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Modelling, analysis and control of coffee berry borer

Yves Fotso Fotso

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PEACE-WORK-FATHERLAND

UNIVERSITY OF DSCHANG

POST GRADUATE SCHOOL



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ÉCOLE DOCTORALE

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TOPIC:

Modelling, analysis and control of coffee berry borer

Thesis defended publicly in fulfilment of the requirements for the award of a
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Speciality: Dynamical Systems

By:

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Registration number: CM04-10SCI1438

Master of Science (MSc.) degree in Mathematics (*Analysis*).

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Dedication

To my parents.

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List of Abbreviations

CBB: Coffee Berry Borer
FAO: Food and Agriculture Organisation
GAS: Glocally Asymptotically Stable
GDP: Gross Domestic Product
ICO: International Coffee Organization
LAS: Locally Asymptotically Stable
ODE: Ordinary Differential Equation
PDE: Partial Differential Equation

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Abstract

Coffee production is an important agricultural activity, in particular in several developing countries. For decades in Cameroon, coffee has been a proven cash crop and continues to be grown despite the fact that production basins have been reduced to make way for food crops. A major threats on coffee production is the serious damage caused by a small insect, *Hypothenemus hampei* called coffee berry borer (CBB). This insect is very difficult to control as it reproduces and spends most of its life cycle inside the coffee berry. It hence causes severe crop losses. The objective of this thesis is to develop mathematical models for the study, understanding and control of the dynamics of CBB in its ecosystem through theoretical and numerical developments. After a brief presentation of the biology of the interaction between coffee berries and CBB, a basic model is proposed. This model is governed by a non-linear system of ordinary differential equations that describes the infestation dynamics of coffee berries by CBB over a cropping season, taking into account berries availability. Based on the knowledge that berry preference is age-dependent during the selection and infestation process of colonising CBB, our basic model is then extended into a system of partial differential equations with an age structure of the coffee berries. Since the ultimate goal of coffee producers is to produce high quality coffee at the best market price and at the lowest cost, we design optimal control strategies based on the use of environmentally friendly methods such as trapping and release of phytopathogenic fungi. The objective of these control strategies is to maximise yield while minimising control costs and the CBB population for the next cropping season. An extended modelling approach based on the basic model taking into account the fruiting period of coffee berries and multiple cropping seasons is then proposed and studied. By using cultural control, which consists of a thorough harvesting of berries at the end of each production period and a cleaning of the plantation during the intercropping, we determine the criteria for extinction of the CBB population over several seasons. On the agronomical front, the main outputs of this thesis consists in the analysis of the dynamics of CBB-berries interaction over several seasons, but also the emphasis on the advantage of using more than one economically friendly CBB control methods simultaneously.

Keywords : epidemiological modelling, *Hypothenemus hampei*, plant-pest interaction, stability, optimal control theory, semi-discrete model, semigroup theory, numerical simulations.

Titre

Modélisation, analyse et contrôle du scolyte
des baies du caféier

Résumé

La production du café est une activité agricole importante, notamment dans de nombreux pays en voie développement. Depuis des décennies au Cameroun, le Café est une culture de rente qui a fait ses preuves et continu malgré les bassins de production qui ont diminué pour laisser la place aux cultures vivrières. L'une des principales menaces à la production du café est le grave dommage causé par un petit insecte, *Hypothenemus hampei* encore appelé scolyte des baies du caféier. Cet insecte est très difficile à contrôler car il se reproduit et passe la majeure partie de son cycle de vie à l'intérieur de la baie du caféier. Il génère ainsi des pertes de récolte importantes. L'objectif de cette thèse est de développer des modèles mathématiques pour l'étude, la compréhension et le contrôle de la dynamique des scolytes dans son écosystème via des approches numériques et analytiques. Après une présentation succincte des préalables biologique sur l'interaction entre baies du caféier et scolytes, un modèle de base est proposé. Ce modèle est régi par un système d'équations différentielles ordinaire non linéaire qui décrit la dynamique d'infestation des baies du caféier par les scolytes au cours d'une saison culturale en tenant compte de la disponibilité des baies. Comme la préférence des baies est fonction de l'âge lors du processus de sélection et d'infestation par les scolytes colonisantes, notre modèle de base est ensuite étendu en un système d'équations aux dérivées partielles avec une structure en âge des baies du caféier. L'objectif ultime des producteurs de café étant de produire un café de haute qualité au meilleur prix du marché et au coût le plus bas, nous concevons ensuite des stratégies optimales de lutte basées sur l'utilisation de méthodes respectueuses de l'environnement telles les pièges et l'introduction de champignons phytopathogènes. Le but de ces stratégies de contrôle étant de maximiser le rendement tout en minimisant les coûts de contrôles et la population des scolytes pour la prochaine saison de culture. Une approche étendue de modélisation basée sur le modèle de base en prenant en compte la période de fructification des baies du caféier et les saisons de culture multiples est ensuite proposée et étudiée. En utilisant la lutte culturale qui consiste à faire une bonne récolte des baies à la fin de chaque période de production et un nettoyage de la plantation pendant l'intersaison, nous déterminons les critères d'extinction de la population des scolytes sur plusieurs saisons. Du point de vue agronomique, les résultats principaux de cette thèse sont l'analyse de la dynamique d'interaction baies-scolytes sur plusieurs saisons, ainsi que la mise en évidence des bénéfices à utiliser plusieurs méthodes de lutte simultanément.

Mots-clés: modélisation épidémiologique, *Hypothenemus hampei*, interaction plante-ravageur, stabilité, théorie du contrôle optimal, modèle semi-discret, théorie des semi-groupes, simulations numériques.

Introduction

Coffee (*Coffea*), originally from Ethiopia, is a shrubby plant cultivated mainly in tropical and subtropical regions. Its cultivation and export play an important role in the economic growth of many developing countries such as Brazil, Vietnam, and Columbia, which are the top producers in the world in decreasing order. As far as Africa is concerned the major producers are Ethiopia, Ivory Coast, Uganda and Cameroon [105]. The meteoric rise of its exploitation worldwide is a consequence of the important use of this fruit for the preparation of stimulating drinks and its role in the manufacture of pastries. Coffee is mainly consumed by developed countries in America, Asia and Europe. There are more than a hundred species of coffee, but only two of these are commercially traded: *Coffea Canephora* (predominantly known as Robusta) and *Coffea Arabica*. Arabica coffee is grown throughout Latin America, Central and East Africa, India and to some extent, Indonesia. Robusta coffee is grown in West and Central Africa throughout South East Asia and to some extent in Brazil. Coffee production has increased in recent years. According to the United States Department of Agriculture [105], the world production is estimated at 175.8 million bags (a bag weighing 60 kg) in 2020–2021 and world exports at 120.3 million bags.

Agriculture plays a predominant role in the Cameroonian economy. It is the country's main source of income, accounting for a quarter of exports and employing around 80% of the working population. Coffee cultivation in Cameroon dates back to 1884 [61]. According to International Coffee Organization (ICO) [61], average coffee production rose from 1 million bags in 1960 to 1.7 million in 1980 and reached a record 2.2 million bags in 1986/1987. From this period onwards, there is a downward trend, down to 409,000 bags in 2015–2016. This is partly due to declining yields and the suspension of some government subsidies on agro-chemical production inputs. Both Arabica and Robusta varieties are produced in Cameroon. Arabica accounts for an average of 10% of production, compared to 90% for Robusta [61]. Both products are grown in 7 regions: Littoral, Centre, East, North-West, West, South and South-West. The Littoral and West regions share the bulk of the country production, accounting for 41% and 36% of this production respectively. The Centre, East and South regions account for around 5% of the production and produce exclusively Robusta. The South-West produces both Arabica and Robusta up to 15% [61]. Coffee cultivation contributes to the creation of added value in the national economy. The share of coffee in the GDP has varied between 0.32% and 0.42% over the last decades [61].

Coffee production is hindered by the presence of pests that affect the yield and quality of coffee. Among these pests, *Hypothenemus hampei* (Coleoptera: Scolytidae) generally called *coffee berry borer* (CBB), is the main insect pest of coffee berries in most coffee-producing countries [29, 73]. It spends almost all of its life cycle inside a coffee berry, which provides food and

adequate conditions for mating and reproduction [97]. The damage caused by CBB are mainly a decrease of coffee yield due to abscission of berries, loss of weight and a decrease on coffee quality. Originally from Central Africa, CBB can be found in most coffee growing countries throughout the world [29, 73]. The infestation levels of the CBB can be extremely high in some regions. For example, they are estimated at 60% in Mexico, 50-90% in Malaysia, 60% in Colombia, 58–85% in Jamaica, 80% in Uganda and 90% in Tanzania [62]. It can cause yield losses of 30-35% and it deteriorates the berry quality with up to 100% of berries perforated at harvest time if no control measure is applied [73]. Damages are estimated at more than U.S. \$500 million annually, affecting a large proportion of more than 25 million rural households involved in coffee production worldwide [107]. The cryptic life cycle inside the berry makes this insect very difficult to control [62, 11, 29, 16], hence the need to develop control strategies.

Control of CBB has long relied on the application of insecticides, which are harmful both for human health and the environment, and whose efficient is limited by insecticide resistance [29, 62]. Alternative methods include cultural control, which mainly consists in removing all remaining berries from the trees and the soil [85, 29, 11], as well as trapping, which allows to capture female CBB when they look for coffee berries to lay their eggs [84, 37, 81]. More recently, research on biological control has led to the use of natural enemies, including parasitoids, nematodes and entomopathogenic fungi such as *Beauveria bassiana* [29, 11, 106]. The latter is present and naturally infects CBB in many coffee growing countries. As a bioinsecticide, it can be safely sprayed in the plantation and significantly reduce CBB infestation (64%–91% CBB infection by *Beauveria bassiana* reported in Colombia) [11]. However, there are still gaps to fill to develop successful biocontrol strategies [106].

To determine how to best apply a biocontrol agent such as this entomopathogenic fungus or trapping control, field experiments are particularly time consuming and expensive [11], as several plots need to be monitored during several months. Moreover, it is difficult to control all factors in the field, such as for instance the initial CBB population. Mathematical modelling and optimal control are hence relevant cost-effective alternatives to assess control strategies and long term dynamics of plant-pest interactions [75]. In the literature, few mathematical models have been developed to get insights into the evolution of CBB. A statistical model analyses the spatio-temporal CBB infestation in Colombia was developed in [94]. This statistical model describes the dispersal of the CBB in a coffee plot taking into account the initial foci of infestation. A tritrophic coffee system of the interactions of coffee, CBB and its three parasitoids was developed by *Gutierrez et al.* [54] using the distributed maturation time demographic model. The system consists of 10 linked age-structured population models in units of numbers, mass or both. Each sub-model is based on the distributed maturation time dynamics model. The coffee model is a canopy model consisting of sub-models for the mass and number of berries. The CBB model on a per tree consist of age-structured models for immature stages developing inside fruits, dispersing colonising females and females in colonised berries. This model has been improved by *Rodriguez et al.* [93] in taking into account the effect of weather on fruiting phenology, the effects of temperature and rainfall on the emergence of CBB. Due to its complexity, this model is analytically untractable, so neither can its qualitative behaviour be investigated, nor its data-dependent outcomes generalised. There is a need for a more generic dynamical

model, which nevertheless takes into account the specificities of the interactions between CBB and coffee berries, and could contribute to design efficient CBB control strategies.

In this thesis, we aim at filling this gap. Our objectives are twofold: firstly, to develop tractable mathematical models describing the infestation dynamics of coffee berries by CBB in a plantation; secondly to propose and implement cost-effective control strategies, using optimal control to maximise the grower's profit.

Organisation of the thesis

This thesis is organised in five chapters, followed by a general conclusion and perspectives.

- Chapter 1 consists of biological and mathematical preliminaries that constitute the basic of the following chapters. Firstly, we present the biology of coffee, the CBB life cycle, and its interaction mechanisms with coffee berries in the plantation. All models presented in the following chapters derive from this biological background. Secondly, we present some insights into optimal control theory of systems governed by nonlinear ordinary differential equations (ODE), used in Chapter 2, and partial differential equations (PDE), used in Chapters 4 and 5.
- Chapter 2 presents our first contribution to the mathematical modelling and control of CBB–coffee berry interactions during a cropping season. We present our baseline model, an ODE dynamical system, and detail our modelling hypotheses based on the biological background exposed in Chapter 1. We then thoroughly analyse this model. Finally, we design optimal control strategies based on the deployment of a biocontrol agent (*e.g.* an entomopathogenic fungus such as *Beauveria bassania*), which is a sustainable and environmentally-friendly alternative to chemical pesticides. Our objective is to maximise the yield, while minimising the cost of the bio-insecticide as well as the CBB population remaining in the plantation at the end of the cropping season, so as to prevent too high an infestation for the next season.

Preliminary results were presented at the 14th CARI Conference in October 2018 and published in the proceedings [44].

- Chapter 3 extends the baseline model exposed in Chapter 2 in a multi-seasonal context. A semi-discrete framework is proposed, which intertwines continuous dynamics and discrete events. The continuous dynamics of berry growth and CBB infestation during a cropping season is followed by a discrete harvest; the following intercropping season, during which no berries are produced and CBB dynamics is slow, ends with a discrete cleaning, consisting in picking most remaining berries before the start of a new cropping season. Before analysing this model, we recall some mathematical results on impulsive differential equations. We then study analytically the existence and stability of the pest-free stationary solution. Finally, considering the harvest and cleaning efforts as control variables, as both remove infested berries and hence CBB from the plantation, we explore numerically their impact on the disappearance or persistence of CBB in the plantation.

- In Chapter 4, we take into account the CBB preference for mature berries, which was neglected in the previous models, and hence introduce a berry age structure in the baseline model. We obtain a PDE model describing the CBB–coffee berry interactions during a cropping season. We address the well-posedness of the model using the semigroup theory. We design optimal control strategies based on the introduction of two environmentally friendly control methods, consisting in applying a bio-insecticide to reduce berry infestation and in trapping colonising CBB. As in Chapter 2, our objective is to maximise the profit while minimising the CBB population for the next cropping season. In the optimisation criterion, we take into account the fact that the berry price depends on its berry age.
- Chapter 5 is based on the PDE model of Chapter 4, but the control strategy differs. We still reduce CBB infestation by means of a bio-insecticide, but we apply it at discrete times, which is more realistic in terms of practical implementation. The well-posedness of the semi-discrete model obtained is addressed and an optimal impulsive control problem is investigated, with the same objective as in Chapters 2 and 4.

Some of the results of this thesis have been published in [44, 48], presented in numerous symposiums and conferences [45, 46, 53] and submit [49, 47].

Preliminaries

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In this chapter, we present the biological background of the coffee production and its pest, the coffee berry borer, that helps us formulate the mathematical models in the following chapters. We also recall some mathematical tools on the control theory of dynamical systems governed by ordinary differential equations and partial differential equations.

1.1 Biological background

1.1.1 Coffee

The coffee tree is a shrub, grown mainly for its berries, which, after processing, can be used to make stimulating drinks and in baking. Coffee is among the most important agricultural commodities on the world market: it is cultivated worldwide on approximately 10.3 million hectares. Its cultivation, processing, trade, transport and marketing employ over 25 million people worldwide [91].

Botanists believe that this plant originated from Africa, more precisely Ethiopia. From a taxonomic point of view, the coffee tree belongs to the *Rubiaceae* family, which includes more than a hundred genus, including the genus *Coffea*. Hundreds of species have been identified. However, there are only two main species that make up the bulk of coffee production in the world: *Coffea arabica* and *Coffea robusta* [31]. Arabica is characterised by its delicate, fruity and



Figure 1: Coffee trees in a plantation: berry picking in Oku Kumbo plantation in the North-West region of Cameroon (left) and zoom on coffee berries on a tree (right) [28, 3].

finely aromatic taste, while Robusta coffee has a strong, full-bodied, spicy, bitter character and contains twice as much caffeine as Arabica. Of these two species, Arabica is the most popular with consumers, while Robusta is mainly used to make soluble coffee, but also in the preparation of Arabica/Robusta blends for cheaper coffees.

The coffee tree usually consists of several trunks and has the appearance of a bush. It can grow up to ten meters in height for some species. However, coffee trees in plantations are generally smaller, two to three meters high, in order to facilitate the cultivation and harvesting of the berries (Figure 1). The development of a coffee crop, specifically flowering, development of berries and harvest time are related to climate. Flowering starts immediately after a dry period followed by rains [66]. The white flowers are clustered of the leaf axils and later develop into coffee berries. The berry successively goes through the green, yellow, red and black colours and becomes mature when it is red, after 7–9 months [108]. Depending on the frequency of rainfall, there may be one, two, or more flowering periods. As a consequence, on a coffee tree, there may be branches without flowers, branches with flowers and branches with berries at all stages of maturation. Several fructifications may hence occur in a single year, but there will be always a main fructification that will concentrate most of the coffee berries [78].

The coffee berry (also called coffee fruit or cherry) contains two ovoid beans or seeds, flattened on one side and separated by a longitudinal groove. Between the seed and the outer skin (also called exocarp) is the pulp, which is a viscous, slightly sweet liquid (also called mesocarp). When a berry is pulped, the coffee bean is enclosed in a transparent semi-rigid shell with a parchment-like appearance (also called endocarp). Once removed, the coffee bean is still surrounded by an adherent silvery skin (Figure 2).

Harvesting is done when the fruit is ripe, *i.e.* when it is red. It should not be delayed for Arabica, whose cherries fall off fairly quickly, unlike Robusta. There are two harvesting methods with different yields and economic costs. Picking consists in harvesting the berries by hand, taking only the ripe ones and going over them several times. This technique is very labour-intensive. Stripping consists in harvesting the whole branch when the majority of the cherries are ripe (about 75%). The latter method is more economically viable, but produces lower quality coffee because of the immature beans harvested.

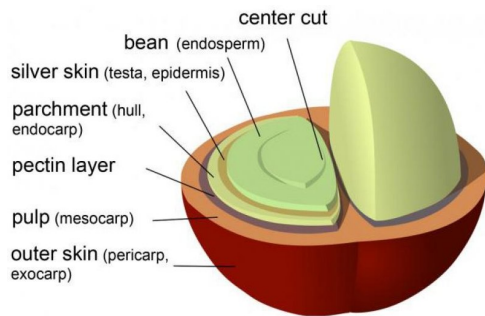


Figure 2: Coffee berry structure: scheme (left) and cross section (right). The two coffee beans are surrounded by the pericarp, composed of three layers: exocarp or skin, mesocarp or pulp and endocarp or parchment; the coffee bean is covered by the silver skin [100, 86].

1.1.2 The coffee berry borer

The coffee berry borer (CBB), whose scientific name is *Hypothenemus Hampei*, is an insect of the *Scolytidae* family that feeds and grows at the expense of the berries. This pest currently threatens all the coffee plantations on the planet from Africa, the continent of origin of its host plant, to South America and Asia. Historically, this insect pest was first reported in Central Africa in 1901, more precisely in Gabon [73]. Accidental introductions of CBB in coffee plantations have led to its rapid geographical expansion worldwide. As it spends most of its life inside a berry, this insect can adapt to a variety of environmental conditions. Nowadays, it is found in almost all coffee-producing countries in the world and has resisted various eradication programmes set up by coffee growers [29, 62, 106].

1.1.2.1 Life cycle

The CBB life cycle consists of five distinct developmental stages which are egg, larva, pre-pupa, pupa and adult [14, 16, 29, 50] (Figure 3).

- The egg is whitish in colour, elliptical in shape and becomes yellowish at maturity. It measures on average 0.6 mm long and 0.3 mm wide, a relatively large size compared to the adult. It usually hatches in 3–10 days.
- The larva is eruciform and measures on average 0.75 mm long and 0.25 mm wide. It has sclerotized mandibles that allow it to attack the seed albumen as soon as it hatches. The larval development consists of two stages for the female, but only one for the male. The larval stage lasts 10 to 26 days.
- When the larva is fully developed, it stops feeding and becomes immobile. This pre-pupa is similar to the larva, but is milky in colour.
- The pupa is whitish and averages 2.1 mm in length. The pre-pupal and pupal stages each last between 5 to 15 days.

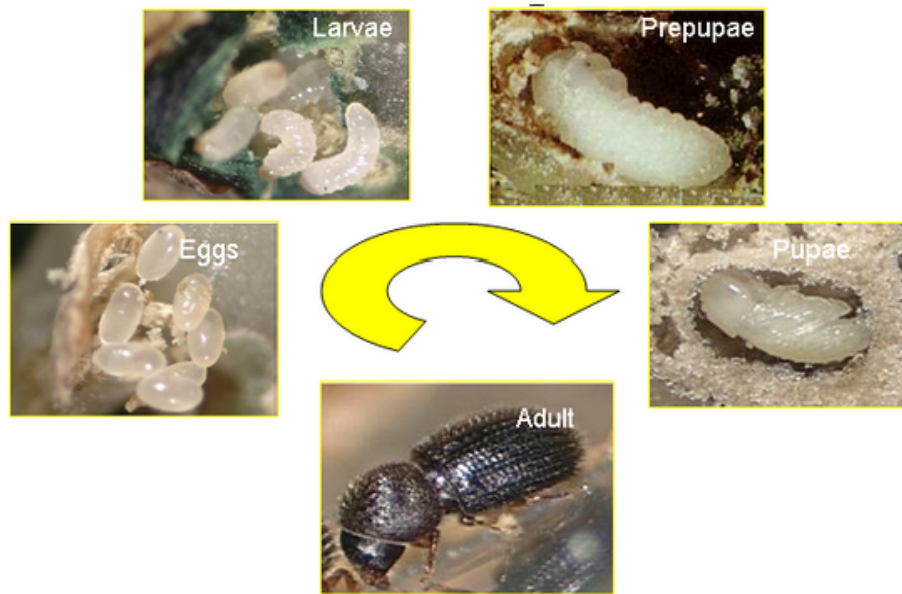


Figure 3: Life cycle of the coffee berry borer *Hypothenemus hampei* [104].

- The adult is a tiny pinhead-sized beetle, cylindrical in shape and dark brown. Its mouthparts are of the grinding type. The female measures between 1.5 and 1.8 mm in length and 0.7 and 0.8 mm in width, while the male measures between 1 and 1.2 mm in length and 0.5 and 0.6 mm in width. Males have vestigial membranous wings and are flightless. The duration of development from egg to adult is positively correlated with temperature and may for example take 28 to 34 days at 27°C [29].

The first four stages are spent inside the coffee berry, in the endosperm, which provides food and shelter. Adult males remain in the berry. Fertilised adult females, which are able to fly, leave the berry to find other berries for oviposition. These females, usually called colonising females are responsible for CBB dispersal in the plantation. They search for an uninfested berry which they enter by drilling a hole. They burrow through the exocarp, mesocarp and endocarp of the berry to reach the endosperm (Figure 2), which may take up to 8 hours. They lay on average 2 to 3 eggs per day during approximately 20 days and remain inside the berry until they die [11, 29]. The sex ratio is skewed towards females and there is a 10:1 female-to-male [97, 29]. The male mate with the young females of their generation and the founder female remains inside the coffee berry after oviposition until she dies. The adult female lifespan ranges from 87 to 282 days, but males are shorter lived with a longevity reported around 40 days [11]. Due to the extended oviposition period of the females, all CBB life stages can be found inside a berry at the same time.

The CBB have several generations per year, for example four generations per year in Tanzania and Colombia, with three generations in Kenya and two generations in Ethiopia. Where environmental conditions allow year-round coffee berry production, such as in Uganda and Ivory Coast, there can be up to eight or nine generations per year [29].

1.1.2.2 Interactions with coffee berries

H. hampei is not specific to coffee. Others plants can provide suitable CBB development conditions, *e.g.* plants of the *Rubiaceae*, *Capparidaceae*, *Passifloraceae*, *Rosaceae*, *Poaceae*, *Sterculiaceae*, *Euphorbiaceae*, *Malvaceae*, *Dioscoreaceae*, *Convolvulaceae*, *Oleaceae*, *Vitaceae* and *Fabaceae* families [29]. However, it is a major pest in coffee plantations worldwide [106].

Many laboratory and field studies have been conducted in several coffee producing zones, indicating that berry ripeness and colour play an important role on the susceptibility of coffee berries to CBB infestation. CBB prefer red and black berries [16, 106, 54, 93]. In addition, the CBB fertility rate is higher in mature berries [81]. However, CBB can also attack small green developing berries that are more than 2–3 months old [13, 62].

After the harvest, CBB survive the intercropping season inside berries that remain on the coffee tree branches or have fallen to the ground. With first rainfall, the coffee tree start to produce new berries and adult females, especially those inside berries lying on the ground, fly off to colonise new unripe berries [36].

The damages caused by CBB are mainly a decrease of coffee yield, due to berry abscission and loss of weight, as well as coffee quality, which impairs its commercial value. If a plantation without any control measure, up to 100% of the berries can be attacked [73, 54]. The infestation levels in untreated plantations are estimated at 60% in Mexico, 60% in Colombia, 50–90% in Malaysia, 58–85% in Jamaica, 80% in Uganda and 90% in Tanzania [62]. In Cameroon, losses caused by CBB on coffee berries were estimated at 10% on a sample of 100 harvested coffee berries [6].

1.1.2.3 Control strategies

Different strategies are implemented to control CBB infestation such as good cultural practices, trapping, chemical and biological control.

Cultural practices Field sanitation is fairly efficient to limit the CBB population. It mainly consists in strip-picking, *i.e.* removing every possible berries remaining in the plantation after harvest and before pruning, berries that either dry on the branches or have dropped on the ground. Indeed, CBB survive in these berries during the intercropping season. However, this control method is regarded as tedious and very labour-intensive, especially picking berries off the ground [11].

Traps Trapping is used for monitoring and pest control. It consists in using attractive traps to capture colonising CBB females during their migration flights. Figure 4 illustrates an example of trap called BROCAP, that was developed by CIRAD, the French agricultural research and cooperation organization working for the sustainable development of tropical and Mediterranean regions, and PROCAFÉ, a coffee research institute in El Salvador. BROCAP is a funnel topped by red blades with a strong attractant at the centre and a transparent recipient bottle at the bottom. Attracted CBB fall inside the funnel connected to the capture recipient containing water and drown. According to [84], a BROCAP trap can capture hundreds up to thousands of CBB colonising females per day in a coffee plantation.

Chemical control It consists in applying synthetic insecticides on the coffee berries in the plantation to kill the CBB, mainly the colonising females that are outside the berries. Undesirable



Figure 4: BROCAP trap, developed by CIRAD and PROCAFÉ [84]. The trap targets CBB colonising females during their migration flight.

side effects include ecological pollution and the development of insecticide resistance in the pest. Endosulfan is the synthetic insecticide most frequently used by coffee growers to control CBB and resistance cases have been observed in New Caledonia [25, 26]. Although still widely used against CBB, chemical control remains very expensive and has limited effectiveness due to the cryptic nature of CBB [29].

Biological control It consists in using CBB natural enemies in order to limit CBB population. The main enemies of CBB are parasitoids, predators and entomopathogenic fungi.

A parasitoid is a living organism that feeds and develops on or within another organism called a host and always causes the death of the host at the end of its own development cycle. The parasitoids of CBB are *Cephalonomia stephanoderis*, *Prorops nasuta* and *Phymastichus coffea* which are all originated from Africa, specifically Uganda, Ivory Coast and Togo respectively. *C. stephanoderis* and *P. nasuta* are small black wasps that enter the CBB infested coffee berries to lay their eggs in CBB larvae; they may also feed on CBB eggs, larvae or young adults [16, 30, 72, 60, 68]. In contrast, *P. coffea* is a small endoparasitoid wasp that attacks CBB females when they bore the coffee berries [106, 29, 63] (Figure 5). This parasite is particularly interesting because it prevents the perforation of coffee berries.

Entomopathogenic fungi are fungi that grow at the expense of insects and kill them during their development. *Beauveria bassiana* is the most-studied fungus for the control of coffee pests, in particular CBB [27, 56, 95, 6, 19, 52]. This fungus has the advantage of not being a harmful pathogen for humans and is able to infect its insect host through a simple contact. It is sprayed on the surface of the coffee berries and kills the colonising CBB females when they select their coffee berry and begin to drill an entry hole (Figure 6). Once the CBB female is infected, the fungus rapidly grows inside of the CBB, feeding on the nutrients present in the CBB body and producing toxins in the process. When the CBB dies, *B. bassiana* covers the carcass with a layer of white mold that produces more infectious spores [52, 95, 106]. The main factors limiting the use of this fungus in the field are abiotic conditions, that, when unfavourable, lead to a loss of efficacy of the fungal inoculum on the plant canopy [106].

IPM Integrated pest management (IPM) aims at keeping pest populations below an

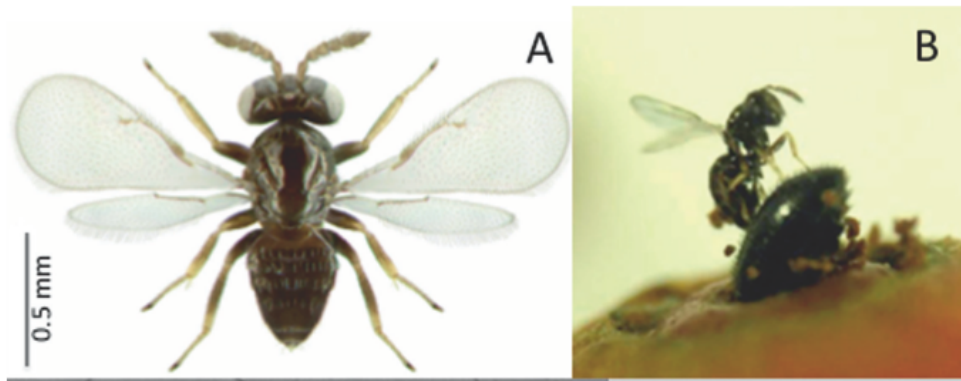


Figure 5: CBB parasitoid wasp *Phymastichus coffea*: (A) adult; (B) adult capturing a CBB colonising female as it is boring a coffee berry [106].

[Credits: (A) G.Goergen, IITA; (B) A. Castillo and F. Infante, ECOSUR]



Figure 6: CBB infected with fungal entomopathogen *Beauveria bassiana* [106]: colonising female infected while boring a hole in a coffee berry (left) and covered with *B. bassiana* white mold.

[Credits: (left) A. Ramírez, Puerto Rico Department of Agriculture; (right) F. Posada]

economically acceptable level, while minimising risks to human health and the environment, in the long run. IPM relies on several monitoring and control methods. Sustainable management of CBB could be based on the combination of traps for the early detection of CBB, good cultural practices such as strip-picking to reduce CBB survival during the intercropping season, post-harvest control to limit re-infestation from harvest facilities, and the application of *B. bassiana* [11].

1.2 Mathematical background

In this section, we present techniques for the analysis of nonlinear optimal control problems, based on systems governed by ordinary and partial differential equations. The optimal control theory is a branch of dynamical optimisation and is a generalisation of calculus of variations. Optimal control addresses the problem of finding a control input for a given system that optimises a certain measure or performance criterion called objective (or cost) functional. There are many disciplines of science and technology in which control theory plays an important role and faces fascinating challenges. The necessary first order conditions to find the optimal control were

developed by Pontryagin and his co-workers [92]. This result is considered as one of the most important mathematical result of the 20th century. Pontryagin introduced the idea of adjoint functions to append the differential equation to the objective functional. Adjoint functions have a similar purpose as Lagrange multipliers in multivariate calculus of variations, which append constraints to the function of several variables to be maximised or minimised. We focus here on the Pontryagin Maximum Principle [92] for nonlinear differential equations and its extension by Feichtinger *et al.* [39] for the partial differential equations with an age structure. These results help us solve the control problems in the following chapters. We end this section with the presentation of numerical methods for optimal control that we also use in the following chapters.

1.2.1 Optimal control for ordinary differential equations

Consider on \mathbb{R}^n the autonomous control system:

$$\dot{x}(t) = f(x(t), u(t)), \quad (1.1)$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$ is C^1 and where the control $u(\cdot)$ is a bounded and measurable function defined on interval $[0, t_f]$ in \mathbb{R}_+ (t_f being fixed) and takes its values in the subset \mathcal{U} of \mathbb{R}^m . Denote by \mathcal{U}_{ad} the set of admissible controls such that the corresponding trajectories steer the system from an initial point $x(0) \in \mathbb{R}^n$. For a such control $u(\cdot)$, the objective (or cost) functional of the corresponding trajectory $x(\cdot)$ is defined by:

$$\mathcal{J}(u) := \int_0^{t_f} f^0(x(t), u(t)) dt + g(x(t_f)), \quad (1.2)$$

where the functions $f^0 : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}$ are C^1 . We investigate the optimal control problem of determining a trajectory $x(\cdot)$ solution of (1.1) associated with a control $u(\cdot)$ on $[0, t_f]$ minimising the objective functional $\mathcal{J}(\cdot)$.

Theorem 1 (Pontryagin Maximum Principle (PMP) [92]) *If the trajectory $x(\cdot)$ associated with the optimal control $u(\cdot)$ on $[0, t_f]$ is optimal, then it is the projection of an extremal $(x(\cdot), p(\cdot), p^0, u(\cdot))$ (called extremal lift), where $p^0 \leq 0$ and $p(\cdot) : [0, t_f] \rightarrow \mathbb{R}^n$ is an absolutely continuous mapping, called adjoint vector, such that the following conditions are satisfied:*

1. *Nontriviality of the multipliers – $(p(\cdot), p^0) \neq (0, 0)$ for all $t \in [0, t_f]$,*
2. *Adjoint equation – the adjoint vector $p(\cdot)$ is a solution to the time-varying linear differential equation:*

$$\dot{p}(t) = -\partial_x \mathcal{H}(x(t), p(t), p^0, u(t)),$$

almost everywhere on $[0, t_f]$, where $\mathcal{H}(x, p, p^0, u(t)) := p^0 f^0(x, u) + \langle p, f(x, u) \rangle$ is called the Hamiltonian function.

3. *Maximum condition – almost everywhere in $[0, t_f]$, we have:*

$$\mathcal{H}(x(t), p(t), p^0, u(t)) = \max_{v \in \mathcal{U}} \mathcal{H}(x(t), p(t), p^0, v).$$

4. *Transversality condition* – at the endpoint of the adjoint vector satisfies:

$$p(t_f) = p^0 \partial_x g(x(t_f)).$$

Definition 1 *If the multipliers $p^0 = 0$, the extremal is called abnormal while it is called normal if $p^0 \neq 0$.*

Definition 2 *The extremal lift $(x(\cdot), p(\cdot), p^0, u(\cdot))$ is said to be singular on an open interval $I \subset [0, t_f]$ if the switching function $\Phi(t) = \partial_u \mathcal{H}(x(t), p(t), p^0, u(t)) = 0$ on I . We say in this case that the control $u(\cdot)$ is singular on I and call the corresponding portion of the controlled trajectory a singular arc.*

Generally, singular controls are determined by differentiating the switching function until the controls explicitly appear and then solving the resulting equations for the control.

Definition 3 *The singular control is said to be of intrinsic order k over an open interval I , if the first $2k-1$ derivatives of the switching function vanish identically and $\partial_u \frac{d^{2k}}{dt^{2k}} (\partial_u \mathcal{H}(x(t), p(t), p^0, u(t)))$ does not vanish on I .*

Singular control arise from the extremal condition of the PMP when the maximisation condition is satisfied trivially because $\partial_u \mathcal{H}(x(t), p(t), p^0, u(t)) = 0$. However, this condition does not distinguish between maximisation and minimisation and thus singular controls can be both minimising or maximising. It is the quantity $\partial_u \frac{d^{2k}}{dt^{2k}} (\partial_u \mathcal{H}(x(t), p(t), p^0, u(t)))$ that distinguishes between these two cases, thanks to the following Legendre-Clebsch condition [24, 96].

Theorem 2 (Legendre-Clebsch condition) *If the controlled trajectory $(x(\cdot), u(\cdot))$ is optimal and the control $u(\cdot)$ is singular of intrinsic order k on an open interval $I \subset [0, t_f]$, then*

$$(-1)^k \partial_u \frac{d^{2k}}{dt^{2k}} (\partial_u \mathcal{H}(x(t), p(t), p^0, u(t))) \leq 0 \quad \text{for all } t \in I$$

1.2.2 Optimal control for age-structured systems

In population dynamics and mathematical epidemiology, models including an age structure by means of partial differential equations are widely used. Optimal control problems for such age-structured systems are of interest in many application fields, such as competing species [41], harvesting control [7, 88], birth control [77] and epidemic disease control [4, 5, 17, 57]. For general mathematical techniques, we can mention the books by [15, 75] and references therein.

We now present the general maximum principle that allows to derive the first order necessary optimality conditions for age-structured systems and that was established by Feichtinger et al [39]. It is used in Chapter 4 to solve our optimal control problem.

Let us consider the following general optimal control problem:

$$\text{minimise } \int_0^A l(a, y(T, a)) da + \int_0^T \int_0^A L(t, a, y(t, a), p(t, a), q(t), u(t, a), v(t), \omega(a)) da dt$$

subject to the following equations:

$$\begin{aligned}\partial_t y(t, a) + \partial_a y(t, a) &= f(t, a, y(t, a), p(t, a), q(t), u(t, a)) \\ p(t, a) &= \int_0^A g(t, a, a', y(t, a'), u(t, a')) da' \\ q(t) &= \int_0^A h(t, a, y(t, a), p(t, a), q(t), u(t, a)) da\end{aligned}$$

with initial conditions:

$$y(0, a) = y^0(a, \omega(a)),$$

boundary conditions:

$$y(t, 0) = \varphi(t, q(t), v(t)), \quad (1.3)$$

and control constraints:

$$u(t, a) \in U, \quad v(t) \in V, \quad \omega(a) \in W. \quad (1.4)$$

Here, the time $t \in [0, T]$, $a \in [0, A]$. Moreover, $y(t, a) = (y_1(t, a), \dots, y_m(t, a)) \in \mathbb{R}^m$, $p(t, a) = (p_1(t, a), \dots, p_n(t, a)) \in \mathbb{R}^n$ and $q(t) = (q_1(t), \dots, q_r(t)) \in \mathbb{R}^r$ are the states for the system. $u(t, a) \in U$, $v(t) \in V$ and $\omega(a) \in W$ are the distributed, boundary and initial controls respectively, where U , V and W are subsets of finite dimensional linear normed spaces. l , L , f , g , h , y^0 and φ are given functions, A and T are positive numbers.

We denote by $D = [0, T] \times [0, A]$ and assume that the set U , V and W are compact, V and W being convex. The functions l , L , f , g , h , y^0 and φ are Carathéodory (that is measurable in t , a , a' and continuous in the rest of the variables) locally essentially bounded, differentiable in (y, p, q, v, ω) , with Lipschitz partial derivatives, uniformly with respect to $u \in U$ and $(t, a) \in D$, $a' \in [0, A]$. The i -th component of function h is independent of q_j for each $i = 1, \dots, r$ and $j \geq i$. An admissible control is any triplet (u, v, ω) with measurable functions $u : D \rightarrow U$, $v : [0, T] \rightarrow V$ and $\omega : [0, \omega] \rightarrow W$.

Let us introduce the following conventional notation:

$$\begin{aligned}L(t, a) &:= L(t, a, y(t, a), p(t, a), q(t), u(t, a), v(t), \omega(a)) \\ f(t, a) &:= f(t, a, y(t, a), p(t, a), q(t), u(t, a)), \\ g(t, a, a') &:= g(t, a, a', y(t, a'), u(t, a')), \\ h(t, a) &:= h(t, a, y(t, a), p(t, a), q(t), u(t, a)).\end{aligned}$$

We denote by ∇_z s the differentiation with respect to the variable z . We introduce the following adjoint system for the adjoint function (ξ, η, ζ) , considered as row-vector functions from

$L^\infty(D; \mathbb{R}^m) \times L^\infty(D; \mathbb{R}^n) \times L^\infty([0, T]; \mathbb{R}^r)$:

$$\begin{aligned} -\partial_t \xi(t, a) - \partial_a \xi(t, a) &= \nabla_y L(t, a) + \xi(t, a) \nabla_y f(t, a) + \zeta(t) \nabla_y h(t, a) \\ &\quad + \int_0^A \eta(t, a') \nabla_y g(t, a, a') da' \\ \xi(T, a) &= \nabla_y l(a, y(T, a)), \quad \xi(t, A) = 0, \\ \eta(t, a) &= \nabla_p L(t, a) + \xi(t, a) \nabla_p f(t, a) + \zeta(t) \nabla_p h(t, a) \\ \zeta(t) &= \xi(t, 0) \nabla_q \varphi(t) + \int_0^A [\nabla_q L(t, a) + \xi(t, a) \nabla_q f(t, a) + \zeta(t) \nabla_q h(t, a)] da \end{aligned}$$

We assume that the following holds:

Condition 1 *There exists an optimal solution $(y, p, q, u, v, \omega) \in L^\infty(D; \mathbb{R}^m) \times L^\infty(D; \mathbb{R}^n) \times L^\infty([0, T]; \mathbb{R}^r) \times L^\infty(D; U) \times L^\infty([0, T]; V) \times L^\infty([0, A]; W)$.*

For the solution (y, p, q) and the corresponding solution (ξ, η, ζ) of the adjoint system, we define the initial, boundary and distributed Hamiltonians:

$$\begin{aligned} H_0(a, \omega) &:= \xi(t, 0) y^0(a, \omega) + \int_0^T L(s, a, \omega) ds \\ H_b(t, v) &:= \xi(t, 0) \varphi + \int_0^A L(t, b, \omega) db \\ H(t, a, u) &:= L(t, a, u) + \xi(t, a) f(t, a, u) + \int_0^A \eta(t, a') g(t, a, a', u) da' + \zeta(t) h(t, a, \omega) \end{aligned}$$

Theorem 3 (Pontryagin Maximum Principle [39]) *Under Condition 1, the adjoint system has a unique solution ξ, η, ζ and for a.e. $t_0 \in [0, T]$, $a_0 \in [0, A]$ and $(t, a) \in D$:*

$$\begin{aligned} \partial_\omega H_0(a_0, \widehat{\omega}(a_0))(\omega - \widehat{\omega}(a_0)) &\geq 0 & \forall \omega \in W, \\ \partial_v H_b(t_0, \widehat{v}(t_0))(v - \widehat{v}(t_0)) &\geq 0 & \forall v \in V, \\ H(t, a, u(t, a)) - H(t, a, \widehat{u}(t, a)) &\geq 0 & \forall u \in U. \end{aligned}$$

1.2.3 Numerical methods in optimal control theory

In general, it is not possible to solve optimal problems analytically. Therefore, we use numerical methods to approximate solutions and display results. There are two types of numerical methods in optimal control: direct and indirect methods. Indirect methods consist in solving numerically the boundary value problem derived from the Pontryagin Maximum Principle. The direct methods are usually less precise than indirect methods, but more robust with respect to the initialisation and they are easier to use.

We present here the direct method implemented in the BOCOP software, used in Chapter 2, and a indirect method called *forward-backward sweep*, used in Chapters 4 and 5.

1.2.3.1 Direct method

Direct methods consist in discretising the time for the state and control variables, as well as the dynamics equations, and thus reducing the optimal control problem to a finite dimensional

optimisation problem with constraints.

