grothendieck topos $\mathcal{E}$ is equivalent to the sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$, then $\mathbb{T}$-models in $\mathcal{E}$ are exactly sheaves of set-valued $\mathbb{T}$-models over $(\mathcal{C}, J)$, that are functors $F : \mathcal{C}^{\text{op}} \to \mathbb{T}[\mathcal{S}]$ such that for any covering family $(s_i : c_i \to c)_{i \in I}$ in J the restrictions maps of F at the s_i exhibits $F(c)$ as a limit in $\mathbb{T}[\mathcal{S}]$

$$F(c) \simeq \lim \left(\prod_{i \in I} F(c_i) \rightrightarrows \prod_{i,j \in I} F(c_i \times_c c_j) \right)$$

This comes from the fact that evaluation at an object preserves limits, hence in particular finite limits.

3.1.1.9. To sum up, the following notions are equivalent:

– a finite limit theory

– a lex category

– a topos of presheaves over a lex category

– a locally finitely presentable category

In the following we shall fix once for all a finite limit theory $\mathbb{T}$, and denote as $\mathcal{B}$ the associated category of set-valued models, and $\mathcal{S}[\mathbb{T}]$ its classifying topos.

Observe that, from what was said, a finite limit theory can be recovered (up to Morita equivalence) from its categories of models in $\mathcal{S}$: this is a situation where we have "enough set-valued models", in the sense that the points of the classifying topos are *jointly conservative*. In some sense, finite limit theories are determined by their semantics in sets.

3.1.1.10. While the notion of free object is very well known in the context of algebraic categories, locally finitely presentable categories also have a notion of free object. For a finite limit theory $\mathbb{T}$, a finite limit extension $\mathbb{T}'$ of $\mathbb{T}$ is a finite limit theory in the same language $\mathcal{L}$ and whose axioms contain the axioms of $\mathbb{T}$. Then models of $\mathbb{T}'$ are in particular models of $\mathbb{T}$ and it can be checked that the inclusion functor $\iota_{\mathbb{T},\mathbb{T}'} : \mathbb{T}'[\mathcal{S}] \hookrightarrow \mathbb{T}[\mathcal{S}]$ is a morphism of locally finitely presentable categories: hence it admits a left adjoint $\iota_{\mathbb{T},\mathbb{T}'}$, assigning to each $\mathbb{T}$-model B a $\mathbb{T}'$-model $\iota^*_{\mathbb{T},\mathbb{T}'}(B)$ which we can see as the *free* $\mathbb{T}'$-model under B. However, for an arbitrary extension of $\mathbb{T}$ that is not a finite limit theory, it is no longer true that such a free construction exists in $\mathcal{S}$. As we shall see, spectra will provide a way to fix this.

3.1.2 Geometric extensions of finite limit theories and failure of the free construction

In this section, we introduce what will play the role of the local data as announced in the introduction. The previous section was devoted to "well behaved" theories. In this section we give generalities about the more general class of theories whose models will be used as local objects in the context of spectra.

Recall that a *site* is a (small, unless explicitly stated) category equiped with a Grothendieck topology; it is qualified as *lex site* if the underlying category is lex, and *standard site* if it is lex and the topology on it is subcanonical.

3.1.2.1. For a Grothendieck topology J on a category $\mathcal{C}$ and a Grothendieck topos $\mathcal{E}$, a functor $F : \mathcal{C} \to \mathcal{E}$ is said to be *J-continuous* if for any J-covering family $(u_i : c_i \to c)_{i\in I}$, the induced arrow

$$\coprod_{i\in I} F(c_i) \xrightarrow{\langle F(u_i)\rangle_{i\in I}} F(c)$$

is an epimorphism.

As well as the Yoneda extension of a lex functor produced the inverse image part of a geometric morphism in the context of presheaf topoi, lex continuous functors also enjoy an extension property, as stated the first time in [86][Proposition 4.9.4]: for a lex site $(\mathcal{C}, J)$ and a Grothendieck topos $\mathcal{E}$, we have an equivalence of categories

$$\mathbf{Lex}_J[\mathcal{C},\mathcal{E}] \simeq \mathbf{Geom}[\mathcal{E}, \mathbf{Sh}(\mathcal{C},J)]$$

where $\mathbf{Lex}_J[\mathcal{C},\mathcal{E}]$ denotes the category of lex J-continuous functors. Indeed it can be shown that a geometric morphism $F : \mathcal{E} \to \widehat{\mathcal{C}}$ factorizes through $\mathbf{Sh}(\mathcal{C},J)$ if and only if the restriction $F^* \text{よ} : \mathcal{C} \to \mathcal{E}$ is J-continuous. Hence, even when the induced functor $a_J \text{よ} : \mathcal{C} \to \mathbf{Sh}(\mathcal{C},J)$ is not full and faithful, any J-continuous lex functor coincides with the restriction of its extension along $a_J \text{よ}$. When J is subcanonical, a_J is full and faithful and this becomes true for any functor. In particular, we have *points* of $\mathbf{Sh}(\mathcal{C},J)$, which are the geometric functors $\mathcal{S} \to \mathbf{Sh}(\mathcal{C},J)$, are obtained as

$$\mathbf{Lex}_J[\mathcal{C},\mathcal{S}] \simeq \mathbf{Geom}[\mathcal{S}, \mathbf{Sh}(\mathcal{C},J)]$$

3.1.2.2. A *geometric theory* is a theory whose sequents involve formulas built from atomic formulas using finite $\wedge$, arbitrary $\exists$ and small $\bigvee$. Beware that the existentials are no longer supposed to

have a proof of uniqueness, and the disjunction may be infinite as long as they are indexed by a set.

For a geometric theory $\mathbb{T}$, the *finite limit part* of $\mathbb{T}$ is the finite limit $\mathbb{T}_0$ theory made of all the sequents of $\mathbb{T}$ involving formulas containing only conjunctions and existentials with proof of uniqueness. Conversely, for $\mathbb{T}$ a finite limit theory, a *geometric extension* of $\mathbb{T}$ is a geometric theory $\mathbb{T}'$ whose finite limit part is $\mathbb{T}$.

Suppose we have a geometric extension $\mathbb{T} \to \mathbb{T}'$. Then $\mathbb{T}'$ can be shown to be built from $\mathbb{T}$ by adding sequents of the form

$$\phi(\overline{x}) \vdash \bigvee_{i \in I} \exists \overline{y}_i \big(\psi(\overline{y}_i) \wedge \theta_i(\overline{y}_i, x) \big)$$

with each $\theta_i(\overline{y}_i, \overline{x})$ $\mathbb{T}$-provably functional from $\psi(\overline{y}_i)$ to $\phi(\overline{x})$.

Then on the lex site $\mathcal{C}_\mathbb{T}$, one can define a topology $J_{\mathbb{T}'}$ whose covering families consist of the corresponding families

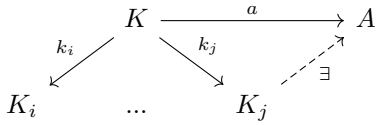
$$\big(\ \{\overline{y}_i, \psi_i\} \xrightarrow{[\theta_i(\overline{y}_i, \overline{x})]_\mathbb{T}} \{\overline{x}, \phi\} \ \big)_{i \in I}$$

Then a model of $\mathbb{T}'$ in a Grothendieck topos $\mathcal{E}$ is a lex functor $F : \mathcal{C}_\mathbb{T} \to \mathcal{S}$ which is moreover $\mathcal{J}_{\mathbb{T}'}$-continuous. In particular, this is a lex functor: this means that we have a full inclusion of categories

$$\mathbb{T}'[\mathcal{S}] \hookrightarrow \mathbb{T}[\mathcal{S}]$$

It will be relevant in the following to characterize models of $\mathbb{T}'$ amongst models of $\mathbb{T}$. The condition of being a model of the geometric extension can be reexpressed by an *injectivity condition*. Observe that, if $\mathcal{B}$ is the locally finitely presentable category $\mathbb{T}[\mathcal{S}]$, those families correspond in $\mathcal{B}^{\mathrm{op}}_\omega$ to families of cone $(f_{\theta_i} : K_\phi \to K_{\psi_i})_{i \in I}$ under the corresponding finitely presented objects. By abuse of notation, we also call J the data of all the families $(f_{\theta_i})_{i \in I}$ in $\mathcal{B}$.

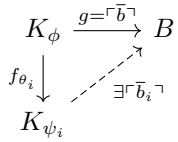
3.1.2.3. For a family J of cones $(k_i : K \to K_i)_{i \in I}$ in $\mathcal{B}$, a *J-local object* (or also *J-injective*) is an object A such that for any cone $(k_i : K \to K_i)_{i \in I}$ in J and any arrow $a : K \to A$, a admits a factorization through some of the k_i



Then for a geometric extension $\mathbb{T}'$ of a finite limit theory $\mathbb{T}$, $\mathbb{T}'$-models are exactly the $J_{\mathbb{T}'}$-local objects. Indeed, in $\mathbb{T}[\mathcal{S}] = \mathbf{Lex}[\mathcal{C}_\mathbb{T}, \mathcal{S}]$, we have $\mathbb{T}[\mathcal{S}](K_\phi, F) \simeq \mathbf{Lex}[\exists_{\{\overline{x}, \phi\}}, F]$ which is $F(\{\overline{x}, \phi\})$ by the Yoneda lemma, hence requiring that $J_{\mathbb{T}'}$-local object exactly amounts to say that for any covering family as above we have a surjection

$$\coprod_{i \in I} F(\{\overline{y}_i, \psi_i\}) \xrightarrow{\langle F([\theta_i(\overline{y}_i, \overline{x})]_\mathbb{T}) \rangle_{i \in I}} F(\{\overline{x}, \phi\})$$

3.1.2.4. Syntactically, this should be understood as follows: for any covering $(f_{\theta_i} : K_\phi \to K_{\psi_i})_{i \in I}$, an arrow $g : K_\phi \to B$ just defines some $\overline{b} \in B$ such that $B \models \phi(\overline{b})$, so as it is a model of $\mathbb{T}_\mathcal{V}$, there is some $i \in I$ and a $\overline{b}_i \in B$ such that $B \models \psi_i(\overline{b}_i)$ and $B \models \theta_i(\overline{b}_i, \overline{b})$). Hence we have the factorization through the name of this witness of ψ_i:



In the other direction, from the equivalence between arrows from finitely presented objects and names of witnesses of their presentation formula in the codomain, it is clear that local objects are models of the geometric extension $\mathbb{T}_J$.

In particular, we have that set-valued models of $\mathbb{T}'$ are the points of the sheaf topos $\mathbf{Sh}(\mathcal{C}_\mathbb{T}, J_{\mathbb{T}'})$.

Conversely, any Grothendieck topology J on $\mathbb{C}_{\mathbb{T}}$ corresponds to a geometric extension of $\mathbb{T}$ whose axioms mirror covering of finitely presented objects:

$$\mathbb{T}_J = \mathbb{T}_{\mathcal{B}} \cup \left\{ \phi(\overline{x}) \vdash \bigvee_{i\in I} \exists \overline{y}_i (\psi_i(\overline{y}_i) \wedge \theta_{f_i}(\overline{y}_i, \overline{x})) \right\}_{\substack{(f_i)_{i\in I} \in J(K_\phi) \\ \{\overline{x},\phi\}\in\mathcal{C}_{\mathbb{T}}}}$$

exhibiting covers as disjunctions of cases for witnesses of domain formulas.

3.1.2.5. Again we can generalize the notion of model of a geometric extension $\mathbb{T}'$ for arbitrary Grothendieck topoi: for a Grothendieck topos $\mathcal{E}$, a $\mathbb{T}'$ *model in* $\mathcal{E}$ is a lex $J_{\mathbb{T}'}$-continuous functor $\mathcal{C}_{\mathbb{T}} \to \mathcal{E}$, while a morphism of $\mathbb{T}'$-models in $\mathcal{E}$ is a natural transformation. Beware that, contrarily to models of finite limit theories, models of arbitrary geometric theories in a topos of sheaves $\mathbf{Sh}(\mathcal{C}, J)$ need not be sheaves of set-valued models over $(\mathcal{C}, J)$. In fact, they are in particular sheaves of set-valued models of their finite limit part, but the additional geometric axioms cannot be enforced objectwisely. However, this can be tested at points: for a point of a topos $p : \mathcal{S} \to \mathcal{E}$ and E in $\mathbb{T}'[\mathcal{E}]$, the composite $p^*E^* : \mathcal{C}_{\mathbb{T}} \to \mathcal{S}$ is $J_{\mathbb{T}'}$-continuous, hence the stalk Fp is in $\mathbb{T}'[\mathcal{S}]$.

We have

$$\mathbb{T}'[\mathcal{E}] = \mathbf{Lex}_{J_{\mathbb{T}'}}[\mathcal{C}_{\mathbb{T}}, \mathcal{E}] \simeq \mathbf{Geom}[\mathcal{E}, \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}'})]$$

where the last equivalence follows from Diaconescu. Moreover, this equivalence is natural in the sense that for any geometric morphism $f : \mathcal{E} \to \mathcal{F}$, post-composition with f induces a functor

$$\mathbb{T}'[\mathcal{F}] \xrightarrow{f^*} \mathbb{T}'[\mathcal{E}]$$

and the same is true for 2-cell of **GTop**. This exhibits $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}'})$ as a representing object for the indexed category

$$\mathbf{GTop}^{\mathrm{op}} \xrightarrow{\mathbb{T}'[-]} \mathbf{Cat}$$

For this reason the sheaf topos $\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}'})$ is called the *classifying topos of* $\mathbb{T}'$ and denoted $\mathcal{S}[\mathbb{T}']$.

In particular we have a subtopos $\mathcal{S}[\mathbb{T}'] \hookrightarrow \mathcal{S}[\mathbb{T}]$ from the inclusion

$$\mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}'}) \hookrightarrow \widehat{\mathcal{C}_{\mathbb{T}}}$$

and for any topos $\mathcal{E}$, the 2-functor $\mathbf{Geom}[\mathcal{E}, -]$ sends this geometric inclusion inclusion into the full inclusion of the categories of models $\mathbb{T}'[\mathcal{E}] \hookrightarrow \mathbb{T}[\mathcal{E}]$. It is well known that for a geometric extension $\mathbb{T} \to \mathbb{T}'$ the inclusion of set-valued models $\mathbb{T}'[\mathcal{S}] \hookrightarrow \mathbb{T}[\mathcal{S}]$ preserves finitely filtered colimits, see for instance [74] or [63].

3.1.2.6. To conclude these prerequisites, observe that the latter inclusion of categories has no longer a left adjoint, contrarily to what we had for extension of finite limit theories. This means that a geometric extension does not enjoy a free construction. Finally, observe also that the inclusion above was full, and in general, it is not possible to axiomatize a non-full subcategory of a category of model of a finite limit theory by a geometric extension. However, our situations of interest will precisely involve both a geometric extension and a choice of maps from a factorization system related by some condition. The next section is devoted to the factorization aspects.

3.2 Geometries

In this section we discuss the relations between the different possible ways to axiomatize the situations producing a notion of spectrum, as they were investigated separately in our main sources [17, 19, 31, 93, 48, 4, 67, 33]. As most of the geometric intuition is already contained in one form or another in each of those axiomatizations, they are in themselves the core of the construction of spectra, which only deploy the geometric information they encode. As we are going to see, these contexts should be classified in three groups:

- [17, 19, 4, 48, 67], which are essentially equivalent and condense in the notion of geometry, differing mostly in the level at which they describe the construction (logical, toposical, geometric), and which depends on the choice of a left-generated factorization system and a Grothendieck topology on a lex category.
- [31, 93] which is more divergent and more "point-set", though taking the form of a purer categorical condition of right multiadjunction into a locally finitely presentable category satisfying an additional condition.

– [33] which relies on the notion of etale class, and which is less dependent on semantics and actually extends to more general situations.

The central condition that enables the construction of the spectrum is a relation entangling factorization and geometric extension. It can be presented in several equivalent ways.

3.2.1 How should geometries be defined ?

The core notion of this chapter, and the one which we later on use in Chapters 5, 6 and 7, is the following. It has been variously called *admissibility triples* in [19] and *geometry* in [67] (in the context of ∞-categories, and with geometric conventions); we shall retain this latter name for its evocative virtue, but the following definition is actually exactly [19][Definition 3.4.2]:

Definition 3.2.1.1. A *geometry* is the data of

– a finite limit theory $\mathbb{T}$

– a saturated class $\mathcal{V}$ in $\mathbb{T}[\mathcal{S}]$

– a Grothendieck pretopology J on $\mathcal{C}_{\mathbb{T}}$ whose covers are made duals of maps in $\mathcal{V}$.

Remark 3.2.1.2. This definition is very close to [67][Definition 1.2.5] and is indeed essentially equivalent in practice, but a few comments should be done here:

– rather than speaking of a finite limit theory $\mathbb{T}$, [67] takes a small lex ($(\infty,1)$-)category: besides it require explicit assumption of Cauchy-completeness because of the wild nature of splittings of idempotents in $(\infty,1)$-categories - which would automatically simplifies for 1-categories - the idea of working directly with a lex category allows a treatment that is more independent of the syntactic details of $\mathbb{T}$: this is more akin to the notion of *Coste-equivalence classes* of geometries as we are going to define them later on at definition 3.4.1.5.

– [67] considers not exactly a saturated class, but only a class of maps (he calls *admissible*) in the lex category that behave mostly like maps of $\mathcal{V}^{\mathrm{op}}$ do in $\mathcal{C}_{\mathbb{T}}$ apart they are only supposed to be closed under composition, left-cancellation, pullback and retract; in our definition, this later condition is an automatic consequence of our assumption that saturated classes are closed under finite colimits in their ambient arrow category. This distinction is actually irrelevant, because what actually mater is the left-generated factorization system associated to this class of maps, and we saw that any class of maps could be closed under this condition without changing the corresponding factorization system.

– in the same vein, [67] refers to a Grothendieck topology generated in the class of maps: we discuss below why this does not change anything in practice.

Remark 3.2.1.3. In fact we could define also a topology as a triple $(\mathbb{T},\mathcal{V},\mathcal{J})$ with $\mathcal{J}$ a Grothendieck topology which is generated by dual maps of $\mathcal{V}$. We cannot of course require all the arrows of a covering sieve to be in $\mathcal{V}^{\mathrm{op}}$, as a sieve is closed under precomposition under arbitrary arrows, while $\mathcal{V}^{\mathrm{op}}$ is not. But would hence require that the Grothendieck topology $\mathcal{J}$ has a basis J made of arrows in $\mathcal{V}^{\mathrm{op}}$ which means that any sieve R in J contains a covering family in the basis J.

In fact we could even start with an arbitrary set of families with common codomain in $\mathcal{V}^{\mathrm{op}}$, without any assumption on their stability, nor requiring each object of $\mathcal{C}_{\mathbb{T}}$ to be the codomain of such a family: this is in general the way the kind of data from which one generates the syntactic topology of a geometric extension - see for instance how the syntactic topology of local rings are generated from just one family over the free object on one element.

Recall first that a *source* is a set of arrows with common domain, while a *sink* is a set of arrows with common codomain. In the following, we shall often refer to the sources in $\mathbb{T}[\mathcal{S}]_\omega$ made of $\mathcal{V}$-maps that are dual to the sinks in $\mathcal{C}_{\mathbb{T}}$ whose covering properties are discussed.

Proposition 3.2.1.4. *Let J_0 be a collection of sinks of $\mathcal{V}^{\mathrm{op}}$-maps in $\mathcal{C}_{\mathbb{T}}$. Then the coverage and the Grothendieck pretopology it generates are made of $\mathcal{V}^{\mathrm{op}}$-arrows, and the Grothendieck topology J it generates defines a geometry $(\mathbb{T},\mathcal{V},J)$*

Proof. Suppose we start from an arbitrary set J_0 of sinks in $\mathcal{V}^{\mathrm{op}}$. Since $\mathcal{V}$ is closed under pushouts along arbitrary finitely presented maps and contains isomorphisms, the coverage J it generates by closing this class is still made of arrows in $\mathcal{V}^{\mathrm{op}}$. Moreover, we can close the coverage it generates under additional axioms without getting out of $\mathcal{V}^{\mathrm{op}}$. Of course, the usual maximality axioms would not be compatible, unless $\mathcal{V}$ contains all finitely presented arrows; however, we can close J_0 under $\mathcal{V}$-*maximality*, that is, by requiring that for each K in $\mathbb{T}[\mathcal{S}]$ the set of objects of $\mathcal{V}_K^{\mathrm{op}}$ consisting of duals of all arrows $n : K \to K'$ in $\mathcal{V}$ is covering. Observe that this is the intersection of the maximal sieve associated to K with the class $\mathcal{V}$. By closure of $\mathcal{V}$ under composition, we also see that one can close J_0 under the transitivity axiom without getting out of $\mathcal{V}^{\mathrm{op}}$. □

Remark 3.2.1.5. In the following we shall abusively denote J the basis of a Grothendieck topology part of a geometry; in particular, we shall only consider the covering families that are in $\mathcal{V}^{\mathrm{op}}$, while the more general sieves may not be considered in practice.

3.2.1.6. In a close manner the notion of *Nisnevich context* was introduced in [4], although this definition does not rely exactly on the same data and is slightly less constricted (beside it is formulated also in the geometric convention, while we follow here the algebraic convention):

Definition 3.2.1.7. A *Nisnevich context* is the data of a factorization system $(\mathbf{Et}, \mathbf{Loc})$ on a locally finite presentable category $\mathcal{B}$ and a class of objects $\mathcal{L}$. Then one defines the $\mathcal{L}$-*localizing families* as the coarsest system of covers $(f_i : B \to B_i)_{i\in I}$ exhibiting $\mathcal{L}$ as local objects, and denote it $J_{\mathcal{L}}$. A Nisnevich context is said to be *compatible* if $\mathcal{L}$ is exactly the class of $J_{\mathcal{L}}$-local objects. Finally, a Nisnevich context is said to be *good* if it is compatible and the factorization system is left-generated.

Remark 3.2.1.8. Beware that there is in general no condition for the localizing topology to be generated from etale maps: this generalization was used for instance to capture geometric situation as the Nisnevich topologies that are not generated from factorization data.

Remark 3.2.1.9. In some sense, Nisnevich contexts arise when the local objects are given before any axiomatization through a geometric extension, and it may happen that the least geometric extension sufficient to axiomatize them has more models than the local objects we started with. In section 2.2 we shall meet similar situation, but from the point of view of the fourth way to axiomatize those situations, that are *Diers context*. However, we do not give their definition right now for we shall devote section 2.2 to their relation with those approaches from which they differ significantly.

We should also make mention of the conditions involved in [33]: here the factorization data have been substituted with an etale class and will be defined for an arbitrary geometric theory:

Definition 3.2.1.10. For $\mathcal{E}$ a Grothendieck topos and $(\mathcal{C}, J)$ a lex, subcanonical site of presentation of $\mathcal{E}$, an etale class $\mathscr{H}$ in $\mathcal{E}$ is said to satisfy the *etale topology condition* if $\mathscr{H}$ is generated from a class of maps $\mathcal{V}$ in $\mathcal{C}$ containing isomorphism, closed under composition and pullback along arbitrary maps, and such that moreover the topology J is generated in $\mathscr{H}$.

Observe this notion is very close, yet slightly stronger, that the notion of geometry. The construction of the spectrum then takes as input the following data:

Definition 3.2.1.11. A *Dubuc context* is the data of a geometric morphism $w : \mathcal{F} \to \mathcal{E}$ and the choice of two presentation sites $(\mathcal{C}_{\mathcal{F}}, J_{\mathcal{F}})$, $(\mathcal{C}_{\mathcal{E}}, J_{\mathcal{E}})$ together with two etale classes $\mathscr{H}$ in $\mathcal{F}$ satisfying the etale topology condition and $\mathcal{B}$ in $\mathcal{E}$ such that $\mathcal{B} \subseteq w(\mathscr{H})$.

Remark 3.2.1.12. Observe that Dubuc contexts are more general than geometries for two reasons:

- $\mathcal{E}$ is not supposed to be the classifying topos of a finite limit theory: it can be the classifier of an arbitrary geometric theory. In particular it is not required to have enough points, so the etale classes may not be induced from a saturated class between finitely presented set-valued models, for those may not even exist.

- A geometric extension $\mathbb{T} \to \mathbb{T}'$ induces geometric inclusion $\mathcal{S}[\mathbb{T}'] \hookrightarrow \mathcal{S}[\mathbb{T}]$; here the geometric morphism is not required to be an inclusion anymore.

3.3 Admissibility condition from a geometry

In fact, the first explicit axiomatization of the relation between ambient and local data predates the different definitions of geometry and was the following:

Definition 3.3.0.1. An *admissibility structure* in the sense of [48][definition 6.57] is the data of

- a finite limit theory $\mathbb{T}$
- a geometric extension $\mathbb{T} \to \mathbb{T}'$
- a 2-functor $\mathbf{Loc} : \mathbf{GTop}^{\mathrm{op}} \to \mathbf{Cat}$ such that for each $\mathcal{E}$ one has $\mathbf{Loc}[\mathcal{E}] \hookrightarrow \mathbb{T}[E]^2$ and $\mathbf{Loc}[\mathcal{E}]$ contains isomorphisms and enjoys left-cancellability.
- and such that we have the condition that for any $\mathcal{E}$ in $\mathbf{GTop}$ and any arrow $f : F \to E$ in $\mathbb{T}[\mathcal{E}]$ with E in $\mathbb{T}'[\mathcal{E}]$ there exists a factorization in $\mathbb{T}[\mathcal{E}]$

$$\begin{array}{ccc} F & \xrightarrow{\quad f \quad} & E \\ & {\scriptstyle n_f} \searrow \quad \nearrow {\scriptstyle u_f} & \\ & H_f & \end{array}$$

  with u_f in $\mathbf{Loc}[\mathcal{E}]$ and H_ϕ in $\mathbb{T}'[\mathcal{E}]$, which is moreover initial amongst all factorizations of f with an arrow in $\mathbf{Loc}[\mathcal{E}]$.

However this definition, while encapsulating in the last condition the key feature that will produce geometry, may seem complicated at first sight. Rather than being something to start with, it is rather what is contained in our base notion.

In the following we deploy the information encoded inside of the notion of geometry and describe how it is sufficient to achieve admissibility. We first focus on the set-valued models and give the topological interpretation of left and right maps, and of the points of the topology. We prove that the category of local objects together with local maps is an accessible multireflective subcategory of the ambient category, and give also a closure theorem under connected limits; we then generalize those results to the categories of models in arbitrary Grothendieck topoi.

Throughout this section, we fix a geometry $(\mathbb{T}, \mathcal{V}, J)$, with J a Grothendieck pretopology. Inspired from [4] terminology, we shall call maps in $\mathrm{Ind}(\mathcal{V})$ (the free inductive completion of $\mathcal{V}$, see remark 1.1.2.3) *etale maps*, and denote their class as $\mathbf{Et}$, while the maps in $\mathcal{V}^\perp$ will be called *local maps* and their class denoted as $\mathbf{Loc}$. They form by proposition 1.1.3.2 a left-generated factorization system denoted as $(\mathbf{Et}, \mathbf{Loc})$. Models of the geometric extension $\mathbb{T}_J$ of $\mathbb{T}$ corresponding to J will be often qualified as *local objects*.

Before going further, we should just give a few terminological remarks: in [19] etale maps are called *extremal* and local maps are called *admissible*; in [67] those are the etale maps of finite presentation that are called *admissible*, and the ones in their pro-completion *pro-admissible*.

3.3.1 Properties of local objects

We begin this section by proving that we have the factorization property of admissibility in $\mathbb{T}[\mathcal{S}]$.

3.3.1.1. Recall that a model of $\mathbb{T}_J$ is an object in $\mathbb{T}[\mathcal{S}]$ which is local relative to the dual in $\mathbb{T}[\mathcal{S}]_\omega$ of covering families in J. But the families dual to cover in J can be extended to the whole category $\mathbb{T}[\mathcal{S}]$ as follows:

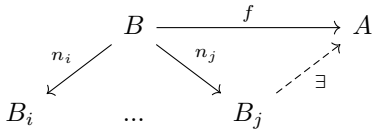
Definition 3.3.1.2. Define the *generalized J-covers* as consisting, for each B in $\mathbb{T}[\mathcal{S}]$, of the families $(n_i : B \to B_i)_{i \in I}$ such that there exists some $a : K \to B$ and some family $(l_i : K \to K_i)_{i \in I}$ dual to a J-cover such that for each $i \in I$ one has a pushout

$$\begin{array}{ccc} K & \xrightarrow{a} & B \\ {\scriptstyle l_i}\downarrow & \ulcorner & \downarrow{\scriptstyle n_i} \\ K_i & \longrightarrow & B_i \end{array}$$

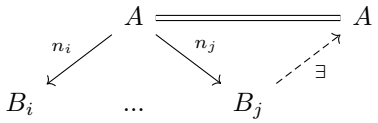
Remark 3.3.1.3. Since we supposed J to be a Grothendieck pretopology, it will be closed under pullback in $\mathcal{C}_{\mathbb{T}}$. Hence in $\mathbb{T}[\mathcal{S}]_\omega$, pushouts of duals of J-covers are still J-covers, hence the generalized coverage does not create new covering families under finitely presented objects.

Then the property of injectiveness of local objects relatively to J-covers extends automatically to those extended covers (which was observed also in [4][Lemma 22]):

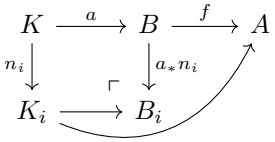
Proposition 3.3.1.4. *An object A in $\mathbb{T}[\mathcal{S}]$ is J-local if and only if for any object B in $\mathbb{T}[\mathcal{S}]$, any arrow $f : B \to A$ and any generalized J-cover $(n_i : B \to B_i)_{i\in I}$, there is a factorization of f for some $i \in I$*



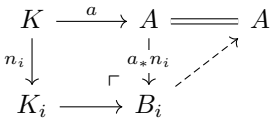
In fact A is J-local if and only if for any generalized cover under it $(n_i : A \to B_i)_{i\in I}$, one has a retraction



Proof. In one sense it is obvious that A is injective relatively to the generalized covers, then in particular it is J-local. The converse is a consequence of the property of the pushout: if A is J-local, then for any generalized family induced from a J cover $(n_i)_{i\in I}$ we have a factorization through some n_i as below



and then an arrow $B_i \to A$ factorizing f by the property of the pushout. Now, if for any generalized cover of A, A is a retract of a member of this cover, then in particular for any $(n_i : K \to K_i)_{i\in I}$ in J and any $a : K \to A$, the pushout of the n_i under A gives a generalized cover of A and we have a factorization for some $i \in I$



so that A is J-local. □

Definition 3.3.1.5. For an object B in $\mathbb{T}[\mathcal{S}]$, a *local form* of B is an etale map $n : B \to A$ toward a J-local object.

Remark 3.3.1.6. Beware that local forms are not required to be finitely presented in general. Etale maps will play the role of saturated compacts of the spectral topology, while finitely presented etale maps will play the role of basic compact opens from which we are going to construct the spectral topology. While this is not apparent in $\mathbb{T}[\mathcal{S}]$ which is "on the algebraic side", this is more intuitive in the opposite category $\mathbb{T}[\mathcal{S}]^{\mathrm{op}}$, whose objects should now be thought of as spaces, where the etale morphisms could be seen as generalized inclusions, and the generalized covers induced from J as covers over objects.

Remark 3.3.1.7. Local objects are like *focal spaces*, that is, spaces with a least point in the specialization order. For instance, in a topological space X and a point $x \in X$, the focal component of X in x is the intersection of all neighborhoods of x, and this is the upset $\uparrow x$ in the specialization order. Local forms behave like inclusions of the form $\uparrow_{\sqsubseteq} x \hookrightarrow X$ as such upsets are unreachable by open covering: indeed, in a cover of $\uparrow_{\sqsubseteq} x$, one open must contain x itself. But as open are up-sets for the specialization order, this open is the whole $\uparrow_{\sqsubseteq} x$. Hence maximal points, as they do not admit non trivial local forms, are alike those x such that $\uparrow_{\sqsubseteq} x = \{x\}$.

In particular, triangles between local forms

$$\begin{array}{ccc} B & \xrightarrow{x_1} & A_1 \\ & \searrow^{x_2} & \downarrow n \\ & & A_2 \end{array}$$

should be seen as coding for specialization order between the corresponding minimal point $x_1 \leq x_2$. Then in $\mathbb{T}[\mathcal{S}]^{\mathrm{op}}$ this will be turned into an inclusion of focal component $\uparrow x_2 \subseteq \uparrow x_1$.

From what was said, we recognize the gliding condition of definition 2.3.3.2 for local objects relatively to local maps:

Lemma 3.3.1.8. *Any object A admitting a local morphism into a local object $u : A \to A_0$ is itself a local object.*

Proof. If $(n_i : A \longrightarrow B_i)_{i \in I}$ is a point covering family of A then its pushout along u is a point covering family for A_0 hence admits a lifting r for some i, so we have a square that diagonalizes because u is local and n_i is in $\mathcal{V}_A$:

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle n_i}\downarrow & \overset{\exists}{\nearrow} & \downarrow{\scriptstyle u} \\ B_i & \xrightarrow[r]{} & A_0 \end{array}$$

□

From this we deduce the following, which is a form of admissibility for set-valued models; a first occurence of this result is for instance [19][Theorem 3.4.1]

Corollary 3.3.1.9. *For any arrow $f : B \to A$ in $\mathbb{T}[\mathcal{S}]$ with A a J-local object, the* $(\mathbf{Et}, \mathbf{Loc})$ *factorization of f*

$$\begin{array}{ccccc} B & & \xrightarrow{f} & & A \\ & {\scriptstyle n_f}\searrow & & \nearrow{\scriptstyle u_f} & \\ & & A_f & & \end{array}$$

returns a J-local object A_f

Hence we know from theorem 2.3.3.3 that the inclusion of local objects and local maps $\iota_{J,\mathbf{Loc}} : \mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}} \to \mathbb{T}[\mathcal{S}]$ is local right adjoint. At first sight it might not be clear whether cones of local units under $\mathbb{T}$-models are small: but in our context accessibility and orthogonality will altogether enhance this result. We have to ensure in a first step accessibility of this inclusion, which reduces in fact to proving that filtered colimits of diagrams of local maps have local colimit inclusions. We shall prove a similar result at theorem 4.1.3.5, yet we choose to give here a first, distinct strategy, in order to see how we can understand it in a functorial way rather than the concrete use of orthogonality as we shall do in theorem 4.1.3.5. Observe that the next lemma is fairly general as it makes no assumption of localness for the objects in the diagrams.

Lemma 3.3.1.10. *The wide subcategory $\mathbb{T}[\mathcal{S}]^{\mathbf{Loc}}$ is closed under filtered colimits in $\mathbb{T}[\mathcal{S}]$.*

Proof. Let $(B_i)_{i \in I}$ be a filtered diagram with local transitions morphisms $(u_d)_{d \in I^2}$ in $\mathbb{T}[\mathcal{S}]^{\mathbf{Loc}}$. Then we want to prove that the colimit inclusions are local: that is, for $n : K \to K'$ in $\mathcal{V}$, we want to prove that for any i the following diagram is a pullback of sets

$$\begin{array}{ccc} B_i(K') & \xrightarrow{(q_i)_{K'}} & \operatorname*{colim}_{i \in I} B_i(K') \\ {\scriptstyle B_i(n)}\downarrow & & \downarrow{\scriptstyle \operatorname*{colim}_{i \in I} B_i(n)} \\ B_i(K) & \xrightarrow[(q_i)_K]{} & \operatorname*{colim}_{i \in I} B_i(K) \end{array}$$

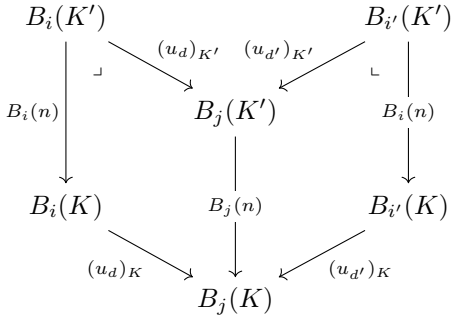
First, as filtered colimits in $\mathbb{T}[\mathcal{S}] \simeq \mathbf{Lex}_J[\mathcal{C}_{\mathbb{T}}, \mathcal{S}]$ are pointwise, $(\mathrm{colim}_{i \in I} B_i)(K)$ is actually for any K the filtered colimit $\mathrm{colim}_{i \in I} B_i(K)$ in $\mathcal{S}$: but for I is filtered we know that this colimit expresses as

$$\operatorname*{colim}_{i \in I} B_i(K) \simeq \coprod_{i \in I} B_i(K)/\theta_K$$

where θ_K consists of all pairs $((i,a),(i',a'))$ with $a \in B_i(K)$, $a' \in B_{i'}(K)$ such that there is $d : i \to j$ and $d' : i' \to j$ such that $u_d(K)(a) = u_{d'}(K)(a')$. Moreover, for $n : K \to K'$, the induced map $\mathrm{colim}_{i\in I} B_i(K') \to \mathrm{colim}_{i\in I} B_i(K)$ is given as

$$\operatorname*{colim}_{i\in I} B_i(n)([(i,a)]_{\theta_{K'}}) = [(i, B_i(n)(a))]_{\theta_K}$$

Then for some $i \in I$ suppose we have some $x \in \mathrm{colim}_{i\in I} B_i(K')$ and $a \in B_i(K)$ such that $(q_i)_K(a) = \mathrm{colim}_{i\in I} B_i(n)(x)$. Take some representing (i',a') for x, so that $x = [(i',a')]_{\theta_{K'}}$. Then our assumption says that $(i,a)\theta_K(i', B_{i'}(n)(a'))$, so there is a span d, d' as above, which induces a span of pullback squares

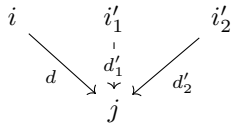


Then we have

$$\begin{aligned}(u_d)_K(a) &= (u_{d'})_K(B_{i'}(n)(a)) \\ &= B_j(n)(u_{d'})_{K'}(a')\end{aligned}$$

But as the square on the left in particular is a pullback (for the u_d are local), the pair $(a, (u_{d'})_{K'}(a'))$ defines a unique b in $B_i(K')$ such that $B_i(n)(b) = a$ and $(u_d)_K(b) = (u_{d'})_{K'}(a')$, then the pair $(a, [(i',a')]_{\theta_{K'}})$ induced uniquely b in a way that does not depend on the choice of the representant (i',a').

If now two pairs $(a_1, [(i'_1,a'_1)]_{\theta_{K'}})$ and $(a_2, [(i'_2,a'_2)]_{\theta_{K'}})$ with $a_1, a_2 \in B_i(K)$ induce the same b in $B_i(K')$, then by filteredness of I there is a triple span



and we have

$$(u_{d'_1})_{K'}(a'_1) = (u_d)_{K'}(b) = (u_{d'_2})_{K'}(a'_2)$$

so that in fact we had $(i'_1,a'_1)\theta_{K'}(i'_2,a'_2)$, while on the other side we have $a_1 = B_i(n)(b) = a_2$.

Conversely, take b_1, b_2 in $B_i(K')$ such that $(i,b_1)\theta_{K'}(i,b_2)$ and $B_i(n)(b_1) = B_i(n)(b_2)$. Then there is some parallel pair $d_1, d_2 : i \rightrightarrows j$ such that $(u_{d_1})_{K'}(b_1) = (u_{d_2})_{K'}(b_2)$. But since I is filtered d_1, d_2 are equalized by a further map $d : j \to j'$, and since the following square is a pullback

$$\begin{array}{ccc} B_i(K') & \xrightarrow{(u_{dd_1})_K} & B_{j'}(K') \\ {\scriptstyle B_i(n)}\downarrow & \lrcorner & \downarrow{\scriptstyle B_{j'}(n)} \\ B_i(K) & \xrightarrow[(u_{dd_2})_{K'}]{} & B_{j'}(K) \end{array}$$

the condition $(u_{dd'_1})_{K'}(b_1) = (u_{dd'_2})_{K'}(b_2)$ and $B_i(n)(b_1) = B_i(n)(b_2)$ jointly force $b_1 = b_2$. This achieves to prove that we have a pullback of sets as expected. □

Lemma 3.3.1.11. *$\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ is closed under filtered colimits in $\mathbb{T}[\mathcal{S}]$.*

Proof. We know that the full subcategory $\mathbb{T}_J[\mathcal{S}]$ is closed under filtered colimits in $\mathbb{T}[\mathcal{S}]$. Hence one has to check that the colimit inclusions of a diagram of local objects and local maps between them are still local, which was proven in the previous lemma. □

Hence we have gathered the necessary results to give the main result of this subsection:

Theorem 3.3.1.12. *Let $(\mathbb{T}, \mathcal{V}, J)$ be a geometry. Then we have a right-multiadjoint*

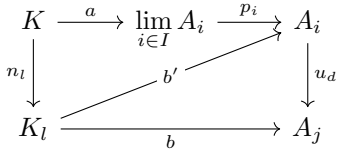
$$\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}} \xhookrightarrow{\iota_{J,\mathbf{Loc}}} \mathbb{T}[\mathcal{S}]$$

Proof. From what precedes we know that $\iota_{J,\mathbf{Loc}}$ is a stable functor and hence is a local right adjoint, which is moreover relatively full and faithful by left cancellation of local maps. Now we have to prove that it satisfies the solution set condition to be right multi-adjoint from proposition 2.1.3.4. But it is sufficient to prove accessibility, as accessible functors always satisfy the solution set condition. This was done in the previous lemma. □

Hence, though local objects are not to be closed under limits in general, we can deduce the following result, though we shall be also able to deduce it from a general statement in the next chapter. However we find relevant to give an explicit proof here to see what are the concrete reason that make it work:

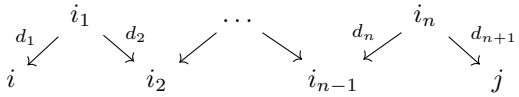
Proposition 3.3.1.13. *$\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ is closed under connected colimits in $\mathbb{T}[\mathcal{S}]$. In fact, more generally, $\mathbb{T}[\mathcal{S}]^{\mathbf{Loc}}$ is also closed under connected limits.*

Proof. Let I be a small connected category and $(A_i)_{i\in I}$ and $(u_d)_{d\in I^2}$ an I-indexed diagram of local objects and local maps. Let $(n_l : K \to K_l)_{l\in L}$ be a J-cover of etale maps: then consider $a : K \to \lim_{i\in I} A_i$: then for each $i \in I$ the composite $p_i a$ with the limit projection p_i factorizes through some n_l. We must prove that one can choose a certain l factorizing all the $p_i a$ to get a cone over the A_i. As the transition morphisms are local, for any $d : i \to i'$ in I, $p_i a$ and $p_j a$ factorize through the same l: indeed, if $p_j a = u_d p_i a$ factorizes as bn_l, one has a diagonalization



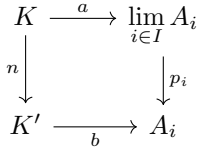
and by uniqueness of b', any other such factorization through n_l of $p_i a$ was equal to b'.

But now, as I is connected, for any two i, j in I there is a finite zigzag



and by what precedes $p_i a$ and $p_{i_1} a$ factorize through the same n_l, but the factorization of $p_{i_1} a$ induces a factorization of $p_{i_2} a$ through n_l, and so on for $1, ..., n$: hence the same n_l factorizes $p_i a$, $p_j a$ for any two i, j, and from what precedes, this defines a cone on the A_i, and hence a morphism $K_l \to \lim_{i\in I} A_i$ as desired.

Now we must prove that the limit projections are local. Take a square as below



with n in $\mathcal{V}$. Then again we can deduce from those data a cone over the A_i with K' as tip: for any other $j \in I$, there is a zigzag connecting it to i, and the same argument as above ensures that one can uniquely lift b to a map $b' : K \to A_j$, and construct a cone as desired inducing a unique lift $K' \to \lim_{i\in I} A_i$ as desired. Observe that this does not depend on the A_i being local: hence it is true in general that a connected limit of a diagram with local transition morphisms has local projections.

Hence $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ has connected limits, which are preserved by $\iota_{J,\mathbf{Loc}}$ as it is a right multi-adjoint. Similarly for the wide subcategory consisting of local maps. □

Remark 3.3.1.14. In the next chapter, we shall recall Diers notion of *locally finitely multipresentable categories*, the categories of the form $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ are examples of, and see the results we identified above are actually part of their general properties.

3.3.2 Etale and local maps between models in arbitrary topoi

Now we want to understand how this admissibility condition induced from a geometry is inherited in arbitrary topoi. The main source for this part is [19].

3.3.2.1. For a geometry $(\mathbb{T}, \mathcal{V}, J)$, with $(\mathbf{Et}, \mathbf{Loc})$ the associated factorization system, the class of finitely presented etale maps $\mathcal{V}$ in $\mathbb{T}[\mathcal{S}]_\omega$ is dual to a class of morphisms in the syntactic category $\mathcal{C}_\mathbb{T}$. That is, an arrow $n : K_\phi \to K_\psi$ (with ϕ, ψ the presentation formulas of the domain and codomains) in $\mathcal{V}$ corresponds to an arrow $[\theta_n(\overline{x}, \overline{y})]_\mathbb{T} : \{\overline{y}, \psi\} \to \{\overline{x}, \phi\}$, which, as a $\mathbb{T}$-provably functional formula, should be seen as a function symbol coding for an operation.

3.3.2.2. Moreover, from the definition of a saturated class, the category $\mathcal{V}^{\mathrm{op}}$ has finite limits, hence codes for a finite limit theory which admits as classifying topos $\widehat{\mathcal{V}^{\mathrm{op}}} = [\mathcal{V}, \mathcal{S}]$ which we denote as $\mathcal{S}[\mathbf{Et}]$. In particular this allows us to define for each Grothendieck topos $\mathcal{E}$ a class of arrows $\mathbf{Et}[\mathcal{E}]$ in $\mathbb{T}[\mathcal{E}]$ as

$$\mathbf{Et}[\mathcal{E}] \simeq \mathbf{Geom}\big[\mathcal{E}, \mathcal{S}[\mathbf{Et}]\big] \simeq \mathbf{Lex}[\mathcal{V}^{\mathrm{op}}, \mathcal{E}]$$

Remark 3.3.2.3. Beware that $\mathbf{Et}[\mathcal{S}] \hookrightarrow \mathbb{T}[\mathcal{S}]^2$ is a full inclusion: that is, seen as a category with etale maps in $\mathbb{T}[\mathcal{S}]$ as objects, its morphisms are squares

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle n_1}\downarrow & & \downarrow{\scriptstyle n_2} \\ E'_1 & \xrightarrow[f']{} & E'_2 \end{array}$$

where f, f' are not assumed to be etale.

Let us look at the counterpart of this for the right class **Loc** of local maps.

Proposition 3.3.2.4. *An arrow $u : B_1 \to B_2$ is in* **Loc** *if and only if it satisfies the following condition:*

$$\textit{if} \begin{cases} \forall n : K_\phi \to K_\psi \textit{ in } \mathcal{V} \\ \forall \overline{a} \in B_1(\{\overline{x}, \phi\}) \\ \forall \overline{b} \in B_2(\{\overline{y}, \psi\}) \textit{ such that } B_2 \models \theta_n(\overline{u(a)}, \overline{b}) \end{cases} \textit{then } \exists! \overline{c} \in A(\{\overline{y}, \psi\}) \begin{cases} B_1 \models \theta_n(\overline{a}, \overline{c}) \\ \overline{u(c)} = \overline{b} \end{cases}$$

Proof. First, note that the situation expressed by the conditions in the characterization just amounts to the commutativity of the following square

$$\begin{array}{ccc} K_\phi & \xrightarrow{\ulcorner a \urcorner} & B_1 \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle u} \\ K_\psi & \xrightarrow[\ulcorner b \urcorner]{} & B_2 \end{array}$$

as the composite $u \circ \ulcorner a \urcorner$ corresponds to the element $\overline{u(a)}$ while the composite $\ulcorner b \urcorner \circ n$ expresses the fact that

$$B_2 \models \phi(\tau_1[\overline{b}/\overline{y}], ..., \tau_m[\overline{b}/\overline{y}])$$

and the condition $B_2 \models \theta_n(\overline{u(a)}, \overline{b})$ means that

$$g(a_1) = \tau_1[\overline{b}/\overline{y}] \wedge ... \wedge g(a_m) = \tau_m[\overline{b}/\overline{y}]$$

hence that the square commutes.

Consequently, if u is local, such a square always diagonalizes uniquely: the diagonal provides us with a witness $\overline{c}$ of ψ in B_1, and the commutation of the down-right triangle expresses the equality $\overline{u(c)} = \overline{b}$ while the up-left one expresses that in B_1 we have

$$a_1 = \tau_1[\overline{c}/\overline{y}] \wedge ... \wedge a_m = \tau_m[\overline{c}/\overline{y}]$$

If conversely u satisfies the syntactic characterization, then the map g defined on the generators as $g(y_1) = c_1, ..., g(y_m) = c_m$ provides a diagonalization

$$\begin{array}{ccc} K_\phi & \xrightarrow{\ulcorner a \urcorner} & B_1 \\ {\scriptstyle n}\downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle u} \\ K_\psi & \xrightarrow[\ulcorner b \urcorner]{} & B_2 \end{array}$$

□

Remark 3.3.2.5. As first explained in [19], the glosis of this result is that local morphisms are those which "reflect" elements produced through the operation coded by finitely presented etale arrows. While morphisms in $\mathcal{V}$ code for function symbols that produce witnesses of their codomain formula from witnesses of their domain formula, local morphisms do not add new witness of propositions presenting the codomain, and reflect the one that already exists.

Observe that the condition above defined a cartesian sequent for the existential in the syntactic characterization was unique. This suggests that local arrows are definable by a finite limit theory.

The process above allows to define a theory for local maps, proving they are axiomatized not only by a geometric, but in fact a cartesian theory.

Remark 3.3.2.6. The condition above says in particular by the Yoneda lemma that

$$\begin{array}{ccc} \mathbb{T}[\mathcal{S}]\big[K_\psi, A\big] & \longrightarrow & \mathbb{T}[\mathcal{S}]\big[K_\psi, B\big] \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{T}[\mathcal{S}]\big[K_\phi, A\big] & \longrightarrow & \mathbb{T}[\mathcal{S}]\big[K_\phi, B\big] \end{array}$$

But observe that

$$\mathbb{T}[\mathcal{S}]\big[K_\psi, A\big] \simeq \mathbf{Lex}[\mathcal{C}_\mathbb{T}, \mathcal{S}]\big[\exists_{\{\overline{y},\psi\}}, A\big] \simeq A(\{\overline{y}, \psi\})$$

and similarly for the other homset. This last formulation still makes sense even if we consider left exact functors with value in other topoi than $\mathcal{S}$: and as we are going to deal with sheaves of local objects in different Grothendieck topoi, this motivates the following generalization of admissibility amongst $\mathbb{T}[\mathcal{S}]$ in arbitrary Grothendieck topoi.

Definition 3.3.2.7. For a Grothendieck topos $\mathcal{E}$, a *local arrow* in $\mathcal{E}$ is a natural transformation

$$\mathcal{C}_\mathbb{T} \;\underset{E}{\overset{F}{\Downarrow u}}\; \mathcal{E}$$

in $\mathbf{Lex}[\mathcal{C}_\mathbb{T}, \mathcal{E}]$ whose naturality square at a morphism $[\theta_n(\overline{x}, \overline{y})]_\mathbb{T} : \{\overline{y}, \psi\} \to \{\overline{x}, \phi\}$ dual of a morphism n in $\mathcal{V}$ is a pullback in $\mathcal{E}$

$$\begin{array}{ccc} F(\{\overline{y}, \psi\}) & \xrightarrow{u_{\{\overline{y},\psi\}}} & E(\{\overline{y}, \psi\}) \\ {\scriptstyle F([\theta_n(\overline{x},\overline{y})]_\mathbb{T})}\downarrow & \lrcorner & \downarrow{\scriptstyle E([\theta_n(\overline{x},\overline{y})]_\mathbb{T})} \\ F(\{\overline{x}, \phi\}) & \xrightarrow[u_{\{\overline{x},\phi\}}]{} & E(\{\overline{x}, \phi\}) \end{array}$$

In the following we denote as $\mathbf{Loc}[\mathcal{E}]$ the class of local morphisms in $\mathcal{E}$.

Remark 3.3.2.8. A terminological remark: local maps are called *admissible* in [19], and *infinitesimal extension* in [33], by analogy with infinitesimal extensions in the theory of formally etale morphisms. Beware that in [67], "admissible map" actually designates the (duals of) etale maps.

Proposition 3.3.2.9. *Local morphisms are stable under inverse image: any geometric morphism $f : \mathcal{F} \to \mathcal{E}$ induces a functor $f^* : \mathbf{Loc}[\mathcal{E}] \to \mathbf{Loc}[\mathcal{F}]$.*

Proof. The inverse image $f^* : \mathcal{E} \to \mathcal{F}$ is lex, hence preserves pullbacks. Then at each morphism n in $\mathcal{V}$, f^* sends the pullback above in $\mathcal{E}$ to a pullback in $\mathcal{F}$. Hence f^*u is local. □

Now, recall from section 1 that models of finite limit theories in sheaf topoi were sheaves of set-valued models over the base site. In particular this suggests the following (which was stated in [19] without proof):

Proposition 3.3.2.10. *If $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}_\mathcal{E}, J_\mathcal{E})$, then a transformation $u : F \to E$ in $\mathcal{E}$ is a local transformation if and only if for any object c in $\mathcal{C}$, the component $u_c : F(c) \to E(c)$ is in $\mathbf{Loc}$.*

Proof. For any etale map $n : K_\phi \to K_\psi$ of finite presentation, we have by naturality a square of morphisms of sheaves on $(\mathcal{C}_\mathcal{E}, J_\mathcal{E})$ as depicted above. Now since evaluation of sheaves creates limits, each $u(c)$ is local as a morphism in $\mathbb{T}[\mathcal{S}]$ if and only if the following square is a pullback

$$\begin{array}{ccc} F(\{\overline{y}, \psi\})(c) & \xrightarrow{u_{\{\overline{y},\psi\}_\mathcal{E}}} & E(\{\overline{y}, \psi\})(c) \\ {\scriptstyle F([\theta_n(\overline{x},\overline{y})]_\mathbb{T})_c}\downarrow & \lrcorner & \downarrow{\scriptstyle E([\theta_n(\overline{x},\overline{y})]_\mathbb{T})_c} \\ F(\{\overline{x}, \phi\})(c) & \xrightarrow[u_{\{\overline{x},\phi\}_c}]{} & E(\{\overline{x}, \phi\})(c) \end{array}$$

But this exactly says that the evaluation at c, that is, the natural transformation

$$\mathcal{C}_{\mathbb{T}} \underset{E(c)}{\overset{F(c)}{\Downarrow u_c}} \mathcal{S}$$

is a local map in $\mathbf{Lex}[\mathcal{C}_{\mathbb{T}}, \mathcal{S}]$. □

Now it appears that the factorization structure $(\mathbf{Et}, \mathbf{Loc})$ generated from $\mathcal{V}$ in $\mathbb{T}[\mathcal{S}]$ is inherited in the category $\mathbb{T}[\mathcal{E}]$ in any Grothendieck topos, and moreover in a functorial and point-wise way. The point-wiseness of this factorization was first established at [19][Theorem 3.6.3].

Proposition 3.3.2.11. *For any Grothendieck topos $\mathcal{E}$ with a standard site of presentation $(\mathcal{C}_{\mathcal{E}}, J_{\mathcal{E}})$, we have a factorization system $(\mathbf{Et}[\mathcal{E}], \mathbf{Loc}[\mathcal{E}])$ in $\mathbb{T}[\mathcal{E}]$. Moreover, for any $f : \mathcal{F} \to \mathcal{E}$ we have adjunctions*

$$\mathbf{Et}[\mathcal{E}] \underset{f_*}{\overset{f^*}{\perp}} \mathbf{Et}[\mathcal{F}] \qquad \mathbf{Loc}[\mathcal{E}] \underset{f_*}{\overset{f^*}{\perp}} \mathbf{Loc}[\mathcal{F}]$$

Proof. In fact the desired factorization is pointwise relatively to the definition site of $\mathcal{E}$. Let $f : F \to E$ be in $\mathbb{T}[\mathcal{E}]$. For any c in $\mathcal{C}_{\mathcal{E}}$, we have a morphism f_c which admits a factorization in $\mathbb{T}[\mathcal{S}]$

$$\begin{array}{ccc} F(c) & \xrightarrow{f_c} & F'(c) \\ & {\scriptstyle n_{f_c}} \searrow \quad \nearrow {\scriptstyle u_{f_c}} & \\ & H_{f_c} & \end{array}$$

with n_{f_c} etale and u_{f_c} local in $\mathbb{T}[\mathcal{S}]$. As the factorization system $(\mathbf{Et}, \mathbf{Loc})$ is functorial, this process is itself functorial and defines a presheaf H_f over $\mathcal{C}_{\mathcal{E}}$ sending c to the intermediate objet H_{f_c}. The desired object is just its sheafification $\mathfrak{a}_{J_{\mathcal{E}}} H_f$, that is, the factorization in $\mathcal{E}$ is obtained as

$$\begin{array}{ccc} F & \xrightarrow{f} & E \\ & {\scriptstyle n_f} \searrow \quad \nearrow {\scriptstyle u_f} & \\ & \mathfrak{a}_{J_{\mathcal{E}}} H_f & \end{array}$$

where n_f, u_f are defined from the data of $(n_{f_c})_{c \in \mathcal{C}_{\mathcal{E}}}$, $(u_{f_c})_{c \in \mathcal{C}_{\mathcal{E}}}$ respectively as the composite in $\widehat{\mathcal{C}_{\mathcal{E}}}$ with the sheafification unit and the induced map through its universal property

$$\begin{array}{ccc} F & \xrightarrow{n_f} & \mathfrak{a}_{J_{\mathcal{E}}} H_f \\ & {\scriptstyle (n_{f_c})_{c \in \mathcal{C}_{\mathcal{E}}}} \searrow \quad \nearrow {\scriptstyle \gamma^{H_f}} & \\ & H_f & \end{array} \qquad \begin{array}{ccc} & \mathfrak{a}_{J_{\mathcal{E}}} H_f & \\ {\scriptstyle \gamma^{H_f}} \nearrow & & \searrow {\scriptstyle u_f} \\ H_f & \xrightarrow[(u_{f_c})_{c \in \mathcal{C}_{\mathcal{E}}}]{} & E \end{array}$$

Beware that subcanonicity of $J_{\mathcal{E}}$ is required to ensure that the composite n_f is in $\mathbf{Sh}(\mathcal{C}_{\mathcal{E}}, J_{\mathcal{E}})$. □

Remark 3.3.2.12. The functoriality of this result might look surprising at first sight. If it is expected that inverse image, as left adjoints, preserves etales maps for they are a left class, and dually, that direct image, as right adjoints, preserve local maps for they are a right class, the preservation of local maps by inverse image and etale map by direct image are consequence of the specific fact that we are in a left-generated factorization system, exhibiting both etale and local maps as models of finite limit theories, hence stable either under direct and inverse images.

Now we should discuss the link with [33]. As like as a saturated class in $\mathbb{T}[\mathcal{S}]_\omega$ induces also a right class in the categories of models $\mathbb{T}[\mathcal{F}]$ for an arbitrary topos $\mathcal{F}$, an etale class in $\mathcal{S}[\mathbb{T}]$ induces a "right-like" class of maps in any category of models $\mathbb{T}[\mathcal{F}]$:

Definition 3.3.2.13. Let $\mathcal{A}$ be an etale class in a Grothendieck topos E. Then a 2-cell $\phi : F \Rightarrow E$ in $\mathbf{GTop}[\mathcal{F}, \mathcal{E}]$ is called an $\mathcal{A}$-*infinitesimal extension* if its naturality square at any h in $\mathcal{A}$ is a pullback.

Observe that this is the same condition as for local arrows, except that it puts no conditions on the topos and tests relatively to the etale class rather than the saturated class. But this makes no difference because of the following:

Proposition 3.3.2.14. *If one has an embedding $\iota : \mathcal{E} \hookrightarrow \mathcal{E}_0$ and an etale class $\mathcal{A}$ in $\mathcal{E}_0$, then $\phi : F \Rightarrow E$ in $\mathbf{GTop}[\mathcal{F}, \mathcal{E}]$ is an $\iota^{-1}(\mathcal{A})$-infinitesimal extension if and only if $\iota * \phi$ is a $\mathcal{A}$-infinitesimal extension in $\mathbf{GTop}[\mathcal{F}, \mathcal{E}_0]$.*

Proposition 3.3.2.15. *If $\mathcal{E}$ has a standard site of presentation $(\mathcal{C}_0, J_0)$ and $\mathcal{A}$ is an etale class generated by a class Λ in $\mathcal{C}_0$ satisfying the etale topology condition. Then we have $\mathcal{A} = \iota_{\mathcal{E}}^{-1}(\mathcal{A}_0)$ where $\mathcal{A}_0$ is the etale class generated in $\widehat{\mathcal{C}_0}$ by $\yoneda(\Lambda)$ and $\iota_{\mathcal{E}} : \mathcal{E} \hookrightarrow \widehat{\mathcal{C}_0}$ is the canonical embedding.*

Moreover for any $\phi : F \Rightarrow E$ in $\mathbf{GTop}[\mathcal{F}, \mathcal{E}]$ the following are equivalent:

- *ϕ is an $\mathcal{A}$-infinitesimal extension*
- *$\iota * \phi$ is a $\mathcal{A}_0$-infinitesimal extension*
- *ϕ is local relatively to Λ.*

3.3.3 Admissibility in arbitrary topoi

We have just seen that the factorization structure is inherited by the category of models in any Grothendieck topos. But the admissibility structure itself is inherited.

Proposition 3.3.3.1. *Let $\mathcal{E}$ be a Grothendieck topos and $u : F \to E$ in $\mathbf{Loc}[\mathcal{E}]$ with E in $\mathbb{T}_J[\mathcal{E}]$: then F itself is in $\mathbb{T}_J[\mathcal{E}]$.*

Proof. F defines in particular a lex functor from $\mathcal{C}_{\mathbb{T}}$ into $\mathcal{E}$. Hence we have to prove that F is also J continuous. Let $(n_i : \{\overline{x}_i, \phi_i\} \to \{\overline{x}, \phi\})_{i \in I} \in J$: those are finitely presented maps in the saturated class $\mathcal{V}$, hence as u is a local transformation, the naturality square in each k_i expresses each $F(\{\overline{x}_i, \phi_i\})$ as a pullback

$$\begin{array}{ccc} F(\{\overline{x}_i, \phi_i\}) & \xrightarrow{F([\theta_i(\overline{x},\overline{x}_i)]_{\mathbb{T}})} & F(\{\overline{x}, \phi\}) \\ {\scriptstyle u_{\{\overline{x}_i,\phi_i\}}}\downarrow & \lrcorner & \downarrow{\scriptstyle u_{\{\overline{x},\phi\}}} \\ E(\{\overline{x}_i, \phi_i\}) & \xrightarrow[E([\theta_i(\overline{x},\overline{x}_i)]_{\mathbb{T}})]{} & E(\{\overline{x}, \phi\}) \end{array}$$

As this inverse image lands in the topos $\mathcal{E}$, where, as in any Grothendieck topos, colimits are stable by pullback, one has

$$\begin{aligned} \coprod_{i \in I} F(\{\overline{x}_i, \phi_i\}) &\simeq \coprod_{i \in I} \Big(E(\{\overline{x}_i, \phi_i\}) \times_{E(\{\overline{x},\phi\})} F(\{\overline{x}, \phi\}) \Big) \\ &\simeq \Big(\coprod_{i \in I} E(\{\overline{x}_i, \phi_i\}) \Big) \times_{E(\{\overline{x},\phi\})} F(\{\overline{x}, \phi\}) \end{aligned}$$

But E, as a local object, transforms covers into jointly epimorphic families. Hence the lower arrow of the corresponding pullback square is an epimorphism

$$\begin{array}{ccc} \coprod_{i \in I} F(\{\overline{x}_i, \phi_i\}) & \xrightarrow{F([\theta_i(\overline{x},\overline{x}_i)]_{\mathbb{T}})} & F(\{\overline{x}, \phi\}) \\ {\scriptstyle \langle u_{\{\overline{x}_i,\phi_i\}} \rangle_{i \in I}}\downarrow & \lrcorner & \downarrow{\scriptstyle u_{\{\overline{x},\phi\}}} \\ \coprod_{i \in I} E(\{\overline{x}_i, \phi_i\}) & \xrightarrow[E([\theta_i(\overline{x},\overline{x}_i)]_{\mathbb{T}})]{} & E(\{\overline{x}, \phi\}) \end{array}$$

But in a Grothendieck topos, pullbacks of epi are epi; hence the upper arrow in this square in also an epimorphism and this confirms that F is also a local object. □

This says that for any Grothendieck topos $\mathcal{E}$, the category of $\mathbb{T}$-models in $\mathcal{E}$ inherits the admissibility structure defined by the geometry $(\mathbb{T}, \mathcal{V}, J)$

Corollary 3.3.3.2. *Let $\mathcal{E}$ be a Grothendieck topos: then for any $f : F \to E$ in $\mathbb{T}[\mathcal{S}]$ with E in $\mathbb{T}_J[\mathcal{E}]$, then in the $(\mathbf{Et}[\mathcal{E}], \mathbf{Loc}[\mathcal{E}])$-factorization*

$$\begin{array}{ccc} F & \xrightarrow{\quad f \quad} & E \\ & {\scriptstyle n_f}\searrow \quad \nearrow{\scriptstyle u_f} & \\ & H_f & \end{array}$$

the middle term is in $\mathbb{T}_J[\mathcal{E}]$.

This proves that a geometry as defined produces an admissibility structure. This can be encapsulated in the following:

Corollary 3.3.3.3. *For any Grothendieck topos $\mathcal{E}$, we have a relatively full and faithful stable inclusion $\iota_{J,\mathbf{Loc}}[\mathcal{E}] : \mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}} \hookrightarrow \mathbb{T}[\mathcal{E}]$.*

We want to strengthen this result to get a generalization of theorem 3.3.1.12, which can be done by proving that the inclusion $\iota_{J,\mathbf{Loc}}[\mathcal{E}]$ is finitely accessible and exploiting then the existence of a solution set condition for accessible functors. In fact those generalizations will proceed from the fact that the corresponding statements are true in the set-valued case and will be tested pointwisely. It is practical to first prove that filtered colimits of diagrams made of local maps have local colimit inclusions as rephrased in the following lemma - where we observe that we need no assumption on the localness of objects:

Lemma 3.3.3.4. *For any Grothendieck topos, the wide subcategory $\mathbb{T}[\mathcal{E}]^{\mathbf{Loc}}$ is closed in $\mathbb{T}[\mathcal{E}]$ under filtered colimits.*

Proof. Let $(F_i)_{i\in I}$ be a filtered diagram of such that all the transitions morphisms $(u_d)_{d\in I^2}$ are local, that is a filtered diagram in $\mathbb{T}[\mathcal{E}]^{\mathbf{Loc}}$. The difficulty there is with reasoning in a sheaf topos $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}_\mathcal{E}, J_\mathcal{E})$ is that colimits (even filtered ones) are not pointwise and are difficult to handle. To prove the diagram above to be a pullback, we must escape from $\mathcal{E}$ to the corresponding presheaf category $\widehat{\mathcal{C}_\mathcal{E}}$ where colimits are pointwise.

Recall that a colimit of sheaves is the sheafification of the corresponding colimit of presheaves. Moreover, this property lifts to the category of $\mathbb{T}$-models which are related through the adjunction

$$\mathbb{T}[\mathcal{E}] \underset{\iota_\mathcal{E}}{\overset{\mathfrak{a}_{J_\mathcal{E}}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{T}[\widehat{\mathcal{C}_\mathcal{E}}]$$

Applying this to the sheaf of $\mathbb{T}$-models we get

$$\operatorname*{colim}_{i\in I} F_i \simeq \mathfrak{a}_{J_\mathcal{E}}(\operatorname*{colim}_{i\in I} \iota_\mathcal{E} F_i)$$

with $\mathfrak{a}_{J_\mathcal{E}} \iota_\mathcal{E} E_i \simeq E_i$. Since $\mathfrak{a}_{J_\mathcal{E}}$ preserves local maps, it is sufficient to prove that the colimit inclusions in $\mathbb{T}[\mathcal{C}_\mathcal{E}]$ are already local.

For each $d : i \to j$ in I, $u_d : E_i \to E_j$ is local in $\mathbb{T}[\mathcal{E}]$. Now recall from proposition 3.3.2.11 that local maps are stable under direct image - as a right class whose left class is stable under inverse image; then the natural transformation between the underlying presheaves $\iota_\mathcal{E}(u_d) : \iota_\mathcal{E} F_i \to \iota_\mathcal{E} F_j$ is still local in $\mathbb{T}[\widehat{\mathcal{C}_\mathcal{E}}]$, which means by proposition 3.3.2.10 that each of its components $\iota_\mathcal{E}(u_d)_c$ at c in $\mathcal{C}_\mathcal{E}$ is local in $\mathbb{T}[\mathcal{S}]$.

Moreover, as colimits are pointwise in presheaf topoi, we have

$$\big(\operatorname{colim}_{i\in I} \iota_\mathcal{E} E_i\big)(c) \simeq \operatorname{colim}_{i\in I} \iota_\mathcal{E} E_i(c)$$

that is, the colimit of the E_i is obtained as the presheaf returning at each c the colimit in $\mathbb{T}[\mathcal{S}]$ of the induced diagram $(\iota_\mathcal{E} E_i(c))_{i\in I}$, which is by what precedes a diagram with local transition morphisms. From lemma 3.3.1.10 we know that at each c the colimits inclusions $(q_i)_c : E_i(c) \to \operatorname{colim}_{i\in I} E_i(c)$ are local, and they are the components at c of the colimit inclusion of the diagram of presheaves $\iota_\mathcal{E} q_i \iota_\mathcal{E} E_i \to \operatorname{colim} \iota_\mathcal{E} E_i$. But the latter are sent by $\mathfrak{a}_{J_\mathcal{E}}$ on the colimit inclusions $q_i : E_i \to \operatorname{colim}_{i\in I} E_i$, which are hence local. □

Proposition 3.3.3.5. *$\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}}$ is closed under filtered colimits in $\mathbb{T}[\mathcal{E}]$.*

Proof. It is a general fact, see for instance [63], that categories of models of geometric theories have filtered colimits, and that geometric inclusion are sent to finitely accessible functors between categories of models. Here this says that $\mathbb{T}_J[\mathcal{E}] \simeq \mathbf{GTop}[\mathcal{E}, \mathcal{S}[\mathbb{T}_J]]$ has filtered colimits that are preserved by the inclusion functor $\iota_J : \mathbb{T}_J[\mathcal{E}] \to \mathbb{T}[\mathcal{E}]$. Hence one has just to check that the colimit inclusions of a diagram of local objects and local maps between them are still local, as this was done in the previous lemma. □

As for theorem 3.3.1.12 we can deduce the following:

Theorem 3.3.3.6. *Any a geometry $(\mathbb{T}, \mathcal{V}, J)$ induces at each Grothendieck topos $\mathcal{E}$ a right multi-adjoint*

$$\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}} \overset{\iota_{J,\mathbf{Loc}}[\mathcal{E}]}{\hookrightarrow} \mathbb{T}[\mathcal{E}]$$

We can also generalize the closure of local objects and local map under connected limits observed in proposition 3.3.1.13:

Proposition 3.3.3.7. *$\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}}$ is closed under connected limits in $\mathbb{T}[\mathcal{E}]$.*

Proof. For $\iota_{J,\mathbf{Loc}}[\mathcal{E}]$ is a right multi-adjoint, it will preserve any connected limits that exists, so that eventual connected limits in $\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}}$ will be actually computed in $\mathbb{T}[\mathcal{E}] \simeq \mathbf{Lex}[\mathcal{C}_{\mathbb{T}}, \mathcal{E}]$, that is as pointwise limits of lex functors: but the latter are also the pointwise limit of the underlying functors. Hence for any eventual connected limit $\lim_{i\in I} E_i$ in $\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}}$, and each $\{\overline{x}, \phi\}$ in $\mathcal{C}_{\mathbb{T}}$, we shall have an isomorphism in $\mathcal{E}$

$$\big(\lim_{i\in I} E_i\big)(\{\overline{x}, \phi\}) \simeq \lim_{i\in I} E_i(\{\overline{x}, \phi\})$$

Moreover, suppose we have a site of presentation $\mathcal{E} \simeq \mathbf{Sh}(\mathcal{C}_{\mathcal{E}}, J_{\mathcal{E}})$: then as well as limits of sheaves are computed pointwise, recall that $\mathbb{T}$-models in $\mathcal{E}$ are sheaves of $\mathbb{T}$-models in $\mathcal{S}$, so that for any eventual connected limit $\lim_{i\in I} E_i$ in $\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}}$, we shall have at each $c \in \mathcal{C}_{\mathcal{E}}$ an isomorphism in $\mathbb{T}[S]$

$$\big(\lim_{i\in I} E_i\big)(c) \simeq \lim_{i\in I} E_i(c)$$

Let I be a small connected category and $(E_i)_{i\in I}$ and $(u_d)_{d\in I^2}$ a I-indexed diagram of local objects and local maps. We first prove that the projections maps $p_i : \lim_{i\in I} E_i \to E_i$ are local. From proposition 3.3.2.10, we know that evaluation creates localness: hence, for each u_d is a local map in $\mathbb{T}[\mathcal{E}]$, each evaluation $(u_d)_c$ is a local map in $\mathbb{T}[\mathcal{S}]$, seeing u_d as a natural transformation between sheaves with value in $\mathbb{T}[\mathcal{S}]$. Hence evaluation at each c defines a connected diagram $(E_i(c))_{i\in I}$ in $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$, and from proposition 3.3.1.13 we can compute the connected limit $\lim_{i\in I} E_i(c)$ at each c, which is local and mainly has local projections: from the preliminary discussion on limits in $\mathbb{T}[\mathcal{E}]$, those projections are the components $(p_i)_c$ at c of the projections p_i of the limit $\lim_{i\in I} E_i$, which are hence local themselves.

Let $([\theta_l]_{\mathbb{T}} : \{\overline{x}_l, \phi_l\} \to \{\overline{x}, \phi\})_{l\in L}$ be a J-cover of etale maps. We want an epimorphism in $\mathcal{E}$

$$\coprod_{l\in L} \big(\lim_{i\in I} E_i\big)(\{\overline{x}_l, \phi_l\}) \overset{\langle \lim_{i\in I} E_i([\theta_l]_{\mathbb{T}})\rangle_{l\in L}}{\twoheadrightarrow} \lim_{i\in I} E_i(\{\overline{x}, \phi\})$$

In fact, it is known that connected limits commute with coproducts in Grothendieck topoi: then we have

$$\coprod_{l\in L} \big(\lim_{i\in I} E_i\big)(\{\overline{x}_l, \phi_l\}) \simeq \coprod_{l\in L} \lim_{i\in I} E_i(\{\overline{x}_l, \phi_l\}) \simeq \lim_{i\in I} \coprod_{l\in L} E_i(\{\overline{x}_l, \phi_l\})$$

Now from the fact that the projections $p_i : \lim_{i\in I} E_i \to E_i$ are local, we have at each $l \in L$ a pullback in $\mathcal{E}$:

$$\begin{array}{ccc} \lim_{i\in I} E_i(\{\overline{x}_l, \phi_l\}) & \xrightarrow{\lim_{i\in I} E_i([\theta_l]_{\mathbb{T}})} & \lim_{i\in I} E_i(\{\overline{x}, \phi\}) \\ {\scriptstyle (p_i)_{\{\overline{x}_l,\phi_l\}}}\downarrow & \lrcorner & \downarrow{\scriptstyle (p_i)_{\{\overline{x},\phi\}}} \\ E_i(\{\overline{x}_l, \phi_l\}) & \xrightarrow[E_i([\theta_l]_{\mathbb{T}})]{} & E_i(\{\overline{x}, \phi\}) \end{array}$$

Then, by stability of coproducts in Grothendieck topoi, the following square is a pullback

$$\begin{array}{ccc} \coprod_{l\in L} \lim_{i\in I} E_i(\{\overline{x}_l, \phi_l\}) & \xrightarrow{\langle \lim_{i\in I} E_i([\theta_l]_{\mathbb{T}})\rangle_{l\in L}} & \lim_{i\in I} E_i(\{\overline{x}, \phi\}) \\ {\scriptstyle \coprod_{l\in L}(p_i)_{\{\overline{x}_l,\phi_l\}}}\downarrow & \lrcorner & \downarrow{\scriptstyle (p_i)_{\{\overline{x},\phi\}}} \\ \coprod_{l\in L} E_i(\{\overline{x}_l, \phi_l\}) & \xrightarrow[\langle E_i([\theta_l]_{\mathbb{T}})\rangle_{l\in L}]{} & E_i(\{\overline{x}, \phi\}) \end{array}$$

and by stability of epimorphisms in Grothendieck topoi, the top arrow is an epimorphism: this ensures localness of the limit. □

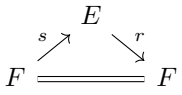
Now some precision on the geometric aspects of local objects: in fact they behave exactly as expected in the sense that they form sheaves of $\mathcal{B}$ objects over the definition site of their base topos and have set valuated local objects as stalks:

Proposition 3.3.3.8. *The stalks of a local object F are set-valued local objects. Evaluated at points, local transformations between local objects return admissible maps between set-valued local objects.*

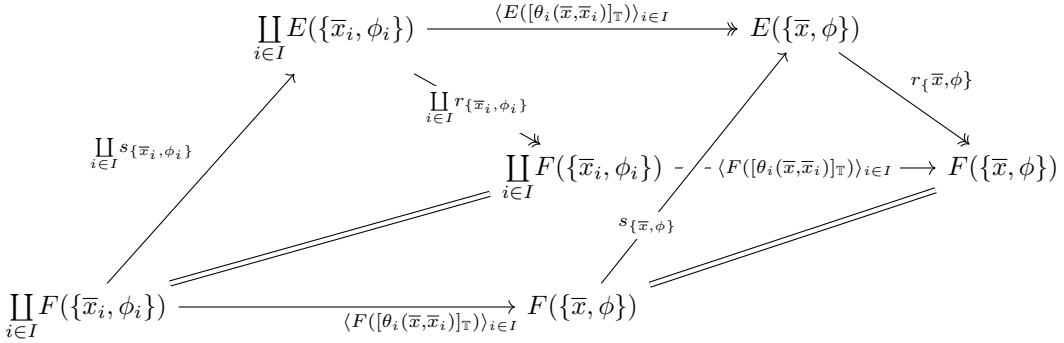
All of this justifies the terminology for local objects and local transformations as our objects of interest behave locally, at points, as such objects. We also have the following closure property under retracts:

Proposition 3.3.3.9. *In any topos $\mathcal{E}$, a retract of a local object still is a local object.*

Proof. For lex functors $F, E : \mathcal{C}_{\mathbb{T}} \to \mathcal{E}$ such that E is moreover local, then by naturality a situation of retraction



induces for any cover in J a roof



where the top arrow is epic since E is local and $r_{\{\overline{x},\phi\}}$ is epic as a retraction, while $\coprod_{i\in I} r_{\{\overline{x}_i,\phi_i\}}$ also is epic as epi are stable by colimits in $\mathcal{S}$, so that the middle horizontal arrow also is epic, exhibiting F as a local object. □

Remark 3.3.3.10. As for set-valued models, we also have that retract of local maps between local objects still are local, as it can be tested point-wisely.

3.4 Transformations of geometries

To conclude this section let us describe the morphisms between geometries and describe briefly the 2-category they form.

3.4.1 The 2-category of geometries

In this we follow a terminology inspired from [67][definition 1.2.6].

Definition 3.4.1.1. A *transformation of geometries* $(\mathbb{T}_1, \mathcal{V}_1, J_1) \to (\mathbb{T}_2, \mathcal{V}_2, J_2)$ consists of a functor $\Phi : \mathcal{C}_{\mathbb{T}_1} \to \mathcal{C}_{\mathbb{T}_2}$ such that

- Φ is lex
- $\Phi(\mathcal{V}_1) \subseteq \mathcal{V}_2$
- Φ induces a morphism of sites $(\mathcal{C}_{\mathbb{T}_1}, J_1) \to (\mathcal{C}_{\mathbb{T}_2}, J_2)$

Definition 3.4.1.2. In the following we denote $\mathfrak{Geom}$ the 2-category whose

- 0-cells are geometries $(\mathbb{T}, \mathcal{V}, J)$
- 1-cells are transformations of geometries $(\mathbb{T}_1, \mathcal{V}_1, J_1) \to (\mathbb{T}_2, \mathcal{V}_2, J_2)$
- and 2-cells between two transformations of geometries Φ, Ψ are natural transformation between the underlying lex functors $\alpha : \Phi \Rightarrow \Psi$.

Remark 3.4.1.3. Observe that $\Phi : \mathcal{C}_{\mathbb{T}_1} \to \mathcal{C}_{\mathbb{T}_2}$ induces a geometric morphism $\Phi^* \dashv \Phi_*$ (we also denote as Φ) which moreover restricts to the sheaf subtopoi as follows:

$$\begin{array}{ccc} \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_2}, J_2) & \xhookrightarrow{\iota_2} & \widehat{\mathcal{C}_{\mathbb{T}_2}} \\ {\scriptstyle\Phi}\downarrow & & \downarrow{\scriptstyle\Phi} \\ \mathbf{Sh}(\mathcal{C}_{\mathbb{T}_1}, J_1) & \xhookrightarrow[\iota_1]{} & \widehat{\mathcal{C}_{\mathbb{T}_1}} \end{array}$$

Moreover, if $\mathscr{H}_i$ is the etale class generated by $\mathcal{V}_i$ in $S[\mathbb{T}_i]$ for $i = 1, 2$, then if Φ satisfies the second condition, one has $\Phi^*(\mathscr{H}_1) \subseteq \mathscr{H}_2$, which means that $\mathscr{H}_1 \subseteq \Phi(\mathscr{H}_2)$.

3.4.1.4. We defined geometries from rather syntactic data, in terms of finite-limit theories and geometric extensions; in fact, when considering collectively geometries, it may be better to work at a more invariant level, as the semantics of those theories actually matters more than their syntactic specifications.

Recall that two geometric theories are said to be *Morita-equivalent* if they share the same classifying topos. If two finite limit theories $\mathbb{T}_1$ and $\mathbb{T}_2$ are Morita-equivalent, then they correspond to equivalent lex categories $\mathcal{C}_{\mathbb{T}_1} \simeq \mathcal{C}_{\mathbb{T}_2}$. It is also known, see [13] for instance, that two geometric extensions of a geometric theory whose syntactic coverages generate the same Grothendieck topology are Morita-equivalent.

However requiring the sole Morita equivalence of the geometric extensions is a bit too coarse for our purpose, as Morita-equivalent geometric theories may have different finite limit parts if they do not live in the same language; in contrast, any notion of equivalence between geometries should also require equivalence of the finite limit parts.

This is why we introduce the following notion, whose name is justified by [18] construction of the classifying topos of a geometric theory as the sheaf topos on the synstactic category of its finite limit part together with a convenient Grothendieck pretopology.

Definition 3.4.1.5. Two geometries $(\mathbb{T}_1, \mathcal{V}_1, J_1)$, $(\mathbb{T}_2, \mathcal{V}_2, J_2)$ will be said to be *Coste-equivalent* if

- $\mathbb{T}_1$ and $\mathbb{T}_2$ have equivalent syntactic categories $\mathcal{C}_{\mathbb{T}_1} \simeq \mathcal{C}_{\mathbb{T}_2}$;
- the equivalence $\mathcal{C}_{\mathbb{T}_1} \simeq \mathcal{C}_{\mathbb{T}_2}$ defines an equivalence of lex sites $(\mathcal{C}_{\mathbb{T}_1}, J_1) \simeq (\mathcal{C}_{\mathbb{T}_2}, J_2)$,
- and the saturated classes $\mathcal{V}_1$, $\mathcal{V}_2$ correspond through the equivalence.

Remark 3.4.1.6. It is in fact more relevant to consider Coste-equivalence classes of geometries than geometries themselves: this would actually be equivalent to defining geometries as the data of $(\mathcal{C}, \mathcal{J}, \mathcal{V})$ with $\mathcal{C}$ a small lex category, $\mathcal{J}$ a Grothendieck *topology* on $\mathcal{C}$ and $\mathcal{V}$ a saturated class in $\mathcal{C}^{\mathrm{op}}$ such that $\mathcal{J}$ is generated in $\mathcal{V}$. In fact we shall see that Coste-equivalent geometries induce the same notion of spectrum. In the following we denote as $\overline{\mathfrak{Geom}}$ the 2-category of Coste-equivalence classes of geometries.

However in the following, for we shall better have to consider Grothendieck pretopologies in practice when constructing the spectral site, we prefer to speak of finite limit theories and geometric extensions rather than directly from their corresponding invariants. Observe that saturated classes are already invariant.

Proposition 3.4.1.7. *The 2-category $\overline{\mathfrak{Geom}}$ has pseudolimits and bicolimits.*

Proof. We prove that the pseudolimits and bicolimits of the 2-category **Lex** are inherited in $\mathfrak{Geom}$. For bicolimit, consider a diagram of geometries and take the bicolimit cocone $(q_i : \mathcal{C}_i \to \operatorname{bicolim}_{i\in I}\mathcal{C}_i)_{i\in I}$. Then $\operatorname{bicolim}_{i\in I}\mathcal{C}_i$ can be equipped with the topology $\langle \bigcup_{i\in I} q_i(J_i)\rangle_{i\in I}$ that makes any inclusion q_i J_i-continuous, while one can consider the class of map $\bigcup_{i\in I} q_i(\mathcal{V}_i)$ and take its associated saturated class. Then we have

$$\operatorname*{bicolim}_{i\in I}\ (\mathcal{C}_i, \mathcal{V}_i, J_i) \simeq \Big(\operatorname*{bicolim}_{i\in I}\ \mathcal{C}_i, \overline{\bigcup_{i\in I} q_i(\mathcal{V}_i))}, \langle \bigcup_{i\in I} q_i(J_i)\rangle_{i\in I}\Big)$$

For pseudolimits, it is well known that pseudolimits in **Lex** are computed as the pseudolimit of the underlying categories: hence $\operatorname{pslim}_{i\in I}\mathcal{C}_i$ has as objects $((C_i)_{i\in I}, (\alpha_d)_{d\in I^2})$ with $\alpha_d : C_j \simeq$

$\Phi_d(C_i)$ an isomorphism for $d : i \to j$, and morphisms $(f_i)_{i\in I}$ compatible with the transition isomorphisms. Then we can equip pslim $_{i\in I}\mathcal{C}_i$ with a finest Grothendieck pretopology J_I making all the projections $p_i : \text{pslim}_{i\in I}\mathcal{C}_i \to \mathcal{C}_i$ morphisms of sites: this is the data of all families $((f_i^k)_{i\in I} : ((C_i^k)_{i\in I}, (\alpha_d^k)_{d\in I^2}) \to ((C_i)_{i\in I}, (\alpha_d)_{d\in I^2}))_{k\in K}$ such that for any $i \in I$, $p_i((f_i^k)_{i\in I})_{k\in I} = (f_i^k)_{k\in K}$ contains a J_i-covering family. Now take $\mathcal{V}_I = \{(f_i)_{i\in I} \mid f_i \in \mathcal{V}_i\}$. Then we have

$$\underset{i\in I}{\text{pslim}}\ (\mathcal{C}_i, \mathcal{V}_i, J_i) \simeq \left(\underset{i\in I}{\text{pslim}}\ \mathcal{C}_i, \mathcal{V}_I, J_I\right)$$

□

Remark 3.4.1.8. For a fixed finite limit theory $\mathbb{T}$, Coste equivalence of geometries on $\mathbb{T}$ form a poset $\mathfrak{Geom}_{\mathbb{T}}$, where $(\mathcal{V}_1, J_1) \leq (\mathcal{V}_2, J_2)$ if

- $\mathcal{V}_1 \subseteq \mathcal{V}_2$ (so that $\mathbf{Et}_{\mathcal{V}_1} \subseteq \mathbf{Et}_{\mathcal{V}_2}$ and $\mathbf{Loc}_{\mathcal{V}_2} \subseteq \mathbf{Loc}_{\mathcal{V}_1}$)
- and $J_1 \leq J_2$, that is any J_1-cover is also a J_2-cover.

Hence in particular $(\mathcal{V}_1, J_2)$ is an admissibility structure. Moreover, if $\mathbb{T}_1$ and $\mathbb{T}_2$ are Coste-equivalent, then their posets of Coste-equivalent geometries are isomorphic.

3.4.1.9. Pseudolimits and bicolimits of (Coste-equivalence classes of) geometries simplify over each $\mathbb{T}$ into a structure of distributive lattice inherited from the lattice structure on saturated classes on one hand, and the lattice of Grothendieck topologies on $\mathcal{C}_{\mathbb{T}}$ - as described in [13]: for any $(\mathcal{V}_1, J_1)$, $(\mathcal{V}_2, J_2)$ on $\mathbb{T}$, we can take

$$(\mathcal{V}_1, J_1) \vee (\mathcal{V}_2, J_2) = (\mathcal{V}_1 \cup \mathcal{V}_2, J_1 \vee J_2)$$

as the join, where $J_1 \vee J_2 = \langle J_1 \bigcup J_2 \rangle$ is the join in the lattice of Grothendieck topologies on $\mathcal{C}_{\mathbb{T}}$, while

$$(\mathcal{V}_1, J_1) \wedge (\mathcal{V}_2, J_2) = (\mathcal{V}_1 \cap \mathcal{V}_2, J_1 \wedge J_2)$$

were $J_1 \wedge J_2 = J_1 \cap J_2$.

Then observe each finite limit theory $\mathbb{T}$, there is a smallest geometry $(\mathbb{T}, \mathbf{Iso}, J_{\text{triv}})$ where J_{triv} is the trivial topology whose covering families are singletons consisting of an isomorphism, and **Iso** is the class of isomorphisms. Then any object is local, and any map is local, and the associated multireflection is the equality of $\mathbb{T}[\mathcal{S}]$.

In fact this generalizes as follows: for any saturated class $\mathcal{V}$, one can always take the trivial topology J_{triv} as above, and for $\mathcal{V}$ contains isomorphisms, $(\mathbb{T}, \mathcal{V}, J_{\text{triv}})$ is the smallest geometry on $\mathbb{T}$ with $\mathcal{V}$ as saturated class. In the following, such geometries will be called *factorization geometries*. They corresponds to the stable inclusions of right class as described in proposition 2.3.3.1.

There is also a biggest geometry $(\mathbb{T}, \mathbb{T}[\mathcal{S}]^2_\omega, J_{\max})$ where $J_{\max}$ is the Grothendieck pretopology where any family is covering, which corresponds to the empty subtopos of $\mathcal{S}[\mathbb{T}]$, hence defines the empty subcategory of $\mathbb{T}[\mathcal{S}]$ which is trivially multireflective.

Remark 3.4.1.10. we shall see in corollary 4.1.3.8 that for each choice of Grothendieck topology J - equivalently, for each class of Coste-equivalent geometric extensions of $\mathbb{T}$ - there is also a smallest saturated class together with it defines a geometry.

3.4.2 Transformations of geometries at the level of models

To conclude, let us discuss how those notions are transferred along transformations of geometries.

3.4.2.1. Consider $\Phi : (\mathbb{T}_1, \mathcal{V}_1, J_1) \to (\mathbb{T}_2, \mathcal{V}_2, J_2)$. Then for any Grothendieck topos $\mathcal{E}$ we have a functor sending a $\mathbb{T}_2$ model corresponding to a lex functor $F : \mathcal{C}_{\mathbb{T}_2} \to \mathcal{E}$ to the precomposite

$$\mathcal{C}_{\mathbb{T}_1} \xrightarrow{\Phi} \mathcal{C}_{\mathbb{T}_2} \xrightarrow{F} \mathcal{E}$$

and a morphism $\phi : F \to G$ to the whiskering $\phi * \Phi$. Observe that this functor has a left adjoint sending a $\mathbb{T}_1$-model $F : \mathcal{C}_{\mathbb{T}_1} \to \mathcal{E}$ to its left Kan extension $\text{lan}_\Phi F : \mathcal{C}_{\mathbb{T}_2} \to \mathcal{E}$. This defines a pair of adjoint functors

$$\mathbb{T}_2[\mathcal{E}] \underset{\Phi[\mathcal{E}]_*}{\overset{\Phi[\mathcal{E}]^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{T}_1[\mathcal{E}]$$

Moreover, from Gabriel-Ulmer duality, we see that in particular $\Phi[\mathcal{S}]^* \dashv \Phi[\mathcal{S}]_*$ is the morphism of locally finitely presentable categories induced by the lex functor Φ. Hence $\Phi[\mathcal{S}]_*$ is continuous and finitary, while $\Phi[\mathcal{S}]^*$ preserves finitely presented objects.

3.4.2.2. Moreover, since Φ is a morphism of site, any J_2-continuous lex functor $F : \mathcal{C}_{\mathbb{T}_1} \to \mathcal{E}$ is sent by precomposition with Φ to a J_1-continuous lex functor $\Phi[\mathcal{E}]_* F = F\Phi$. Hence the direct image part of the adjunction above restricts to local objects

$$\begin{array}{ccc} \mathbb{T}_{J_2}[\mathcal{E}] & \xrightarrow{\iota_2[\mathcal{E}]} & \mathbb{T}_2[\mathcal{E}] \\ \vdots & & \downarrow{\scriptstyle \Phi[\mathcal{E}]_*} \\ \mathbb{T}_{J_1}[\mathcal{E}] & \xrightarrow[\iota_1[\mathcal{E}]]{} & \mathbb{T}_1[\mathcal{E}] \end{array}$$

Now as well as local objects are transferred along the induced functors between categories of models, we have the following concerning local maps:

Proposition 3.4.2.3. *$\Phi[\mathcal{S}]_* : \mathbb{T}_2[\mathcal{S}] \to \mathbb{T}_1[\mathcal{S}]$ sends $\mathcal{V}_2$-local maps to $\mathcal{V}_1$-local maps, that is*

$$\Phi[\mathcal{S}]_*(\mathbf{Loc}_1[\mathcal{S}]) \subseteq \mathbf{Loc}_2[\mathcal{S}]$$

Proof. For any $n : K \to K'$ in $\mathcal{V}_1$, $\Phi(n)$ is $\mathcal{V}_2$, hence for a $\mathcal{V}_2$-local arrow $u : F \Rightarrow G$ in $\mathbf{Loc}_2[\mathcal{S}]$ we have a pullback square

$$\begin{array}{ccc} F\Phi(K) & \xrightarrow{(u*\Phi)_K} & G\Phi(K) \\ {\scriptstyle F\Phi(n)}\downarrow & \lrcorner & \downarrow{\scriptstyle G\Phi(n)} \\ F\Phi(K') & \xrightarrow[(u*\Phi)_{K'}]{} & G\Phi(K') \end{array}$$

and this proves that the corresponding arrow $u * \Phi$ is in $\mathbf{Loc}_1[\mathcal{S}]$. □

Recall that localness can be tested evaluation-wisely as stated in proposition 3.3.2.10. Hence we have the following:

Corollary 3.4.2.4. *For any Grothendieck topos $\mathcal{E}$, we have $\Phi[\mathcal{E}]_*(\mathbf{Loc}_1[\mathcal{E}]) \subseteq \mathbf{Loc}_2[\mathcal{E}]$*

3.4.2.5. It is also worth describing concretely the action of the direct image functor $\Phi[\mathcal{E}]_*$ at a given $\mathbb{T}_2$-model in $\mathcal{E}$. For a F in $\mathbb{T}_2[\mathcal{E}]$ seen as a sheaf of $\mathbb{T}_2[\mathcal{S}]$-objects on $(\mathcal{C}_\mathcal{E}, J_\mathcal{E})$, since $\mathbb{T}_2$ is a finite limit theory, it is sufficient to apply evaluation-wisely $\Phi[\mathcal{S}]_*$ to values of F, that is in each c of $\mathcal{C}_\mathcal{E}$

$$(\Phi[\mathcal{E}]_* F)(c) = \Phi[\mathcal{S}]_*(F(c))$$

Now recall that $\Phi[\mathcal{S}]_*$ is finitary: hence for any point $x : \mathcal{S} \to \mathcal{E}$, since the element category $(\int x^*)^{\mathrm{op}}$ is filtered, we have

$$\begin{aligned} \Phi[\mathcal{S}]_*(x^* F) &\simeq \Phi[\mathcal{S}]_*(\operatorname*{colim}_{(c,a)\in(\int x^*)^{\mathrm{op}}} F(c)) \\ &\simeq \operatorname*{colim}_{(c,a)\in(\int x^*)^{\mathrm{op}}} \Phi[\mathcal{S}]_*(F(c)) \\ &\simeq \operatorname*{colim}_{(c,a)\in(\int x^*)^{\mathrm{op}}} (\Phi[\mathcal{E}]_* F)(c) \\ &\simeq x^*(\Phi[\mathcal{E}]_* F) \end{aligned}$$

Chapter 4

Admissibility and stability

This chapter investigates the relation between the two previous ones. We saw at theorem 3.3.1.12 that any geometry induced an accessible multireflection of local objects and local maps in the category of ambient objects. Moreover this idea of multi-reflection efficiently encapsulated the admissibility condition. Here we are interested in the inverse problem of constructing a geometry from the data of a right multi-adjunction assuming a few reasonable conditions.

In the first section, we focus on right multi-adjoints in an accessible context. After recalling Diers theory of locally finitely multipresentable categories, we give some results about accessible right multi-adjoints (in particular an "accessible multi-adjoint theorem" at theorem 4.1.2.1) and detail their connection with finitely multipresentable categories. Though most of those results were actually present in Diers thesis, again, because of its restricted availability and for the importance of those aspects in this work, we choose to be quite prolix on this point, and, by the way, use different strategies or prove slightly generalized forms of the results. In particular theorem 4.1.3.5 is a more general version of a very interesting result from [25] which could be read as a method to produce a "minimal geometry" associated to a Grothendieck topology on a finite limit theory, see corollary 4.1.3.8. We also relate this to [52] remarks on the relations between locally finitely multipresentable categories and *disjunctive theories*.

The second section is devoted to the construction of a geometry from a Diers context through a notion of *localizing pretopology*, whose associated accessible multi-right adjunction could be thought of as a "geometric envelope" of the initial right multi-adjoint, see corollary 4.2.2.5. We also describe a 2-dimensional adjunction between geometries and Diers contexts at theorem 4.2.5.1.

4.1 Diers condition and multi-presentable categories

4.1.1 Locally finitely multi-presentable categories and Diers duality

Locally multipresentable categories were introduced by Diers in [59] (under the name "catégories localisantes") as a generalization of locally presentable categories encompassing a wide class of non-locally presentable categories, as local rings, fields, integral domains, local lattices... They are defined in the same way as locally presentable categories, but in the language of multi-colimits. Because of their tight relation with multiadjointness and their recurrent role as the categories of local objects in spectral situations, we recall Diers theory of locally multipresentable categories, also present in [26].

This section will make important use of the notion of multilimits, multicolimits and connected limits, as the functors preserving them, as defined in definition 2.1.3.9 and definition 2.1.3.10.

Definition 4.1.1.1. A category $\mathcal{A}$ is said to be *locally finitely multipresentable* if it is finitely accessible and multicocomplete

As locally finitely presentable categories can be characterized as the finitely accessible categories that are moreover either complete or cocomplete, locally finitely multipresentable categories admit the following characterization:

Proposition 4.1.1.2. *Locally finitely multipresentable categories are exactly the finitely accessible categories with connected limits, where finite connected limits commute with filtered colimits.*

Remark 4.1.1.3. Recall that a category is complete if and only if it has connected limits and a terminal object. Then a locally finitely multipresentable category is a locally finitely presentable category if and only if it has a terminal object.

Remark 4.1.1.4. If $\mathcal{A}$ is locally finitely multipresentable, then:

- the arrow category $\mathcal{A}^2$ is also locally finitely multipresentable
- for any object A in $\mathcal{A}$ the coslice $A \downarrow \mathcal{A}$ is also finitely multipresentable and moreover, the codomain functor $\mathrm{cod} : A \downarrow \mathcal{A} \to \mathcal{A}$ is finitary and right multiadjoint.

Proof. For the first item, observe that the category $\mathcal{A}^2_\omega$ is a generator of finitely presented objects. Now multicolimits in $\mathcal{A}^2$ are computed as follows: for I a finite category and $(f_i : A_i \to A'_i)_{i\in I}$ a I-indexed diagram in $\mathcal{A}^2$, the multicolimit $(g'_{ij} : A'_i \to B'_j)_{i\in I, j\in J}$ of the codomains induces a multicocone $(g'_{ij} f_i : A_i \to B_j)_{i\in I, j\in J}$, which defines in each $j \in J$ a cocone over the $(A_i)_{i\in I}$, and if we choose a multicolimit $(g_{ik} : A_i \to B_k)_{i\in I, k\in K}$, then for each $i \in I$ and $j \in J$ there is a unique $k \in K$ such that the cocone $(g'_{ij} f_i)_{i\in I}$ factorizes uniquely through g_{ik}

$$\begin{array}{ccc} A_i & \xrightarrow{f_i} & A'_i \\ {\scriptstyle g_{ik}}\downarrow & & \downarrow{\scriptstyle g'_{ij}} \\ B_k & \xrightarrow[\overline{f_j}]{} & B'_j \end{array}$$

and the family $(\overline{f}_j : B_k \to B'_j)_{j\in J}$ is a multicolimit in $\mathcal{A}^2$. Then in particular as $\mathcal{A}_\omega$ is closed under finite multicolimits, so is$\mathcal{A}^2_\omega$.

For the second item, we refer to [25][Proposition 8.4] concerning existence of multicolimits. However we emphasize the following: for any object A' in $\mathcal{A}$, the canonical cone of local units relatively to the codomain functor $A \downarrow \mathcal{A} \to \mathcal{A}$ is made of the inclusions $(q_i : A' \to B_i)_{i\in I}$ into a multicoproduct of A' with A. In particular, as we shall see in proposition 4.1.2.4, this allows us to exhibit the generator of finitely presented objects $(A \downarrow \mathcal{A})_\omega$ as consisting of members of the multicoproducts of A with finitely presented objects of $\mathcal{A}$. □

For the sake of completeness, we recall here elements of (finite) *Diers duality*, which is the multi-version of (finite) Gabriel-Ulmer duality.

4.1.1.5. Recall from definition 2.1.3.9 that a multilimit is a family of cones $(p^j_i : L_j \to F(i))_{i\in I, j\in J}$ that are locally universal, which can be encapsulated for any other X in $\mathcal{B}$, in the following isomorphism

$$\lim_{i\in I} \mathcal{B}[X, F(i)] \simeq \coprod_{j\in J} \mathcal{B}[X, L_j]$$

A functor $F : \mathcal{C} \to \mathcal{D}$ is multicontinuous if for any multilimit $(p^j_i)_{i\in I, j\in J}$ in $\mathcal{C}$ there exists another multilimit $(q^k_i : M_k \to UF(i))_{i\in I, k\in K}$ in $\mathcal{B}$ that exhibits for each $k \in K$ the object M_k as a coproduct $M_k \simeq \coprod_{j\in J_k} U(L_j)$ where J_k is the set of $j \in J$ such that the cone $(U(p^j_i) : U(L_j) \to UF(i))_{i\in I}$ factorizes through M_k. In particular, whenever $\mathcal{D}$ has limits, this says that

$$\lim_{i\in I} UF(i) \simeq \coprod_{j\in J} U(L_j)$$

Observe that this implies that the codomain category $\mathcal{D}$ has some amount of coproducts - though not necessarily all. In particular any continuous functor is multicontinuous.