Let us consider the following optimal control problem, noting x the state and u the control:

$$\left\{ \begin{array}{ll} \min_{u \in \mathcal{U}} \left\{ \mathcal{J}(x(\cdot), u(\cdot)) = \int_0^{t_f} f^0(x(t), u(t)) dt + g(x(t_f)) \right\} & \text{Objective functional} \\ \text{s.t. } \dot{x}(t) = f(x(t), u(t)) \quad \forall t \in [0, t_f] & \text{Dynamics} \\ u(t) \in \mathcal{U} \quad \text{for a.e. } t \in [0, t_f] & \text{Admissible controls} \\ \phi(x(t), u(t)) \leq 0 & \text{Path constraints} \\ c(x(0), x(t_f)) = 0 & \text{Boundary conditions} \end{array} \right.$$

Summary of the time discretisation, using for example the Euler formula:

$$\begin{array}{ll} t \in [0, t_f] & \longrightarrow \{t_0 = 0, \dots, t_N = t_f\} \\ x(\cdot), u(\cdot) & \longrightarrow X = \{x_0, \dots, x_N, u_0, \dots, u_{N-1}\} \\ \text{Criterion} & \longrightarrow \min h \sum_{i=0}^{N-1} f^0(x_i, u_i) + g(x_N) \\ \text{Dynamics} & \longrightarrow x_{i+1} = x_i + f(x_i, u_i), \quad i = 0, \dots, N-1 \\ \text{Controls} & \longrightarrow u_i \in \mathcal{U}, \quad i = 0, \dots, N \\ \text{Path constraints} & \longrightarrow \phi(x_i, u_i) \leq 0, \quad i = 0, \dots, N \\ \text{Boundary conditions} & \longrightarrow c(x_0, u_N) = 0 \end{array}$$

We therefore obtain a nonlinear programming problem on the discretised state and control variable:

$$(NLP) \quad \left\{ \begin{array}{l} \min F(X) \\ LB \leq C(X) \leq UB. \end{array} \right.$$

BOCOP is an open-source toolbox for solving optimal control problem (optimisation of dynamical systems governed by differential equations) by the direct method [23, 103]. It is supported by the Team Commands of the INRIA-Saclay France. This toolbox implements a local optimisation method, where the optimal control problem is approximated by a finite dimensional optimisation problem (NLP) using the time discretisation. The NLP is solved by the software IPOPT solver [109] that implements a primal-dual interior point algorithm. The derivatives required for the optimization are computed by the automatic differentiation tool CppAD [18].

1.2.3.2 Forward-backward sweep

The forward-backward sweep method is a numerical technique for solving optimal control problems. The technique is one of the indirect method in which the differential equations from the maximum principle are numerically solved. Initially designed to solve control problem associated with ordinary differential equations, it has been extended to partial differential equations [75, 8].

This method can be summarised by the following steps:

Step 1: Make an initial guess for control variable values over time.

Step 2: Given initial conditions for the state variables and the current control values, approximate the state variables by integrating the state dynamics, using for example the finite differential forward sweep method (see [8] for the age-structured systems).

Step 3: Given the state solutions from the previous step and the final conditions for the adjoint system, approximate the adjoint variables, using for example finite backward sweep method.

Step 4: Update the control variable values by the new values arising from the control characterisation.

Step 5: Repeat Steps 2–4 until successive values of all states, adjoint and control variables are sufficiently close, that is:

$$\frac{\|z - z_{old}\|}{\|z\|} \leq \epsilon,$$

where ϵ is the accepted tolerance, z is the state vector of current values (state, adjoint and control variables), z_{old} is the vector from the previous iteration and finally $\|\cdot\|$ refers to the norm of vector z .

Convergence of the forward-backward sweep method has been established in [83].

Modelling and optimal control strategy of coffee berry borer

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2.1 Introduction

In this chapter, we first formulate and study a mathematical model governed by nonlinear differential equations to describe the infestation dynamics of coffee berries by CBB during a cropping season, taking into account berry availability. It constitutes the baseline model of this thesis. Secondly, we propose and implement an optimal control strategy based on the release of entomopathogenic fungus *B. Bassiana* in order to maximise the yield while minimising the cost, as well as the CBB population at the end of the cropping season to prevent too high infestations for the next season.

This control problem was first introduced in [44], but the control used was non persistent and a quadratic objective function was considered. In this work, we propose a more realistic optimal control problem, by introducing an extra state variable to represent the dynamics of *B. bassiana*: the fungus is released and persists in the plantation. Moreover, we consider a linear objective function which represents the cost of fungus application and hence the profit better than a quadratic function. Our control problem is solved analytically using Pontryagin's maximum principle (PMP) [92] and numerically using a direct method implemented in the BOCOP software.

2.2 Model formulation

The model describes the infestation dynamics of coffee berries by CBB during a cropping season in a plantation, taking into account the CBB life cycle and the berry availability. At each time t , the coffee berries in the plantation are subdivided into two compartments: the healthy berries represented by their number $s(t)$ and the infested coffee berries represented by $i(t)$. As only fertilised females are responsible for CBB dispersal, we only consider this development stage in the model. We subdivide these females into two compartments: the colonising females represented by their number $y(t)$, which correspond to the flying fertilised females looking for their host, and the infesting females represented by their number $z(t)$, which correspond to the females that are laying eggs inside the berries. Figure 7 illustrates the infestation dynamics of coffee berries by CBB.

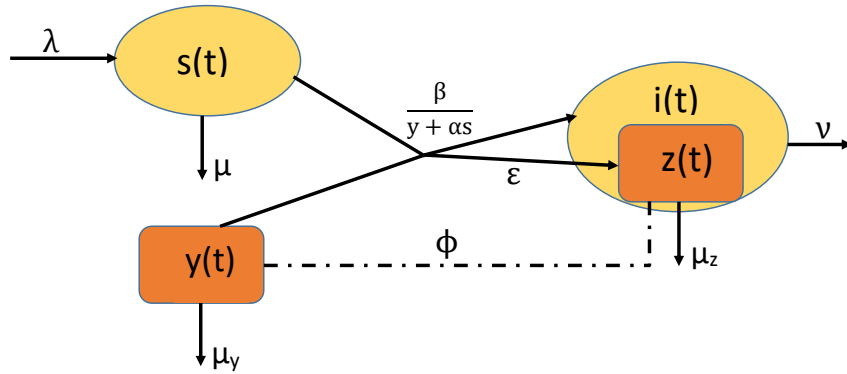


Figure 7: Model diagram. Healthy berries are produced at a constant rate (λ). CBB colonising females (y) enter healthy berries (s) with ratio-dependent rate (parameters α, β and berry-CBB conversion rate ϵ). Colonising CBB become infesting females (z) and healthy berries become infested (i). Infesting females lay eggs inside the berries, from which colonising females emerge (at rate ϕ). All CBB and berry stages undergo mortality (at rates μ, ν, μ_y, μ_z).

In accordance with Figure 7, we derive the following system of nonlinear ordinary differential

equations:

$$\begin{cases} \dot{s}(t) = \lambda - \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu s(t), \\ \dot{i}(t) = \beta \frac{sy}{y + \alpha s} - \nu i, \\ \dot{y}(t) = \phi z - \varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_y y(t), \\ \dot{z}(t) = \varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_z z(t). \end{cases}$$

The model hypotheses are described below.

- We assume that flowering occurs throughout the season, which translates into a constant healthy berry production rate λ .
- Although the sex ratio is largely female biased, we assume that there are enough males in each generation to fertilise young females (no fertility issues were raised in the field).
- We represent the infestation through a ratio-dependent function $f(s, y) = \beta \frac{sy}{y + \alpha s}$. Ratio-dependence [9, 10, Chapters 1 and 6] is particularly adapted to this pathosystem as there is a one-to-one interaction between CBB and berries. Ratio-dependent interactions are invariant to a proportional change of all abundances in the system. Moreover, classical Holling type II functional response implements a saturation term on the resource (due to handling time or satiety), which is ill-adapted to this case, especially when CBB are abundant. With a ratio-dependent function, when healthy berries are largely outnumbered by colonising females ($s \ll y$), berries determine the infestation response ($f(s, y) \approx \beta s$), and vice versa. The definition of f is extended to $f(0, 0) = 0$ [67]. Parameter β represents the infestation rate per unit of time and α the infestation half saturation constant.
- Super-parasitism is rarely observed in a plantation: colonising females can usually detect infested coffee berries, thanks to the hole drilled in the berry by the previous CBB, and they rather choose another healthy berry to lay their eggs [54, 93]. Hence, a healthy coffee berry is usually infested by a single colonising female. The parameter ε , which is the number of colonising females that can infest a healthy coffee berry and become infesting females, is hence equal to 1 CBB per berry.
- The number ϕ of colonising females emerging from the berries per infesting female and per time unit is supposed constant, as infesting females lay eggs at a constant rate over their fertility period.
- The mortality rates of healthy and infested coffee berries are μ and ν respectively, with $\mu < \nu$ as CBB infestation damages the berries.
- μ_y is the natural mortality rate of colonising females; the average life expectancy $1/\mu_y$ of colonising females is much shorter than the average adult lifespan, as colonising females do not have access to coffee berries, which are their sole food source. Infesting females survive their egg-laying period for quite some time but, as they then do not contribute to the infestation dynamics anymore, they are not accounted for in the z -compartment. Hence, $1/\mu_z$ corresponds to the average egg-laying period of infesting females.

Since variable $i(t)$ has no impact on the dynamics of the three remaining variables, then its dynamics is determined by those of the other variables. we can reduce our analysis to the following model:

$$\begin{cases} \dot{s}(t) = \lambda - \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu s(t) \\ \dot{y}(t) = \phi z - \varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_y y(t) \\ \dot{z}(t) = \varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_z z(t) \end{cases} \quad (2.1)$$

with the initial condition:

$$s(0) \geq 0, \quad y(0) > 0 \quad \text{and} \quad z(0) \geq 0. \quad (2.2)$$

2.3 Model analysis

2.3.1 Positivity

For system (2.1) to be mathematically and biologically meaningful, it is important to verify that all state variables remain nonnegative and are bounded for nonnegative initial conditions and for all $t \geq 0$. We have the following result.

Proposition 1 *The compact domain*

$$\Gamma := \left\{ (s, y, z) \in \mathbb{R}_+^3 : s \leq \frac{\lambda}{\mu}, \varepsilon s + z \leq \frac{\varepsilon \lambda}{\zeta}, y \leq \frac{\varepsilon \phi \lambda}{\zeta \mu_y} \right\},$$

where $\zeta = \min\{\mu, \mu_z\}$, is positively invariant for the flow of system (2.1).

Proof : In order to prove positivity of the solution of this system, we can consider each state variable at 0 (with the other taking non-negative values) and check that it cannot become non positive. Since we have $\dot{s} = \lambda > 0$ when $s = 0$, $\dot{y} = \phi z \geq 0$ when $y = 0$ and $\dot{z} = \varepsilon \beta \frac{sy}{y+\alpha s} \geq 0$ when $z = 0$, the first orthant is invariant, so that all solutions of system (2.1) remain in \mathbb{R}_+^3 at all time $t \in \mathbb{R}_+$ for any initial condition (2.2). Using the nonnegativity of the state variables of system (2.1), one first gets:

$$\dot{s} \leq \lambda - \mu s \quad (2.3)$$

so that, when $s = \frac{\lambda}{\mu}$, $\dot{s} \leq 0$. We also have:

$$\varepsilon \dot{s} + \dot{z} \leq \varepsilon \lambda - \varepsilon \mu s - \mu_z z \leq \varepsilon \lambda - \zeta (\varepsilon s + z) \quad (2.4)$$

so that, when $\varepsilon s + z = \frac{\varepsilon \lambda}{\zeta}$, then $\varepsilon \dot{s} + \dot{z} \leq 0$. Finally, we have:

$$\dot{y} \leq \phi z - \mu_y y \leq \phi \frac{\varepsilon \lambda}{\zeta} - \mu_y y, \quad (2.5)$$

when $z \leq \frac{\varepsilon \lambda}{\zeta}$, so that, when $y = \frac{\varepsilon \phi \lambda}{\zeta \mu_y}$, one has $\dot{y} \leq 0$. Therefore, the compact set Γ is positively invariant. It is noticed that the right hand side of system (2.1) is locally Lipschitz continuous on the non-negative orthant, this system has a unique solution for any non-negative initial condition

(2.2). Thus, by the classical results of dynamical systems, for any initial condition in (2.2) the Cauchy problem associated with system (2.1) has a unique solution in $C(\mathbb{R}_+, \mathbb{R}_+^3)$. \square

2.3.2 Asymptotic behaviour

In this section, we study the existence and stability of the model equilibria. Although the model is only valid over a cropping season $[0, t_f]$, such an asymptotic analysis gives an idea of the system behaviour during this finite period, especially if the transient dynamics are fast, which can be expected with CBB that very efficiently colonise plantations. System (2.1) has a pest-free steady state $\mathcal{E}^0 = (s^0, 0, 0)$ with $s^0 = \frac{\lambda}{\mu}$. We compute the basic reproduction number, \mathcal{R}_0 , using the next generation operator approach [35]. Let $F = [\phi z, 0]$ be the emergence of CBB and $V = \left[\frac{\varepsilon\beta sy}{y+\alpha s} + \mu_y y, -\frac{\varepsilon\beta sy}{y+\alpha s} + \mu_z z \right]$ the outflows minus the inflows of the CBB compartments, emergence excepted. The Jacobian matrices of F and V at the pest-free steady state \mathcal{E}^0 are respectively given by:

$$\mathcal{F} = \begin{bmatrix} 0 & \phi \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \mathcal{V} = \begin{bmatrix} \varepsilon\beta\frac{1}{\alpha} + \mu_y & 0 \\ -\varepsilon\beta\frac{1}{\alpha} & \mu_z \end{bmatrix}.$$

The basic reproduction number is defined as the spectral radius of the next generation matrix $\mathcal{F}\mathcal{V}^{-1}$:

$$\mathcal{R}_0 = \rho(\mathcal{F}\mathcal{V}^{-1}) = \frac{\phi\varepsilon\beta\frac{1}{\alpha}}{\mu_z(\varepsilon\beta\frac{1}{\alpha} + \mu_y)}. \quad (2.6)$$

Remark 1 *The biological interpretation of the basic reproduction number \mathcal{R}_0 is the following. The factor $\frac{\varepsilon\beta\frac{1}{\alpha}}{\varepsilon\beta\frac{1}{\alpha} + \mu_y}$ is the average number of infesting females produced by a colonising female during its lifespan, whereas the factor $\frac{\phi}{\mu_z}$ is the average number of colonising females produced by an infesting female. Then the basic reproduction number \mathcal{R}_0 , corresponding to the product of these two factors, represents the average number of new infesting (or colonising) females originated from a single infesting (or colonising) female during its lifespan.*

Based on the basic reproduction number, we obtain the following stability results.

Theorem 4 *The pest-free steady state \mathcal{E}^0 is globally asymptotically stable (GAS) if $\mathcal{R}_0 \leq 1$.*

Proof : In order to prove the global asymptotical stability of the pest-free equilibrium point, we first show attractivity in finite time of a set that is slightly larger than Γ defined in Proposition 1. We will show that any solution initiated in \mathbb{R}_+^3 outside of

$$\Gamma_\delta := \left\{ (s, y, z) \in \mathbb{R}_+^3 : s \leq \frac{\lambda}{\mu} + \delta, \varepsilon s + z \leq \frac{\varepsilon\lambda}{\zeta} + \delta, y \leq \frac{\varepsilon\phi\lambda}{\zeta\mu_y} + \frac{\phi\delta}{\mu_y} + \delta \right\}$$

for some $\delta > 0$, enters Γ_δ in finite time. In fact, if $(s(0), y(0), z(0))$ is outside Γ_δ , it violates at least one of the inequalities defining Γ_δ . If we first suppose that $s > \frac{\lambda}{\mu} + \delta$, equation (2.3) yields:

$$\dot{s} \leq \lambda - \mu s < \lambda - \mu \left(\frac{\lambda}{\mu} + \delta \right) = -\mu\delta < 0. \quad (2.7)$$

$\dot{s} < -\mu\delta$ implies that:

$$s(t) < s(0) - \mu\delta t. \quad (2.8)$$

If we suppose that $s(t)$ never reaches $\lambda/\mu + \delta$, we can write inequality (2.8) at time $t_\delta = (s(0) - \lambda/\mu - \delta)/(\mu\delta) \geq 0$. It then reads $s(t_\delta) < \lambda/\mu + \delta$, which is in contradiction with the hypothesis that $s(t)$ never reaches $\lambda/\mu + \delta$. Hence, $s(t)$ reaches $\lambda/\mu + \delta$ in finite time and will stay below this value afterwards.

Similarly, if we now suppose that $\varepsilon s + z > \frac{\varepsilon\lambda}{\zeta} + \delta$, equation (2.4) yields:

$$\varepsilon\dot{s} + \dot{z} \leq \varepsilon\lambda - \zeta(\varepsilon s + z) < \varepsilon\lambda - \zeta\left(\frac{\varepsilon\lambda}{\zeta} + \delta\right) = -\zeta\delta$$

and $\varepsilon s + z$ will reach $\frac{\varepsilon\lambda}{\zeta} + \delta$ in finite time and stay below this value afterwards.

Finally, supposing now that $y > \frac{\varepsilon\phi\lambda}{\zeta\mu_y} + \frac{\phi\delta}{\mu_y} + \delta$ with $\varepsilon s + z \leq \frac{\varepsilon\lambda}{\zeta} + \delta$ (which may be valid at time 0 or after the previously described convergence in finite time), equation (2.5) yields:

$$\dot{y} \leq \phi z - \mu_y y < \phi\left(\frac{\varepsilon\lambda}{\zeta} + \delta\right) - \mu_y\left(\frac{\varepsilon\phi\lambda}{\zeta\mu_y} + \frac{\phi\delta}{\mu_y} + \delta\right) = -\delta\mu_y$$

and y will reach $\frac{\varepsilon\phi\lambda}{\zeta\mu_y} + \frac{\phi\delta}{\mu_y} + \delta$ in finite time and stay below this value afterwards. Hence, the compact set Γ_δ is reached in finite time and is positively invariant.

Locally asymptotically stability of the pest-free steady state is readily obtained when $\mathcal{R}_0 < 1$ as a consequence of the construction of \mathcal{R}_0 through the next generation matrix approach of [35]. Also this equilibrium is unstable if $\mathcal{R}_0 > 1$.

For global attractivity of the equilibrium, we will now distinguish two cases depending on the value taken by the threshold $\mathcal{R} = \frac{\phi}{\mu_z}$.

If $\mathcal{R} < 1$ (which implies $\mathcal{R}_0 < 1$), let us consider the total population of female CBB $w = y + z$. Its dynamics is governed by:

$$\dot{w} = -\mu_z(1 - \mathcal{R})z - \mu_y y \leq -\kappa w,$$

where $\kappa = \min\{\mu_z(1 - \mathcal{R}), \mu_y\}$. So using the comparison principle, we have $w(t) \leq w(0) \exp(-\kappa t)$ which converges to 0, when $t \rightarrow +\infty$. As a consequence, the variable $s(t)$ will converge to $\frac{\lambda}{\mu}$ when $t \rightarrow +\infty$, and the pest-free equilibrium is globally asymptotically stable.

If $\mathcal{R} \geq 1$ and $\mathcal{R}_0 \leq 1$, let us consider the following Lyapunov candidate function $\mathcal{V}(s, y, z) = \mu_y y + \phi z$, whose time derivative verifies:

$$\begin{aligned} \dot{\mathcal{V}}(s, y, z) &= -\mu_z \mu_y y + (-\mu_z + \phi) \frac{\varepsilon\beta s y}{y + \alpha s} \\ &= -\mu_z \mu_y y + (-\mu_z + \phi) \varepsilon\beta \frac{1}{\alpha} y - (-\mu_z + \phi) \varepsilon\beta \frac{1}{\alpha} y + (-\mu_z + \phi) \frac{\varepsilon\beta s y}{y + \alpha s} \\ &= \left(-\mu_z \left(\varepsilon\beta \frac{1}{\alpha} + \mu_y\right) + \varepsilon\beta \frac{1}{\alpha} \phi\right) y - (-\mu_z + \phi) \varepsilon\beta \frac{y}{\alpha} \left(1 - \frac{\alpha s}{y + \alpha s}\right) \\ &= - \left[\mu_z (\mu_y + \varepsilon\beta \frac{1}{\alpha}) (1 - \mathcal{R}_0) + (\mathcal{R} - 1) \mu_z \varepsilon\beta \frac{1}{\alpha} \frac{y}{y + \alpha s} \right] y. \end{aligned}$$

As $\mathcal{R} \geq 1$ and $\mathcal{R}_0 \leq 1$, $\dot{\mathcal{V}} \leq 0$, so \mathcal{V} is indeed a Lyapunov function for the pest-free steady state defined in Γ_δ . This proves that \mathcal{E}^0 is stable. It remains to prove that \mathcal{E}^0 is attractive. We shall use the LaSalle invariant principle. First of all notice that when $\mathcal{R}_0 = 1$, then $\mathcal{R} = 1 + \frac{\alpha\mu_y}{\varepsilon\beta} > 1$,

such that it is impossible to have both \mathcal{R}_0 and \mathcal{R} equal to 1 simultaneously. This implies that the expression inside the square brackets is always positive unless $y = 0$. Finally, the largest invariant set contained in $M = \{(s, y, z) \in \Gamma_\delta : \dot{\mathcal{V}} = 0\} = \{(s, 0, z) \in \Gamma_\delta\}$ is the pest-free steady state \mathcal{E}^0 . Indeed, if $z \neq 0$, we have $\dot{y} > 0$, so that the corresponding solution leaves M . Also, if $s \neq \lambda/\mu$, the s dynamics in backward time (y and z remaining zero) bring the solution outside Γ_δ . Thus, using the LaSalle invariance principle [71], we conclude that the pest-free steady state \mathcal{E}^0 is attractive inside Γ_δ which, combined with the global attractivity of Γ_δ and the local stability, ensures that \mathcal{E}^0 is globally asymptotically stable. \square

Theorem 5 *When $\mathcal{R}_0 > 1$, there exists a unique coexistence steady state $\bar{\mathcal{E}} = (\bar{s}, \bar{y}, \bar{z})$ for system (2.1) where \bar{s} , \bar{y} and \bar{z} are defined as follows:*

$$\bar{s} = \frac{\lambda \left(\frac{1}{\mu_y} (\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \alpha \right)}{\frac{1}{\mu_y} (\mu + \beta)(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \mu\alpha}; \quad \bar{y} = \frac{1}{\mu_y} (\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1)\bar{s}; \quad \bar{z} = \frac{\varepsilon\beta}{\mu_z} \frac{\bar{y}\bar{s}}{\bar{y} + \alpha\bar{s}}. \quad (2.9)$$

Moreover, this unique steady state $\bar{\mathcal{E}}$ is locally asymptotically stable.

Proof : Let $\bar{\mathcal{E}} = (\bar{s}, \bar{y}, \bar{z})$ be the positive coexistence steady state of system (2.1). Then, the components of $\bar{\mathcal{E}}$ solve for positive values the following system:

$$\begin{cases} \lambda - \beta \frac{\bar{s}\bar{y}}{\bar{y} + \alpha\bar{s}} - \mu\bar{s} = 0, \\ \phi\bar{z} - \varepsilon\beta \frac{\bar{s}\bar{y}}{\bar{y} + \alpha\bar{s}} - \mu_y\bar{y} = 0, \\ \varepsilon\beta \frac{\bar{s}\bar{y}}{\bar{y} + \alpha\bar{s}} - \mu_z\bar{z} = 0. \end{cases}$$

Multiplying the first equation by $\bar{y} + \alpha\bar{s}$ and eliminating \bar{z} between the last two equations, we obtain, after some computations:

$$\begin{cases} (\lambda - \mu\bar{s})(\bar{y} + \alpha\bar{s}) - \beta\bar{s}\bar{y} = 0, \\ (\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1)\bar{s} - \mu_y\bar{y} = 0. \end{cases}$$

Eliminating \bar{y} between these two equations, we obtain:

$$\bar{s} = \frac{\lambda \left(\frac{1}{\mu_y} (\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \alpha \right)}{\frac{1}{\mu_y} (\mu + \beta)(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \mu\alpha},$$

from which we build the other two components of $\bar{\mathcal{E}}$ as follows:

$$\bar{y} = \frac{1}{\mu_y} (\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1)\bar{s}; \quad \bar{z} = \frac{\varepsilon\beta}{\mu_z} \frac{\bar{y}\bar{s}}{\bar{y} + \alpha\bar{s}}.$$

To investigate the stability of the coexistence steady state $\bar{\mathcal{E}}$, we compute the Jacobian matrix of

system (2.1) evaluated at the steady state $\bar{\mathcal{E}}$:

$$\mathcal{J} = \begin{pmatrix} -\frac{\beta\bar{y}^2}{(\bar{y}+\alpha\bar{s})^2} - \mu & -\frac{\beta\alpha\bar{s}^2}{(\bar{y}+\alpha\bar{s})^2} & 0 \\ -\frac{\varepsilon\beta\bar{y}^2}{(\bar{y}+\alpha\bar{s})^2} & -\frac{\varepsilon\beta\alpha\bar{s}^2}{(\bar{y}+\alpha\bar{s})^2} - \mu_y & \phi \\ \frac{\varepsilon\beta\bar{y}^2}{(\bar{y}+\alpha\bar{s})^2} & \frac{\varepsilon\beta\alpha\bar{s}^2}{(\bar{y}+\alpha\bar{s})^2} & -\mu_z \end{pmatrix}.$$

Its characteristic polynomial is then given by:

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where:

$$\begin{aligned} a_2 &= \mu_y + \mu + \mu_z + \frac{\varepsilon\beta\alpha + \beta\frac{1}{\mu_y^2}(\varepsilon\beta + \alpha\mu_y)^2(\mathcal{R}_0 - 1)^2}{\left[\frac{1}{\mu_y}(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \alpha\right]^2}, \\ a_1 &= \frac{\mu(\mu_z + \mu_z) \left(\frac{\varepsilon\beta(\phi - \mu_z)}{\mu_y\mu_z}\right)^2 + (\mu_y + \mu_z)\frac{\beta}{\mu_y^2}(\varepsilon\beta + \alpha\mu_y)^2(\mathcal{R}_0 - 1)^2}{\left[\frac{1}{\mu_y}(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \alpha\right]^2} \\ &\quad + \frac{\frac{\varepsilon\beta}{\mu_y}(\phi - \mu_z)(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \mu\varepsilon\beta\alpha}{\left[\frac{1}{\mu_y}(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \alpha\right]^2}, \\ a_0 &= \frac{\mu\varepsilon\beta\frac{(\phi - \mu_z)}{\mu_y}(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \frac{\mu_z\beta}{\mu_y}(\varepsilon\beta + \alpha\mu_y)^2(\mathcal{R}_0 - 1)^2}{\left[\frac{1}{\mu_y}(\varepsilon\beta + \alpha\mu_y)(\mathcal{R}_0 - 1) + \alpha\right]^2}. \end{aligned}$$

The Routh-Hurwitz stability criterion imposes that $a_2 > 0$, $a_1 > 0$, $a_0 > 0$ and $a_2a_1 - a_0 > 0$. The first three are trivial when $\mathcal{N} > 1$, and the last one is verified as easy computations show term compensations between the negative terms of $-a_0$ and some terms within a_1a_2 . \square

Remark 2 *Theorems 4 and 5 have a biological meaning: the CBB population disappears from the coffee plantation if the basic reproduction number $\mathcal{R}_0 \leq 1$, so no control is needed, except to hasten the CBB decline. In contrast, when $\mathcal{R}_0 > 1$, the CBB population persists in the coffee plantation. Hence, control measures should aim at modifying model parameters such that \mathcal{R}_0 is brought below 1, in order to reduce the damages and the CBB population at the end of the cropping season.*

2.3.3 Time-dependent effective reproduction number

The basic reproduction number gives information on the onset of the epidemic in an entirely susceptible population. The status of an epidemic in a partially infected population is given by the effective reproduction number $\mathcal{R}_e(t)$, whose value compared to 1 indicates if the epidemic is expanding, endemic or contracting. At time t , it is defined as the average number of secondary cases per infectious case present at time t . A number of formulas have been proposed to approximate $\mathcal{R}_e(t)$, but they mostly require that the total population of susceptible and infected is constant, and are defined in the case where infection is directly transmitted from an infectious individual to a susceptible [51]. Neither is satisfied in our case, as the host population strongly increases during the season and as the infection is vector-transmitted.

Therefore, we propose an exact formula for $\mathcal{R}_e(t)$. Within our model, we can compute the number of secondary colonising females produced in average by a colonising female present at time t (the same can be done for infesting females). We identify the number of colonising females present at time t and follow their decrease, through mortality and transfer to the infesting female compartment. These infesting females are then computed and used to calculate the secondary colonising females. Even though the actual season is limited to a length t_f , we evaluate $\mathcal{R}_e(t)$ by computing all the secondary colonising females generated over an infinite horizon. Doing so, we include those that would have been produced after the end of the season, so that $\mathcal{R}_e(t)$ is not influenced by the proximity to the end of the actual season.

We compute the number of secondary colonising females produced by the colonising females present at time t_r . For that, we first follow the evolution of these colonising females that are present at time t_r ; we define these by \hat{y} , whose dynamics are only decreasing, through mortality μ_y and transformation into infesting females (at rate $\varepsilon\beta\frac{s}{y+\alpha s}$):

$$\dot{\hat{y}} = - \left(\varepsilon\beta \frac{s(t)}{y(t) + \alpha s(t)} + \mu_y \right) \hat{y} \quad (2.10)$$

and then, for $t \geq t_r$

$$\hat{y}(t) = y(t_r) e^{-\varepsilon\beta \int_{t_r}^t \frac{s(r)}{y(r) + \alpha s(r)} dr - \mu_y(t-t_r)}.$$

The dynamics of the infesting females produced by the latter colonising females $\hat{y}(t)$ are then:

$$\begin{aligned} \dot{\hat{z}} &= \varepsilon\beta \frac{s}{y + \alpha s} \hat{y} - \mu_z \hat{z} \\ &= \varepsilon\beta \frac{s(t)}{y(t) + \alpha s(t)} e^{-\varepsilon\beta \int_{t_r}^t \frac{s(r)}{y(r) + \alpha s(r)} dr - \mu_y(t-t_r)} y(t_r) - \mu_z \hat{z} \\ &= \eta(t) y(t_r) - \mu_z \hat{z} \end{aligned} \quad (2.11)$$

whose solution is:

$$\hat{z}(t) = \hat{z}(t_r) e^{-\mu_z(t-t_r)} + \int_{t_r}^t \eta(\hat{t}) y(t_r) e^{-\mu_z(t-\hat{t})} d\hat{t}.$$

Since we only consider infesting females that are initiated by the colonising females present at time t_r , we have $\hat{z}(t_r) = 0$, so that the secondary colonising females are given by:

$$\dot{\bar{y}} = \phi \hat{z} \quad (2.12)$$

and

$$\begin{aligned} \bar{y}(T) &= \int_{t_r}^T \phi \hat{z}(\tau) d\tau \\ &= \int_{t_r}^T \phi \int_{t_r}^{\tau} \eta(\hat{t}) e^{-\mu_z(\tau-\hat{t})} d\hat{t} d\tau y(t_r) \\ &= \int_{t_r}^T \phi \int_{t_r}^{\tau} \varepsilon\beta \frac{s(\hat{t})}{y(\hat{t}) + \alpha s(\hat{t})} e^{-\varepsilon\beta \int_{t_r}^{\hat{t}} \frac{s(r)}{y(r) + \alpha s(r)} dr - \mu_y(\hat{t}-t_r)} e^{-\mu_z(\tau-\hat{t})} d\hat{t} d\tau y(t_r). \end{aligned}$$

If we pick $T = t_f$ in order to take into account the actual season length, $\bar{y}(T) \rightarrow 0$ as t_r nears the end of the season, whatever the status of the infection. It hides a situation that might be troublesome for the onset of the next season. Hence, we opt to have $T \rightarrow +\infty$ so that all

potential offspring are taken into account. We then obtain $\mathcal{R}_e(t)$ by renaming t_r as t and taking the limit for T going to $+\infty$ of $\frac{\bar{y}(T)}{y(t)}$:

$$\mathcal{R}_e(t) = \int_t^{+\infty} \int_t^\tau \phi \varepsilon \beta \frac{s(\hat{t})}{y(\hat{t}) + \alpha s(\hat{t})} e^{-\varepsilon \beta \int_t^{\hat{t}} \frac{s(r)}{y(r) + \alpha s(r)} dr - \mu_y(\hat{t} - t) - \mu_z(\tau - \hat{t})} d\hat{t} d\tau \quad (2.13)$$

This closed-form formula has the advantage of being self-contained. In practice, in order to obtain $\mathcal{R}_e(t)$ for some t , we simulate system (2.1) until time t and then simulate (2.1)-(2.10)-(2.11)-(2.12) for a long duration from time t (until some large time T). In the end, we obtain $\mathcal{R}_e(t)$ from the ratio $\frac{\bar{y}(T)}{y(t)}$.

We note that, when $y(0) \approx 0$ and $s(0)$ is at the pest-free equilibrium, $\frac{s(t)}{y(t) + \alpha s(t)} \approx \frac{1}{\alpha}$ at all times, so that $\mathcal{R}_e(0) = \mathcal{R}_0$. Moreover, at the endemic equilibrium (2.9), $\mathcal{R}_e = 1$.

An approximation of the time-dependent effective reproduction number may be obtained from the basic reproduction number (2.6), by replacing $\frac{1}{\alpha}$ by $\frac{s(t)}{y(t) + \alpha s(t)}$:

$$\mathcal{R}(t) = \frac{\phi \varepsilon \beta \frac{s(t)}{y(t) + \alpha s(t)}}{\mu_z \left(\varepsilon \beta \frac{s(t)}{y(t) + \alpha s(t)} + \mu_y \right)} = \frac{\phi \varepsilon \beta}{\mu_z} \frac{s(t)/y(t)}{(\varepsilon \beta + \mu_y \alpha) s(t)/y(t) + \mu_y}, \quad (2.14)$$

where $s(t)$ and $y(t)$ are the solutions of system (2.1). Assuming that the population does not vary too much, which determines the validity of this approximation, the time-dependent approximated reproduction number can be interpreted as the basic reproduction number at time t , that is the product of the average number of infesting females produced by a colonising female during its lifespan $\frac{\varepsilon \beta \frac{s(t)}{y(t) + \alpha s(t)}}{\varepsilon \beta \frac{s(t)}{y(t) + \alpha s(t)} + \mu_y}$ and the average number of colonising females produced by an infesting female $\frac{\phi}{\mu_z}$. As for the exact effective reproduction number, at the pest-free equilibrium $\mathcal{R} = \mathcal{R}_0$ and at the endemic equilibrium $\mathcal{R} = 1$.

2.4 Optimal control strategy

2.4.1 Problem statement

In this section, we formulate a continuous optimal control problem which consists in maximising the profit, by means of an entomopathogenic fungus.

Entomopathogenic fungi grow at the expense of insects and usually kill them. *Beauveria bassiana* is the most studied fungus to control coffee pests, especially CBB [6, 52, 95, 56, 19, 32, 27]. This fungus has the advantage of not being harmful to humans, of having no negative impact on coffee berries, and of being infective through a simple contact [11, 52, 95, 56, 19]. It is sprayed on the surface of the coffee berries and it infects the colonising CBB females when they begin to drill an entry hole into a coffee berry. The fungus then rapidly grows, feeding on the nutrients in the CBB body and producing toxins in the process. When the CBB dies after 3 to 5 days, *B. bassiana* covers the carcass in a layer of white mold [6]. Abiotic factors limit the efficacy of the fungus on the plant canopy, leading to a progressive decay of the inoculum [11]

Let us denote by $v(t)$ the fungus load at time t . The dynamics of the entomopathogenic

fungus on the berries is:

$$\dot{v}(t) = -\gamma v(t) + h(t),$$

with initial condition $v(0) \in \mathbb{R}_+$. Parameter γ denotes the natural decay rate of the fungus. The function $h(t)$ represents the fungus application rate at time t , with an upper bound $h_M > 0$. The effect of the fungus is limited and saturates at $\xi \in (0, 1)$ with the fungus load half saturation constant k :

$$\sigma(v) = \frac{\xi v}{v + k},$$

So we obtain the following controlled system:

$$\begin{cases} \dot{s}(t) = \lambda - (1 - \sigma(v(t)))\beta \frac{s(t)y(t)}{y + \alpha s(t)} - \mu s(t) \\ \dot{y}(t) = \phi z - \varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_y y(t) \\ \dot{z}(t) = (1 - \sigma(v))\varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_z z(t) \\ \dot{v}(t) = -\gamma v(t) + h(t) \end{cases} \quad (2.15)$$

with the initial condition:

$$s(0) \geq 0, \quad y(0) > 0 \quad z(0) \geq 0 \text{ and } v(0) \geq 0. \quad (2.16)$$

The set \mathcal{U} of admissible control functions is defined as:

$$\mathcal{U} = \{h \in L^\infty(0, t_f) : 0 \leq h(t) \leq h_M, \quad \forall t \in [0, t_f]\}. \quad (2.17)$$

The well-posedness of the controlled system, which is a biological requirement, is handled by the following proposition.

Proposition 2 *For all nonnegative initial conditions $(s(0), y(0), z(0), v(0)) \in \mathbb{R}_+^4$, the solutions $(s(t), y(t), z(t), v(t))$ of system (2.15) remain nonnegative and are bounded for all time $t > 0$. Moreover, the bounded set*

$$\Omega = \Gamma \times \left\{ v \in \mathbb{R}_+ : v(t) \leq \frac{h_M}{\gamma} \right\}$$

is positively invariant.

Proof : Since h is a nonnegative function, v remains nonnegative at all times for any nonnegative initial condition $v(0)$. Then, as $\sigma(v) \leq 1$ and proceeding as in the proof of Proposition 1, the other variables also remain nonnegative and bounded. This implies that all trajectories stay in the first quadrant for all initial conditions in \mathbb{R}_+^4 . Moreover, Γ also remains positively invariant for the three first state variables. According to the standard comparison principle [110] and using the fact that $h(t) \leq h_M$, $v(t)$ is positively invariant in $\left\{ v \in \mathbb{R}_+ / v(t) \leq \frac{h_M}{\gamma} \right\}$. Therefore, system (2.15) is positively invariant in Ω . \square

Our primary goal is to maximise the profit generated by the berry yield, by applying an entomopathogenic fungus to control the CBB population. An option would be to bring the CBB population down as fast as possible in order to increase the yield, but we want to keep the control

cost in check. Furthermore, as CBB survive between cropping seasons, we also want to reduce the CBB population at the end of the season considered, so as not to unduly burden the next season yield. So we aim at maximising the berry yield, while minimising the control cost as well as the CBB population at the end of the cropping season. We assume that the harvest takes place at the end of the cropping season, denoted by t_f . We also assume that almost all berries, and hence almost all infesting females z that are inside berries, are picked at t_f , so that only colonising females y remain in the plantation after harvest. The application cost C of a fungus unit is considered constant and only healthy coffee berries are sold, at a constant berry price per berry ζ_s . The optimal control problem is then formulated as follows.

Problem 1 Find an admissible fungus application strategy $h^*(.) \in \mathcal{U}$ and the corresponding state variable $x^*(.) = (s^*(.), y^*(.), z^*(.), v^*(.))^\top$ maximising the following objective functional:

$$\mathcal{J}(x, h) = \zeta_s s(t_f) - \int_0^{t_f} Ch(t)dt - \zeta_y y(t_f), \quad (2.18)$$

subject to the control system (2.15) and with initial condition in \mathbb{R}_+^4 .

The first term of the objective functional represents the crop yield, that is the harvest of healthy coffee berries at the end of cropping season multiplied by the coffee berry price ζ_s . The second term represents the control cost, with C the price of a fungus unit. The first two terms hence represent the financial gain \mathcal{G} . Finally, the last term represents the CBB population that remains after harvest in the plantation, consisting of colonising females, weighted by parameter ζ_y . So \mathcal{J} represents the profit, penalised by the final CBB population.

Let us consider the vectors $x(t) = (s(t), y(t), z(t), v(t))^\top$ and $e_4 = (0, 0, 0, 1)^\top$. We define $f : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ as follows:

$$f(x) = \begin{pmatrix} \lambda - (1 - \sigma(v))\beta \frac{sy}{y+\alpha s} - \mu s \\ \phi z - \varepsilon \beta \frac{sy}{y+\alpha s} - \mu_y y \\ (1 - \sigma(v))\varepsilon \beta \frac{sy}{y+\alpha s} - \mu_z z \\ -\gamma v \end{pmatrix}$$

Then the model (2.15) can be rewritten in the following compact form:

$$\dot{x}(t) = f(x(t)) + h(t)e_4.$$

2.4.2 Necessary optimal conditions

Based on Corollary III.4.1 of Theorem III.4.1 in Fleming and Rishel [42] (see also [33]), we establish the existence of an optimal strategy that maximises the objective functional (2.18) in the following theorem.

Theorem 6 The optimal control (1) admits a solution, i.e. there exists an optimal strategy $h^*(.) \in \mathcal{U}$ with associated optimal solution $x^*(.) = (s^*(.), y^*(.), z^*(.), v^*(.))^\top$ which maximises the objective functional $\mathcal{J}(x, h)$.

Proof : Corollary III.4.1 in Fleming and Rishel [42] imposes three conditions:

1. \mathcal{G} , denoting the class of all $(x(0), h)$ with $x(0) \in \mathbb{R}_+^4$ and $h \in \mathcal{U}$, such that the corresponding solution of system (2.15) is defined over $[0, t_f]$, is nonempty.
2. The admissible control set \mathcal{U} is closed and convex.
3. The integrand of objective functional $\mathcal{J}(x, h)$ is concave on \mathcal{U} and bounded above by $-C_1|h|^\kappa + C_2$ for some $C_1 > 0$ and $\kappa > 1$.

Firstly, let us choose some $x(0) \in \mathbb{R}_+^4$ and $h \in \mathcal{U}$. Since system (2.15) has bounded coefficients and any solution is bounded on the finite interval time $[0, t_f]$, then by the Caratheodory theorem (see for instance in [76, p. 82]), there exists a unique solution of system (2.15) with initial condition $x(0)$ and control h . This implies that the domain \mathcal{G} is nonempty. By definition, the admissible set control \mathcal{U} is closed and convex, hence the second condition is satisfied. For the third condition, it is easy to see that the integrand function of the objective functional is concave since it is linear in control $h(t)$. Moreover, taking $C_1 = \frac{C}{h_M}$, $\kappa = 2$ and $C_2 = 0$, the last condition is satisfied. Therefore, there exists an optimal strategy. \square

We now apply the Pontryagin's Maximum Principle [92] to establish the first-order necessary optimality conditions of Problem 1. The Hamiltonian associated to Problem 1 is defined as:

$$\begin{aligned} \mathcal{H}(x, h, p, p^0) &= \langle p, f(x) + h e_4 \rangle - p^0 C h \\ &= p_1 \left[\lambda - (1 - \sigma(v)) \beta \frac{sy}{y + \alpha s} - \mu s \right] + p_2 \left[\phi z - \varepsilon \beta \frac{sy}{y + \alpha s} - \mu_y y \right] \\ &\quad + p_3 \left[(1 - \sigma(v)) \varepsilon \beta \frac{sy}{y + \alpha s} - \mu_z z \right] + p_4 [-\gamma v + h] - p^0 C h, \end{aligned} \quad (2.19)$$

where $\langle \cdot \rangle$ represents the scalar product in \mathbb{R}^4 , $p(\cdot) = (p_1(\cdot), p_2(\cdot), p_3(\cdot), p_4(\cdot))^\top \in \mathbb{R}^4$ and $p^0 \in \mathbb{R}$.

According to the Pontryagin's Maximum Principle, if $h^*(\cdot) \in \mathcal{U}$ is an optimal strategy and $x^*(\cdot)$ is its associated trajectory, there exists an absolutely continuous vector mapping $p(\cdot) : [0, t_f] \rightarrow \mathbb{R}^4$ called the *adjoint vector* and a real scalar $p^0 \geq 0$ that satisfy the following conditions:

- (i). The vector $(p(\cdot), p^0)$ is non trivial.
- (ii). The adjoint vector $p(\cdot)$ satisfies the adjoint equation $\dot{p}(t) = -\nabla_x \mathcal{H}(x(t), h(t), p(t), p^0)$ for a.e. $t \in [0, t_f]$, that is:

$$\begin{cases} \dot{p}_1 = [(1 - \sigma(v))(p_1 - \varepsilon p_3) + \varepsilon p_2] \frac{\beta y^2}{(y + \alpha s)^2} + \mu p_1 \\ \dot{p}_2 = [(1 - \sigma(v))(p_1 - \varepsilon p_3) + \varepsilon p_2] \frac{\beta \alpha s^2}{(y + \alpha s)^2} + \mu_y p_2 \\ \dot{p}_3 = -\phi p_2 + \mu_z p_3 \\ \dot{p}_4 = (-p_1 + \varepsilon p_3) \frac{d\sigma}{dv} \frac{\beta sy}{y + \alpha s} + \gamma p_4. \end{cases} \quad (2.20)$$

- (iii). The following maximisation condition holds true for a.e. $t \in [0, t_f]$:

$$h^*(t) = \operatorname{argmax}_{\omega \in [0, h_M]} \mathcal{H}(x(t), \omega, p(t), p^0). \quad (2.21)$$

(iv). The following transversality conditions, obtained by deriving the terminal terms of the objective functional (2.18) relative to each system variable and applied at final point t_f , are verified:

$$p_1(t_f) = \zeta_s p^0, \quad p_2(t_f) = -\zeta_y p^0, \quad p_3(t_f) = p_4(t_f) = 0. \quad (2.22)$$

By definition, an *extremal trajectory* of the control problem 1 is a quadruplet $(x(\cdot), h(\cdot), p(\cdot), p^0)$ satisfying equations (2.15) and (2.19)–(2.22). If $p^0 = 0$, then the terminal condition of adjoint vector are given by $p(t_f) = 0$ and thus the vector $p(t)$ vanish identically as solutions of a system of homogenous linear differential equations. This contradicts condition (i) above. Hence $p^0 \neq 0$ and the extremal is called *normal* in this case. In the latter, we may assume that $p^0 = 1$. The p^0 argument is then dropped from the Hamiltonian expression and based on homogeneity, we write $(x(\cdot), h(\cdot), p(\cdot))$ in the place of $(x(\cdot), h(\cdot), p(\cdot), 1)$. We now study the structure of the optimal control. As \mathcal{H} is linear with respect to control $h(t)$, maximising \mathcal{H} depends on the sign of

$$\partial_h \mathcal{H}(x(t), h(t), p(t)) = -C + p_4(t).$$

So we introduce the *switching function*:

$$\Phi(t) = -C + p_4(t), \quad \forall t \in [0, t_f],$$

and from the maximality condition (2.21), the optimal control $h^*(t)$ is characterised by:

$$h^*(t) = \begin{cases} 0 & \text{if } \Phi(t) < 0 \\ \text{undefined} & \text{if } \Phi(t) = 0 \\ h_M & \text{if } \Phi(t) > 0. \end{cases}$$

If $h^*(t)$ is equal to 0 or h_M for a.e. $t \in I \subset [0, t_f]$, the optimal solution is called a *bang-bang* solution.

Remark 3 *At the end of cropping season, according to equation (2.22), $\Phi(t_f) = -C < 0$, so the control vanishes, that is $h(t_f) = 0$.*

2.4.3 Singular control

A control is called *singular* if there exists a nonempty interval $[t_1, t_2] \subset [0, t_f]$ such that $\Phi(t) = 0$ for every $t \in [t_1, t_2]$.

Theorem 7 *If there exists a singular control $h_{sing}^*(t)$ over $[t_1, t_2]$, it is of order 1 and locally maximising. Moreover, it is given as a function of the state and adjoint variables by:*

$$h_{sing}^*(t) = -\frac{\mathcal{S}(x, p)}{\mathcal{Q}(x)},$$

where

$$\begin{aligned} \mathcal{S}(x, p) &= \left[-(1 - \sigma(v))\gamma C - \varepsilon p_2 \frac{d\sigma}{dv} \frac{\beta sy}{y + \alpha s} \right] \frac{\beta y^2}{(y + \alpha s)^2} - \frac{2\gamma^2 C v}{v + k} - \mu \gamma C \\ &\quad + \frac{d\sigma}{dv} \frac{\beta sy}{y + \alpha s} [-\varepsilon \phi p_2 + \varepsilon(\mu_z - \mu)p_3] - \frac{\gamma C y}{s(y + \alpha s)} \left(\lambda - (1 - \sigma(v))\beta \frac{sy}{y + \alpha s} - \mu s \right) \\ &\quad - \frac{\gamma C \alpha s}{y(y + \alpha s)} \left(\phi z - \varepsilon \beta \frac{sy}{y + \alpha s} - \mu_y y \right), \\ \mathcal{Q}(x) &= \frac{2\gamma C}{v + k}. \end{aligned}$$

Proof : Assuming that a singular control exists for all $t \in [t_1, t_2]$, we have $\Phi(t) = 0$. Then differentiating the switching function with respect to time t along the singular control and adjoint system (2.20), we obtain:

$$\dot{\Phi}(t) = \dot{p}_4(t) = (-p_1 + \varepsilon p_3) \frac{d\sigma}{dv} \beta \frac{sy}{y + \alpha s} + \gamma p_4(t),$$

which does not depend explicitly on control h . So we differentiate $\Phi(t)$ a second time:

$$\begin{aligned} \ddot{\Phi}(t) &= \left[(1 - \sigma(v))(-p_1 + \varepsilon p_3) \frac{d\sigma}{dv} \beta \frac{sy}{y + \alpha s} - \varepsilon p_2 \frac{d\sigma}{dv} \frac{sy}{y + \alpha s} \right] \beta \frac{y^2}{(y + \alpha s)^2} \\ &\quad + [-\mu p_1 - \varepsilon \phi p_2 + \mu_z p_3] \frac{d\sigma}{dv} \beta \frac{sy}{y + \alpha s} - (-\gamma v + h) \frac{2}{v + k} (-p_1 + \varepsilon p_3) \frac{d\sigma}{dv} \beta \frac{sy}{y + \alpha s} \\ &\quad + (-p_1 + \varepsilon p_3) \frac{d\sigma}{dv} \beta \frac{sy}{y + \alpha s} \left(\dot{s} \frac{y}{s(y + \alpha s)} + \dot{y} \frac{\alpha s}{y(y + \alpha s)} \right) + \gamma \dot{p}_4(t). \end{aligned}$$

Then, as $\Phi(t) = 0$ and $\dot{\Phi}(t) = 0$ for all $t \in [t_1, t_2]$, we obtain:

$$p_4(t) = C, \quad (-p_1 + \varepsilon p_3) \frac{d\sigma}{dv} \beta \frac{sy}{y + \alpha s} = -\gamma C \quad \forall t \in [t_1, t_2].$$

Substituting the equations above into the expression of $\ddot{\Phi}(t)$, we obtain:

$$\ddot{\Phi}(t) = \mathcal{S}(x, p) + h \mathcal{Q}(x).$$

As $\ddot{\Phi}(t) = 0$ along the singular trajectory, we obtain the singular control expression:

$$h_{sing}^*(t) = -\frac{\mathcal{S}(x, p)}{\mathcal{Q}(x)}.$$

The number of derivatives of the switching function required to determine the singular control is 2, which implies that the singular control is of order 1. For the singular control to be maximising, the classical Legendre Clebsch condition [33, 65] requires that:

$$(-1) \partial_h \left[\frac{d^2}{dt^2} \partial_h \mathcal{H}(x(\cdot), h(\cdot), p(\cdot)) \right] \leq 0.$$

Since $\frac{d^2}{dt^2} \partial_h \mathcal{H}(x(\cdot), h(\cdot), p(\cdot)) = \ddot{\Phi}(t) = \mathcal{S}(x, p) + h \mathcal{Q}(x)$, we have $(-1) \partial_h [\ddot{\Phi}(t)] = -\mathcal{Q}(x) \leq 0$, because the state variable v is nonnegative for all time $t \geq 0$. Then the singular control $h_{sing}^*(\cdot)$

is a locally maximising strategy. □

We can summarise the results above in the following theorem.

Theorem 8 *Let $h^* \in \mathcal{U}$ be an optimal control that maximises the objective functional (2.18). Then, there exists an adjoint vector function $p(t) \in \mathbb{R}^4$ satisfying (2.20) with transversality conditions (2.22). Moreover, the structure of the optimal control is given by:*

$$h^*(t) = \begin{cases} 0 & \text{if } \Phi(t) < 0, \\ h_{sing}^*(t) & \text{if } \Phi(t) = 0 \text{ for } t \in [t_1, t_2], \\ h_M & \text{if } \Phi(t) > 0. \end{cases} \quad (2.23)$$

2.5 Numerical simulations

In this section, we investigate numerically the effect of the optimal strategy on the infestation process of coffee berries by CBB.

2.5.1 Parameter values

Some parameters, particularly those associated with CBB, are estimated from the literature and expert opinions. For instance, the mortality of females is chosen such that $1/\mu_y + 1/\mu_z$ plus the larval development time of CBB lies within the interval 87–282 days, which represents the female lifespan [11]. The emergence rate of females is estimated from the average number of eggs laid by females, which is one to three egg per day [11]. The entomopathogenic fungus is supposed to be viable up to a few months. Fungus cost and application rate are derived from biocontrol products on the market. For the small family plantation considered in this study (0.12 ha), the recommended application rate ranges between 60 and 180 g. To translate this amount into a daily maximal rate, we make the hypothesis that there will be no more than one application per week, so the maximal daily rate is a little less than the value we conservatively chose. The coffee cost is derived from market values. The duration of a coffee cropping season is taken from [11]. The remaining parameter values are assumed. All parameter values are given in Table 1. With these parameter values, the basic reproduction number $\mathcal{R}_0 = 12$ for the model without control described in system (2.1), which is consistent with the fast CBB dynamics observed in the field. However, we illustrate the optimal control strategy when $\mathcal{R}_0 < 1$ and the pest-free equilibrium is stable.

2.5.2 Optimal control

To solve our optimal control problem numerically, we use the BOCOP software [103, 23], an open source toolbox in which the optimal control problem is approximated by a finite dimensional optimisation problem. In BOCOP we choose the Lobatto discretisation method (implicit, 4 stages, order 6), 2000 time steps, and a 10^{-5} tolerance.

We use the following initial conditions. At the beginning of a cropping season, there are no berries, so $s(0) = 0$. Moreover, we assume that there are only colonising females, as infesting females were eliminated from the plantation when berries were manually removed during the

Table 1: Model and control parameter values

Symbol	Description	Value
λ	Production rate of new coffee berries	1200 berries.day ⁻¹
μ	Natural mortality rate of healthy coffee berries	0.002 day ⁻¹
ϕ	Emergence rate of new colonising females	2 day ⁻¹
β	Infestation rate	0.01 day ⁻¹
α	Infestation half saturation constant	0.7 female.berry ⁻¹
ε	Conversion rate from coffee berries to CBB	1 female.berry ⁻¹
μ_y	Natural mortality rate of colonising females	1/20 day ⁻¹
μ_z	Natural mortality rate of infesting females	1/27 day ⁻¹
ξ	Fungus load maximal effectiveness	0.8
k	Fungus load half saturation constant	200 g.day ⁻¹
γ	Fungus decay rate	1/50 day ⁻¹
ζ_s	Price of coffee berries	2.10 ⁻³ US\$.berry ⁻¹
ζ_y	Cost of remaining colonising females	10 ⁻⁴ US\$.female ⁻¹
C	Cost of fungus	0.022 US\$.g ⁻¹
h_M	Maximal fungus application rate in the plantation	30 g.day ⁻¹
t_f	Duration of a cropping season	250 days

preceding harvest, so $z(0) = 0$. We also assume that the entomopathogenic fungus is absent in the plantation, so the initial load $v(0) = 0$. We consider several initial conditions for the colonising females: $y(0) = 10^2$ in Figure 8, a fairly low number of females corresponding to a new infestation of the plantation; $y(0) = 10^6$ in Figure 12, which is closed to the coexistence equilibrium value without control and hence corresponds to a worst case scenario, where all females have survived since the end of the preceding cropping season; $y(0) = 10^3$ in Figure 9, $y(0) = 10^4$ in Figure 10, and $y(0) = 10^5$ in Figure 11, which are intermediate infestation values.

There are six panels on each figure. Panel (a) represents the evolution of the healthy coffee berries $s(t)$ during the cropping season, without CBB (black), with initial CBB in the plantation but no control (blue), and with initial CBB and optimal fungus application (red). Similarly, panels (b) and (c) represent the dynamics of the colonising $y(t)$ and infesting $z(t)$ CBB females, respectively, without and with optimal control. Panel (e) represents the optimal fungus application $h(t)$, the control that maximises the penalised profit \mathcal{J} defined in equation (2.18). Panel (d) represents the dynamics of the fungus load $v(t)$ in the plantation and panel (e) the effect of this load, that saturates through function $\sigma(v) = \frac{\xi v}{v+k}$. Finally in panels (a–c), the horizontal dotted blue lines represent the coexistence equilibrium without control.

In Figures 8–10(b), the trajectories of the colonising females increase, both in the uncontrolled (dashed blue curve) and controlled (red curve) cases. In contrast, for higher initial values in Figure 12(b) and to a lesser extent Figure 11(b), the colonising female trajectories first decrease, due to the lack of healthy coffee berries at the beginning of the season, and later increase.

For all initial conditions, at the end of the cropping season, the uncontrolled solution (dashed blue curves in subfigures (a–c)) is very closed to the coexistence steady state: $\bar{s} = 1.06 \times 10^5$, $\bar{y} = 10.5 \times 10^5$, $\bar{z} = 0.27 \times 10^5$ (dotted blue horizontal lines). Since the basic reproduction number $\mathcal{R}_0 = 12 > 1$, according to Theorem 5, the CBB population persists in the plantation and the coexistence steady state is locally stable. Without control, the berry production at the end of the cropping season is more than halved, compared to the pest-free case (black curves).

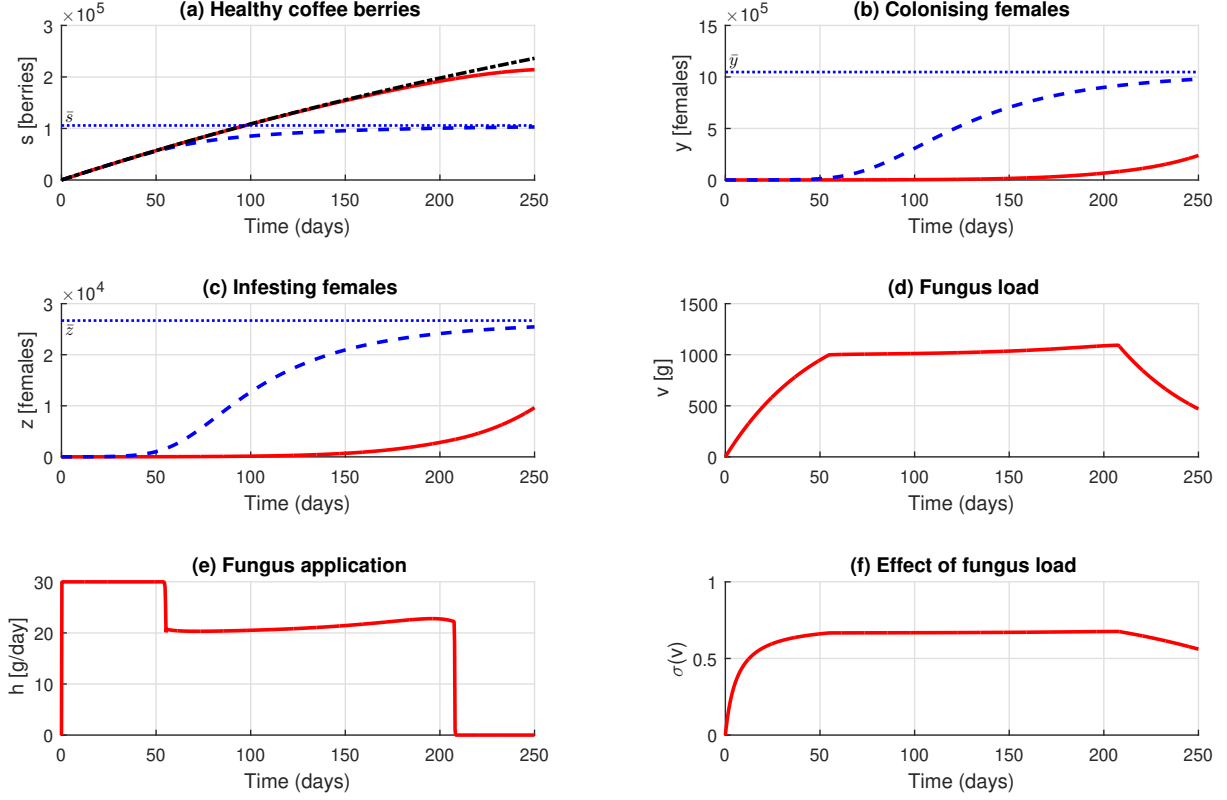


Figure 8: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal control (2.23) (plain red curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^2$ colonising females. The healthy berry growth without pest is also represented in (a) (dash-dotted black curve). The coexistence steady state without control is indicated in (a–c) (dotted blue lines).

The optimal control depends on the initial condition. In Figures 8–10(e) the control structure is *bang(30)–singular–bang(0)*, whereas in Figures 11–12(e) it is *bang(0)–singular–bang(0)*; both are consistent with the theoretical results in Remark 3 and Theorem 7. In the latter case, the colonising female population is initially higher and first decreases. As a consequence, the fungus is first applied later than for lower initial conditions. Interestingly, the switch between the *bang(30)* and the *singular* control occurs earlier as the initial number of colonising females increases, as shown in Figures 8–10(e). In parallel, the switch between the *bang(0)* and the *singular* control occurs later as the initial number of colonising females increases, as shown in Figures 11–12(e). In all cases, more effort is devoted earlier to control the borers when the initial population is lower, which allows to maintain the population at lower levels. The control pay-off is higher for lower pest populations, which justifies more important control efforts. When the CBB initial population is high, as in Figures 11 and 12, the CBB population first decreases, so there is initially no control effort. In all cases, once the fungus is applied, its effect (subfigures (f)) rapidly increases but remains below its maximum value ($\xi = 0.8$). The colonising (subfigures (b), red curves) and infesting (subfigures (c), red curves) female trajectories (subfigures (c), red curves) are hence much lower than in the case without control (dashed blue curves).

Table 2 summarises the plantation state at the end of the cropping season (for $t_f = 250$

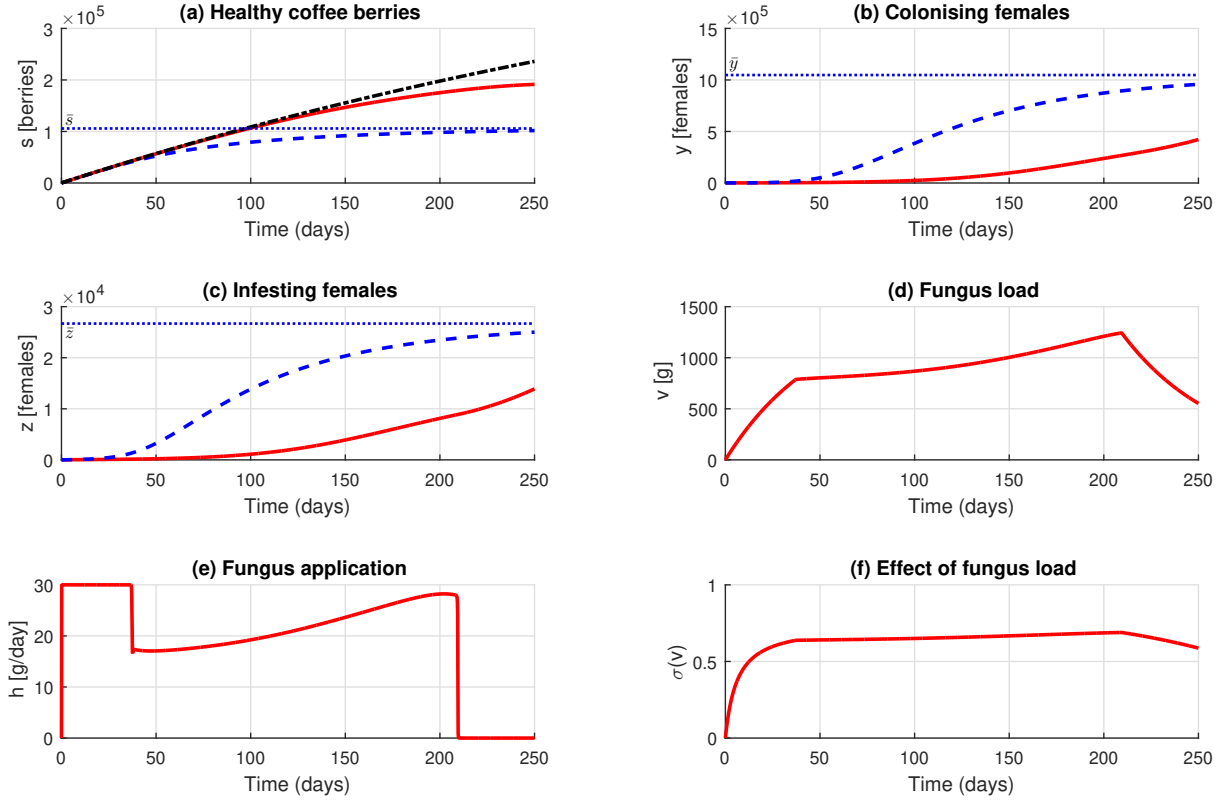


Figure 9: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal control (2.23) (plain red curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^3$ colonising females. The healthy berry growth without pest is also represented in (a) (dash-dotted green curve). The coexistence steady state without control is indicated in (a–c) (dotted blue lines).

days), with and without optimal control, depending on the initial conditions. As we assume that all berries have been harvested, only colonising females (y) remain in the plantation. The financial gain \mathcal{G} corresponds to the coffee berry yield, minus the control cost (when applicable). The penalised profit \mathcal{J} , defined in equation (2.18), corresponds to the coffee berry yield, minus the control cost (when applicable) and the remaining CBB “cost”, as we also aim at reducing the CBB population at the end of the cropping season, so as not to unduly burden the next season yield. This partly explains why the penalised profit in the pest-free case is much higher than when CBB are present in the plantation: the remaining CBB “cost” is around 95 US\$ without control, and 49 US\$ with optimal control. The control cost is around 60–80 US\$ and also impacts the profit. At the end of the cropping season, regardless the initial condition, the remaining CBB population, the financial gain, and the penalised profit are very similar in the case without control. It is expected, as the CBB dynamics are very fast and trajectories get very close to the coexistence equilibrium before the end of the cropping season. In the case with control, there are notable differences. The financial gain and the penalised profit decrease with the initial CBB population size, especially for the lower values ($y(0) = 10^2$ to 10^4 females). For higher values ($y(0) = 10^4$ to 10^6 females), the lack of susceptible berries at the beginning of the cropping season regulates the number of CBB, so the impact of the initial number of CBB is limited. Remaining

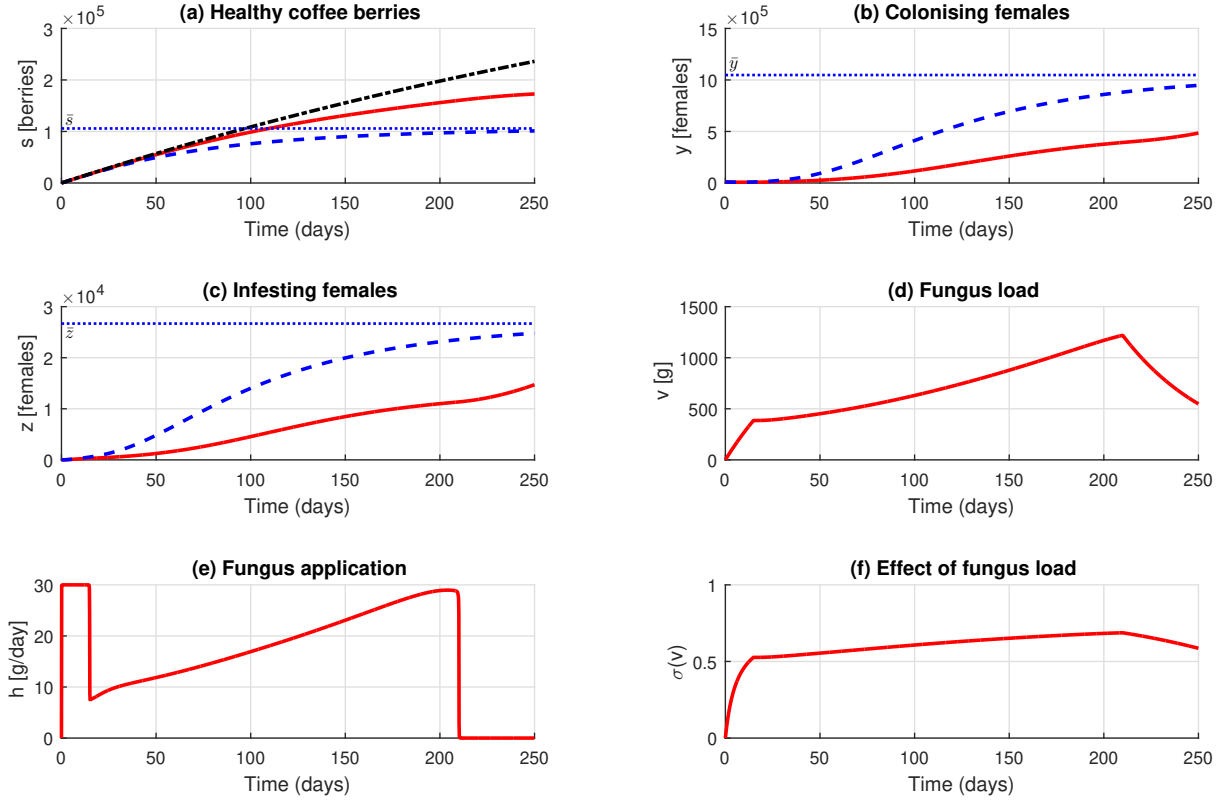


Figure 10: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal control (2.23) (plain red curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^4$ colonising females. The healthy berry growth without pest is also represented in (a) (dash-dotted black curve). The coexistence steady state without control is indicated in (a–c) (dotted blue lines).

CBB population, financial gain, and penalised profit vary when the entomopathogenic fungus is applied, compared to the case without control: the remaining CBB population is more or less half as abundant, the monetary gain is notably improved and the penalised profit is at worst almost twice as high (82% increase) and at best almost three times higher (174% increase).

Coffee berry borers infest berries very efficiently, so the simulations we performed in this section, based on the best estimates we could find for the model parameters given in Table 1, correspond to a situation in which the pest-free equilibrium is unstable and the basic reproduction number $\mathcal{R}_0 = 12 \gg 1$. Nevertheless, for the sake of completeness, we also compute the optimal control when $\mathcal{R}_0 < 1$, that is when the pest-free equilibrium is stable, by drastically reducing the infestation rate value to $\beta = 0.0005 \text{ day}^{-1}$ which corresponds to $\mathcal{R}_0 = 0.78 < 1$. We observe in Figure 13 that, with or without optimal control, the CBB population declines and the healthy coffee berry dynamics are very closed to the pest-free case. Moreover, the penalised profit $\mathcal{J} = 472 \text{ US\$}$ is similar to the financial gain in the pest-free case. The optimal control is singular and very little entomopathogenic fungus is applied on the coffee berries to accelerate the decline of the CBB population. In this unrealistic case, the optimal control is hence basically useless.

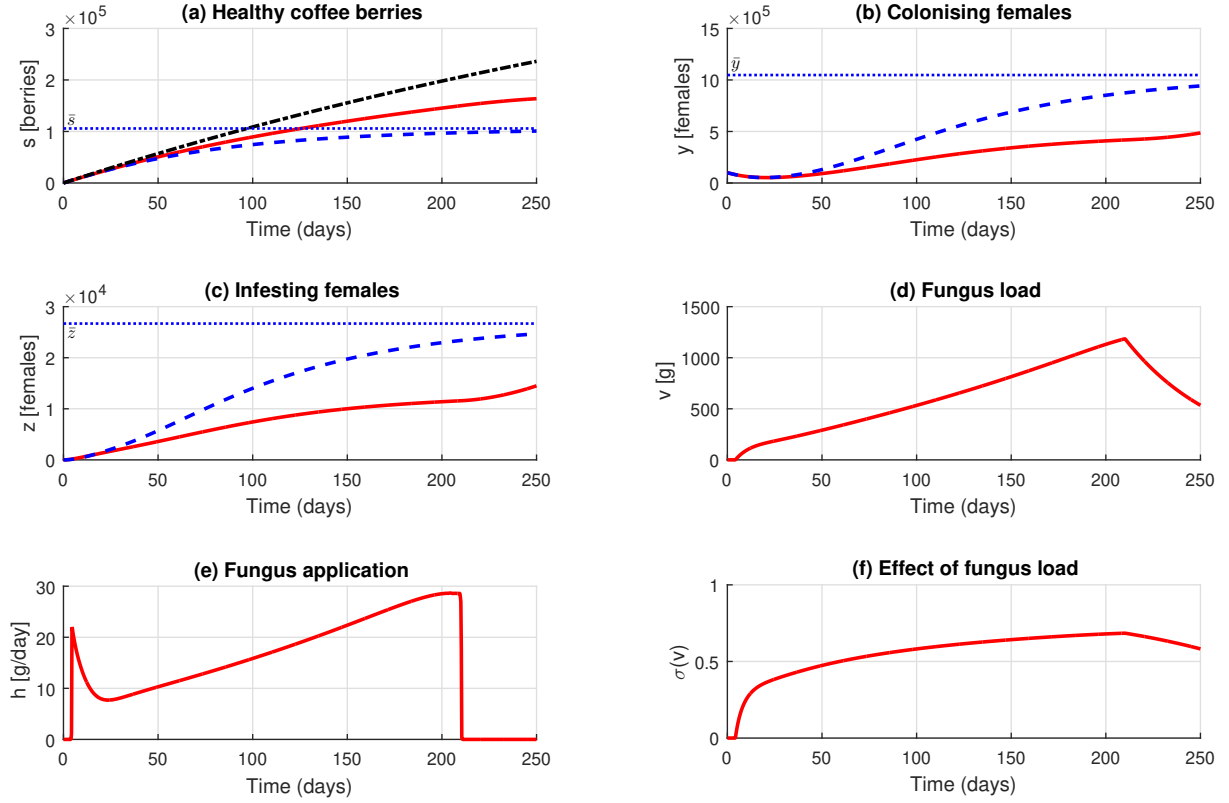


Figure 11: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal control (2.23) (plain red curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^5$ colonising females. The healthy berry growth without pest is also represented in (a) (dash-dotted green curve). The coexistence steady state without control is indicated in (a–c) (dotted blue lines).

2.5.3 Effective reproduction numbers

We also compute the exact (2.13) and approximated (2.14) time-dependent effective reproduction numbers defined for system (2.1) without control. For system (2.15) with optimal control (2.23), the approximated time-dependent reproduction number is similarly defined as follows:

$$\mathcal{R}^*(t) = \frac{\phi \varepsilon \beta [1 - \sigma(v(t))] \frac{s(t)}{(y(t) + \alpha s(t))}}{\mu_z \left(\varepsilon \beta \frac{s(t)}{y(t) + \alpha s(t)} + \mu_y \right)} = \frac{\phi \varepsilon \beta [1 - \sigma(v(t))]}{\mu_z} \frac{s(t)/y(t)}{(\varepsilon \beta + \mu_y \alpha) s(t)/y(t) + \mu_y}, \quad (2.24)$$

where $s(t)$, $y(t)$ and $v(t)$ are the solutions of system (2.15) with optimal control (2.23). Note that $\mathcal{R}^*(t)$, just as $\mathcal{R}(t)$ in equation (2.14), increases with the healthy berries per capita ratio $s(t)/y(t)$. The exact time-dependent reproduction number cannot be defined in the case with control. Indeed, to compute the average number of colonising females produced by a colonising female present at time t , we would need to know the optimal control strategy after the end of the cropping season (to estimate the offspring of the CBB that appear close to the end of the season), but the control is not defined. Results are shown in Figure 14.

In the case without control, the exact and approximated time-dependent reproduction numbers (blue curves) are reasonably close. As there are no berries at initial time $t = 0$,

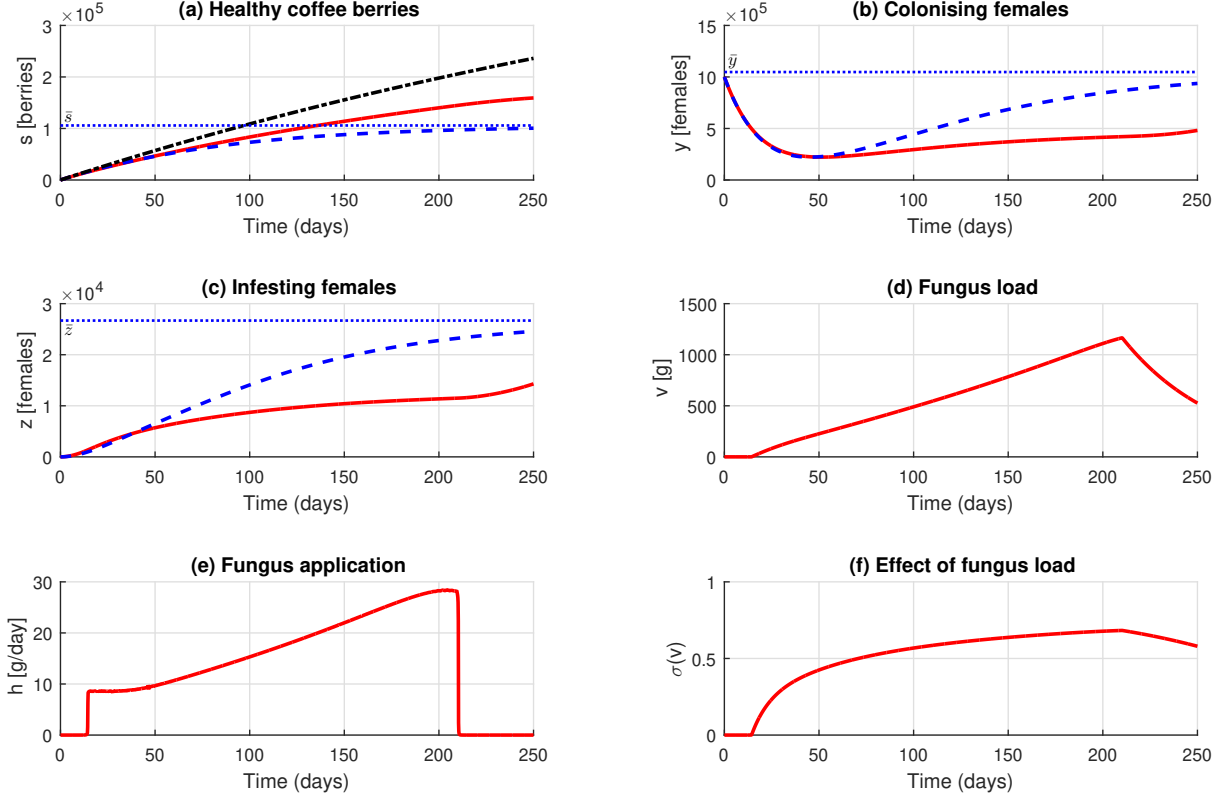


Figure 12: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal control (2.23) (plain red curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^6$ colonising females. The healthy berry growth without pest is also represented in (a) (dash-dotted black curve). The coexistence steady state without control is indicated in (a–c) (dotted blue lines).

the approximated numbers (plain blue curves) are equal to zero, but not the exact numbers as colonising females are present. With few initial colonising females in Figure 14(a), which corresponds to Figure 8, shortly after $t = 0$, both exact and approximated numbers are closed to the basic reproduction number $\mathcal{R}_0 = 12$. Indeed, as healthy berries (Figure 8(a)) grow faster than colonising females (Figure 8(b)), the plantation is closed to a pest-free state, with few CBB compared to healthy berries. The time-dependent reproduction numbers then decrease as colonising females become abundant compared to healthy berries. This decrease is faster for the exact calculation, as the female offspring of a colonising female present at time t will be borne at a later and less favourable time in terms of healthy berries per capita ($s(t)/y(t)$). Both time-dependent reproduction numbers tend towards 1, as the system gets close to the endemic equilibrium, in which the plantation can only sustain the replacement of a colonising female by a single colonising female.

With more initial colonising females in Figure 14(b–c), which corresponds to Figures 10 and 12, both exact and approximated time-dependent reproduction numbers first increase but remain much lower than the basic reproduction number $\mathcal{R}_0 = 12$. Indeed, the system is quite far from the pest-free state, especially for the higher value of initial colonising females (Figure 14(c)). In this case, the exact time-dependent reproduction number starts below the threshold value 1,

Table 2: Penalised profit, financial gain, and colonising CBB females remaining in the plantation after harvest at the end of the cropping season ($t_f = 250$ days), without control and with optimal fungus application, for several initial abundances of colonising females.

$y(0)$	Penalised profit \mathcal{J} (US\$)		Financial gain \mathcal{G} (US\$)		Remaining CBB $y(t_f)$ ($\times 10^5$)	
	No control	Optimal	No control	Optimal	No control	Optimal
10^2	108	296	206	320	9.79	2.38
10^3	108	233	204	275	9.58	4.22
10^4	108	207	203	256	9.48	4.85
10^5	107	199	201	248	9.42	4.86
10^6	107	195	201	243	9.37	4.81

Penalised profit/financial gain in the pest-free case: $\mathcal{J} = \mathcal{G} = 472$ US\$.

as there are not enough berries to sustain the very high CBB population, which first declines (Figure 12(b)). For both initial conditions, the time-dependent reproduction numbers increase, as healthy berries become available and then decreases towards 1, as the system gets close to the endemic equilibrium.

The optimal control lowers the peaks of the approximated effective reproduction number (red curves). At the beginning of the simulation, the effective reproduction numbers with control are lower than without control, thanks to the control term $[1 - \sigma(v(t))]$ in equation (2.24) which represents the effect of fungus application. However, the population dynamics are fairly different with and without control: the healthy berries per capita ratio is much larger with control, so the effective reproduction numbers with control decrease less. With few initial colonising females, the effective reproduction number with control remains notably higher than 1 (Figure 14(a)), as healthy berries per capita are still high thanks to the control efficiency; the system with control is fairly closed to the system without pest (Figure 8). With many initial colonising females, the healthy berries per capita are particularly low (Figure 12), hence the very low effective reproduction number with control (Figure 14(c)); the increase at the end is due to the interruption of fungus application.

2.5.4 Discrete control strategies

From a practical standpoint, it is more realistic to apply the entomopathogenic fungus at discrete times. To pursue this idea, we consider that the fungus is applied periodically at discrete times $t_n = nT$, $n \in \mathbb{N}^*$ satisfying $t_0 = 0 < t_1 < \dots < t_n < \dots < t_f$. The indice n denotes the n^{th} application event and T is the period between two successively fungus applications. Based on these frameworks, we consider the following impulsive controlled system:

$$\left\{ \begin{array}{l} \dot{s}(t) = \lambda - (1 - \sigma(v(t)))\beta \frac{s(t)y(t)}{y + \alpha s(t)} - \mu s(t) \\ \dot{y}(t) = \phi z - \varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_y y(t) \\ \dot{z}(t) = (1 - \sigma(v))\varepsilon \beta \frac{s(t)y(t)}{y(t) + \alpha s(t)} - \mu_z z(t) \\ \dot{v}(t) = -\gamma v(t), \quad t \neq t_{n-1}, \quad t_{n-1} < t_n < t_f \\ v(t_{n-1}^+) = v(t_{n-1}) + h_{n-1}, \quad n \in \mathbb{N}^*, \end{array} \right. \quad (2.25)$$

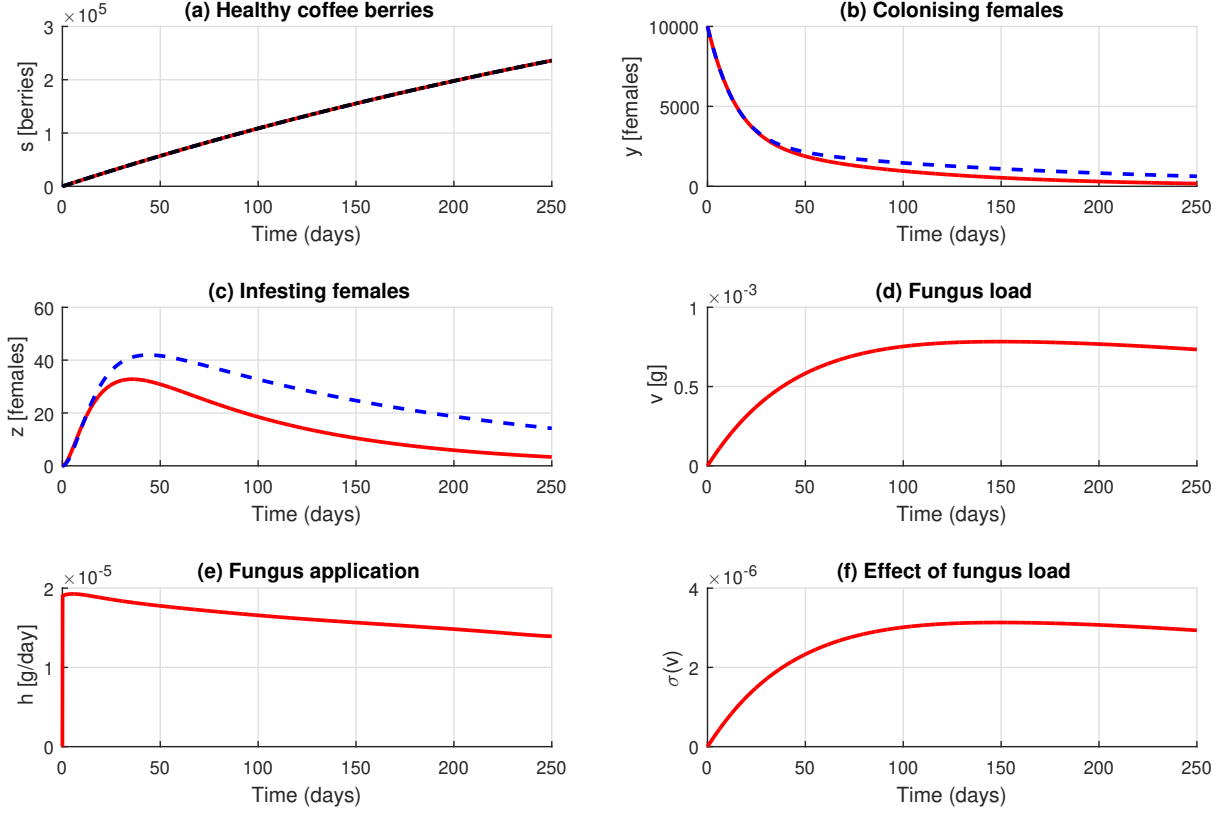


Figure 13: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal control (2.23) (plain red curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^4$ colonising females, when the infestation rate $\beta = 0.0005 \text{ day}^{-1}$ which corresponds to $\mathcal{R}_0 = 0.78 < 1$. The healthy berry growth without pest is also represented in (a) (dash-dotted black curve).

where the piecewise continuous function $v(t)$ has jumps at t_n , the value $v(t_n^+)$ denotes its right limit at t_n . h_n denotes the fungus quantity applied at discrete time t_n . We integrate on fixed time intervals (t_n, t_{n+1}) , the optimal fungus application obtained in the continuous case and apply the resulting cumulated fungus loads at the beginning of each corresponding interval ie $h_{n-1} = \int_{t_{n-1}}^{t_n} h^*(t) dt$. With this consideration, the total quantity of fungus applied and the control costs are hence similar in the continuous and discrete cases. See numerical results displayed in Figure 15 and 16 for $T = 7$ days (1 week) and $T = 14$ days (2 weeks) respectively.