Remark 4.1.1.6. A category has *finite multilimits*, or is *finitely multicomplete*, or also *mlex*, if any finite family has a multilimit; similarly we will speak of *mlex* functors for functors preserving finite multi-limits. We denote **Mlex** the 2-category of small mlex categories and multicontinuous functors between them.

In the following, we fix a small mlex category $\mathcal{K}$.

Remark 4.1.1.7. Observe that the corepresentables of the form $\mathcal{K}[K, -]$ are always multicontinuous, as they are continuous. Hence we have an inclusion

$$\mathcal{K}^{\mathrm{op}} \xhookrightarrow{\exists} \mathbf{Mlex}[\mathcal{K}, \mathcal{S}]$$

sending each K to the corresponding corepresentable.

As we could characterize flat functors in the context of lex categories as those preserving the finite limits, we have the following:

Proposition 4.1.1.8. *For a small mlex category $\mathcal{K}$, a functor $F : \mathcal{K} \to \mathcal{S}$ is mlex if and only if it is flat.*

Proof. Suppose that F is mlex: then we must prove that its category of elements $\int F$ is cofiltered: but this is because in $\mathcal{S}$ we have (finite) limits and coproducts. For (X_1, a_1) and (X_2, a_2) in $\int F$, the pair X_1, X_2 has a multilimit $(p_i^j : L_j \to X_i)_{i=1,2, j\in J}$ and since F is mlex and $F(X) = [\mathcal{K}, \mathcal{S}][\exists_X, F]$ we have

$$F(X_1) \times F(X_2) \simeq \coprod_{j\in J} F(L_j)$$

so that the pair $(a_1, a_2) \in F(X_1) \times F(X_2)$ lies in some $F(L_j)$ for some $j \in J$. Similarly for a parallel pair $f, g : (X_1, a_1) \rightrightarrows (X_2, a_2)$ we have a multilimit of $X_1 \rightrightarrows X_2$ whose coproduct gives the equalizer of $F(f), F(g)$.

Conversely if F is flat - so that F decomposes as the filtered colimit $F \simeq \operatorname{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} \exists_X$, and $(p_i^j : L_j \to X_i)_{i\in I, j\in J}$ is a finite multilimit (with I finite), we have the following

$$\begin{aligned}
\lim_{i\in I} F(X_i) &\simeq \lim_{i\in I} [\mathcal{K}, \mathcal{S}](\exists_{X_i}, F) \\
&\simeq \lim_{i\in I} [\mathcal{K}, \mathcal{S}][\exists_{X_i}, \operatorname*{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} \exists_X] \\
&\simeq \lim_{i\in I} \operatorname*{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} [\mathcal{K}, \mathcal{S}][\exists_{X_i}, \exists_X] \\
&\simeq \operatorname*{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} \lim_{i\in I} \mathcal{K}[X, X_i] \\
&\simeq \operatorname*{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} \coprod_{j\in J} \mathcal{K}[X, L_j] \\
&\simeq \coprod_{j\in J} [\mathcal{K}, \mathcal{S}][L_j, F] \simeq \coprod_{j\in J} F(L_j)
\end{aligned}$$

□

Proposition 4.1.1.9. ***Mlex***$[\mathcal{K}, \mathcal{S}]$ *is closed in* $[\mathcal{K}, \mathcal{S}]$ *under filtered colimits. In particular filtered colimit of representable are mlex.*

Proof. For $(F_d)_{d\in D}$ a filtered diagram in $\mathbf{MLex}[\mathcal{K}, \mathcal{S}]$ and $(p_i^j : L_j \to X_i)_{i\in I, j\in J}$ a finite multilimit, since both limits and filtered colimits are pointwise in the functor category $[\mathcal{K}, \mathcal{S}]$, we have the following sequence of isomorphisms

$$\begin{aligned}
\big(\lim_{i\in I} \operatorname*{colim}_{d\in D} F_d\big)(X_i) &\simeq \lim_{i\in I} \operatorname*{colim}_{d\in D} F_d(X_i) \\
&\simeq \operatorname*{colim}_{d\in D} \lim_{i\in I} F_d(X_i) \\
&\simeq \operatorname*{colim}_{d\in D} \coprod_{j\in J} F_d(L_j) \\
&\simeq \coprod_{j\in J} (\operatorname*{colim}_{d\in D} F_d)(L_j)
\end{aligned}$$

□

Proposition 4.1.1.10. ***Mlex***$[\mathcal{K}, \mathcal{S}]$ *is closed in* $[\mathcal{K}, \mathcal{S}]$ *under filtered colimits. Moreover finite connected limits commute with filtered colimits.*

Proof. For $(F_d)_{d\in D}$ a connected diagram in $\mathbf{MLex}[\mathcal{K}, \mathcal{S}]$ and $(p_i^j : L_j \to X_i)_{i\in I, j\in J}$ a finite multilimit, we can use the fact that connected limits commute with coproducts in $\mathcal{S}$, and that moreover limits are computed pointwisely in the category of functors, so that the connected limit of the F_d

is mlex:

$$\begin{aligned}(\lim_{i\in I}\lim_{d\in D} F_d)(X_i) &\simeq \lim_{i\in I}\lim_{d\in D} F_d(X_i)\\ &\simeq \lim_{d\in D}\lim_{i\in I} F_d(X_i)\\ &\simeq \lim_{d\in D}\coprod_{j\in J} F_d(L_j)\\ &\simeq \coprod_{j\in J}(\lim_{d\in D} F_d)(L_j)\end{aligned}$$

□

Proposition 4.1.1.11. *The full inclusion* $\boldsymbol{Mlex}[\mathcal{K},\mathcal{S}] \hookrightarrow [\mathcal{K},\mathcal{S}]$ *is multireflective.*

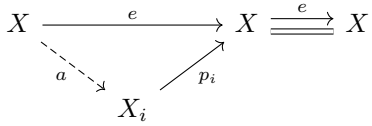
Proof. From what precedes, $\mathbf{Mlex}[\mathcal{K},\mathcal{S}] \hookrightarrow [\mathcal{K},\mathcal{S}]$ is accessible and preserves connected limits. From [74] it is known that accessible functors satisfy the solution set condition. Moreover, preservation of connected limits implies preservation of wide pullbacks - in fact, those conditions are equivalent - so by proposition 2.1.3.7 we know that $\mathbf{Mlex}[\mathcal{K},\mathcal{S}] \hookrightarrow [\mathcal{K},\mathcal{S}]$ is a full right multiadjoint. □

Corollary 4.1.1.12. $\boldsymbol{Mlex}[\mathcal{K},\mathcal{S}]$ *is multicocomplete, hence is locally finitely multipresentable.*

Moreover, we can control the generator of finitely presented object in $\mathbf{Mlex}[\mathcal{K},\mathcal{S}]$. The strategy is the same as in Gabriel-Ulmer duality and comes from the following lemma. Recall that a category is said to be *Cauchy complete* if any idempotent has a splitting.

Lemma 4.1.1.13. *Any mlex category is Cauchy-complete.*

Proof. Let $e : X \to X$ be an idempotent in $\mathcal{K}$. Then consider a multilimit $(p_i : X_i \to X)$ of the parallel pair $1_X, e : X \rightrightarrows X$. Then for $e = ee = e1_X$ we have a factorization of e through one member of the multilimit



But then the singleton $p_i : X_i \to X$ is a limit of the parallel pair, and (a, p_i) is a split of the idempotent e. □

Proposition 4.1.1.14. *For a mlex small category* $\mathcal{K}$, $\boldsymbol{Mlex}[\mathcal{K},\mathcal{S}]$ *is locally finitely multi-presentable and we have an equivalence* $\mathcal{K}^{\mathrm{op}} \simeq \boldsymbol{Mlex}[\mathcal{K},\mathcal{S}]_\omega$

Proof. We saw in proposition 4.1.1.8 that $\mathbf{Mlex}[\mathcal{K},\mathcal{S}] \simeq \mathbf{Flat}[\mathcal{K},\mathcal{S}]$. But any category of flat functors is accessible, hence so is $\mathbf{Mlex}[\mathcal{K},\mathcal{S}]$. But from what precedes, we know that $\mathbf{Mlex}[\mathcal{K},\mathcal{S}]$ has also multicolimits - equivalently connected limits. Hence it is locally finitely multi-presentable.

We always have that representable are finitely presented, for filtered colimits are computed pointwisely in $\mathbf{Mlex}[\mathcal{K},\mathcal{S}]$. But conversely, it is a general fact that finitely presented objects in category of flat functors are retracts of representable. But from the fact that $\mathcal{K}$ is Cauchy complete as a mlex category, and splitting of idempotent are absolute, any such retract must actually be representable. □

Now we consider the converse process where mlex categories are retrieved from locally finitely multipresentable categories.

Proposition 4.1.1.15. *Any finite multicolimit of finitely presented objects is made of finitely presented objects. In particular for a locally finitely multipresentable category* $\mathcal{A}$, $\mathcal{A}_\omega^{\mathrm{op}}$ *is mlex.*

Proof. Suppose that $F : I \to \mathcal{A}_\omega$ is a finite diagram and take $(q_i^j : F(i) \to X_j)_{i\in I, j\in J}$ a multicolimit. Then for any filtered diagram $S : D \to \mathcal{A}$, we have

$$\begin{aligned}\coprod_{j\in J}\mathcal{A}[X_j, \operatorname*{colim}_{d\in D} S(d)] &\simeq \lim_{i\in I}\mathcal{A}[F(i), \operatorname*{colim}_{d\in D} S(d)]\\ &\simeq \lim_{i\in I}\operatorname*{colim}_{d\in D}\mathcal{A}[F(i), S(d)]\\ &\simeq \operatorname*{colim}_{d\in D}\lim_{i\in I}\mathcal{A}[F(i), S(d)]\\ &\simeq \operatorname*{colim}_{d\in D}\coprod_{j\in J}\mathcal{A}[X_j, S(d)]\end{aligned}$$

and hence for each $j \in J$ we have a filtered colimit

$$\mathcal{A}[X_j, \operatorname*{colim}_{d\in D} S(d)] \simeq \operatorname*{colim}_{d\in D} \mathcal{A}[X_j, S(d)]$$

hence the finite presentedness. □

4.1.1.16. Now, since $\mathcal{A}$ is finitely accessible, the embedding $\iota_{\mathcal{A}} : \mathcal{A}_\omega \hookrightarrow \mathcal{A}$ is dense, so that the corresponding nerve $\mathcal{N}_{\mathcal{A}} = \mathcal{A}[\iota_{\mathcal{A}}, -]$ is full and faithful, and the nerve $\mathcal{N}_{\mathcal{A}}$ exhibits $\mathcal{A}$ as category of flat functors $\mathcal{A} \simeq \mathbf{Flat}[\mathcal{A}_\omega^{\mathrm{op}}, \mathcal{S}]$. But from the previous proposition, as $\mathcal{A}_\omega^{\mathrm{op}}$ is mlex, those are exactly the mlex functors $\mathbf{Mlex}[\mathcal{A}_\omega^{\mathrm{op}}, \mathcal{S}]$.

Now we give the 2-categorical statement of Diers duality. The same way we defined the 2-category **Mlex** of small mlex categories, mlex functors and natural transformations, we can define a 2-category of locally finitely multipresentable categories. First let us give the notion of morphisms:

Definition 4.1.1.17. A *morphism of locally finitely multipresentable categories* is a accessible functor $G : \mathcal{A} \to \mathcal{B}$ with a left adjoint G^* between locally finitely multipresentable categories. Denote **LFMP** the 2-category of locally finitely multipresentable categories, morphisms of locally finitely multipresentable categories and natural transformations between them.

Remark 4.1.1.18. The condition of having a global left adjoint seems rather surprising in this context, where we would have expected a condition of multi-right adjointness. But as in Gabriel-Ulmer duality, we really need to extract a left adjoint restricting to the generators of finitely presented objects to have a contravariant duality of the kind we want. In fact we have the following:

Lemma 4.1.1.19. *If $F : \mathcal{C} \to \mathcal{D}$ is a mlex functor between small mlex categories, then the extension $G^* = \operatorname{lan}_G$ is mlex.*

Proof. Recall that G^* can also be defined as defined as the left Kan extension $\operatorname{lan}_\exists \exists G^{\mathrm{op}}$ as below

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{op}} & \xrightarrow{G} & \mathcal{D}^{\mathrm{op}} \\ {\scriptstyle\exists}\downarrow & & \downarrow{\scriptstyle\exists} \\ [\mathcal{C}, \mathcal{S}] & \xrightarrow[G^*]{} & [\mathcal{D}, \mathcal{S}] \end{array}$$

So that in particular, since $\exists$ is full and faithful, for any X in $\mathcal{C}$ we know $G^* \exists_X = \exists_{G(X)}$. Now, recall that left Kan extensions commute with filtered colimit: as any mlex functor $F : \mathcal{C} \to \mathcal{D}$ is flat, hence is the filtered colimit $\operatorname{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} \exists_X$, we have

$$\begin{aligned} G^* F &= \operatorname{lan}_G F \\ &\simeq \operatorname*{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} \operatorname{lan}_G \exists_X \\ &\simeq \operatorname*{colim}_{(X,a)\in(\int F)^{\mathrm{op}}} \exists_{G(X)} \end{aligned}$$

But filtered colimits of representables are mlex: hence G^*F is mlex. □

Remark 4.1.1.20. Conversely, recall that if $G^* \dashv G_*$ is a pair of adjoint functors such that G_* is finitary, then G^* maps finitely presented objects to finitely presented objects.

Proposition 4.1.1.21 (Diers duality)**.** *There is a 2-equivalence of 2-categories*

$$\boldsymbol{Mlex}^{\mathrm{op}} \simeq \boldsymbol{LMFP}$$

sending a mlex category $\mathcal{K}$ to $\boldsymbol{Mlex}[\mathcal{K}, \mathcal{S}]$ and a mlex functor $G : \mathcal{K}_1 \to \mathcal{K}_2$ to $G_* : \boldsymbol{Mlex}[\mathcal{K}_2, \mathcal{S}] \to \boldsymbol{Mlex}[\mathcal{K}_1, \mathcal{S}]$, and a locally finitely multipresentable category $\mathcal{A}$ to $\mathcal{A}_\omega^{\mathrm{op}}$ and a morphism of locally finitely multipresentable multicategory $G^* \dashv G_*$ to the restriction $G^*|_\omega$.

4.1.2 Accessible right multi-adjoints

Another analog to locally presentable categories is the relation with adjointness. First, there is a well known variant of the adjoint functor theorem saying that a functor between locally presentable categories has a left adjoint if and only if it is accessible and preserves limits. Here we provide an analogous statement for right multiadjoint with a locally multipresentable domain:

Theorem 4.1.2.1. *Let $U : \mathcal{A} \to \mathcal{B}$ be a functor with $\mathcal{A}$ and $\mathcal{B}$ locally finitely multipresentable categories. Then U is a right multiadjoint if and only if it is accessible and preserves connected limits.*

Remark 4.1.2.2. Beware that we cannot control the rank of accessibility of U, which will end up λ-accessible for some $\lambda \geq \aleph_0$ we do not know, even though $\mathcal{A}$ and $\mathcal{B}$ are finitely accessible. In the following proof, we make use of the following general fact: for $\kappa \leq \lambda$ two cardinals, λ-filtered categories are in particular κ-filtered; hence if a category has κ-filtered colimits, it has in particular λ-filtered colimits, and similarly, a functor preserving κ-filtered colimits preserves in particular λ-filtered colimits. In particular, κ-presented objects are also λ-presented.

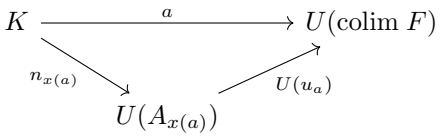
Proof. For the indirect sense, it is known (see [74] and [85]) that accessible functors between accessible categories satisfy the solution set condition. As $\mathcal{A}$ and $\mathcal{B}$ are in particular (finitely) accessible and U is accessible, U satisfies the solution set condition, and if moreover it preserves connected limits, then by proposition 2.1.3.7 it is right multi-adjoint.

For the direct sense, we propose this adaptation of [2][Theorem 1.66] in the context of multi-adjointness and multipresentability. Suppose that U is right multiadjoint (and hence preserves connected limits by proposition 2.1.3.7). As $\mathcal{B}_\omega$ is small and $\mathcal{A}$ is accessible, we can choose a cardinal $\lambda \geq \aleph_0$ such that for any K in $\mathcal{B}_\omega$ and any local unit $n_x : K \to U(A_x)$ with $x \in I_K$, the object A_x is λ-presented in $\mathcal{A}$. Then we prove that U is λ-accessible as follows.

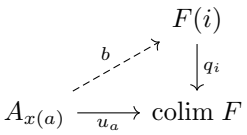
Let $F : I \to \mathcal{A}$ be a λ-filtered diagram: in particular, I is finitely filtered, and the colimit $(q_i : F(i) \to \operatorname{colim} F)_{i \in I}$ exists in $\mathcal{A}$. We prove that $(U(q_i) : F(i) \to U(\operatorname{colim} F))_{i \in I}$ is a colimit in $\mathcal{B}$ - and in particular, for it is λ-filtered, this colimit is also finitely filtered. By [2][Exercise 1.o(1)] we know that it is sufficient to check that

- for any finitely presented object K in $\mathcal{B}$ and $f : K \to U(\operatorname{colim} F)$, there is some lift $a : K \to UF(i)$ of a along q_i for some $i \in I$
- and that for any two such lifts $b : K \to UF(i)$ and $b' : K \to F(i')$ there is a common refinement $d : i \to i''$ and $d' : i' \to i''$ such that $UF(d)b = UF(d')b'$.

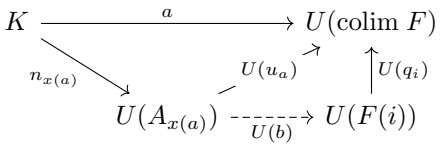
For $a : K \to U(\operatorname{colim} F)$, consider the local factorization



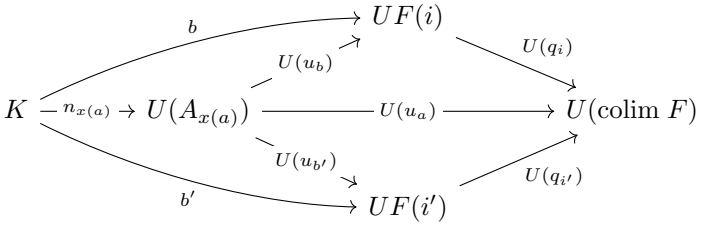
As K is finitely presented, $A_{x(a)}$ is λ-presented in $\mathcal{A}$, and as I is λ-filtered, there is a lift



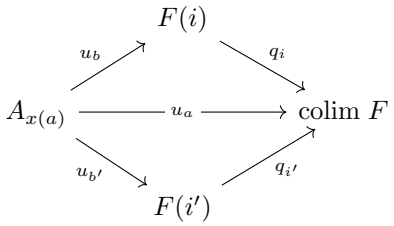
whose image along U provides the desired lift by precomposing with the local unit $n_{x(a)}$



Now for two such lifts $b : K \to UF(i)$, $b' : K \to F(i')$ of a, as the image of the inclusions $U(q_i)$, $U(q_{i'})$ are in the range of U, we know by theorem 2.1.1.5 that up to canonical isomorphism, b and b' factorize through the same local unit as a, that is $x(b) = x(a) = x(b')$ and we have the commutation below



This returns in $\mathcal{A}$ the following commutation



But now, as $A_{x(a)}$ is λ-presented and I is λ-directed, the lifts u_b and $u_{b'}$ factorize through a common refinement $d : i \to i''$ and $d' : i' \to i''$ such that $F(d)u_b = F(d')u_{b'}$, which provides in $\mathcal{B}$ the desired common refinement for b and b'.

Then, we know that the cocone $(U(q_i) : UF(i) \to U(\text{colim } F))_{i \in I}$ is a colimit in $\mathcal{B}$; this proves that U preserves λ-filtered colimits, that is, is λ-accessible. □

Remark 4.1.2.3. In proposition 4.1.2.4, we are going to prove that if we suppose that U is *finitely* accessible, then the localizations A_x for $x \in I_K$ and K finitely presented in $\mathcal{B}$ are finitely presentable. In fact, in the proof above, λ was dependent of the rank of presentability of the localizations of finitely presented objects: if all the A_x for $x \in I_K$ and K in $\mathcal{B}_\omega$ happen to be finitely presented, then the functor U can be certified as finitely accessible.

Recall that any locally finitely presentable functor is a right adjoint preserving filtered colimits; the last condition amounts to saying that its left adjoint sends finitely presented objects to finitely presented objects. Now recall from [2][Theorem 1.39] that a full, reflective subcategory $\mathcal{A}$ of a locally finitely presentable category $\mathcal{B}$ which is moreover closed under filtered colimits is locally finitely presentable - and the inclusion functor is locally finitely presentable; moreover, one could use as the generator of finitely presented objects in $\mathcal{A}$ the reflections of the finitely presented objects of $\mathcal{B}$. In [25][Theorem 8.3.1] is given the following analog - we propose here a slightly different adaptation of.

Proposition 4.1.2.4. *Let $\mathcal{B}$ be a locally finitely presentable category and $U : \mathcal{A} \hookrightarrow \mathcal{B}$ such that*

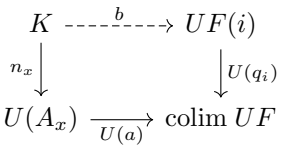
- *U is a right multiadjoint*
- *U is moreover faithful, and relatively full and faithful,*
- *$\mathcal{A}$ closed under filtered colimits and U preserves them.*

Then $\mathcal{A}$ is a locally finitely multipresentable category. Moreover, the full, dense generator $\mathcal{A}_\omega$ of finitely presented object of $\mathcal{A}$ has as objects local reflections of finitely presented objects, that is

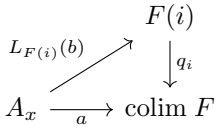
$$|\mathcal{A}_\omega| = \coprod_{K \in \mathcal{B}_\omega} \{A_x \mid n_x : K \to U(A_x) \in I_K\}$$

where I_K is the set of local units of K for U.

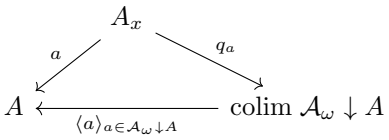
Proof. First, let us prove that $\mathcal{A}$ is finitely accessible. Since it is supposed to have filtered colimits, we have to check that $\mathcal{A}_\omega$ is a small dense generator of finitely presented objects. Let K be in $\mathcal{B}_\omega$ and $n_x : K \to U(A_x)$ a local unit under K. We first check that A_x is finitely presented in $\mathcal{A}$: let $F : I \to \mathcal{A}$ be a filtered diagram, and an arrow $a : A_x \to \text{colim}F$. Then for U is supposed to preserves filtered colimits and K is finitely presented, there is some i in I such that we have a lift



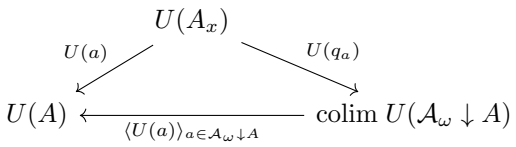
Then the local reflection of b provides us with a filler and moreover we have a commutative triangle



Hence A_x is finitely presented. Now we prove that for each A in $\mathcal{A}$, we have a filtered colimit $A \simeq \mathcal{A}_\omega \downarrow A$. First, observe that there exists a canonical arrow in $\mathcal{A}$ as below



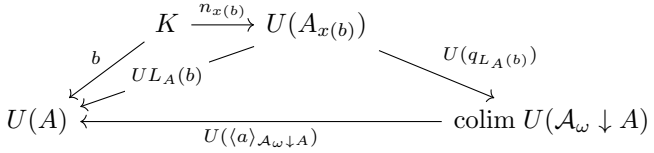
induced from the fact that the diagram dom : $\mathcal{A}_\omega \downarrow A \to \mathcal{A}$ is equipped with a canonical cocone of tip A. Moreover, since U preserves filtered colimit, this triangle is sent by U to a triangle



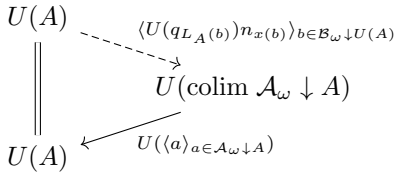
But in $\mathcal{B}$, we have a filtered colimit $U(A) \simeq \text{colim } \mathcal{B}_\omega \downarrow U(A)$, where each $b : K \to U(A)$ is the inclusion at the index it defines in this canonical colimit. Moreover the local adjoint of U over A restricts to finitely presented objects as a functor

$$\mathcal{B}_\omega \downarrow U(A) \to \mathcal{A}_\omega \downarrow A$$

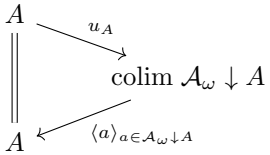
sending any $b : K \to U(A)$ to its local part $L_A(b) : A_{x(b)} \to A$ for the unique $x(b)$ in I_K corresponding to the local factorization of b. For each b the corresponding $L_A(b)$ is equipped with a canonical inclusion $q_{L_A(b)} : A_{x(b)} \to \text{colim } \mathcal{A}_\omega \downarrow A$ and one has $L_A(b) = \langle a \rangle_{\mathcal{A}_\omega \downarrow A} q_{L_A(b)}$. Then one has a factorization of the inclusion b



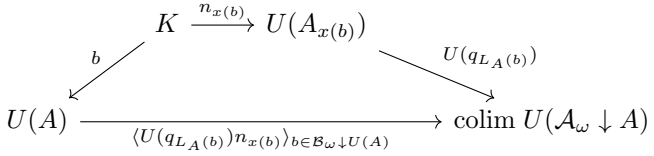
which entails by the universal property of the colimit that we have a retraction



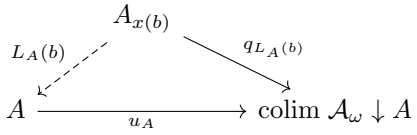
Moreover, since U is relatively full and faithful, the map $\langle U(q_{L_A(b)}) n_{x(b)} \rangle_{b \in \mathcal{B}_\omega \downarrow U(A)}$ must actually come from a unique section u_A in $\mathcal{A}$



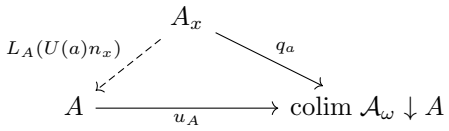
On the other hand, the composites $U(q_{L_A(b)}) n_x : K \to \text{colim } U(\mathcal{A}_\omega \downarrow A)$ define altogether a universal map



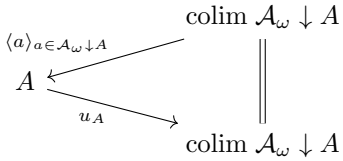
But as we have $\langle U(q_{L_A(b)}) n_{x(b)} \rangle_{b \in \mathcal{B}_\omega \downarrow U(A)} = U(u_A)$ and for each $b : K \to U(A)$, $n_{x(b)}$ is a local unit, and the local factorization provides a diagonalization of this square



U being relatively full and faithful, we know that each U_A is full and faithful, so that the counits of the local adjunctions are isomorphisms, and for any $a : A_x \to A$ in $\mathcal{A}_\omega \downarrow A$, we have $a \simeq L_A(U_A(a))$, and hence $a = L_A(U_A(a)n_x)$, and $q_a = q_{L_A(U_A(a))}$. Hence for any $a \in \mathcal{A}_\omega \downarrow A$ we have a factorization of the inclusion q_a given as

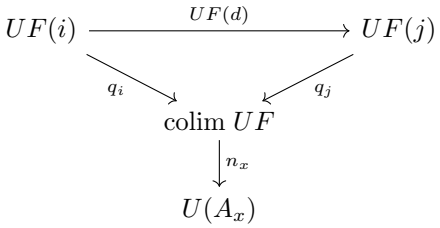


which entails by the universal property of the colimit that we have a retraction

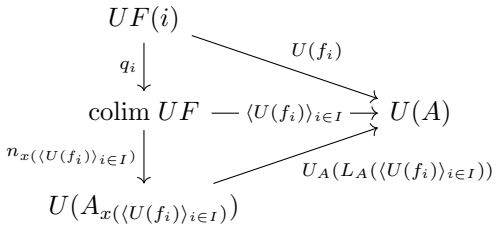


Hence we have an isomorphism $A \simeq \operatorname{colim} \mathcal{A}_\omega \downarrow A$ as desired.

Now we prove that $\mathcal{A}$ has multicolimits. Take $F : I \to \mathcal{A}$. Then, since $\mathcal{B}$ is locally finitely presentable, we can compute in $\mathcal{B}$ the colimit colim UF; now consider the local units under colim UF, and observe that for each of those local unit $x \in I_{\operatorname{colim} UF}$ we have a cocone over UF



and we claim that each of those cocones comes uniquely from a cocone in $\mathcal{A}$, which altogether form the multicolimit. For a given cocone $(f_i : F(i) \to A)_{i \in I}$ in $\mathcal{A}$, take the universal map $\langle U(f_i) \rangle_{i \in I} : \operatorname{colim} UF \to U(A)$; for U is right multi-adjoint, this arrow factorizes uniquely through one of the local unit of index $x(\langle U(f_i) \rangle_{i \in I}) \in I_{\operatorname{colim} UF}$. Then for each $i \in I$ we have a factorization of $U(f_i)$



which forces the composite $n_{x(\langle U(f_i) \rangle_{i \in I})} q_i$ to come from a unique arrow $\gamma_i^x : F(i) \to A_{x(\langle U(f_i) \rangle_{i \in I})}$ in $\mathcal{A}$ for U is relatively full and faithful. Hence in particular each composite $n_x q_i$ for $i \in I$ and $x \in I_{\operatorname{colim} UF}$ comes from a unique $\gamma_i^x : F(i) \to A_x$ in $\mathcal{A}$, and the family of cocones

$$((\gamma_i^x : F(i) \to A_x)_{i \in I})_{x \in I_{\operatorname{colim} UF}}$$

is exhibited as a multicolimit for F in $\mathcal{A}$. Hence $\mathcal{A}$ is multicocomplete, and being finitely accessible, it is locally finitely multipresentable.

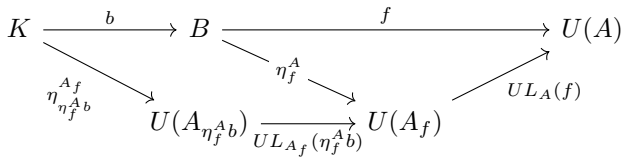
In particular observe that $\mathcal{A}$ has connected limits - and observe that U preserves them as a right multi-adjoint, so that they are computed as limits in $\mathcal{B}$. □

In this context, we can in particular compute explicitly the factorization through local units associated to U by using the canonical cone of its domain:

Corollary 4.1.2.5. *With the hypothesis of proposition 4.1.2.4, for any $f : B \to U(A)$, we have*

$$A_f \simeq \operatorname*{colim}_{b \in \mathcal{B}_\omega \downarrow B} A_{fb} \qquad \eta_f^A = \operatorname*{colim}_{b \in \mathcal{B}_\omega \downarrow B} \eta_{fb}^A \qquad L_A(f) = \langle L_A(fb) \rangle_{b \in \mathcal{B}_\omega \downarrow B}$$

Proof. We have $B \simeq \operatorname{colim} \mathcal{B}_\omega \downarrow B$, so that $f = \langle fb \rangle_{b \in \mathcal{B}_\omega \downarrow B}$. Now for a given $b : K \to B$, consider the factorization



but by theorem 2.1.1.5 we know that actually

$$A_{\eta^A_f b} \simeq A_{fb} \qquad \eta^{A_f}_{\eta^A_f b} = \eta^A_{fb} \qquad L_A(f) L_{A_f}(\eta^A_f b) = L_A(fb)$$

□

4.1.3 Multireflection through cone-orthogonality

In [28], a specific process to construct locally finitely multipresentable categories is provided, which encompasses most of the interesting examples and enjoys Diers condition. Though this process is originally done inside an ambient locally multipresentable category, we consider here the special case of an ambient locally presentable category, where the construction simplifies slightly and meets the version of the small object argument presented above.

In the following, let $\mathcal{B}$ be a locally finitely presentable category. We consider a small class Γ of small cones in $\mathcal{B}_\omega$, with

$$V_\Gamma = \{g_i \mid \ (g_i : K \to K_i)_{i \in I} \in \Gamma, i \in I\}$$

the set of all arrows involved in cones of Γ.

Definition 4.1.3.1. An object A in $\mathcal{B}$ is said to be Γ-*local* if for any cone $(g_i : K \to K_i)_{i \in I}$ in Γ, we have a surjection

$$\coprod_{i \in I} \mathcal{B}[K_i, A] \xrightarrow{\langle \mathcal{B}[g_i, A] \rangle_{i \in I}} \mathcal{B}[K, A]$$

Remark 4.1.3.2. Observe that this notion is slightly weaker than the one considered in [25][Section 8.6], which should bear the name of *strongly* Γ-*local* and add an injectiveness condition on the maps $\mathcal{B}[g_i, A]$. Our notion, beside being more general, is better suited to encompass the situations corresponding to geometric extensions of the finite limit theory behind $\mathcal{B}$ as we shall see below - where the injectiveness condition is generally absent.

We have also a notion of local morphism, which is actually the notion of right map relatively to the class of all maps involved in the cones of Γ; together with local objects they form a certain subcategory of $\mathcal{B}$ we are going to focus on:

Definition 4.1.3.3. A morphism is said to be Γ-*local* if it is in $V_\Gamma^\perp$. We denote as $\mathcal{B}_\Gamma$ the category of Γ-local objects and Γ-local morphisms between them, equipped with a faithfull, injective on objects inclusion

$$\mathcal{B}_\Gamma \xhookrightarrow{U_\Gamma} \mathcal{B}$$

Remark 4.1.3.4. Observe that the category hence obtained is not a full subcategory: this allows in particular to select a distinguished class of morphisms between models of an extension of the theory behind $\mathcal{B}$, while considering only a finite limit extension of the theory of $\mathcal{B}$ in the same signature does not restrict morphisms between the models of the extension, producing always a full subcategory: this is because morphisms are determined by the signature and not by the axioms of the theory.

We know then from the small object argument that Γ-local morphisms are the right class of a left generated factorization system $(\mathrm{Ind}(\overline{V_\Gamma}), V_\Gamma^\perp)$ where $\overline{V_\Gamma}$ is the saturated class generated from V_Γ.

Then we have the following property, which is slightly stronger than [28][Theorem 3.2]- and whose proof simplifies also a bit thanks to the prior treatment of the small object argument:

Theorem 4.1.3.5. *Let Γ be any small class of small cones in $\mathcal{B}_\omega$. Then the inclusion U_Γ is*

– *accessible,*

– *relatively full and faithful,*

– *right multi-adjoint,*

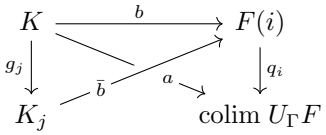
– *and satisfies Diers condition*

In particular it exhibits $\mathcal{B}_\Gamma$ as a locally finitely multipresentable category.

Proof. First observe that U_Γ is faithful; moreover, it is easy to see it is relatively full and faithful as a consequence of left cancellation that local maps enjoy as a right class.

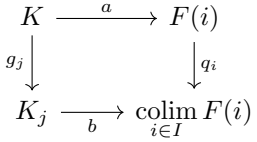
We now prove that $\mathcal{B}_\Gamma$ has filtered colimits and that they are preserved by U_Γ.

Let $F : I \to \mathcal{B}_\Gamma$ be a small diagram with I filtered. Consider its filtered colimit $\mathrm{colim} U_\Gamma F$ in $\mathcal{B}$. Now take a cone $(g_j : K \to K_j)_{j\in J}$ in Γ. Now for the joint surjectivity, consider an arrow $a : K \to \mathrm{colim}\, U_\Gamma F$; as K is finitely presented, we have a lift $b : K \to F(i)$ for some $i \in I$, and by localness of $F(i)$ there is some j in J and a factorization

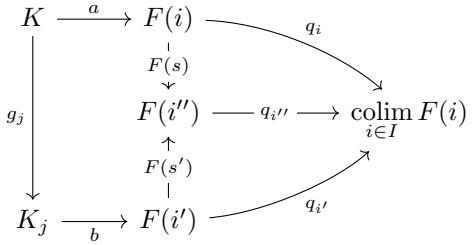


and we have $a = q_i b = q_i \bar{b} g_j$. Hence the localness of $\mathrm{colim}\, U_\Gamma F$.

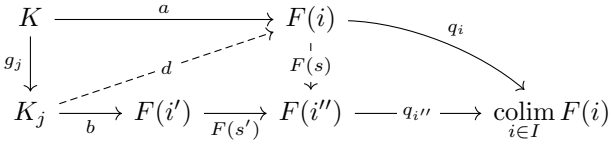
We must now prove the inclusions q_i are Γ-local. Consider a square as below, for $(g_j : K \to K_j)_{j\in I}$ and $j \in J$:



Then for K_j is finitely presented and I filtered, there is some i' such that b factorizes through $q_{i'}$ as $b = q_{i'} b'$. But then $q_i a = b g_j = q_{i'} b' g_j$, so that there exists some span $s : i \to i''$, $s' : i' \to i''$ in I such that we have a factorization as follows

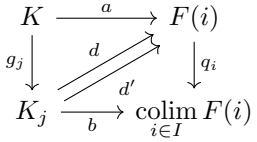


But then for g_j is in V_Γ and $F(s)$ is Γ-local we have a unique diagonalization as below



where d provides also a diagonalization of the original square above.

Moreover such a diagonalization must always be unique: suppose indeed one has two parallel diagonalizations



Then g_j equalize d, d'; but we also have that q_i coequalizes d, d', and from the fact that K_j is finitely presented and I is filtered, this implies there exists some arrow $s : i \to i'$ in I such that

$F(s)$ coequalizes d, d': but $F(s)$ is Γ-local, that is, in $V_\Gamma^\perp$, so by corollary 1.1.1.4, this forces d, d' to be equal. This finishes to prove that $\mathcal{B}_\Gamma$ has filtered colimit and that U_Γ preserves them, hence is accessible.

Now we have to prove U_Γ is a right multi-adjoint. We consider as already constructed the factorization system $(\mathrm{Ind}(\overline{V_\Gamma}), V_\Gamma^\perp)$. We prove that Γ-local objects have the gliding property relatively to local maps (see definition 2.3.3.2). Suppose that A is Γ-local and C is an object equipped with a local arrow $u : C \to A$. Now for any cone $(g_i : K \to K_i)_{i\in I}$ in Γ and any $f : K \to C$ we have a factorization for some g_i:

$$\begin{array}{ccc} K & \xrightarrow{f} & C \\ {\scriptstyle g_i}\downarrow & & \downarrow{\scriptstyle u} \\ K_i & \overset{a}{\dashrightarrow} & A \end{array}$$

Then for $g_i \perp u$ this induces a unique factorization $K_i \to C$ of f as desired. Then C is Γ-local.

Hence from theorem 2.3.3.3 we know that U_Γ is stable, as $\mathcal{B}_\Gamma$ consists precisely of a class of objects that have the gliding property to a right class. Hence for any arrow $f : B \to A$ with A a Γ-local object, the $(\mathrm{Ind}(\overline{V_\Gamma}), V_\Gamma^\perp)$-factorization

$$\begin{array}{ccccc} B & & \xrightarrow{\quad f \quad} & & A \\ & {\scriptstyle n_f}\searrow & & \nearrow{\scriptstyle u_f} & \\ & & A_f & & \end{array}$$

returns a local object A_f. Moreover, as U_Γ is relatively full and faithful, arrow $n : B \to A$ in $\mathrm{Ind}(\overline{V_\Gamma})$ with A Γ-local are exactly the candidate for U: hence U_Γ automatically satisfies Diers condition.

By theorem 2.1.4.6, U_Γ is local right adjoint. But now, as U_Γ is accessible, it satisfies the solution set condition: then by proposition 2.1.3.4, U_Γ is a right multi-adjoint. As a consequence, $\mathcal{B}_\Gamma$ is locally finitely multipresentable, from proposition 4.1.2.4. □

Remark 4.1.3.6. Beware however that without additional assumption, U_Γ may not be diagonally axiomatizable; it may actually happen that the class of Γ-local object is empty, or at least does not contain enough objects to ensure that ${}^\perp U_\Gamma(\mathcal{B}_\Gamma^2) = \mathrm{Ind}(\overline{V_\Gamma})$.

Moreover, we also must restrict to the local maps between local objects in order to get a multireflective subcategory, for there is no reason for local objects to have the gliding property along arbitrary maps: hence in general the full subcategory of local objects is not multireflective in $\mathcal{B}$, and hence not locally multipresentable.

4.1.3.7. Observe that this result returns actually a situation of geometry. If $\mathcal{B}$ is $\mathbb{T}[\mathcal{S}]$ with $\mathbb{T}$ a finite limit theory, then any class of finitely presented cones Γ in $\mathcal{B}_\omega \simeq \mathcal{C}_\mathbb{T}^{\mathrm{op}}$ corresponds to a system of covers and hence generates a Grothendieck pretopology J_Γ. Conversely, any Grothendieck pretopology J on $\mathcal{C}_\mathbb{T}$ coding for a geometric extension $\mathbb{T}_J$ of $\mathbb{T}$ defines a class of cones Γ_J in $\mathcal{B}_\omega$. Moreover, for a Grothendieck pretopology J on $\mathcal{C}_\mathbb{T}$, the class V_{Γ_J} is already saturated by the axioms of a Grothendieck pretopology.

Corollary 4.1.3.8. *For any finite limit theory $\mathbb{T}$ and any Grothendieck pretopology J on $\mathcal{C}_\mathbb{T}$, $(\mathbb{T}, J, \mathcal{V}_{\Gamma_J})$ is the smallest geometry on $\mathbb{T}$ with $\mathbb{T}_J[\mathcal{S}]$ as local objects.*

Remark 4.1.3.9. Beware however that this construction is not Morita-invariant, in the sense that it depends on the choice of a pretopology: if J_1, J_2 are two pretopologies generating a same topology, then, though Γ_{J_1}-local objects and Γ_{J_2}-local objects coincide, we cannot prove that Γ_{J_1}-local maps and Γ_{J_2}-local maps coincide. Then, even if J_1 and J_2 are Morita equivalent, $(\mathbb{T}, J_1, \mathcal{V}_{\Gamma_{J_1}})$ and $(\mathbb{T}, J_2, \mathcal{V}_{\Gamma_{J_2}})$ may not be Morita-equivalent geometries.

4.1.4 Disjunctive theories

In [52] was given a syntactic approach to locally finitely multipresentable categories, expliciting a link with *disjunctive theories*. It was proven that any locally finitely multipresentable category is equivalent to the category of set-valued models of a disjunctive theory, and conversely that any

disjunctive theory defines a locally finitely multipresentable category which is multireflective in the category of models of its finite-limit part. However in general this relation is not as strict as the relation between locally finitely presentable categories and finite-limit theory as we shall see. In this subsection we recall mostly material from [52].

Recall that a disjunction $\phi(\overline{x}) \vee \psi(\overline{x})$ is *provably disjoint* if one has $\phi(\overline{x}) \wedge \psi(\overline{x}) \vdash_{\overline{x}} \bot$.

Definition 4.1.4.1. A *disjunctive theory* is a theory whose formulas involve

- finite conjunctions
- existentials with proof of uniqueness
- strict (infinite) disjunctions

Remark 4.1.4.2. Beware that we put the same restrictions on the existentials as in the case of finite limit theories; on the other hand there is no restriction on size of disjunctions, as soon as they are provably disjoint.

Then disjunctive theories are convenient to axiomatize locally finitely multipresentable categories:

4.1.4.3. Let $\mathcal{C}$ be a small mlex category. Then we can construct as in [52][Proposition 3.1] a disjunctive theory $\mathbb{T}_{\mathcal{C}}$ as follows. Define the language $\mathcal{L}_{\mathcal{C}}$ as consisting for each object C of $\mathcal{C}$ defines a sort S_C, and for each morphism $f : C_1 \to C_2$ a function symbol $\theta_f : S_{C_1} \to S_{C_2}$. Then define $\mathbb{T}_C$ as consisting of the usual functoriality axiom $\vdash_x \theta_{fg}(x) = \theta_f(\theta_g(x))$ and identity axiom $\vdash_x \theta_{1_C}(x) = x$, plus, for each finite multilimit $(p_i^j : L_j \to X_i)_{i\in I, j\in J}$ over a finite diagram $(X_i)_{i\in I}$ with transition morphisms $f_d : X_i \to X_{i'}$ for $d : i \to i'$ in $\mathcal{I}^2$, the following axioms:

- an axiom expressing that the function symbols coded by projections in a local cone of the multilimit are jointly injective:

$$\bigwedge_{i\in I} \theta_{p_i^j}(\overline{y}_j) = \theta_{p_i^j}(\overline{y}'_j) \vdash_{\overline{y}_j \overline{y}_j : S_{L_j}} \overline{y}_j = \overline{y}'_j$$

- an axiom expressing that no term can be the solution of the equation coded by the function symbols coding the projections of two distinct local cones at j, j' distinct

$$\bigwedge_{i\in I} \theta_{p_i^j}(\overline{y}_j) = \theta_{p_i^{j'}}(\overline{y}_{j'}) \vdash_{\substack{\overline{y}_j : S_{L_j} \\ \overline{y}_{j'} : S_{L_{j'}}}} \bot$$

- an axiom expressing factorization of each cone trough one member of the multilimit

$$\bigwedge_{d\in I^2} \theta_{f_d}(\overline{x}_i) = \overline{x}_{i'} \vdash_{\substack{\overline{x}_i : S_{X_i} \\ \overline{x}_{i'} : S_{X_{i'}}}} \bigvee_{j\in J} \exists \overline{y}_j : S_{L_j} \, \Big(\bigwedge_{i\in I} \overline{x}_i = \theta_{p_i^j}(\overline{y}_j) \Big)$$

Lemma 4.1.4.4. *The theory $\mathbb{T}_{\mathcal{C}}$ is disjunctive.*

Proof. The only disjunctions involved in $\mathbb{T}$ are those of the third class of axioms: but observe that for any j, j' which are distinct in J the second class of axioms forces

$$\bigwedge_{i\in I} \overline{x}_i = \theta_{p_i^j}(\overline{y}_j) = \theta_{p_i^{j'}}(\overline{y}_{j'}) \vdash_{\substack{\overline{x}_i : S_{X_i} \\ \overline{y}_j : S_{L_j}, \overline{y}_{j'} : S_{L_{j'}}}} \bot$$

Hence the disjunction in the third axioms are strict. Moreover the existential are strict from the first class of axioms. □

Proposition 4.1.4.5. *Let $\mathcal{C}$ be a small mlex category. Then the disjunctive theory $\mathbb{T}_{\mathcal{C}}$ such that $\mathbb{T}_{\mathcal{C}}[\mathcal{S}] \simeq \boldsymbol{Mlex}[\mathcal{C}, \mathcal{S}]$.*

Proof. Recall that any functor $F : \mathcal{C} \to \mathcal{S}$ can be seen as a $\mathcal{L}_{\mathcal{C}}$-structure, where the sort S_C is interpreted as $F(C)$, and the function symbol $\theta_f : S_{C_1} \to S_{C_2}$ by $F(f) : F(C_1) \to F(C_2)$. The functoriality and identity axioms are then automatically satisfied by the functoriality of F. But also recall from the Yoneda lemma that $F(C) \simeq [\mathcal{C}, \mathcal{S}][\exists_C, F]$.

Now for a formula of the form $\theta_f(\overline{x}) = \overline{y}$ with $\overline{x} : S_{C_1}$ and $\overline{y} : S_{C_2}$, and elements $\overline{b}$ in $F(C_1)$, $a \in F(C_2)$, one can write $F \models \theta_f(\overline{b}) = \overline{a}$ if $F(f)(b) = a$. From Yoneda lemma this means that we have a commutation

$$\begin{array}{ccc} \exists_{C_2} & \xrightarrow{a} & F \\ {\scriptstyle \exists_f}\downarrow & \nearrow_{b} & \\ \exists_{C_1} & & \end{array}$$

Now, let $(p_i^j : L_j \to X_i)_{i\in I, j\in J}$ be a finite multilimit in $\mathcal{C}$. Then the first axiom says that for any b, b' in $F(L_j)$, if for all $i \in I$ one has $F(p_i^j)(b) = F(p_i^j)(b')$ then $b = b'$: In term of cones, this means that any two parallel arrows

$$\exists_{L_j} \underset{b'}{\overset{b}{\rightrightarrows}} F$$

that induce the same cone $(b\,\exists_{p_i^j} = b'\,\exists_{p_i^j})_{i\in I}$ are actually equal, so that each local cone $(\exists_{p_i^j})_{i\in I}$ behaves locally as a colimit.

The second axiom says that there are never two distinct j, j' and $b \in F(L_j)$, $b' \in F(L_{j'})$ such that one has simultaneously for all $i \in I$ $F(p_i^j)(b) = F(p_i^{j'})(b')$: this means that a cone $(b_i : \exists_{X_i} \to F)_{i\in I}$ never admits simultaneously two factorizations through two distincts local cocone.

But the last axioms says that any such cone $(b_i : \exists_{X_i} \to F)_{i\in I}$ still induces a factorization for some $j \in J$

$$\begin{array}{ccc} \exists_{X_i} & \xrightarrow{b_i} & F \\ {\scriptstyle \exists_{p_i^j}}\downarrow & \nearrow_{b} & \\ \exists_{L_j} & & \end{array}$$

In fact from the previous axioms, there is only one $j \in J$, and only one such b satisfying this condition.

Hence we see that those conditions are exactly equivalent to requiring F to be mlex, as we know from remark 2.1.3.11 that this condition says that for any multilimit

$$\lim_{i\in I} F(X_i) \simeq \coprod_{j\in J} F(L_j)$$

But this is exactly what is expressed by the axioms above. □

Corollary 4.1.4.6. *Any finitely locally multipresentable category can be axiomatized by a disjunctive theory. In the following we call this theory* $\mathbb{T}$ *its* disjunctive axiomatization.

Remark 4.1.4.7. Beware that this results does not say that $\mathcal{A}_\omega^{\mathrm{op}}$ is a syntactic category for $\mathbb{T}$ - it is not clear what notion of syntactic category should be involved here.

Any disjunctive theory is geometric: hence it admits a finite-limit part $\mathbb{T}_0$ consisting of all its finite limit sequents of $\mathbb{T}$. The syntactic category of $\mathbb{T}_0$ is lex, and $\mathbb{T}_0[\mathcal{S}]$ is locally finitely presentable. We also have the converse result that the category of models of any disjunctive theory is locally finitely multipresentable:

Proposition 4.1.4.8. *Let* $\mathbb{T}$ *be a disjunctive theory. Then* $\mathbb{T}[\mathcal{S}]$ *is locally finitely multipresentable.*

Proof. It can be shown, see [52][corollary 3.5] than any formula of $\mathbb{T}$ can be chosen as being of the form $\phi(\overline{x}) \vdash \bigvee_{i\in I} \exists \overline{y}_i \psi(\overline{y}_i)$, with the existential with proof of uniqueness and the disjunction strict, and both ϕ and the ψ_i finite-limite formulas with $\psi_i(\overline{y}_i) \vdash \phi(\overline{x})$.

Then those disjunctions code for cones in the syntactic category of the finite-limit part $\mathcal{C}_{\mathbb{T}_0}$, and we just have to take the corresponding class of cocone in $\mathbb{T}_0$

$$\Gamma_{\mathbb{T}} = \{(\langle \overline{x}\rangle/\phi(\overline{x}) \to \langle \overline{y}_i\rangle/\psi_i(\overline{y}_i) \mid \phi(\overline{x}) \vdash_{\mathbb{T}} \bigvee_{i\in I} \exists \overline{y}_i \psi(\overline{y}_i)\}$$

Then from theorem 4.1.3.5, we know that the category of local objects and local maps $(\mathbb{T}_0[\mathcal{S}])_{\Gamma_\mathbb{T}}$ in $\mathbb{T}_0[\mathbb{S}]$ is locally finitely multipresentable, multireflective in $\mathbb{T}[\mathcal{S}]$. Moreover, $\Gamma_\mathbb{T}$-local objects are exactly $\mathbb{T}$-models in $\mathbb{T}_0$ and moreover any $\Gamma_\mathbb{T}$-local object is automatically strongly $\Gamma_\mathbb{T}$-local.

Then we have to check that the inclusion $(\mathbb{T}_0[\mathcal{S}])_{\Gamma_\mathbb{T}} \hookrightarrow \mathbb{T}_0[\mathcal{S}]$ is full, that is, that any arrow $f : A_1 \to A_2$ between local objects is $\Gamma_\mathbb{T}$-local. Take any $f_{\theta_i} : \langle\overline{x}\rangle/\phi(\overline{x}) \to \langle\overline{y}_i\rangle/\psi_i(\overline{y}_i)$ for $\theta_i(\overline{x},\overline{y}_i)$ coding for $\psi_i(\overline{y}_i) \vdash \phi(\overline{x})$. Then consider a square

$$\begin{array}{ccc} \langle\overline{x}\rangle/\phi(\overline{x}) & \xrightarrow{a} & A_1 \\ {\scriptstyle f_{\theta_i}}\downarrow & & \downarrow{\scriptstyle f} \\ \langle\overline{y}_i\rangle/\psi_i(\overline{y}_i) & \xrightarrow[b]{} & A_2 \end{array}$$

Then as A_1 is $\Gamma_\mathbb{T}$-local, there is some i in I and some factorization

$$\begin{array}{ccc} \langle\overline{x}\rangle/\phi(\overline{x}) & \xrightarrow{a} & A_1 \\ {\scriptstyle f_{\theta_i}}\downarrow & \nearrow_{c} & \\ \langle\overline{y}_i\rangle/\psi_i(\overline{y}_i) & & \end{array}$$

But then this provides two parallel factorizations of the same arrow $fa = fcf_{\theta_j} = bf_{\theta_i}$

$$\begin{array}{ccc} \langle\overline{x}\rangle/\phi(\overline{x}) & \xrightarrow{f_{\theta_j}} & \langle\overline{y}_j\rangle/\psi_j(\overline{y}_j) \\ {\scriptstyle f_{\theta_i}}\downarrow & & \downarrow{\scriptstyle fc} \\ \langle\overline{y}_i\rangle/\psi_i(\overline{y}_i) & \xrightarrow[b]{} & A_2 \end{array}$$

which forces $i = j$ as A_2, as a $\Gamma_\mathbb{T}$-local object, is a $\mathbb{T}$-models. Hence as local objects are automatically strongly local, one also has $fc = b$, and this proves that c is a diagonalization as expected.

Finally, being two theories in the same language, $\mathbb{T}[\mathcal{S}] \hookrightarrow \mathbb{T}_0[\mathcal{S}]$ must be a full inclusion: but the inclusion $(\mathbb{T}_0[\mathcal{S}])_{\Gamma_\mathbb{T}} \hookrightarrow \mathbb{T}_0[\mathcal{S}]$ being also full with the same objects, we have

$$(\mathbb{T}_0[\mathcal{S}])_{\Gamma_\mathbb{T}} \simeq \mathbb{T}[\mathcal{S}]$$

Hence $\mathbb{T}[\mathcal{S}]$ is locally finitely multipresentable. □

4.1.5 Remarks on the locally multipresentable categories induced from geometries

We end this section with some remarks on the accessible aspects involved in geometries, summing up what was said in the previous chapters.

4.1.5.1. In theorem 3.3.1.12, we saw that for a geometry $(\mathbb{T}, \mathcal{V}, J)$, local objects and local maps formed altogether a relatively full and faithful inclusion

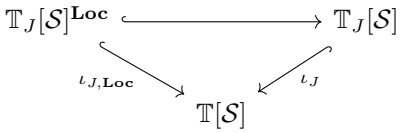
$$\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}} \xrightarrow{\iota[\mathcal{S}]} \mathbb{T}[\mathcal{S}]$$

which was deduced to be stable from the gliding of local objects along local maps in lemma 3.3.1.8.

Moreover we had the following observations concerning local objects. From **??**, we know that the full subcategory $\iota_J : \mathbb{T}_J[\mathcal{S}] \hookrightarrow \mathbb{T}[\mathcal{S}]$ is accessible: hence it has a small dense generator of κ-presented objects for some κ. Moreover, $\mathbb{T}_J[\mathcal{S}]$ has finitely filtered colimits, and the inclusion ι_J preserves them, so that they are calculated in $\mathbb{T}[\mathcal{S}]$.

Moreover we can prove with the same argument as in theorem 4.1.3.5 that the non full inclusion $\iota_{J,\mathbf{Loc}} : \mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}} \hookrightarrow \mathbb{T}$ exhibits $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ as closed under filtered colimits in $\mathbb{T}[\mathcal{S}]$. Then we deduced in theorem 3.3.1.12 that the inclusion $\iota_{J,\mathbf{Loc}}$ was not only stable but actually a right multi-adjoint. Hence from proposition 4.1.2.4, we know $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ to be *finitely accessible*, even though $\mathbb{T}_J[\mathcal{S}]$ was only accessible for some unknown rank. We saw in particular that one could exhibit as a small dense generator of finitely presented objects for $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ all the codomains A_x of the local units $n_x : K \to U(A_x)$ for $x \in I_K$ at finitely presented objects K of $\mathbb{T}[\mathcal{S}]$.

4.1.5.2. This surprising result deserves some precision. We have a triangle of finitely accessible inclusions



where in particular the horizontal inclusion is bijective on objects, so that all the finitely presented objects of $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ are in particular in $\mathbb{T}_J[\mathcal{S}]$ and still form a dense generator of $\mathbb{T}_J[\mathcal{S}]$, which could lead to think that $\mathbb{T}_J[\mathcal{S}]$ ought to also have a small generator of finitely presented objects. But this is in fact not the case, as the inclusion $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}} \hookrightarrow \mathbb{T}[\mathcal{S}]$ does not send finitely presented objects to finitely presented objects: in other words, being finitely presented in $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ does not mean being finitely presented in $\mathbb{T}_J[\mathcal{S}]$, since there may be filtered colimits of local objects with non-local transition morphisms the finitely presented objects of $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ may not be finitely presented relatively to.

4.1.5.3. For $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$ is locally finitely multipresentable, proposition 4.1.4.5 tells us there is a disjunctive theory $\mathbb{T}_{(\mathbb{T},\mathcal{V},\mathcal{J})}$ such that

$$\mathbb{T}_{(\mathbb{T},\mathcal{V},J)}[\mathcal{S}] \simeq \mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}}$$

However one should not make this result to tell more than it does:

- First, the disjunctive theory $\mathbb{T}_{(\mathbb{T},\mathcal{V},J)}$ may not classify categories of the form $\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}}$ for arbitrary Grothendieck toposes $\mathcal{E}$: those results were only viable in $\mathcal{S}$. In some topos those theories may cease to be equivalent, for instance see the distinction between the theory of local rings and strong local ring in [46].

- Secondly, the Grothendieck topology we started with J was never supposed to code for a disjunctive extension of $\mathbb{T}$, its finite-limit part: in fact the disjunctive theory we end up with $\mathbb{T}_{(\mathbb{T},\mathcal{V},J)}$ is not an extension of $\mathbb{T}$, having new sorts, and its finite-limit part is a new finite-limit theory.

- Finally, the disjunctive theory $\mathbb{T}_{(\mathbb{T},\mathcal{V},J)}$ also has a finite limit part $(\mathbb{T}_{(\mathbb{T},\mathcal{V},J)})_0$, and the inclusion $\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}} \simeq \mathbb{T}_{(\mathbb{T},\mathcal{V},J)}[\mathcal{S}] \hookrightarrow (\mathbb{T}_{(\mathbb{T},\mathcal{V},J)})_0[\mathcal{S}]$ is a full, accessible multireflection, while the inclusion $\iota_{J,\mathbf{Loc}} : \mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}} \hookrightarrow \mathbb{T}$ was only a relatively full and faithful - but non full - accessible multireflection. Beware that the theory $(\mathbb{T}_{(\mathbb{T},\mathcal{V},J)})_0$ is not in the same language as $\mathbb{T}$.

4.2 Geometric envelope of a stable functor

In the last section we saw how a class of cone of finite presentation in a locally finitely presentable category induced a multireflective locally finitely multipresentable subcategory. In this section we turn to a converse process, constructing a geometry from an arbitrary stable functor

4.2.1 Cone-injectivity relatively to local units

In this subsection we first discuss some cone-orthogonality and injectivity conditions. First we consider injectivity relatively to the canonical cone of local units under a given object, and see why this intermediate notion is not convenient; then we consider injectivity relatively to a notion of localizing families, and see how, though failing to returns a geometric envelope, is an intermediate notion which can be improved as done in the next subsection.

Definition 4.2.1.1. For $\mathcal{C}$ a class of cones

- The cone-injectivity class for $\mathcal{C}$ is the class $\mathcal{C}^{\perp}$ of objects A such that for any cone $(f_i : B \to B_i)_{i\in I}$ in $\mathcal{C}$, any $f : B \to A$ from the summit of the cone factorizes through some arrow f_i in the cone.

- The cone-orthogonality class for $\mathcal{C}$ is the class $\mathcal{C}^{inj}$ of objects A such that for any cone $(f_i : B \to B_i)_{i\in I}$ in $\mathcal{C}$, any $f : B \to A$ from the summit of the cone factorizes uniquely through exactly one f_i in the cone.

Remark 4.2.1.2. As detailed in [2], if $\mathcal{C}$ is a class of cones made of arrows of finite presentation, then the corresponding injectivity class is an accessibly embedded, accessible *full* subcategory of $\mathcal{B}$, while the corresponding orthogonality class is a *full* multireflective subcategory of $\mathcal{B}$.

Moreover, it should be pointed out that in an admissibility structure, the full subcategory of local objects, which are the models of a geometric extension, have no reason in general to be multireflective in the ambient locally finitely presentable category, though it should be closed under filtered colimits. We saw in fact above a result telling us morally that full, accessibly embedded multireflective subcategories of locally finitely presentable categories are models of *disjunctive* extensions. Though it encompasses certain prominent examples, our situations of interests are not restricted to disjunctive extensions, hence multireflectivity of the class of local objects should in general appear only after restriction to local maps.

In a Diers context the considered stable functor usually does not define a full subcategory. It is so exactly when the specialization order between local objects is trivial. But in the general case, we have to distinguish the considered non full subcategory defined by the essential image of U from the full multireflective subcategory given by the class of cone orthogonality it is expected to generate.

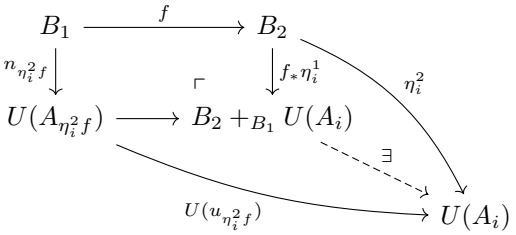
4.2.1.3. Consider the class of cones of local units under arbitrary objects:

$$\mathcal{C}_U = \{(B \overset{\eta_A(f)}{\to} UL_A(f))_{B \overset{f}{\to} U(A) \in B\downarrow U} \mid B \in \mathcal{B}\}$$

If we denote $(B \overset{\eta_i}{\to} U(A_i))_{i \in I_B}$ the cone of (isomorphism classes of) all local units of B, then it generates in some way a cocoverage in $\mathcal{B}$ in the sense that for any $f : B_1 \to B_2$, one has for any $i \in I_{B_2}$ the following factorization

$$\begin{array}{ccc} B_1 & \xrightarrow{f} & B_2 \\ {\scriptstyle n_{\eta_i^2 f}}\downarrow & & \downarrow{\scriptstyle \eta_i^2} \\ U(A_{\eta_i^2 f}) & \xrightarrow[U(u_{\eta_i^2 f})]{} & U(A_i) \end{array}$$

However the cones of local units are not stable by pushout: indeed, for $i \in I_{B_1}$ even though the $f_*\eta_i^1 : B_2 \to B_2 +_{B_1} U(A_i)$ is still diagonally universal, its codomain needs not be in the range of U. But pushouts of cones of local units are still covering families in the sense that "they still cover the local units" of the codomain. By this we mean that all its local units factorize through one of those pushouts at least. This results from the property of pushout as visualized in the following:



This must be generalized by considering cones of etales maps jointly factorizing local units under a given object. Those were considered in [30].

Definition 4.2.1.4. For a stable functor $U : \mathcal{A} \to \mathcal{B}$, we can consider the *$U$-localizing families* as the families of U-etale morphisms jointly factorizing local units of U, that are all the families of the form $(B \overset{n_i}{\to} B_i)_{i \in I}$ with n_i in $\mathbf{Et}_B$ such that for each local unit $x \in X_B$ there is some $i \in I$ providing a factorization

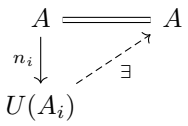
$$\begin{array}{ccc} B & \xrightarrow{\eta_x} & U(A_x) \\ & {\scriptstyle n_i}\searrow & \nearrow{\scriptstyle \exists} \\ & B_i & \end{array}$$

In particular one can define *U-local objects* as those A such that for any B and any U-localizing family $(n_i : B \to B_i)_{i \in I}$ in $\mathbf{Et}_U$ together with an arrow $f : B \to A$, we have a factorization of f through some n_i.

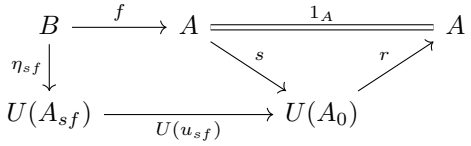
Proposition 4.2.1.5. *For any object A in $\mathcal{B}$, the following are equivalent:*

1. *A is U-local,*
2. *A is a retract of an object in the range of U,*
3. *A is in $\mathcal{C}_U^{inj}$*
4. *The $(\mathbf{Et}_U, \mathbf{Loc}_U)$ factorization of any $f : B \to A$ returns a retract of an object in the essential image of U.*

Proof. First, observe that if A is U-local, then in particular 1_A lifts through its own unit cone: that is we have a retraction for some $i \in I_A$:



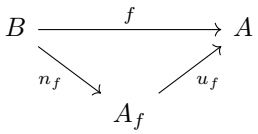
Now if A is a retract of some $U(A_0)$ with $1_A = rs$, for any $f : B \to A$ the factorization of sf provides a factorization of f itself through $rU(u_{sf})$:



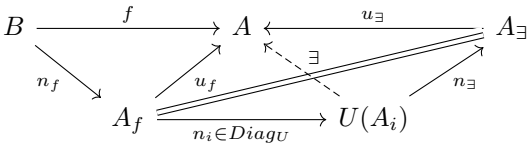
So that A is cone injective.

Now, if A is cone injective and $(B \to B_i)_{i \in I}$ is a cover, any $f : B \to A$ lifts through some unit $n_i : B \to U(A_i)$ which is itself factorized by some member of the cover, thus so is f, exhibiting A as a U-local object.

Next, for A a U-local object, let

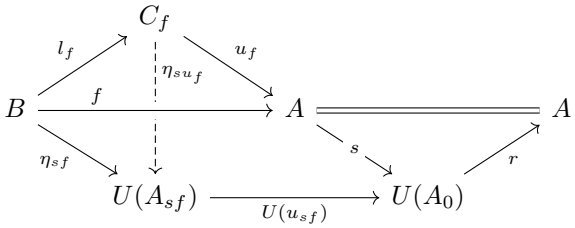


be the $(\mathbf{Et}_U, \mathbf{Loc}_U)$ factorization of $f : B \to A$; then A_f itself possesses a cone of local units under it, which is in particular trivially a U-localizing family: hence the local part u_f of the factorization itself lifts through one of them as A is U-local. But now the factorization of the lift returns the same object as the factorization of f by uniqueness of factorization, as we have a chain of diagonally universal maps on the left:

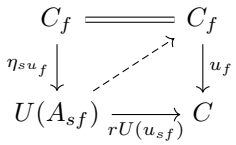


Hence A_f is a retract of some object in the range of U.

We can also prove directly that the second and the fourth item are equivalent. If A is a retract of an object in the range of U, then for $f : B \to A$ we can compare the left part of the factorization of A with the local unit

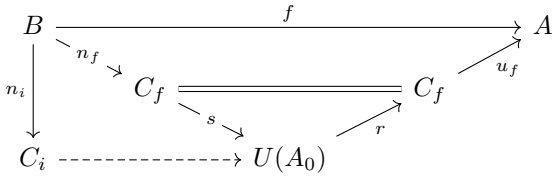


but now the following square below admits some filler



exhibiting C_f as a retract of $U(A_{sf})$. Proving that the last item implies the second is trivial by applying the result the identity of A

Finally, suppose that any arrow toward A factorizes through a retract of an object in the range of U. Then for any $f : B \to A$ and any U-localizing family $(n_i)_{i\in I}$ of B, we have a factorization as below

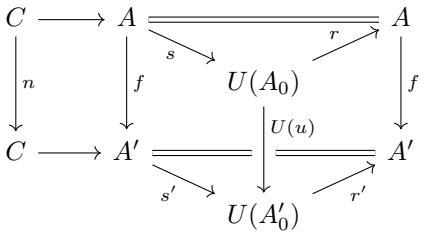


ensuring that A is U-local. □

Proposition 4.2.1.6. *We have a similar property of local maps:*

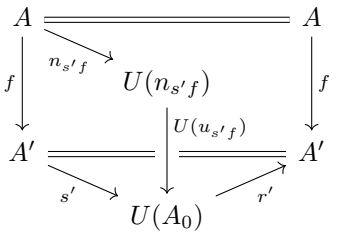
- *A map between retracts of objects in the essential image of U is U-local (that is right orthogonal to the maps in $\mathbf{Diag}_U$) if and only if it is a retract of a map in the range of U.*
- *In particular, any object sending a local map toward a retract of an object in the range of U is itself such a retract, as is the local map.*

Proof. For a retract of a map in the range of U and an arbitrary square as in the following with n in ${}^{\perp}(U(A^2))$:

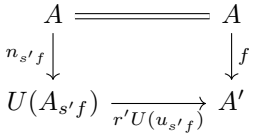


the composite square with n on the left and $U(u)$ on the right admits a lifting, whose composite with r constitutes a lifting of the leftmost square.

For the converse we have to prove the second item. Consider a local map whose codomain is a retract, then consider the factorization of the composite map, as in the following:



This provides us with the following square



whose diagonalization exhibits both A as a retract of $U(A_{s'f})$ and f as a retract of $U(u_{s'f})$. □

Corollary 4.2.1.7. *The U-local objects and U-local maps between them form a (non full) multireflective subcategory*

$$U\text{-}\mathbf{Loc}^{\mathbf{Loc}_U} \hookrightarrow \mathcal{B}$$

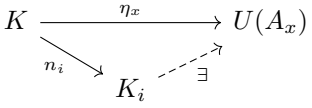
This means that for any stable functor, the closure by retracts of its essential image image is multireflective. Its full closure in $\mathcal{B}$ is the cone injectivity class $\mathcal{C}_U^{inj}$.

However, beware that the class thus obtained is not yet adequate to provide an admissibility: in fact, we cannot yet see the U-local objects as all the models of a geometric theory, for the injectivity condition involved are too constraining. Indeed, the U-localizing families were defined under arbitrary objects, and injectivity had to be tested hence relatively to a large class of cover; similarly, the cone-injectivity condition was tested relatively to a large family of cone outside the generator of finitely presented objects, and hence we cannot ensure the resulting class to be the class of models of a geometric extension.

4.2.2 Localizing families and geometric envelope

To produce the desired admissibility structure, we have to restrict the localness test relatively to finitely presented object. Hence we have to restrict the U-localizing families to the finitely presented objects and the finitely presented etale morphisms under them:

Definition 4.2.2.1. Define the *U-localizing pretopology* J_U on $\mathcal{B}_\omega^{\mathrm{op}}$ as having as covering families the duals of the families $(n_i : K \to K_i)_{i\in I}$ in $\mathbf{Et}_\omega$ such that for each $x \in X_K$, there is some $i \in I$ and a factorization



Then from what was said in 1.2, J_U codes for a geometric extension of the finite limit theory whose category of set-valued models is $\mathbb{T}$:

Definition 4.2.2.2. We denote as $\mathbb{T}_U$ the corresponding geometric extension, and call it the *geometric axiomatization* of U.

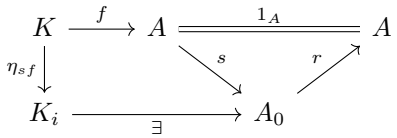
Remark 4.2.2.3. Observe that any U-local object is trivially J_U, hence is a model of $\mathbb{T}_U$. This is because a J_U-local object lifts in particular the basic cover of finitely presented objects.
However, as pointed out in [4], the converse is not true in general: there may be $\mathbb{T}_U$-models that fail to be J_U-local. In that sense the notion defined in [30] is not yet sufficient to axiomatize our local data in a universal way.

The rest of this section is devoted to prove that the data of $(J_U, \mathbf{Et}_U)$ define an admissibility structure with $\mathbb{T}_U[\mathcal{S}]$ as the local objects.
Any object in the essential image of U is in particular a model of $\mathbb{T}$. However, not only are there additional U-local objects, but there are even more $\mathbb{T}_U$-models. We have the following:

Proposition 4.2.2.4. *We have the following:*

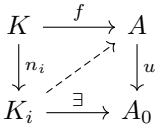
- *The subcategory $\mathbb{T}_U[\mathcal{S}]$ is closed in $\mathcal{B}$ by retracts.*
- *$\mathbb{T}_U$-models have the gliding property relatively to local maps.*

Proof. For the first item, the proof is the same as for a retract of objects in the range of U: if A is a retract of a $\mathbb{T}_U$-model A_0 then for any $f : K \to A$ and covering family of K in $J_U\mid_\omega$ one has



So that A itself is $J_U\mid_\omega$-flat by the Yoneda lemma, hence is a $\mathbb{T}_U$-model.

For the second item, let $u : A \to A_0$ be local with A_0 a $\mathbb{T}_U$-object. Then for any $f : K \to A$ and covering family of K in $J_U\mid_\omega$ the composite $uf : K \to A_0$ extends through some n_i; but as those n_i are in $\mathbf{Diag}_U$, this extension induces a diagonalization



Hence A is a $\mathbb{T}_U$-model. □

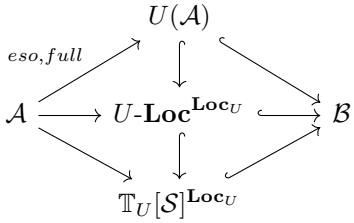
Corollary 4.2.2.5. *$\mathbb{T}_U$-models and local maps between them form altogether a (non full) multireflective subcategory*

$$\mathbb{T}_U[\mathcal{S}]^{\mathbf{Loc}} \hookrightarrow \mathcal{B}$$

In particular the full subcategory $\mathbb{T}_U[\mathcal{S}]$ can be seen as a geometric envelope of the functor U.

Remark 4.2.2.6. Observe that $\mathcal{A}$ is full in $J_U - \mathbf{Loc}^{\mathbf{Loc}}$ and $\mathbb{T}_U^{\mathbf{Loc}}$ as for a local map $u : U(A_1) \to U(A_2)$ the diagonally universal part is trivial while the left part is in the range of U, so is u.

To sum up, we have the following factorization of U, where all the faithful functors into $\mathcal{B}$ are also multireflective



Remark 4.2.2.7. More generally, one could ask, for a topology J, why it is worth to restrict us to a certain class of local maps between J-local objects rather that considering just general maps. The reason is that etale maps would have to be iso, although cover generating J are not, so this would not define an admissibility structure.

4.2.3 2-category of Diers contexts

We would also ask for the functoriality of this process. In [81][Definition 3.1] we introduced the following definition of the 2-category of *Diers contexts*:

Definition 4.2.3.1. We define the 2-category $\mathfrak{Diers}$ of *Diers Contexts* as having

- as 0-cells, triples $(U, \mathcal{A}, \mathcal{B})$ locally right adjoint functors $U : \mathcal{A} \to \mathcal{B}$ with $\mathcal{B}$ locally finitely presentable and satisfying Diers condition
- as 1-cells $(U_1, \mathcal{A}_1, \mathcal{B}_1) \to (U_2, \mathcal{A}_2, \mathcal{B}_2)$, triples of the form (F, G, θ) with $F : \mathcal{A}_1 \to \mathcal{A}_2$ a functor and $G : \mathcal{B}_1 \to \mathcal{B}_2$ a morphism of locally finitely presentable categories, and θ an invertible 2-cell (which we will leave implicit in general)

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{U_1} & \mathcal{B}_1 \\ {\scriptstyle F}\downarrow & \overset{\theta}{\simeq} & \downarrow{\scriptstyle G} \\ \mathcal{A}_2 & \xrightarrow[U_2]{} & \mathcal{B}_2 \end{array}$$

- and, as 2-cells $(F, G, \theta) \Rightarrow (F', G', \theta')$, pairs of natural transformations (α, β)

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{U_1} & \mathcal{B}_1 \\ F \Big(\overset{\alpha}{\Rightarrow}\Big) F' & & G \Big(\overset{\beta}{\Rightarrow}\Big) G' \\ \mathcal{A}_2 & \xrightarrow{U_2} & \mathcal{B}_2 \end{array}$$

  such that the two whiskering coincide through the provided iso : $\theta' U_{2*}\alpha = U_1^*\beta\theta$.

4.2.3.2. There we should give a remark about 1-cells: we recall that a morphism of locally finitely presentable categories is a functor $G : \mathcal{B}_1 \to \mathcal{B}_2$ preserving filtered colimits and admitting a left adjoint $G^* : \mathcal{B}_2 \to \mathcal{B}_1$. It can be shown that this left adjoint sends finitely presented objects of $\mathcal{B}_2$ to finitely presented object of $\mathcal{B}_1$ - this is equivalent to preservation of filtered colimits by G.

Moreover, observe that, for a morphism (F, G, θ), we have in each $A \in \mathcal{A}_1$ a pseudo commutative square

$$\begin{array}{ccc} \mathcal{A}_1/_A & \xrightarrow{U_1/_A} & \mathcal{B}_1/_{U_1(A)} \\ {\scriptstyle F/_A}\downarrow & \overset{\theta_A}{\simeq} & \downarrow{\scriptstyle G/_{U_1(A)}} \\ \mathcal{A}_2/_{F(A)} & \xrightarrow[U_2/_{F(A)}]{} & \mathcal{B}_2/_{U_2F(A)} \end{array}$$

which is equipped with a canonical mate $\sigma^A : L^2_{F(A)} \circ G \Rightarrow F \circ L^1_A$ as depicted below

$$\begin{array}{ccc} \mathcal{A}_1/_A & \xleftarrow{L^1_A} & \mathcal{B}_1/_{U_1(A)} \\ {\scriptstyle F/_A}\downarrow & \swarrow_{\sigma_A} & \downarrow{\scriptstyle G/_{U_1(A)}} \\ \mathcal{A}_2/_{F(A)} & \xleftarrow[L^2_{F(A)}]{} & \mathcal{B}_2/_{U_2F(A)} \end{array}$$

As we shall see, this mate acts as a comparison map between the U_2-factorization of the image by G and the image by G of the U_1-factorization.

4.2.3.3. Even though we shall not have major use of it, it is worth giving some precision on 2-cells, especially their interaction with the canonical mate. First observe that on the slices α induces a transformation

$$\begin{array}{ccccc} & & \mathcal{A}_1/A & & \\ & {\scriptstyle F/A}\swarrow & \overset{\alpha/A}{\Rightarrow} & \searrow{\scriptstyle F'/A} & \\ \mathcal{A}_2/F(A) & & \xrightarrow[\alpha_{A!}]{} & & \mathcal{A}_2/F'(A) \end{array}$$

where $\alpha_{A!}$ is the postcomposition with the component of α in A, $\alpha_A : F(A) \to F'(A)$, and in any $u \in \mathcal{A}_1/A$, α/A has for component $(\alpha/A)_u$, which is indeed an arrow in $\mathcal{A}_2/F'(A)$ from the commutativity of the naturality square of α. We have the same for β and this induces the following diagram

$$\begin{array}{ccccc} \mathcal{A}_1/A & & \xrightarrow{U_1/A} & & \mathcal{B}_1/U_1(A) \\ & {\scriptstyle F/A}\swarrow & & \searrow{\scriptstyle G/U_1(A)} & \\ \mathcal{A}_2/F(A) & \xrightarrow[\alpha_{A!}]{\overset{\alpha/A}{\Rightarrow}} & \mathcal{A}_2/F'(A) \; {\scriptstyle F'(A)}\downarrow \; \xrightarrow{U_2/F(A)} & \xrightarrow[\beta_{U_1(A)!}]{\overset{\beta/_{U_1(A)}}{\Rightarrow}} & \mathcal{B}_2/GU_1(A) \; \downarrow{\scriptstyle G'/U_1(A)} \\ & & \mathcal{A}_2/F'(A) & \xrightarrow{U_2/F'(A)} & \mathcal{B}_2/G'U_1(A) \end{array}$$

where the lower square commutes from $GU_1(A) = U_2F(A)$ (resp $G'U_1(A) = U_2F'(A)$) and the equality of whiskering. In fact this square does even satisfy Beck-Chevalley condition, that is the following square also commutes :

$$\begin{array}{ccc} \mathcal{A}_2/F(A) & \xleftarrow{L^2_{F(A)}} & \mathcal{B}_2/GU_1(A) \\ {\scriptstyle \alpha_{A!}}\downarrow & & \downarrow{\scriptstyle \beta_{U_1(A)!}} \\ \mathcal{A}_2/F'(A) & \xleftarrow[L^2_{F'(A)}]{} & \mathcal{B}/G'U_1(A) \end{array}$$

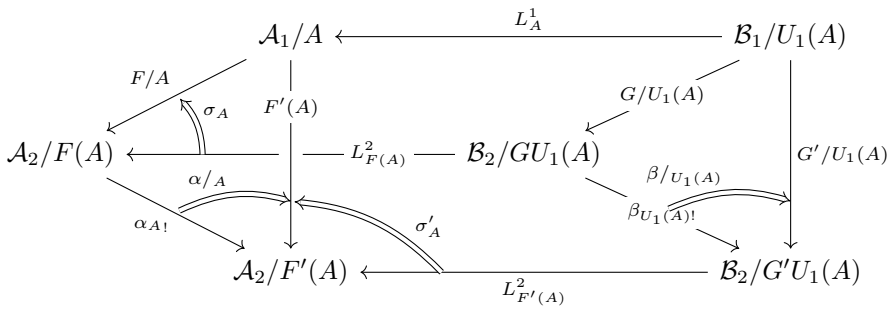
This comes from the fact that post composing with an arrow in the range of a stable functor does not modify the candidate part of the factorization (up to isomorphism). Hence for any arrow $f : B \to U_2F(A)$ in $\mathcal{B}_2$, we have a canonical isomorphism between the factorizations

$$\begin{array}{ccccccc} B & & \xrightarrow{x} & & GU_1(A) \simeq U_2F(A) & \xrightarrow{U_2(\alpha_A)} & G'U_1(A) \simeq U_2F'(A) \\ & {\scriptstyle \eta_f^{F(A)}}\searrow & & \nearrow{\scriptstyle U_2(u_x)} & & & \nearrow{\scriptstyle U_2(u_{U_2(\alpha_A)x})} \\ & & U_2L^2_{F(A)}(A_f) & \xrightarrow{\simeq} & U_2L^2_{F'(A)}(U_2(\alpha_A)x) & & \end{array}$$

This provides an equality between the pasting of the sliced transformations with the respective mates :

$$\alpha/_A * \sigma_A = \sigma'_A * \beta/_{U_1(A)}$$

making the following diagram to commute through a 2-dimensional equality



4.2.4 Functoriality of the envelope construction

In the middle of this section, we saw how to construct the geometric envelope of a given Diers context; here we would like to functorialize this process. For a morphism of Diers context $(F, G) : (U_1, \mathcal{A}_1, \mathcal{B}_1) \to (U_2, \mathcal{A}_2, \mathcal{B}_2)$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are categories of set valued models of finite limit theories $\mathbb{T}_1$ and $\mathbb{T}_2$; moreover the morphism of locally finitely presentable categories G restricts as a lex functor $(G^*)^{\mathrm{op}}_\omega : (\mathcal{B}_2)^{\mathrm{op}}_\omega \to (\mathcal{B}_1)^{\mathrm{op}}_\omega$ by Gabriel-Ulmer duality. In the following, recall that for $U : \mathcal{A} \to \mathcal{B}$ we denote $\mathcal{D}_U$ the class of diagonally universal morphisms of finite presentation $\mathbf{Diag}_U \cap \mathcal{B}^2_\omega$.

Proposition 4.2.4.1. *Let $(F, G) : (U_1, \mathcal{A}_1, \mathcal{B}_1) \to (U_2, \mathcal{A}_2, \mathcal{B}_2)$ be a morphism of Diers contexts. Then $(G^*)^{\mathrm{op}}_\omega$ defines a transformation of geometries*

$$(\mathbb{T}_2, \mathcal{D}_{U_2}, J_{U_2}) \xrightarrow{(G^*)^{\mathrm{op}}_\omega} (\mathbb{T}_1, \mathcal{D}_{U_1}, J_{U_1})$$

Proof. We have to prove that $(G^*)^{\mathrm{op}}_\omega$ preserves etale maps and sends J_{U_2}-covers on J_{U_1}-covers. For the first item, if $n : K \to K'$ is a U_2-etale map of finite presentation, observe that any square in $\mathcal{B}_1$ as below

$$\begin{array}{ccc} G^*(K) & \xrightarrow{f} & U_1(A) \\ {\scriptstyle G^*(n)}\downarrow & & \downarrow{\scriptstyle U_1(u)} \\ G^*(K') & \xrightarrow[g]{} & U_1(A') \end{array}$$

with $u : A \to A'$ in $\mathcal{A}_1$ corresponds uniquely to another square in $\mathcal{B}_2$ as below

$$\begin{array}{ccc} K & \xrightarrow{\overline{f}} & G_*U_1(A) \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle G_*U_1(u)} \\ K' & \xrightarrow[\overline{g}]{} & G_*U_1(A') \end{array}$$

But now from $G_*U_1 \simeq U_2F$ and n is U_2-etale, there exists a unique lift $d : K' \to G_*U_1(A)$ which corresponds by adjunction to a lift $\overline{d} : K \to U_1(A)$, ensuring that $G^*(n)$ is left orthogonal to any morphism in the range of U_1.

Now for local objects, take a J_2-cover $(n_i : K \to K_i)_{i\in I}$, and a U_1-unit $n : G^*(K) \to U_1(A)$. Then again by adjunction this unit corresponds uniquely to a morphism $\overline{n} : K \to G_*U_1(A) \simeq U_2F(A)$, whose unit $\eta_{\overline{n}} : B \to U(A_{\overline{n}})$ factorizes uniquely through some n_i. Hence by adjunction this factorization ensures that n factorizes through some $G^*(n_i)$, which are hence in $J_{U_1}(G^*(K))$. □

4.2.4.2. Conversely, from what was said in proposition 3.4.2.3 we know that a transformation of geometries $\Phi : (\mathbb{T}_1, \mathcal{V}_1, J_1) \to (\mathbb{T}_2, \mathcal{V}_2, J_2)$ induces a morphism of the corresponding Diers contexts

$$\begin{array}{ccc} \mathbb{T}_{J_2}[\mathcal{S}]^{\mathbf{Loc}_2[\mathcal{S}]} & \xrightarrow{\iota_2[\mathcal{S}]} & \mathbb{T}_2[\mathcal{S}] \\ \downarrow & & \downarrow{\scriptstyle \Phi[\mathcal{S}]_*} \\ \mathbb{T}_{J_1}[\mathcal{S}]^{\mathbf{Loc}_1[\mathcal{S}]} & \xrightarrow[\iota_1[\mathcal{S}]]{} & \mathbb{T}_1[\mathcal{S}] \end{array}$$

Moreover, the 2-dimensional aspects of this correspondence trivially procedes from the 2-dimensionality of Gabriel-Ulmer duality. Therefore it appears that we can formalize the correspondence between Diers contexts and geometries.

4.2.5 Coste-Diers Adjunction

Theorem 4.2.5.1. *We have an adjunction between the 2-category of geometries and the 2-category of Diers contexts*

$$\begin{array}{ccc} \mathfrak{Geom}^{\mathrm{op}} & \overset{\longleftarrow}{\underset{\longrightarrow}{\perp}} & \mathfrak{Diers} \\ (\mathbb{T},\mathcal{V},J) & \longmapsto & (\iota[\mathcal{S}],\mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}[\mathcal{S}]},\mathbb{T}[\mathcal{S}]) \\ (\mathbb{T},\mathcal{D}_U,J_U) & \longmapsto & (U,\mathcal{A},\mathcal{B}) \end{array}$$

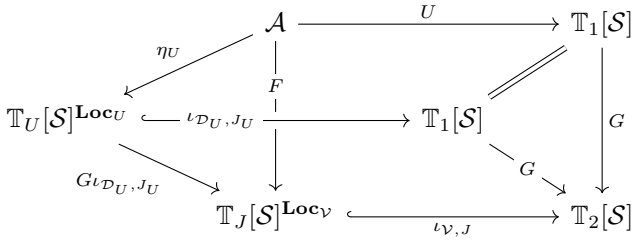
Proof. Consider a situation as below

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathbb{T}_1[\mathcal{S}] \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle G} \\ \mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}_{\mathcal{V}}} & \xrightarrow[\iota_{\mathcal{V},J}]{} & \mathbb{T}_2[\mathcal{S}] \end{array}$$

with $G = G^* \dashv G_*$ a morphism of locally finitely presentable categories. Then, for $G_*U = \iota_{\mathcal{V},J}$, we have that G^* sends finitely presented maps in $\mathcal{V}$ to U-diagonally universal morphisms of finite presentation. Moreover, as any object in $\mathcal{A}$ is sent to a J-local object, any J-cover is then U-localizing. Hence we have a transformation of geometries

$$(\mathbb{T}_2,\mathcal{V},J) \xrightarrow{G^*_\omega} (\mathbb{T}_1,\mathcal{D}_U,J_U)$$

Moreover, we have a factorization in the 2-category of Diers contexts



Conversely, for a morphism of geometry

$$(\mathbb{T}_2,\mathcal{V},J) \xrightarrow{\Phi} (\mathbb{T}_1,\mathcal{D}_U,J_U)$$

we have that any J cover is U-localizing: then the J_U-local objects are in particular J-locals, that is, are $\mathbb{T}_J$-models, and in particular, $\Phi[\mathcal{S}]_*U$ must land in the category of $\mathbb{T}_J$-models. Similarly, any $\mathcal{V}$-map is in particular diagonally universal, so that $\Phi[\mathcal{S}]*$ maps U-local maps to $\mathcal{V}$-local maps, so that we have a restriction of $\Phi[\mathcal{S}]\iota_{\mathcal{D}_U,J_U}$ along $\iota_{\mathcal{V},J}$:

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{U} & \mathbb{T}_1[\mathcal{S}] \\ \downarrow & & \downarrow{\scriptstyle \Phi[\mathcal{S}]_*} \\ \mathbb{T}_J[\mathcal{S}]^{\mathbf{Loc}_{\mathcal{V}}} & \xrightarrow[\iota_{\mathcal{V},J}]{} & \mathbb{T}_2[\mathcal{S}] \end{array}$$

This induces a factorization in the 2-category of geometries

$$\begin{array}{ccc} (\mathbb{T}_2,\mathcal{V},J) & \xrightarrow{\Phi} & (\mathbb{T}_1,\mathcal{D}_U,J_U) \\ {\scriptstyle \epsilon_{(\mathbb{T}_2,\mathcal{V},J)}}\downarrow & \nearrow_{\Phi} & \\ (\mathbb{T}_2,\mathcal{D}_{\iota_{\mathcal{V},J}},J_{\iota_{\mathcal{V},J}}) & & \end{array}$$

Indeed, J_U contains all U-localizing families; and any $J_{\iota_{\mathcal{V},J}}$-cover is sent by Φ to a J_U-cover because J_U is the finest topology localizing U, hence contains all U-localizing families, and $J_{\iota_{\mathcal{V},J}}$-cover are U-localizing as $\Phi[\mathcal{S}]_*U$ restricts to $\mathbb{T}_J[\mathcal{S}]$. Similar argument for maps. □

Remark 4.2.5.2. This correspondence can be restricted over each choice of $\mathcal{B}$. For a fixed locally finitely presentable category $\mathcal{B}$ with underlying theory $\mathbb{T}$, we can define a poset of Diers contexts on $\mathcal{B}$ by defining that $(U_1, \mathcal{A}_1) \leq (U_2, \mathcal{A}_2)$ if

$$\begin{array}{ccc} \mathcal{A}_1 & \xrightarrow{U_1} & \mathcal{B} \\ \downarrow & \nearrow_{U_2} & \\ \mathcal{A}_2 & & \end{array}$$

Though this looses some information for arbitrary stable functors, multireflectives subcategories on $\mathcal{B}$ really form a poset. We denote by $\mathfrak{Diers}_{\mathcal{B}}$ and $\mathfrak{Geom}_{\mathcal{B}}$ those respective poset.

Then the correspondence above restricts to a Galois connexion at $\mathcal{B}$

$$\begin{array}{ccc} \mathfrak{Geom}_{\mathcal{B}}^{\mathrm{op}} & \underset{\longrightarrow}{\overset{\longleftarrow}{\bot}} & \mathfrak{Diers}_{\mathcal{B}} \\ (\mathcal{V}, J) & \longmapsto & (\iota, \mathbb{T}_J[\mathcal{S}]^{Loc_{\mathcal{V}}}) \\ (\mathcal{D}_U, J_U) & \longleftarrow & (U, \mathcal{A}) \end{array}$$

Where the closure operation on Diers contexts corresponded to the unit of this adjunction.

4.2.5.3. Alas it does not seem possible to characterize geometries and Diers contexts for which this 2-adjunction reduces to an equivalence. The problem arises at the level of the factorization data:

- for a given right multi-adjoint $U : \mathcal{A} \to \mathcal{B}$, even if accessible and relatively full and faithful, we cannot ensure it to be axiomatic, so that the left class in the left generated factorization system generated from diagonally universal morphisms may not encompass all the diagonally universal morphisms, so dually the class $\mathbf{Diag}_U^{\perp}$ may contain fewer maps than the U-local maps: hence we cannot infer that the unit $\mathcal{A} \hookrightarrow \mathbb{T}_U[\mathcal{S}]^{\mathbf{Loc}_U}$ is full.
- for a given geometry $(\mathbb{T}, \mathcal{V}, J)$, even if J had enough points, so we had that $J_{\iota_{\mathcal{V},J}} = J$, the class of local objects may not contain enough objects to reconstruct the whole original factorization system by orthogonality.

Part II

Spectrum

Chapter 5

The spectrum as a classifier

This chapter is devoted to two different, yet related constructions, namely Cole's construction of [17] and Dubuc's construction of [33]. Those two methods construct the spectrum through sequences of universal properties, exhibiting it as the classifier of the local forms under a given model in a topos.

Throughout this chapter, we emphasize the presentation of models of a theory as geometric morphisms into its classifier, and the construction we consider will operate on those representing morphisms and the different geometric morphisms relating intermediate classifiers in the bicategory of Grothendieck topoi. For this reason, we shall work at the level of (the opposite of) the oplax slices of the bicategory of Grothendieck topoi over the different classifiers and some non full sub-bicategories of them.

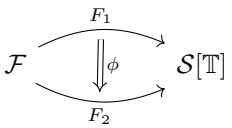
In the first two sections we present Cole's construction of the spectrum through a sequence of finite bilimits, for which we recall first a provision of useful finite bilimits in **GTop** we shall use as the building blocks of the construction. We try to provide a description as detailed as possible of the process, which was left largely implicit in [17] - though some elements until some step were clarified in [33]. We shall see that it coincides with the description of the spectrum as the left adjoint to the inclusion between categories of modelled topoi detailed in the next chapter - where however it will be stated in a different way of visualizing the involved bicategories of modelled topoi.

We split the exposition of [17] into two sections, one dealing with the case of factorization geometries with no specification of local objects, where we already can construct a first spectral bi-adjunction at theorem 5.1.2.8. Then we refine it to include the local data, which gives a first form of the spectral adjunction at theorem 5.2.1.7. We also give at the end of this section a site description of the generic classifiers involved in the construction, which prepare both some elements of [33] method and the description of the spectral site which will be the topic of a further chapter.

The third section is devoted to [33], which is partially similar to [17], yet diverges at some point in order to make use of the construction of the topos of *etale objects*, which is not obtained as a bilimit but actually as a coinverter. Though this construction is more general as it applies to Dubuc contexts, we shall also specify how it coincides with the previous one for geometries.

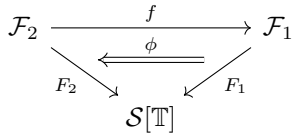
5.0.1 Preliminary remarks on the oplax slices of GTop

Recall that, for a geometric theory $\mathbb{T}$, a $\mathbb{T}$-model in a Grothendieck topos is the same as a geometric morphism $\mathcal{E} \to \mathcal{S}[\mathbb{T}]$, while a homomorphism of $\mathbb{T}$-models in $\mathcal{E}$ is a globular 2-cell



However, there is no way to construct such a thing as the spectrum of a model if we do not manipulate models in a more universal way, that is, *up to inverse images*. In fact, rather that restricting us to morphisms of models inside the category of models in a fixed topos, we must allow models to circulate along inverse image parts of geometric morphisms between topoi and consider morphisms between models that do not live in the same topos: only at this level will appear an

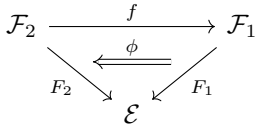
object with the desired universal property of the spectrum. Concretely, we must also consider 2-cells of the form



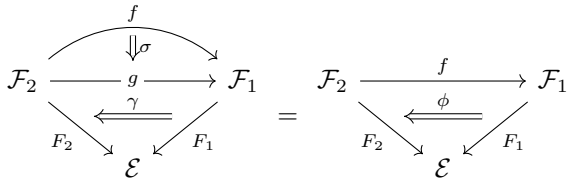
which code for a morphism of $\mathbb{T}$-models $f^*F_1 \to F_2$ in $\mathbb{T}[\mathcal{F}_2]$.

In this section, to emphasize the 2-dimensional aspects, we refer to objects of the *oplax slice* at $\mathcal{S}[\mathbb{T}]$ as $\mathbb{T}$-modelled topoi, with, beware, the following arrow convention:

Definition 5.0.1.1. For a Grothendieck topos $\mathcal{E}$, we call the *oplax slice* the bicategory $\mathbf{GTop}//\mathcal{E}$ whose 0-cells are pairs $(\mathcal{F}, F)$ consisting of a geometric morphisms $F : \mathcal{F} \to \mathcal{E}$, 1-cells $(\mathcal{F}_1, F_1) \to (\mathcal{F}_2, F_2)$ are 2-cells in $\mathbf{GTop}$ of the form



and 2-cells $(f, \phi) \to (g, \gamma)$ are equalities of 2-cells



In the following, we are going to consider in particular, for a geometry $(\mathbb{T}, \mathcal{V}, J)$, the oplax slices $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]$ and $\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]$ (as well as some non-full sub-bicategories of them) - which will be examined from another point of view in the next chapter. Hence we choose to introduce right now the following terminology:

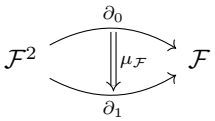
Definition 5.0.1.2. For a geometry $(\mathbb{T}, \mathcal{V}, J)$, an object $(\mathcal{F}, F)$ of $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]$ with $F : \mathcal{F} \to \mathcal{S}[\mathbb{T}]$ will be called a $\mathbb{T}$-*modelled topos*, and an object $(\mathcal{F}, F)$ in $\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]$ a $\mathbb{T}_J$-*locally modelled topos*. In the following we shall denote as w the non full inclusion of

Remark 5.0.1.3. Beware that with this convention, 1-cells in our notion of oplax slice have the same direction as their 2-cells part in $\mathbf{GTop}$, and the *opposite* direction of their underlying 1-cell. We choose this convention in order to allow us to see $\mathbb{T}$-$\mathbf{GTop}$, as it will be defined in the next chapter, as the oplax slice $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]$, and in order to ensure that in the spectral bi-adjunction theorem 5.2.1.7 the spectrum really is a *left* bi-adjoint to the inclusion - it would be a right bi-adjoint with the other convention, which would make the main result of this chapter uselessly opaque.

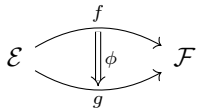
5.1 Factorization geometries

In this first section we construct the classifiers of the etale and local arrows and produce a first kind of spectral adjunction corresponding to *factorization geometries*, that are geometries without specification of local objects. We shall see that the general case can be obtained from this one in an simple way.

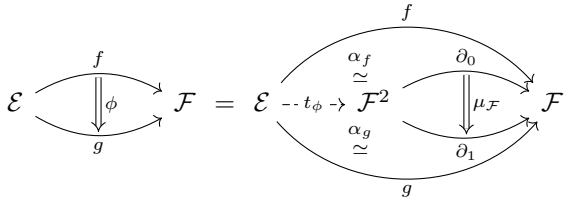
Recall that the bicategory $\mathbf{GTop}$ of Grothendieck topoi has finite bilimits. In particular, it possesses bipullbacks, bicomma, and bipower with 2. In particular the *bipower* of a topos $\mathcal{E}$ with 2 is equipped with its universal 2-cell



such that any other 2-cell



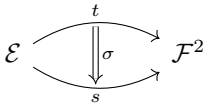
induces a universal arrow $\mathcal{E} \to \mathcal{F}^2$, unique up to unique invertible 2-cell, equipped with two invertible two cells α_f, α_g such that we have an equality of 2-cells



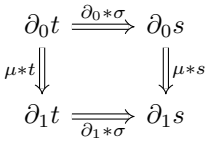
In other words, the bipower of $\mathcal{F}$ with 2 satisfies the universal property

$$\mathbf{Cat}\big[2, \mathbf{GTop}[\mathcal{E}, \mathcal{F}]\big] \simeq \mathbf{GTop}[\mathcal{E}, \mathcal{F}^2]$$

In particular, whenever $\mathcal{F} \simeq \mathcal{S}[\mathbb{T}]$ for a geometric theory $\mathbb{T}$, the bipower $\mathcal{S}[\mathbb{T}]^2$ classifies morphisms between $\mathbb{T}$-models. Moreover, this construction is functorial in the sense that any 2-cell



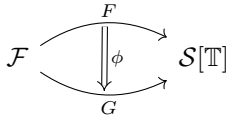
induces a square in $\mathbf{Geom}\big[\mathcal{E}, \mathcal{S}[\mathbb{T}]\big] \simeq \mathbb{T}[\mathcal{E}]$



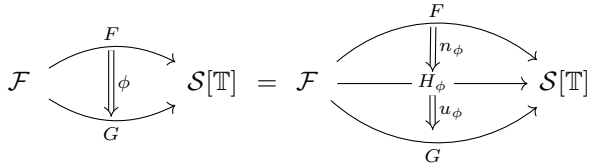
which codes for a commutative square in $\mathbb{T}[\mathcal{E}]$, that is, a morphism $\mu * t \to \mu * s$ in $\mathbb{T}[\mathcal{E}]^2$. Conversely, any square in $\mathbb{T}[\mathcal{E}]$, seen as a morphism between objects of $\mathbb{T}[\mathcal{E}]^2$, induces uniquely such a 2-cell.