Figure 15 shows that with weekly fungus applications, the healthy berry production and the colonising female population at the end of the season are very closed to those observed with the optimal continuous control. The penalised profit is hence only slightly lower in the discrete case ($\mathcal{J} = 200 \text{ US\$}$) than in the optimal continuous case ($\mathcal{J} = 207 \text{ US\$}$).

When the fungus is applied every two weeks, as shown in Figure 16, the discrete control (plain magenta curve) is more efficient than the optimal continuous control (plain red curve): the healthy berry production is higher (subfigure (a)) and the colonising female population is lower (subfigure (b)) at the end of the season. Therefore, the penalised profit with the discrete fungus application ($\mathcal{J} = 242 \text{ US\$}$) is greater than with the optimal continuous fungus application ($\mathcal{J} = 207 \text{ US\$}$). This is probably due to the higher loads present in the field when the fungus is

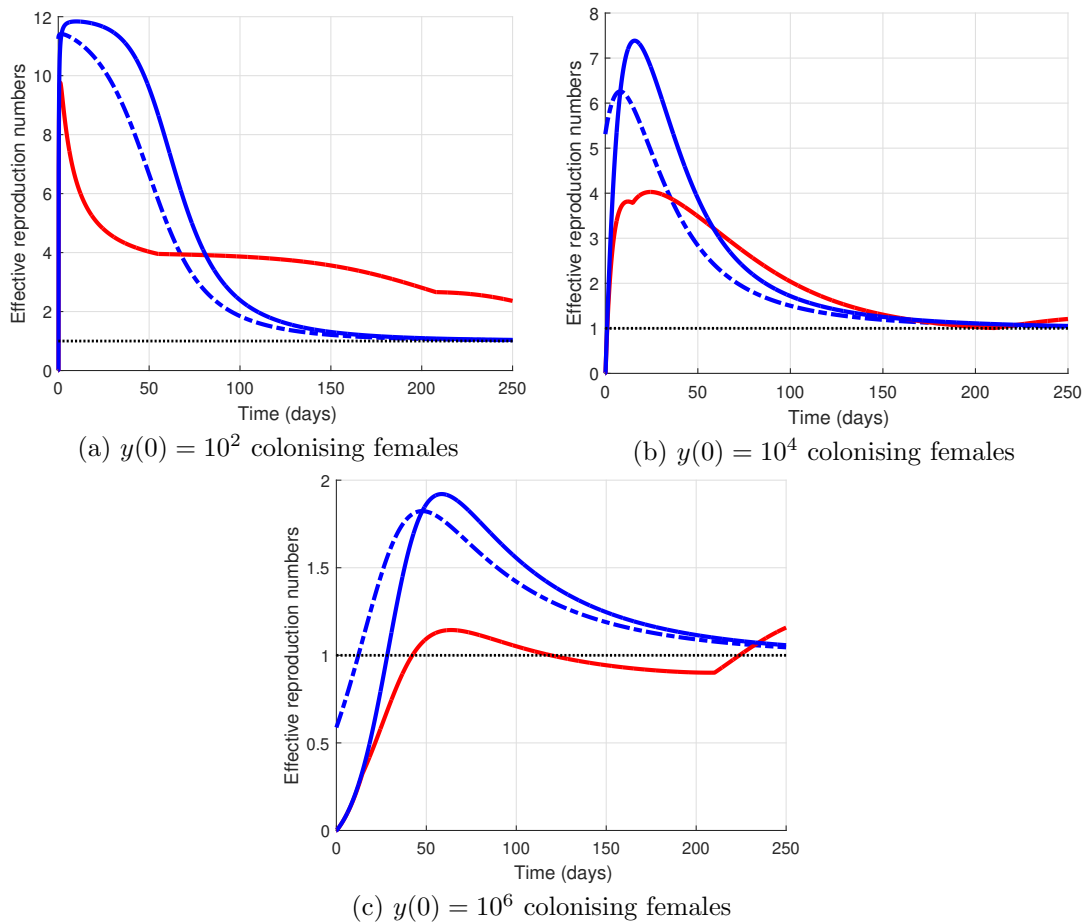


Figure 14: Exact (dashed curves) and approximated (plain curves) time-dependent reproduction numbers for system (2.1) without control (blue curves) and system (2.15) with optimal control (2.23) (red curves), for initial conditions $s(0) = z(0) = v(0) = 0$ and various values of $y(0)$. In the case without control, the exact reproduction number $\mathcal{R}_e(t)$ (dashed blue line) is computed according to equation (2.13), the approximated number $\mathcal{R}(t)$ (plain blue line) corresponds to equation (2.14). In the case with optimal control the approximated number $\mathcal{R}^*(t)$ (plain red line), corresponds to equation (2.24).

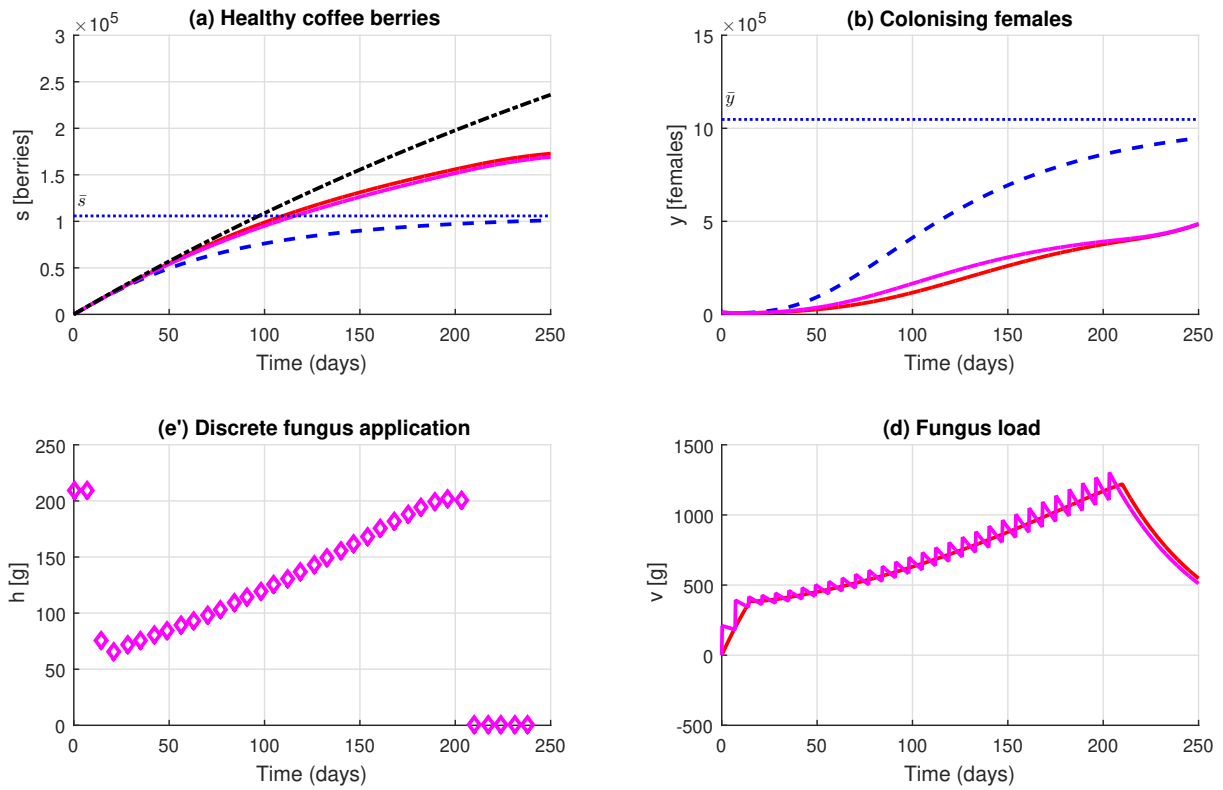


Figure 15: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal continuous control (2.23) (plain red curves) and weekly discrete control (plain magenta curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^4$ colonising females. The healthy berry growth without pest is also represented in (a) (dash-dotted black curve). The coexistence steady state without control is indicated in (a–b) (dotted blue lines).

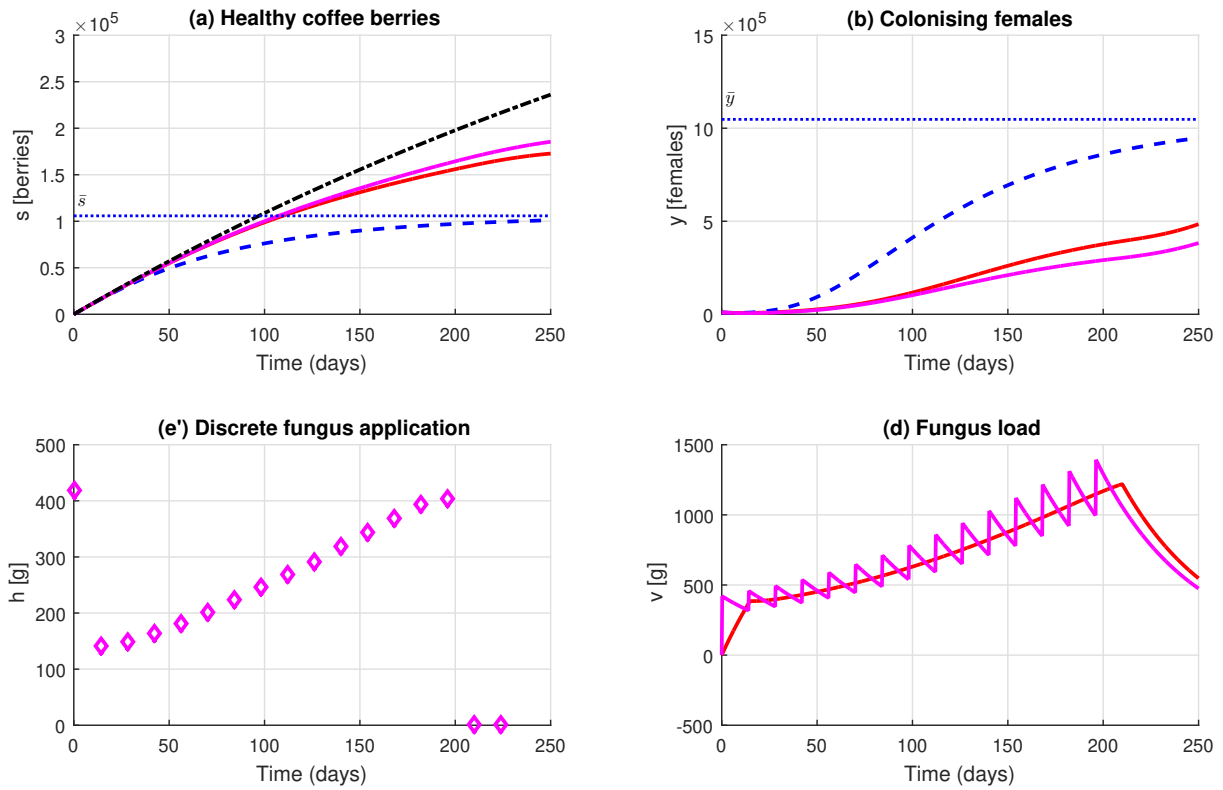


Figure 16: Simulation of system (2.1) without control (dashed blue curves) and system (2.15) with optimal continuous control (2.23) (plain red curves) and with discrete control applied every 2 weeks (plain magenta curves), for initial condition $s(0) = z(0) = v(0) = 0$ and $y(0) = 10^4$ colonising females. The healthy berry growth without pest is also represented in (a) (dash-dotted black curve). The coexistence steady state without control is indicated in (a–b) (dotted blue lines).

applied (subfigure (d)). As no bounds were set on the discrete control, the fungus applications may be fairly large in this case (subfigure (e')).

2.6 Conclusion

In this chapter, we have developed a baseline mathematical model to study the infestation dynamics of coffee berries by CBB in a plantation during a cropping season. This model is based on the CBB life cycle and takes into account the availability of healthy coffee berries. Our theoretical analysis shows that the CBB dynamics varies according to a threshold parameter called the basic reproduction \mathcal{R}_0 : when $\mathcal{R}_0 < 1$, the CBB population decreases until extinction; when $\mathcal{R}_0 > 1$ the pest-free steady state becomes unstable and a locally stable coexistence steady state appears. In the latter case, the CBB population needs to be controlled.

To reduce the infestation in the plantation, we designed an optimal control problem based on the use of an entomopathogenic fungus, which attacks colonising CBB as they drill into the berries. The objective of this control problem was to maximise the healthy coffee berry yield at the end of cropping season, while minimising the cost of control as well as the remaining CBB population for the next cropping season. We showed that an optimal control exists and that it can be characterised using the Pontryagin's Maximum Principle. In our numerical simulations, we obtained bang-singular-bang optimal controls that are consistent with our theoretical results. Moreover, we showed that applying the entomopathogenic fungus efficiently controls the CBB population, which is halved, and doubles the penalised profit, compared to the case without control. However, because of the control cost, the profit is still less than half of the pest-free profit (without control, the profit is only slightly more than a fifth of the pest-free profit).

We believe that for a start, the powerful tools of optimal control on the continuous system give us a first idea about the optimal strategy, depicted in Figures 8-12. However, impulsive control, consisting in discrete and regular fungus applications, is more realistic and may yield better results than optimal continuous control, provided that fairly large quantities can be applied at a time. Chapter 5 considers such impulsive control.

Using synthetic pesticides instead of entomopathogenic fungi would reduce the control cost and might also entail a better short-term efficiency, but their extensive use leads to the development of resistance and has deleterious consequences of the environment and public health [26, 25]. Biopesticides such as *Beauveria bassiana* may be more costly to produce, but they do not cause such problems [11, 52, 95]. Moreover, coffee produced with biopesticides can be sold on the organic, where it would fetch premium prices that could compensate the higher entomopathogenic fungus cost.

We considered a single cropping season. Nevertheless, as CBB survive between cropping seasons, we incorporated in our optimisation criterion a term that penalises the CBB population at the end of the season, so as not to unduly burden the next season yield. The control we obtain is hence still relevant in a multi-seasonal perspective, as it does not completely disregard what happens after the cropping season. The next step is to extend our model beyond a single cropping season, using a semi-discrete formalism to account for discrete events such as harvest or control methods such as dropped berry removal between cropping seasons. This is the aim of Chapter 3.

A multi-seasonal model of coffee berries-CBB interaction

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3.1 Introduction

In this chapter, we extend our modelling approach proposed in Chapter 2 by taking into account several cropping seasons. The model is based on a semi-discrete formalism, *i.e.* continuous dynamics over time intervals, with discrete events occurring at fixed times. Such a formalism has already been used in the literature by many authors in the life sciences [80] more particularly in plant-parasitic interactions [55, 21, 87, 101], or plant-diseases modeling [79]. The extinction conditions of CBB on multiple cropping seasons are investigated. Finally, considering the harvesting effort rate of coffee berries at the end of each cropping season and the cleaning effort of the plantation during the intercropping season as control parameters, we show numerically the parameter regions for which the CBB population disappears in the plantation.

3.2 Model formulation

We assume that the cultivation of coffee takes place in a geographical areas where there is only one annual harvesting, *i.e.* areas where the climate has two well marked seasons: a dry one and a rainy one. We consider that the year is subdivided into two time periods: the cropping season,

the rainy season during which the coffee berries are produced and grow in the plantation, and an intercropping season period, which defines the period just after harvesting and corresponds to the dry season. Let T denote the year length, $\tau < T$ the length of the cropping season, and $\varsigma < \tau$ the flowering period. Thus $T - \tau$ is the intercropping season length. Throughout this chapter, we denote $t_k = kT$ where k denotes a year index. The cropping and intercropping periods are separated by crop harvest of berries, and the intercropping season is followed by plantation cleaning, which are considered discrete events. For this reason, we define by t^+ the instant just after time t . Figure 17 illustrates the different interactions between coffee berries and CBB females during the cropping season at continuous time $(t_k, t_k + \tau)$ and during the intercropping season at continuous time $(t_k + \tau, t_{k+1})$, as well as the discrete events at times $t_k + \tau$ and t_{k+1} .

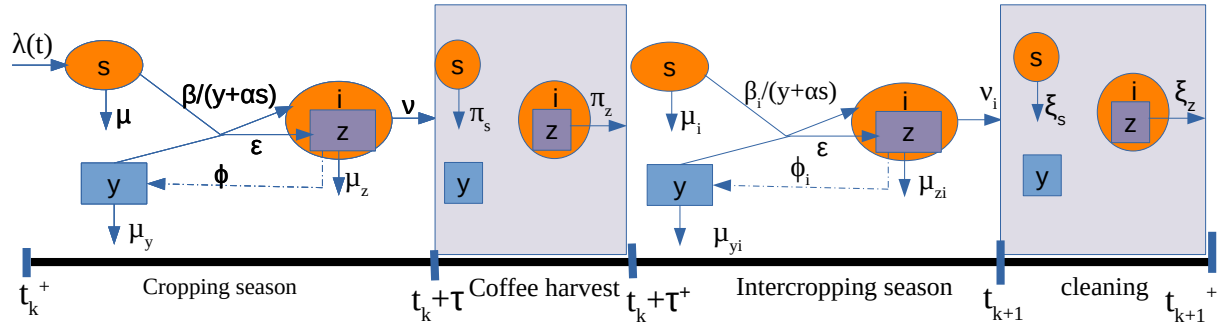


Figure 17: Schematic representation of the infestation dynamics of coffee berries by CBB from one cropping season to the next through four phases: cropping season, transition from cropping season to intercropping season (harvesting), intercropping season, and transition from intercropping season to cropping season (cleaning); those are detailed in the text. The orange ellipses represent the berry variables, while the blue rectangles concern the CBB.

In accordance with the Figure 17, we derive the semi-discrete model in several steps, for $k \in \mathbb{N}$ by:

1. **During the cropping season** *i.e.* when $t \in (t_k, t_k + \tau)$, we consider that the infestation dynamics of coffee berries by CBB is described as in system (2.1) proposed in the previous chapter with only the difference that the coffee berry production rate depends on time. That is:

$$\begin{cases} \dot{s}(t) = \lambda(t) - \frac{\beta s(t)y(t)}{y(t) + \alpha s(t)} - \mu s(t) \\ \dot{y}(t) = \phi z(t) - \varepsilon \frac{\beta s(t)y(t)}{y(t) + \alpha s(t)} - \mu_y y(t) \\ \dot{z}(t) = \varepsilon \frac{\beta s(t)y(t)}{y(t) + \alpha s(t)} - \mu_z z(t). \end{cases} \quad (3.1)$$

with the initial condition:

$$s(0^+) \geq 0, \quad y(0^+) > 0, \quad z(0^+) \geq 0. \quad (3.2)$$

This initial conditions assume that the first cropping season begins with CBB colonising females.

Colonising females (y) infest healthy coffee berries (s) with ratio-dependent rate (parameters

α , β and berry to CBB conversion rate ε). This leads to the transfer of coffee berries from healthy to infested berries $i(t)$ and simultaneously, for CBB female from colonising to infesting females $z(t)$. The infesting females lay eggs inside the coffee berries, which go through their development cycle until their emergence of new fertilised colonising females at rate ϕ . All CBB and berry stages undergo mortality at rates μ , ν , μ_y and μ_z respectively. During the whole flowering period, the white flowers fade in a very short time (approximately 2 days) and become the coffee berries [98]. This is why we assume that the coffee berries production rate $\lambda(t)$ is constant λ during the flowering period ς and is equal to zero on $(t_k + \varsigma, t_{k+1}]$. Then, we derive the birth function of berries:

$$\lambda(t) = \begin{cases} \lambda & \text{if } t_k \leq t \leq t_k + \varsigma \\ 0 & \text{if } t_k + \varsigma < t \leq t_{k+1}. \end{cases} \quad (3.3)$$

2. **Transition from cropping season to intercropping season.** At time $t = t_k + \tau^+$, the coffee berries are harvested. We assume that the quantity of healthy coffee berries and infesting CBB females remaining in the plantation after harvesting, is proportional to the healthy coffee berries and infesting CBB females present in the plantation at the end of cropping season at rates π_s and π_z respectively, while colonising females are not influenced by harvesting. This is described by the following discrete equations:

$$s(t_k + \tau^+) = \pi_s s(t_k + \tau), \quad y(t_k + \tau^+) = y(t_k + \tau), \quad z(t_k + \tau^+) = \pi_z z(t_k + \tau). \quad (3.4)$$

3. **During the intercropping season.** No new berries are produced and the CBB infest the remaining (dry) berries or those on the ground, to survive during the intercropping season. A part of the colonising females are doomed to die due to the lack of resources. We make the assumption that there is no early flowering during the intercropping season. The infestation process remains the same on the forgotten berries with modified parameters due to environmental conditions ($\lambda = 0$, and parameters β_i , ϕ_i , μ_i , μ_{yi} , μ_{zi} and ν_i). During this period, the climatic and environmental conditions are not favorable for the development of the CBB, that is why we assume that the mortality rate of CBB colonising and infesting females denoted by μ_{yi} and μ_{zi} respectively are such that $\mu_{yi} > \mu_y$ and $\mu_{zi} > \mu_z$.

A study carried out in a laboratory on the frequency of the emergence of colonising females as a function of physiological or abiotic factors was done in [82], and showed that the emergence of colonising CBB females from the berries depends on sunlight and the presence or absence of berries. This study showed that, the emergence of colonising females is more important in the presence of abundant coffee berries than when berries is few or absent in the plantation. For this fact, we assume that the emergence rate of colonising females is denoted by ϕ_i and is such that $\phi_i < \phi$ and defined by β_i the infestation rate during the

intercropping season. Thus, for $t \in (t_k + \tau, t_{k+1})$, we have the continuous system:

$$\begin{cases} \dot{s}(t) = -\frac{\beta_i s(t)y(t)}{y(t) + \alpha s(t)} - \mu_i s(t) \\ \dot{y}(t) = \phi_i z(t) - \varepsilon \frac{\beta_i s(t)y(t)}{y(t) + \alpha s(t)} - \mu_{yi} y(t) \\ \dot{z}(t) = \varepsilon \frac{\beta_i s(t)y(t)}{y(t) + \alpha s(t)} - \mu_{zi} z(t). \end{cases} \quad (3.5)$$

4. **Transition from intercropping season to new cropping season** i.e for $t = t_{k+1}^+$. Towards the end of intercropping season, the coffee growers clean the plantation by removing old berries on the trees and on the ground to favour a good flowering and also to reduce the quantity of CBB at the start of the next cropping season. This practice is included in the cultural control strategy [11, 29]. Let ξ_s and ξ_z be the proportion of coffee berries and infesting CBB females remaining in the plantation at begin the new cropping season, after cleaning. The discrete system corresponding to cleaning is thus:

$$s(t_{k+1}^+) = \xi_s s(t_{k+1}), \quad y(t_{k+1}^+) = y(t_{k+1}), \quad z(t_{k+1}^+) = \xi_z z(t_{k+1}). \quad (3.6)$$

3.3 Overview of impulsive differential equations

Impulsive differential equations are adequate mathematical tools for the description of evolution process characterised by the combination of continuous dynamics and jump events. They arise from the real world problems to describe the dynamics of processes in which sudden, discontinuous jumps occurs. Such processes are naturally seen in biology, physics, engineering etc..; We introduce here basic notions that we will use to analyse the semi-discrete system (3.1)-(3.6).

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$, $I_k : \Omega \longrightarrow \mathbb{R}^n$ for each $k \in \mathbb{Z}$. We consider the following impulsive differential equation with fixed moments:

$$\begin{cases} \dot{x}(t) = f(t, x) & t \neq t_k, t \in \mathbb{R} \\ \Delta x(t_k) = x(t_k^+) - x(t_k) = I_k(x(t_k)) & t = t_k, k \in \mathbb{Z}, \\ x(t_0^+) = x_0, \end{cases} \quad (3.7)$$

where the numbers t_k are called instants of impulse, I_k represents the jump of state at each t_k , whereas $x(t_k^+)$ and $x(t_k^-)$ represent the right limit and the left limit respectively of the state at $t = t_k$. We assume that the time sequence satisfies $t_0 < t_1 < \dots < t_k < t_{k+1} < \dots$

Theorem 9 [69] *Let the function $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$ be continuous in the sets $(t_k, t_{k+1}] \times \Omega$ ($k \in \mathbb{Z}$) and for each $k \in \mathbb{Z}$ and $x \in \Omega$. Suppose that there exists a finite limit of $f(t, y)$ as $(t, y) \longrightarrow (t_k, x)$, $t > t_k$. Then for each $(t_0, x_0) \in \mathbb{R} \times \Omega$, there exists $\beta > t_0$ and a solution $x : (t_0, \beta) \longrightarrow \mathbb{R}^n$ of the initial value problem (3.7). If, moreover, the function f is locally Lipschitz continuous with respect to x in $\mathbb{R} \times \Omega$, then this solution is unique.*

Definition 4 *A map $x : \mathbb{R} \longrightarrow \mathbb{R}^n$ is said to be a periodic solution of system (3.7) with period T if:*

1. It satisfies the system (3.7) and is a piecewise continuous map with jump discontinuities.
2. It satisfies $x(t + T) = x(t)$ for $t \neq t_k$ and $x(t_k + T^+) = x(t_k^+)$ for $k \in \mathbb{Z}$.

We now introduce a few basic results regarding Floquet theory for impulsive systems of ordinary differential equations, that will be used in the next section to discuss the stability of the pest-free periodic solution of semi-discrete system (3.1)-(3.6). Let us consider the linear T -periodic impulsive system:

$$\begin{cases} \dot{x}(t) = A(t)x(t) & t \neq t_k, t \in \mathbb{R} \\ \Delta x = x(t_k^+) - x(t_k) = B_k x(t_k) & t = t_k, t_k < t_{k+1}, k \in \mathbb{Z}, \end{cases} \quad (3.8)$$

under the following hypotheses.

- (H1) $A(\cdot) \in PC(\mathbb{R}, M_n(\mathbb{C}))$ and $A(t + T) = A(t)$ for all $t \in \mathbb{R}$.
- (H2) $B_k \in M_n(\mathbb{C})$, $\det(I_n + B_k) \neq 0$, $t_k < t_{k+1}$ for $k \in \mathbb{Z}$.
- (H3) There exists a $q \in \mathbb{N}$ such that $B_{k+q} = B_k$, $t_{k+q} = t_k + T$ for $k \in \mathbb{Z}$.

Without loss the generality, we assume that $t_0 = 0$ and $0 < t_1$. The following theorem is a generalisation of Floquet's theorem.

Theorem 10 [12] *Suppose that condition (H1) – (H3) hold. Then each fundamental matrix of (3.8) can be represented in the form $X(t) = \phi(t)e^{\Lambda t}$, ($t \in \mathbb{R}$) for a non-singular, T -periodic matrix $\phi(\cdot) \in PC^1(\mathbb{R}, M_n(\mathbb{C}))$ and a constant matrix $\Lambda \in M_n(\mathbb{C})$.*

Let $X(t)$ be a fundamental matrix of (3.8). Then the matrix $X(t + T)$ is also fundamental, and there corresponds a unique non-singular matrix $M \in M_n(\mathbb{C})$ such that

$$X(t + T) = X(t)M,$$

for all $t \in \mathbb{R}$. Here, M is called *monodromy matrix* of equation (3.8) corresponding to the fundamental matrix $X(t)$. All monodromy matrices of (3.8) are similar and have the same eigenvalues. The eigenvalues p_1, \dots, p_n of M are called *Floquet multipliers* of equation (3.8).

In the order to calculate the Floquet multipliers of (3.8), we have to choose an arbitrary fundamental matrix $X(t)$ of (3.8) and calculate the eigenvalues of the matrix $M = X(t_0 + T)X^{-1}(t_0)$ where $t_0 \in \mathbb{R}$ is fixed.

If $X(0) = I$ (or $X(0^+) = I$), then we can choose $M = X(T)$ ($M = X(T^+)$) as the monodromy matrix of (3.8). It follows that for any monodromy matrix M of (3.8), we have [12]

$$\det M = p_1 \cdot p_2 \cdot \dots \cdot p_n = \prod_{k=1}^q \det(I + B_k) \exp \left(\int_0^T \text{Tr} A(s) ds \right). \quad (3.9)$$

Under these hypotheses, the following stability result holds.

Theorem 11 [12] *Suppose that the conditions (H1) – (H3) hold. Then the linear T -periodic impulsive equation (3.8) is:*

1. stable if and only if all Floquet multipliers p_j , ($j = 1, \dots, n$) of equation (3.8) satisfy $|p_j| \leq 1$ and moreover, to those p_j for which $|p_k| = 1$, there corresponds a simple elementary divisor.
2. asymptotically stable if and only if all Floquet multipliers p_j , ($j = 1, \dots, n$) of the equation (3.8) satisfy the inequality $|p_j| < 1$.
3. unstable if there is a Floquet multiplier p_j , ($j = 1, \dots, n$) such that $|p_j| > 1$.

This theory can be used to evaluate the stability of a periodic solution of $x^*(t)$ of (3.7). Indeed, if we define the variable $\tilde{x}(t) = x(t) - x^*(t)$, we obtain the system

$$\begin{cases} \dot{\tilde{x}}(t) = f(t, x) - f(t, x^*) = f(t, \tilde{x}(t) + x^*(t)) - f(t, x^*(t)) & t \neq t_k, t \in \mathbb{R} \\ \Delta \tilde{x}(t_k) = x(t_k^+) - x^*(t_k^+) - x(t_k) + x^*(t_k) = I_k(x(t_k)) - I_k(x^*(t_k)) \\ \quad = I_k(\tilde{x}(t_k) + x^*(t_k)) - I_k(x^*(t_k)) & t = t_k, k \in \mathbb{Z}, \\ \tilde{x}(t_0^+) = x_0 - x^*(t_0), \end{cases} \quad (3.10)$$

We can now linearize (3.10) around its equilibrium in $\tilde{x} = 0$. Using the jacobian matrix $J_{f,x^*}(t)$ of the vector field $f(t, \tilde{x}(t) + x^*(t)) - f(t, x^*(t))$ around $\tilde{x} = 0$, which also is the jacobian matrix of f around $x^*(t)$; and using the jacobian matrix J_{I_k,x^*} of I_k around $x^*(t_k)$, the linearised version of (3.10) becomes

$$\begin{cases} \dot{\tilde{x}}(t) = J_{f,x^*}(t)\tilde{x} & t \neq t_k, t \in \mathbb{R} \\ \Delta \tilde{x}(t_k) = J_{I_k,x^*}\tilde{x}(t_k) & t = t_k, k \in \mathbb{Z}, \\ \tilde{x}(t_0^+) = x_0 - x^*(t_0), \end{cases} \quad (3.11)$$

which is exactly in the form of (3.8). We then just need to compute the monodromy matrix of (3.11) to analyse the local stability of (3.7).

3.4 Existence of a pest-free periodic solution

This section is devoted to show the existence of the pest-free periodic solution of semi-discrete system (3.1)-(3.6). The structure of system (3.1)-(3.6) shows that the semi-discrete system (3.1)-(3.6) has a unique solution that remains nonnegative for nonnegative initial conditions. This make sure that the semi-discrete model is biologically well defined.

Lemma 1 *The system formed by the equations (3.1) - (3.6) has a unique pest-free periodic solution $(s^*(t), 0, 0)$ of period T corresponding to the periodic solution of system without CBB population, where:*

$$s^*(t) = \begin{cases} \lambda/\mu - [\lambda/\mu - s(0^+)] e^{-\mu(t-t_k)}, & \text{if } t \in (t_k, t_k + \varsigma] \\ [\lambda/\mu - (\lambda/\mu - s(0^+)) e^{-\mu\varsigma}] e^{-\mu(t-t_k-\varsigma)}, & \text{if } t \in (t_k + \varsigma, t_k + \tau] \\ \pi_s e^{-\mu(\tau-\varsigma)} [\lambda/\mu - (\lambda/\mu - s(0^+)) e^{-\mu\varsigma}] e^{-\mu_i(t-t_k-\tau)}, & \text{if } t \in (t_k + \tau, t_{k+1}] \end{cases} \quad (3.12)$$

and

$$s^*(0^+) = \frac{\xi_s \pi_s \lambda [e^{-\mu(\tau-\varsigma)} - e^{-\mu\tau}] e^{-\mu_i(T-\tau)}}{\mu [1 - \xi_s \pi_s e^{-(\mu-\mu_i)\tau} e^{-\mu_i T}]} \quad (3.13)$$

Proof : By definition, the pest-free periodic solution represents the solution of semi-discrete system (3.1)-(3.6) without CBB ie $y(t) = z(t) = 0$. Then, the dynamics of coffee berries without CBB in the plantation is described by the following semi-discrete system:

$$\begin{cases} \dot{s}(t) = \lambda(t) - \mu s(t) & \text{if } t_k^+ \leq t \leq t_k + \tau \\ \dot{s}(t) = -\mu_i s(t) & \text{if } t_k + \tau^+ \leq t \leq t_{k+1} \\ s(t_k + \tau^+) = \pi_s s(t_k + \tau) & s(t_{k+1}^+) = \xi_s s(t_{k+1}). \end{cases} \quad (3.14)$$

Then integrating the first two equations of the semi-discrete system (3.14), we obtain:

$$s(t) = \begin{cases} \lambda/\mu - [\lambda/\mu - s(t_k^+)] e^{-\mu(t-t_k)} & \text{if } t \in (t_k, t_k + \varsigma] \\ [\lambda/\mu - (\lambda/\mu - s(t_k^+)) e^{-\mu\varsigma}] e^{-\mu(t-t_k-\varsigma)}, & \text{if } t \in (t_k + \varsigma, t_k + \tau] \end{cases} \quad (3.15)$$

and also

$$s(t) = \pi_s s(t_k + \tau) e^{-\mu_i(t-t_k-\tau)} \text{ for } t \in (t_k + \tau, t_{k+1}]. \quad (3.16)$$

From the third equation of system (3.14) and replacing in the equations (3.15) and (3.16) respectively, we obtain after some manipulations the discrete equation:

$$s(t_{k+1}^+) = \xi_s \pi_s [\lambda/\mu - (\lambda/\mu - s(t_k^+)) e^{-\mu\varsigma}] e^{-\mu(\tau-\varsigma)} e^{-\mu_i(T-\varsigma)}. \quad (3.17)$$

The solution $s(t)$ will be T -periodic if and only if $s(t_{k+1}^+) = s(t_k^+)$. Setting this equality into (3.17), we get:

$$s(t_k^+) = \xi_s \pi_s [\lambda/\mu - (\lambda/\mu - s(t_k^+)) e^{-\mu\varsigma}] e^{-\mu(\tau-\varsigma)} e^{-\mu_i(T-\varsigma)}.$$

Therefore,

$$s(t_k^+) = \frac{\xi_s \pi_s \lambda [e^{-\mu(\tau-\varsigma)} - e^{-\mu\tau}] e^{-\mu_i(T-\tau)}}{\mu [1 - \xi_s \pi_s e^{-(\mu-\mu_i)\tau} e^{-\mu_i T}]} > 0,$$

as announced for the value of $s^*(0^+)$ in (3.13).

Then, the pest-free periodic solution searched for does indeed exist, it is strictly positive and from (3.15)-(3.16), its analytic expression is given by (3.12)-(3.13). \square

Moreover, in the discrete linear dynamic equation (3.17), the coefficient of $s(t_k^+)$ is given by the expression $\xi_s \pi_s e^{-\mu\tau} e^{-\mu_i(T-\varsigma)}$, which is less than one in magnitude. The fixed point $s^*(0^+)$ is then globally asymptotically stable for (3.17).

Remark that for $\pi_s = 0$, ie all healthy coffee berries are harvested at the end of each cropping season, then the pest-free periodic solution of semi-discrete system (3.1)-(3.6) is reduced to $(s^*(t), 0, 0)$ with:

$$s^*(t) = \begin{cases} \lambda/\mu (1 - e^{-\mu(t-t_k)}) & \text{if } t \in (t_k, t_k + \varsigma) \\ \lambda/\mu (1 - e^{-\mu\varsigma}) e^{-\mu(t-t_k-\varsigma)}, & \text{if } t \in (t_k + \varsigma, t_k + \tau] \\ 0, & \text{if } t \in (t_k + \tau, t_{k+1}]. \end{cases} \quad (3.18)$$

Due to the ratio-dependent force of infestation, we will see that the cases where $s^*(t) = 0$ give

rise to very distinct situations. We will now analyse the stability of the pest-free periodic solution of the semi-discrete system (3.1)-(3.6) and distinguish two possible situations according to the values taken by the parameter $\pi_s > 0$ and $\pi_s = 0$.

3.5 Stability of the pest-free periodic solution

3.5.1 Stability of the pest-free periodic solution for $\pi_s > 0$

In this section, we study the local stability of the pest-free periodic solution $(s^*(t), 0, 0)$ for $\pi_s \neq 0$. Let us denote $u(t) = s(t) - s^*(t)$, $v(t) = y(t)$ and $w(t) = z(t)$, where $u(t)$, $v(t)$ and $w(t)$ form the \tilde{x} vector that we defined earlier and are to be understood as small amplitude perturbations. Computing the Jacobian matrix of system (3.1), we get

$$J = \begin{pmatrix} -\frac{\beta y^2(t)}{(y+\alpha s)^2} - \mu & -\frac{\beta \alpha s^2(t)}{(y(t)+\alpha s)^2} & 0 \\ -\frac{\epsilon \beta y^2(t)}{(y+\alpha s)^2} & -\frac{\epsilon \beta \alpha s^2(t)}{(y(t)+\alpha s)^2} - \mu_y & \phi \\ \frac{\epsilon \beta y^2(t)}{(y+\alpha s)^2} & \frac{\epsilon \beta \alpha s^2(t)}{(y(t)+\alpha s)^2} & -\mu_z \end{pmatrix} \quad (3.19)$$

which, evaluated in $(s^*(t), 0, 0)$ yields the linearised system at pest-free periodic solution by setting $\tilde{x}(t) = (u(t), v(t), w(t))^\top$:

$$\dot{\tilde{x}}(t) = \begin{pmatrix} -\mu & -\frac{\beta}{\alpha} & 0 \\ 0 & -\epsilon \frac{\beta}{\alpha} - \mu_y & \phi \\ 0 & \epsilon \frac{\beta}{\alpha} & -\mu_z \end{pmatrix} \tilde{x}(t), \text{ for } t \in (t_k, t_k + \tau].$$

Integrating this linear system in $(t_k, t_k + \tau)$, we get:

$$\tilde{x}(t) = \begin{pmatrix} e^{-\mu(t-t_k)} & * \\ 0_{\mathbb{R}^2} & e^{\mathbb{A}(t-t_k)} \end{pmatrix} \tilde{x}(t_k^+), \quad (3.20)$$

where the matrix \mathbb{A} is defined as follows:

$$\mathbb{A} = \begin{pmatrix} -\epsilon \frac{\beta}{\alpha} - \mu_y & \phi \\ \epsilon \frac{\beta}{\alpha} & -\mu_z \end{pmatrix}.$$

Now, we investigate the exponential matrix application $t \in (t_k, t_k + \tau) \mapsto e^{\mathbb{A}(t-t_k)}$. Let $\theta \in \mathbb{C}$ be the eigenvalue of matrix \mathbb{A} , then the characteristic polynomial associated to \mathbb{A} is given by:

$$P(\theta) = \theta^2 - Tr(\mathbb{A})\theta + \det(\mathbb{A}),$$

where the trace of matrix \mathbb{A} is given by $Tr(\mathbb{A}) = -\left(\epsilon \frac{\beta}{\alpha} + \mu_y + \mu_z\right)$ and its determinant is $\det(\mathbb{A}) = -\mu_z \left(\epsilon \frac{\beta}{\alpha} + \mu_y\right) (\mathcal{R}_0 - 1)$, \mathcal{R}_0 corresponding to the basic reproduction number of the single season model defined in equation 2.6. The discriminant to determine the number of roots in a quadratic equation $p(\theta) = 0$ is given by:

$$\Delta = Tr(\mathbb{A})^2 - 4 \det(\mathbb{A}) = \left(\epsilon \frac{\beta}{\alpha} + \mu_y - \mu_z\right)^2 + 4\epsilon \phi \frac{\beta}{\alpha} > 0.$$

Then the characteristic polynomial has two real eigenvalues given by

$$\theta_{\pm} = \frac{1}{2} \left[\text{Tr}(\mathbb{A}) \pm \sqrt{\text{Tr}(\mathbb{A})^2 - 4 \det(\mathbb{A})} \right].$$

Remark that, when $\mathcal{R}_0 < 1$, then $\theta_+ < 0$ and $\theta_- < 0$ and it follows that the CBB population disappear in the plantation during the cropping season as it has been demonstrated in Chapter 2. Throughout what follows, we assume that $\mathcal{R}_0 > 1$, so that $\det(\mathbb{A}) < 0$, and then $\theta_+ > 0$ and $\theta_- < 0$, which represents the case where the CBB-free solution is a saddle point, and the CBB population persists in the plantation during the cropping season.

The eigenvectors associated to eigenvalues θ_+ and θ_- are respectively given by

$$v_+ = \begin{pmatrix} \phi \\ \varepsilon \frac{\beta}{\alpha} + \mu_y + \theta_+ \end{pmatrix} \quad \text{and} \quad v_- = \begin{pmatrix} \phi \\ \varepsilon \frac{\beta}{\alpha} + \mu_y + \theta_- \end{pmatrix}.$$

Then the exponential of matrix \mathbb{A} is defined as

$$M(t - t_k) := e^{\mathbb{A}(t-t_k)} = \begin{pmatrix} M_{11}(t - t_k) & M_{12}(t - t_k) \\ M_{21}(t - t_k) & M_{22}(t - t_k) \end{pmatrix}.$$

where

$$\begin{aligned} M_{11}(t - t_k) &= (\mu_z + \theta_+) \frac{e^{\theta_+(t-t_k)} - e^{\theta_-(t-t_k)}}{\theta_+ - \theta_-} + e^{\theta_-(t-t_k)}; \\ M_{12}(t - t_k) &= 2(\mu_z + \theta_+) \frac{e^{\theta_+(t-t_k)} - e^{\theta_-(t-t_k)}}{\theta_+ - \theta_-}; \\ M_{21}(t - t_k) &= \left(\varepsilon \frac{\beta}{\alpha} + \mu_y + \theta_+ \right) \frac{e^{\theta_+(t-t_k)} - e^{\theta_-(t-t_k)}}{\theta_+ - \theta_-}; \\ M_{22}(t - t_k) &= \left(\varepsilon \frac{\beta}{\alpha} + \mu_y + \theta_+ \right) \frac{e^{\theta_+(t-t_k)} - e^{\theta_-(t-t_k)}}{\theta_+ - \theta_-} + e^{\theta_-(t-t_k)}. \end{aligned}$$

Note that, when $\mathcal{R}_0 > 1$, all components of the matrix M are nonnegative (ie $M_{ij} \geq 0$ for all $i, j \in \{1, 2\}$), since $\theta_+ > 0$.