5.1.1 Classifier of etale and local maps

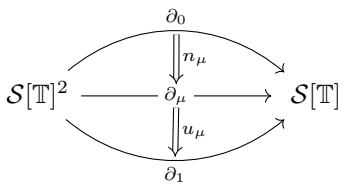
5.1.1.1. Following [17], we now describe how to use those universal constructions in order to exhibit the spectrum as the classifier of local form under a given modelled topos. Let $(\mathbb{T}, \mathcal{V}, J)$ be an admissibility structure with $\iota_J : \mathcal{S}[\mathbb{T}_J] \hookrightarrow \mathcal{S}[\mathbb{T}]$ the embedding of the classifier of local objects into the classifier of $\mathbb{T}$. As the category of $\mathbb{T}$-models $\mathbb{T}[\mathcal{E}] \simeq \mathbf{Geom}\big[\mathcal{E}, \mathcal{S}[\mathbb{T}]\big]$ in any topos $\mathcal{E}$ inherits a factorization system $(Et_{\mathcal{E}}, \mathcal{L}oc_{\mathcal{E}})$ from $\mathcal{V}$, any two cell



coding for a morphism $\phi : F \to G$ between $\mathbb{T}$-models in $\mathcal{F}$ admits an admissible factorization



In particular consider the admissible factorization of the canonical 2-cell of the bipower of $\mathcal{S}[\mathbb{T}]$



Proposition 5.1.1.2. *Let $\phi : F \to G$ be a morphism in $\mathbb{T}[\mathcal{F}]$, with $t_\phi : \mathcal{F} \to \mathcal{S}[\mathbb{T}]^2$ the associated universal map. Then choose $n_\phi = n_\mu * t_\phi$ and $u_\phi = u_\mu * t_\phi$.*

Proof. As the factorization system is left generated, the category of etale maps between $\mathbb{T}$-models is locally finitely presentable, being classified by the preseaf topos $\widehat{\mathcal{V}^{\mathrm{op}}}$. Hence they are stable under inverse image along geometric morphisms. In particular, as $n_\mu : \partial_0 \to \partial_\mu$ is an etale map in $\mathbb{T}[\mathcal{S}[\mathbb{T}]^2]$, its inverse image $n_\phi * t_\phi : \partial_0 t_\phi \to \partial_\mu t_\phi$ is an etale map in $\mathbb{T}[\mathcal{F}]$, as well as its composite with the isomorphism $\alpha_F : F \simeq \partial_0 t_\phi$.

Now recall that local maps are also stable under inverse image. Hence, as $u_\mu : \partial_\mu \to \partial_1$ is a local map in $\mathbb{T}[\mathcal{S}[\mathbb{T}]^2]$, its inverse image $u_\phi * t_\phi : \partial_\mu t_\phi \to \partial_1 t_\phi$ is a local map in $\mathbb{T}[\mathcal{F}]$, as well as its composite with the isomorphism $\alpha_G : \partial_0 t_\phi \simeq G$. Moreover, by functoriality of whiskering, we have $\phi = \alpha_G u_\phi * t_\phi n_\phi * t_\phi \alpha_F$, providing an admissible factorization of ϕ. □

Corollary 5.1.1.3. *The classifiers of etales and local maps between $\mathbb{T}$-models are respectively the inverters of u_μ and n_μ:*

$$\mathcal{S}[\mathbf{Et}] \xhookrightarrow{\iota_{\mathbf{Et}}} \mathcal{S}[\mathbb{T}]^2 \underset{\partial_1}{\overset{\partial_\mu}{\Downarrow u_\mu}} \mathcal{S}[\mathbb{T}]$$

$$\mathcal{S}[\mathbf{Loc}] \xhookrightarrow{\iota_{\mathbf{Loc}}} \mathcal{S}[\mathbb{T}]^2 \underset{\partial_\mu}{\overset{\partial_0}{\Downarrow n_\mu}} \mathcal{S}[\mathbb{T}]$$

Similarly one can define the classifier of local maps between local objects as the following inverter

$$\mathcal{S}[\mathbf{Loc}_J] \xhookrightarrow{\iota_{\mathbf{Loc}_J}} \mathcal{S}[\mathbb{T}_J]^2 \underset{\partial_\mu}{\overset{\partial_0}{\Downarrow n_\mu * \iota_J}} \mathcal{S}[\mathbb{T}_J]$$

Proof. As for any orthogonal factorization system, etale maps are those whose local part is invertible and dually, local maps are those whose etale part is invertible. Then a morphism between $\mathbb{T}$-models $\phi : F \to G$ in a topos $\mathcal{F}$ is an etale map in $\mathbb{T}[\mathcal{F}]$ if and only if the whiskering $u_\mu * t_\phi$ is invertible, and is a local map if and only if $n_\phi * t_\phi$ is invertible. Each of those conditions is equivalent to say that t_ϕ factorizes through the corresponding inverter. □

5.1.1.4. In the following we denote the canonical 2-cells of those three classifiers as

$$\mathcal{S}[\mathbf{Et}] \underset{\partial_1^{\mathbf{Et}}}{\overset{\partial_0^{\mathbf{Et}}}{\Downarrow \nu}} \mathcal{S}[\mathbb{T}] \qquad \mathcal{S}[\mathbf{Loc}] \underset{\partial_1^{\mathbf{Loc}}}{\overset{\partial_0^{\mathbf{Loc}}}{\Downarrow \upsilon}} \mathcal{S}[\mathbb{T}] \qquad \mathcal{S}[\mathbf{Loc}_J] \underset{\partial_1^{\mathbf{Loc}_J}}{\overset{\partial_0^{\mathbf{Loc}_J}}{\Downarrow \upsilon_J}} \mathcal{S}[\mathbb{T}_J]$$

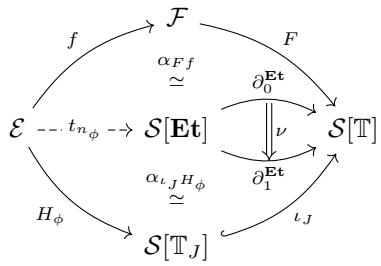
5.1.1.5. Now any morphism of modelled topoi $(\mathcal{F}, F) \to (\mathcal{E}, E)$ with $(\mathcal{E}, E)$ a locally modelled topos defines a 2-cell in **GTop**

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ {\scriptstyle E}\downarrow & \overset{\phi}{\swarrow} & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbb{T}_J] & \xhookrightarrow[\iota_J]{} & \mathcal{S}[\mathbb{T}] \end{array}$$

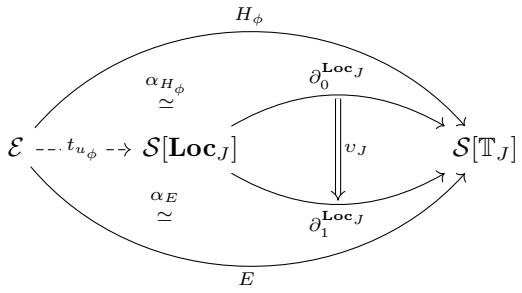
But considering the admissible factorization

$$\begin{array}{ccc} f^*F & \xrightarrow{\phi} & E \\ & {\scriptstyle n_\phi}\searrow \quad \nearrow{\scriptstyle u_\phi} & \\ & H_\phi & \end{array}$$

of the inverse image part $\phi^\flat : f^*F \Rightarrow E$, where H_ϕ is a local object in $\mathcal{E}$ we have on the left a universal map classifying the etale part n_ϕ together with the 2-cell



while we have on the right a universal map classifying u_ϕ (seen as a local map between local objects) together with the 2-cell



5.1.2 Classifier of etale maps under a fixed modelled topos

5.1.2.1. Before constructing the spectrum, we must first construct two different objects classifying on one side the factorization data, and on the other side, the local data the spectrum classifies. Indeed, in the following we are going to exhibit the spectrum as a classifier of local forms under a given $\mathbb{T}$-model, that are its etale maps toward a local object. Observe that from this very definition, it suffices to have both an object classifying etale maps under a fixed object, and another one classifying etale arrows toward arbitrary local objects.

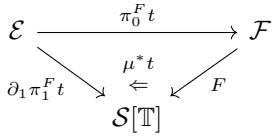
Consider the pullback of F along the universal domain of the classifier of etale maps

$$\begin{array}{ccc} \mathcal{S}[F, \mathbf{Et}] & \xrightarrow{\pi_0^F} & \mathcal{F} \\ {\scriptstyle \pi_1^F}\downarrow & \lrcorner & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbf{Et}] & \xrightarrow[\partial_0^{\mathbf{Et}}]{} & \mathcal{S}[\mathbb{T}] \end{array}$$

Any geometric morphism $t : \mathcal{E} \to \mathcal{S}[F, \mathbf{Et}]$ is the name of an etale map in $\mathbb{T}[\mathcal{E}]$

$$\mu^* t : \pi_0^F F t \simeq \partial_0 \pi_1^F t \to \partial_1 \pi_1^F t$$

with domain $F\pi_0^F t$, so any such morphism defines uniquely a morphism



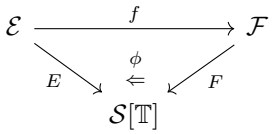
We also consider the bicategory $\mathbf{GTop}//\mathcal{S}[\mathbf{Et}]^{\mathbf{Loc}}$, the (0,2)-full sub-bicategory of the oplax slice $\mathbf{GTop}//\mathcal{S}[\mathbf{Et}]$ with again 1-cells consisting of a local map as the underlying 2-cell in $\mathbf{GTop}$.

Proposition 5.1.2.2. *The assignment sending a $\mathbb{T}$-modelled topos $(\mathcal{F}, F)$ to the arrow $\pi_0^F : \mathcal{S}[F, \mathbf{Et}] \to \mathcal{S}[\mathbf{Et}]$ defines a pseudofunctor*

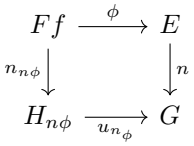
$$\mathbf{GTop}//\mathcal{S}[\mathbb{T}] \xrightarrow{\partial_0^{\mathbf{Et}\,*}} \mathbf{GTop}//\mathcal{S}[\mathbf{Et}]^{\mathbf{Loc}}$$

Remark 5.1.2.3. Restricted to the pseudoslices, the functoriality of $\partial_0^{\mathbf{Et}^*} = \mathcal{S}[-, \mathbf{Et}]$ is obvious as it coincides with the base change functor along $\partial_0^{\mathbf{Et}}$. However we are interested here in oplax slices, where the functoriality of this process is not trivially induced by the universal property of bipullback, but by the factorization data involved with etales maps, through a more complex sequence of universal properties. In fact, we are going to construct a morphism between the corresponding classifiers of etale maps from the "generic admissible factorization after precomposition" as below.

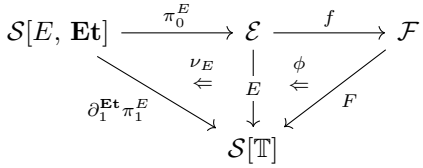
Proof. Consider a morphism of $\mathbb{T}$-modelled topoi, that is, a morphism in the oplax slice



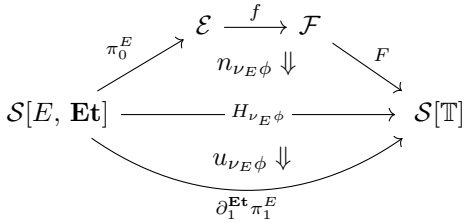
and its admissible factorization $\phi = u_\phi n_\phi$ in $\mathbb{T}[\mathcal{E}]$. Observe that for any etale arrow $n : E \to G$ under E in $\mathbb{T}[\mathcal{E}]$, we have an admissible factorization



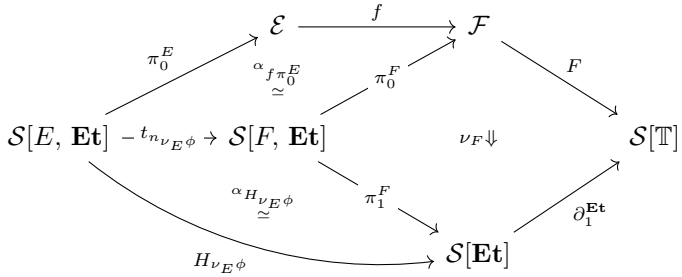
We are going to consider the generic version of this process. Consider the composite 2-cell



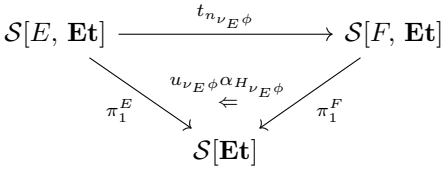
and its admissible factorization



The etale part defines in particular a universal arrow $t_{n_{\nu_E \phi}} : \mathcal{S}[E, \mathbf{Et}] \to \mathcal{S}[F, \mathbf{Et}]$ together with invertible 2-cells $\alpha_{f\pi_0^E}$ and $\alpha_{H_{\nu_E\phi}}$ such that $n_{\nu_E\phi}$ decomposes as the following 2-cell

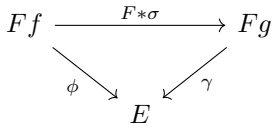


Now observe that the composite $\pi_1^F t_{n_{\nu_E\phi}}$ classifies the same map $t_{n_{\nu_E\phi}}$ but seen as an etale map, forgetting its domain; then we end with the following 2-cell

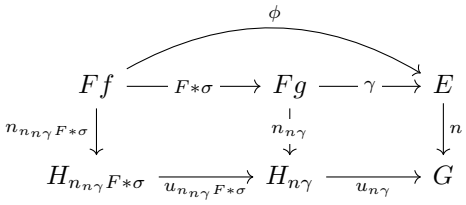


were $u_{\nu_E\phi}\alpha_{H_{\nu_E\phi}}$ is still local since $u_{\nu_E\phi}$ is so and $\alpha_{H_{\nu_E\phi}}$ is invertible. This produces the desired morphism in the bicategory $\mathbf{GTop}//\mathcal{S}[\mathbf{Et}]^{\mathbf{Loc}}$.

For the sake of the precision, let us describe how this processes on the 2-cells. Consider $\sigma : (f, \phi) \Rightarrow (g, \gamma)$ between $(\mathcal{F}, F)$ and $(\mathcal{E}, E)$. Then we have the 2-dimensional equation



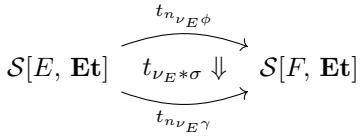
Then for any etale arrow $n : E \to G$ in $\mathbb{T}[\mathcal{E}]$, the admissible factorizations of $n\phi$ and $n\gamma$ are related as follows: consider the successive factorizations



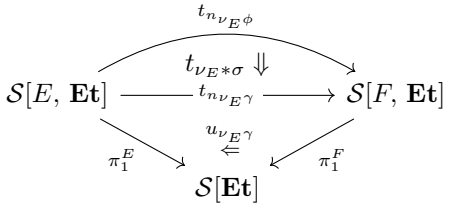
then by uniqueness of the admissible factorization, one has

$$n_{n\phi} = n_{n_{n\gamma}F*\sigma} \qquad H_{n\phi} = H_{n_{n\gamma}F*\sigma} \qquad u_{n\phi} = u_{n\gamma}u_{n_{n\gamma}F*\sigma}$$

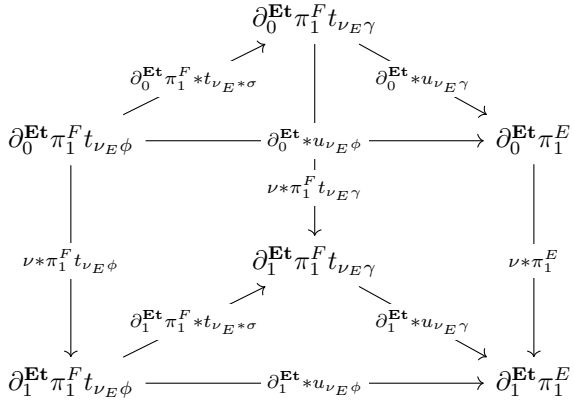
Applying this property to the generic etale map, this says that there is a generic transition 2-cell



such that the 2-cell $u_{\nu_E\phi}$ factorizes as



This decomposition produces after whiskering with the canonical 2-cell of $\mathcal{S}[\mathbf{Et}]$ the commutative diagram

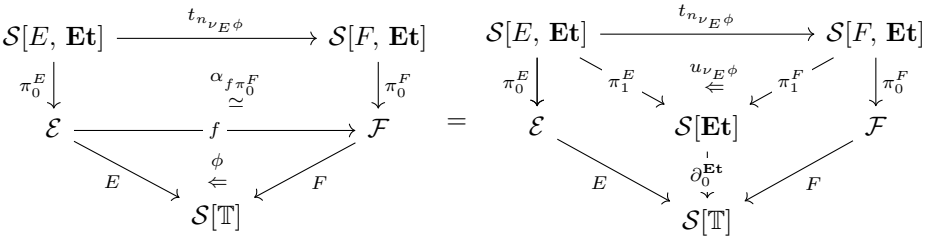


which is the generic form of the relations between the factorization of $n\phi$ and n_γ above, but for the generic etale map under E, knowing that the top triangle is actually, up to the canonical iso, the equation $\gamma F * \sigma = \phi$, while the lower triangle expresses the equation between the local parts of the factorizations, and the vertical arrows are the etale parts of the factorization, with $\nu * \pi_1^E$ the generic etale map. □

Remark 5.1.2.4. Observe that in this process we obtained in particular a canonical invertible 2-cell

$$\begin{array}{ccc} \mathcal{S}[E,\,\mathbf{Et}] & \xrightarrow{t_{n_{\nu_E\phi}}} & \mathcal{S}[F,\,\mathbf{Et}] \\ {\scriptstyle \pi_0^E}\downarrow & \overset{\alpha_{f\pi_0^F}}{\simeq} & \downarrow{\scriptstyle \pi_0^F} \\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{F} \end{array}$$

satisfying the following 2-dimensional equality



This invertible 2-cell will be involved in proving the functoriality of the spectral construction. This says that one returns the original arrow ϕ by whiskering with the domain functor $\partial_0^{\mathbf{Et}*}$.

Remark 5.1.2.5. The 2-cell in the classifier of etale maps under F codes for the "generic etale map under F" in the sense that a random etale map under F is obtained as whiskering a geometric morphism with this 2-cell; observe that π_1^F can also be seen as an etale map in $\mathbb{T}[\mathcal{S}[F,\,\mathbf{Et}]]$. Now observe that for a morphism $(f,\phi):(\mathcal{F},F)\to(\mathcal{E},E)$ in $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]$, exploiting the functoriality of the universal property of $\mathcal{S}[\mathbf{Et}]$ returns a square between those universal etale arrows

$$\begin{array}{ccc} \partial_0^{\mathbf{Et}} F t_{n_{\nu_E\phi}} & \xrightarrow{\partial_0 * u_{\nu_E\phi}} & \partial_0^{\mathbf{Et}} E \\ {\scriptstyle \nu * F t_{n_{\nu_E\phi}}}\downarrow & & \downarrow{\scriptstyle \nu * E} \\ \partial_1^{\mathbf{Et}} F t_{n_{\nu_E\phi}} & \xrightarrow[\partial_1 * u_{\nu_E\phi}]{} & \partial_1^{\mathbf{Et}} E \end{array}$$

which is the "generic admissible factorization of the composite of an etale map under E with ϕ"; the top row is indeed equal up to canonical iso to ϕ while $\partial_1^{\mathbf{Et}} F t_{n_{\nu_E\phi}} \simeq H_{\nu_E\phi}$ we have $\partial_1 * u_{\nu_E\phi}$, being local by stability of local maps along inverse image, is the local part of the generic factorization.

5.1.2.6. The following proposition serves as an intermediate step in the construction of the spectrum for the admissibility structure $(\mathbb{T},\mathcal{V},J)$. In fact, it constitutes the spectral adjunction for the admissibility structure $(\mathbb{T},T,\mathcal{V})$ with the same factorization data $\mathcal{V}$ but without choice of local objects, that is with the trivial topology T corresponding to the trivial geometric extension of $\mathbb{T}$ into itself. To this purpose we introduce the (0,2)-full sub-bicategory $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]^{\mathbf{Loc}}$ whose objects are $\mathbb{T}$-modelled topos, and whose 1-cells are 2-cells

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{f} & \mathcal{F} \\ & {\scriptstyle E}\searrow \quad \overset{\upsilon}{\Leftarrow} \quad \swarrow{\scriptstyle F} & \\ & \mathcal{S}[\mathbb{T}] & \end{array}$$

such that $\upsilon : Ff \to E$ is a local map in $\mathbb{T}[\mathcal{E}]$, with no restrictions on 2-cells. Observe that those morphisms can also be visualized by the 2-cell

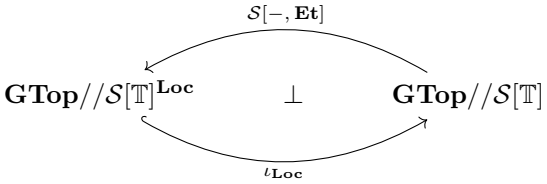
5.1.2.7. In particular we have a pseudofunctor

$$\mathbf{GTop}//\mathcal{S}[\mathbf{Et}]^{\mathbf{Loc}} \xrightarrow{\partial_1^{\mathbf{Et}!}} \mathbf{GTop}//\mathcal{S}[\mathbb{T}]^{\mathbf{Loc}} \xrightarrow{\iota_{\mathbf{Loc}}} \mathbf{GTop}//\mathcal{S}[\mathbb{T}]$$

sending any $n:\mathcal{E}\to\mathcal{S}[\mathbf{Et}]$ to its domain $\partial_0^{\mathbf{Et}} n$, and any morphism (f,υ) to the whiskering $\partial_0 * \upsilon$.

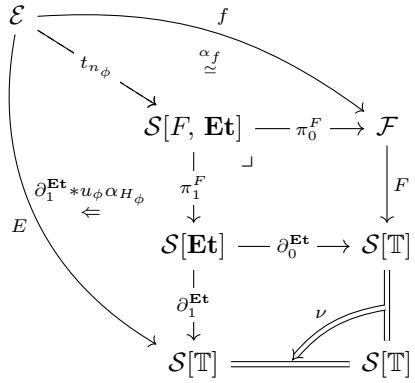
We are going to prove that the arrow $\pi_0^F : \mathcal{S}[F,\,\mathbf{Et}] \to \mathcal{F}$ together with the isomorphism $F\pi_0^F \simeq \partial^{\mathbf{Et}}\pi_1^F$ is the unit of this biadjunction (the invertibility of the 2-dimensional part follows from the fact that it already serves as a unit for the restricted biadjunction between pseudo-slices).

Theorem 5.1.2.8. *We have a biadjunction*



where the pseudofunctor $\mathcal{S}[-, \mathbf{Et}]$ *sends* $(\mathcal{F}, F)$ *to the composite* $\partial_1^{\mathbf{Et}}\pi_1^F : \mathcal{S}[F, \mathbf{Et}] \to \mathcal{S}[\mathbb{T}]$.

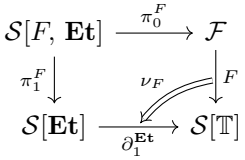
Proof. Let $(f, \phi) : (\mathcal{F}, F) \to (\mathcal{E}, E)$ be a morphism in $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]$. Then the etale part of the admissible factorization of ϕ in $\mathbb{T}[\mathcal{E}]$ produces a canonical arrow $t_{n_\phi} : \mathcal{E} \to \mathcal{S}[F, \mathbf{Et}]$ which together with the local part of ϕ produces a morphism $(t_{n_\phi}, \partial_1^{\mathbf{Et}} * u_\phi \alpha_{H_\phi}) : (\mathcal{S}[F, \mathbf{Et}], \partial_1^{\mathbf{Et}}\pi_1^F) \to (\mathcal{E}, E)$ such that the 2-cell ϕ decomposes as the pasting



We left the process relatively to the 2-cell as an exercise for the reader. As those data uniquely determine (f, ϕ) and were produced out of universal properties, we have then an equivalence of category

$$\mathbf{GTop}//\mathcal{S}[\mathbb{T}]^{\mathbf{Loc}}\big[(\mathcal{S}[F, \mathbf{Et}], \partial_1^{\mathbf{Et}}\pi_1^F), (\mathcal{E}, E)\big] \simeq \mathbf{GTop}//\mathcal{S}[\mathbb{T}]\big[(\mathcal{F}, F), (\mathcal{E}, E)\big]$$

proving the desired adjunction. In particular the 2-cell



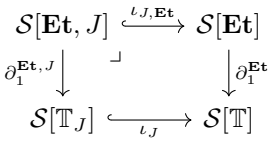
where $\nu_F = \nu * \pi_1^F$ is the unit of the bi-adjunction, and is the generic etale map under F. It is easy to see that the converse process of composing a 2-cell $(g, u) : (\mathcal{S}[F, \mathbf{Et}], \partial_1^{\mathbf{Et}}\pi_1^F) \to (\mathcal{E}, E)$ simply has to be pasted with the unit 2-cell to return a morphism $(\mathcal{F}, F) \to (\mathcal{E}, E)$, whose local part gets back u. □

5.2 The general case

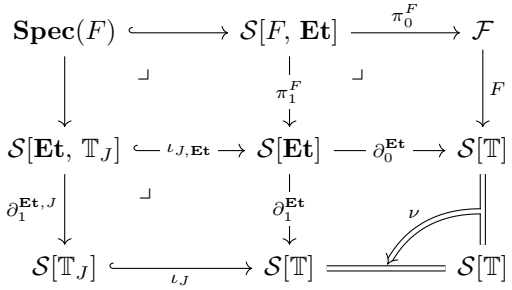
The previous part was aimed at constructing the spectrum for a purely factorial admissibility structure, for instance the one we get from an arbitrary admissibity structure by forgetting the specification of geometric extension of $\mathbb{T}$. In this section we consider the general case involving also local data.

5.2.1 The Spectrum as the classifier of local forms

5.2.1.1. A local form is an etale arrow toward a local object: then local forms can be classified by the pullback

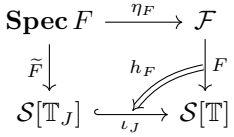


Definition 5.2.1.2. Let $(\mathbb{T}, \mathcal{V}, J)$ be a geometry. Then define the *(Cole) spectrum* of an $\mathbb{T}$-modelled topos $(\mathcal{F}, F)$ as the following sequence of pullbacks



In the following we denote as $\eta_F : \mathbf{Spec}(F) \to \mathcal{F}$ the composite of the top side of this square, and $\widetilde{F} : \mathbf{Spec}(F) \to \mathcal{S}[\mathbb{T}_J]$ the composite of the left side, and $h_F : F\eta_F \Rightarrow \iota_J \widetilde{F}$ the whiskering of ν with the map induced from the top-left pullback.

5.2.1.3. The spectrum of $(\mathcal{F}, F)$ is equipped with a canonical 2-cell

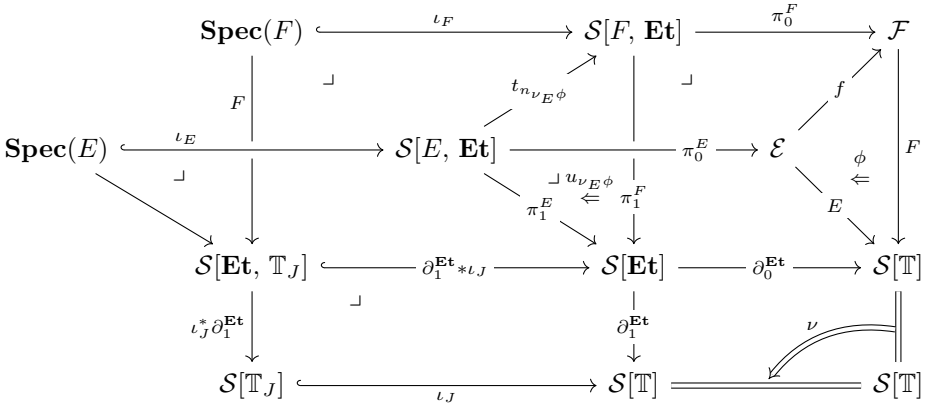


In the following we shall refer to this canonical 2-cell as *the spectral comma*.

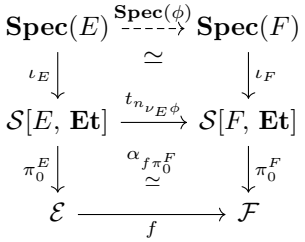
In particular observe the following, which is just unravelling the universal property through which the spectrum is constructed:

Proposition 5.2.1.4. *For any Grothendieck topos $\mathcal{E}$, the category $\mathbf{Geom}[\mathcal{E}, \mathbf{Spec}(F)]$ has for objects pairs (f, x) with $f : \mathcal{E} \to \mathcal{F}$ and $x : Ff \to E_x$ in $\mathbf{Et}[\mathcal{E}]$ with E in $\mathbb{T}_J[\mathcal{E}]$. Conversely, any such datum (f, x) in $\mathcal{E}$ defines uniquely a geometric morphism $t_{(f,x)} : \mathcal{E} \to \mathbf{Spec}(F)$. In particular a point of $\mathbf{Spec}(F)$ is the data of a point $p : \mathcal{S} \to \mathcal{F}$ and a set-valued local form $x : Fp \to E$.*

5.2.1.5. Now some precision on the functoriality of this process. Let us consider a morphism of $\mathbb{T}$-modelled topoi $(f, \phi) : (\mathcal{F}, F) \Rightarrow (\mathcal{E}, E)$. Then one can construct the following diagram, where the same generic data are used to specify local objects in the bottom part, while one use respectively the classifiers of etale maps under the domain and codomain objects in the top row:



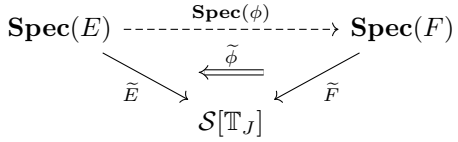
To complete this diagram with a 2-cell between the spectra, observe that we have in $\mathcal{E}$ a normal form under F with $n_\phi : f^*F \to H_\phi$, which is sent to a normal form $\iota_E^* {\pi_0^E}^* n_\phi$ in $\mathbf{Spec}(E)$ inducing a unique arrow $\mathbf{Spec}(\phi) : \mathbf{Spec}(E) \to \mathbf{Spec}(F)$ which moreover makes the top left square to commute, so that we have a composite invertible 2-cell



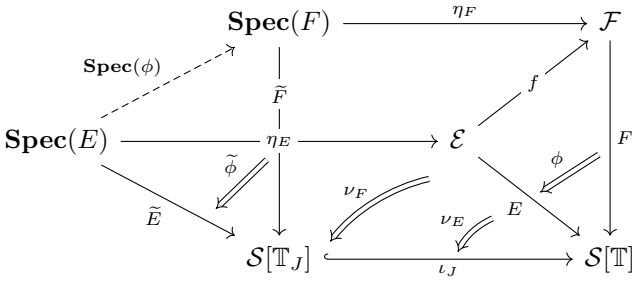
And pasting this upper invertible 2-cell with $u_{\nu_E\phi}$ provides an invertible 2-cell $\pi_1^F \iota_F \mathbf{Spec}(\phi) \simeq \pi_1^F t_{n_{\nu_E\phi}} \iota_E \Rightarrow \pi_1^E \iota_E$, and composing it again with the pullbacks intertible 2-cells returns a 2-cellwhich itself must come uniquely from a 2-cell

$$(\partial_1^{\mathbf{Et}} * \iota_J)^* \pi_1^F \overset{\widetilde{\phi}}{\Rightarrow} (\partial_1^{\mathbf{Et}} * \iota_J)^* \pi_1^E \mathbf{Spec}(\phi)$$

because $\partial_1^{\mathbf{Et}} * \iota_J$ is an embedding and hence is a fully faithful morphism. This provides the desired 2-cell



related to ϕ through the following diagram



where we have the equality of 2-cells $\nu_E\phi = \iota_J * \widetilde{\phi}\nu_F$ encapsulating the generic admissible factorization of the precomposition of the generic local form under E with ϕ.

Remark 5.2.1.6. Observe that the last step extracting the 1-cell $(\mathbf{Spec}(\phi), \phi)$ in $\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]^{\mathbf{Loc}}$ is the only step where we made explicit use of the condition that $\mathbb{T}_J \hookrightarrow \mathbb{T}$ is an embedding. In the next section, we shall see how to get rid of this assumption through another method.

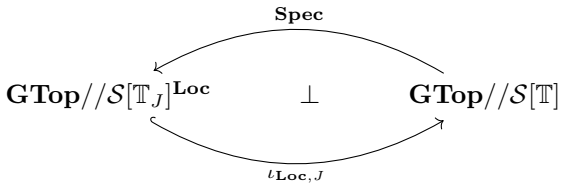
Now, as before, we consider the (0,2)-full sub-bicategory

$$\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]^{\mathbf{Loc}} \hookrightarrow \mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]$$

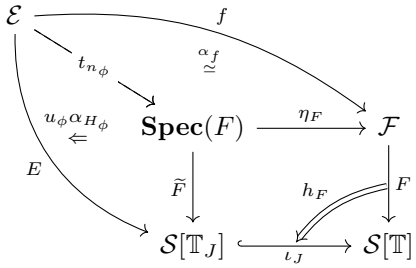
consisting of 1-cells with a local map as underlying 2-cell. It is equipped with an inclusion

$$\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]^{\mathbf{Loc}} \overset{\iota_{\mathbf{Loc},J}}{\hookrightarrow} \mathbf{GTop}//\mathcal{S}[\mathbb{T}]$$

Theorem 5.2.1.7 (Cole)**.** *We have a bi-adjunction*



Proof. At this point, this central result has become a corollary of the intermediate result theorem 5.1.2.8. For a morphism of modelled topoi $(f, \phi) : (\mathcal{F}, F) \to (\mathcal{E}, \iota_J E)$ with $(\mathcal{E}, E)$ locally modelled, the admissible factorization of $\phi^\flat : f^*F \to E$ produces an etale map $n_\phi : f^*F \to H_\phi$ toward a local object by the admissibility condition, and we have a factorization as below



and the data of the universal map t_{n_ϕ} and the right part u_ϕ of the factorization defines a morphism of locally modelled topos

$$(t_{n_\phi}, u_\phi \alpha_{H_\phi}) : (\mathbf{Spec}(F), \widetilde{F}) \to (\mathcal{E}, E)$$

□

Proposition 5.2.1.8. *Let $(\mathcal{F}, F)$ be a $\mathbb{T}$-modelled topos. Then for any geometric morphism $f : \mathcal{G} \to \mathcal{F}$ one has a bipullback*

$$\begin{array}{ccc} \mathbf{Spec}(Ff) & \xrightarrow{\eta_{Ff}} & \mathcal{G} \\ {\scriptstyle \mathbf{Spec}(1_{Ff})}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathbf{Spec}(F) & \xrightarrow[\eta_F]{} & \mathcal{F} \end{array}$$

with $\widetilde{Ff} = \widetilde{F}\mathbf{Spec}(1_{Ff})$.

Proof. This is just a property of left-cancellation of pullbacks applied to the 2-dimensional equality:

$$\begin{array}{ccccc} \mathbf{Spec}(Ff) & \longrightarrow & \mathcal{S}[Ff, \mathbf{Et}] & \xrightarrow{\pi_0^{Ff}} & \mathcal{G} \\ {\scriptstyle \mathbf{Spec}(1_{Ff})}\downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ \mathbf{Spec}(F) & \hookrightarrow & \mathcal{S}[F, \mathbf{Et}] & \xrightarrow{\pi_0^F} & \mathcal{F} \\ \downarrow & \lrcorner & {\scriptstyle \pi_1^F}\downarrow & \lrcorner & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbf{Et}, \mathbb{T}_J] & \xhookrightarrow[\iota_{j,\mathbf{Et}}]{} & \mathcal{S}[\mathbf{Et}] & \xrightarrow[\partial_0^{\mathbf{Et}}]{} & S[\mathbb{T}] \end{array} = \begin{array}{ccccc} \mathbf{Spec}(Ff) & \hookrightarrow & \mathcal{S}[Ff, \mathbf{Et}] & \xrightarrow{\pi_0^{Ff}} & \mathcal{G} \\ \downarrow & \lrcorner & {\scriptstyle \pi_1^{Ff}}\downarrow & \lrcorner & \downarrow{\scriptstyle Ff} \\ \mathcal{S}[\mathbf{Et}, \mathbb{T}_J] & \xhookrightarrow[\iota_{j,\mathbf{Et}}]{} & \mathcal{S}[\mathbf{Et}] & \xrightarrow[\partial_0^{\mathbf{Et}}]{} & S[\mathbb{T}] \end{array}$$

□

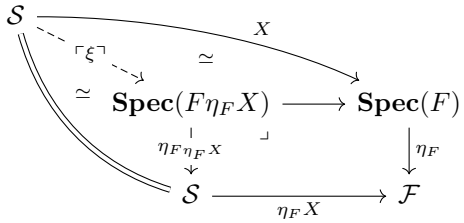
Proposition 5.2.1.9. *Points of $\mathbf{Spec}(F)$ are local forms under stalks of F: that is, we have*

$$\mathbf{pt}(\mathbf{Spec}(F)) \simeq \operatorname*{oplaxcolim}_{x \in \mathbf{pt}(\mathcal{F})} \mathbf{pt}(Fx)$$

Proof. Applying at any point $x : \mathcal{S} \to \mathcal{F}$ we have a bipullback

$$\begin{array}{ccc} \mathbf{Spec}(Fx) & \xrightarrow{\eta_{Fx}} & \mathcal{S} \\ {\scriptstyle \mathbf{Spec}(1_{Fx})}\downarrow & \lrcorner & \downarrow{\scriptstyle x} \\ \mathbf{Spec}(F) & \xrightarrow[\eta_F]{} & \mathcal{F} \end{array}$$

Now any point $X : \mathcal{S} \to \mathbf{Spec}(F)$ has an underlying point $\eta_F X : \mathcal{S} \to \mathcal{F}$, and the pair $((X, \eta_F X), \eta_F X = \eta_F X)$ induces uniquely factorization through the bipullback



and this factorization is the name of some local form ξ of $F\eta_F X$. Conversely for any point $x : \mathcal{S} \to \mathcal{F}$ and $\xi : \mathcal{S} \to \mathbf{Spec}(F)$, we have a point $\mathbf{Spec}(1_{Fx})^{\ulcorner \xi \urcorner}$ of $\mathbf{Spec}(F)$ whose underlying point is isomorphic to x. □

Remark 5.2.1.10. One could object that actually points of each bipullback $\mathbf{Spec}(Fx)$ are pairs of points (X, x) together with an isomorphism $x \simeq \eta_F X$ which may not be an equality. But as local forms are classified only up to isomorphisms - or even more exactly, *are* isomorphism classes of etale maps toward local objects - we consider this distinction to be irrelevant.

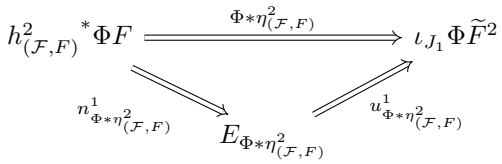
Remark 5.2.1.11. Observe that, as an immediate consequence, the spectrum of a modelled topos has no points at all whenever the underlying topos has no points.

5.2.2 Functoriality relatively to transformations of geometries

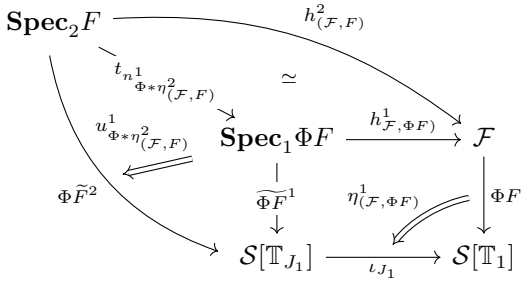
5.2.2.1. Now let us examine functoriality of this construction. For $\Phi : (\mathbb{T}_1, \mathcal{V}_1, J_1) \to (\mathbb{T}_2, \mathcal{V}_2, J_2)$ a transformation of geometries, the spectral comma of a modelled topos $(\mathcal{F}, F)$ is sent by Φ to the following square

$$\begin{array}{ccc} \mathbf{Spec}_2 F & \xrightarrow{h^2_{(\mathcal{F},F)}} & \mathcal{F} \\ {\scriptstyle \Phi\widetilde{F}^2}\downarrow & \swarrow{\scriptstyle \Phi * \eta^2_{(\mathcal{F},F)}} & \downarrow{\scriptstyle \Phi F} \\ \mathcal{S}[\mathbb{T}_{J_1}] & \xrightarrow[\iota_{J_1}]{} & \mathcal{S}[\mathbb{T}_1] \end{array}$$

But now the whiskering $\Phi * \eta^2_{(\mathcal{F},F)}$ factorizes uniquely in $\mathbb{T}_1[\mathbf{Spec}(F)]$ as



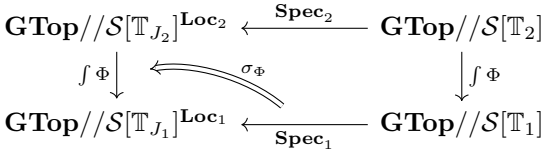
which induces a unique factorization through the spectral comma of ΦF for the $(\mathbb{T}_1, J_1, \mathcal{V}_1)$-geometry



Then we get a canonical morphism of $\mathbb{T}_{J_2}$-locally modelled topoi

$$(\mathbf{Spec}_1\Phi F, \widetilde{\Phi F}^1) \xrightarrow{(t_{n^1_{\Phi*\eta^2_{(\mathcal{F},F)}}}, u^1_{\Phi*\eta^2_{(\mathcal{F},F)}})} (\mathbf{Spec}_2 F, \Phi\widetilde{F}^2)$$

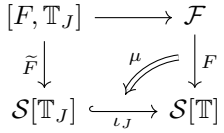
which defines a mate to the induced square of functors between categories of locally modelled topoi



We shall discuss an explicit description of this mate σ_Φ as a morphism of modelled topos at the end of 7.6.1.1.

5.2.3 Why do we need factorization data ?

This small subsection is aimed at justifying the construction above. We saw we could consider purely factorization geometries without choice of local objects: for a factorization system, we can classify etale maps under a given modelled topos - all of them are local forms as all objects are local - and the inclusion 2-functor $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]^{\mathbf{Loc}} \hookrightarrow \mathbf{GTop}//\mathcal{S}[\mathbb{T}]$ had a left adjoint, which was the spectrum of the associated factorization geometry. Then we saw how this process could be refined to take into account a choice of local objects - as long as they satisfied admissibility. Now for a choice of local object, that is, for a geometric extension $\mathbb{T} \to \mathbb{T}_J$, we can also construct a classifier of arrows toward local objects as the comma



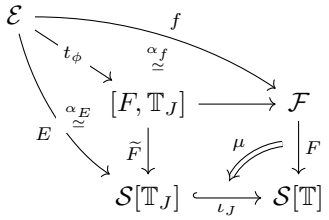
and this does not involve factorization data at all. Then one could ask if this construction defines a spectrum in the sense of a left biadjoint to an inclusion of the bicategory of locally modelled topoi into in the bicategory of modelled topoi. There are two ways to answer this question:

– either one considers that "no factorization data" means that any map is local so that

$$\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]^{\mathbf{Loc}} = \mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]$$

Then the only etale maps are isomorphisms. But then we do not have admissibility as the factorization of an arrow $\phi : F \to E$ with E local returns a local objects $H_\phi \simeq F$ only if F is already local, and the pseudofunctor $[-, \mathbb{T}_J]$ cannot be a left bi-adjoint to the inclusion

$\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J] \hookrightarrow \mathbf{GTop}//\mathcal{S}[\mathbb{T}]$: indeed, for a morphism $(f,\phi) : (\mathcal{F},F) \to (\mathcal{E},\iota_J E)$, the universal property of the comma category returns a universal arrow t_ϕ and two invertible 2-cells

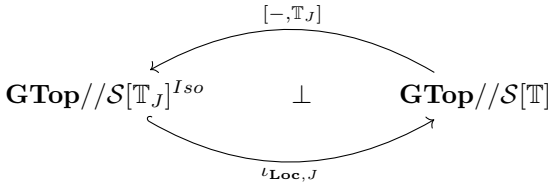


where $(t_\phi,\alpha_E) : (\mathbf{Spec}(F),\widetilde{F}) \to (\mathcal{E},E)$ is the corresponding morphism of locally modelled topos and α_E is always an iso. This defines an adjunction between homcategories

$$\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]\big[(\mathbf{Spec}(F),\widetilde{F}),(\mathcal{E},E)\big] \quad \bot \quad \mathbf{GTop}//\mathcal{S}[\mathbb{T}]\big[(\mathcal{F},F),(\mathcal{E},\iota_J E)\big]$$

where the right adjoint sends (g,ψ) to the wiskering $(\pi_1 g, \iota_J * \psi\mu * g)$ and the left adjoint sends (f,ϕ) on (t_ϕ,α_E) as above. But as an arbitrary (g,ψ) in $\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]\big[(\mathbf{Spec}(F),\widetilde{F}),(\mathcal{E},E)\big]$ has not necessarily an invertible ψ, the counit of this adjunction seldom is an isomorphism, and this adjunction does not induce an equivalence of categories: hence $[-,\mathbb{T}_J]$ is not left adjoint to the inclusion;

– either one considers that "no factorization data" means that any map is etale; hence local maps are all iso, and we trivially have admissibility. Then, from what was said above, since any morphism of locally modelled topos has now an invertible algebraic part, we actually have a biadjunction



But such a geometry is very rigid and of limited interest as any morphism of locally modelled topos hence only consists in a restriction along a geometric morphism. In some sense this is the most discrete geometry associated with $\mathbb{T}$ and J.

5.2.4 Site presentations of the classifiers

In this subsection we describe standard sites associated to the different generic classifiers constructed in the previous sections. The techniques used here allow to construct such classifiers of arrows, arrows toward local objects, etale arrows and local forms, but *before* fixing their domain: it does not in particular permit yet to access the spectral site of a given ambient object, which would be the classifier of local forms under it - this will be the point of chapter 7. Here the sites are constructed from the arrow category, the etale generator and convenient topologies on them derived from the Grothendieck pretopology provided by a choice of geometry.

5.2.4.1. Before embarking into those constructions, let us give a word on the ambivalence of some limits in **Lex**. It was observed first in [17] that in **Lex** pseudopowers coincide with bitensors, so that in fact

$$(\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega)^2 \simeq \mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega \otimes 2$$

This oddity comes from the fact that the power projections come equiped with retractions, provided by the identity arrow, which behave as tensor inclusions. Though we will not make use of it, let us also recall here that similarly, pseudoproducts in **Lex** are also bicoproducts, and that the pseudoterminal object 1 is also bi-initial. Those properties generalize similar statements about the category of $\wedge$-semilattices.

5.2.4.2. Recall that $(\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega)^2$ is the syntactic site of the theory of morphisms of $\mathbb{T}$-models. We have indeed that

$$\begin{aligned}\mathbf{pt}(\widehat{(\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}}}) &\simeq \mathbf{Lex}[(\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}}, \mathcal{S}]\\ &\simeq \mathrm{Ind}(\mathbb{T}[\mathcal{S}]^2_\omega)\\ &\simeq \mathbb{T}[\mathcal{S}]^2\end{aligned}$$

From the previous remark the equivalence above generalizes to arbitrary Grothendieck topoi as

$$\begin{aligned}\mathbf{GTop}[\mathcal{E}, \widehat{(\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega)^2}] &\simeq \mathbf{Lex}[(\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega)^2, \mathcal{E}]\\ &\simeq \mathbf{Lex}[\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega \otimes 2, \mathcal{E}]\\ &\simeq \mathbb{T}[\mathcal{E}]^2\end{aligned}$$

This exhibits $\widehat{(\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega)^2}$ as the bipower $\mathcal{S}[\mathbb{T}]^2$ in **GTop**, that is, as the classifier of morphisms between $\mathbb{T}$-models.

5.2.4.3. Similarly, from the factorization system $(\mathbf{Et}, \mathbf{Loc})$ was left generated from $\mathcal{V}$, we have

$$\begin{aligned}\mathbf{Et} &= \mathrm{Ind}(\mathcal{V})\\ &\simeq \mathbf{Lex}[\mathcal{V}^{\mathrm{op}}, \mathcal{S}]\\ &\simeq \mathbf{pt}(\widehat{\mathcal{V}^{\mathrm{op}}})\end{aligned}$$

so that $\widehat{\mathcal{V}^{\mathrm{op}}}$ is the classifier of etale maps $\mathcal{S}[\mathbf{Et}]$.

Moreover, the factorization can be constructed as follows from a site theoretic point of view. As $\mathcal{V}$ is closed in $\mathbb{T}[\mathcal{S}]^2_\omega$ under finite colimits, the inclusion $\mathcal{V}^{\mathrm{op}} \hookrightarrow (\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}}$ is lex. Hence for a lex functor $\ulcorner n \urcorner : \mathcal{V}^{\mathrm{op}} \to \mathcal{E}$ coding for an etale map in $\mathbb{T}[\mathcal{E}]$, the left Kan extension

$$\begin{array}{ccc}\mathcal{V}^{\mathrm{op}} & \xrightarrow{\ulcorner n \urcorner} & \mathcal{E}\\ {\scriptstyle \iota^{\mathrm{op}}_{\mathcal{V}}}\downarrow & \simeq & \nearrow {\scriptstyle \mathrm{lan}_{\iota^{\mathrm{op}}_{\mathcal{V}}} \ulcorner n \urcorner}\\ (\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}} & & \end{array}$$

is the name of $\iota_{\mathbf{Et}}(n)$, that is, of the image of n along the inclusion $\mathbf{Et} \hookrightarrow \mathbb{T}[\mathcal{S}]^2$. On the other hand, for a lex functor $f : (\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}} \to \mathcal{E}$ coding for an arbitrary map, then its restriction along $\iota^{\mathrm{op}}_{\mathcal{V}}$ codes for the etale part of its factorization, that is

$$\ulcorner f \urcorner \iota^{\mathrm{op}}_{\mathcal{V}} = \ulcorner n_f \urcorner$$

while the counit of $\ulcorner f \urcorner$

$$\begin{array}{ccc}\mathcal{V}^{\mathrm{op}} & \xrightarrow{\ulcorner f \urcorner \iota^{\mathrm{op}}_{\mathcal{V}}} & \mathcal{E}\\ {\scriptstyle \iota^{\mathrm{op}}_{\mathcal{V}}}\downarrow & \nearrow {\scriptstyle \ulcorner f \urcorner} & \\ (\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}} & \xrightarrow[\mathrm{lan}_{\iota^{\mathrm{op}}_{\mathcal{V}}} \ulcorner f \urcorner \iota^{\mathrm{op}}_{\mathcal{V}}]{} & \end{array} \qquad \epsilon_{\ulcorner f \urcorner}$$

is the name of the local part, that is

$$\epsilon_{\ulcorner f \urcorner} = u_f$$

which is obtained by whiskering $\epsilon_{\ulcorner f \urcorner}$ with the morphism of site provided by the identity functor

$$(\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega, J) \xrightarrow{\mathrm{id}} ((\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}}, J^2)$$

Remark 5.2.4.4. Observe that the full faithfulness of $\iota^{\mathrm{op}}_{\mathcal{V}}$ ensures that $\ulcorner n \urcorner \simeq \mathrm{lan}_{\iota^{\mathrm{op}}_{\mathcal{V}}} \ulcorner n \urcorner \iota^{\mathrm{op}}_{\mathcal{V}}$, that is, that an etale map is its own etale part.

5.2.4.5. Now, we can equip $(\mathbb{T}[\mathcal{S}]^{\mathrm{op}}_\omega)^2$ with the following pretopology J^2: for an arrow $k : K \to K'$, a covering family for $J^2(k)$ is a family of squares

$$\begin{array}{ccc}K & \xrightarrow{u_i} & K_i\\ {\scriptstyle k}\downarrow & & \downarrow{\scriptstyle k_i}\\ K' & \xrightarrow[v_i]{} & K'_i\end{array}$$

such that $(n_i : K \to K_i)_{i\in I}$ is in $J(K)$. Then observe that $(n'_i : K' \to K'_i)_{i\in I}$ is automatically covering by the stability axiom of pretopologies. In particular one can choose the squares in a family in J^2 to be the pushouts squares along k of a covering family $(n_i : K \to K_i)_{i\in I}$ in $J(K)$.

Then the adjunction

$$\mathbb{T}[\mathcal{S}]^2_\omega \underset{\text{id}}{\overset{\text{cod}}{\rightleftarrows}} \mathbb{T}[\mathcal{S}]_\omega \quad (\perp)$$

defines an adjunction between morphisms of sites

$$((\mathbb{T}[\mathcal{S}]^2_\omega)^{\text{op}}, J^2) \underset{\text{cod}}{\overset{\text{id}}{\rightleftarrows}} ((\mathbb{T}[\mathcal{S}]_\omega)^{\text{op}}, J) \quad (\perp)$$

as id sends J-covers to J^2-covers.

Proposition 5.2.4.6. *The topos* $\mathbf{Sh}((\mathbb{T}[\mathcal{S}]^{\text{op}}_\omega)^2, J^2)$ *classifies morphisms between* $\mathbb{T}_J$*-models, that is*

$$\mathbf{Sh}((\mathbb{T}[\mathcal{S}]^{\text{op}}_\omega)^2, J^2) \simeq \mathcal{S}[\mathbb{T}_J]^2$$

5.2.4.7. Now we can also construct an intermediate topology in such a way that the associated sheaf topos classifies maps whose codomain is local - and the domain arbitrary. Define the pretopology J_{cod} on $(\mathbb{T}[\mathcal{S}]^2_\omega)^{\text{op}}$ whose families consist of those squares as above, but where only the codomain family $(n'_i : K' \to K'_i)_{i\in I}$ are supposed to be J-covering.

Then in the adjunction

$$\mathbb{T}[\mathcal{S}]^2_\omega \underset{\text{cod}}{\overset{!_{(-)}}{\rightleftarrows}} \mathbb{T}[\mathcal{S}]_\omega \quad (\perp)$$

where $!_{(-)}$ sends K to the initial map $!_K : 0 \to K$ - as 0 is always finitely presented - defines an adjunction of morphisms of sites

$$((\mathbb{T}[\mathcal{S}]^2_\omega)^{\text{op}}, J_{\text{cod}}) \underset{!^{\text{op}}_{(-)}}{\overset{\text{cod}^{\text{op}}}{\rightleftarrows}} (\mathbb{T}[\mathcal{S}]^{\text{op}}_\omega, J) \quad (\perp)$$

On one hand, the codomain functor sends J_{cod} families to J families by construction; on the other hand, if $(n_i : K \to K_i)_{i\in I}$ is in $J(K)$, then the family consisting of those squares

$$\begin{array}{ccc} 0 & = & 0 \\ {\scriptstyle !_K}\downarrow & & \downarrow{\scriptstyle !_{K_i}} \\ K & \xrightarrow[n_i]{} & K_i \end{array}$$

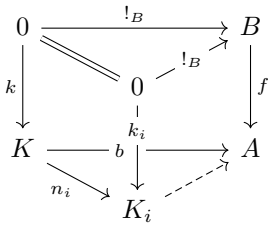
is in $J_{\text{cod}}(!_K)$.

Proposition 5.2.4.8. *The sheaf topos* $\mathbf{Sh}((\mathbb{T}[\mathcal{S}]^2_\omega)^{\text{op}}, J_{\text{cod}})$ *classifies morphisms toward* $\mathbb{T}_J$*-models.*

Proof. Let $\ulcorner f \urcorner : (\mathbb{T}[\mathcal{S}]^2_\omega)^{\text{op}} \to \mathcal{E}$ be the name of a morphism $f : B \to A$ of $\mathbb{T}$-models in $\mathcal{E}$. Then by Yoneda lemma, requiring it to be J_{cod}-continuous amounts to saying that for any $k : K \to K'$ and any family in $J_{\text{cod}}(k)$, any square $(a, b) : k \to f$ factorizes as below for some i

$$\begin{array}{ccccc} K & & \xrightarrow{\quad a \quad} & & B \\ & \searrow^{n_i} & & \nearrow & \\ {\scriptstyle k}\downarrow & & K_i & & \downarrow{\scriptstyle f} \\ & & \vdots{\scriptstyle k_i} & & \\ K' & \xrightarrow{\quad b \quad} & \downarrow & \nearrow & A \\ & \searrow_{n'_i} & & \nearrow & \\ & & K'_i & & \end{array}$$

In particular, for any $(n_i : K \to K_i)_{i\in I}$, we have a factorization



so that A is actually J-local. □

5.2.4.9. Now we would need to present the topos classifying local forms, that are, etale maps toward local objects. We saw that $\mathcal{V}^{\mathrm{op}}$ was a site for the classifier of etale maps: then we claim it suffices to restrict the codomain topology to get the classifier of etale forms. As J is generated in $\mathcal{V}$, we can consider in $\mathcal{V}^{\mathrm{op}}$ the restricted pretopology $J_{\mathrm{cod}}\mid_{\mathcal{V}^{\mathrm{op}}}$ consisting of squares of finitely presented maps:

$$\begin{array}{ccc} K & \xrightarrow{n_i} & K_i \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m_i} \\ K' & \xrightarrow[n'_i]{} & K'_i \end{array}$$

such that the bottom families $(n'_i : K' \to K'_i)_{i\in I}$ are in $J(K')$. Beware that $\mathcal{V}$ is a full subcategory of $\mathbb{T}[\mathcal{S}]^2_\omega$ so that it also contain squares with non-etale top and bottom arrows, but the generation condition makes we actually consider squares entirely made of $\mathcal{V}$ maps for the pretopology.

Then the inclusion of the basic etale maps defines a morphism of site

$$(\mathcal{V}^{\mathrm{op}}, J_{\mathrm{cod}}\mid_{\mathcal{V}^{\mathrm{op}}}) \xrightarrow{\iota_{\mathcal{V}}^{\mathrm{op}}} ((\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}}, J_{\mathrm{cod}})$$

Proposition 5.2.4.10. *The sheaf topos* $\mathbf{Sh}(\mathcal{V}^{\mathrm{op}}, J_{\mathrm{cod}}\mid_{\mathcal{V}^{\mathrm{op}}})$ *is the classifier of local forms, that is, in the notation of 5.2.1.1*

$$\mathbf{Sh}(\mathcal{V}^{\mathrm{op}}, J_{\mathrm{cod}}\mid_{\mathcal{V}^{\mathrm{op}}}) \simeq \mathcal{S}[\mathbf{Et}, \mathbb{T}_J]$$

Moreover we have a local geometric morphism

$$\mathbf{Sh}(\mathcal{V}^{\mathrm{op}}, J_{\mathrm{cod}}\mid_{\mathcal{V}^{\mathrm{op}}}) \longrightarrow \mathbf{Sh}((\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}}, J_{\mathrm{cod}})$$

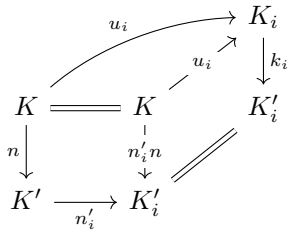
whose center is the geometric morphism $\mathbf{Sh}(\iota_{\mathcal{V}}^{\mathrm{op}})$.

Proof. This is a combination of the previous remarks. A $J_{\mathrm{cod}}\mid_{\mathcal{V}^{\mathrm{op}}}$-continuous $\ulcorner n \urcorner : \mathcal{V}^{\mathrm{op}} \to \mathcal{E}$ is the name of an etale map, and from 5.2.4.3, we can consider its left Kan extension $\mathrm{lan}_{\iota_{\mathcal{V}}^{\mathrm{op}}} \ulcorner n \urcorner$, which is J_{cod}-continuous as $\iota_{\mathcal{V}}$ is a morphism of site: hence its codomain is a local object, so that $\ulcorner n \urcorner$ is the name of a local form.

Moreover, the morphism of site $\iota_{\mathcal{V}^{\mathrm{op}}}$ happens to be also a comorphism of site: indeed, for any $n : K \to K'$ in $\mathcal{V}$ and $((u_i, n'_i) : n \to k_i)_{i\in I}$ a cover in $J_{\mathrm{cod}}(\iota_{\mathcal{V}}^{\mathrm{op}}(n)$ - with, beware, u_i and k_i arbitrary finitely presented maps - we have that $(n'_i)_{i\in I}$ is a family of etale maps so each composite $n'_i n$ is an etale map and hence the family consisting of the squares

$$\begin{array}{ccc} K & = & K \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle n'_i n} \\ K' & \xrightarrow[n'_i]{} & K'_i \end{array}$$

is a $J_{\mathrm{cod}}\mid_{\mathcal{V}^{\mathrm{op}}}$-cover in $\mathcal{V}$, and we have a factorization at each i



ensuring that the image of this cover along $\iota_{\mathcal{V}}^{\mathrm{op}}$ refines $(u_i, n_i')_{i\in I}$: hence $\iota_{\mathcal{V}}^{\mathrm{op}}$ has the cover lifting property, and is a comorphism of site. Hence, being also continuous, it defines from [15][Theorem 7.20, (iii)] a local geometric morphism consisting of a triple of adjoints

$$\mathbf{Sh}(\mathcal{V}^{\mathrm{op}}, J_{\mathrm{cod}}\,|_{\mathcal{V}^{\mathrm{op}}}) \underset{\mathbf{Sh}(\iota_{\mathcal{V}}^{\mathrm{op}})^!}{\overset{\mathbf{Sh}(\iota_{\mathcal{V}}^{\mathrm{op}})^*}{\leftarrow \mathbf{Sh}(\iota_{\mathcal{V}}^{\mathrm{op}})_* -}} \mathbf{Sh}((\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}}, J_{\mathrm{cod}})$$

□

Remark 5.2.4.11. Observe that admissibility can be detected at this level of description by combining the previous lemma with 5.2.4.3. Indeed, suppose that $\ulcorner f \urcorner : (\mathbb{T}[\mathcal{S}]^2_\omega)^{\mathrm{op}} \to \mathcal{E}$ is J_{cod} continuous. Then its precomposition along the morphism of site $\iota_{\mathcal{V}}^{\mathrm{op}}$, which codes for its etale maps, is also $J_{\mathrm{cod}}\,|_{\mathcal{V}^{\mathrm{op}}}$-continuous: this means that the etale part of the factorization of a morphism with local codomain still has a local codomain.

5.3 Spectrum from an etale class

Here we sum up Dubuc's construction of the spectrum as done in [33], which is at first sight completely divergent from Cole's method. We saw moreover that Dubuc contexts, which this construction takes as input, are more general than admissibility structures for they are not restricted to geometric extensions of finite limit theories. As the source concerning this work is more explicit and contains more detailed proofs than [17] or also [19], we allow ourselves to be more synthetic than in the previous section, as the details ensuring the viability of this construction are already written down.

5.3.1 Some bicolimits of Grothendieck topoi

First, we must describe the construction of the topos of etale objects associated to an etale classe. This construction was also considered in [86][IV, exercice 4.10.6] and also in [4][Proposition 39] and is related to the notions of *big* and *little* topos. While the previous constructions involved (finite) bilimits in **GTop** and in particular bipower with 2 and bi-inverters, the following involves (finite) bicolimits and in particular bitensor and bicoinverters in **GTop** we describe before going further.

5.3.1.1. Let $\mathcal{E}$ be a Grothendieck topos: then the *bitensor of* $\mathcal{E}$ *with* 2 is the topos $\mathcal{E}\otimes 2$ such that we have an equivalence of categories

$$\mathbf{Cat}\big[2, \mathbf{GTop}[\mathcal{E}, \mathcal{F}]\big] \simeq \mathbf{GTop}[\mathcal{E}\otimes 2, \mathcal{F}]$$

Moreover, alike the bipower, the bitensor comes equipped with its universal 2-cell

$$\mathcal{E} \underset{p_1}{\overset{p_0}{\overset{\Downarrow\rho}{\rightrightarrows}}} \mathcal{E}\otimes 2$$

through which any 2-cell out of $\mathcal{E}$ factorizes uniquely. The bitensor with 2 admits a more concrete description: observe that when $\mathcal{F} \simeq \mathcal{S}[\mathbb{O}]$ the previous universal property becomes

$$\mathbf{Cat}\big[2, \mathbf{GTop}[\mathcal{E}, \mathcal{S}[\mathbb{O}]\big] \simeq \mathbf{GTop}[\mathcal{E}\otimes 2, \mathcal{S}[\mathbb{O}]]$$

and this expresses the topos $\mathcal{E}\otimes 2$ as the arrow category $\mathcal{E}^2$ whose objects are arrows between objects of $\mathcal{E}$ and morphisms are commutative squares.

5.3.1.2. Similarly we have a notion of *bicoinverter* of a 2-cell $\phi : F \Rightarrow G$ in $\mathbf{GTop}[\mathcal{E}, \mathcal{F}]$ in **GTop** which is the 2-cell

$$\mathcal{E} \underset{G}{\overset{F}{\overset{\Downarrow\phi}{\rightrightarrows}}} \mathcal{F} \xrightarrow[c_\phi]{} \mathrm{coInv}(\phi)$$

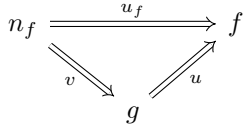
initial amongst geometric morphisms out of $\mathcal{F}$ whose whiskering with ϕ is invertible.

5.3.2 The topos of etale objects and the generic infinitesimal extension

Definition 5.3.2.1. Let $\mathscr{H}$ be an etale class in a Grothendieck topos $\mathcal{E}$. An object H of $\mathcal{E}$ is said to be *etale* if its terminal map $!_H : H \to 1$ is in $\mathscr{H}$. Etale objects and etale maps between them form a full subcategory $\mathcal{E}_{\mathscr{H}} \hookrightarrow \mathcal{E}$. In particular, if $\mathcal{E}$ has a small lex site of presentation we can consider the full subcategory $\mathcal{C}_{\mathscr{H}} \hookrightarrow \mathcal{C}$ of *basic etale objects* which are the H in $\mathcal{C}$ whose terminal map $!_H : H \to 1$ is in $\mathscr{H} \cap \mathcal{C}^2$.

Remark 5.3.2.2. Beware that in all generality it is not known whether the category of etale objects is a Grothendieck topos, for there is no way to ensure the existence of a generator. Additional conditions are required. The following lemma, which is [33][Proposition 5.2], is crucial to ensure this property in the case of an etale class satisfying an etale topology condition (cf. definition 3.2.1.10):

Proposition 5.3.2.3. *For $\mathcal{E}$ with a standard presentation $(\mathcal{C}, J)$ and an etale class $\mathscr{H}$ in $\mathcal{E}$ satisfying the etale topology condition with respect to J. Then for each geometric morphism $f : \mathcal{F} \to \mathcal{E}$, the category of $\mathscr{H}$-infinitesimal extensions above f admits an initial object, that is there exists an infinitesimal extension $u_f : n_f \Rightarrow f$ in $\mathbf{GTop}[\mathcal{F}, \mathcal{E}]$ such that any other infinitesimal extension $u : g \Rightarrow f$ there is a factorization through an unique v as below (which is also an infinitesimal extension)*



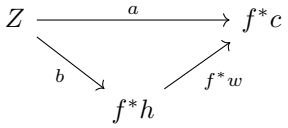
Proof. We only give the global strategy of the proof, which done in detail in [33][5.2]. The domain n_f of this infinitesimal extension is constructed as follows: we define a functor

$$\mathcal{F}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{N} \mathcal{S}$$

sending a pair (Z, c) with Z in $\mathcal{F}$ and c in $\mathcal{C}$ to the coend

$$N(Z, c) = \int^{h \in \mathcal{C}_{\mathscr{H}}} \mathcal{F}[Z, f^* h] \times \mathcal{C}[h, c]$$

indexing all the possible factorizations in $\mathcal{F}$ of the form



of all Z-indexed element of $f^*(c)$ modulo the zigzag relation; as f^* is a morphism of site, it is representably flat, so each $(F \downarrow f^*)^{\mathrm{op}}$ is filtered and the coend above is also expressible as a filtered colimit

$$N(Z, c) = \operatorname*{colim}_{(h,x) \in (Z \downarrow f^*)^{\mathrm{op}}} \mathcal{C}(h, c)$$

From this expression one can prove that for each Z the induced functor

$$\mathcal{C} \xrightarrow{N(Z,-)} \mathcal{S}$$

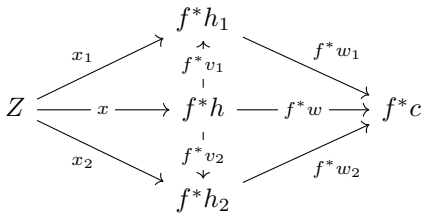
is lex and J-continuous. The associated functor

$$\mathcal{C} \xrightarrow{\overline{N}} [\mathcal{F}^{\mathrm{op}}, \mathcal{S}]$$

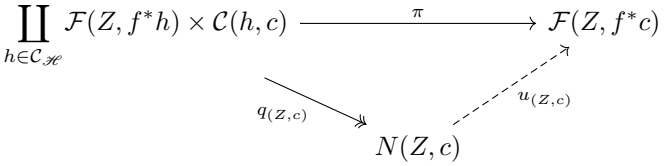
can be composed with the sheafification for the canonical topology landing in $\mathbf{Sh}(\mathcal{F}^{\mathrm{op}}, J_{can})$ which is equivalent to $\mathcal{F}$. This composite can be proved to be also lex, J-continuous as each $N(Z, -)$ is. This defines hence a lex site morphism $n_f^* : \mathcal{C} \to \mathcal{F}$ as desired.

Then the infinitesimal extension u_f is obtained by the composition functor sending a triple (x, h, w) to its composite $f^*(w)x : Z \to f^*(c)$: this functors factorizes indeed through $N(Z, c)$ as

any two triples (x_1, h_1, w_1) and (x_2, h_2, w_2) induce the same composite if and only if they related by a zigzag as below

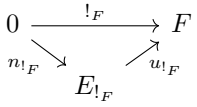


which exactly says that they are identified in the coend $N(Z, c)$. Hence one has a natural map $u_{(Z,c)}$ as below

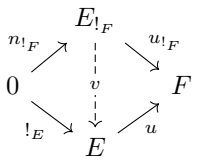


and the $u_{(Z,c)}$ define altogether a natural transformation $\overline{u_f} : N \Rightarrow \mathcal{F}[-, f^*]$ corresponding to a natural transformation $u_f : n_f \Rightarrow f^*$ as desired. □

Remark 5.3.2.4. At first sight, this result might seem both technical and mysterious; in particular one could be misled into thinking, in the context of a geometry $(\mathbb{T}, \mathcal{V}, J)$, that it is meant to be applied to geometric morphisms $F : \mathcal{F} \to \mathcal{S}[\mathbb{T}]$ coding for a $\mathbb{T}$-model, and would say something about existence of an initial infinitesimal extension with codomain F. The only way to interpret this (in the context of a geometry, where $\mathbb{T}$ being finite-limit, $\mathbb{T}[\mathcal{F}]$ has an initial object) is to see u_f as the local part of the (etale, local) factorization of the initial map



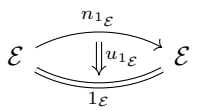
Hence indeed, as any local map with codomain F is a factorization of $!_F$, the initialness of this factorization amongst local factorizations ensures the existence of a unique factorization



Hence we could read the result above as a generalization of this fact in the general case of a geometry, where the initial object is not required to exist. However this is not the way it is meant to be read.

When applied to the relevant context, this result says something both deep and familiar with our topic: it constitutes a very generalized form of admissibility. Indeed, it is meant to be applied to the topos $\mathcal{S}[F, \mathbb{T}_J]$ constructed later on at 5.3.4.3, classifying arrows from a given object to local objects, where it will ensure that any such arrow $f : F \to E$ admits an initial infinitesimal extension $u_f : n_f \to f$ above it in the category of arrows: but then the domain of this infinitesimal extension n_f is nothing but the corresponding local form, the etale part of the admissible factorization $f = u_f n_f$. Before coming to this, we must mention the main consequences of this fact regarding the construction of the etale topos.

5.3.2.5. In the condition of the previous proposition, we can consider in particular the initial infinitesimal extension below the identity of $\mathcal{E}$



We should call it the *generic infinitesimal extension*. Its concrete expression is obtained as follows. As the geometric morphism $1_{\mathcal{E}}$ is obtained as the Yoneda extension of its own embedding $n^*_{1_{\mathcal{E}}}$ is constructed from the functor $\mathcal{E}$

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\text{よ}} & \mathcal{E} \\ {\scriptstyle \text{よ}}\downarrow & \simeq & \| \\ \mathcal{E} & & {\scriptstyle 1_{\mathcal{E}}} \end{array}$$

Hence $n_{1_{\mathcal{E}}}$ corresponds to the functor $\mathcal{E}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{S}$ sending (E, c) to the category of all factorizations of arrows in $\mathcal{E}[E, \text{よ}_c]$ through an etale object in $\mathcal{C}_{\mathscr{H}}$

$$\begin{array}{ccc} E & \xrightarrow{\quad a \quad} & \text{よ}_c \\ & {\scriptstyle x}\searrow \quad \nearrow{\scriptstyle \text{よ}_v} & \\ & \text{よ}_h & \end{array}$$

Then for an object c of $\mathcal{C}$, we can consider the category $\mathcal{C}_{\mathscr{H}} \downarrow c$ of all basic etale morphisms over c, and take its colimit in $\mathcal{E}$. By the properties of etale maps, we can see in the diagram below

$$\begin{array}{ccccc} \text{よ}_h & & \xrightarrow{\quad q_{(h,v)} \quad} & & \underset{(h,v)\in\mathcal{C}_{\mathscr{H}}\downarrow c}{\operatorname{colim}} \text{よ}_h \\ & {\scriptstyle v}\searrow & & \nearrow{\scriptstyle \langle v\rangle_{(h,v)\in\mathcal{C}_{\mathscr{H}}\downarrow c}} & \\ & & \text{よ}_c & & \\ {\scriptstyle !_h}\searrow & & \downarrow{\scriptstyle !_{\text{よ}_c}} & \swarrow & {\scriptstyle \langle !_h\rangle_{(h,v)\in\mathcal{C}_{\mathscr{H}}\downarrow c}} \\ & & 1 & & \end{array}$$

that the induced map $\langle !_h \rangle_{(h,v)\in\mathcal{C}_{\mathscr{H}}\downarrow c}$ is etale by proposition 1.2.1.4, so that the colimit is an etale object. This category is filtered as the coproduct of a pair of etale objects is etale.

Then we have a filtered colimit

$$n^*_{1_{\mathcal{E}}}(c) = \underset{(h,v)\in\mathcal{C}_{\mathscr{H}}\downarrow c}{\operatorname{colim}} \text{よ}_h$$

and the inverse image part of the generic infinitesimal extension

$$\begin{array}{ccc} & {\scriptstyle n^*_{1_{\mathcal{E}}}} & \\ \mathcal{C} & \Downarrow{\scriptstyle u_{1_{\mathcal{E}}}} & \mathcal{E} \\ & {\scriptstyle \text{よ}} & \end{array}$$

has as component at c the induced universal map

$$(u_{1_{\mathcal{E}}})_c = \langle v \rangle_{(h,v)\in\mathcal{C}_{\mathscr{H}}\downarrow c}$$

This formula extends without change into the same colimit formula for arbitrary objects in $\mathcal{E}$.

5.3.2.6. As a consequence, the inverse image part of $n_{1_{\mathcal{E}}}$ lands in $\mathcal{E}_{\mathscr{H}}$, for the colimit of etale objects above is in particular an etale object. Hence we have a factorization

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad {\scriptstyle n^*_{1_{\mathcal{E}}}} \quad} & \mathcal{E} \\ & \searrow \quad \nearrow{\scriptstyle \iota_{\mathscr{H}}} & \\ & \mathcal{E}_{\mathscr{H}} & \end{array}$$

Moreover, this induced functor is right adjoint to the inclusion functor $\iota_{\mathscr{H}}$: indeed, we have a bijection

$$\mathcal{E}[\iota_{\mathscr{H}}(h), E] \simeq \mathcal{E}_{\mathscr{H}}[\text{よ}_h, \underset{(h',v)\in\mathcal{C}_{\mathscr{H}}\downarrow E}{\operatorname{colim}} \text{よ}'_h]$$

sending $v : h \to E$ on the corresponding colimit inclusion $q_{(h,v)}$ and a map into the colimit to the composite with $\langle v \rangle_{(h,v)\in\mathcal{C}_{\mathscr{H}}\downarrow E}$. Hence $\mathcal{E}_{\mathscr{H}}$ is a coreflective subcategory of $\mathcal{E}$. In particular it is closed under colimits.

Proposition 5.3.2.7. *If $\mathcal{H}$ satisfies the etale topology condition relatively to $(\mathcal{C}, J)$ then the category $\mathcal{E}_{\mathscr{H}}$ is a Grothendieck topos, and we have*

$$\mathcal{E}_{\mathscr{H}} \simeq \mathbf{Sh}(\mathcal{C}_{\mathscr{H}}, J_{\mathscr{H}})$$

Proof. As a coreflective subcategory of $\mathcal{E}$, $\mathcal{E}_{\mathscr{H}}$ is closed under colimits and also inherits (finite) limits (through they are not calculated as in $\mathcal{E}$ a priori). In particular its coproducts are disjoint as they are so in $\mathcal{E}$. Moreover, from its definition as a filtered colimit, we know that $n_{1_\mathcal{E}}^*$ preserves finite limits. Hence in particular coproducts are pullback-stable in $\mathcal{E}_{\mathscr{H}}$. Moreover, as the coequalizer of arrows into an etale object has an etale codomain, and the latter is preserved by the inclusion $\iota_{\mathscr{H}}$, $\mathcal{E}_{\mathscr{H}}$ inherits exactness of $\mathcal{E}$. The only thing to prove is the existence of the small generator.

But from the following retraction

$$\begin{array}{ccccc} & & \mathcal{E} & & \\ & {\scriptstyle \iota_{\mathscr{H}}} \nearrow & & \searrow {\scriptstyle n_{1_\mathcal{E}}^*} & \\ \mathcal{E}_{\mathscr{H}} & & = & & \mathcal{E}_{\mathscr{H}} \end{array}$$

we know that for any H in $\mathcal{E}_{\mathscr{H}}$ we have

$$\begin{aligned} H &\simeq \iota_{\mathscr{H}} n_{1_\mathcal{E}}^*(H) \\ &\simeq \operatorname*{colim}_{(h,v)\in \mathcal{C}_{\mathscr{H}} \downarrow H} H \end{aligned}$$

but this later condition exactly means that $\mathcal{C}_{\mathscr{H}}$ is a dense generator for $\mathcal{E}_{\mathscr{H}}$. Hence by the Giraud axioms, $\mathcal{E}_{\mathscr{H}}$ is a Grothendieck topos and we have a geometric morphism

$$\mathcal{E} \underset{n_{1_\mathcal{E}}^*}{\overset{\iota_{\mathscr{H}}}{\underset{\longleftarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{E}_{\mathscr{H}}$$

□

5.3.2.8. Since a famous exercise affectionately nicknamed *médaille en chocolat* of [86][IV, exercise 4.10.6], this process of constructing a topos from a class of maps $\mathscr{H}$ in a site $(\mathcal{C}, J)$ with J generated in $\mathscr{H}$ has been known to produce a local geometric morphism

$$\mathbf{Sh}(\mathcal{C}, J) \underset{\iota_{\mathscr{H}}^!}{\overset{\iota_{\mathscr{H}}^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}}} \mathbf{Sh}(\mathcal{C}_{\mathscr{H}}, J_{\mathscr{H}}) \quad (\iota_{\mathscr{H}*})$$

which can moreover be relativized under arbitrary object (though this plays no role in our purpose). In such a context, the sheaf topos $\mathbf{Sh}(\mathcal{C}, J)$ is traditionally called the *gros topos* and $\mathbf{Sh}(\mathcal{C}_{\mathscr{H}}, J_{\mathscr{H}})$ the *petit topos* - though the name of gros topos makes more sense in the relativized version.

Corollary 5.3.2.9. *The inclusion $\mathcal{E}_{\mathscr{H}} \hookrightarrow \mathcal{E}$ is the inverse image part of a local geometric morphism*

$$\mathcal{E} \underset{\iota_{\mathscr{H}}^!}{\overset{\iota_{\mathscr{H}}^*}{\underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}}}} \mathcal{E}_{\mathscr{H}} \quad (\iota_{\mathscr{H}*})$$

To conclude this subsection, we would like to add this result, generalizing a result of [33][Theorem 3.2] which was however stated only in a more specific context:

Proposition 5.3.2.10. *If $\mathcal{E}$ has a small lex site of presentation $(\mathcal{C}, J)$ such that $\mathscr{H}$ satisfies the etale topology condition with respect to J. Then the geometric morphism $\iota^* \dashv \iota_*$ exhibits $\mathcal{E}_{\mathscr{H}}$ as the bicoinverter of the generic infinitesimal extension*

$$\mathcal{E} \underset{1_\mathcal{E}}{\overset{n_{1_\mathcal{E}}}{\underset{\Downarrow u_{1_\mathcal{E}}}{\rightrightarrows}}} \mathcal{E} \xrightarrow{q_{u_{1_\mathcal{E}}}} \mathcal{E}_{\mathscr{H}} \simeq \mathrm{coInv}(u_{1_\mathcal{E}})$$

Proof. Suppose $f : \mathcal{E} \to \mathcal{F}$ to be coinverting $u_{1_\mathcal{E}}$, hence factorizing through the coinverter. Then by definition of bicoinverters as inverters of the inverse images part in **Cat**, this means that f^* inverses $u_{1_\mathcal{E}} : n^*_{1_\mathcal{E}} \Rightarrow 1^*_\mathcal{E}$. But on one hand we know that $n^*_{1_\mathcal{E}}$ lands in $\mathcal{E}_\mathscr{H}$, and as $1^*_\mathcal{E}$ is the identity, this means that f^* factorizes through the inclusion of $\mathcal{E}_\mathscr{H}$; but this means that f factorizes through $\mathcal{E} \to \mathcal{E}_\mathscr{H}$. Moreover, such a factorization is unique up to equivalence as $\mathcal{E} \to \mathcal{E}_\mathscr{H}$ is a split retract, hence is an eso-morphism in **GTop**. This exhibits $\mathcal{E}_\mathscr{H}$ as the bicoinverter of $u_{1_\mathcal{E}}$. □

5.3.3 The spectrum as a topos of etale objects

Though the first steps of both constructions look similar, [33][section 9] proceeds in a different strategy to [17]. We could have considered the classifier of etale maps and then constructed the classifiers of local forms under a fixed object, as the bipullback of the classifier of etale map under this object together with the classifier of etale maps toward local objects. Instead, it proceeds by constructing first the classifier of arbitrary maps from a fixed object toward local objects, and then construct from it the classifier of local forms under this object as the topos of etale objects. We should also acknowledge that this method is more general for two distinct reasons:

- it allows to consider a preexisting etale class independent from the one associated to the admissible factorization.
- it takes as input a Dubuc context, where the specification of local objects - equivalently, of a geometric extension - is replaced with an arbitrary geometric morphism which is not anymore supposed to be an inclusion of topos. This later condition was important to ensure functoriality in Cole method.

In this section we describe succinctly [33][Section 8 and 9] results, to which one can refer for the precise argument at each step.

5.3.3.1. Suppose we have a Dubuc context $(\mathbb{T}, \mathbb{T}', \mathscr{H}, \mathscr{H}', j)$, that is $j : \mathcal{S}[\mathbb{T}'] \to \mathcal{S}[\mathbb{T}]$ with $\mathscr{H} \subseteq j(\mathscr{H}')$. Denote as $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]^\mathscr{H}$, resp. $\mathbf{GTop}//\mathcal{S}[\mathbb{T}']^{\mathscr{H}'}$ the sub-bicategories of the corresponding oplax coslices, where the 1-cells are restricted to triangles consisting of $\mathscr{H}$, resp. $\mathscr{H}'$-infinitesimal extensions.

Recall that the bipower object $\mathcal{S}[\mathbb{T}]^2$ comes equipped naturally with a canonical 2-cell $\mu_\mathbb{T} : \partial_0 \Rightarrow \partial_1$ sending in particular arrows of $\mathscr{H}$ to an $\mathscr{H}$-indexed family of squares in $\mathcal{S}[\mathbb{T}]^2$

$$\begin{array}{ccc} \partial_0^* X & \xrightarrow{\partial_0^* h} & \partial_0^* Y \\ {\scriptstyle (\mu_\mathbb{T})_X}\downarrow & & \downarrow{\scriptstyle (\mu_\mathbb{T})_Y} \\ \partial_1^* X & \xrightarrow[\partial_1^* h]{} & \partial_1^* Y \end{array}$$

A priori, the generic morphism $\mu_\mathbb{T}$ has no reason to be $\mathscr{H}$-infinitesimal - which would amount to requiring all the squares above to be pullbacks. But then we can consider the smallest lex localization $\mathcal{S}[\mathscr{H}] \hookrightarrow \mathcal{S}[\mathbb{T}]^2$ forcing those squares to become pullbacks: this is the *classifier of $\mathscr{H}$-infinitesimal extensions*, which comes equipped with a canonical generic infinitesimal extension

$$\mathcal{S}[\mathscr{H}] \;\underset{\partial_1^\mathscr{H}}{\overset{\partial_0^\mathscr{H}}{\Downarrow u_\mathscr{H}}}\; \mathcal{S}[\mathbb{T}]$$

5.3.3.2. Then for any $F : \mathcal{F} \to \mathcal{F}$ one can consider the classifier of infinitesimal $\mathscr{H}$-extensions toward $\mathbb{T}'$-models

$$\begin{array}{ccc} \mathcal{S}[F, \mathscr{H}, \mathbb{T}'] & \xrightarrow{\pi_F} & \mathcal{F} \\ {\scriptstyle \partial_1^{F,\mathscr{H},\mathbb{T}'}}\downarrow & \swarrow{\scriptstyle u_{F,\mathscr{H},\mathbb{T}'}} & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbb{T}'] & \xrightarrow[j]{} & \mathcal{S}[\mathbb{T}] \end{array}$$

Then $\mathcal{S}[F, \mathscr{H}, \mathbb{T}']$ comes equipped with two etale classes, $(\partial_1^\mathscr{H})^{-1}(\mathscr{H}')$ and $(F\pi_F)^{-1}(\mathscr{H})$, and one can consider the etale class $\mathscr{H}_F$ they jointly generate. It can be proven, see [33][Appendice A] that this class satisfies the etale topology condition relatively to a certain site of presentation.

Remark 5.3.3.3. The last argument relies on the possibility to construct an explicit site presentation for the bicomma topos; such process exists for instance in [1]. However it involves three sites such that the morphisms we compute a bicomma of are induced from site morphisms between those sites. Though it seems reasonable to require j to be induced as a morphism between the syntactic sites of $\mathbb{T}$, $\mathbb{T}'$, the morphism F will not be induced as a morphisms between underlying small sites: only as a site morphism $F^* : (\mathcal{C}_\mathbb{T}, J_\mathbb{T}) \to (\mathcal{F}, J_{can})$, which has a large site as codomain. This may not be a real issue, as often are size issues, nevertheless we prefer not to enter into the detail of the construction it would lead us to, and take for granted the existence of a site for $\mathcal{S}[F, \mathscr{H}, \mathbb{T}']$ satisfying the etale topology condition relatively to $\mathscr{H}_F$.

Definition 5.3.3.4. The *(Dubuc) Spectrum* of F is the topos of $\mathscr{H}_F$-etale objects

$$\mathbf{Spec}(F) = \mathcal{S}[F, \mathscr{H}, \mathbb{T}']_{\mathscr{H}_F}$$

Theorem 5.3.3.5. *For a Dubuc context as above, we have a bi-adjunction*

$$\mathbf{GTop}//\mathcal{S}[\mathbb{T}']^{\mathscr{H}'} \underset{\iota_{j,\mathscr{H}'}}{\overset{\mathbf{Spec}}{\rightleftarrows}} \mathbf{GTop}//\mathcal{S}[\mathbb{T}]^{\mathscr{H}} \quad (\perp)$$

5.3.4 Comparison between Cole's and Dubuc's constructions

We conclude this chapter by a comparison of [33] and [17] constructions.

5.3.4.1. It is clear that any geometry $(\mathbb{T}, \mathcal{V}, J)$ is a special instance of a Dubuc context, satisfying the following properties:

- the theory $\mathbb{T}$ is a finite-limit theory
- $\mathbb{T}'$ is a geometric extension of $\mathbb{T}$, so that j is a geometric embedding $\mathcal{S}[\mathbb{T}'] \hookrightarrow \mathcal{S}[\mathbb{T}]$
- the second etale class $\mathscr{H}$ in $\mathcal{S}[\mathbb{T}]$ is actually trivial, that is, it consists only of isomorphisms: hence any oplax 1-cell is an infinitesimal extension, and $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]^{\mathscr{H}}$ is the whole oplax slice over $\mathcal{S}[\mathbb{T}]$.

Remark 5.3.4.2. Though we suppose $\mathscr{H}$ to be trivial, remark that the etale class we are going to consider $\mathscr{H}'$ will be constructed from $\mathcal{V}$ which is at the begining a saturated class in $\mathcal{S}[\mathbb{T}]$. But recall that, as $\mathcal{S}[\mathbb{T}'] \simeq \mathbf{Sh}(\mathcal{C}_\mathbb{T}, J)$ and J is generated in $\mathcal{V}$, we can also see $\mathscr{H}'$ as natively defined in $\mathcal{S}[\mathbb{T}']$, where the generation condition exactly amounts to the etale topology condition.

5.3.4.3. Let $(\mathbb{T}, \mathcal{V}, J)$ be a geometry, with $\mathscr{H}$ the etale class in $\mathcal{S}[\mathbb{T}]$ associated to $\mathcal{V}$. We saw that $\mathscr{H}$ satisfies the etale topology condition. Then for each $F : \mathcal{F} \to \mathcal{S}[\mathbb{T}]$, we can construct the classifier of $\mathbb{T}$-morphisms under F

$$\begin{array}{ccc} \mathcal{S}[F, \mathbb{T}] & \xrightarrow{\pi_0^F} & \mathcal{F} \\ {\scriptstyle \pi_1^F}\downarrow & \lrcorner \simeq & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbb{T}]^2 & \xrightarrow[\partial_0]{} & \mathcal{S}[\mathbb{T}] \end{array}$$

We can refine this construction to force the codomain of such morphisms to be local by taking the bipullback:

$$\begin{array}{ccccc} \mathcal{S}[F, \mathbb{T}_J] & \longrightarrow & \mathcal{S}[F, \mathbb{T}] & \xrightarrow{\pi_0^F} & \mathcal{F} \\ \downarrow & \lrcorner & {\scriptstyle \pi_1^F}\downarrow & \lrcorner & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbb{T}, \mathbb{T}_J] & \longrightarrow & \mathcal{S}[\mathbb{T}]^2 & \xrightarrow{\partial_0} & \mathcal{S}[\mathbb{T}] \\ \downarrow & \lrcorner & {\scriptstyle \partial_1}\downarrow & \overset{\mu}{\swarrow} & \| \\ \mathcal{S}[\mathbb{T}_J] & \xrightarrow[\iota_J]{} & \mathcal{S}[\mathbb{T}] & = & \mathcal{S}[\mathbb{T}] \end{array}$$

where $\mathcal{S}[\mathbb{T}, \mathbb{T}_J]$ classifies morphisms of $\mathbb{T}$-models towards $\mathbb{T}_J$-models. In the following we denote as

$$\begin{array}{ccc} \mathcal{S}[F, \mathbb{T}_J] & \xrightarrow{\pi_F} & \mathcal{F} \\ {\scriptstyle \partial_{1,J}}\downarrow & \overset{\mu_{F,J}}{\swarrow} & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbb{T}_J] & \xrightarrow[\iota_J]{} & \mathcal{S}[\mathbb{T}] \end{array}$$

the induced canonical 2-cell above. $\mu_{F,J}$ is there the generic morphism from F to $\mathbb{T}_J$-models.

5.3.4.4. From the existence of admissible factorizations, the generic morphism from F to $\mathbb{T}_J$-models factorizes as below

$$\begin{array}{ccc} \mathcal{S}[F,\mathbb{T}_J] & \longrightarrow & \mathcal{F} \\ {\scriptstyle \partial_{F,J}}\downarrow & \overset{u_{\mu_{F,J}}}{\Leftarrow} \quad \overset{n_{\mu_{F,J}}}{\Leftarrow} & \downarrow{\scriptstyle F} \\ \mathcal{S}[\mathbb{T}_J] & \underset{i_J}{\hookrightarrow} & \mathcal{S}[\mathbb{T}] \end{array}$$

and by commutation of bilimits, we can also recover the Cole spectrum of F as the bi-inverter of the local part of the factorization above

$$\mathrm{Inv}(u_{\mu_{F,J}}) \simeq \mathbf{Spec}(F) \xhookrightarrow{\iota_F} \mathcal{S}[F,\mathbb{T}_J] \underset{\partial_{F,J}}{\overset{H_F}{\Downarrow u_{\mu_{F,J}}}} \mathcal{S}[\mathbb{T}_J]$$

5.3.4.5. The topos $\mathcal{S}[F,\mathbb{T}_J]$ comes equipped with an etale class $\partial_{F,J}(\mathscr{H}')$. Then we can consider its generic infinitesimal extension - its generic local map with the terms of admissibility structures

$$\mathcal{S}[F,\mathbb{T}_J] \underset{1_{\mathcal{S}[F,\mathbb{T}_J]}}{\overset{n_{1_{\mathcal{S}[F,\mathbb{T}_J]}}}{\Downarrow u_{1_{\mathcal{S}[F,\mathbb{T}_J]}}}} \mathcal{S}[F,\mathbb{T}_J]$$

Observe that $1_{\mathcal{S}[F,\mathbb{T}_J]}$ is the very name of the generic morphism from F toward $\mathbb{T}_J$-models. Now from admissibility, it happens that the etale part $n_{\mu_{F,J}}$ of the admissible factorization of the generic morphism $\mu_{F,J}$ has a $\mathbb{T}_J$-model H_F as codomain; in the category $\mathbb{T}[\mathcal{S}[F,\mathbb{T}_J]]^2$ this factorization is materialized as a map

$$n_{\mu_{F,J}} \xrightarrow{u_{\mu_{F,J}}} 1_{\mathcal{S}[F,\mathbb{T}_J]}$$

which is an infinitesimal extension as it is a local map.

Lemma 5.3.4.6. *The generic infinitesimal extension coincides with the name of the local part of the generic factorization, that is*

$$\ulcorner n_{\mu_{F,J}} \urcorner = n_{1_{\mathcal{S}[F,\mathbb{T}_J]}} \qquad \ulcorner u_{\mu_{F,J}} \urcorner = u_{1_{\mathcal{S}[F,\mathbb{T}_J]}}$$

Proof. This comes from a coincidence of two universal properties. Indeed $u_{1_{\mathcal{S}[F,\mathbb{T}_J]}}$ is the initial infinitesimal extension above $1_{\mathcal{S}[F,\mathbb{T}_J]}$, which is the name of the generic morphism $\mu_{F,J}$: but on the other hand, in the admissible factorization $(n_{\mu_{F,J}}, u_{\mu_{F,J}})$, $n_{\mu_{F,J}}$ is terminal amongst factorizations through etales maps, while $u_{\mu_{F,J}}$ is initial amongst factorizations through local maps, and as local maps coincide with infinitesimal extensions, this exactly says that $u_{\mu_{F,J}}$ the the initial object in the category of infinitesimal extensions above $1_{\mathcal{S}[F,\mathbb{T}_J]}$, and its domain is $n_{\mu_{F,J}}$. □

5.3.4.7. Now, by uniqueness of bi-adjoints (equivalently, by their common universal property), Dubuc and Cole spectra must be equivalent. By proposition 5.3.2.10 Dubuc spectrum is obtained as the coinverter of the generic infinitesimal extension $u_{1_{\mathcal{S}[F,\mathbb{T}_J]}}$, which is also the topos of etale objects. On the other hand, Cole spectrum is obtained as the inverter of the universal local map: this means that the spectrum of a modelled topos $(\mathcal{F}, F)$ is *simultaneously an inverter and a coinverter*:

$$\begin{aligned} \mathbf{Spec}(F) &\simeq \mathrm{Inv}(u_{\mu_{F,J}}) \\ &\simeq \mathrm{coInv}(u_{1_{\mathcal{S}[F,\mathbb{T}_J]}}) \\ &\simeq \mathcal{S}[F,\mathbb{T}_J]_{\partial_{F,J}(\mathscr{H}')} \end{aligned}$$

where the inverter structure is given by the topos inclusion $\iota_{\mathscr{H}_F *} \dashv \iota^!_{\mathscr{H}}$ while the coinverter structure is given by the local geometric morphism $\iota^*_{\mathscr{H}} \dashv \iota_{\mathscr{H}_F *}$.

Chapter 6

(Locally) modelled topoi

In the last chapter, we constructed directly from a geometry a notion of spectrum by means of finite bilimits in the bicategory of Grothendieck topoi. This method relied on the vision of models of a theory in a topos as geometric morphisms toward its classifier, and morphisms between models as oplax 1-cells. This lead us to consider the oplax slices of **GTop** over the corresponding classifiers; restrictions at the level of morphisms of models were realized by restricting the oplax 1-cells in the oplax slices. It is well known that the oplax slices are the Grothendieck construction associated to the representable 2-functors. In this chapter we give another description of those oplax categories grounded on this approach and in a more geometric spirit, generalizing the classical notion of *(locally) ringed spaces* of algebraic geometry.

In the first section we give the general properties of the bicategory of (locally) modelled topoi, as well as of the pseudofunctors relating them.

A particular interest is given in section 2 to the calculus of bilimits and bicolimits, see proposition 6.2.1.1 and proposition 6.2.2.1.

However the main result on bilimit is only obtained in section 3 at theorem 6.3.2.3 after some provisional discussion about pseudomonads, proving first at theorem 6.3.1.6 that pseudoalgebras of a pseudomonad inherit bilimits, and at lemma 6.3.1.8 that in the case of a pseudomonad induced from a (non full) bireflective sub-bicategory satisfying a certain condition this can force inheritance of bilimits in the subcategory; it happens that this result applies to the *spectral pseudomonad* induced form the spectral bi-adjunction.

6.1 Bicategories of (locally) modelled topoi

First we discus an alternative way to discribe the bicategories of (locally) modelled topoi, using the fact that the oplax slices are the Grothendieck construction associated to representable 2-functors. Throughout this section, we fix a geometry $(\mathbb{T}, \mathcal{V}, J)$.

6.1.1 $\mathbb{T}$-Modelled topoi

6.1.1.1. We have a pseudofunctor, which is representable by $\mathcal{S}[\mathbb{T}]$

$$\mathbf{GTop}^{\mathrm{op}} \xrightarrow{\mathbb{T}[-]} \mathbf{Cat}$$

and is defined as follows:

- for 0-cells, it returns the category $\mathbb{T}[\mathcal{E}]$ of $\mathbb{T}$-models
- for a 1-cell $f : \mathcal{F} \to \mathcal{E}$, it returns the inverse image functor

$$\mathbb{T}[\mathcal{E}] \xrightarrow{f^*} \mathbb{T}[\mathcal{F}]$$

  which is moreover lex and cocontinuous, being left adjoint to the direct image functor

$$\mathbb{T}[\mathcal{F}] \xrightarrow{f_*} \mathbb{T}[\mathcal{E}]$$

– on a 2-cell $\alpha : f \Rightarrow g$ it returns the natural transformation also denoted as α

$$\mathbb{T}[\mathcal{E}] \;\underset{g^*}{\overset{f^*}{\Downarrow \alpha}}\; \mathbb{T}[\mathcal{F}]$$

This defines a 2-fibration with 1-truncated fibers

$$\int \mathbb{T}[-] \xrightarrow{p_{\mathbb{T}}} \mathbf{GTop}$$

This bicategory can be considered as a category of all models of $\mathbb{T}$ regardless of their base topos. An object in this category is a *modelled topos*, that is, a pair $(\mathcal{E}, E)$ with $\mathcal{E}$ a Grothendieck topos and E in $\mathbb{T}[E]$. However in the following, we choose to work with an *algebraic convention* in the sense that we want morphisms between modelled topoi to have the orientation of the morphisms of models rather than the orientation of the underlying geometric morphism. To this end, we use the following, which is nothing but the *direct fibration* associated to $\mathbb{T}[-]$:

Definition 6.1.1.2. The bicategory $\mathbb{T}$-**GTop** of $\mathbb{T}$-modelled topoi has

– for 0-cells, modelled topoi $(\mathcal{E}, E)$

– for 1-cells, $(f, \phi) : (\mathcal{F}, F) \to (\mathcal{E}, E)$ with $f : \mathcal{E} \to \mathcal{F}$ a geometric morphism and ϕ consisting of a pair $(\phi^\flat, \phi^\sharp)$ with

$$\phi^* F \xrightarrow{f^\flat} E \quad \text{and} \quad F \xrightarrow{\phi^\sharp} f_* E$$

mates along the adjunction $f^* \dashv f_*$

– for 2-cells $\alpha : (f, \phi) \to (g, \psi)$, 2-cell $\alpha : f \to g$ in **GTop**.

From its construction, this bicategory is equipped with a forgetful functor which is both a fibration and an opfibration $p_{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{GTop} \to \mathbf{GTop}^{\mathrm{op}}$.

6.1.1.3. Before turning to local objects, it is worth giving some details about the morphisms in this category, which will be shown enlightening in Section 4.3. The 2-category $\mathbb{T}$-**GTop** inherits from its fibrational and opfibrational properties over $\mathbf{GTop}^{\mathrm{op}}$ two factorization systems

(Vertical, Cartesian) and (Cocartesian, Vertical)

as seen below in the two alternative factorizations of a same morphism

$$\begin{array}{ccc} (\mathcal{F}_1, F_1) & \xrightarrow{(f,(1_{f^*F_1},\eta_{F_1}))} & (\mathcal{F}_2, f^*F_1) \\ {\scriptstyle (1_{F_1},(\phi^\sharp,\phi^\sharp))}\downarrow & \searrow^{(f,\phi)} & \downarrow{\scriptstyle (1_{F_2},(\phi^\flat,\phi^\flat))} \\ (\mathcal{F}_1, f_*F_2) & \xrightarrow[(f,(\epsilon_{F_2},1_{f_*F_2}))]{} & (\mathcal{F}_2, F_2) \end{array}$$

provided by the respective unit-counit triangles for the $f^* \dashv f_*$-adjunction

$$\begin{array}{ccc} & f^*f_*F_2 & \\ {\scriptstyle f^*\phi^\sharp}\nearrow & \downarrow{\scriptstyle \epsilon_{F_2}} & \\ f^*F_1 & \xrightarrow[\phi^\flat]{} & F_2 \end{array} \qquad \begin{array}{ccc} f_*f^*F_1 & & \\ {\scriptstyle \eta_{F_1}}\uparrow & \searrow^{f_*\phi^\flat} & \\ F_1 & \xrightarrow[\phi^\sharp]{} & f_*F_2 \end{array}$$

6.1.2 $\mathbb{T}_J$-locally modelled topoi

In the previous chapter, the category of locally modelled topoi was a non-full sub-bicategory of the oplax slice over the classifier of local objects: in fact this can be obtained as the Grothendieck construction associated to a sub-pseudofunctor of the representable.