Similarly, the linearisation of system (3.1) around the pest-free periodic solution during the intercropping season is given by

$$\dot{\tilde{x}}(t) = \begin{pmatrix} -\mu_i & -\frac{\beta_i}{\alpha} & 0 \\ 0 & -\varepsilon \frac{\beta_i}{\alpha} - \mu_{yi} & \phi_i \\ 0 & \varepsilon \frac{\beta_i}{\alpha} & -\mu_{zi} \end{pmatrix} \tilde{x}(t) \quad \text{for } t \in (t_k + \tau, t_{k+1}].$$

In proceeding by analogy with the previous case, since the matrix as the same form, we have the submatrix :

$$\widehat{\mathbb{A}} = \begin{pmatrix} -\varepsilon \frac{\beta_i}{\alpha} - \mu_{yi} & \phi_i \\ \varepsilon \frac{\beta_i}{\alpha} & -\mu_{zi} \end{pmatrix},$$

and also there exists a threshold:

$$\widehat{\mathcal{R}}_0 = \frac{\varepsilon \phi_i \frac{\beta_i}{\alpha}}{\mu_{zi} \left(\varepsilon \frac{\beta_i}{\alpha} + \mu_{yi} \right)}.$$

Then the characteristic polynomial associated to matrix $\widehat{\mathbb{A}}$ has two roots $\widehat{\theta}_{\pm}$ and the exponential of the matrix $\widehat{\mathbb{A}}$ is given by \widehat{M} whose expressions are in the same form as M , where parameters β , μ_y , μ_z and ϕ are replaced with β_i , μ_{yi} , μ_{zi} and ϕ_i . Then we have:

$$\tilde{x}(t) = \begin{pmatrix} e^{-\mu_i(t-t_k-\tau)} & * \\ 0_{\mathbb{R}^2} & e^{\widehat{\mathbb{A}}(t-t_k-\tau)} \end{pmatrix} \tilde{x}(t_k + \tau^+), \quad t \in (t_k + \tau, t_{k+1}]. \quad (3.21)$$

Therefore, we obtain from (3.20) and (3.21) and using the linearising of the semi-discrete equation (3.4) and (3.6) around the periodic pest-free solution, the discrete equations

$$\tilde{x}(t_{k+1}^+) = \begin{pmatrix} \xi_s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_z \end{pmatrix} \begin{pmatrix} \pi_s e^{-\mu_i(T-\tau)} & * \\ 0 & \widehat{M}(T-\tau) \begin{pmatrix} 1 & 0 \\ 0 & \pi_z \end{pmatrix} \end{pmatrix} \tilde{x}(t_k + \tau), \quad (3.22)$$

and

$$\tilde{x}(t_k + \tau) = \begin{pmatrix} e^{-\mu\tau} & * \\ 0 & M(\tau) \end{pmatrix} \tilde{x}(t_k^+). \quad (3.23)$$

Since the above matrix in the systems (3.22) and (3.23) are triangular and using the fact that the term $\xi_s \pi_s e^{-\mu_i(T-\tau)} e^{-\mu\tau} < 1$, then the stability of system depends of the submatrix:

$$\mathbb{K} := \begin{pmatrix} 1 & 0 \\ 0 & \xi_z \end{pmatrix} \widehat{M}(T-\tau) \begin{pmatrix} 1 & 0 \\ 0 & \pi_z \end{pmatrix} M(\tau),$$

that is

$$\mathbb{K} = \begin{pmatrix} \widehat{M}_{11}M_{11} + \pi_z \widehat{M}_{12}M_{21} & \widehat{M}_{11}M_{12} + \pi_z \widehat{M}_{12}M_{22} \\ \xi_z [\widehat{M}_{21}M_{11} + \pi_z \widehat{M}_{22}M_{21}] & \xi_z [\widehat{M}_{21}M_{12} + \pi_z \widehat{M}_{22}M_{22}] \end{pmatrix}$$

According to the Floquet theory (see Theorem 11), the pest-free periodic solution of system (3.1)-(3.6) is locally asymptotically stable if and only if the module of all eigenvalues of matrix \mathbb{K} is less than one. This property is satisfied if the matrix \mathbb{K} satisfies the Jury conditions [64] which are given as:

$$\text{Cond 1 : } \Psi_1(\xi_z, \pi_z) \equiv 1 - \text{Tr}(\mathbb{K}) + \det(\mathbb{K}) > 0,$$

$$\text{Cond 2 : } \Psi_2(\xi_z, \pi_z) \equiv 1 + \text{Tr}(\mathbb{K}) + \det(\mathbb{K}) > 0,$$

$$\text{Cond 3 : } \Psi_3(\xi_z, \pi_z) \equiv 1 - \det(\mathbb{K}) > 0.$$

We have by the direct calculation:

$$\begin{aligned}\det(\mathbb{K}) &= \pi_z \xi_z \left[\widehat{M}_{11} \widehat{M}_{22} - \widehat{M}_{12} \widehat{M}_{21} \right] [M_{11} M_{22} - M_{12} M_{21}] \\ &= \xi_z \pi_z \det(\widehat{M}) \det(M) = \xi_z \pi_z e^{(\widehat{\theta}_+ + \widehat{\theta}_-)(T-\tau)} e^{(\theta_+ + \theta_-)\tau} \\ &= \xi_z \pi_z e^{Tr(\widehat{\mathbb{A}})(T-\tau)} e^{Tr(\mathbb{A})\tau} < 1,\end{aligned}$$

since $Tr(\mathbb{A}) < 0$ and $Tr(\widehat{\mathbb{A}}) < 0$. Therefore the third Jury condition is satisfied. The trace and determinant of matrix \mathbb{K} are nonnegative since the components of matrix M and \widehat{M} are all nonnegative, then $1 + Tr(\mathbb{K}) + \det(\mathbb{K}) > 0$. Hence, the Jury condition 2 is satisfied. It is not easy to verify the first Jury condition and the expression of function $\Psi_1(., .)$ is given analytically by

$$\Psi_1(\xi_z, \pi_z) = 1 - \widehat{M}_{11} M_{11} - \pi_z \widehat{M}_{12} M_{21} - \xi_z \widehat{M}_{21} M_{12} - \xi_z \pi_z \left(\widehat{M}_{22} M_{22} - e^{Tr(\widehat{\mathbb{A}})(T-\tau)} e^{Tr(\mathbb{A})\tau} \right).$$

Therefore, the pest-free periodic solution is locally asymptotically stable if and only if the function $\Psi_1(\xi_z, \pi_z)$ is positive. To verify that the function $\Psi_1(\xi_z, \pi_z)$ is positive, we distinguish several particular situations:

1. If $\pi_z = 0$ and $\xi_z = 0$, then the first Jury condition is satisfied if and only if $\Psi_1(0, 0) = 1 - \widehat{M}_{11} M_{11} > 0$. Thus, the pest-free periodic solution of system (3.1)-(3.6) is locally asymptotically if and only if

$$\mathcal{R}_1 := \widehat{M}_{11} M_{11} < 1. \quad (3.24)$$

The threshold, \mathcal{R}_1 , represents the average number of colonising females originated from a single infesting female during a multiple cropping seasons. The expression of \mathcal{R}_1 is the product of two crucial factors, the first one \widehat{M}_{11} represents the quantity of colonising females at the end of an intercropping season linked to the presence of 1 colonising female at the beginning of the intercropping season; the second term M_{11} denotes the same thing for the cropping season. The product of the two then represents the quantity of colonising females at the beginning of a cropping season linked to the presence of 1 colonising female at the beginning of the previous season, via colonising females between the cropping and intercropping season (which are the only path of transfer since $\pi_z = \xi_z = 0$).

2. If we assume that $\pi_z \neq 0$ and $\xi_z = 0$, then the first Jury condition is satisfied if and only if $\Psi_1(0, \pi_z) = 1 - \widehat{M}_{11} M_{11} - \pi_z \widehat{M}_{12} M_{21} > 0$. Therefore, the pest-free periodic solution of system (3.1)-(3.6) is locally asymptotically stable if and only if

$$\mathcal{R}_2 := \mathcal{R}_1 + \pi_z \widehat{M}_{12} M_{21} < 1. \quad (3.25)$$

Note that, the threshold \mathcal{R}_2 has the same meaning as \mathcal{R}_1 : the quantity of colonising females at the beginning of a cropping season linked to the presence of 1 colonising female at the beginning of the previous season, except that, in this case, there are two paths of transfer: the colonising females at the end of the cropping season, but also the infesting females at the end of the cropping season. Remark that $\mathcal{R}_2 < 1$ implies that $\mathcal{R}_1 < 1$. Then from

(3.25), there exists a critical value of the parameter π_z defined by

$$\pi_z^{crit} = \frac{1 - \mathcal{R}_1}{\widehat{M}_{12}M_{21}}, \quad (3.26)$$

such that the pest-free periodic solution of system (3.1)-(3.6) is locally asymptotically stable for all $\pi_z \in [0, \pi_z^{crit})$.

3. If the parameters $\pi_z > 0$ and $\xi_z > 0$, then it is not easy to verify analytically the stability of the pest-free periodic solution.

3.5.2 Numerical simulations when $\pi_s > 0$

All parameters of the semi-discrete system (3.1)-(3.6) are nonnegative and given in Table 3.

Table 3: Parameter values of semi-discrete system (3.1)-(3.6). Parameter values (*) are based on biological data collected in Table 1, the value (**) is taken in [98] and the rest is assumed.

Symbol	Description	Value
λ	Production rate of new coffee berries	1200 berries.day ⁻¹ *
μ	Natural mortality rate of healthy coffee berries	0.002 day ⁻¹ *
ϕ	Emergence rate of new colonising females	2 day ⁻¹ *
β	Infestation rate	0.0084 day ⁻¹
β_i	Infestation rate during intercropping season	0.0001 day ⁻¹
ε	Conversion rate from coffee berries to CBB	1 female.berry ⁻¹ *
α	Infestation half saturation	0.7 *
μ_y	Natural mortality rate of colonising females	1/20 day ⁻¹ *
μ_z	Natural mortality rate of infesting females	1/27 day ⁻¹ *
ϕ_i	Intercropping emergence rate of colonising females	0.02 day ⁻¹
μ_{yi}	Intercropping mortality rate of infesting females	0.23 day ⁻¹
μ_{zi}	Intercropping mortality rate of infesting females	0.21 day ⁻¹
τ	Duration of a cropping season	250 days *
ς	Duration of the flowering period	4 months **

Using the parameters of Table 3, we plot the diagram in Figure 18 to show the regions of asymptotic stability and unstability of the pest-free periodic solution in varying the values of parameters $\pi_z > 0$ and $\xi_z > 0$.

In Figure 18, we observe that it is extremely difficult to control the CBB when berries harvesting and cleaning of the plantation during the intercropping season is poorly done, since the asymptotic stability of pest-free periodic solution holds for the very small values of π_z and ξ_z . This result can be explained biologically by the presence of healthy coffee berries in the plantation during the intercropping season which become the main reservoir of CBB and favour the survival during the intercropping season of CBB that infested the new coffee berries of the next cropping season [29, 11].

We consider the following initial conditions for numerical simulations. At the beginning of the first cropping season, there are no berries so that $s(0^+) = 0$, $\varsigma = 120$ days and we consider 10 seasons. Moreover, we assume that there are only colonising females, so $y(0^+) = 10^4$ and no infesting females, so that $z(0^+) = 0$ and the rest of parameter values are given in Table 3.

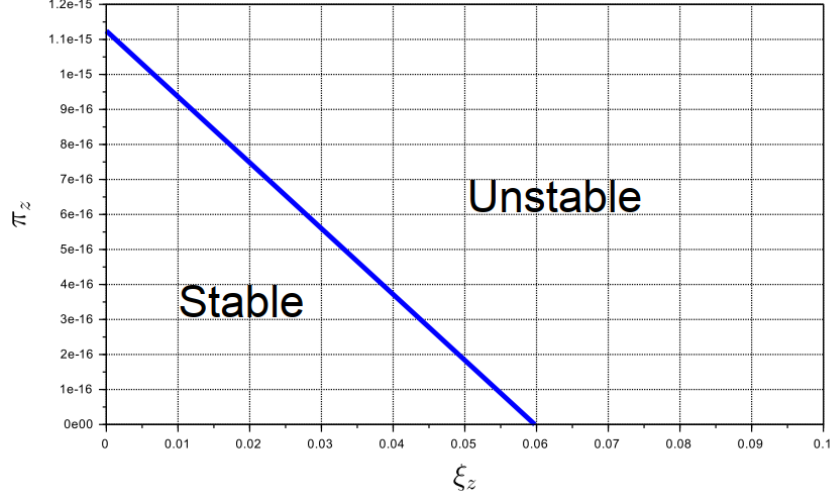


Figure 18: Stability of the pest-free periodic solution according to the harvesting π_z and cleaning ξ_z parameter values of infested berries, for imperfect harvesting of healthy berries $\pi_s = 0.1 > 0$.

Figure 19 is an illustration of the particular situation where the parameter values $\pi_z = \xi_z = 0$, showing the asymptotic stability of the pest-free periodic solution of semi-discrete system (3.1)-(3.6). It illustrates that the CBB population disappears in the plantation when the threshold $\mathcal{R}_1 < 1$. On the other hand, we can see in the Figure 20 that the CBB persist in the plantation when the threshold $\mathcal{R}_1 > 1$.

3.5.3 Stability of pest-free periodic solution for $\pi_s = 0$

In this section, we analyse the stability of the pest-free periodic solution for $\pi_s = 0$. In this case, the healthy coffee berries are assumed absent during the intercropping season and so the colonising females that emerge are those from the infesting females left in the plantation. The parameter $\pi_s = 0$, that implies that $s(t + \tau^+) = 0$. Therefore, the system of equations (3.5) during the intercropping season can be reduced for $t \in (t_k + \tau, t_{k+1})$ to the continuous system:

$$s(t) = 0, \quad \dot{y}(t) = \phi_i z(t) - \mu_{yi} y(t), \quad \dot{z}(t) = -\mu_{zi} z(t) \quad (3.27)$$

Denoting by $\tilde{x}(t) = (u(t), v(t), w(t))^T$ and we obtain the corresponding linearisation system during the intercropping season of (3.27):

$$\dot{\tilde{x}}(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\mu_{yi} & \phi_i \\ 0 & 0 & -\mu_{zi} \end{pmatrix} \tilde{x}(t), \quad t \in (t_k + \tau, t_{k+1}). \quad (3.28)$$

Integrating the system (3.28) and using the transition linearised discrete system

$$\tilde{x}(t_k + \tau^+) = \begin{pmatrix} \pi_s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pi_z \end{pmatrix} \tilde{x}(t_k + \tau).$$

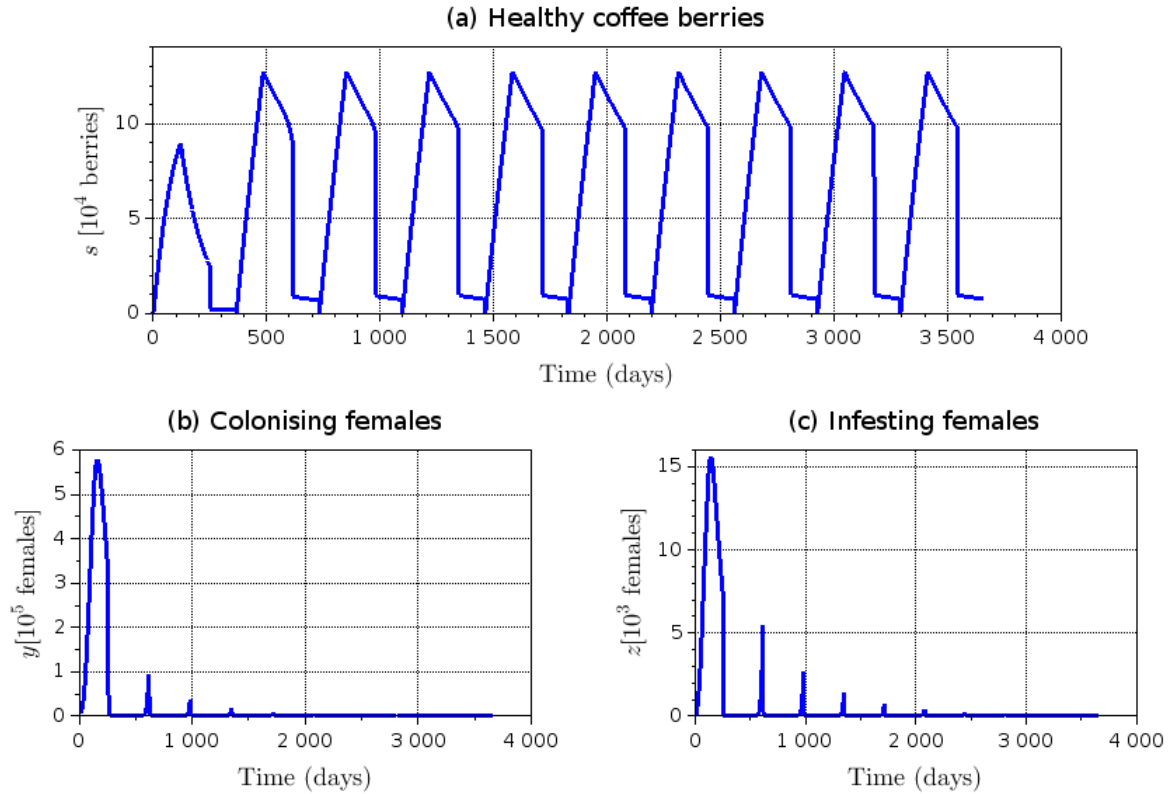


Figure 19: Simulation of semi-discrete system (3.1)-(3.6): (a) healthy coffee berries, (b) colonising females and (c) infesting females. Harvesting parameters are $\pi_s = 0.1$ and $\pi_z = 0$, cleaning parameter is $\xi_z = 0$; remaining parameters are given in Table 3. The resulting threshold is $\mathcal{R}_1 = 0.5 < 1$, which indicates that the pest-free periodic solution is asymptotically stable.

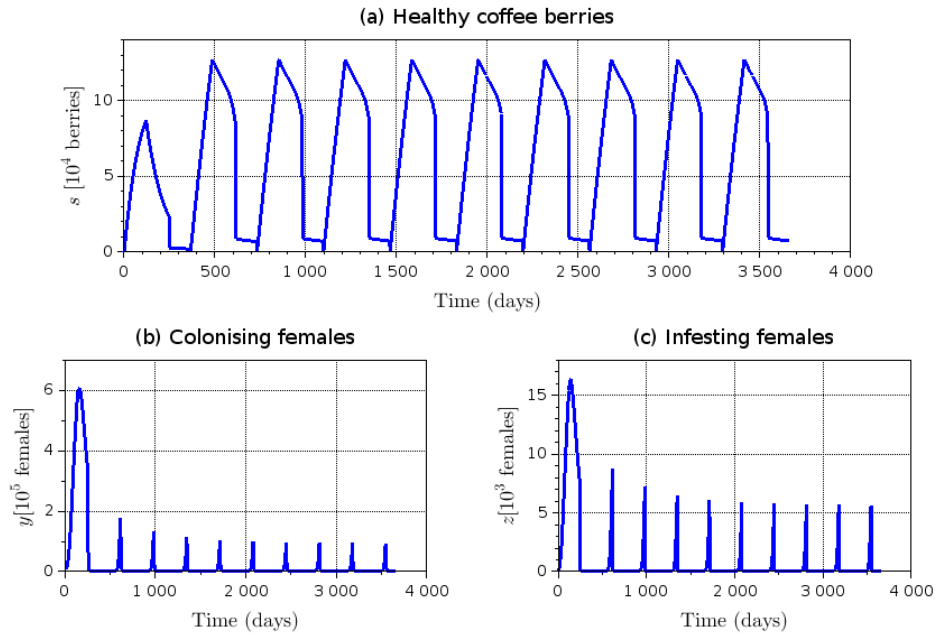


Figure 20: Simulation of semi-discrete system (3.1)-(3.6): (a) healthy coffee berries, (b) colonising females and (c) infesting females. Harvesting parameters are $\pi_s = 0.1$ and $\pi_z = 0$, cleaning parameter is $\xi_z = 0$; remaining parameters are given in Table 3, except $\beta = 0.009$. The resulting threshold is $\mathcal{R}_1 = 1.7 > 1$, which indicates that the pest-free periodic solution is unstable.

Then we obtain the following solution for $t \in [t_k + \tau, t_{k+1})$,

$$\begin{aligned} u(t) &= 0, \quad w(t) = \pi_z w(t_k + \tau) e^{-\mu_{zi}(t-t_k-\tau)} \\ v(t) &= y(t_k + \tau) e^{-\mu_{yi}(t-t_k-\tau)} + \frac{\phi_i \pi_z w(t_k + \tau)}{\mu_{yi} - \mu_{zi}} \left[e^{-\mu_{zi}(t-t_k-\tau)} - e^{-\mu_{yi}(t-t_k-\tau)} \right]. \end{aligned}$$

Using the fact that these above solutions are continuous, we can able to evaluate the value of each variable at the end of intercropping season. For this, let define the following threshold

$$B = \frac{\phi_i}{\mu_{yi} - \mu_{zi}} \left[e^{-\mu_{zi}(T-\tau)} - e^{-\mu_{yi}(T-\tau)} \right].$$

The linearisation of transition initial condition system at pest-free periodic solution is given by

$$\begin{aligned} \tilde{x}(t_{k+1}^+) &= \begin{pmatrix} \xi_s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_z \end{pmatrix} \tilde{x}(t_{k+1}) \\ &= \begin{pmatrix} \xi_s & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \xi_z \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-\mu_{yi}(T-\tau)} & \pi_z B \\ 0 & 0 & \pi_z e^{-\mu_{zi}(T-\tau)} \end{pmatrix} \tilde{x}(t_k + \tau), \end{aligned}$$

that is

$$\tilde{x}(t_{k+1}^+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-\mu_{yi}(T-\tau)} & \pi_z B \\ 0 & 0 & \xi_z \pi_z e^{-\mu_{zi}(T-\tau)} \end{pmatrix} \tilde{x}(t_k + \tau). \quad (3.29)$$

Therefore, we obtain from (3.29) and (3.20) the following discrete system:

$$\tilde{x}(t_{k+1}^+) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e^{-\mu_{yi}(T-\tau)} & \pi_z B \\ 0 & 0 & \xi_z \pi_z e^{-\mu_{zi}(T-\tau)} \end{pmatrix} \begin{pmatrix} e^{-\mu\tau} & * \\ 0_{\mathbb{R}^2} & e^{\mathbb{A}\tau} \end{pmatrix} \tilde{x}(t_k^+). \quad (3.30)$$

Since the above matrices in the system (3.30) are triangular and the first term of matrix is zero, then the stability of system depends of the submatrix:

$$\widehat{\mathbb{K}} := \begin{pmatrix} e^{-\mu_{yi}(T-\tau)} & \pi_z B \\ 0 & \xi_z \pi_z e^{-\mu_{zi}(T-\tau)} \end{pmatrix} M(\tau).$$

That is

$$\widehat{\mathbb{K}} = \begin{pmatrix} M_{11} e^{-\mu_{yi}(T-\tau)} + M_{21} \pi_z B & M_{12} e^{-\mu_{yi}(T-\tau)} + M_{22} \pi_z B \\ M_{21} \xi_z \pi_z e^{-\mu_{zi}(T-\tau)} & M_{22} \xi_z \pi_z e^{-\mu_{zi}(T-\tau)} \end{pmatrix}$$

Hence, the pest-free periodic solution of system (3.1)-(3.6) is locally asymptotically stable if and only if the matrix $\widehat{\mathbb{K}}$ satisfies the Jury conditions which are given as:

$$\begin{aligned} \text{Cond 1 : } & \widehat{\Psi}_1(\xi_z, \pi_z) \equiv 1 - Tr(\widehat{\mathbb{K}}) + \det(\widehat{\mathbb{K}}) > 0, \\ \text{Cond 2 : } & \widehat{\Psi}_2(\xi_z, \pi_z) \equiv 1 + Tr(\widehat{\mathbb{K}}) + \det(\widehat{\mathbb{K}}) > 0, \\ \text{Cond 3 : } & \widehat{\Psi}_3(\xi_z, \pi_z) \equiv 1 - \det(\widehat{\mathbb{K}}) > 0. \end{aligned}$$

We have by the direct calculation:

$$\begin{aligned} \det(\widehat{\mathbb{K}}) &= \det(M)\xi_z\pi_z e^{-(\mu_{yi}+\mu_{zi})(T-\tau)} \\ &= \xi_z\pi_z e^{Tr(\mathbb{A})\tau} e^{-(\mu_{yi}+\mu_{zi})(T-\tau)} < 1, \end{aligned}$$

since $Tr(\mathbb{A}) < 0$. Therefore the Jury condition 3 is satisfied. The trace of matrix $\widehat{\mathbb{K}}$ is given by:

$$Tr(\widehat{\mathbb{K}}) = M_{11}(\tau)e^{-\mu_{yi}(T-\tau)} + M_{21}(\tau)\pi_z B + M_{22}(\tau)\xi_z\pi_z e^{-\mu_{zi}(T-\tau)}.$$

The values of trace and determinant of matrix $\widehat{\mathbb{K}}$ are always nonnegative since the components $M_{ij} \geq 0$ for all $i, j \in \{1, 2\}$, then $1 + Tr(\widehat{\mathbb{K}}) + \det(\widehat{\mathbb{K}}) > 0$. Hence, Jury condition 2 is satisfied. The expression of function $\widehat{\Psi}_1$ is given analytically by:

$$\widehat{\Psi}_1(\xi_z, \pi_z) = 1 - M_{11}(\tau)e^{-\mu_{yi}(T-\tau)} - \pi_z M_{21}(\tau)B - \pi_z \xi_z e^{-\mu_{zi}(T-\tau)} \left(M_{22}(\tau) - e^{Tr(\mathbb{A})\tau - \mu_{yi}(T-\tau)} \right).$$

Therefore according to the Floquet theory, the pest-free periodic solution is locally asymptotically stable if and only if the above function is positive, which leads to a discussion similar to that with $\pi_s > 0$:

1. If $\pi_z = 0$, the pest-free periodic solution of system (3.1)-(3.6) is locally asymptotically stable if and only if

$$\widehat{\mathcal{R}}_1 := M_{11}(\tau)e^{-\mu_{yi}(T-\tau)} < 1. \quad (3.31)$$

From this relation, there exists a critical threshold $\mu_{yi}^* = \frac{\ln(M_{11}(\tau))}{T-\tau}$ such that the CBB disappear in the plantation whenever $\mu_{yi} > \mu_{yi}^*$. This result indicates that a control strategy of CBB population in this case is to capture the colonising females by placing for example the insect traps during the intercropping season to increase the natural mortality rate in such a way that the new mortality rate of CBBs is above the threshold μ_{yi}^* .

2. If $\pi_z \neq 0$ and $\xi_z = 0$, then the first Jury condition is satisfied if and only if $\widehat{\Psi}_1(0, \pi_z) = 1 - M_{11}(\tau)e^{-\mu_{yi}(T-\tau)} - M_{21}(\tau)\pi_z B > 0$. Therefore, the pest-free periodic solution of system (3.1)-(3.6) is asymptotically stable if and only if:

$$\widehat{\mathcal{R}}_2 := \widehat{\mathcal{R}}_1 + M_{21}(\tau)\pi_z B < 1.$$

Remark that $\widehat{\mathcal{R}}_2 < 1$ implies that $\widehat{\mathcal{R}}_1 < 1$ and then from (3.25), there exists a critical value of the parameter π_z defined by:

$$\widehat{\pi}_z^{crit} = \frac{1 - \widehat{\mathcal{R}}_1}{M_{21}(\tau)B},$$

such that the pest-free periodic solution of semi-discrete system (3.1)-(3.6) is stable for all $\pi_z \in [0, \widehat{\pi}_z^{crit})$.

3.5.4 Numerical simulations when $\pi_s = 0$

Figure 21 illustrates the regions of parameter values π_z and ξ_z for which the pest-free solution is asymptotically stable and unstable. This figure shows that there exists a critical value of rate π_z denoted $\pi_z^* \approx 0.05$ such that for all $\pi_z \leq \pi_z^*$, the pest-free periodic solution is asymptotically stable independently of the value taken by ξ_z . This result means biologically that only with a very good harvesting of coffee berries at the end of each cropping season (perfect harvesting of healthy berries $\pi_s = 0$ and good harvesting of infested berries $\pi_z \leq \pi_z^*$), the CBB population can be controlled without cleaning during the intercropping period.

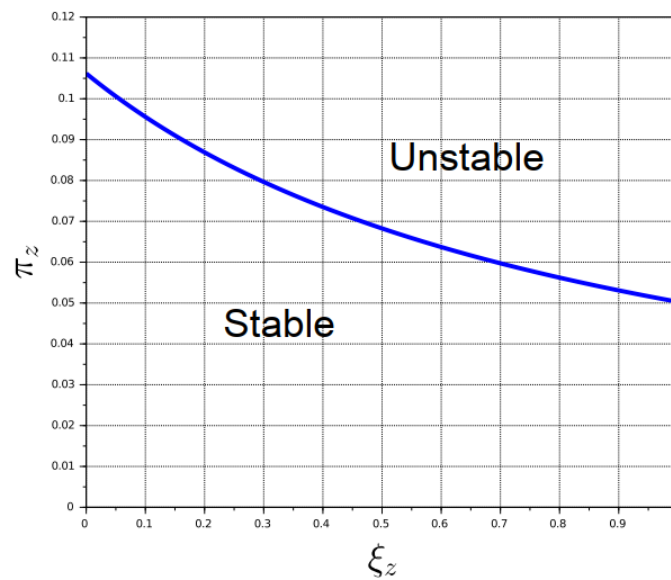


Figure 21: Stability of the pest-free periodic solution according to the values of parameters π_z and ξ_z for $\pi_s = 0$.

Figure 22 illustrates the evolution of state variables of semi-discrete system (3.1)-(3.6), showing the asymptotic stability of the pest-free periodic solution for $\pi_z = \xi_z = 0$ and $\widehat{\mathcal{R}}_1 < 1$: the CBB population disappears quickly in the plantation. However, for a small perturbation of the natural mortality rate of the colonising females during the intercropping season, as shown in Figure 23, ($\mu_{yi} = 0.22$ and $\widehat{\mathcal{R}}_1 = 1.5 > 1$), the CBB persist in the plantation. Perfect harvesting and cleaning are hence not sufficient to eradicate the CBB during multiple cropping seasons.

3.6 Conclusion

Motivated by the huge damage caused by CBB population in the coffee plantations, this study focused on the infestation dynamics of the coffee berries by CBB. In this chapter, we formulated a semi-discrete model based on impulsive ordinary differential equations for describe the coffee berries-CBB interaction on multiple cropping seasons. We computed the pest-free periodic solution corresponding to the solution of the system without CBB and established some criteria

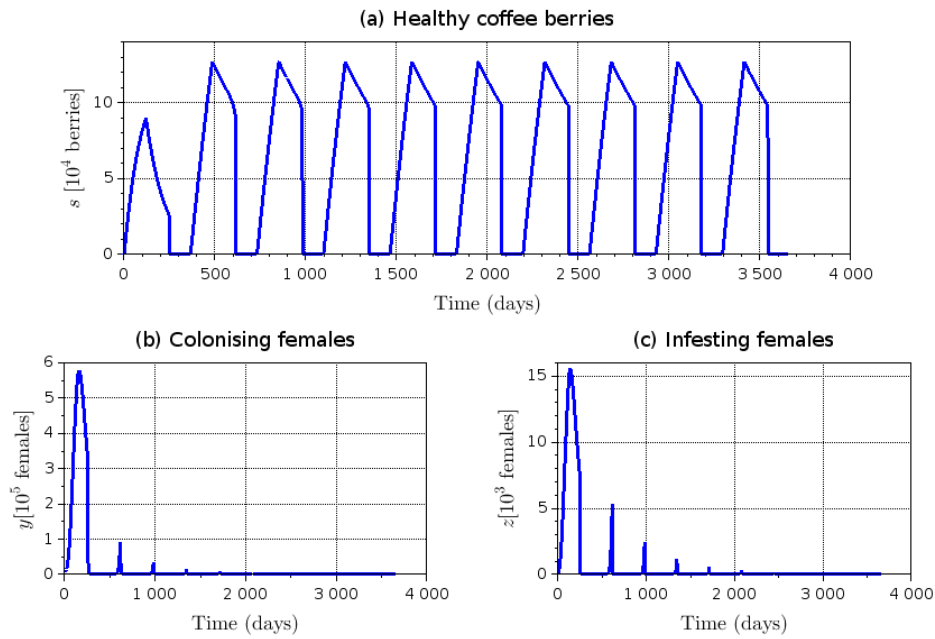


Figure 22: Simulation of semi-discrete system (3.1)-(3.6): (a) healthy coffee berries, (b) colonising females and (c) infesting females. Harvesting and cleaning are perfect, that is $\pi_s = \pi_z = \xi_z = 0$; remaining parameters are given in Table 3. The resulting thresholds are $\mathcal{R}_0 = 10.4$ and $\widehat{\mathcal{R}}_1 = 0.45 < 1$, which indicates that the pest-free periodic solution is asymptotically stable.

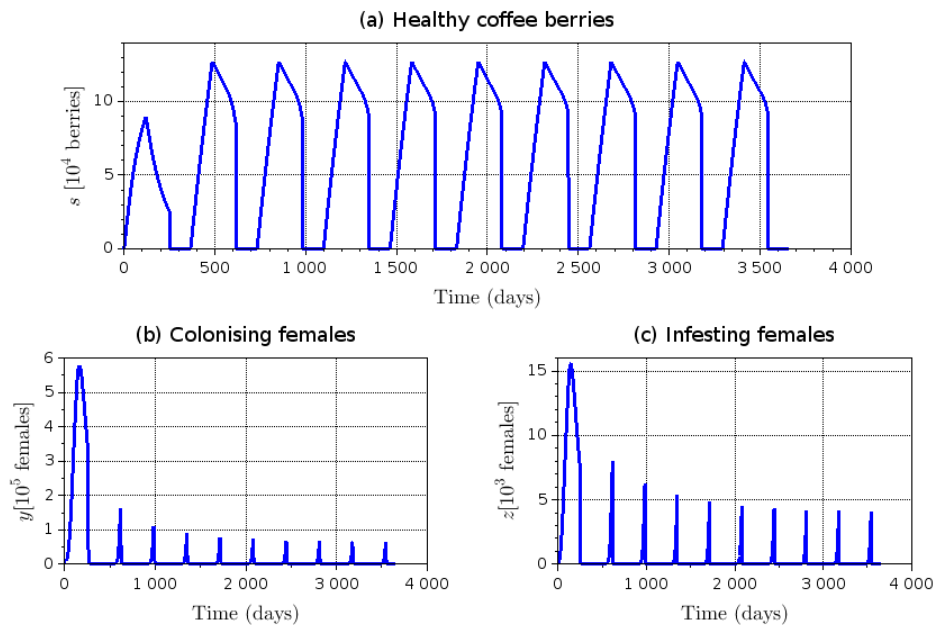


Figure 23: Simulation of semi-discrete system (3.1)-(3.6): (a) healthy coffee berries, (b) colonising females and (c) infesting females. Harvesting and cleaning are perfect, that is $\pi_s = \pi_z = \xi_z = 0$; remaining parameters are given in Table 3, except $\mu_{yi} = 0.22$. The resulting thresholds are $\mathcal{R}_0 = 10.4$, and $\widehat{\mathcal{R}}_1 = 1.5 > 1$, which indicates that the pest-free periodic solution is unstable.

for which this solution is locally asymptotically stable *i.e.* where the CBB population disappears through two considered control parameters corresponding to the harvesting effort of coffee berries at the end of each cropping season and the cleaning effort of the plantation during the intercropping season. Our results indicate that the harvesting of coffee berries contributes to the reduction of CBB notably the infesting females and reduces the quantity of coffee berries which plays the role of reservoir of CBB for their survival during the intercropping season. In addition, the cleaning of the plantation makes it possible to collect the berries remaining in the plantation after the harvesting, thus reducing CBB infesting females at the end of intercropping season [29, 11]. However, it is extremely difficult to eradicate CBB population when harvesting and cleaning is poorly done. On the other hand, a perfect harvesting does not directly ensure the extinction of the CBB population. Indeed, in this case there is an indicator $\widehat{\mathcal{R}}_1$ such that the CBB disappear when this indicator is lower than one and persist when it is greater than one.

Controlling CBB in an age-structured coffee berry model

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This Chapter is based on the works submitted to *Applied Mathematical Modelling* [49] and to *journal of optimization theory and applications* [47].

4.1 Introduction

Although CBB may infest immature berries and cause them to fall prematurely, they preferably target mature berries, which provide them a better environment to feed and reproduce [16, 106, 54, 93]. This biological aspect was not taken into account in previous chapters, neither in Chapter 2 which introduce the baseline CBB-berry interaction model, nor in Chapter 3 which extends the baseline model to several cropping seasons. This is why, in this chapter, we formulate an age-structured model describing the infestation dynamics of coffee berries by CBB, that extends the model proposed in Chapter 2, taking into account the preference of CBB for mature

berries. The model consists in two partial differential equations describing the age-structured dynamics of berries and two ordinary differential equations depicting the CBB life-cycle. We show the well-posedness of this PDE model and determine optimal control strategies based on the introduction of two environmentally friendly control methods, consisting in applying a bio-insecticide to reduce berry infestation, such as *Beauveria bassania* which was implemented in Chapter 2, and in trapping the colonising CBB. As in Chapter 2, the objective is to maximise the gain while minimising both the control costs and the CBB population for the next cropping season. However, in the optimisation criterion, we take into account the fact that the berry price depends on the berry age. Some basic properties of the control problem are shown and its characterisation is given based on Pontryagin maximum principle for general age-structured systems on a finite time horizon provided by Feichtinger *et al.* [39]. The numerical simulations provide discussion points.

4.2 Model presentation

To take into account the CBB preference for mature berries, a berry age-structured model is formulated. The berry population density is subdivided into two compartments: the healthy berry density at time t and age a denoted by $s(t, a)$ and the infested berry density denoted by $i(t, a)$. The berry age being defined as a “classical” age, we have $da = dt$. We denote by a_{\dagger} the maximal age of the coffee berries. As in Chapter 2, males are considered as not limiting to fertilise young females and not included; the female population is divided into two compartments: the colonising females denoted by $y(t)$, which correspond to the flying females looking for new berries, and the infesting females denoted by $z(t)$, which correspond to the females that are laying eggs inside the berries. Figure 24 provides a schematic and simplified summary of the interactions between coffee berries and CBB.

Extending the baseline model found in Chapter 2 to incorporate the berry age, the diagram in Figure 24 can be translated into the following mathematical age-structured system:

$$\begin{cases} \partial_t s(t, a) + \partial_a s(t, a) = -\mathcal{W}(B, y) \beta(a) s(t, a) - \mu(a) s(t, a), \\ \partial_t i(t, a) + \partial_a i(t, a) = +\mathcal{W}(B, y) \beta(a) s(t, a) - \nu(a) i(t, a), \\ \dot{y}(t) = -\varepsilon \mathcal{W}(B, y) \int_0^{a_{\dagger}} \beta(a) s(t, a) da - \mu_y y(t) + \phi z(t), \\ \dot{z}(t) = +\varepsilon \mathcal{W}(B, y) \int_0^{a_{\dagger}} \beta(a) s(t, a) da - \mu_z z(t), \end{cases} \quad (4.1)$$

where $B(t) = \int_0^{a_{\dagger}} s(t, a) da$ represents the total number of healthy berries at time t .

System (4.1) is supplemented by the following boundary and initial conditions:

$$\begin{aligned} s(t, 0) &= \lambda(t), & i(t, 0) &= 0, \\ s(0, a) &= s_0(a), & i(0, a) &= i_0(a), & y(0) &= y_0, & z(0) &= z_0. \end{aligned} \quad (4.2)$$

Model (4.1,4.2) relies on the assumptions of the baseline model listed in Section 2.2 and the extra assumptions below:

1. Many factors such as climatic and environmental conditions influence the development of the coffee crop. We do not explicitly represent these factors but we assume that healthy

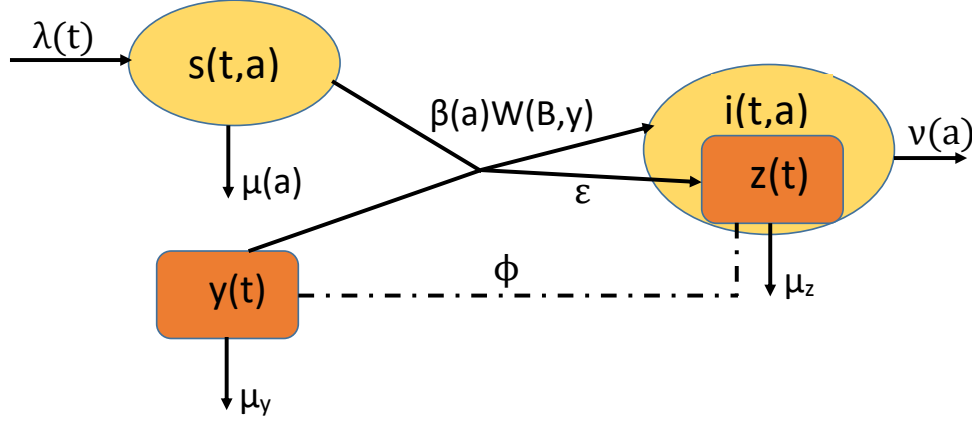


Figure 24: Schematic representation of the coffee berry–CBB interaction model. Healthy coffee berries $s(t, 0)$ are produced at the time-dependent rate $\lambda(t)$. Colonising females $y(t)$ infest healthy coffee berries $s(t, a)$ at rate $\beta(a)\mathcal{W}(B, y)$, where $B(t)$ represents the number of healthy berries and \mathcal{W} the interaction function. This induces a transfer of berry densities, from healthy to infested berries $i(t, a)$, and its counterpart for CBB females, from colonising to infesting females $z(t)$. Infesting females lay eggs inside the coffee berries, which go through their development cycle until the emergence of new fertilised colonising females at rate ϕ . Colonising and infesting CBB females undergo mortality at rates μ_y and μ_z , healthy and infested coffee berries at rates $\mu(a)$ and $\nu(a)$.

coffee berries are produced at time-dependent rate $\lambda(t)$.

2. The force of infestation is modelled by $\beta(a)\mathcal{W}(B, y)$, where $\beta(\cdot)$ represents the age-specific infestation rate per unit of time, while the interaction between CBB and berries is described by function \mathcal{W} which verifies the following hypotheses:
 - Function \mathcal{W} is a bounded, C^1 -Lipschitz function for both positive arguments, *i.e.* there exists a positive real number M such that:

$$|\mathcal{W}(B_1, y_1) - \mathcal{W}(B_2, y_2)| \leq M (|B_1 - B_2| + |y_1 - y_2|),$$

for all variables (B_1, y_1) and (B_2, y_2) in \mathbb{R}_+^2 .

- Moreover, $\mathcal{W}(B, 0) = 0$ and function \mathcal{W} is non decreasing in y (for $B > 0$). Indeed, infestation can only occur when colonising females are present and for a given number of berries, the more the colonising females, the more severe the infestation.
- Finally, \mathcal{W} is non increasing in B (for $y > 0$) and for all $y \in \mathbb{R}_+^*$, $\lim_{B \rightarrow +\infty} B\mathcal{W}(B, y)$ is finite. The latter hypothesis means that infestation is limited by the number of colonising females. As a consequence, $\lim_{B \rightarrow +\infty} \mathcal{W}(B, y) = 0$.

These hypotheses are compatible with density- or ratio-dependent interactions, as implemented in Chapter 2.

3. The age-dependent mortality rates of healthy and infested coffee berries are respectively $\mu(a)$ and $\nu(a)$. We assume $\nu(a) > \mu(a)$ since CBB infestation leads to the premature fall of coffee berries in the plantation [29, 11]. μ_y and μ_z represent the natural mortality rates for colonising and infesting females respectively. We assume that an infesting female may

pursue its development if its berry falls down. In turn, even if an infesting female dies, its berry may survive.

It is worth noticing that, if the age-dependent and time-dependent functions are chosen as positive constants, *i.e.* $\mu(a) = \mu$, $\beta(a) = \beta$, $\nu(a) = \nu$ and $\lambda(t) = \lambda$, then by integrating the first two equations of system (4.1) on $[0, a_+]$, we obtain a nonlinear ordinary differential system similar to the models proposed in [44] and in Chapter 2.

As in Chapter 2, the age-dependent variable $i(t, a)$ does not appear in the equations of the other variables in system (4.1), so one can ignore the dynamics of $i(t, a)$ and study the following reduced model given for $t \geq 0$ and $a \in [0, a_+]$ by:

$$\begin{cases} \partial_t s(t, a) + \partial_a s(t, a) = -\mathcal{W}(B, y) \beta(a) s(t, a) - \mu(a) s(t, a), \\ \dot{y}(t) = -\varepsilon \mathcal{W}(B, y) \int_0^{a_+} \beta(a) s(t, a) da - \mu_y y(t) + \phi z(t), \\ \dot{z}(t) = +\varepsilon \mathcal{W}(B, y) \int_0^{a_+} \beta(a) s(t, a) da - \mu_z z(t), \\ s(t, 0) = \lambda(t), \quad s(0, a) = s_0(a), \quad y(0) = y_0, \quad z(0) = z_0. \end{cases} \quad (4.3)$$

Let us denote by $L_+^1(I)$ and $L_+^\infty(I)$ the spaces of functions $L^1(I)$ and $L^\infty(I)$ respectively which are nonnegatives over the set $I \subset \mathbb{R}$. We make the following realistic assumptions on the positivity and smoothness of the parameters and functions involved in model (4.3).

Assumption 1 *Throughout this paper, we assume that:*

1. Parameters ϕ , ε , μ_y , μ_z and initial conditions y_0 and z_0 are nonnegative.
2. $\lambda(\cdot) \in L_+^\infty(0, \infty)$, $s_0(\cdot) \in L_+^1(0, a_+)$ and $\beta(\cdot) \in L_+^\infty(0, a_+)$.
3. $\mu(\cdot) \in L_+^\infty(0, a_+)$ and there exists a real number $\tilde{\mu} > 0$ satisfying, $\mu(a) \geq \tilde{\mu}$ for almost every $a \in [0, a_+]$.

4.3 Well-posedness

In this section, we aim at proving the existence and uniqueness of a nonnegative solution for system (4.3). The preliminary step consists in finding the equivalent system with zero boundary conditions, using the ideas of [70, 90] and writing the new model as an abstract Cauchy problem, which is easier to handle. Then we prove the existence and uniqueness of a mild solution to this Cauchy problem in the nonnegative cone of a Banach space, by adapting the arguments of [89]. Finally, we deduce the existence and uniqueness of a mild solution for the initial system (4.3).

4.3.1 Abstract Cauchy problem

Let us define the survival probability of healthy coffee berries until age a by $\pi(a) = e^{-\int_0^a \mu(x) dx}$. For all $t \geq 0$, we introduce the new functional:

$$\Phi_t(\cdot) : \xi \longrightarrow \Phi_t(\xi) := \begin{cases} \pi(\xi) \lambda(t - \xi) & \text{if } t \geq \xi \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (4.4)$$

Based on these definitions and Assumption 1, we have the following instrumental lemma which can be easily established [70].

Lemma 2

1. The function $\pi(\cdot) \in L^1_+(0, a_+)$ and satisfies the differential equation:

$$\dot{\pi}(a) = -\mu(a)\pi(a).$$

Moreover, for all $a \in [0, a_+]$ and any $t \leq a$, $0 \leq \pi(a) \leq e^{-\tilde{\mu}a}$ and $\frac{\pi(a)}{\pi(a-t)} \leq \pi(t)$.

2. The function $t \mapsto \Phi_t(\cdot)$ is in $\mathcal{C}(\mathbb{R}_+, L^1_+(0, a_+))$. Moreover, for all $t \geq 0$, $\Phi_t(0) = \lambda(t)$ and $\Phi_t(\cdot)$ satisfies the partial differential equation:

$$\partial_t \Phi_t(a) + \partial_a \Phi_t(a) = -\mu(a)\Phi_t(a). \quad (4.5)$$

Considering the transformation $s(t, a) \mapsto \widehat{s}(t, a) := s(t, a) - \Phi_t(a)$, then system (4.3) is equivalent to the following system with zero boundary conditions:

$$\left\{ \begin{array}{l} \partial_t \widehat{s}(t, a) + \partial_a \widehat{s}(t, a) = -\mathcal{W}(\widehat{B}, y) \beta(a) [\widehat{s}(t, a) + \Phi_t(a)] - \mu(a) \widehat{s}(t, a) \\ \dot{y}(t) = -\varepsilon \mathcal{W}(\widehat{B}, y) \|\beta(\widehat{s}(t, \cdot) + \Phi_t(\cdot))\| - \mu_y y(t) + \phi z(t) \\ \dot{z}(t) = +\varepsilon \mathcal{W}(\widehat{B}, y) \|\beta(\widehat{s}(t, \cdot) + \Phi_t(\cdot))\| - \mu_z z(t) \\ \widehat{s}(t, 0) = 0, \quad \widehat{s}(0, a) = s_0(a), \quad y(0) = y_0, \quad z(0) = z_0. \end{array} \right. \quad (4.6)$$

where: $\widehat{B}(t) = \int_0^{a_+} [\widehat{s}(t, a) + \Phi_t(a)] da$ and $\|\beta(\widehat{s}(t, \cdot) + \Phi_t)\| = \int_0^{a_+} [\beta(a)\widehat{s}(t, a) + \Phi_t(a)] da$.

Let \mathcal{A}_s and \mathcal{A}_I denote the differential operators defined by:

$$\begin{aligned} \mathcal{A}_s : D(\mathcal{A}_s) \subset L^1(0, a_+) &\longrightarrow L^1(0, a_+) \\ \Psi &\longmapsto \mathcal{A}_s \Psi(a) = -\Psi'(a) - \mu(a)\Psi(a), \\ \mathcal{A}_I : D(\mathcal{A}_I) \subset \mathbb{R}^2 &\longrightarrow \mathbb{R}^2 \\ \Psi &\longmapsto \mathcal{A}_I \Psi = \text{diag}(-\mu_y, -\mu_z)\Psi, \end{aligned}$$

where $D(\mathcal{A}_s) := \{\Psi \in W^{1,1}(0, a_+) : \Psi(0) = 0\}$ and $D(\mathcal{A}_I) = \mathbb{R}^2$, with $W^{1,1}(0, a_+)$ denoting the Sobolev space. Biologically speaking, the linear differential operators \mathcal{A}_s and \mathcal{A}_I represent the mortality processes related to the healthy coffee berries and females CBB respectively. Let us define the Banach space $\mathcal{X} = L^1(0, a_+) \times \mathbb{R} \times \mathbb{R}$ equipped by the usual norm $\|\cdot\|_{\mathcal{X}}$. The nonnegative cone of the Banach space \mathcal{X} is $\mathcal{X}_+ = L^1_+(0, a_+) \times \mathbb{R}_+ \times \mathbb{R}_+$. Let us define the linear differential operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \longrightarrow \mathcal{X}$ where $D(\mathcal{A}) = D(\mathcal{A}_s) \times D(\mathcal{A}_I)$ such that

$$\mathcal{A} := \text{diag}(\mathcal{A}_s, \mathcal{A}_I). \quad (4.7)$$

Let $u(t) = (\widehat{s}(t, a), y(t), z(t))$ denote the solution of system (4.6) with its initial condition $u_0 = (\widehat{s}_0(a), y_0, z_0)$. We define a nonlinear perturbation map $\mathcal{H} : \mathbb{R}_+ \times D(\mathcal{A}) \subset \mathbb{R}_+ \times \mathcal{X} \longrightarrow \mathcal{X}$

as follows:

$$\mathcal{H}(t, u(t)) = \begin{pmatrix} -\beta(a)\mathcal{W}(\widehat{B}(t), y(t))[\widehat{s}(t, a) + \Phi_t(a)] \\ -\varepsilon\mathcal{W}(\widehat{B}(t), y(t))\|\beta(\widehat{s}(t, \cdot) + \Phi_t(\cdot))\| + \phi z(t) \\ \varepsilon\mathcal{W}(\widehat{B}(t), y(t))\|\beta(\widehat{s}(t, \cdot) + \Phi_t(\cdot))\| \end{pmatrix}. \quad (4.8)$$

Then, we can transform system (4.6) into the following abstract Cauchy problem in the Banach space \mathcal{X} :

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t) + \mathcal{H}(t, u(t)) & \text{for } t \geq 0 \\ u(0) = u_0 \in D(\mathcal{A}). \end{cases} \quad (4.9)$$

To prove the existence and uniqueness of a solution to the abstract Cauchy problem (4.9), we begin by showing that the linear autonomous problem associated to system (4.9) has a unique solution. Then we prove the existence of a unique solution for the non-autonomous and nonlinear system (4.9) using the perturbation theory of linear evolution equations (see [89, 114] for more details).

4.3.2 The linear problem

This section is devoted to the definition of the semigroup generated by the differential operator \mathcal{A} and the existence and uniqueness of a solution to the linear problem associated to the system (4.9). Let us define the resolvent set $\rho(\mathcal{A}_s)$ of the linear operator \mathcal{A}_s as the set of all complex number ζ for which $(\zeta I - \mathcal{A}_s)$ is invertible, *i.e.* $(\zeta I - \mathcal{A}_s)^{-1}$ is a bounded linear operator in $L^1(0, a_+)$. Let Δ denotes the subset of the complex number defined by $\Delta := \{\zeta \in \mathbb{R} : \zeta > -\widetilde{\mu}\}$.

Lemma 3 ([70]) *The linear operator $(\mathcal{A}_s, D(\mathcal{A}_s))$ is such that $D(\mathcal{A}_s)$ is dense in $L^1(0, a_+)$ and the resolvent set $\rho(\mathcal{A}_s) \supset \Delta$. Moreover, the family of the bounded linear operators $R_\zeta := (\zeta I - \mathcal{A}_s)^{-1}$, called resolvent, is given for each $\zeta \in \Delta$ by:*

$$R_\zeta(\Psi)(a) = \int_0^a e^{\zeta(\eta-a)} \frac{\pi(a)}{\pi(\eta)} \Psi(\eta) d\eta, \quad \forall \Psi \in L^1(0, a_+). \quad (4.10)$$

Furthermore, R_ζ satisfies $\|R_\zeta\| \leq \frac{1}{\zeta + \widetilde{\mu}}, \forall \zeta \in \Delta$.

Lemma 4 *The differential operator \mathcal{A} is closed.*

Proof : According to [20], it is sufficient to show that $D(\mathcal{A})$ with norm $\|x\|_{D(\mathcal{A})} = \|x\|_{\mathcal{X}} + \|\mathcal{A}x\|_{\mathcal{X}}, \forall x \in D(\mathcal{A})$ is a Banach space. It is easy to verify that $\|\cdot\|_{D(\mathcal{A})}$ is a norm on $D(\mathcal{A})$. For show that $(\mathcal{A}, D(\mathcal{A}))$ is complete. we take (θ_n, y_n, z_n) to be a Cauchy sequence of points of $D(\mathcal{A})$. Thus $\theta_n \in W^{1,1}$ which is a Banach space (*i.e.* complete), so $\theta_n \xrightarrow[n \rightarrow \infty]{} \theta_*$ in $W^{1,1}$, *i.e.* that $\theta_n \xrightarrow[n \rightarrow \infty]{} \theta_*$ and $\theta'_n \xrightarrow[n \rightarrow \infty]{} \theta'_*$ in L^1 . Moreover, $y_n \xrightarrow[n \rightarrow \infty]{} y_* \in \mathbb{R}$ and $z_n \xrightarrow[n \rightarrow \infty]{} z_* \in \mathbb{R}$ since \mathbb{R} is complete. Then $\|\mu\theta_n - \mu\theta_*\| \leq \|\mu\|_\infty \|\theta_n - \theta_*\| \xrightarrow[n \rightarrow \infty]{} 0$ since $\|\mu\|_\infty$ is finite. Therefore, $\mu\theta_n \xrightarrow[n \rightarrow \infty]{} \mu\theta_*$ in L^1 . We can thus see that the sequence (θ_n, y_n, z_n) converges in $D(\mathcal{A})$ to (θ_*, y_*, z_*) . In fact, $\|(\theta_n, y_n, z_n) - (\theta_*, y_*, z_*)\|_{D(\mathcal{A})} \leq \|\theta_n - \theta_*\| + \|y_n - y_*\| + \|z_n - z_*\| + \|\mu(\theta_n - \theta_*) + (\theta'_n - \theta'_*)\| + \mu_y \|y_n - y_*\| + \mu_z \|z_n - z_*\|$. Thus any Cauchy sequence is convergent and the space $D(\mathcal{A})$ with its norm $\|\cdot\|_{D(\mathcal{A})}$ is complete. Hence the operator \mathcal{A} is closed \square

The properties of the linear operator $(\mathcal{A}, D(\mathcal{A}))$ are precised by the following proposition.

Proposition 3 *The linear operator $(\mathcal{A}, D(\mathcal{A}))$ is an infinitesimal generator of a strongly positive C_0 -semigroup of contraction $T_{\mathcal{A}}(t) = \text{diag}(T_{\mathcal{A}_s}(t), T_{\mathcal{A}_I}(t))$ such that for every $t \geq 0$ and $\Psi \in L^1(0, a_{\dagger})$:*

$$T_{\mathcal{A}_s}(t)\Psi(a) = \begin{cases} \frac{\pi(a)}{\pi(a-t)}\Psi(a-t) & \text{if } t \leq a \leq a_{\dagger}, \\ 0 & \text{otherwise,} \end{cases} \quad (4.11)$$

and for every $(x, y) \in \mathbb{R}^2$:

$$T_{\mathcal{A}_I}(t)(x, y) = \text{diag}(e^{-\mu_y t}, e^{-\mu_z t})(x, y) = (xe^{-\mu_y t}, ye^{-\mu_z t}). \quad (4.12)$$

Moreover, the semigroup $\{T_{\mathcal{A}}(t)\}_{t \geq 0}$ is exponentially stable.

Proof: From Lemma 3, it is easy to verify that $\|(\lambda I - \mathcal{A})\| \leq \frac{1}{\lambda + \omega}$ where $\omega = \min\{\tilde{\mu}, \mu_y, \mu_z\}$ for $\lambda > -\omega$. Then from Lemma 4 and According to the Hille–Yosida Theorem [89], the linear differential operator $(\mathcal{A}, D(\mathcal{A}))$ is infinitesimal generator of a C_0 -semigroup of contraction. Moreover, the resolvent given in (4.10) being positive in $L^1(0, a_{\dagger})$, then the linear differential operator $(\mathcal{A}, D(\mathcal{A}))$ generates a positive C_0 -semigroup of contraction $T_{\mathcal{A}}(t)$ [89, 114]. Hence, performing a change of variable $\xi = a - \eta$ in equation (4.10) for the operator \mathcal{A}_s and computing the exponential of the operator \mathcal{A}_I , we obtain the desired expressions for the semigroups given by (4.11) and (4.12) respectively. Finally, the estimation is a direct consequence of Lemma 2 which implies that $\|T_{\mathcal{A}}(t)\| = \sup\{\|T_{\mathcal{A}_s}(t)\|, \|T_{\mathcal{A}_I}(t)\|\} \leq \max\{\pi(t), e^{-\mu_y t}, e^{-\mu_z t}\} \leq e^{-\theta t}$, with $\theta = \min\{\tilde{\mu}, \mu_y, \mu_z\}$ for all $t > 0$. Then, the operator $\|T_{\mathcal{A}}(t)\|$ converges towards zero when $t \rightarrow +\infty$. Therefore, the semigroup $\{T_{\mathcal{A}}(t)\}_{t \geq 0}$ is positive and exponentially stable. \square

As a consequence of Proposition 3 and according to [89, 59], we obtain the following result regarding the existence of a solution to the linear abstract Cauchy problem.

Theorem 12 *For the given initial condition $u(0) \in D(\mathcal{A})$, $u(t) = T_{\mathcal{A}}(t)u_0$ for all $t \geq 0$ is a unique classical solution of the linear abstract Cauchy system $\dot{u}(t) = \mathcal{A}u(t)$ in \mathcal{X} .*

4.3.3 Existence and uniqueness of the solution for the nonlinear system

Having established the existence and uniqueness of a solution to the linear problem given by Theorem 12, we are now interested in establishing the existence and uniqueness of a solution to the nonlinear evolution problem (4.9). To do so, we opt for the Lipschitz perturbation approach for a semigroup. The map $\mathcal{H}(\cdot, u(\cdot))$ in the abstract system (4.9) is considered as a perturbation of the semigroup generated by the linear differential operator \mathcal{A} . For every constant $\delta > 0$, \mathcal{B}_{δ} denotes the ball of centre 0 and radius δ in space \mathcal{X} , $\mathcal{B}_{\delta} := \{x \in \mathcal{X} : \|x\|_{\mathcal{X}} \leq \delta\}$. We show that the nonautonomous map $\mathcal{H}(\cdot, u(\cdot))$ is a Lipschitzian perturbation of the continuous semigroup $\{T_{\mathcal{A}}(t)\}_{t \geq 0}$. We begin with the following lemma.

Lemma 5 *The perturbation map \mathcal{H} is a locally Lipschitz continuous function in $u(\cdot)$, uniformly in $t \geq 0$: for all $\delta > 0$, there exists a constant depending on δ denoted by M_{δ} such that:*

$$\|\mathcal{H}(t, u_1) - \mathcal{H}(t, u_2)\|_{\mathcal{X}} \leq M_{\delta}\|u_1 - u_2\|_{\mathcal{X}}, \quad \forall (u_1, u_2) \in (\mathcal{B}_{\delta})^2.$$

Proof : It is easy to verify that under the Assumption 1, we have $\|\Phi_t\| \leq \frac{1}{\mu}\|\lambda\|_\infty$. Let $\delta > 0$ and for any $u_1 = (\hat{s}^1, y^1, z^1)$, $u_2 = (\hat{s}^2, y^2, z^2) \in \mathbb{B}_\delta$, then we get the relation:

$$\begin{aligned} \beta(a)\mathcal{W}(\hat{B}^1, y^1)(\hat{s}^1 + \Phi_t) &- \beta(a)\mathcal{W}(\hat{B}^2, y^2)(\hat{s}^2 + \Phi_t) \\ &= \beta(a)(\mathcal{W}(\hat{B}^1, y^1) - \mathcal{W}(\hat{B}^2, y^2))(\hat{s}^2 + \Phi_t) \\ &+ \beta(a)\mathcal{W}(\hat{B}^1, y^1)(\hat{s}^1 - \hat{s}^2). \end{aligned} \quad (4.13)$$

Using the fact that the function $\mathcal{W}(\cdot, \cdot)$ is bounded and Lipschitz continuous with respect to its arguments, a straightforward computation using the relation (4.13) shows that there is a constant $K_1 > 0$ which depends of constants M , $\tilde{\mu}$, $\|\beta\|_\infty$, $\|\lambda\|_\infty$ and δ such that:

$$\|\beta\mathcal{W}(\hat{B}^1, y^1)(\hat{s}^1 + \Phi_t) - \beta\mathcal{W}(\hat{B}^2, y^2)(\hat{s}^2 + \Phi_t)\| \leq K_1 (\|\hat{s}^1 - \hat{s}^2\| + |y^1 - y^2|), \quad (4.14)$$

Proceeding similarly, we can show that there is a positive constant K_2 , which also depends on constants M , $\tilde{\mu}$, $\|\beta\|_\infty$, $\|\lambda\|_\infty$ and δ , such that:

$$|\mathcal{W}(\hat{B}^1, y^1)\|\beta(\hat{s}^1 + \Phi_t)\| - \mathcal{W}(\hat{B}^2, y^2)\|\beta(\hat{s}^2 + \Phi_t)\|| \leq K_2 (\|\hat{s}^1 - \hat{s}^2\| + |y^1 - y^2|). \quad (4.15)$$

Hence, from (4.14)-(4.15), it follows by direct computation that there exists a $M_\delta > 0$ depending of K_1 , K_2 , ϕ and δ such that the following inequality holds: $\|\mathcal{H}(t, u_1) - \mathcal{H}(t, u_2)\|_{\mathcal{X}} \leq M_\delta\|u_1 - u_2\|_{\mathcal{X}}$. This achieves the proof. \square

We now define the concept of mild solution and give the main results related to well-posedness of abstract Cauchy problem (4.9).

Definition 5 ([89]) *A mild solution of abstract Cauchy problem (4.9) on $[0, T]$ is a function $u \in \mathcal{C}([0, T], \mathcal{X})$ that satisfies*

$$u(t) := T_{\mathcal{A}}(t)u_0 + \int_0^t T_{\mathcal{A}}(t - \varsigma)\mathcal{H}(\varsigma, u(\varsigma))d\varsigma, \quad \forall t \geq 0. \quad (4.16)$$

We establish the existence of the solution of system (4.9) in the following theorem.

Theorem 13 *For any initial condition in \mathcal{X} , there exists an interval of time $[0, t_{max})$ in which the abstract Cauchy problem (4.9) has a unique mild solution.*

Proof : We use a fixed point method by adapting the ideas of [89, 90, 70]. Let us denote $\hat{\delta} := 2\|u_0\|_{\mathcal{X}} \sup_{t \in [0, 1]} \|T_{\mathcal{A}}(t)\|$ and $\tau := \min \left\{ 1, \frac{\|u_0\|_{\mathcal{X}}}{\delta M_\delta} \right\}$ where $u_0 \in \mathcal{X}$ is the initial condition of system (4.9) and M_δ is the local Lipschitz constant for the map $\mathcal{H}(\cdot, \cdot)$ defined in Lemma 5. Let us consider the ball defined by $\mathcal{B}_\delta^c := \left\{ u \in \mathcal{C}([0, \tau], \mathcal{X}) : \|u(t)\|_{\mathcal{X}} \leq \hat{\delta} \text{ for all } t \in [0, \tau] \right\}$. Next, consider the nonlinear mapping:

$$\begin{aligned} \mathbb{G} : \mathcal{C}([0, \tau], \mathcal{X}) &\longrightarrow \mathcal{C}([0, \tau], \mathcal{X}) \\ u(\cdot) &\longmapsto \mathbb{G}(u(t)) = T_{\mathcal{A}}(t)u_0 + \int_0^t T_{\mathcal{A}}(t - \varsigma)\mathcal{H}(\varsigma, u(\varsigma))d\varsigma. \end{aligned}$$

Standard boundedness techniques, as performed in [89, page 187], prove that \mathbb{G} maps the ball \mathcal{B}_δ^c into itself and is a contraction mapping of \mathcal{B}_δ^c with Lipschitz constant 1/2. Consequently,

the Banach fixed point theorem implies that map \mathbb{G} possesses a unique fixed point $u \in \mathcal{B}_\delta^c$. This fixed point is the desired mild solution on interval $[0, \tau]$ of the evolution system (4.9). We repeat all previous arguments, but now with initial condition $u(\tau)$ instead of $u(0)$, $\tau_1 = \tau + \gamma$ with $\gamma > 0$, together with space $\mathcal{C}([\tau, \tau_1], \mathcal{X})$ and mapping \mathbb{G}_1 defined in a similar way to \mathbb{G} . The same arguments applied to the latter case also show that \mathbb{G}_1 is a strict contraction with Lipschitz constant $1/2$. This gives once more a unique fixed point which extends the previous solution on interval $[0, \tau_1]$. By proceeding successively, we can extend the solution on $[0, t_{max})$, so that $u \in \mathcal{C}([0, t_{max}), \mathcal{X})$ is a mild solution of system (4.9). \square

Since we have shown that the abstract Cauchy problem (4.9) has a unique mild solution $u \in \mathcal{C}([0, t_{max}), \mathcal{X})$, the transformation $u(t) \mapsto u(t) + (\Phi_t(\cdot), 0, 0) := (s(t, \cdot), y(t), z(t))$ leads to the existence and uniqueness of a mild solution $(s, y, z) \in \mathcal{C}([0, t_{max}), \mathcal{X})$ of system (4.3). We only obtain a local solution, so it is important to show that this solution is actually global by showing that the solution is bounded. This is provided by the following lemma.

Lemma 6 *Under Assumption 1, the solution of system (4.3) remains nonnegative and bounded for all time $t > 0$. Moreover, the domain:*

$$\Sigma := \left\{ (s, y, z) \in \mathcal{X}_+ : \varepsilon \|s(t, \cdot)\| + z \leq \frac{\varepsilon \|\lambda\|_\infty}{\xi}, y \leq \frac{\varepsilon \phi \|\lambda\|_\infty}{\mu_y \xi} \right\},$$

where $\xi := \min\{\hat{\mu}, \mu_z\}$, is positively invariant under the flow of system (4.3).

Proof : Let $n(t, r) = \mu(r) + \beta(r)\mathcal{W}(B(t), y(t))$ and $\eta(t, r) = e^{-\int_0^a n(t, r) dr}$. The application of Volterra formulation [111, 58] solves the $s(t, a)$ -equation in system (4.3) along the characteristic $t - a = \text{constant}$ as follows:

$$s(t, a) = \begin{cases} s_0(a - t)e^{-\int_{a-t}^a n(t, r) dr} = s_0(a - t) \frac{\eta(t, a)}{\eta(t, a-t)} & \text{if } a > t \\ \lambda(t - a)e^{-\int_0^a n(t, r) dr} = \lambda(t - a)\eta(t, a) & \text{if } a \leq t. \end{cases}$$

It is clear that $s(t, a)$ remains nonnegative for nonnegative initial conditions. Moreover, since $\dot{y} = \phi z \geq 0$ when $y = 0$ and $\dot{z} = \varepsilon \mathcal{W}(B, y) \|\beta s(t, \cdot)\| \geq 0$ when $z = 0$, the state variables $y(t)$ and $z(t)$ are nonnegative for nonnegative initial conditions. Then the first quadrant of \mathbb{R}_+^2 is positively invariant and the solution of system (4.3) remains nonnegative at any time $t > 0$.

Let $(s(t, \cdot), y(t), z(t)) \in \mathcal{X}$ denote a solution of system (4.3). Adding the integral of first equation and the third equation of system (4.3) yields:

$$\frac{d}{dt} (\varepsilon \|s(t, \cdot)\| + z(t)) \leq \varepsilon \|\lambda\|_\infty - \xi (\varepsilon \|s(t, \cdot)\| + z(t)),$$

where $\xi = \min\{\tilde{\mu}, \mu_z\}$. Hence, using the Gronwall inequality, we have:

$$\varepsilon \|s(t, \cdot)\| + z(t) \leq \frac{\varepsilon \|\lambda\|_\infty}{\xi} + \left(\varepsilon \|s_0\| + z(0) - \frac{\varepsilon \|\lambda\|_\infty}{\xi} \right) e^{-\xi t}. \quad (4.17)$$

Thus, for all $t \geq 0$:

$$\varepsilon \|s(t, \cdot)\| + z(t) \leq \max \left(\varepsilon \|s_0\| + z(0), \frac{\varepsilon \|\lambda\|_\infty}{\xi} \right) =: D.$$

The colonising female population can be bounded similarly, since:

$$\dot{y}(t) \leq \phi z(t) - \mu_y y(t) \leq \phi D - \mu_y y(t).$$

Thus, using the Gronwall inequality once more, we obtain:

$$y(t) \leq \frac{\phi D}{\mu_y} + \left(y(0) - \frac{\phi D}{\mu_y} \right) e^{-\mu_y t}, \quad (4.18)$$

Hence, for all $t \geq 0$:

$$y(t) \leq \max \left(y(0), \frac{\phi D}{\mu_y} \right). \quad (4.19)$$

Therefore, the solutions of system (4.3) are bounded. Furthermore, inequalities (4.17) and (4.18) show that the set Σ is positively invariant under the flow of system (4.3). \square

A direct consequence of Lemma 6 is that $\|(s(t, \cdot), y(t), z(t))\|_{\mathcal{X}}$ is bounded for all time $t > 0$ in bounded intervals, which means that $t_{max} = +\infty$. Therefore, we have the following result.

Corollary 1 *Let Assumption 1 be satisfied. Then for every nonnegative initial condition, there exists a unique mild solution $(s(t, \cdot), y(t), z(t)) \in \mathcal{C}([0, +\infty), \mathcal{X}_+)$ for system (4.3).*

Let us introduce the total CBB population $w(t) = y(t) + z(t)$. By adding the second and third equations of system (4.3), we obtain the total population dynamics $\dot{w}(t) = -\mu_z(1 - \mathcal{R})z(t) - \mu_y y(t)$, where $\mathcal{R} = \frac{\phi}{\mu_z}$ corresponds to the average number of emerging colonising females produced by a single infesting female during its lifespan. If $\mathcal{R} < 1$, the total population $w(t) \leq w(0)e^{-\zeta t}$ with $\zeta = \min\{\mu_z(1 - \mathcal{R}), \mu_y\}$, so the CBB population disappears in the plantation. However, with realistic parameter values $\mathcal{R} \gg 1$ (cf. Table 4).

4.3.4 Asymptotic behaviour for a particular case

In this section, we study the existence and stability of the pest-free equilibrium point for a constant berry production rate $\lambda(t) = \lambda$ and determine the analytical expression of basic reproduction number.

System (4.3) always has a pest-free steady state $\mathcal{E}(a) = (s_b^0(a), 0, 0)$, where $s_b^0(a) = \lambda\pi(a)$, corresponding to the equilibrium without CBB. We now compute the basic reproduction number, \mathcal{R}_0 , using the next generation operator approach [59, 34]. Linearising the system around the pest-free steady state with $x(t, a)$, $v(t)$ and $w(t)$ being the perturbations, that is $x(t, a) = s_b(t, a) - s_b^0(a)$, $v(t) = y(t)$ and $w(t) = z(t)$, we obtain the following linear system written in the abstract form:

$$\dot{\psi}(t) = \mathcal{A}\psi(t) + D\mathcal{H}(\mathcal{E}(a))(\psi(t)), \quad \psi(0) = 0 \in D(\mathcal{A}). \quad (4.20)$$

where $\psi(t) = (x(t, \cdot), v(t), w(t))^\top$, $B^0 = \int_0^{a_i} s_0(a) da$ and the linear operator $D\mathcal{H}(\mathcal{E}(a)) : \mathcal{X} \rightarrow \mathcal{X}$ is defined by:

$$D\mathcal{H}(\mathcal{E}(a))(\psi(t)) = \begin{pmatrix} -\mathcal{W}_y(B^0, 0)[\beta s^0]v \\ \phi w - \varepsilon \mathcal{W}_y(B^0, 0)\|\beta s^0\|v \\ \varepsilon \mathcal{W}_y(B^0, 0)\|\beta s^0\|v \end{pmatrix}.$$

The next generator operator is defined by $\mathcal{B}(-\mathcal{A})^{-1}$, where $\mathcal{B} := D\mathcal{H}(\mathcal{E}(a))$. Consider the vectors $(z_1, z_2, z_3)^\top \in \mathcal{X}_0$ and $(\Phi, y, z)^\top \in \mathcal{X}$, then we have $(-\mathcal{A})^{-1}(z_1, z_2, z_3)^\top = (\Phi, y, z)^\top$ which implies that

$$z_1(a) = z_1(0)e^{-\int_0^a \mu(\sigma)d\sigma} - \int_0^a \Phi(\sigma)e^{-\int_\sigma^a \mu(l)dl}d\sigma, \quad y = -\frac{1}{\mu_y}z_2, \quad z = -\frac{1}{\mu_z}z_3.$$

Therefore, we get by direct calculation:

$$\mathcal{B}(-\mathcal{A})^{-1} = \begin{pmatrix} 0 & -\frac{1}{\mu_y}\mathcal{W}_y(B^0, 0)\|\beta s^0\| & 0 \\ 0 & -\frac{\varepsilon}{\mu_y}\mathcal{W}_y(B^0, 0)\|\beta s^0\| & \frac{\phi}{\mu_z} \\ 0 & \frac{\varepsilon}{\mu_y}\mathcal{W}_y(B^0, 0)\|\beta s^0\| & 0 \end{pmatrix}. \quad (4.21)$$

Based on [59], the basic reproduction number is defined as the spectral radius of the next generation operator $\mathcal{B}(-\mathcal{A})^{-1}$, that is:

$$\mathcal{R}_0 = -\frac{1}{2}\frac{\varepsilon}{\mu_y}\mathcal{W}_y(B^0, 0)\|\beta s^0\| + \frac{1}{2}\left(\left(\frac{\varepsilon}{\mu_y}\mathcal{W}_y(B^0, 0)\|\beta s^0\|\right)^2 + 4\frac{\phi}{\mu_y\mu_z}\varepsilon\mathcal{W}_y(B^0, 0)\|\beta s^0\|\right)^{\frac{1}{2}}.$$

Remark 4 1. Using the computation approach proposed in [35], we obtain the following basic reproduction number

$$\widetilde{\mathcal{R}}_0 = \mathcal{R}\frac{\varepsilon\mathcal{W}_y(B^0, 0)\|\beta s^0\|}{\varepsilon\mathcal{W}_y(B^0, 0)\|\beta s^0\| + \mu_y}. \quad (4.22)$$

2. Set $\mathcal{T} = \frac{\varepsilon}{\mu_y}\mathcal{W}_y(B^0, 0)\|\beta s^0\|$, then $\widetilde{\mathcal{R}}_0 = \frac{\mathcal{R}_0(\mathcal{T} + \mathcal{R}_0)}{\mathcal{T} + 1}$. Simple calculations show that $\mathcal{R}_0 < 1 (= 1, > 1)$ is equivalent to $\widetilde{\mathcal{R}}_0 < 1 (= 1, > 1)$.

3. Threshold \mathcal{R}_0 is difficult to interpret biologically. The biological interpretation of threshold $\widetilde{\mathcal{R}}_0$ is more straightforward:

- The term $\frac{\varepsilon\mathcal{W}_y(B^0, 0)\|\beta s^0\|}{\varepsilon\mathcal{W}_y(B^0, 0)\|\beta s^0\| + \mu_y}$ represents the average number of infesting females produced by a colonising female.
- So \mathcal{R}_0 can be defined as the average number of new infesting females originated from a single infesting female in the coffee plantation during its lifespan.

4. If $\mathcal{R} = \frac{\phi}{\mu} < 1$, which corresponds to unrealistic parameter values leading to the extension of the CBB population, then $\widetilde{\mathcal{R}}_0 < 1$.

Before establishing some results about stability, let us recall the notion of spectrum, growth bound and essential growth bound in spectral theory and some of their properties.

Denoting $\mathcal{L}(X)$ the set of bounded linear operators on X and $\mathcal{K}(X)$ the subset of compact operators on X . We defined the *essential norm* $\|T\|_{ess}$ of $T \in \mathcal{L}(X)$ by

$$\|T\|_{ess} := \inf_{B \in \mathcal{K}(X)} \|T - B\|_X.$$

We recall that the quotient $\mathcal{L}(X)/\mathcal{K}(X)$ is called the *Calkin algebra* with when proving the norm $\|\widehat{T}\| = \|T\|_{ess}$ where $\widehat{T} = T + \mathcal{K}(X)$ is a Banach algebra with unit, see for instance [38] for more

details. Let K be the infinitesimal generator of the strongly continuous semigroup $\{T_K(t)\}_{t \geq 0}$ on a Banach space X . The spectrum of K denoted by $\sigma(K)$ and the spectral bound denoted by $s(K)$ are defined by $\sigma(K) = \mathbb{C} \setminus \rho(K)$ and $s(K) = \sup\{Re\zeta, \zeta \in \sigma(K)\}$ respectively. We denote by $\omega_0(K) \in [-\infty, +\infty)$ the *growth bound* of a semigroup $\{T_K(t)\}_{t \geq 0}$ defined by

$$\omega_0(K) := \lim_{t \rightarrow +\infty} t^{-1} \ln(\|T_K(t)\|),$$

and the *essential growth bound* $\omega_{ess}(K) \in [-\infty, +\infty)$ of K is

$$\omega_{ess}(K) := \lim_{t \rightarrow +\infty} t^{-1} \ln(\|T_K(t)\|_{ess}).$$

Then it is proved in [38] that:

$$\omega_0(K) = \max\{\omega_{ess}(K), s(K)\},$$

and if C is a bounded linear operator in X , then $K + C$ is the infinitesimal generator of a strongly semigroup. Moreover, if C is also compact, then

$$\omega_{ess}(K) = \omega_{ess}(K + C).$$

Proposition 4 *Under Assumption 1 and when λ is constant, if $\widetilde{\mathcal{R}}_0 < 1$, then the pest-free steady state $\mathcal{E}(a)$ is locally asymptotically stable (LAS) and it becomes unstable when $\widetilde{\mathcal{R}}_0 > 1$.*

Proof: According to [111], the steady state $\mathcal{E}(a)$ is LAS if $\omega_0(\mathcal{A} + D\mathcal{H}(\mathcal{E})) < 0$ and becomes unstable when $\omega_0(B + D\mathcal{H}(E)) > 0$ and $\omega_{ess}(B + D\mathcal{H}(E)) < 0$.

On the one hand, the differential operator $(\mathcal{A}, D(\mathcal{A}))$ is an infinitesimal generator of a strongly continuous semigroup $\{T_{\mathcal{A}}(t)\}_{t \geq 0}$ in \mathcal{X} satisfying $\|T_{\mathcal{A}}(t)\| \leq e^{-\theta t}$, $\forall t \geq 0$ and, on the other hand, we see that $D\mathcal{H}(\mathcal{E}) : D(\mathcal{A}) \subset \mathcal{X} \rightarrow \mathcal{X}$ is finite dimensional, so $D\mathcal{H}(\mathcal{E})$ is compact bounded operator, It follows that

$$\omega_{ess}(\mathcal{A} + D\mathcal{H}(\mathcal{E})) \leq \omega_{ess}(\mathcal{A}) \leq \omega_0(\mathcal{A}) = \lim_{t \rightarrow +\infty} t^{-1} \ln(e^{-\theta t}) = -\theta.$$

We can observe that the variable $\tilde{x}(t, a)$ has no impact on the dynamics of the two remaining variables in system (4.20). Then it suffices to use the second and third equations of (4.20) to determine the stability criterion of system (4.3) around the pest-free steady state. To this end, we seek solutions with an exponential form setting $\tilde{y}(t) = \tilde{y}^0 e^{\zeta t}$ and $\tilde{z}(t) = \tilde{z}^0 e^{\zeta t}$ where \tilde{y}^0 and \tilde{z}^0 are real numbers which can be estimated and ζ is either a real or a complex number. Substituting these exponential forms in system (4.20), we obtain the linear homogeneous system $(\zeta I - \mathbb{P}(\mathcal{E}))\tilde{X} = 0$ where $\tilde{X} = (\tilde{y}^0, \tilde{z}^0)^\top$, I is identity matrix and

$$\mathbb{P}(\mathcal{E}) = \begin{pmatrix} -\varepsilon \mathcal{W}_y(B^0, 0) \|\beta s^0\| - \mu_y & \phi \\ \varepsilon \mathcal{W}_y(B^0, 0) \|\beta s^0\| & -\mu_z \end{pmatrix}.$$

Hence, $s(\mathcal{A} + D\mathcal{H}(\mathcal{E})) < 0$ if and only if all roots $\zeta \in \mathbb{C}$ of the characteristic equation $\det(\zeta I -$

$\mathbb{P}(\mathcal{E}) = 0$ have a negative real part. The trace $Tr(\mathbb{P}) = -\varepsilon\mathcal{W}_y(B^0, 0)\|\beta s^0\| - \mu_y - \mu_z$ is negative. When $\widetilde{\mathcal{R}}_0 < 1$, the determinant $\det(\mathbb{P}) = \varepsilon\phi\mathcal{W}_y(B^0, 0)\|\beta s^0\| \left(\frac{1}{\widetilde{\mathcal{R}}_0} - 1\right)$ is positive and it becomes negative when $\widetilde{\mathcal{R}}_0 > 1$. Then all the roots ζ of the characteristic equation associated with matrix $\mathbb{P}(\mathcal{E})$ have a negative real part if $\widetilde{\mathcal{R}}_0 < 1$. Therefore, if $\widetilde{\mathcal{R}}_0 < 1$, then $\mathbf{s}(\mathcal{A} + D\mathcal{H}(\mathcal{E})) < 0$ and $\omega_{ess}(\mathcal{A} + D\mathcal{H}(\mathcal{E})) < 0$, hence $\omega_0(\mathcal{A} + D\mathcal{H}(\mathcal{E})) < 0$. Thus, the pest-free steady state is *LAS* if $\widetilde{\mathcal{R}}_0 < 1$. When $\widetilde{\mathcal{R}}_0 > 1$, then there exists at least one root ζ_0 with a positive real part. This implies that $\mathbf{s}(\mathcal{A} + D\mathcal{H}(\mathcal{E})) = \mathcal{R}e\zeta_0 > 0$. Therefore $\omega_0(\mathcal{A} + D\mathcal{H}(\mathcal{E})) > 0$ and it follows that $\mathcal{E}(\cdot)$ is unstable. This achieves the proof. \square

Proposition 4 means that the CBB population could disappear from the coffee plantation if $\widetilde{\mathcal{R}}_0 < 1$ and if the initial CBB population was in the basin of attraction of the pest-free steady state. In this case, it would not be necessary to control CBB. However, as previously stated, with realistic parameter values $\mathcal{R} \gg 1$ (cf. Table 4).

4.4 Control strategies

In this section, we describe and study an optimal control problem designed to maximise the yield of healthy coffee berries, while minimising the control costs and the CBB population for the next cropping season.

4.4.1 Definition of the control problem

We consider a fixed time interval $[0, t_f]$, where t_f denotes the end of the cropping season. We assume that all coffee berries are picked at time t_f . Two interventions strategies, called controls, are included into initial system (4.3). These controls are functions of time, constrained by lower and upper bounds, and described as follows:

- First, control $u(t) \in [0, 1]$ denotes the effort made to reduce the infestation of healthy coffee berries. In practise, this control can represent the action of synthetic insecticides, which are widely used in some coffee growing countries, but pollute the environment and favour the development of CBB resistance [25, 26]. It can also represent the action of the entomopathogenic fungus *Beauveria Bassiana*, which is as an environmentally friendly bio-insecticide, not toxic to workers, that is sprayed on the coffee berries and kills CBB when they drill their entry hole in the berries [27, 52]. We denote by ξ the efficiency of this control $u(t)$, so $(1 - \xi u(t))\beta(a)$ represents the reduced infestation rate.
- Second, control $v(t) \in [0, 1]$ consists in reducing the colonising female population. It can represent traps set in the plantation to capture colonising females during their migration flight [84, 37, 81]. We denote by η the efficiency of control $v(t)$, so $\eta v(t)$ represents the fraction of colonising females captured per unit of time.