6.1.2.1. The local data associated to $(\mathcal{V}, J)$ also define a pseudofunctor

$$\mathbf{GTop}^{\mathrm{op}} \xrightarrow{\mathbb{T}_J[-]^{\mathbf{Loc}}} \mathbf{Cat}$$

and again we can consider the associated direct 2-fibration; but for each topos $\mathcal{E}$, we have a non-full inclusion of category

$$\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}[\mathcal{E}]} \hookrightarrow \mathbb{T}[\mathcal{E}]$$

inducing the following non-full sub-bicategory of $\mathbb{T}$-**GTop** defined:

Definition 6.1.2.2. The bicategory $\mathbb{T}_J$-**GTop**$^{\mathbf{Loc}}$ of *$\mathbb{T}_J$-locally modelled topoi* has

- for 0-cells pairs $(\mathcal{E}, E)$ with E in $\mathbb{T}_J[\mathcal{E}]$,
- for 1-cells pairs $(f, \phi) : (\mathcal{F}, F) \to (\mathcal{E}, E)$ with $f : \mathcal{E} \to \mathcal{F}$ and ϕ is such that $\phi^\flat : f^*F \to E$ is in $\mathbf{Loc}[\mathcal{E}]$,
- and the same 2-cells as $\mathbb{T}$-**GTop**.

In particular this inclusion is a strict morphism of opfibrations

$$\begin{array}{ccc} \mathbb{T}_J\text{-}\mathbf{GTop}^{\mathbf{Loc}} & \xrightarrow{\iota_{J,\mathbf{Loc}}} & \mathbb{T}\text{-}\mathbf{GTop} \\ & \overset{p_{J,\mathbf{Loc}}}{\searrow} \quad = \quad \overset{p_{\mathbb{T}}}{\swarrow} & \\ & \mathbf{GTop}^{\mathrm{op}} & \end{array}$$

so that in the following, we may innocently write $\iota_{J,\mathbf{Loc}}(\mathcal{E}, E)$ as $(\mathcal{E}, wE)$ where we denote abusively w for the faithful (but non-full) inclusion $w_{\mathcal{E}} : \mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}[\mathcal{E}]} \hookrightarrow \mathbb{T}[\mathcal{E}]$: in other words, $\iota_{J,\mathbf{Loc}}$ just forget the localness of models and arrows without modifying the underlying topos. However beware that $p_{J,\mathbf{Loc}}$ does not inherit the fibration structure of $p_{\mathbb{T}}$ as $\mathbb{T}_J$ models are not stable under direct image of geometric morphisms.

Lemma 6.1.2.3. *The inclusion functor $\iota_{J,\mathbf{Loc}}$ creates equivalences.*

Proof. Suppose we have two $\mathbb{T}_J$-locally modelled topoi $(\mathcal{E}_1, E_1)$, $(\mathcal{E}_2, E_2)$ and an equivalence $(f, \phi) : \iota_{J,\mathbf{Loc}}(\mathcal{E}_1, E_1) \simeq \iota_{J,\mathbf{Loc}}(\mathcal{E}_2, E_2)$ in $\mathbb{T}$-**GTop**. Then recall that at each $\mathcal{E}$ the inclusion $w_{\mathcal{E}}$ is conservative and creates isomorphisms as **Loc** contains already all isomorphisms. Hence in our case we have both a geometric equivalence $f : \mathcal{E}_2 \simeq \mathcal{E}_1$ and an isomorphism $\phi^\flat : f^*wF_1 \simeq wF_2$: but this latter isomorphism is in particular local, hence our equivalence already was an equivalence of locally modelled topoi. □

6.1.2.4. To conclude this section, we just have to rephrase the spectral adjunction obtained in the previous chapter at theorem 5.2.1.7, which was stated as a bi-adjunction between bicategories obtained from the oplax slices, but now in terms of locally modelled topoi:

$$\mathbb{T}_J\text{-}\mathbf{GTop}^{\mathbf{Loc}} \underset{\iota_{J,\mathbf{Loc}}}{\overset{\mathbf{Spec}}{\rightleftarrows}} \mathbb{T}\text{-}\mathbf{GTop} \quad (\perp)$$

6.2 Bilimits and bicolimits of modelled topoi

Now we should give a word about bilimits and bicolimits of modelled topoi. The main idea in both cases is that we first compute the bicolimit, resp. the bilimit - as long as it exists - of the corresponding underlying diagram of Grothendieck topoi. Then this provides a truncated diagram of direct, reps. inverse images we can compute the limit, resp. colimit of.

We shall also discuss why there should morally be no problem with computing bilimits of locally modelled topoi as it actually requires to only compute connected limits of local objects and local maps, which we know to exist from proposition 3.3.3.7.

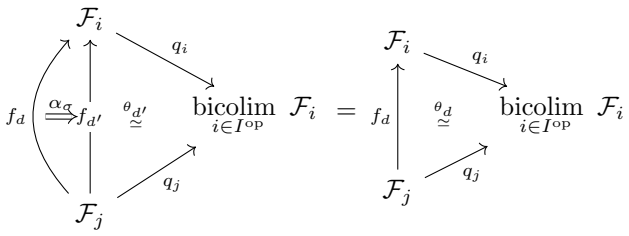
6.2.1 Bilimits of (locally) modelled topoi

Proposition 6.2.1.1. *The bicategory* $\mathbb{T}$-**GTop** *has small bilimits.*

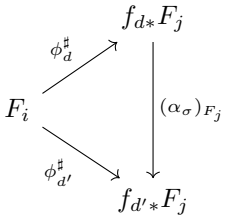
Proof. We split here the case of conical and weighted bilimits. Let us first examine the conical case. Let $\mathbb{F} : I \to \mathbb{T}$-**GTop** be a 2-functor with $\mathbb{F}(i) = (\mathcal{F}_i, F_i)$ and at $d : i \to j$ the transition morphism denoted as (f_d, ϕ_d). Then, since Grothendieck topoi have small bicolimits, one can compute a bilimiting bicocone

$$\begin{array}{ccc} \mathcal{F}_i & \xrightarrow{q_i} & \underset{i \in I^{\mathrm{op}}}{\mathrm{bicolim}}\ \mathcal{F}_i \\ {\scriptstyle f_d}\uparrow & \overset{\theta_d}{\simeq} & \\ \mathcal{F}_j & \xrightarrow{q_j} & \end{array}$$

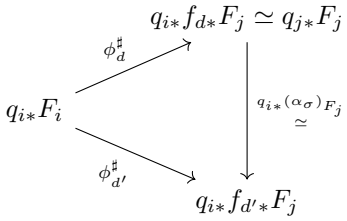
while a 2-cell $\sigma : d \Rightarrow d'$ in I induces an equality of 2-cells



so that in particular each whiskering $q_i * \alpha_\sigma$ has to be a natural isomorphism. Then for any morphism $d : i \to j$ we have a natural isomorphism $(\theta_d)_{F_j} : q_{i*} f_{d*} F_j \simeq q_{j*} F_j$, while for a 2-cell $\sigma : d \Rightarrow d'$, the triangle in $\mathcal{F}_i$



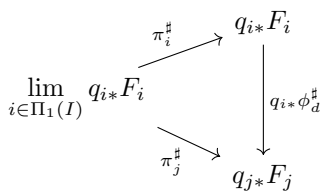
is sent in bicolim $_{i\in I}\mathcal{F}_i$ to a triangle with invertible transition



so that all 2-dimensional data in I are collapsed into a 1-dimensional diagram

$$\Pi_1(I) \xrightarrow{q_{(-)*}F_{(-)}} \operatorname*{bicolim}_{i\in I} \mathcal{F}_i$$

where $\Pi_1(I)$ is the 1-category whose objects are objects of I and for i, j in I, $\Pi_1(I)[i, j]$ is the set of connected components of $I[i, j]$. Hence we get a $\Pi_1(I)$-indexed diagram made of the $q_{i*}F_i$ in bicolim $_{i\in I}\mathcal{F}_i$ and we can compute its limit in $\mathbb{T}$[bicolim $_{i\in I}\mathcal{F}_i$]



Now suppose one has a bicone $((f_i, \phi_i) : (\mathcal{F}, F) \to (\mathcal{F}_i, F_i)_{i\in I}$ over $\mathbb{F}$: then $p_{\mathbb{T}}$ sends it to a bicocone in the underlying bicategory of Grothendieck topoi $(f_i : \mathcal{F}_i \to \mathcal{F})_{i\in I}$ inducing a universal map $\langle f_i\rangle_{i\in I} : \text{bicolim }_{i\in I}\mathcal{F}_i \to \mathcal{F}$. Its direct image part preserves limits, so that

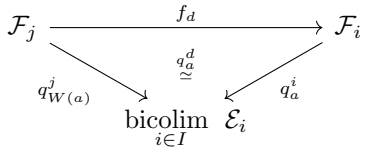
$$\begin{aligned}(\langle f_i\rangle_{i\in I})_*(\lim_{i\in\Pi_1(I)} q_{i*}F_i) &\simeq \lim_{i\in\Pi_1(I)} (\langle f_i\rangle_{i\in I})_* q_{i*}F_i \\ &\simeq \lim_{i\in\Pi_1(I)} f_{i*}F_i\end{aligned}$$

and the cone in $\mathcal{F}$ given by the $\phi_i^\sharp : F \to f_{i_*}F_i$ induces a unique map $(\phi_i^\sharp)_{i\in I} : F \to \lim_{i\in\Pi_1(I)} f_{i*}F_i$, which is the direct image part of a universal morphism $(\mathcal{F}, F) \to (\text{bicolim }_{i\in I}, \lim q_{i*}F_i)$ as desired. Hence pose

$$\operatorname*{bilim}_{i\in I} \mathbb{F} = (\operatorname*{bicolim}_{i\in I^{\mathrm{op}}} \mathcal{F}_i, \lim_{i\in\Pi_1(I)} q_{i_*}F_i)$$

Now we want to generalize this result to weighted bilimits. Let $W : I \to \mathbf{Cat}$ be a weight. Though the weighted bicolimit of Grothendieck topoi bicolim $^W_{i\in I^{\mathrm{op}}}\mathcal{F}_i$ exists without further complications, we must understand how the 2-dimensional data encoded by the arrows in the weights

are collapsed to 1-dimensional data in the diagram of models one gets in the bicolimit topos. For $i \in I$ and a in $W(i)$ we have an inclusion $q^i_a : \mathcal{F}_i \to \text{bicolim}_{i\in I}\mathcal{F}_i$, creating a disinct direct image of F_i; for $t : a \to b$ in $W(i)$ we have a 2-cell $q^i_t : q^i_a \Rightarrow q^i_b$ in **GTop** defining itself a natural transformation between the lift of the inclusion to the categories of models, which manifests at the corresponding F_i as a morphism $(q^i_t)_{F_i} : q^i_{b*}F_i \to q^i_{a*}F_i$. On the other hand for a morphism $d : i \to j$ in I we have an invertible 2-cell



while a 2-cell $\sigma : d \to d'$ must again be inverted by each q^a_i; but we know have also a component $W_\sigma : W(d) \Rightarrow W(d')$ defining at each a of $W(i)$ a morphism $(W_d)_\sigma : W(d)(a) \to W(d')(a)$ in $W(j)$.

The weight defines an indexed category we can consider the associated opfibration of, $\int W \to I$; its oplax colimit is the underlying category of this 2-category. We have a diagram

$$\Pi_1(\int W) \xrightarrow{\mathbb{F}\otimes W} \underset{i\in I^{\text{op}}}{\text{bicolim}}\ \mathcal{F}_i$$

sending

- each (i, a) to the direct image $q^i_{a*}(F_i)$
- each cocartesian morphism $(d, 1_{W(d)(a)}) : (i, a) \to (j, W(d)(a))$ to the morphism $q^i_{a*}(\phi^\sharp_d) : q^i_{a*}(F_i) \to q^i_{a*}f_{d*}(F_j) \simeq q^j_{W(a)*}(F_j)$
- each vertical morphism $(i, t) : (i, a) \to (i, b)$ on the morphism $(q^i_t)_{F_i} : q^i_{b*}F_i \to q^i_{a*}F_i$

Then we claim that the weighted bilimit of the diagram of modelled topoi is obtained as

$$\underset{i\in I}{\text{bilim}}\ ^W \mathbb{F} = (\underset{i\in I^{\text{op}}}{\text{bicolim}}\ \mathcal{F}_i, \lim_{i\in \Pi_1(\int W)} q^i_{a_*}F_i)$$

□

6.2.1.2. Before turning to bicolimits, we would like to prove that actually the 2-category of locally modelled topoi is actually closed under bilimits. It is known from [20][I, 1.1.6] that in the context of algebraic geometry, that is, for the *geometry of commutative rings, local rings and localizations* (see section 9.1.1), *locally ringed spaces* have limits (actually colimits in the spatial convention as it is traditionally stated), but such results are proven by ad hoc considerations on the specific properties of local rings. Here we are going to give some elements about why the general version of this statement is true for any geometry. This could seem surprising at first sight, as the category of local objects and local maps in a given topos has not all limits: but we saw in proposition 3.3.3.7 that they always have connected limits, and this will be sufficient to prove existence of limits of locally modelled topoi indexed by arbitrary diagrams. In fact the discrete and connected aspects of the construction will be supported separately:

- discrete diagrams do not require to compute any limit of local objects because of the nature of bicolimits of Grothendieck topoi. In particular bicoproducts are actually the pseudoproducts of the underlying categories of sheaves. Hence the local object component of a biproduct of locally modelled topoi can be defined as the family of local objects of each member of the biproduct.
- and for connected limits, we shall have a connected diagram of local objects in each underlying topos of the colimit, whose limit exists in the category of local objects and local maps.

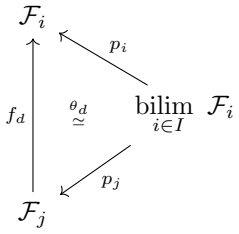
Hence it should be sufficient for a given diagram to calculate separately the limit in each connected component, and construct a family whose members are determined from the connected limit associated to the corresponding connected component. However, in practice, it reveals surprisingly difficult to prove directly that locally modelled topoi inherit bilimits, because of the unmanageable complications that arise from the sequences of inverse-direct images which impeach to construct a diagram of local objects in the colimit topos. We shall propose an abstract proof of this result through a pseudomonadic argument in the next chapter.

6.2.2 Bicolimits of (locally) modelled topoi

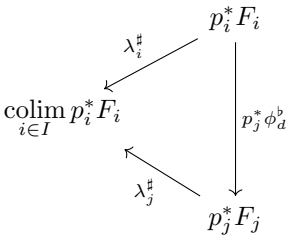
In fact, colimits also exist and are dually computed from inverse images part along an underlying bilimiting cone of topoi: however there are some restrictions there, since only specific kinds of bilimits are ensured to exist in the bicategory of Grothendieck topoi:

Proposition 6.2.2.1. *The bicategory* $\mathbb{T}$-**GTop** *has finite bicolimits over diagrams whose duals have bilimits in* **GTop**. *In particular it has finite bicolimits and filtered bicolimits.*

Proof. Suppose that I is a shape of diagram such that any 2-functor $I \to$ **GTop** has a bilimit. The previous construction dualizes: for $\mathbb{F} : I \to \mathbb{T}$-**GTop**, one can consider the bilimit



with the commutation of the inverses images $(\theta_d)_{F_i} : p_j^* f_d^* F_i \simeq p_i^* F_i$. Again for each 2-cell $\sigma : d \Rightarrow d'$ we have equality of 2-cells $\theta'_d \alpha_\sigma^* p_j^* = \theta_d$, so that $p_j^* f_d^* F_i \simeq p_j^* f_{d'}^* F_i$ and the 2-dimensional information of I collapses. Hence we can compute the colimit of this diagram in bilim $_{i\in I}\mathcal{F}_i$



And again, using this time preservation of colimits by inverse images, we see that we can pose

$$\operatorname*{bicolim}_{i\in I} \mathbb{F} = (\operatorname*{bilim}_{i\in I} \mathcal{F}_i, \operatorname*{colim}_{i\in I} p_i^* F_i)$$

□

Now we could ask which of those bicolimits are inherited by locally modelled topoi: this depends mostly on the kind of colimits under which local objects are closed in their category of ambient objects, which are generally quite restricted unless very particular case. In general, local objects only have finitely filtered colimits, as the category $\mathbb{T}_J[\mathcal{S}]$, being the category of points of a Grothendieck topos, is accessible (of unknown rank of accessibility) with finitely filtered colimits.

Corollary 6.2.2.2. *The bicategory* $\mathbb{T}_J$-$\mathbf{GTop}^{\mathbf{Loc}}$ *is closed under filtered bicolimits in* $\mathbb{T}$-**GTop**.

Proof. First observe that $\iota_{J,\mathbf{Loc}}$ is a morphism of fibrations, as it does not modify the underlying topos: hence if a locally modelled topos has a bilimit - or also a bicolimit - as its underlying topos, so has its image along $\iota_{J,\mathbf{Loc}}$. Moreover, $\mathbb{T}_J[\mathcal{F}]^{\mathbf{Loc}}$ is closed under filtered colimits in $\mathbb{T}[\mathcal{F}]$. Moreover we saw in lemma 6.1.2.3 that $\iota_{J,\mathbf{Loc}}$ is conservative, hence reflects colimits that it preserves. □

Remark 6.2.2.3. we shall see later that more can be said concerning limits in the category of locally modelled topoi: however those results will rely on the spectral adjunction, which is the topic of the next section, where we construct the spectrum as the left adjoint of this inclusion $\iota_{J,\mathbf{Loc}}$.

6.3 The spectral pseudomonad and pseudolimits of locally modelled topoi

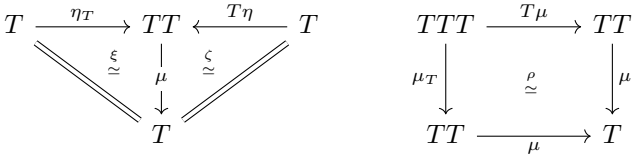
In the last section we had some insight why the bicategory of locally modelled topoi should inherit bilimits of modelled topoi, even though local objects *in a fixed* topos have only connected limits. However it seemed difficult to prove it concretely. In this section, we do a pseudomonadic

detour to ensure this in an abstract method. Generalizing the well known result that the 2-category of algebras and pseudomorphisms of a 2-monad inherits pseudolimits, we prove that the bicategory of pseudoalgebras of a pseudomonad inherit bilimit. Then we consider a special situation of a pseudomonad induced from a (non-full) bireflection where the forgetful functor factorizes through the right bi-adjoint: we prove that this forces the underlying bicategory to be closed under bilimits. Observing that this is the case of the spectral adjunction, we prove the bicategory of locally modelled topoi is closed under bilimits.

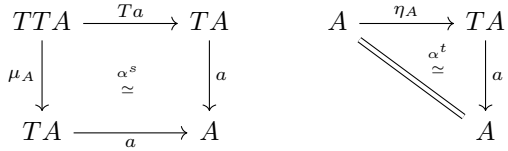
6.3.1 Generalities on pseudomonads and their algebras

Here we give prerequisites on pseudomonads and their different flavors of algebras and morphisms.

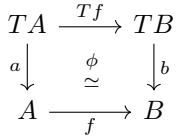
Definition 6.3.1.1. A *pseudomonad* on a 2-category $\mathcal{K}$ is a pseudofunctor $T : \mathcal{K} \to \mathcal{K}$ equipped two pseudonatural transformations *unit* $\eta : 1 \Rightarrow T$ and a *multiplication* $\mu : TT \Rightarrow T$ together with canonical invertible 2-cells (ξ, ζ) and ρ such that for any object A in $\mathcal{K}$ the following diagrams are respectively a strict equalizer and commutative up to identity:



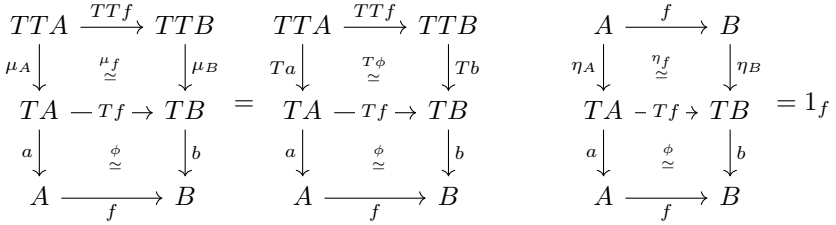
Definition 6.3.1.2. A *pseudo-algebra* of a pseudomonad T is a pair $(A, a, (\alpha^t, \alpha^s))$ with A an object of $\mathcal{K}$ and $a : A \to TA$ a 1-cell in $\mathcal{K}$ and (α^t, α^s) is a pair to 2-cells as below



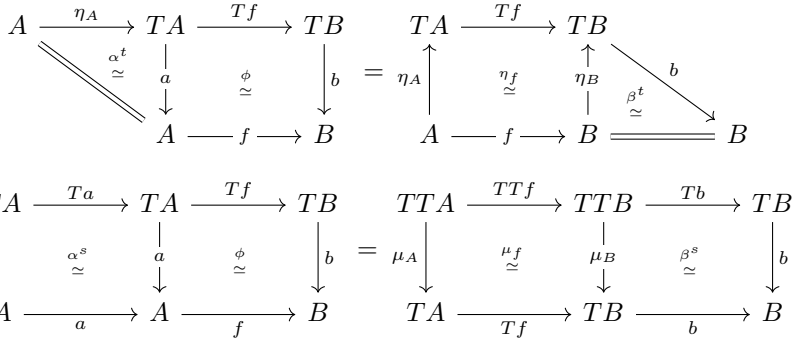
Definition 6.3.1.3. Let T be a pseudomonad and $(A, a, (\alpha^t, \alpha^s))$, $(B, b, (\beta^t, \beta^s))$ two pseudoalgebras: then a *pseudomorphism* $(A, a, (\alpha^t, \alpha^s)) \to (B, b, (\beta^t, \beta^s))$ is a pair (f, ϕ) with an arrow $f : A \to B$ in $\mathcal{K}$ and ϕ an invertible 2-cell as below



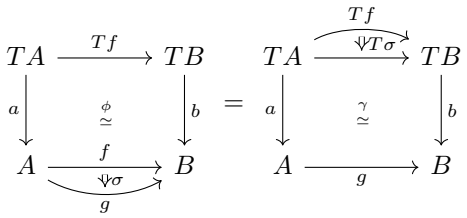
satisfying moreover the following compatibility conditions:



and also compatibility conditions for the triangle parts and square parts



Definition 6.3.1.4. Let $(f,\phi),(g,\gamma) : (A,a,(\alpha^t,\alpha^s)) \rightrightarrows (B,b,(\beta^t,\beta^s))$ be two pseudomorphisms of pseudo-algebras with same domain and codomain; then a transformation between them is a 2-cell $\alpha : f \to g$ in $\mathcal{K}$ such that

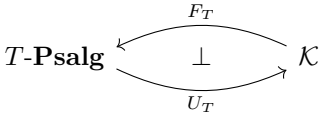


This defines three distinct 2-categories of strict T-algebras T-**Psalg**, whose 0-cells are pseudo-algebras, 1-cells are pseudomorphisms of T-pseudoa-algebras, and 2-cells are transformations between them.

6.3.1.5. For a pseudomonad T, we have then a forgetful functor

$$T\text{-}\mathbf{Psalg} \xrightarrow{U_T} \mathcal{K}$$

sending an algebra $(A,a,(\alpha^t,\alpha^s))$ on the underlying A and (f,ϕ) on f. This functor is right pseudo-adjoint to the associated free functor sending A to the pseudo-algebra $(TA,\mu_A,(\xi_A,\rho_A))$ and f to (f,μ_f) with μ_f the naturality square of μ at f: it is standard calculation so see that we have a pseudo-adjunction



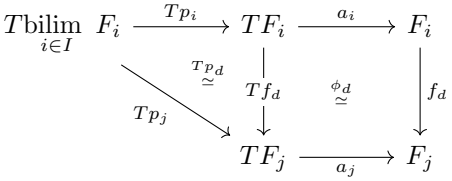
Now let us give a pseudo version of [7][Theorem 2.6], which can also be seen a specific case of the main result of [92] from which we inspire us.

Theorem 6.3.1.6. *Let T be a pseudomonad on a 2-category $\mathcal{K}$. Then the forgetful functor U_T creates bilimits.*

Proof. Let $\mathbb{F} : I \to T$-**Psalg** be a 2-functor, and for each i denote $\mathbb{F}(i) = (F_i,a_i,(\alpha_i^t,\alpha_i^s))$, for $d : i \to j$ denote $\mathbb{F}(d) = (f_d,\phi_d)$ and for $\sigma : d \Rightarrow d'$ denote $\mathbb{F}(\sigma) = \theta_\sigma$. Then if $\mathcal{K}$ has bilimits we have a bilimiting cone $((p_i : \mathrm{bilim}_{\,i\in I} F_i \to F_i)_{i\in I},(p_d : f_d p_i \simeq p_j)_{d\in I^2})$. Consider then the pseudocone consisting for each i in I of the composite

$$T\operatorname*{bilim}_{i\in I} F_i \xrightarrow{Tp_i} TF_i \xrightarrow{a_i} F_i$$

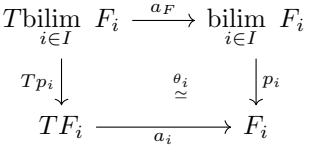
and at each $d : i \to j$ of the 2-cell



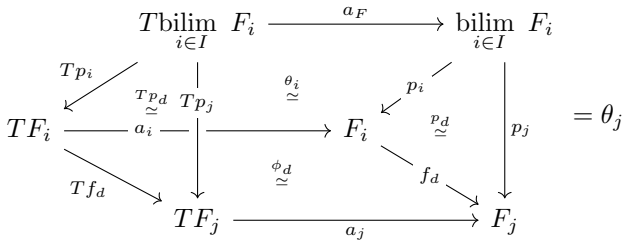
Then by the universal property of the bilimit, we are given a universal arrow

$$T\operatorname*{bilim}_{i\in I} F_i \xrightarrow{a_F} \operatorname*{bilim}_{i\in I} F_i$$

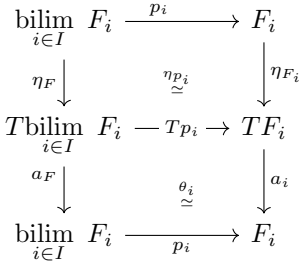
together with an invertible 2-cell at each i



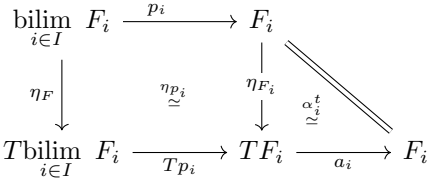
satisfying moreover at each $d : i \to j$ the equality



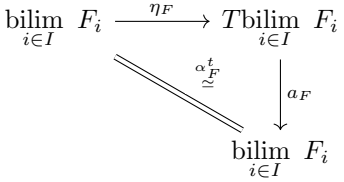
We must equip a_F with the triangle and square 2-cells to make it part of a structure of pseudoalgebra. Whiskering the aforementioned pseudocone with the unit η_F gives a pseudocone $((a_i Tp_i \eta_F)_{i\in I}, (\phi_d * Tp_i\, a_j * Tp_d) * \eta_F)_{d\in I^2})$ and the pasting below



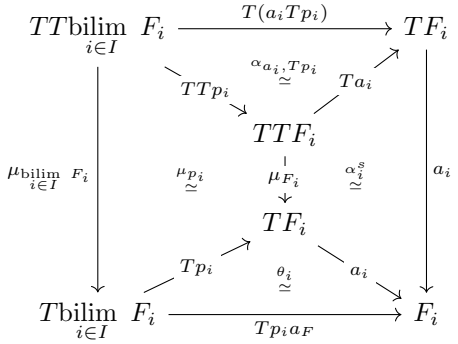
exhibits $a_F \eta_F$ as the universal map induced from this pseudocone. But also, at each i, the triangle 2-cell of the pseudoalgebra structure at i gives us an invertible 2-cell



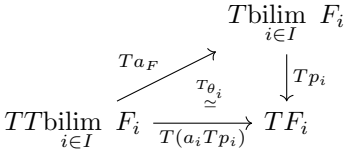
so that $a_i Tp_i \eta_F \simeq p_i$, but the pseudocone constituted of the p_i is the unique one up to equivalence to induce the equality bilim $_{i\in I} F_i =$ bilim $_{i\in I} F_i$: hence there exists a unique invertible 2-cell as desired



For the square part, consider the invertible 2-cell obtained as the pasting



where α_{a_i,Tp_i} is the canonical invertible 2-cell of the pseudofunctoriality of T at the composite $a_i Tp_i$. Then it suffices to paste it along $T(a_i Tp_i)$ with the invertible 2-cell



This returns at each i an invertible 2-cell

$$\begin{array}{ccc} TT\underset{i\in I}{\text{bilim}}\ F_i & \xrightarrow{Ta_F} & T\underset{i\in I}{\text{bilim}}\ F_i \\ {\scriptstyle \mu_{\underset{i\in I}{\text{bilim}}\ F_i}}\downarrow & \overset{\omega_i}{\simeq} & \downarrow{\scriptstyle a_i T p_i} \\ T\underset{i\in I}{\text{bilim}}\ F_i & \xrightarrow[p_i a_F]{} & F_i \end{array}$$

Then the maps $(p_i a_F \mu_{\text{bilim}\ _{i\in I}F_i} : TT\text{bilim}\ _{i\in I}F_i \to F_i)_{i\in I}$ and $(a_i T p_i T a_F : TT\text{bilim}\ _{i\in I}F_i \to F_i)_{i\in I}$ define two equivalent pseudocones, and from the universal property of the bilimit, this equivalence of pseudocones induces the desired invertible 2-cell as below

$$\begin{array}{ccc} TT\underset{i\in I}{\text{bilim}}\ F_i & \xrightarrow{Ta_F} & T\underset{i\in I}{\text{bilim}}\ F_i \\ {\scriptstyle \mu_{\underset{i\in I}{\text{bilim}}\ F_i}}\downarrow & \overset{\alpha^s_F}{\simeq} & \downarrow{\scriptstyle a_F} \\ T\underset{i\in I}{\text{bilim}}\ F_i & \xrightarrow[a_F]{} & \underset{i\in I}{\text{bilim}}\ F_i \end{array}$$

Hence we constructed a structure of pseudoalgebra $(\text{bilim}\ _{i\in I}F_i, a_F, (\alpha^t_F, \alpha^s_F))$ as desired, as well as morphisms of algebras (p_i, θ_i) which are ensured of satisfying the compatibility conditions from the very construction of α^t_F and α^s_F.

Now we must check that this pseudo-algebra has the property of the bilimit in T-**Psalg**. Take a pseudocone

$$\begin{array}{ccc} & (A, a, (\alpha^t, \alpha^s)) & \\ {\scriptstyle (g_i,\psi_i)}\swarrow & \overset{g_d}{\simeq} & \searrow{\scriptstyle (g_j;\psi_j)} \\ (F_i, a_i, (\alpha^t_i, \alpha^s_i)) & \xrightarrow[(f_d,\phi_d)]{} & (F_j, a_j, (\alpha^t_j, \alpha^s_j)) \end{array}$$

Then the pseudocone in $\mathcal{K}$ consisting of the $((g_i : A \to F_i)_{i\in I}, (g_d)_{d\in I^2})$ defines a universal map $g : A \to \text{bilim}\ _{i\in I}$ together with a canonical invertible 2-cell at each i

$$\begin{array}{ccc} A & \xrightarrow{g} & \underset{i\in I}{\text{bilim}}\ F_i \\ & {\scriptstyle g_i}\searrow \quad \overset{\tau_i}{\simeq} & \downarrow{\scriptstyle p_i} \\ & & F_i \end{array}$$

On the other hand we get at each i an invertible 2-cell as below by pasting the canonical invertible 2-cell $\alpha_{p_i,g}$ given by pseudofunctoriality of T with the pseudomorphism 2-cell ψ_i

$$\begin{array}{ccccc} TA & \xrightarrow{Tg} & T\underset{i\in I}{\text{bilim}}\ F_i & \xrightarrow{Tp_i} & TF_i \\ {\scriptstyle a}\downarrow & & \overset{\psi_i\ a_i * \alpha_{p_i,g}}{\simeq} & & \downarrow{\scriptstyle a_i} \\ A & & \xrightarrow[g_i]{} & & F_i \end{array}$$

Hence by the uniqueness up to invertible 2-cell condition of the universal property of the bilimit we have an invertible 2-cell

$$\begin{array}{ccc} TA & \xrightarrow{Tg} & T\underset{i\in I}{\text{bilim}}\ F_i \\ {\scriptstyle a}\downarrow & \overset{\psi}{\simeq} & \downarrow{\scriptstyle a_F} \\ A & \xrightarrow[g_i]{} & \underset{i\in I}{\text{bilim}}\ F_i \end{array}$$

which equips g with the structure of pseudomorphism and provides a factorization

$$\begin{array}{ccc} (A, a, (\alpha^t, \alpha^s)) & \xrightarrow{(g,\psi)} & (\underset{i\in I}{\text{bilim}}\ F_i, a_F, (\alpha^t_F, \alpha^s_F)) \\ & {\scriptstyle (g_i,\psi_i)}\searrow \quad \overset{\tau_i}{\simeq} & \downarrow{\scriptstyle (p_i,\theta_i)} \\ & & (F_i, a_i, (\alpha^t_i, \alpha^s_i)) \end{array}$$

□

6.3.1.7. Now we are going to discuss creation of bilimits in the case of a pseudomonad induced from a pseudo-adjunction

$$\mathcal{M} \underset{R}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathcal{K}$$

with the unit and counit equipped with canonical 2-cells

$$\begin{array}{ccc} R & \xrightarrow{\eta_R} & RLR \\ & \overset{\rho}{\simeq} & \downarrow{\scriptstyle R(\epsilon)} \\ & & R \end{array} \qquad \begin{array}{ccc} L & & \\ {\scriptstyle \epsilon_L}\downarrow & \overset{\lambda}{\simeq} & \\ LRL & \xrightarrow[L(\eta)]{} & L \end{array}$$

Then the composite $RL : \mathcal{K} \to \mathcal{K}$ defines a pseudomonad, with η as unit and the multiplication defined as $\mu = R\epsilon_L$. Moreover, any object M in $\mathcal{M}$ is equipped in $\mathcal{K}$ with a canonical structure of RL-pseudo-algebra given by $(R(M), R(\epsilon_M), (\rho_M, R(\epsilon_{\epsilon_M})))$ where ϵ_{ϵ_M} is the naturality square of the unit at its own component at M

$$\begin{array}{ccc} LRLRM & \xrightarrow{LR(\epsilon_M)} & LRM \\ {\scriptstyle \epsilon_{LRM}}\downarrow & \overset{\epsilon_{\epsilon_M}}{\simeq} & \downarrow{\scriptstyle \epsilon_M} \\ LRM & \xrightarrow[\epsilon_M]{} & M \end{array}$$

This defines a *comparison pseudofunctor*

$$\mathcal{M} \xrightarrow{\iota} RL\text{-}\mathbf{Psalg}$$

Moreover from [62][Proposition 1.7] we know that the comparison functor is equipped with pseudo-natural transformations factorizing both the left and right adjoints as below

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\iota} & RL\text{-}\mathbf{Psalg} \\ {\scriptstyle L}\uparrow & \simeq & \nearrow{\scriptstyle F_{RL}} \\ \mathcal{K} & & \end{array} \qquad \begin{array}{ccc} \mathcal{M} & \xrightarrow{\iota} & RL\text{-}\mathbf{Psalg} \\ {\scriptstyle R}\downarrow & \simeq & \swarrow{\scriptstyle U_{RL}} \\ \mathcal{K} & & \end{array}$$

In the following we are going to describe a situation where the induced forgetful functor U_{RL} factorizes itself through the underlying right adjoint R in the converse sense.

Lemma 6.3.1.8. *Suppose that we have an 2-adjunction $L \dashv R$ such that $R : \mathcal{M} \hookrightarrow \mathcal{K}$ is $(1,2)$ faithfull and 2-full and faithful with $\mathcal{K}$ having bilimits. Suppose moreover that one has an adjoint retract to the comparison functor*

$$\begin{array}{ccc} & & RL\text{-}\boldsymbol{Psalg} \\ & \overset{\iota}{\nearrow} & \overset{\alpha}{\simeq} \downarrow{\scriptstyle S} \\ \mathcal{M} & = & \mathcal{M} \end{array}$$

such that the forgetfull functor factorizes as

$$\begin{array}{ccc} RL\text{-}\boldsymbol{Psalg} & & \\ {\scriptstyle S}\downarrow & \searrow{\scriptstyle U_{RL}} & \\ \mathcal{M} & \overset{\beta}{\simeq} \xrightarrow[R]{} & \mathcal{K} \end{array}$$

Then R creates bilimits, and exhibits $\mathcal{M}$ as closed in $\mathcal{K}$ under bilimits.

Proof. Let $F : I \to \mathcal{M}$ be a 2-functor. Then $RF : I \to \mathcal{K}$ has a bilimit in $\mathcal{K}$, as well as $\iota F : I \to RL\text{-}\mathbf{Psalg}$ and one has

$$\begin{aligned} \text{bilim } RF &\simeq \text{bilim } U_{RL}\iota F \\ &\simeq U_{RL} \,\text{bilim } \iota F \\ &\simeq RS \,\text{bilim } \iota F \end{aligned}$$

As a (1,2)-faithful functor, R reflects bilimits, exhibiting $S\text{bilim } F$ as a bilimit of F in $\mathcal{M}$. □

Though this situation may seem rather specific, we are going to see that it exactly concerns the spectral adjunctions.

6.3.2 Algebras of the spectral monad

Recall that we have for a given geometry $(\mathbb{T}, \mathcal{V}, J)$ two pseudo-adjunctions constructed in theorem 5.1.2.8 and theorem 5.2.1.7, namely the spectral adjunction associated to the underlying factorization geometry $(\mathbb{T}, J_{\mathrm{Triv}}, \mathcal{V})$

$$\mathbb{T}\text{-}\mathbf{GTop}^{\mathbf{Loc}} \underset{\iota_{\mathbf{Loc}}}{\overset{\mathcal{S}[\text{-},\mathbf{Et}]}{\perp}} \mathbb{T}\text{-}\mathbf{GTop}$$

from the bicategory of $\mathbb{T}$-modelled topos with local morphisms between them (which was called $\mathbf{GTop}//\mathcal{S}[\mathbb{T}]^{\mathbf{Loc}}$ in our previous notations) sending a $\mathbb{T}$-modelled topos on its classifier of etale maps, which defines a pseudomonad $T_{\mathbf{Loc}} = \iota_{\mathbf{Loc}}\mathcal{S}[-, \mathbf{Et}]$ we call the *local map pseudomonad*.

We have also the spectral adjunction

$$\mathbb{T}_J\text{-}\mathbf{GTop}^{\mathbf{Loc}} \underset{\iota_{J,\mathbf{Loc}}}{\overset{\mathbf{Spec}}{\perp}} \mathbb{T}\text{-}\mathbf{GTop}$$

from the bicategory of $\mathbb{T}_J$-locally modelled topoi (that was $\mathbf{GTop}//\mathcal{S}[\mathbb{T}_J]^{\mathbf{Loc}}$ in our previous notations) sending a modelled topos on its spectrum seen as the classifier of local forms. We call the associated pseudomonad $T_{J,\mathbf{Loc}} = \iota_{J,\mathbf{Loc}}\mathbf{Spec}$ the *spectral monad*.

Proposition 6.3.2.1. *The forgetful pseudofunctor $T_{J,\mathbf{Loc}}$-**Psalg** $\to \mathbb{T}$-**GTop** factorizes through the inclusion $\iota_{J,\mathbf{Loc}}$ as a pseudofunctor*

$$T_{J,\mathbf{Loc}}\text{-}\boldsymbol{Psalg} \longrightarrow \mathbb{T}_J\text{-}\mathbf{GTop}^{\mathbf{Loc}}$$

Proof. An algebra structure for a modelled topos will be the same thing as a retraction of its unit:

$$\begin{array}{ccc} (\mathcal{F}, F) & \xrightarrow{(h_F, \eta_F)} & (\mathbf{Spec}(F), w\widetilde{F}) \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle (a,\alpha)} \\ & & (\mathcal{F}, F) \end{array}$$

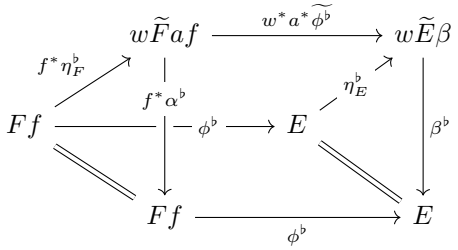
which consists of a section of the underlying geometric morphism in the unit and a morphism exhibiting F as a retract of a local object in $\mathcal{F}$:

$$\begin{array}{ccc} \mathcal{F} & \xleftarrow{h_F} & \mathbf{Spec}(F) \\ {\scriptstyle F}\downarrow & \swarrow{\scriptstyle \eta_F^\flat} \quad \nearrow{\scriptstyle w\widetilde{F}} & \uparrow{\scriptstyle a} \\ \mathbb{T} & \xleftarrow[F]{\quad\alpha^\flat\quad} & \mathcal{F} \end{array} \qquad \begin{array}{ccc} a^*h_F^*F \simeq F & \xrightarrow{a^*h_F^\flat} & a^*w\widetilde{F} \\ & \searrow\!\!\searrow & \downarrow{\scriptstyle \alpha^\flat} \\ & & F \end{array}$$

But we know that this forces F to be itself a local object in $\mathcal{F}$. Similarly for maps, for a map of algebra, that is a (f, ϕ) intertwining those retractions:

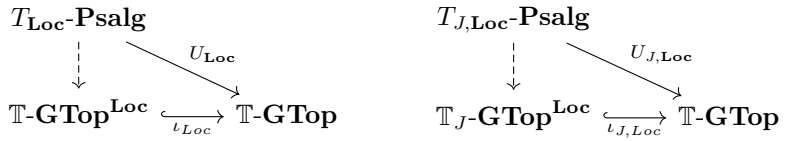
$$\begin{array}{ccccc} & & (\mathbf{Spec}F, w\widetilde{F}) & \xrightarrow{(\mathbf{Spec}(\phi), \widetilde{\phi})} & (\mathbf{Spec}(E), w\widetilde{E}) \\ & {\scriptstyle (h_F, \eta_F^\flat)}\nearrow & \downarrow{\scriptstyle (a,\alpha)} & & {\scriptstyle (h_E, \eta_E^\flat)}\nearrow \quad \downarrow{\scriptstyle (b,\beta)} \\ (\mathcal{F}, F) & \longrightarrow & \quad\downarrow & {\scriptstyle (f,\phi)}\longrightarrow & (\mathcal{E}, E) \\ & \searrow\!\!\searrow & \downarrow & & \searrow\!\!\searrow \\ & & (\mathcal{F}, F) & \xrightarrow[(f,\phi)]{} & (\mathcal{E}, E) \end{array}$$

which gives at the level of $\mathbb{T}$-models in $\mathcal{E}$:



This exhibits $\phi^\flat$ as a retract of $w^*\alpha^*\widetilde{\phi^\flat}$ which is a local arrow: hence from proposition 3.3.3.9, $\phi^\flat$ has to be local. □

6.3.2.2. Then we have (strict) factorizations of the forgetful functors



This gives us alternative proofs of the following theorem:

Theorem 6.3.2.3. *Let $(\mathbb{T}, \mathcal{V}, J)$ be a geometry. Then both $\mathbb{T}$-**GTop**$^{\mathbf{Loc}}$ and $\mathbb{T}_J$-**GTop**$^{\mathbf{Loc}}$ are closed in $\mathbb{T}$-**GTop** under bilimits.*

6.4 Functoriality of the construction

6.4.0.1. Let us say a word about functoriality. For $\Phi : (\mathbb{T}_1, \mathcal{V}_1, J_1) \to (\mathbb{T}_2, \mathcal{V}_2, J_2)$ a transformation of geometries, recall that we had in each $\mathcal{E}$ a restriction of Φ to local objects and local maps between them. Then to any $\mathbb{T}_2$-modelled topos $(\mathcal{E}, E)$ we have a $\mathbb{T}_1$-modelled topos with the same underlying space $(\mathcal{E}, \Phi[\mathcal{E}]_*(E))$ with $\Phi[\mathcal{E}]_*(E) = E\Phi$. Whenever $(\mathcal{E}, E)$ is $\mathbb{T}_{J_2}$-locally modelled, then $(\mathcal{E}, \Phi[\mathcal{E}]_*(E))$ is $\mathbb{T}_{J_1}$-locally modelled since $\Phi[\mathcal{E}]_*$ restricts to local objects.

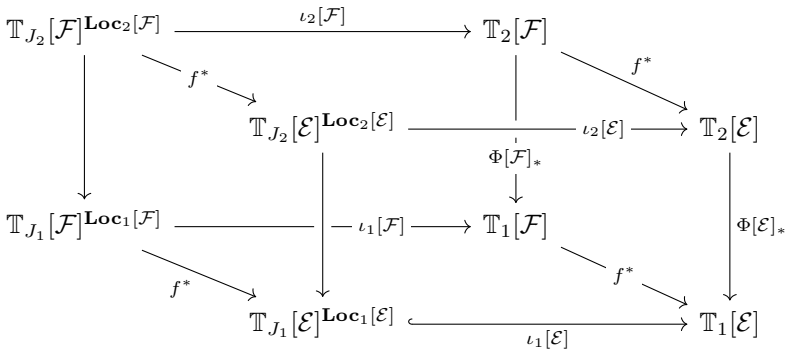
Let us examine what happens at the level of morphisms; 2-naturality of $\Phi[-]_*$ in $\mathcal{E}$ gives at each geometric morphism $f : \mathcal{E} \to \mathcal{F}$ a strictly commutative square

$$\begin{array}{ccc} \mathbb{T}_2[\mathcal{F}] & \xrightarrow{f^*} & \mathbb{T}_2[\mathcal{E}] \\ {\scriptstyle \Phi[\mathcal{F}]_*}\downarrow & & \downarrow{\scriptstyle \Phi[\mathcal{F}]_*} \\ \mathbb{T}_1[\mathcal{F}] & \xrightarrow[f^*]{} & \mathbb{T}_1[\mathcal{E}] \end{array}$$

Indeed for a $\mathbb{T}_2$-model F in $\mathcal{F}$, the equality above at F is obtained as the associativity $f^*(F\Phi) = (f^*F)\Phi$ of the following composite

$$\mathcal{C}_{\mathbb{T}_1} \xrightarrow{\Phi} \mathcal{C}_{\mathbb{T}_2} \xrightarrow{F} \mathcal{F} \xrightarrow{f^*} \mathcal{E}$$

Moreover, whenever F is a J_2-local object in $\mathcal{F}$, then f^*F is J_2-local in $\mathcal{E}$, and hence the composite $f^*F\Phi$ is J_1-local in $\mathcal{E}$, so the square above restricts to local objects. Finally we also have preservations of local maps by f^* as f^* preserves pullbacks, being lex. Those observations are encoded in the pseudocommutativity of the cube below



Hence, for $(f, \phi) : (\mathcal{F}, F) \to (\mathcal{E}, E)$ a morphism in $\mathbb{T}_2$-**GTop**, the whiskering

$$\begin{array}{ccccc} \mathcal{C}_{\mathbb{T}_1} & \xrightarrow{\Phi} & \mathcal{C}_{\mathbb{T}_2} & \xrightarrow{F^*} & \mathcal{F} \\ & & & \overset{\phi^\flat}{\swarrow\!\!\swarrow} & \downarrow{\scriptstyle f^*} \\ & & & {\scriptstyle E^*}\searrow & \mathcal{E} \end{array}$$

is the induced morphism $\Phi[\mathcal{E}]_*(\phi^\flat)$ in $\mathbb{T}_1[\mathcal{E}]$ (while $\Phi[\mathcal{E}]_*(\phi^\sharp)$ is in $\mathbb{T}_1[\mathcal{F}]$), and whenever $\phi^\flat$ is in $\mathbf{Loc}_2[\mathcal{E}]$, then we know $\Phi[\mathcal{E}]_*(\phi^\flat)$ to be in $\mathbf{Loc}_1[\mathcal{E}]$. Similarly, the action of Φ on 2-cells does not raise any problem, so we see this process is pseudofunctorial. To sum up, this defines a pseudocommutative square

$$\begin{array}{ccc} \mathbb{T}^{\mathbf{Loc}_2}_{J_2}\text{-}\mathbf{GTop} & \xhookrightarrow{\iota_{J_2,\mathcal{V}_2}} & \mathbb{T}_2\text{-}\mathbf{GTop} \\ {\scriptstyle \int \Phi}\downarrow & & \downarrow{\scriptstyle \int \Phi} \\ \mathbb{T}^{\mathbf{Loc}_1}_{J_1}\text{-}\mathbf{GTop} & \xhookrightarrow[\iota_{J_1,\mathcal{V}_1}]{} & \mathbb{T}_1\text{-}\mathbf{GTop} \end{array}$$

where moreover both of the following triangles are morphisms of fibrations

$$\begin{array}{ccc} \mathbb{T}_2\text{-}\mathbf{GTop} & \xrightarrow{\int \Phi} & \mathbb{T}_1\text{-}\mathbf{GTop} \\ & {\scriptstyle p_{\mathbb{T}_2}}\searrow \quad \swarrow{\scriptstyle p_{\mathbb{T}_1}} & \\ & \mathbf{GTop}^{\mathrm{op}} & \end{array} \qquad \begin{array}{ccc} \mathbb{T}^{\mathbf{Loc}_2}_{J_2}\text{-}\mathbf{GTop} & \xrightarrow{\int \Phi} & \mathbb{T}^{\mathbf{Loc}_1}_{J_1}\text{-}\mathbf{GTop} \\ & {\scriptstyle p_{J_2,\mathbf{Loc}_2}}\searrow \quad \swarrow{\scriptstyle p_{J_1,\mathbf{Loc}_1}} & \\ & \mathbf{GTop}^{\mathrm{op}} & \end{array}$$

Chapter 7

The spectral site

This chapter, which will be the longest and densest of this thesis, provides an in-depth description of the geometric properties of the spectrum associated with a geometry. In chapter 5, we detailed two very abstract methods to construct the spectrum through 2-dimensional universal properties, focusing on its classifying purpose, without actually any insight on the spatial aspects of the spectrum. In this chapter, we present the site-theoretic approach of the construction of the spectrum. Though we start from the recipe given in [19] when defining the spectral site, most of this chapter is done in a new manner, developing many aspects that were left mostly implicit or unnoticed in our sources.

First we give the spectral site of set-valued models, which gathers the finitely presented etale maps under it, and proves it possesses the desired universal property of the spectrum: in particular it induces a restricted spectral adjunction theorem 7.1.6.2; then we provide a pseudolimit decomposition theorem 7.1.2.7 for the spectrum of arbitrary etale maps, and give some geometric properties of the spectra, proving in particular that the spectrum turns etale maps into etale geometric morphisms (proposition 7.1.2.2) and local objects into local topoi (proposition 7.1.3.6). We also give a carefull examination of the universal property of the structural sheaf and its associated *generic etale map* and *generic local form*, from which we deduce a restricted spectral adjunction for set-valued models.

In the second section, we recall the notion of fibered site and fibered topos as introduced in [86], and introduce a variation of this notion to take account of a topology on the basis, using the notion of *comorphism of site*. From such a notion of fibered site, we construct certain site equipped with a composite (pre)topology made of *horizontal and vertical covers*, and exhibit the associated sheaf topos as the topos of continuous sections of the associated *direct fibration* at theorem 7.2.3.7, generalizing Grothendieck result about the limit topos as a topos of cartesian sections.

In section 3, we provide a detailed account of the spectral site of an arbitrary modelled topos. It is was already guessed in [19] how to construct it, in a quite *ad hoc* way: here we inscribe this construction in the more general pattern of fibered sites and fibered topoi. Combine the two previous sections, we prove that the spectrum of an arbitrary modelled topos is a topos of continuous sections of a fibered topos, see theorem 7.3.3.2. We also use fibrational techniques to prove the structural sheaf of the spectrum to be a local object. We also describe the geometric and sheaf data associated to the canonical fibration, which will be proven to be the unit of the spectral adjunction.

Section 4 is devoted to proving the pseudofunctoriality of the spectrum: though this aspect was somewhat swept under the rug in most sources, this problem is actually highly non trivial and requires a large amount of work, involving some techniques of extensions of models along the equivalence between a sheaf topos and its own category of sheaves for its canonical topology, and splitting the problem between the *horizontal and vertical* morphisms of modelled topoi as they were distinguished at 6.1.1.3.

Section 5 is devoted to proving the spectral adjunction in the general case, for arbitrary modelled topoi, combining the two previous sections. The main lemma is to prove the existence of a retraction of the unit for locally modelled topoi (see proposition 7.5.1.2), which generalizes also the localness of the spectrum of a locally modelled topos, but this time *over its base topos* - see corollary 7.5.1.4.

Finally, section 6 describes the functoriality of this method relative to transformations of geometry, and how we get a *comparison* functor between the spectra associated with two different geometries related by a transformation.

7.1 Spectral site of a set-valued model

In the previous chapters, the spectrum was constructed in an abstract manner through 2-dimensional universal properties. Now we would like to give a concrete site presentation of the spectrum. Following essentially [19], this chapter is devoted to the construction of the spectral site for a set-valued model.

Throughout this chapter we fix a geometry $(\mathbb{T}, \mathcal{V}, J)$ and denote as **Et** and **Loc** the associated factorization system in $\mathbb{T}[\mathcal{S}]$.

7.1.1 Spectral topology on the etale generator

Recall that for any set-valued $\mathbb{T}$-model B we denote as $\mathcal{V}_B$ the etale generator of B, which consists of arrows obtained as pushouts under B of finitely presented etale maps in $\mathcal{V}$.

Definition 7.1.1.1. For any B in $\mathbb{T}[\mathcal{S}]$, the opposite category of the etale generator $\mathcal{V}_B^{\mathrm{op}}$ can be equipped with the *spectral pretopology*, which is the Grothendieck pretopology J_B defined from the duals of the families

$$\left(\begin{array}{ccc} B & \xrightarrow{n} & C \\ & \searrow^{n_i} & \downarrow{\scriptstyle m_i} \\ & & C_i \end{array} \right)_{i \in I}$$

such that there exists some arrow $b : K \to C$ and a covering family $(l_i : K \to K_i)_{i \in I}$ dual of a covering family in J such that for each $i \in I$ we have

$$\begin{array}{ccccc} K & \xrightarrow{b} & C & \xleftarrow{n} & B \\ {\scriptstyle l_i}\downarrow & & \downarrow{\scriptstyle m_i} & \swarrow_{n_i} & \\ K_i & \longrightarrow & C_i & & \end{array}$$

Remark 7.1.1.2. In particular, in the case of a finitely presented object K, we know $\mathcal{V}_K$ to be made of finitely presented etale arrows in $\mathcal{V}$, that is, $\mathcal{V}_K \hookrightarrow \mathcal{V}$ for $\mathcal{V}$ is closed under pushouts between finitely presented objects. Then, in particular, any map n in $\mathcal{V}_K$ has a finitely presented codomain. But now, a covering family in $J_K(n)$ is induced by pushout from some $(n_i : K_0 \to K_i')_{i \in I}$ corresponding to a covering family in J as follows

$$\begin{array}{ccccc} K & \xrightarrow{n} & K' & \xleftarrow{k} & K_0 \\ & \searrow^{n_i} & \downarrow{\scriptstyle m_i} & & \downarrow{\scriptstyle l_i} \\ & & K_i & \longleftarrow & K_i' \end{array}$$

But J, as a Grothendieck coverage, is closed under pullback, and hence the pushout family $(m_i : K' \to K_i)_{i \in I}$ is dual to a covering family in $J(K')$. Conversely, any family $(m_i : K' \to K_i)_{i \in I}$ dual to a cover of K' in J induces trivially a covering family of l in J_K as it is a pushout of itsefl along the identity $1_{K'}$. Hence a familly $(m_i : n \to n_i)_{i \in I}$ is covering in J_K if and only if $(m_i : \mathrm{cod}(n) \to \mathrm{cod}(n_i))_{i \in I}$ is dual to a covering family in $J(\mathrm{cod}(n))$.

Definition 7.1.1.3. For a set-valued model B in $\mathbb{T}[\mathcal{S}]$, $(\mathcal{V}_B^{\mathrm{op}}, J_B)$ is called the *spectral site* of B. Then define the *(Coste) spectrum* of B as

$$\mathbf{Spec}(B) = \mathbf{Sh}(\mathcal{V}_B^{\mathrm{op}}, J_B)$$

In the following we denote as

$$\widehat{\mathcal{V}_B^{\mathrm{op}}} \xrightarrow{\mathfrak{a}_{J_B}} \mathbf{Spec}(B)$$

the associated sheafification functor left adjoint to the inclusion

$$\mathbf{Spec}(B) \xhookrightarrow{\iota_B} \widehat{\mathcal{V}_B^{\mathrm{op}}}$$

Remark 7.1.1.4. Observe that the sheafification functor $\mathfrak{a}_{J_B}$ extends into a functor between categories of $\mathbb{T}$-models and $\mathbb{T}_J$-models. In fact we have a pseudocommutative square

$$\begin{array}{ccc} \mathbb{T}[\widehat{\mathcal{V}_B^{\mathrm{op}}}] & \xrightarrow{\mathfrak{a}^*_{J_B}} & \mathbb{T}[\mathbf{Spec}(B)] \\ \uparrow & \simeq & \uparrow \\ \mathbb{T}_J[\widehat{\mathcal{V}_B^{\mathrm{op}}}] & \xrightarrow[\mathfrak{a}^*_{J_B}]{} & \mathbb{T}_J[\mathbf{Spec}(B)] \end{array}$$

which is the pseudonaturality square of the natural transformation

$$\mathbf{Geom}[\widehat{\mathcal{V}_B^{\mathrm{op}}}, -] \xRightarrow{\mathbf{Geom}[\iota_B, -]} \mathbf{Geom}[\mathbf{Spec}(B), -]$$

at the inclusion $\iota_J : \mathcal{S}[\mathbb{T}_J] \hookrightarrow \mathcal{S}[\mathbb{T}]$.

7.1.1.5. Now we turn to the functoriality of the construction. For a morphism $f : B_1 \to B_2$ in $\mathbb{T}[\mathcal{S}]$, the geometric morphism $\mathbf{Spec}(f)$ is computed from the pushout functor along f

$$\mathcal{V}_{B_1} \xrightarrow{f_*} \mathcal{V}_{B_2}$$

sending a finitely presented etale arrow to its pushout

$$\begin{array}{ccc} B_1 & \xrightarrow{f} & B_2 \\ {\scriptstyle n}\downarrow & \ulcorner & \downarrow{\scriptstyle f_*n} \\ \mathrm{cod}(n) & \xrightarrow[n_*f]{} & \mathrm{cod}(f_*n) \end{array}$$

But now, observe that the pushout functor sends finite colimits of $\mathcal{V}_{B_1}$ to finite colimits in $\mathcal{V}_{B_2}$ hence defines a lex functor $\mathcal{V}_{B_1}^{\mathrm{op}} \to \mathcal{V}_{B_2}^{\mathrm{op}}$. Moreover, this functor is J_{B_1}-continuous, by composition of pushouts. Hence Diaconescu applies and returns an extension

$$\begin{array}{ccc} \mathcal{V}_{B_1}^{\mathrm{op}} & \xrightarrow{f_*} & \mathcal{V}_{B_2}^{\mathrm{op}} \\ {\scriptstyle \mathfrak{a}_{J_{B_1}} \exists}\downarrow & \simeq & \downarrow{\scriptstyle \mathfrak{a}_{J_{B_2}} \exists} \\ \mathbf{Spec}(B_1) & \xrightarrow[\mathrm{lan}_{\mathfrak{a}_{J_{B_1}} \exists}\, \mathfrak{a}_{J_{B_2}}]{\exists\, f_*} & \mathbf{Spec}(B_2) \end{array}$$

which is the inverse image part of $\mathbf{Spec}(f)$.

7.1.2 Etale maps produce (pro-)etale geometric morphisms

7.1.2.1. $\mathcal{V}_B^{\mathrm{op}}$ is a lex site coding for "basic compact open inclusions". Objects of the sheaf topos $\mathbf{Spec}(B) \hookrightarrow [\mathcal{V}_B, \mathcal{S}]$ should be seen as generalized opens of the spectral topology, while objects of $\mathrm{Ind}(\mathcal{V}_B)$, which are arbitrary etale arrows under B, should be seen as saturated compacts of the spectral topology. In particular, the embedding $\mathcal{V}_B^{\mathrm{op}} \hookrightarrow \mathbf{Spec}(B)$ exibits $\mathcal{V}_B$ as a basis of *basic compact sets that are open* - and $\mathcal{V}_B^{\mathrm{op}}$ as a basis of *open sets that are compacts*.

The following observation motivates the name for etale arrows:

Proposition 7.1.2.2. *Finitely presented etale arrows $n : B \to C$ under B correspond to etale geometric morphisms of the form:*

$$\mathbf{Spec}(C) \simeq \mathbf{Spec}(B)/\mathfrak{a}_{J_B} \exists_n \xrightarrow{\mathbf{Spec}(n)} \mathbf{Spec}(B)$$

Proof. For $n : B \to C$ in $\mathcal{V}_B$ we have an equivalence of categories

$$\mathcal{V}_C \simeq n \downarrow \mathcal{V}_B$$

sending $m : C \to D$ to the triangle

$$\begin{array}{ccc} B & & \\ {\scriptstyle n}\downarrow & \searrow^{mn} & \\ C & \xrightarrow[m]{} & D \end{array}$$

and conversely any triangle $l = mn$ in $n \downarrow \mathcal{V}_B$ to the underlying arrow m. In particular we have

$$\widehat{\mathcal{V}_C^{\mathrm{op}}} \simeq (\widehat{n \downarrow \mathcal{V}_B})^{\mathrm{op}} \simeq \widehat{\mathcal{V}_B^{\mathrm{op}}}/\exists_n$$

But also by the expression of slices in a sheaf category (see [86][III, Proposition 5.4]), we know that the topology induced on $n \downarrow \mathcal{V}_B$ by J_B is the same as J_C - this is the corresponding topology J' corresponding to J_C through the equivalence above - and we have

$$\mathbf{Spec}(C) \simeq \mathbf{Sh}(n \downarrow \mathcal{V}_B^{\mathrm{op}}, J') \simeq \mathbf{Spec}(B)/\mathfrak{a}_{J_B}\exists_n$$

□

Remark 7.1.2.3. Observe that we have a 2-pullback square in the bicategory of Grothendieck topoi

$$\begin{array}{ccc} \mathbf{Spec}(B)/\mathfrak{a}_{J_B}\exists_n & \longrightarrow & \widehat{\mathcal{V}_B^{\mathrm{op}}}/\exists_n \\ {\scriptstyle \mathbf{Spec}(n)}\downarrow & \lrcorner & \downarrow{\scriptstyle \overline{\exists_n}} \\ \mathbf{Spec}(B) & \xhookrightarrow{\iota_B} & \widehat{\mathcal{V}_B^{\mathrm{op}}} \end{array}$$

exhibiting the etale geometric morphism $\mathbf{Spec}(n)$ as the 2-pullback of the etale geometric morphism associated to $\exists_n$.

Remark 7.1.2.4. The further left adjoint of the inverse image will be induced from the postcomposition functor $\mathcal{V}_C^{\mathrm{op}} \to \mathcal{V}_B^{\mathrm{op}}$ sending $m : C \to D$ to the composite $mn : B \to D$ which is in $\mathcal{V}_B$. This functor defines a left adjoint

$$\mathcal{V}_B^{\mathrm{op}} \underset{n_*}{\overset{n^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{V}_C^{\mathrm{op}}$$

The intuition that objects of $\mathcal{V}_B$ are compact can be formalized thanks to the following property. Recall that a geometric morphism is said to be *tidy* if its direct image part preserves filtered colimits. From [77][Theorem 4.8] we know that tidy geometric morphisms are stable under 2-pullback.

Proposition 7.1.2.5. *For $n : B \to C$ in $\mathcal{V}_B$, the geometric morphism $\mathbf{Spec}(n) : \mathbf{Spec}(C) \to \mathbf{Spec}(B)$ is tidy.*

Proof. Recall we can express $\mathbf{Spec}(n)$ as the pullback of the etale geometric morphism $\widehat{\mathcal{V}_B^{\mathrm{op}}}/\exists_n \to \widehat{\mathcal{V}_B^{\mathrm{op}}}$ along ι_B. But we know that $\exists_n$ is a finitely presented object in the presheaf topos $\widehat{\mathcal{V}_B^{\mathrm{op}}}$, so that the associated internal hom functor $(-)^{\exists_n}$ preserves filtered colimits: hence its associated etale geometric morphism is tidy, and hence its pullback $\mathbf{Spec}(n)$ also is. □

Remark 7.1.2.6. Arbitrary etale arrows are not in the topos $\mathbf{Spec}(B)$, but rather from the side of points and saturated compacts. Hence they do not correspond to etale geometric morphisms in general. In fact observe that an arbitrary etale map $l : B \to C$ is an object of $\mathrm{Ind}(\mathcal{V}_B)$, for the factorization system $(\mathbf{Et}, \mathbf{Loc})$ was left generated from $\mathcal{V}$; but we have $\mathrm{Ind}(\mathcal{V}_B) \simeq \mathrm{Pro}(\mathcal{V}_B^{\mathrm{op}})^{\mathrm{op}}$, which is the pro-completion of $\mathcal{V}_B^{\mathrm{op}}$, whose objects are those functors $\mathcal{V}_B^{\mathrm{op}} \to \mathcal{S}$ which are cofiltered limits of representables: this mimics the fact that arbitrary etale maps are constructed as cofiltered intersection of basic open compact sets. For this reason, [4] says *pro-etale* for what we call arbitrary etale, reserving "etale" for the basic ones. This can be formalized into the following result:

Theorem 7.1.2.7. *Let $l : B \to C$ be an arbitrary etale arrow under B. Then $\mathbf{Spec}(C)$ decomposes as a cofiltered pseudolimit*

$$\mathbf{Spec}(C) \simeq \underset{(n,a)\in \int \nmid_l}{\mathrm{bilim}} \mathbf{Spec}(\mathrm{cod}(n))$$

Proof. From theorem 1.1.4.3, we know that

$$\mathcal{V}_C \simeq \underset{(n,a)\in \int \nmid_l}{\mathrm{pscolim}} \mathcal{V}_{\mathrm{cod}(n)}$$

Now recall that for each (n, a) in $\int \nmid_l$, the opposite category of the etale generator $\mathcal{V}_{\mathrm{cod}(n)}$ can be equipped with a pretopology $J_{\mathrm{cod}(n)}$, and the opposite category of the pseudocolimit

$$(\mathrm{pscolim}_{(n,a)\in \int \nmid_l} \mathcal{V}_{\mathrm{cod}(n)})^{\mathrm{op}}$$

can be equipped with the coarsest topology

$$\langle \bigcup_{(n,a)\in\int \lambda_l} q_{(n,a)}(J_{\mathrm{cod}(n)})\rangle$$

making the canonical inclusion continuous

$$q^{\mathrm{op}}_{(n,a)} : \mathcal{V}^{\mathrm{op}}_{\mathrm{cod}(n)} \to (\mathrm{pscolim}_{(n,a)\in\int \lambda_l} \mathcal{V}_{\mathrm{cod}(n)})^{\mathrm{op}}$$

From [86] and [34], and also the general version of proposition 7.2.2.10 on cofiltered pseudolimits of Grothendieck topoi, we know that the corresponding sheaf topos is the pseudolimit

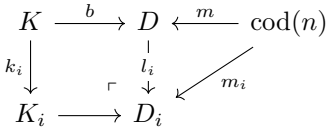
$$\mathbf{Sh}\big((\mathrm{pscolim}_{(n,a)\in\int \lambda_l} \mathcal{V}_{\mathrm{cod}(n)})^{\mathrm{op}}, \langle \bigcup_{(n,a)\in\int \lambda_l} q_{(n,a)}(J_{\mathrm{cod}(n)})\rangle\big) \simeq \mathrm{bilim}_{(n,a)\in\int \lambda_l} \mathbf{Spec}(\mathrm{cod}(n))$$

and moreover, this topology is exactly the image of the induced topology on the pseudocolimit along the equivalence of categories above with $\mathcal{V}_C$. Now we can also glue the image of the $J_{\mathrm{cod}(n)}$-covering families along the pushout functors a_* to generate a topology on $\mathcal{V}^{\mathrm{op}}_C$

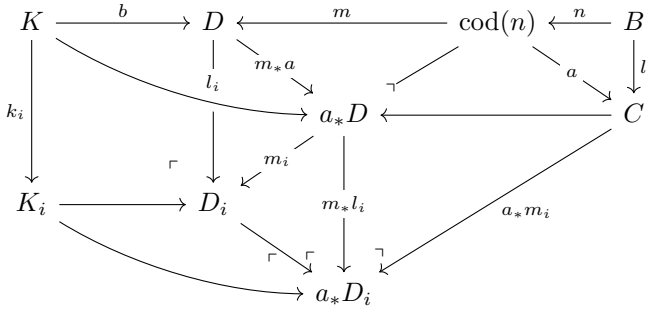
$$\langle \bigcup_{(n,a)\in\int \lambda_l} a_*(J_{\mathrm{cod}(n)})\rangle$$

We must prove that any covering family $a_*(J_{\mathrm{cod}(n)})$ is covering in J_C, and conversely that any J_C covering family is covering in the topology induced from the colimit.

First, let us prove that a cover in the jointly generated pretopology is covering in J_C. For (n,a) in $\int \lambda_l$, m an object of $\mathcal{V}_{\mathrm{cod}(n)}$ and a covering family $(l_i)_{i\in I}$ induced as

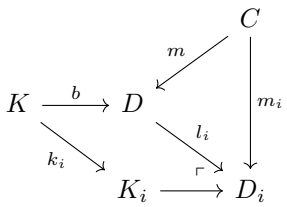


in $J_{\mathrm{cod}(n)}(m)$, consider the composition of pushouts as in the diagram below

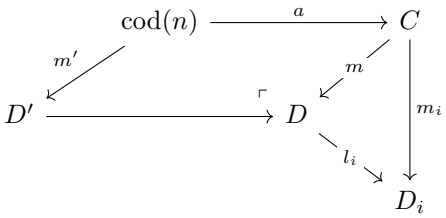


Then we see that for each $i \in I$ the arrow a_*l_i is also the pushout $(m_*a\, b)k_i$, and this exhibits the pushout family $(m_*l_i)_{i\in I}$ as a pushout of a family in J, and hence as a covering family of J_C.

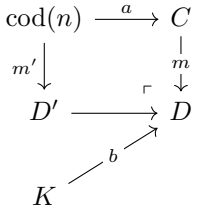
Conversely let us prove that a J_C-cover is covering in the jointly generated pretopology. For a covering family of some m in $\mathcal{V}_C$ for J_C



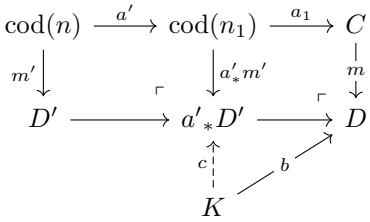
we know from the essential surjectivity of the equivalence result in theorem 1.1.4.3 that m is induced as some a_*m' for some $(n,a) \in \int \lambda_l$



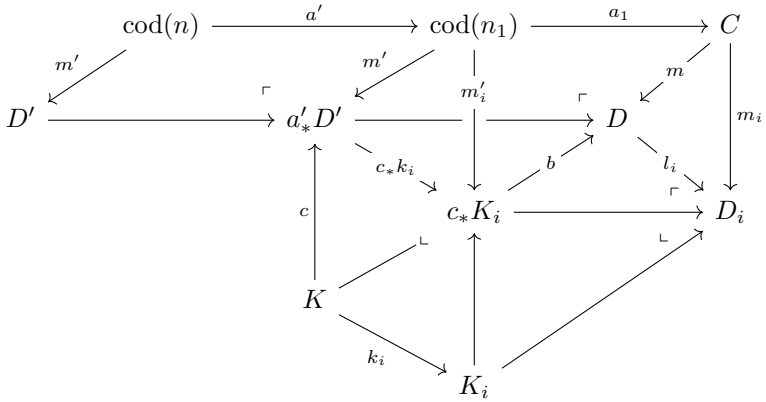
Now observe that from the situation below



there exists from lemma 1.1.2.7 some (n_1, a_1) in $\int \nleftarrow_l$ and a factorization of b through the intermediate pushout as below



and now we can pushout the family $(k_i)_{i\in I}$ of J along c to get a covering family $(c_*k_i)_{i\in I}$ in $J_{\mathrm{cod}(n)_1}$ of the object a'_*m' in $\mathcal{V}_{\mathrm{cod}(n_1)}$



Moreover, the objects $((n,a),m')$ and $((n_1,a_1),a'_*m')$ are identified in the pseudocolimit for they are related through an opcartesian morphism, so that $(c_*k_i)_{i\in I}$ is both covering for the class of $((n,a),m')$ in the induced topology on the pseudocolimit $\mathrm{colim}_{(n,a)\in\int \nleftarrow_l}\mathcal{V}_{\mathrm{cod}(n)}$, and is sent to the covering family $(l_i)_{i\in I}$ of J_C through the pushout functor a_{1*}. Hence the J_C-cover $(l_i)_{i\in I}$ is in the induced topology $a_{1*}(J_{\mathrm{cod}(n_1)})$, and hence in the jointly generated topology. This proves that the equivalence of categories of theorem 1.1.4.3 induces an equivalence of sites

$$\Big((\underset{(n,a)\in\int \nleftarrow_l}{\mathrm{pscolim}}\ \mathcal{V}_{\mathrm{cod}(n)})^{\mathrm{op}}, \langle \bigcup_{(n,a)\in\int \nleftarrow_l} q_{(n,a)}(J_{\mathrm{cod}(n)})\rangle\Big) \simeq \big((\mathcal{V}_C^{\mathrm{op}}, J_C\big)$$

so that they induce the same sheaf topos, which proves the desired limit decomposition of $\mathbf{Spec}(C)$. □

Remark 7.1.2.8. Observe that we also have directly from theorem 1.1.4.3

$$\widehat{\mathcal{V}_C^{\mathrm{op}}} \simeq \underset{(n,a)\in\int \nleftarrow_l}{\mathrm{bilim}} \widehat{\mathcal{V}_B^{\mathrm{op}}}/\exists_n$$

Since each $\mathbf{Spec}(\mathrm{cod}(n))$ for $(n,a) \in \int \text{ɟ}_l$ expresses as an etale geometric morphism obtained as a pullback as in remark 7.1.2.3, and pullback commutes with limits, we have

$$\begin{array}{ccc} \underset{(n,m)\in\int \text{ɟ}_l}{\mathrm{bilim}} \mathbf{Spec}(B)/_{\mathfrak{a}_J} \exists_n & \longrightarrow & \underset{(n,m)\in\int \text{ɟ}_l}{\mathrm{bilim}} \widehat{\mathcal{V}_B^{\mathrm{op}}}/\exists_n \\ \downarrow & \lrcorner & \downarrow \\ \mathbf{Spec}(B) & \xrightarrow[\iota_B]{} & \widehat{\mathcal{V}_B^{\mathrm{op}}} \end{array}$$

so that we have a pullback in **GTop**

$$\begin{array}{ccc} \mathbf{Spec}(C) & \xhookrightarrow{\iota_C} & \widehat{\mathcal{V}_C^{\mathrm{op}}} \\ {\scriptstyle \mathbf{Spec}(n)}\downarrow & \lrcorner & \downarrow \\ \mathbf{Spec}(B) & \xhookrightarrow[\iota_B]{} & \widehat{\mathcal{V}_B^{\mathrm{op}}} \end{array}$$

This means that the natural inclusion $\mathbf{Spec}(-) \hookrightarrow \widehat{\mathcal{V}_{(-)}^{\mathrm{op}}}$ is cartesian at etale maps $l \in \mathbf{Et}$.

7.1.3 Spectrum and local data

In particular we have the following, for local forms are etale arrows (that are seldom finitely presented):

Proposition 7.1.3.1. *Points of* $\mathbf{Spec}(B)$ *correspond to local forms* $x : B \to A$.

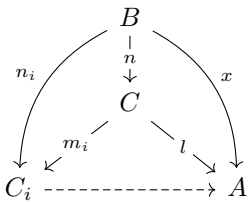
Proof. First observe that a point of the spectrum, that is a J_B-continuous lex functor x in $\mathbf{Lex}[\mathcal{V}_B^{\mathrm{op}}, \mathcal{S}]$ is in particular an object of the ind-completion of $\mathcal{V}_B$, hence an object of $B \downarrow \mathbb{T}[\mathcal{S}]$, so we can write x as an etale arrow $x : B \to A$; now the condition of continuity says that for a covering $(m_i^{\mathrm{op}} : n_i \to n)_{i\in I}$ in $\mathcal{V}_B^{\mathrm{op}}$ one has

$$\coprod_{i\in I} x(n_i) \xrightarrow{\langle x(m_i)\rangle_{i\in I}} x(n)$$

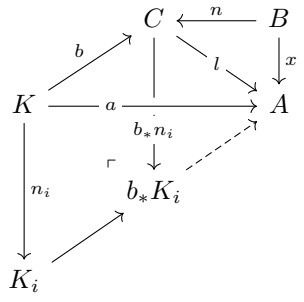
But Yoneda tells us that

$$\begin{aligned} x(n) &\simeq \mathrm{Ind}(\mathcal{V}_B)[\text{ɟ}_n, x] \\ &= \{l : C \to A \mid \ ln = x\} \end{aligned}$$

and the surjectivity property above expresses the existence of the dashed arrow in the following diagram for some $i \in I$:



We must prove that A is a local object: let $(n_i : K \to K_i)_{i\in I}$ be a J-cover and $a : K \to A$. Recall first that x, as an etale map, is the filtered colimit of the finitely presented etale maps under B over it: $x = \mathrm{colim}_{(n,l)\in V_B\downarrow x}$, so is $A = \mathrm{colim}_{(n,l)\in V_B\downarrow x}\mathrm{cod}(n)$. But then for K is finitely presented, $a : K \to A$ lift through some $l : C \to A$ as $b : K \to C$ for some factorization of x as above. Then we can consider the pushout family $(b_* n_i)_{i\in I}$ which is a J_B-cover of n in $\mathcal{V}_B$: but then we see in the following diagram



that there must be a dashed arrow as provided by J_B-injectiveness of x, which ensures existence of a lift for A. Hence A is local and x is a local form. □

Remark 7.1.3.2. More generally, geometric morphisms $x : \mathcal{E} \to \mathbf{Spec}(F)$ will correspond to etale arrows $x : !_{\mathcal{E}}^* B \to E$ in $\mathbf{Et}[\mathcal{E}]$ with E in $\mathbb{T}_J[\mathcal{E}]$: however this correspondence can only be seen through the structural sheaf, which will be described later in this section. For now let us focus on the set-valued points.

7.1.3.3. At the level of points, an etale map of finite presentation $n : B \to C$ produces a discrete fibration

$$\mathbf{pt}(\mathbf{Spec}(C)) \simeq \mathbf{pt}(\mathbf{Spec}(B)/\mathfrak{a}_{J_B}(\exists_n)) \to \mathbf{pt}(\mathbf{Spec}(B))$$

More generally, this remains true for an arbitrary etale map $l : B \to C$; cartesian lifts are computed as follows: if x is a local form under B and x' a local form under C, then a morphism $x \to x'l$ is a square as below

$$\begin{array}{ccc} B & \xrightarrow{l} & C \\ {\scriptstyle x}\downarrow & & \downarrow{\scriptstyle x'} \\ A & \xrightarrow[f]{} & A' \end{array}$$

with f an arbitrary arrow; but the etale-local factorization of this very map comes equiped with a diagonalization of the square below

$$\begin{array}{ccccc} B & & \xrightarrow{\quad l \quad} & & C \\ {\scriptstyle x}\downarrow & & & \swarrow{\scriptstyle x''} & \downarrow{\scriptstyle x'} \\ A & \xrightarrow[n_f]{} & A_f & \xrightarrow[u_f]{} & A' \end{array}$$

where the diagonal $x'' : C \to A_f$ is a local form equiped with a lift $x'' \to x'$ given by u_f. Moreover, as the functor of points $\mathbf{pt} \simeq \mathbf{Geom}[\mathcal{S}, -]$ preserves pseudolimits, we have a pseudolimit of category

$$\mathbf{pt}(\mathbf{Spec}(C)) \simeq \operatorname*{pslim}_{(n,a)\in \int \nleftarrow_l} \mathbf{pt}(\mathbf{Spec}(\mathrm{cod}(n))$$

Remark 7.1.3.4. Observe that arbitrary etale maps under B correspond to points of the presheaf topos $\widehat{\mathcal{V}_B^{\mathrm{op}}}$ as $\mathbf{Lex}[\mathcal{V}_B^{\mathrm{op}}, \mathcal{S}] \simeq \mathrm{Ind}(\mathcal{V}_B)$.

7.1.3.5. As the term "etale" was justified by the fact that finitely presented etale morphisms in $\mathcal{V}_B$ where sent to etale geometric morphisms by **Spec**, the name of "local" for objects is justified as follows. Recall that a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ is said to be *local* if its inverse image part f^* is full and faithful and moreover the direct image part f_* has a further right adjoint $f^!$. In particular, a Grothendieck topos $\mathcal{E}$ is said to be *local* if the global section functor $\Gamma : \mathcal{E} \to \mathcal{S}$ has a further right adjoint - the full-and-faithfulness condition being automatic in this context.

The prototypical example of local geometric morphism is the universal domain map $\partial_0 : \mathcal{E}^2 \to \mathcal{E}$ of a topos. Now for any point $p : \mathcal{S} \to \mathcal{E}$, we can consider the *Grothendieck Verdier localization at* p, which is defined as the pseudopullback

$$\begin{array}{ccc} \mathcal{E}_p & \longrightarrow & \mathcal{E}^2 \\ {\scriptstyle p^*\partial_0}\downarrow & \lrcorner & \downarrow{\scriptstyle \partial_0} \\ \mathcal{S} & \xrightarrow[p]{} & \mathcal{E} \end{array}$$

Its universal property is that for any Grotendieck topos $\mathcal{F}$, we have an equivalence with the cocomma category

$$\mathbf{Geom}[\mathcal{F}, \mathcal{E}_p] \simeq p\, !_{\mathcal{F}} \downarrow \mathbf{Geom}[\mathcal{F}, \mathcal{E}]$$

In particular, when $\mathcal{F}$ is $\mathcal{S}$, we have an equivalence of categories

$$\mathbf{pt}(\mathcal{E}_p) \simeq p \downarrow \mathbf{pt}(\mathcal{E})$$

From [51][Theorem 3.7] we know that if $\mathcal{E}$ has $(\mathcal{C}, J)$ as a lex site of definition, then $\mathcal{E}_p$ can be expressed as a cofiltered pseudolimit of etale geometric morphisms

$$\mathcal{E}_p \simeq \operatorname*{bilim}_{(C,a)\in \int p^*} \mathcal{E}/\mathfrak{a}_J \exists_C$$

where $\int p^*$ is the cofiltered category of elements of the J-flat functor $p^* : \mathcal{C} \to \mathcal{S}$.

Proposition 7.1.3.6. *Let A be a local object in $\mathbb{T}[\mathcal{S}]$: then $\mathbf{Spec}(A)$ is a local topos.*

Proof. Suppose that A is in $\mathbb{T}_J[\mathcal{S}]$. Then in particular we know that A is J-local for the generalized covers associated to J in $\mathbb{T}[\mathcal{S}]$ in the sense of definition 3.3.1.2. Then in $\mathcal{V}_A^{\mathrm{op}}$, the terminal object 1_A is J_A-local as it lifts through all its covers. Then the functor $\yoneda_{1_A} : \mathcal{V}_A^{\mathrm{op}} \longrightarrow \mathcal{S}$ is lex J_A-continuous and defines a point p_{1_A}. But now consider the Grothendieck-Verdier localization at this point

$$\mathbf{Spec}(A)_{p_{1_A}} \simeq \underset{(n,a)\in \int p_{1_A}^*}{\mathrm{bilim}} \mathbf{Spec}(A)/\mathfrak{a}_{J_A}\, \exists_n$$

Observe then that $\int p_{1_A}^*$ has an initial object $(1_A, 1_{1_A})$, corresponding to the identity triangle of 1_A in $\mathcal{V}_A$. This initial object makes the limit above to reduces to $\mathbf{Spec}(A)_{p_{1_A}} \simeq \mathbf{Spec}(A)/\mathfrak{a}_{J_A}\, \exists_{1_A}$, but 1_A was the initial object of $\mathcal{V}_A$, and $\exists$ turns it into the terminal object of $\widehat{\mathcal{V}_A^{\mathrm{op}}}$, which is preserved by $\mathfrak{a}_{J_A}$. Therefore $\mathbf{Spec}(A)_{p_{1_A}} \simeq \mathbf{Spec}(A)$, exhibiting $\mathbf{Spec}(A)$ as a local topos. □

We also have this corollary from theorem 7.1.2.7:

Corollary 7.1.3.7. *For $x : B \to A_x$ a local form under B, the geometric morphism $\mathbf{Spec}(x) : \mathbf{Spec}(A_x) \to \mathbf{Spec}(B)$ is the Grothendieck-Verdier localization of $\mathbf{Spec}(B)$ at the point $p_x : \mathcal{S} \to \mathbf{Spec}(B)$.*

To conclude this section, we characterize the class of geometric morphisms the spectrum sends local maps to. Let us first introduce the following notion, which is also related to [15][Section 4.7]:

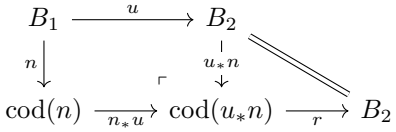
Definition 7.1.3.8. A geometric morphism $f : \mathcal{E} \to \mathcal{F}$ is said to be *terminally connected* if the inverse image part f^* lifts global elements, that is, if for any F in $\mathcal{F}$ one has

$$\mathcal{E}[1, f^*(F)] \simeq \mathcal{F}[1, F]$$

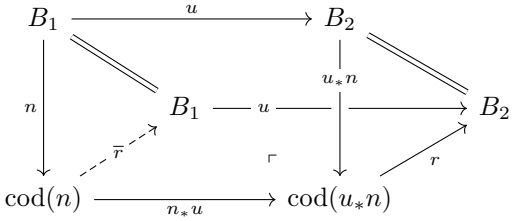
Lemma 7.1.3.9. *Let $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ be a morphism of sites between standard sites: if f lifts global elements then $\mathbf{Sh}(f) : \mathbf{Sh}(\mathcal{D}, J) \to \mathbf{Sh}(\mathcal{C}, J)$ is terminally connected.*

Proposition 7.1.3.10. *Let $u : B_1 \to B_2$ be a local map in $\mathbb{T}[\mathcal{S}]$. Then $\mathbf{Spec}(u) : \mathbf{Spec}(B_2) \to \mathbf{Spec}(B_1)$ is terminally connected.*

Proof. Recall that $\mathbf{Spec}(u)^*$ restricts as a site morphism $u_* : (\mathcal{V}_{B_1}^{\mathrm{op}}, J_{B_1}) \to (\mathcal{V}_{B_2}^{\mathrm{op}}, J_{B_2})$ given by the pushout functor along u. Now, suppose one has, for some etale map $n : B_1 \to \mathrm{cod}(n)$ in $\mathcal{V}_{B_1}$, a global element of u_*n, which is the same as a retraction as below

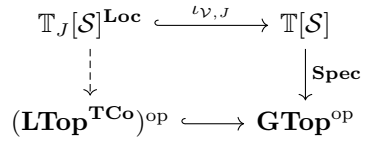


so that we have a factorization $u = rn_*un$. But then, for u is local and n is etale, we know by lemma 1.1.1.1 that n must retract on 1_{B_1} for the etale part of u is invertible, which produces a retraction $\overline{r}$ as below



and we have by cancellation of pushout that $u_*(\overline{r}) = r$; moreover such an $\overline{r}$ has to be unique: for any other choice r' of retraction with $u_*(r') = r$, one would have $u(r') = rn_*u = u\overline{r}$ and $r'n = \overline{r}n$, so that r' and $\overline{r}$ would be simultaneously equalized by n and coequalized by u: but from corollary 1.1.1.4 this forces $r' = \overline{r}$. Hence u_* lifts global elements, and by the previous lemma, this prove $\mathbf{Spec}(u)$ to be terminally connected. □

Remark 7.1.3.11. This proves that, for any geometry $(\mathbb{T}, \mathcal{V}, J)$, the functor sending a set-valued $\mathbb{T}$-model in $\mathbb{T}[\mathcal{S}]$ to the underlying topos of its spectrum (which we will abusively denote $\mathbf{Spec}$ in this remark) restricts as below



where $\mathbf{LTop}^{\mathbf{TCo}}$ denotes the bicategory of local Grothendieck topoi and terminally connected geometric morphisms between them. Moreover, one can show that terminally connected geometric morphisms are left orthogonal in a 2-categorical sense to etale geometric morphisms: this suggests that a kind of "universal" 2-dimensional notion of geometry on the bicategory of Grothendieck topoi is involved in any instance of geometry. This will be investigated in [82][Proposition 5.4.11].

7.1.4 The structural sheaf

Now we turn to the structural sheaf, which is obtained from the codomain functor modulo sheafification:

Definition 7.1.4.1. For a set-valued model B in $\mathbb{T}[\mathcal{S}]$, the *structural sheaf* of B is the sheaf of set-valued $\mathbb{T}$-models $\widetilde{B}$ obtained as

$$\widetilde{B} = \mathfrak{a}_{J_B}\mathrm{cod}$$

7.1.4.2. The structural sheaf can also be described as follows: recall that any $\mathbb{T}$-model B is in $\mathrm{Ind}(\mathbb{T}[\mathcal{S}]_\omega) \simeq \mathbf{Lex}[\mathcal{C}_\mathbb{T}, \mathcal{S}]$. Moreover, we can consider the *conerve* of the codomain functor

$$\begin{aligned} \mathbb{T}[\mathcal{S}]^{\mathrm{op}} &\longrightarrow \widehat{\mathcal{V}_B^{\mathrm{op}}} \\ B &\longmapsto \mathbb{T}[\mathcal{S}]\big[B, \mathrm{cod}\big] \end{aligned}$$

which can be composed along the embedding $\mathcal{C}_\mathbb{T} \hookrightarrow \mathbb{T}[\mathcal{S}]^{\mathrm{op}}$

$$\begin{array}{ccccc} \mathcal{V}_B^{\mathrm{op}} & \xrightarrow{\mathrm{cod}} & \mathbb{T}[\mathcal{S}]^{\mathrm{op}} & \xleftarrow{\exists} & \mathcal{C}_\mathbb{T} \\ \downarrow & \swarrow_{\chi} & & \swarrow_{\mathrm{cod}_*} & \\ \widehat{\mathcal{V}_B^{\mathrm{op}}} & & & & \end{array}$$

to produce a lex functor

$$\begin{aligned} \mathcal{C}_\mathbb{T} &\longrightarrow \widehat{\mathcal{V}_B^{\mathrm{op}}} \\ \{\overline{x}, \phi\} &\longmapsto \mathbb{T}[\mathcal{S}]\big[K_\phi, \mathrm{cod}\big] \end{aligned}$$

which we can now compose with the lex localization $\mathfrak{a}_{J_B} : \widehat{\mathcal{V}_B^{\mathrm{op}}} \to \mathbf{Spec}(B)$ to get a lex functor

$$\begin{aligned} \mathcal{C}_\mathbb{T} &\xrightarrow{\widetilde{B}} \mathbf{Spec}(B) \\ \{\overline{x}, \phi\} &\longmapsto \mathfrak{a}_{J_B}\mathbb{T}[\mathcal{S}]\big[K_\phi, \mathrm{cod}\big] \end{aligned}$$

Proposition 7.1.4.3. *The structural sheaf $\widetilde{B}$ is in $\mathbb{T}_J[\mathbf{Spec}(B)]$. In particular for any point $x : \mathcal{S} \to \mathbf{Spec}(B)$ the stalk $x^*\widetilde{B}$ is in $\mathbb{T}_J[\mathcal{S}]$.*

Proof. We have to prove that for any J-cover $(\theta_i : \{\overline{x}_i, \phi_i\} \to \{\overline{x}, \phi\})_{i\in I}$ we have an epimorphism in the category of sheaves $\mathbf{Spec}(B) = \mathbf{Sh}(\mathcal{V}_B^{\mathrm{op}}, J_B)$

$$\coprod_{i\in I} \mathfrak{a}_{J_B}\mathbb{T}[\mathcal{S}]\big[K_{\phi_i}, \mathrm{cod}\big] \xrightarrow{\langle \mathfrak{a}_{J_B}\mathbb{T}[\mathcal{S}]\left[f_{\theta_i}, \mathrm{cod}\right]\rangle_{i\in I}} \mathfrak{a}_{J_B}\mathbb{T}[\mathcal{S}]\big[K_\phi, \mathrm{cod}\big]$$

But proving some morphism to be epic in a sheaf topos is a local condition, which however here would be made impossible to test because of the expression of the sheafification. We are in fact going to prove that before sheafification, the map $\langle \mathbb{T}[\mathcal{S}]\big[f_{\theta_i}, \mathrm{cod}\big]\rangle_{i\in I}$ is a *local epimorphism*, that is, a map that is sent to an epimorphism after sheafification - though not yet being itself an epimorphism in the presheaf category. For sheafification is a left adjoint, it preserves coproducts, that is

$$\coprod_{i\in I} \mathfrak{a}_{J_B}\mathbb{T}[\mathcal{S}]\big[K_{\phi_i}, \mathrm{cod}\big] \simeq \mathfrak{a}_{J_B}\coprod_{i\in I}\mathbb{T}[\mathcal{S}]\big[K_{\phi_i}, \mathrm{cod}\big]$$

and moreover, the localness condition is preserved after sheafification, and makes

$$\mathfrak{a}_{J_B}\langle \mathbb{T}[\mathcal{S}]\big[f_{\theta_i}, \mathrm{cod}\big]\rangle_{i\in I} \simeq \langle \mathfrak{a}_{J_B}\mathbb{T}[\mathcal{S}]\big[f_{\theta_i}, \mathrm{cod}\big]\rangle_{i\in I}$$

an epimorphism in $\mathbf{Spec}(B)$.

Let us prove the localness condition for $\langle \mathbb{T}[\mathcal{S}]\big[f_{\theta_i}, \mathrm{cod}\big]\rangle_{i\in I}$, which amounts to it to be a local surjection - see for instance [69]. Take some $b : K_\phi \to \mathrm{cod}(n)$; then one can push the J-cover $(f_{\theta_i} : K_\phi \to K_{\phi_i})_{i\in I}$ along b so we get a cover of n in $\mathcal{V}_B^{\mathrm{op}}$

$$\begin{array}{ccccc} K_\phi & \xrightarrow{b} & \mathrm{cod}(n) & \xleftarrow{n} & B \\ {\scriptstyle f_{\theta_i}}\downarrow & & {\scriptstyle b_* f_{\theta_i}}\downarrow & \swarrow_{nb_*f_{\theta_i}} & \\ K_{\phi_i} & \longrightarrow & b_* K_{\phi_i} & & \end{array}$$

Then for each $i \in I$, we have

$$\begin{aligned} \mathbb{T}[\mathcal{S}]\big[K_\phi, b_* f_{\theta_i}\big](b) &= b_* f_{\theta_i}\, b \\ &= f_{\theta_i *} b\, f_{\theta_i} \\ &= \mathbb{T}[\mathcal{S}]\big[f_{\theta_i}, K_{\phi_i}\big](f_{\theta_i *} b) \end{aligned}$$

which exactly says that the restriction of b along each member of the cover has an antecedent along $\langle \mathbb{T}[\mathcal{S}]\big[f_{\theta_j}, b_* K_{\phi_i}\big]\rangle_{j\in I}$: hence the natural transformation $\langle \mathbb{T}[\mathcal{S}]\big[f_{\theta_j}, \mathrm{cod}\big]\rangle_{j\in I}$ is a local epimorphism relative to the Grothendieck topology J_B, hence its sheafification is an epimorphism in $\mathbf{Spec}(B)$. □

Remark 7.1.4.4. Beware that the structural sheaf $\widetilde{B} = \mathfrak{a}_{J_B}\mathrm{cod}$ needs not to return local objects as values; in particular, whenever the topology generated by J_B is subcanonical, $\widetilde{B} = \mathrm{cod}$, but the codomains of basic etale arrows have no reason to be local objects. This is because local objects are models of a geometric extension of $\mathbb{T}$, being a model of which is a local notion that does not hold globally. Nevertheless, as we saw above, stalks of the structural sheaf are set-valued local objects; but in general having local objects at stalks may not be sufficient to ensure localness of a sheaf itself.

7.1.4.5. Now we address the functoriality of the spectrum at the level of the structural sheaves. For $f : B_1 \to B_2$ in $\mathbb{T}[\mathcal{S}]$, we saw in 7.1.1.5 how $\mathbf{Spec}(f)$ was obtained from the pushout functor $f_* : \mathcal{V}_{B_1} \to \mathcal{V}_{B_2}$. Now to get the direct image part, the bottom arrows at each n in $\mathcal{V}_{B_1}$ of the pushout square $n_* f : \mathrm{cod}(n) \to \mathrm{cod}(f_* n)$ define altogether a natural transformation

$$\mathrm{cod}_1 \xrightarrow{\nu_f} \mathbf{Spec}(f)_* \mathrm{cod}_2$$

which is sent after sheafification to a morphism in $\mathbb{T}[\mathbf{Spec}(F_1)]$

$$\widetilde{F_1} \xrightarrow{\mathfrak{a}_{J_{B_1}}(\nu_f)} \mathbf{Spec}(f)_* \widetilde{F_2}$$

so we have to put

$$\widetilde{f^\sharp} = \mathfrak{a}_{J_{B_1}}(\nu_f)$$

while we get automatically from the adjunction $\mathbf{Spec}(f)$ a mate

$$\mathbf{Spec}(f)^* \widetilde{F_1} \xrightarrow{\widetilde{\phi}^\flat} \widetilde{F_2}$$

There is also a more concrete way to construct this morphism, which moreover expresses its connection to the (etale, locale factorization), exhibiting it as induced from the local part of precompositions with f:

Proposition 7.1.4.6. *For each $f : B_1 \to B_2$ in $\mathbb{T}[\mathcal{S}]$, $\widetilde{f}^\flat$ is in $\mathbf{Loc}[\mathbf{Spec}(B_2)]$.*

Proof. Recall that inverse images commute with sheafification, so we can first compute the inverse image of the codomain functor $\mathbf{Spec}(f)^*\mathrm{cod}$ and then apply sheafification to get the inverse image $\mathbf{Spec}(f)^*\widetilde{F_1}$. But now the inverse image is computed as a left Kan extension $\mathrm{lan}_{f_*}\mathrm{cod}$, which expresses at each n of $\mathcal{V}_{B_2}$ as the filtered colimit

$$\mathrm{lan}_{f_*}\mathrm{cod}(n) \simeq \operatorname*{colim}_{f_* \downarrow n} \mathrm{cod}(m)$$

But in fact, by the universal propert of the pushout - which can be summed up in the adjunction $f_* \dashv f^!$ where $f^!$ is precomposition with f, we have an equivalence of categories $f_* \downarrow n \simeq \mathcal{V}_{B_1} \downarrow nf$. This means that the colimit above can be seen equivalently as ranging over all factorizations

$$\begin{array}{ccc} B_1 & \xrightarrow{f} & B_2 \\ {\scriptstyle n}\downarrow & \ulcorner & \downarrow{\scriptstyle n} \\ \mathrm{cod}(m) & \xrightarrow[g]{} & \mathrm{cod}(n) \end{array}$$

while the data of all the $g : \mathrm{cod}(m) \to \mathrm{cod}(n)$ induce a natural arrow $(u_f)_n : \mathrm{lan}_{f_*}\mathrm{cod}(n) \to \mathrm{cod}(n)$. But we saw in proposition 1.1.3.2 that this is exactly how the (etale, locale) factorization of the composite $nf : B_1 \to \mathrm{cod}(n)$ is computed as the colimit of all its factorizations through finitely presented etale maps under B_1, which are precisely the objects of the spectral site of B_1. Hence in the factorization

$$\begin{array}{ccccc} B_1 & \xrightarrow{f} & B_2 & \xrightarrow{n} & \mathrm{cod}(n) \\ & {\scriptstyle \mathrm{colim}\,\mathcal{V}_{B_1}\downarrow nf}\searrow & & \nearrow{\scriptstyle (u_f)_n} & \\ & & \mathrm{lan}_{f_*}\mathrm{cod}(n) & & \end{array}$$

$\mathrm{colim}\,\mathcal{V}_{B_1} \downarrow nf$ is etale while $(u_f)_n$ is local. Moreover from naturality of this process, this defines a natural transformation

$$\mathbf{Spec}(f)^*\mathrm{cod} \xrightarrow{u_f} \mathrm{cod}$$

whose sheafification for J_{F_2} is the desired inverse image part $\widetilde{\phi}^\flat = \mathfrak{a}_{J_{B_2}}(u_f)$: since sheafification preserves localness, this exhibits $\widetilde{\phi}^\flat$ as a local map. □

7.1.5 The generic etale map and classifying properties of the spectrum

This section is actually the doorstep of the spectral adjunction: here are described the classifying property of the spectrum - where the structural sheaf and a certain canonical map are crucially involved - from which the later adjunction will appear as a corollary.

7.1.5.1. The structural sheaf comes associated with a canonical morphism of sheaves whose component indexes basic etale arrows under the object. For B in $\mathbb{T}[\mathcal{S}]$, the spectrum of B has a terminal geometric morphism $!_{\mathbf{Spec}(B)}$ to $\mathcal{S}$ with the direct image part sending a sheaf $F : \mathcal{V}_B \to \mathcal{S}$ to its set of global sections $\Gamma(F) = F(1_B)$ and the inverse image $!^*_{\mathbf{Spec}(B)}$ sending a set X to the X-indexed coproduct $\coprod_X 1_{\mathbf{Spec}(B)}$. This adjunction lifts to the categories of $\mathbb{T}$-models

$$\mathbb{T}[\mathbf{Spec}(B)] \underset{\Gamma}{\overset{!^*_{\mathbf{Spec}(B)}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbb{T}[\mathcal{S}]$$

Moreover this adjunction also exists at the level of the presheaf topos $\widehat{\mathcal{V}_B^{\mathrm{op}}}$, and we have $!_{\mathbf{Spec}(B)} = !_{\widehat{\mathcal{V}_B^{\mathrm{op}}}}\iota_B$. Now the identity of B defines a map $1_B : B \to B = \mathrm{cod}_*(1_B)$, and the latter object is actually the global section object of the codomain functor: by the version of the adjunction above relative to the presheaf topos, this defines a canonical map

$$!^*_{\widehat{\mathcal{V}_B^{\mathrm{op}}}} B \xrightarrow{\nu_B} \mathrm{cod}_*$$

which is sent after sheafification to a composite map

$$!^*_{\mathbf{Spec}(B)} B \xrightarrow{\eta^\flat_B} w\widetilde{B}$$

This very map corresponds itself to a comparison map

$$B \xrightarrow{\eta^\sharp_B} \Gamma w\widetilde{B}$$

Those two maps are going to be part of the unit of a restricted $\mathbf{Spec} \dashv \Gamma$ adjunction below.