We derive the following controlled system:

$$\begin{cases} \partial_t s(t, a) + \partial_a s(t, a) = -\mathcal{W}(B, y)(1 - \xi u(t))\beta(a)s(t, a) - \mu s(t, a) \\ \dot{y}(t) = -\varepsilon \mathcal{W}(B, y) \|\beta s(t, \cdot)\| - \mu_y y(t) - \eta v(t)y(t) + \phi z(t) \\ \dot{z}(t) = +\varepsilon \mathcal{W}(B, y)(1 - \xi u(t)) \|\beta s(t, \cdot)\| - \mu_z z(t) \\ s(t, 0) = \lambda(t), \quad s(0, a) = s_0(a), \quad y(0) = y_0, \quad z(0) = z_0. \end{cases} \quad (4.23)$$

The existence and uniqueness of a solution of controlled system (4.23) over a finite time interval is obtained by proceeding as in the case without control. Moreover, the solution remains nonnegative for nonnegative initial conditions and as controlled system (4.23) is upper-bounded by system (4.3), the domain Σ remains positively invariant for controlled system (4.23).

Let \mathcal{K} be the space of admissible controls defined by:

$$\mathcal{K} := \{(u, v) \in (L^\infty(0, t_f))^2 : u(\cdot) : [0, t_f] \rightarrow [0, 1], v(\cdot) : [0, t_f] \rightarrow [0, 1]\}.$$

The optimal control problem is then formulated below.

Problem 2 Find an admissible control pair $(u^*, v^*) \in \mathcal{K}$ maximising the following objective functional:

$$\mathcal{J}(u, v) = \int_0^{a_\dagger} \Theta(a)s(t_f, a)da - \int_0^{t_f} [C_u u^2(t) + C_v v^2(t)] dt - C_y y(t_f). \quad (4.24)$$

The first term of the objective functional represents the crop yield, where the coffee berry price $\Theta(\cdot)$ is an increasing, bounded, continuous function depending on berry age. We assume that infested berries do not contribute to the yield, as their economic return is negligible compared to healthy berries. The second integral term corresponds to the control implementation costs, with coefficients C_u and C_v being the maximal costs per unit of time of controls u and v , respectively. Quadratic expressions of the controls are included to indicate the nonlinearity of the implementation costs, as it is more costly to increase the control efficiency when it is already high. Finally, the last term is a penalty on the CBB population that remains in the plantation after harvest, weighted by parameter C_y . The remaining population consists only of colonising females, as we assume that all berries are picked to limit the infestation of the next cropping season.

4.4.2 Existence of an optimal control pair

This section is devoted to the existence of the solution of the optimal control Problem 2. Consider the function:

$$\chi(t, a, B, y, u) = e^{-\int_0^a [\mu(\eta) + (1 - \xi u(\theta))\beta(\eta)\mathcal{W}(B(\theta), y(\theta))]_{\theta=t-a+\eta} d\eta}.$$

Setting $Q := [0, t_f] \times [0, a_\dagger]$, then the following lemma holds.

Lemma 7 Under Assumption 1, function χ is Lipschitz in the following sense: for (B^i, y^i, u^i) , $i \in \{1, 2\}$ and $(t, a) \in Q$, there exists a constant $K > 0$ such that:

$$|\chi(t, a, B^1, y^1, u^1) - \chi(t, a, B^2, y^2, u^2)| \leq K(|B^1 - B^2| + |y^1 - y^2| + |u^1 - u^2|).$$

Proof: Lemma 7 is obtained by direct computation, using the fact that $|e^{-m} - e^{-p}| \leq |m - p|$ for all $m, p \in \mathbb{R}_+$, that the state variables $B(t)$ and $y(t)$ are uniformly bounded and that $u \in \mathcal{U}$. \square

Using the method of characteristics, we get an explicit solution of the $s(t, a)$ -equation in system (4.23) as follows:

$$s(t, a) = \begin{cases} s_0(a-t) \frac{\chi(t, a, B, y, u)}{\chi(t, a-t, B, y, u)} & \text{if } a > t, \\ \lambda(t-a) \chi(t, a, B, y, u) & \text{if } a \leq t. \end{cases} \quad (4.25)$$

For $(t, a) \in Q$, we obtain from equation (4.25) the boundedness of the age-specific density of healthy coffee berries as follows:

$$|s(t, a)| \leq \max \left\{ \sup_{a \in [0, a_\dagger]} |s_0(a)|; \|\lambda\|_\infty \right\}. \quad (4.26)$$

By integrating the $s(t, a)$ -equation of controlled system (4.23) on the age interval $[0, a_\dagger]$, one gets:

$$\dot{B}(t) = \lambda(t) - (1 - \xi u) \mathcal{W}(B, t) \|\beta s(t, \cdot)\| - \|\mu s(t, \cdot)\| \leq \|\lambda\|_\infty - \tilde{\mu} B(t). \quad (4.27)$$

Therefore, it follows that $B(t) \leq \max \left\{ B(0), \frac{\|\lambda\|_\infty}{\tilde{\mu}} \right\}$ for all $t \in [0, t_f]$. Since the state variables $B(t)$, $y(t)$ and $z(t)$ are bounded for $t \in [0, t_f]$, it follows from (4.23) and (4.27) that:

$$|\dot{B}(t)| + |\dot{y}(t)| + |\dot{z}(t)| \leq \mathcal{M}, \quad (4.28)$$

where \mathcal{M} is a constant that does not neither depend on time nor on the control pair.

We now prove that there exists an optimal control strategy that maximises the objective functional (4.24) subject to the age-structured model (4.23).

Theorem 14 *Under Assumption 1, the optimal control Problem 2 admits a solution, i.e. there exists an optimal control pair $(u^*, v^*) \in \mathcal{K}$ with associated optimal solution (s^*, y^*, z^*) which maximises the objective functional $\mathcal{J}(u, v)$.*

Proof: Since the state variables and the controls are uniformly bounded, then we have $\sup \{ \mathcal{J}(u, v) : (u, v) \in \mathcal{K} \}$ is finite. Thus there exists a maximising sequence $(u_n, v_n)_n \subset \mathcal{K}$ such that:

$$\lim_{n \rightarrow +\infty} \mathcal{J}(u_n, v_n) = \sup \{ \mathcal{J}(u, v) : (u, v) \in \mathcal{K} \}.$$

Since sequence $(u_n, v_n)_n$ is bounded, there exists a subsequence, still denoted (u_n, v_n) , that converges to the limit (u^*, v^*) in the weak- \star topology of $L^\infty(0, t_f) \times L^\infty(0, t_f)$. The limit (u^*, v^*) belongs to \mathcal{K} , since \mathcal{K} is a closed convex subset of $L^\infty(0, t_f) \times L^\infty(0, t_f)$ and so it is weakly closed. Let (s_n, y_n, z_n) be the state variables associated with control pair (u_n, v_n) and $B_n(t) = \int_0^{a_\dagger} s_n(t, a) da$. From equation (4.25), these variables are also related to each other by:

$$s_n(t, a) = \begin{cases} s_0(a-t) \frac{\chi(t, a, B_n, y_n, u_n)}{\chi(t, a-t, B_n, y_n, u_n)} & \text{if } a > t, \\ \lambda(t-a) \chi(t, a, B_n, y_n, u_n) & \text{if } a \leq t. \end{cases} \quad (4.29)$$

Thanks to inequality (4.28), the $B_n(t)$, $y_n(t)$ and $z_n(t)$ sequences are uniformly Lipschitz continuous, so they are bounded and equicontinuous. According to Arzela-Ascoli's theorem, we can extract a subsequence, still denoted $B_n(t)$, $y_n(t)$ and $z_n(t)$, which converges uniformly to the limit $B^*(t)$, $y^*(t)$ and $z^*(t)$ respectively in $\mathcal{C}(0, t_f)$. Using inequality (4.26) and the uniqueness of the limit, we have $s_n(t, a) \rightarrow s^*(t, a)$ when $n \rightarrow \infty$ almost everywhere in Q and $B^*(t) = \int_0^{a^\dagger} s^*(t, a) da$. As a consequence of Lemma 7, we have the convergence $\chi(t, a, B_n, y_n, u_n) \rightarrow \chi(t, a, B^*, y^*, u^*)$ as $n \rightarrow \infty$ almost everywhere in Q . So we can pass to the limit in equation (4.29) to obtain:

$$s^*(t, a) = \begin{cases} s_0(a-t) \frac{\chi(t, a, B^*, y^*, u^*)}{\chi(t, a-t, B^*, y^*, u^*)} & \text{if } a > t, \\ \lambda(t-a) \chi(t, a, B^*, y^*, u^*) & \text{if } a \leq t. \end{cases} \quad (4.30)$$

Moreover, passing to the limit in the differential equations satisfied by the sequences $y_n(t)$ and $z_n(t)$ in controlled system (4.23), we obtain:

$$\begin{aligned} \dot{y}^*(t) &= \phi z^*(t) - \varepsilon \mathcal{W}(B^*, y^*) \|\beta s^*(t, \cdot)\| - \mu_y y^*(t) - \eta v^*(t) y^*(t), \\ \dot{z}^*(t) &= (1 - \xi u^*(t)) \varepsilon \mathcal{W}(B^*, y^*) \|\beta s^*(t, \cdot)\| - \mu_z z^*(t). \end{aligned}$$

By using the continuity and boundedness of function $\Theta(\cdot)$ and passing to the limit in the objective functional (4.24), we obtain $\lim_{n \rightarrow +\infty} \mathcal{J}(u_n, v_n) = \mathcal{J}(u^*, v^*)$. So the pair (u^*, v^*) satisfies Problem 2. Hence, there exists an optimal control pair $(u^*, v^*) \in \mathcal{K}$ and the corresponding state variables s^* , y^* and z^* that maximise the objective functional \mathcal{J} in \mathcal{K} . \square

4.4.3 Necessary optimality conditions

We use the maximum principle for general age-structured systems on a finite time horizon provided by Feichtinger *et al.* [39, 40] and references cited therein, to derive the first-order necessary conditions and characterise the optimal control of Problem 2.

We set state vector $x(t) = (y(t), z(t))^\top$ to rewrite system (4.23) in the following compact form:

$$\begin{cases} \partial_t s(t, a) + \partial_a s(t, a) = f(t, a, x(t), q(t), s(t, a), u(t)) =: f(t, a) \\ \dot{x}(t) = g(t, x(t), q(t), u(t), v(t)) =: g(t) \\ s(t, 0) = \lambda(t), \quad s(0, a) = s_0(a), \quad x(0) = (y(0), z(0))^\top, \end{cases} \quad (4.31)$$

where $q(t) = (q_1(t), q_2(t))^\top$ derives from $h(t, a) = (h_1(t, a), h_2(t, a))^\top$, with $h_1(t, a) = s(t, a)$ and $q_1(t) = \int_0^{a^\dagger} h_1(t, a) da$, $h_2(t, a) = \beta(a) s(t, a)$ and $q_2(t) = \int_0^{a^\dagger} h_2(t, a) da$. Functions $f(t, a)$ and $g(t) = (g_1(t), g_2(t))^\top$ represent the right hand side of system (4.23) for the s -, y - and z -compartment respectively. Let ∇_x denotes the differentiation with respect to state variable x . We introduce the adjoint functions $\Phi_s(t, a)$ and $\Phi_x(t) = (\Phi_y(t), \Phi_z(t))$ corresponding to the state

variable $s(t, a)$ and vector $x(t)$, respectively. Then, from [39], the adjoint system is given by:

$$\begin{cases} -\partial_t \Phi_s(t, a) - \partial_a \Phi_s(t, a) = \Phi_s(t, a) \nabla_s f(t, a) + \Phi_x(t) \nabla_q g(t) \nabla_s h(t, a) \\ \quad + \int_0^{a^\dagger} \Phi_s(t, a) \nabla_q f(t, a) da \nabla_s h(t, a) \\ \Phi_s(t_f, a) = \Theta(a), \quad \Phi_s(t, a^\dagger) = 0 \\ -\dot{\Phi}_x(t) = \Phi_x(t) \nabla_x g(t) + \int_0^{a^\dagger} \Phi_s(t, a) \nabla_x f(t, a) da \\ \Phi_x(t_f) = (-C_y, 0). \end{cases}$$

This adjoint system can be rewritten as:

$$\begin{cases} \partial_t \Phi_s + \partial_a \Phi_s = (1 - \xi u) [\beta \mathcal{W}(B, y) (\Phi_s - \varepsilon \Phi_z) + \mathcal{W}_B(B, y) (\|\beta s \Phi_s\| \\ \quad - \varepsilon \|\beta s \Phi_z\|) + [\varepsilon \beta \mathcal{W}(B, y) + \varepsilon \|\beta s \mathcal{W}_B(B, y)\] \Phi_y + \mu \Phi_s \\ \dot{\Phi}_y = (1 - \xi u) \mathcal{W}_y(B, y) [-\varepsilon \|\beta s \Phi_z\| + \|\beta s \Phi_s\|] + \varepsilon \|\beta s \mathcal{W}_y(B, y)\] \Phi_y \\ \quad + (\mu_y + \eta v) \Phi_y \\ \dot{\Phi}_z = -\phi \Phi_y + \mu_z \Phi_z \end{cases} \quad (4.32)$$

with the following transversality conditions associated with the adjoint state variables for $(t, a) \in Q$:

$$\Phi_s(t_f, a) = \Theta(a), \quad \Phi_s(t, a^\dagger) = 0, \quad \Phi_y(t_f) = -C_y, \quad \Phi_z(t_f) = 0. \quad (4.33)$$

From the solution $s(t, a)$, $y(t)$ and $z(t)$ of system (4.23) and the corresponding solution $\Phi_s(t, a)$, $\Phi_y(t)$ and $\Phi_z(t)$ of adjoint system (4.32), we define the Hamiltonian functional associated with the control problem 2 by:

$$\mathcal{H}(t, u(t), v(t)) := -C_u u^2(t) - C_v v^2(t) + \Phi_y g_1(t) + \Phi_z g_2(t) + \int_0^{a^\dagger} \Phi_s(t, a) f(t, a) da. \quad (4.34)$$

Applying Pontryagin's maximum principle [39], by solving $\partial_u \mathcal{H}(t, u, v) = \partial_v \mathcal{H}(t, u, v) = 0$ and taking into account the boundaries of each control, we obtain:

$$u^*(t) = \min\{\max\{0, u_{sing}(t)\}, 1\}, \quad v^*(t) = \min\{\max\{0, v_{sing}(t)\}, 1\}. \quad (4.35)$$

where

$$u_{sing}(t) = -\frac{\xi \mathcal{W}(B(t), y(t)) (-\|\beta s(t, \cdot) \Phi_s(t, \cdot)\| + \varepsilon \|\beta s(t, \cdot) \Phi_z(t)\|)}{2C_u}, \quad v_{sing}(t) = -\frac{\eta y(t) \Phi_y(t)}{2C_v}.$$

4.5 Numerical results

4.5.1 Numerical methods

To solve our optimal control problem numerically, we begin by giving a brief description of the numerical scheme used to plot the states variables of system (4.3). This scheme is based on semi-implicit finite differential equations for the variables $y(t)$ and $z(t)$ and on the hyperbolic partial differential equation verified by $s(t, a)$ [8]. The integral terms are discretised using an

explicit composite trapezoidal rule. Let $\Delta a > 0$ be the discretisation step for the interval $[0, a_{\dagger}]$, with $N = \frac{a_{\dagger}}{\Delta a}$ the number of age sub-intervals, and let $\Delta t > 0$ be the discretisation step for the interval $[0, t_f]$, with $K = \frac{t_f}{\Delta t}$ the number of time sub-intervals. Replacing the derivative in time with a backward difference and the derivative in age with a forward difference, $\partial_t x(t, a)$ and $\partial_a x(t, a)$ are evaluated at time t_n and age a_j by:

$$\partial_t x(t_n, a_j) \simeq \frac{x_{j+1}^{n+1} - x_{j+1}^n}{\Delta t} \quad \text{and} \quad \partial_a x(t_n, a_j) \simeq \frac{x_{j+1}^n - x_j^n}{\Delta a},$$

where n and j denote the time and age index, respectively, and x_j^n denotes the solution approximation at time $t_n := n\Delta t$ and age $a_j := j\Delta a$. Since berry age evolves at the same speed as chronological time, choosing $\Delta a = \Delta t$ ensures that the necessary CFL (Courant–Friedrichs–Lewy) stability condition, *i.e.* $\frac{\Delta t}{\Delta a} \leq 1$, holds [1, 99].

Our numerical approach to approximate the optimal solution is based on the extension of the forward-backward sweep method [75]. Firstly, the state variables of system (4.23) are approximated using the forward semi-implicit finite difference scheme in time and backward semi-implicit difference in age, with an initial guess for the control pair $(u(t), v(t))$, for instance 1-valued functions. Secondly, the adjoint system (4.33) is solved by using the backward difference in time and forward difference in age, using the solutions of the state equations. After these two steps, the control function values are updated with the new values of the state and adjoint variables, thanks to equations (4.35) that characterise the optimal control. This procedure is repeated until convergence is achieved *i.e.* until successive values of all states, adjoint and control variables are sufficiently close, that is

$$\frac{\|(s, y, z) - (s_{old}, y_{old}, z_{old})\|_{\mathcal{X}} + \|u - u_{old}\|}{\|(s, y, z)\|_{\mathcal{X}}} \leq \epsilon,$$

where $\epsilon = 10^{-6}$ is the tolerance and $(s_{old}, y_{old}, z_{old}, u_{old})$ is the vector from the previous iteration.

4.5.2 Parameterisation

In the numerical simulations, we assume that flowering occurs in the plantation throughout the cropping season, so that we can consider a constant berry production rate $\lambda(t) \equiv \lambda$. Since coffee berries become mature after 6–9 months [11], the maximum berry age of is set at $a_{\dagger} = 365$ days. We found no data on age-dependent mortality for non infested berries, so we apply a constant rate μ to all berries up to the berry maximum age a_{\dagger} .

CBB colonising females infest healthy berries of all ages, but with a preference for more mature berries [54, 93]. According to [13], the infestation of berries younger than 90 days is low. So we represent the infestation rate as a constant function for berry ages lower than a_{β} and an increasing function for older ages, according to the following function:

$$\beta(a) = \begin{cases} \beta_{\min} & 0 \leq a < a_{\beta}, \\ \beta_{\min} + \beta_a(1 - e^{-k_{\beta}(a-a_{\beta})}), & a_{\beta} \leq a \leq a_{\dagger} \end{cases} \quad (4.36)$$

with $a_{\beta} = 90$ days. Moreover, parameters are chosen such that the average value $\bar{\beta} =$

$$\frac{1}{a_{\dagger}} \int_0^{a_{\dagger}} \beta(a) da \approx 0.023 \text{ day}^{-1}.$$

The CBB–berry interaction function \mathcal{W} is modelled by:

$$\mathcal{W}(B, y) = \frac{y}{y + \alpha B + 1},$$

in order to ensure that both coffee berries and colonising females are limiting variables for the infestation process. As generally $y \gg 1$, \mathcal{W} introduces a ratio-dependent-like interaction between CBB and coffee berries.

We choose a sigmoid function to model the price of healthy coffee berries according to berry age:

$$\Theta(a) = \frac{\bar{\Theta} a^n}{a^n + a_{\Theta}^n}, \quad (4.37)$$

where $\bar{\Theta}$ is the asymptotic price of healthy coffee berries, a_{Θ} is the age at which berries are at half asymptotic price and $n \in \mathbb{N}^*$ is the Hill constant. Note that with a finite age bounded by a_{\dagger} , the maximum price that mature berries can reach is $\frac{\bar{\Theta} a_{\dagger}^n}{a_{\dagger}^n + a_{\Theta}^n}$. We assume that the cost C_u of the infestation control, associated with the bio-insecticide, is higher than the cost C_v of traps, as regular spraying is more time consuming and labour intensive. All parameter values used in the simulations are given in Table 4; infestation and price functions are depicted in Figure 25.

Table 4: Model and control parameter values. Parameter values are based on biological data collected in the literature [11]; more information is available in Table 1 in Chapter 2.

Symbol	Description	Value
t_f	Duration of a cropping season	250 days
a_{\dagger}	maximum age of coffee berry	250 days
λ	Production rate of new coffee berries	1200 berries.day ⁻¹
μ	Natural mortality rate of healthy coffee berries	0.002 day ⁻¹
ε	Colonising CBB per berry (scaling factor)	1 female.berry ⁻¹
$\beta(a)$	Infestation function (4.36):	day ⁻¹
β_{\min}	minimum infestation rate	0.004 day ⁻¹
β_a	age-dependent extra infestation rate	0.036 day ⁻¹
k_{β}	infestation coefficient	0.035 day ⁻¹
a_{β}	infestation threshold age	90 days
α	CBB–berry interaction constant	0.7 female.berry ⁻¹
ϕ	Emergence rate of new colonising females	2 day ⁻¹
μ_y	Natural mortality rate of colonising females	1/20 day ⁻¹
μ_z	Natural mortality rate of infesting females	1/27 day ⁻¹
ξ	Maximum efficiency of control u (bio-insecticide)	0.75 –
η	Maximum efficiency of control v (traps)	0.08 day ⁻¹
C_u	Weight of control u	5 \$.day ⁻¹
C_v	Weight of control v	0.65 \$.day ⁻¹
$\Theta(a)$	Coffee berry price function (4.37):	\$.berry ⁻¹
$\bar{\Theta}$	coffee berry asymptotic price	0.025 \$.berry ⁻¹
a_{Θ}	berry age at half asymptotic price	120 days
n	Hill constant	7 –
C_y	Cost of remaining colonising females	10 ⁻⁴ \$.female ⁻¹

Moreover, we use the following initial conditions. At the beginning of the cropping season, flowering has not started yet. We also assume that there are only colonising females, as infesting females were eliminated from the plantation when berries were picked during the preceding harvest. So there are initially neither coffee berries nor infesting females, *i.e.* $s_0(a) = 0$ for all

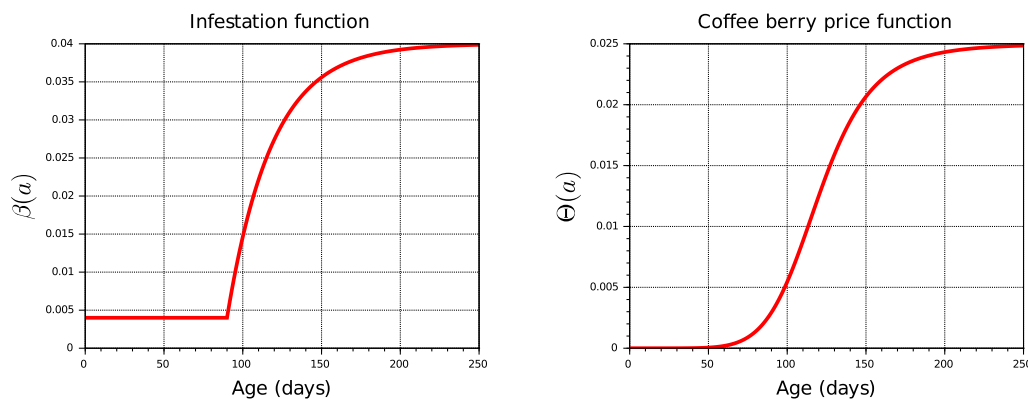


Figure 25: Representation of the infestation (left panel) and coffee berry price (right panel) as functions of coffee berry age for the parameter values given in Table 4.

$a \in [0, a_{\dagger}]$ and $z(0) = 0$. The number of initial colonising females is set at $y(0) = 10^4$ females.

4.5.3 Simulations

Using the parameters given in Table 4 and the initial conditions described above, we numerically solved optimal control Problem 2 associated with system (4.23). Results are illustrated in Figure 26. Panels (a–c) represent the dynamics of the (integrated) state variables, *i.e.* the total healthy coffee berries B , the colonising y and infesting z CBB females (plain red curves); panel (d) represents the evolution of the optimal controls u^* (plain magenta curve) and v^* (plain orange curve). For comparison purposes, the solution of system (4.3) without control is also drawn in panels (a–c) (dashed blue curves), as well as the pest-free solution obtained with $y(0) = 0$ female in panel (a) (dash-dotted black curve).

These results show that the optimal application of both controls, the bio-insecticide that hampers infestation (u) and the traps that capture colonising females (v), significantly reduce the CBB population (Figure 26(b) and (c), plain red curves), compared to the case without control (dashed blue curves). Moreover, the total healthy coffee berries regain the dynamics obtained in the pest-free case (Figure 26(a), dash-dotted black curve). During most of the cropping season, both controls are at their highest level so as to severely decrease the CBB population. From day 190, the population increases again, but remains several orders of magnitude below the case without control. When we decompose the impact of both controls on the CBB dynamics, the bio-insecticide and the traps (see Figure 27), we observe that neither control is sufficient to reduce the CBB population when applied alone. In both cases, the CBB population growth is slowed down compared to the uncontrolled case, but similar values are reached at the end of the cropping season. As a consequence, the healthy berries decline at the end of season, which reduces the harvest compared to the optimal case or pest-free case depicted in Figure 26(d). This shows that the controls have a synergistic effect.

To further explore the impact of each control, we solve a modified version of control Problem 2: we still aim at maximising the objective functional (4.24), but we use a single control, either u or v , setting the other control to zero. The optimal controls obtained are denoted by u^{\dagger} and v^{\dagger} . Results are shown in Figure 28. In both cases, the optimal strategy consists in applying the

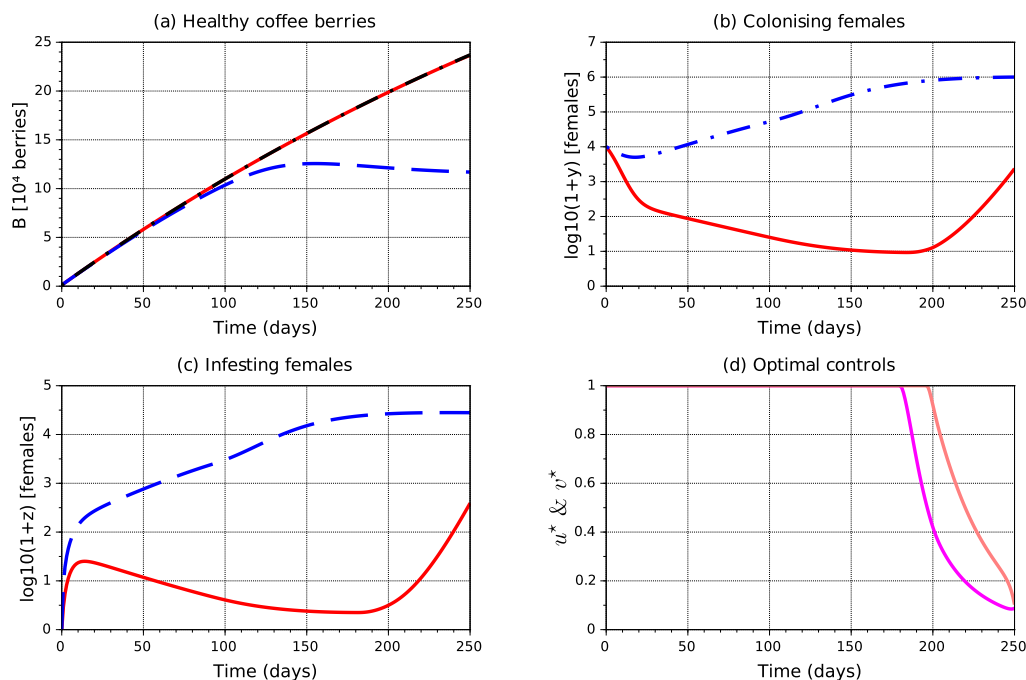


Figure 26: (a–c) Simulation of system (4.23) with optimal control pair (u^*, v^*) (plain red curves), without controls ($u = v = 0$, dashed blue curves) and without pest ($y(0) = z(0) = 0$, dash-dotted black curve). (d) Evolution of optimal controls u^* (plain magenta curve) and v^* (plain orange curve). Parameter values are given in Table 4. Zero initial conditions are set, except for colonising CBB: $y(0) = 10^4$ females.

control at its maximal value during almost (for the bio-insecticide u) or all (for the traps v) the cropping season. A single optimal control is less efficient to contain the CBB population than the optimal control pair (Figure 26), especially when only traps (v^\dagger) are deployed. In this latter case, $v^\dagger(t) = 1$ at all times, and the final healthy coffee berries are less abundant than in the optimal or pest-free cases depicted in Figure 26(d), which has an impact on the harvest. These results show that the bio-insecticide (u) alone can preserve the harvest, but not the traps (v). However, to control the CBB population, both controls are needed.

This synergistic effect of the controls appears even more clearly in Table 5, which splits the penalised profit \mathcal{J} defined in equation (4.24) into its three components, for the various control strategies depicted above (the optimal strategies aiming at maximising this penalised profit). The penalised profit \mathcal{J} is based on the berry financial yield \mathcal{Y} , minus the control costs \mathcal{C} and a penalty based on the colonising CBB population remaining after harvest, so as to limit the damages for the next cropping season. The bio-insecticide, especially when optimised (u^\dagger), is fairly efficient to preserve the yield and reduce the colonising CBB population after harvest, but its cost is high. In contrast, the traps (v) alone are poorly efficient, but not too costly. However, when applying traps (v^*) on top of the bio-insecticide (u^*), their effect is particular notable on all components of the penalised profit, which highlights the synergistic effect of both controls.

The yield is computed as the integral of the healthy berry age distribution at the end of the cropping season multiplied by the age-dependent sigmoid-shaped price (4.37). These final distributions are depicted in Figure 29. As the season starts with no berries, which are then produced at a constant rate, there are less mature than young berries at the end of the season.

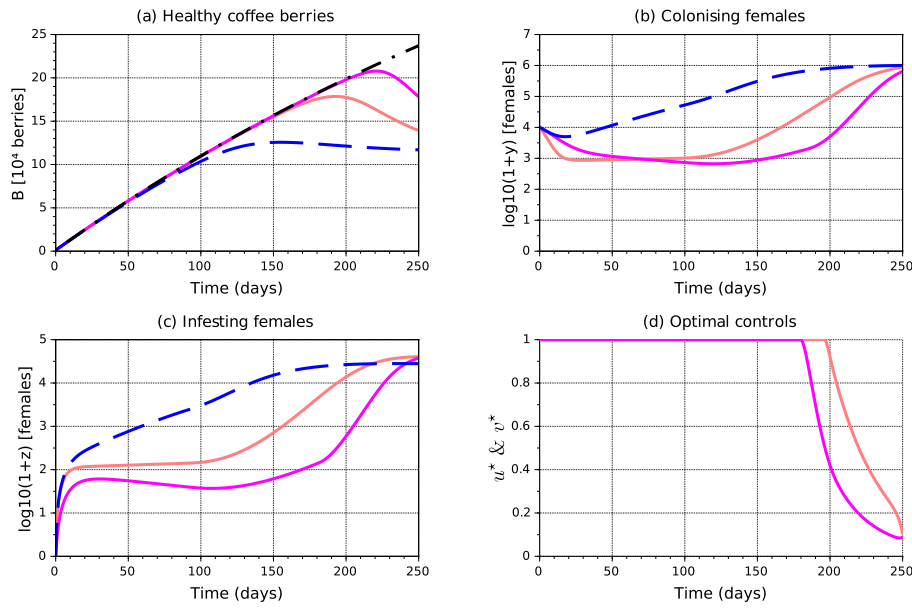


Figure 27: (a–c) Simulation of system (4.23) when only applying control u^* ($v = 0$, plain magenta curves) or control v^* ($u = 0$, plain orange curves) from optimal control pair (u^*, v^*) , without controls ($u = v = 0$, dashed blue curves) and without pest ($y(0) = z(0) = 0$, dash-dotted black curve). (d) Evolution of controls u^* and v^* . Parameter values are given in Table 4. Zero initial conditions are set, except for colonising CBB: $y(0) = 10^4$ females.

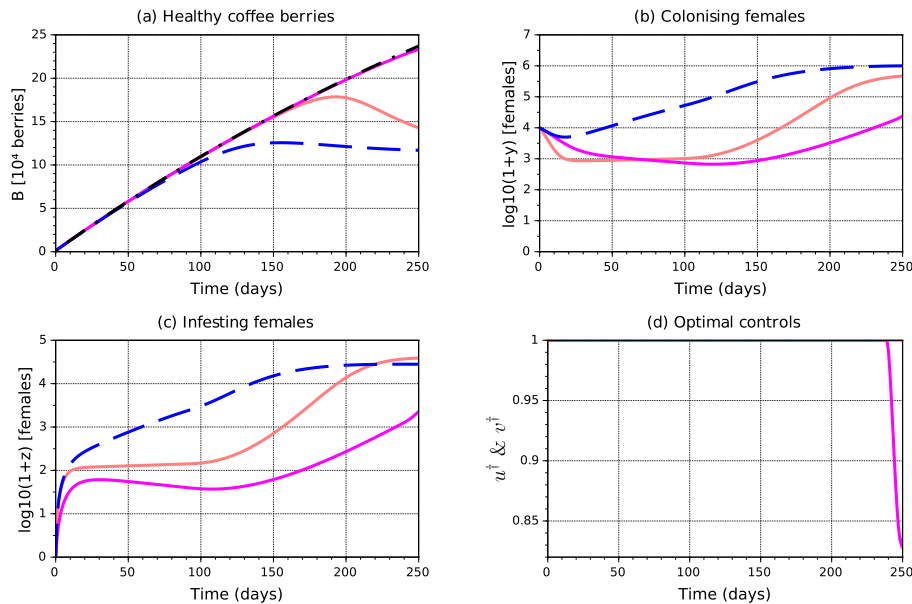


Figure 28: (a–c) Simulation of system (4.23) with optimal control u^\dagger ($v = 0$, plain magenta curves), optimal control v^\dagger ($u = 0$, plain orange curves), without control ($u = v = 0$, dashed blue curves) and without pest ($y(0) = z(0) = 0$, dash-dotted black curve). (d) Evolution of optimal controls u^\dagger and v^\dagger . Parameter values are given in Table 4. Zero initial conditions are set, except for colonising CBB: $y(0) = 10^4$ females.

Table 5: Penalised profit (\mathcal{J} in US\$), yield (\mathcal{Y} in US\$) control costs (\mathcal{C} in US\$) and colonising CBB females remaining after harvest at the end of the cropping season ($t_f = 250$ days). “control u^* only” corresponds to the application of $u = u^*$ from optimal control pair (u^*, v^*) with $v = 0$; “optimal control u^\dagger ” corresponds to the optimisation of u when $v = 0$ (and vice versa for v).

Intervention	\mathcal{J}	\mathcal{Y}	\mathcal{C}	$y(t_f)$
no control	311	412	0	1002008
optimal controls (u^*, v^*)	1528	2635	1106	2356
control u^* only	471	1503	967	649030
control v^* only	553	782	139	891815
optimal control u^\dagger	1316	2563	1244	23695
optimal control v^\dagger	628	838	163	469589

Penalised profit/financial gain in the pest-free case: $\mathcal{J} = \mathcal{Y} = 2644$ US\$

Young berries do not contribute to the yield. As expected from the results above, the final berry and price distributions are similar in the pest-free (dash-dotted black curves) and optimal control cases when both controls are used (plain red curves). When the bio-insecticide only is optimised (u^\dagger , plain magenta curves), there are slightly less mature berries, which is reflected in the yield (Table 5). When only traps are used (v^\dagger , plain orange curves), there are markedly less berries, but still more than without control (dashed blue curves), hence a better yield. In these two latter cases, the age preference for berries older than 90 days is notable and reflected by the sharper decline occurring at this age. In the case without control (blue curves), we can compare the results with CBB preference for mature berries through function (4.36) (dotted curves) and without preference, setting the infestation rate β to a fixed value between β_{\min} and β_{\max} (shaded area). Using the mean value of function (4.36) results in a drastic reduction of berries of (almost) all ages at the end of the season and very low revenues (plain blue curves). Indeed, without preference, colonising females infest more immature berries. However, for lower values of the infestation rate β , there are more mature berries at the end of season, which generate notably higher revenues. It is therefore particularly relevant to take into account the CBB preference for mature berries.

Figure 30 is obtained by numerically solving optimal control Problem 2 to obtain the optimal control pair (u^*, v^*) , but then applying only one of the two controls and setting the other one to zero to integrate system (4.23). We hence observe the impact of the bio-insecticide (u^*) alone and the traps (v^*) alone on the age distribution of the healthy coffee berries and their price at the end of the simulation.

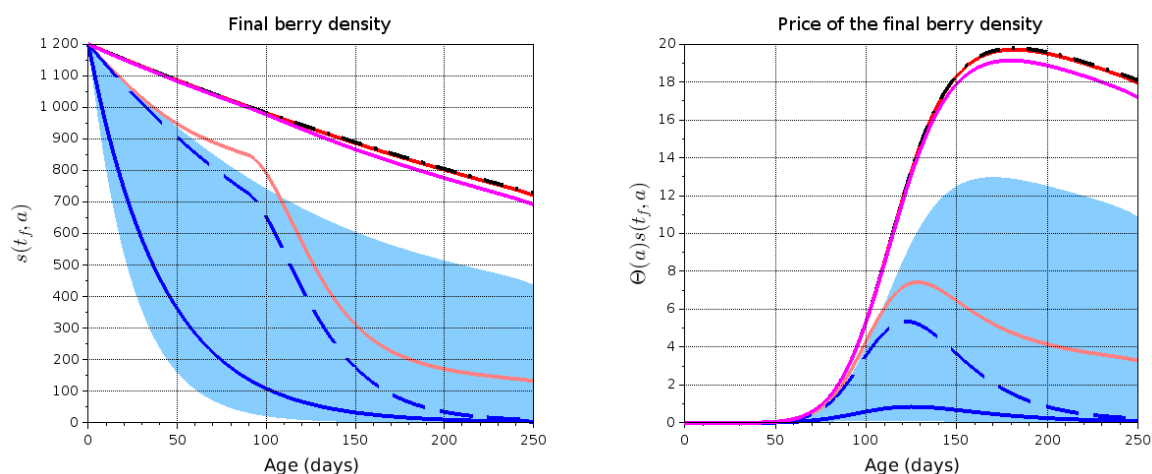


Figure 29: Age distribution of the healthy coffee berries and their price (yield) at the end of the simulation ($t = t_f$) with optimal control pair (u^*, v^*) (plain red curves), optimal control u^\dagger ($v = 0$, plain magenta curves), optimal control v^\dagger ($u = 0$, plain orange curves), without control ($u = v = 0$, dashed blue curves) and without pest ($y(0) = z(0) = 0$, dash-dotted black curves). Cases without control and with constant infestation rates β are also represented (blue-shaded areas delimited by β_{\min} and $\beta_{\max} = \beta_{\min} + \beta_a$, plain blue curves for mean value $\bar{\beta}$). Parameter values are given in Table 4. Zero initial conditions are set, except for colonising CBB: $y(0) = 10^4$ females.

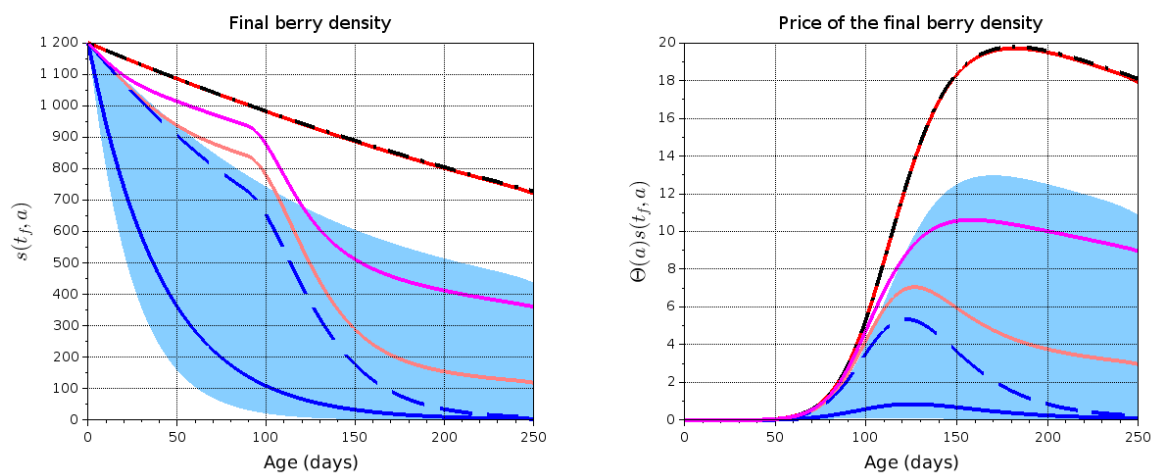


Figure 30: Age distribution of the healthy coffee berries and their price at the end of the simulation ($t = t_f$) with optimal control pair (u^*, v^*) (plain red curves), when only u^* ($v = 0$, plain magenta curves) or v^* ($u = 0$, plain orange curves) is applied, without control ($u = v = 0$, dashed blue curves) and without pest ($y(0) = z(0) = 0$, dash-dotted black curves). Cases without control and with constant infestation rates β are also represented (blue-shaded areas delimited by β_{\min} and $\beta_{\max} = \beta_{\min} + \beta_a$, plain blue curves for mean value $\bar{\beta}$). Parameter values are given in Table 4. Zero initial conditions are set, except for colonising CBB: $y(0) = 10^4$ females.

4.6 Conclusion

In this chapter, we formulated and analysed an original PDE model describing the infestation dynamics of coffee berries by CBB during a cropping season. It is an extension of the model studied in Chapter 2, taking into account the important role played by the coffee berry age in the CBB infestation process. Indeed, CBB colonising females preferably infest mature coffee berries and we show that this preference has an impact on the coffee berry yield at harvest. The existence and uniqueness of a solution was established via the semigroup perturbation theory. An optimal control problem was then formulated, aiming at maximising the yield while minimising the control costs and the CBB population for the next cropping season. Two levers were considered to achieve this aim: a control which reduces the berry infestation, by spraying a bio-insecticide such as the entomopathogenic fungus *Beauveria Bassiana*, and a control which increases the colonising female mortality, based on traps. We established the existence of an optimal solution, which we characterised using the maximum principle for general age-structured systems. The problem was solved numerically using a forward-backward sweep method.

Our main result is the synergistic effect of both controls: the bio-insecticide is efficient but costly, the traps alone are poorly efficient but less costly; combining both allows to reach a cost-effective optimum.

To increase their efficiency over several cropping seasons, these control methods should be coupled with field sanitation [11, 29]. This cultural practice is introduced in Chapter 3, at harvest when infested berries are picked together with healthy berries, and again before the following cropping season when the plantation is cleaned. It would be interesting to implement and optimise these CBB complementary controls during and between cropping seasons and to assess their efficiency over longer time horizons.

Optimal impulsive control of an age-structured model of coffee berries-CBB interaction

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5.1 Introduction

In this chapter, we first formulate here a more realistic semidiscrete age-structured model describing the infestation dynamics of coffee berries by CBB including a control strategy based to the application of bio-insecticide such entomopathogenic fungus at discrete time during the cropping season. This control is included in the PDE model 4.3 proposed in the previous Chapter that can persist over the time, hence the introduction of an extract state variable as in Chapter 2. We address an optimal control problem with the objective to maximise the profit while minimising the CBB population for the next cropping season. The first-order necessary optimality condition of control problem is carefully established. We conclude with numerical simulations and discussions.

5.2 Model formulation

We extend the model (4.3) formulated in the Chapter 4 by including the possibility to apply at discrete time the bio-insecticides on the coffee berries in the plantation. The aim of this control is to prevent the CBB to infest the healthy berries and kill them by contact. We shall

call throughout this chapter the bio-insecticides, *control*. It can be represented in the practice for example by the entomopathogenic fungus, *Beauveria Bassiana*. Following the formulation approach made in Chapter 2, we define the state variable $v(t)$ the control load in the plantation at time t . Based on the age-structured model (4.3) formulated in the Chapter 4, we consider here the following semidiscrete age-structured model describing the infestation dynamics of coffee berries by CBB in the plantation with control application at discrete time:

$$\left\{ \begin{array}{l} \partial_t s(t, a) + \partial_a s(t, a) = -\beta(a)q(v(t))\mathcal{W}(B(t), y(t))s(t, a) - \mu(a)s(t, a) \\ \dot{y}(t) = -\varepsilon\mathcal{W}(B(t), y(t))\|\beta s(t, \cdot)\| - \mu_y y(t) + \phi z(t) \\ \dot{z}(t) = \varepsilon q(v(t))\mathcal{W}(B(t), y(t))\|\beta s(t, \cdot)\| - \mu_z z(t) \\ \dot{v}(t) = -\gamma v(t) \quad t \neq t_n \\ \Delta v(t_n) = v(t_n^+) - v(t_n) = h_n, \quad t = t_n, \quad n \in \mathbb{N} \\ s(t, 0) = \lambda(t), \quad s(0, a) = s_0(a), \quad y(0) = y_0, \quad z(0) = z_0, \quad v(0^+) = v_0. \end{array} \right. \quad (5.1)$$

In the model (5.1), we consider that the control is applied periodically on the coffee berries in the plantation at discrete times by the increasing sequence $(t_n)_{n \in \mathbb{N}}$ satisfying $t_0 = 0 < t_1 < t_2 < \dots < t_n < \dots$. The instant $t_n = nT_{per}$, where the indice n denotes the n^{th} control application event and T_{per} is the length of period between two successives control applications. The t_n^+ notation depicts the instant just after t_n and then $s(t_n^+, a) = \lim_{j \rightarrow 0^+} s(t_n + j, a)$, $y(t_n^+) = \lim_{j \rightarrow 0^+} y(t_n + j)$ and $z(t_n^+) = \lim_{j \rightarrow 0^+} z(t_n + j)$ and $v(t_n^+) = \lim_{j \rightarrow 0^+} v(t_n + j)$ are the density of healthy coffee berries, the colonising females, infesting females and control load instantly after application of control at instant t_n respectively. The parameter γ denotes the natural decay rate of control. This decay is favorised by the abiotic factors including the environmental conditions. The constant $h_n \equiv h(t_n) \geq 0$ represents the control application rate at discrete time t_n , for any $n \in \mathbb{N}$ and the initial condition $v(0^+)$ is nonnegative. The aim of control being to reduce the infestation of coffee berries by CBB and kill the colonising females by contact. The function $q(\cdot) : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ represents the effort that prevents the failure of control and is explicitly as in the Chapter 2 by:

$$q(v) = 1 - \frac{\xi v}{v + k}. \quad (5.2)$$

The parameter $\xi \in (0, 1)$ denotes the maximal effectiveness of control load and the positive constant k represents the control load half saturation. This function means that, the effectiveness of control load increases when the control is sprayed at high quantity. The function $q(\cdot)$ verifies the basic properties: $q(0) = 1$, $\lim_{v \rightarrow +\infty} q(v) = 1 - \xi$, it is \mathcal{C}^1 -Lipschitz continuous and decreasing function follows its argument. We assume that the Assumption 1 holds true in this chapter and the initial condition of fungus load $v(0)$ is nonnegative.

5.3 An optimisation problem

5.3.1 Problem statement

In this section, we investigate the optimal impulsive control problem in order to maximise the healthy coffee berries yield at the end of cropping season at lower cost of control. To this end, we

use the optimal control theory for impulsive system in the framework by [112, 113]. Impulsive and continuous control problem, in which a continuously evolving state is modified by discrete control actions, have several applications in epidemiology, medicine, engineering and ecology [43, 2, 102, 22]. The main goal of coffee growers is to produce a good quality of coffee berries with lower control costs at the end of cropping season. To this fact, We assume that the harvest takes place at the end of the cropping season, denoted by t_f . We also assume that almost all berries, and hence almost all infesting females $z(t)$ that are inside berries, are picked at t_f , so that only colonising females $y(t)$ remain in the plantation after harvest. We assume that $t_f \in [t_n, t_{n+1})$. Denoting by N_f the number of control application during a cropping season $[0, t_f]$ and that the harvesting take place at the end of cropping season t_f . We define the class of admissible controls as

$$\mathcal{M} := \{h = (h_n) \in \mathbb{R}^{N_f} : 0 \leq h_n \leq h_{\max}, \forall n \in \{0, 1, 2, \dots, N_f\}\}, \quad (5.3)$$

where h_{\max} represents the upper bound of control apply at discrete time t_n for any $n \in \{0, 1, 2, \dots, N_f\}$. The objective functional is defined for a control $h = (h_n)_{n \in [1, N_f] \cap \mathbb{N}^*} \in \mathcal{M}$ as follows:

$$\mathcal{J}(h) = \int_0^{a^\dagger} \Theta(a)s(t_f, a)da - \sum_{n=0}^{N_f} C(h_n) - C_y y(t_f), \quad (5.4)$$

subject to the impulsive evolution System (5.1). In our objective functional (5.4), the first term is the healthy coffee berries yield where the function $\Theta(\cdot)$ denotes the weight of berries which is increasing continuous follows age-specific of berries and bounded. The second term represents the cost control, where $C(\cdot)$ is the continuous function of each cost for control application h_n . Finally, the penalised last term is the CBB population that remains after harvest in the plantation consisting of colonising females, weighed in the constant C_y . We assume in this objective functional that the infected coffee berries are not monetisables.

Our purpose is to maximise the objective functional over the admissible class of controls, ie seek $h^* = (h_n^*)_{n \in [0, N_f] \cap \mathbb{N}^*}$ belongs to \mathcal{M} so that

$$\mathcal{J}(h^*) = \max_{h \in \mathcal{M}} \mathcal{J}(h). \quad (5.5)$$

5.3.2 Basic properties

By the Theorem 9 of Chapter 3 the impulsive system

$$\begin{cases} \dot{v}(t) = -\gamma v(t) & t \neq t_n \\ \Delta v(t_n) = v(t_n^+) - v(t_n) = h_m, & t = t_n, n = 0, 1, 2, \dots \end{cases} \quad (5.6)$$

has a unique nonnegative solution.

Let $h_m = \max\{h_n, \forall n = 0, 1, 2, \dots, N_f\}$, we have the inequality $v(t_n^+) \leq v(t_n) + h_m$. Let $w(t)$ be the solution of the impulsive system

$$\begin{cases} \dot{w}(t) = -\gamma w(t) & t \neq t_n \\ \Delta w(t_n) = w(t_n^+) - w(t_n) = h_m, & t = t_n, n = 0, 1, 2, \dots \end{cases} \quad (5.7)$$

Then, for $t \in (t_n^+, t_{n+1})$, we have from the first equation of system above $w(t) = w(t_n^+)e^{-\gamma(t-t_n)}$. Therefore we get $w(t_{n+1}^+) = w(t_n^+)e^{-\gamma\tau} + h_m$. The coefficient of discrete state $w(t_n^+)$ that is given by the expression $e^{-\gamma\tau}$ which is less than one in magnitude. We deduce that the sequence $(w(t_n^+))_n$ converges to a fixed point, that is $w(t_n^+) \rightarrow \frac{h_m}{1-e^{-\gamma\tau}}$. Finally, we obtain the following periodic solution of $w(t) = \frac{h_m e^{-\gamma(t-t_n)}}{1-e^{-\gamma\tau}} \leq \frac{h_m}{1-e^{-\gamma\tau}}$. Therefore, from the comparison principle [69], we get $v(t) \leq w(t) \leq \frac{h_m}{1-e^{-\gamma\tau}}$. Hence the the state v is bounded. Since the function $1 - \xi \leq q(v) \leq 1$ then by the comparison principle and using the same approach as in the chapter 4, the subsystem defines by the s , y and z variables has a unique nonnegative and bounded solution. Consequently, we have the following result:

Theorem 15 *The system (5.1) has a unique nonnegative and bounded solution on $[0, t_f]$.*

5.3.3 Optimality conditions

In this section, we begin by show the existence of an optimal solution of optimisation control problem (5.5) and derive the characterisation of an optimal solution using the maximum principle method [112, 113]. The existence of such optimal control application is guaranteed by the following result:

Theorem 16 *There exists an optimal control application $h^* = (h_n^*)_{n \in [0, N_f] \cap \mathbb{N}^*}$ belongs to \mathcal{M} , which maximises the objective functional $\mathcal{J}(\cdot)$ subject to System (5.1).*

Proof : Since the state variables of System (5.1) are bounded, then it follows that the objective functional $\mathcal{J}(\cdot)$ is finite. So, it is possible to define by $d = \sup_{h \in \mathcal{M}} \mathcal{J}(h)$ and thus, there is a maximising sequence $(h^j)_{j \in \mathbb{N}}$ so that the sequence $(\mathcal{J}(h^j))_{j \in \mathbb{N}}$ converges to d . Since the set \mathcal{M} is compact, then there exists a subsequence still denoted by $(h^j)_{j \in \mathbb{N}}$ that converges to $h^* = (h_n^*)_{n \in [0, N_f] \cap \mathbb{N}^*}$ so that h^* belongs to \mathcal{M} . As the objective functional $\mathcal{J}(\cdot)$ is continuous, then it follows that $\mathcal{J}(h^*) = d$. This achieves the proof. \square

Follows the framework in [113, 112], we derive the optimal control application through the combination of the state and the adjoints variables. We determine the adjoint equations by first introducing the sensitivity functions. Let us denotes by $X = (s, y, z, v)$ and define the solution map: $h \rightarrow X(h)$. The sensitivity functions are defined by the Gâteaux derivative, $(Q_s, Q_y, Q_z, Q_v) = \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} [X(h + \epsilon \bar{h}) - X(h)]$, for $\bar{h} \in \mathbb{R}^{N_f}$. Consequently, the sensitivity functions $Q_s(t, a)$, $Q_y(t)$, $Q_z(t)$ and $Q_v(t)$ corresponding to the state variables $s(t, a)$, $y(t)$, $z(t)$ and $v(t)$ satisfy the following equations:

$$\left\{ \begin{array}{l} \partial_t Q_s + \partial_a Q_s = -q(v)[\beta(a)\mathcal{W}(B, y)Q_s + \beta s \mathcal{W}_B(B, y)\|Q_s\| + \beta s \mathcal{W}_y(B, y)Q_y] \\ \quad + q_v(v)\mathcal{W}(B, y)\beta s Q_v - \mu Q_s \\ \dot{Q}_y = \phi Q_z - \varepsilon \mathcal{W}(B, y)\|\beta Q_s\| - \varepsilon \|\beta s\| \mathcal{W}_y(B, y)Q_y - \varepsilon \|\beta s\| \mathcal{W}_B(B, y)\|Q_s\| - \mu_y Q_y \\ \dot{Q}_z = q(v)[\varepsilon \mathcal{W}(B, y)\|\beta Q_s\| + \varepsilon \|\beta s\| \mathcal{W}_y(B, y)Q_y + \varepsilon \|\beta s\| \mathcal{W}_B(B, y)\|Q_s\|] \\ \quad - q_v(v)\varepsilon \mathcal{W}(B, y)\|\beta s\| Q_v - \mu_z Q_z \end{array} \right. \quad (5.8)$$

and

$$\left\{ \begin{array}{l} \dot{Q}_v = -\gamma Q_v \quad t \neq t_n \\ Q_v(t_n^+) = Q_v(t_n) + \bar{h}_n \quad t = t_n, \quad n = 0, 1, 2, \dots, N_f \end{array} \right. \quad (5.9)$$

The functions $\mathcal{W}_y(\cdot, \cdot)$, $\mathcal{W}_B(\cdot, \cdot)$ and $q_v(\cdot)$ represent the partial derivative of the function $\mathcal{W}(\cdot, \cdot)$ and $q(\cdot)$ respectively with respect to their arguments y , B and v . The sensitivity functions respect the initial and boundary conditions:

$$\mathcal{Q}_s(t, 0) = 0, \quad \mathcal{Q}_s(0, a) = 0, \quad \mathcal{Q}_y(0) = \mathcal{Q}_z(0) = \mathcal{Q}_v(0) = 0. \quad (5.10)$$

Next, we introduce by $\Phi_s(t, a)$, $\Phi_y(t)$, $\Phi_z(t)$ and $\Phi_v(t)$ the adjoint variables corresponding to the state variables $s(t, a)$, $y(t)$, $z(t)$ and $v(t)$ respectively. Then the adjoint equations are derived by using the adjoint operator associated with the sensitivity equations (5.8)-(5.9) together with appropriate transversality and boundary conditions. Specifically, the adjoint equations are given by:

$$\begin{cases} \partial_t \Phi_s + \partial_a \Phi_s = q(v)[\beta \mathcal{W}(B, y)(\Phi_s - \varepsilon \Phi_z) + \mathcal{W}_B(B, y)(\|\beta s \Phi_s\| - \varepsilon \|\beta s\| \Phi_z)] \\ \quad + [\varepsilon \beta \mathcal{W}(B, y) + \varepsilon \|\beta s\| \mathcal{W}_B(B, y)] \Phi_y + \mu \Phi_s \\ \dot{\Phi}_y = q(v) \mathcal{W}_y(B, y) [-\varepsilon \|\beta s\| \Phi_z + \|\beta s \Phi_s\|] + \varepsilon \|\beta s\| \mathcal{W}_y(B, y) \Phi_y + \mu_y \Phi_y \\ \dot{\Phi}_z = -\phi \Phi_y + \mu_z \Phi_z \\ \dot{\Phi}_v = q_v(v) \mathcal{W}(B, y) [-\|\beta s \Phi_s\| + \varepsilon \|\beta s\| \Phi_z] + \gamma \Phi_v, \end{cases} \quad (5.11)$$

and:

$$\Phi_v(t_n^+) = \Phi_v(t_n) \quad \text{for } n = 0, 1, 2, \dots, N_f, \quad (5.12)$$

with the transversality conditions associated to the adjoint variables by

$$\Phi_s(t_f, a) = \Theta(a); \quad \Phi_s(t, a_\dagger) = 0; \quad \Phi_y(t_f) = -C_y; \quad \Phi_y(t_f) = \Phi_v(t_f) = 0. \quad (5.13)$$

Therefore, we have the following result about the characterisation of each optimal control application strategy.

Theorem 17 *For any optimal control application $h = (h_n)_{n \in [0, N_f] \cap \mathbb{N}^*} \in \mathcal{M}$, then the Gâteaux derivative of $\mathcal{J}(\cdot)$ denoted $D_{\bar{h}} \mathcal{J}(h) = \sum_{n=0}^{N_f} [\Phi_v(t_n^+) - C'(h_n)] \bar{h}_n$, where $C'(\cdot)$ represents the derivation of the function $C(\cdot)$ with respect to its argument \bar{h}_n and $\bar{h} \in \mathcal{V}_h$, with the set $\mathcal{V}_h := \{\bar{h} \in \mathbb{R}^{N_f+1} : \exists \epsilon \in [0, 1]; h + \epsilon \bar{h} \in \mathcal{M}\}$.*

Proof : For any given $\bar{h} \in \mathcal{V}_h$, there is $\epsilon > 0$ small enough, such that $h^\epsilon := h + \epsilon \bar{h} \in \mathcal{M}$. Let $s^\epsilon(t, a)$, $y^\epsilon(t)$, $z^\epsilon(t)$, and $v^\epsilon(t)$ be the solution of system (5.1) corresponding to the control strategy h^ϵ . Then, the directional derivative of objective functional $\mathcal{J}(\cdot)$ is given by

$$\begin{aligned} D_{\bar{h}} \mathcal{J}(h) &= \lim_{\epsilon \rightarrow 0^+} \epsilon^{-1} [\mathcal{J}(h + \epsilon \bar{h}) - \mathcal{J}(h)] \\ &= \int_0^{a_\dagger} \Theta(a) \mathcal{Q}_s(t_f, a) da - \sum_{n=1}^{N_f} C'(h_n) \bar{h}_n - \xi_y \mathcal{Q}_y(t_f). \end{aligned} \quad (5.14)$$

Multiplying the equations of the sensitivity system (5.8) by the adjoint state variables $\Phi_s(t, a)$, $\Phi_y(t)$, $\Phi_z(t)$ and $\Phi_v(t)$ respectively, and also the equations (5.11) by the sensitivities $\mathcal{Q}_v(t, a)$,

$\mathcal{Q}_y(t)$, $\mathcal{Q}_z(t)$ and $\mathcal{Q}_v(t)$ respectively. We derive the following relation:

$$\begin{aligned}
 \int_Q \mathcal{Q}_s [\partial_t \Phi_s + \partial_a \Phi_s] dadt &+ \int_0^{t_f} \mathcal{Q}_y \dot{\Phi}_y dt + \int_0^{t_f} \mathcal{Q}_z \dot{\Phi}_z dt + \int_0^{t_f} \mathcal{Q}_v \dot{\Phi}_v dt \\
 &+ \int_Q \Phi_s [\partial_t \mathcal{Q}_s + \partial_a \mathcal{Q}_s] dadt + \int_0^{t_f} \Phi_y \dot{\mathcal{Q}}_y dt + \int_0^{t_f} \Phi_z \dot{\mathcal{Q}}_z dt \\
 &+ \int_0^{t_f} \Phi_v \dot{\mathcal{Q}}_v dt = 0.
 \end{aligned} \tag{5.15}$$

We have the following relation:

$$\begin{aligned}
 \int_0^{t_f} \Phi_v \dot{\mathcal{Q}}_v dt &= \sum_{n=0}^{N_f-1} \int_{t_n}^{t_{n+1}} \Phi_v \dot{\mathcal{Q}}_v dt + \int_{t_{N_f}}^{t_f} \Phi_v \dot{\mathcal{Q}}_v dt \\
 &= \sum_{n=0}^{N_f-1} [\Phi_v \mathcal{Q}_v]_{t_n^+}^{t_{n+1}^+} - \sum_{n=0}^{N_f-1} \int_{t_n}^{t_{n+1}} \dot{\Phi}_v \mathcal{Q}_v dt + [\Phi_v \mathcal{Q}_v]_{t_{N_f}^+}^{t_f} - \int_{t_{N_f}}^{t_f} \dot{\Phi}_v \mathcal{Q}_v dt \\
 &= \sum_{n=0}^{N_f-1} \Phi_v(t_{n+1}) \mathcal{Q}_v(t_{n+1}) - \sum_{n=0}^{N_f} \Phi_v(t_n^+) \mathcal{Q}_v(t_n^+) - \int_0^{t_f} \dot{\Phi}_v \mathcal{Q}_v dt \quad (\text{since } \Phi_v(t_f) = 0) \\
 &= \sum_{n=0}^{N_f-1} \Phi_v(t_{n+1}) \mathcal{Q}_v(t_{n+1}) - \sum_{n=0}^{N_f} \Phi_v(t_n^+) [\mathcal{Q}_v(t_n) + \bar{h}_n] - \int_0^{t_f} \dot{\Phi}_v \mathcal{Q}_v dt \\
 &= - \sum_{n=0}^{N_f} \Phi_v(t_n^+) \bar{h}_n - \int_0^{t_f} \dot{\Phi}_v \mathcal{Q}_v dt \quad (\text{since } \Phi_v(t_n) = \Phi_v(t_n^+) \text{ and } \Phi_v(t_0) = 0).
 \end{aligned} \tag{5.16}$$

Integrating the equation (5.15) and taking into account the initial and boundary conditions of equations (5.8)–(5.11) and also the relation (5.16), we get the relation:

$$\int_0^{a^\dagger} \Theta(a) \mathcal{Q}_s(t_f, a) da - C_y \mathcal{Q}_y(t_f) - \sum_{n=0}^{N_f} \Phi_v(t_n^+) \bar{h}_n = 0. \tag{5.17}$$

Therefore,

$$\int_0^{a^\dagger} \Theta(a) \mathcal{Q}_s(t_f, a) da - \sum_{n=0}^{N_f} C'(h_n) \bar{h}_n - C_y \mathcal{Q}_y(t_f) = - \sum_{n=0}^{N_f} C'(h_n) \bar{h}_n + \sum_{n=0}^{N_f} \Phi_v(t_n^+) \bar{h}_n. \tag{5.18}$$

Hence $D_{\bar{h}} \mathcal{J}(h) = \sum_{n=0}^{N_f} [\Phi_v(t_n^+) - C'(h_n)] \bar{h}_n$ for all $\bar{h} \in \mathcal{V}_h$. This achieves the proof. \square

The method used to prove the theorem above is called *the impulsive maximum principle* which gives the first order necessary optimality conditions of control application strategy [112, 113], but does not provide an efficient way to compute it. We know suppose that the cost of control effort, *i.e.* ηh_n where η is the cost of a control unit, is proportional to the control effort at each instant of application t_n which is expressed by the quantity of control applied at each instant t_n . Then, we assume in this case that, the cost function $C(h_n) = \eta h_n^2$ and the objective functional (5.4) is quadratic in control application. For this particular consideration of cost function $C(\cdot)$, the following proposition provides a means for computing the structure of optimal control strategy.

Proposition 5 *There exists an optimal control application $h^* = (h_n^*)_{n \in [0, N_f] \cap \mathbb{N}^*} \in \mathcal{M}$ in which*

corresponds the optimal states $s^*(t, a)$, $y^*(t)$, $z^*(t)$ and $v^*(t)$ maximising the objective functional $\mathcal{J}(\cdot)$ defined in (5.4). Moreover the characterisation of optimal control h^* is given by

$$h_n^* = \max \left\{ 0, \min \left\{ \frac{\Phi_{v^*}(t_n^+)}{2\eta}, h_{\max} \right\} \right\}, \quad \forall n \in [0, N_f] \cap \mathbb{N}^*. \quad (5.19)$$

Proof : If $h^* = (h_n^*)_{n \in [0, N_f] \cap \mathbb{N}^*} \in \mathcal{M}$ is an optimal impulsive control application which maximises the objective functional $\mathcal{J}(\cdot)$ and denote by $s^*(t, a)$, $y^*(t)$, $z^*(t)$ and $v^*(t)$ be the state variables corresponding to the control application strategy h^* . Then, for an arbitrary but fixed $\bar{h} \in \mathcal{V}_{h^*}$ and there is $\epsilon > 0$ small enough such that $h^* + \epsilon \bar{h} \in \mathcal{V}$, we have $\mathcal{J}(h^* + \epsilon \bar{h}) \leq \mathcal{J}(h^*)$. It is derived from Theorem 17 that

$$D_{\bar{h}} \mathcal{J}(h^*) = \sum_{n=0}^{N_f} [\Phi_{v^*}(t_n^+) - 2\eta h_n^*] \bar{h}_n \leq 0. \quad (5.20)$$

In the case where $\Phi_{v^*}(t_n^+) < 2\eta h_n^*$, then inequality (5.20) requires that $\bar{h}_n \in \mathbb{R}_+$, which necessarily leads to $h^* = 0$. Similarly, if $\Phi_{v^*}(t_n^+) > 2\eta h_n^*$, then the relation (5.20) requires that $\bar{h}_n \in \mathbb{R}_-$, which corresponds to the impulsive control strategy $h^* = h_{\max}$. Finally, for $0 < h_n^* < h_{\max}$, for all $n \in \{0, 1, 2, \dots, N_f\}$, the variation \bar{h}_n can be modified and still stay inside the bounds. Thus in this case, we obtain $\Phi_{v^*}(t_n^+) - 2\eta h_n^* = 0$, so that $h_n^* = \frac{\Phi_{v^*}(t_n^+)}{2\eta}$. By taking the lower and upper bounds into account, we obtain the desired optimal impulsive control (5.19). This achieves the proof. \square

5.4 Numerical simulations

In this section, we present the numerical simulations illustrating the effect of the optimal control strategy on the infestation process of coffee berries by CBB in the plantation. Our numerical approach is based on an extension of the forward-backward sweep method [74], originally proposed to solve the optimal control of continuous ordinary differential equations with impulsive controls. The procedure consists of the following steps: first, the state equations in (5.1) are solved using the forward semi-implicit finite difference in time and backward difference in age, with an initial guess for the control variable. Second, the adjoint equations in (5.11)-(5.13) are solved by backward semi-implicit finite difference in time and forward difference in age, using the solutions of the state equations. Next, the control is updated with the new values of the state and adjoint solutions given by Proposition 5. The algorithm is repeated until the states and control converge.

In the numerical simulations, we assume that the control used here represents the entomopathogenic fungus *Beauveria Bassiana*. We consider that the infestation function and coffee berry price are the same as in Chapter 4. Most parameter values used in the simulations are given in Table 1 and in Table 6.

Moreover, we use the following initial conditions. At the beginning of the cropping season, flowering has not started yet. We also assume that there are only colonising females, as infesting females were eliminated from the plantation when berries were picked during the preceding harvest. So there are initially neither coffee berries nor infesting females, *i.e.* $s_0(a) = 0$ for all $a \in [0, a_+]$ and $z(0) = 0$. The number of initial colonising females is set at $y(0) = 10^4$ females. We also assume that the control is absent in the plantation so the initial load $v(0) = 0$. We

Table 6: Model and control parameter values (collected from Table 1 in Chapter 2 and Table 4 in Chapter 4).

Symbol	Description	Value
t_f	Duration of a cropping season	250 days
a_{\dagger}	maximum age of coffee berry	250 days
λ	Production rate of new coffee berries	1200 berries.day ⁻¹
μ	Natural mortality rate of healthy coffee berries	0.002 day ⁻¹
ε	Colonising CBB per berry (scaling factor)	1 female.berry ⁻¹
$\beta(a)$	Infestation function (4.36):	day ⁻¹
β_{\min}	minimum infestation rate	0.004 day ⁻¹
β_a	age-dependent extra infestation rate	0.036 day ⁻¹
k_{β}	infestation coefficient	0.035 day ⁻¹
a_{β}	infestation threshold age	90 days
α	CBB–berry interaction constant	0.7 female.berry ⁻¹
ϕ	Emergence rate of new colonising females	2 day ⁻¹
μ_y	Natural mortality rate of colonising females	1/20 day ⁻¹
μ_z	Natural mortality rate of infesting females	1/27 day ⁻¹
ξ	Fungus load maximal effectiveness	0.8
k	Fungus load half saturation constant	200 g.day ⁻¹
γ	Fungus decay rate	1/50 day ⁻¹
η	Cost of fungus	0.0015 \$.g ⁻²
τ	period of the control application	variable
h_{\max}	Maximal fungus application rate in the plantation	180 g
$\Theta(a)$	Coffee berry price function (4.37):	\$.berry ⁻¹
$\bar{\Theta}$	coffee berry asymptotic price	0.025 \$.berry ⁻¹
a_{Θ}	berry age at half asymptotic price	120 days
n	Hill constant	7 –
C_y	Cost of remaining colonising females	10 ⁻⁴ \$.female ⁻¹

consider two periods of fungus application in the plantation during the cropping season: $T_{per} = 7$ days (1 week) in Figure 31 and $T_{per} = 14$ days (2 weeks) in Figure 32.

There are six panels in Figures 31 and 32. Panel (a) represents the evolution of cumulative healthy coffee berries $B(t) = \int_0^{a_{\dagger}} s(t, a) da$ during the cropping season, without CBB (black curves), with CBB and optimal impulsive control application (red and magenta) and with CBB and without control. Similarly, panels (b) and (c) represent the dynamics of colonising $y(t)$ and infesting $z(t)$ CBB females respectively, with (red and magenta curves) and without (blue curves) optimal fungus application. Panel (d) represents the evolution of the fungus load $v(t)$ in the plantation. Panel (e) represents the optimal fungus application h and finally the panel (f) the effect of fungus load that saturates through function $\sigma(v) = 1 - q(v) = \frac{\xi v}{k+v}$.

In Figures 31 and 32, trajectories of colonising and infesting females increase, both in the uncontrolled (dashed blue curves) and controlled (red and magenta curves) case. Due to the impact of age preference of coffee berries by CBB, this growth of the females starts slowly. In all cases, the fungus (subfigures (e)) is applied at its maximal value at the beginning of the cropping season and decreases progressively. Its effect (subfigures (f)) rapidly increases but remains below its maximum value ($\xi = 0.8$). The colonising (subfigures (b), red curves) and infesting (subfigures (c), red curves) female trajectories are much lower than in the case without control (dashed blue curves). Consequently the healthy coffee berries (subfigures (a), red curve for Figure 31 and magenta curve for Figure 32) increase considerably the end of cropping season. The control is more effective when applied weekly (Figure 31) than every two weeks (Figure 32), both in

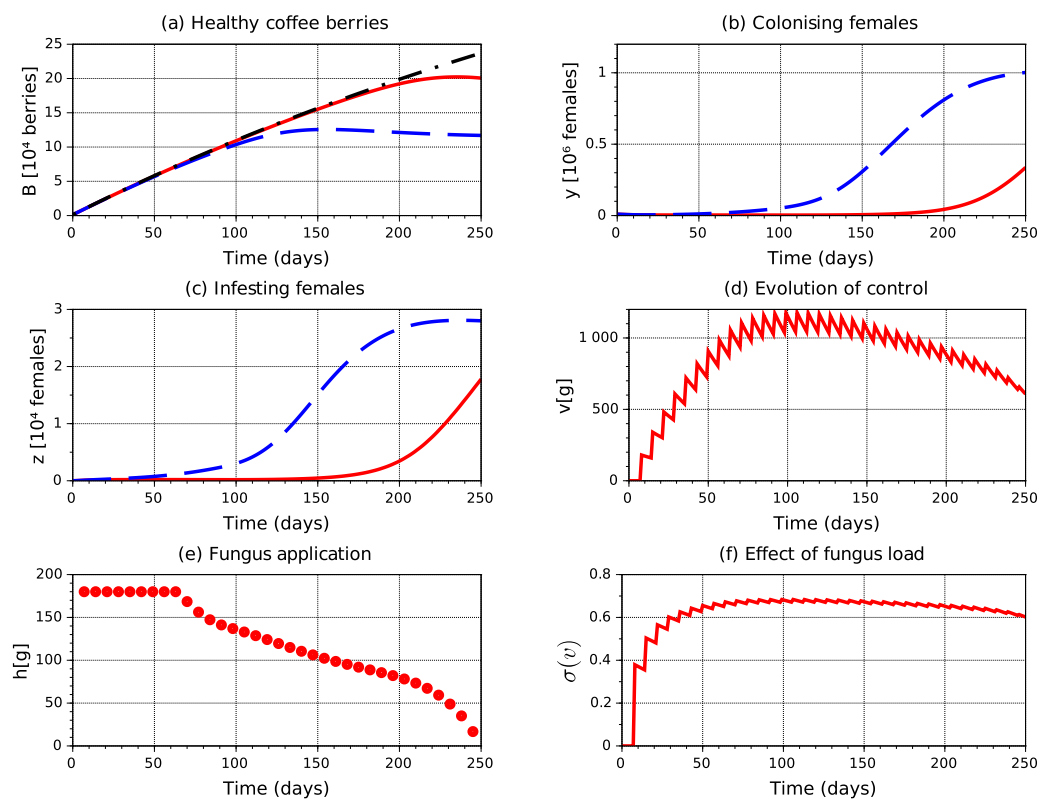


Figure 31: Simulation of system (4.3) without control (dashed blue curves) and system (5.1) with weekly discrete optimal fungus application (plain red curves). The healthy berry growth without pest is also represented in (a) (dash-dotted black curve).

terms of healthy berries (subfigures (a)) and colonising females (subfigures (b)) at the end of the cropping season.

In Table 7, we summarise the plantation state at the end of the cropping season (for $t_f = 250$ days) without and with optimal control. The optimal fungus application considerably increases the penalised profit and the yield, for weekly and every two weeks applications. We also aim at reducing the CBB population at the end of the cropping season in order to reduce the impact of CBB for the next cropping season yield. The optimal application of the fungus significantly reduces the final CBB colonising female population by 66.35% and 38.4% for the fungus application every week and 2 weeks respectively. The effect of the fungus load on the CBB population increases when the period of application is lower and consequently provides a more effective protection of the coffee berries and a significant increase of the penalised profit and yield.

Table 7: Penalised profit (\mathcal{J}), yield (\mathcal{Y}), control cost (\mathcal{C}) and colonising CBB females remaining in the plantation after harvest at the end of the cropping season ($t_f = 250$ days), without control and with optimal fungus application every week and 2 weeks.

Intervention	\mathcal{J} (US\$)	\mathcal{Y} (US\$)	\mathcal{C} (US\$)	$y(t_f)$
Without control	311	412	0	1002008
With control $T_{per} = 1$ week	1865	1905	7	337129
With control $T_{per} = 2$ weeks	1313	1378	4	617363

Penalised profit/financial gain in the pest-free case: $\mathcal{J} = \mathcal{Y} = 2644.1$ US\$.

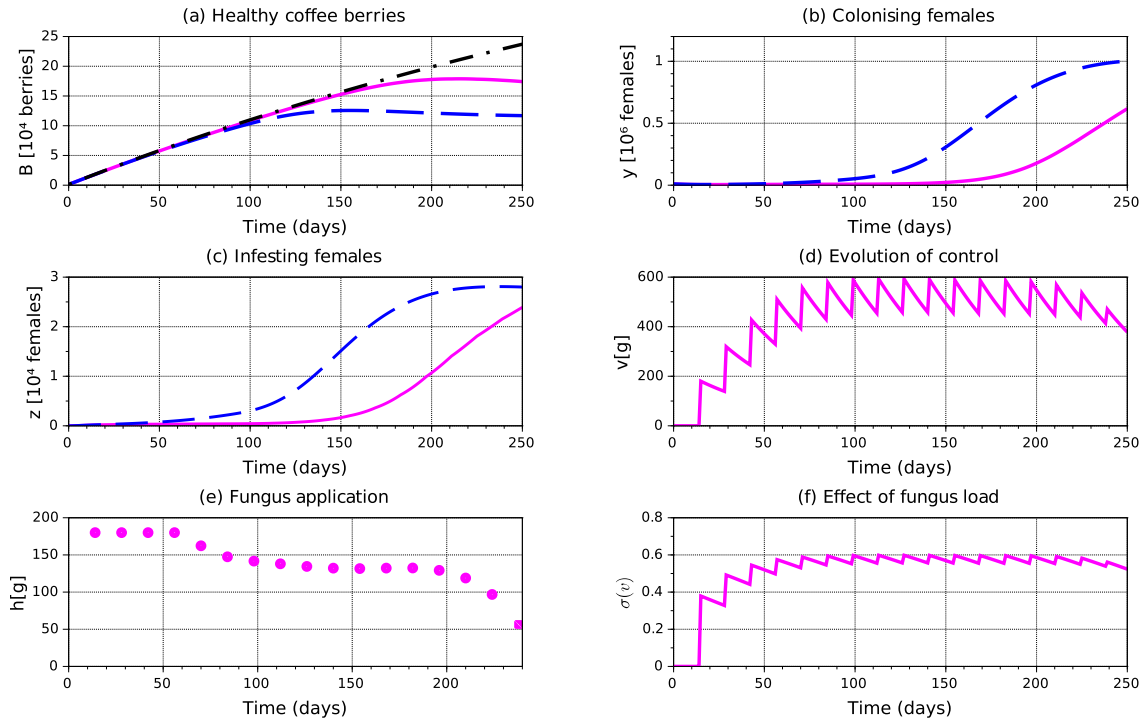


Figure 32: Simulation of System (4.3) without control (dashed blue curves) and System (5.1) with discrete optimal fungus application every 2 weeks (plain magenta curves). The healthy berry growth without pest is also represented in (a) (dash-dotted black curve).

Figure 33 illustrates the evolution of the final berry density and price of coffee berries as functions of the berry age at harvest for fungus application every week (red curves) and every 2 weeks (magenta curves). We observe a strong decrease of the mature coffee berries (≥ 90 days) without and with control (left panel) due to the berry age preference of CBB. However, the price distribution of berries increases with berry age but remains considerably lower than in the case without CBB (right panel).

5.5 Conclusion

In view of the extent of damage caused by CBB in the plantations, we formulated in this work a mathematical model to control the dynamics of the infestation of coffee berries by CBB. The control is based on the application at discrete and periodic moments of the bio-pesticide. The role of this control is to prevent CBB from infesting healthy berries and killing them during the infestation process. The model with this control is a semidiscrete system with a continuous berry age structure to represent the CBB preference for mature berries. An optimisation problem was designed in order to maximise the yield of healthy coffee berries while minimising the cost of control as well as the remaining CBB population for the next cropping season. We showed the existence of an optimal control and gave its characterisation using the maximum principle. Numerical simulations indicate that the application of the bio-pesticide effectively controls the CBB and increases considerably the yield and profit at the end of cropping season. Weekly applications are more efficient than applications every two weeks, both in terms of CBB

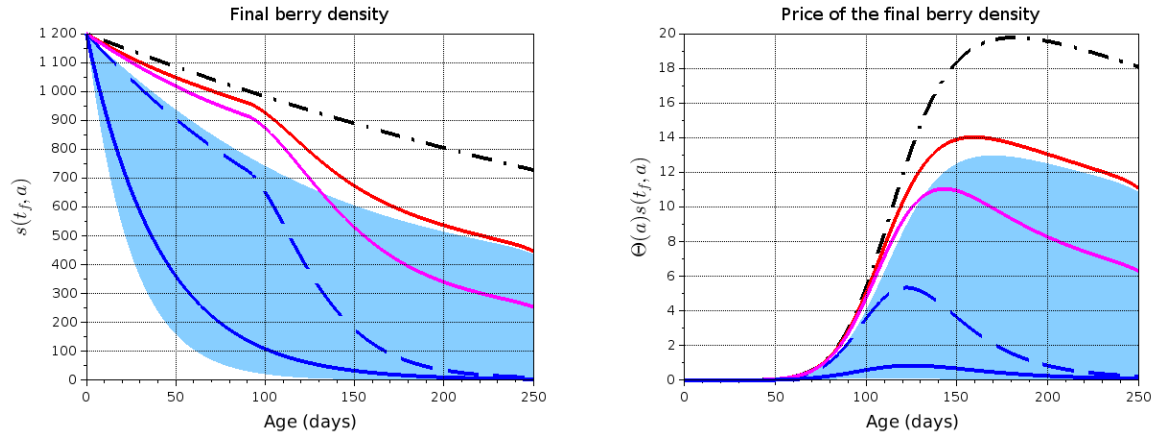


Figure 33: Age distribution of the healthy coffee berries and their price (yield) at the end of the simulation ($t = t_f$) with optimal fungus application (plain red curves when $T_{per} = 1$ week and plain magenta curves when $T_{per} = 2$ weeks), without control (dashed blue curves) and without pest (dash-dotted black curves). Cases without control and with constant infestation rate β are also represented (blue shaded area delimited by β_{\min} and $\beta_{\max} = \beta_{\min} + \beta_a$ plain curves for mean value $\bar{\beta}$).

population reduction and profit maximisation.

For more realism, it would be interesting to consider that the cost function $C(h_n) = \eta h_n$, *i.e.* the objective functional (5.4) is linear in control. Then, proceeding as the proof of the Proposition 5, we obtain the directional derivative $D_{\bar{h}} \mathcal{J}(h^*) = \sum_{n=1}^{N_f} [\Phi_{v^*}(t_n^+) - \eta] \bar{h}_n$ which does not depend explicitly of control h_n^* . In this case the optimal control $h_n^* = h_{\max}$ when $\Phi_{v^*}(t_n^+) > \eta$ and $h_n^* = 0$ when $\Phi_{v^*}(t_n^+) < \eta$. If the optimal control switches between its maximum value and minimum value on interval $I \subset [0, t_f]$ or vice versa, then the control is then said to be *bang-bang*. Otherwise, it is very complex to analyse the case where $D_{\bar{h}} \mathcal{J}(h^*) = 0$, which could imply a *singular* control. This case remains an open question.

General conclusion and perspectives

The aim of this thesis was to develop mathematical models that describe the infestation of coffee berries by CBB and their impact on the coffee production, in order to design cost-effective control strategies. To achieve this goal, we developed several dynamical models of growing complexity, that describe the berry-CBB interactions and take into account the CBB life cycle and berry growth. The baseline model (Chapter 2) is an ODE system that represents the coffee berry-CBB dynamics during a cropping season. It was extended to represent several cropping seasons using a semi-discrete formalism (Chapter 3), or to include the CBB preference for mature berries by means of a berry age structure and PDE formalism (Chapters 4 and 5). All these models were tailored for the pathosystem of interest, but kept tractable and thoroughly analysed to study their qualitative behaviour. Promising control results were obtained, in particular when a bio-insecticide reducing berry infestation was combined with traps that capture colonising CBB (Chapter 4).

To limit the CBB population, we chose in this thesis to focus on environmentally friendly control methods. Hence, during the cropping seasons, biopesticides such as the entomopathogenic fungus *Beauveria bassiana* (Chapters 2, 4 and 5) and traps (Chapter 4) are preferred to chemical insecticides such as endosulfan. These latter may be less costly and more efficient in the short term, but they have severely negative consequences on the environment as well as public health and their extensive use induces insecticide resistance [26, 25]. Moreover, biopesticides are compatible with organic farming and coffee hence produced can be sold at a higher price.

Other sustainable control methods can be applied to limit the CBB population and their damages on the coffee berries [62]. Several biocontrol agents have been identified, in particular parasitoid wasps originated from Africa, the CBB being also native of this continent. *Phymastichus coffea*, for instance, lays its eggs in CBB colonising females as they bore their hole to enter the berry and hence can effectively reduce infestation [63]. *Cephalonomia stephanoderis* and *Prorops nasuta* penetrate inside the infested berries to lay their eggs in CBB larvae [68, 72, 60]; they may also feed on CBB eggs, larvae or young adults. Their interactions with CBB are somehow different from the operating mode of entomopathogenic fungi such as *Beauveria Bassiana*, as they target immature stages within the berries, so the controls would need to be adapted in the model.

As the spraying of entomopathogenic fungi or the introduction of biocontrol agents cannot realistically be performed continuously, such controls should rather be applied periodically at discrete times, as in Chapter 5. Moreover, we showed in Chapter 5 (Section 2.5.4) that optimising

a continuous control may be a good starting point, but that discrete control may induce larger profits, provided that fairly large quantities can be applied at a time.

All these biocontrol methods limit the CBB spread during the cropping seasons. They should be complemented by good cultural practises. First and foremost, an effective field sanitation should be implemented to remove infected berries remaining in the plantation after harvest [11, 29], as in the multi-seasonal model of Chapter 3.

Based on these results, it would be interesting to introduce environmentally friendly control methods during the cropping seasons in the multi-seasonal model, to optimise the CBB complementary controls during and between cropping seasons, and to assess their efficiency over longer time horizons. This would allow to determine sustainable management strategies. Our aim would still be to maximise the profit, *i.e.* the berry yield minus the control costs, but over several cropping seasons. Whether the final CBB population should still be minimised in this case remains open. This penalty was introduced when optimising over a single cropping season (Chapters 2, 4 and 5), to prevent too high an infestation for the next season. On the one hand, it might still be necessary to avoid relaxing the control effort at the end of the time horizon. On the other hand, if the horizon is long enough to represent the whole production period of the plantation (up to 50 years or so), the penalty might be avoided. Optimising on the longer run, we might need to introduce a discount rate on the profit, to reflect grower's impatience or decreasing marginal values.

In this thesis, open loop optimal control was implemented. Closed loop control would be preferable for robustness purposes, but state variables, in particular CBB population levels are not easily accessible. Through trapping though, one might get a fair idea of the colonising CBB density, which could be used to design closed loop (optimal) controls.

Last but not least, it would be interesting to confront our model to field or experimental data. We conducted a thorough literature survey to calibrate our model and incorporated all the qualitative and quantitative knowledge we could gather, but CBB-related epidemiological data are relatively scarce. To go further, we would need to work with field experts, who could critically review our modelling assumptions and parameterisation. If a direct application of our promising theoretical results on CBB control is not realistic, together with these experts, we could for instance set up experiments in the field to implement and assess some strategies that we identified as cost-effective. Moreover, the optimisation criterion that we used could probably be refined, together with economic experts or stakeholders.

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