Proposition 7.1.5.2. *The map $\eta_B^\flat$ is etale in $\mathbb{T}[\mathbf{Spec}(B)]$*

Proof. The inverse image presheaf $!^*_{\widehat{\mathcal{V}_B^{\mathrm{op}}}} B$ can also be described as the constant sheaf returning B everywhere; hence the map ν_B can also be described as the natural transformation

$$\mathcal{V}_B \;\underset{\mathrm{cod}}{\overset{!^*_{\widehat{\mathcal{V}_B^{\mathrm{op}}}} B}{\Downarrow \nu_B}}\; \mathbb{T}[\mathcal{S}]$$

whose component at n is n itself, which is etale: hence ν_B is an etale map as it is pointwise etale, and so is its sheafification $\eta_B^\flat$. □

Definition 7.1.5.3. In the following $\nu_B^\flat$ will be called the *generic etale map under B*, while $\eta_B^\flat$ will be called the *generic local form under B*.

The generic etale map gathers all the etale maps under B you need to compute the etale-locale factorizations of maps under inverse images of B, as you can extract the etale part of any morphism of sheaves from the inverse image of this morphism. Similarly the generic local form gathers all admissible factorizations under B, so that any morphism into a local object will now factorize through this very map: that is how admissible factorizations through different local units will be turned into an ordinary unit-like factorization. We split the process in two steps encoding first the factorization aspects and then the local data - in the same way we did in the fifth chapter.

7.1.5.4. Recall that the presheaf topos $\widehat{\mathcal{V}_B^{\mathrm{op}}}$ is endowed with the codomain presheaf $\mathrm{cod} : \mathcal{V}_B \to \mathbb{T}[\mathcal{S}]$ together with the canonical etale map $\nu_B : !^*_{\widehat{\mathcal{V}_B^{\mathrm{op}}}} B \to \mathrm{cod}_*$ whose component at n is n itself. Then any geometric morphism $x : \mathcal{E} \to \widehat{\mathcal{V}_B^{\mathrm{op}}}$ defines an etale map in $\mathcal{E}$

$$!^*_{\mathcal{E}} B \xrightarrow{x^* \eta_B^\flat} w x^* \mathrm{cod}$$

Similarly, any geometric morphism $x : \mathcal{E} \to \mathbf{Spec}(B)$ transfers the canonical map $\eta_B^\flat$ in $\mathcal{E}$ to an etale map

$$!^*_{\mathcal{E}} B \xrightarrow{x^* \eta_B^\flat} w x^* \widetilde{B}$$

where $x^* \widetilde{B}$ is a local object.

Now we want to describe the converse process: we want to prove that

- the presheaf topos $\widehat{\mathcal{V}_B^{\mathrm{op}}}$ classifies etale maps under B;
- the sheaf topos $\mathbf{Spec}(B)$ classifies local forms under B.

7.1.5.5. To any morphism of sheaves $\phi^\flat : !^*_{\mathcal{E}} B \to E$ in $\mathbb{T}[\mathcal{E}]$ - corresponding to a morphism $\phi^\sharp : B \to \Gamma E$ in $\mathbb{T}[\mathcal{S}]$, we want to associate a geometric morphism $\mathcal{E} \to \mathbf{Spec}(B)$. Suppose that $\mathcal{E}$ has a standard site of definition $(\mathcal{C}_\mathcal{E}, J_\mathcal{E})$. Define the functor $x^*_\phi : \mathcal{V}_B^{\mathrm{op}} \to \widehat{\mathcal{C}_\mathcal{E}}$ as sending a finitely presented etale arrow $n : B \to \mathrm{cod}(n)$ to the presheaf $x^*_\phi(n) : \mathcal{C}_\mathcal{E}^{\mathrm{op}} \to \mathcal{S}$ which associates to each c of $\mathcal{C}_\mathcal{E}$ the set of all possible factorizations

$$\begin{array}{ccc} B & \xrightarrow{\phi^\sharp} & \Gamma w E \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle E(!_c)} \\ \mathrm{cod}(n) & \xrightarrow[a]{} & E(c) \end{array}$$

of the composite $E(!_c)\phi^\sharp$ (where $!_c : c \to 1_{\mathcal{C}_\mathcal{E}}$ is the terminal map of c in $\mathcal{C}_\mathcal{E}$) through the finitely presented map n, and $s : c_1 \to c_2$ to postcomposition of a with $E(s)$.

Lemma 7.1.5.6. *For each ϕ as above, the functor x^*_ϕ lands in $\mathcal{E}$.*

Proof. We must prove that for each n in $\mathcal{V}_B$, the presheaf $x^*_\phi(n)$ is a sheaf for $J_\mathcal{E}$. But this is a consequence of E being a sheaf: for a family $(s_i : c_i \to c)_{i \in I}$ in $J_\mathcal{E}$, we have $E(c)$ is the limit of the $E(c_i)$ for the descent diagram for $(s_i)_{i \in I}$: hence, any matching family $(a_i)_{i \in I}$ with $a_i : n \to E(s_i)E(!_c)\phi^\sharp$ is in particular a family of arrows $a_i : \mathrm{cod}(n) \to E(c_i)$ satisfying the commutations of the descent diagram, and hence induces uniquely a map $(a_i)_{i \in I} : n \to E(c)$. □

The functor constructed above is actually involved in the universal factorization of ϕ.

Lemma 7.1.5.7. *For any ϕ as above, x_ϕ^* defines a geometric morphism $\mathcal{E} \to \widehat{\mathcal{V}_B^{\mathrm{op}}}$.*

Proof. This amounts to proving that x_ϕ^* is lex. Recall that $\mathcal{V}_B$ is closed under finite colimits in $B \downarrow \mathbb{T}[\mathcal{S}]$. Moreover, as (finite) limits in sheaf topoi are pointwise, it suffices to prove that for each c in $\mathcal{C}_\mathcal{E}$ the functor $x_\phi^*(-)(c) : \mathcal{V}_B^{\mathrm{op}} \to \mathbf{Set}$ is lex: but in fact $x_\phi^*(-)(c)$ is nothing but the composite of the functor $B \downarrow \mathbb{T}[\mathcal{S}][-, E(!_c)\phi^\sharp]$, which, as a representable, turns colimits into limits, along the inclusion of $\mathcal{V}_B$ into the coslice $B \downarrow \mathbb{T}[\mathcal{S}]$. Hence x_ϕ^* is lex.

□

In fact, this is because x_ϕ^* "points" to the *etale part* of ϕ as classified by the presheaf topos $\widehat{\mathcal{V}_B^{\mathrm{op}}}$. Let us precise this intuition - for which the geometric morphism x_ϕ will be called the *classifying morphism* of ϕ in the following.

Theorem 7.1.5.8. *For any morphism of the form $\phi^\flat : !_\mathcal{E}^* B \to E$ in a Grothendieck topos $\mathcal{E}$ we have a universal factorization as below*

$$\begin{array}{ccc} !_\mathcal{E}^* B & \xrightarrow{\quad\phi^\flat\quad} & E \\ & {\scriptstyle x_\phi^*(\nu_B^\flat)} \searrow \quad \nearrow {\scriptstyle u_{\phi^\flat}} & \\ & x_\phi^* \mathrm{cod} & \end{array}$$

with $u_{\phi^\flat}$ a local map.

Proof. Our strategy is to construct first a pointwise factorization for the underlying presheaves and prove it coincides with inverse image - forgetting first that x_ϕ^* land in the sheaf topos $\mathcal{E}$. We have at any object c of $\mathcal{C}_\mathcal{E}$ a lex functor

$$\mathcal{V}_B^{\mathrm{op}} \xrightarrow{x_\phi^*(-)(c)} \mathcal{S}$$

Its category of elements $\int x_\phi^*(-)(c)$ is hence cofiltered and is also the category of all factorizations of the composite $E(!_c)x^\sharp$ through an etale arrow on the left: but recall this is the category indexing the filtered colimit from which we constructed the etale part of the etale-local factorization ! In other words, we have

$$n_{E(!_c)\phi^\sharp} \simeq \operatorname*{colim}_{(n,a)\in(\int x_\phi^*(-)(c))^{\mathrm{op}}} n$$

But the latter coincides also with the expression of the inverse image (in the presheaf topos) $x_\phi^* \mathrm{cod}$, which is the left Kan extension

$$\begin{array}{ccc} \mathcal{V}_B & \xrightarrow{\mathrm{cod}} & \mathbb{T}[\mathcal{S}] \\ {\scriptstyle (x_\phi^*)^{\mathrm{op}}}\downarrow & \overset{q}{\Rightarrow} \nearrow {\scriptstyle \mathrm{lan}_{(x_\phi^*)^{\mathrm{op}}}\mathrm{cod}} & \\ (\widehat{\mathcal{C}_\mathcal{E}})^{\mathrm{op}} & & \end{array}$$

whose computation at each c returns

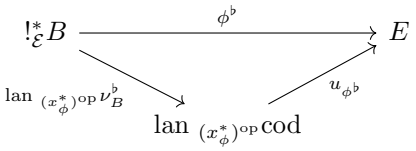
$$\begin{aligned} \mathrm{lan}_{(x_\phi^*)^{\mathrm{op}}}\mathrm{cod}(c) &\simeq \operatorname*{colim}_{(n,a)\in(\not\perp_c \downarrow x_\phi^*)^{\mathrm{op}}} \mathrm{cod}(n) \\ &\simeq \mathrm{cod} \operatorname*{colim}_{(n,a)\in(\int x_\phi^*(-)(c))^{\mathrm{op}}} n \\ &\simeq \mathrm{cod}(n_{E(!_c)\phi^\sharp}) \end{aligned}$$

In other words, the following square

$$\begin{array}{ccc} B & \xrightarrow{\quad\phi^\sharp\quad} & \Gamma E \\ {\scriptstyle \operatorname*{colim}_{(n,a)\in(\int x_\phi^*(c))^{\mathrm{op}}} n}\downarrow & & \downarrow{\scriptstyle E(!_c)} \\ \mathrm{lan}_{(x_\phi^*)^{\mathrm{op}}}\mathrm{cod}(c) & \xrightarrow[\langle a\rangle_{(n,a)\in(\int x_\phi^*(c))^{\mathrm{op}}}]{} & E(c) \end{array}$$

coincides with the etale-locale factorization of $E(!_c)\phi^\sharp$, whose local part is the induced map $u_{E(!_c)\phi^\sharp} = \langle a \rangle_{(n,a)\in(\int x_\phi^*(c))^{\mathrm{op}}}$.

Since the factorization of presheaves morphisms is pointwise, this exactly says that in $\mathbb{T}[\widehat{\mathcal{C}_\mathcal{E}}]$ the etale-locale factorization is given as



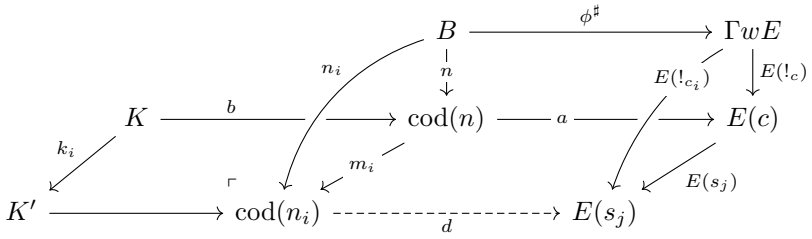
Then the desired factorization in $\mathbb{T}[\mathcal{S}]$ is obtained by the sheafification $\mathfrak{a}_{J_\mathcal{E}}$, which preserves the etale-local factorization, and turns the left Kan extension into the inverse image functor $x^*_\phi = \mathfrak{a}_{J_\mathcal{E}}\mathrm{lan}_{(x^*)^{\mathrm{op}}}$. □

Proposition 7.1.5.9. *Let ϕ be as above, and such that moreover E is a local object in $\mathbb{T}[\mathcal{S}]$. Then x^*_ϕ defines a geometric morphism $x_\phi : \mathcal{E} \to \mathbf{Spec}(B)$.*

Proof. J_B-continuity of x^*_ϕ results from localness of E: let $(m_i : n \to n_i)_{i\in I}$ be a J_B-cover in $\mathcal{V}_B$ induced from some J-cover $(k_i : K \to K_i)_{i\in I}$ along some map $b : K \to \mathrm{cod}(n)$. We have to prove that the following morphism in $\mathcal{E}$

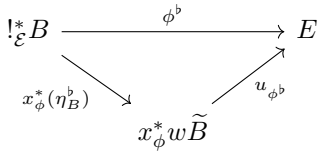
$$\coprod_{i\in I} x^*_\phi n_i \xrightarrow{\langle x^*_\phi m_i\rangle_{i\in I}} x^*_\phi n$$

is an epimorphism in $\mathcal{E}$, that is, a local surjection for $J_\mathcal{E}$. Let c be in $\mathcal{C}_\mathcal{E}$ and a in $x^*_\phi n(c)$: then the composite $ab : K \to E(c)$ is also an object of $E(K)(c)$ for E seen as a J-continuous functor $\mathcal{C}_\mathbb{T} \to \mathcal{E}$, so that $\langle E(m_i)\rangle_{i\in I}$ is itself a local surjection: hence there is a $J_\mathcal{E}$-cover $(s_j : c_j \to c)_{i\in I'}$ such that for any $j \in I'$, $E(s_j)(K)(b)$ comes from some $E(c_j)(K_i)$ for some $i \in I$: but this exactly says that for each j there is a i together with factorization d as below



and such a d is in particular an element of $x^*_\phi(n_i)(c_j)$ such that $x^*_\phi(n)(s_i)(a) = x^*_\phi(m_i)(c_j)(b)$. Whence continuity of x^*_ϕ. □

Corollary 7.1.5.10. *For ϕ as above with E a local object, the universal factorization obtained at theorem 7.1.5.8 coincides with*



Proof. This is because the inverse image part x^*_ϕ factorizes through the sheaf topos $\mathbf{Spec}(B) \simeq \mathbf{Sh}(\mathcal{V}^{\mathrm{op}}, J_B)$ via the sheafification functor $\mathfrak{a}_{J_B}$: hence doing the inverse image along x^*_ϕ localizes the sheafification map γ_B, whence

$$\begin{aligned} x^*_\phi \mathrm{cod} &\simeq x^*_\phi \mathfrak{a}_{J_B}\mathrm{cod} \\ &\simeq x^*_\phi w\widetilde{B} \end{aligned}$$

and the same for the generic etale map. □

Remark 7.1.5.11. This corollary is in fact a manifestation of admissibility: it says that the universal factorization of a map into a local object has its middle term local itself, which, in this case, says that the universal factorization above identifies the codomain functor with the structural sheaf, or that in other terms, the classifying morphism x^*_ϕ "sees" the codomain functor as a local object.

Now proposition 7.1.3.1 generalizes to the following more universal form, where points are replaced with arbitrary geometric morphisms and local form are considered up to inverse images:

Theorem 7.1.5.12. *For B in $\mathbb{T}[\mathcal{S}]$ and any Grothendieck topos, geometric morphisms $\mathcal{E} \to$ $\mathbf{Spec}(B)$ correspond to etale maps $!^*_{\mathcal{E}}B \to E$ with E a local object in $\mathbb{T}[\mathcal{E}]$.*

Proof. We saw above how one can toggle between geometric morphisms into the spectrum and local forms. We must prove that this correspondence is actually an equivalence of categories.

In one direction, consider a local form $\phi^\flat : !^*_{\mathcal{E}}B \to E$ with E local. Take its classifying morphism $x_\phi : \mathcal{E} \to \mathbf{Spec}(B)$ and then the inverse image of the generic etale map: this is exactly the left part of the factorization of $\phi^\flat$: but hence, if $\phi^\flat$ was already etale, then its etale part is an isomorphism, exhibiting

$$\phi^\flat \simeq x_\phi^*(\eta_B^\flat) \qquad E \simeq x_\phi^* w\widetilde{B}$$

In the other direction, take a geometric morphism $x : \mathcal{E} \to \mathbf{Spec}(B)$, then the induced local form $x^*(\eta_B^\flat)$, and then back the classifying morphism of the later $x^*_{x^*(\eta_B^\flat)} : \mathcal{V}_B^{\mathrm{op}} \to \mathcal{E}$. We must prove that for any n in $\mathcal{V}_B^{\mathrm{op}}$ we have an isomorphism of sheaves $x^*_{x^*(\eta_B^\flat)}(n) \simeq x^*(n)$, which amounts to natural bijections $x^*_{x^*(\eta_B^\flat)}(n)(c) \simeq x^*(n)(c)$ for each c of $\mathcal{C}_{\mathcal{E}}$. Recall that, on one hand, $x^*_{x^*(\eta_B^\flat)}(n)(c)$ is the set of all factorizations

$$\begin{array}{ccc} B & \xrightarrow{(x^*(\eta_B^\flat))^\sharp} & \Gamma x^* w\widetilde{B} \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle x^* w\widetilde{B}(!_c)} \\ \mathrm{cod}(n) & \xrightarrow[a]{} & x^* w\widetilde{B}(c) \end{array}$$

while on the other hand $x^*(n)(c)$ is by Yoneda lemma the set of all arrows $\yoneda_c \to x^*(n)$ in $\mathcal{E}$. Since x defined a morphism in $\mathbf{Spec}(B)$, its inverse image x^* factorizes through the sheafification $\mathfrak{a}_{J_B}$, so that we had a natural isomorphism between inverse images $x^* w\widetilde{B} \simeq x^*\mathrm{cod}$. The latter is the $J_{\mathcal{E}}$-sheafification of the left Kan extension $\mathrm{lan}_{(x^*)^{\mathrm{op}}}\mathrm{cod}$, whose value at c is computed as the filtered colimit $\mathrm{lan}_{(x_\phi^*)^{\mathrm{op}}}\mathrm{cod}(c) \simeq \mathrm{colim}_{(n,a)\in \yoneda_c\downarrow x_\phi^*}\mathrm{cod}(n)$: but the indexing set of this colimit is precisely $x^*(n)(c)$. Hence any factorization as above exactly corresponds to its colimit inclusion, whence the bijection. □

7.1.6 Spec ⊣ Γ-adjunction for set-valued models and sheaf representation

7.1.6.1. Now recall that $\mathcal{S}$ is terminal amongst Grothendieck topoi, and for any Grothendieck topos $\mathcal{E}$, the terminal geometric morphism $!_{\mathcal{E}} : \mathcal{E} \to \mathcal{S}$ has for direct image part $\Gamma = \mathcal{E}(1, -)$. Now as $\mathbb{T}$ is a finite limit theory, it is stable under direct image, so that Γ induces a functor

$$\mathbb{T}[\mathcal{E}] \xrightarrow{\Gamma} \mathbb{T}[\mathcal{S}]$$

In particular for any locally $\mathbb{T}_J$-modelled topos $(\mathcal{E}, E)$, we can apply $!_{\mathcal{E}*}$ to wE to get a set-valued $\mathbb{T}$-model ΓE, and for a morphism of locally $\mathbb{T}_J$-modelled topoi $(f, \phi) : (\mathcal{E}, E) \to (\mathcal{F}, F)$, we have a morphism $\Gamma\phi^\sharp : \Gamma E \to \Gamma F$, as $\Gamma f_* F = \Gamma F$ since direct images commute with global sections. Moreover, for a 2-cell $\alpha : (f, \phi) \Rightarrow (g, \psi)$, the equality $\psi^\flat F * \alpha^\flat = \phi^\flat$ corresponds to an equality $\phi^\sharp = \alpha^\sharp * F\psi^\sharp$ with $\alpha^\sharp : g_* \Rightarrow f_*$; but Γ sends $F * \alpha^\sharp$ into an equality, so that α is collapsed into the equality of the morphism $\Gamma\phi^\sharp = \Gamma\psi^\sharp$ in $\mathbb{T}[\mathcal{S}]$. This defines a 2-functor

$$\mathbb{T}_J^{\mathbf{Loc}}\text{-}\mathbf{GTop} \xrightarrow{\Gamma} \mathbb{T}[\mathcal{S}]$$

Theorem 7.1.6.2. *We have an adjunction*

$$\mathbb{T}[\mathcal{S}] \underset{\Gamma}{\overset{\mathbf{Spec}}{\perp}} \mathbb{T}_J^{\mathbf{Loc}}\text{-}\mathbf{GTop}$$

Proof. Let $(f, \phi) : (\mathbf{Spec}(B), \widetilde{B}) \to (\mathcal{E}, E)$ be a morphism of locally modelled topoi, that is the data of a geometric morphism $f : \mathcal{E} \to \mathbf{Spec}(B)$ and a local map $\phi^\flat : f^*\widetilde{B} \to E$. Then from

theorem 7.1.5.12, f defines uniquely a local form $f^*(\eta_B^\flat) : !_{\mathcal{E}}^* B \to f^*\widetilde{B}$ in $\mathbb{T}[\mathcal{E}]$, which we can compose with the local map $\phi^\flat$ to get a morphism into E

$$\begin{array}{ccc} !_{\mathcal{E}}^* B & \dashrightarrow & E \\ & {\scriptstyle f^*(\eta_B^\flat)} \searrow \quad \nearrow {\scriptstyle \phi^\flat} & \\ & f^*\widetilde{B} & \end{array}$$

which corresponds uniquely to a morphism

$$B \xrightarrow{(\phi^\flat f^*(\eta_B^\flat))^\sharp} \Gamma E$$

For the converse, any $f : B \to \Gamma E$ with $(\mathcal{E}, E)$ a locally modelled topos defines uniquely a morphism $f^\flat : !_{\mathcal{E}}^* B \to E$ in $\mathcal{E}$, whose etale-local factorization - given in corollary 7.1.5.10 defines uniquely a morphism of modelled topoi

$$(\mathbf{Spec}(B), \widetilde{B}) \xrightarrow{(x_{n_{f^\flat}}, u_{f^\flat})} (\mathcal{E}, E)$$

where $x_{n_{f^\flat}} : \mathcal{E} \to \mathbf{Spec}(B)$ is the classifying morphism of the etale part $n_{\phi^\flat} \simeq x_{n_{f^\flat}}^*(\eta_B^\flat)$ and $u_{f^\flat}$ is the local part of the factorization. □

This process can be explicitly generalized for $\mathbb{T}[\mathcal{S}]$-objects in arbitrary Grothendieck topoi. But this will be better understood in the light of the concepts of *fibered sites* and *fibered topoi*. The next section is devoted to some prerequisites on this notion, but also contains a new variation of it and some results allowing to adapt it to our situation.

However before turning to the general case and our auxiliary results about fibered topoi, we should end this section with a *representability criterion*, explaining when the adjunction above has invertible units, that is, when a set valued model is representable as the global section object of its own structural sheaf.

7.1.6.3. Recall that a Grothendieck topology is *subcanonical* if representable presheaves are sheaves, equivalently if the sheafified Yoneda embedding still is fully faithful. This also amounts that, for any J-cover $(u_i : c_i \to c)_{i\in I}$, the nerve diagram

$$\coprod_{i,j\in I} \text{ɟ}_{c_i \times_c c_j} \rightrightarrows \coprod_{i\in I} \text{ɟ}_{c_i} \xrightarrow{\langle \text{ɟ}_{u_i}\rangle_{i\in I}} \text{ɟ}_c$$

exhibits ɟ_c as the coequalizer of the left parallel pair, not only after sheafification but already in in the presheaf category.

Recall we defined at definition 3.3.1.2 the notion of extended J-covers as those $(n_i : B \to C_i)_{i\in I}$ obtained as pushouts of J-cover under an object B. In particular J-covers are extended J-covers. Now for an extended J-cover we can define its nerve

$$B \xrightarrow{\langle n_i\rangle_{i\in I}} \coprod_{i\in I} C_i \rightrightarrows \coprod_{i,j\in I} C_i \times_B C_j$$

Theorem 7.1.6.4. *Let $(\mathbb{T}, \mathcal{V}, J)$ be a geometry, with $\mathbf{Spec} \dashv \Gamma$ the associated adjunction; then the following are equivalents:*

- *any extended J-cover $(n_i : B \to C_i)_{i\in I}$ exhibits B as the limit of its nerve;*
- *for any B in $\mathbb{T}[\mathcal{S}]$ the structural presheaf* cod *is a sheaf for J_B;*
- *for any B in $\mathbb{T}[\mathcal{S}]$ the unit $\eta_B^\sharp : B \to \Gamma\widetilde{B}$ is invertible;*
- *Γ is full and faithful.*

Proof. Observe that for any J_B cover

$$\begin{array}{ccc} B & \xrightarrow{n} & \mathrm{cod}(n) \\ & \searrow^{n_i} & \downarrow{\scriptstyle m_i} \\ & & \mathrm{cod}(n_i) \end{array}$$

the family $(m_i : \mathrm{cod}(n) \to \mathrm{cod}(n_i))_{i\in I}$ is an extended J-cover by the very definition of J_B. Hence if the first condition is satisfied, we have

$$\mathrm{cod}(n) \simeq \lim \big(\prod_{i\in I} \mathrm{cod}(n_i) \rightrightarrows \prod_{i,j\in I} \mathrm{cod}(n_i +_n n_j) \big)$$

which exactly says that cod is a sheaf for J_B.

As a consequence, the sheafification of the codomain functor provides with an isomorphism $\mathrm{cod} \simeq \mathfrak{a}_{J_B}\mathrm{cod} = \widetilde{B}$ in $\mathbb{T}_J\big[[\mathcal{V}_B, \mathcal{S}]\big]$, so that the structural sheaf coincides with the codomain functor. In particular we have

$$\Gamma\widetilde{B} = \mathrm{cod}(1_B) \simeq B$$

which exhibits the unit as an isomorphism. The last item is equivalent by generality on adjunctions. □

Definition 7.1.6.5. We say that $(\mathbb{T}, \mathcal{V}, J)$ has *sheaf representation* if it satisfies one, hence all of the equivalent conditions of theorem 7.1.6.4.

Proposition 7.1.6.6. *If $(\mathbb{T}, \mathcal{V}, J)$ has sheaf representation, then the topology J is subcanonical.*

Proof. Take a J-cover $(\theta_i : \{\overline{x}_i, \phi_i\} \to \{\overline{x}, \phi\})_{i\in I}$ in $\mathcal{C}_{\mathbb{T}}$, with $(n_{\theta_i} : K_\phi \to K_{\phi_i})_{i\in I}$ the corresponding family in $\mathbb{T}[\mathcal{S}]_\omega$. Then for any $\{\overline{y}, \psi\}$ one has $\text{ⱡ}_{\{\overline{y},\psi\}}(\{\overline{x}, \phi\}) \simeq \mathbb{T}[\mathcal{S}](K_\psi, K_\phi)$. Then assuming that extended J-covers are limiting, we know that

$$K_\phi \simeq \lim \big(\prod_{i\in I} K_{\phi_i} \rightrightarrows \prod_{i,j\in I} K_{\phi_i} +_{K_\phi} K_{\phi_j} \big)$$

Then, since the covariant representable preserve limits, we have

$$\text{ⱡ}_{\{\overline{y},\psi\}}(\{\overline{x}, \phi\}) \simeq \lim \big(\prod_{i\in I} \text{ⱡ}_{\{\overline{y},\psi\}}(\{\overline{x}_i, \phi_i\}) \rightrightarrows \prod_{i,j\in I} \text{ⱡ}_{\{\overline{y},\psi\}}(\{\overline{x}, \phi\})(\{\overline{x}_i, \phi_i\} \times_{\{\overline{x},\phi\}} \{\overline{x}_j, \phi_j\}) \big)$$

This exactly says that the representable $\text{ⱡ}_{\{\overline{y},\psi\}}$ is a sheaf for J, hence that J is subcanonical. □

7.2 Fibered sites and fibered topoi

We first recall, and adapt slightly, [86][VI, part 7] results on the notion of *fibered site* and *fibered topos*. In this source, they were introduced over a base category without topology; we shall propose in this section an adaptation for the case of a topology on the base category, as it shall be used in the next section.

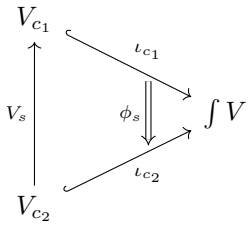
7.2.1 Fibered sites as oplax colimits

Definition 7.2.1.1. A *fibered lex site* on a small category $\mathcal{C}$ is an indexed category $V : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ such that in each $c \in \mathcal{C}$, V_c is lex and equipped with a Grothendieck topology J_c such that for each $s : c_1 \to c_2$ the corresponding transition functor $V_s : V_{c_2} \to V_{c_1}$ is a morphism of lex site.

For a fibered site on small, lex category $\mathcal{C}$, one can consider the Grothendieck construction

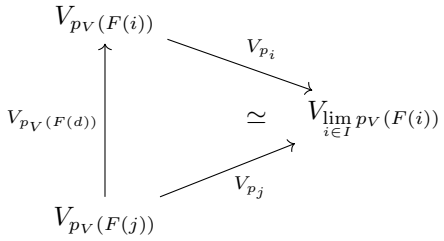
$$\int V \xrightarrow{p_V} \mathcal{C}$$

at the indexed category V, which is the oplax colimit of V, equipped with the canonical oplax cocone



Lemma 7.2.1.2. *For a fibered lex site $V : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ over a lex category $\mathcal{C}$, the oplax colimit $\int V$ is lex, as well as the fibration p_V.*

Proof. The finite limit of a finite diagram $F : I \to \int V$ is constructed as follows: first take the limit $\lim_{i\in I} p_V(F(i))$ in $\mathcal{C}$; then, we have a pseudococone diagram in $\mathcal{C}$



producing a finite diagram $(V_{p_i}(F(i)))_{i\in I}$ in $V_{\varinjlim_{i\in I} p_V(F(i))}$, where one can take the limit $\lim_{i\in I} V_{p_i}(F(i))$. Then we have

$$\lim F \simeq (\varinjlim_{i\in I} p_V(F(i)), \lim_{i\in I} V_{p_i}(F(i)))$$

□

Remark 7.2.1.3. In the case of the oplax colimits we need the indexing category to have itself finite limits; when considering pseudocolimits, it will be sufficient to require it to be cofiltered, as we wont need the underlying cone to be limiting anymore thanks to localizations along cartesian morphisms.

7.2.1.4. Now we can equip $\int V$ with a coarsest topology, the *fibered topology*, making the inclusions ι_c lex continuous: define the topology

$$J_V = \langle \bigcup_{c\in\mathcal{C}} \iota_c(J_c) \rangle$$

as generated by the inclusion of all fiber topologies, then trivially each $\iota_c : (V_c, J_c) \hookrightarrow (\int V, J_V)$ is a morphism of lex sites; the covers in the $\iota_c(J_c)$ are called *horizontal families*.

7.2.2 Fibered topos and topos of sections

Now one can ask for a fibered site as seen above what is corresponding notion of sheaf topos. This is the purpose of the following notion

Definition 7.2.2.1. A *fibered topos* on a category $\mathcal{C}$ is an indexed category $\mathcal{E}_{(-)} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ such that for any $c \in \mathcal{C}$ the fiber $\mathcal{E}_c$ is a Grothendieck topos, and for each $s : c_1 \to c_2$ the transition functor $\mathcal{E}_s : \mathcal{E}_{c_2} \to \mathcal{E}_{c_1}$ is the inverse image part of a geometric functor $f_s : \mathcal{E}_{c_1} \to \mathcal{E}_{c_2}$.

Then we can also consider the Grothendieck construction at a fibered topos and define the fibration

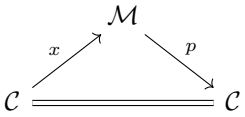
$$\int \mathcal{E}_{(-)} \xrightarrow{p_{\mathcal{E}}} \mathcal{C}$$

Remark 7.2.2.2. Observe that we consider here inverse image part, so that this Grothendieck construction is in particular an oplax colimit *in* **Cat** of the underlying categories of the fiber topoi.

7.2.2.3. Now, to a fibered site V, we can canonically associate a fibered topos whose fiber at c is the sheaf topos $\mathbf{Sh}(V_c, J_c)$ and the transition at $s : c_1 \to c_2$ is the inverse image of the geometric morphism $\mathbf{Sh}(V_s) : \mathbf{Sh}(V_{c_1}, J_{c_1}) \to \mathbf{Sh}(V_{c_2}, J_{c_2})$ induced by V_s. Then we can consider the Grothendieck construction associated to the fibered topos

$$\int \mathbf{Sh}(V_{(-)}, J_{(-)}) \xrightarrow{\overline{p_V}} \mathcal{C}$$

Definition 7.2.2.4. For a Grothendieck fibration $p : \mathcal{M} \to \mathcal{C}$, we denote as $\Gamma(p)$ the category whose objects are *sections*



and whose arrows are natural transformations between those sections. In particular, *cartesian sections* are sections x sending any arrow $s : c_1 \to c_2$ to a cartesian morphism $x(s) : x(c_1) \to x(c_2)$.

7.2.2.5. Moreover, observe that any fibered topos $\mathcal{E}_{(-)} : \mathcal{C} \to \mathbf{Cat}$ defines also a Grothendieck fibration on $\mathcal{C}^{\mathrm{op}}$ thanks to the adjunctions $f_s^* \dashv f_{s*}$, where the fiber at c is still $\mathcal{E}_c$ but the transition morphism at $s : c_1 \to c_2$ is now the direct image functor $f_{s*} : \mathcal{E}_{c_1} \to \mathcal{E}_{c_2}$. We denote as

$$\int \mathcal{E}'_{(-)} \xrightarrow{p'_{\mathcal{E}}} \mathcal{C}^{\mathrm{op}}$$

the associated fibration. In the following we call this fibration the *direct fibration* of the fibered topos $\mathcal{E}_{(-)}$.

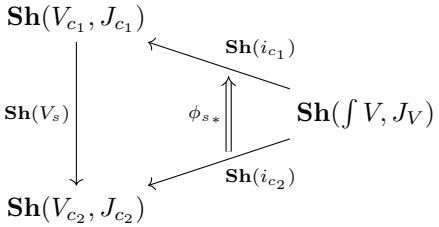
Then in particular a section $X : \mathcal{C} \to \int \mathcal{E}'_{(-)}$ of the direct fibration of a fibered topos returns at each object c an object X_c of the topos $\mathcal{E}_c$ and at an arrow $s : c_1 \to c_2$ a morphism $(s, X_s) : (c_2, X_{c_2}) \to (c_1, X_{c_1})$ with $X_s : X_{c_2} \to f_{s*}X_{c_1}$ with f_{s*} the direct image of the transition geometric morphism $\mathcal{E}_s$.

The following, which is [86][Proposition 7.4.7], says that the sheaf topos over the indexed site is the category of sections of the direct fibration associated to the fibered topos constructed by sheafification of the fibers.

Proposition 7.2.2.6. *Let $V : \mathcal{C} \to \mathbf{Cat}$ a fibered lex site over a small lex category; then we have an equivalence of categories*

$$\mathbf{Sh}(\int V, J_V) \simeq \Gamma(\overline{p_V}')$$

7.2.2.7. Moreover, sheafification turns the oplax cone made of the inclusions $(\iota_c)_{c\in\mathcal{C}}$ into an oplax cocone of direct image part of geometric morphisms

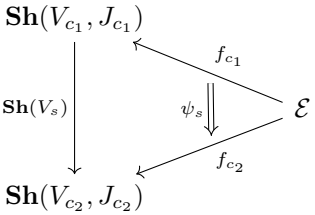


where ϕ_{s*} is the mate of the transformation $\phi_s^* : \mathbf{Sh}(\iota_{c_1})^*\mathbf{Sh}(V_s)^* \Rightarrow \mathbf{Sh}(\iota_{c_2})^*$ induced from the cocone ϕ_s. In the bicategory of Grothendieck topoi, where 2-cells direction is the one of their inverse image part, this becomes a lax cone. Then we also have the following:

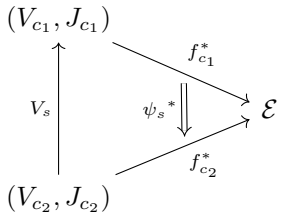
Proposition 7.2.2.8. *We have in the bicategory of Grothendieck topoi that*

$$\mathbf{Sh}(\int V, J_V) \simeq \operatorname*{laxlim}_{c\in\mathcal{C}} \mathbf{Sh}(V_c, J_c)$$

Proof. It would actually be expectable for oplaxcolimits of lex sites to be turned into oplaxlimits of topoi, as well as finite colimits of sites are turned into finite limits and filtered colimits into cofiltered limits. To see this, observe that a lax cone



is the same as an lax cocone of lex-continuous functors



which factorizes uniquely in **Cat** through the oplax colimit $f^* : \int V \to \mathcal{E}$, and this functor is both lex and J_V continuous since all its restrictions at fibers are lex continuous, so that it defines a geometric morphism $f : \mathcal{E} \to \mathbf{Sh}(\int V, J_V)$. □

7.2.2.9. From the construction above of the oplax limit, we can construct the pseudolimit in **Cat** of a fibered topos. It can be shown that the cartesian morphisms in the oplax colimit $\int V$ have a left calculus of fractions in the sense of Gabriel-Zisman, as explained in [86][Proposition 6.4], so that we can consider the localization of the oplax colimit $\int V$ at the cartesian morphisms; we know that this localization is the pseudocolimit of the pseudofunctor V in **Cat**, that is,

$$\text{pscolim } V \simeq \int V[\Sigma_V^{-1}]$$

where Σ_V denotes the class of cartesian morphisms. Moreover, the topology J_V is transferred to the localization - we still denote the induced topology as J_V. This provides the following expression of pseudolimits:

Proposition 7.2.2.10. *If V is a fibered site on a small lex category $\mathcal{C}$, we have in the bicategory of Grothendieck topoi that*

$$\mathbf{Sh}(\underset{\mathcal{C}}{\text{pscolim}}\ V, J_V) \simeq \underset{c\in\mathcal{C}}{\text{pslim}}\ \mathbf{Sh}(V_c, J_c)$$

Moreover the latter is also equivalent to the category $\Gamma_{\mathbf{Cart}}(p'_V)$ of cartesian sections of the direct fibration.

Remark 7.2.2.11. In [86][Theorem 8.2.3] the underlying category is just supposed to be cofiltered for the equivalence above to hold; in our case this condition is automatically satisfied as we supposed $\mathcal{C}$ to be lex. Beware that the results above are not necessarily true for an arbitrary small category $\mathcal{C}$. In fact, it is not known whether the bicategory of Grothendieck topoi has arbitrary small pseudolimits.

As this is the content of [86][Sections 6 and 8, in particular 8.2.3] (and has also its bilimit version for bifiltered diagrams at [34][Theorem 2.4]) we do not prove it again.

7.2.3 Topos of continuous sections

We now focus rather on the following adaptation in the case where the underlying category is endowed with a Grothendieck topology, asking for a way to make the fibration in a fibered site a comorphism of sites.

7.2.3.1. First observe that any fibered lex site has a terminal section $1_{(-)} : \mathcal{C} \to \int V$ associating to each c the terminal object 1_{V_c} of V_c and to each $s : c_1 \to c_2$. In particular this is a cartesian section as the transitions morphisms V_s for each $s : c_1 \to c_2$ are lex so that $V_s(1_{V_{c_2}}) = 1_{V_{c_1}}$, so the value of this section at $s : c_1 \to c_2$ is the cartesian lift $\overline{s} : (c_1, V_s(1_{V_{c_2}}) \to (c_2; 1_{V_{c_2}})$. Now for a topology J on $\mathcal{C}$, and $\mathcal{V} : \mathcal{C}^{\text{op}} \to \mathbf{Cat}$ a fibered site, observe that any covering family $(c_i \to c)_{i\in I}$ can be lifted to a family $(\overline{s}_i : (c_i, 1_{V_{c_i}}) \to (c, 1_{V_c}))_{i\in I}$: we call such families *horizontal*. Then consider the topology

$$J_{V,J} = \langle J_V \cup 1_{V_{(-)}}(J) \rangle$$

where $1_{(-)}(J)$ consists of all families of the form $(\overline{s}_i : (c_i, 1_{V_{c_i}}) \to (c, 1_{V_c}))_{i\in I}$.

Remark 7.2.3.2. In the context of geometries, we consider pretopologies on the fibers, and want actually to manipulate covers of a spectral pretopology rather than sieves of a topology: then it is worth precising that the result above can be rephrased in term of pretopology: if the fibers V_c

are equipped with a Grothendieck pretopology J'_c, and if one chooses a basis J_0 for the topology J on the indexing category $\mathcal{C}$ (for instance the maximal basis associated to J), then we can first generate a pretopology $J'_{V,J}$ on the oplaxcolimit from the image of those basis $\bigcup_{c\in\mathcal{C}} \iota_c(J'_c)$ and $1_{V_{(-)}}(J_0)$, which are the vertical and horizontal covers. Then observe that

– the pretopology jointly generated by the fiber basis is itself a basis for the topology on the oplax colimit, or even more directly:

$$J_V = \langle \bigcup_{c\in\mathcal{C}} \iota_c(J'_c) \rangle$$

– the pretopology jointly generated by the images $\iota_c(J'_c)$ and $1_{V_{(-)}}(J_0)$ is a basis for $J_{V,J}$:

$$J_{V,J} = \langle J'_{V,J} \rangle = \langle \bigcup_{c\in\mathcal{C}} \iota_c(J'_c) \cup 1_{V_{(-)}}(J_0) \rangle$$

We used the terminal element to canonically lift J-covers in $\mathcal{C}$ to covers in $\int V$; but once the topology is generated from those data, we get actually horizontal covers by lifting J-covers at any object in a fiber. To see this, use the following general lemma expressing that cartesian lifts of an arrow form altogether a cartesian transformation:

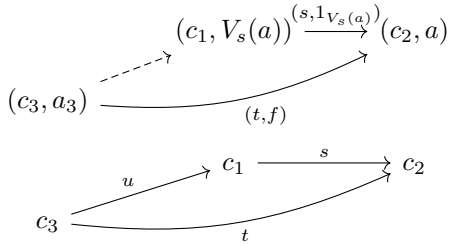
Lemma 7.2.3.3. *Let $V : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ be a lex fibration. Then for any $s : c_1 \to c_2$ the following square is a pullback*

$$\begin{array}{ccc} (c_1, V_s(a_1)) & \xrightarrow{(s,1_{V_s(a_1)})} & (c_2, a) \\ {\scriptstyle (1_{c_2},!_{V_s(a)})}\downarrow & \lrcorner & \downarrow{\scriptstyle (1_{c_2},!_a)} \\ (c_1, 1_{V_{c_1}}) & \xrightarrow[(s,1_{V_s(a_2)})]{} & (c_2, 1_{V_{c_2}}) \end{array}$$

Proof. In any other square

$$\begin{array}{ccc} (c_3, a') & \xrightarrow{(t,f)} & (c_2, a) \\ {\scriptstyle (u,!_{a'})}\downarrow & & \downarrow{\scriptstyle (1_{c_2},!_a)} \\ (c_1, 1_{V_{c_1}}) & \xrightarrow[(s,1_{V_s(a_2)})]{} & (c_2, 1_{V_{c_2}}) \end{array}$$

the vertical component of the left map is forced to be the terminal map $!_{a'}$ as V_u preserves the terminal element. But then, such a square is the same as a situation testing the cartesianness of the lift $(s, 1_{V_s(a)})$ which always produces a unique map as the dashed arrow below



which provides in particular the desired factorization. □

As a consequence, from we know that Grothendieck topologies are closed under pullback of covering families, it appears the following:

Corollary 7.2.3.4. *Let V be a fibered site on a lex category $\mathcal{C}$ and J a Grothendieck topology on $\mathcal{C}$. Then for any J-cover $(s_i : c_i \to c)_{i\in I}$ and any $a \in V_c$, the family $((s_i, 1_{V_{s_i}(a)}) : (c_i, V_{s_i}(a)) \to (c, a))_{i\in I}$ is a covering family in $J_{V,J}$.*

As defined above, the topology $J_{V,J}$ is the simplest that allows to induce a geometric morphism toward $\mathbf{Sh}(C, J)$. In fact:

Proposition 7.2.3.5. *The topology $J_{V,J}$ is the coarsest topology such that we have simultaneously the two following conditions:*

– *for each c in $\mathcal{C}$, the inclusion $\iota_c : (V_{c_1}, J_{c_1}) \to (\int V, J_{V,J})$ is a morphism of sites;*

– *the fibration* $p_V : (\int V, J_{V,J}) \to (\mathcal{C}, J)$ *is a comorphism of sites.*

Moreover, as this topology refines the topology J_V, there is a corresponding inclusion of topos

$$\mathbf{Sh}(\int V, J_{V,J}) \hookrightarrow \mathbf{Sh}(\int V, J_V)$$

exhibiting objects of $\mathbf{Sh}(\int V, J_{V,J})$ as a specific kind of sections, because the right topos was the topos of sections. The intuition is that some notion of continuity relative to the base topology is involved. We introduce the following notion to this end:

Definition 7.2.3.6. Let $\mathcal{E}_{(-)}$ be a fibered topos on a lex site $(\mathcal{C}, J)$. Then by a *continuous section of* $\mathcal{E}_{(-)}$ we mean a section of the associated direct fibration

$$\begin{array}{ccc} & \int \mathcal{E}'_{(-)} & \\ {\scriptstyle X}\nearrow & & \searrow{\scriptstyle p'_{\mathcal{E}}} \\ \mathcal{C}^{\mathrm{op}} & = & \mathcal{C}^{\mathrm{op}} \end{array}$$

such that for any J-covering family $(s_i : c_i \to c)_{i\in I}$ the lifting $(X_{s_i} : X_c \to f_{s_i *}X_{c_i})_{i\in I}$ exhibits X_c a limit in the fiber $\mathcal{E}_c$ of the diagram

$$X_c = \lim_{i\in I}\Big(\prod_{i\in I} f_{s_i *}X_{c_i} \rightrightarrows \prod_{i,j\in I} f_{s_i *}f_{s^i_{ij}*}X_{c_{ij}}\Big)$$

where the double arrow is induced from the transitions $(f_{s_i *}(X_{s^i_{ij}}) : f_{s_i *}X_{c_i} \to f_{s_i *}f_{s^i_{ij}*}X_{c_{ij}})_{i,j\in I}$ over the nerve of the cover $(s_i : c_i \to c)_{i\in I}$. We denote as $\Gamma_J(\overline{p_E}')$ the category of continuous sections and natural transformations between them.

Theorem 7.2.3.7. *Let* $(\mathcal{C}, J)$ *be a small lex site,* $V : \mathcal{C}^{\mathrm{op}} \to \mathbf{Cat}$ *a fibered lex site on* $\mathcal{C}$ *with* $\overline{p_V} : \int \mathbf{Sh}(V_{(-)}, J_{(-)}) \to \mathcal{C}$ *the associated fibered topos and* $\overline{p_V}' : \int \mathbf{Sh}(V_{(-)}, J_{(-)})' \to \mathcal{C}^{\mathrm{op}}$ *the corresponding direct fibration. Then one has an equivalence of categories*

$$\mathbf{Sh}(\int V, J_{V,J}) = \Gamma_J(\overline{p_V}')$$

Proof. For one direction, observe that any sheaf X in $\mathbf{Sh}(\int V, J_{V,J})$ can be composed with the duals of the fiber inclusions

$$\begin{array}{ccccc} V^{\mathrm{op}}_{c_1} & & \xrightarrow{\iota^{\mathrm{op}}_{c_1}} & & \\ {\scriptstyle V^{\mathrm{op}}_s}\uparrow & & {\scriptstyle \phi^{\mathrm{op}}_s}\Downarrow & & \int V^{\mathrm{op}} \xrightarrow{X} \mathcal{S} \\ V^{\mathrm{op}}_{c_2} & & \xrightarrow[\iota^{\mathrm{op}}_{c_2}]{} & & \end{array}$$

and as the topology of each fiber is part of the topology $J_{V,J}$ the restriction $X\iota^{\mathrm{op}}_c$ is a sheaf for the topology J_c, hence is an object of the sheaf topos $\mathbf{Sh}(V_c, J_c)$; moreover, for each $s : c_1 \to c_2$, the diagram below provides us by whiskering with a transformation

$$X * \phi^{\mathrm{op}}_s : X\iota^{\mathrm{op}}_{c_2} \to X\iota^{\mathrm{op}}_{c_1}V^{\mathrm{op}}_s$$

But precomposition with V_s is what the direct image $\mathbf{Sh}(V_s)_*$ consists in. So we can associate to X the section $\widehat{X} : \mathcal{C}^{\mathrm{op}} \to \int \mathcal{E}$ with $\widehat{X}_c = X\iota^{\mathrm{op}}_c = \mathbf{Sh}(\iota_c)_* X$ and $\widehat{X}_s = X * \phi_s$. Now we have to check that X is a continuous section. But we know that X is a sheaf for the horizontal topology, and also relative to any horizontal cover $((s_i, 1_{V_{s_i}(a)}) : (c_i, V_{s_i}(a)) \to (c, a))_{i\in I}$, which hence is sent on a limit diagram

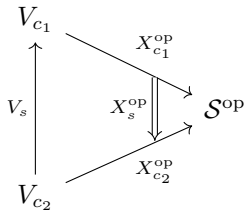
$$\widehat{X}_c(a) = \lim_{i\in I}\Big(\prod_{i\in I} \widehat{X}_{c_i}(V_{s_i}(a)) \rightrightarrows \prod_{i,j\in I} \widehat{X}_{c_{ij}}(V_{s^i_{ij}}V_{s_i}(a))\Big)$$

where

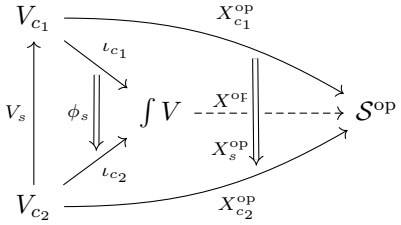
$$V_{s^i_{ij}}V_{s_i}(a) = V_{s^j_{ij}}V_{s_j}(a)$$

for each $i, j \in I$: but from the fact that limits are pointwise in categories of sheaves, this is sufficient to ensure that $\widehat{X}$ is a continuous section.

For the converse, suppose we have a continuous section $\widehat{X} : \mathcal{C}^{\mathrm{op}} \to \int \mathcal{E}'_{(-)}$. Then each X_c is a sheaf on (V_c, J_c); by the property of the oplaxcolimit, the data of the cocone



induces a unique functor



whose restriction at each fiber coincides with X_c and whose whiskering with some ϕ_s^{op} is X_s. Hence X is a sheaf for the topology $\langle \bigcup_{c \in \mathcal{C}} J_c \rangle$, with values $X(c, a) = X_c(a)$ for $a \in V_c$, and $X(s, u)$ is obtained as the composite

$$\begin{array}{ccc} X_{c_2}(a_2) & & \\ {\scriptstyle (X_s)_{a_2}}\downarrow & \searrow^{X(s,u)} & \\ X_{c_1}(V_s(a_2)) & \xrightarrow[X_{c_1}(u)]{} & X_{c_1}(a_1) \end{array}$$

where $(X_s)_{a_2}$ is the component at a_2 of the natural transformation $X_{c_2} \to \mathbf{Sh}(V_s)_* X_{c_1}$ and $X_{c_1}(u)$ is the restriction map of X_{c_1} at $u : a_1 \to V_s(a_2)$. But now, being continuous as a section, for any J-cover $(s_i : c_i \to c)_{i \in I}$ the transitions $(X_{s_i} : X_c \to \mathbf{Sh}(V_s)_* X_{c_i})_{i \in I}$ form a limit diagram in $\mathbf{Sh}(V_c, J_c)$, and as limits are pointwise in categories of sheaves, this means in particular that in 1_{V_c}, the induced horizontal cover $((s_i, \overline{s_i}) : (c_i, 1_{V_{c_i}}) \to (c, 1_{V_c})$ where $1_{V_{c_i}} = V_{s_i}(1_{V_c}))_{i \in I}$ is sent to a limit diagram

$$X(c, 1_{V_c}) = \lim_{i \in I} \Big(\prod_{i \in I} X(c_i, 1_{V_{c_i}}) \rightrightarrows \prod_{i,j \in I} X(c_{ij}, 1_{V_{c_{ij}}}) \Big)$$

which exactly means that X is a sheaf for the topology $1_{V_{(-)}}(J)$.

Now it is clear that those two processes are mutually inverse from the fact that $\int V$ is the oplaxcolimit, X and $\widehat{X}$ are mutually determined as functors. □

7.3 The spectral site of a modelled topos is a fibered site

In this section we describe the spectral site of an arbitrary $\mathbb{T}$-modelled topos for a fixed geometry $(\mathbb{T}, \mathcal{V}, J)$. While this definition of the spectral site was given in [19], we think it is worth giving new details about the geometric and fibrational aspects involved in this construction. As we shall see, the spectral site of a sheaf of $\mathbb{T}[\mathcal{S}]$-models will be constructed as a fibered site made of the spectral site of its values at basic opens of the topos it lives in. Then from the results of the previous section, the spectrum will be exhibited as a topos of global sections of the direct fibration of the fibered topos constituted of the spectra of the local values. We shall also describe the structural sheaf in this context.

Previously we proved that, in the case of set-valued models, the functor returning the topos of sheaves over the spectral site together with its structural sheaf was left adjoint to a certain global section functor. In this section, we shall see that the 2-functor associating to an arbitrary modelled topos the topos constructed from the spectral site together with its structural sheaf also defines a left adjoint, this time to the inclusion of locally modelled topoi. Equivalently, we prove that the

spectral topos classifies local forms modulo inverse images. Having therefore the same universal property as the object constructed in definition 5.2.1.2, the two constructions will be naturally equivalent.

An important result of this section is the localness of the spectrum of a local object over its base topos, generalizing proposition 7.1.3.6. We also describe how the spectrum operates relative to local values and stalks of its input sheaf of $\mathbb{T}$-models.

7.3.1 The spectral site as a fibered site

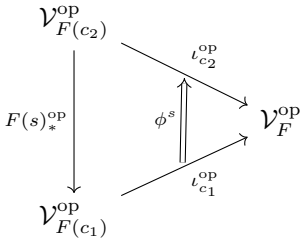
7.3.1.1. For a Grothendieck topos $\mathcal{F}$ with a lex site of presentation $\mathbf{Sh}(\mathcal{C}_\mathcal{F}, J_\mathcal{F})$ and F in $\mathbb{T}[\mathcal{F}]$, each local value $F(c)$ at basic open c of $\mathcal{C}_\mathcal{F}$ is a set valued $\mathbb{T}$-model, while for a morphism $s : c_1 \to c_2$ we have a morphism of $\mathbb{T}$-models $F(s) : F(c_2) \to F(c_1)$. This defines a pseudofunctor

$$\begin{array}{rcl} \mathcal{C}^{\mathrm{op}} & \overset{\mathcal{V}^{\mathrm{op}}_{F(-)}}{\to} & \mathbf{Lex} \\ c & \mapsto & \mathcal{V}^{\mathrm{op}}_{F(c)} \\ c_1 \overset{s}{\to} c_2 & \mapsto & \mathcal{V}^{\mathrm{op}}_{F(c_2)} \overset{(s_*)^{\mathrm{op}}}{\to} \mathcal{V}^{\mathrm{op}}_{F(c_1)} \end{array}$$

At an arrow $s : c_1 \to c_2$ in $\mathcal{C}_\mathcal{F}$, one has a pushout functor

$$\mathcal{V}_{F(c_2)} \overset{F(s)_*}{\to} \mathcal{V}_{F(c_1)}$$

relating the fiber inclusions through a natural transformation



whose component ϕ^s_n at an object $n : F(c_2) \to C$ is given by the opposite of the pushout map $(n_* F(s))^{\mathrm{op}}$ as below

$$\begin{array}{ccc} F(c_2) & \xrightarrow{F(s)} & F(c_1) \\ {\scriptstyle n}\downarrow & \ulcorner & \downarrow{\scriptstyle F(s)_* n} \\ C & \xrightarrow[n_* F(s)]{} & F(s)_* C \end{array}$$

The data of the fibers inclusions $\iota_c : \mathcal{V}^{\mathrm{op}}_{F(c)} \hookrightarrow \mathcal{V}^{\mathrm{op}}_F$ together with those transformations define an oplax cocone exhibiting $\mathcal{V}^{\mathrm{op}}_F$ as the oplax colimit of the indexed category $\mathcal{V}^{\mathrm{op}}_{F(-)} : \mathcal{C}^{\mathrm{op}}_\mathcal{F} \to \mathbf{Cat}$. Beware that the cocone made of the etale generator themselves $\mathcal{V}_{F(-)} : \mathcal{C}^{\mathrm{op}}_\mathcal{F} \to \mathbf{Cat}$ is a *lax cocone*.

Remark 7.3.1.2. Beware here that the indexed site has as value $\mathcal{V}^{\mathrm{op}}_{F(c)}$ and not $\mathcal{V}_{F(c)}$ itself. The applications of the previous section to this one will hence require carefull substitution of $\mathcal{V}^{\mathrm{op}}$ to V...

Definition 7.3.1.3. Define the category $\mathcal{V}_F$ as the following lax colimit in **Lex**

$$\mathcal{V}_F = \operatorname*{laxcolim}_{c \in \mathcal{C}^{\mathrm{op}}_\mathcal{F}} \mathcal{V}_{F(c)}$$

Remark 7.3.1.4. Observe we have then an oplax colimit

$$\mathcal{V}^{\mathrm{op}}_F = \operatorname*{oplaxcolim}_{c \in \mathcal{C}^{\mathrm{op}}_\mathcal{F}} \mathcal{V}^{\mathrm{op}}_{F(c)}$$

which will be the underlying category of the spectral site.

More explicitely, $\mathcal{V}_F$ has as objects the pairs (c, n) with $c \in \mathcal{C}_\mathcal{F}$ and n a morphism in $\mathcal{V}_{F(c)}$, that is some $n : F(c) \to \mathrm{cod}(n)$ gotten as a pushout of map in $\mathcal{V}$. As morphisms, it has the pairs

$(s,h) : (c_1, n_1) \to (c_2, n_2)$ with $s : c_2 \in c_1$ and $f : \mathrm{cod}(n_1) \to \mathrm{cod}(n_2)$ related by a commutative square

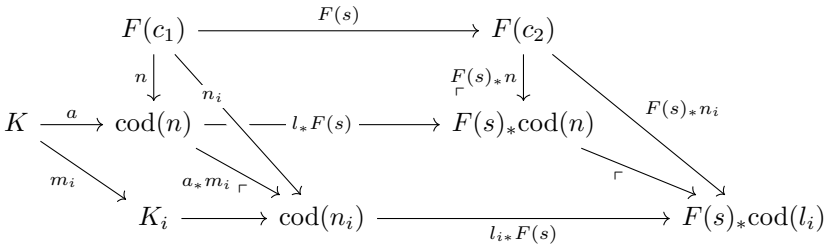
$$\begin{array}{ccc} F(c_1) & \xrightarrow{F(s)} & F(c_2) \\ {\scriptstyle l_1}\downarrow & & \downarrow{\scriptstyle n_2} \\ \mathrm{cod}(n_1) & \xrightarrow[f]{} & \mathrm{cod}(n_2) \end{array}$$

Now recall that each $\mathcal{V}_{F(c)}$ is part of the spectral site $(\mathcal{V}^{\mathrm{op}}_{F(c)}, J_{F(c)})$ of $F(c)$. Moreover we have the following:

Proposition 7.3.1.5. *For each $s : c_1 \to c_2$ we have a morphism of lex sites*

$$(\mathcal{V}^{\mathrm{op}}_{F(c_2)}, J_{F(c_2)}) \xrightarrow{F(s)_*} (\mathcal{V}^{\mathrm{op}}_{F(c_1)}, J_{F(c_1)})$$

Proof. For a family in $\mathcal{V}_{F(c')}$ obtained as a pushout along some a of a covering family $(n_i : K \to K_i)$ of finite presentation under $\mathrm{cod}(n)$, then composition of pushouts ensures that this family is transferred into a covering family of $F(s)_* n$ in $\mathcal{V}_{F(c_2)}$ as visualised below



□

7.3.1.6. From lemma 7.2.1.2 we know that the oplax colimit $\mathcal{V}^{\mathrm{op}}_{F(-)}$ inherits finite limits of the fibers $\mathcal{V}^{\mathrm{op}}_{F(c)}$ and the basis $\mathcal{C}$: for a finite diagram $((c_i, n_i))_{i\in I}$ one first computes the finite limit $(p_i : \lim_{i\in I} c_i \to c_i)_{i\in I}$ in $\mathcal{C}_{\mathcal{F}}$, which is sent to a cocone $((F(p_i) : F(c_i) \to F(\lim_{i\in I} c_i))_{i\in I}$. Then pushing the n_i along the transition functor $F(p_i)$ defines a finite diagram $(F(p_i)_* n_i)_{i\in I}$ in $F(\lim_{i\in I})$, and since $\mathcal{V}_{F(\lim_{i\in I})}$ has finite colimits, we have

$$\lim_{i\in I} (c_i, n_i) \simeq (\lim_{i\in I} c_i, \operatorname*{colim}_{i\in I} F(p_i)_* n_i)$$

In particular, pullbacks are computed as follows:

$$\begin{array}{ccc} (c_1 \times_c c_2, F(p_1)_* F(u_1)_* n_1 +_{F(p)_* n} F(p_2)_* F(u_2)_* n_2) & \longrightarrow & (c_2, n_2) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (u_2, f_2)} \\ (c_1, n_1) & \xrightarrow[(u_1, f_1)]{} & (c, n) \end{array}$$

where $F(p_1)_* F(u_1)_* n_1 +_{F(p)_* n} F(p_2)_* F(u_2)_* n_2)$ is the pushout in $\mathcal{V}_{F(c_1 \times_c c_2)}$ of the span

$$\begin{array}{ccc} & F(p)_* n & \xrightarrow{F(p)_* \langle n_2, f_2 \rangle} F(p_2)_* F(u_2)_* n_2 \\ {\scriptstyle F(p)_* \langle n_1, f_1 \rangle}\downarrow & & \\ F(p_1)_* F(u_1)_* n_1 & & \end{array}$$

7.3.2 The spectrum from the fibered spectral site

7.3.2.1. Observe that the fibration $p_F : \mathcal{V}^{\mathrm{op}}_F \to \mathcal{C}_{\mathcal{F}}$ has a left adjoint

$$\mathcal{C}_{\mathcal{F}} \xhookrightarrow{\iota_F} \mathcal{V}^{\mathrm{op}}_F$$

sending an object c to the pair $(c, 1_{F(c)})$ with the identity arrow $1_{F(c)} : F(c) \to F(c)$, which is etale and finitely presented (and is the terminal object of $\mathcal{V}^{\mathrm{op}}_{F(c)}$).

7.3.2.2. Applying what was done in the previous section, if we fix $J_{\mathcal{F}}$ as a subcanonical pretopology such that $\mathcal{F} \simeq \mathbf{Sh}(\mathcal{C}_{\mathcal{F}}, J_{\mathcal{F}})$, we can equip the oplax colimit $\mathcal{V}_F$ with the pretopology induced jointly from the spectral pretopologies of the local values and the underlying pretopology, that is,

$$J_F = \langle \iota_F(J_{\mathcal{F}}) \cup \bigcup_{c \in \mathcal{C}_{\mathcal{F}}} \iota_c(J_{F(c)}) \rangle$$

In other words this pretopology is the pretopology jointly generated from *horizontal* families of the form

$$((c, 1_c) \xrightarrow{(s_i, F(s_i))} (c_i, 1_{c_i}))_{i \in I} \text{ with } (c \xrightarrow{s_i} c_i)_{i \in I} \in J^{\mathrm{op}}_{\mathcal{F}}(c)$$

and *vertical families* of the form

$$((c, n) \xrightarrow{(1_c, m_i)} (c, n_i))_{i \in I} \text{ with } \left(\begin{array}{ccc} F(c) & \xrightarrow{n} & \mathrm{cod}(n) \\ & \searrow^{n_i} & \downarrow m_i \\ & & \mathrm{cod}(l_i) \end{array} \right)_{i \in I} \in J^{\mathrm{op}}_{F(c)}(n)$$

Remark 7.3.2.3. In general, it can be quite difficult to give an explicit description of a topology generated this way; however the pretopology generated is more easy to handle. In this context we can make the following remarks. Recall that in a fibered category, any map factorizes uniquely as a vertical arrow followed by an horizontal, cartesian arrow: here this corresponds to

$$\begin{array}{ccc} (d, m) & \xrightarrow{(s,f)} & (c, n) \\ & \searrow^{(1_d, \langle f, m \rangle)} \quad \nearrow_{(s, 1_{F(s)_* n})} & \\ & (d, F(s)_* n) & \end{array}$$

where $\langle f, m \rangle$ is the map induced from the pushout property.

By the pullback axiom of pretopology, observe that if $(s_i : c_i \to c)_{i \in I}$ is covering in $J_{\mathcal{F}}$ and n is in $\mathcal{V}_{F(c)}$ then the family of cartesian lifts

$$(c_i, F(s_i)_* n) \xrightarrow{(s_i, 1_{F(s_i)_* n})} (c, n)$$

is a cover for J_F as being the pullback of the family $\iota_F((s_i)_{i \in I}$ along the morphism $(c, n) \to (c, 1_{F(c)})$ by lemma 7.2.3.3.

For $s : d \to c$ in $\mathcal{C}_{\mathcal{F}}$ and n in $\mathcal{V}_{F(c)}$, then the identity arrow $1_{F(s)_* n}$ is a $J_{F(d)}$ cover of $F(s)_* n$ as it is induced by pushing out the identity map of K for any $a : K \to F(c)$. Then a cartesian morphism is a covering singleton as soon as s itself is a covering singleton in $J_{\mathcal{F}}$.

Now, for any $(s_i : c_i \to c)_{i \in I}$ in $J_{\mathcal{F}}$ and $(m_j : n \to n_j)_{j \in I'}$ a cover in $J_{F(c)}$, then for each $i \in I$ the pushout family $(F(s_i)_* m_j : F(s_i)_* n \to F(s_i)_* n_j)_{j \in I'}$ is a cover in $J_{F(c_i)}$.

But now, by transitivity axiom, we have a particular kind of covering families in J_F constructed as follows. Take a covering family $(s_i : c_i \to c)_{i \in I}$ in $J_{\mathcal{F}}(c)$, an object n in $\mathcal{V}_{F(c)}$, and for each $i \in I$ a covering family $(m_{ij} : F(s_i)_* n \to n_{ij})_{i \in I_i}$ in $J_{F(c_i)}$: then the induced family

$$((s_i, m_{ij} n_* F(s)) : (c_i, n_{ij}) \to (c, n))_{i \in I, j \in I_i}$$

must be covering in J_F.

However beware that it may happen that there will be families $(s_i, d_i)_{i \in I}$ in J_F whose underlying family $(s_i)_{i \in I}$ is not in $J_{\mathcal{F}}$. Otherwise the projection p_F would be a morphism of site, which is not true in general.

Definition 7.3.2.4. Define the *spectral site* of the $\mathbb{T}$-modelled topos $(\mathcal{F}, F)$ as the site $(\mathcal{V}_F^{\mathrm{op}}, J_F)$. Then define the *(Coste) spectrum* as

$$\mathbf{Spec}(\mathcal{F}, F) = \mathbf{Sh}(\mathcal{V}_F^{\mathrm{op}}, J_F)$$

In particular, this construction generalizes the construction of the spectrum of a set valued $\mathbb{T}$-model, as those ones just are the modelled topos of the form $(*, B)$ with B seen as the constant sheaf of $\mathbb{T}$-models on the point.

7.3.2.5. It is worth understanding how the spectrum $\mathbf{Spec}(F)$ and its structural sheaf $\widetilde{F}$ compare to the spectra $\mathbf{Spec}(F(c))$ and structural sheaves $\widetilde{F(c)}$ of the local values. Recall that the oplax colimit $\mathcal{V}_F^{\mathrm{op}} \simeq \mathrm{oplaxcolim}_{c\in\mathcal{C}_{\mathcal{F}}^{\mathrm{op}}} \mathcal{V}_{F(c)}^{\mathrm{op}}$ is canonically equipped with an induced pretopology

$$J_{\mathcal{V}_{F(-)}^{\mathrm{op}}} = \langle \bigcup_{c\in\mathcal{C}_{\mathcal{F}}} \iota_c(J_{F(c)})\rangle$$

such that we have a geometric equivalence in the bicategory of Grothendieck topoi

$$\mathbf{Sh}(\mathcal{V}_F, J_{\mathcal{V}_{F(-)}^{\mathrm{op}}}) \simeq \underset{c\in\mathcal{C}_{\mathcal{F}}}{\mathrm{laxlim}}\ \mathbf{Spec}(F(c))$$

Now from the topology J_F is generated from $J_{\mathcal{V}_{F(-)}^{\mathrm{op}}}$ together with $\iota_F(J_{\mathcal{F}})$ we have a geometric inclusion

$$\mathbf{Spec}(F) \xhookrightarrow{\mathfrak{i}_F} \underset{c\in\mathcal{C}_{\mathcal{F}}}{\mathrm{laxlim}}\ \mathbf{Spec}(F(c))$$

whose inverse image part is the sheafification functor $\mathfrak{a}_{\iota_F(\mathcal{F})}$: intuitively, this functor "corrects" sheaves for the generated pretopology $J_{\mathcal{V}_{F(-)}^{\mathrm{op}}}$ into sheaves for the spectral pretopology J_F.

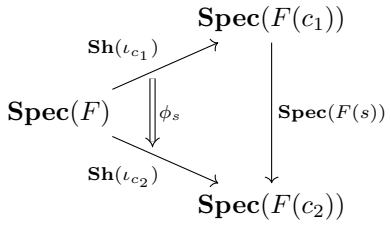
Precomposing $\mathfrak{i}_F$ together with the limiting projections

$$\underset{c\in\mathcal{C}_{\mathcal{F}}}{\mathrm{laxlim}}\ \mathbf{Spec}(F(c)) \xrightarrow{p_c} \mathbf{Spec}(F(c))$$

produces a lax cone

$$(\mathbf{Spec}(F) \xrightarrow{p_c\mathfrak{i}_F} \mathbf{Spec}(F(c)))_{c\in\mathcal{C}_{\mathcal{F}}}$$

where $p_c\mathfrak{i}_F = \mathbf{Sh}(\iota_c)$. But now recall that $\mathbf{Sh}(F(s)_*) = \mathbf{Spec}(F(s))$, seeing $F(s) : F(c_2) \to F(c_1)$ as a morphism in $\mathbb{T}[\mathcal{S}]$: hence the transition morphisms of this cone are given as



7.3.3 The spectrum as a topos of continuous sections and the canonical fibration

7.3.3.1. Now we turn to the relation between the spectrum and its base topos. We have a fibered lex site $\mathcal{V}_{F(-)}^{\mathrm{op}} : \mathcal{C}^{\mathrm{op}} \to \mathbf{Lex}$. Now recall that each of the sites $(\mathcal{V}_{F(c)}^{\mathrm{op}}, J_{F(c)})$ is the spectral site of $F(c)$, that is, $\mathbf{Spec}(F(c)) = \mathbf{Sh}(\mathcal{V}_{F(c)}^{\mathrm{op}}, J_{F(c)})$. Then the associated fibered topos has the spectrum of the $F(c)$ as fiber, that is, we have a fibered topos

$$\int \mathbf{Spec}(F(-)) \xrightarrow{\overline{p_F}} \mathcal{C}$$

whose transition morphisms are the inverse image $\mathbf{Spec}(F(s))^* : \mathbf{Spec}(F(c_2)) \to \mathbf{Spec}(F(c_1))$ for $s : c_1 \to c_2$. We can consider its associated direct fibration

$$\int \mathbf{Spec}(F(-))' \xrightarrow{\overline{p_F}'} \mathcal{C}^{\mathrm{op}}$$

Recognize now in the definition of the topology J_F that it is generated from the horizontal and vertical families as done in the previous section. Applying theorem 7.2.3.7 we have the following:

Theorem 7.3.3.2. *The spectrum of F is the topos of continuous sections of the direct fibration associated to the fibered spectral topos:*

$$\mathbf{Spec}(F) \simeq \Gamma_J(\overline{p_F}')$$

7.3.3.3. We saw above that the fibration of site p_F is equipped here with a left adjoint ι_F; moreover from the way J_F was defined, we know trivially ι_F to define a morphism of site. Conversely the projection p_F, though not being a morphism of site, is a comorphism of site. Hence we are in a situation

$$(\mathcal{V}_F^{\mathrm{op}}, J_F) \underset{\iota_F}{\overset{p_F}{\rightleftarrows}} (\mathcal{C}_{\mathcal{F}}, J_{\mathcal{F}}) \quad \bot$$

with a comorphism of sites right adjoint to a morphism of sites. Hence from [15][Proposition 3.14] or also [69][Theorem VII.10.5] they both induce a same geometric morphism

$$\mathbf{Spec}(F) \xrightarrow{h_F} \mathcal{F}$$

which moreover lifts to an adjoint functor between categories of $\mathbb{T}$-models

$$\mathbb{T}[\mathbf{Spec}(F)] \underset{h_{F*}}{\overset{h_F^*}{\rightleftarrows}} \mathbb{T}[\mathcal{F}] \quad \bot$$

with the inverse and direct part respectively given by

$$h_F^* = \mathfrak{a}_{J_F}((-) \circ p_F^{\mathrm{op}}) = \mathrm{lan}_{\iota_F^{\mathrm{op}}}(-)$$

$$h_{F*} = \mathfrak{a}_{J_{\mathcal{F}}}(\mathcal{C}_{\mathcal{F}}(p_F^{\mathrm{op}}, -)) = (-) \circ \iota_F^{\mathrm{op}}$$

Beware that in the computation of the inverse image h_F^*, precomposing a $J_{\mathcal{F}}$-sheaf with the comorphism p_F^{op} does not return a J_F sheaf (since the vertical families have no reason to be taken into account by this process), hence the necessity of the further sheafification. Observe that we must op the functors ι_F and p_F as the involved sheaves of $\mathbb{T}$-objects have $\mathcal{V}_F$ and $\mathcal{C}^{\mathrm{op}}$ as domains.

7.3.4 The structural sheaf

Again, the structural sheaf is obtained as the sheafification of the codomain functor: observe this latter is the functor

$$\mathcal{V}_F \xrightarrow{\mathrm{cod}_F} \mathbb{T}[\mathcal{S}]$$

defined as acting fiberwisely as the codomain functor, that is, sending (c, n) on $\mathrm{cod}_c(n)$, seeing n as an object of $\mathcal{V}_{F(c)}$, and $(s, f) : (c_1, n_1) \to (c_2, n_2)$ as the underlying map $f : \mathrm{cod}(n_1) \to \mathrm{cod}(n_2)$. In other word cod_F is the functor induced by the laxcolimit property of $\mathcal{V}_F$ from the laxcocone

$$\begin{array}{ccc} \mathcal{V}_{F(c_2)} & & \\ \downarrow{\scriptstyle F(s)_*} & \overset{\mathrm{cod}_{c_2}}{\searrow}\ \Downarrow{\scriptstyle \phi_s}\ \underset{\mathrm{cod}_{c_1}}{\nearrow} & \mathbb{T}[\mathcal{S}] \\ \mathcal{V}_{F(c_1)} & & \end{array}$$

Definition 7.3.4.1. The *structural sheaf* of the modelled topos $(\mathcal{F}, F)$ is the sheaf

$$\widetilde{F} = \mathfrak{a}_{J_F} \mathrm{cod}_F$$

obtained as the sheafification of the presheaf of $\mathbb{T}$-objects defined on $\mathcal{V}_F$ as the codomain functor

$$\overline{F} : (c, n) \mapsto \mathrm{cod}(n)$$

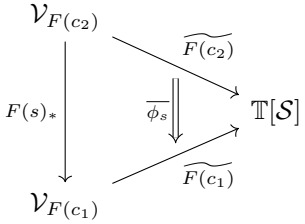
where we denote as $\gamma : \mathrm{cod} \to \mathfrak{a}_{J_F} \mathrm{cod}$ the unit of the sheafification at cod.

Remark 7.3.4.2. Hence this structural sheaf is obtained as applying successively the sheafification of vertical, then horizontal families.

7.3.4.3. Now at the level of the structural sheaves, observe that the data of all the $(\widetilde{F(c)})_{c\in\mathcal{C}_\mathcal{F}}$ induce a lax cocone

$$(\mathcal{V}_{F(c)} \xrightarrow{\widetilde{F(c)}} \mathbb{T}[\mathcal{S}])_{c\in\mathcal{C}_\mathcal{F}}$$

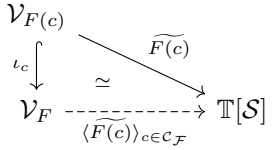
whose transition 2-cells are given by



where $\overline{\phi_s}$ is the morphism of sheaves induced from the comparison functor obtained from the pushout maps $\phi_s = (n_*F(s))_{n\in\mathcal{V}_{F(c_2)}}$ after sheafification

$$\begin{array}{ccc} \mathrm{cod}_{c_2} & \xRightarrow{\phi_s} & F_{s*}\mathrm{cod}_{c_1} \simeq \mathrm{cod}_{c_1}F_{s*} \\ {\scriptstyle\gamma_{c_2}}\Downarrow & & \Downarrow{\scriptstyle\gamma_{c_1}} \\ \widetilde{F(c_2)} & \xRightarrow[\overline{\phi_s}]{} & \mathbf{Spec}(F_s)_*\widetilde{F(c_2)} \end{array}$$

Then by the universal property of the oplax colimit in **Cat** this induces uniquely a canonical functor



Or in other terms, we know that

$$p_{c*}\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}} \simeq \widetilde{F(c)}$$

Moreover, for each $\widetilde{F}(c)$ is a $J_{F(c)}$ sheaf of $\mathbb{T}[\mathcal{S}]$-objects, the induced $\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}$ is a $J^{\mathrm{op}}_{\mathcal{V}_{F(-)}}$-sheaf.

Recalling that the lax limit topos is also the category of all sections of the direct fibration $\Gamma(\overline{p_F}')$, this object can also be described as the functor

$$\mathcal{C}_\mathbb{T} \xrightarrow{\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}} \operatorname*{laxlim}_{c\in\mathcal{C}_\mathcal{F}} \mathbf{Spec}(F(c))$$

sending each $\{\overline{x},\phi\}$ to the section of the direct fibration

$$\mathcal{C}_\mathcal{F} \xrightarrow{\widetilde{F(-)}(\{\overline{x},\phi\})} \int \mathbf{Spec}(F(-))'$$

sending c to $\widetilde{F(c)}(\{\overline{x},\phi\})$ in $\mathbf{Spec}(F(c))$.

Beware however that, at this step, this induced sheaf is not yet a sheaf for J_F as the $\iota_F(J_\mathcal{F})$-families are not considered in $J^{\mathrm{op}}_{\mathcal{V}_{F(-)}}$.

Lemma 7.3.4.4. *The structural sheaf is the sheafification for the horizontal topology of the induced presheaf*

$$\widetilde{F} \simeq \mathfrak{a}_{\iota_\mathcal{F}}(\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}})$$

Proof. Each $\widetilde{F(c)}$ is the sheafification of the codomain functor for the spectral topology at c; but the topology in the lax limit topos is exactly the topology jointly generated by the fiberwise spectral topology, while the transition functors are continuous: hence being a sheaf for this topology amounts to being locally a sheaf for the fiberwise topologies, and we have

$$\begin{aligned} \langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}} &= \langle\mathfrak{a}_{J_{F(c)}}\mathrm{cod}_c\rangle_{c\in\mathcal{C}_\mathcal{F}} \\ &\simeq \mathfrak{a}_{\bigcup_{c\in\mathcal{C}_\mathcal{F}} \iota_c(J_{F(c)})}\langle\mathrm{cod}_c\rangle_{c\in\mathcal{C}_\mathcal{F}} \end{aligned}$$

but then it suffice to apply the sheafification for the horizontal topology to get the structural sheaf. □

7.3.4.5. In particular, there is a canonical way to compare the global structural sheaf to the structural sheaves associated to local values. At any c the codomain functor over $\mathcal{V}_{F(c)}$ and the restriction along ι_c of the codomain functor over $\mathcal{V}_F$ coincide. Yet however, this is not sufficient for the structural sheaves to coincide after restriction. As a direct image, $\mathbf{Sh}(\iota_c)_*$ does not commute with sheafification: hence $\mathbf{Sh}(\iota_c)_*\widetilde{F} = \mathbf{Sh}(\iota_c)_*\mathfrak{a}_{J_F}\mathrm{cod}$ needs not be the sheafification of cod for $J_{F(c)}$, though it is a sheaf for $J_{F(c)}$ because ι_c is a morphism of site. Hence by the universal property of the sheafification, we have a factorization in $\mathbb{T}[\mathbf{Spec}(F(c))]$

$$\begin{array}{ccc} \mathrm{cod} & = & \iota_{c*}\mathrm{cod} \\ {\scriptstyle \gamma_{\mathrm{cod}}}\downarrow & & \downarrow{\scriptstyle \iota_{c*}(\gamma_{\mathrm{cod}})} \\ \mathfrak{i}_{F(c)}\mathfrak{a}_{J_{F(c)}}\mathrm{cod} & \dashrightarrow & \iota_{c*}\mathfrak{i}_F\mathfrak{a}_{J_F}\mathrm{cod} \end{array}$$

where $\iota_{c*}\mathfrak{i}_F = \mathfrak{i}_F\mathbf{Sh}(\iota_c)_*$ and $\mathfrak{i}_{F(c)}$ is the geometric inclusion $\mathbf{Spec}(F(c)) \hookrightarrow [\mathcal{V}_{F(c)}, \mathcal{S}]$. We denote this morphism as

$$\widetilde{F(c)} \xrightarrow{\zeta_c^\sharp} \mathbf{Sh}(\iota_c)_*\widetilde{F}$$

and its mate through the $\mathbf{Sh}(\iota_c)$ adjunction as

$$\mathbf{Sh}(\iota_c)^*\widetilde{F(c)} \xrightarrow{\zeta_c^\flat} \widetilde{F}$$

Theorem 7.3.4.6. *The structural sheaf $\widetilde{F}$ is a local object in $\mathbb{T}[\mathbf{Spec}(F)]$.*

Proof. In fact we are going to prove that the structural sheaf is already a local object in the lax limit topos even before sheafification, that is, we prove that $\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}$ is already a local object in laxlim $_{c\in\mathcal{C}_\mathcal{F}}\mathbf{Spec}(F(c))$, We claim this functor to be a local object. For each $(n_i : K_\phi \to K_{\phi_i})_{i\in I}$ in J, we must display an epimorphism in the category of sections

$$\coprod_{i\in I}\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}(\{\overline{x}_i, \phi_i\}) \xrightarrow{\langle\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}(\theta_{n_i})\rangle_{i\in I}} \langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}(\{\overline{x}, \phi\})$$

As in the set-valued case, this is obtained by proving that this morphism comes from a local epimorphism in $\widehat{\mathcal{V}_F^{\mathrm{op}}}$ given by

$$\coprod_{i\in I}\mathbb{T}[\mathcal{S}][(K_{\phi_i}, \mathrm{cod}_F])) \xrightarrow{\langle\mathbb{T}[\mathcal{S}][(n_i,\mathrm{cod}_F])\rangle_{i\in I}} \mathbb{T}[\mathcal{S}][(K_\phi, \mathrm{cod}_F])$$

For each (c, n), we know that the restriction $\langle\mathbb{T}[\mathcal{S}][(n_i, \mathrm{cod}_c])\rangle_{i\in I}$ at c is a local epimorphism, as seen in proposition 7.1.4.3 since, for each $a : K_\phi \to \mathrm{cod}(n)$, the cover induced as the pushout $(a_*n_i)_{i\in I}$ produced a family of antecedents of a ensuring local surjectivity: but then the family $((c, n) \to (c, a_*n_in))_{i\in I}$ is itself convenient in $\widehat{V_F^{\mathrm{op}}}$, and being a family of for a vertical cover, this proves $\langle\mathbb{T}[\mathcal{S}][(n_i, \mathrm{cod}_c])\rangle_{i\in I}$ to be a local epimorphism relative to the vertical topology $\bigcup_{c\in\mathcal{C}_\mathcal{F}} \iota_c(J_{F(c)})$.

Since $\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}$ is the sheafification of cod_F for the vertical topology, this proves $\langle\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}(\theta_{n_i})\rangle_{i\in I}$ to be an epimorphism in the lax limit topos. But now, from lemma 7.3.4.4, for the structural sheaf is the sheafification of $\langle\widetilde{F(c)}\rangle_{c\in\mathcal{C}_\mathcal{F}}$ along the horizontal topology, and sheafification preserves epimorphisms, we deduce $\widetilde{F}$ to be a local object in $\mathbf{Spec}(F)$ □

7.3.4.7. Now we turn to the canonical morphisms of sheaves associated with the structural sheaf. At an object (c, n) of the spectral site $\mathcal{V}_F^{\mathrm{op}}$ we are provided with a canonical arrow given by the composite of n with the sheafification

$$\begin{array}{ccc} F(c) & \xrightarrow{(g_F)_{(c,n)}} & (\mathfrak{a}_{J_F}\mathrm{cod})(n) \\ & {\scriptstyle n}\searrow \quad \nearrow{\scriptstyle \gamma_{(c,n)}} & \\ & \mathrm{cod}(n) & \end{array}$$

This map can be shown to be natural in (c, n), and having in mind that $F(c) = F(p_F^{\mathrm{op}}(c, n))$, this defines a natural transformation in the category of presheaves $[\mathcal{V}_F, \mathcal{S}]$:

$$F \circ p_F^{\mathrm{op}} \xrightarrow{g_F} \mathfrak{i}_{F*}(w\widetilde{F})$$

(where we also denote $\mathfrak{i}_F : \mathbf{Spec}(F) \hookrightarrow [\mathcal{V}_F, \mathcal{S}]$), which in turns factorizes uniquely through the sheafification in $\mathbf{Spec}(F)$

$$h_F^* F \xrightarrow{\eta_F^\flat} \widetilde{F}$$

Moreover, its mate

$$F \xrightarrow{\eta_F^\sharp} h_{F*} w\widetilde{F}$$

along the adjunction $h_F^* \dashv h_{F*}$ indexes the values of $\eta_F^\flat$ at the top element of the fibers, as we have $h_{F*} w\widetilde{F}(c) = w\widetilde{F}(\iota_F(c)) = w\widetilde{F}(c, 1_{F(c)})$, so that we have

$$(\eta_F^\sharp)_c = (\eta_F^\flat)_{(c, 1_{F(c)})}$$

Proposition 7.3.4.8. *The natural transformation $\eta_F^\flat$ is an etale map in $\mathbb{T}[\mathbf{Spec}(F)]$, while $\eta_F^\sharp$ is etale in $\mathbb{T}[\mathcal{F}]$.*

Proof. By naturality of the unit of the sheafification $\gamma : \mathrm{cod} \Rightarrow \widetilde{F}$, the triangles above are natural in (c, n) and exhibit a factorization of η_F in the category of $\mathbb{T}$-models in the presheaf topos $[\mathcal{V}_F, \mathcal{S}]$

$$\begin{array}{ccc} F \circ p_F^{\mathrm{op}} & \xrightarrow{g_F} & \mathfrak{i}_{F*} w\widetilde{F} \\ & {\scriptstyle \nu_F} \searrow \quad \nearrow {\scriptstyle \gamma_{\mathrm{cod}}} & \\ & \mathrm{cod} & \end{array}$$

where ν_F is the canonical map induced from all the $n : F(c) \to \mathrm{cod}(n)$ for (c, n) in $\mathcal{V}_F^{\mathrm{op}}$, which is etale as it is pointwise etale. But now, as $\mathfrak{i}_{F*}$ is full and faithful with left adjoint $\mathfrak{a}_{J_F}$, we have a natural isomorphism $\mathfrak{a}_{J_F} \mathfrak{i}_{F*} \simeq 1$, so that the aforementioned triangle is sent after sheafification to the following triangle in $\mathbb{T}[\mathbf{Spec}(F)]$

$$\begin{array}{ccc} h_F^* F & \xrightarrow{\eta_F^\flat} & w\widetilde{F} \\ & {\scriptstyle \mathfrak{a}_{J_F}\nu_F} \searrow \quad \nearrow {\scriptstyle \mathfrak{a}_{J_F}\gamma_{\mathrm{cod}}} & \\ & \mathfrak{a}_{J_F}\mathrm{cod} & \end{array}$$

where $\mathfrak{a}_{J_F}\gamma_{\mathrm{cod}}$ is an isomorphism, while $\mathfrak{a}_{J_F}\nu_F$ still is etale as etale maps are stable under inverse images. Hence, factorizing it as an etale map followed by an isomorphism, we have proven $\eta_F^\flat$ to be an etale arrow in $\mathbb{T}[\mathbf{Spec}(F)]$. From the equality $(\eta_F^\sharp)_c = (\eta_F^\flat)_{(c, 1_{F(c)})}$, we have that $\eta_F^\sharp$ is pointwise etale, hence is etale in $\mathbb{T}[\mathcal{F}]$. □

7.3.4.9. These data define a morphism of modelled topoi

$$(\mathcal{F}, F) \xrightarrow{(h_F, \eta_F^\flat)} (\mathbf{Spec}(F), w\widetilde{F})$$

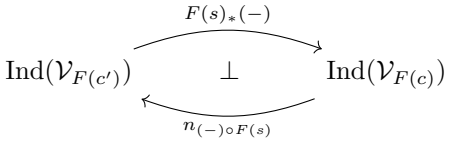
In fact we shall see that $\eta_F^\flat$ plays the same role as - and actually *is* - the generic etale map η_F defined in definition 5.2.1.2.

7.3.5 Bifibration of Ind-objects

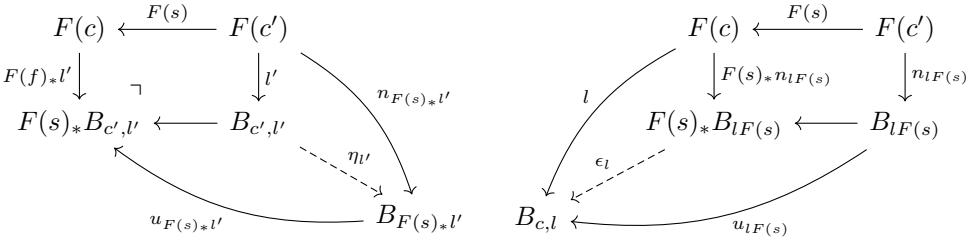
7.3.5.1. In the construction above, we considered as transition functor between the fibers of the spectral site the pushouts functors. On the other hand, taking the etale part of the composite of an etale map with the restriction map should define some kind of adjoint of those transition functors:

$$\begin{array}{ccc} F(c) & \xleftarrow{F(s)} & F(c') \\ {\scriptstyle l}\downarrow & & \downarrow{\scriptstyle n_{lF(s)}} \\ B_{c,l} & \xleftarrow[u_{lF(s)}]{} & B_{lF(s)} \end{array}$$

However the etale part $n_{lF(s)}$ in such factorization needs not be a finitely presented map even if l was. This can be fixed at the level of the indcompletion of the underlying site in which live all such factorizations. Actually we have the following adjunction for each $s : c \to c'$



Where the unit and counit are defined as

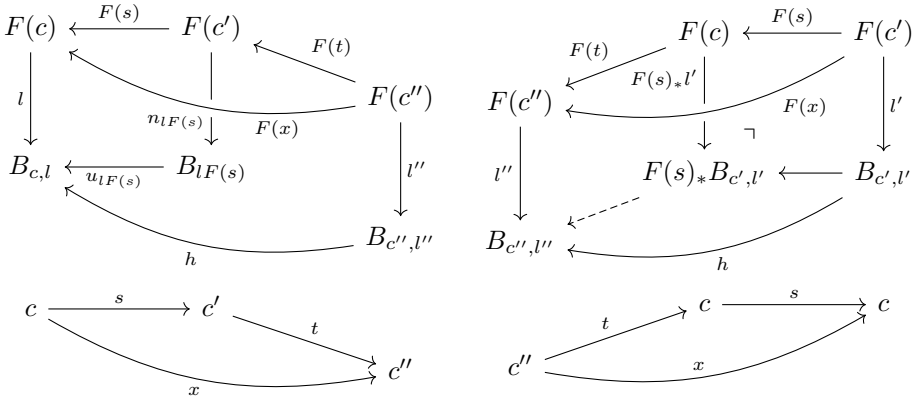


where η'_l is induced from the (etale,locale) factorization being the terminal one amongst those with an etale map on the left, while the counit is induced by property of pushout. Beware that this adjunction will be actually flipped upside down between the actual underlying sites of the spectrum.

Observe that the right adjoint $n_{(-)\circ F(s)}$ is finitary as its left adjoints send finitely presented object into finitely presented objects. It is in fact the morphism of locally finitely presentable categories in induced by $F(s)^{\mathrm{op}}_*$ seen as a left exact functor $\mathcal{V}^{\mathrm{op}}_{F(c')} \to \mathcal{V}^{\mathrm{op}}_{F(c)}$

Proposition 7.3.5.2. *We have a bifibration defined by the projection* $\mathrm{Ind}(\mathcal{V}^{\mathrm{op}}_F) \to \mathcal{C}_{\mathcal{F}}$ *and its right adjoint given by the inclusion of* $\mathcal{C}_{\mathcal{F}}$*.*

Proof. We have to prove the cartesianness and cocartesiannes of the projection. First, for a given (c, l) and an arrow $c \to c'$, the cleavage comes from the factorization by taking the etale part of $lF(s)$ as pullback object $s^*(c, l) = (c', n_{lF(s)})$ and the local part as the lifting. This choice is cocartesian: indeed, for any situation as the following left diagram:



we get two factorizations of $lF(s)F(t)$: one with an etale part on the left hl'' and one with a local part on the right $u_{lF(s)}n_{lF(s)}F(t)$ so by the property of admissibility there exists a composite arrow between them

$$B_{c'',l''} \to B_{lF(s)F(t)} \to B_{lF(s)}$$

providing the desired lifting. On the other hand, cartesianness is obvious from the use of pushout as depicted in the right diagram. □

7.4 Functoriality of Spec

Let us examine functoriality of **Spec**. As we shall see, this is far from a trivial fact and is obtained through several steps; those complications were left totally implicit in the other sources on the topic, as they gave no precision on the functoriality: we think it justifies to devote some

effort to clarify this aspect.

We saw in 6.1.1.3 that $\mathbb{T}$-**GTop** admitted two factorizations systems (Vertical, Cartesian) and (Cocartesian, Vertical), where the vertical morphisms could be seen as "algebraic morphisms" with trivial geometric part, while the other were given respectively from the counit and unit of the codomain and domain sheaf along the direct and inverse images part of a geometric morphism; in some sense, algebraic data in the cartesian and cocartesian morphisms were trivial, so they could be seen as two sorts of "geometric morphisms" of modelled topoi. In the following we shall construct separately the spectrum of vertical, cartesian and cocartesian morphisms of modelled topoi, and define the spectrum of an arbitrary morphism as the composite of the spectrum of its vertical and cartesian part - or equivalently, its cocartesian and vertical part.

7.4.1 Functoriality relative to vertical morphisms

Recall the vertical morphisms are those of the form $(1_{\mathcal{F}}, \phi) : (\mathcal{F}, F_1) \to (\mathcal{F}, F_2)$, that is, whose underlying geometric morphism part is an identity - which makes the inverse and direct image part $\phi^\flat$ and $\phi^\sharp$ coincide. Vertical morphisms over $\mathcal{F}$ correspond exactly to morphisms in $\mathbb{T}[\mathcal{F}]$, and we shall see the construction of the construction of their spectrum generalizes the set valued case which corresponded to vertical morphisms over $\mathcal{S}$.

7.4.1.1. For $\mathcal{F}$ a Grothendieck topos, let $\phi : F_1 \to F_2$ be a morphism in $\mathbb{T}[\mathcal{F}]$. Then we are provided with a natural transformation of pseudofunctors

$$\mathcal{C}_{\mathcal{F}}^{\mathrm{op}} \;\underset{\mathcal{V}_{F_2(-)}}{\overset{\mathcal{V}_{F_1(-)}}{\Downarrow \phi_*}}\; \mathbf{Cat}$$

defined at c in $\mathcal{C}_{\mathcal{F}}$ as the pushout functor

$$\begin{array}{ccc} \mathcal{V}_{F_1(c)} & \xrightarrow{(\phi_c)_*} & \mathcal{V}_{F_2(c)} \\ n & \longmapsto & (\phi_c)_* n \end{array}$$

whose naturality square at $u : c_1 \to c_2$

$$\begin{array}{ccc} \mathcal{V}_{F_1(c_2)} & \xrightarrow{(\phi_{c_2})_*} & \mathcal{V}_{F_2(c_2)} \\ {\scriptstyle F_1(s)_*}\downarrow & \simeq & \downarrow{\scriptstyle F_2(s)_*} \\ \mathcal{V}_{F_1(c_1)} & \xrightarrow[(\phi_{c_1})_*]{} & \mathcal{V}_{F_2(c_1)} \end{array}$$

is given by the commutation of pushouts functors along the two sides of the naturality square

$$\begin{array}{ccc} F_1(c_2) & \xrightarrow{\phi_{c_2}} & F_2(c_2) \\ {\scriptstyle F_1(s)}\downarrow & & \downarrow{\scriptstyle F_2(s)} \\ F_1(c_1) & \xrightarrow[\phi_{c_1}]{} & F_2(c_1) \end{array}$$

By functoriality of the Grothendieck construction, that is, by naturality of oplaxcolimits, the natural transformation ϕ_* induces a morphism of fibrations

$$\begin{array}{ccc} \mathcal{V}_{F_1}^{\mathrm{op}} & \xrightarrow{\int \phi_*} & \mathcal{V}_{F_2}^{\mathrm{op}} \\ & {\scriptstyle p_{F_1}}\searrow \quad \swarrow{\scriptstyle p_{F_2}} & \\ & \mathcal{C}_{\mathcal{F}} & \end{array}$$

where $\int \phi_*(c, n) = (c, (\phi_c)_* n)$ for (c, n) in $\mathcal{V}_{F_1}$. Moreover, by composition of pushouts, each functor $(\phi_c)_*$ defines a morphism of lex site

$$(\mathcal{V}_{F_1(c)}^{\mathrm{op}}, J_{F_1(c)}) \xrightarrow{(\phi_c)_*} (\mathcal{V}_{F_2(c)}^{\mathrm{op}}, J_{F_2(c)})$$

so that the induced functor $\int \phi_*$ is a morphism of lex site. Hence $\int \phi_*$ is a lex functor and is continuous for the jointly generated pretopologies $\langle \bigcup_{c\in\mathcal{C}_\mathcal{F}} \iota_c(J_{F_1(c)})\rangle$ and $\langle \bigcup_{c\in\mathcal{C}_\mathcal{F}} \iota_c(J_{F_2(c)})\rangle$. Moreover, since identities are preserved by pushouts, we also have the following commutation

$$\begin{array}{ccc} \mathcal{V}^{\mathrm{op}}_{F_1(c)} & \xrightarrow{(\phi_c)_*} & \mathcal{V}^{\mathrm{op}}_{F_2(c)} \\ & {\scriptstyle \iota_{F_1}}\nwarrow \quad \nearrow{\scriptstyle \iota_{F_2}} & \\ & \mathcal{C}_\mathcal{F} & \end{array}$$

Hence $\int \phi_*$ sends $\iota_{F_1}(J_\mathcal{F})$-families to $\iota_{F_2}(J_\mathcal{F})$-families. This provides us with a morphism of lex sites

$$(\mathcal{V}^{\mathrm{op}}_{F_1}, J_{F_1}) \xrightarrow{\int \phi_*} (\mathcal{V}^{\mathrm{op}}_{F_2}, J_{F_2})$$

and consequently with a geometric morphism

$$\mathbf{Spec}(F_2) \xrightarrow{\mathbf{Spec}(\phi)} \mathbf{Spec}(F_1)$$

7.4.1.2. Now, the associated morphisms of sheaves are defined as follows. For the direct image part, the data at each (c, n) of the map $n_*\phi_c$ as obtained in the following pushout

$$\begin{array}{ccc} F_1(c) & \xrightarrow{\phi_c} & F_2(c) \\ {\scriptstyle n}\downarrow & \ulcorner & \downarrow{\scriptstyle (\phi_c)_* n} \\ \mathrm{cod}(n) & \xrightarrow[n_*\phi_c]{} & \mathrm{cod}((\phi_c)_* n) \end{array}$$

define altogether a natural transformation

$$\mathrm{cod}_1 \xrightarrow{\nu_\phi} \mathbf{Spec}(\phi)_* \mathrm{cod}_2$$

which is sent after sheafification to a morphism in $\mathbb{T}[\mathbf{Spec}(F_1)]$

$$\widetilde{F_1} \xrightarrow{\mathfrak{a}_{J_{F_1}}(\nu_\phi)} \mathbf{Spec}(\phi)_* \widetilde{F_2}$$

Hence we have to define

$$\widetilde{\phi}^\sharp = \mathfrak{a}_{J_{F_1}}(\nu_\phi)$$

while the existence of its mate

$$\mathbf{Spec}(\phi)^* \widetilde{F_1} \xrightarrow{\widetilde{\phi}^\flat} \widetilde{F_2}$$

is ensured through the adjunction defining $\mathbf{Spec}(\phi)$. However, it admits a more concrete construction, which is related to the (etale, locale) factorization in the sense that it is induced from taking the local part after precomposition with ϕ_c, in the same vein as proposition 7.1.4.6:

Proposition 7.4.1.3. *The morphism $\widetilde{\phi}^\flat$ is in* $\mathbf{Loc}[\mathbf{Spec}(F_2)]$

Proof. Recall that inverse images and sheafification commutes, so we can first compute the inverse image of the codomain functor over $\mathcal{V}_{F_1}$ along $\mathbf{Spec}(\phi)$ before sheafifying into $\mathbf{Spec}(\phi)^*\widetilde{F_1}$. Recall that the inverse image of the codomain functor (as a presheaf) is obtained as the left Kan extension

$$\begin{array}{ccc} \mathcal{V}_{F_1} & \xrightarrow{\mathrm{cod}} & \mathbb{T}[\mathcal{S}] \\ {\scriptstyle \int \phi_*}\downarrow & \overset{\zeta}{\Rightarrow} \nearrow{\scriptstyle \mathrm{lan}_{\int \phi_*}\mathrm{cod}} & \\ \mathcal{V}_{F_2} & & \end{array}$$

For each (c, n) in $\mathcal{V}_{F_2}$, this left Kan extension is computed as the filtered colimit

$$(\mathrm{lan}_{\int \phi_*}\mathrm{cod})(c, n) \simeq \operatorname*{colim}_{\int \phi_* \downarrow (c,n)} \mathrm{cod}(m)$$

ranging over all $(u, f) : (d, (\phi_d) * m) \to (c, n)$ where $(d, (\phi_d)_* m) = \int \phi_*(d, m)$. Beware that we use there the native orientation of $\mathcal{V}_{F_2}$, so that this corresponds to an underlying morphism $u : c \to d$ in $\mathcal{C}_\mathcal{F}$. But observe now that the subcategory of $\int \phi_* \downarrow (c, n)$ admits a cofinal subcategory consisting

of all the morphisms of the form $(1_c, f) : (c, (\phi_c)_* m) \to (c, n)$, since any object in $\int \phi_* \downarrow (c, n)$ factorizes in $\mathcal{V}_{F_2}$ as

$$\begin{array}{ccc} (d, (\phi_c)_* m) & \xrightarrow{(u,f)} & (c, n) \\ & {\scriptstyle (f, 1_{F_2(u)_*(\phi_c)_* m})} \searrow \quad \nearrow {\scriptstyle (1_c, \langle n, f \rangle)} & \\ & (c, F_2(u)_*(\phi_c)_* m) & \end{array}$$

while any two parallel factorizations can be merged by a pushout. Hence the colimit reduces to a colimit over those $(1_c, f)$, so we can restrict in some sense at a computation in the fiber at c. But alike what was said at proposition 7.1.4.6, remark that $(\phi_c)^*$ is left adjoint to the precomposition functor $(\phi_c)^! : \mathcal{V}_{F_2(c)} \to F_1(c) \downarrow \mathbb{T}[\mathcal{S}]$, so that the cofinal category we exhibited above is equivalent to $\mathcal{V}_{F_1(c)} \downarrow n\phi_c$. Hence we have

$$\operatorname{lan}_{\int \phi_*} \operatorname{cod}(c, n) \simeq \operatorname*{colim}_{\mathcal{V}_{F_1(c)} \downarrow n\phi_c} \operatorname{cod}(m)$$

From proposition 1.1.3.2, this exhibits $\operatorname{lan}_{\int \phi_*} \operatorname{cod}(c, n)$ as the middle term in the (etale, locale) factorization of $n(\phi_c)_*$, whose local part shall be denoted $(u_\phi)_c : \operatorname{lan}_{\int \phi_*} \operatorname{cod}(c, n) \to \operatorname{cod}(c, n)$. Now by naturality this can be gathered into a local map in the category of $\mathbb{T}$-models in the presheaf topos $[\mathcal{V}_{F_2}, \mathcal{S}]$

$$\mathbf{Spec}(\phi)^* \operatorname{cod} \xrightarrow{u_f} \operatorname{cod}$$

whose sheafification returns $\widetilde{\phi^\flat}$. □

Remark 7.4.1.4. Observe that we exploited morally the fact that (etale, local) factorization in arbitrary topoi are done pointwisely (modulo sheafification) as observed in proposition 3.3.2.11, since we constructed the inverse image part from factorizations in the fibers before gathering them into a morphism of sheaf. From the triangle of geometric morphisms

$$\begin{array}{ccc} \mathbf{Spec}(F_2) & \xrightarrow{\mathbf{Spec}(\phi)} & \mathbf{Spec}(F_1) \\ & {\scriptstyle h_{F_2}} \searrow \quad \swarrow {\scriptstyle h_{F_1}} & \\ & \mathcal{F} & \end{array}$$

we got a square in $\mathbf{Spec}(F_2)$

$$\begin{array}{ccccc} \mathbf{Spec}(\phi)^* h_{F_1}^* F_1 & & \xrightarrow{\mathbf{Spec}(\phi)^* \eta^\flat_{F_1}} & & \mathbf{Spec}(\phi)^* \widetilde{wF_1} \\ \| & & & & \downarrow {\scriptstyle w\widetilde{\phi^\flat}} \\ h_{F_2}^* F_1 & \xrightarrow[h_{F_2}^* \phi]{} & h_{F_2}^* F_2 & \xrightarrow[\eta^\flat_{F_2}]{} & \widetilde{wF_2} \end{array}$$

In fact, we can see this square encodes the (etale, locale) factorization of the composite

$$h_{F_2}^* F_1 \xrightarrow{h_{F_2}^* \phi \eta^\flat_{F_2}} \widetilde{wF_2}$$

through a local form, as $\widetilde{F_2}$ is a local object, $\widetilde{\phi^\flat}$ is a local map, and $\eta^\flat_{F_1}$ is the etale map coding the universal etale form under F_1.

7.4.2 Functoriality relative to horizontal morphisms

Recall that there are actually two kinds of horizontal morphisms, respectively the cartesian obtained through direct images and the cocartesian obtained through inverse images. We shall see it is sufficient to have only the cartesian together with the vertical ones to compute the spectrum of any morphism.

We turn to the cartesian morphisms and the construction of the spectrum of an arbitrary morphism. In fact the main difficulty is that one needs to consider the underlying geometric morphisms of a morphism of modelled topos as induced in a manner or another from a morphism of site in order to perform the pushout of basic etales map along it. Of course this is not possible in general to find simultaneously two *small* sites of presentation for the domain and codomain topos

inducing a geometric morphism between them. But we can always induce it from a site morphism landing in the domain topos, which, together with its canonical topology, can be seen as a *small generated site*, for which there exist also far enough theory for our needs, as in [15]. We shall show we can extend canonically a $\mathbb{T}$-model seen as a sheaf of $\mathbb{T}$-objects over a presentation site to a sheaf over the whole topos with the canonical topology.

7.4.2.1. Recall, if $\mathcal{F}$ admits a lex subcanonical site of presentation $(\mathcal{C}_\mathcal{F}, J_\mathcal{F})$ then we have a geometric equivalence

$$\mathbf{Sh}(\mathcal{C}_\mathcal{F}, J_\mathcal{F}) \simeq \mathbf{Sh}(\mathbf{Sh}(\mathcal{C}_\mathcal{F}, J_\mathcal{F}), J_{\text{can}})$$

sending a sheaf X over $\mathcal{C}_\mathcal{F}$ to its left Kan extension

$$\begin{array}{ccc} \mathcal{C}_\mathcal{F}^{\text{op}} & \xrightarrow{X} & \mathcal{S} \\ {\scriptstyle \text{ﻉ}^{\text{op}}}\downarrow & \overset{\simeq}{\nearrow}_{\overline{X}=\text{lan }_{\text{ﻉ}^{\text{op}}} X} & \\ \mathcal{F}^{\text{op}} & & \end{array}$$

where ﻉ is full and faithful and lands in the category of sheaves as $J_\mathcal{F}$ is subcanonical; in the other direction, a sheaf for the canonical topology is sent to its restriction along ﻉ^{op}. Moreover, this construction extends to categories of sheaves in locally finitely presentable categories, in particular for $\mathbb{T}[\mathcal{S}]$, so we have an equivalence of categories

$$\mathbb{T}[\mathcal{F}] \simeq \mathbb{T}[\mathbf{Sh}(\mathcal{F}, J_{\text{can}})]$$

7.4.2.2. Hence from the construction above, any $\mathbb{T}$-modelled topos $(\mathcal{F}, F)$ defines canonically another $\mathbb{T}$-modelled topos $(\mathbf{Sh}(\mathcal{F}, J_{\text{can}}), \overline{F})$, which will be called its *extended modelled topos*, which is actually equipped with an invertible morphism of modelled topoi for $\mathcal{F} \simeq \mathbf{Sh}(\mathcal{F}, J_{\text{can}})$ while $F \simeq \text{lan }_{\text{ﻉ}} F \circ \text{ﻉ}$ (by subcanonicity) and $\overline{F} = \text{lan }_{\text{ﻉ}} F$. Hence we expect those two modelled topoi to have the same spectrum, but from the construction of Coste sprectral site, this is not obvious - especially since we have not yet achieved functoriality. We shall in fact need this result in the following to prove functoriality for horizontal morphisms.

In fact, we can see that the spectral sites themselves of F and $\overline{F}$ are not equivalent. At each c in $\mathcal{C}$, we have an isomorphism $F(c) \simeq \overline{F}(\text{ﻉ}_c)$, and hence an equivalence between the fibers at object of $\mathcal{C}$

$$\mathcal{V}_{F(c)} \simeq \mathcal{V}_{\overline{F}(\text{ﻉ}_c)}$$

This induces a canonical inclusion between the oplax colimits

$$\mathcal{V}_F \simeq \underset{c \in \mathcal{C}_\mathcal{F}}{\text{oplaxcolim}}\ \mathcal{V}_{F(c)} \xhookrightarrow{i_F} \underset{X \in \mathcal{F}}{\text{oplaxcolim}}\ \mathcal{V}_{\overline{F}(X)} \simeq \mathcal{V}_{\overline{F}}$$

sending (c, n) to $(\text{ﻉ}_c, n)$. However, though we have a dense inclusion $\text{ﻉ} : \mathcal{C}_\mathcal{F} \hookrightarrow \mathcal{F}$, the inclusion i_F is by no mean an equivalence itself, as the oplax colimits cannot be contracted - as a pseudocolimit may have been with an argument of cofinality. However, the following notion will help us to fix this at the level of induced topoi. Recall the following definition:

Definition 7.4.2.3. A morphism of site $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is said to be K-*dense* if it satisfies the following conditions:

- f creates covers, that is, a family $(u_i : c_i \to c)_{i \in I}$ is J-covering if and only if its image $(f(u_i))_{i \in I}$ is a K-covering;
- for any object d in $\mathcal{D}$ there exists a K-cover of the form $(f_i : f(c_i) \to d)_{i \in I}$ with c_i in $\mathcal{C}$;
- for any c_1, c_2 in $\mathcal{C}$ and $g : f(c_1) \to f(c_2)$, there exists a J-cover $(f_i : c_i \to c)_{i \in I}$ and a family of arrows $(g_i : c_i \to c_2)_{i \in I}$ such that $gf(f_i) = f(g_i)$.

Then it is known from [87][Theorem 11.14](see also [15][Remark 5.2]) that if $f : (\mathcal{C}, J) \to (\mathcal{D}, K)$ is a K-dense morphism of site, then the induced functor $\mathbf{Sh}(f)$ is a geometric equivalence $\mathbf{Sh}(\mathcal{D}, K) \simeq \mathbf{Sh}(\mathcal{C}, J)$. Observe that in particular that, if J is subcanonical, then the dense inclusion $\text{ﻉ} : (\mathcal{C}, J) \to (\mathbf{Sh}(\mathcal{C}, J), J_{\text{can}})$ is a dense morphism of site, and moreover that restricting back the canonical topology to $\mathcal{C}$ defines a dense morphism of sites $(\mathcal{C}, J) \to (\mathcal{C}, J_{\text{can}} \mid_\mathcal{C})$ and also a dense morphism of site $(\mathcal{C}, J) \to (\text{ﻉ}(\mathcal{C}), J_{\text{can}} \mid_{\text{ﻉ}(\mathcal{C})})$.

Lemma 7.4.2.4. *We have a geometric equivalence* $\mathbf{Spec}(F) \simeq \mathbf{Spec}(\overline{F})$.

Proof. We prove that the functor i_F above defines a $J_{\overline{F}}$-dense morphism of site

$$(\mathcal{V}_F^{\mathrm{op}}, J_F) \xhookrightarrow{i_F} (\mathcal{V}_{\overline{F}}^{\mathrm{op}}, J_{\overline{F}})$$

For the first condition, it suffices to prove separately that i_F creates vertical and horizontal families:

- for vertical families, observe that in each c in $\mathcal{C}_{\mathcal{F}}$, the fibers are equivalent as well as the topologies $J_{F(c)}$ and $J_{\overline{F}(\text{よ}_c)}$: whence the creation of vertical families;
- for horizontal families, this is a consequence of the fact that $(\mathcal{C}_{\mathcal{F}}, J_{\mathcal{F}}) \to (\text{よ}(\mathcal{C}_{\mathcal{F}}), J_{\mathrm{can}} \mid_{\text{よ}(\mathcal{C}_{\mathcal{F}})})$ is a dense morphism of sites, so that a family $(\text{よ}_{u_i} : \text{よ}_{c_i} \to \text{よ}_c)_{i\in I}$ is in J_{can} if and only if $(u_i : c_i \to c)_{i\in I}$ is in $J_{\mathcal{F}}$. Hence a family $(\text{よ}_{u_i}, f) : (\text{よ}_{c_i}, n_i) \to (\text{よ}_c, n))_{i\in I}$ is $J_{\overline{F}}$-covering if and only if $(u_i, f) : (c_i, n_i) \to (c, n))_{i\in I}$ is J_F-covering.

For the second condition, recall that よ is a dense functor so that for any X in $\mathcal{F}$ we have a colimit $X \simeq \text{よ} \downarrow X$ (where beware the colimit is computed in the category of sheaves). Hence $\text{よ} \downarrow X$ is never empty, and in fact, from the definition of colimits in topoi and their relation with the canonical topology, the family $(a : \text{よ}_c \to X)_{\text{よ}\downarrow X}$ is a cover in J_{can}. Hence one can take for each object (X, n) of $\mathcal{V}_{\overline{F}}$ with $n : \overline{F}(X) \to \mathrm{cod}(n)$ the horizontal $J_{\overline{F}}$-cover

$$(\text{よ}_c, \overline{F}(a)_* n) \xrightarrow{(a, n_* \overline{F}(a))} (X, n)$$

where $(\text{よ}_c, \overline{F}(a)_* n) \simeq i_F(c, \overline{F}(a)_* n)$.

For the third item, we use that, by subcanonicity of $J_{\mathcal{F}}$, よ is full and faithful, so that for any (c_1, n_1), (c_2, n_2) and $(f, g) : (\text{よ}_{c_1}, n_1) \to (\text{よ}_{c_2}, n_2)$, the arrow f must come from some $f = \text{よ}_u$ with $u : c_1 \to c_2$ and hence we had already $(u, g) : (c_1, n_1) \to (c_2, n_2)$. □

7.4.2.5. We saw that $(\mathcal{F}, F)$ is actually equivalent as a modelled topos to its extended form $(\mathbf{Sh}(\mathcal{F}_2, J_{can}), \overline{F})$. Consequently, we expect their respective spectrum to be equivalent not only at the level of the underlying topoi but at the level of the structural sheaves. This can be seen as follows: at each (c, n) of $\mathcal{V}_F$ we have $i_F(c, n) = (\text{よ}_c, n)$ as $\overline{F}(\text{よ}_c) = F(c)$ - so that n is both an object of $\mathcal{V}_{F(c)}$ and $\mathcal{V}_{\overline{F}(\text{よ}_c)}$; then we have a pointwise equality $i_{F*}\mathrm{cod}(c, n) = \mathrm{cod}(\text{よ}_c, n) = n$, so that we get an isomorphism after sheafification relative to $J_{\overline{F}}$

$$\widetilde{\overline{F}} \simeq \mathbf{Sh}(i_F)_* \widetilde{F}$$

whose mate between inverse images is also an isomorphism as the unit and counit of the adjoint equivalence $\mathbf{Sh}(i_F)^* \dashv \mathbf{Sh}(i_F)_*$ are so.

7.4.2.6. Now for a morphism $(f, \phi) : (\mathcal{F}_1, F_1) \to (\mathcal{F}_2, F_2)$, even though f does not necessarily arise form a morphism of site $(\mathcal{C}_{\mathcal{F}_1}, J_{\mathcal{F}_1}) \to (\mathcal{C}_{\mathcal{F}_2}, J_{\mathcal{F}_2})$, f is still induced as a morphism of sites $f^* : \mathcal{C}_{\mathcal{F}_1} \to \mathcal{F}_2$ sending $J_{\mathcal{F}_1}$-covers to covers for the canonical topology J_{can} on $\mathcal{F}_2$ seen as a small-generated standard site, and the direct image can be computed explicitly at c in $\mathcal{C}_1$ as

$$f_* F_2(c) \simeq \overline{F_2}(f^*(c))$$

For this later is an ordinary set-based $\mathbb{T}$-model, we can compute its spectral site $\mathcal{V}_{\overline{F_2}(f^*(c))}$, and for each c we have a transition functor

$$\mathcal{V}_{F_1(c)} \xrightarrow{(\phi_c^\sharp)_*} \mathcal{V}_{\overline{F_2}(f^*(c))}$$

which induces by naturality of the oplax colimit a functor

$$\mathcal{V}_{F_1} \xrightarrow{\int \phi_*^\sharp} \operatorname*{oplaxcolim}_{c\in\mathcal{C}_{\mathcal{F}_1}} \mathcal{V}_{\overline{F_2}(f^*(c))}$$

where the later oplax colimit is $\mathcal{V}_{f_* F_2}$. Then we can apply 7.4.1.1 to the vertical morphism

$$(\mathcal{F}_1, F_1) \xrightarrow{(1_{\mathcal{F}_1}, (\phi^\sharp, \phi^\sharp))} (\mathcal{F}_1, f_* F_2)$$

so we get in particular a geometric morphism

$$\mathbf{Spec}(f_* F_2) \xrightarrow{\mathbf{Spec}(\phi^\sharp)} \mathbf{Spec}(F_1)$$

7.4.2.7. On the other hand, for each $f^*(c)$ is in particular an object of the sheaf topos $\mathcal{F}_2$, reindexing along $f^* : \mathcal{C}_1 \to \mathcal{F}_2$ induces a canonical inclusion between the oplax colimits

$$\operatorname*{oplaxcolim}_{c\in\mathcal{C}_{\mathcal{F}_1}} \mathcal{V}_{\overline{F_2}(f^*(c))} \xrightarrow{q_{(f,\phi)}} \operatorname*{oplaxcolim}_{X\in\mathcal{F}_2} \mathcal{V}_{\overline{F_2}(X)}$$

which is moreover trivially a morphism of sites. Hence it induces a geometric morphism

$$\mathbf{Spec}(\overline{F}_2) \xrightarrow{\mathbf{Sh}(q_{(f,\phi)})} \mathbf{Spec}(f_*F_2)$$

Hence from lemma 7.4.2.4, we can finally compose all those data into a geometric morphism $\mathbf{Spec}(\phi)$ as below

$$\begin{array}{ccc} \mathbf{Spec}(F_2) & \xrightarrow{\mathbf{Spec}(\phi)} & \mathbf{Spec}(F_1) \\ {\scriptstyle \mathbf{Sh}(i_{F_2})}\downarrow{\scriptstyle \simeq} & & \uparrow{\scriptstyle \mathbf{Spec}(\phi^\sharp)} \\ \mathbf{Spec}(\overline{F}_2) & \xrightarrow[\mathbf{Sh}(q_{(f,\phi)})]{} & \mathbf{Spec}(f_*F_2) \end{array}$$

7.4.2.8. We conclude with the computation of the sheaf data associated to $\mathbf{Spec}(\phi)$. This will be done by combining easy to compute data associated to each part of the decomposition $\mathbf{Spec}(\phi) = \mathbf{Spec}(\phi^\sharp)\mathbf{Sh}(q_{(f,\phi)})\mathbf{Sh}(i_{F_2})$.

From 7.4.2.5 we know that $\widetilde{\overline{F_2}} \simeq \mathbf{Sh}(i_{F_2})_*\widetilde{F_2}$. On the other side, the vertical morphism $(1, \phi^\sharp) : (\mathcal{F}_1, F_1) \to (\mathcal{F}_1, f_*F_2)$ is sent to a morphism of locally modelled topos

$$(\mathbf{Spec}(F_1), \widetilde{F_1}) \xrightarrow{(\mathbf{Spec}(\phi),\widetilde{\phi})} (\mathbf{Spec}(f_*F_2), \widetilde{f_*F_2})$$

We are going to see that this part contains actually all the sheaf information of the composite above.

Indeed, the intermediate part acts like restriction, that is,

$$\widetilde{f_*F_2} \simeq \mathbf{Sh}(q_{f,\phi *})\widetilde{\overline{F_2}}$$

This is because $\mathcal{V}_{f_*F_2}$ is a subcategory of $\mathcal{V}_{\overline{F}_2}$ with the $\mathcal{V}_{F_2(f^*(c))}$ as fibers, and again, one can apply sheafification to the equality between codomain functors. This returns a cartesian morphism

$$(\mathbf{Spec}(f_*F_2), \widetilde{f_*F_2}) \xrightarrow{(\mathbf{Sh}(q_{(f,\phi)}, 1_{\widetilde{f_*F_2}})} (\mathbf{Spec}(\overline{F_2}), \widetilde{\overline{F_2}})$$

Hence we can define the direct image part $\widetilde{\phi}^\sharp$ of $\widetilde{\phi}$ as the composite in $\mathbf{Spec}(F_1)$

$$\widetilde{F_1} \xrightarrow{\widetilde{\phi^\sharp}} \widetilde{f_*F_2} \simeq \mathbf{Sh}(q_{(f,\phi)})_*\widetilde{\overline{F_2}} \simeq \mathbf{Spec}(\phi)_*F_2$$

Summing up those considerations, we have a decomposition in $\mathbb{T}_J$-$\mathbf{GTop}^{\mathbf{Loc}}$

$$\begin{array}{ccc} (\mathbf{Spec}(F_1), \widetilde{F_1}) & \xrightarrow{(\mathbf{Spec}(\phi),\widetilde{\phi})} & (\mathbf{Spec}(F_2), \widetilde{F_2}) \\ {\scriptstyle (\mathbf{Spec}(\phi),\widetilde{\phi})}\downarrow & & \uparrow{\scriptstyle (\mathbf{Sh}(i_{F_2}), 1_{\mathbf{Sh}(i_{F_2})_*\widetilde{F_2}})} \\ (\mathbf{Spec}(f_*F_2), \widetilde{f_*F_2}) & \xrightarrow[(\mathbf{Sh}(q_{(f,\phi)}, 1_{\widetilde{f_*F_2}})]{} & (\mathbf{Spec}(\overline{F_2}), \widetilde{\overline{F_2}}) \end{array}$$

7.4.3 Spectrum of 2-cells

For the sake of exhaustiveness, we give here to the treatment of 2-cells.

7.4.3.1. Take a 2-cell in $\mathbb{T}$-$\mathbf{GTop}$

$$(\mathcal{F}_1, F_1) \underset{(g,\gamma)}{\overset{(f,\phi)}{\Downarrow \sigma}} (\mathcal{F}_2, F_2)$$

that is, a 2-cell in **GTop**

$$\mathcal{F}_2 \underset{g}{\overset{f}{\Downarrow \sigma}} \mathcal{F}_1$$

such that we have one, hence both of the following commutations between inverse and direct images in $\mathcal{F}_2$ and $\mathcal{F}_1$ respectively respectively

$$\begin{array}{ccc} f^*F_1 & \xrightarrow{\sigma^\flat_F} & g^*F_1 \\ & {\scriptstyle\phi^\flat}\searrow \quad \swarrow{\scriptstyle\gamma^\flat} & \\ & F_2 & \end{array} \qquad \begin{array}{ccc} & F_1 & \\ & {\scriptstyle\gamma^\sharp}\swarrow \quad \searrow{\scriptstyle\phi^\sharp} & \\ g_*F_2 & \xrightarrow[\sigma^\sharp_{F_2}]{} & f_*F_2 \end{array}$$

The later, as a triangle of vertical morphisms of modelled topoi, induces an invertible 2-cell between the corresponding spectra

$$\begin{array}{ccc} & \mathbf{Spec}(F_1) & \\ {\scriptstyle\mathbf{Spec}(\gamma^\sharp)}\nearrow & \simeq & \nwarrow{\scriptstyle\mathbf{Spec}(\phi^\sharp)} \\ \mathbf{Spec}(g_*F_2) & \xrightarrow[\mathbf{Spec}(\sigma^\sharp_{F_2})]{} & \mathbf{Spec}(f_*F_1) \end{array}$$

which induces in $\mathbf{Spec}(F_1)$ a triangle toward direct images

$$\begin{array}{ccc} F_1 & \xrightarrow{\widetilde{\gamma^\sharp}} & \mathbf{Spec}(\gamma^\sharp)_*\widetilde{g_*F_2} \\ & {\scriptstyle\widetilde{\phi^\sharp}}\searrow & \downarrow{\scriptstyle\mathbf{Spec}(\phi^\sharp)(\widetilde{\sigma^\sharp_{F_2}}^\sharp)} \\ & & \mathbf{Spec}(\phi^\sharp)_*\widetilde{f_*F_2} \end{array}$$

where the middle arrow comes from the direct image part of $\widetilde{\sigma^\sharp_{F_2}}$

$$\widetilde{g_*F_2} \xrightarrow{\widetilde{\sigma^\sharp_{F_2}}^\sharp} \mathbf{Spec}(\sigma^\sharp_{F_2})_*(\widetilde{f_*F_2})$$

7.4.3.2. Now recall that, though neither f nor g are supposed to be induced from a morphism of site, we can use the extended sheaf $\overline{F_2}$ associated to F_2 which is at each c of $\mathcal{C}_{\mathcal{F}_1}$ respectively $g_*F_2(c) = \overline{F_2}(g^*(c))$ and $f_*F_2(c) = \overline{F_2}(f^*(c))$. Then the component at c of the direct image part $\sigma^\sharp_{F_2}$ is also the image along $\overline{F_2}$ of the component of $\sigma^\sharp : g_* \Rightarrow f_*$ at c by Yoneda lemma, that is

$$(\sigma^\sharp_{F_2})_c = \overline{F_2}(\sigma^\sharp_c)$$

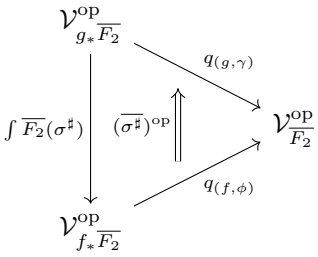
along which we can consider the pushout functor

$$\begin{array}{ccc} \overline{F_2}(g^*(c)) & \xrightarrow{\overline{F_2}(\sigma^\sharp_c)} & \overline{F_2}(f^*(c)) \\ {\scriptstyle n}\downarrow & \ulcorner & \downarrow{\scriptstyle\overline{F_2}(\sigma^\sharp_c)_* n} \\ \mathrm{cod}(n) & \xrightarrow[n_*\overline{F_2}(\sigma^\sharp_c)]{} & \mathrm{cod}(\overline{F_2}(\sigma^\sharp_c)_* n) \end{array}$$

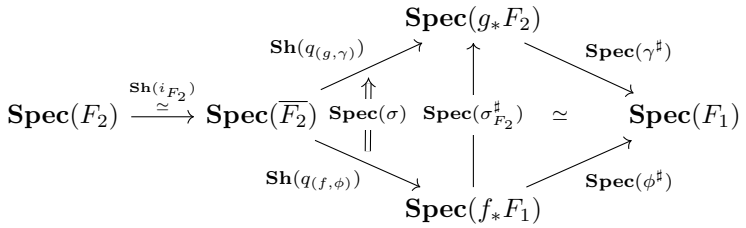
This defines at each c a 2-cell

$$\begin{array}{ccc} \mathcal{V}^{\mathrm{op}}_{\overline{F_2}(g^*(c))} & & \\ & {\scriptstyle\iota^{\mathrm{op}}_{g^*(c)}}\searrow & \\ {\scriptstyle\overline{F_2}(\sigma^\sharp_c)_*}\downarrow & {\scriptstyle(\overline{\sigma^\sharp_c})^{\mathrm{op}}}\Uparrow & \mathcal{V}^{\mathrm{op}}_{\overline{F_2}} \\ & {\scriptstyle\iota^{\mathrm{op}}_{f^*(c)}}\nearrow & \\ \mathcal{V}^{\mathrm{op}}_{\overline{F_2}(f^*(c))} & & \end{array}$$

Those data induce a morphism of fibrations between the fibered sites of the inverse images



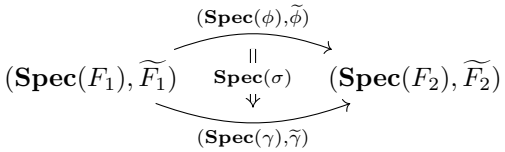
7.4.3.3. Then to get the desired 2-cell in $\mathbb{T}_J$-$\mathbf{GTop}^{\mathbf{Loc}}$, one has to paste the associated 2-cell between the spectra of inverse images together with the spectrum of the triangle of vertical morphism



We would have to check that this 2-cell satisfies the compatibility condition of morphisms of modelled topos. But this comes from the very definition of $\mathbf{Spec}(\sigma)$ as induced from $\overline{\sigma^\sharp}$ so that its component at $\mathbf{Sh}(i_{F_2})_*\widetilde{F_2}$ satisfies

$$\mathbf{Spec}(\sigma)_{\mathbf{Sh}(i_{F_2})_*\widetilde{F_2}} = \widetilde{\sigma^\sharp_{F_2}}^\sharp$$

This achieves to produce a 2-cell of locally modelled topoi



7.4.3.4. To sum up this section, we have shown how to make the spectrum into a pseudofunctor

$$\mathbb{T}_J\text{-}\mathbf{GTop}^{\mathbf{Loc}} \xrightarrow{\mathbf{Spec}} \mathbb{T}\text{-}\mathbf{GTop}$$

The next section will now prove it to be the desired left adjoint of the inclusion $\iota_{J,\mathbf{Loc}}$.

7.5 Site-theoretic version of the spectral adjunction

In this section we shall prove that the notion of spectrum as constructed in this chapter has the desired universal property announced in the previous chapter and is part of the desired spectral adjunction. In [19], there was already such a proof, yet we shall follow a different strategy based on the in-depth description of the spectral site and some reflections on locally modelled topoi.

7.5.1 Spectra of locally modelled topoi are local

7.5.1.1. Consider now a $\mathbb{T}_J$-locally modelled topos $(\mathcal{E}, E)$ with $\mathcal{E} = \mathbf{Sh}(\mathcal{C}_\mathcal{E}, J_\mathcal{E})$. Recall that the condition that E is a J-local objects in $\mathbb{T}[\mathcal{E}]$ amounts to saying that the corresponding J-continuous functor

$$\mathcal{C}_\mathbb{T} \xrightarrow{E^*} \mathcal{E}$$

sends any J-family $([\theta_j]_\mathbb{T} : \{\overline{x_j}, \phi_j\} \to \{\overline{x}, \phi\})_{j\in I}$ to an epimorphism

$$\coprod_{j\in I} E(\{\overline{x_j}, \phi_j\}) \xrightarrow{\langle E([\theta_i]_\mathbb{T})\rangle_{j\in I}} E(\{\overline{x}, \phi\})$$

But the property of being an epimorphism in the category of sheaves over $(\mathcal{C}_{\mathcal{E}}, J_{\mathcal{E}})$ says that for any $c \in \mathcal{C}_{\mathcal{E}}$ and any $a \in E(\{\overline{x}, \phi\})(c)$, there is a $J_{\mathcal{E}}$-cover $(u_i : c_i \to c)_{i \in I'}$ such that the image $E(\{\overline{x}, \phi\})(u_i)(a)$ is in the range of $\langle E([\theta_j]_{\mathbb{T}})\rangle_{j \in I}(c_i)$. But each $E(c)$ is a $\mathbb{T}[\mathcal{S}]$-model, and we have

$$E(\{\overline{x}, \phi\})(c) \simeq \mathbb{T}[\mathcal{S}](K_\phi, E(c))$$

so a is some $a : K_\phi \to E(c)$. Then this means that each of the composite $E(u_i)a$ factorizes through some $f_{\theta_j} : K_\phi \to K_{\phi_j}$ for some $j \in J$, and then we have a factorization of the identiy of $E(c_i)$ through the corresponding pushout as follows:

$$\begin{array}{ccccccc} K_\phi & \xrightarrow{a} & E(c) & \xrightarrow{E(u_i)} & E(c_i) & = & E(c_i) \\ {\scriptstyle f_{\theta_j}}\downarrow & & \downarrow{\scriptstyle a_* f_{\theta_j}} & & \downarrow{\scriptstyle E(u_i)_* a_* f_{\theta_j}} & \nearrow{\scriptstyle b} & \\ K_{\phi_j} & \longrightarrow & a_* K_{\phi_j} & \longrightarrow & E(u_i)_* a_* K_{\phi_j} & & \end{array}$$

This leads to the following:

Proposition 7.5.1.2. *If E is a J-local object in $\mathbb{T}[\mathcal{E}]$, then $\iota_{wE} : (\mathcal{C}_{\mathcal{E}}, J_{\mathcal{E}}) \hookrightarrow (\mathcal{V}_{wE}^{\mathrm{op}}, J_{wE})$ is also a comorphism of site, and defines a geometric morphism $e_E = \mathbf{Sh}(\iota_{wE})_* \dashv \mathbf{Sh}(\iota_{wE})^!$ which is moreover a section of h_E*

$$\begin{array}{ccc} & \mathbf{Spec}(E) & \\ {\scriptstyle e_E}\nearrow & & \searrow{\scriptstyle h_E} \\ \mathcal{E} & = & \mathcal{E} \end{array}$$

Proof. For any c in $\mathcal{C}_{\mathcal{E}}$, a $\iota_c(J_{F(c)})$-cover $(1_c, m_j) : (c, n_j) \to (c, 1_{F(c)}))_{j \in J}$ coming from $(m_j : E(c) \to \mathrm{cod}(n_j))_{j \in I}$ is obtained as

$$\begin{array}{ccc} K_\phi & \xrightarrow{a} & E(c) \\ {\scriptstyle f_{\theta_j}}\downarrow & & \downarrow{\scriptstyle n_j} \\ K_{\phi_j} & \longrightarrow & \mathrm{cod}(n_i) \end{array}$$

with $([\theta_j]_{\mathbb{T}} : \{\overline{x_j}, \phi_j\} \to \{\overline{x}, \phi\})_{j \in I}$ a J-family. But from what precedes, as a is an element of $E(\{\overline{x}, \phi\})(c)$, there is some $(u_i : c_i \to c)_{i \in I'}$ such that for each $i \in I'$, there is $j \in I$ and a factorization

$$\begin{array}{ccccc} K_\phi & \xrightarrow{a} & E(c) & \xrightarrow{E(u_i)} & E(c_i) \\ {\scriptstyle f_{\theta_j}}\downarrow & & \downarrow{\scriptstyle n_j} & & \| \\ K_{\phi_j} & \longrightarrow & \mathrm{cod}(n_j) & \xrightarrow[b]{} & E(c_i) \end{array}$$

In other word, for any $i \in I'$ there is some $j \in I$ and a factorization in $\mathcal{V}_{wE}^{\mathrm{op}}$

$$\begin{array}{ccc} & & (c, n_j) \\ & {\scriptstyle (u_i, b)}\swarrow & \downarrow{\scriptstyle (1_c, n_i)} \\ (c_i, 1_{E(c_i)}) & \xrightarrow[(u_i, E(u_i))]{} & (c, 1_{F(c)}) \end{array}$$

But in the diagram above, $(u_i, E(u_i))_{i \in I'}$ is an horizontal family in $\iota_{wE}(J_{\mathcal{E}})$, while $(1_c, n_i)$ is a vertical family in $\iota_c(J_{F(c)})$ refining it; in particular, the vertical family must hence be contained in the sieve generated by the horizontal family, and so is the sieve its generates. Hence any $J_{\mathcal{V}_{E(-)}}$-covering sieve is in particular a $\iota_{wE}(J_{\mathcal{E}})$-covering sieve: the topology $J_{\mathcal{V}_{E(-)}}$ is coarser than $\iota_{wE}(J_{\mathcal{E}})$, and hence the inclusion ι_{wE} becomes a comorphism of sites. □

Now recall that in the case of set-based models, the spectrum of a local object was a local topos: this result will generalize as follows. To this end we invoke [15][Theorem 7.20] characterization of local geometric morphisms: a continuous comorphism of site $F : (\mathcal{C}, J) \to (\mathcal{D}, K)$ which is moreover a morphism of site induces a triple of adjoints $\mathbf{Sh}(F)^* \dashv \mathbf{Sh}(F)_* \dashv \mathbf{Sh}(F)^!$, which can be proven to be moreover local if F is full and faithful.

Remark 7.5.1.3. Actually, [15][Theorem 7.20, (iii)] requires the functor F to be *J-full and J-faithful* for the condition to hold. Though we cannot give the definition of those notions which involves a notion of local equality of morphisms in the sheaf topos, we have in [15][Proposition 5.16] the following characterization: F is J-full and J-faithul if and only if for each c in $\mathcal{C}$, we have an isomorphism

$$\mathfrak{a}_J \not\downarrow_c \simeq \mathfrak{a}_J(\mathcal{D}[F(-), F(c)])$$

But then observe that if F is full and faithful in the usual sense, then for each d we have $\mathcal{C}[d, c] \simeq \mathcal{D}[F(d), F(c)]$, and hence there is a natural isomorphism $\not\downarrow_c \simeq \mathcal{D}[F(-), F(c)]$ which still is an isomorphism after sheafification, so that F is also J-full and J-faithful.

Corollary 7.5.1.4. *If $(\mathcal{E}, E)$ is a $\mathbb{T}_J$-locally modelled topos, then $h_{wE} : \mathbf{Spec}(E) \to \mathcal{E}$ is a local geometric morphism, with center e_E. In particular, whenever $\mathcal{E}$ was itself a local topos with center x, then $\mathbf{Spec}(E)$ is a local topos with center $e_E x$.*

Proof. We saw in the previous part that ι_{wE} becomes in this case a continuous comorphism, while still being also a morphism of site: hence it induces a triple of adjoints functors

$$\mathbf{Spec}(wE) \underset{\mathbf{Sh}(\iota_{wE})_*}{\overset{\mathbf{Sh}(\iota_{wE})^*}{\underset{\mathbf{Sh}(\iota_{wE})^!}{\rightleftarrows}}} \mathcal{E}$$

Moreover, as ι_E is full and faithful, then by the remark above, this triple of adjoints functors defines a local geometric morphism. Observe that we have the equalities

$$\underbrace{\mathbf{Sh}(\iota_{wE})^* = \mathbf{Sh}(p_{wE})_!}_{h^*_{wE}} \dashv \underbrace{\mathbf{Sh}(\iota_{wE})_* = \mathbf{Sh}(p_{wE})^*}_{h_{wE*}=e^*_E} \dashv \underbrace{\mathbf{Sh}(\iota_{wE})^! = \mathbf{Sh}(p_{wE})_*}_{e_{E*}}$$

where both h^*_E and e_{E*} are full and faithful. In particular h_E is a connected geometric morphism. □

7.5.1.5. Now our goal is to complete the geometric morphism e_E with a local map $\epsilon^\flat_E : e^*_E\widetilde{wE} \to E$ in $\mathbb{T}[\mathcal{E}]$ defining a retraction

$$\begin{array}{ccc} & e^*_E\widetilde{wE} & \\ {\scriptstyle we^*_E\eta^\flat_{wE}}\nearrow & & \searrow{\scriptstyle \epsilon^\flat_E} \\ wE & = & wE \end{array}$$

(where we used the fact that $E \simeq e^*_E h^*_{wE} E \simeq \mathbf{Sh}(\iota_{wE})_*\mathbf{Sh}(\iota_{wE})^* E$ from the fact that $\mathbf{Sh}(\iota_{wE})^*$ is full and faithful, hence has invertible unit). But we can already guess that actually such a map $\epsilon^\flat_E$ has to be an isomorphism: indeed, we know from proposition 7.3.4.8 that $\eta^\flat_{wE}$ has to be an etale map, as well as its inverse image $e^*_E(\eta^\flat_{wE})$, while $\epsilon^\flat_E$ must be defined in such a way it is local: but as we hence have a factorization of the identity of E, those maps form an (etale, local) factorization of the identity, hence both are isomorphisms, and we then use that w is conservative.

Actually, we can see directly which is this isomorphism: recall that sheafification commutes with inverse images, so that we have

$$\begin{aligned} e^*_E\widetilde{wE} &\simeq e^*_E\mathfrak{a}_J\mathrm{cod} \\ &\simeq \mathfrak{a}_J\mathrm{cod} \circ \iota_E \end{aligned}$$

But we have at each c of $\mathcal{C}_\mathcal{E}$ that $\mathrm{cod}(\iota_E(c)) = \mathrm{cod}(c, 1_{E(c)}) = E(c)$: so we have a pointwise equality $\mathrm{cod} \circ \iota_E = E$ which is sent to an isomorphism after sheafification $\epsilon^\sharp_E : we^*_E\widetilde{wE} \simeq E$ for $\mathfrak{a}_J\iota_\mathcal{E}E \simeq E$ as E is already a sheaf.

Now the mate $\epsilon^\sharp_E$ coincides up to a canonical iso with the unit of the $e^*_E \dashv e_{E*}$ adjunction as one has

$$\begin{array}{ccc} \widetilde{wE} & \xrightarrow{\epsilon^\sharp_E} & e_{E*}E \\ \downarrow & \nearrow_{\simeq}{\scriptstyle e^*(\epsilon^\flat_E)} & \\ e_{E*}e^*_E\widetilde{wE} & & \end{array}$$

But as h_{wE} is a local geometric geometric morphism with $h^!_{wE} \simeq e_{E*}$, e_{E*} is full and faithful, so its unit is an isomorphism, and as a consequence $\epsilon^\sharp_E$ is itself an isomorphism.

This exhibits the counit as an horizontal morphism

$$(\mathbf{Spec}(wE), \widetilde{wE}) \xrightarrow{(e_E,(1_E,1_{\widetilde{wE}}))} (\mathcal{E}, E)$$

Remark 7.5.1.6. Beware that e_{E*} is not just precomposition with the projection p_{wE} as one needs further sheafification over the vertical families.

Remark 7.5.1.7. When E lives in $\mathcal{S}$, then $(\mathcal{S}, E)$ being a locally modelled topos means for E to be a set-valued local object. Then e_E defines an initial point of $\mathbf{Spec}(wE)$, and as $e^*(*) = 1_{\mathbf{Spec}(E)}$ one has

$$\begin{aligned}\Gamma\widetilde{wE} &\simeq \widetilde{wE}(1_E)\\ &\simeq h_{wE*}E\\ &\simeq e_E^*\widetilde{wE}\end{aligned}$$

Therefor e_E corresponds to the local form $1_E : E \to E$ and we have a representation

$$E \simeq e_E^*\widetilde{wE} \simeq \Gamma\widetilde{wE}$$

In the case of a set based local object, it would just express the fact that global sections are determined by the stalk at the focal point of the spectrum of the local object. The general case means the same thing but in term of $\mathcal{E}$-indexed point, and enforces that the inverse image of the structural sheaf along e_E^* - equivalently, the direct image along h_{E*} - returns the original sheaf. Hence local objects enjoy sheaf representation "for free" - which however will require additional assumption for arbitrary objects.

7.5.2 The spectral bi-adjunction

In this subsection we prove the morphisms of modelled topos (h_F, η_F) and the morphisms of locally modelled topos (e_E, ϵ_E) are respectively units and counits of a bi-adjunction $\mathbf{Spec} \dashv \iota_{J,\mathbf{Loc}}$. Of course, this result is redundant with theorem 5.2.1.7, yet the strategy is totally different - and also differs totally from a previous proof in [19] - as it exploits the concrete site presentation of the spectrum.

Theorem 7.5.2.1. *We have a biadjunction*

$$\mathbb{T}_J\text{-}\mathbf{GTop}^{\mathbf{Loc}} \underset{\iota_{J,\mathbf{Loc}}}{\overset{\mathbf{Spec}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathbb{T}\text{-}\mathbf{GTop}$$

Proof. Let $(\mathcal{F}, F)$ be a $\mathbb{T}$-modelled topos and $(\mathcal{E}, E)$ a $\mathbb{T}_J$-locally modelled topos. We must construct an equivalence of homcategories

$$\mathbb{T}\text{-}\mathbf{GTop}\big[(\mathcal{F}, F), (\mathcal{E}, wE)\big] \simeq \mathbb{T}_J\text{-}\mathbf{GTop}^{\mathbf{Loc}}\big[(\mathbf{Spec}(F), \widetilde{F}), (\mathcal{E}, E)\big]$$

Let $(f, \phi) : (\mathcal{F}, F) \to (\mathcal{E}, E)$ be in $\mathbb{T}$-$\mathbf{GTop}$. Then by pseudonaturality of $(h, \eta) : 1 \Rightarrow \iota_{J,\mathbf{Loc}}\mathbf{Spec}$ we have a pseudocommutative square in $\mathbf{GTop}$

$$\begin{array}{ccc}\mathbf{Spec}(wE) & \xrightarrow{\mathbf{Spec}(\phi)} & \mathbf{Spec}(F)\\ {\scriptstyle h_{wE}}\downarrow & \overset{\eta_f}{\simeq} & \downarrow{\scriptstyle h_F}\\ \mathcal{E} & \xrightarrow[f]{} & \mathcal{F}\end{array}$$

and in particular a natural isomorphism between inverse images

$$\mathbf{Spec}(\phi)^* h_F^* \overset{\eta_f}{\simeq} h_{wE}^* f^*$$

Hence we have a commutative square between inverse images in $\mathbf{Spec}(wE)$

$$\begin{array}{ccccc} \mathbf{Spec}(\phi)^* h_F^* F & \xrightarrow{(\eta_f)_F}_{\simeq} & h_{wE}^* f^* F & \xrightarrow{h_{wE}^* \phi^\flat} & h_{wE}^* wE \\ {\scriptstyle \mathbf{Spec}(\phi)^* \eta_F^\flat}\downarrow & & & & \downarrow{\scriptstyle \eta_{wE}^\flat} \\ \mathbf{Spec}(\phi)^* w\widetilde{F} & & \xrightarrow[w\widetilde{\phi}^\flat]{} & & w\widetilde{E} \end{array}$$

Pasting the diagram of geometric morphisms above with the retraction we obtained at proposition 7.5.1.2, we get

$$\begin{array}{ccccc} \mathcal{E} & \xrightarrow{e_E} & \mathbf{Spec}(wE) & \xrightarrow{\mathbf{Spec}(\phi)} & \mathbf{Spec}(F) \\ & \overset{\simeq}{\searrow} & {\scriptstyle h_{wE}}\downarrow & \overset{\eta_f}{\simeq} & \downarrow{\scriptstyle h_F} \\ & & \mathcal{E} & \xrightarrow[f]{} & \mathcal{F} \end{array}$$

so that, from $e_E^* h_{wE}^* \simeq 1$, the commutative square of inverse images above is sent to

$$\begin{array}{ccc} f^* F & \xrightarrow{\phi^\flat} & wE \\ {\scriptstyle e_E^* \mathbf{Spec}(\phi)^* \eta_F^\flat}\downarrow & & \downarrow{\scriptstyle e_E^* \eta_{wE}^\flat} \\ e_E^* \mathbf{Spec}(\phi)^* w\widetilde{F} & \xrightarrow[e_E^* w\widetilde{\phi}^\flat]{} & e_E^* w\widetilde{E} \end{array}$$

But from 7.5.1.5 we know that $e_E^* \eta_{wE}^\flat$ is an isomorphism with inverse $\epsilon_E^\flat$ as E is local, so this actually defines a morphism in $\mathbb{T}[\mathcal{E}]$ between local objects

$$e_E^* \mathbf{Spec}(\phi)^* w\widetilde{F} \xrightarrow{\epsilon_E^\flat e_E^* w\widetilde{\phi}^\flat} wE$$

But as $\widetilde{\phi}^\flat$ is local as well as the isomorphism $\epsilon_E^\flat$ this morphism is local, and comes uniquely from a morphism in $\mathbb{T}_J[\mathcal{E}]^{\mathbf{Loc}}$

$$e_E^* \mathbf{Spec}(\phi)^* \widetilde{F} \xrightarrow{\epsilon_E^\flat e_E^* \widetilde{\phi}^\flat} E$$

This provides us with a morphism of locally modelled topoi which satisfies by its very construction the following pseudocommutation

$$\begin{array}{ccc} (\mathcal{F}, F) & \xrightarrow{(f,\phi)} & (\mathcal{E}, E) \\ {\scriptstyle (h_F, \eta_F)}\downarrow & \overset{\simeq}{\nearrow_{(\mathbf{Spec}(\phi) e_E, \epsilon_E^\flat e_E^* \widetilde{\phi}^\flat)}} & \\ (\mathbf{Spec}(F), \widetilde{F}) & & \end{array}$$

The converse direction follow similar argument. A morphism of locally modelled topoi

$$(\mathbf{Spec}(F), \widetilde{F}) \xrightarrow{(g,\gamma)} (\mathcal{E}, E)$$

can have its image pasted with with the unit of $(\mathcal{F}, F)$ to get a morphism as desired; this will return in $\mathcal{E}$ a composite

$$\begin{array}{ccc} g^* h_F^* F & \xrightarrow{g^*(\eta_F^\flat)} & g^* w\widetilde{F} \\ & {\scriptstyle \gamma^\flat(\eta_F^\flat)}\searrow & \downarrow{\scriptstyle \gamma^\flat} \\ & & wE \end{array}$$

which is actually uniquely determined from $\eta_F^\flat$ and $\gamma^\flat$ as the first one is etale - as well as its inverse image - while the second one is local. Hence no other morphism $g^* F \to E$ induces the same composite as it would provide two distinct local parts for a same map, which is impossible by the uniqueness of the factorization. Proving that $(g, \gamma)(h_F, \eta_F)$ induces back the same (g, γ) after applying the spectrum and pasting it with the canonical retraction of $(\mathcal{E}, E)$ is routine. □

Remark 7.5.2.2. Observe that in both directions, we actually compute the etale-local factorization of the inverse image of the morphism of sheaves, where the etale part is indexed by the unit $\eta_F^\flat$ of F in a universal way, in the sense that $f^* \eta_F^\flat$ still indexes the etale parts of morphisms from $f^* F$ toward local objects in $\mathcal{E}$.

7.5.2.3. To conclude this section, we shall use the result above to describe points of $\mathbf{Spec}(F)$ and how they are related to points of the underlying topos $\mathcal{F}$. Observe that each point $x : \mathcal{S} \to \mathcal{F}$ induces a stalk x^*F which is in $\mathbb{T}[\mathcal{S}]$. From the general theory of sheaves, we know this stalk to be expressed as the filtered colimit

$$x^*F \simeq \operatorname*{colim}_{(c,a)\in(\int x^*)^{\mathrm{op}}} F(c)$$

where the colimit inclusions are the restrictions functors $\rho_x^{(c,a)} : F(c) \to x^*F$ - this colimit being filtered for $x^* : \mathcal{C}_{\mathcal{F}} \to \mathcal{S}$ is flat. Moreover, x determines a morphism of modelled topoi

$$(\mathcal{F}, F) \xrightarrow{(x, 1_{x^*F})} (\mathcal{S}, x^*F)$$

which is sent to a morphism of locally modelled topoi

$$(\mathbf{Spec}(F), \widetilde{F}) \xrightarrow{(\mathbf{Spec}(1_{x^*F}), \widetilde{1_{x^*F}})} (\mathbf{Spec}(x^*F), \widetilde{x^*F})$$

Theorem 7.5.2.4. *For any $x : \mathcal{S} \to \mathcal{F}$ we have a pseudolimit decomposition*

$$\mathbf{Spec}(x^*F) \simeq \operatorname*{bilim}_{(c,a)\in\int x^*} \mathbf{Spec}(F(c))$$

Proof. From the bi-adjunction obtained in theorem 7.1.6.2, we know that **Spec** preserves bicolimits of set-valued models. Hence we have

$$\mathbf{Spec}(x^*F, \widetilde{x^*F}) \simeq \operatorname*{bicolim}_{(c,a)\in\int x^*} (\mathbf{Spec}(F(c)), \widetilde{F(c)})$$

Now from the expression of colimits of modelled topoi, and the fact that $\iota_{J,\mathbf{Loc}}$ is a morphism of fibration, we know that the underlying Grothendieck topos is a cofiltered bilimit as desired. □

7.6 Naturality of the spectral adjunction

As usual, we conclude this chapter with functoriality aspects.

7.6.1 Comparison of spectral sites

7.6.1.1. For a transformation of geometries Φ we want to describe explicitly the mate

$$\sigma_\Phi : \int \Phi \mathbf{Spec}_2 \Rightarrow \mathbf{Spec}_1 \int \Phi$$

where $\int \Phi$ denotes the morphism of opfibrations between the bicategories of modelled topoi for the respective geometries and also its restriction to the bicategories of locally modelled topoi.

For $(\mathcal{F}, F)$ a $\mathbb{T}_2$-modelled topos, recall first that we have in each c in $\mathcal{C}_{\mathcal{F}}$ an equality $\Phi[\mathcal{F}]_* F(c) = \Phi[\mathcal{S}]_*(F(c))$. Then observe that for (c, n) in the site $\mathcal{V}^2_{\Phi F}$, any choice of pushout square for n in $\mathbb{T}_2[\mathcal{S}]$

$$\begin{array}{ccc} K & \xrightarrow{m} & K' \\ {\scriptstyle a}\downarrow & & \downarrow \\ \Phi[\mathcal{S}]_*(F(c)) & \xrightarrow[n]{} & B \end{array}$$

corresponds uniquely to a pushout square in $\mathbb{T}_1[\mathcal{S}]$

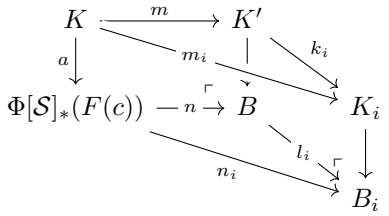
$$\begin{array}{ccc} \Phi[S]^*(K) & \xrightarrow{\Phi[S]^*(m)} & \Phi[S]^*(K') \\ {\scriptstyle \overline{a}}\downarrow & & \downarrow \\ F(c) & \xrightarrow[\overline{a}_*\Phi[S]^*(m)]{} & \overline{a}_*\Phi[S]^*(K') \end{array}$$

But recall that restricting back $\Phi[\mathcal{S}]^*$ to finitely presented objects of $\mathbb{T}_1[\mathcal{S}]$ returns finitely presented objects in $\mathbb{T}_2[\mathcal{S}]$ $\Phi[S]^*(K)$, and sends finitely presented $\mathcal{V}_2$ etale maps to finitely presented $\mathcal{V}_1$-etale maps, so that $\Phi^*[\mathcal{S}](m)$ is in $\mathcal{V}_2$, and $\overline{a}_*\Phi^*[\mathcal{S}](m)$ is in $\mathcal{V}^2_F$. This process defines hence a functor

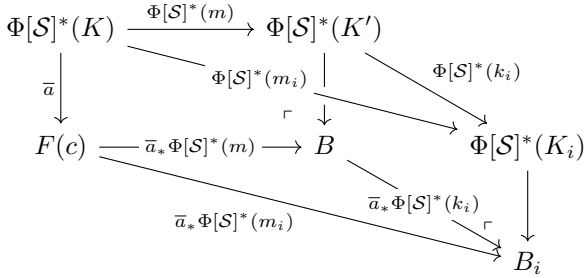
$$\begin{array}{ccc} \mathcal{V}^1_{\Phi[\mathcal{F}]_* F} & \xrightarrow{\Phi_F} & \mathcal{V}^2_F \\ (c, n) & \longmapsto & (c, \overline{a}_*\Phi[S]^* m) \end{array}$$

Lemma 7.6.1.2. *The functor Φ_F defines a morphism of site $({\mathcal{V}^1_{\Phi[\mathcal{F}]_* F}}^{\mathrm{op}}, J^1_{\Phi[\mathcal{F}]_* F}) \to ({\mathcal{V}^2_F}^{\mathrm{op}}, J^2_F)$ between the spectral sites.*

Proof. We can process by proving that Φ_F sends separately horizontal families and vertical families of $J^1_{\Phi[\mathcal{F}]_* F}$ to horizontal and vertical families in J^2_F. For horizontal families, this is just a consequence of the definition of $\Phi[\mathcal{F}]_* F$. For vertical families, let c be in $\mathcal{C}_{\mathcal{F}}$ and $(l_i : n \to n_i)_{i\in I}$ in $\mathcal{V}^1_{\Phi[\mathcal{S}]_*(F(c))}$; then from lemma 1.1.2.9 we can find some $a : K \to \Phi[\mathcal{S}]_*(F(c))$ and a family $(k_i : m \to m_i$ in $J_1(K)$ such that all the squares in the diagram below are pushouts



Now again from the adjunction we can consider the following pushouts



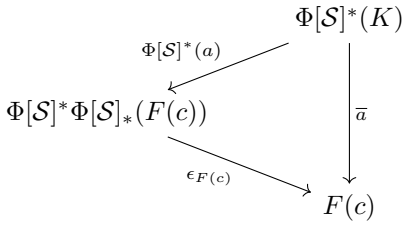
But from the definition of a transformation of geometry, we know that Φ sends J_1-covering families to J_2-covering families, so that the triangle above $\Phi[\mathcal{S}]^*(k_i) : \Phi[\mathcal{S}]^*(K') \to \Phi[\mathcal{S}]^*(K_i)$ is a J_2-cover, exhibiting $\overline{a}_*\Phi[\mathcal{S}]^*(k_i)$ as a $J^2_{\Phi[\mathcal{S}]^*(F(c))}$-cover of $\Phi_F(c,n)$. Hence Φ_F sends covers to covers. □

All of this provides us with a geometric morphism

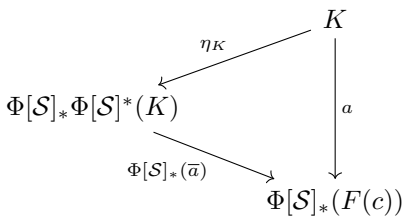
$$\mathbf{Spec}_2(F) \xrightarrow{\mathbf{Sh}(\Phi_F)} \mathbf{Spec}_1(\Phi[\mathcal{F}]_*(F))$$

7.6.2 Comparison of structural sheaves

7.6.2.1. Now let us describe the induced morphism of sheaves between the corresponding structural sheaves. Observe that for any (c,n) and any pair (a,m) inducing n, recall that the $\overline{a}$ produced by the adjunction $\Phi[\mathcal{S}]^* \dashv \Phi[\mathcal{S}]_*$ is obtained as the composite

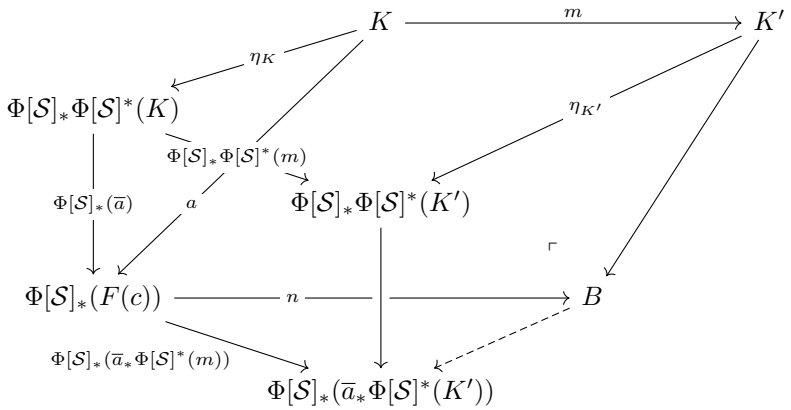


and conversely a can be retrieved from $\overline{a}$ as the composite



Now, while $\Phi[\mathcal{S}]_*$ does not preserve pushouts, we can consider the image of the pushout $\Phi_F(n) = \overline{a}_*\Phi[\mathcal{S}]^*(m)$ along $\Phi[\mathcal{S}]_*$, and we see that the expression of n as a pushout induces a factorization

as depicted below



This dashed arrow $B \to \Phi[\mathcal{S}]_*(\overline{a}_*\Phi[\mathcal{S}]^*(K'))$ can be shown to be natural in n. Moreover, the codomain functor cod lives in the presheaf topos $\widehat{\mathcal{V}_F^{2\,\mathrm{op}}}$, and $\Phi[\mathcal{S}]_*(\overline{a}_*\Phi[\mathcal{S}]^*(K'))$ coincides with $\mathrm{cod}\,\Phi_F(c,n)$ which is the value at (c,n) of the direct image $\mathbf{Sh}(\Phi_F)_*\mathrm{cod}$. Moreover we have

$$\Phi[\mathcal{S}]_*(\overline{a}_*\Phi[\mathcal{S}]^*(m)) = (\Phi[\widehat{\mathcal{V}_F^{2\,\mathrm{op}}}]_*\mathrm{cod}\,\Phi_F)(c,n)$$

Hence we are provided with a morphism of presheaves

$$\mathrm{cod} \Rightarrow \Phi[\widehat{\mathcal{V}_F^{2\,\mathrm{op}}}]_*\mathrm{cod}\,\Phi_F$$

But now, naturality of $\Phi[-]$ at the inclusion $\mathbf{Spec}_2(F) \hookrightarrow \widehat{\mathcal{V}_F^{2\,\mathrm{op}}}$ gives the commutation

$$\begin{array}{ccc} \mathbb{T}_{J_2}[\mathbf{Spec}_2] & \xrightarrow{\mathfrak{a}_{J_F^2}} & \mathbb{T}_{J_2}[\widehat{\mathcal{V}_F^{2\,\mathrm{op}}}] \\ {\scriptstyle \Phi[\mathbf{Spec}_2]_*}\downarrow & & \downarrow{\scriptstyle \Phi[\widehat{\mathcal{V}_F^{2\,\mathrm{op}}}]_*} \\ \mathbb{T}_{J_1}[\mathbf{Spec}_2] & \xrightarrow[\mathfrak{a}_{J_F^2}]{} & \mathbb{T}_{J_1}[\widehat{\mathcal{V}_F^{2\,\mathrm{op}}}] \end{array}$$

Therefore we have an isomorphism of sheaves

$$\begin{aligned} \mathfrak{a}_{J_F^2}\Phi[\widehat{\mathcal{V}_F^{2\,\mathrm{op}}}]_*\mathrm{cod}\,\Phi_F &\simeq \Phi[\mathbf{Spec}_2]_*\mathfrak{a}_{J_F^2}\mathrm{cod}\,\Phi_F \\ &\simeq \Phi[\mathbf{Spec}_2]_*\widetilde{F^2}\,\Phi_F \\ &\simeq \mathbf{Sh}(\Phi_F)_*\Phi[\mathbf{Spec}_2]_*\widetilde{F^2} \end{aligned}$$

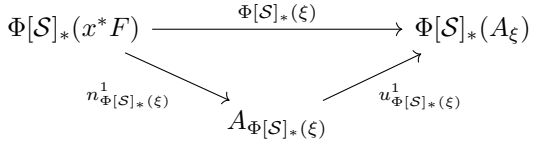
Hence the morphism of presheaves above provides us with a canonical morphism of sheaves in $\mathbf{Spec}_1(\Phi[\mathcal{F}]_*F)$

$$\widetilde{\Phi[\mathcal{F}]_*F}^{\,1} \xRightarrow{\sigma_\Phi^\sharp} \mathbf{Sh}(\Phi_F)_*\Phi[\mathbf{Spec}_2]_*\widetilde{F^2}$$

Moreover by adjunction we know this corresponds uniquely to a morphism of sheaves in $\mathbf{Spec}_2(F)$

$$\mathbf{Sh}(\Phi_F)^*\widetilde{\Phi[\mathcal{F}]_*F}^{\,1} \xRightarrow{\sigma_\Phi^\flat} \Phi[\mathbf{Spec}_2]_*\widetilde{F^2}$$

whose component at a point (x,ξ) of $\mathbf{Spec}_2(F)$ is the local part of the $(\mathbf{Et}_1, \mathbf{Loc}_1)$-factorization



Chapter 8

Point-set construction of the spectrum

The three different topos-theoretic methods we saw in the previous chapters rivaled in terms of esotericism; however in practice it may be useful to have a more concrete recipe to compute the spectrum as a topological space, as it is in a lot of important examples where the spectrum is better known from its point and topology than as a site presentation. To this end, Diers proposed in [31] an independent, point-wise construction suited for its multi-adjoint approach of geometries. This is the topic of the present chapter.

The first part of the chapter is devoted to recalling Diers construction and its version of the spectral adjunction for set-valued models, see theorem 8.1.5.5. The construction of the spectrum as a topological space is emphasized.

The second part proposes at theorem 8.2.4.3 a generalization of this adjunction to arbitrary modelled spaces closer to the other forms of the spectral adjunction presented in this thesis; in particular the connection with proposition 2.2.2.3 is investigated at proposition 8.2.4.5.

The last part of this chapter proposes an axiomatization of the kind of dualities one gets from Diers contexts after implementing the spectrum construction; in particular, we prove at theorem 8.3.4.1 that any such duality gets back a Diers context.

8.1 Spectrum from a Diers context

In this first section, we recall and detail as explicitly as possible Diers's construction of the spectra of an object in the context of a right multi-adjoint $U : \mathcal{A} \to \mathcal{B}$. In the following, we may refer to objects in $\mathcal{B}$ as *ambient objects*, and to objects and maps of $\mathcal{A}$ as *local objects and maps*.

8.1.1 The basis of diagonally universal arrows

In several papers following earlier works by Diers, the set X_B indexing the local units under an object B in the context of a right multi-adjoint has been called "the spectrum of B". If it is true that Diers's approach is point-set contrarily to other ways to construct spectra, it is abusive to reduce the spectrum of an object to the set of local units, since it corresponds more exactly to the set of points of the spectrum. As explained in [31], the spectrum in itself is far more than just a set of points: it is a topological space equipped with a structural sheaf of $\mathcal{B}$-objects with stalks in $\mathcal{A}$, where the diagonally universal morphisms under a given objects correspond to the open of its spectrum.

8.1.1.1. Recall that a Diers context is the data of a functor $U : \mathcal{A} \to \mathcal{B}$ satisfying the following conditions:

- U is right multi-adjoint
- $\mathcal{B}$ is locally finitely presentable
- any local unit under B is the filtered colimit of all the diagonally universal morphisms from B of finite presentation factorizing it.

The last condition has both a topological and sheaf theoretic interpretation. Topologically, as local units will play the role of the points of the spectrum and diagonally universal morphisms of finite presentation are a basis for the topology, it says that the focal component at a point is the intersection of all its basic open neighborhoods. From a sheaf theoretic aspect, it ensures the correct relation between the codomain of a local unit and the stalks of a certain structural sheaf we are going to construct over the spectrum.

However, beware that this condition is not automatically fulfilled in a general context. It is fulfilled when the arrows in $\mathcal{A}$ are induced from a right class of a left-generated factorization system.

8.1.1.2. For each object B, the slice category $B \downarrow \mathcal{B}$ is also locally finitely presentable. Now recall from 2.3.2.1 that $\mathcal{D}$ denotes the category of diagonally universal morphisms between finitely presented objects in $\mathcal{B}$. It is a small category with finite colimits. For any B, consider the category $\mathcal{D}_B$ whose objects are pushouts of morphisms in $\mathcal{D}$ under B; again, we saw that this category has finite colimits.

Diers' approach states that the set X_B of local units under a given object B, while insufficient in itself to convey the topological and geometric information encoded in a Diers context, can be equipped with a topology defined from $\mathcal{D}_B$. However, in order to get a topological space, we need to forget about the categorical structure of $\mathcal{D}_B$ and see it as a poset. This is the purpose of the following.

Definition 8.1.1.3. The set $\mathcal{D}_B$ can be equipped with the *order of factorization*: for $n_1 : B \to C_1$ and $n_2 : B \to C_2$, define $n_1 \leq n_2$ if and only if n_2 factorizes through n_1 for some n

$$\begin{array}{ccc} B & \xrightarrow{n_2} & C_2 \\ {\scriptstyle n_1}\downarrow & \nearrow_{\exists n} & \\ C_1 & & \end{array}$$

This order extends to arbitrary diagonally universal morphisms under B.

8.1.1.4. In particular, this order extends to the set X_B of local units under B: that is, for $x_1 : B \to U(A_1)$ and $x_2 : B \to U(A_2)$, we say $x_1 \leq x_2$ if there exists some $n : U(A_1) \to U(A_2)$ such that

$$\begin{array}{ccc} B & \xrightarrow{x_2} & U(A_2) \\ {\scriptstyle x_1}\downarrow & \nearrow_{\exists n} & \\ U(A_1) & & \end{array}$$

Remark 8.1.1.5. By left cancellation of diagonally universal morphisms, any map n as in the diagram above must be diagonally universal. Moreover, $\mathcal{D}_B$ inherits posetal analogs of its categorical structure. In particular, $\mathcal{D}_B$ has finite joins: for $n_1 : B \to C_1$ and $n_2 : B \to C_2$, let $n_1 \vee n_2$ denote the map induced under B by the pushout

$$\begin{array}{ccc} B & \xrightarrow{n_2} & C_2 \\ {\scriptstyle n_1}\downarrow & \ulcorner & \downarrow \\ C_1 & \longrightarrow & C_1 +_B C_2 \end{array}$$

Then $n_1 \vee n_2$ is really the join in the poset $\mathcal{D}_B$ as for any $n : B \to C$, if $n_1, n_2 \leq C$, then there are m_1, m_2 such that $m_1 n_1 = n = m_2 n_2$, so that m_1 and m_2 produce a commutative square, hence a map $\langle m_1, m_2 \rangle : C_1 +_B C_2 \to n$ attesting that $n_1 \vee n_2 \leq n$, and conversely. In particular, the opposite poset $\mathcal{D}_B^{\mathrm{op}}$ is a $\wedge$-semilattice with top element.

Definition 8.1.1.6. For each $n \in \mathcal{D}_B$, define the set

$$D_n = \{x \in X_B \mid n \leq x\}$$

This definition extends naturally for arbitrary diagonally universal morphisms. Conversely, for each $x \in X_B$, define the set

$$V_x = \{n \in \mathcal{D}_B \mid n \leq x\}$$

Remark 8.1.1.7. Then in a Diers context, any candidate x decomposes as a colimit in the category $B \downarrow \mathcal{B}$:

$$B \xrightarrow{x} U(A_x) = \operatorname*{colim}_{n \in V_x} B \xrightarrow{n} C$$

Then from the expression of filtered colimits in the coslice, we have that

$$U(A_x) \simeq \operatorname*{colim}_{n \in V_x} \mathrm{cod}(n)$$

We also list here some obvious, yet meaningful properties of D:

Proposition 8.1.1.8. *We have the following, for any object B of $\mathcal{B}$:*

- *If $n_1 \leq n_2$ in $\mathcal{D}_B$, then $D_{n_2} \subseteq D_{n_1}$*
- *If $x_1 \leq x_2$ in X_B, then $V_{x_1} \subseteq V_{x_2}$*
- *$D(1_B) = X_B$*
- *for $n_1, n_2 \in \mathcal{D}_n$, we have $D(n_1) \cap D(n_2) = D(n_1 \vee n_2)$*

Remark 8.1.1.9. The fourth item says in particular that $\{D_n \mid n \in \mathcal{D}_B\}$ is a basis for a topology as it is closed under intersection.

Definition 8.1.1.10. The *spectral topology* of B is the topology on the set X_B generated as

$$\tau_B = \langle \{D_n \mid n \in \mathcal{D}_n\} \rangle$$

In particular we have a monotone, $\wedge$-preserving map $D : \mathcal{D}_B^{\mathrm{op}} \to \tau_B$.

Remark 8.1.1.11. One could also process in a point-free way as follows: from the poset $\mathcal{D}_B$, generated the free frame $\mathcal{F}_B = \langle \mathcal{D}_B^{\mathrm{op}} \rangle_{\mathcal{F}rm}$, then equip it with the cover J_B defined as

$$J_B(u) = \{(n_i)_{i \in I} \mid \bigcup_{i \in I} D_{n_i} = u\}$$

Beware however that, without some specific hypothesis, the spectrum (X_B, τ_B) may not be T_0-separated and hence not sober, so that X_B may not coincide with $\mathbf{pt}(\mathcal{D}_B, J_B)$. Conversely, the frame $\mathcal{F}_B$ may not be spatial if the map D fails to be injective, or to be order reflecting.

Remark 8.1.1.12. Beware that, for an arbitrary Diers context, the category of diagonally universal morphisms may not be locally finitely presentable, and the orthogonality structure generated may not be left generated. In this case some diagonally universal morphisms may not be obtained as filtered colimit of finitely presented one: then the inclusion

$$\mathrm{Ind}({}^{\perp}U(\mathcal{A}^2) \cap \mathcal{B}^2_{fp}) \subseteq {}^{\perp}U(\mathcal{A}^2)$$

is strict, and the factorization system induced by the small object argument as in [4] returns a wider class on the right

$$({}^{\perp}U(\mathcal{A}^2))^{\perp} \subseteq ({}^{\perp}U(\mathcal{A}^2) \cap \mathcal{B}^2_{fp})^{\perp}$$

So in the general case, we shall have at some point to distinguish between arbitrary diagonally universal morphisms and those that are obtained as filtered colimits of basic ones. We call morphisms in $\mathrm{Ind}({}^{\perp}U(\mathcal{A}^2) \cap \mathcal{B}^2_{fp})$ *axiomatisable diagonally universal morphisms*, because we saw earlier they are models of a finite-limit theory. We also consider $\mathrm{Ind}(\mathcal{D}_B)$, which are the axiomatisable diagonally universal morphisms under B.

Recall that we said that a Diers context U to be *diagonally axiomatisable* when one, hence both, of the two inclusions above are equalities. Then in this case, the category $\mathbf{Diag} = \mathrm{Ind}({}^{\perp}U(\mathcal{A}^2) \cap \mathcal{B}^2_{fp})$ of diagonally universal morphisms is locally finitely presented, and for each B in $\mathcal{B}$, so is the category $\mathbf{Diag}_B$ of diagonally universal morphisms under B, in this case we do have $\mathbf{Diag}_B = \mathrm{Ind}(\mathcal{D}_B)$.

8.1.1.13. Now observe that for each B, we can extend D to axiomatisable diagonally universal morphisms by computing the pointwise left Kan extension of the functor D

$$\begin{array}{ccc} \mathcal{D}_B & \xrightarrow{D} & \tau_B^{\mathrm{op}} \\ {\scriptstyle \iota}\downarrow & \simeq & \nearrow {\scriptstyle \mathrm{lan}_{\iota} D} \\ \mathrm{Ind}(\mathcal{D}_B) & & \end{array}$$

where the canonical 2-cell is invertible by full faithfulness of the dense inclusion. This extension expresses as a codirected intersection

$$\mathrm{lan}_{\iota} D(l) = \bigcap_{\mathcal{D}_B \downarrow l} D_n$$

ensuring that the canonical cocone of l in $\mathrm{Ind}(\mathcal{D}_B)$ is sent to a canonical intersection of basic neighborhoods. In the following, we also abusively denote $\mathrm{lan}_{\iota} D(l)$ as D_l. The fact that the intersection ranges over the canonical cone does not modify the intersection as intersections are idempotent, so we could also rewrite this identity as

$$D_l = \bigcap_{\substack{n \leq l \\ n \in \mathcal{D}_B}} D_n$$

that is, by considering only the order on diagonally universal morphisms rather than their categorical structure.

Remark 8.1.1.14. The equality above can be understood as follows: suppose that $x \in \bigcap_{\substack{m:n\to l \\ n\in\mathcal{D}_B}} D_n$, that is, that for each $m : n \to l$ we have a $q_m : n \to x$; then as $l = \mathrm{colim}_{m\in\mathcal{D}_B\downarrow l}\mathrm{dom}(m)$, we have a canonical morphism $\langle q_m\rangle_{\mathcal{D}_B\downarrow l} : l \to x$ ensuring that $l \leq x$, that is, $x \in D_l$. One has to compose any $l \to x$ with the canonical cone of f to see the converse inclusion.

Remark 8.1.1.15. The induced specialization order on X_B is exactly the restriction of the order of factorization $\leq$ on the local units because if $x_1 \leq x_2$ in X_B, then for any $n \in \mathcal{D}_n$, $x_1 \in D_n$ implies $x_2 \in D_n$, that is, $V_{x_1} \subseteq V_{x_2}$, and as the D_n are a basis for the topology, so the same is true for any open. Remark we can also restrict D at any V_x.

Remark 8.1.1.16. Observe that one could extend the functor D to arbitrary diagonally universal morphisms under B. In the case where they are left generated, they are filtered colimits of diagonally universal morphisms of B.

8.1.2 The Diers Space

Definition 8.1.2.1. For a Diers context $U : \mathcal{A} \to \mathcal{B}$, any object B of $\mathcal{B}$, define the *(Diers) spectrum of B* as the space $\mathbf{Spec}(B) = (X_B, \tau_B)$. We call also this space the *Diers space of B*.

Now let us explicitly describe the functorial aspects of this construction. Let $f : B_1 \to B_2$ in $\mathcal{B}$. Then for any point $x : B_2 \to U(A)$ the factorization of the composite

$$\begin{array}{ccccc} B_1 & \xrightarrow{f} & B_2 & \xrightarrow{x} & U(A) \\ & \searrow^{\eta^A_{xf}} & & \nearrow_{U(L_A(xf))} & \\ & & UL_A(xf) & & \end{array}$$

defines uniquely a point $\eta_{xf} : B_1 \to UL_A(nf)$ of X_{B_1} and we have to pose

$$\begin{array}{rccc} \mathbf{Spec}(f) : & X_{B_2} & \to & X_{B_1} \\ & B_2 \xrightarrow{x} U(A) & \longmapsto & B_1 \xrightarrow{\eta_{xf}} U(L_A(xf)) \end{array}$$

Proposition 8.1.2.2. *For any $f : B_1 \to B_2$ in $\mathcal{B}$ and any n in $\mathcal{D}_{B_1}$, we have $\mathbf{Spec}(f)^{-1}(D^{B_1}_n) = D^{B_2}_{f_*n}$, that is, $\mathbf{Spec}(f)$ restricts to the basis of the topology and we have*

$$\begin{array}{ccc} \mathcal{D}^{\mathrm{op}}_{B_1} & \xrightarrow{(f_*)^{\mathrm{op}}} & \mathcal{D}^{\mathrm{op}}_{B_2} \\ {\scriptstyle D^{B_1}}\downarrow & & \downarrow{\scriptstyle D^{B_2}} \\ \tau_{B_1} & \xrightarrow[\mathbf{Spec}(f)^{-1}]{} & \tau_{B_2} \end{array}$$

In particular, the map $\mathbf{Spec}(f) : \mathbf{Spec}(B_2) \to \mathbf{Spec}(B_1)$ is continuous.

Proof. Now for a diagonally universal morphism of finite presentation $n \in \mathcal{D}_{B_1}$ one has $n \leq \eta_{xf}$ if and only if $f_*n \leq x$. That is, $\mathbf{Spec}(f)(x) \in D^{B_1}_n$ if $x \in D^{B_2}_{f_*n}$. This proves that $\mathbf{Spec}(f)$ restricts to the basis as stated above. But testing that inverse image of basic opens are open is sufficient for continuity. □

Definition 8.1.2.3. The spaces of the form (X_B, τ_B) for B in $\mathcal{B}$ are collectively called *U-spectral spaces* or *the Diers spaces of U*, while the maps of the form $\mathbf{Spec}(f)$ are called *U-spectral maps*.

The power of Diers' approach resides also in the way one can characterize topological properties of the Diers spaces from purely functorial properties of U. We list hereafter the following characterizations from [31][Section 7].

Proposition 8.1.2.4. *Let $U : \mathcal{A} \to \mathcal{B}$ be a Diers context. Then:*

1. *Diers spaces are T_0 iff U is conservative*
2. *Diers spaces are T_1 iff U is full and faithful*
3. *Diers spaces are T_2 iff they are zero-dimensional iff U is full and faithful and moreover any object B such that $X_B = \emptyset$ is a finitely presented object in $\mathcal{B}$*
4. *Diers spaces are compact iff U lifts ultraproducts*
5. *Diers spaces are boolean iff U is full and faithful and lifts ultraproducts*

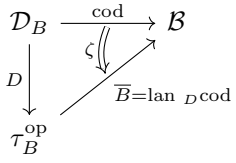
Remark 8.1.2.5. In several examples of this construction, considering only Diers spaces together with spectral maps between them is sufficient to give rise to a duality, as long as one can characterize their topological structure and how it is preserved by spectral maps. This process is suited in particular for situations where diagonally universal morphisms are quotients in correspondence with some specific kind of congruences or ideals, forming posets in a certain variety, so that the corresponding Diers spaces can be characterized through algebraic properties of the basis of their spectral topology. This is more in the spirit of concrete dualities, though such concrete dualities always seem to correspond to a Diers context. However in this paper, following Diers' approach and more generally a vision of spectra more related to algebraic geometry than concrete dualities, the kind of dualities we are going to obtain makes use of geometric information attached to the Diers space to reconstruct algebras, rather than extracting them from information in their topology. In particular, the spectrum of an object will not just correspond to a Diers space, but needs also to bear a structural sheaf to achieve the universal property expected from the spectrum.

We have constructed for each object B in $\mathcal{B}$ a space from the data associated to the right multi-adjoint U. This space was defined as having as set of points the set indexing its local units under B, and as basis for the topology the opposite of the underlying poset of the category of diagonally universal morphisms of finite presentation under B. But observe that this construction does not retain enough informations in some sense, as it forgets about the objects it was defined from: in particular, while we record the set indexing the local units (and the specialization order induced by the topology), the precise value of the codomains of the local units and the local units themselves are not remembered. So we need to attach those data to the Diers space of B. This is the purpose of the *structural sheaf of B*.

8.1.3 The structural sheaf

Now we turn to the construction of the structural sheaf on the Diers space (X_B, τ_B) for an object B in $\mathcal{B}$.

Definition 8.1.3.1. For a given $B \in \mathcal{B}$ the *structural presheaf* is defined as the left Kan extension of the codomain functor along the functor D :



That is for any $u \in \tau_B$:

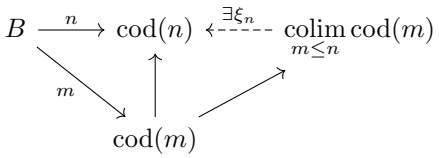
$$\overline{B}(u) = \operatorname*{colim}_{u \subseteq D_n} \mathrm{cod}(n)$$

and it is equipped with its universal natural transformation $\zeta : \mathrm{cod} \Rightarrow \overline{B}D$ defined as the collection of maps

$$\zeta = (\zeta_n : \mathrm{cod}(n) \to \operatorname*{colim}_{D_n \subseteq D_m} \mathrm{cod}(m))_{n \in \mathcal{D}_B}$$

defined as a subcocone of the pointwise colimit expression of the left Kan extension ranging over the elements $D_n = D_n$ amongst all the inequalities $D_m \supseteq D_n$.

Remark 8.1.3.2. Beware that in general the natural transformation ζ may not be a pointwise isomorphism: this will be the case if and only if D is order-reflecting as a poset map. Then in this case, we can also define a natural transformation in the opposite direction from the fact that $D_n \subseteq D_m$ iff n factorizes through m, inducing a map from the property of colimits



which is natural in n by universal property of colimits, and this provides a natural inverse to ζ.

Definition 8.1.3.3. The *structural sheaf* for B is the sheafification $\overline{B} \xrightarrow{\gamma} \widetilde{B} = \mathfrak{a}_\tau \overline{B}$, and we have for each $n : B \to C$ a morphism in $\mathbb{B}$

$$C \xrightarrow{\zeta_n} \overline{B}(D_n) \xrightarrow{\gamma_n} \widetilde{B}(D_n)$$

and, in particular, a universal map

$$B \overset{\zeta_{1_B}}{\to} \overline{B}(1_B) \overset{\gamma_{1_B}}{\to} \widetilde{B}(1_B) = \Gamma\widetilde{B}$$

we denote as $\eta_B : B \to \Gamma\widetilde{B}$

8.1.3.4. Now we come to the local behavior of this structural sheaf by defining separately the restriction of the structural presheaf to the neighborhood. Recall that the upset of x for the specialization order is the saturated compact obtained as the intersection of all open neighborhoods - the poset of which we denote $\mathcal{V}_x$:

$$\uparrow_\sqsubseteq x = \bigcap \mathcal{V}_x$$

This subset of X_B is called the *focal component* of X_B at x. The restriction codomain of arrows to V_x comes equipped with a colimiting cone with submit $U(A_x)$

$$\mathrm{cod}\,|_{V_x} \overset{\phi}{\Rightarrow} \Delta_{U(A_x)}$$

from the colimit decomposition of Diers condition exhibiting $U(A_x) = \mathrm{colim}_{n \in V_x} \mathrm{cod}(n)$. Then observe that $D\,|_x \colon V_x \hookrightarrow \mathcal{V}_x^{\mathrm{op}}$ is cofinal as any neighborhood of x contains a basic neighborhood of the form D_n with $n \leq x$.

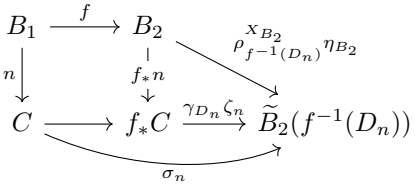
Proposition 8.1.3.5. *For any B in $\mathcal{B}$ and $x \in X_B$, $\widetilde{B}\,|_x = U(A_x)$*

Proof. This comes from the expression of Kan extension as colimit and the condition in the Diers context that candidates are obtained as filtered colimits of morphism of finite presentation. First recall that sheafification does not modify the stalks, so that $\widetilde{B}\,|_x = \overline{B}\,|_x$. Now we have

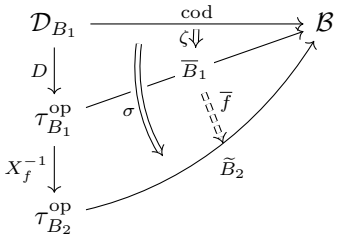
$$\overline{B}\,|_x = \operatorname*{colim}_{n \leq x} \overline{B}(D_n) = \operatorname*{colim}_{n \leq x} \operatorname*{colim}_{D_n \subseteq D_m} \mathrm{cod}(m)$$

But the diagram made of all the $n \leq x$ is cofinal in the indexing diagram of this colimit, and then it coincides with the colimit $\mathrm{colim}_{n \leq x} \mathrm{cod}(n)$ which is $U(A_x)$ by Diers condition. □

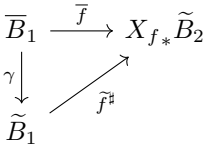
8.1.3.6. Now, for a morphism $f : B_1 \to B_2$ in $\mathcal{B}$, we construct a morphism of sheaf between the structural sheaves associated to B_1 and B_2. For any n in $\mathcal{D}_{B_1}$ we have a map $\gamma_{D_n}\zeta_n$ producing a composite map σ_n as below



and the data of all the $(\sigma_n : \mathrm{cod}(n) \to \widetilde{B}_2(X_f^{-1}(D_n))_{n\in\mathcal{D}_{B_1}}$ can be shown to provide a natural transformation $\mathrm{cod} \Rightarrow X_{f*}\widetilde{B}_2$, so the left Kan extension produces a factorization



and as the direct image $X_{f*}\widetilde{B}_2$ is still a sheaf, this induced map $\overline{f}$ itself factorizes through the sheafification as



This morphism $\widetilde{f^\sharp}$ now corresponds itself to a morphism of sheaf $\widetilde{f^\flat} : {X_f}^*\widetilde{B}_1 \to \widetilde{B}_2$.

8.1.4 The category of U-spaces

Now, to turn this construction into a functor, we need first to determine where it would land. Consider what we constructed: a space equipped with a distinguished structural sheaf of $\mathcal{B}$-objects with stalks in the range of U.

Definition 8.1.4.1. Define the category U-**Spaces** of *U-spaces* as the category whose

- objects consist of triples $((X,\tau), \mathbb{A}, (A_x)_{x\in X})$ where (X,τ) is a topological space, $\mathbb{A}$ is a sheaf of $\mathcal{B}$-objects over (X,τ), and $(A_x)_{x\in X}$ is a family of objects in $\mathcal{A}$ such that the stalks of $\mathbb{A}$ satisfy $\mathbb{A}\,|_x \simeq U(A_x)$ for any $x \in X$
- morphisms $((X_1,\tau_1), \mathbb{A}_1, (A_x^1)_{x\in X_1}) \to ((X_2,\tau_2), \mathbb{A}_2, (A_x^2)_{x\in X_2})$ consist of triples $(f, \phi, (u_x)_{x\in X_1})$ where $f : (X_2,\tau_2) \to (X_1,\tau_1)$ is continuous, ϕ consists of a pair of morphisms of sheaves

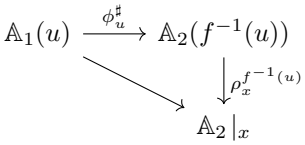
  $$(\phi^\flat : f^*\mathbb{A}_1 \to \mathbb{A}_2,\ \phi^\sharp : \mathbb{A}_1 \to f_*\mathbb{A}_2)$$

  corresponding through the adjunction $f^* \dashv f_*$ and $u_x : A^1_{f(x)} \to A^2_x$ is an arrow in $\mathcal{A}$ such we have at stalks $\phi^\flat_x = U(u_x)$ for each $x \in X_1$. The morphisms $(\phi^\flat, \phi^\sharp)$ are called the *inverse and direct comorphism parts* of the morphism of U-spaces.

Remark 8.1.4.2. In this definition, U-spaces are not just spaces with sheaves that have their stalks in the essential image of U: in fact we attach to them a specification of which objects of $\mathcal{A}$ their stalks come from. Similarly for morphisms where we impose that their inverse image part comes from an arrow in $\mathcal{A}$.

Remark 8.1.4.3. For any morphism of U-spaces (f,ϕ), the inverse image $f^*\mathbb{A}$ still has its stalks in $\mathcal{A}$ as for any point in X_2, we have $f^*\mathbb{A}_1\,|_x = \mathbb{A}_1\,|_{f(x)}$. Moreover it is a standard result that the inverse image preserves finite limits. However we cannot control the stalks of the direct image $f_*\mathbb{A}_2$, which may not be in the range of U. This is related to the fact that, while sheaves $\mathcal{B}$-objects, as object of a locally presentable categories, are stable under inverse and direct image, objects of $\mathcal{A}$ may be in a more wild class of objects, and in general sheaves of objects in non locally finitely presentable categories are not anymore stable under direct image. This is typically true when objects in $\mathcal{A}$ are model of a geometric sketch with non trivial inductive part that cannot be preserved by direct image since they are right adjoints and hence need not preserve colimits.

Remark 8.1.4.4. In a morphism of U-spaces $((X_1,\tau_1), \mathbb{A}_1, (A_x^1)_{x\in X_1}) \to ((X_2,\tau_2), \mathbb{A}_2, (A_x^2)_{x\in X_2})$, the inverse and direct image parts are related as follows. For any $x \in X_2$ and $u \in \tau_1$ such that $f(x) \in u$, that is, $x \in f^{-1}(u)$, then we have $f_*\mathbb{A}_2(u) = \mathbb{A}(f^{-1}(u))$, and we have the composite map



But now the stalk of $\mathbb{A}_1$ at $f(x)$ is the filtered colimit $\mathbb{A}_1\,|_{f(x)}= \mathrm{colim}_{f(x)\in u}\mathbb{A}_1(u)$, so the cone made of all the maps above factorizes uniquely through this colimits, and this produces the desired value of the inverse image part of ϕ at x, so that we have a commutation

$$\begin{array}{ccc} \mathbb{A}_1(u) & \xrightarrow{\phi_u^\sharp} & \mathbb{A}_2(f^{-1}(u)) \\ {\scriptstyle \rho^u_{f(x)}}\downarrow & & \downarrow{\scriptstyle \rho_x^{f^{-1}(u)}} \\ \mathbb{A}_1\,|_{f(x)} & \xrightarrow[\phi_x^\flat]{} & \mathbb{A}_2\,|_x \end{array}$$

where the inverse image part is exhibited as the universal map

$$\phi_x^\flat = \langle \rho_x^{f^{-1}(u)} \phi_u^\sharp \rangle_{f(x)\in u}$$

8.1.5 The spectral adjunction of a Diers context

To sum up, in the first part of this section, we defined for each object B in $\mathcal{B}$ a U-space $((X_B,\tau_B),\widetilde{B},(A_x)_{x\in X_B})$, and for each $f : B_1 \to B_2$, a morphism of U-space $(\mathbf{Spec}(f),\widetilde{f})$.

Definition 8.1.5.1. The construction above defines a functor called *the spectrum of U*

$$\mathcal{B} \xrightarrow{\mathbf{Spec}} U\text{-}\mathbf{Spaces}$$

8.1.5.2. Now we look at a functor going in the converse direction. Let $((X,\tau),\mathbb{A},(A_x)_{x\in X})$ be a U-space. As $\mathcal{B}$ is locally finitely presentable, the category of sheaves over (X,τ) with value in $\mathcal{B}$ is equipped with a global section functor $\Gamma : \mathbf{Sh}_{\mathbb{B}}(X,\tau) \to \mathcal{B}$ sending a sheaf $\mathbb{A}$ to the $\mathcal{B}$-object of global sections $\Gamma\mathbb{A} = \mathbb{A}(X)$. For any $u \in \tau$, we denote as $q_u^X : \Gamma\mathbb{A} \to \mathbb{A}(u)$, and for any $x \in X$, the stalk at x is obtained as a filtered colimit in $\mathcal{B}$

$$\mathbb{A}\,|_x = \operatorname*{colim}_{x\in u} \mathbb{A}(u)$$

and the induced map $q_x^X : \Gamma\mathbb{A} \to \mathbb{A}\,|_x$ is the filtered colimit of the maps q_u^X in the slice $\Gamma\mathbb{A}\downarrow\mathcal{B}$.

8.1.5.3. Now for a morphism of U-spaces $(f,\phi,(u_x)_{x\in X}) : ((X_1,\tau_1),\mathbb{A}_1,(A_x^1)_{x\in X_1}) \to ((X_2,\tau_2),\mathbb{A}_2,(A_x^2)_{x\in X_2})$ the direct image part of the morphism of sheaf $\phi^\sharp$ takes at X_1 the value $\phi^\sharp_{X_1} : \mathbb{A}_1(X_1) \to \mathbb{A}_2(f^{-1}(X_1))$, but as $f^{-1}(X_1) = X_2$, this defines a morphism in $\mathcal{B}$

$$\Gamma\mathbb{A}_1 \xrightarrow{\Gamma\phi} \Gamma\mathbb{A}_2$$

Definition 8.1.5.4. These data allow us to define a functor

$$U\text{-}\mathbf{Spaces} \xrightarrow{\Gamma} \mathcal{B}^{\mathrm{op}}$$

sending a U-space $((X,\tau),\mathbb{A},(A_x)_{x\in X})$ to the $\mathcal{B}$-object of global sections $\Gamma\mathbb{A}$ and a morphism $(f,\phi,(u_x)_{x\in X})$ to $\Gamma\phi$.

Theorem 8.1.5.5 (Diers)**.** *Let $U : \mathcal{A} \to \mathcal{B}$ define a Diers context. Then there is an adjunction*

$$\mathcal{B} \underset{\Gamma}{\overset{\mathbf{Spec}}{\rightleftarrows}} U\text{-}\boldsymbol{Spaces} \qquad (\perp)$$

Proof. This proof follows [31][3.6.1]; as it was done only in french and without the help of diagrams, we chose to give here a reformulation of it. Let B be in $\mathcal{B}$, $((X,\tau),\mathbb{A},(A_x)_{x\in X})$ a U-space and $\phi : B \to \Gamma\mathbb{A}$. Then for any $x \in X$ we have $\mathbb{A}\,|_x = U(A_x)$, so we can consider the following factorization

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \Gamma\mathbb{A} \\ {\scriptstyle \eta^{A_x}_{q_x^X\phi}}\downarrow & & \downarrow{\scriptstyle q_x^X} \\ U(A_{q_x^X\phi}) & \xrightarrow[U(L_{A_x}(q_x^X\phi))]{} & U(A_x) \end{array}$$

Then the following map

$$\begin{array}{ccc}(X,\tau) & \xrightarrow{f} & (X_B,\tau_B)\\ x & \longmapsto & \eta^{A_x}_{q^X_x\phi}\end{array}$$

is continuous. Indeed, for any finitely presented diagonally universal morphism n, we have

$$f^{-1}(D_n) = \left\{x \in X \mid n \leq \eta^{A_x}_{q^X_x\phi}\right\}$$

where the latter condition says that we have a factorization

$$\begin{array}{ccccc} B & \xrightarrow{\phi} & \Gamma\mathbb{A} & & \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \phi_* n} & \searrow^{q^X_x} & \\ C & \xrightarrow[n_*\phi]{} & \phi_* C & \xrightarrow{s} & \mathbb{A}|_x \end{array}$$

But pushouts of finitely presented diagonally universal morphisms are still finitely presented, that is, $\phi_* n$ is finitely presented as a diagonally universal morphism under $\Gamma\mathbb{A}$: but as $q^X_x = \mathrm{colim}_{x\in u} q^X_x$ in the slice $\Gamma\mathbb{A}\downarrow\mathcal{B}$, we have a factorization through some q^X_u for some u in τ such that $x \in u$

$$\begin{array}{ccc} \Gamma\mathbb{A} & \xrightarrow{q^X_u} & \mathbb{A}(u) \\ {\scriptstyle \phi_* n}\downarrow & \overset{s_u}{\nearrow} & \downarrow{\scriptstyle q^u_x} \\ \phi_* C & \xrightarrow{s} & \mathbb{A}|_x \end{array}$$

and moreover, any two such factorizations are equalized by a third one: that is, if there are two opens u, v with $x \in u, v$ such that s factorizes through q^u_x, q^v_x as s_u, s_v, then there is some $w \subseteq u, v$ and $x \in w$ such that s factorizes through w as $q^w_x s_w$. And moreover, for any $y \in u$, we still have that $\phi_* n$ factorizes q^X_x, so that

$$n \leq \eta^{A_y}_{q^X_y\phi}$$

and hence $y \in f^{-1}(D_n)$. This means that $u \subseteq f^{-1}(D_n)$. To sum up, we can choose for each $x \in f^{-1}(D_n)$ an open u_x with $x \in u_x$ and $u_x \subseteq f^{-1}(D_n)$, and hence we have

$$f^{-1}(D_n) = \bigcup_{x\in f^{-1}(D_n)} u_x$$

which is hence open in τ.

Now we construct a morphism of sheaves ψ as follows. We first construct the direct image part. As seen above, for any $n \in \mathcal{D}_B$, we can exhibit a cover of $f^{-1}(D_n)$ with a family $(u_x)_{x\in f^{-1}(D_n)}$ such that $x \in u_x$ and we have a factorization

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \Gamma\mathbb{A} \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle q^X_{u_x}} \\ C & \xrightarrow[s_x]{} & \mathbb{A}(u_x) \end{array}$$

Now, as $\mathbb{A}$ is a sheaf, and $f^{-1}(D_n)$ is open, we have the limit decomposition

$$\mathbb{A}(f^{-1}(D_n)) = \lim\Big(\prod_{x\in f^{-1}(D_n)} \mathbb{A}(u_x) \rightrightarrows \prod_{x,y\in f^{-1}(D_n)} \mathbb{A}(u_x \cap u_y) \Big)$$

So we have to check that the data of the arrows $(s_x)_{x\in X}$ define a cone over the diagram above, which amounts to checking that for any $x, y \in X$ the following diagram commutes

$$\begin{array}{ccccc} & & \mathbb{A}(u_x) & & \\ & \overset{s_x}{\nearrow} & & \overset{q^{u_x}_{u_x\cap u_y}}{\searrow} & \\ C & & & & \mathbb{A}(u_x \cap u_y) \\ & \underset{s_y}{\searrow} & & \underset{q^{u_y}_{u_x\cap u_y}}{\nearrow} & \\ & & \mathbb{A}(u_y) & & \end{array}$$

For any $z \in u_x \cap u_y$ we have

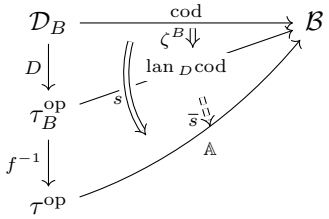
$$q_z^{u_x} s_x n = q_z^{u_x} q_{u_x}^X \phi = q_z^{u_y} q_{u_y}^X \phi = q_z^{u_y} s_y n$$

hence s_x and s_y produce factorizations of q_z^X, so by what was said before as a consequence of finite presentability of n, those two factorizations have to be merged into some $v_z \subseteq u_x \cap u_y$: but this is true for any $z \in u_x \cap u_y$, so the opens v_z cover the intersection, and hence the square above commutes.

Then the universal property of the limit provides a unique arrow

$$\begin{array}{ccc} B & \xrightarrow{\phi} & \Gamma\mathbb{A} \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \rho^X_{f^{-1}(D_n)}} \\ C & \xrightarrow[s_n]{} & \mathbb{A}(f^{-1}(D_n)) \end{array}$$

and the data of all those arrows $(s_n : \mathrm{cod}(n) \to \mathbb{A}(f^{-1}(D_n)))_{n \in \mathcal{D}_B}$ defines a natural transformation $s : \mathrm{cod} \Rightarrow f_*\mathbb{A}$: now the universal property of the left Kan extension produces a universal arrow as below



But now, as $\mathbb{A}$ is a sheaf, so is the direct image $f_*\mathbb{A}$; hence the natural transformation $\overline{s}$ factorizes through the sheafification of $\mathrm{lan}_D \mathrm{cod}$, that is

$$\begin{array}{ccc} \mathrm{lan}_D \mathrm{cod} & \xrightarrow{\overline{s}} & f_*\mathbb{A} \\ {\scriptstyle \gamma}\downarrow & \nearrow_{\psi^\sharp} & \\ \widetilde{B} & & \end{array}$$

This returns the desired direct image part $\psi^\sharp$.

For the inverse image part, the adjunction $f^* \dashv f_*$ associates a unique mate $\psi^\flat$ to $\psi^\sharp$. Now we want to check that this mate behaves correctly at the stalks, that is, that $\psi^\flat_x = U(L_{A_x}(q_x^X \phi))$ for any point x. For any $x \in X$ and any $n \in \mathcal{D}_B$, we have from remark 8.1.4.4 an commutative square

$$\begin{array}{ccc} \widetilde{B}(D_n) & \xrightarrow{\psi^\sharp_{D_n}} & \mathbb{A}(f^{-1}(D_n)) \\ {\scriptstyle \rho^{D_n}_{f(x)}}\downarrow & & \downarrow{\scriptstyle \rho_x^{f^{-1}(D_n)}} \\ \widetilde{B}|_{f(x)} & \xrightarrow[\psi^\flat_x]{} & \mathbb{A}|_x \end{array}$$

But the stalk of $\widetilde{B}$ at $f(x) = \eta^{A_x}_{\rho_x^X \phi}$ is both the unit of the factorization of $\rho_x^X \phi$ and the filtered colimit

$$\widetilde{B}|_{f(x)} = \operatorname*{colim}_{n \leq \eta^{A_x}_{\rho_x^X \phi}} \widetilde{B}(D_n)$$

inducing $\phi^\flat_x$ as the universal map

$$\langle \rho_x^{f^{-1}(D_n)} \psi^\sharp_{D_n} \rangle_{n \leq \eta^{A_x}_{\rho_x^X \phi}}$$

But from Diers condition, this later must be the right part of the factorization, so that we have the desired equality. Now we gather all the maps $(L_{A_x}(q_x^X \phi))_{x \in X}$ to complete the data of the morphism of U-space

$$((X_B, \tau_B), \widetilde{B}, (A_x)_{x \in X_B}) \xrightarrow{(f, \psi, (L_{A_x}(\rho_x^X \phi))_{x \in X})} ((X, \tau), \mathbb{A}, (A_x)_{x \in X})$$

□

Remark 8.1.5.6. While suited for a large range of examples where the spectrum is expected to be spatial, this construction of the spectrum as a topological space causes some loss of information as it equips the set of local units with a structure of poset, mimicking the specialization order of the spectral topology, while in general local units under a given object and diagonally universal morphisms between them form a category. In some sense Diers only considers the localic reflection of a small site made of finitely presentable diagonally universal morphisms, though the topos associated to this site may not be localic, for instance in the case of the étale spectrum of a ring. This problem would be fixed by introducing a variant of Diers' construction in terms of ionad, where the link with the other notions of spectrum would appear more clearly; as this would require a more involved notion of topoi and ionads, we choose to postpone this task to a later paper devoted to the synthesis between different approaches of the construction of spectra from a more topos-theoretic point of view.

8.2 Spectrum of an arbitrary $\mathcal{B}$-space

In the previous section we recalled Diers' original construction of the spectrum of an object of $\mathcal{B}$ for a Diers context. Observe that the spectral functor thus defined is left adjoint to a global sections functor sending a U-space to the $\mathcal{B}$-object of global sections of its structural sheaf. This process is somewhat reminiscent of the relative adjoint $\mathcal{B} \to \Pi\mathcal{A}$ constructed in proposition 2.2.2.3, which returns the points of the spectrum of a given object. Hence the fact that this relative adjoint could extend into a functor $\Pi\mathcal{B} \to \Pi\mathcal{A}$ left adjoint to ΠU invites us to extend Diers spectral functor to a larger class of $\mathcal{B}$-spaces, amongst which the objects of $\mathcal{B}$ embed as $\mathcal{B}$-spaces with the point as underlying space. This generalized spectrum will be left adjoint to the a functor sending a U-space to the corresponding $\mathcal{B}$-space one gets by applying stalk-wise the functor U. The process in this part is also totally point-set, hence both quite concrete and somewhat "handmade", in contrast to the more abstract, yet purer point-free approach in the topos-theoretic methods of the previous chapters.

Throughout this section, we fix a right multi-adjoint $U : \mathcal{A} \to \mathcal{B}$ satisfying Diers conditions.

8.2.1 $\mathcal{B}$-spaces

Definition 8.2.1.1. We define a *$\mathcal{B}$-space* as the data of a topological space (X, τ) together with a sheaf of $\mathcal{B}$-object $\mathbb{B} : \tau^{\mathrm{op}} \to \mathcal{B}$. A *morphism of $\mathcal{B}$-spaces* $((X_1, \tau_1), \mathbb{B}_1) \to ((X_2, \tau_2), \mathbb{B}_2)$ is the data of a continuous map $f : X_2 \to X_1$ and a pair of morphisms of sheaves

$$(\phi^\flat : f^*\mathbb{B}_1 \to \mathbb{B}_2,\ \phi^\sharp : \mathbb{B}_1 \to f_*\mathbb{B}_2)$$

corresponding through the adjunction $f^* \dashv f_*$. We denote as $\mathcal{B}$-**Spaces** the category of $\mathcal{B}$-spaces and morphisms between them.

Proposition 8.2.1.2. *There is a functor*

$$U\text{-}\boldsymbol{Spaces} \xrightarrow{\iota_u} \mathcal{B}\text{-}\boldsymbol{Spaces}$$

sending a U-space $((X, \tau), \mathbb{A}, (A_x)_{x\in X})$ to the $\mathcal{B}$-space $((X, \tau), \mathbb{A})$, and a morphism of U-spaces to the induced morphism of $\mathcal{B}$-space.

Remark 8.2.1.3. Beware that, without additional assumption on U, this functor may not be faithful, nor even injective on objects. Its action on objects is to forget which A_x of $\mathcal{A}$ the stalk of $\mathbb{A}\mid_x =$ $U(A_x)$ comes from. In some sense, the structural sheaf of $\iota_U((X, \tau), \mathbb{A}, (A_x)_{x\in X})$ has its stalks in the essential image of U, so that two U-spaces over a same topological space $((X, \tau), \mathbb{A}_1, (A^1_x)_{x\in X})$ and $((X, \tau), \mathbb{A}_2, (A^2_x)_{x\in X})$ may become isomorphic in $\mathcal{B}$-**Spaces** if $U(A^1_x) \simeq U(A^2_x)$ in each x, though some A^1_x and A^2_x were not isomorphic in $\mathcal{A}$. The same phenomenon happens for arrows.

Moreover, in most cases, ι_U is not full, unless U was itself full, but this corresponds to a specific kind of geometries returning T_1 spaces. In general we do not desire the functor ι_U to be full as the morphisms $(f, (\phi^\flat, \phi^\sharp), (u_x)_{x\in X})$ in U-**Spaces** have the additional data $(u_x)_{x\in X}$ attached to the $\phi^\flat$ part of the morphism of sheaf, forcing the map at stalks $\phi^\flat_x : \mathbb{A}_1 \mid_{f(x)} \to \mathbb{A}_2 \mid_x$ to be equal to $U(u_x) : U(A^1_{f(x)}) \to U(A^2_x)$, hence in the range of U, while there is no such condition for general morphisms of $\mathcal{B}$-spaces.

8.2.2 The Diers space of a $\mathcal{B}$-space

In the following we fix a $\mathcal{B}$-space $((X,\tau),\mathbb{B})$. We are going to process by gluing the spectra of the stalks of the structural sheaf $\mathbb{B}$ at points of X, and equip it with a topology generated from the finitely presented diagonally universal morphisms under the values of the sheaf $\mathbb{B}$ at opens of τ.

Definition 8.2.2.1. We define the *Diers space* of $((X,\tau),\mathbb{B})$ as the space $(X_{\mathbb{B}},\tau_{\mathbb{B}})$ with

$$X_{\mathbb{B}} = \big\{(x,\xi) \mid x \in X,\ \xi \in X_{\mathbb{B}|_x}\big\}$$

equipped with the topology generated as

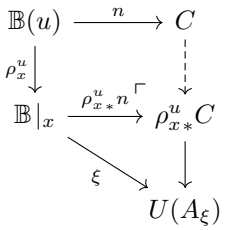
$$\tau_{\mathbb{B}} = \langle D_{(u,n)}\rangle_{(u,n)\in\mathcal{D}_{\mathbb{B}}}$$

where $\mathcal{D}_{\mathbb{B}} = \{(u,n) \mid u \in \tau,\ n \in \mathcal{D}_{\mathbb{B}(u)}\}$ and

$$D_{(u,n)} = \big\{(x,\xi) \mid x \in u,\ \rho^u_{x*}n \leq \xi\big\}$$

where $\rho^u_n : \mathbb{B}(u) \to \mathbb{B}\mid_x$ is the restriction map.

Remark 8.2.2.2. The condition defining the basic open set $D_{(u,n)}$ concerns the finitely presented diagonally universal morphism n under $\mathbb{B}(u)$. Each $\mathbb{B}(u)$ is an object in $\mathcal{B}$, as well as each stalk $\mathbb{B}\mid_x$, which is the filtered colimit of all $\mathbb{B}(u)$ such that $x \in u$. Hence for such a neighborhood u there is a canonical inclusion $\rho^u_x : \mathbb{B}(u) \to \mathbb{B}\mid_x$, and we can push n along this inclusion to get the finitely presented diagonally universal morphism $\rho^u_{x*}n$ in $\mathcal{D}_{\mathbb{B}|_x}$. The condition then requires that we have a factorization



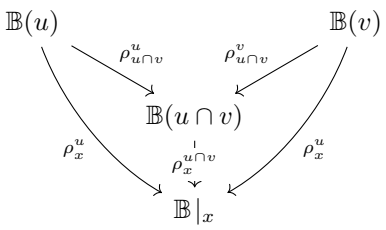
Remark 8.2.2.3. Also it is interesting to note that, if τ admits a basis $\tau_0 \hookrightarrow \tau$, then one can generate $\tau_{\mathcal{B}}$ from the open $D_{(u,n)}$ with $u \in \tau_0$. But this is not that important in this point-set context as τ has anyway to be a small set.

It is worth checking that this is a basis:

Lemma 8.2.2.4. *For each (u,n) and (v,m) in $\mathcal{D}_{\mathbb{B}}$, we have*

$$D_{(u,n)} \cap D_{(v,m)} = D_{(u\cap v,\rho^u_{u\cap v\,*}n \vee \rho^v_{u\cap v\,*}m)}$$

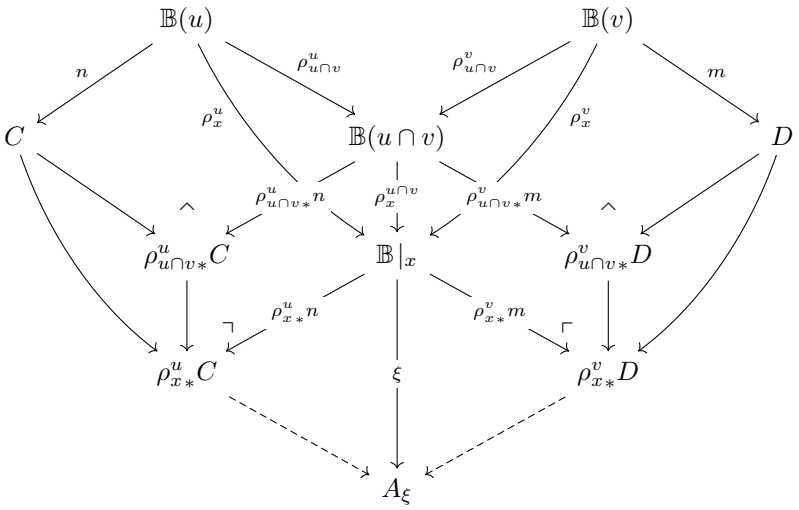
Proof. First, it is clear that if $(x,\xi) \in D_{(u,n)} \cap D_{(v,m)}$ then $x \in u \cap v$. Then the projections ρ^u_x and ρ^v_x factorize as



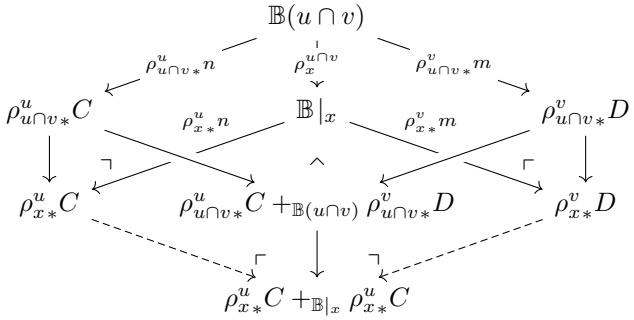
Now by left cancellation of pushouts we have that

$$\rho^u_{x*}n = \rho^{u\cap v}_x{}_*(\rho^u_{u\cap v*}n) \text{ and } \rho^v_{x*}m = \rho^{v\cap v}_x{}_*(\rho^v_{u\cap v*}m)$$

as depicted in the following diagram



Moreover commutation of pushouts ensures that all squares in the following cube are pushouts



But we have

$$\rho^u_{x*}n \vee \rho^v_{x*}m \leq \xi$$

and hence

$$\rho^u_{u\cap v*}(\rho^u_{u\cap v*}n \vee \rho^v_{u\cap v*}m) \leq \xi$$

which ensures that $(x,\xi) \in D_{(u\cap v, \rho^u_{u\cap v*}n \vee \rho^v_{u\cap v*}m)}$ □

Proposition 8.2.2.5. *We have a continuous map*

$$(X_{\mathbb{B}}, \tau_{\mathbb{B}}) \xrightarrow{\eta} (X, \tau)$$

$$(x,\xi) \longmapsto x$$

which is moreover open.

Proof. For $u \in \tau$, we have $\eta^{-1}(u) = \{(x,\xi) \mid x \in u,\ \xi \in X_{\mathbb{B}|_x}\}$ which coincides with the basic open $D_{(u,1_{\mathbb{B}(u)})}$ as $q^u_{x*}(1_{\mathbb{B}(u)}) = 1_{\mathbb{B}|_x}$ is the initial object of $\mathcal{D}_{\mathbb{B}|_x}$ hence factorizes any local unit ξ under $\mathbb{B}\ |_x$. It is moreover open because the direct image of an open $D_{(u,n)}$ along η is the underlying open of τ, that is $\eta(D_{(u,n)}) = u$. □

Proposition 8.2.2.6. *We have in each $x \in X$ a continuous map*

$$(X_{\mathbb{B}|_x}, \tau_{\mathbb{B}|_x}) \xhookrightarrow{\iota_x} (X_{\mathbb{B}}, \tau_{\mathbb{B}})$$

$$\xi \longmapsto (x,\xi)$$

exhibiting the Diers space as the disjoint union of the Diers spaces of the stalks

$$X_{\mathbb{B}} = \coprod_{x\in X} X_{\mathbb{B}|_x}$$

and η as the induced map $\langle c_x \rangle_{x\in X}$ where $c_x : X_{\mathbb{B}|_x} \to X$ is the constant map equal to x.

Proof. Observe that if $x \in u$ we have $\iota_x^{-1}(D_{(u,n)}) = D_{q^u_{x*}n}$ in $\tau_{\mathbb{B}|_x}$ by property of the pushout; otherwise we have $\iota_x^{-1}(u) = 0$ if $x \notin u$. □

At each $u \in \tau$ we have an open $D_{(u,1_{\mathbb{B}(u)})}$, and this defines an open subspace

$$(D_{(u,1_{\mathbb{B}(u)})}, \tau_{\mathbb{B}}\mid_{D_{(u,1_{\mathbb{B}(u)}}}) \xhookrightarrow{\iota_{(u,1_{\mathbb{B}(u)})}} (X_{\mathbb{B}}, \tau_{\mathbb{B}})$$

Observe that we also have, for any $u \in \tau$, a canonical map

$$D_{(u,1_{\mathbb{B}(u)})} \xrightarrow{p_u} X_{\mathbb{B}(u)}$$

$$(x,\xi) \longmapsto \eta^{A_\xi}_{\xi\rho^u_x}$$

sending any (x,ξ) with $x \in u$ to the unit under $\mathbb{B}(u)$ factorizing the composite

$$\begin{array}{ccc} \mathbb{B}(u) & \xrightarrow{\eta^{A_\xi}_{\xi\rho^u_x}} & U(A_{\xi\rho^u_x}) \\ {\scriptstyle \rho^u_x}\downarrow & & \downarrow{\scriptstyle U(L_{A_\xi}(\xi\rho^u_x))} \\ \mathbb{B}\mid_x & \xrightarrow[\xi]{} & U(A_\xi) \end{array}$$

Moreover Diers condition ensures that this map is continuous for $U(A_{\xi\rho^u_x}) = \mathrm{colim}_{n \leq \xi\rho^u_x} \mathrm{cod}(n)$ and $n \leq \xi\rho^u_x$ if and only if $\rho^u_{x*}n \leq \xi$. Hence in particular p_u^{-1} coincides on the basis $\mathcal{D}_{\mathbb{B}(u)}$ with the pushout along ρ^u_x, that is we have

$$\mathcal{D}_{\mathbb{B}(u)} \xhookrightarrow{p_u^{-1}} \mathcal{D}_{\mathbb{B}}$$

$$n \longmapsto (u,n)$$

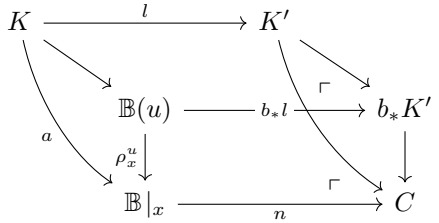
Before going further, we need the following key observation, which is a point-set version of theorem 7.5.2.4:

Lemma 8.2.2.7. *For any $x \in X$ and any finitely presented diagonally universal morphism n in $\mathcal{D}_{\mathbb{B}|_x}$, there is some u with $x \in u$ and m in $\mathcal{D}_{\mathbb{B}(u)}$ such that $n = \rho^u_{x*}m$.*

Proof. Recall that diagonally universal morphisms of finite presentation under $\mathbb{B}\mid_x$ are those induced as pushouts

$$\begin{array}{ccc} K & \xrightarrow{l} & K' \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle l_*a} \\ \mathbb{B}\mid_x & \xrightarrow[n]{} & C \end{array}$$

whith K, K' in $\mathcal{B}_\omega$ and l diagonally universal. But now, as we have a filtered colimit decomposition $\mathbb{B}\mid_x = \mathrm{colim}_{x\in u}\mathbb{B}(u)$ and K is finitely presented, a factorizes through some ρ^u_x, and by left cancellation of pushouts, the front square in the following diagram is a pushout since the back and top are so



so that $n = \rho^u_{x*}b_l$, but b_*l is in $\mathcal{D}_{\mathbb{B}(u)}$. □

In some sense, this result says that the categories $\mathcal{D}_{\mathbb{B}(u)}$ for $x \in u$ are jointly essentially surjective on $\mathcal{D}_{\mathbb{B}|_x}$. This has also the following consequence which will be central to control the stalks of the structural sheaf we are going to construct:

Theorem 8.2.2.8. *For any $(x,\xi) \in X_{\mathbb{B}}$, we have a filtered colimit decomposition of ξ in the slice $\mathcal{B}^2$:*

$$\xi = \operatorname*{colim}_{x\in u} \rho^u_{x*}\eta^{A_\xi}_{\xi\rho^u_x}$$

As a consequence, the cocone $(U(L_{A_\xi}(\xi\rho^u_x)))_{x\in u}$ exhibits $U(A_\xi)$ as a filtered colimit

$$U(A_\xi) \simeq \operatorname*{colim}_{x\in u} \rho^u_{x*}U(A_{\xi\rho^u_x})$$

Proof. First, for any u with $x \in u$, we have $n \leq \eta^{A_\xi}_{A_{\xi\rho^u_x}}$ if and only if $\rho^u_{x*} n \leq \xi$. But Diers condition at $\mathbb{B}\,|_x$ says that $\xi = \mathrm{colim}_{n \leq \xi} n$, and from the previous lemma, we know that we can precise this colimit as

$$\begin{aligned}
\xi &\simeq \operatorname*{colim}_{\substack{x \in u \\ n \in \mathcal{D}_{\mathbb{B}(u)} \ n \leq \eta^{A_\xi}_{A_{\xi\rho^u_x}}}} \rho^u_{x*} n \\
&\simeq \operatorname*{colim}_{x \in u} \rho^u_{x*} \Big(\operatorname*{colim}_{n \leq \eta^{A_\xi}_{A_{\xi\rho^u_x}}} \mathrm{cod}(n) \Big) \\
&\simeq \operatorname*{colim}_{x \in u} \rho^u_{x*} \eta^{A_\xi}_{\xi\rho^u_x}
\end{aligned}$$

where the last isomorphism comes from Diers condition at $\eta^{A_\xi}_{\xi\rho^u_x}$. The second item is deduced from the fact that cod commutes with filtered colimits. □

8.2.3 The structural sheaf

Now we describe the process to construct a structural sheaf $\widetilde{\mathbb{B}}$ on $(X_{\mathbb{B}}, \tau_{\mathbb{B}})$ associated to $\mathbb{B}$. The poset $\mathcal{D}_{\mathbb{B}} = \{D_{(u,n)} \mid u \in \tau,\, n \in \mathcal{D}_{\mathbb{B}(u)}\}$ is equipped with a functor

$$\begin{aligned}
\mathcal{D}_{\mathbb{B}} &\xrightarrow{\mathrm{cod}\pi_1} \mathcal{B} \\
(u, n) &\longmapsto \mathrm{cod}(n)
\end{aligned}$$

and then consider its Kan extension along the inclusion into the spectral topology

$$\begin{array}{ccc}
\mathcal{D}_{\mathbb{B}} & \xrightarrow{\mathrm{cod}\pi_1} & \mathcal{B} \\
{\scriptstyle \iota_{\mathbb{B}}}\downarrow & \overset{\zeta^{\mathbb{B}}}{\Rightarrow} & \nearrow {\scriptstyle \mathrm{lan}_{\iota_{\mathbb{B}}} \mathrm{cod}\pi_1} \\
\tau_{\mathbb{B}} & &
\end{array}$$

and define $\widetilde{\mathbb{B}}$ as the sheafification for the topology $\tau_{\mathbb{B}}$

$$\widetilde{\mathbb{B}} = \mathfrak{a}_{\tau_{\mathbb{B}}}(\mathrm{lan}_{\iota_{\mathbb{B}}} \mathrm{cod}\pi_1)$$

At this point it is worth giving some detail of the behavior on the sheaf $\widetilde{\mathbb{B}}$ and its relation with the spectra of the values at open $\mathbb{B}(u)$. For each $u \in \tau$ the object $\mathbb{B}(u)$ is in $\mathcal{B}$, hence has itself a spectrum

$$(X_{\mathbb{B}(u)}, \tau_{\mathbb{B}(u)}, \widetilde{\mathbb{B}(u)})$$

Then the structural sheaf $\widetilde{\mathbb{B}}$ can be compared to the structural sheaves of the form $\widetilde{\mathbb{B}(u)}$ as follows: indeed we had in each $u \in \tau$ an inclusion on one side along which one can restrict the structural sheaf $\widetilde{\mathbb{B}}$ into $\iota^*_{(u,1_{\mathbb{B}(u)})}\widetilde{\mathbb{B}}$. But after restriction we could define a continuous map p_u

$$\begin{array}{ccc}
(D_{(u,1_{\mathbb{B}(u)})}, \tau_{\mathbb{B}}\,|_{D_{(u,1_{\mathbb{B}(u)})}}) & \xrightarrow{p_u} & (X_{\mathbb{B}(u)}, \tau_{\mathbb{B}(u)}) \\
{\scriptstyle \iota_{(u,1_{\mathbb{B}(u)})}}\big\downarrow & & \\
(X_{\mathbb{B}}, \tau_{\mathbb{B}}) & &
\end{array}$$

Lemma 8.2.3.1. *For each $u \in \tau$, the restriction of $\widetilde{\mathbb{B}}$ along p_u is related to $\widetilde{\mathbb{B}(u)}$ through a canonical morphism of sheaves*

$$\widetilde{\mathbb{B}(u)} \xrightarrow{\widetilde{p_u}^\sharp} p_{u*}\widetilde{\mathbb{B}}\,|_{D_{(u,1_{\mathbb{B}(u)})}}$$

Proof. This can be tested open-wisely on the basis for each $n \in \mathcal{D}_{\mathbb{B}(u)}$, and at the level of the structural presheaves: indeed we have $p_u^{-1}(D_n) = D_{(u,n)}$ and we have

$$\overline{\mathbb{B}(u)}(D_n) = \operatorname*{colim}_{D_n \subseteq D_m} \mathrm{cod}(m)$$

But observe that $D_n \subseteq D_m$ implies that $D_{(u,n)} \subseteq D_{(u,m)}$: indeed, the first condition says that for any $\zeta \in X_{\mathbb{B}(u)}$, $n \leq \zeta$ if and only if $m \leq \zeta$; but then, as for any $x \in u$ and any $\xi \in X_{\mathbb{B}|_x}$, by

theorem 8.2.2.8 we have that $\xi = \mathrm{colim}_{x\in v}\rho^v_{x*}U(A_{\xi\rho^v_x})$, then $n \leq \eta^{A_\xi}_{\xi\rho^u_x}$ implies $m \leq \eta^{A_\xi}_{\xi\rho^u_x}$, but this condition also means that $\rho^u_{x*}n \leq \xi$ implies $\rho^u_{x*}m \leq \xi$. Hence the implication. But now recall that the restriction of $\overline{\mathbb{B}}$ has the same values as $\overline{\mathbb{B}}$ itself for opens of $D_{(u,1_{\mathbb{B}(u)})}$. But $\overline{\mathbb{B}}$ is itself computed as

$$\overline{\mathbb{B}}(D_{(u,n)}) = \operatorname*{colim}_{D_{(u,n)}\subseteq D_{(v,m)}} \mathrm{cod}(m)$$

Hence the first colimit ranges over a subset of the indexing set of the second colimit, inducing a canonical arrow

$$\overline{\mathbb{B}(u)}(D_n) \xrightarrow{(\overline{p_u})_n} \overline{\mathbb{B}}(D_{(u,n)})$$

and those arrows, being induced by universal properties, define altogether a natural transformation. Now the desired $\widetilde{p_u}^\sharp$ is the induced morphism of sheaves after sheafification. □

Now let us look at the local behavior of the structural sheaf.

Proposition 8.2.3.2. *The stalk of $\widetilde{\mathbb{B}}$ at (x,ξ) is given as*

$$\widetilde{\mathbb{B}}|_{(x,\xi)} = U(A_\xi)$$

Proof. We just have to compute the stalk of the structural presheaf $\overline{\mathbb{B}}$ at (x,ξ):

$$\overline{\mathbb{B}}|_{(x,\xi)} \simeq \operatorname*{colim}_{\substack{x\in u\\ \rho^u_{x*}m\leq\xi}} \overline{\mathbb{B}}(D_{(u,n)}) = \operatorname*{colim}_{\substack{x\in u\\ \rho^u_{x*}m\leq\xi}} \operatorname*{colim}_{D_{(u,n)}\subseteq D_{(v,m)}} \mathrm{cod}(m)$$

But we have that the set of all the $D(u,\rho^u_{v*}m)$ for $u\subseteq v$ and $m\in\mathcal{D}_{\mathbb{B}(v)}$ is cofinal in the indexing set of the inner colimit, so we have

$$\overline{\mathbb{B}}|_{(x,\xi)} \simeq \operatorname*{colim}_{\substack{x\in u\\ \rho^u_{x*}m\leq\xi}} \operatorname*{colim}_{\substack{D_{(u,n)}\subseteq D_{(u,\rho^v_{u*}m)}\\ u\leq v,\ m\in\mathcal{D}_{\mathbb{B}(v)}}} \mathrm{cod}(\rho^u_{x*}m)$$

But, on one hand, we have $\mathrm{cod}(\rho^u_{x*}m) = \rho^u_{x*}(\mathrm{cod}(m))$, and on the other hand, the indexing set above is cofinal in the set of all $D_{(u,n)}$ with $x\in u$ and $\rho^u_{x*}n\leq\xi$: this entails that

$$\overline{\mathbb{B}}|_{(x,\xi)} \simeq \operatorname*{colim}_{x\in u} \operatorname*{colim}_{\rho^u_{x*}m\leq\xi} \rho^u_{x*}(\mathrm{cod}(m))$$

But by theorem 8.2.2.8, we know that this latter colimit is $U(A_\xi)$. □

Remark 8.2.3.3. To come back to the comparison morphism $\widetilde{p_u}^\sharp$, we can see now that the corresponding inverse image part $\widetilde{p_u}^\flat$ is obtained from the universal property of the colimit in the square

$$\begin{array}{ccc} \widetilde{\overline{\mathbb{B}(u)}}(D_n) & \xrightarrow{\widetilde{p_u}^\sharp_n} & \widetilde{\mathbb{B}}(D_{(u,n)}) \\ {\scriptstyle \eta^{A_\xi}_{\xi\rho^u_x}}\downarrow & & \downarrow{\scriptstyle \rho^{D_{(u,n)}}_\xi} \\ U(A_{\xi\rho^u_x}) & \xrightarrow[=U(L_{A_\xi}(\xi\rho^u_x))]{\widetilde{p_u}^\flat_\xi} & U(A_\xi) \end{array}$$

Now we can also compare the structural sheaf $\widetilde{\mathbb{B}}$ and the structural sheaves $\widetilde{\mathbb{B}|_x}$:

Lemma 8.2.3.4. *For each $(x,\xi)\in X_{\mathbb{B}}$, we have an isomorphism*

$$\iota_x^*\widetilde{\mathbb{B}} \simeq \widetilde{\mathbb{B}|_x}$$

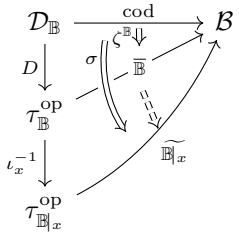
Proof. For each $u\in\tau$ such that $x\in u$ and each $n\in\mathcal{D}_{\mathbb{B}(u)}$ we have a canonical map

$$\sigma_{(u,n)} : \mathrm{cod}(n) \to \widetilde{\mathbb{B}|_x}(D_{\rho^u_{x*}n})$$

provided by the composite of the right vertical maps

$$\begin{array}{ccc} \mathbb{B}(u) & \xrightarrow{n} & C \\ {\scriptstyle \rho^u_x}\downarrow & & \downarrow \\ \mathbb{B}|_x & \xrightarrow{\rho^u_{x*}n} & \rho^u_{x*}C \\ & \searrow & \downarrow{\scriptstyle \gamma_{\rho^u_{x*}n}\zeta^{\mathbb{B}|_x}_{\rho^u_{x*}n}} \\ & & \widetilde{\mathbb{B}|_x}(D_{\rho^u_{x*}n}) \end{array}$$

In the case where $x \notin u$, then $\iota_x^{-1}(u) = \emptyset$ and then $\widetilde{\mathbb{B}|_x}(\emptyset) = 1$ is always the terminal object of $\mathcal{B}$: then for each $n \in \mathcal{D}_{\mathbb{B}(u)}$ put $\sigma_{(u,n)} =!_{\mathrm{cod}(n)}$. The data of the $(\sigma_{(u,n)})_{(u,n)\in\mathcal{D}_{\mathbb{B}}}$ define a natural transformation $\sigma : \mathrm{cod} \Rightarrow \overline{\mathbb{B}}$ and we can exploit the universal property of the left Kan extension to deduce a natural transformation $\overline{\mathbb{B}} \to \iota_{x*}\widetilde{\mathbb{B}|_x}$.



which factorizes through the sheafification $\widetilde{\mathbb{B}}$. The mate of this defines a natural map $\iota_x^*\widetilde{\mathbb{B}} \to \widetilde{\mathbb{B}|_x}$, but we have at each point ξ of $X_{\mathbb{B}|_x}$ an isomorphism

$$\iota_x^*\widetilde{\mathbb{B}}\,|_\xi \simeq \widetilde{\mathbb{B}}\,|_{(x,\xi)} \simeq U(A_\xi) \simeq \widetilde{\mathbb{B}|_x}\,|_\xi$$

But as $X_{\mathbb{B}|_x}$ has enough points, this suffices to have isomorphism of sheaves. □

8.2.4 The extended spectral adjunction of a Diers context

Definition 8.2.4.1. The *(Diers) spectrum* of a $\mathcal{B}$-space $((X,\tau),\mathbb{B})$ is the U-space

$$((X_{\mathbb{B}},\tau_{\mathbb{B}}),\widetilde{\mathbb{B}},(A_\xi)_{(x,\xi)\in X_{\mathbb{B}}})$$

Now for a morphism of $(f,\phi) : \mathcal{B}$-space $((X_1,\tau_1),\mathbb{B}_1) \to ((X_2,\tau_2),\mathbb{B}_2)$, the inverse and direct image parts of ϕ are defined respectively on points $(x,\xi) \in X_{\mathbb{B}_2}$ and basic opens $D_{(u,n)}$ of $\tau_{\mathbb{B}_1}$ by factorization and puhsouts as follows

$$\begin{array}{ccc} \mathbb{B}_1|_{f(x)} & \xrightarrow{\phi_x^\flat} & \mathbb{B}_2|_x \\ {\scriptstyle n_{\xi\phi_x^\flat}}\downarrow & & \downarrow{\scriptstyle \xi} \\ U(A_{\xi\phi_x^\flat}) & \xrightarrow[U(L_{A_\xi}(\xi\phi_x^\flat))]{} & U(A_\xi) \end{array} \qquad \begin{array}{ccc} \mathbb{B}_1(u) & \xrightarrow{\phi_u^\sharp} & \mathbb{B}_2(f^{-1}(u)) \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \phi^\sharp_{u*}n} \\ C & \xrightarrow[n^*\phi_u^\sharp]{} & \phi^\sharp_{u*}C \end{array}$$

Proposition 8.2.4.2. *We have a morphism of U-spaces*

$$((X_{\mathbb{B}_1},\tau_{\mathbb{B}_2}),\widetilde{\mathbb{B}_1},(A_\xi)_{(x,\xi)\in X_{\mathbb{B}_1}}) \longrightarrow ((X_{\mathbb{B}_2},\tau_{\mathbb{B}_2}),\widetilde{\mathbb{B}_2},(A_\xi)_{(x,\xi)\in X_{\mathbb{B}_2}})$$

whose underlying continuous map is

$$\begin{array}{ccc} (X_{\mathbb{B}_2},\tau_{\mathbb{B}_2}) & \xrightarrow{\mathbf{Spec}(\phi)} & (X_{\mathbb{B}_1},\tau_{\mathbb{B}_2}) \\ (x,\xi) & \longmapsto & (f(x),n_{\xi\phi_x^\flat}) \end{array}$$

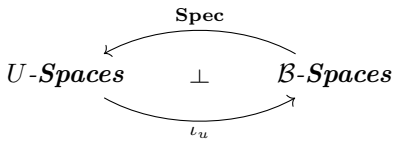
which is moreover spectral, and where the sheaf morphism $\widetilde{\phi}$ has respectively the following inverse and direct image parts at points $(x,\xi) \in X_{\mathbb{B}_2}$ and basic opens $D_{(u,n)}$ of $\tau_{\mathbb{B}_1}$

$$\widetilde{\phi}^\flat_{(x,\xi)} = L_A(\xi\phi_x^\flat) \qquad \widetilde{\phi}^\sharp_{D_{(u,n)}} = \gamma_{\mathbb{B}_2}\zeta^{\mathbb{B}_2}_{D_{(u,n)}}n^*\phi_u^\sharp$$

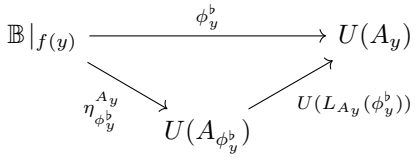
where $\gamma_{\mathbb{B}_2} : \mathrm{lan}_{D_{\mathbb{B}_2}}\mathrm{cod}\pi_1 \Rightarrow \widetilde{\mathbb{B}_2}$ is the sheafification map and $\zeta^{\mathbb{B}_2} : \mathrm{cod}\pi_1 \Rightarrow \mathrm{lan}_{D_{\mathbb{B}_2}}\mathrm{cod}\pi_1$ is the canonical natural transformation of the left Kan extension.

Proof. For a basic open $D_{(u,n)}$ in $\tau_{\mathbb{B}_1}$ we have $\mathbf{Spec}(\phi)^{-1}(D_{(u,n)}) = D_{(f^{-1}(u),\phi^\sharp_{u*}n)}$ which is a basic open of $\tau_{\mathbb{B}_2}$. □

Theorem 8.2.4.3. *We have an adjunction*



Proof. Suppose we have a morphism of $\mathbb{B}$-spaces $(f, \phi) : ((X, \tau), \mathbb{B}) \to \iota_U((Y, \sigma), \mathbb{A}, (A_y)_{y \in Y})$. Then in each $y \in Y$ we have a factorization

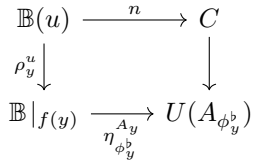


producing a local form under the stalk $\mathbb{B}|_{f(y)}$ so we can define

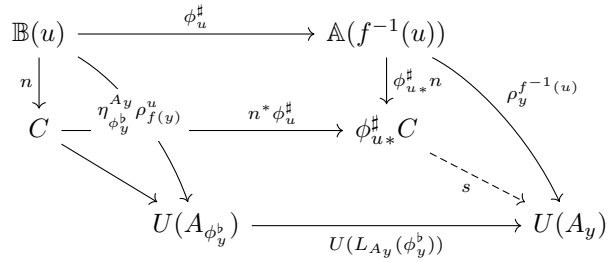
$$\begin{array}{ccc} Y & \xrightarrow{\ g\ } & X_{\mathbb{B}} \\ y & \longmapsto & (f(y), \eta^{A_y}_{\phi_y^\flat}) \end{array}$$

From this point, the strategy is similar to the set valued case, although we have to work under the values of the structural sheaf under an open rather than just considering a map produced by global section.

We prove that g is continuous. Take $u \in \tau$ and $n \in \mathcal{D}_{\mathbb{B}(u)}$. We have that $g^{-1}(D_{(u,n)})$ consists of all those $y \in Y$ such that we have a factorization



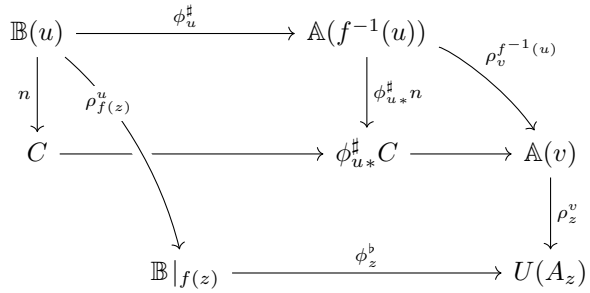
Now for each such $y \in g^{-1}(D_{(u,n)})$ we have a factorization as the dashed arrow below



But we have a filtered colimit

$$\rho_y^{f^{-1}} = \operatorname*{colim}_{\substack{y \in v \\ v \subseteq f^{-1}(u)}} \rho_v^{f^{-1}(u)}$$

in the slice $\mathbb{A}(f^{-1}(u)) \downarrow \mathcal{B}$ for $\mathbb{A}$ is a sheaf and the set $\{v \in \sigma \mid v \subseteq f^{-1}(u), y \in v\}$ is cofinal in the set of neighborhood of y in σ. Hence, for $\phi^\sharp_{u*}n$ is still finitely presented as a diagonally universal map, there is some neighborhood v of y in σ such that $v \subseteq f^{-1}(u)$ and the arrow s factorizes through the restriction ρ^v_y, and moreover, any two such factorizations can be equalized by a third one. But now any point z in v is also in $g^{-1}(D_{(u,n)})$: indeed, if we have a factorization as below



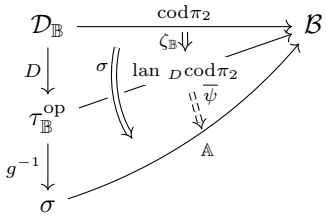
then $\rho^u_{f(z)*}n$ factorizes $\phi_z^\flat$, and hence by Diers condition of the local unit of a morphism toward U being the filtered colimit of all finitely presented diagonally universal maps factorizing

it, we have that ${\rho^u_{f(z)}}_* n \leq \eta^{A_z}_{\phi^\flat_z}$, so that $z \in g^{-1}(D_{(u,n)})$ for $f(z) \in u$ and the last condition. Hence v is included in $g^{-1}(D_{(u,n)})$ and as before such open v could have be chosen for each point of $g^{-1}(D_{(u,n)})$, which is hence open: this ensures the continuity of g. Moreover observe that $g^{-1}(D_{(u,n)}) \subseteq f^{-1}(u) = g^{-1}(D_{(u,1_{\mathbb{B}(u)})})$.

The remains of the proof is similar to the proof of the point-based case: the opens $v \in \sigma$ constructed above defined a covering of $g^{-1}(D_{(u,n)})$, and the universal property of the limit from the sheaf condition of $\mathcal{A}$ at this cover provides a map as the dashed arrow below

$$\begin{array}{ccc} \mathbb{B}(u) & \xrightarrow{\phi^\sharp_u} & \mathbb{A}(f^{-1}(u)) \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \rho^{f^{-1}(u)}_{g^{-1}(D_{(u,n)})}} \\ C & \xrightarrow[\sigma_{(u,n)}]{} & \mathbb{A}(g^{-1}(D_{(u,n)})) \end{array}$$

Again, the data of all the $(\sigma_{(u,n)})_{u\in\tau,\, n\in\mathcal{D}_{\mathbb{B}(u)}}$ provide a natural transformation $\sigma : \mathrm{cod}\pi_2 \Rightarrow g_*\mathbb{A}$ and we apply the property of left Kan extension

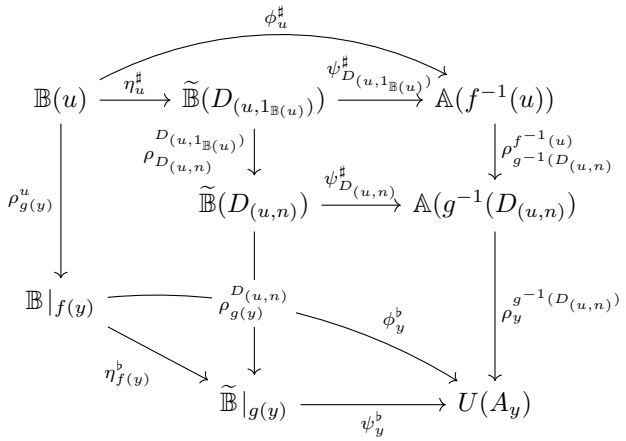


and again this $\overline{\psi}$ factorizes through the sheafification $\widetilde{\mathbb{B}} \to g_*\mathbb{A}$.

Now concerning the inverse image part, take the mate of $\psi^\sharp$ and apply again Diers condition and theorem 8.2.2.8. For any $y \in Y$ and $(u,n) \in \mathcal{D}_{\mathbb{B}}$ with $g(y) \in D_{(u,n)}$, the following square

$$\begin{array}{ccc} \widetilde{\mathbb{B}}(D_{(u,n)})) & \xrightarrow{\psi^\sharp_{D_{(u,n)}}} & \mathbb{A}(g^{-1}(D_{(u,n)}) \\ {\scriptstyle \rho^{D_{(u,n)}}_{g(y)}}\downarrow & & \downarrow{\scriptstyle \rho^{g^{-1}(D_{(u,n)})}_{y}} \\ \widetilde{\mathbb{B}}|_{g(y)} & \xrightarrow[\psi^\flat_y]{} & U(A_y) \end{array}$$

is part of the following diagram expressing the factorization through the unit



where the arrow $\psi^\flat_y$ (the bottom one) is induced by the universal property of the filtered colimit

$$\widetilde{\mathbb{B}}|_{g(y)} = \operatorname*{colim}_{n \leq \eta^{A_y}_{\phi^\flat_y}} \widetilde{\mathbb{B}|_{f(y)}}(D_n) \simeq U(A_{\phi^\flat_y})$$

as the map $U(L_{A_y}(\phi^\flat_y))$. □

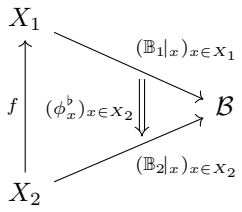
8.2.4.4. To conclude this part, let us explain how this adjunction is related to the adjunction existing at the level of the free product completion as exposed in proposition 2.2.2.3. Recall that a right multi-adjoint $U : \mathcal{A} \to \mathcal{B}$ induced an adjunction

$$\Pi\mathcal{A} \underset{\Pi U}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \Pi\mathcal{B}$$

Now we can define a "functor of stalks" $\pi_{\mathcal{B}} : \mathcal{B}\text{-}\mathbf{Spaces} \to \Pi\mathcal{B}$ sending a $\mathcal{B}$-space $((X,\tau),\mathbb{B})$ to the family of stalks

$$X \xrightarrow{(\mathbb{B}|_x)_{x\in X}} \mathcal{B}$$

and a morphism $(f,\phi) : ((X_1,\tau_1),\mathbb{B}_1) \to ((X_2,\tau_2),\mathbb{B}_2)$ to



Similarly, we can define a functor $\pi_{\mathcal{A}} : U\text{-}\mathbf{Spaces} \to \mathcal{A}$ sending a U-space on the specified family of $\mathcal{A}$-objects attached to its stalks, and a morphism of U-space to the specified maps in $\mathcal{A}$ attached to the inverse image part at the stalks. Then the following result is obvious from the way we defined the left adjoint $L \dashv \Pi U$ in proposition 2.2.2.3:

Proposition 8.2.4.5. *The following square satisfies the Beck-Chevalley condition*

$$\begin{array}{ccc} U\text{-}\boldsymbol{Spaces} & \underset{\iota_U}{\overset{\mathbf{Spec}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \mathcal{B}\text{-}\boldsymbol{Spaces} \\ \downarrow{\scriptstyle \pi_{\mathcal{A}}} & & \downarrow{\scriptstyle \pi_{\mathcal{B}}} \\ \Pi\mathcal{A} & \underset{\Pi U}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} & \Pi\mathcal{B} \end{array}$$

that is we have both that $\Pi U \pi_{\mathcal{A}} = \pi_{\mathcal{B}} \iota_U$ *and* $\pi_{\mathcal{A}}\mathbf{Spec} = L\pi_{\mathcal{B}}$

8.2.4.6. In some sense, the functors $\pi_{\mathcal{B}}$ and $\pi_{\mathcal{A}}$ forget about the topological data attached to structured spaces and remind only the local data at stalks. But in a converse process, we could see families in the free product completion as *discrete* structured spaces: indeed we can define two functors

$$\iota_{\mathcal{B}} : \Pi\mathcal{B} \hookrightarrow \mathcal{B}\text{-}\mathbf{Spaces}$$

sending a family $(B_i)_{i\in I}$ to the $\mathcal{B}$-space $((I,\mathcal{P}(I)),\mathbb{B})$ where the set I is now equipped with the discrete topology and the sheaf $\mathbb{B}$ is defined as $\mathbb{B}(J) = \prod_{i\in J} B_i$ for $J \subseteq I$. Similarly we define a functor

$$\iota_{\mathcal{A}} : \Pi\mathcal{A} \hookrightarrow U\text{-}\mathbf{Spaces}$$

sending a family $(A_i)_{i\in I}$ to the discrete U-space $((I,\mathcal{P}(I)),\mathbb{A},(A_i)_{i\in I})$ where $\mathbb{A}(J) = \prod_{i\in I} U(A_i)$ for $J \subseteq I$.

8.3 2-Functoriality of Diers construction

In definition 4.2.3.1 we saw that Diers contexts formed altogether a 2-category. This leads us to ask what an abstraction of the adjunctions one gets from Diers construction and its generalization to arbitrary structured spaces could be, in order to construct a 2-functor sending a Diers context to the associated spectral adjunction.

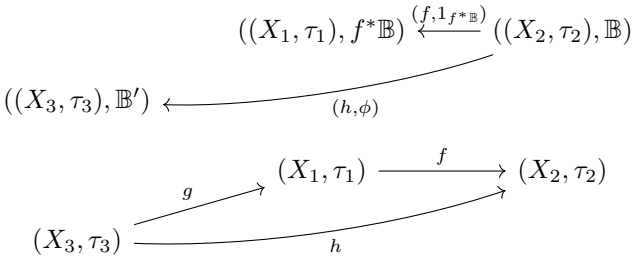
8.3.1 The fibration of $\mathcal{B}$-spaces

To guess what should be at the other end of our construction, let us give some observation on the nature of the categories of U-spaces and $\mathbb{B}$-spaces.

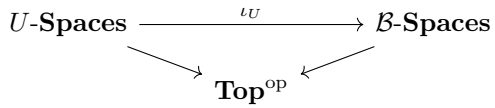
8.3.1.1. First, remark that $\mathcal{B}$-**Spaces** is the opposite category of the Grothendieck construction associated to the pseudofunctor

$$\begin{array}{rcl} \mathbf{Top}^{\mathrm{op}} & \to & \mathbf{Cat} \\ (X,\tau) & \mapsto & \mathbf{Sh}_{\mathcal{B}}(X,\tau) \\ (X_1,\tau_1) \xrightarrow{f} (X_2,\tau_2) & \mapsto & \mathbf{Sh}_{\mathcal{B}}(X_2,\tau_2) \xrightarrow{f^*} Sh_{\mathcal{B}}(X_1,\tau_1) \end{array}$$

and hence has the structure of a fibration: that is for each continuous map $f : (X_1,\tau_1) \to (X_2,\tau_2)$, we have a cartesian lift $((X_2,\tau_2),\mathbb{B}) \to ((X_1,\tau_1), f^*\mathbb{B})$, as for any situation as below



we have $h^* = g^*f^*$ so that $\phi : h^*\mathbb{B} = g^*f^*\mathbb{B} \to \mathbb{B}'$ provides itself automatically a lift $(g,\phi) : ((X_3,\tau_3),\mathbb{B}') \to ((X_1,\tau_1), f^*\mathbb{B})$. Moreover this structure of fibration restricts to U-**Spaces** as the inverse image preserves stalks, and the cartesian lifts are identities between sheaves, hence have identities at stalks, which are in the range of U. Hence we have a morphism of fibrations



for the forgetful functor ι_U does not modify the underlying topological space. Observe that $\mathbb{B}$-**Space** has also a structure of opfibration thanks to direct image, but this structure is not preserved by the restriction of U-spaces.

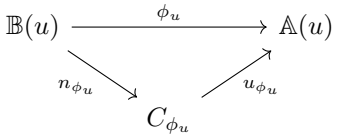
The following is also worth of interest for it is akin to the topos theoretic approach, corresponding to theorem 3.3.3.6. However, it does not concern all Diers contexts but only axiomatisable ones, that are those where the class of diagonally universal morphisms is finitely accessible, so that the stable factorization coincides with the factorization $(\mathrm{Ind}(\mathcal{D}), \mathcal{D}^{\perp})$:

Proposition 8.3.1.2. *Let $(U,\mathcal{A},\mathcal{B})$ be an axiomatisable Diers context; then for each (X,τ), we have a right multiadjoint*

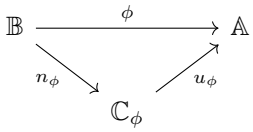
$$U\text{-}\boldsymbol{Spaces}_{(X,\tau)} \xrightarrow{\iota_U|_{(X,\tau)}} \mathbf{Sh}_{\mathcal{B}}(X,\tau)$$

where U-$\boldsymbol{Spaces}_{(X,\tau)}$ is the category of U-spaces whose underlying space is (X,τ).

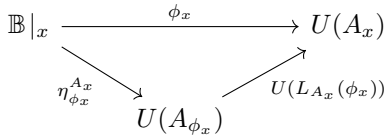
Proof. For any $(1_X,\phi) : ((X,\tau),\mathbb{B}) \to \iota_U\,|_{(X,\tau)}\,((X,\tau),\mathbb{A})$, with hence the identity as underlying map, the inverse and direct parts of ϕ coincide into a same morphism of sheaves $\mathbb{B} \to \mathbb{A}$, and in each $u \in U$ we have a $(\mathrm{Ind}(\mathcal{D}), \mathcal{D}^{\perp})$ factorization



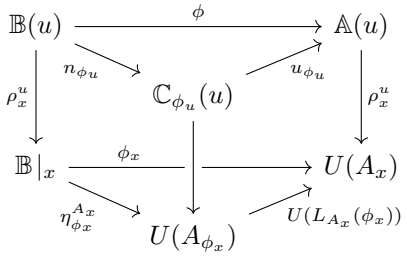
and we can define a presheaf $\tau^{\mathrm{op}} \to \mathbb{B}$ from the data $u \mapsto C_{\phi_u}$, which gives a factorization amongst sheaves



And moreover, in each $x \in X$ we have also the stable factorization



But as we supposed U to be diagonally axiomatisable, this stable factorization coincides with the $(\mathrm{Ind}(\mathcal{D}), \mathcal{D}^{\perp})$ factorization, and by functoriality of such factorization, we have that the following diagram

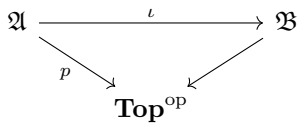


But now apply theorem 8.2.2.8 and commutation of colimits and pushouts to see that $U(A_\phi) \simeq \mathrm{colim}_{x \in u} C_{\phi_u}$, which exhibit it as the stalk of the presheaf from which $\mathbb{C}_\phi$ was induced from, and hence of $\mathbb{C}_\phi$ itself, so $\mathcal{C}_\phi$ is a sheaf with stalks in the range of U. Now we can check that n_ϕ is a candidate for each n_{ϕ_u} is a diagonally universal. □

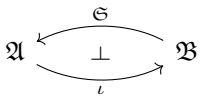
8.3.2 2-category of dualities

Definition 8.3.2.1. We define the 2-category $\mathfrak{Dual}$ of *spectral dualities* as having:

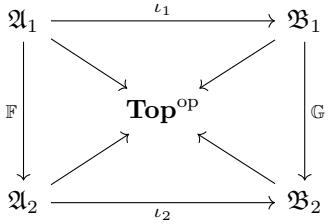
– as 0-cells, quadruples $(\mathfrak{A}, \mathfrak{B}, \iota, \mathfrak{S})$ with $\mathfrak{A}$ and $\mathfrak{B}$ fibered categories over $\mathbf{Top}^{\mathrm{op}}$ such that moreover the fibration of $\mathfrak{B}$ is also an opfibration over $\mathbf{Top}$, ι a strict morphism of fibrations



and an adjunction

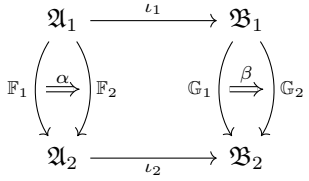


– as 1-cells $(\mathfrak{A}_1, \mathfrak{B}_1, \iota_1, \mathfrak{S}_1) \to (\mathfrak{A}_2, \mathfrak{B}_2, \iota_2, \mathfrak{S}_2)$, triples $(\mathbb{F}, \mathbb{G}, \theta)$ forming a squares of morphisms of fibrations



where all triangles commute strictly and θ is an invertible 2-cell $\mathbb{G}\iota_1 \simeq \iota_2\mathbb{F}$.

– as 2-cells $(\alpha, \beta) : (\mathbb{F}_1, \mathbb{G}_1, \theta_1) \Rightarrow (\mathbb{F}_2, \mathbb{G}_2, \theta_2)$ such that we have equality of whiskering



Remark 8.3.2.2. In the following, for an object $\mathbb{B}$ in $\mathfrak{B}$ we denote as $|\mathbb{B}|$ for the underlying object (similarly, $|f|$ for an arrow), and this notation is transfered to $\mathfrak{A}$ as $|\mathbb{A}| = |\iota\mathbb{A}|$. Beware that in our condition we require that this fibration $|-|$ be itself also an opfibration.

Observe that any morphism of duality $(\mathbb{F}, \mathbb{G}, \theta)$ admits a mate

$$\begin{array}{ccc} \mathfrak{A}_1 & \xleftarrow{\mathfrak{S}_1} & \mathfrak{B}_1 \\ {\scriptstyle\mathbb{F}}\downarrow & \swarrow\!\!\!\swarrow {\scriptstyle\sigma} & \downarrow{\scriptstyle\mathbb{G}} \\ \mathfrak{A}_2 & \xleftarrow[\mathfrak{S}_2]{} & \mathfrak{B}_2 \end{array}$$

In the following we are going to give some interest to the information carried by this mate. Moreover, as well as we can recover the classical $\Gamma \dashv \mathbf{Spec}$ adjunction by looking at the fiber over $*$, we can exploit the condition that $\mathfrak{B}$ is also an opfibered category to compute a "global section functor". For any $\mathbb{A}$ in $\mathfrak{A}$, with projection $|\ \mathbb{A}\ |$ in **Top**, we have the opcartesian lift in $\mathfrak{B}$

$$\begin{array}{ccc} \iota(\mathbb{A}) & \xleftarrow{\overline{!_{|\mathbb{A}|}}} & !_{|\mathbb{A}|_*}\iota(\mathbb{A}) \\ |\mathbb{A}| & \xrightarrow{\quad !\quad} & * \end{array}$$

and for any morphism $f : \mathbb{A}_1 \to \mathbb{A}_2$ we end up with two distinct lifts over the point related by the following arrow

$$\begin{array}{ccccc} & & \iota(\mathbb{A}_1) & \xleftarrow{\overline{!_{|\mathbb{A}_1|}}} & !_{|\mathbb{A}_1|_*}\iota(\mathbb{A}_1) \\ & {\scriptstyle\iota(f)}\swarrow & & & \downarrow{\scriptstyle !_{|\mathbb{A}_1|_*}(\iota(f))} \\ \iota(\mathbb{A}_2) & & \xleftarrow[\overline{!_{|\mathbb{A}_2|}}]{} & & !_{|\mathbb{A}_2|_*}\iota(\mathbb{A}_2) \simeq !_{|\mathbb{A}_1|_*}|f|_*\iota(\mathbb{A}_2) \\ & & |\mathbb{A}_1| & \xrightarrow{!_{|\mathbb{A}_1|}} & * \\ & {\scriptstyle|f|}\nearrow & & & \\ |\mathbb{A}_2| & & \xrightarrow[!_{|\mathbb{A}_2|}]{} & & \end{array}$$

This allows us to define a Global sections functor into the fiber of $\mathbb{B}$ over the point

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\ \Gamma\ } & \mathfrak{B}_* \\ \mathbb{A} & \longmapsto & !_{|A|_*}\iota(\mathbb{A}) \end{array}$$

where $i_* : \mathfrak{B}_* \hookrightarrow \mathfrak{B}$ is the inclusion of the fiber at $*$.

Lemma 8.3.2.3. *For a duality $(\mathfrak{A}, \mathfrak{B}, \iota, \mathfrak{S})$, we have an adjunction $\mathfrak{S}i_* \dashv \Gamma$*

Proof. For one direction, let $\mathbb{B}$ be an object over the point, that is, $|\mathbb{B}| = *$, and $f : \mathbb{B} \to \Gamma\mathbb{A}$. Then composing with the cartesian lift $\overline{!_{|\mathbb{A}|}} : \Gamma\mathbb{A} \to \iota_(\mathbb{A})$ induces from $\mathfrak{S} \dashv \iota$ a unique map $\mathfrak{S}B \to \mathbb{A}$ in $\mathfrak{A}$. Conversely, for any $f : \mathfrak{S}B \to \mathbb{A}$, if we have $\Gamma f : \Gamma\mathfrak{S}\mathbb{B} \to \Gamma\mathbb{A}$. But observe that the unit $\eta_{\mathbb{B}} : \mathbb{B} \to \iota\mathfrak{S}\mathbb{B}$ induces a unique dashed arrow below from the opcartesiannes of $\overline{!_{|\iota\mathfrak{S}\mathbb{B}|}}$:

$$\begin{array}{ccccc} \Gamma\mathfrak{S}\mathbb{B} & & & & \\ & \nwarrow & & \searrow & \\ * & \xleftarrow[!_{|\iota\mathfrak{S}\mathbb{B}|}]{} & \mathbb{B} & \xrightarrow[\eta_{\mathbb{B}}]{} & \iota\mathfrak{S}\mathbb{B} \\ & & * & \xleftarrow[!_{|\iota\mathfrak{S}\mathbb{B}|}]{} & |\iota\mathfrak{S}\mathbb{B}| \end{array}$$

and this is the unit of the restricted adjunction $\mathfrak{S}i_* \dashv \Gamma$. Indeed, we can then compose $\Gamma f : \Gamma\mathfrak{S}\mathbb{B} \to \Gamma\mathbb{A}$ with $B \to \Gamma\mathfrak{S}$ to get a map $f : \mathbb{B} \to \Gamma\mathbb{A}$. □

Remark 8.3.2.4. This lemma shows that the fact we could restrict the spectral adjunction as developped in section 2 to the original version of Diers as recalled in section 1 is in fact inherent to the situation of duality and processes essentially from fibrational and opfibrational aspects, which are enacted through inverse and direct images in the concrete situations.

8.3.3 From Diers contexts to dualities

Theorem 8.3.3.1. *The construction above defines a 2-functor*

$$\mathfrak{Diers} \xrightarrow{\mathfrak{Spec}} \mathfrak{Dual}$$

assigning to each Diers context $(U, \mathcal{A}, \mathcal{B})$ *the adjunction* $\mathbf{Spec}_U \dashv \iota_U$.

Proof. The action of this functor on 0-cells was described in the second part. The remaining part of this section is aimed at making explicit how a morphism of Diers context could induce a 1-cell in the 2-category of geometric dualities.

In the following we fix a morphism of Diers contexts $(F, G) : (U_1, \mathcal{A}_1, \mathcal{B}_1) \to (U_2, \mathcal{A}_2, \mathcal{B}_2)$. We construct a pair of functors $\mathbb{F} : U_1\textbf{-Spaces} \to U_2\textbf{-Spaces}$ and $\mathbb{G} : \mathcal{B}_1\textbf{-Spaces} \to \mathcal{B}_2\textbf{-Spaces}$ as follows. For each $\mathcal{B}_1$ space $((X,\tau),\mathbb{B})$, the sheaf $\mathbb{B}$ of $\mathcal{B}_1$ objects can be turned into a sheaf of $\mathcal{B}_2$ objects thanks to G, not only without changing the base space, but also in a quite smooth manner: we have from composition with G a presheaf $G\mathbb{B} : \tau^{\mathrm{op}} \to \mathcal{B}_2$ which acts as $u \mapsto G(\mathbb{B}(u))$; but in fact this presheaf is already a sheaf, for G is a right adjoint as a morphism of locally finitely presentable categories, so that for each u and each $(u_i)_{i\in I}$ with $u = \bigcup_{i\in I} u_i$ the limit in the descent property for $\mathbb{B}$ is preserved by G, that is,

$$G\mathbb{B}(u) = \lim\Big(\prod_{i\in I} G\mathbb{B}(u_i) \rightrightarrows \prod_{i,j\in I} G\mathbb{B}(u_i \cap u_j)\Big)$$

Hence this defines a sheaf of $\mathcal{B}_2$ objects $\mathbb{G}(\mathbb{B})$ on (X,τ), and for a morphism of sheaf, we define $\mathbb{G}(\phi)$ as the whiskering

$$\tau^{\mathrm{op}} \underset{\mathbb{B}_2}{\overset{\mathbb{B}_1}{\Downarrow \phi}} \mathcal{B}_1 \xrightarrow{G} \mathcal{B}_2$$

This defines a functor $\mathbb{G}$ sending a $\mathcal{B}$-space $((X,\tau),\mathbb{B})$ on $((X,\tau),\mathbb{G}(\mathbb{B}))$ This functor $\mathbb{G}$ conveniently restricts along ι_{U_1} thanks to its relation with F. Let $((X,\tau),\mathbb{A},(A_x)_{x\in X})$ be a U_1-space; then $((X,\tau),\mathbb{G}(\mathbb{A}))$ canonically defines a U_2-space: indeed, at each stalk x we have $\mathbb{A}\,|_x = U_1(A_x)$; but now, as G is finitary, we have

$$\begin{aligned}
\mathbb{G}(\mathbb{A})\,|_x &= \operatorname*{colim}_{x\in u} G(\mathbb{A}(u)) \\
&\simeq G(\operatorname*{colim}_{x\in u} \mathbb{A}(u)) \\
&\simeq G(\mathbb{A}\,|_x) \\
&= GU_1(A_x) \\
&\simeq U_2F(A_x)
\end{aligned}$$

where the last isomorphism comes from the natural isomorphism $\theta : GU_1 \simeq U_2F$: hence $\mathbb{G}(\mathbb{A})$ has $F(A_x)_{x\in X}$ as the local data attached to its stalks, and we can do the same for arrows. Hence we have to put $\mathbb{F}((X,\tau),\mathbb{A},(A_x)_{x\in X}) = ((X,\tau),\mathbb{G}(\mathbb{A}),(F(A_x))_{x\in X})$. This proves we have an invertible 2-cell

$$\begin{array}{ccc}
U_1\textbf{-Spaces} & \xrightarrow{\iota_{U_1}} & \mathcal{B}_1\textbf{-Spaces} \\
{\scriptstyle\mathbb{F}}\downarrow & \overset{\theta}{\simeq} & \downarrow{\scriptstyle\mathbb{G}} \\
U_2\textbf{-Spaces} & \xrightarrow[\iota_{U_2}]{} & \mathcal{B}_2\textbf{-Spaces}
\end{array}$$

2-functoriality can be left as an exercice, for it is more tiresome than enlightening and does not seem to correspond to any known concrete situation. □

Remark 8.3.3.2. In the following, we need to distinguish data relative to U_1 and data relative to U_2. To this end we put the corresponding index into exponent to precise if a spectral data is relative to U_1 or U_2: for instance the set D^1_B is the set of U_1-diagonally universal morphisms under B in $\mathcal{B}_1$; in particular we denote as $\mathcal{D}^1_{\mathbb{B}}$ the category of basic opens in the U_1-spectral topology for a sheaf of $\mathcal{B}_1$-objects $\mathbb{B}$ as defined in section 2; similarly $D^1 : \mathcal{D}^1_{\mathbb{B}} \to \tau^{\mathrm{op}}_{\mathbb{B}}$ is the corresponding basis of the U_1 spectral topology, and so on...

8.3.3.3. Now we turn to the description of the mate σ, which entangles the action of the associated spectra $\mathbf{Spec}_{U_1}$ and $\mathbf{Spec}_{U_2}$. We want to exhibit a canonical 2-cell

$$\begin{array}{ccc} U_1\text{-}\mathbf{Spaces} & \xleftarrow{\mathbf{Spec}_{U_1}} & \mathcal{B}_1\text{-}\mathbf{Spaces} \\ {\scriptstyle\mathbb{F}}\downarrow & \overset{\sigma}{\Leftarrow} & \downarrow{\scriptstyle\mathbb{G}} \\ U_2\text{-}\mathbf{Spaces} & \xleftarrow[\mathbf{Spec}_{U_2}]{} & \mathcal{B}_2\text{-}\mathbf{Spaces} \end{array}$$

In the following we fix a $\mathcal{B}$-space $((X,\tau),\mathbb{B})$. First, let us look at the the underlying map. Observe that G may not preserve diagonally universal morphisms: hence in each $(x,\xi) \in X_{\mathbb{B}}$, we have a factorization

$$\begin{array}{ccccc} G(\mathbb{B}|_x) & & \xrightarrow{\quad G(\xi) \quad} & & GU_1(A_\xi) \simeq U_2F(A_\xi) \\ & {\scriptstyle \eta^{A_\xi}_{G(\xi)}} \searrow & & \nearrow {\scriptstyle U_2L_{A_\xi}(G(\xi))} & \\ & & U_2(A_\xi) & & \end{array}$$

Moreover we also have the following general property concerning the way G interacts with stable factorization in $\mathcal{B}_1$:

Lemma 8.3.3.4. *For any $f : B \to U_1(A)$ we have that $n_{G(\eta^A_f)} = \eta^{F(A)}_{G(f)}$.*

Proof. Beware that there is no reason for G to preserve diagonally universal morphisms. However we have the following factorization

$$\begin{array}{ccc} G(B) & \xrightarrow{\quad G(f) \quad} & U_2F(A) \\ {\scriptstyle n_{G(f)}}\downarrow & \overset{U_2(u_{G(f)})}{\searrow} \quad \overset{G(n_f)}{\nearrow} & \uparrow {\scriptstyle GU_1(u_f)=U_2F(u_f)} \\ U_2(L^2_{F(A)}(G(f))) & \xrightarrow[U_2(\sigma^A_f)]{} & U_2F(L^1_A(f)) \end{array}$$

And postcomposing with $GU_1(u_f) = U_2F(u_f)$ does not modify the factorization while $U(\sigma^A_f)$ is a local morphism, so that $n_{G(f)}$ is also the candidate for $G(n_f)$. □

Also make the following observation about the left adjoint of G:

Lemma 8.3.3.5. *G^* sends U_2-diagonally universal morphisms of finite presentation to U_1-diagonally universal morphisms of finite presentation.*

Proof. Let $n : B \to C$ be in $\mathcal{D}^2_B$ for B in $\mathcal{B}_2$, and $u : A \to A'$ in $\mathcal{A}_1$. Then for a square

$$\begin{array}{ccc} G^*(B) & \xrightarrow{f} & U_1(A) \\ {\scriptstyle G^*(n)}\downarrow & & \downarrow{\scriptstyle U_1(u)} \\ G^*(C) & \xrightarrow[g]{} & U_1(A') \end{array}$$

then by adjunction this square corresponds to a unique square in $\mathcal{B}_2$ which admits a filler as below

$$\begin{array}{ccc} B & \xrightarrow{\overline{f}} & GU_1(A) \simeq U_2F(A) \\ {\scriptstyle n}\downarrow & \overset{d}{\nearrow} & \downarrow{\scriptstyle U_2F(A)} \\ C & \xrightarrow[\overline{g}]{} & GU_1(A') \simeq U_2F(A') \end{array}$$

and the map $C \to GU_1(A)$ itself corresponds again uniquely to a map $\overline{d} : G^*C \to U_1(A)$ wich is a filler in $\mathcal{B}_1$. □

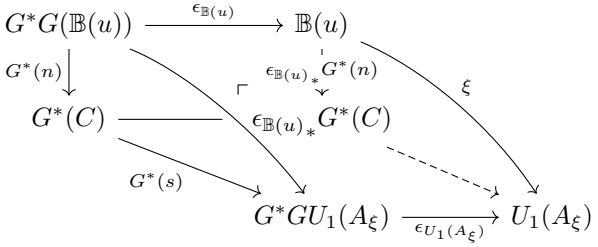
Proposition 8.3.3.6. *For any $((X,\tau),\mathbb{B})$ following map is continuous*

$$\begin{array}{ccc} \mathbf{Spec}_{U_1}(\mathbb{B}) & \xrightarrow{\mathfrak{s}_{((X,\tau),\mathbb{B})}} & \mathbf{Spec}_{U_2}(\mathbb{G}(\mathbb{B})) \\ (x,\xi) & \longmapsto & (x, \eta^{F(A_\xi)}_{G(\xi)}) \end{array}$$

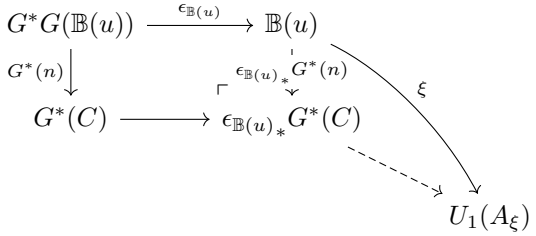
Proof. Take $(u,n) \in \mathcal{D}^2_{\mathbb{GB}}$ and (x,ξ) in $\mathfrak{s}^{-1}_{((X,\tau),\mathbb{B})}(D^2_{(u,n)})$: that is, suppose we have a factorization

$$\begin{array}{ccc} G(\mathbb{B}(u)) & \xrightarrow{\rho^u_x} & G(\mathbb{B}|_x) \\ {\scriptstyle n}\downarrow & & \downarrow{\scriptstyle \eta^{F(A_\xi)}_{G(\xi)}} \\ C & \xrightarrow[s]{} & U_2(A_{G(\xi)}) \end{array}$$

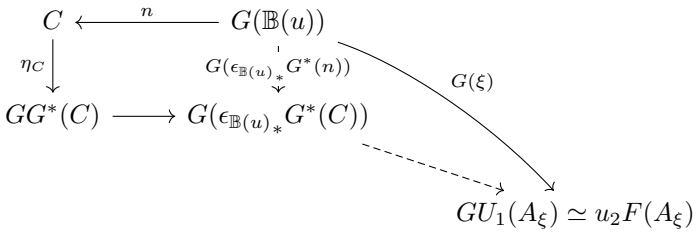
Then, for the left adjoint G^* preserves finitely presentedness and diagonally universal morphisms by the previous lemma, we know that $G^*(n) : G^*G(\mathbb{B}(u)) \to G^*(C)$ is U_1-diagonally universal of finite presentation in $\mathcal{B}_1$. But then one can push along the $G^* \dashv G$-counit, we have a factorization of ξ through this pushout



This defines an open $D^1_{(u,\epsilon_{\mathbb{B}(u)_*}G^*(n))}$ containing (x,ξ). But now any ξ' in $D^1_{(u,\epsilon_{\mathbb{B}(u)_*}G^*(n))}$ must also be sent inside $D^2_{(u,n)}$: indeed, if we have a factorization as below



then applying again G and composing with the unit ensures that $n \leq G(\xi)$ as visualized below



But then in particular one must have $n \leq \eta^{F(A_\xi)}_{G(\xi)}$ by Diers condition. This proves that

$$\mathfrak{s}^{-1}_{((X,\tau),\mathbb{B})}(D^2_{(u,n)}) = D^1_{(u,\epsilon_{\mathbb{B}(u)_*}G^*(n))}$$

ensuring the continuity of $\mathfrak{s}_{((X,\tau),\mathbb{B})}$. □

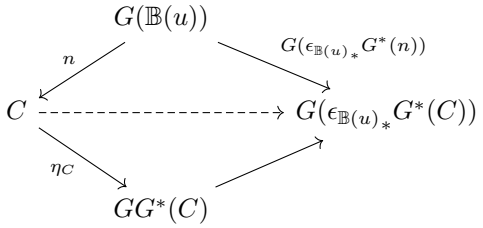
8.3.3.7. Now we turn to the construction of a morphism of sheaves $\sigma^\sharp : \widetilde{\mathbb{GB}} \to \mathfrak{s}_{((X,\tau),\mathbb{B})_*}\mathbb{G}\widetilde{\mathbb{B}}$. First, consider the expression of the structural presheaf $\overline{\mathbb{B}}$ at $D^2_{(u,n)}$ and apply G:

$$\begin{aligned} G\overline{\mathbb{B}}\mathfrak{s}^{-1}_{((X,\tau),\mathbb{B})}(D^2_{(u,n)}) &\simeq G\Big(\operatorname*{colim}_{\mathfrak{s}^{-1}_{((X,\tau),\mathbb{B})}(D^2_{(u,n)}) \subseteq D^1(v,m)} \mathrm{cod}(m)\Big) \\ &\simeq \operatorname*{colim}_{\mathfrak{s}^{-1}_{((X,\tau),\mathbb{B})}(D^2_{(u,n)}) \subseteq D^1(v,m)} G(\mathrm{cod}(m)) \end{aligned}$$

where the last iso comes form G is finitary. The indexing set enumerates the pairs such that

$$D^1_{(u,\epsilon_{\mathbb{B}(u)_*}G^*(n))} \subseteq D^1(v,m)$$

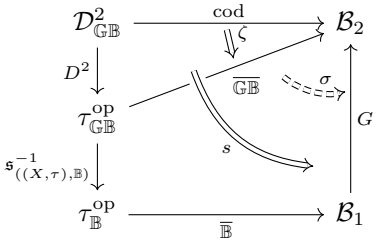
But we always have $n \leq G(\epsilon_{\mathbb{B}(u)_*}G^*(n)))$ thanks to the unit as seen in the following diagram



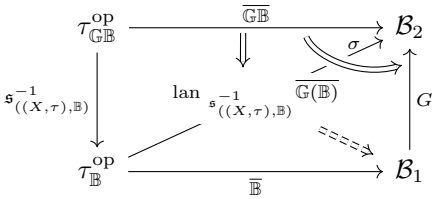
Hence the colimit can restrict to the $D^1_{(v,m)}$ with $u = v$ ad $m = \epsilon_{\mathbb{B}(u)_*}G^*(C)$, and hence the cocone of the colimit, composed with the map above, defines a natural transformation

$$(s_{(u,n)} : \mathrm{cod} \Rightarrow G\mathbb{B}\mathfrak{s}^{-1}_{((X,\tau),\mathbb{B})}D)_{(u,n)\in\mathcal{D}^2_{\mathbb{GB}}}$$

Hence we can invoke the universal property of the left Kan extension as below



Then the desired direct image part is obtained after factorizing through the sheafification. Now, to retrieve the inverse image part, recall that inverse images are also defined through left Kan extensions, and in our case, we have $\mathfrak{s}^*_{((X,\tau),\mathbb{B})}\overline{\mathbb{G}(\mathbb{B})} = \mathrm{lan}_{\mathfrak{s}^{-1}_{((X,\tau),\mathbb{B})}}\overline{\mathbb{G}(\mathbb{B})}$, so we have a natural map in the diagram below



which induces after sheafification a morphism $\sigma^\flat : {\mathfrak{s}_{((X,\tau),\mathbb{B})}}^*\widetilde{\overline{\mathbb{GB}}} \to \mathbb{G}\widetilde{\mathbb{B}}$.

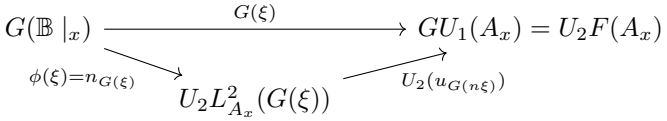
Remark 8.3.3.8. At the level of stalks, the inverse image comorphism happens to coincide with the mate. Indeed, as stalks are unchanged by sheafification, for a point (x, ξ) of $Spec_{U_1}(\mathbb{B})$ corresponding to a candidate $\xi : \mathbb{B}|_x \to U_1(A_\xi)$:

$$\begin{aligned}{\mathfrak{s}_{((X,\tau),\mathbb{B})}}^*\overline{\mathbb{GB}}|_{(x,\xi)} &= \overline{\mathbb{GB}}|_{\eta^{F(A_\xi)}_{G(\xi)}} \\ &= U_2(L^2_{F(A_x)}(G(\xi)))\end{aligned}$$

while we have

$$\begin{aligned}G(\overline{\mathbb{B}}_{(x,\xi)}) &= GU_1(A_\xi) \\ &= U_2F(A_\xi)\end{aligned}$$

And the value of the comorphism at the stalk is actually the right part of the factorization



that is $\sigma^\sharp_{(x,\xi)} = U_2(u_{G(\xi)})$, which comes uniquely from the mate $u_{G(\xi)} = \sigma^{A_x}_\xi : L^2_{A_x}(G(\xi)) \to F(A_x)$. In particular this ensures that $(\mathfrak{s}_{((X,\tau),\mathbb{B})}, \sigma)$ is a morphism of U_2-locally structured spaces as it behaves correctly at stalks.

8.3.4 From spectral dualities to Diers contexts

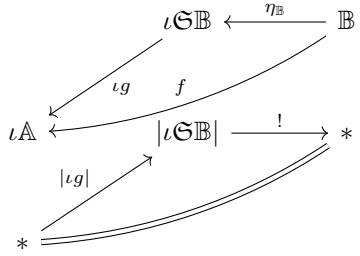
Let us conclude with the following partial inverse result, which allows one to reconstruct right multi-adjoints from dualities by taking the fibers over the point:

Theorem 8.3.4.1. *Let $(\mathfrak{A}, \mathfrak{B}, \iota, \mathfrak{S})$ be a duality. Then the functor*

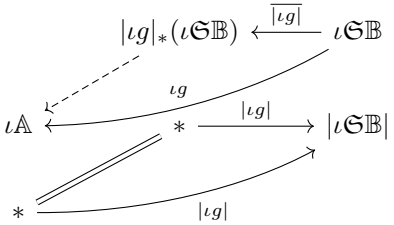
$$\mathfrak{A}_* \xrightarrow{\iota_*} \mathfrak{B}_*$$

is a right multi-adjoint.

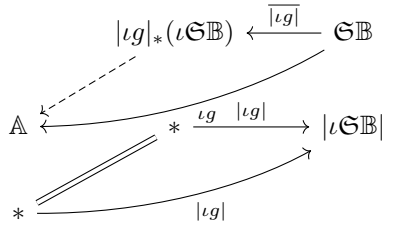
Proof. This exploits the properties of fibrations. Let $\mathbb{B}$ be an object of the fiber $\mathfrak{B}_*$: we prove that the comma category $\mathbb{B} \downarrow \iota_*$ has a small initial family. For any $\mathbb{A}$ in $\mathfrak{A}$, a morphism $f : \mathbb{B} \to \iota\mathbb{A}$ factorizes uniquely through $\mathfrak{S}\mathbb{B}$ as



so that f defines a point $|\iota g| : * \to |\iota\mathfrak{S}\mathbb{B}|$. Now, using the cartesianness of the fibration, there is a canonical lifting of this point, factorizing f through the dashed vertical map as below:



and moreover this lift is actually in $\mathfrak{A}$ as it is itself equipped with a fibration, so we have in $\mathfrak{A}$



Hence the functor associating to any point its cartesian lifts

$$\begin{array}{rcl} |\,\mathfrak{S}\mathbb{B}\,| & \longrightarrow & \mathfrak{S}\mathbb{B} \downarrow \iota_* \\ p & \longmapsto & \overline{p} : \mathfrak{S}\mathbb{B} \to p_*\mathfrak{S}\mathbb{B} \end{array}$$

indexes an initial family in the comma under $\mathbb{B}$. This exhibits ι_* as a right multi-adjoint. □

Chapter 9

Examples of geometries

In this chapter we gather diverse examples from the literature.

Commutative ring-based geometries were first introduced in [42] in order to formalize the classical construction of spectrum of a commutative ring, and although it predates the first general attempt to construct spectra, this seminal work contained a large amount of the ideas and techniques that were met again in the subsequent works. One can also find a very complete study of those examples in [4][Section 4], which will be summed up there.

The treatment of Stone-like dualities for ordered structures was first done in [10], while various independent constructions were proposed for Heyting algebras, MValgebras, rigs... We also provide the first spectral account of Jipsen-Moshier duality for $\wedge$-semilattices.

In order to organize the various examples we are going to list, we give at fig. 9.1 the following atlas of the geometries listed in this chapter in term of the corresponding Diers contexts (where the name of the different categories are each defined in the corresponding section).
In this atlas, all horizontal arrows are stable functors - and here all of them are actually faithful except for the localic reflection, left adjoint to the inclusion of frames into the opposite bicategory of Grothendieck topoi in the bottom square. Each of the Diers contexts above, the spectrum induce a terminal morphism of Diers context toward the geometry of local topoi and terminally connected geometric morphisms (at the very bottom); for each of those examples except for the inclusions of the strictly henselian local rings and henselian maps, this spectrum functor factorizes through the inclusion of local frames and 1-conservative morphisms.

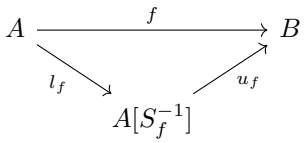
We should emphasize this bottom square is not strictly speaking in the 2-category of Diers contexts, for neither frames nor the opposite bicategory of Grothendieck topoi are accessible.

9.1 Geometries for commutative rings

9.1.1 Zariski geometry

This is the Ur-example of geometries, for it was from its original statement presented comprising the sheaf data - contrarily to the usual statement of Stone duality involving only topological data. This was also the leading example in [42].

9.1.1.1. The category **CRing** of commutative rings is locally finitely presentable and its first order algebraic axiomatization is well known. The factorization system is $(\mathbf{Loc}, \mathbf{Cons})$ where **Loc** is the class of ring localization - beware that, despite their name, they are the etale class of this geometry - and **Cons** is the class of conservative ring homomorphisms, that are, those that reflect invertibility. The factorization of a ring homomorphism is



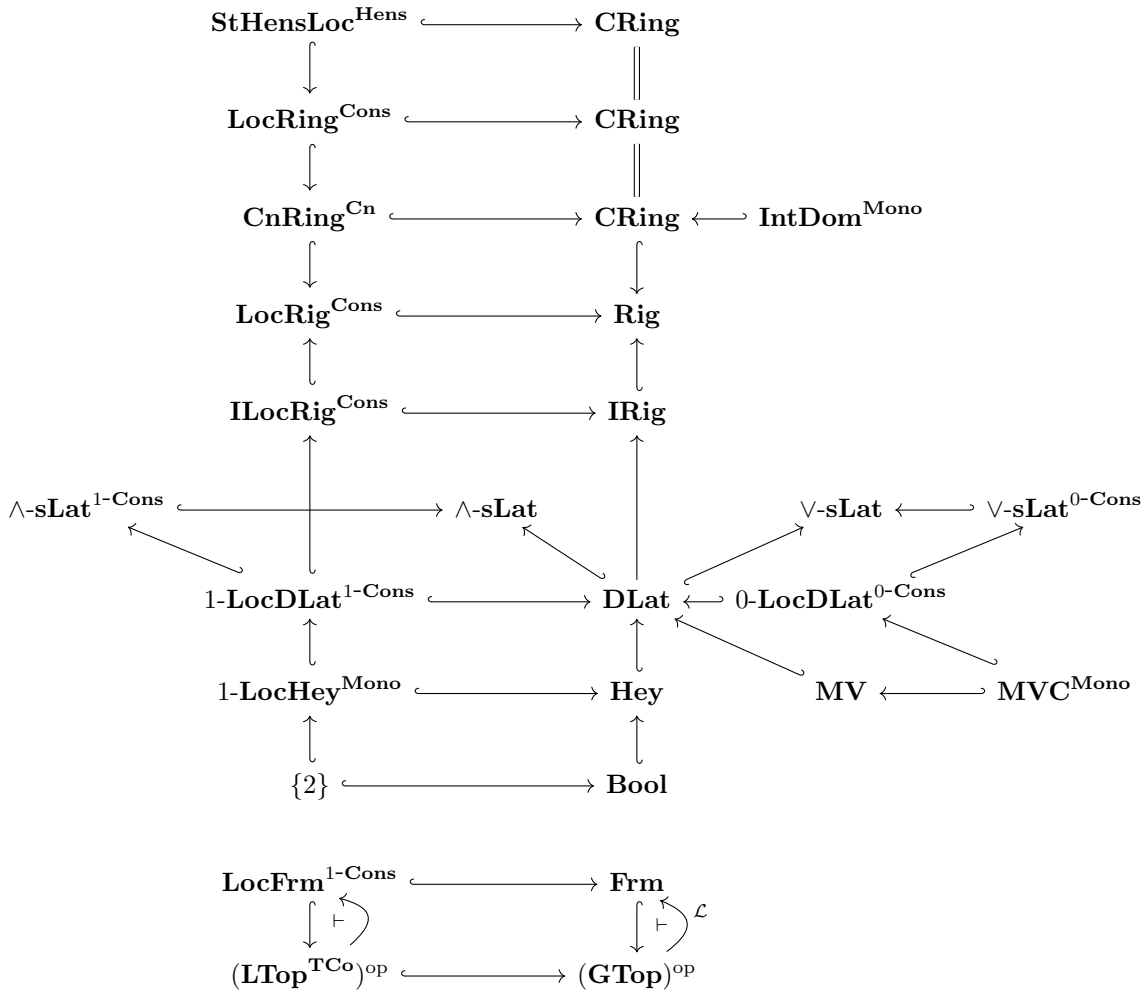


Figure 9.1: An atlas of various Diers Contexts

where S_f is the set of element of A such that $f(a)$ is invertible in B.

9.1.1.2. For a radical ideal I of A, one can consider the localization *away from* I

$$A \xrightarrow{n_I} A[(A \setminus I)^{-1}]$$

localizing elements that do not belong to I. Equivalently, this is obtained as

$$A \xrightarrow{n_I} A[(X_a)_{a \in A \setminus I}]/(aX_a - 1)$$

where one first adds freely a new element for each element a out of I

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{!} & \mathbb{Z}[(X_a)_{a \in A \setminus I}] \\ {\scriptstyle !}\downarrow & \ulcorner & \downarrow \\ A & \xrightarrow[n_I]{} & A[(X_a)_{a \in A \setminus I}] \end{array}$$

and then forces those elements to be inverses of those a.

It is interesting to note that localizations are both monomorphisms (as they are injective) and epimorphisms. Indeed, localizations are totally determined by the elements one adjoins a formal inverse to: for a fork

$$A \xrightarrow{n_s} A[S^{-1}] \underset{g}{\overset{f}{\rightrightarrows}} B$$

if one has $f(a) = g(a)$ for each a in A, then

$$f(aa^{-1} - 1) = f(a)f(a^{-1}) - 1 = g(a)f(a^{-1}) - 1 = 0$$

$$g(aa^{-1} - 1) = g(a)g(a^{-1}) - 1 = f(a)g(a^{-1}) - 1 = 0$$

so that both $f(a^{-1})$ and $g(a^{-1})$ are inverse to $f(a) = g(a)$ and are hence equal by uniqueness of inverse whenever it exists. Yet, because **CRing** is not balanced, localizations are far from being invertible. However, being epimorphic, localization away from a commutative ring A form a poset. In particular the etale generator $\mathcal{V}_A$ of finitely presented localizations is a poset.

Finitely presented localizations of a ring A are well known to be pushouts of finitely presented localizations. In particular for any element a the principal localization at a is

$$\begin{array}{ccc} \mathbb{Z}[X] & \longrightarrow & \mathbb{Z}[X,Y]/(XY-1) \\ {\scriptstyle \ulcorner a \urcorner}\downarrow & \ulcorner & \downarrow \\ A & \xrightarrow[n_a]{} & A[a^{-1}] \end{array}$$

which is the same as localizing the multiplicative part generated by a, that is, containing all its powers a^n for $n \in \mathbb{N}$. Such principal localizations are sufficient to generate arbitrary finitely presented localizations under A by pushouts, as

$$\begin{array}{ccc} A & \xrightarrow{n_{a_1}} & A[a_1^{-1}] \\ {\scriptstyle n_{a_2}}\downarrow & \ulcorner & \downarrow \\ A[a_2^{-1}] & \longrightarrow & A[a_1^{-1}, a_2^{-1}] \end{array}$$

Those finitely presented localizations will be the finitely presented etale maps under a ring.

9.1.1.3. Local objects in this geometry are *local rings* - the term of *local objects* derives from their name: those commutative rings such that for any element a one has either a or $1 - a$ invertible (but not both). The theory of local rings is then axiomatized by adding the following identities to the theory of commutative rings

$$0 = 1 \vdash \bot \qquad \vdash_x \exists y\, xy = 1 \vee \exists y\, (1-x)y = 1$$

where the latter can also be reformulated as

$$x + y = 1 \vdash_{x,y} \exists z\, xz = 1 \vee \exists z\, yz = 1$$

It is also well known those are exactly the commutative rings admitting exactly one maximal ideal, which is exactly the ideal of all non invertible elements. It is also standard that an ideal x of a commutative ring A is prime if and only if the localization A_x of A away from x is a local ring.

9.1.1.4. The admissibility of local ring relatively to the factorization system (localization, conservative) comes from the following fact: if $f : A \to L$ is a ring homomorphism and L is local with $\mathfrak{m}_{\mathfrak{L}}$ its maximal ideal of non-invertible elements, then, as maximal ideals are prime, $f^{-1}(\mathfrak{m}_L)$ is a prime ideal of A containing all elements of A that are not made invertible by f; hence elements of $S_f = A \setminus f^{-1}(\mathfrak{m}_L)$ are exactly those that are made invertible, and hence the factorization

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & L \\ & {\scriptstyle n_f}\searrow \quad \nearrow{\scriptstyle u_f} & \\ & A[A \setminus f^{-1}(\mathfrak{m}_L)] & \end{array}$$

returns a local ring. In particular one has a right multi-adjoint

$$\mathbf{LocRing}^{\mathbf{Cons}} \hookrightarrow \mathbf{CRing}$$

Here the smallness of the cone of local unit comes from the fact that a commutative ring only has a set of prime ideals.

9.1.1.5. It is common to construct directly the spectrum as the topological space $\mathbf{Spec}(A) = (\mathcal{I}_A^{prime}, \tau_{Zar})$ where τ_{Zar} is the topology generated by Zariski basic compact open sets $D_a = \{x \in \mathcal{I}_A^{prime} \mid a \notin x\}$. This is in fact the Diers space of A.

Formally, take the poset of finitely presented localization. Then the localizing pretopology is generated from the single cover

$$\begin{array}{ccc} & \mathbb{Z}[X,Y] & \\ \swarrow & & \searrow \\ \mathbb{Z}[X,Y]/(XY-1) & & \mathbb{Z}[X,Y]/((X-1)Y-1) \end{array}$$

Covers for the Zariski pretopology J_A are generated by pushing out this very cover; they can also be described as $(n_{a_i} : A \to A[a_i])_{i\in I}$ for I finite and $\sum_{i\in I} a_i = 1$. Then in fact $\mathbf{Spec}(A) = \mathbf{Sh}((\mathcal{V}_A^{Zar})^{op}, J_A^{Zar})$ and the Diers space of A is exactly $\mathbf{pt}(\mathbf{Sh}((\mathcal{V}_A^{Zar})^{op}, J_A^{Zar}))$.

9.1.1.6. It is well known that commutative rings have the pullback-pushout lemma, which is equivalent to the so-called *Chinese remainder theorem*:

$$\begin{array}{ccc} A[(a+b)^{-1}] & \longrightarrow & A[b^{-1}] \\ \downarrow & \lrcorner & \downarrow \\ A[a^{-1}] & \longrightarrow & A[(ab)^{-1}] \end{array}$$

Hence commutative rings have sheaf representation by global section of their structural sheaf.

Ambient objects	Commutative Rings
Local objects	Local Rings
Local maps	Conservative morphisms
Etale maps	Localizations away from radical ideals
Local forms	Localizations away from prime ideals $A \to A[(A \setminus x)^{-1}]$
Spectral site	Principal localization away from principal radical ideal
Saturated compacts	Radical ideals
Spectral topology	Localic, coherent
Structure of points	DCPO
Sheaf representability	Yes

Table 9.1: Zariski geometry for Commutative Rings

9.1.2 Pierce geometry

Zariski geometry is not the only geometry one can put on commutative rings: other choices of local objects and factorization data are available. While Zariski geometry is centered on the progressive "elimination" of non-invertible elements through localizations, *Pierce geometry* is centered on the elimination of idempotent elements through quotienting. The main source for this example is [47][Chapter V, section 2].

Recall that for a commutative ring A, the set of idempotents elements form a boolean algebra $I(A)$ where $\wedge$ by $\cdot$ and the supremum is computed as

$$e \wedge e' = e + e' - ee'$$

Complement is given by $\neg e = 1 - e$. Hence in particular this defines a functor

$$\mathbf{CRing} \xrightarrow{I} \mathbf{Bool}$$

9.1.2.1. Recall that the only element that is simultaneously idempotent and invertible is 1. The factorization system is the following: etale maps are the ind-completion of all localization of a commutative ring A at an idempotent element e - which are isomorphic to its quotient at $e-1$

$$A \xrightarrow{n_e} A[e^{-1}] \simeq A/(e-1)$$

Moreover, for $e \leq e'$ in $I(A)$ one has a factorization

$$\begin{array}{ccc} & A & \\ {\scriptstyle n_{e'}}\swarrow & & \searrow{\scriptstyle n_e} \\ A/(1-e') & \longrightarrow & A/(1-e) \end{array}$$

Then $\mathcal{V}_A^{\mathrm{op}}$ is a poset isomorphic to the boolean algebra of idempotent elements $I(A)$.

Local maps are those that are injective on idempotents: let us call them *connected* ring homomorphisms. Then the admissible factorization is obtained as

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle n_f}\searrow \quad \nearrow{\scriptstyle u_f} & \\ & A[E_f^{-1}] & \end{array}$$

where E_f is the set of idempotent elements of A that are localized by f.

9.1.2.2. A commutative ring is *connected* if it has no non-trivial idempotent element, which is axiomatized through the following sequents

$$0 = 1 \vdash \bot \qquad x^2 = x \vdash_x x = 0 \vee x = 1$$

This is encoded in the basic cover

$$\begin{array}{ccc} & \mathbb{Z}[X]/(X^2-X) & \\ \swarrow & & \searrow \\ \mathbb{Z}[X]/(X-1) & & \mathbb{Z}[X]/X \end{array}$$

from which is generated Pierce topology J_{Pierce}.

Then admissibility of the factorization of a morphism $f : A \to B$ with B connected follows from the fact that all idempotents of A are sent to 1, so the localization $A[E_f^{-1}]$ localizes all idempotent elements in A: hence $A[E_f^{-1}]$ is itself connected.

9.1.2.3. Finally sheaf representability is ensured by the following general fact that the following square is a pullback-pushout

$$\begin{array}{ccc} A/(1-e)(1-e') & \longrightarrow & A/(1-e) \\ \downarrow & \lrcorner & \downarrow \\ A/(1-e') & \longrightarrow & A/(1-ee') \end{array}$$

where $(1-e)(1-e') = 1-e-e'+ee' = 1-e\vee e'$.

Ambient objects	Commutative Rings
Local objects	Connected Rings
Local maps	Connected morphisms
Etale maps	Localizations at idempotent
Local forms	Localizations at maximal filters of $I(A)$
Spectral site	Boolean algebra $I(A)$ of idempotents
Saturated compacts	Filters of $I(A)$
Spectral topology	Localic, boolean, compact Hausdorff
Structure of points	Discrete set
Sheaf representability	Yes

Table 9.2: Pierce geometry for Commutative Rings

9.1.3 Integral domains geometry

The following geometry is interesting for being both quite natural and a counterexample to sheaf representability. Here this geometry is centered on the progressive elimination of zero-divisor by quotient, and its local objects are integral domains.

9.1.3.1. The category of commutative rings is not balanced: in particular we saw that localizations are both monomorphisms and epimorphisms. However those epimorphisms are not regular, hence not surjective. Here we are interested into *regular epimorphisms*, which are the ones of the form $q_\theta : A \twoheadrightarrow A/\theta$ for θ a ring congruence.
Recall that in a ring A, a *zero divisor* is some a in $\mathcal{A}$ such that there is some b such that $ab = 0$. An *integral domain* is a commutative ring that does not contain any zero divisor. An ideal x is prime if and only if the corresponding quotient A/x at x is an integral domain. Integral domains are the local objects relatively to the topology on $\mathbf{CRing}_\omega^{\mathrm{op}}$ generated by the basic cover

$$\begin{array}{ccc} & \mathbb{Z}[X,Y]/XY & \\ \swarrow & & \searrow \\ \mathbb{Z}[X,Y]/X & & \mathbb{Z}[X,Y]/Y \end{array}$$

Recall that for a ring homomorphism $f : A \to B$ the image factorization is obtained as the quotient at the preimage of 0

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle q_f}\searrow \quad \nearrow{\scriptstyle m_f} & \\ & A/f^{-1}(0) & \end{array}$$

In particular B is an integral domain if and only if $f^{-1}(0)$ is prime, that is, if an only if $A/f^{-1}(0)$ is a prime ideal. Whence the admissible factorization for the (regular epi, mono) factorization of **CRing** relatively to integral domain; we have a right multi-adjoint

$$\mathbf{Dom}^{\mathbf{Mono}} \hookrightarrow \mathbf{CRing}$$

9.1.3.2. For an element a one can consider the complement of the Zariski open D_a, that is $V_a = \{x \in \mathcal{I}_A^{prime} \mid a \in x\}$. We have $x \in V_a$ if and only if there is a factorization of quotients

$$\begin{array}{ccc} & A & \\ & {\scriptstyle q_a}\swarrow \quad \searrow{\scriptstyle q_x} & \\ A/\sqrt{a} & \twoheadrightarrow & A/x \end{array}$$

where $\sqrt{a}$ is the radical ideal at a, and moreover, the quotient A/x exactly is the filtered colimits of all such principal quotients. If now I is a radical ideal one can consider similarly the quotient

$q_I : A \twoheadrightarrow A/I$. We can define $V_I = \{x \in \mathcal{I}_A^{prime} \mid \exists a \in I, a \in x\}$ - this is the complement of the Zariski compact-open set $D_I = \bigcup_{a\in I} D_a$, that is $V_I = \bigcup_{a\in I} V_a$. If we equip $\mathcal{I}_A^{prime}$ with the topology generated by finite intersections of the sets V_a we get the Diers space of A for the right multi-adjoint above. We can also take the site made of all principal quotients of A at the radical ideal generated from a single element $q_a : A \to A/\sqrt{a}$, that are exactly the pushouts

$$\begin{array}{ccc} \mathbb{Z}[X] & \longrightarrow & \mathbb{Z}[X]/\sqrt{X} \\ \downarrow & \ulcorner & \downarrow \\ A & \xrightarrow[q_a]{} & A/\sqrt{a} \end{array}$$

There, the spectral site is then the category $\mathcal{V}_A^{Int}$ consisting of quotient at finitely generated radical ideals $q_{a_1,\dots,a_n} : A \to A/\sqrt{a_1,\dots,a_n}$.

9.1.3.3. Beware that this geometry does not enjoy sheaf representation. The following square

$$\begin{array}{ccc} A/ab & \longrightarrow & A/a \\ \downarrow & & \downarrow \\ A/b & \longrightarrow & A/\langle a,b\rangle \end{array}$$

not always is a pullback-pushout square, for instance if a, b are zero divisors with $ab = 0$ with $a, b \neq 0$. In fact we have that $\Gamma\widetilde{A} \simeq A/\sqrt{0}$ where $\sqrt{0}$ is the nilradical of A, that is, the set of a such that there is some n with $a^n = 0$. Of course if A is an integral domain, the nilradical of A is reduced to 0 and we have sheaf representation - this is a special instance of the automatic representation result for the local objects. In fact there is a more general class of commutative ring enjoying sheaf representation for this geometry, yet their characterization is involved, see [56].

Ambient objects	Commutative Rings
Local objects	Integral domains
Local maps	Monomorphisms
Etale maps	Regular epimorphisms
Local forms	Quotients at prime ideals $A \twoheadrightarrow A/x$
Spectral site	Principal quotients $A \twoheadrightarrow A/\theta_{(a,0)}$
Saturated compacts	Radical ideals
Spectral topology	Spatial, coherent
Structure of points	DCPO
Sheaf representability	No : $\Gamma\widetilde{A} \simeq A/\sqrt{0}$

Table 9.3: Integral domain geometry

9.1.4 Etale geometry

This geometry was first treated also in [42], but we base ourself on the description given in [4]. It is interesting as the main example of geometry where the spectral site is not localic, and the spectrum non spatial.

A morphism of commutative ring is said to be *etale* if it is flat and unramified. Local objects in this geometries are *strictly henselian local rings*.

Localizations are in particular ind-etale morphisms. Conversely, henselian morphisms are conservative. Hence we have a morphism of Diers contexts

$$\begin{array}{ccc} \mathbf{StHensLoc}^{\mathbf{Hens}} & \hookrightarrow & \mathbf{CRing} \\ \downarrow & & \| \\ \mathbf{LocRing}^{\mathbf{Cons}} & \hookrightarrow & \mathbf{CRing} \end{array}$$

For more details about this geometry, refer to [4][Section 4.3]

Ambient objects	Commutative Rings
Local objects	Strictly henselian local rings
Local maps	Henselian morphisms
Etale maps	Etale morphisms
Local forms	Strictly Henselian and local algebras
Spectral site	Finitely presented etale maps
Saturated compacts	?
Spectral topology	Non-spatial
Structure of points	Have non-trivial automorphisms
Sheaf representability	Yes

Table 9.4: Etale geometry

9.2 Geometries for propositional dualities

9.2.1 Stone geometry for distributive lattices

In general, Stone duality is considered from the point of view of concrete dualities. However it is possible to provide a spectral account of it, or at least to reconstruct the Stone dual of a distributive lattice as the spectra of a certain geometry. But in this process the stone dual is also endowed with a structural sheaf which is not considered in classical Stone duality, and the spectrum will be adjoint to the global section functor rather than the "compact open" functor, which, in Stone duality, returns the basis of compact open set of Stone spaces to which it is hence restricted.

As for Stone duality, this construction can be done in two manners, equipping the Stone dual with either the *Zariski* or the *coZariski* topology: this depends on the way we define the admissibility structure.

9.2.1.1. We recall here the admissibility structure for Zariski. Our ambient locally finitely presentable category is **DLat**, the category of bounded distributive lattices. Recall that distributive lattices are not 1-regular, that is, for any given ideal there are several congruences whose class in 1 is this ideal. For etale maps one can choose *1-minimal quotients*: those are morphisms $A \twoheadrightarrow A/\theta$ with θ minimal amongst congruences whose class in 1 is $[1]_\theta$. One can easily prove this class is closed by composition and colimits, contains iso and is left-cancellative. For a lattice D finitely presented 1-minimal quotients are of the form $D \to D/\theta_{(a,1)}$

Then it can be shown that one has a factorization system $(1\text{-}\mathbf{Quo}, 1\text{-}\mathbf{Cons})$ on **DLat**, where 1-**Cons** is the class of maps that reflect the top element, that is those $f : D \to D'$ such that $f^{-1}(1) = \{1\}$.

Then define the category 1-$\mathbf{LocDLat}^{1\text{-}\mathbf{Cons}}$ having:

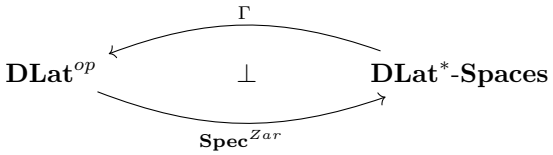
- as objects *local distributive lattices*, where $\{1\}$ is prime filter
- as morphisms 1-**Cons**ervative morphisms f.

Then 1-$\mathbf{LocDLat}^{1\text{-}\mathbf{Cons}} \hookrightarrow \mathbf{DLat}$ is a multireflection. But we can also axiomatize the category of local lattices as follows: define J_1 on $\mathbf{DLat}_{fp}^{op}$ generated by $(f_i : D \twoheadrightarrow D/\theta_i)$ such that $\bigcap_{i\in I} \theta_i = \Delta_D$. Now observe that a distributive lattice D is J_1-local if an only if $\{1\}$ is a prime ideal, that is, D has a minimal point $L \to 2$ sending any $a \neq 1$ on 0. Local lattices are the points of the topos $\mathbf{Sh}(\mathbf{DLat}_{fp}^{\mathrm{op}}, J_1)$.

9.2.1.2. The associated Diers spectrum for D is

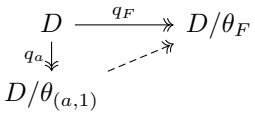
$$(\mathbf{Spec}(D) = (\mathcal{F}_D^{Prime}, \tau_D^{Zariski}), \widetilde{D})$$

with $\widetilde{D}$ defined on the basis as $\widetilde{D}(U_a^{coZar}) = D/\theta_{(a,0)}$ for any $a \in D$. Then we have an adjunction



Then one recovers the Stone spaces as the underlying spaces of affine **DLat**-spaces.

The spectral site of a distributive lattice D is $(Zar_D^{op}, J_1(D))$ where Zar_D consists of finitely presented 1-minimal quotients of D; in particular for a filter F of D, a factorization as below $D \twoheadrightarrow D/\theta_{(a,1)}$, and a factorization



expresses the fact that a lies in F.

Now, at a distributive lattice D, the induced topology $J_1(D)$ consists of finite families $(D \twoheadrightarrow D/\theta_{(a_i,1)})_{i\in I}$ with $\bigvee a_i = 1$. Being made of epi, Zar_D is a poset and $Zar_D \simeq D^{\mathrm{op}}$ and we have $D \hookrightarrow \tau_{Zar}$. $J_1(D)$ coincides with the coherent topology on D. The spectrum is spatial and is equipped with the Zariski topology which is the frame of filters $\mathcal{F}_D$.

Opens of Zariski topology form the frame $\tau_{Zar} = \mathbf{Sh}(Zar_D^{op}, J_1(D)) = \mathbf{Sh}(I_D)$: Zariski opens correspond to ideals of D and $D \hookrightarrow I_D$ is a base of compact open of Zariski topology. On the other side, $D \hookrightarrow (\mathcal{F}_D)^{op}$, but a filter F of D defines a filtered diagram whose colimit is the 1-minimal quotient at F

$$S \twoheadrightarrow S/\theta_F^{min} = \operatorname*{colim}_{a\in F} S/\theta(a,1)$$

Those filters are *saturated compact* of Zariski topology. A prime filter x corresponds to the 1-quotient $D \twoheadrightarrow D/\theta_x$, which is the saturated compact $\uparrow x$, the focal component in x.

Ambient objects	Bounded Distributive Lattices
Local objects	1-Local lattices
Local maps	1-Conservative morphisms
Etale maps	1-Minimal quotients
Local forms	1-Minimal quotients at primes filters
Spectral site	Principal 1-Minimal quotients $D \twoheadrightarrow D/\theta_{(a,1)}$
Saturated compacts	Filters
Spectral topology	Stone spaces
Structure of points	DCPO
Sheaf representability	Yes

Table 9.5: Zariski geometry for distributive lattices

9.2.1.3. However Zariski geometry is not the only way to retrieve Stone duality. One could have either defined the factorization system (0-**MinQuo**, 0-**Cons**), whith 0-**MinQuo** as the minimal quotient with a fixed ideal, and 0-**Cons** the morphisms f such that $f^{-1}(0) = \{0\}$, and could have taken as local objects those distributive lattices with $\{0\}$ prime.

The CoZariski site would have been $(coZar_D^{op}, J_0(D))$ with $coZar_D$ made of the minimal quotients $D \twoheadrightarrow D/\theta_{(a,0)}$ and $J_0(D)$ defined by $(a_i)_{i\in I}$ such that $\bigwedge_{i\in I} a_i = 0$.

Then $D \simeq coZar_D$, so that $D^{op} \hookrightarrow \tau_{coZar} = \mathbf{Sh}(coZar_D^{op}, J_0(D)) \simeq (\mathcal{F}_D)^{op}$. Then filters are the closed subsets of coZariski topology. On their sides ideals I_D define filtered colimits of finitely

presented minimal quotients maps in $coZar_D$, hence correspond to saturated compacts. Observe that the existence of both a Zariski and a coZariski topology for a distributive lattice, which is called *Hochster duality*, is in fact nothing but an instance of Isbell duality.

Ambient objects	Bounded Distributive Lattices
Local objects	0-Local lattices
Local maps	0-Conservative morphisms
Etale maps	0-Minimal quotients
Local forms	0-Minimal quotients at primes ideals
Spectral site	Principal 0-Minimal quotients $D \twoheadrightarrow D/\theta_{(a,0)}$
Saturated compacts	Ideals
Spectral topology	Stone spaces
Structure of points	DCPO
Sheaf representability	Yes

Table 9.6: coZariski geometry for distributive lattices

9.2.2 Geometry for boolean algebras

The geometry of boolean algebras is a restriction of both Zariski and coZariski geometries, which happens to coincide for a boolean algebra.

9.2.2.1. For an element a in a distributive lattice, a *complement* is an element, unique whenever it exists, $\neg a$ such that $a \vee \neg a = 1$, and $a \wedge \neg a = 0$. Observe that a prime ideal x of a distributive lattice D contains always either a or $\neg a$ for any complemented element a in D from $a \wedge \neg a = 0$, while an ideal never contains simultaneously an element and its complement unless it is trivial; the corresponding statements are true for filters and prime filters. Any morphism of distributive lattices preserves complement.

A *boolean algebra* is a distributive lattice where any element is complemented. Boolean algebras form a full, reflective subcategory $\mathbf{Bool} \hookrightarrow \mathbf{DLat}$. Boolean algebras have better exactness properties than distributive lattices. First, they form a regular categor: any ideal I (resp. a filter F) in a boolean algebra, there is exactly one congruence θ_I such that $[0]_{\theta_I} = I$ (resp. one congruence θ_F such that $[1]_{\theta_F} = F$). In particular, there is no more distinction of 1-minimal or 0-minimal quotient amongst quotient. Moreover, any epimorphism of boolean algebra is a regular quotient, and the category of boolean algebras is balanced. Finally, recall that any prime ideal (or filter) of a boolean algebra is maximal.

9.2.2.2. Now the only 1-local (or 0-local) boolean algebra is the two elements algebra 2: indeed, if $\{1\}$ is a prime filter in A, then $x \vee y = 1$ implies $x = 1$ or $y = 1$; but then for $x \vee \neg x = 1$, any x in A must actually be 0 or 1.

Then the Diers context for both the Zariski and coZariski geometries of distributive lattices reduces on the inclusion

$$\begin{array}{ccc} \{2\} & \hookrightarrow & \mathbf{Bool} \\ \downarrow & & \downarrow \\ \text{1-}\mathbf{LocDLat} & \hookrightarrow & \mathbf{DLat} \end{array}$$

Ambient objects	Boolean algebras
Local objects	The two-element algebra 2
Local maps	Monomorphisms
Etale maps	Epimorphisms
Local forms	Quotients at maximal filters/ideals
Spectral site	Principal Quotients $D \twoheadrightarrow D/\theta_{(a,1)}$ - or $D \twoheadrightarrow D/(\neg a, 0)$
Saturated compacts	Closed sets; either filters or ideals
Spectral topology	Boolean Stone spaces, Hausdorff
Structure of points	Discrete set
Sheaf representability	Yes

Table 9.7: Geometry for boolean algebras

9.2.3 Esakia geometry for Heyting algebras

Esakia duality for Heyting algebras, as other Stone-like dualities, was introduced from a concrete duality point of view and without reference to structural sheaves. However in [24] was introduced a new presentation including sheaf theoretic consideration, and in particular a sheaf representation theorem. This geometry is also a restriction of Zariski geometry to Heyting algebras, being intermediate between Zariski and the geometry of boolean algebras.

However, factorization data simplify as they do in boolean algebras. Heyting algebras are 1-regular, so that for any filter F there is a unique congruence θ_F such that $[1]_{\theta_F} = F$. Hence 1-minimal quotient are not distinguished from other quotient, as they are determined from the preimage of 1.

Local maps can still be defined as conservative morphisms, but in fact they coincide then with monomorphisms. Local forms are quotient at primes filters.

Ambient objects	Heyting Algebras
Local objects	1-Local Heyting Algebras
Local maps	Monomorphisms
Etale maps	Regular epimorphisms
Local forms	Quotient at prime filters
Spectral site	Principal quotients
Saturated compacts	Filters
Spectral topology	Spatial, Esakia spaces
Sheaf representability	Yes

Table 9.8: Esakia geometry for Heyting algebras

Remark 9.2.3.1. Beware that coZariski is not suited for Heyting algebras, as they are not 0-regular; however one could dually define a *coEsakia geometry* for coHeyting algebras, which are 0-regular, and use ideal instead of filters.

9.2.4 Jipsen-Moshier geometry for semilattices

Jipsen-Moshier duality is an example of an geometry without specification of local objects, so that any object is actually local.

9.2.4.1. In a space X with specialization order $\sqsubseteq$, a compact open filter is an upset F for $\sqsubseteq$ which is both open and compact. For X denote $\mathbf{KOF}(X)$ its set of compact open filters. A point x is basic compact open if $\uparrow$ is a compact open filter. *Hofmann-Mislove-Stralka spaces* - for short, *HMS*

spaces - are sober spaces X such that $\mathbf{KOF}(X)$ is a basis closed under finite intersection.

Denote **HMS** the category of HMS spaces with continuous maps $f : X \to Y$ such that f^{-1} restricts to $\mathbf{KOF}(Y) \to \mathbf{KOF}(X)$. Any compact open filter of a HMS space has a focal point. Moreover, in a HMS space, any point is a directed join of basic compact open points. The specialization order makes $(X, \sqsubseteq)$ a complete lattice, and there are simultaneously an initial point and terminal point in such a X.

9.2.4.2. Then, one can recover the spectra of Jispen and Moshier duality for $\wedge$-semilattices with unit:

$$\wedge\text{-}\mathbf{SLat}^{\mathrm{op}} \simeq \mathbf{HMS}$$

Define $\mathbf{Spec}(S) = (\mathcal{F}_S, \downarrow S)$. For X HMS, $\mathbf{KOF}(X)$ is a $\wedge$-semilattice. Moreover, for a semillatice one has $S \simeq \mathbf{KOF}(Spec(S))$, while for a HMS space, one has $X = \mathbf{Spec}(\mathbf{KOF}(X))$.

If S is a $\wedge$-semilattice, $\mathcal{F}_S \simeq (\mathcal{I}_S^{prime})^{\mathrm{op}}$ is a complete lattice. Moreover any filter of a $\wedge$-semilattice is trivially prime. This says that $\mathbf{Spec}(S) = \wedge\text{-}\mathbf{SLat}[S, 2]$.

9.2.4.3. Then the geometry associated to Jispen and Moshier duality can be constructed as follows: take (1-**MinQuo**, 1-**Cons**) also is a factorization system on $\wedge$-**SLat**, and no topology. Finitely presented etale maps under a $\wedge$-semilattice S are principal 1-minimal quotients

$$S \twoheadrightarrow S/\theta(a, 1)$$

and they always define a basic compact open point. Conversely for any filter F one has a minimal quotient

$$S \twoheadrightarrow S/\theta_F^{min} = \operatorname*{colim}_{a \in F} S/\theta(a, 1)$$

where θ_F^{min} is the congruence in F given as $\theta_F^{min} = \bigcap\{\theta \mid [1]_\theta = F\}$. This defines a point of $\mathbf{Spec}(S)$, and any saturated compact actually has a focal point. Observe that this is a situation of the purely factorization geometries as in theorem 5.1.2.8, which is induced from the stable inclusion of the right class of a factorization system 1-**Cons** $\hookrightarrow \wedge$-**SLat**.

Finally, sheaf representation is trivial as the topology here is the trivial topology, for which the codomain functor already is a sheaf.

Ambient objects	$\wedge$-semilattices with unit
Local objects	All objects are local
Local maps	1-Conservative morphisms
Etale maps	1-Minimal quotients
Local forms	1-Minimal quotients at any filter
Spectral site	Princial 1-minimal quotients $S \twoheadrightarrow S/\theta_{(a,1)}$
Saturated compacts	Filter
Spectral topology	Hoffman-Mislove-Stralka topology
Structure of points	Complete lattice
Sheaf representability	Yes

Table 9.9: Jipsen-Moshier Geometry

9.2.5 Castiglioni-Menni-Zuluaga-Botero geometry for rigs

Here we sum up the results of [16], which develop a geometry for integral rigs subsuming both Zariski geometry of commutative rings and Stone geometry of distributive lattices.

9.2.5.1. Recall that a *rig* (aka "ring without negative") is the data of $(A, +, \cdot, 0, 1)$ such that

– $(A, +, 0)$ and $(A, \cdot, 1)$ are commutative monoids

– both of the generic equations hold $x \cdot 0 = 0$ and $(x + y) \cdot z = x \cdot z + y \cdot z$.

Any rig can be equipped with a pre-order $\leq$ where $x \leq y$ if there exists some z such that $x + z = y$. In general this relation is only a pre-order. A rig is said to be *integral* if it satisfies the generic equation $x + 1 = 1$; then one has $x \leq 1$ for any x. As a consequence, the operation $+$ is idempotent in an integral rig, as

$$x + x = (1 + 1) \cdot x = 1 \cdot x = x$$

so that the monoid $(A, +, 0)$ is then a $\vee$-semilattice. In particular a distributive lattice is an integral rig where $\cdot$ is also idempotent.

Similarly, one can define a commutative ring in the signature of rigs as a rig satisfying the cartesian sequent $\vdash_x \exists y x + y = 0$. By uniqueness of such a solution, the theory of commutative rings is a finite limit extension of the theory of rigs.

9.2.5.2. An element a of a rig A is said to be *invertible* if there exists b in A such that $a \cdot b = 1$. Denote as $\mathrm{Inv}(A)$ the set of pairs (a, b) of A such that $a \cdot b = 1$: this is the pullback

$$\begin{array}{ccc} \mathrm{Inv}(A) & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow{\scriptstyle \ulcorner 1 \urcorner} \\ A \times A & \xrightarrow[\cdot]{} & A \end{array}$$

Moreover we have two canonical projections $\mathrm{Inv}(A) \to A$ which are monomorphisms and code for the same subobject.

Then, similarly to the case of rings, one can consider *conservative maps* of rigs (they are directly qualified of local in [16]) as maps that reflect invertibility, that are those morphisms of rigs $f : A \to B$ such that one has a pullback

$$\begin{array}{ccc} \mathrm{Inv}(A) & \longrightarrow & \mathrm{Inv}(B) \\ \downarrow & \lrcorner & \downarrow \\ A & \xrightarrow[f]{} & B \end{array}$$

9.2.5.3. On the other hand, a *multiplicative submonoid* is a $F \subseteq A$ which is a submonoid of $(A, \cdot, 1)$. At a multiplicative submonoid F one can consider the *localization* $n_F : A \to A[F^{-1}]$ forcing each element of F to become invertible; in particular, one can define for each a in A the *principal localization at* a $n_a : A \to A[a^{-1}]$ which corresponds to the localization at the multiplicative submonoid $\{a^n \mid n \in \mathbb{N}\}$. A *filter* of a rig is a multiplicative submonoid F such that $a \in F$ implies that $a + b \in F$ for all b in A. An *ideal of rigs* is a $I \subseteq A$ which is a submonoid of $(A, +, 0)$ and such that for each a in A, $ax \in I$ whenever $x \in I$. For each multiplicative submonoid F the complement $A \setminus F$ is an ideal, and conversely. A filter F of rig is prime if $a + b \in F$ implies that either $a \in F$ or $b \in F$; similarly an ideal of rig is prime if $ab \in I$ implies that either $a \in I$ or $b \in I$.

When A is an integral rig, then $\mathrm{Inv}(A) \simeq 1$, and localizing at an element amounts to forcing it to become 1, as $a \leq 1$ implies that $1 \leq a^{-1}$ if the latter exists. Then the localization $A \to A[a^{-1}]$ is exhibited as the quotient $q_a : A \to A/\theta_a$ where $\theta_a = \{(x, y) \mid ax = ay\}$; in particular $(a, 1) \in \theta_a$. More generally for a mutiplicative submonoid F one can define the congruence $\theta_F = \{(x, y) \mid \exists z \in F, xz = yz\}$, and the localization $A \to A[F^{-1}]$ coincides with the quotient $q_F : A \twoheadrightarrow A/\theta_F$ contracting F on 1. In particular localization of integral rigs are regular epimorphisms; observe however that, contrarily to ring localizations, they are not monomorphisms.

One has a factorization system $(\mathbf{Loc}, \mathbf{Cons})$ in the category **IRig** of integral local rigs, where the factorization is obtained as the quotient

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ & {\scriptstyle q_f} \searrow \quad \nearrow {\scriptstyle u_f} & \\ & A/f^{-1}(1) & \end{array}$$

9.2.5.4. A rig is *really local* if it satisfies

$$0 = 1 \vdash \bot \qquad \text{and} \qquad x + y = 1 \vdash_{x,y} x = 1 \vee y = 1$$

Now observe that really local integral rigs have the gliding property relatively to conservative maps: indeed for a conservative morphism $u : A \to B$ with B really local, if one has $a + b = 1$ in A then either a or b is in $u^{-1}(1)$, which is $\{1\}$.

An integral rig is really local if and only if $\{1\}$ is a prime filter. Observe that in this case $A \setminus \{1\}$ is an ideal, for it becomes closed under $+$, and it is then automatically prime. In fact from this, a really local rig is a rig where $A \setminus \{1\}$ defines a unique maximal ideal.

A filter F is prime if and only if $A[F^{-1}] \simeq A/\theta_F$ is really local: indeed, if $a + b$ is in F, then $[a + b]_F = [a]_F + [b]_F = 1$, so A is really local if and only if F is prime.

Hence we have a stable inclusion

$$\mathbf{ILocRig}^{\mathbf{Cons}} \hookrightarrow \mathbf{IRig}$$

where the local units of a rig A are exactly its localization at prime filters.

Observe that we have a morphism of Diers contexts comparing Stone geometry to the geometry for Rigs

$$\begin{array}{ccc} \text{1-}\mathbf{LocLat}^{\text{1-}\mathbf{Cons}} & \hookrightarrow & \mathbf{DLat} \\ \downarrow & & \downarrow \\ \mathbf{ILocRig}^{\mathbf{Cons}} & \hookrightarrow & \mathbf{IRig} \end{array}$$

As well as a morphism of Diers contexts comparing the geometry of commutative rings to the geometry of rigs

$$\begin{array}{ccc} \mathbf{LocCRing}^{\mathbf{Cons}} & \hookrightarrow & \mathbf{CRing} \\ \downarrow & & \downarrow \\ \mathbf{LocRig}^{\mathbf{Cons}} & \hookrightarrow & \mathbf{Rig} \end{array}$$

9.2.5.5. Finally, it is proved at [16][Lemma 4.2] that for any a, b one has a pullback-pushout square

$$\begin{array}{ccc} A[(a+b)^{-1}] & \longrightarrow & A[b^{-1}] \\ \downarrow & \lrcorner & \downarrow \\ A[a^{-1}] & \longrightarrow & A[(ab)^{-1}] \end{array}$$

Hence we have a sheaf representation for integral rig.

Ambient objects	(Integral) Rigs
Local objects	Really local (integral) Rigs
Local maps	Conservative morphisms
Etale maps	Localizations
Local forms	Localizations at prime filters
Spectral site	Principal localizations $A \twoheadrightarrow A[a^{-1}]$
Saturated compacts	Filters
Spectral Topology	Localic, coherent
Structure of points	DCPO
Sheaf representability	Yes

Table 9.10: Castiglione-Menni-Zualaga-Botero Geometry

9.2.6 Dubuc-Poveda geometry for MV-algebras

Spectra of MV-Algebras were introduced in [32] and later studied in further works on residuated lattices as [39]. We list here the main results of this work.

9.2.6.1. MV-algebra can be defined in various choices of signatures, either emphasizing their residuated lattice nature, or from their specific operations, those data being mutually determined. An MV-algebra is the data of $(A, \oplus, 0, \neg)$ such that

- $(A, \oplus, 0)$ is a commutative monoid
- $\neg$ is an involution $\neg\neg x = x$
- and we have the generic identities

$$x \oplus \neg 0 = \neg 0$$
$$\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$$

It is also natural to define the following other operations

$$x \odot y = \neg(\neg x \oplus \neg y)$$
$$x \ominus y = x \odot \neg y$$
$$1 = \neg 0$$

Then we end up with the following identities:

$$x \oplus y = \neg(\neg x \odot \neg y)$$
$$x \oplus \neg x = 1$$
$$\neg 1 = 0$$

Any morphism of MV-algebra preserves automatically those operations and constants, for they are defined from the ones in the signature. Denote as **MV** the category of MV-algebras and their morphisms.

Moreover we can also equip A with an order $\leq$ with $x \leq y$ if there exists z such that $x \oplus z = y$, which can be chosen as $y \ominus x$. Then the negation $\neg$ is order reversing, while the operations are monotonic. Then this order defines a structure of distributive lattice where

$$x \vee y = (x \odot \neg y) \oplus y = (x \ominus y) \oplus y$$
$$x \wedge y = x \odot (\neg x \oplus y)$$

In particular one always have $x \vee y \leq x \oplus y$ and $x \odot y \leq x \wedge y$. Any morphism of MV-algebras is also a morphism of distributive lattices, and is hence order preserving. The category of MV-algebras has hence in particular a faithful functor $\iota : \mathbf{MV} \to \mathbf{DLat}$ sending a MV-algebra on its undelying distributive lattice. The functor ι is moreover a morphism of locally finitely presentable category, for both filtered colimits and limits of MV-algebras and distributive lattices are computed from the underlying set; the left adjoint of ι is the functor ι_* sending a distributive lattice D with a generator and relations presentation $D = \mathbf{DLat}[X]/\theta$ to the MV-algebra presented as

$$\iota^* D = \mathbf{MV}[X]/\overline{\theta}$$

where θ is the free MV-congruence generated by θ in $\mathbf{MV}[1]$. We see this functor sends finitely presented distributive lattices on finitely presented algebras.

9.2.6.2. We can define the *distance* between two elements in a MV-algebra as

$$d(x, y) = (x \ominus y) \oplus (y \ominus x)$$

In particular one has $d(x, y) = 0$ iff $x = y$. The following identity is always satisfied in an MV-algebra:

$$(x \ominus y) \wedge (y \ominus x) = 0$$

Observe that $x \ominus y = 0$ iff $x \leq y$, hence $d(x, y) = y \ominus x$ if $x \leq y$. In general not all elements are pairwise comparable; a totally ordered MV-algebra is called an *MV-chain*.

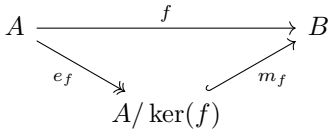
An *ideal of MV-algebra* of A is a lattice ideal of $(A, \wedge, \vee, 0, 1)$ which is moreover closed under the $\oplus$ operation. For $S \subseteq A$ a subset, the ideal $(S]$ generated by S is its the closure under $\oplus$ of the downset $\downarrow S$. In particular, an ideal is said to be *principal* if it is of the form $(a]$ for some a in A. For any pair a, b we have

$$(a, b] = (a \oplus b] = (a \vee b]$$
$$(a \wedge b] = (a] \cap (b]$$

Hence any finitely generated ideal is principal. If $f : A \to B$ is a morphism of MV-algebras, then $\ker(f) = f^{-1}(0)$ is an ideal in A. A morphism of MV-algebras is a monomorphism if and only if $\ker(f) = \{0\}$. A congruence θ of MV-algebra defines an ideal $[0]_\theta$; conversely, for any I, define the congruence θ_I as

$$\theta_I = \{(a, b) \in A^2 \mid d(a, b) \in I\}$$

In particular MV-algebras are 0-regular, as any congruence is determined by the class of 0. We denote as $\mathcal{I}_A^{\mathbf{MV}}$ the set of MV-ideals of A. Moreover, $\mathbf{MV}$ is a regular category, and has its (Epi,Mono) factorization obtained as



Moreover, we have that for any MV-ideal I in A, I is directed as a subset for it is closed under $\oplus$, and we have a filtered colimit

$$A/p \simeq \operatorname*{colim}_{a \in p} A/(a]$$

Beware that not all lattice ideals are MV-ideals, and not all lattice congruences are MV-congruences. However MV-algebras have reticulation: we can define the relation $a \simeq b$ if $(a] = (b]$, which is a lattice congruence on the underlying lattice $\iota(A)$, and the quotient $\beta(A) = A/\simeq$, called the *Belluce lattice* of A, satisfies

$$\mathcal{I}_A^{\mathbf{MV}} \simeq \mathcal{I}_{\beta(A)}^{\mathbf{DLat}}$$

Moreover this construction is functorial as any morphism of MV-algebras induces a unique factorization in $\mathbf{DLat}$

$$\begin{array}{ccc} \iota(A) & \xrightarrow{\iota(f)} & \iota(B) \\ {\scriptstyle \sigma_A}\downarrow & & \downarrow{\scriptstyle \sigma_B} \\ \beta(A) & \xrightarrow[\beta(f)]{} & \beta(B) \end{array}$$

This defines a functor $\beta : \mathbf{MV} \to \mathbf{DLat}$.

9.2.6.3. A non trivial MV-ideal p is *prime* if for any x, y either $x \ominus y$ or $y \ominus x$ is in p. In particular an MV-ideal is prime if and only if the underlying lattice ideal is prime. Moreover, we also have that an MV-ideal is prime if and only if the quotient A/I is an MV-chain. Conversely, one can prove that A is an MV-chain if and only if $\{0\}$ is a prime ideal. Prime MV-ideals are stable under inverse image: if p is prime in B and $f : A \to B$ is a morphism of MV-algebra, then $f^{-1}(p)$ is prime. Any ideal that contains a prime ideal is prime, and for each prime ideal p the set $\{I \in \mathcal{I}_A^{\mathbf{MV}} \mid p \subseteq I\}$ is totally ordered. Hence the set of prime ideals Z_A of A is a *root system* for the inclusion, that is, satisfies that for any p in X_A, the upset in p is totally ordered.

Hence an MV-chain have the gliding property along monomorphisms of MV-algebras: if one has a monomorphism $m : A \hookrightarrow L$ with L an MV-chain, then A itself is an MV-chain. Hence we have a local right adjoint

$$\mathbf{MVC}^{\mathbf{Mono}} \hookrightarrow \mathbf{MV}$$

where $\mathbf{MVC}^{\mathbf{Mono}}$ is the category of MV-chains and monomorphisms. From the fact that the (epi, mono) factorization is left generated, we know this defines a Diers context.

9.2.6.4. Moreover, observe that MV-chains are local as a distributive lattice, while mono are conservative, so that we have a morphism of Diers Contexts

$$\begin{array}{ccc} \mathbf{MVC}^{\mathbf{Mono}} & \hookrightarrow & \mathbf{MV} \\ \downarrow & & \downarrow{\scriptstyle \iota} \\ 0\text{-}\mathbf{LocDLat}^{0\text{-}\mathbf{Cons}} & \hookrightarrow & \mathbf{DLat} \end{array}$$

9.2.6.5. A local unit under a MV-algebra A is a quotient at a prime ideal $n_p : A \twoheadrightarrow A/p$, and A/p is an MV-chain. One can construct the Diers space of A pointwisely by defining Z_A as the set of prime MV-ideals of A; for any a, one has $a \in p$ if and only if one has the factorization

$$\begin{array}{ccc} A & \xrightarrow{q_p} & A/p \\ {\scriptstyle q_a}\downarrow & \nearrow & \\ A/(a] & & \end{array}$$

One can define coZariski basic open

$$W_a = \{p \in Z_A \mid a \in p\}$$

We have in particular

$$\begin{aligned} W_{a \oplus b} &= W_a \cap W_b \\ W_0 &= Z_A \end{aligned}$$

so that we have a basis $(W_a)_{a \in A}$ for a topology τ_A on Z_A; moreover observe that $W_a = W_b$ if and only if $a \simeq b$, so that the basis is actually $\beta(A)$. We can extend the definition to arbitrary open by defining $W_I = \{p \in Z_A \mid I \subseteq p\}$, and we have

$$W_I = \bigcap_{a \in I} W_a$$

so that $\mathcal{I}_A^{\mathbf{MV}}$ indices the set of saturated compact of the topology on Z_A. Observe that (Z_A, τ_A) coincides with the Stone dual of the Belluce lattice $(X_{\beta(A)}, \tau_{\beta(A)}^{coZar})$ together with its coZariski topology.

9.2.6.6. The structural sheaf can be constructed explicitely as follows. Consider the bundle $p_A : E_A \to Z_A$ where

$$E_A = \coprod_{p \in Z_A} A/p$$

Observe that elements of A defines families of sections

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & \prod_{p \in Z_A} A/p \\ & \searrow & \downarrow \\ & & A/p \end{array}$$

where η_A sends a to the section $\widehat{a} = ([a]_p)_{p \in A}$ of p_A. Then one equip E_A with the etale topology $\langle \widehat{a}(W_b) \rangle_{a,b \in A}$, and one just has to define $\widetilde{A} = \Gamma(p_A)$.

Moreover it is known that MV-algebras satisfy the pullback-pushout lemma:

$$\begin{array}{ccc} A/(a \wedge b] & \longrightarrow & A/(a] \\ \downarrow & \lrcorner & \downarrow \\ A/(b] & \ulcorner & A/(a \oplus b] \end{array}$$

Hence the structural sheaf $\widetilde{A}$ is flabby on the basis and we have a sheaf representation theorem $A \simeq \Gamma \widetilde{A}$; moreover this means that any MV-algebras is a subdirect product of MV-chains.

One can also process in a pointfree way as follows. The distributive lattice $\beta(A)$ indexing the basis can be equipped with its coherent topology to get a site $(\beta(A), J_{Zar})$, and we can set $\mathbf{Spec}(A) = \mathbf{Sh}(\beta(A), J_{Zar})$; for the topology J_{Zar} is finitary, we know $\mathbf{Spec}(A)$ to have enough points, which are the prime lattice ideals of $\beta(A)$, we know to be the prime MV-ideals of A.

Remark 9.2.6.7. In fact, one can recover the Belluce quotient of A as the comparison transformation

$$\begin{array}{ccc} \mathbf{MV} & \xrightarrow{\mathbf{Spec}^{\mathbf{MV}}} & \mathbb{T}_{\mathbf{MV}, J_{\mathbf{MVC}, \mathcal{M}}}\text{-}\mathbf{GTop} \\ {\scriptstyle \iota}\downarrow \swarrow {\scriptstyle \sigma} & & \downarrow {\scriptstyle \int \iota} \\ \mathbf{DLat} & \xleftarrow[\Gamma^{\mathbf{DLat}}]{} & \mathbb{T}_{\mathbf{DLat}, J_{\mathbf{Zar}, \mathcal{V}_{\iota\text{-}\mathbf{Cons}}}}\text{-}\mathbf{GTop} \end{array}$$

Ambient objects	MV-algebras
Local objects	MV-chains
Local maps	Monomorphisms
Etale maps	Epimorphisms
Local forms	Quotients at prime MV-ideals
Spectral site	Quotients at principal MV-ideals
Saturated compacts	Quotients at MV-ideals
Spectral topology	Spatial, Spectral
Structure of points	DCPO, root system
Sheaf representability	Yes

Table 9.11: Dubuc-Poveda geometry for MV-algebras

9.3 The terminal geometry

Here we investigate two quasi-examples which have all the expected properties of Diers contexts except for the accessibility condition; this latter defect does not prevent them from actually playing a central role in the spectral construction.

9.3.1 Isbell "geometry" for frames

Zariski geometry for distributive lattices restricts actually to the category of frames and frames homomorphisms

$$\begin{array}{ccc} \mathbf{LocFrm}^{\text{1-Cons}} & \hookrightarrow & \mathbf{Frm} \\ \downarrow & & \downarrow \\ \text{1-}\mathbf{LocDLat}^{\text{1-Cons}} & \hookrightarrow & \mathbf{DLat} \end{array}$$

(where, beware, the inclusion $\mathbf{Frm} \hookrightarrow \mathbf{DLat}$ is not full as frame homomorphisms have moreover to preserve arbitrary joins). Here 1-local frames are those such that the singleton $\{1\}$ is *completely* prime, that is, such that $\bigvee_I a_i = 1$ implies that some $a_i = 1$; 1-conservative frame morphisms do not raise problems. Observe there that the admissible factorization is a restriction of Grothendieck-Verdier localization for localic topoi.

Because frames do not form an accessible category, we cannot innocently consider the spectral construction for this "geometry". However, there is already a corresponding spectral-like functor: the functor of points $\mathbf{pt} : \mathbf{Frm}^{\text{op}} \to \mathbf{Top}$ sending a frame to its space of points. The topos theoretic spectrum would simply here be the localic topos over the corresponding locale, which, in term of site, can be coded by the site of all principal 1-minimal quotients. We name this Diers-like context the "Isbell geometry" because the famous $\Omega \dashv \mathbf{pt}$ adjunction is sometimes called "Isbell duality" - although this usually refers to another unrelated duality.

Ambient objects	Frames
Local objects	1-local frames
Local maps	1-conservative morphisms
Etale maps	1-minimal quotient
Local forms	1-minimal quotient at completely prime filters
Spectral site	The corresponding locale
Saturated compacts	Filters
Spectral topology	Sober
Structure of points	DCPO, possibly empty
Sheaf representability	Yes

Table 9.12: Isbell "geometry" for frames

9.3.2 The terminal "geometry" for Grothendieck topoi

We made mention at remark 7.1.3.11 of an inclusion

$$(\mathbf{FocGTop}^{\mathbf{TCo}})^{\mathrm{op}} \hookrightarrow \mathbf{GTop}^{\mathrm{op}}$$

of the opposite bicategory of local Grothendieck topoi and terminally connected geometric morphisms into the opposite bicategory of Grothendieck topoi (also known as the bicategory of *logoi*). Because any geometry admitted a canonical morphism of Diers contexts into this one given by the spectrum, we qualify it as terminal (even as a geometry, although it is not a geometry and would be initial in the convention of morphisms of geometries).

In [82][Lemma 5.4.3] we prove that local topoi have the gliding property along terminally connected geometric morphisms (up to formal duality); the etale data here seem to correspond to pro-etale geometric morphisms and there seems to be a (terminally connected, pro-etale) factorization enjoying admissibility relative to local topoi.

Here, we lack again a formal method to construct a notion of spectrum, for the bicategory of logoi, beside being by no means accessible, really has a 2-dimensional content which cannot be correctly conveyed by a 1-categorical construction. Nevertheless, we can guess what the spectrum should be like, and as for Isbell geometry, the answer is very simple: the spectrum of a logos is the corresponding topos ! Indeed recall that etale geometric morphisms are discrete; moreover

$$\mathbf{Et}/\mathcal{E} \simeq \mathcal{E}$$

The spectral topology is expected to be the *etale topology* in $\mathbf{Et}/\mathcal{E}$ externalizing the canonical topology in $\mathcal{E}$. But recall that

$$\mathcal{E} \simeq \mathbf{Sh}(\mathcal{E}, J_{can})$$

Moreover, Isbell geometry just is a restriction of this geometry along the inclusion of the category of frames into the bicategory of logoi sending a frame on the corresponding localic sheaf topos. A 1-local frame can easily be seen as inducing a local localic topos, as its focal point defines an initial point - this is also visible from the fact that $\{1\}$ is completely prime, which exactly says that $L[1,-]$ preserves colimits - here, arbitrary joins. Similarly, requiring a localic geometric morphism to be terminally connected amounts to requiring the corresponding frame morphism to be 1-connected as a global element for $1 \to f^*(a)$ is nothing but a witness of $f^*(a) = 1$. This gives us something we would like to see as a morphism of Diers contexts (disregarding accessibility issues)

$$\begin{array}{ccc} \mathbf{LocFrm}^{\mathbf{1\text{-}Cons}} & \hookrightarrow & \mathbf{Frm} \\ \downarrow & & \downarrow \\ (\mathbf{LTop}^{\mathbf{TCo}})^{\mathrm{op}} & \hookrightarrow & (\mathbf{GTop})^{\mathrm{op}} \end{array}$$

In fact, in almost all examples above, the spectral site is made of epimorphisms, hence form a poset: the spectrum is hence localic, so that actually it factorizes through this morphism of Diers

contexts. The only counter-example to this is the Etale Geometry for commutative rings, since etale morphisms of commutative rings under a given ring do not form a poset.

Moreover, the localic reflection also defines a kind of morphism of Diers contexts: the localic reflection of a local topos is a 1-local frames, because if the terminal object in a Grothendieck topos 1 preserves colimits, then in particular it preserves joins of subobjects, which exactly means that 1 is inaccessible by infinite joins. Hense we also have

$$\begin{array}{ccc} \mathbf{LocFrm}^{\mathbf{1\text{-}Cons}} & \hookrightarrow & \mathbf{Frm} \\ \uparrow & & \uparrow \mathcal{L} \\ (\mathbf{LTop}^{\mathbf{TCo}})^{\mathrm{op}} & \hookrightarrow & (\mathbf{GTop})^{\mathrm{op}} \end{array}$$

which would be the only example here of a non-faithful morphism of Diers contexts, if it was a morphism of Diers contexts...

In this case, we cannot ensure what the analogs of the "saturated compacts" for a logos $\mathcal{E}$ are, but in some sense, we conjecture they correspond to *cosheaves*, that are cocontinuous functors $\mathcal{E} \to \mathcal{S}$.

This geometry should be somehow involved in the main result of [65]. In particular it is known that the category of points of a Grothendieck topos is an accessible category with filtered colimits - whenever it is not empty ! The topological information is recorded in terms of a bounded ionad attached to the category of points, see [65] - we shall also give a word on this in the epilogue.

Its associated sheaf representation result should also be closely connected to Awodey theorem exhibiting any topos as the topos of global sections of a stack in local topoi [6] - the nature of the underlying space not being yet clear to us.

Ambient objects	Logoi
Local objects	Local logoi
Local maps	(Dual of) terminally connected geometric morphism
Etale maps	(Dual of) Pro-etale geometric morphisms ?
Local forms	Grothendieck-Verdiers localizations
Spectral site	The dual topos
Saturated compacts	Cosheaves ?
Spectral topology	The ionad of points
Structure of points	Accessible category with filtered colimits (or empty)
Sheaf representability	Yes ? (Awodey)

Table 9.13: The terminal geometry for logoi

Chapter 10

Epilogue: Toward a geometry of semantics

In this last chapter, we discuss the spectral-like nature of first order syntax-semantics adjunctions, guided by the spectral account we gave of their corresponding propositional versions in the last chapter.

In the first section, we discuss the content of this correspondence and give a topological description of the semantics of first order dualities, driven by the intuition from ionads and our comparison with the different geometries for Stone-like dualities.

In the second section, we describe a special and seemingly previously unknown bifactorization system on **Lex**, the *(focalization, terminally connected)* system, which we construct from a close examination of Grothendieck-Verdier localizations - whose semantics value is also discussed. We conjecture this system to be part of a notion of 2-geometry on **Lex**, and that it may also be involved in an eventual notion of 2-geometries for the other syntax-semantics dualities.

However, though some material is already present in several parallel ongoing works, we shall not develop here the complete theory of 2-spectrum as it would represent too huge an amount of material to this thesis, and moreover needs still important efforts of clarification and stabilisation.

10.1 Syntax-semantics adjunctions

In this section we recall the stratification of logics, not only in different fragments characterized by theirs connectors and inference rules, but also in term of dimension, detailing in particular the correspondence between the propositional and first order logics. We also give an in-depth description of the semantics of those fragments from a geometric point of view. We saw in the last chapter that the algebraic semantics of propositional logics were all associated with a geometry and a spectral construction accounting for their semantics: here, we try to exhibit similar spatial behavior from which one could deduce the data involved in appropriate notions of 2-geometries categorifying the propositional ones.

10.1.1 Correspondence between propositional and first order dualities

10.1.1.1. Each fragment of propositional logic comes with a corresponding *algebraic semantics*, a correspondence between propositional theories and objects in a variety of ordered structures whose generic identities mirror the deduction rules of a the fragment of logic they live in:

- *cartesian* propositional theories, whose formulas are built from finite conjunctions $\wedge$ and the truth value $\top$, correspond $\wedge$-*semilattices*;
- *coherent* propositional theories, whose formulas involve both finite conjunctions $\wedge$ and disjunctions $\vee$ as well as both truth values $\top$ and $\bot$, correspond to *bounded distributive lattices*;
- *first order* propositional theories are coherent propositional theories involving also the implication symbol $\Rightarrow$, and they correspond to *Heyting algebras*;

– *boolean coherent* propositional theories are those coherent theories that also involve the negation symbol $\neg$, and they correspond to *boolean algebras*;

– *geometric* propositional theories, whose formulas involve finite conjunctions, arbitrary disjunctions $\bigvee$, and both truth values, correspond to *frames*.

10.1.1.2. But each of those varieties is well known to be at one side of a corresponding *propositional duality*, with, on the other side, a certain category of topological spaces one can see as the spaces of models of those theories:

– the category of $\wedge$-semillatices is dual to the category of *Hofmann-Mislove-Stralka spaces* as explained at section 9.2.4: this is the content of *Jipsen-Moshier duality*

– the category of bounded distributive lattices is dual to the category of *Stones spaces*, this is the content of *Stone duality*.

– the category of Heyting algebras is dual to the category of *Esakia spaces*: this is the content of *Esakia duality*.

– the category of boolean algebras is dual to the category of *boolean spaces*

– the category of frames is related to the category of topological spaces through the so called "omega-point" adjunction; in particular, it reduces on a duality between the subcategories of spatial frames and sober topological spaces.

10.1.1.3. Except for the omega-point duality, all the duality above can be described through a corresponding geometry and spectral duality as detailed in the second part of our previous chapter. In fact, we saw that those geometries were all related by morphisms of geometries, and derived from Jipsen-Moshier duality. The same factorization data were used, consisting of 1-minimal quotients for etale maps and 1-conservative morphisms for local maps; though no specification of local objects was done in Jipsen-Moshier duality, the other one used each time a notion of local lattice, with a unique minimal prime filter. Local forms determined focal components of the spectrum at the corresponding point, and basic etale maps corresponded to the inclusion of basic compact open of the spectral topology.

Remark 10.1.1.4. Though it is clearly spectral in nature, the case of the omega-point adjunction is more subtle; in fact, the category of frames is not locally finitely presentable, not even accessible, as its function symbols involve unbounded arities. However the geometric data of Stone geometry seems to be suited to give a geometric account of this duality: this lets think that they are an instance of a generalized spectral construction allowing the category of ambient objects to be more wild. Because frames are monadic over set, we conjecture that this would involve a monadic spectral construction.

10.1.1.5. Now, it is well known that all those propositional dualities have a *first order* counterpart, consisting of formulas with free variables in a given signature; in fact, one may observe moreover that the list diversifies with at least two new noticeable fragments. Each of those first order dualities comes with its notion of *syntactic category*, which belongs to a certain 2-category:

– *finite limit theories*, as we have met them repeatedly in this work, are those whose formulas involve only finite conjunctions, true and existentials with proofs of uniqueness. Their syntactic categories are the small lex categories **Lex**; their propositional counterpart are cartesian propositional theories;

– *cartesian theories* (also known as *Lawvere theories*) are those whose formulas involve only only finite conjunctions and true; their syntactic categories are small categories with finite product **Cart**.

– *regular* theories are those whose formulas involve only finite conjunctions, true and *arbitrary* existentials; their syntactic categories are *small regular categories*, whose 2-category will be denoted **Reg**

– *coherent* theories are those whose formulas involve only finite conjunctions and disjunction, existential, and both truth values. Their syntactic categories are *coherent categories*, whose 2-category will be denoted **Coh**

– *first order* theories are coherent theories involving also the implication symbol $\Rightarrow$; their syntactic categories are the *small Heyting categories*

– *geometric* theories are those involving finite conjunction, arbitrary disjunction, existentials and both truth values. Their syntactic categories are *small geometric categories*.

10.1.1.6. It is conjectured that each first-order duality has a corresponding *syntax-semantics duality* classifying the 2-categories of models of its theories. Those are already known in some case:

– as we saw this at the begining of this thesis, **Lex** is dual to the 2-category of locally finitely presentable categories as detailed in Gabriel-Ulmer duality.

– models of small cartesian theories form *algebraic varieties*; in particular, this reduces to a 2-categorical duality between *Cauchy-complete* small cartesian categories and algebraic varieties, this is Lawvere-Adamek-Rosicky-Vitale duality, see [3]

– there exists also a duality-like result for regular categories which may refer to as Kuber-Rosicky duality, see [58]

– although there is not currently an explicit description of the class of accessible categories corresponding to categories of models of coherent theories, it is the topic of several constructions as *Awodey-Forssell duality*, see [5], which is in some sense the Galois theory associated to this duality, or Makkai duality describing categories of models of coherent theories as *ultracategories*, see [71] and [**lurie2018ultracategories**] - this seems to correspond to a 2-categorical *Priestley duality* as ultracategories are the categorification of compact Hausdorff spaces.

– more recently, the omega-point adjunction was categorified into a 2-adjunction between the 2-categories of Grothendieck topoi and *bounded ionads*, and we shall refer to this duality as *Di Liberti duality*, see [65].

Stone-like dualities	Syntax-semantics dualities
Jipsen-Moshier: $\wedge$-$\mathbf{SLat}_1^{\mathrm{op}} \simeq \mathbf{HMS}$	Gabriel-Ulmer: $\mathbf{Lex}^{\mathrm{op}} \simeq \mathbf{LFP}$
-	Lawvere-Adamek-Rosicky-Vitale: $\mathbf{Cart}_{cc}^{\mathrm{op}} \simeq \mathbf{Alg}$
-	Kuber-Rosický for $\mathbf{Reg}/\mathbf{Ex}$
Stone: $\mathbf{DLat}^{\mathrm{op}} \simeq \mathbf{Stone}$	Awodey-Forsell, Lurie-Makkai for coherent theories
Esakia: $\mathbf{Heyt}^{\mathrm{op}} \simeq \mathbf{Esa}$	Duality for Heyting categories ?
Duality for frames: $\mathbf{Frm}^{\mathrm{op}} \leftrightarrows \mathbf{Top}$	Di Liberti duality: $\mathbf{BIon} \leftrightarrows \mathbf{GTop}$

Table 10.1: Correspondence between propositional and first order Syntax-semantics dualities

Remark 10.1.1.7. As the 1-categorical, first order world is richer than the posetal propositional world, additional fragments of first order logic appear that have seemingly no propositional counterpart as the duality between cartesian cartegories and algebraic categories or the duality for regular categories. In the first case, as both finite products and finite limits reduce on meets, while on the other hand siftedness and filteredness both coincide with directedness for posets, Gabriel-Ulmer and Adamek-Lawere-Rosicky-Vitale both restrict on Jipsen-Moshier duality in the posetal world. For Kuber-Rosicky duality, it is not clear whether there is a notion of "regular $\wedge$-semilattice" distinguished from ordinary semilattices and if this corresponds to something different of Jipsen-Moshier.

10.1.2 First order topologies

10.1.2.1. Let us give some topological interpretation - which is also part of a certain formulation of the *ionadic* approach - of the semantics associated to first order theories.

Suppose that $\mathbb{T}$ is some geometric theory classified by a subcanonical site $(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$ so that $\mathcal{S}[\mathbb{T}] \simeq \mathbf{Sh}(\mathcal{C}_{\mathbb{T}}, J_{\mathbb{T}})$. Then we have an *evaluation functor*

$$\mathcal{C}_{\mathbb{T}} \xrightarrow{\mathrm{ev}} [\mathbb{T}[\mathcal{S}], \mathcal{S}]$$

sending any formula in context $\{\overline{x}, \phi\}$ to the functor

$$\mathbb{T}[\mathcal{S}] \xrightarrow{\mathrm{ev}_{\{\overline{x},\phi\}}} \mathcal{S}$$

associating to a model X seen as a $J_{\mathbb{T}}$-continuous lex functor $X : \mathcal{C}_{\mathbb{T}} \to \mathcal{S}$ the evaluation $X(\{\overline{x}, \phi\})$ at $\{\overline{x}, \phi\}$, and a morphism of $\mathbb{T}$-models $f : X_1 \to X_2$ seen as a natural transformation the associated component $f_{\{\overline{x},\phi\}} : X_1(\{\overline{x}, \phi\}) \to X_2(\{\overline{x}, \phi\})$ at $\{\overline{x}, \phi\}$.

10.1.2.2. Then by Grothendieck construction any object $\{\overline{x}, \phi\}$ in $\mathcal{C}_{\mathbb{T}}$ defines a discrete opfibration over $\mathbb{T}[\mathcal{S}]$

$$\int \mathrm{ev}_{\{\overline{x},\phi\}} \to \mathbb{T}[\mathcal{S}]$$

which is the discrete opfibration of the points of the etale geometric morphism

$$\mathcal{S}[\mathbb{T}]/\text{ɬ}_{\{\overline{x},\phi\}} \to \mathcal{S}[\mathbb{T}]$$

Moreover this construction extends to arbitrary objects of $\mathcal{S}[\mathbb{T}]$, coding for the inclusion of opens and more generally for the etale geometric morphisms over $\mathcal{S}[\mathbb{T}]$. In this interpretation, we can see elements a of the evaluations $X(\{\overline{x}, \phi\})$ as "witnesses that X is in ϕ", seeing X as a point, and $\{\overline{x}, \phi\}$ as a compact open in a basis for a topology on the space of models.

Here the lifting property in the opfibration explicitly says that (basic) opens are up-closed for the specialization order :

$$(X, a) \xrightarrow{f} (X', \mathrm{ev}_{\{\overline{x},\phi\}}(f)(a))$$

$$X \xrightarrow{\quad f \quad} X'$$

This expresses indeed that a witness a of ϕ in X - or topologically, that "X is in ϕ", is sent to a witness that X' is in ϕ.

Then the *contravariant* representables $\text{ɬ}_{\{\overline{x},\phi\}}$ are to be seen as basic compact opens generating the topology on models, and $\mathcal{C}_{\mathbb{T}}$ as an indexing category indexing the basis through the covariant Yoneda embedding

$$\mathcal{C}_{\mathbb{T}} \xhookrightarrow{\text{ɬ}} \mathcal{S}[\mathbb{T}]$$

Remark 10.1.2.3. Observe we need $J_{\mathbb{T}}$ to be subcanonical to ensure that the Yoneda embedding corestricts correctly to the sheaf topos. But this is not restrictive as any Grothendieck topos admits a lex, subcanonical small site of presentation. So we can suppose that $\mathbb{T}$ is chosen in such a way that its syntactic site is one of those sites. In fact, though we started with a choice of a geometric theory, the topological properties we are discussing exists in fact in the underlying topos, and hence are Morita-invariant.

10.1.2.4. In particular, let us recall the basis of the topologies associated to theories in the different fragment of logic:

- A finite limit $\mathbb{T}$ is classified by some $\widehat{L}$ with L a small lex category; no syntactic topology is needed, and models are
$$\mathbf{pt}(\widehat{L}) = \mathbf{Lex}[L, \mathcal{S}]$$
- A regular theory $\mathbb{T}$ is classified by some regular topos $\mathbf{Sh}(\mathcal{C}, J_{reg})$ with $\mathcal{C}$ a small regular category and J_{reg} the associated regular coverage generated by single regular epimorphisms. Models are regular functors, that are lex functors preserving regular epimorphisms:
$$\mathbf{pt}(\mathbf{Sh}(\mathcal{C}, J_{reg})) = \mathbf{Reg}[\mathcal{C}, \mathcal{S}]$$
- A coherent theory $\mathbb{T}$ is classified by some $\mathbf{Sh}(\mathcal{C}, J_{coh})$ with $\mathcal{C}$ a coherent category and J_{coh} the coherent coverage generated by finite jointly regular-epimorphic families. Models are coherent functors, that are those lex functors preserving regular epimorphisms and finite coproducts
$$\mathbf{pt}(\mathbf{Sh}(\mathcal{C}, J_{coh})) = \mathbf{Coh}[\mathcal{C}, \mathcal{S}]$$

All those sites are subcanonical: hence any $\{\overline{x}, \phi\}$ of $\mathcal{C}$ can be seen as a open through the representable $\text{ɬ}_{\{\overline{x},\phi\}}$ which is always a sheaf for each of the three cases above.

10.1.2.5. Moreover, an important question of syntax-semantics dualities is the reverse problem of reconstructing the syntactic site - hence the theory up to Morita equivalence - from the category of models.

– For a small lex category $\mathcal{C}$, Gabriel-Ulmer duality says that $\mathcal{C}$ can be retrieved from $\mathbf{Lex}[\mathcal{C},\mathcal{S}]$:

$$\mathcal{C} \simeq \mathbf{LFP}[\mathbf{Lex}[\mathcal{C},\mathcal{S}],\mathcal{S}]$$

But locally finitely presentable functors are those that preserve filtered colimits and small limits. The latter condition says they must be representable, that is why we can retrieve $\mathcal{C}$ from a subcategory of models, while the former condition precises which ones: the finitely presented ones. The classifying topos is the presheaf topos $[\mathcal{C}^{\mathrm{op}},\mathcal{S}]$.

– For Lawvere theories, one can get back (up to Cauchy completion) $\mathcal{C}$ with

$$\mathcal{C} = \mathbf{Alg}[\mathbf{Cart}[\mathcal{C},\mathcal{S}],\mathcal{S}]$$

where **Alg** is for functors preserving small limits and sifted colimits, and **Cart** for functors preserving finite products. Then those are the representables amongst functors preserving sifted colimits, which are strictly more than finitary functors; as $\mathbf{Cart}[\mathcal{C},\mathcal{S}] \simeq \mathbf{Sind}(\mathcal{C}^{\mathrm{op}})$ is the sifted colimit-completion, the classifying topos is

$$\mathbf{Sift}[\mathbf{Cart}[\mathcal{C},\mathcal{S}],\mathcal{S}] \simeq [\mathcal{C}^{\mathrm{op}},\mathcal{S}]$$

because filtered colimits are in particular sifted.

– For an exact category $\mathcal{C}$, the category of models is $\mathbf{Reg}[\mathcal{C},\mathcal{S}]$, and by Kuber-Rosický duality we can retrieve $\mathcal{C}$ as

$$\mathcal{C} \simeq \mathbf{Fin}^{\prod}[\mathbf{Reg}[\mathcal{C},\mathcal{S}],\mathcal{S}]$$

while the classifying topos is $\mathbf{Sh}(\mathcal{C}, J_{reg})$. This is an instance of the *Barr embedding theorems*.

– For a coherent category $\mathcal{C}$, the category of set-valued models is the category of coherent functors $\mathbf{Coh}[\mathcal{C},\mathcal{S}]$. From [68] it inherits a pointwise *ultrastructure*. Hence for a set S in $\mathcal{S}$ and $\mu \in \beta S$ and a family M_s in $Mod(\mathcal{C})$, $\int_S M_s d\mu = \mathrm{colim}_{S' \in \mu} \prod_{s \in S'} M_s$. Lurie formula expresses $\mathcal{C}$ as the category of ultrafunctors (that are functors preserving ultraproduct)

$$\mathcal{C} \simeq \mathbf{Ult}[\mathbf{Coh}[\mathcal{C},\mathcal{S}],\mathcal{S}]$$

while the classifying topos is $\mathbf{Sh}(\mathcal{C}, J_{Coh}) \simeq \mathbf{LUlt}[\mathbf{Coh}[\mathcal{C},\mathcal{S}],\mathcal{S}]$, the category of left ultrafunctors. Beware the following subtleties:

  – while in the case above, we took *full* subcategories of the functor categories, the category $\mathbf{LUlt}[\mathbf{Coh}[\mathcal{C},\mathcal{S}],\mathcal{S}]$ has only specific kind of natural transformations as morphisms. Moreover, those left ultrafunctors are not just functors with property, they are functors with *structure* given by their left map

$$F(\int_S M_s d\mu) \overset{\sigma_{M_s}}{\to} \int_S F(M_s) d\mu$$

  However this restriction is trivialized for natural transformations of ultrafunctors, so that $\mathbf{Ult}[\mathbf{Coh}[\mathcal{C},\mathcal{S}],\mathcal{S}]$ is a full subcategory of the category of functors.

  – Secondly, ultrafunctors are not required to preserve filtered colimit (nor product); if they do so, they are ultrafunctors because the ultraproducts of of $Mod(\mathcal{C})$ are pointwise, but there may be more ultrafunctors than functors preserving filtered colimits and products. The situation is even worse concerning the classifying topos because now its object are not functors preserving something but something strictly weaker.

10.1.2.6. In the case of limit-doctrines as **Lex** or **Cart**, each formula in context $\{\overline{x},\phi\}$ of the syntactic site $\mathcal{C}_{\mathbb{T}}$ not only defines a basic compact open $\not\perp_{\{\overline{x},\phi\}}$, but also a point

$$\mathcal{C}_{\mathbb{T}} \overset{\exists_{\{\overline{x},\phi\}}}{\longleftarrow} \mathcal{S}$$

as the corepresentable is automatically left exact.

Hence one could ask more generally when objects of the syntactic site also define *points*, that is, when a corepresentable $\exists_{\{\overline{x},\phi\}}$ is a point. Whenever it is, it defines a finitely presented model K_ϕ, which could be seen as a "compact point". More exactly, it will be a case of a *saturated compact* that is also open, while the basic open $\not\perp_{\{\overline{x},\phi\}}$ is a basic open which is also a saturated compact and has a focal point.

10.1.2.7. Let us analyse in each fragment of logics which ones of the basic compact opens have a focal point:

- In **Lex**, any corepresentable $\exists_c$ is lex. Hence any compact open has a focal point, which is compact, because they correspond to morphisms of locally finitely presentable categories toward $\mathcal{S}$, and each of them is representable by a finitely presented object.
- In **Cart**, basic compact opens are algebraic functors toward $\mathcal{S}$, which are also representable by a perfectly presented object.
- In **Reg**, $\exists_c$ is regular when c is projective in C: only compact opens associated to a projective c have a focal point. In particular there is an initial model when 1 is projective. Moreover, not all basic compact opens are representable, as regular functors do not necessarily preserve all small limits.
- In **Coh**, $\exists_c$ is coherent if c is *indecomposable* (connected projective): only compact opens for a indecomposable c have a focal point. In particular, there is an initial model when 1 is indecomposable. Again, not all basic compact open are representable.

This can be summed up as the following:

$\mathbb{D}$	$\mathcal{S}[\mathbb{T}]$	$\mathbb{T}[\mathcal{S}]$	Basic compact opens	Basic points	Basic opens with focal points
Lex	$[C^{op}, \mathcal{S}]$	$\mathbf{Lex}[C, \mathcal{S}]$	$C \simeq \mathbf{LFP}[\mathbf{Lex}[C, \mathcal{S}], \mathcal{S}]$	f.p. $\exists_c$	All
Cart	$[C^{op}, \mathcal{S}]?$	$\mathbf{Cart}[C, \mathcal{S}]$	$\mathcal{C} \simeq \mathbf{Alg}[\mathbf{Cart}[\mathcal{C}, \mathcal{S}], \mathcal{S}]$	p.p. $\exists_c$	All
Reg	$\mathbf{Sh}(C, J_{reg})$	$\mathbf{Reg}[C, \mathcal{S}]$	$\mathcal{C} \simeq \mathbf{Fin}^{\prod}[\mathbf{Reg}[\mathcal{C}, \mathcal{S}], \mathcal{S}]$	?	c projective
Coh	$\mathbf{Sh}(C, J_{coh})$	$\mathbf{Coh}[C, \mathcal{S}]$	$\mathcal{C} \simeq \mathbf{Ult}[\mathbf{Coh}[\mathcal{C}, \mathcal{S}], \mathcal{S}]$	?	c indecomposable

Table 10.2: First order topologies and their basic compact opens

A theory $\mathbb{T}$ has an initial model when the terminal object 1 of C is local for the corresponding syntactic topology. Then the classifying topos is a *local* topos. This notion is related to the notion of Grothendieck-Verdier localization we have already met when studying the spectral site. We want now to extract "etale"-like information form the semantics of lex

10.2 Focalizations

In this section, we develop a bifactorization system on **Lex** deduced from a peculiar approach to the Grothendieck-Verdier localizations. We introduce here the notion of *focalization*, which will be to Grothendieck-Verdier localizations what etale arrows are to local forms. We explain then how this bifactorization system is related to the semantics of finite limit theories and Gabriel-Ulmer duality. The way we deduce a 2-dimensional factorization system from Gabriel-Ulmer - and which suspect to be involved in the semantics of the other fragment of logics - should be compared with the way the factorization system (1-Minimal quotient, 1-conservative) was obtained in the study of Jipsen-Moshier duality and then reused for the other Stone-like dualities.

In this section, we will speak of *bifactorization systems*. Here we mean factorization up to an equivalence. A (stricter) definition of 2-factorization system was introduced in [35], to which one can refer; to avoid too much formal definitions and technicalities in this epilogue of a more gnostic purpose, we deliberately cut the theory of bifactorization systems we originally planed to be part of this work: one can find it in [82][Section 3], as well as elements for a 2-dimensional theory of stable functors.

10.2.1 Grothendieck-Verdier localization

We saw that local objects returned as spectrum a local topos over their base. This is connected to the idea that the identity of a local object is an initial point in the category of local forms under it. In our topological interpretation of syntax-semantics dualities, a similar connection appears in the sense that any model of a theory will define through its comma a localization of the category

of models by forcing it to become initial.

10.2.1.1. Let $\mathcal{E}$ be a Grothendieck topos with presentation site $(\mathcal{C}, J)$ and $x : \mathcal{S} \to \mathcal{E}$ a point. Then x^*, as a flat functor, has a cofiltered category of elements $\int x^*$ whose objects are (c, a) with c in $\mathcal{C}$ and $a \in x^*(c)$. Moreover, at each such element, one has a factorization

$$\begin{array}{ccc} & & \mathcal{E}/\text{ɬ}_c \\ & \overset{\ulcorner a \urcorner}{\nearrow} & \downarrow c_* \\ \mathcal{S} & \xrightarrow[x]{} & \mathcal{E} \end{array}$$

induced from the section of the pullback coding for the name of the element a

$$\begin{array}{ccccc} & & \mathcal{S}/x^*(c) & \longrightarrow & \mathcal{E}/\text{ɬ}_c \\ & \overset{\ulcorner a \urcorner}{\nearrow} & \downarrow & \lrcorner & \downarrow c_* \\ \mathcal{S} & = & \mathcal{S} & \xrightarrow[x]{} & \mathcal{E} \end{array}$$

Then the Grothendieck-Verdiers localization is obtained as the cofiltered pseudolimit

$$\mathcal{E}_x \simeq \underset{(c,a)\in\int x^*}{\text{pslim}} \mathcal{E}/\text{ɬ}_c$$

The universal cone given by the $\int x^*$-indexed etale maps $\mathcal{E}/\text{ɬ}_c \to \mathcal{E}$ produces moreover a universal map $p_x : \mathcal{E}_x \to \mathcal{E}$ which is part of a factorization

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\quad x \quad} & \mathcal{E} \\ & \overset{1_x}{\searrow} \quad \simeq \quad \overset{p_x}{\nearrow} & \\ & \mathcal{E}_x & \end{array}$$

It satisfies the universal property that its points form the comma category under the point x

$$\mathbf{pt}(\mathcal{E}_x) \simeq x \downarrow \mathbf{pt}(\mathcal{E})$$

and the factorization above is send to the following diagram in the 2-category $\mathbf{Acc}_\omega$ of accessible categories with finitely filtered colimits together with finitely accessible functors

$$\begin{array}{ccc} * & \xrightarrow{\quad x \quad} & \mathbf{pt}(\mathcal{E}) \\ & \overset{1_x}{\searrow} \quad \simeq \quad \overset{\text{cod}}{\nearrow} & \\ & x \downarrow \mathbf{pt}(\mathcal{E}) & \end{array}$$

which corresponds to the comprehensive factorization of the name of x, as 1_x is initial (as pointing the initial object of $x \downarrow \mathbf{pt}(\mathcal{E})$) while cod is a discrete opfibration.

Remark 10.2.1.2. If now $\mathcal{E} \simeq \mathcal{S}[\mathbb{T}]$ with $(\mathcal{C}_\mathbb{T}, J_\mathbb{T})$ the corresponding syntactic category, a point B is a set-valued model in $\mathbb{T}[\mathcal{S}]$, and its category of elements contains all the elements in interpretation of formulas in contexts $[\![\overline{x}, \phi]\!]_B$, that is $\int B$ consists in the pairs $(\{\overline{x}, \phi\}, \overline{a})$ with $B \models \phi(\overline{a})$.

10.2.1.3. In the doctrine **Lex** of finite limit theory, Grothendieck-Verdier localization takes the following form. The classifying topos has the form of a presheaf topos on a small lex category $\mathcal{S}[\mathbb{T}] \simeq \widehat{\mathcal{C}}$ and points are objects of the locally finitely presented category $\mathbb{T}[\mathcal{S}] \simeq \mathbf{Lex}[\mathcal{C}, \mathcal{S}]$.

At a point $A : \mathcal{C} \to \mathcal{S}$, we saw that the coslice $A \downarrow \mathbb{T}[\mathcal{S}]$ still is locally finitely presentable, and has as generator of finitely presented objects pushouts under A of finitely presented maps. Moreover the codomain functor

$$A \downarrow \mathbb{T}[\mathcal{S}] \xrightarrow{\text{cod}} \mathbb{T}[\mathcal{S}]$$

is a morphism of locally finitely presentable categories and its left adjoint is the coproduct functor sending B in $\mathbb{T}[\mathcal{S}]$ to the coproduct inclusion $q_B^A : A \to A + B$. In particular this functor restricts

to the categories of finitely presented objects as 0 is finitely presented so for any finitely presented $\exists_c$ the pushout inclusion

$$\begin{array}{ccc} 0 & \xrightarrow{!} & \exists_c \\ {\scriptstyle !}\downarrow & \ulcorner & \downarrow \\ A & \xrightarrow[q_K^A]{} & A + \exists_c \end{array}$$

is in $(A \downarrow \mathbb{T}[\mathcal{S}])_\omega$. In the following we denote the opposite category $(A \downarrow \mathbb{T}[\mathcal{S}])_\omega^{\mathrm{op}}$ as $\mathcal{C}_A$: this is a lex category, and $A \downarrow \mathbb{T}[\mathcal{S}] \simeq \mathbf{Lex}[\mathcal{C}_A, \mathcal{S}]$.

Then the lex functor A admits a factorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{A} & \mathcal{S} \\ & {\scriptstyle n_A}\searrow \quad \simeq \quad \nearrow{\scriptstyle \ulcorner 1_A \urcorner} & \\ & \mathcal{C}_A & \end{array}$$

where $\ulcorner 1_A \urcorner$ is the name of 1_A, the initial object of $A \downarrow \mathbb{T}[\mathcal{S}]$, or equivalently, the the initial object of the hom category $\mathbf{Lex}[\mathcal{C}_A, \mathcal{S}]$, which sends an object of $\mathcal{C}_A$ of the form

$$\begin{array}{ccc} \exists_{c_1} & \xrightarrow{\exists_u} & \exists_{c_2} \\ {\scriptstyle a}\downarrow & \ulcorner & \downarrow \\ A & \xrightarrow[a_* \exists_u]{} & a_* \exists_{c_2} \end{array}$$

(with $u : c_2 \to c_1$ in $\mathcal{C}$) to the set

$$\begin{aligned} \exists_{1_{C_A}}(a_* k) &= \mathcal{C}_{\mathcal{A}}[1_{\mathcal{C}_A}, a_* k] \\ &= A \downarrow \mathbf{Lex}[\mathcal{C}, \mathcal{S}][a_* k, 1_A] \end{aligned}$$

of sections of the pushout map $a_* k$

$$\begin{array}{ccc} \exists_{c_1} & \xrightarrow{\exists_u} & \exists_{c_2} \\ {\scriptstyle a}\downarrow & \ulcorner & \downarrow \\ A & \xrightarrow{a_* \exists_u} & a_* \exists_{c_2} \\ & \searrow & \downarrow{\scriptstyle b} \\ & & A \end{array}$$

In particular, $\ulcorner 1_A \urcorner$ sends coproducts inclusions $A \to A + \exists_c$ to the set of their sections, which are exactly in bijections with the set of maps $a : \exists_c \to A \simeq A(c)$: whence the invertible 2-cell $\ulcorner 1_A \urcorner n_A \simeq A$.

Remark 10.2.1.4. Beware that the factorization above does not occur in **Lex**, but in a larger category **LEX** of large lex categories, so that Gabriel-Ulmer duality does not apply to the right part of this triangle as $\mathcal{S}$ is not the generator of a locally finitely presentable category.

10.2.1.5. In the case where we consider a finitely presented model, that is for $\exists_c : \mathcal{C} \to \mathcal{S}$ for c in $\mathcal{C}$, then the generator of finitely presented object simplifies. Indeed, for c in $\mathcal{C}$, any pushout as below

$$\begin{array}{ccc} \exists_{c_2} & \xrightarrow{\exists_u} & \exists_{c_1} \\ {\scriptstyle \exists_v}\downarrow & \ulcorner & \downarrow \\ \exists_c & \longrightarrow & \exists_{v*} \exists_{c_1} \end{array}$$

is in fact a finitely presented object as

$$\exists_c +_{\exists_{c_2}} \exists_{c_1} \simeq \exists_{c \times_{c_2} c_1}$$

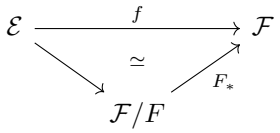
that is, is the image of the underlying pullback: hence any object of $\mathcal{C}_{\exists_c}$ arises from an arrow in $\mathcal{C}/c$. Hence we have an equivalence

$$\mathcal{C}_{\exists_c} \simeq \mathcal{C}/c$$

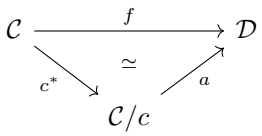
This is in particular to relate with the expression of the etale geometric morphism at a representable in the presheaf topos

$$\widehat{\mathcal{C}}/\text{ɥ}_c \simeq \widehat{\mathcal{C}/c}$$

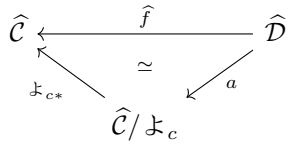
10.2.1.6. In order to determine the way the bifactorization processes, we must first understand what data are encoded in bifactorizations along the pullback functors c^*. But we know from the property of etale geometric morphisms that bifactorizations



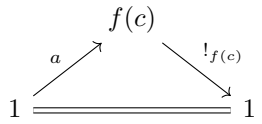
code for global elements $a : 1 \to f^*F$. In particular, applying this to presheaf topoi over small lex categories, we get that a bifactorization



inducing a bifactorization between the corresponding presheaf toposes



which exhibits a as the name of a global element $a : 1 \to f^*(c)$ in $\widehat{D}$, which comes uniquely by fullness of the Yoneda embedding from a section of the terminal map

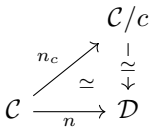


To sum up, we have the following isomorphism of homsets

$$\mathcal{C} \downarrow \mathbf{Lex}[c^*, f] \simeq \mathcal{C}[1, f(c)]$$

In particular, this induces that the lex functors $c^* : \mathcal{C} \to \mathcal{C}/c$ are discrete morphisms in **Lex**, which mirrors the fact that etale geometric morphisms are discrete morphisms in **GTop**.

Definition 10.2.1.7. Let A be in $\mathbf{Lex}[\mathcal{C}, \mathcal{S}]$; the *focalization at* A is the induced lex functor $n_A : \mathcal{C} \to \mathcal{C}_A$. More generally, a lex functor n will be called a *focalization* if there is an invertible 2-cell



with an equivalence $\mathcal{C}/c \simeq \mathcal{D}$ and $n_A : \mathcal{C} \to \mathcal{C}_A$ the focalization at some model A.

Proposition 10.2.1.8. *The focalization at a lex functor $A : \mathcal{C} \to \mathcal{S}$ is the filtered pseudocolimit in $\mathcal{C} \downarrow \mathbf{Lex}$*

$$n_A \simeq \underset{(c,a)\in(\int A)^{\text{op}}}{\text{pscolim}} \; c^*$$

Consequently, we have a 1-filtered pseudocolimit in **Lex**

$$\mathcal{C}_A \simeq \underset{(c,a)\in(\int A)^{\text{op}}}{\text{pscolim}} \; \mathcal{C}/c$$

Proof. This is the underlying lex content of Grothendieck-Verdier localization. In fact, this is a special instance of theorem 1.1.4.3 where $\mathcal{V}^{\text{op}} = \mathcal{C}^2$. □

10.2.1.9. In particular, pullback functors under a lex category are focalizations: indeed, as we saw above, any c induce a finitely presented object $\exists_c$ in $\mathbf{Lex}[\mathcal{C}, \mathcal{S}]$, and we have $\mathcal{C}/c \simeq \mathcal{C}_{\exists_c}$; in the following we call them the *basic focalizations*. In fact, they will form the "etale generator" of focalizations:

Proposition 10.2.1.10. *The basic focalizations $c^* : \mathcal{C} \to \mathcal{C}/c$ are finitely presented in $\mathcal{C} \downarrow \mathbf{Lex}$.*

Proof. Let I be a bifiltered 2-category and $f_{(-)} : I \to \mathcal{C} \downarrow \mathbf{Lex}$ a 2-functor, together with a bifactorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\underset{i\in I}{\text{bicolim}}\ f_i} & \underset{i\in I}{\text{bicolim}}\ \mathcal{C}_i \\ & {\scriptstyle c^*} \searrow \quad \simeq \quad \nearrow {\scriptstyle a} & \\ & \mathcal{C}/c & \end{array}$$

Then we have in bicolim $_{i\in I}\mathcal{C}_i$ a global element

$$1 \xrightarrow{a} (\underset{i\in I}{\text{bicolim}}\ f_i)(c)$$

But we know that filtered bicolimit in **Lex** are computed as in **Cat** - where we can choose the pseudocolimit which is the localization of the oplaxcolimit. The terminal object of bicolim $_{i\in I}\mathcal{C}_i$ admits any $(i, 1_{\mathcal{C}_i})$ as a representent as the transitions functors are lex, hence preserve terminal objects. On the other side, (bicolim $_{i\in I} f_i)(c)$ is the class of some $(i, f_i(c))$ for some i in I; and then a arises from some $a : 1_{\mathcal{C}_i} \to f_i(c)$ in $\mathcal{C}_i$, which is exactly the name of a bifactorization

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f_i} & \mathcal{C}_i \\ & {\scriptstyle c^*} \searrow \quad \simeq \quad \nearrow {\scriptstyle a} & \\ & C/c & \end{array}$$

Suppose now we have two lifts of a, that is

$$\begin{array}{ccccccc} & & & & \mathcal{C}_{i_1} & & \\ & & & & \uparrow & \searrow^{q_i} & \\ & & & \overset{f_{i_1}}{\nearrow} & \underset{a_{i_1}}{|} & \overset{\alpha_{i_1}}{\simeq} & \\ \mathcal{C} & \xrightarrow{c^*} & C/c & \xrightarrow{a} & \underset{i\in I}{\text{bicolim}}\ \mathcal{C}_i & & \\ & & & \underset{f_{i_2}}{\searrow} & \underset{a_{i_2}}{|} & \overset{\alpha_{i_2}}{\simeq} & \\ & & & & \downarrow & \nearrow_{q_{i_2}} & \\ & & & & \mathcal{C}_{i_2} & & \end{array}$$

Then we have two global elements $(i_1, a_1) : (i_1, 1_{\mathcal{C}_{i_1}}) \to (i_1, f_{i_1}(c))$ and $(i_2, a_2) : (i_2, 1_{\mathcal{C}_{i_2}}) \to (i_2, f_{i_2}(c))$, which moreover are identified with the global element coded by a in the pseudocolimit. This means there is a span $d_1 : i_1 \to i$, $d_2 : i_2 \to i$ in I such that we have a span of cartesian morphisms in oplaxcolim $_{i\in I}\mathcal{C}_i$

$$\begin{array}{ccccc} (i_1, 1_{\mathcal{C}_{i_1}}) & & & & (i_2, 1_{\mathcal{C}_{i_2}}) \\ \downarrow{\scriptstyle (i_1,a_1)} & \searrow^{(d_1,!)} & & \swarrow_{(d_2,!)} & \downarrow{\scriptstyle (i_2,a_2)} \\ (i_1, f_{i_1}(c)) & & (i, 1_{\mathcal{C}_i}) & & (i_2, f_{i_2}(c)) \\ & \searrow_{(d_1,1)} & \downarrow & \swarrow^{(d_2,1)} & \\ & & (i, f_i(c)) & & \end{array}$$

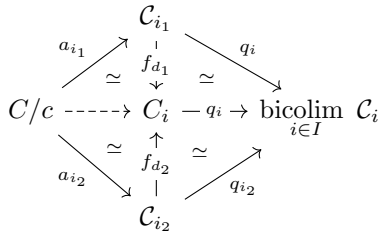
where

$$f_{d_1} f_{i_1}(c) \simeq f_i(c) \simeq f_{d_2} f_{i_2}(c)$$

and

$$f_{d_1}(a_1) = f_{d_2}(a_2)$$

Hence we have an intermediate factorization



There is no lifting property relative to 2-cell to examine as we saw that the functors $c^* : \mathcal{C} \to \mathcal{C}/c$ are discrete. □

Proposition 10.2.1.11. *Basic focalizations are exaclty focalizations that are finitely presented in the pseudocoslice $\mathcal{C} \downarrow \mathbf{Lex}$.*

Proof. This is because any focalization under $\mathcal{C}$ is of the form $n_A : \mathcal{C} \to \mathcal{C}_A$, □

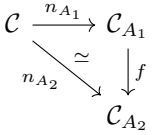
Now, we turn to the left-like properties of the focalizations, which are innocent categorifications of properties of left maps in ordinary factorization systems:

Proposition 10.2.1.12. *Focalizations have the following properties:*

- *Equivalences are focalizations;*
- *Focalizations are closed under invertible 2-cells and composition;*
- *Focalizations are right cancellative up to invertible 2-cells.*

Proof. For any Lex category $\mathcal{C}$ we have a canonical equivalence $\mathcal{C} \simeq \mathcal{C}/1$ which is also the focalization at the initial model $\mathcal{C}_{\exists_1}$. Closure under invertible 2-cell is automatic from the definition of focalizations.

For right pseudocancellation: suppose we have



Then f comes uniquely from a morphism $A_1 \to A_2$ in $\mathbf{Lex}[\mathcal{C}, \mathcal{S}]$. Seeing f as an object of the comma $A_1 \downarrow \mathbf{Lex}[\mathcal{C}, \mathcal{S}]$, we have

$$\mathcal{C}_{A_2} \simeq (\mathcal{C}_{A_1})_f$$

Hence f is also a focalization. □

Lemma 10.2.1.13. *Focalizations are discrete morphisms in* **Lex**. *Hence focalizations under $\mathcal{C}$ are discrete objects in the pseudoslice $\mathcal{C} \downarrow \mathbf{Lex}$.*

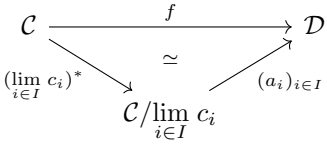
Proof. We saw that for each c of $\mathcal{C}$ morphisms $c^* \to f$ was the name of some $a : 1 \to f(c)$: but $\mathcal{D}[1, f(c)]$ is a set. Now as focalizations are filtered colimits of basic focalizations, they are discrete as well. □

Corollary 10.2.1.14. *Basic focalizations under $\mathcal{C}$ are closed under finite bicolimits in $\mathcal{C} \downarrow \mathbf{Lex}$.*

Proof. We can see directly that, for a finite 1-truncated diagram $F : I \to \mathcal{C}$, the finite bicolimit of the induced diagram made of the c_i^* is

$$\operatorname*{pscolim}_{i\in I} c_i^* \simeq (\lim_{i\in I} c_i)^*$$

Indeed a pseudococone $(a_i : c_i^* \to f)_{i\in I}$ in $\mathcal{C} \downarrow \mathbf{Lex}$ is the same as a I-indexed diagram $(a_i : 1 \to f(c_i))_{i\in I}$, which induces uniquely $(a_i)_{i\in I} : 1 \to \lim_{i\in I} f(c_i) \simeq f(\lim_{i\in I} c_i)$, which the the name of a factorization



Now as basic focalizations are discrete, we know that the bipower of a basic focalization is equivalent to it, that is for any c, we have $c \simeq c^* \otimes 2$. From this we deduce that basic focalization are closed under finitely weighted bilimits in $\mathcal{C} \downarrow \mathbf{Lex}$. □

Lemma 10.2.1.15. *Basic focalizations are stable under bipushout.*

Proof. In fact, for $f : \mathcal{C}_1 \to \mathcal{C}_2$ and c in $\mathcal{C}_1$ we have a pseudopushout of lex

$$\begin{array}{ccc} \mathcal{C}_1 & \xrightarrow{f} & \mathcal{C}_2 \\ {\scriptstyle c^*}\downarrow & \ulcorner & \downarrow{\scriptstyle f(c)^*} \\ \mathcal{C}_1/c & \xrightarrow[\ulcorner \delta_{f(c)} \urcorner]{} & \mathcal{C}_2/f(c) \end{array}$$

where $\delta_{f(c)}$ is the name of the diagonal $f(c) \to f(c) \times f(c)$. □

Remark 10.2.1.16. This result mirrors the fact that etale geometric morphisms are stable under bipullbacks

$$\begin{array}{ccc} \widehat{\mathcal{C}_2}/\!\!\downarrow_{f(c)} & \longrightarrow & \widehat{\mathcal{C}_1}/\!\!\downarrow_{c} \\ {\scriptstyle f(c)_*}\downarrow & \lrcorner & \downarrow{\scriptstyle c_*} \\ \widehat{\mathcal{C}_2} & \xrightarrow[\widehat{f}]{} & \widehat{\mathcal{C}_1} \end{array}$$

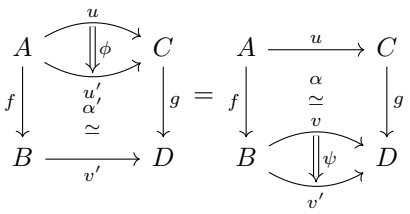
10.2.2 The (Focalization, Terminally connected) bifactorization

We saw that focalizations had left-like properties and were involved in a canonical factorization of lex functors into $\mathcal{S}$. Here we introduce their associated right class and explain how the factorization involved in the Grothendieck-Verdier localization generalize to arbitary lex functors. In fact we have already met, under another form, the right maps when computing the spectrum of local maps; but before revealing their identity, let us recall a minimal amount of what we need of 2-dimensional factorization systems:

10.2.2.1. For $f : A \to B$ and $g : C \to D$ 1-cells in $\mathcal{C}$, a pseudocommutative square in f, g is the data of a triple $(u, v, \alpha) : f \Rightarrow g$ whith $\alpha : gu \simeq vf$ an invertible 2-cell as below,

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & \overset{\alpha}{\simeq} & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

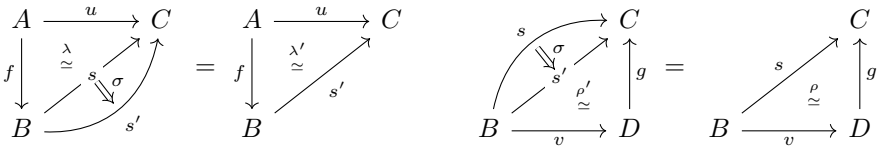
A morphism of pseudocommutative squares $(\phi, \psi) : (u, v, \alpha) \Rightarrow (u', v', \alpha')$ is the data of $\phi : u \Rightarrow u'$ and $\psi : v \Rightarrow v'$ such that $\alpha' g^* \phi = \psi^* f \alpha$ as below



Definition 10.2.2.2. A *filler* for a pseudocommutative square (u, v, α) between f, g is the data of a diagonal map $s : B \to C$ and a pair of invertible natural transformations $\lambda : sf \xrightarrow{\simeq} u$ and $\rho : gs \xrightarrow{\simeq} v$

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & \overset{\lambda}{\simeq}\ s\ \overset{\rho}{\simeq} & \uparrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

while a morphism of filler $(s, \lambda, \rho) \to (s', \lambda', \rho')$ is the data of a 2-cell $\sigma : s \Rightarrow s'$ such that $\lambda' \sigma^* f = \lambda$ and $\rho' g^* \sigma = \rho$ as visualized below:



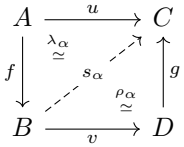
In particular, for any such σ the whiskerings $\sigma^* f$ and $g^*\sigma$ necessarily are invertible by cancellation of invertible cells. But in general we cannot infer that a morphism of filler is invertible.

Definition 10.2.2.3. We say that $f \perp g$, or that f and g are respectively *left and right bi-orthogonal* if the following square is a bipullback in Cat

$$\begin{array}{ccc} \mathcal{C}[B,C] & \xrightarrow{\mathcal{C}[f,C]} & \mathcal{C}[A,C] \\ {\scriptstyle \mathcal{C}[B,g]}\downarrow & \simeq & \downarrow{\scriptstyle \mathcal{C}[A,g]} \\ \mathcal{C}[B,D] & \xrightarrow[\mathcal{C}[f,D]]{} & \mathcal{C}[A,D] \end{array}$$

Remark 10.2.2.4. This means that

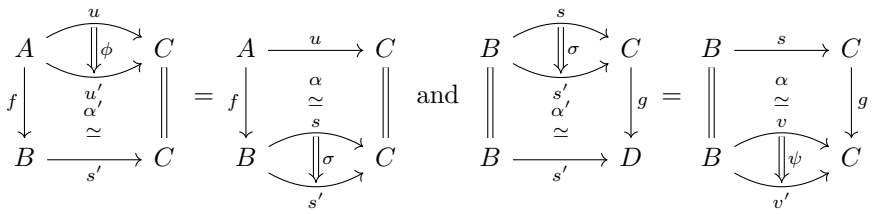
– for any pseudocommutative square (u, v, α) there is an universal filler $(s_\alpha, \lambda_\alpha, \rho_\alpha)$



whith the property that any other filler (s, λ, ρ) comes equipped with a unique, invertible morphism of fillers $\omega : s \simeq s_\alpha$

– and for a morphism of pseudosquares $(\phi, \psi) : (u, v, \alpha) \Rightarrow (u', v', \alpha')$ there exists a unique 2-cell $\sigma : s_\alpha \Rightarrow s_{\alpha'}$ such that

$$\alpha'\phi = \sigma^* f\alpha \quad \text{and} \quad \alpha' g^*\sigma = \psi\alpha$$



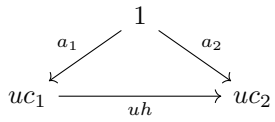
Actually those conditions are synthetised by the existence of an equivalence of categories

$$\mathcal{C}[B,C] \simeq \mathbf{ps}[2,\mathcal{C}](f,g)$$

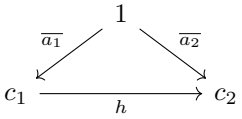
sending any arrow $s : B \to C$ to the canonical isomorphisms induced by composition with f and g $(sf, gs, \alpha_{sf}\alpha_{gs})$, whith the choice of universal filler as pseudoinverse. This says that a universal filler exists up to unique invertible 2-cell, and that any filler is universal.

Now let us apply this notion to our context in **Lex**:

Definition 10.2.2.5. We say that a lex functor $u : \mathcal{C} \to \mathcal{D}$ is *terminally connected* if it lifts global elements, an does it in a functorial way, that is, if any $a : 1 \to u(c)$ arises uniquely from a global element $\overline{a} : 1 \to c$ such that $u(\overline{a}) = a$, and moreover, for any triangle as below



the unique lifts already satisfy the commutation



In the following we denote as 1-**Con** the class of terminally connected lex functors.

Remark 10.2.2.6. This is a condition of "full faithfulness" right to 1: indeed it can be rephrased to a natural isomorphism of homsets

$$\mathcal{C}[1, u(c)] \simeq \mathcal{C}[1, c]$$

Proposition 10.2.2.7. *Terminally connected morphisms are exactly the lex functors that are right bi-orthogonal to the basic focalizations.*

Proof. This proposition is almost tautological: a filler for a pseudosquare

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ {\scriptstyle c^*}\downarrow & \simeq & \downarrow{\scriptstyle u} \\ \mathcal{C}/c & \xrightarrow[a]{} & \mathcal{D}' \end{array}$$

where a is the name of a global element $a : 1 \to uf(c)$ is exactly the name of an antecedent $\overline{a} : 1 \to f(c)$ of a, whose uniqueness amounts the uniqueness of the filler. Now the 2-dimensional part in the bi-orthogonality condition is to be read as follows: a morphism of pseudosquare

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f_1} & \mathcal{D} \\ {\scriptstyle c^*}\downarrow & {\scriptstyle \alpha_1}\simeq & \downarrow{\scriptstyle u} \\ \mathcal{C}/c & \xrightarrow[a_2]{a_1 \Downarrow \tau} & \mathcal{D}' \end{array} = \begin{array}{ccc} \mathcal{C} & \overset{f_1}{\underset{f_2}{\Downarrow \sigma}} & \mathcal{D} \\ {\scriptstyle c^*}\downarrow & {\scriptstyle \alpha_2}\simeq & \downarrow{\scriptstyle u} \\ \mathcal{C}/c & \xrightarrow[a_2]{} & \mathcal{D}' \end{array}$$

codes for a triangle

$$\begin{array}{ccccc} & & 1 & & \\ & {\scriptstyle a_1}\swarrow & & \searrow{\scriptstyle a_2} & \\ uf_1(c) & & \xrightarrow[u\sigma_c]{} & & uf_2(c) \end{array}$$

Then requiring u to be terminally connected amounts to saying that a_1 and a_2 have unique antecedent satisfying

$$\begin{array}{ccccc} & & 1 & & \\ & {\scriptstyle \overline{a_1}}\swarrow & & \searrow{\scriptstyle \overline{a_2}} & \\ f_1(c) & & \xrightarrow[\sigma_c]{} & & f_2(c) \end{array}$$

which is exactly the name of a morphism of fillers

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f_1} & \mathcal{D} \\ {\scriptstyle c^*}\downarrow & {\scriptstyle \overline{a}_1}\nearrow {\scriptstyle \sigma_c}\Downarrow {\scriptstyle \overline{a}_2} & \downarrow{\scriptstyle u} \\ \mathcal{C}/c & \xrightarrow[a_2]{} & \mathcal{D}' \end{array}$$

□

10.2.2.8. Observe that we can consider more generally terminally connected morphisms in the 2-category **LEX** of large lex categories. We do not know whether focalizations still make sense in **LEX**, as they are intrinsically related to the semantics of Gabriel-Ulmer which does not apply to large lex categories. Yet we are interested in terminally connected morphisms with $\mathcal{S}$ as codomain.

Proposition 10.2.2.9. *For any small lex category $\mathcal{C}$, there is exactly one terminally connected lex morphism $\mathcal{C} \to \mathcal{S}$: the corepresentable $\exists_1$.*

Proof. If a lex functor $A : \mathcal{C} \to \mathcal{S}$ is terminally connected, then for each c in $\mathcal{C}$ we have a natural isomorphism

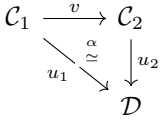
$$\begin{aligned} A(c) &\simeq \mathcal{S}[1, A(c)] \\ &\simeq \mathcal{C}[1, c] \\ &\simeq \exists_1(c) \end{aligned}$$

□

Proposition 10.2.2.10. *The class of terminally connected lex functors has the following properties:*

- *Equivalences are terminally connected;*
- *Terminally connected functors are closed under composition and invertible 2-cells;*
- *Terminally connected functors are left-cancellative up to invertible 2-cells*

Proof. Consider a triangle as below

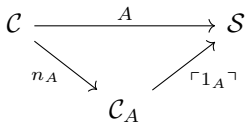


Suppose that both v and u_2 are terminally connected. Then for $a : 1 \to u_2 v(c)$ we have successively lifts along u_2 and v which ensures that $u_2 v$ is terminally connected.

If we now suppose u_1 and u_2 to be terminally connected. Let c be in $\mathcal{C}_1$ and $a : 1 \to v(c)$ in $\mathcal{C}_2$. Then we have $u_2(a) : u_2(1) \simeq 1 \to u_2 v(c) \simeq u_1(c)$ in $\mathcal{D}$, which comes uniquely from some $\overline{u_2(a)} : 1 \to c$ in $\mathcal{C}$, and by uniqueness of lifts of global elements along u_2, we must have $v(\overline{u_2(a)}) = a$. Whence left cancellation. □

Lemma 10.2.2.11. *Any lex functor of the form $A : \mathcal{C} \to \mathcal{S}$ has a (focalization, terminally connected) factorization.*

Proof. This is the Grothendieck-Verdier localization: indeed in the localization

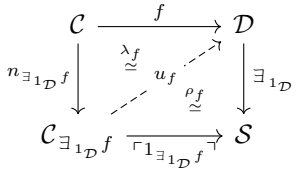


the name of the identity of A point at the initial object of the comma $A \downarrow \mathbf{Lex}[\mathcal{C}, \mathcal{S}]$, but we know that the initial object in a locally finitely presentable category A coincides always with the corepresentable at the terminal object of the underlying lex site: in this case, this says that $\ulcorner 1_A \urcorner = \exists_{1_{\mathcal{C}_A}}$, which is terminally connected. □

Theorem 10.2.2.12. *We have a bifactorization (**Foc**, 1-**Con**) system in **Lex** where left maps are focalizations and right maps are terminally connected lex functors.*

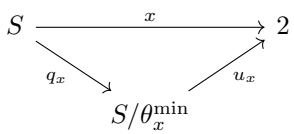
Proof. We could directly invoke a small object argument to prove that, as the completions of basic focalizations under bifiltered bicolimit, the class of focalizations coincides with the double bi-orthogonal. However, in absence of a formal 2-dimensional small object argument, we find both preferable and enlightening to describe how this factorizations processes.

Take $f : \mathcal{C} \to \mathcal{D}$. Then one can consider the composite with the name of the initial model $\exists_{1_{\mathcal{D}}}$ of $\mathcal{D}$: this returns a lex functor $\exists_{1_{\mathcal{D}}} f : \mathcal{C} \to \mathcal{S}$, and we can take its factorization as in lemma 10.2.2.11: but then, for $\exists_{1_{\mathcal{D}}}$ is terminally connected and $n_{\exists_{1_{\mathcal{D}}} f}$ is a focalization, we have a diagonalization



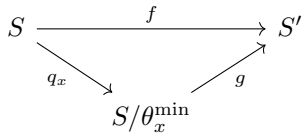
and by left-cancellation of terminally connected morphisms, u_f must itself be terminally connected. □

10.2.2.13. This should be compared to the factorization of a morphism of $\wedge$-semilattice into 2

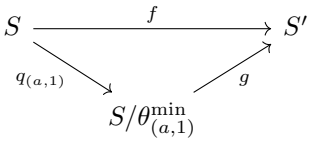


where q_x is the minimal quotient at the filter x collapsing only elements of x on 1 while u_x is the name of the minimal point of X_S, the HMS-space of S, or equivalently, the quotient of $S/\theta_x^{\min}$ collapsing all element distinct from 1 on 0.

More generally, observe that for any morphism f of $\wedge$-semillatices, we have factorization



if and only if f collapses x on 1. In particular for each element a in S, we have a factorization



if and only if $f(a) = 1$.

This is exactly the posetal analog of the focalization. We saw that the bifactorizations of a lex morphism $f : \mathcal{C} \to \mathcal{D}$ through the basic focalization at c of a small lex $\mathcal{C}$ corresponds to global element $a : 1 \to f(c)$. If $\mathcal{C}$ and $\mathcal{D}$ are posetal, hence are $\wedge$-semilattices, then there is at most one such global element, and it witnesses that $1 \leq c$, but this forces that $1 = c$: hence focalizations correspond to 1-minimal quotients. On their side, terminally connected functors restrict exactly to the 1-conservative morphisms, as reflecting global elements amounts not to collapsing elements on 1.

10.3 Elements for a future account on first order geometries

From the considerations above, we conjecture that a 2-dimensional spectral construction will provide a general template of the different syntax-semantics adjunction. Such a notion of 2-spectrum will be constructed as in the 1-dimensional case from a notion of 2-geometry, which we conjecture to be constituted of the data of a small bilex 2-category $\mathcal{C}$ together with a lax-generic, pseudofunctorial bifactorization system $(\mathcal{L}, \mathcal{R})$ satisfying some convenient 2-dimensional 2-accessibility condition, and a Street topology on $\mathcal{C}$ in the sense of [90], which will also satisfy some generation condition relatively to $\mathcal{L}$.

However this requires the development of three theories that still have to be done:

– A theory of 2-dimensional analogs of locally finitely presentable categories. Such a notion is investigated in [9], yet in an enriched way - and moreover, it seems that **Lex**, **Reg** and **Coh** are not *finitely* locally presentable with this formalism, which is an hindrance to our purpose. A future work will be dedicated to a theory of *sigma-accessible* and *sigma-presentable* 2-categories using the notion of *sigma-filtered 2-categories* in the sense of [23] and a corresponding notion of *sigma-compact* objects. Although the theory in itself seems satisfying, we still lack a proof that our 2-categories of interest **Lex**, **Reg**, **Coh** are of this kind.

– A systematic account of 2-dimensional Grothendieck topoi; several categorifications of Grothendieck topoi exist depending of the considered level of strictness, see [89] and [90]. A 2-dimensional Diaconescu theorem for left biKan extensions of flat pseudofunctors was moreover recently proposed in [23], and an ongoing work will propose a refinement of this result taking in account a Street topology, defining a notion of local object relatively to a Street topology.

– A first categorification of the notion of stable functor is provided by [95] notion of *lax familial pseudofunctor*; on the other hand, we develop in [82] a notion of *bistable pseudofunctor*, and study alternative laxness conditions. We conjecture that, for a good notion of 2-geometry, the 2-category of local objects and right maps will form a bistable inclusion encoding a 2-dimensional form of admissibility.

Once developed, this formalism may provide a satisfying notion of 2-geometry, and the spectrum may be constructed as the 2-topos of stacks over a small 2-site made of "finitely presented"

left maps together with an induced Street topology.

In particular, each doctrine may come equipped with its 2-geometry. We conjecture that the (focalization, terminally connected) factorization system will provide the desired factorization system for **Lex**, **Reg**, **Coh**.

The effort we did above to construct a factorization system on **Lex** are motivated by analogy between Gabriel-Ulmer duality and Jipsen-Moshier duality. As well as principal minimal quotients under a fixed $\wedge$-semilattice corresponded to the basic compact open of its spectrum, or equivalently, as the basis for its spectral site - which was localic as made of epi - we expect that basic focalization under a fixed category may provide a way to construct a notion of spectrum.

We saw above that focalizations are discrete morphisms in **Lex**. Hence the 2-category of focalizations under a fixed small lex category $\mathcal{C}$ is in fact 1-truncated, as well as its generator of basic focalizations. In fact, they are nothing but "externalization" of $\mathcal{C}$ and its category of models over $\mathcal{C}$ in **Lex**, as well as etale geometric morphisms "externalize" a topos $\mathcal{E}$ as a 1-truncated sub-bicategory of geometric morphisms over $\mathcal{E}$.

In particular, from the observation above on **Lex** and the fact that all representable defines models, we conjecture that the 2-geometry for Gabriel-Ulmer will only consist in factorization data, provided by the (focalization, totally connected) bifactorization system. If we define the *Gabriel-Ulmer 2-spectrum* of a small lex category $\mathcal{C}$ for the Gabriel-Ulmer geometry as the 2-category of strict 2-functors and pseudonatural transformations

$$\mathbf{Spec}(\mathcal{C}) = [\mathcal{V}_{\mathcal{C}}, \mathbf{Cat}]_p$$

with $\mathcal{V}_{\mathcal{C}}$ the 2-category of basic focalizations under $\mathcal{C}$, or using the retract equivalence between a 2-category of strict 2-functors with pseudonatural equivalence and the corresponding 2-category of pseudofunctors, we could equivalently define

$$\mathbf{Spec}(\mathcal{C}) \simeq \mathbf{ps}[\mathcal{V}_{\mathcal{C}}, \mathbf{Cat}]$$

Because basic focalizations are discrete morphisms, this 2-topos of stacks has a small generator of discrete objects. Hence the Gabriel-Ulmer spectrum of a small lex category $\mathcal{C}_{\mathbb{T}}$ classifying a finite limit theory $\mathbb{T}$ is 1-truncated, and the classifying topos topos $\mathcal{S}[\mathbb{T}] \simeq \widehat{\mathcal{C}_{\mathbb{T}}}$ is the generator of 1-truncated objects, that is

$$\mathbf{Spec}(\mathcal{C}_{\mathbb{T}}) \simeq [\mathcal{C}_{\mathbb{T}}^{\mathrm{op}}, \mathbf{Cat}]_p \simeq \mathbf{St}[\widehat{\mathcal{C}_{\mathbb{T}}}, J_{Can}]$$

and

$$\mathcal{S}[\mathbb{T}] \simeq \tau_{\leq 1}(\mathbf{Spec}(\mathcal{C}_{\mathbb{T}}))$$

For a finite limit theory $\mathbb{T}$, $\mathbb{T}$-models in $\mathcal{S}$ are bipoints of the Gabriel-Ulmer spectrum of $\mathcal{C}_{\mathbb{T}}$:

$$\mathbb{T}[\mathcal{S}] \simeq \mathbf{biGeom}[\mathbf{Cat}, \mathbf{Spec}(\mathcal{C}_{\mathbb{T}})]$$

In particular the structural stack of a lex category $\mathcal{C}$ coincides with the self-indexation

$$\mathcal{C} \xrightarrow{\mathcal{C}/(-)} \mathbf{Lex}$$

Similarly, we conjecture that the regular (resp. coherent) topologies on a regular (resp. a coherent) small category $\mathcal{C}$ will be externalized into a Street topology on the generator of basic focalizations under $\mathcal{C}$, and we shall recover the spectrum as the corresponding 2-topos of stacks. Moreover, for the self indexation of a regular (resp. coherent) category is always a stack for the regular (resp. coherent) topology, we conjecture the self-indexation to be the structural stack associated to this notion of 2-spectrum.

To be continued...

Index

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